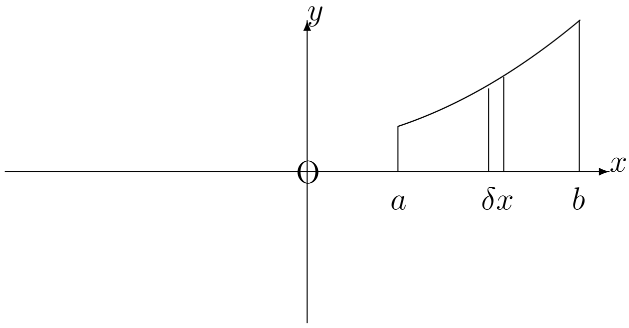


The region may divided up into small elements by using a network, consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

But all of the elements in a narrow 'strip' of width δx and height y (parallel to the y-axis) have the same perpendicular distance, x, from the y-axis.

Hence the first moment of this strip about the y-axis is x times the area of the strip; that is, $x(y\delta x)$, implying that the total first moment of the region about the y-axis is given by

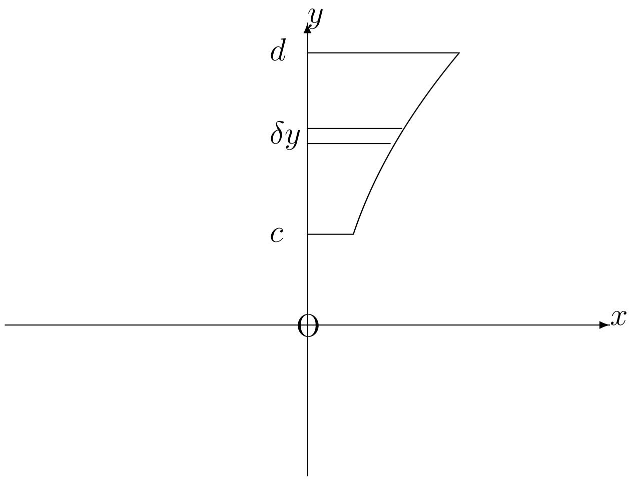
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy \, \mathrm{d}x.$$

Note:
First moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment about the x-axis is given by

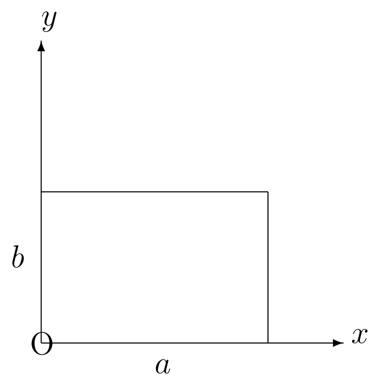
$$\int_c^d yx \, \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a rectangular region, with sides of lengths a and b about the side of length b.

   Solution



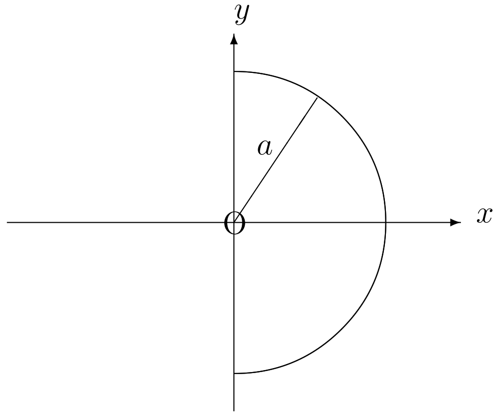
The first moment about the y-axis is given by

$$\int_0^a xb \, \mathrm{d}x = \left[\frac{x^2 b}{2}\right]_0^a = \frac{1}{2}a^2 b.$$

2. Determine the first moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the first moment about the y-axis is given by

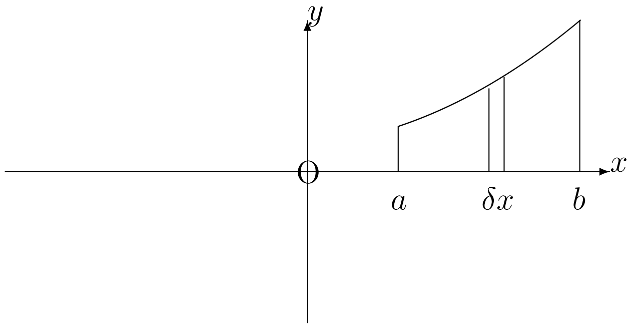
$$2\int_0^a x\sqrt{a^2 - x^2}\ \mathrm{d}x = \left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}a^3.$$

Note:
Although first moments about the x-axis will be discussed mainly in the next section of this Unit, we note that the symmetry of the above region shows that its first moment about the x-axis would be zero; this is because, for each $y(x\delta y)$, there will be a corresponding $-y(x\delta y)$ in calculating the first moments of the strips parallel to the x-axis.

13.7.3 FIRST MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the first moment of a rectangular region about one of its sides. This result may now be used to determine the first moment about the x-axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow strip, of width δx and height y, is regarded as approximately a rectangle, its first moment about the x-axis is $\frac{1}{2}y^2\delta x$. Hence, the first moment of the whole region about the x-axis is given by

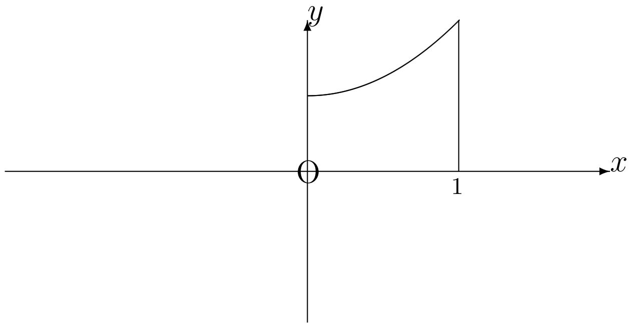
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{2}y^2\delta x = \int_a^b \frac{1}{2}y^2 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the first moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution

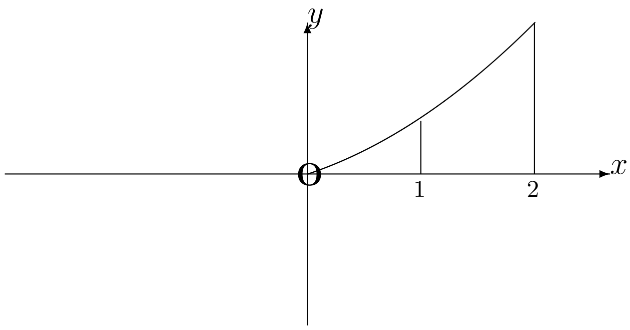


First moment $= \int_0^1 \frac{1}{2}(x^2+1)^2 \, \mathrm{d}x = \frac{1}{2}\int_0^1 (x^4+2x^2+1) \, \mathrm{d}x = \frac{1}{2}\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^1 = \frac{28}{15}.$

2. Determine the first moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the lines $x = 1$, $x = 2$ and the curve

$$y = xe^x.$$

Solution



$$\text{First moment } = \int_1^2 \frac{1}{2}x^2e^{2x} \, \mathrm{d}x$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \int_1^2 xe^{2x}\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \left[x\frac{e^{2x}}{2}\right]_1^2 + \int_1^2 \frac{e^{2x}}{2}\,\mathrm{d}x\right).$$

That is,

$$\frac{1}{2}\left[x^2\frac{e^{2x}}{2} - x\frac{e^{2x}}{2} + \frac{e^{2x}}{4}\right]_1^2 = \frac{5e^4 - e^2}{8} \simeq 33.20$$

13.7.4 THE CENTROID OF AN AREA

Having calculated the first moments of a two dimensional region about both the x-axis and the y-axis, it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $A\overline{x}$, where A is the total area of the region;

and

(b) The first moment about the x-axis is given by $A\overline{y}$, where A is the toal area of the region.

The point is called the **"centroid"** or the **"geometric centre"** of the region and, in the case of a region bounded, in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b xy\,\mathrm{d}x}{\int_a^b y\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b \frac{1}{2}y^2\,\mathrm{d}x}{\int_a^b y\,\mathrm{d}x}.$$

Notes:

(i) The first moment of an area, about an axis through its centroid will, by definition, be zero. In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δA, to the y-axis, the first moment about the given axis will be

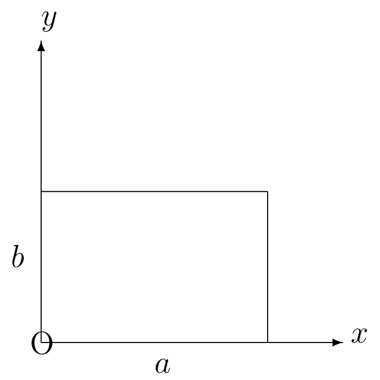
$$\sum_{\mathrm{R}} (x - \overline{x})\delta A = \sum_{\mathrm{R}} x\delta A - \overline{x}\sum_{\mathrm{R}} \delta A = A\overline{x} - A\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole area at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass for a thin plate with uniform density, whose shape is that of the region which we have been considering.

EXAMPLES

1. Determine the position of the centroid of a rectangular region with sides of lengths, a and b.

   Solution



The area of the rectangle is ab and, from Example 1 in section 13.7.2, the first moments about the y-axis and the x-axis are $\frac{1}{2}a^2b$ and $\frac{1}{2}b^2a$, respectively.

Hence,

$$\overline{x} = \frac{\frac{1}{2}a^2b}{ab} = \frac{1}{2}a$$

and

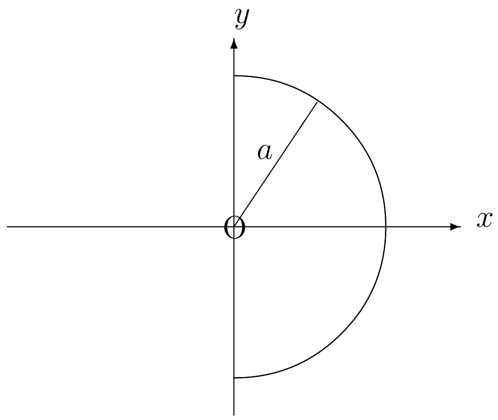
$$\overline{y} = \frac{\frac{1}{2}b^2a}{ab} = \frac{1}{2}b,$$

as we would expect for a rectangle.

2. Determine the position of the centroid of the semi-circular region bounded, in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



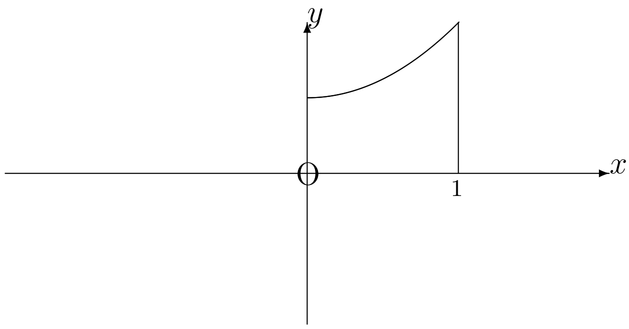
The area of the semi-circular region is $\frac{1}{2}\pi a^2$ and so, from Example 2, in section 13.7.2,

$$\overline{x} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \quad \text{and} \quad \overline{y} = 0.$$

3. Determine the position of the centroid of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



The first moment about the y-axis is given by

$$\int_0^1 x(x^2+1)\ \mathrm{d}x = \left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^1 = \frac{3}{4}.$$

The area is given by

$$\int_0^1 (x^2+1)\ \mathrm{d}x = \left[\frac{x^3}{3}+x\right]_0^1 = \frac{4}{3}.$$

Hence,

$$\overline{x} = \frac{3}{4} \div \frac{4}{3} = 1.$$

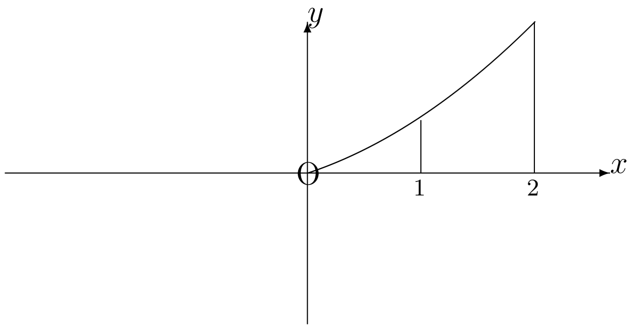
The first moment about the x-axis is $\frac{28}{15}$, from Example 1 in section 13.7.3; and, therefore,

$$\overline{y} = \frac{28}{15} \div \frac{4}{3} = \frac{7}{5}.$$

4. Determine the position of the centroid of the region bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = xe^x.$$

Solution



The first moment about the y-axis is given by

$$\int_1^2 x^2e^x \, \mathrm{d}x = \left[x^2e^x - 2xe^x + 2e^x\right]_1^2 \simeq 12.06,$$

using integration by parts (twice).
The area is given by

$$\int_1^2 xe^x \, \mathrm{d}x = [xe^x - e^x]_1^2 \simeq 7.39$$

using integration by parts (once).
Hence,

$$\overline{x} \simeq 12.06 \div 7.39 \simeq 1.63$$

The first moment about the x-axis is approximately 33.20, from Example 2 in section 13.7.3; and so,

$$\overline{y} \simeq 33.20 \div 7.39 \simeq 4.47$$

13.7.5 EXERCISES

Determine the position of the centroid of each of the following regions of the xy-plane:

1. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$x^2 + y^2 = a^2.$$

2. Bounded by the line $x = 1$ and the semi-circle whose equation is

$$(x-1)^2 + y^2 = 4, \ \ x > 1.$$

3. Bounded in the fourth quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 2x^2 - 1.$$

4. Bounded in the first quadrant by the x-axis and the curve whose equation is

$$y = \sin x.$$

5. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = xe^{-2x}.$$

13.7.6 ANSWERS TO EXERCISES

1.

$$\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right).$$

2.

$$\left(\frac{11}{3\pi}, 0\right).$$

3.

$$\left(\frac{3\sqrt{2}}{16}, -\frac{13}{20}\right).$$

4.

$$\left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

5.

$$(0.28, 0.04).$$

"JUST THE MATHS"

UNIT NUMBER

13.8

INTEGRATION APPLICATIONS 8
(First moments of a volume)

by

A.J.Hobson

UNIT 13.8 - INTEGRATION APPLICATIONS 8

FIRST MOMENTS OF A VOLUME

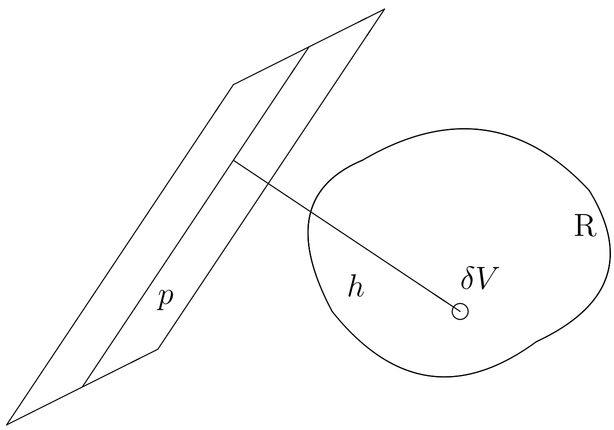
13.8.1 INTRODUCTION

Suppose that R denotes a region of space (with volume V) and suppose that δV is the volume of a small element of this region.

Then the **"first moment"** of R about a fixed plane, p, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h\delta V,$$

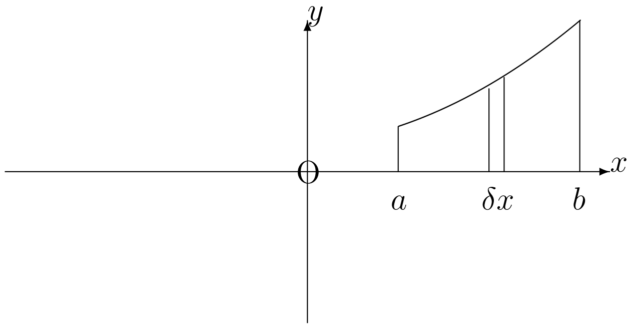
where h is the perpendicular distance, from p, of the element with volume, δV.



13.8.2 FIRST MOMENT OF A VOLUME OF REVOLUTION ABOUT A PLANE THROUGH THE ORIGIN, PERPENDICULAR TO THE X-AXIS.

Let us consider the volume of revolution about the x-axis of a region, in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



For a narrow 'strip' of width, δx, and height, y, parallel to the y-axis, the volume of revolution will be a thin disc with volume $\pi y^2 \delta x$ and all the elements of volume within it have the same perpendicular distance, x, from the plane about which moments are being taken.

Hence the first moment of this disc about the given plane is x times the volume of the disc; that is, $x(\pi y^2 \delta x)$, implying that the total first moment is given by

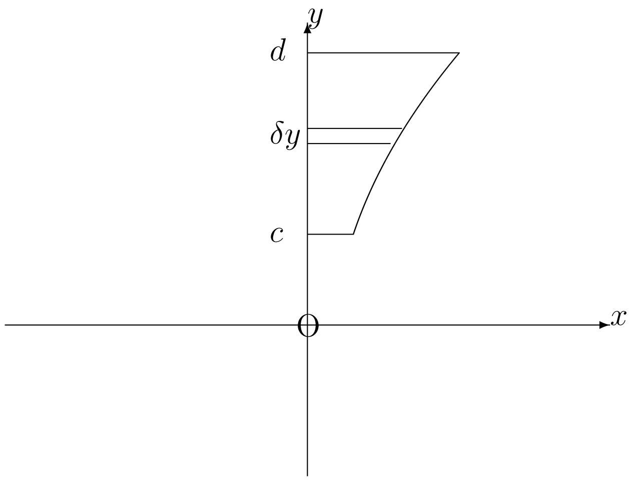
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi x y^2 \delta x = \int_a^b \pi x y^2 \ \mathrm{d}x.$$

Note:
For the volume of revolution about the y-axis of a region in the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment of the volume about a plane through the origin, perpendicular to the y-axis, is given by

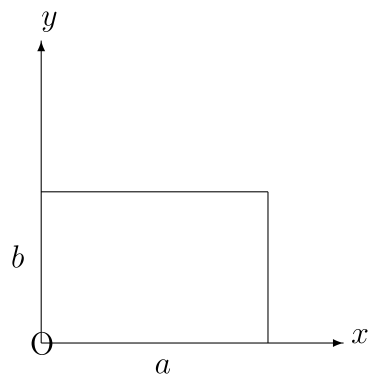
$$\int_c^d \pi y x^2 \ \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a solid right-circular cylinder with height, a and radius b, about one end.

   Solution



Let us consider the volume of revolution about the x-axis of the region, bounded in the first quadrant of the xy-plane, by the x-axis, the y-axis and the lines $x = a$, $y = b$.

The first moment of the volume about a plane through the origin, perpendicular to the x-axis, is given by

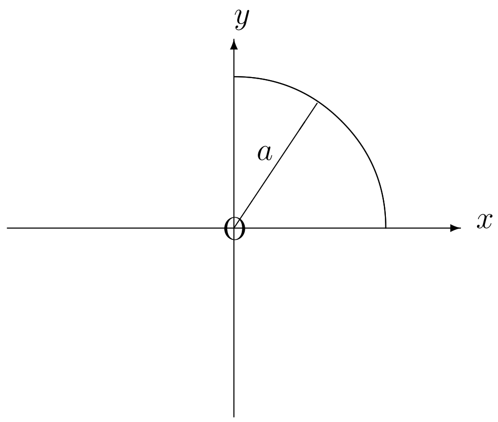
$$\int_0^a \pi x b^2 \, \mathrm{d}x = \left[\frac{\pi x^2 b^2}{2}\right]_0^a = \frac{\pi a^2 b^2}{2}.$$

2. Determine the first moment of volume, about its plane base, of a solid hemisphere with radius a.

   Solution

   Let us consider the volume of revolution about the x-axis of the region, bounded in the first quadrant, by the x-axis, y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$



The first moment of the volume about a plane through the origin, perpendicular to the x-axis is given by

$$\int_0^a \pi x(a^2 - x^2) \, \mathrm{d}x = \left[\pi(\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)\right]_0^a = \pi\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{\pi a^4}{4}.$$

Note:
The symmetry of the solid figures in the above two examples shows that their first moments about a plane through the origin, perpendicular to the y-axis would be zero. This is because, for each $y\delta V$ in the calculation of the total first moment, there will be a corresponding $-y\delta V$.

In much the same way, the first moments of volume about the xy-plane (or indeed any plane of symmetry) would also be zero.

13.8.3 THE CENTROID OF A VOLUME

Suppose R denotes a volume of revolution about the x-axis of a region of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$

Having calculated the first moment of R about a plane through the origin, perpendicular to the x-axis (assuming that this is not a plane of symmetry), it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $V\overline{x}$, where V is the total volume of revolution about the x-axis.

The point is called the **"centroid"** or the **"geometric centre"** of the volume, and $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b \pi x y^2 \, \mathrm{d}x}{\int_a^b \pi y^2 \mathrm{d}x} = \frac{\int_a^b x y^2 \, \mathrm{d}x}{\int_a^b y^2 \mathrm{d}x}.$$

Notes:

(i) The centroid effectively tries to concentrate the whole volume at a single point for the purposes of considering first moments. It will always lie on the line of intersection of any two planes of symmetry.

(ii) In practice, the centroid corresponds to the position of the centre of mass for a solid with uniform density, whose shape is that of the volume of revolution which we have been considering.

(iii) For a volume of revolution about the y-axis, from $y = c$ to $y = d$, the centroid will lie on the y-axis, and its distance, $\overline{y}$, from the origin will be given by

$$\overline{y} = \frac{\int_c^d \pi y x^2 \, \mathrm{d}y}{\int_c^d \pi x^2 \mathrm{d}y} = \frac{\int_c^d y x^2 \, \mathrm{d}y}{\int_c^d x^2 \mathrm{d}y}.$$

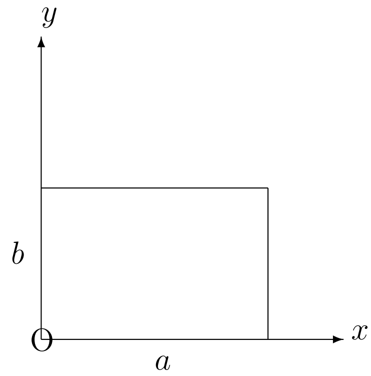
(iv) The first moment of a volume about a plane through its centroid will, by definition, be zero. In particular, if we take the plane through the y-axis, perpendicular to the x-axis to be parallel to the plane through the centroid, with x as the perpendicular distance from an element, δV, to the plane through the y-axis, the first moment about the plane through the centroid will be

$$\sum_{\mathrm{R}} (x - \overline{x})\delta V = \sum_{\mathrm{R}} x\delta V - \overline{x}\sum_{\mathrm{R}} \delta V = V\overline{x} - V\overline{x} = 0.$$

EXAMPLES

1. Determine the position of the centroid of a solid right-circular cylinder with height, a, and radius, b.

   Solution



Using Example 1 in Section 13.8.2, the centroid will lie on the x-axis and the first moment about a plane through the origin, perpendicular to the x-axis is $\frac{\pi a^2 b^2}{2}$.

Also, the volume is $\pi b^2 a$.

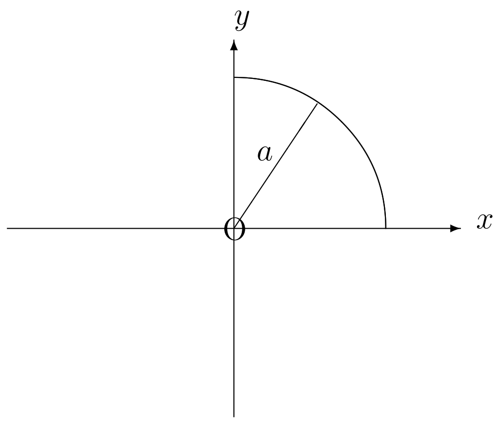
Hence,

$$\overline{x} = \frac{\frac{\pi a^2 b^2}{2}}{\pi b^2 a} = \frac{a}{2},$$

as we would expect for a cylinder.

2. Determine the position of the centroid of a solid hemisphere with base-radius, a.

Solution



Let us consider the volume of revolution about the x-axis of the region bounded in the first quadrant by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2$$

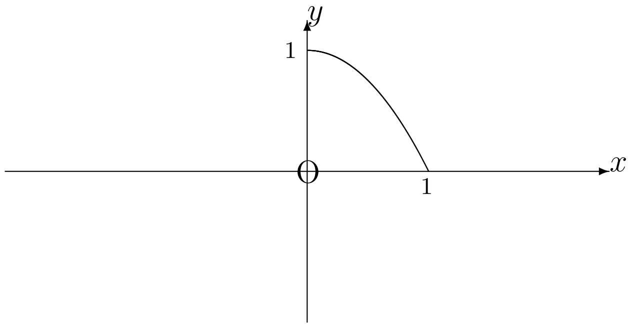
From Example 2 in Section 13.8.2, the centroid will lie on the x-axis and the first moment of volume about a plane through the origin, perpendicular to the x-axis is $\frac{\pi a^4}{4}$. Also, the volume of the hemisphere is $\frac{2}{3}\pi a^3$ and so,

$$\overline{x} = \frac{\frac{2}{3}\pi a^3}{\frac{\pi a^4}{4}} = \frac{3a}{8}.$$

3. Determine the position of the centroid of the volume of revolution about the y-axis of region bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - x^2.$$

Solution



Firstly, by symmetry, the centroid will lie on the y-axis.

Secondly, the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\int_0^1 \pi y(1-y)\ \mathrm{d}y = \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}.$$

Thirdly, the volume is given by

$$\int_0^1 \pi(1-y)\ \mathrm{d}y = \left[y - \frac{y^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

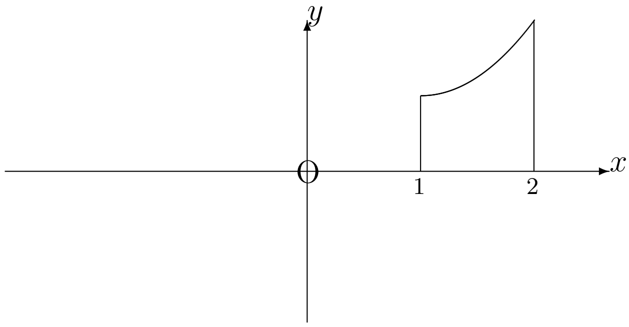
Hence,

$$\overline{y} = \frac{\pi}{6} \div \frac{\pi}{2} = \frac{1}{3}.$$

4. Determine the position of the centroid of the volume of revolution about the x-axis of the region, bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = e^x.$$

Solution



Firstly, by symmetry, the centroid will lie on the x axis.

Secondly, the First Moment about a plane through the origin, perpendicular to the x-axis is given by

$$\int_1^2 \pi x e^{2x}\ \mathrm{d}x = \pi\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]_1^2 \simeq 122.84,$$

using integration by parts.

The volume is given by

$$\int_1^2 \pi e^{2x}\ \mathrm{d}x = \pi\left[\frac{e^{2x}}{2}\right]_1^2 \simeq 74.15$$

Hence,

$$\overline{x} \simeq 122.84 \div 74.15 \simeq 1.66$$

13.8.4 EXERCISES

1. Determine the position of the centroid of the volume obtained when each of the following regions of the xy-plane is rotated through 2π radians about the x-axis:

   (a) Bounded in the first quadrant by the x-axis, the line $x = 1$ and the quarter-circle represented by

   $$(x-1)^2 + y^2 = 4,\ \ x > 1, y > 0;$$

(b) Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - 2x^2;$$

(c) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = \frac{\pi}{2}$ and the curve whose equation is

$$y = \sin x;$$

(d) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = \sqrt{x}e^{-x}.$$

2. A solid right-circular cone, whose vertex is at the origin, has, for its central axis, the part of the y-axis between $y = 0$ and $y = h$. Determine the position of the centroid of the cone.

13.8.5 ANSWERS TO EXERCISES

1. (a)

$$\overline{x} = 1.75;$$

(b)

$$\overline{x} \simeq 0.22;$$

(c)

$$\overline{x} \simeq 1.10;$$

(d)

$$\overline{x} \simeq 0.36$$

2.

$$\overline{y} = \frac{3h}{4}.$$

"JUST THE MATHS"

UNIT NUMBER

13.9

INTEGRATION APPLICATIONS 9
(First moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.9 - INTEGRATION APPLICATIONS 9

FIRST MOMENTS OF A SURFACE OF REVOLUTION

13.9.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"first moment"** about a plane through the origin, perpendicular to the x-axis, is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} 2\pi x y \delta s,$$

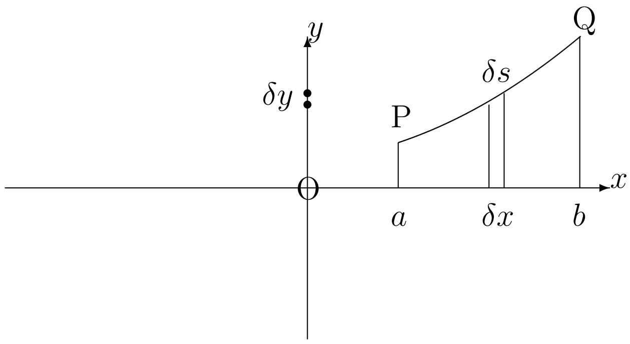
where x is the perpendicular distance, from the plane of moments, of the thin band, with surface area $2\pi y \delta s$, so generated.

13.9.2 INTEGRATION FORMULAE FOR FIRST MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x;$$

so that, for the surface of revolution of the arc about the x-axis, the first moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi xy\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b 2\pi xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the first moment about the plane through the origin, perpendicular to the x-axis is given by

$$\text{First Moment} = \pm\int_{t_1}^{t_2} 2\pi xy\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

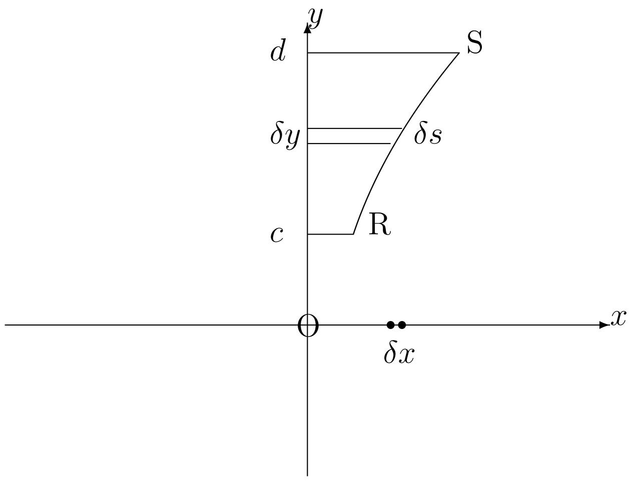
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\int_c^d 2\pi yx\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the first moment about a plane through the origin, perpendicular to the y-axis, is given by

$$\text{First moment} = \pm \int_{t_1}^{t_2} 2\pi yx \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

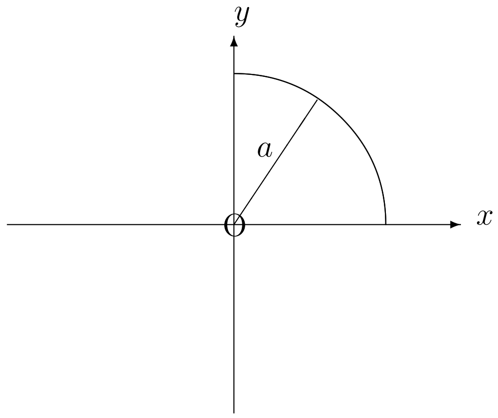
EXAMPLES

1. Determine the first moment about a plane through the origin, perpendicular to the x-axis, for the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.
The first moment about the specified plane is therefore given by

$$\int_0^a 2\pi xy\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi xy\sqrt{\frac{x^2+y^2}{y^2}}\ \mathrm{d}x.$$

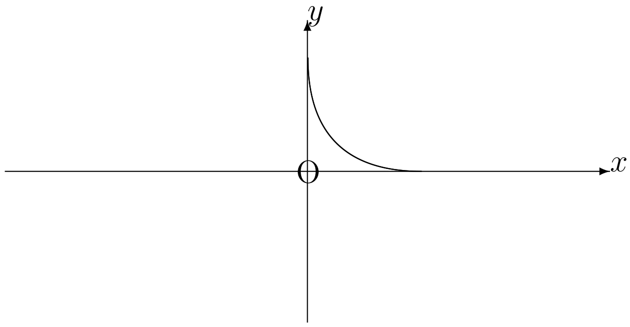
But $x^2 + y^2 = a^2$, and so the first moment becomes

$$\int_0^a 2\pi ax\ \mathrm{d}x = [\pi ax^2]_0^a = \pi a^3.$$

2. Determine the first moments about planes through the origin, (a) perpendicular to the x-axis and (b) perpendicular to the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi xy\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^2\cos^3\theta\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 6\pi a^3\cos^4\theta\sin^4\theta\ \mathrm{d}\theta.$$

Using $2\sin\theta\cos\theta \equiv \sin 2\theta$, the integral reduces to

$$\frac{3\pi a^3}{8}\int_0^{\frac{\pi}{2}} \sin^4 2\theta\ \mathrm{d}\theta,$$

which, by the methods of Unit 12.7, gives

$$\frac{3\pi a^3}{32}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right)\ \mathrm{d}\theta = \frac{3\pi a^3}{32}\left[\frac{3\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16}\right]_0^{\frac{\pi}{2}} = \frac{9\pi a^3}{128}.$$

By symmetry, or by direct integration, the first moment about a plane through the origin, perpendicular to the y-axis is also $\frac{9\pi a^3}{128}$.

13.9.3 THE CENTROID OF A SURFACE OF REVOLUTION

Having calculated the first moment of a surface of revolution about a plane through the origin, perpendicular to the x-axis, it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $S\overline{x}$, where S is the total surface area.

The point is called the **"centroid"** or the **"geometric centre"** of the surface of revolution and, for the surface of revolution of the arc of the curve whose equation is $y = f(x)$, between $x = a$ and $x = b$, the value of $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b \ 2\pi xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \ \mathrm{d}x}{\int_a^b \ 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x} \quad = \frac{\int_a^b \ xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x}{\int_a^b \ y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \ \mathrm{d}x}.$$

Note:
The centroid effectively tries to concentrate the whole surface at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass of a thin sheet, for example, with uniform density.

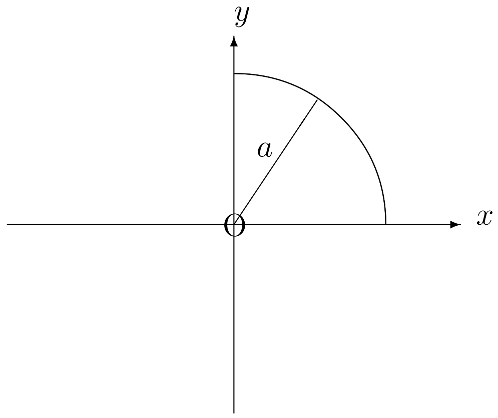
EXAMPLES

1. Determine the position of the centroid of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 of Section 13.9.2, we know that the first moment of the surface about a plane through the origin, perpendicular to the the x-axis is equal to πa^3.

Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2,$$

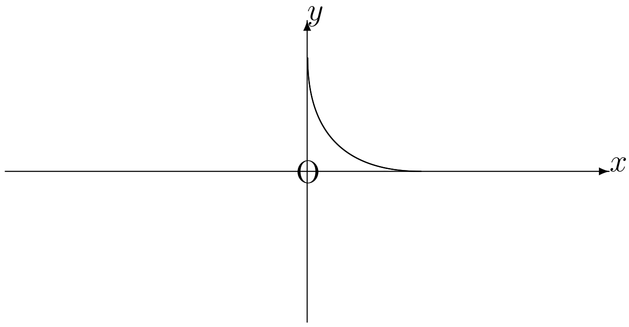
which implies that

$$\overline{x} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}.$$

2. Determine the position of the centroid of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 of Section 13.9.2 that that the first moment of the surface about a plane through the origin, perpendicular to the x-axis is equal to $\frac{9\pi a^3}{128}$.
Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a\sin^3\theta.3a\cos\theta\sin\theta \, \mathrm{d}\theta = \int_{0}^{\frac{\pi}{2}} 3a^2\sin^4\theta\cos\theta \, \mathrm{d}\theta = 3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$\overline{x} = \frac{15a}{128}.$$

13.9.4 EXERCISES

1. Determine the first moment, about a plane through the origin, perpendicular to the x-axis, of the surface of revolution (about the x-axis) of the straight-line segment joining the origin to the point $(3, 4)$.

2. Determine the first moment about a plane through the origin, perpendicular to the x-axis, of the surface of revolution (about the x-axis) of the arc of the curve whose equation is

$$y^2 = 4x,$$

lying between $x = 0$ and $x = 1$.

3. Determine the first moment about a plane through the origin, perpendicular to the y-axis, of the surface of revolution (about the y-axis) of the arc of the curve whose equation is

$$y^2 = 4(x - 1),$$

lying between $y = 2$ and $y = 4$.

4. Determine the first moment, about a plane through the origin, perpendicular to the y-axis, of the surface of revolution (about the y-axis) of the arc of the curve whose parametricc equations are

$$x = 2\cos t, \quad y = 3\sin t,$$

joining the point $(2, 0)$ to the point $(0, 3)$.

5. Determine the position of the centroid of a hollow right-circular cone with height h.

6. For the curve whose equation is

$$9y^2 = x(3 - x)^2,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$

Hence, show that the centroid of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x-axis lies at the point $\left(\frac{5}{4}, 0\right)$.

13.9.5 ANSWERS TO EXERCISES

1.

$$40\pi.$$

2.

$$4\pi\left[\frac{12\sqrt{2}}{5}-\frac{4}{15}\right]\simeq 39.3$$

3.

$$\left[\frac{8\pi}{5}\left(1+\frac{y^2}{4}\right)^{\frac{5}{2}}\right]_2^4\simeq 41.98$$

4.

$$\left[-\frac{4\pi}{5}\left(4+5\cos^2 t\right)^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}}\simeq 47.75$$

5. Along the central axis, at a distance of $\frac{2h}{3}$ from the vertex.

6.

$$\text{First moment}=\frac{15\pi}{4}\quad\text{Surface Area}=3\pi.$$

"JUST THE MATHS"

UNIT NUMBER

13.10

INTEGRATION APPLICATIONS 10
(Second moments of an arc)

by

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UNIT 13.10 - INTEGRATION APPLICATIONS 10

SECOND MOMENTS OF AN ARC

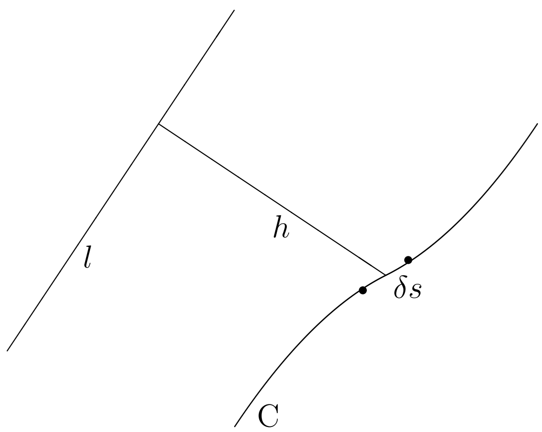
13.10.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then the **"second moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} h^2 \delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

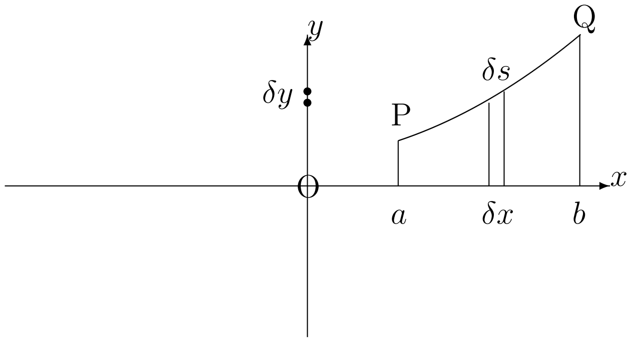


13.10.2 THE SECOND MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The second moment of each element about the y-axis is x^2 times the length of the element; that is, $x^2\delta s$, implying that the total second moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\text{C}} x^2 \delta s.$$

But, by Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

so that the second moment of arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b x^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the second moment of the arc about the y-axis is given by

$$\pm \int_{t_1}^{t_2} x^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.10.3 THE SECOND MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the second moment about the x-axis will be

$$\int_a^b y^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the second moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

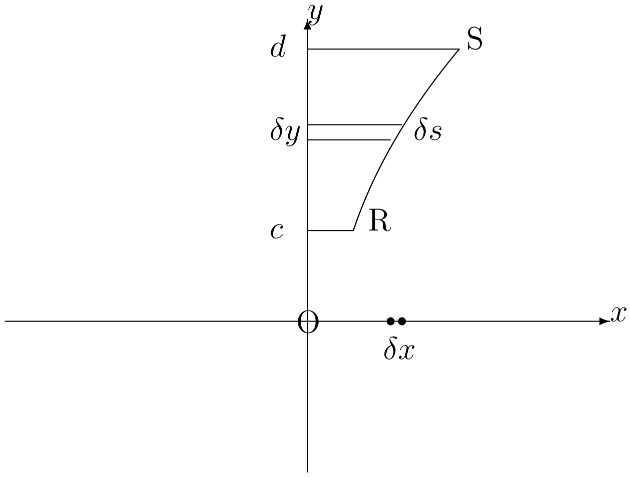
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.10.2 so that the second moment about the x-axis is given by

$$\int_c^d y^2 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the second moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

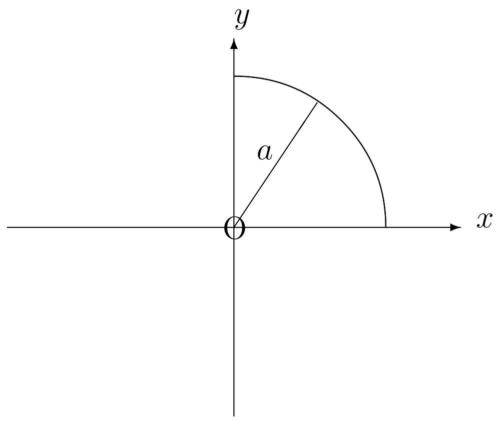
EXAMPLES

1. Determine the second moments about the x-axis and the y-axis of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.
The second moment about the y-axis is therefore given by

$$\int_0^a x^2\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a \frac{x^2}{y}\sqrt{x^2+y^2}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$ and, hence,

$$\text{second moment} = \int_0^a \frac{ax^2}{y}\mathrm{d}x.$$

Making the substitution $x = a\sin u$ gives

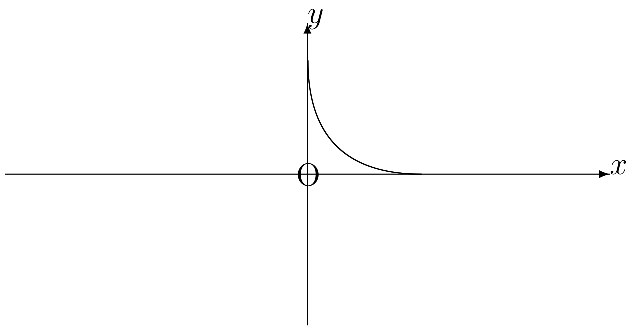
$$\text{second moment} = \int_0^{\frac{\pi}{2}} a^3\sin^2 u \; \mathrm{d}u = a^3\int_0^{\frac{\pi}{2}} \frac{1-\cos 2u}{2}\; \mathrm{d}u = a^3\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^3}{4}.$$

By symmetry, the second moment about the x-axis will also be $\frac{\pi a^3}{4}$.

2. Determine the second moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \;\text{ and }\; \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the second moment about the y-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} x^2\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\; \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a^2\cos^6\theta.3a\cos\theta\sin\theta\; \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \cos^7\theta \sin\theta \ \mathrm{d}\theta$$

$$= 3a^2 \left[-\frac{\cos^8\theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8}.$$

Similarly, the second moment about the x-axis is given by

$$\int_0^{\frac{\pi}{2}} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} a^2\sin^6\theta.(3a\cos\theta\sin\theta) \ \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \sin^7\theta\cos\theta \ \mathrm{d}\theta = 3a^3 \left[\frac{\sin^8\theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8},$$

though, again, this second result could be deduced, by symmetry, from the first.

13.10.4 THE RADIUS OF GYRATION OF AN ARC

Having calculated the second moment of an arc about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by sk^2, where s is the total length of the arc.

We simply divide the value of the second moment by s in order to obtain the value of k^2 and, hence, the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole arc at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

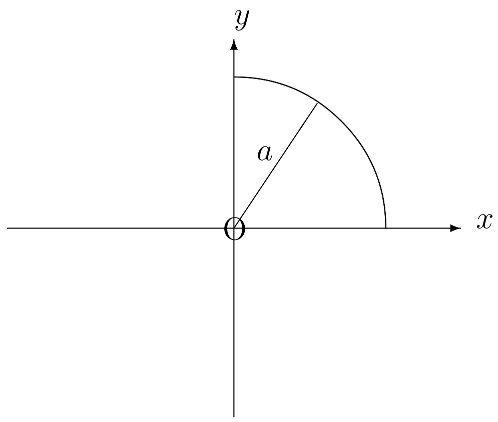
EXAMPLES

1. Determine the radius of gyration, about the y-axis, of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 in Section 13.10.3, we know that the Second Moment of the arc about the y-axis is equal to $\frac{\pi a^3}{4}$.

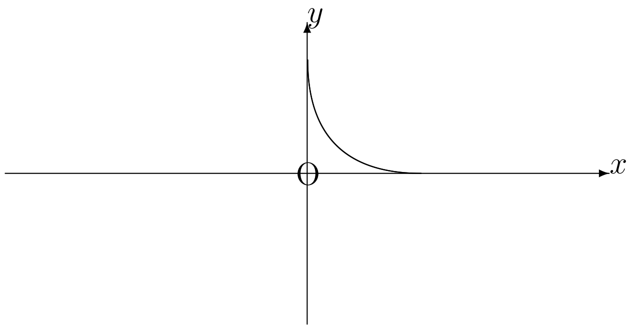
Also, the length of the arc is $\frac{\pi a}{2}$, which implies that the radius of gyration is

$$\sqrt{\frac{\pi a^3}{4} \times \frac{2}{\pi a}} = \frac{a}{\sqrt{2}}.$$

2. Determine the radius of gyration, about the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in Section 13.10.3, we know that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta \sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

and that the second moment of the arc about the y-axis is equal to $\frac{3a^3}{8}$. Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus, the radius of gyration is

$$\sqrt{\frac{3a^3}{8} \times \frac{2}{3a}} = \frac{a}{2}.$$

13.10.5 EXERCISES

1. Determine the second moments about (a) the x-axis and (b) the y-axis of the straight line segment with equation

$$y = 2x + 1,$$

lying between $x = 0$ and $x = 3$.

2. Determine the second moment about the y-axis of the first-quadrant arc of the curve whose equation is

$$25y^2 = 4x^5,$$

lying between $x = 0$ and $x = 2$.

3. Determine, correct to two places of decimals, the second moment, about the x-axis, of the arc of the curve whose equation is

$$y = e^x,$$

lying between $x = 0.1$ and $x = 0.5$.

4. Given that

$$\int \sqrt{x^2 - 1}\ \mathrm{d}x = \frac{1}{2}\left(x\sqrt{x^2 - 1} - \ln(x + \sqrt{x^2 - 1})\right) + C,$$

determine, correct to two places of decimals, the second moment, about the x-axis, of the arc of the curve whose equation is

$$y^2 = 8x,$$

lying between $x = 0$ and $x = 1$.

5. Verify, using integration, that the radius of gyration, about the y-axis, of the straight line segment defined by the equation

$$y = 3x + 2,$$

from $x = 0$ to $x = 1$ is $\frac{1}{\sqrt{3}}$.

6. Determine the radius of gyration about the x-axis of the arc of the circle given parametrically by

$$x = 5\cos\theta, \quad y = 5\sin\theta,$$

from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

7. For the curve whose equation is

$$9y^2 = x(3-x)^2,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-x}{2\sqrt{x}}.$$

Hence, determine, correct to three significant figures, the radius of gyration, about the y-axis, of the first quadrant arch of this curve.

13.10.6 ANSWERS TO EXERCISES

1.

$$\text{(a)} \ \frac{9\sqrt{5}}{2} \quad \text{(b)} \ 12\sqrt{5}.$$

2.

$$\frac{52}{9} \simeq 5.78$$

3.

$$1.29$$

4.

$$8.59$$

5.

$$\text{Second moment} = \frac{\sqrt{10}}{3} \quad \text{Length} = \sqrt{10}.$$

6.

$$k = \sqrt{\frac{125(\pi-2)}{10\pi}} \simeq 2.13$$

7.

$$k \simeq 1.68$$

"JUST THE MATHS"

UNIT NUMBER

13.11

INTEGRATION APPLICATIONS 11
(Second moments of an area (A))

by

A.J.Hobson

UNIT 13.11 - INTEGRATION APPLICATIONS 11

SECOND MOMENTS OF AN AREA (A)

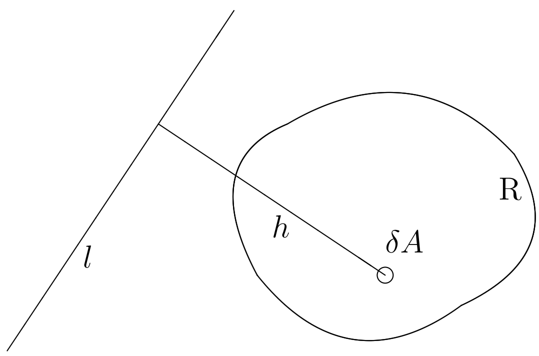
13.11.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy-plane in cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, **not necessarily in the plane of** R, is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h^2 \delta A,$$

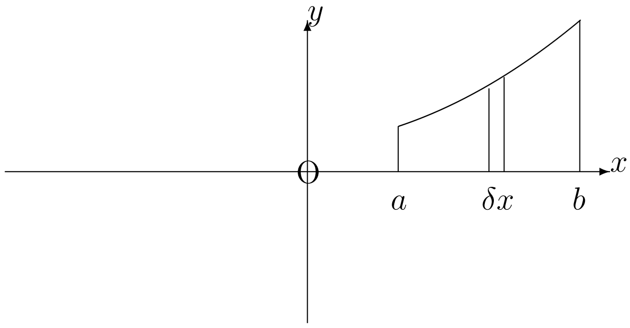
where h is the perpendicular distance from l of the element with area, δA.



13.11.2 THE SECOND MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may be divided up into small elements by using a network consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

But all of the elements in a narrow 'strip', of width δx and height y (parallel to the y-axis), have the same perpendicular distance, x, from the y-axis.

Hence the second moment of this strip about the y-axis is x^2 times the area of the strip; that is, $x^2(y\delta x)$, implying that the total second moment of the region about the y-axis is given by

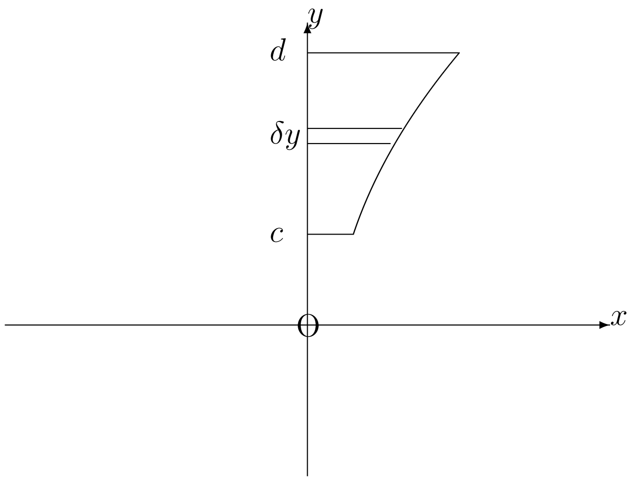
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 y \delta x = \int_a^b x^2 y \, \mathrm{d}x.$$

Note:
Second moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

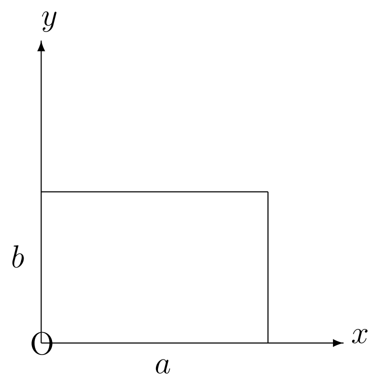
$$\int_c^d y^2 x \, \mathrm{d}y.$$



EXAMPLES

1. Determine the second moment of a rectangular region with sides of lengths, a and b, about the side of length b.

   Solution



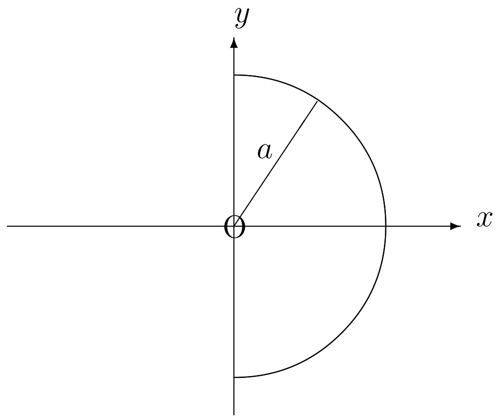
The second moment about the y-axis is given by

$$\int_0^a x^2 b \, \mathrm{d}x = \left[\frac{x^3 b}{3}\right]_0^a = \frac{1}{3}a^3 b.$$

2. Determine the second moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the second moment about the y-axis is given by

$$2\int_0^a x^2\sqrt{a^2 - x^2}\ \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} a^2\sin^2\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

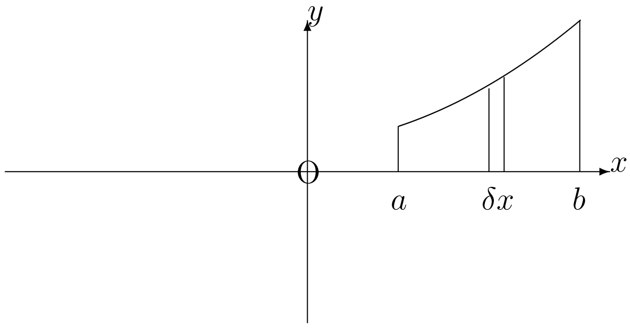
if we substitute $x = a\sin\theta$.
This simplifies to

$$2a^4\int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4}\ \mathrm{d}\theta = \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2}\ \mathrm{d}\theta$$

$$= \frac{a^4}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{8}.$$

13.11.3 THE SECOND MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the second moment of a rectangular region about one of its sides. This result may now be used to determine the second moment about the x-axis, of a region enclosed, in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is $y = f(x)$.



If a narrow strip of width δx and height y is regarded, approximately, as a rectangle, its second moment about the x-axis is $\frac{1}{3}y^3\delta x$. Hence the second moment of the whole region about the x-axis is given by

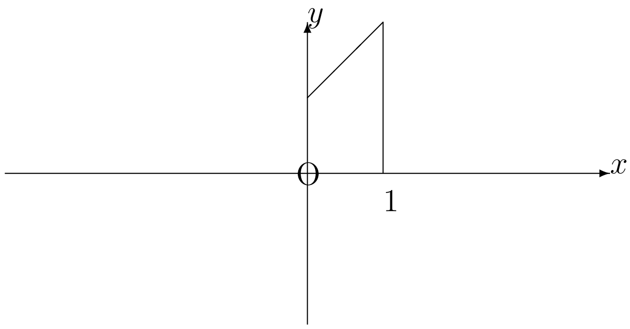
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{3}y^3\delta x = \int_a^b \frac{1}{3}y^3 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the second moment about the x-axis of the region bounded, in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{1}{3}(x+1)^3 \, \mathrm{d}x$$

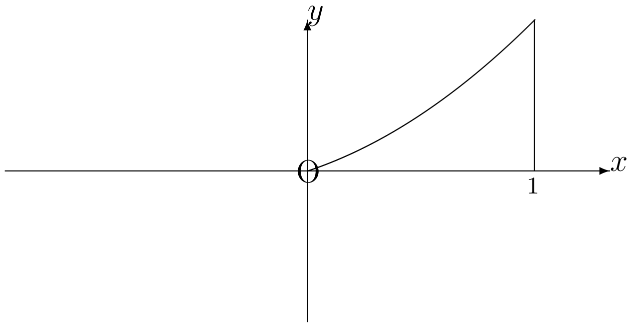
$$= \frac{1}{3}\int_0^1 (x^3 + 3x^2 + 3x + 1) \, \mathrm{d}x = \frac{1}{3}\left[\frac{x^4}{4} + x^3 + + \frac{3x^2}{2} + x\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{4} + 1 + \frac{3}{2} + 1\right) = \frac{5}{4}.$$

2. Determine the second moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve

$$y = xe^x.$$

Solution



$$\text{Second moment } = \int_0^1 \frac{1}{3}x^3 e^{3x}\ \mathrm{d}x$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \int_0^1 x^2 e^{3x}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \int_0^1 2x\frac{e^{3x}}{3}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \frac{2xe^{3x}}{9} - \frac{2}{3}\int_0^1 \frac{e^{3x}}{3}\ \mathrm{d}x\right).$$

That is,

$$\frac{1}{3}\left[x^3\frac{e^{3x}}{3} - x^2\frac{e^{3x}}{3} + \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27}\right]_0^1 = \frac{4e^3+2}{81} \simeq 1.02$$

Note:
The Second Moment of an area about a certain axis is closely related to its **"moment of inertia"** about that axis. In fact, for a thin plate with uniform density, ρ, the moment of inertia is ρ times the second moment of area, since multiplication by ρ, of elements of area, converts them into elements of mass.

13.11.7 EXERCISES

Determine the second moment of each of the following regions of the xy-plane about the axis specified:

1. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
$$y = 1 - 2x^2.$$
Axis: The y-axis.
2. Bounded in the first quadrant by the x-axis and the curve whose equation is
$$y = \sin x.$$
Axis: The x-axis.
3. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is
$$y = e^{-2x}.$$
Axis: The x-axis
4. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is
$$y = e^{-2x}.$$
Axis: The y-axis.

13.11.8 ANSWERS TO EXERCISES

1. $$\frac{\sqrt{2}}{30}.$$
2. $$\frac{4}{9}.$$
3. $$0.055, \text{ approximately.}$$
4. $$0.083, \text{ approximately.}$$

"JUST THE MATHS"

UNIT NUMBER

13.12

INTEGRATION APPLICATIONS 12
(Second moments of an area (B))

by

A.J.Hobson

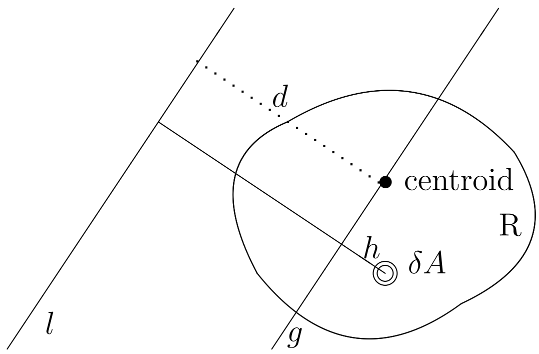
UNIT 13.12 - INTEGRATION APPLICATIONS 12

SECOND MOMENTS OF AN AREA (B)

13.12.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis, in the same plane as R and having a perpendicular distance of d from the first axis.



We have

$$M_l = \sum_{\mathrm{R}} (h+d)^2 \delta A = \sum_{\mathrm{R}} (h^2 + 2hd + d^2).$$

That is,

$$M_l = \sum_{\mathrm{R}} h^2 \delta A + 2d \sum_{\mathrm{R}} h\delta A + d^2 \sum_{\mathrm{R}} \delta A = M_g + Ad^2,$$

since the summation, $\sum_{\mathrm{R}} h\delta A$, is the first moment about the an axis through the centroid and therefore zero; (see Unit 13.7, section 13.7.4).

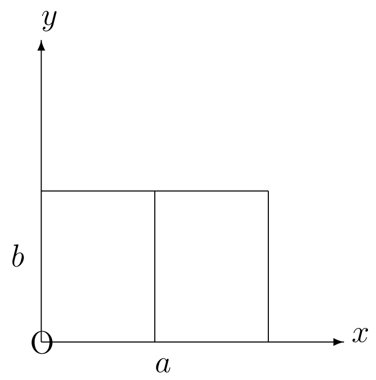
The Parallel Axis Theorem states that

$$M_l = M_g + Ad^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region about an axis through its centroid, parallel to one side.

   Solution



For a rectangular region with sides of length a and b, the second moment about the side of length b is $\frac{a^3b}{3}$ from Example 1 in the previous Unit, section 13.11.2.

The perpendicular distance between the two axes is then $\frac{a}{2}$, so that the required second moment, M_g is given by

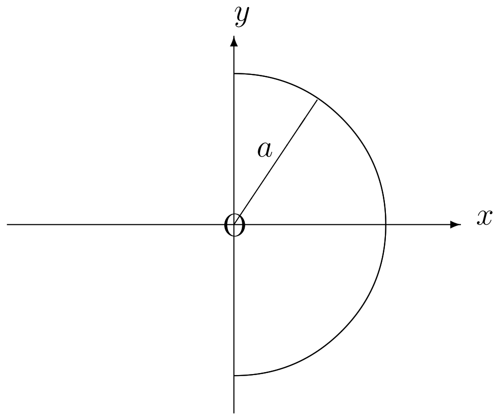
$$\frac{a^3b}{3} = M_g + ab\left(\frac{a}{2}\right)^2 = M_g + \frac{a^3b}{4}$$

Hence,

$$M_g = \frac{a^3b}{12}.$$

2. Determine the second moment of a semi-circular region about an axis through its centroid, parallel to its diameter.

Solution



The second moment of the semi-circular region about its diameter is $\frac{\pi a^4}{8}$, from Example 2 in the previous Unit, section 13.11.2.

Also the position of the centroid, from Example 2 in Unit 13.7, section 13.7.4, is a distance of $\frac{4a}{3\pi}$ from the diameter, along the radius which perpendicular to it.

Hence,

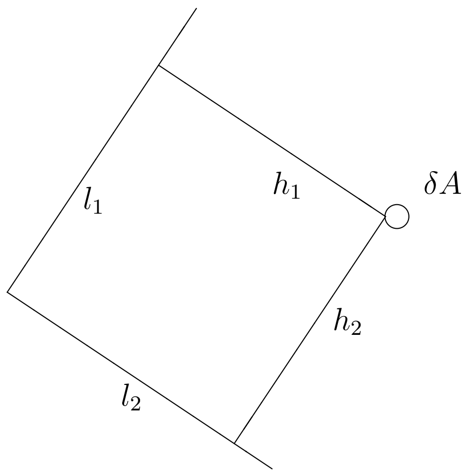
$$\frac{\pi a^4}{8} = M_g + \frac{\pi a^2}{2}.\left(\frac{4a}{3\pi}\right)^2 = M_g + \frac{8a^4}{9\pi^2}.$$

That is,

$$M_g = \frac{\pi a^4}{8} - \frac{8a^4}{9\pi^2}.$$

13.12.2 THE PERPENDICULAR AXIS THEOREM

Suppose l_1 and l_2 are two straight lines, at right-angles to each other, in the plane of a region R with area A and suppose h_1 and h_2 are the perpendicular distances from these two lines, respectively, of an element δA in R.



The second moment about l_1 is given by

$$M_1 = \sum_R h_1^2 \delta A$$

and the second moment about l_2 is given by

$$M_2 = \sum_R h_2^2 \delta A.$$

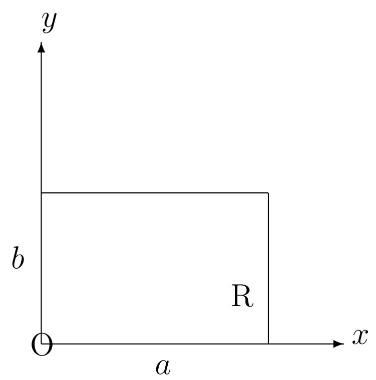
Adding these two together gives the second moment about an axis, perpendicular to the plane of R and passing through the point of intersection of l_1 and l_2. This is because the square of the perpendicular distance, h_3, of δA from this new axis is given, from Pythagoras's Theorem, by

$$h_3^2 = h_1^2 + h_2^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region, R, with sides of length a and b, about an axis through one corner, perpendicular to the plane of R.

Solution

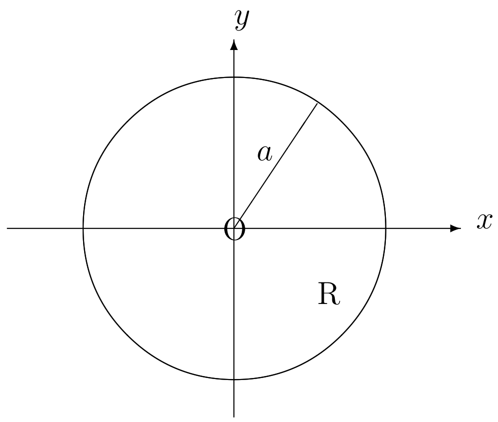


Using Example 1 in the previous Unit, section 13.11.2, the required second moment is

$$\frac{1}{3}a^3b + \frac{1}{3}b^3a = \frac{1}{3}ab(a^2 + b^2).$$

2. Determine the second moment of a circular region, R, with radius a, about an axis through its centre, perpendicular to the plane of R.

Solution



The second moment of R about a diameter is, from Example 2 in the previous Unit, section 13.11.2, equal to $\frac{\pi a^4}{4}$; that is, twice the value of the second moment of a semi-circular region about its diameter.

The required second moment is thus

$$\frac{\pi a^4}{4} + \frac{\pi a^4}{4} = \frac{\pi a^4}{2}.$$

13.12.3 THE RADIUS OF GYRATION OF AN AREA

Having calculated the second moment of a two dimensional region about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Ak^2, where A is the total area of the region.

We simply divide the value of the second moment by A in order to obtain the value of k^2 and hence the value of k.

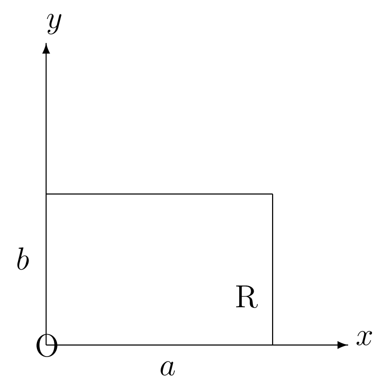
The value of k is called the **"radius of gyration"** of the given region about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole area at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a rectangular region, R, with sides of lengths a and b about an axis through one corner, perpendicular to the plane of R.

   Solution



Using Example 1 from the previous section, the second moment is

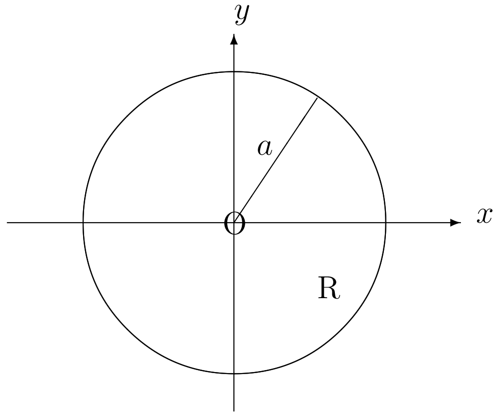
$$\frac{1}{3}ab(a^2+b^2)$$

and, since the area itself is ab, we obtain

$$k=\sqrt{a^2+b^2}.$$

2. Determine the radius of gyration of a circular region, R, about an axis through its centre, perpendicular to the plane of R.

Solution



From Example 2 in the previous section, the second moment about the given axis is $\frac{\pi a^4}{2}$ and, since the area itself is πa^2, we obtain

$$k=\frac{a}{\sqrt{2}}.$$

13.12.4 EXERCISES

Determine the radius of gyration of each of the following regions of the xy-plane about the axis specified:

1. Bounded in the first quadrant by the x-axis, the y-axis and the lines $x = a$, $y = b$.
   Axis: Through the point $\left(\frac{a}{2}, \frac{b}{2}\right)$, perpendicular to the xy-plane.
2. Bounded in the first quadrant by the x-axis, the y-axis and the lines $x = a$, $y = b$.
   Axis: The line $x = 4a$.
3. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$x^2 + y^2 = a^2.$$
   Axis: Through the origin, perpendicular to the xy-plane.
4. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$x^2 + y^2 = a^2.$$
   Axis: The line $x = a$.

13.12.5 ANSWERS TO EXERCISES

1. $$\frac{1}{12}\left(a^2 + b^2\right).$$
2. $$\frac{7a}{\sqrt{3}}.$$
3. $$\frac{a}{\sqrt{2}}.$$
4. $$\frac{a\sqrt{5}}{2}.$$

"JUST THE MATHS"

UNIT NUMBER

13.13

INTEGRATION APPLICATIONS 13
(Second moments of a volume (A))

by

A.J.Hobson

UNIT 13.13 - INTEGRATION APPLICATIONS 13

SECOND MOMENTS OF A VOLUME (A)

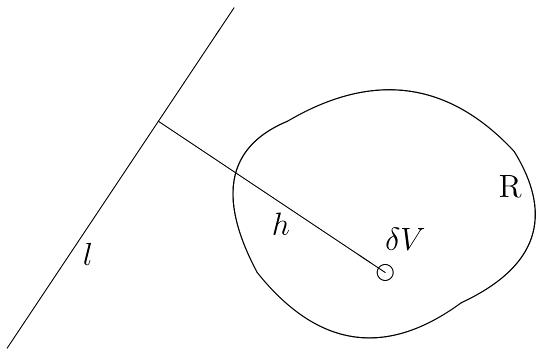
13.13.1 INTRODUCTION

Suppose that R denotes a region (with volume V) in space and suppose that δV is the volume of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h^2 \delta V,$$

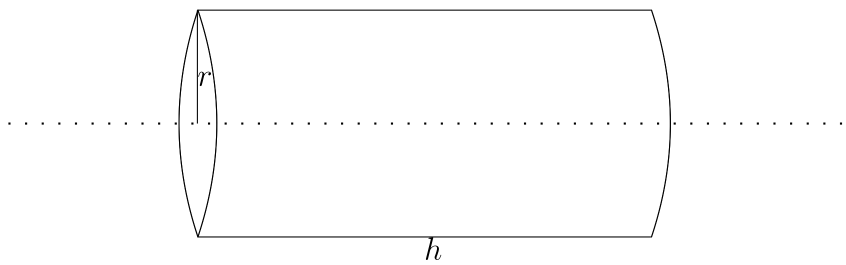
where h is the perpendicular distance from l of the element with volume δV.



EXAMPLE

Determine the second moment, about its own axis, of a solid right-circular cylinder with height, h, and radius, a.

Solution



In a thin cylindrical shell with internal radius, r, and thickness, δr, all of the elements of volume have the same perpendicular distance, r, from the axis of moments.

Hence the second moment of this shell will be the product of its volume and r^2; that is, $r^2(2\pi r h \delta r)$.

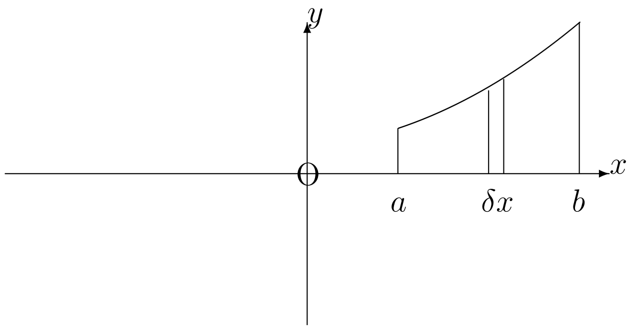
The total second moment is therefore given by

$$\lim_{\delta r \to 0} \sum_{r=0}^{r=a} r^2(2\pi r h \delta r) = \int_0^a 2\pi h r^3 \ \mathrm{d}r = \frac{\pi a^4 h}{2}.$$

13.13.2 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution of a narrow 'strip', of width, δx, and height, y, (parallel to the y-axis), is a cylindrical 'shell', of internal radius x, height, y, and thickness, δx.

Hence, from the example in the previous section, its second moment about the y-axis is $2\pi x^3 y\delta x$.

Thus, the total second moment about the y-axis is given by

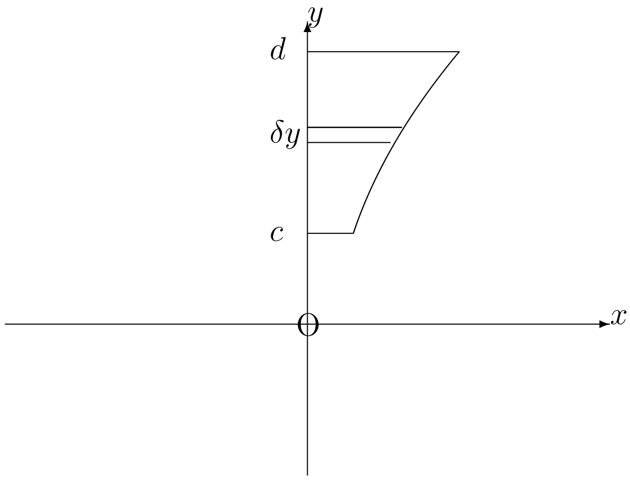
$$\lim_{\delta x\to 0}\sum_{x=a}^{x=b} 2\pi x^3 y\delta x = \int_a^b 2\pi x^3 y \ \mathrm{d}x.$$

Note:
Second moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for the volume of revolution, about the x-axis, of a region in the first quadrant bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

$$\int_c^d 2\pi y^3 x \ \mathrm{d}y.$$



EXAMPLE

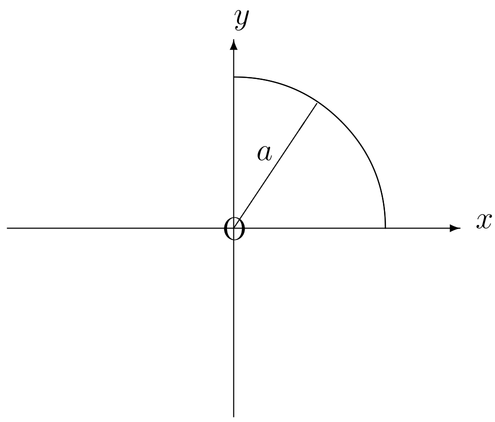
Determine the second moment, about a diameter, of a solid sphere with radius a.

Solution

We may consider, first, the volume of revolution about the y-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2,$$

then double the result obtained.



The total second moment is given by

$$2\int_0^a 2\pi x^3\sqrt{a^2-x^2}\ \mathrm{d}x = 4\pi\int_0^{\frac{\pi}{2}} a^3\sin^3\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

This simplifies to

$$4\pi a^5\int_0^{\frac{\pi}{2}} \sin^3\theta\ \cos^2\theta\mathrm{d}\theta = 4\pi\int_0^{\frac{\pi}{2}} \left(\cos^2\theta - \cos^4\theta\right)\sin\theta\mathrm{d}\theta,$$

if we make use of the trigonometric identity

$$\sin^2\theta \equiv 1 - \cos^2\theta.$$

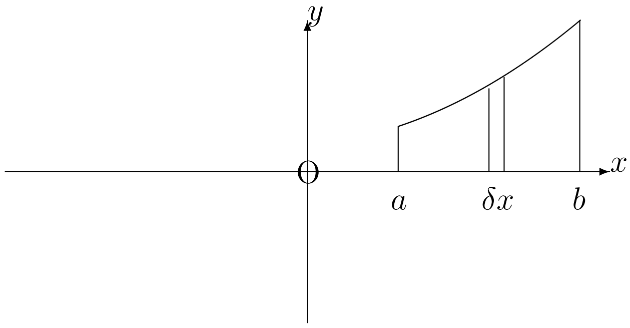
The total second moment is now given by

$$4\pi a^5\left[-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}} = 4\pi a^5\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8\pi a^5}{15}.$$

13.13.3 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE X-AXIS

In the introduction to this Unit, a formula was established for the second moment of a solid right-circular cylinder about its own axis. This result may now be used to determine the second moment about the x-axis for the volume of revolution about this axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution about the x-axis of a narrow strip, of width δx and height y, is a cylindrical 'disc' whose second moment about the x-axis is $\frac{\pi y^4 \delta x}{2}$. Hence the second moment of the whole region about the x-axis is given by

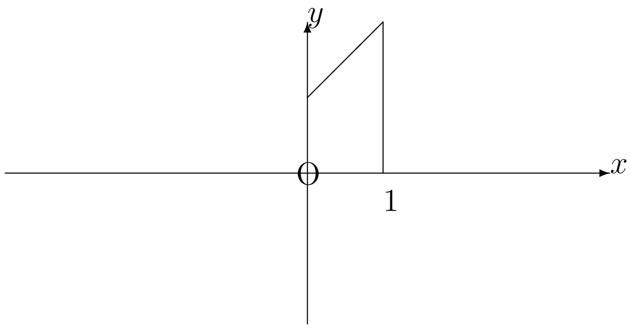
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{\pi y^4}{2} \delta x = \int_a^b \frac{\pi y^4}{2} \, \mathrm{d}x.$$

EXAMPLE

Determine the second moment about the x-axis for the volume of revolution about this axis of the region bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{\pi(x+1)^4}{2}\,\mathrm{d}x = \left[\pi\frac{(x+1)^4}{10}\right]_0^1 = \frac{31\pi}{10}.$$

Note:
The second moment of a volume about a certain axis is closely related to its **"moment of inertia"** about that axis. In fact, for a solid, with uniform density, ρ, the Moment of Inertia is ρ times the second moment of volume, since multiplication by ρ of elements of volume converts them into elements of mass.

13.13.4 EXERCISES

1. Determine the second moment about a diameter of a circular disc with small thickness, t, and radius, a.
2. Determine the second moment, about the axis specified, for the volume of revolution of each of the following regions of the xy-plane about this axis:
   (a) Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$y = 1 - 2x^2.$$
   Axis: The y-axis.
   (b) Bounded in the first quadrant by the x-axis and the curve whose equation is
   $$y^2 = \sin x.$$
   Axis: The x-axis.

(c) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The x-axis

(d) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The y-axis.

13.13.5 ANSWERS TO EXERCISES

1.

$$\frac{\pi a^4 t}{4}.$$

2. (a)

$$\frac{\pi}{24}.$$

(b)

$$\frac{\pi^2}{4}.$$

(c)

$$0.196, \text{ approximately.}$$

(d)

$$0.337, \text{ approximately.}$$

"JUST THE MATHS"

UNIT NUMBER

13.14

INTEGRATION APPLICATIONS 14
(Second moments of a volume (B))

by

A.J.Hobson

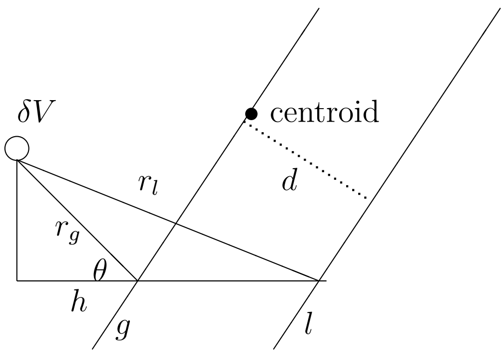
UNIT 13.14 - INTEGRATION APPLICATIONS 14

SECOND MOMENTS OF A VOLUME (B)

13.14.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis and has a perpendicular distance of d from the first axis.



In the above **three-dimensional** diagram, we have

$$M_l = \sum_{\rm R} r_l^2 \delta V \text{ and } M_g = \sum_{\rm R} r_g^2 \delta V.$$

But, from the Cosine Rule,

$$r_l^2 = r_g^2 + d^2 - 2r_g d\cos(180^\circ - \theta) = r_g^2 + d^2 + 2r_g d\cos\theta.$$

Hence,

$$r_l^2 = r_g^2 + d^2 + 2dh$$

and so

$$\sum_R r_l^2 \delta V = \sum_R r_g^2 \delta V + \sum_R d^2 \delta V + 2d \sum_R h \delta V.$$

Finally, the expression

$$\sum_R h \delta V$$

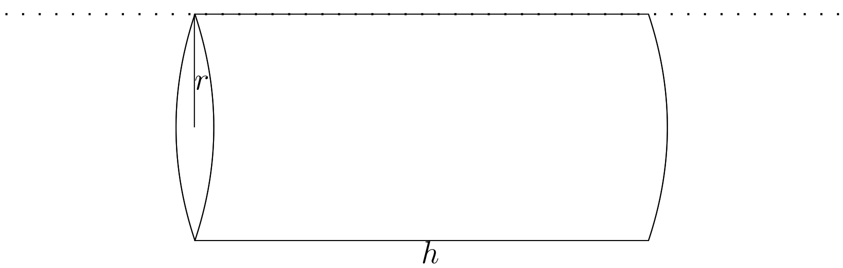
represents the first moment of R about a plane through the centroid, which is perpendicular to the plane containing l and g. Such first moment will be zero and hence,

$$M_l = M_g + V d^2.$$

EXAMPLE

Determine the second moment of a solid right-circular cylinder about one of its generators (that is, a line in the surface, parallel to the central axis).

Solution



The second moment of the cylinder about the central axis was shown, in Unit 13.13, section 13.13.2, to be $\frac{\pi a^4 h}{2}$; and, since this axis and the generator are a distance a apart, the required second moment is given by

$$\frac{\pi a^4 h}{2} + (\pi a^2 h) a^2 = \frac{3\pi a^4 h}{2}.$$

13.14.2 THE RADIUS OF GYRATION OF A VOLUME

Having calculated the second moment of a three-dimensional region about a certain axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Vk^2, where V is the total volume of the region.

We simply divide the value of the second moment by V in order to obtain the value of k^2 and hence, the value of k.

The value of k is called the **"radius of gyration"** of the given region about the given axis.

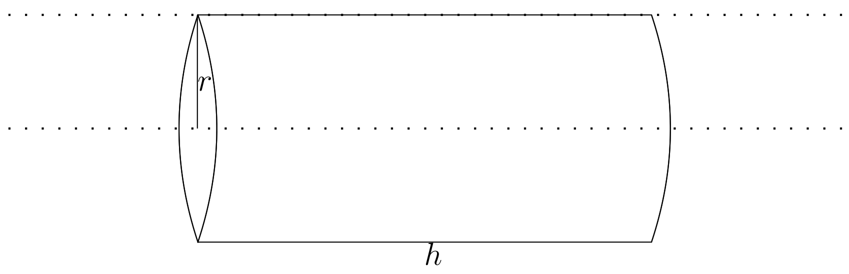
Note:

The radius of gyration effectively tries to concentrate the whole volume at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a solid right-circular cylinder with height, h, and radius, a, about (a) its own axis and (b) one of its generators.

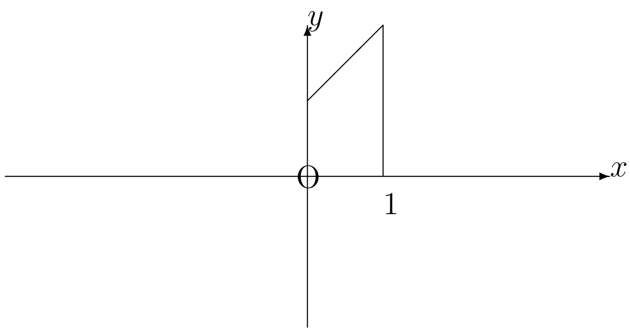
   Solution

   Using earlier examples, together with the volume, $V = \pi a^2 h$, the required radii of gyration are (a) $\sqrt{\frac{\pi a^4 h}{2} \div \pi a^2 h} = \frac{a}{\sqrt{2}}$ and (b) $\sqrt{\frac{3\pi a^4 h}{2} \div \pi a^2 h} = a\sqrt{\frac{3}{2}}$.

2. Determine the radius of gyration of the volume of revolution about the x-axis of the region, bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is
$$y = x + 1.$$

Solution



From Unit 13.13, section 13.13.3, the second moment about the given axis is $\frac{31\pi}{10}$. The volume itself is given by

$$\int_0^1 \pi(x+1)^2 \, \mathrm{d}x = \left[\pi\frac{(x+1)^3}{3}\right]_0^1 = \frac{7\pi}{3}.$$

Hence,

$$k^2 = \frac{31\pi}{10} \times \frac{3}{7\pi} = \frac{93}{70}.$$

That is,

$$k = \sqrt{\frac{93}{70}} \simeq 1.15$$

13.14.3 EXERCISES

1. Determine the radius of gyration of a hollow cylinder with internal radius, a, and external radius, b, about
   (a) its central axis;
   (b) a generator lying in its outer surface.
2. Determine the radius of gyration of a solid hemisphere, with radius a, about
   (a) its base-diameter;
   (b) an axis through its centroid, parallel to its base-diameter.
3. For a solid right-circular cylinder with height, h, and radius, a, determine the radius of gyration about
   (a) a diameter of one end;
   (b) an axis through the centroid, perpendicular to the axis of the cylinder.
4. For a solid right-circular cone with height, h, and base-radius, a, determine the radius of gyration about
   (a) the axis of the cone;
   (b) a line through the vertex, perpendicular to the axis of the cone;
   (c) a line through the centroid, perpendicular to the axis of the cone.

13.14.4 ANSWERS TO EXERCISES

1. (a)

$$\sqrt{\frac{a^2+b^2}{2}};$$

   (b)

$$\sqrt{\frac{3b^2+a^2}{2}}.$$

2. (a)

$$a\sqrt{\frac{2}{5}};$$

   (b)

$$a\sqrt{\frac{173}{320}}.$$

3. (a)

$$\sqrt{\frac{3a^2+4h^2}{12}};$$

   (b)

$$\sqrt{\frac{3a^2+7h^2}{12}}.$$

4. (a)

$$a\sqrt{\frac{3}{10}};$$

   (b)

$$\sqrt{\frac{3a^2}{20}+\frac{3h^2}{5}};$$

   (c)

$$\sqrt{\frac{3a^2}{20}+\frac{3h^2}{80}}.$$

"JUST THE MATHS"

UNIT NUMBER

13.15

INTEGRATION APPLICATIONS 15
(Second moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.15 - INTEGRATION APPLICATIONS 15

SECOND MOMENTS OF A SURFACE OF REVOLUTION

13.15.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"second moment"** about the x-axis, is given by

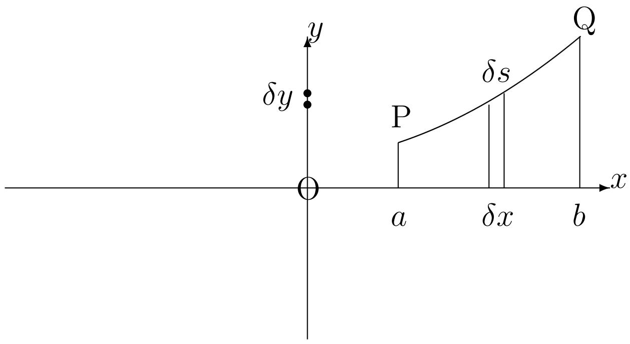
$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} y^2.2\pi y \delta s.$$

13.15.2 INTEGRATION FORMULAE FOR SECOND MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

so that, for the surface of revolution of the arc about the x-axis, the second moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y^3 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{t_1}^{t_2} 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the second moment about the plane through the origin, perpendicular to the x-axis, is given by

$$\text{second moment} = \pm\int_{t_1}^{t_2} 2\pi y^3\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

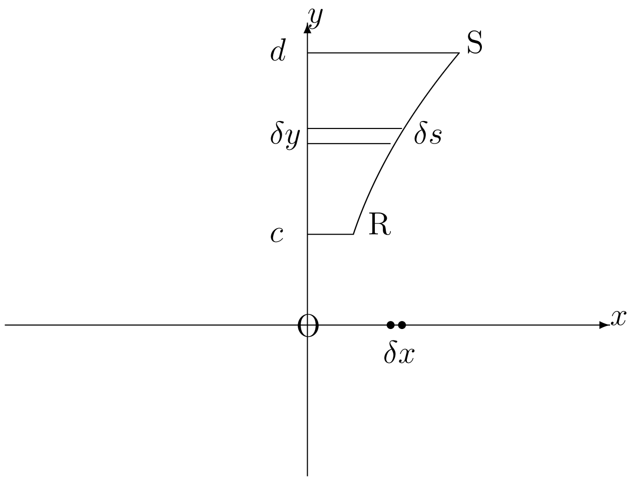
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the second moment about the y-axis is given by

$$\int_c^d 2\pi x^3\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the second moment about the y-axis is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi x^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

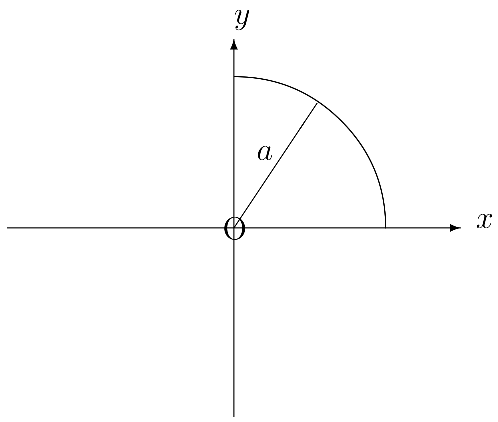
EXAMPLES

1. Determine the second moment about the x-axis of the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and, hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The second moment about the x-axis is therefore given by

$$\int_0^a 2\pi y^3\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi y^3\sqrt{\frac{x^2+y^2}{y^2}}\ \mathrm{d}x.$$

But $x^2+y^2=a^2$, and so the second moment becomes

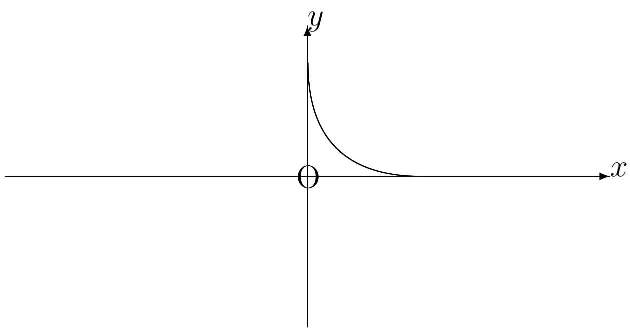
$$\int_0^a 2\pi a(a^2-x^2)\ \mathrm{d}x = 2\pi a\left[a^2x-\frac{x^3}{3}\right]_0^a = \frac{4\pi a^4}{3}.$$

2. Determine the second moment about the axis of revolution, when the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta,\quad y = a\sin^3\theta$$

is rotated through 2π radians about (a) the x-axis and (b) the y-axis.

Solution



(a) Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the second moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi y^3\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta+\sin^2\theta\equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^3 \sin^{27}\theta . 3a\cos\theta\sin\theta \ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 6\pi a^4 \sin^{28}\theta\cos\theta \ \mathrm{d}\theta$$

$$= 6\pi a^4 \int_0^{\frac{\pi}{2}} \sin^{28}\theta\cos\theta \ \mathrm{d}\theta,$$

which, by the methods of Unit 12.7 gives

$$6\pi a^4 \left[\frac{\sin^{29}\theta}{29}\right]_0^{\frac{\pi}{2}} = \frac{6\pi a^4}{29}.$$

(b) By symmetry, or by direct integration, the second moment about the y-axis is also $\frac{6\pi a^4}{29}$.

13.15.3 THE RADIUS OF GYRATION OF A SURFACE OF REVOLUTION

Having calculated the second moment of a surface of revolution about a specified axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Sk^2, where S is the total surface area of revolution.

We simply divide the value of the second moment by S in order to obtain the value of k^2 and hence the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole surface at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

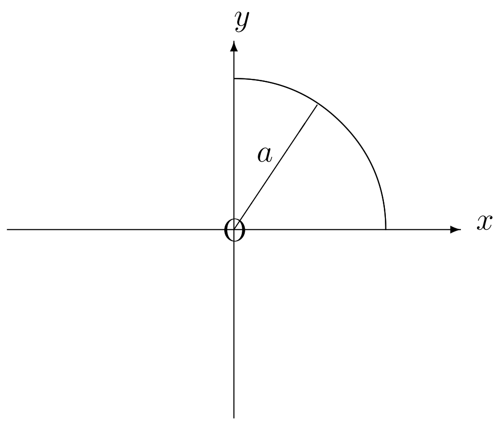
EXAMPLES

1. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From an Example 1 in section 13.15.2, we know that the second moment of the surface about the x-axis is equal to $\frac{4\pi a^4}{3}$.
Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2,$$

which implies that

$$k^2 = \frac{4\pi a^4}{3} \times \frac{1}{2\pi a^2} \quad = \frac{2a^2}{3}.$$

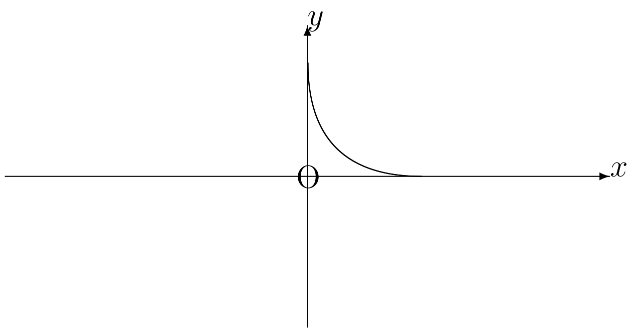
The radius of gyration is thus given by

$$k = a\sqrt{\frac{2}{3}}.$$

2. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 in section 13.15.2, that that the second moment of the surface about the x-axis is equal to $\frac{6\pi a^4}{29}$.
Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a\sin^3\theta.3a\cos\theta\sin\theta \ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 3a^2\sin^4\theta\cos\theta \ \mathrm{d}\theta = 3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$k^2 = \frac{6\pi a^4}{29} \times \frac{5}{3\pi a^2} = \frac{10a^2}{29}.$$

13.15.4 EXERCISES

1. Determine the second moment, about the x-axis, of the surface of revolution (about the x-axis) of the straight-line segment joining the origin to the point $(2, 3)$.
2. Determine the second moment about the x-axis, of the surface of revolution (about the x-axis) of the first quadrant arc of the curve whose equation is $y^2 = 4x$, lying between $x = 0$ and $x = 1$.
3. Determine, correct to two places of decimals, the second moment about the y-axis, of the surface of revolution (about the y-axis) of the first quadrant arc of the curve whose equation is $3y = x^3$, lying between $x = 1$ and $x = 2$.

4. Determine, correct to two places of decimals, the second moment, about the y-axis, of the surface of revolution (about the y-axis) of the arc of the circle given parametrically by
$$x = 2\cos t, \quad y = 2\sin t,$$
joining the point $(\sqrt{2}, \sqrt{2})$ to the point $(0, 2)$.

5. Determine the radius of gyration of a hollow right-circular cone with maximum radius, a, about its central axis.

6. For the curve whose equation is $9y^2 = x(3-x)^2$, show that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-x}{2\sqrt{x}}.$$
Hence, show that the radius of gyration about the y-axis of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x-axis is 4, correct to the nearest whole number.

13.15.5 ANSWERS TO EXERCISES

1.
$$\frac{\pi 27\sqrt{13}}{2}.$$

2.
$$\frac{32\pi}{5}[4-\sqrt{2}] \simeq 51.99$$

3.
$$70.44$$

4.
$$0.73$$

5.
$$k = \frac{a}{\sqrt{2}}.$$

6.
$$\text{Second moment} \simeq 139.92, \quad \text{surface area} \simeq 9.42$$

"JUST THE MATHS"

UNIT NUMBER

13.16

INTEGRATION APPLICATIONS 16
(Centres of pressure)

by

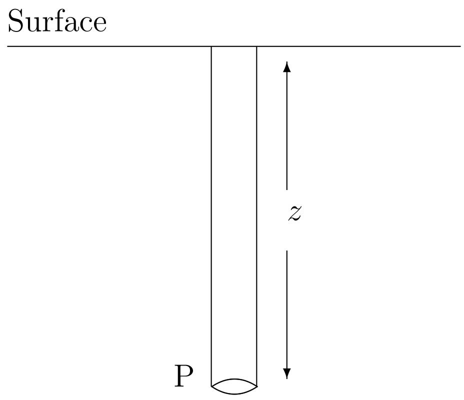
A.J.Hobson

UNIT 13.16 - INTEGRATION APPLICATIONS 16

CENTRES OF PRESSURE

13.16.1 THE PRESSURE AT A POINT IN A LIQUID

In the following diagram, we consider the pressure in a liquid at a point, P, whose depth below the surface of the liquid is z.



Ignoring atmospheric pressure, the pressure, p, at P is measured as the thrust acting upon unit area and is due to the weight of the column of liquid with height z above it.

Hence,

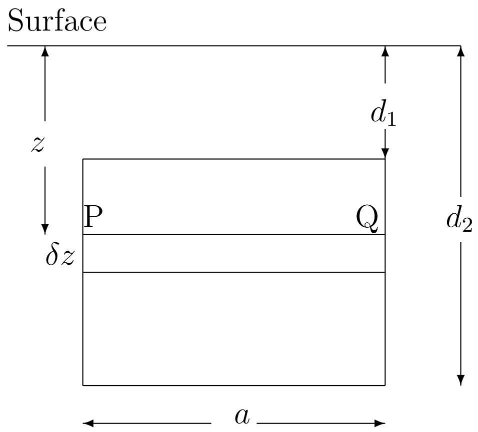
$$p = wz$$

where w is the weight, per unit volume, of the liquid.

Note:
The pressure at P is directly proportional to the depth of P below the surface; and we shall assume that the pressure acts equally in all directions at P.

13.16.2 THE PRESSURE ON AN IMMERSED PLATE

We now consider a rectangular plate, with dimensions a and $(d_2 - d_1)$, immersed vertically in a liquid as shown below.



For a thin strip, PQ, of width, δz, at a depth, z, below the surface of the liquid, the thrust on PQ will be the pressure at P multiplied by the area of the strip; that is, $wz \times a\delta z$.

The total thrust on the whole plate will therefore be

$$\sum_{z=d_1}^{z=d_2} waz\delta z.$$

Allowing δz to tend to zero, the total thrust becomes

$$\int_{d_1}^{d_2} waz \text{ d}z = \left[\frac{waz^2}{2}\right]_{d_1}^{d_2} = \frac{wa}{2}(d_2^2 - d_1^2).$$

This may be written

$$wa(d_2 - d_1)\left(\frac{d_2 + d_1}{2}\right),$$

where, in this form, $a(d_2 - d_1)$ is the area of the plate and $\frac{d_2+d_1}{2}$ is the depth of the centroid of the plate.

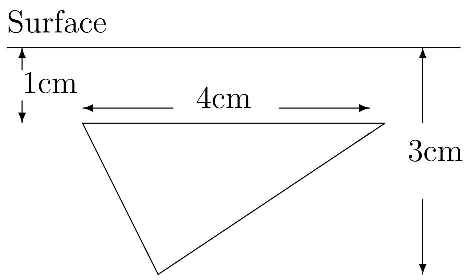
We conclude that

$$\text{Total Thrust} = \text{Area of Plate} \times \text{Pressure at the Centroid.}$$

Note:
It may be shown that this result holds whatever the shape of the plate is; and even when the plate is not vertical.

EXAMPLES

1. A triangular plate is immersed vertically in a liquid for which the weight per unit volume is w. The dimensions of the plate and its position in the liquid is shown in the following diagram:



Determine the total thrust on the plate as a multiple of w.

Solution

The area of the plate is given by

$$\text{Area} = \frac{1}{2} \times 4 \times 2 = 4\text{cm}^2$$

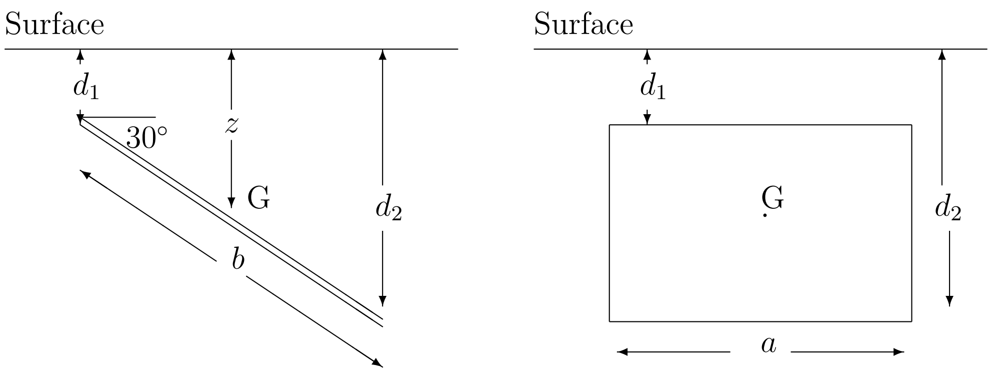
The centroid of the plate, which is at a distance from its horizontal side equal to one third of its perpendicular height, will lie at a depth of

$$\left(1 + \frac{1}{3} \times 2\right) \text{cms.} = \frac{5}{3}\text{cms.}$$

Hence, the pressure at the centroid is $\frac{5w}{3}$ and we conclude that

$$\text{Total thrust} = 4 \times \frac{5w}{3} = \frac{20w}{3}.$$

2. The following diagram shows a rectangular plate immersed in a liquid for which the weight per unit volume is w; and the plate is inclined at $30°$ to the horizontal:



Determine the total thrust on the plate as a multiple of w.

Solution

The depth, z, of the centroid, G, of the plate is given by

$$z = d_1 + \frac{b}{2}\sin 30° = d_1 + \frac{b}{4}.$$

Hence, the pressure, p, at G is given by

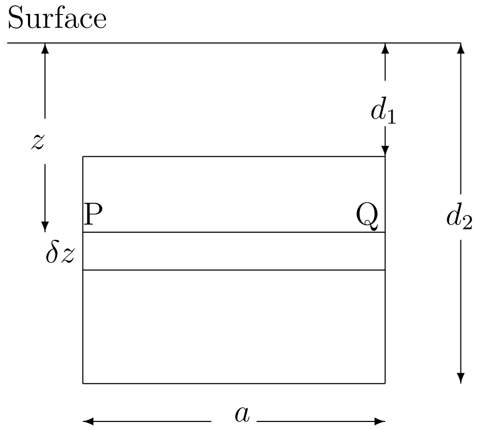
$$p = \left(d_1 + \frac{b}{4}\right)w;$$

and, since the area of the plate is ab, we obtain

$$\text{Total thrust} = ab\left(d_1 + \frac{b}{4}\right)w.$$

13.16.3 THE DEPTH OF THE CENTRE OF PRESSURE

In this section, we consider again an earlier diagram for a rectangular plate, immersed vertically in a liquid whose weight per unit volume is w.



We have already seen that the total thrust on the plate is

$$\int_{d_1}^{d_2} waz \text{ d}z = w\int_{d_1}^{d_2} az \text{ d}z$$

and is the resultant of varying thrusts, acting according to depth, at each level of the plate.

But, by taking first moments of these thrusts about the line in which the plane of the plate intersects the surface of the liquid, we may determine a particular depth at which the total thrust may be considered to act.

This depth is called **"the depth of the centre of pressure"**.

The Calculation

In the diagram, the thrust on the strip PQ is $waz\delta z$ and its first moment about the line in the surface is $waz^2\delta z$ so that the sum of the first moments on all such strips is given by

$$\sum_{z=d_1}^{z=d_2} waz^2\delta z = w\int_{d_1}^{d_2} az^2 \text{ d}z$$

where the definite integral is, in fact, the <u>second</u> moment of the plate about the line in the surface.

Next, we define the depth, C_p, of the centre of pressure to be such that

$$\text{Total thrust} \times C_p = \text{sum of first moments of strips like PQ.}$$

That is,

$$w\int_{d_1}^{d_2} az\ \mathrm{d}z \times C_p = w\int_{d_1}^{d_2} az^2\ \mathrm{d}z$$

and, hence,

$$C_p = \frac{\int_{d_1}^{d_2} az^2\ \mathrm{d}z}{\int_{d_1}^{d_2} az\ \mathrm{d}z},$$

which may be interpreted as

$$C_p = \frac{Ak^2}{A\overline{z}} = \frac{k^2}{\overline{z}},$$

where A is the area of the plate, k is the radius of gyration of the plate about the line in the surface of the liquid and $\overline{z}$ is the depth of the centroid of the plate.

Notes:

(i) It may be shown that the formula

$$C_p = \frac{k^2}{\overline{z}}$$

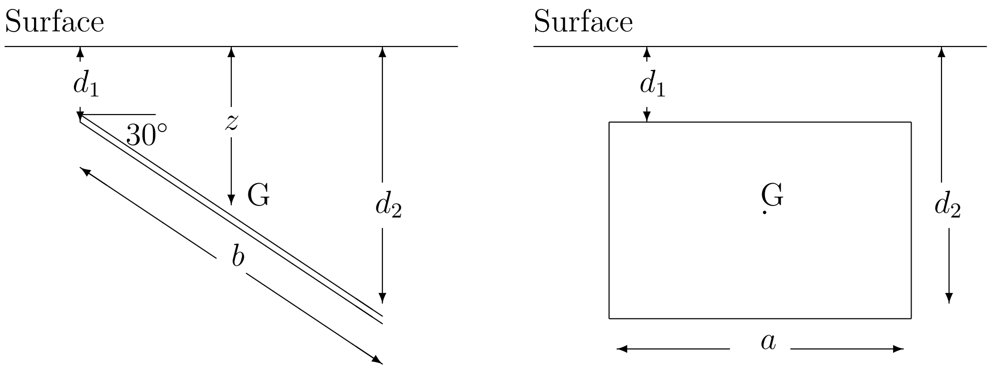
holds for any shape of plate immersed at any angle.

(ii) The phrase, "centre of pressure" suggests a particular point at which the total thrust is considered to act; but this is simply for convenience. The calculation is only for the depth of the centre of pressure.

EXAMPLE

Determine the depth of the centre of pressure for the second example of the previous section.

Solution



The depth of the centroid is

$$d_1 + \frac{b}{4}$$

and the square of the radius of gyration of the plate about an axis through the centroid, parallel to the side with length a is $\frac{a^2}{12}$.

Furthermore, the perpendicular distance between this axis and the line of intersection of the plane of the plate with the surface of the liquid is

$$\frac{b}{2} + \frac{d_1}{\sin 30^\circ} = \frac{b}{2} + 2d_1.$$

Hence, the square of the radius of gyration of the plate about the line in the surface is

$$\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2,$$

using the Theorem of Parallel Axes.

Finally, the depth of the centre of pressure is given by

$$C_p = \frac{\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2}{d_1 + \frac{b}{4}}.$$

13.16.4 EXERCISES

1. A thin equilateral triangular plate is immersed vertically in a liquid for which the weight per unit volume is w, with one edge on the surface. If the length of each side is a, determine the total thrust on the plate.
2. A thin plate is bounded by the arc of a parabola and a straight line segment of length 1.2m perpendicular to the axis of symmetry of the parabola, this axis being of length 0.4m.

   If the plate is immersed vertically in a liquid with the straight edge on the surface, determine the total thrust on the plate in the form lw, where w is the weight per unit volume of the liquid and l is in decimals, correct to two places.
3. A thin rectangular plate, with sides of length 10cm and 20cm is immersed in a liquid so that the sides of length 10cm are horizontal and the sides of length 20cm are incline at 55° to the horizontal. If the uppermost side of the plate is at a depth of 13cm, determine the total thrust on then plate in the form lw, where w is the mass per unit volume of the liquid.
4. A thin circular plate, with diameter 0.5m is immersed vertically in a tank of liquid so that the uppermost point on its circumference is 2m below the surface. Determine the depth of the centre of pressure. correct to two places of decimals.
5. A thin plate is in the form of a trapezium with parallel sides of length 1m and 2.5m, a distance of 0.75m apart, and the remaining two sides inclined equally to either one of the parallel sides.

   If the plate is immersed vertically in water with the side of length 2.5m on the surface, calculate the depth of the centre of pressure, correct to two places of decimals.

13.16.5 ANSWERS TO EXERCISES

1.

$$\text{Total thrust} = \frac{wa^3}{8}.$$

2.

$$\text{Total Thrust} = 5.12w.$$

3.

$$C_p \simeq 2.26\text{m}.$$

4.

$$C_p \simeq 0.46\text{m}.$$

"JUST THE MATHS"

UNIT NUMBER

14.1

PARTIAL DIFFERENTIATION 1
(Partial derivatives of the first order)

by

A.J.Hobson

UNIT 14.1 - PARTIAL DIFFERENTIATION 1 - PARTIAL DERIVATIVES OF THE FIRST ORDER

14.1.1 FUNCTIONS OF SEVERAL VARIABLES

In most scientific problems, it is likely that a variable quantity under investigation will depend (for its values), not only on **one** other variable quantity, but on **several** other variable quantities.

The type of notation used may be indicated by examples such as the following:

1. $$z = f(x, y),$$

   which means that the variable, z, depends (for its values) on two variables, x and y.

2. $$w = F(x, y, z),$$

   which means that the variable, w, depends (for its values) on three variables, x, y and z.

   Normally, the variables on the right-hand side of examples like those above may be chosen independently of one another and, as such, are called the **"independent variables"**. By contrast, the variable on the left-hand side is called the **"dependent variable"**.

Notes:

(i) Some relationships between several variables are not stated as an **explicit** formula for one of the variables in terms of the others.

An illustration of this type would be $x^2 + y^2 + z^2 = 16$.

In such cases, it may be necessary to specify separately which is the dependent variable.

(ii) The variables on the right-hand side of an explicit formula, giving a dependent variable in terms of them, may not actually be independent of one another. This would occur if those variables were already, themselves, dependent on a quantity not specifically mentioned in the formula.

For example, in the formula

$$z = xy^2 + \sin(x - y),$$

suppose it is also known that $x = t - 1$ and $y = 3t + 2$.

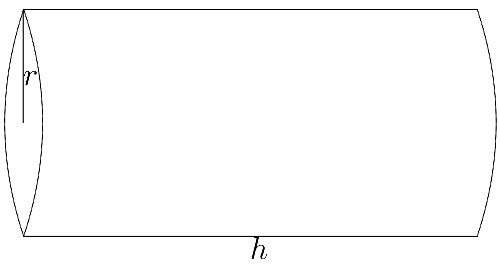
Then the variables x and y are not independent of each other. In fact, by eliminating t, we obtain

$$y = 3(x + 1) + 2 = 3x + 5.$$

14.1.2 THE DEFINITION OF A PARTIAL DERIVATIVE

ILLUSTRATION

Consider the formulae for the volume, V, and the surface area, S, of a solid right-circular cylinder with radius, r, and height, h.



The relevant formulae are

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r^2 + 2\pi rh,$$

so that both V and S are functions of the two variables, r and h.

But suppose it were possible for r to be held constant while h is allowed to vary. Then the corresponding rates of increase of V and S with respect to h are given by

$$\left[\frac{\mathrm{d}V}{\mathrm{d}h}\right]_{r \text{ const.}} = \pi r^2$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}h}\right]_{r \text{ const.}} = 2\pi r.$$

These two expressions are called the **"partial derivatives of V and S with respect to h"**.

Similarly, suppose it were possible for h to be held constant while r is allowed to vary. Then the corresponding rates of increase of V and S with respect to r are given by

$$\left[\frac{\mathrm{d}V}{\mathrm{d}r}\right]_{h \text{ const.}} = 2\pi rh$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}r}\right]_{h \text{ const.}} = 4\pi r + 2\pi h.$$

These two expressions are called the **"partial derivatives of V and S with respect to r"**.

THE NOTATION FOR PARTIAL DERIVATIVES

In the defining illustration above, the notation used for the partial derivatives of V and S was an adaptation of the notation for what will, in future, be referred to as **ordinary** derivatives.

It was, however, rather cumbersome; and the more standard notation which uses the symbol ∂ rather than d is indicated by restating the earlier results as

$$\frac{\partial V}{\partial h} = \pi r^2, \quad \frac{\partial S}{\partial h} = 2\pi r$$

and

$$\frac{\partial V}{\partial r} = 2\pi rh, \quad \frac{\partial S}{\partial r} = 4\pi r + 2\pi h.$$

In this notation, it is understood that each independent variable (except the one with respect to which we are differentiating) is held constant.

EXAMPLES

1. Determine the partial derivatives of the following functions with respect to each of the independent variables:

   (a)
$$z = (x^2 + 3y)^5;$$

   Solution
$$\frac{\partial z}{\partial x} = 5(x^2 + 3y)^4.2x = 10x(x^2 + 3y)^4$$
   and
$$\frac{\partial z}{\partial y} = 5(x^2 + 3y)^4.3 = 15(x^2 + 3y)^4.$$

   (b)
$$w = ze^{3x-7y};$$

   Solution
$$\frac{\partial w}{\partial x} = 3ze^{3x-7y},$$
$$\frac{\partial w}{\partial y} = -7ze^{3x-7y},$$
   and
$$\frac{\partial w}{\partial z} = e^{3x-7y}.$$

   (c)
$$z = x\sin(2x^2 + 5y).$$

   Solution
$$\frac{\partial z}{\partial x} = \sin(2x^2 + 5y) + 4x^2\cos(2x^2 + 5y)$$
   and
$$\frac{\partial z}{\partial y} = 5x\cos(2x^2 + 5y).$$

2. If

$$z = f(x^2 + y^2),$$

show that

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

Solution

$$\frac{\partial z}{\partial x} = 2xf'(x^2 + y^2)$$

and

$$\frac{\partial z}{\partial y} = 2yf'(x^2 + y^2).$$

Hence,

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

3. Given the formula

$$\cos(x + 2z) + 3y^2 + 2xyz = 0$$

as an implicit relationship between two independent variables x and y and a dependent variable z, determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x, y and z.

Solution

Differentiating the formula partially with respect to x and y in turn, we obtain

$$-\sin(x + 2z).\left(1 + 2\frac{\partial z}{\partial x}\right) + 2y\left(x\frac{\partial z}{\partial x} + y\right) = 0$$

and

$$-\sin(x + 2z).2\frac{\partial z}{\partial y} + 6y + 2x\left(y\frac{\partial z}{\partial y} + z\right) = 0,$$

respectively.

Thus,

$$\frac{\partial z}{\partial x} = \frac{\sin(x + 2z) - 2y^2}{2yx - 2\sin(x + 2z)}$$

and

$$\frac{\partial z}{\partial y} = \frac{2xz + 6y}{2\sin(x + 2z) - 2xy} = \frac{xz + 3y}{\sin(x + 2z) - xy}.$$

14.1.3 EXERCISES

1. Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in the following cases:

   (a)
   $$z = 2x^2 - 4xy + y^3;$$

   (b)
   $$z = \cos(5x - 3y);$$

   (c)
   $$z = e^{x^2+2y^2};$$

   (d)
   $$z = x\sin(y - x).$$

2. If
   $$z = (x + y)\ln\left(\frac{x}{y}\right),$$
   show that
   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

3. Determine $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ in the following cases:

   (a)
   $$w = x^5 + 3xyz + z^2;$$

   (b)
   $$w = ze^{2x-3y};$$

   (c)
   $$w = \sin(x^2 - yz).$$

14.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\partial z}{\partial x} = 4x - 4y \quad \text{and} \quad \frac{\partial z}{\partial y} = -4x + 3y^2;$$

(b)

$$\frac{\partial z}{\partial x} = -5\sin(5x - 3y) \quad \text{and} \quad \frac{\partial z}{\partial y} = 3\sin(5x - 3y);$$

(c)

$$\frac{\partial z}{\partial x} = 2xe^{x^2+2y^2} \quad \text{and} \quad \frac{\partial z}{\partial y} = 4ye^{x^2+2y^2};$$

(d)

$$\frac{\partial z}{\partial x} = \sin(y - x) - x\cos(y - x) \quad \text{and} \quad \frac{\partial z}{\partial y} = x\cos(y - x).$$

2.

$$\frac{\partial z}{\partial x} = \ln\left(\frac{x}{y}\right) + \frac{x + y}{x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \ln\left(\frac{x}{y}\right) - \frac{x + y}{y}.$$

3. (a)

$$\frac{\partial w}{\partial x} = 5x^4 + 3yz, \quad \frac{\partial w}{\partial y} = 3xz, \quad \frac{\partial w}{\partial z} = 3xy + 2z;$$

(b)

$$\frac{\partial w}{\partial x} = 2ze^{2x-3y}, \quad \frac{\partial w}{\partial y} = -3ze^{2x-3y}, \frac{\partial w}{\partial z} = e^{2x-3y};$$

(c)

$$\frac{\partial w}{\partial x} = 2x\cos(x^2 - yz), \quad \frac{\partial w}{\partial y} = -z\cos(x^2 - yz), \quad \frac{\partial w}{\partial z} = -y\cos(x^2 - yz).$$

"JUST THE MATHS"

UNIT NUMBER

14.2

PARTIAL DIFFERENTIATION 2
(Partial derivatives of order higher than one)

by

A.J.Hobson

UNIT 14.2 - PARTIAL DIFFERENTIATION 2

PARTIAL DERIVATIVES OF THE SECOND AND HIGHER ORDERS

14.2.1 STANDARD NOTATIONS AND THEIR MEANINGS

In Unit 14.1, the partial derivatives encountered are known as partial derivatives of the **first order**; that is, the dependent variable was differentiated only **once** with respect to each independent variable.

But a partial derivative will, in general contain **all** of the independent variables, suggesting that we may need to differentiate again with respect to **any** of those variables.

For example, in the case where a variable, z, is a function of two independent variables, x and y, the possible partial derivatives of the second order are

(i)

$$\frac{\partial^2 z}{\partial x^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right);$$

(ii)

$$\frac{\partial^2 z}{\partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right);$$

(iii)

$$\frac{\partial^2 z}{\partial x \partial y}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right);$$

(iv)

$$\frac{\partial^2 z}{\partial y \partial x}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

The last two can be shown to give the same result for all elementary functions likely to be encountered in science and engineering.

Note:

Occasionally, it may be necessary to use partial derivatives of order higher than two, as illustrated, for example, by

$$\frac{\partial^3 z}{\partial x \partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]$$

and

$$\frac{\partial^4 z}{\partial x^2 \partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]\right).$$

EXAMPLES

Determine all the first and second order partial derivatives of the following functions:

1.

$$z = 7x^3 - 5x^2y + 6y^3.$$

Solution

$$\frac{\partial z}{\partial x} = 21x^2 - 10xy; \qquad \frac{\partial z}{\partial y} = -5x^2 + 18y^2;$$

$$\frac{\partial^2 z}{\partial x^2} = 42x - 10y; \qquad \frac{\partial^2 z}{\partial y^2} = 36y;$$

$$\frac{\partial^2 z}{\partial y \partial x} = -10x; \qquad \frac{\partial^2 z}{\partial x \partial y} = -10x.$$

2.

$$z = y \sin x + x \cos y.$$

Solution

$$\frac{\partial z}{\partial x} = y \cos x + \cos y; \qquad \frac{\partial z}{\partial y} = \sin x - x \sin y;$$

$$\frac{\partial^2 z}{\partial x^2} = -y \sin x; \qquad \frac{\partial^2 z}{\partial y^2} = -x \cos y;$$

$$\frac{\partial^2 z}{\partial y \partial x} = \cos x - \sin y; \qquad \frac{\partial^2 z}{\partial x \partial y} = \cos x - \sin y.$$

3.

$$z = e^{xy}(2x - y).$$

Solution

$$\frac{\partial z}{\partial x} = e^{xy}[y(2x - y) + 2] = e^{xy}[2xy - y^2 + 2]; \qquad \frac{\partial z}{\partial y} = e^{xy}[x(2x - y) - 1] = e^{xy}[2x^2 - xy - 1];$$

$$\frac{\partial^2 z}{\partial x^2} = e^{xy}[y(2xy - y^2 + 2) + 2y] = e^{xy}[2xy^2 - y^3 + 4y]; \qquad \frac{\partial^2 z}{\partial y^2} = e^{xy}[x(2x^2 - xy - 1) - x] = e^{xy}[2x^3 - x^2y - 2x];$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^{xy}[x(2xy - y^2 + 2) + 2x - 2y] = e^{xy}[2x^2y - xy^2 + 4x - 2y]; \qquad \frac{\partial^2 z}{\partial x \partial y} = e^{xy}[y(2x^2 - xy - 1) + 4x - y] = e^{xy}[2x^2y - xy^2 + 4x - 2y].$$

14.2.2 EXERCISES

1. Determine all the first and second order partial derivatives of the following functions:

   (a)

   $$z = 5x^2y^3 - 7x^3y^5;$$

   (b)

   $$z = x^4 \sin 3y.$$

2. Determine all the first and second order partial derivatives of the function

   $$w \equiv z^2 e^{xy} + x\cos(y^2 z).$$

3. If

   $$z = (x + y)\ln\left(\frac{x}{y}\right),$$

   show that

   $$x^2\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial y^2}.$$

4. If

$$z = f(x + ay) + F(x - ay),$$

show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial y^2}.$$

14.2.3 ANSWERS TO EXERCISES

1. (a) The required partial derivatives are as follows:

$$\tfrac{\partial z}{\partial x} = 10xy^3 - 21x^2y^5; \qquad \tfrac{\partial z}{\partial y} = 15x^2y^2 - 35x^3y^4;$$

$$\tfrac{\partial^2 z}{\partial x^2} = 10y^3 - 42xy^5; \qquad \tfrac{\partial^2 z}{\partial y^2} = 30x^2y - 140x^3y^3;$$

$$\tfrac{\partial^2 z}{\partial y \partial x} = 30xy^2 - 105x^2y^4; \quad \tfrac{\partial^2 z}{\partial x \partial y} = 30xy^2 - 105x^2y^4.$$

(b) The required partial derivatives are as follows:

$$\tfrac{\partial z}{\partial x} = 4x^3 \sin 3y; \qquad \tfrac{\partial z}{\partial y} = 3x^4 \cos 3y;$$

$$\tfrac{\partial^2 z}{\partial x^2} = 12x^2 \sin 3y; \qquad \tfrac{\partial^2 z}{\partial y^2} = -9x^4 \sin 3y;$$

$$\tfrac{\partial^2 z}{\partial y \partial x} = 12x^3 \cos 3y; \qquad \tfrac{\partial^2 z}{\partial x \partial y} = 12x^3 \cos 3y.$$

2. The required partial derivatives are as follows:

$$\frac{\partial w}{\partial x} = yz^2e^{xy} + \cos(y^2z); \quad \frac{\partial w}{\partial y} = z^2xe^{xy} - 2xyz\sin(y^2z); \quad \frac{\partial w}{\partial z} = 2ze^{xy} - xy^2\sin(y^2z);$$

$$\frac{\partial^2 w}{\partial x^2} = y^2z^2e^{xy}; \quad \frac{\partial^2 w}{\partial y^2} = z^2x^2e^{xy} - 2xz\sin(y^2z) + 4xy^2z^2\cos(y^2z); \frac{\partial^2 w}{\partial z^2} = 2e^{xy} - xy^4\cos(y^2z);$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x} = z^2 e^{xy} + z^2 xy e^{xy} - 2yz \sin(y^2 z);$$

$$\frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 w}{\partial z \partial y} = 2zx e^{xy} - 2xy \sin(y^2 z) - 2xy^3 z \cos(y^2 z);$$

$$\frac{\partial^2 w}{\partial z \partial x} = \frac{\partial^2 w}{\partial x \partial z} = 2zy e^{xy} - y^2 \sin(y^2 z).$$

3.

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x} - \frac{y}{x^2} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = -\frac{1}{y} + \frac{x}{y^2}.$$

4.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + F''(x-ay) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 F''(x-ay).$$

"JUST THE MATHS"

UNIT NUMBER

14.3

PARTIAL DIFFERENTIATION 3
(Small increments and small errors)

by

A.J.Hobson

UNIT 14.3 - PARTIAL DIFFERENTIATION 3

SMALL INCREMENTS AND SMALL ERRORS

14.3.1 FUNCTIONS OF ONE INDEPENDENT VARIABLE - A RECAP

For functions of **one** independent variable, a discussion of small increments and small errors has already taken place in Unit 11.6.

It was established that, if a dependent variable, y, is related to an independent variable, x, by means of the formula

$$y = f(x),$$

then

(a) The **increment**, δy, in y, due to an increment of δx, in x is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x;$$

and, in much the same way,

(b) The **error**, δy, in y, due to an error of δx in x, is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

14.3.2 FUNCTIONS OF MORE THAN ONE INDEPENDENT VARIABLE

Let us consider, first, a function, z, of two independent variables, x and y, given by the formula

$$z = f(x, y).$$

If x is subject to a small increment (or a small error) of δx, while y remains constant, then the corrresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x.$$

Similarly, if y is subject to a small increment (or a small error) of δy, while x remains constant, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial y}\delta y.$$

It seems reasonable to assume, therefore, that, when x is subject to a small increment (or a small error) of δx **and** y is subject to a small increment (or a small error) of δy, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

It may be shown that, to the first order of approximation, this is indeed true.

Notes:

(i) To prove more rigorously that the above result is true, use would have to be made of the result known as **"Taylor's Theorem"** for a function of two independent variables.

In the present case, where $z = f(x, y)$, it would give

$$f(x+\delta x, y+\delta y) = f(x, y) + \left(\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right) + \left(\frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\delta x\delta y + \frac{\partial^2 z}{\partial y^2}(\delta y)^2\right) + \ldots,$$

which shows that

$$\delta z = f(x + \delta x, y + \delta y) - f(x, y) \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

to the first order of approximation.

(ii) The formula for a function of two independent variables may be extended to functions of a greater number of independent variables by simply adding further appropriate terms to the right hand side.

For example, if

$$w = F(x, y, z),$$

then

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

EXAMPLES

1. A rectangle has sides of length $x$cms. and $y$cms.

   Determine, approximately, in terms of x and y, the increment in the area, A, of the rectangle when x and y are subject to increments of δx and δy, respectively.

   Solution

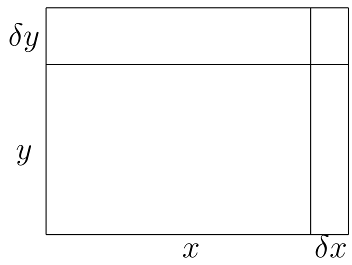
   The area, A, is given by

   $$A = xy,$$

   so that

   $$\delta A \simeq \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y = y\delta x + x\delta y.$$

   Note:
   The exact value of δA may be seen in the following diagram:



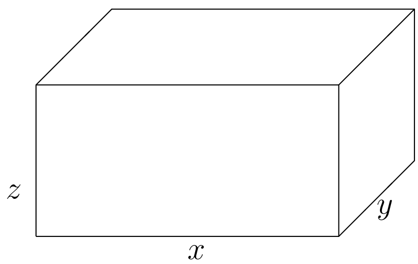
The difference between the approximate value and the exact value is represented by the area of the small rectangle having sides δxcms. and δycms.

2. In measuring a rectangular block of wood, the dimensions were found to be 10cms., 12cms and 20cms. with a possible error of $\pm 0.05cms.$ in each.

   Calculate, approximately, the greatest possible error in the surface area, S, of the block and the percentage error so caused.

   Solution

   First, we may denote the lengths of the edges of the block by x, y and z.



The surface area, S, is given by

$$S = 2(xy + yz + zx),$$

which has the value 1120cms^2 when $x = 10$cms., $y = 12$cms. and $z = 20$cms.

Also,

$$\delta S \simeq \frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y + \frac{\partial S}{\partial z}\delta z,$$

which gives

$$\delta S \simeq 2(y+z)\delta x + 2(x+z)\delta y + 2(y+x)\delta z;$$

and, on substituting $x = 10$, $y = 12$, $z = 20$, $\delta x = \pm 0.05$, $\delta y = \pm 0.05$ and $\delta z = \pm 0.05$, we obtain

$$\delta S \simeq \pm 2(12+20)(0.05) \pm 2(10+20)(0.05) \pm 2(12+10)(0.05).$$

The greatest error will occur when all the terms of the above expression have the same sign. Hence, the greatest error is given by

$$\delta S_{\max} \simeq \pm 8.4\text{cms.}^2;$$

and, since the originally calculated value was 1120, this represents a percentage error of approximately

$$\pm\frac{8.4}{1120} \times 100 = \pm 0.75$$

3. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

We have

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

That is,

$$\delta w \simeq \frac{3x^2 z}{y^4}\delta x - \frac{4x^3 z}{y^5}\delta y + \frac{x^3}{y^4}\delta z,$$

where

$$\delta x = -\frac{3x}{100}, \quad \delta y = \frac{y}{100} \quad \text{and} \quad \delta z = \frac{2z}{100}.$$

Thus,

$$\delta w \simeq \frac{x^3 z}{y^4}\left[-\frac{9}{100} - \frac{4}{100} + \frac{2}{100}\right] = -\frac{11w}{100}.$$

The percentage error in w is given approximately by

$$\frac{\delta w}{w} \times 100 = -11.$$

That is, w is too small by approximately 11%.

14.3.3 THE LOGARITHMIC METHOD

In this section we consider again examples where it is required to calculate either a percentage increment or a percentage error.

We may conveniently use logarithms if the right hand side of the formula for the dependent variable involves a product, a quotient, or a combination of these two in which the independent variables are separated. This would be so, for instance, in the final example of the previous section.

The method is to take the natural logarithms of both sides of the equation before considering any partial derivatives; and we illustrate this, firstly, for a function of **two** independent variables.

Suppose that

$$z = f(x, y)$$

where $f(x, y)$ is the type of function described above.

Then,

$$\ln z = \ln f(x, y);$$

and, if we temporarily replace $\ln z$ by w, we have a new formula

$$w = \ln f(x, y).$$

The increment (or the error) in w, when x and y are subject to increments (or errors) of δx and δy respectively, is given by

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y.$$

That is,

$$\delta w \simeq \frac{1}{f(x,y)}\frac{\partial f}{\partial x}\delta x + \frac{1}{f(x,y)}\frac{\partial f}{\partial y}\delta y = \frac{1}{f(x,y)}\left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right].$$

In other words,

$$\delta w \simeq \frac{1}{z}\left[\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right].$$

We conclude that

$$\delta w \simeq \frac{\delta z}{z},$$

which means that the fractional increment (or error) in z approximates to the actual increment (or error) in $\ln z$. Multiplication by 100 will, of course, convert the fractional increment (or error) into a percentage.

Note:
The logarthmic method will apply equally well to a function of more than two independent variables where it takes the form of a product, a quotient, or a combination of these two.

EXAMPLES

1. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

Taking the natural logarithm of both sides of the given formula,

$$\ln w = 3 \ln x + \ln z - 4 \ln y,$$

giving

$$\frac{\delta w}{w} \simeq 3\frac{\delta x}{x} + \frac{\delta z}{z} - 4\frac{\delta y}{y},$$

where

$$\frac{\delta x}{x} = -\frac{3}{100}, \quad \frac{\delta y}{y} = \frac{1}{100} \quad \text{and} \quad \frac{\delta z}{z} = \frac{2}{100}.$$

Hence,

$$\frac{\delta w}{w} \times 100 = -9 + 2 - 4 = -13.$$

Thus, w is too small by approximately 11%, as before.

2. In the formula,

$$w = \sqrt{\frac{x^3}{y}},$$

x is subjected to an increase of 2%. Calculate, approximately, the percentage change needed in y to ensure that w remains unchanged.

Solution

Taking the natural logarithm of both sides of the formula,

$$\ln w = \frac{1}{2}[3 \ln x - \ln y].$$

Hence,

$$\frac{\delta w}{w} \simeq \frac{1}{2}\left[3\frac{\delta x}{x} - \frac{\delta y}{y}\right],$$

where $\frac{\delta x}{x} = 0.02$, and we require that $\delta w = 0$.
Thus,

$$0 = \frac{1}{2}\left[0.06 - \frac{\delta y}{y}\right],$$

giving

$$\frac{\delta y}{y} = 0.06,$$

which means that y must be approximately 6% too large.

14.3.4 EXERCISES

1. A triangle is such that two of its sides (of length 6cms. and 8cms.) are at right-angles to each other.

   Calculate, approximately, the change in the length of the hypotenuse of the triangle when the shorter side is lengthened by 0.25cms. and the longer side is shortened by 0.125cms.

2. Two sides of a triangle are measured as $x = 150$cms. and $y = 200$cms. while the angle included between them is measured as $\theta = 60°$. Calculate the area of the triangle.

   If there are possible errors of ± 0.2cms. in the measurement of the sides and $\pm 1°$ in the angle, determine, approximately, the maximum possible error in the calculated area of the triangle.

   State your answers correct to the nearest whole number.

   (**Hint** use the formula, Area $= \frac{1}{2}xy\sin\theta$).

3. Given that the volume of a segment of a sphere is $\frac{1}{6}x(x^2 + 3y^2)$ where x is the height and y is the radius of the base, obtain, in terms of x and y, the percentage error in the volume when x is too large by 1% and y is too small by 0.5%.

4. If

   $$z = kx^{0.01}y^{0.08},$$

   where k is a constant, calculate, approximately, the percentage change in z when x is increased by 2% and y is decreased by 1%.

5. If
$$w = \frac{5xy^4}{z^3},$$
calculate, approximately, the maximum percentage error in w if x, y and z are subject to errors of $\pm 3\%$, $\pm 2.5\%$ and $\pm 4\%$, respectively.

6. If
$$w = 2xyz^{-\frac{1}{2}},$$
where x and z are subject to errors of 0.2%, calculate, approximately, the percentage error in y which results in w being without error.

14.3.5 ANSWERS TO EXERCISES

1. 0.05cms.
2. 12990cms.^2 and 161cms.^2
3. $\frac{3x^2}{x^2+3y^2}$.
4. z decreases by 0.06%.
5. 25%.
6. -0.1%.

"JUST THE MATHS"

UNIT NUMBER

14.4

PARTIAL DIFFERENTIATION 4
(Exact differentials)

by

A.J.Hobson

UNIT 14.4 - PARTIAL DIFFERENTIATION 4

EXACT DIFFERENTIALS

14.4.1 TOTAL DIFFERENTIALS

In Unit 14.3, use was made of expressions of the form,

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

as an approximation for the increment (or error), δf, in the function, $f(x, y, ...)$, when x, y etc. are subject to increments (or errors) of δx, δy etc., respectively.

The expression may be called the **"total differential"** of $f(x, y, ...)$ and may be denoted by $\mathrm{d}f$, giving

$$\mathrm{d}f \simeq \delta f.$$

OBSERVATIONS

Consider the formula,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ \ldots$$

(a) In the special case when $f(x, y, ...) \equiv x$, we may conclude that $\mathrm{d}f = \delta x$ or, in other words,

$$\mathrm{d}x = \delta x.$$

(b) In the special case when $f(x, y, ...) \equiv y$, we may conclude that $\mathrm{d}f = \delta y$ or, in other words,

$$\mathrm{d}y = \delta y.$$

(c) Observations (a) and (b) imply that the total differential of each **independent** variable is the same as the small increment (or error) in that variable; but the total differential of the **dependent** variable is only approximately equal to the increment (or error) in that variable.

(d) All of the previous observations may be summarised by means of the formula

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \ldots$$

14.4.2 TESTING FOR EXACT DIFFERENTIALS

In general, an expression of the form

$$P(x, y, ...)\mathrm{d}x + Q(x, y, ...)\mathrm{d}y + \ldots$$

will not be the total differential of a function, $f(x, y, ...)$, unless the functions, $P(x, y, ...)$, $Q(x, y, ...)$ etc. can be identified with $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc., respectively.

If this **is** possible, then the expression is known as an **"exact differential"**.

RESULTS

(i) The expression

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$$

is an exact differential if and only if

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

Proof:

(a) If the expression,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y,$$

is an exact differential, $\mathrm{d}f$, then

$$\frac{\partial f}{\partial x} \equiv P(x, y) \text{ and } \frac{\partial f}{\partial y} \equiv Q(x, y).$$

Hence, it must be true that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \left(\equiv \frac{\partial^2 f}{\partial x \partial y} \right).$$

(b) Conversely, suppose that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

We can certainly say that

$$P(x, y) \equiv \frac{\partial u}{\partial x}$$

for some function $u(x, y)$, since $P(x, y)$ could be integrated partially with respect to x.

But then,

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y} \equiv \frac{\partial^2 u}{\partial y \partial x};$$

and, on integrating partially with respect to x, we obtain

$$Q(x, y) = \frac{\partial u}{\partial y} + A(y),$$

where $A(y)$ is an **arbitrary** function of y.

Thus,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = \frac{\partial u}{\partial x}\mathrm{d}x + \left(\frac{\partial u}{\partial y} + A(y) \right) \mathrm{d}y;$$

and the right-hand side is the exact differential of the function,

$$u(x, y) + \int A(y) \, \mathrm{d}y.$$

(ii) By similar reasoning, it may be shown that the expression

$$P(x,y,z)\mathrm{d}x + Q(x,y,z)\mathrm{d}y + R(x,y,z)\mathrm{d}z$$

is an exact differential, provided that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

ILLUSTRATIONS

1.

$$x\mathrm{d}x + y\mathrm{d}y = \mathrm{d}\left[\frac{1}{2}\left(x^2 + y^2\right)\right].$$

2.

$$y\mathrm{d}x + x\mathrm{d}y = \mathrm{d}[xy].$$

3.

$$y\mathrm{d}x - x\mathrm{d}y$$

is not an exact differential since

$$\frac{\partial y}{\partial y} = 1 \quad \text{and} \quad \frac{\partial(-x)}{\partial x} = -1.$$

4.

$$2\ln y\mathrm{d}x + (x+z)\mathrm{d}y + z^2\mathrm{d}z$$

is not an exact differential since

$$\frac{\partial(2\ln y)}{\partial y} = \frac{2}{y}, \quad \text{and} \quad \frac{\partial(x+z)}{\partial x} = 1.$$

14.4.3 INTEGRATION OF EXACT DIFFERENTIALS

In section 14.4.2, the second half of the proof of the condition for the expression,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y,$$

to be an exact differential suggests, also, a method of determining which function, $f(x, y)$, it is the total differential of. The method may be illustrated by the following examples:

EXAMPLES

1. Verify that the expression,

$$(x + y\cos x)\mathrm{d}x + (1 + \sin x)\mathrm{d}y,$$

is an exact differential, and obtain the function of which it is the total differential.
Solution
Firstly,

$$\frac{\partial}{\partial y}(x + y\cos x) \equiv \frac{\partial}{\partial x}(1 + \sin x) \equiv \cos x;$$

and, hence, the expression is an exact differential.
Secondly, suppose that the expression is the total differential of the function, $f(x, y)$.
Then,

$$\frac{\partial f}{\partial x} \equiv x + y\cos x \quad - - - - - - - - - - (1)$$

and

$$\frac{\partial f}{\partial y} \equiv 1 + \sin x. \quad - - - - - - - - - - -(2)$$

Integrating (1) partially with respect to x gives

$$f(x, y) \equiv \frac{x^2}{2} + y\sin x + A(y),$$

where $A(y)$ is an **arbitrary** function of y only.

Substituting this result into (2) gives

$$\sin x + \frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1 + \sin x.$$

That is,

$$\frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1;$$

and, hence,

$$A(y) \equiv y + \text{constant}.$$

We conclude that

$$f(x, y) \equiv \frac{x^2}{2} + y \sin x + y + \text{constant}.$$

2. Verify that the expression,

$$(yz + 2)\mathrm{d}x + (xz + 6y)\mathrm{d}y + (xy + 3z^2)\mathrm{d}z,$$

is an exact differential and obtain the function of which it is the total differential.

Solution

Firstly,

$$\frac{\partial}{\partial y}(yz + 2) \equiv \frac{\partial}{\partial x}(xz + 6y) \equiv z,$$

$$\frac{\partial}{\partial z}(xz + 6y) \equiv \frac{\partial}{\partial y}(xy + 3z^2) \equiv x,$$

and

$$\frac{\partial}{\partial x}(xy + 3z^2) \equiv \frac{\partial}{\partial z}(yz + 2) \equiv y,$$

so that the given expression is an exact differential.

Suppose it is the total differential of the function, $F(x, y, z)$.
Then,

$$\begin{aligned}\frac{\partial F}{\partial x} &\equiv yz + 2, \quad ---------(1)\\ \frac{\partial F}{\partial y} &\equiv xz + 6y, \quad --------(2)\\ \frac{\partial F}{\partial z} &\equiv xy + 3z^2. \quad --------(3)\end{aligned}$$

Integrating (1) partially with respect to x gives

$$F(x, y, z) \equiv xyz + 2x + A(y, z),$$

where $A(y, z)$ is an arbitrary function of y and z only.
Substituting this result into both (2) and (3) gives

$$\begin{aligned}xz + \frac{\partial A}{\partial y} &\equiv xz + 6y,\\ xy + \frac{\partial A}{\partial z} &\equiv xy + 3z^2.\end{aligned}$$

That is,

$$\begin{aligned}\frac{\partial A}{\partial y} &\equiv 6y, \quad -------(4)\\ \frac{\partial A}{\partial z} &\equiv 3z^2. \quad -------(5)\end{aligned}$$

Integrating (4) partially with respect to y gives

$$A(y, z) \equiv 3y^2 + B(z),$$

where $B(z)$ is an arbitrary function of z only.
Substituting this result into (5) gives

$$\frac{\mathrm{d}B}{\mathrm{d}z} \equiv 3z^2,$$

which implies that

$$B(z) \equiv z^3 + \text{constant}.$$

We conclude that

$$F(x, y, z) \equiv xyz + 2x + 3y^2 + z^3 + \text{constant}.$$

14.4.4 EXERCISES

1. Verify which of the following are exact differentials and integrate those which are:

   (a)
   $$5x + 12y - 9)\mathrm{d}x + (2x + 5y - 4)\mathrm{d}y;$$
   (b)
   $$12x + 5y - 9)\mathrm{d}x + (5x + 2y - 4)\mathrm{d}y;$$
   (c)
   $$(3x^2 + 2y + 1)\mathrm{d}x + (2x + 6y^2 + 2)\mathrm{d}y;$$
   (d)
   $$(y - e^x)\mathrm{d}x + x\mathrm{d}y;$$
   (e)
   $$\frac{1}{x}\mathrm{d}x - \left(\frac{y}{x^2} + 2x\right)\mathrm{d}y;$$
   (f)
   $$\cos(x + y)\mathrm{d}x + \cos(y - x)\mathrm{d}y;$$
   (g)
   $$(1 - \cos 2x)\mathrm{d}y + 2y \sin 2x\mathrm{d}x.$$

2. Verify that the expression,

   $$3x^2\mathrm{d}x + 2yz\mathrm{d}y + y^2\mathrm{d}z,$$

   is an exact differential and obtain the function of which it is the total differential.

3. Verify that the expression,

$$e^{xy}[y \sin z\mathrm{d}x + x \sin z\mathrm{d}y + \cos z\mathrm{d}z],$$

is an exact differential and obtain the function of which it is the total differential.

14.4.5 ANSWERS TO EXERCISES

1. (a) Not exact;
   (b)
   $$6x^2 + 5xy - 9x + y^2 - 4y + \text{constant};$$
   (c)
   $$x^3 + 2xy + x + 2y^3 + 2y + \text{constant};$$
   (d)
   $$xy - e^x + \text{constant};$$
   (e) Not exact;
   (f) Not exact;
   (g)
   $$y(1 - \cos 2x) + \text{constant}.$$
2. $$x^3 + y^2 z;$$
3. $$e^{xy} \sin z.$$

"JUST THE MATHS"

UNIT NUMBER

14.5

PARTIAL DIFFERENTIATION 5
(Partial derivatives of composite functions)

by

A.J.Hobson

UNIT 14.5 - PARTIAL DIFFERENTIATION 5

PARTIAL DERIVATIVES OF COMPOSITE FUNCTIONS

14.5.1 SINGLE INDEPENDENT VARIABLES

In this Unit, we shall be concerned with functions, $f(x, y...)$, of two or more variables in which those variables are not independent, but are themselves dependent on some other variable, t.

The problem is to calculate the rate of increase (positive or negative) of such functions with respect to t.

Let us suppose that the variable, t, is subject to a small increment of δt, so that the variables $x, y...$ are subject to small increments of $\delta x,\ \delta y,\ \ldots$, respectively. Then the corresponding increment, δf, in $f(x, y...)$ is given by

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

where we note that no label other than f is being used, here, for the function of several variables. That is, it is not essential to use a specific **formula**, such as $w = f(x, y..)$.

Dividing throughout by δt gives

$$\frac{\delta f}{\delta t} \simeq \frac{\partial f}{\partial x}.\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}.\frac{\delta y}{\delta t} + \ldots$$

Allowing δt to tend to zero, we obtain the standard result for the **"total derivative"** of $f(x, y..)$ with respect to t, namely

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \ldots$$

This rule may be referred to as the **"chain rule"**, but more advanced versions of it will appear later.

EXAMPLES

1. A point, P, is moving along the curve of intersection of the surface whose cartesian equation is

$$\frac{x^2}{16} - \frac{y^2}{9} = z \quad \text{(a Paraboloid)}$$

and the surface whose cartesian equation is

$$x^2 + y^2 = 5 \quad \text{(a Cylinder)}.$$

If x is increasing at 0.2 cms/sec, how fast is z changing when $x = 2$?

Solution

We may use the formula

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t},$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2 \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

But, from the equation of the paraboloid,

$$\frac{\partial z}{\partial x} = \frac{x}{8} \ \text{ and } \ \frac{\partial z}{\partial y} = -\frac{2y}{9};$$

and, from the equation of the cylinder,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Substituting $x = 2$ gives $y = \pm 1$ on the curve of intersection, so that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{2}{8}\right)(0.2) + \left(-\frac{2}{9}\right)(\pm 1)(0.2)\left(\frac{-2}{\pm 1}\right) = 0.2\left(\frac{1}{4} + \frac{4}{9}\right) = \frac{5}{36} \text{ cms/sec.}$$

2. Determine the total derivative of u with respect to t in the case when

$$u = xy + yz + zx, \quad x = e^t, \quad y = e^{-t} \text{ and } z = x + y.$$

Solution

We may use the formula

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}.\frac{\mathrm{d}z}{\mathrm{d}t},$$

where

$$\frac{\partial u}{\partial x} = y + z, \quad \frac{\partial u}{\partial y} = z + x, \quad \frac{\partial u}{\partial z} = x + y$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^t = x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -e^{-t} = -y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = e^t - e^{-t} = x - y.$$

Hence,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (y + z)x - (z + x)y + (x + y)(x - y)$$

$$= -zy + zx + x^2 - y^2$$

$$= z(x - y) + (x - y)(x + y).$$

That is,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (x - y)(x + y + z).$$

14.5.2 SEVERAL INDEPENDENT VARIABLES

We may now extend the work of the previous section to functions, $f(x, y..)$, of two or more variables in which $x, y..$ are each dependent on two or more variables, $s, t..$

Since the function, $f(x, y..)$, is dependent on $s, t..$, we may wish to determine its **partial** derivatives with respect to any one of these (independent) variables.

The result previously established for a **single** independent variable may easily be adapted as follows:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial s} + \ldots$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial t} + \ldots$$

Again, this is referred to as the **"chain rule"**.

EXAMPLES

1. Determine the first-order partial derivatives of z with respect to r and θ in the case when

$$z = x^2 + y^2, \quad \text{where} \quad x = r\cos\theta \quad \text{and} \quad y = r\sin 2\theta.$$

Solution
We may use the formulae

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \theta}.$$

These give

$$\text{(i)} \quad \frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin 2\theta$$

$$= 2r\left(\cos^2\theta + \sin^2 2\theta\right)$$

and

$$\text{(ii)}\ \frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) + 2y(2r\cos 2\theta)$$

$$= 2r^2\left(2\cos 2\theta\sin 2\theta - \cos\theta\sin\theta\right).$$

2. Determine the first-order partial derivatives of w with respect to u, θ and ϕ in the case when

$$w = x^2 + 2y^2 + 2z^2,$$

where

$$x = u\sin\phi\cos\theta,\quad y = u\sin\phi\sin\theta\quad \text{and}\quad z = u\cos\phi.$$

Solution

We may use the formulae

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \theta}$$

and

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \phi}.$$

These give

$$\text{(i)}\ \frac{\partial w}{\partial u} = 2x\sin\phi\cos\theta + 4y\sin\phi\sin\theta + 4z\cos\phi$$

$$= 2u\sin^2\phi\cos^2\theta + 4u\sin^2\phi\sin^2\theta + 4u\cos^2\phi;$$

$$\text{(ii)} \quad \frac{\partial w}{\partial \theta} = -2xu\sin\phi\sin\theta + 4yu\sin\phi\cos\theta$$

$$= -2u^2\sin^2\phi\sin\theta\cos\theta + 4u^2\sin^2\phi\sin\theta\cos\theta$$

$$= 2u^2\sin^2\phi\sin\theta\cos\theta;$$

$$\text{(iii)} \quad \frac{\partial w}{\partial \phi} = 2xu\cos\phi\cos\theta + 4yu\cos\phi\sin\theta - 4zu\sin\phi$$

$$= 2u^2\sin\phi\cos\phi\cos^2\theta + 4u^2\sin\phi\cos\phi\sin^2\theta - 4u^2\sin\phi\cos\phi$$

$$= 2u^2\sin\phi\cos\phi\left(\cos^2\theta + 2\sin^2\theta - 2\right).$$

14.5.3 EXERCISES

1. Determine the total derivative of z with respect to t in the cases when

   (a)
   $$z = x^2 + 3xy + 5y^2, \quad \text{where} \quad x = \sin t \quad \text{and} \quad y = \cos t;$$

   (b)
   $$z = \ln(x^2 + y^2), \quad \text{where} \quad x = e^{-t} \quad \text{and} \quad y = e^t;$$

   (c)
   $$z = x^2y^2 \quad \text{where} \quad x = 2t^3 \quad \text{and} \quad y = 3t^2.$$

2. If $z = f(x, y)$, show that, when y is a function of x,
   $$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x}.$$

   Hence, determine $\frac{\mathrm{d}z}{\mathrm{d}x}$ in the case when $z = xy + x^2y$ and $y = \ln x$.

3. The base radius, r, of a cone is decreasing at a rate of 0.1cms/sec while the perpendicular height, h, is increasing at a rate of 0.2cms/sec. Determine the rate at which the volume, V, is changing when $r = 2$cm and $h = 3$cm. (**Hint:** $V = (\pi r^2 h)/3$).

4. A rectangular solid has sides of lengths 3cms, 4cms and 5cms. Determine the rate of increase of the length of the diagonal of the solid if the sides are increasing at rates of $\frac{1}{3}$cms./sec, $\frac{1}{4}$cms./sec and $\frac{1}{5}$cms/sec, respectively.

5. If
$$z = (2x + 3y)^2 \quad \text{where} \quad x = r^2 - s^2 \quad \text{and} \quad y = 2rs,$$
determine, in terms of r and s the first-order partial derivatives of z with respect to r and s.

6. If
$$z = f(x, y) \quad \text{where} \quad x = e^u \cos v \quad \text{and} \quad y = e^u \sin v,$$
show that
$$\frac{\partial z}{\partial u} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial z}{\partial v} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

7. If
$$w = 5x - 3y^2 + 7z^3 \quad \text{where} \quad x = 2s + 3t, \quad y = s - t \quad \text{and} \quad z = 4s + t,$$
determine, in terms of s and t, the first order partial derivatives of w with respect to s and t.

14.5.4 ANSWERS TO EXERCISES

1. (a)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 3\cos 2t - 4\sin 2t;$$
(b)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 2\left[\frac{e^{4t} - 1}{e^{4t} + 1}\right];$$
(c)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 360t^9.$$

2.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^2 + 2xy + 1 + x.$$

3. The volume is decreasing at a rate of approximately 0.42 cubic centimetres per second.
4. The diagonal is increasing at a rate of approximately 0.42 centimetres per second.
5.

$$\frac{\partial z}{\partial r} = 8(r^2 - s^2 + 3rs)(2r + 3s) \quad \text{and} \quad \frac{\partial z}{\partial s} = 8(r^2 - s^2 + 3rs)(3r - 2s).$$

6. Results follow immediately from the formulae

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}.$$

7.

$$\frac{\partial w}{\partial s} = 1344s^2 + 672st + 84t^2 - 6s + 6t + 10$$

and

$$\frac{\partial w}{\partial t} = 336s^2 + 168st + 21t^2 + 6s - 6t + 15.$$

"JUST THE MATHS"

UNIT NUMBER

14.6

PARTIAL DIFFERENTIATION 6
(Implicit functions)

by

A.J.Hobson

UNIT 14.6 - PARTIAL DIFFERENTIATION 6

IMPLICIT FUNCTIONS

14.6.1 FUNCTIONS OF TWO VARIABLES

The chain rule, encountered earlier, has a convenient application to implicit relationships of the form,

$$f(x,y) = \text{constant},$$

between two independent variables, x and y.

It provides a means of determining the total derivative of y with respect to x.

Explanation

Taking x as the single independent variable, we may interpret $f(x,y)$ as a function of x and y in which both x and y are functions of x.

Differentiating both sides of the relationship, $f(x,y) = \text{constant}$, with respect to x gives

$$\frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

In other words,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

EXAMPLES

1. If
$$f(x,y) \equiv x^3 + 4x^2y - 3xy + y^2 = 0,$$
determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Solution
$$\frac{\partial f}{\partial x} = 3x^2 + 8xy - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x^2 - 3x + 2y.$$
Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 + 8xy - 3y}{4x^2 - 3x + 2y}.$$

2. If
$$f(x,y) \equiv x\sin(2x - 3y) + y\cos(2x - 3y),$$
determine and expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Solution
$$\frac{\partial f}{\partial x} = \sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)$$
and
$$\frac{\partial f}{\partial y} = -3x\cos(2x - 3y) + \cos(2x - 3y) + 3y\sin(2x - 3y).$$
Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)}{3x\cos(2x - 3y) - \cos(2x - 3y) - 3y\sin(2x - 3y)}.$$

14.6.2 FUNCTIONS OF THREE VARIABLES

For relationships of the form,

$$f(x,y,z) = \text{constant},$$

let us suppose that x and y are independent of each other.

Then, regarding $f(x, y, z)$ as a function of x, y and z, where x, y and z are **all** functions of x and y, the chain rule gives

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0.$$

But,

$$\frac{\partial x}{\partial x} = 1 \quad \text{and} \quad \frac{\partial y}{\partial x} = 0.$$

Hence,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0,$$

giving

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}};$$

and, similarly,

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

EXAMPLES

1. If

$$f(x, y, z) \equiv z^2xy + zy^2x + x^2 + y^2 = 5,$$

determine expressions for $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$.

Solution

$$\frac{\partial f}{\partial x} = z^2y + zy^2 + 2x,$$

$$\frac{\partial f}{\partial y} = z^2x + 2zyx + 2y$$

and

$$\frac{\partial f}{\partial z} = 2zxy + y^2x.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{z^2y + zy^2 + 2x}{2zxy + y^2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{z^2x + 2zyx + 2y}{2zxy + y^2x}.$$

2. If

$$f(x, y, z) \equiv xe^{y^2+2z},$$

determine expressions for $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$.

Solution

$$\frac{\partial f}{\partial x} = e^{y^2+2z},$$

$$\frac{\partial f}{\partial y} = 2yxe^{y^2+2z},$$

and

$$\frac{\partial f}{\partial z} = 2xe^{y^2+2z}.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{e^{y^2+2z}}{2xe^{y^2+2z}} = -\frac{1}{2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{2yxe^{y^2+2z}}{2xe^{y^2+2z}} = -y.$$

14.6.3 EXERCISES

1. Use partial differentiation to determine expressions for $\frac{dy}{dx}$ in the following cases:

   (a)
   $$x^3 + y^3 - 2x^2y = 0;$$

   (b)
   $$e^x \cos y = e^y \sin x;$$

   (c)
   $$\sin^2 x - 5 \sin x \cos y + \tan y = 0.$$

2. If
   $$x^2y + y^2z + z^2x = 10,$$
   where x and y are independent, determine expressions for
   $$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

3. If
   $$xyz - 2\sin(x^2 + y + z) + \cos(xy + z^2) = 0,$$
   where x and y are independent, determine expressions for
   $$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

4. If
   $$r^2 \sin\theta = (r\cos\theta - 1)z,$$
   where r and θ are independent, determine expressions for
   $$\frac{\partial z}{\partial r} \text{ and } \frac{\partial z}{\partial \theta}.$$

14.6.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4xy - 3x^2}{3y^2 - 2x^2};$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x \cos y - e^y \cos x}{x^x \sin y + e^y \sin x};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5 \cos x \cos y - 2 \sin x \cos x}{5 \sin x siny + \sec^2 y}.$$

2.

$$\frac{\partial z}{\partial x} = -\frac{2xy + z^2}{y^2 + 2zx}$$

and

$$\frac{\partial z}{\partial y} = -\frac{x^2 + 2yz}{y^2 + 2zx}.$$

3.

$$\frac{\partial z}{\partial x} = -\frac{yz - 4x \cos(x^2 + y + z) - y \sin(xy + z^2)}{xy - 2 \cos(x^2 + y + z) - 2z \sin(xy + z^2)}$$

and

$$\frac{\partial z}{\partial y} = -\frac{xz - 2 \cos(x^2 + y + z) - x \sin(xy + z^2)}{xy - 2 \cos(x^2 + y + z) - 2z \sin(xy + z^2)}.$$

4.

$$\frac{\partial z}{\partial r} = \frac{2r \sin \theta - z \cos \theta}{r \cos \theta - 1}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{r^2 \cos \theta + rz \sin \theta}{r \cos \theta - 1}.$$

"JUST THE MATHS"

UNIT NUMBER

14.7

PARTIAL DIFFERENTIATION 7
(Change of independent variable)

by

A.J.Hobson

UNIT 14.7 - PARTIAL DIFFERENTIATON 7

CHANGE OF INDEPENDENT VARIABLE

14.7.1 ILLUSTRATIONS OF THE METHOD

In the theory of **"partial differential equations"** (that is, equations which involve partial derivatives) it is sometimes required to express a given equation in terms of a new set of independent variables. This would be necessary, for example, in changing a discussion from one geometrical reference system to another. The method is an application of the chain rule for partial derivatives and we illustrate it with examples.

EXAMPLES

1. Express, in plane polar co-ordinates, r and θ, the following partial differential equations:

   (a)

$$\frac{\partial V}{\partial x} + 5\frac{\partial V}{\partial y} = 1;$$

   (b)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

   Solution

   Both differential equations involve a function, $V(x, y)$, where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

   Hence,

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial r},$$

   or

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cos\theta + \frac{\partial V}{\partial y}\sin\theta$$

   and

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial \theta},$$

or

$$\frac{\partial V}{\partial \theta} = -\frac{\partial V}{\partial x} r\sin\theta + \frac{\partial V}{\partial y} r\cos\theta.$$

Now, we may eliminate, first $\frac{\partial V}{\partial y}$, and then $\frac{\partial V}{\partial x}$ to obtain

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}$$

and

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}.$$

Hence, differential equation, (a), becomes

$$(\cos\theta + 5\sin\theta)\frac{\partial V}{\partial r} + \left(\frac{5\cos\theta}{r} - \sin\theta\right)\frac{\partial V}{\partial \theta} = 1.$$

In order to find the <u>second</u>-order derivatives of V with respect to x and y, it is necessary to write the formulae for the first-order derivatives in the form

$$\frac{\partial}{\partial x}[V] = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)[V]$$

and

$$\frac{\partial}{\partial y}[V] = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)[V].$$

From these, we obtain

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right),$$

which gives

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Adding these together gives the differential equation, (b), in the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0.$$

2. Express the differential equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

(a) in cylindrical polar co-ordinates,

and

(b) in spherical polar co-ordinates.

Solution

(a) Using

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z,$$

we may use the results of the previous example to give

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

(b) Using

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta, \quad \text{and} \quad z = u\cos\phi,$$

we could write out three formulae for $\frac{\partial V}{\partial u}$, $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi}$ and then solve for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$; but this is complicated.

However, the result in part (a) provides a shorter method as follows:

Cylindrical polar co-ordinates are expressible in terms of spherical polar co-ordinates by the formulae

$$z = u\cos\phi, \quad r = u\sin\phi, \quad \theta = \theta.$$

Hence, by using the previous example with z, r, θ in place of x, y, z respectively and u, ϕ in place of r, θ, respectively, we obtain

$$\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial r^2}=\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}.$$

Therefore, to complete the conversion we need only to consider $\frac{\partial V}{\partial r}$; and, by using r, u, ϕ in place of y, r, θ, respectively, the previous formula for $\frac{\partial V}{\partial y}$ gives

$$\frac{\partial V}{\partial r}=\sin\phi\frac{\partial V}{\partial u}+\frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}.$$

The given differential equation thus becomes

$$\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{1}{u^2\sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{u\sin\phi}\left[\sin\phi\frac{\partial V}{\partial u}+\frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}\right]=0.$$

That is,

$$\frac{\partial^2 V}{\partial u^2}+\frac{2}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\cot\phi}{u^2}\frac{\partial V}{\partial \phi}+\frac{1}{u^2\sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}=0.$$

14.7.2 EXERCISES

1. Express the partial differential equation,

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial x}=0,$$

in plane polar co-ordinates, r and θ, where

$$x=r\cos\theta \quad \text{and} \quad y=r\sin\theta.$$

2. Express the differential equation,

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=\frac{\partial V}{\partial z},$$

in spherical polar co-ordinates u, θ and ϕ, where

$$x=u\sin\phi\cos\theta, \quad y=u\sin\phi\sin\theta \quad \text{and} \quad z=u\cos\phi.$$

3. A function $\phi(x,t)$ satisfies the partial differential equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{k^2}\frac{\partial^2\phi}{\partial t^2},$$

where k is a constant.

Express this equation in terms of new independent variables, u and v, where

$$x = \frac{1}{2}(u+v) \quad \text{and} \quad t = \frac{1}{2k}(u-v).$$

4. A function $\theta(x,y)$ satisfies the partial differential equation,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Express this equation in terms of new independent variables, s and t, where

$$x = \ln u \quad \text{and} \quad y = \ln v.$$

Determine, also, an expression for $\frac{\partial^2\theta}{\partial x\partial y}$ in terms of θ, u and v.

14.7.3 ANSWERS TO EXERCISES

1.

$$\frac{\partial V}{\partial r} = 0.$$

2.

$$(\sin\phi - \cos\phi)\frac{\partial V}{\partial u} - \frac{\cos\phi + \sin\phi}{u}\frac{\partial V}{\partial\phi} = 0.$$

3.

$$\frac{\partial^2\phi}{\partial u\partial v} = 0.$$

4.

$$u^2\frac{\partial^2\theta}{\partial x^2} + v^2\frac{\partial^2\theta}{\partial v^2} + u\frac{\partial\theta}{\partial u} + v\frac{\partial\theta}{\partial v} = 0,$$

and

$$\frac{\partial^2\theta}{\partial x\partial y} = uv\frac{\partial^2\theta}{\partial u\partial v}.$$

"JUST THE MATHS"

UNIT NUMBER

14.8

PARTIAL DIFFERENTIATION 8
(Dependent and independent functions)

by

A.J.Hobson

UNIT 14.8 - PARTIAL DIFFERENTIATION 8

DEPENDENT AND INDEPENDENT FUNCTIONS

14.8.1 THE JACOBIAN

Suppose that

$$u \equiv u(x,y) \quad \text{and} \quad v \equiv v(x,y)$$

are two functions of two independent variables, x and y; then, in general, it is not possible to express u solely in terms of v, nor v solely in terms of u.

However, on occasions, it may be possible, as the following illustrations demonstrate:

ILLUSTRATIONS

1. If

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y},$$

then

$$u \equiv 1 + \frac{x}{y} \quad \text{and} \quad v \equiv \frac{x}{y} - 1,$$

which gives

$$(u-1)(v+1) \equiv \frac{x}{y}.\frac{y}{x} \equiv 1.$$

Hence,

$$u \equiv 1 + \frac{1}{v+1} \quad \text{and} \quad v \equiv \frac{1}{u-1} - 1.$$

2. If

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2,$$

then

$$v \equiv u^2 \quad \text{and} \quad u \equiv \pm\sqrt{v}.$$

If u and v are **not** connected by an identical relationship, they are said to be **"independent functions"**.

THEOREM

Two functions, $u(x,y)$ and $v(x,y)$, are independent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \not\equiv 0.$$

Proof:

We prove an equivalent statement, namely that $u(x,y)$ and $v(x,y)$ are dependent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0.$$

(a) Suppose that v is dependent on u by virtue of the relationship

$$v \equiv v(u).$$

By expressing $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we shall establish that the determinant, J, is identically equal to zero.

We have

$$\frac{\partial v}{\partial x} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial y}.$$

Thus,

$$\frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}} \equiv \frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}$$

or

$$\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}.\frac{\partial u}{\partial y} \equiv 0,$$

which means that the determinant, J, is identically equal to zero.

(b) Secondly, let us suppose that

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0,$$

and attempt to prove that $u(x, y)$ and $v(x, y)$ are dependent.

In theory, we could express v in terms of u and x by eliminating y between $u(x, y)$ and $v(x, y)$.

We shall assume that

$$v \equiv A(u, x).$$

By expressing $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we may show that $A(u, x)$ does not contain x.

We have

$$\left(\frac{\partial v}{\partial x}\right)_y \equiv \left(\frac{\partial A}{\partial u}\right)_x . \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u$$

and

$$\left(\frac{\partial v}{\partial y}\right)_x \equiv \left(\frac{\partial A}{\partial u}\right)_x . \left(\frac{\partial u}{\partial y}\right)_x .$$

Hence, if the determinant, J, is identically equal to zero, we may say that

$$\begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial A}{\partial u}\right)_x . \left(\frac{\partial u}{\partial y}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u & \left(\frac{\partial A}{\partial u}\right)_x . \left(\frac{\partial u}{\partial y}\right)_x \end{vmatrix} \equiv 0;$$

and, on expansion, this gives

$$\left(\frac{\partial u}{\partial y}\right)_x . \left(\frac{\partial A}{\partial x}\right)_u \equiv 0.$$

If the first of these two is equal to zero, then u contains only x and, hence, x could be expressed in terms of u, giving v as a function of u only. If the second is equal to zero, then A contains no x's and, again, v is a function of u only.

Notes:

(i) The determinant

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

may also be denoted by

$$\frac{\partial(u,v)}{\partial(x,y)}$$

and is called the **"Jacobian determinant"** or simply the **"Jacobian"** of u and v with respect to x and y.

(ii) Similar Jacobian determinants may be used to test for the dependence or independence of three functions of three variables, four functions of four variables, and so on.

For example, the three functions

$$u \equiv u(x,y,z), \quad v \equiv v(x,y,z) \ \text{ and } \ w \equiv w(x,y,z)$$

are independent if and only if

$$J \equiv \frac{\partial(u,v,w)}{\partial(x,y,z)} \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \not\equiv 0.$$

ILLUSTRATIONS

1.

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y}$$

are **not** independent, since

$$J \equiv \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} \equiv \frac{1}{xy} - \frac{1}{xy} \equiv 0$$

2.

$$u \equiv x+y \quad \text{and} \quad v \equiv x^2+2xy+y^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 1 & 1 \\ 2x+2y & 2x+2y \end{vmatrix} \equiv 0.$$

3.

$$u \equiv x^2+2y \quad \text{and} \quad v \equiv xy$$

are independent, since

$$J \equiv \begin{vmatrix} 2x & 2 \\ y & x \end{vmatrix} \equiv 2x^2-2y \not\equiv 0.$$

4.

$$u \equiv x^2-2y+z, \quad v \equiv x+3y^2-2z, \quad \text{and} \quad w \equiv 5x+y+z^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 2x & -2 & 1 \\ 1 & 6y & -2 \\ 5 & 1 & 2z \end{vmatrix} \equiv 24xyz+4x-30y+4z+25 \not\equiv 0.$$

14.8.2 EXERCISES

1. Determine which of the following pairs of functions are independent:

   (a)
$$u \equiv x\cos y \quad \text{and} \quad v \equiv x\sin y;$$

   (b)
$$u \equiv x + y \quad \text{and} \quad v \equiv \frac{y}{x+y};$$

   (c)
$$u \equiv x - 2y \quad \text{and} \quad v \equiv x^2 + 4y^2 - 4xy + 3x - 6y;$$

   (d)
$$u \equiv x + 2y \quad \text{and} \quad v \equiv x^2 - y^2 + 2xy - x.$$

2. Show that
$$u \equiv x + y + z, \quad v \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$
and
$$w \equiv x^3 + y^3 + z^3 - 3xyz$$
are dependent.
Show also that w may be expressed as a linear combination of u^3 and uv.

3. Given that
$$x + y + z \equiv u, \quad y + z \equiv uv \quad \text{and} \quad z \equiv uvw,$$
express x and y in terms of u, v and w.
Hence, show that
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv u^2 v.$$

14.8.3 ANSWERS TO EXERCISES

1. (a) Independent, since $J \equiv x$;
   (b) Independent, since $J \equiv 1/(x+y)$;
   (c) Dependent, since $J \equiv 0$;
   (d) Independent, since $J \equiv 2 - 2x - 6y$.

2.
$$w \equiv \frac{1}{4}\left[u^3 + 3uv\right].$$

3.
$$x \equiv u - uv \quad \text{and} \quad y \equiv uv - uvw.$$

"JUST THE MATHS"

UNIT NUMBER

14.9

PARTIAL DIFFERENTIATION 9
(Taylor's series)
for
(Functions of several variables)

by

A.J.Hobson

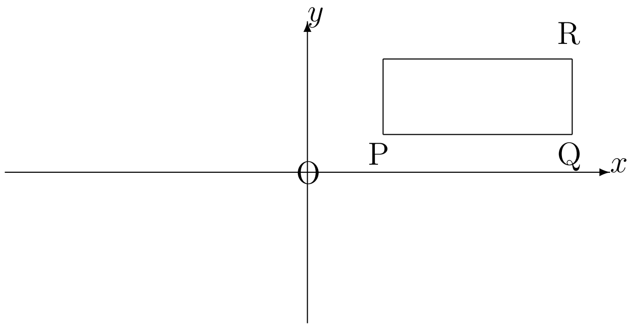
UNIT 14.9 - PARTIAL DIFFERENTIATION 9

TAYLOR'S SERIES FOR FUNCTIONS OF SEVERAL VARIABLES

14.9.1 THE THEORY AND FORMULA

Initially, we shall consider a function, $f(x, y)$, of **two** independent variables, x, y, and obtain a formula for $f(x + h, y + k)$ in terms of $f(x, y)$ and its partial derivatives.

Suppose that P,Q and R denote the points with cartesian co-ordinates, (x, y), $(x + h, y)$ and $(x + h, y + k)$, respectively.



(a) As we move in a straight line from P to Q, y remains constant so that $f(x, y)$ behaves as a function of x only.

Hence, by Taylor's theorem for one independent variable,

$$f(x + h, y) = f(x, y) + f_x(x, y) + \frac{h^2}{2!} f_{xx}(x, y) + \ldots,$$

where $f_x(x, y)$ and $f_{xx}(x, y)$ mean $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ respectively, with similar notations encountered in what follows.

In abbreviated notation,

$$f(\mathrm{Q}) = f(\mathrm{P}) + hf_x(\mathrm{P}) + \frac{h^2}{2!} f_{xx}(\mathrm{P}) + \ \ldots$$

(b) As we move in a straight line from Q to R, x remains constant so that $f(x, y)$ behaves as a function of y only.

Hence,

$$f(x+h, y+k) = f(x+h, y) + kf_x(x+h, y) + \frac{k^2}{2!}f_{xx}(x+h, y) + \ldots;$$

or, in abbreviated notation,

$$f(\mathrm{R}) = f(\mathrm{Q}) + kf_y(\mathrm{Q}) + \frac{k^2}{2!}f_{yy}(\mathrm{Q}) + \ldots$$

(c) From the result in (a)

$$f_y(\mathrm{Q}) = f_y(\mathrm{P}) + hf_{yx}(\mathrm{P}) + \frac{h^2}{2!}f_{yxx}(\mathrm{P}) + \ldots$$

and

$$f_{yy}(\mathrm{Q}) = f_{yy}(\mathrm{P}) + hf_{yyx}(\mathrm{P}) + \frac{h^2}{2!}f_{yyxx}(\mathrm{Q}) + \ldots$$

(d) Substituting the results into (b) gives

$$f(\mathrm{R}) = f(\mathrm{P}) + hf_x(\mathrm{P}) + kf_y(\mathrm{P}) + \frac{1}{2!}\left[h^2 f_{xx}(\mathrm{P}) + 2hkf_{yx}(\mathrm{P}) + k^2 f_{yy}(\mathrm{P})\right] + \ldots$$

It may be shown that the complete result can be written as

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x, y) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x, y) + \ \ldots$$

Notes:

(i) The equivalent of this result for a function of three variables would be

$$f(x+h,y+k,z+l) = f(x,y,z) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right) f(x,y,z) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(x,y,z) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^3 f(x,y,z) + \ \ldots$$

(ii) Alternative versions of Taylor's theorem may be obtained by interchanging $x, y, z...$ with $h, k, l....$

For example,

$$f(x+h,y+k) = f(h,k) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(h,k) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

(iii) Replacing x with $x-h$ and y with $y-k$ in (ii) gives the formula,

$$f(x,y) = f(h,k) + \left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^2 f(h,k) + \frac{1}{3!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

This is called the **"Taylor expansion of $f(x,y)$ about the point (a,b)"**

(iv) A special case of Taylor's series (for two independent variables) is obtained by putting $h = 0$ and $k = 0$ in (ii) to give

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \ldots$$

This is called a **"MacLaurin's series"** but is also the Taylor expansion of $f(x,y)$ about the point $(0,0)$.

EXAMPLE

Determine the Taylor series expansion of the function $f\left(x+1, y+\frac{\pi}{3}\right)$ in ascending powers of x and y when

$$f(x,y) \equiv \sin xy,$$

neglecting terms of degree higher than two.

Solution

We use the result that

$$f\left(x+1, y+\frac{\pi}{3}\right) = f\left(1,\frac{\pi}{3}\right) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f\left(1,\frac{\pi}{3}\right) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f\left(1,\frac{\pi}{3}\right) + \ldots,$$

in which the first term on the right has value $\sqrt{3}/2$.

The partial derivatives required are as follows:

$$\frac{\partial f}{\partial x} \equiv y\cos xy \ \ \text{giving} \ \ -\frac{\pi}{6} \ \ \text{at} \ \ x = 1, \ \ y = \frac{\pi}{3};$$

$$\frac{\partial f}{\partial y} \equiv x\cos xy \ \ \text{giving} \ \ \frac{1}{2} \ \ \text{at} \ \ x = 1, \ \ y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x^2} \equiv -y^2\sin xy \ \ \text{giving} \ \ -\frac{\pi^2\sqrt{3}}{18} \ \ \text{at} \ \ x = 1, \ \ y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \cos xy - xy \sin xy \quad \text{giving} \quad \frac{1}{2} - \frac{\pi\sqrt{3}}{6} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial y^2} \equiv -x^2 \sin xy \quad \text{giving} \quad -\frac{\sqrt{3}}{2} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3}.$$

Neglecting terms of degree higher than two, we have

$$\sin xy = \frac{\sqrt{3}}{2} + \frac{\pi}{6}x + \frac{1}{2}y - \frac{\sqrt{3}\pi^2}{36}x^2 + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)xy - \frac{\sqrt{3}}{4}y^2 + \ldots$$

14.9.2 EXERCISES

1. If $f(x, y) \equiv x^3 - 3xy^2$, show that

$$f(2 + h, 1 + k) = 2 + 9h - 12k + 6\left(h^2 - hk - k^2\right) + h^3 - 3hk^2.$$

2. If $f(x, y) \equiv \sin x \cosh y$, evaluate all the partial derivatives of $f(x, y)$ up to order five at the point, $(x, y) = (0, 0)$, and, hence, show that

$$\sin x \cosh y = x - \frac{1}{6}\left(x^3 - 3xy^2\right) + \frac{1}{120}\left(x^5 - 10x^3y^2 + 5xy^4\right) + \ldots$$

3. If z is a function of two independent variables, x and y, where $y \equiv z - x \sin z$, evaluate all the partial derivatives of $z(x, y)$ up to order three at the point, $(x, y) = (0, 0)$, and, hence, show that

$$z(x, y) = y + xy + x^2y + \ldots$$

"JUST THE MATHS"

UNIT NUMBER

14.10

PARTIAL DIFFERENTIATION 10
(Stationary values)
for
(Functions of two variables)

by

A.J.Hobson

UNIT 14.10 - PARTIAL DIFFERENTIATION 10

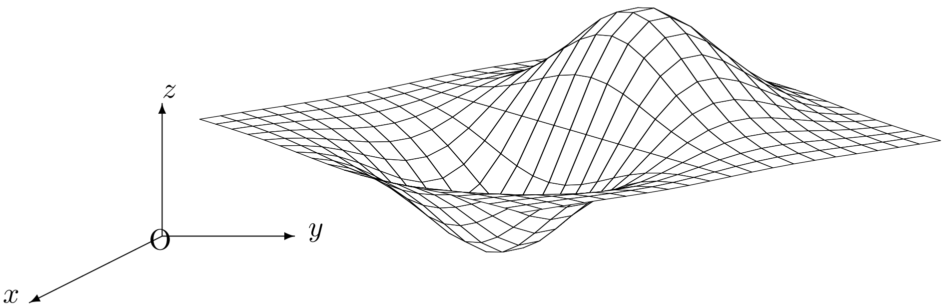
STATIONARY VALUES FOR FUNCTIONS OF TWO VARIABLES

14.10.1 INTRODUCTION

If $f(x, y)$ is a function of the two independent variables, x and y, then the equation,

$$z = f(x, y),$$

will normally represent some surface in space, referred to cartesian axes, Ox, Oy and Oz.



DEFINITION 1

The **"stationary points"**, on a surface whose equation is $z = f(x, y)$, are defined to be the points for which

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0.$$

DEFINITION 2

The function, $z = f(x, y)$, is said to have a **"local maximum"** at a point, $\mathrm{P}(x_0, y_0, z_0)$, if z_0 is larger than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

DEFINITION 3

The function $z = f(x, y)$ is said to have a **"local minimum"** at a point, $\mathrm{P}(x_0, y_0, z_0)$, if z_0 is smaller than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

Note:
At a stationary point, $P(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, each of the planes, $x = x_0$ and $y = y_0$, intersect the surface in a curve which has a stationary point at P.

14.10.2 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA

A complete explanation of the conditions for a function, $z = f(x, y)$, to have a local maximum or a local minimum at a particular point require the use of Taylor's theorem for two variables.

At this stage, we state the standard set of sufficient conditions without proof.

(a) Sufficient conditions for a local maximum

A point, $P(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local maximum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} < 0 \text{ or } \frac{\partial^2 z}{\partial y^2} < 0$$

and

$$\frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

(b) Sufficient conditions for a local minimum

A point, $P(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local minimum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} > 0 \text{ or } \frac{\partial^2 z}{\partial y^2} > 0$$

and

$$\frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

Notes:

(i) If $\frac{\partial^2 z}{\partial x^2}$ is positive (or negative) and also $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, then $\frac{\partial^2 z}{\partial y^2}$ is automatically positive (or negative).

(ii) If it turns out that $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ is **negative** at P, we have what is called a **"saddle-point"**, irrespective of what $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ themselves are.

(iii) The values of z at the local maxima and local minima of the function, $z = f(x, y)$, may also be called the **"extreme values"** of the function, $f(x, y)$.

EXAMPLES

1. Determine the extreme values and the co-ordinates of any saddle-points of the function,

$$z = x^3 + x^2 - xy + y^2 + 4.$$

Solution

(i) First, we determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = 3x^2 + 2x - y \quad \text{and} \quad \frac{\partial z}{\partial y} = -x + 2y.$$

(ii) Secondly, we solve the equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ for x and y.

$$\begin{aligned} 3x^2 + 2x - y &= 0, - - - - - - - - - - -(1) \\ -x + 2y &= 0. - - - - - - - - - - - (2) \end{aligned}$$

Substituting equation (2) into equation (1) gives

$$3x^2 + 2x - \frac{1}{2}x = 0.$$

That is,

$$6x^2 + 3x = 0 \quad \text{or} \quad 3x(2x + 1) = 0.$$

Hence, $x = 0$ or $x = -\frac{1}{2}$, with corresponding values, $y = 0$, $z = 4$ and $y = -\frac{1}{4}$, $z = -\frac{65}{16}$, respectively.

The stationary points are thus $(0, 0, 4)$ and $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$.

(iii) Thirdly, we evaluate $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at each stationary point.

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

(a) At the point $(0, 0, 4)$,

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} > 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 3 > 0$$

and, therefore, the point, $(0, 0, 4)$, is a local minimum, with z having a corresponding extreme value of 4.

(b) At the point $\left(-\frac{1}{2}, -\frac{1}{4}, \frac{65}{16}\right)$,

$$\frac{\partial^2 z}{\partial x^2} = -1, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} < 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -3 < 0$$

and, therefore, the point, $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$, is a saddle-point.

2. Determine the stationary points of the function,

$$z = 2x^3 + 6xy^2 - 3y^3 - 150x,$$

and determine their nature.

Solution

Following the same steps as in the previous example, we have

$$\frac{\partial z}{\partial x} = 6x^2 + 6y^2 - 150 \quad \text{and} \quad \frac{\partial z}{\partial y} = 12xy - 9y^2.$$

Hence, the stationary points occur where x and y are the solutions of the simultaneous equations,

$$\begin{aligned} x^2 + y^2 &= 25, - - - - - - - - - - (1) \\ y(4x - 3y) &= 0. - - - - - - - - - - - (2) \end{aligned}$$

From the second equation, $y = 0$ or $4x = 3y$.

Putting $y = 0$ in the first equation gives $x = \pm 5$ and, with these values of x and y, we obtain stationary points at $(5, 0, -500)$ and $(-5, 0, 500)$.

Putting $x = \frac{3}{4}y$ into the first equation gives $y = \pm 4$, $x = \pm 3$ and, with these values of x and y, we obtain stationary points at $(3, 4, -300)$ and $(-3, -4, 300)$.

To classify the stationary points we require

$$\frac{\partial^2 z}{\partial x^2} = 12x, \quad \frac{\partial^2 z}{\partial y^2} = 12x - 18y, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = 12y,$$

and the conclusions are given in the following table:

Point	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial y^2}$	$\frac{\partial^2 z}{\partial x \partial y}$	$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$	Nature
$(5, 0, -500)$	60	60	0	positive	minimum
$(-5, 0, 500)$	-60	-60	0	positive	maximum
$(3, 4, -300)$	36	-36	48	negative	saddle-point
$(-3, -4, 300)$	-36	36	-48	negative	saddle-point

Note:

The conditions used in the examples above are only **sufficient** conditions; that is, if the conditions are satisfied, we may make a conclusion. But it may be shown that there are stationary points which do **not** satisfy the conditions.

Outline proof of the sufficient conditions

From Taylor's theorem for two variables,

$$f(a+h,b+k)-f(a,b) = hf_x(a,b)+kf_y(a,b)+\frac{1}{2}\left(h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right)+\ldots,$$

where h and k are small compared with a and b, f_x means $\frac{\partial f}{\partial x}$, f_y means $\frac{\partial f}{\partial y}$, f_{xx} means $\frac{\partial^2 f}{\partial x^2}$, f_{yy} means $\frac{\partial^2 f}{\partial y^2}$ and f_{xy} means $\frac{\partial^2 f}{\partial x \partial y}$.

If $f_x(a,b) = 0$ and $f_y(a,b) = 0$, the conditions for a local minimum at the point $(a, b, f(a,b))$ will be satisfied when the second term on the right-hand side is positive; and the conditions for a local maximum at this point are satisfied when the second term on the right is negative.

We assume, here, that later terms of the Taylor series expansion are negligible.

Also, it may be shown that a quadratic expression of the form

$$Lh^2 + 2Mhk + Nk^2$$

is positive when $L > 0$ or $N > 0$ and $LN - M^2 > 0$; but negative when $L < 0$ or $N < 0$ and $LN - M^2 > 0$.

If it happens that $LN - M^2 < 0$, then it may be shown that the quadratic expression may take both positive and negative values.

Finally, replacing L, M and N by $f_{xx}(a,b)$, $f_{yy}(a,b)$ and $f_{xy}(a,b)$ respectively, the sufficient conditions for local maxima, local minima and saddle-points follow.

14.10.3 EXERCISES

1. Show that the function,

$$z = 3x^3 - y^3 - 4x + 3y,$$

has a local minimum value when $x = \frac{2}{3}$, $y = -1$ and calculate this minimum value. What other stationary points are there, and what is their nature ?

2. Determine the smallest value of the function,

$$z = 2x^2 + y^2 - 4x + 8y.$$

3. Show that the function,

$$z = 2x^2y^2 + x^2 + 4y^2 - 12xy,$$

has three stationary points and determine their nature.

4. Investigate the local extreme values of the function,

$$z = x^3 + y^3 + 9(x^2 + y^2) + 12xy.$$

5. Discuss the stationary points of the following functions and, where possible, determine their nature:

(a)

$$z = x^2 - 2xy + y^2;$$

(b)

$$z = xy.$$

Note:
It will not be possible to use all of the standard conditions; and a geometrical argument will be necessary.

14.10.4 ANSWERS TO EXERCISES

1. $\left(\frac{2}{3},-1,-\frac{34}{9}\right)$ is a local minimum;
   $\left(-\frac{2}{3},1,\frac{34}{9}\right)$ is a local maximum;
   $\left(\frac{2}{3},1,\frac{2}{9}\right)$ is a saddle-point;
   $\left(-\frac{2}{3},-1,-\frac{2}{9}\right)$ is a saddle-point.

2. The smallest value is -18, since there is a single local minimum at the point $(1,-4,-18)$.

3. $(0,0,0)$ is a saddle-point;
   $(2,1,-8)$ is a local minimum; (**Hint:** try $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$)
   $(-2,-1,-8)$ is a local minimum.

4. $(0,0,0)$ is a local minimum;
   $(-10,-10,1000)$ is a local maximum; (**Hint**: try $\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}$)
   $(-4,2,28)$ is a saddle-point;
   $(2,-4,28)$ is a saddle-point.

5. (a) Points $(\alpha,\alpha,0)$ are such that $\frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$; but $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=0$. In fact the surface is a "parabolic cylinder" which contains the straight line $x=y,\ z=0$ and is symmetrical about the plane $x=y$.

   (b) $(0,0,0)$ is a saddle-point since z may have both positive and negative values in the neighbourhood of this point.

"JUST THE MATHS"

UNIT NUMBER

14.11

PARTIAL DIFFERENTIATION 11 (Constrained maxima and minima)

by

A.J.Hobson

UNIT 14.11 - PARTIAL DIFFERENTIATION 11

CONSTRAINED MAXIMA AND MINIMA

Having discussed the determination of local maxima and local minima for a function, $f(x, y, ...)$, of several independent variables, we shall now consider that an additional constraint is imposed in the form of a relationship, $g(x, y, ...) = 0$.

This would occur, for example, if we wished to construct a container with the largest possible volume for a fixed value of the surface area.

14.11.1 THE SUBSTITUTION METHOD

The following examples illustrate a technique which may be used in elementary cases:

EXAMPLES

1. Determine any local maxima or local minima of the function,

$$f(x, y) \equiv 3x^2 + 2y^2,$$

subject to the constraint that $x + 2y - 1 = 0$.

Solution

In this kind of example, it is possible to eliminate either x or y by using the constraint. If we eliminate x, for instance, we may write $f(x, y)$ as a function, $F(y)$, of y only.

In fact,

$$f(x, y) \equiv F(y) \equiv 3(1 - 2y)^2 + 2y^2 \equiv 3 - 12y + 14y^2.$$

Using the principles of maxima and minima for functions of a single independent variable, we have

$$F'(y) \equiv 28y - 12 \quad \text{and} \quad F'' \equiv 28$$

and, hence, a local minimum occurs when $y = 3/7$ and hence, $x = 1/7$.

The corresponding local minimum value of $f(x, y)$ is

$$3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{21}{49} = \frac{3}{7}.$$

2. Determine any local maxima or local minima of the function,

$$f(x,y,z) \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Eliminating x, we may write $f(x,y,z)$ as a function, $F(y,z)$, of y and z only.
In fact,

$$f(x,y,z) \equiv F(y,z) \equiv (1 - 2y - 3z)^2 + y^2 + z^2.$$

That is,

$$F(y,z) \equiv 1 - 4y - 6z + 12yz + 5y^2 + 10z^2.$$

Using the principles of maxima and minima for functions of two independent variables we have,

$$\frac{\partial F}{\partial y} \equiv -4 + 12z + 10y \quad \text{and} \quad \frac{\partial F}{\partial z} \equiv -6 + 12y + 20z,$$

and a stationary value will occur when these are both equal to zero.
Thus,

$$\begin{aligned} 5y + 6z &= 2, \\ 6y + 10z &= 3, \end{aligned}$$

which give $y = 1/7$ and $z = 3/14$, on solving simultaneously.
The corresponding value of x is $1/14$, which gives a stationary value, for $f(x,y,z)$, of $14/(14)^2 = \frac{1}{14}$.
Also, we have

$$\frac{\partial^2 F}{\partial y^2} \equiv 10 > 0, \quad \frac{\partial^2 F}{\partial z^2} \equiv 20 > 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial y \partial z} \equiv 12,$$

which means that

$$\frac{\partial^2 F}{\partial y^2} . \frac{\partial^2 F}{\partial z^2} - \left(\frac{\partial^2 F}{\partial y \partial z}\right)^2 = 200 - 144 > 0.$$

Hence there is a local minimum value, $\frac{1}{14}$, of $x^2+y^2+z^2$, subject to the constraint that $x+2y+3z=1$, at the point where

$$x=\frac{1}{14}, \quad y=\frac{1}{7}, \quad \text{and} \quad z=\frac{3}{14}.$$

Note:
Geometrically, this example is calculating the square of the shortest distance from the origin onto the plane whose equation is $x+2y+3z=1$.

14.11.2 THE METHOD OF LAGRANGE MULTIPLIERS

In determining the local maxima and local minima of a function, $f(x,y,...)$, subject to the constraint that $g(x,y,...)=0$, it may be inconvenient (or even impossible) to eliminate one of the variables, $x,y,....$

An alternative method may be illustrated by means of the following steps for a function of two independent variables:

(a) Suppose that the function, $z\equiv f(x,y)$, is subject to the constraint that $g(x,y)=0$.

Then, since z is effectively a function of x only, its stationary values will be determined by the equation

$$\frac{\mathrm{d}z}{\mathrm{d}x}=0.$$

(b) From Unit 14.5 (Exercise 2), the total derivative of $z\equiv f(x,y)$ with respect to x, when x and y are not independent of each other, is given by the formula,

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(c) From the constraint that $g(x,y)=0$, the process used in **(b)** gives

$$\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

and, hence, for all points on the surface with equation, $g(x,y)=0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}.$$

Thus, throughout the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} - \left(\frac{\partial f}{\partial y}\right)\frac{\left(\frac{\partial g}{\partial x}\right)}{\left(\frac{\partial g}{\partial y}\right)}.$$

(d) Stationary values of z, subject to the constraint that $g(x, y) = 0$, will, therefore, occur when

$$\frac{\partial f}{\partial x}.\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}.\frac{\partial g}{\partial x} = 0.$$

But this may be interpreted as the condition that the two equations,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 \ \text{ and } \ \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0,$$

should have a common solution for λ.

(e) Suppose that

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y).$$

Then $\phi(x, y, \lambda)$ would have stationary values whenever its first order partial derivatives with respect to x, y and λ were equal to zero.

In other words,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0, \ \ \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0, \ \ \text{and} \ \ g(x, y) = 0.$$

Conclusion

The stationary values of the function, $z \equiv f(x,y)$, subject to the constraint that $g(x,y) = 0$, occur at the points for which the function

$$\phi(x,y,\lambda) \equiv f(x,y) + \lambda g(x,y)$$

has stationary values.

The number, λ, is called a **"Lagrange multiplier"**.

Notes:

(i) In order to determine the nature of the stationary values of z, it will usually be necessary to examine the geometrical conditions in the neighbourhood of the stationary points.

(ii) The Lagrange multiplier method may also be applied to functions of three or more independent variables.

EXAMPLES

1. Determine any local maxima or local minima of the function,

$$z \equiv 3x^2 + 2y^2,$$

subject to the constraint that $x + 2y - 1 = 0$.

Solution

Firstly, we write

$$\phi(x,y,\lambda) \equiv 3x^2 + 2y^2 + \lambda(x + 2y - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 6x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 4y + 2\lambda \text{ and } \frac{\partial \phi}{\partial \lambda} \equiv x + 2y - 1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 6x + \lambda &= 0, \\ 2y + \lambda &= 0. \end{aligned}$$

Eliminating λ shows that $6x - 2y = 0$, or $y = 3x$; and, if we substitute this into the constraint, we obtain $7x - 1 = 0$.

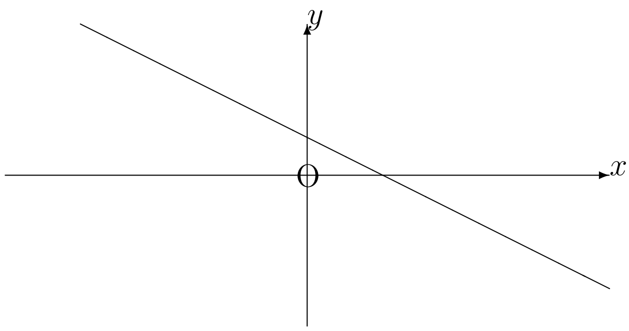
Hence,

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \text{ and } \lambda = -\frac{6}{7}.$$

A single stationary point therefore occurs at the point where

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \text{ and } z = 3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{3}{7}.$$

Finally, the geometrical conditions imply that the stationary value of z occurs at a point on the straight line whose equation is $x + 2y - 1 = 0$.



The stationary point is, in fact, a **minimum** value of z, since the function, $3x^2 + 2y^2$, has values larger than $3/7 \simeq 0.429$ at any point either side of the point, $(1/7, 3/7) = (0.14, 0.43)$, on the line whose equation is $x + 2y - 1 = 0$.

For example, at the points, $0.12, 0.44)$ and $(0.16, 0.42)$, on the line, the values of z are 0.4304 and 0.4296, respectively.

2. Determine the maximum and minimum values of the function, $z \equiv 3x + 4y$, subject to the constraint that $x^2 + y^2 = 1$.

Solution

Firstly, we write

$$\phi(x, y, \lambda) \equiv 3x + 4y + \lambda(x^2 + y^2 - 1).$$

Then,

$$\frac{\partial\phi}{\partial x} \equiv 3 + 2\lambda x, \quad \frac{\partial\phi}{\partial y} \equiv 4 + 2\lambda y \quad \text{and} \quad \frac{\partial\phi}{\partial\lambda} \equiv x^2 + y^2 - 1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 3 + 2\lambda x &= 0, \\ 2 + \lambda y &= 0. \end{aligned}$$

Thus,

$$x = -\frac{3}{2\lambda} \quad \text{and} \quad y = \frac{2}{\lambda},$$

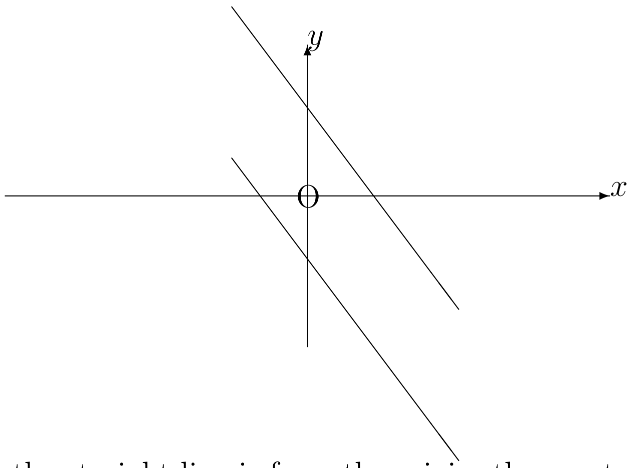
which we may substitute into the constraint to give

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1.$$

That is,

$$9 + 16 = 4\lambda^2 \quad \text{and} \quad \text{hence} \quad \lambda = \pm\frac{5}{2}.$$

We may deduce that $x = \pm\frac{3}{5}$ and $y = \pm\frac{4}{5}$, giving stationary values, ± 5, of z.
Finally, the geometrical conditions suggest that we consider a straight line with equation $3x + 4y = c$ (a constant) moving across the circle with equation $x^2 + y^2 = 1$.



The further the straight line is from the origin, the greater is the value of the constant, c.

The maximum and minimum values of $3x+4y$, subject to the constraint that $x^2+y^2=1$ will occur where the straight line touches the circle; and we have shown that these are the points, $(3/5, 4/5)$ and $(-3/5, -4/5)$.

3. Determine any local maxima or local minima of the function,

$$w \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Firstly, we write

$$\phi(x, y, z, \lambda) \equiv x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 2x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 2y + 2\lambda, \quad \frac{\partial \phi}{\partial z} \equiv 2z + 3\lambda \ \text{ and } \ \frac{\partial \phi}{\partial \lambda} \equiv x + 2y + 3z - 1.$$

The fourth of these is already equal to zero; but we equate the first three to zero, giving

$$\begin{aligned} 2x + \lambda &= 0, \\ y + \lambda &= 0, \\ 2z + 3\lambda &= 0. \end{aligned}$$

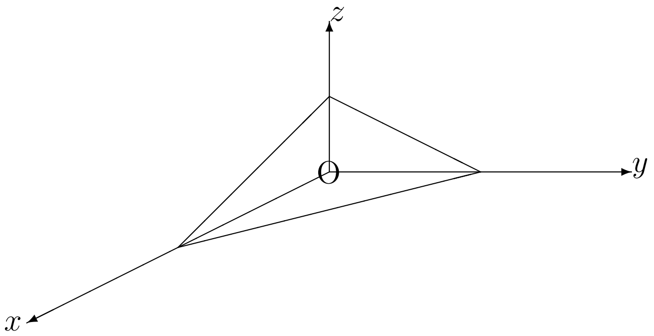
Eliminating λ shows that $2x - y = 0$, or $y = 2x$, and $6x - 2z = 0$, or $z = 3x$.
Substituting these into the constraint gives $14x = 1$.
Hence,

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \text{ and } \lambda = -\frac{1}{7}.$$

A single stationary point therefore occurs at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \text{ and } w = \left(\frac{1}{14}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2 = \frac{1}{14}.$$

Finally, the geometrical conditions imply that the stationary value of w occurs at a point on the plane whose equation is $x + 2y + 3z = 1$.



The stationary point must give a **minimum** value of w since the function, $x^2 + y^2 + z^2$, represents the square of the distance of a point, (x, y, z), from the origin; and, if the point is constrained to lie on a plane, this distance is bound to have a minimum value.

14.11.3 EXERCISES

1. In the following exercises, use both the substitution method and the Lagrange multiplier method:

   (a) Determine the minimum value of the function,

   $$z \equiv x^2 + y^2,$$

   subject to the constraint that $x + y = 1$.

   (b) Determine the maximum value of the function,

   $$z \equiv xy,$$

   subject to the constraint that $x + y = 15$.

(c) Determine the maximum value of the function,

$$z \equiv x^2 + 3xy - 5y^2,$$

subject to the constraint that $2x + 3y = 6$.

2. In the following exercises, use the Lagrange multiplier method:

(a) Determine the maximum and minimum values of the function,

$$w \equiv x - 2y + 5z,$$

subject to the constraint that $x^2 + y^2 + z^2 = 30$.

(b) If $x > 0$, $y > 0$ and $z > 0$, determine the maximum value of the function,

$$w \equiv xyz,$$

subject to the constraint that $x + y + z^2 = 16$.

(c) Determine the maximum value of the function,

$$w \equiv 8x^2 + 4yz - 16z + 600,$$

subject to the constraint that $4x^2 + y^2 + 4z^2 = 16$.

14.11.4 ANSWERS TO EXERCISES

1. (a) The minimum value is $z = 1/2$, and occurs when $x = y = 1/2$;

(b) The maximum value is $z \simeq 56.25$, and occurs when $x = y = 15/2$;

(c) The maximum value is $z = 9$, and occurs when $x = 3$ and $y = 0$.

2. (a) The maximum value is 30, and occurs when $x = 1$, $y = -2$ and $z = 5$;
The minimum value is -30, and occurs when $x = -1$, $y = 2$ and $z = -5$;

(b) The maximum value is

$$\frac{4096}{25\sqrt{5}} \simeq 73.27,$$

and occurs when $x = 32/\sqrt{5}$, $y = 32/\sqrt{5}$ and $z = 4/\sqrt{5}$;

(c) The maximum value is approximately 613.86, and occurs when $x = 0$, $y = -2$ and $z = \sqrt{3}$.

"JUST THE MATHS"

UNIT NUMBER

14.12

PARTIAL DIFFERENTIATION 12
(The principle of least squares)

by

A.J.Hobson

UNIT 14.12 - PARTIAL DIFFERENTIATION 12

THE PRINCIPLE OF LEAST SQUARES

14.12.1 THE NORMAL EQUATIONS

Suppose two variables, x and y, are known to obey a **"straight line law"**, of the form $y = a + bx$, where a and b are constants to be found.

Suppose also that, in an experiment to test this law, we obtain n pairs of values, (x_i, y_i), where i = 1,2,3,...,n.

If the values x_i are **assigned** values, they are likely to be free from error, whereas the **observed** values, y_i, will be subject to experimental error.

The principle underlying the straight line of **"best fit"** is that, in its most likely position, the sum of the squares of the y-deviations, from the line, of all observed points is a minimum.

The Calculation

The y-deviation, ϵ_i, of the point, (x_i, y_i), is given by

$$\epsilon_i = y_i - (a + bx_i).$$

Hence,

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} [y_i - (a + bx_i]^2 = P \text{ say.}$$

Regarding P as a function of a and b, it will be a minimum when

$$\frac{\partial P}{\partial a} = 0, \quad \frac{\partial P}{\partial b} = 0, \quad \frac{\partial^2 P}{\partial a^2} > 0 \text{ or } \frac{\partial^2 P}{\partial b^2} > 0, \text{ and } \frac{\partial^2 P}{\partial a^2}.\frac{\partial^2 P}{\partial b^2} - \left(\frac{\partial^2 P}{\partial a \partial b}\right)^2 > 0.$$

For these conditions, we have

$$\frac{\partial P}{\partial a} = -2\sum_{i=1}^{n} [y_i - (a + bx_i] \text{ and } \frac{\partial P}{\partial b} = -2\sum_{i=1}^{n} x_i[y_i + bx_i],$$

and these will be zero when

$$\sum_{i=1}^{n} [y_i - (a + bx_i] = 0 \quad - - - (1)$$

and

$$\sum_{i=1}^{n} x_i[y_i + bx_i] = 0 \quad - - - (2).$$

From (1),

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - \sum_{i=1}^{n} bx_i = 0.$$

That is,

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \quad - - - (3).$$

From (2),

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \quad - - - (4).$$

The statements (3) and (4) are two simultaneous equations which may be solved for a and b.

They are called the **"normal equations"**

A simpler notation for the normal equations is

$$\Sigma\ y = na + b\Sigma\ x;$$

$$\Sigma xy = a\Sigma x + b\Sigma\ x^2.$$

By eliminating a and b in turn, we obtain the solutions

$$a = \frac{\Sigma\, x^2.\Sigma\, y - \Sigma x.\Sigma\, xy}{n\Sigma\, x^2 - (\Sigma\, x)^2} \quad \text{and} \quad b = \frac{n\Sigma\, xy - \Sigma\, x.\Sigma\, y}{n\Sigma\, x^2 - (\Sigma\, x)^2}.$$

With these values of a and b, the straight line with equation, $y = a + bx$, is called the **"regression line of y on x"**.

Note:
To verify that the y-deviations from the regression line have indeed been minimised, we also need the results that

$$\frac{\partial^2 P}{\partial a^2} = \sum_{i=1}^{n} 2 = 2n, \quad \frac{\partial^2 P}{\partial b^2} = \sum_{i=1}^{n} 2x_i^2, \quad \text{and} \quad \frac{\partial^2 P}{\partial a \partial b} = \sum_{i=1}^{n} 2x_i.$$

The first two of these are clearly positive; and it may be shown that

$$\frac{\partial^2 P}{\partial a^2}.\frac{\partial^2 P}{\partial b^2} - \left(\frac{\partial^2 P}{\partial a \partial b}\right)^2 > 0.$$

EXAMPLE

Determine the equation of the regression line of y on x for the following data, which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

x	y	xy	x^2
45	6.53	293.85	2025
42	6.30	264.60	1764
56	9.52	533.12	3136
48	7.50	360.00	2304
42	6.99	293.58	1764
35	5.90	206.50	1225
58	9.49	550.42	3364
40	6.20	248.00	1600
39	6.55	255.45	1521
50	8.72	436.00	2500
455	73.70	3441.52	21203

The regression line of y on x thus has equation $y = a + bx$ where

$$a = \frac{(21203)(73.70) - (455)(3441.52)}{(10)(21203) - (455)^2} \simeq -0.645$$

and

$$b = \frac{(10)(3441.52) - (455)(73.70)}{(10)21203) - (455)^2} \simeq 0.176$$

Thus,

$$y = 0.176x - 0.645$$

14.12.2 SIMPLIFIED CALCULATION OF REGRESSION LINES

A simpler method of determining the regression line of y on x for a given set of data, is to consider a temporary change of origin to the point $(\overline{x}, \overline{y})$, where $\overline{x}$ is the arithmetic mean of the values x_i and $\overline{y}$ is the arithmetic mean of the values y_i.

RESULT

The regression line of y on x contains the point $(\overline{x}, \overline{y})$.

Proof:

From the first of the normal equations,

$$\frac{\Sigma\ y}{n} = a + b\frac{\Sigma\ x}{n}.$$

That is,

$$\overline{y} = a + b\overline{x}.$$

A change of origin to the point $(\overline{x}, \overline{y})$, with new variables X and Y, is associated with the formulae

$$X = x - \overline{x} \text{ and } Y = y - \overline{y}$$

and, in this system of reference, the regression line will pass through the origin.

Its equation is therefore

$$Y = BX,$$

where

$$B = \frac{n\Sigma\ XY - \Sigma\ X.\Sigma\ Y}{n\Sigma\ X^2 - (\Sigma\ X)^2}.$$

However,

$$\Sigma\, X = \Sigma\, (x - \overline{x}) = \Sigma\, x - \Sigma\, \overline{x} = n\overline{x} - n\overline{x} = 0$$

and

$$\Sigma\, Y = \Sigma\, (y - \overline{y}) = \Sigma\, y - \Sigma\, \overline{y} = n\overline{y} - n\overline{y} = 0.$$

Thus,

$$B = \frac{\Sigma\, XY}{\Sigma\, X^2}.$$

Note:
In a given problem, we make a table of values of x_i, y_i, X_i, Y_i, X_iY_i and X_i^2.

The regression line is then

$$y - \overline{y} = B(x - \overline{x}) \quad \text{or} \quad y = BX + (\overline{y} - B\overline{x});$$

though, there may be slight differences in the result obtained compared with that from the earlier method.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

The arithmetic mean of the x values is $\overline{x} = 45.5$

The arithmetic mean of the y values is $\overline{y} = 7.37$

This gives the following table:

x	y	$X = x - \overline{x}$	$Y = y - \overline{y}$	XY	X^2
45	6.53	-0.5	-0.84	0.42	0.25
42	6.30	-3.5	-1.07	3.745	12.25
56	9.52	10.5	2.15	22.575	110.25
48	7.50	2.5	0.13	0.325	6.25
42	6.99	-3.5	-0.38	1.33	12.25
35	5.90	-10.5	-1.47	15.435	110.25
58	9.49	12.5	2.12	26.5	156.25
40	6.20	-5.5	-1.17	6.435	30.25
39	6.55	-6.5	-0.82	5.33	42.25
50	8.72	4.5	1.35	6.075	20.25
455	73.70			88.17	500.5

Hence,

$$B = \frac{88.17}{500.5} \simeq 0.176$$

and so the regression line has equation

$$y = 0.176x + (7.37 - 0.176 \times 45.5).$$

That is,

$$y = 0.176x - 0.638$$

14.12.3 EXERCISES

1. For the following tables, determine the regression line of y on x, assuming that $y = a+bx$.

(a)

x	0	2	3	5	6
y	6	-1	-3	-10	-16

(b)

x	0	20	40	60	80
y	54	65	75	85	96

(c)

x	1	3	5	10	12
y	58	55	40	37	22

2. To determine the relation between the normal stress and the shear resistance of soil, a shear-box experiment was performed, giving the following results:

Normal Stress, x p.s.i.	11	13	15	17	19	21
Shear Stress, y p.s.i.	15.2	17.7	19.3	21.5	23.9	25.4

If $y = a + bx$, determine the regression line of y on x.

3. Fuel consumption, y miles per gallon, at speeds of x miles per hour, is given by the following table:

x	20	30	40	50	60	70	80	90
y	18.3	18.8	19.1	19.3	19.5	19.7	19.8	20.0

Assuming that

$$y = a + \frac{b}{x},$$

determine the most probable values of a and b.

14.12.4 ANSWERS TO EXERCISES

1. (a)

$$y = 6.46 - 3.52x;$$

(b)

$$y = 54.20 + 0.52x;$$

(c)

$$y = 60.78 - 2.97x.$$

2.

$$y = 4.09 + 1.03x.$$

3.

$$a \simeq -42 \quad \text{and} \quad b \simeq 20.$$

"JUST THE MATHS"

UNIT NUMBER

15.1

ORDINARY DIFFERENTIAL EQUATIONS 1 (First order equations (A))

by

A.J.Hobson

UNIT 15.1 - ORDINARY DIFFERENTIAL EQUATIONS 1

FIRST ORDER EQUATIONS (A)

15.1.1 INTRODUCTION AND DEFINITIONS

1. An **ordinary differential equation** is a relationship between an independent variable (such as x), a dependent variable (such as y) and one or more ordinary derivatives of y with respect to x.

   There is no discussion, in Units 15, of **partial** differential equations, which involve partial derivatives (see Units 14). Hence, in what follows, we shall refer simply to "differential equations".

   For example,

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-2x}, \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = y, \quad x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$$

   are differential equations.

2. The **"order"** of a differential equation is the order of the highest derivative which appears in it.

3. The **"general solution"** of a differential equation is the most general algebraic relationship between the dependent and independent variables which satisfies the differential equation.

   Such a solution will not contain any derivatives; but we shall see that it will contain one or more arbitrary constants (the number of these constants being equal to the order of the equation). The solution need not be an explicit formula for one of the variables in terms of the other.

4. A **"boundary condition"** is a numerical condition which must be obeyed by the solution. It usually amounts to the substitution of particular values of the dependent and independent variables into the general solution.

5. An **"initial condition"** is a boundary condition in which the independent variable takes the value zero.

6. A **"particular solution"** (or **"particular integral"**) is a solution which contains no arbitrary constants.

   Particular solutions are usually the result of appplying a boundary condition to a general solution.

15.1.2 EXACT EQUATIONS

The simplest kind of differential equation of the first order is one which has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

It is an elementary example of an **"exact differential equation"** because, to find its solution, all that it is necessary to do is integrate both sides with respect to x.

In other cases of exact differential equations, the terms which are not just functions of the independent variable only, need to be recognised as the exact derivative with respect to x of some known function (possibly involving both of the variables).

The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 5,$$

subject to the boundary condition that $y = 2$ when $x = 1$.

Solution

By direct integration, the general solution is

$$y = x^3 - 3x^2 + 5x + C,$$

where C is an arbitrary constant.

From the boundary condition,

$$2 = 1 - 3 + 5 + C, \text{ so that } C = -1.$$

Thus the particular solution obeying the given boundary condition is

$$y = x^3 - 3x^2 + 5x - 1.$$

2. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3,$$

subject to the boundary condition that $y = 4$ when $x = 2$.

Solution

The left hand side of the differential equation may be recognised as the exact derivative with respect to x of the function xy.

Hence, we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3;$$

and, by direct integration, this gives

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

That is,

$$y = \frac{x^3}{4} + \frac{C}{x}.$$

Applying the boundary condition,

$$4 = 2 + \frac{C}{2},$$

which implies that $C = 4$ and the particular solution is

$$y = \frac{x^3}{4} + \frac{4}{x}.$$

3. Determine the general solution to the differential equation

$$\sin x + \sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

The second and third terms on the right hand side may be recognised as the exact derivative of the function $x\sin y$; and, hence, we may write

$$\sin x + \frac{\mathrm{d}}{\mathrm{d}x}(x \sin y) = 0.$$

By direct integration, we obtain

$$-\cos x + x \sin y = C,$$

where C is an arbitrary constant.

This result counts as the general solution without further modification; but an explicit formula for y in terms of x may, in this case, be written in the form

$$y = \mathrm{Sin}^{-1}\left[\frac{C + \cos x}{x}\right].$$

15.1.3 THE METHOD OF SEPARATION OF THE VARIABLES

The method of this section relates to differential equations of the first order which may be written in the form

$$P(y)\frac{\mathrm{d}y}{\mathrm{d}x} = Q(x).$$

Integrating both sides with respect to x gives

$$\int P(y)\frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = \int Q(x)\ \mathrm{d}x.$$

But, from the formula for integration by substitution in Units 12.3 and 12.4, this simplifies to

$$\int P(y)\ \mathrm{d}y = \int Q(x)\ \mathrm{d}x.$$

Note:
The way to remember this result is to treat $\mathrm{d}x$ and $\mathrm{d}y$, in the given differential equation, as if they were separate numbers; then rearrange the equation so that one side contains only y while the other side contains only x; that is, we **separate the variables**. The process is completed by putting an integral sign in front of each side.

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

subject to the boundary condition that $y = 6$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x};$$

and, hence,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x,$$

giving

$$\ln y = \ln x + C.$$

Applying the boundary condition,

$$\ln 6 = \ln 2 + C,$$

so that

$$C = \ln 6 - \ln 2 = \ln\left(\frac{6}{2}\right) = \ln 3.$$

The particular solution is therefore

$$\ln y = \ln x + \ln 3 \quad \text{or} \quad y = 3x.$$

Note:

In a general solution where most of the terms are logarithms, the calculation can be made simpler by regarding the arbitrary constant itself as a logarithm, calling it $\ln A$, for instance, rather than C. In the above example, we would then write

$$\ln y = \ln x + \ln A \quad \text{simplifying} \quad \text{to} \quad y = Ax.$$

On applying the boundary condition, $6 = 2A$, so that $A = 3$ and the particular solution is the same as before.

2. Solve the differential equation

$$x(4-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to the boundary condition that $y = 7$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x(4-x)}.$$

Hence,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x(4-x)}\,\mathrm{d}x;$$

or, using the theory of partial fractions,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \left[\frac{\frac{1}{4}}{x} + \frac{\frac{1}{4}}{4-x}\right]\,\mathrm{d}x.$$

The general solution is therefore

$$\ln y = \frac{1}{4}\ln x - \frac{1}{4}\ln(4-x) + \ln A$$

or

$$y = A\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

Applying the boundary condition, $7 = A$, so that the particular solution is

$$y = 7\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

15.1.4 EXERCISES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^5 + 3e^{-2x}.$$

2. Given that differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \sin x$$

is exact, determine its general solution.

3. Given that the differential equation

$$\tan x\frac{\mathrm{d}y}{\mathrm{d}x} + y\sec^2 x = \cos 2x$$

is exact, determine the particular solution for which $y = 1$ when $x = \frac{\pi}{4}$.

4. Use the method of separation of the variables to determine the general solution of each of the following differential equations:

(a)

$$\frac{\mathrm{d}x}{\mathrm{d}y} = (x-1)(x+2);$$

(b)

$$x(y-3)\frac{\mathrm{d}y}{\mathrm{d}x} = 4y.$$

5. Use the method of separation of the variables to solve the following differential equations subject to the given boundary condition:

(a)

$$(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y,$$

where $y = 2$ when $x = 1$;

(b)

$$x^3 + (y+1)^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

where $y = 0$ when $x = 0$.

15.1.5 ANSWERS TO EXERCISES

1. $$y = \frac{x^6}{6} - \frac{3e^{-2x}}{2} + C.$$

2. $$y = \frac{C - \cos x}{x^2}.$$

3. $$y = \frac{3}{2}\cot x - \cos^2 x.$$

4. (a) $$y = \ln\left[A\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}}\right];$$

   (b) $$y = \ln[Ax^4y^3].$$

5. (a) $$y^3 = 4(1 + x^3);$$

   (b) $$4[1 - (y+1)^3] = 3x^4.$$

"JUST THE MATHS"

UNIT NUMBER

15.2

ORDINARY DIFFERENTIAL EQUATIONS 2 (First order equations (B))

by

A.J.Hobson

UNIT 15.2 - ORDINARY DIFFERENTIAL EQUATIONS 2

FIRST ORDER EQUATIONS (B)

15.2.1 HOMOGENEOUS EQUATIONS

A differential equation of the first order is said to be **"homogeneous"** if, on replacing x by λx and y by λy in all the parts of the equation except $\frac{\mathrm{d}y}{\mathrm{d}x}$, λ may be removed from the equation by cancelling a common factor of λ^n, for some integer n.

Note:
Some examples of homogeneous equations would be

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x-y) = 0, \quad \text{and} \quad 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2+y^2) = 0,$$

where, from the first of these, a factor of λ could be cancelled and, from the second, a factor of λ^2 could be cancelled.

15.2.2 THE STANDARD METHOD

It turns out that the substitution

$$\boxed{y = vx} \quad \left(\text{giving} \ \ \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right),$$

always converts a homogeneous differential equation into one in which the variables can be separated. The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = x + 2y,$$

subject to the condition that $y = 6$ when $x = 6$.

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$x\left(v + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = x + 2vx.$$

That is,

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 2v$$

or

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v.$$

On separating the variables,

$$\int \frac{1}{1+v}\ \mathrm{d}v = \int \frac{1}{x}\ \mathrm{d}x,$$

giving

$$\ln(1+v) = \ln x + \ln A,$$

where A is an arbitrary constant.
An alternative form of this solution, without logarithms, is

$$Ax = 1 + v$$

and, substituting back $v = \frac{y}{x}$, the solution becomes

$$Ax = 1 + \frac{y}{x}$$

or

$$y = Ax^2 - x.$$

Finally, if $y = 6$ when $x = 1$, we have $6 = A - 1$ and, hence, $A = 7$.
The required particular solution is thus

$$y = 7x^2 - x.$$

2. Determine the general solution of the differential equation

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x - y) = 0.$$

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$(x + vx)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (4x - vx) = 0.$$

That is,

$$(1 + v)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (4 - v) = 0$$

or

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{v + 1}.$$

On further rearrangement, we obtain

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{v + 1} - v = \frac{-4 - v^2}{v + 1};$$

and, on separating the variables,

$$\int \frac{v + 1}{4 + v^2}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$$

or

$$\frac{1}{2}\int \left[\frac{2v}{4 + v^2} + \frac{2}{4 + v^2}\right]\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x.$$

Hence,

$$\frac{1}{2}\left[\ln(4 + v^2) + \tan^{-1}\frac{v}{2}\right] = -\ln x + C,$$

where C is an arbitrary constant.

Substituting back $v = \frac{y}{x}$, gives the general solution

$$\frac{1}{2}\left[\ln\left(4 + \frac{y^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{2x}\right)\right] = -\ln x + C.$$

3. Determine the general solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2) = 0.$$

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$2vx^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (x^2 + v^2x^2) = 0.$$

That is,

$$2v\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (1 + v^2) = 0$$

or

$$2vx\frac{\mathrm{d}v}{\mathrm{d}x} = -(1 + 3v^2).$$

On separating the variables, we obtain

$$\int \frac{2v}{1 + 3v^2}\,\mathrm{d}x = -\int \frac{1}{x}\,\mathrm{d}x,$$

which gives

$$\frac{1}{3}\ln(1 + 3v^2) = -\ln x + \ln A,$$

where A is an arbitrary constant.
Hence,

$$(1 + 3v^2)^{\frac{1}{3}} = \frac{A}{x}$$

or, on substituting back $v = \frac{y}{x}$,

$$\left(\frac{x^2 + 3y^2}{x^2}\right)^{\frac{1}{3}} = Ax,$$

which can be written

$$x^2 + 3y^2 = Bx^5,$$

where $B = A^3$.

15.2.3 EXERCISES

Use the substitution $y = vx$ to solve the following differential equations subject to the given boundary condition:

1.
$$(2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + y,$$
where $y = 3$ when $x = -2$.

2.
$$(x^2 - y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$
where $y = 5$ when $x = 0$.

3.
$$x^3 + y^3 = 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x},$$
where $y = 1$ when $x = 2$.

4.
$$x(x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^3,$$
where $y = 2$ when $x = 1$.

5.
$$x\frac{\mathrm{d}y}{\mathrm{d}x} - (y + \sqrt{x^2 - y^2}) = 0,$$
where $y = 0$ when $x = 1$.

15.2.4 ANSWERS TO EXERCISES

1.

$$y^2 - xy - x^2 = 11.$$

2.

$$y = 5e^{-\frac{x^2}{2y^2}}.$$

3.

$$x^3 - 2y^3 = 3x.$$

4.

$$3x^2y = 2(y^2 - x^2).$$

5.

$$e^{\sin^{-1}\frac{y}{x}} = x.$$

"JUST THE MATHS"

UNIT NUMBER

15.3

ORDINARY DIFFERENTIAL EQUATIONS 3 (First order equations (C))

by

A.J.Hobson

UNIT 15.3 - ORDINARY DIFFERENTIAL EQUATIONS 3

FIRST ORDER EQUATIONS (C)

15.3.1 LINEAR EQUATIONS

For certain kinds of first order differential equation, it is possible to multiply the equation throughout by a suitable factor which converts it into an exact differential equation.

For instance, the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2$$

may be multiplied throughout by x to give

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3.$$

It may now be written

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3$$

and, hence, it has general solution

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

Notes:

(i) The factor, x which has multiplied both sides of the differential equation serves as an **"integrating factor"**, but such factors cannot always be found by inspection.

(ii) In the discussion which follows, we shall develop a formula for determining integrating factors, in general, for what are known as **"linear differential equations"**.

DEFINITION

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

is said to be **"linear"**.

RESULT

Given the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

the function

$$e^{\int P(x) \ \mathrm{d}x}$$

is always an integrating factor; and, on multiplying the differential equation throughout by this factor, its left hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{\int P(x) \ \mathrm{d}x}\right].$$

Proof

Suppose that the function, $R(x)$, is an integrating factor; then, in the equation

$$R(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)P(x)y = R(x)Q(x),$$

the left hand side must be the exact derivative of some function of x.

Using the formula for differentiating the product of two functions of x, we can **make** it the derivative of $R(x)y$ provided we can arrange that

$$R(x)P(x) = \frac{\mathrm{d}}{\mathrm{d}x}[R(x)].$$

But this requirement can be interpreted as a differential equation in which the variables $R(x)$ and x may be separated as follows:

$$\int \frac{1}{R(x)}\ \mathrm{d}R(x) = \int\ P(x)\ \mathrm{d}x.$$

Hence,

$$\ln R(x) = \int\ P(x)\ \mathrm{d}x.$$

That is,

$$R(x) = e^{\int\ P(x)\ \mathrm{d}x},$$

as required.

The solution is obtained by integrating the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times R(x)] = R(x)P(x).$$

Note:
There is no need to include an arbitrary constant, C, when $P(x)$ is integrated, since it would only serve to introduce a constant factor of e^C in the above result, which would then immediately cancel out on multiplying the differential equation by $R(x)$.

EXAMPLES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2.$$

Solution
An integrating factor is

$$e^{\int\ \frac{1}{x}\ \mathrm{d}x} = e^{\ln x} = x.$$

On multiplying throughout by the integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times x] = x^3;$$

and so,

$$yx = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2e^{-x^2}.$$

Solution
An integrating factor is

$$e^{\int\ 2x\ \mathrm{d}x} = e^{x^2}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{x^2}\right] = 2,$$

giving

$$ye^{x^2} = 2x + C,$$

where C is an arbitrary constant.

15.3.2 BERNOUILLI'S EQUATION

A similar type of differential equation to that in the previous section has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n.$$

It is called **"Bernouilli's Equation"** and may be converted to a linear differential equation by making the substitution

$$z = y^{1-n}.$$

Proof

The differential equation may be rewritten as

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x).$$

Also,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence the differential equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}x} + P(x)z = Q(x).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)P(x)z = (1-n)Q(x),$$

which is a linear differential equation.

Note:
It is better not to regard this as a standard formula, but to apply the method of obtaining it in the case of particular examples.

EXAMPLES

1. Determine the general solution of the differential equation

$$xy - \frac{\mathrm{d}y}{\mathrm{d}x} = y^3e^{-x^2}.$$