The order of the factors in the terms on the RHS of (10.6) is, of course, just as important as it is in the original vector product.

►*A particle of mass m with position vector* $\mathbf{r}$ *relative to some origin O experiences a force* $\mathbf{F}$*, which produces a torque (moment)* $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ *about O. The angular momentum of the particle about O is given by* $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$*, where* $\mathbf{v}$ *is the particle's velocity. Show that the rate of change of angular momentum is equal to the applied torque.*

The rate of change of angular momentum is given by

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}).$$

Using (10.6) we obtain

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d\mathbf{r}}{dt} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v})\\ &= \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v})\\ &= \mathbf{0} + \mathbf{r} \times \mathbf{F} = \mathbf{T},\end{aligned}$$

where in the last line we use Newton's second law, namely $\mathbf{F} = d(m\mathbf{v})/dt$. ◄

If a vector $\mathbf{a}$ is a function of a scalar variable s that is itself a function of u, so that $s = s(u)$, then the chain rule (see subsection 2.1.3) gives

$$\frac{d\mathbf{a}(s)}{du} = \frac{ds}{du}\frac{d\mathbf{a}}{ds}. \tag{10.7}$$

The derivatives of more complicated vector expressions may be found by repeated application of the above equations.

One further useful result can be derived by considering the derivative

$$\frac{d}{du}(\mathbf{a} \cdot \mathbf{a}) = 2\mathbf{a} \cdot \frac{d\mathbf{a}}{du};$$

since $\mathbf{a} \cdot \mathbf{a} = a^2$, where $a = |\mathbf{a}|$, we see that

$$\mathbf{a} \cdot \frac{d\mathbf{a}}{du} = 0 \qquad \text{if } a \text{ is constant.} \tag{10.8}$$

In other words, if a vector $\mathbf{a}(u)$ has a constant magnitude as u varies then it is perpendicular to the vector $d\mathbf{a}/du$.

10.1.2 Differential of a vector

As a final note on the differentiation of vectors, we can also define the *differential* of a vector, in a similar way to that of a scalar in ordinary differential calculus. In the definition of the vector derivative (10.1), we used the notion of a small change $\Delta\mathbf{a}$ in a vector $\mathbf{a}(u)$ resulting from a small change Δu in its argument. In the limit $\Delta u \to 0$, the change in $\mathbf{a}$ becomes infinitesimally small, and we denote it by the differential $d\mathbf{a}$. From (10.1) we see that the differential is given by

$$d\mathbf{a} = \frac{d\mathbf{a}}{du}\, du. \tag{10.9}$$

Note that the differential of a vector is also a vector. As an example, the infinitesimal change in the position vector of a particle in an infinitesimal time dt is

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}\,dt = \mathbf{v}\,dt,$$

where $\mathbf{v}$ is the particle's velocity.

10.2 Integration of vectors

The integration of a vector (or of an expression involving vectors that may itself be either a vector or scalar) with respect to a scalar u can be regarded as the inverse of differentiation. We must remember, however, that

(i) the integral has the same nature (vector or scalar) as the integrand,
(ii) the constant of integration for indefinite integrals must be of the same nature as the integral.

For example, if $\mathbf{a}(u) = d[\mathbf{A}(u)]/du$ then the indefinite integral of $\mathbf{a}(u)$ is given by

$$\int \mathbf{a}(u)\,du = \mathbf{A}(u) + \mathbf{b},$$

where $\mathbf{b}$ is a constant vector. The definite integral of $\mathbf{a}(u)$ from $u = u_1$ to $u = u_2$ is given by

$$\int_{u_1}^{u_2} \mathbf{a}(u)\,du = \mathbf{A}(u_2) - \mathbf{A}(u_1).$$

►*A small particle of mass m orbits a much larger mass M centred at the origin O. According to Newton's law of gravitation, the position vector* $\mathbf{r}$ *of the small mass obeys the differential equation*

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Show that the vector $\mathbf{r} \times d\mathbf{r}/dt$ *is a constant of the motion.*

Forming the vector product of the differential equation with $\mathbf{r}$, we obtain

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{r^2}\mathbf{r} \times \hat{\mathbf{r}}.$$

Since $\mathbf{r}$ and $\hat{\mathbf{r}}$ are collinear, $\mathbf{r} \times \hat{\mathbf{r}} = \mathbf{0}$ and therefore we have

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{0}. \qquad (10.10)$$

However,

$$\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = \mathbf{0},$$

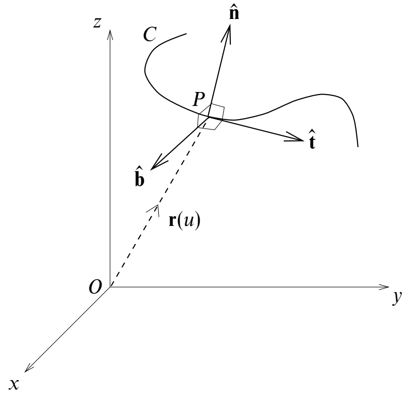


Figure 10.3 The unit tangent $\hat{\mathbf{t}}$, normal $\hat{\mathbf{n}}$ and binormal $\hat{\mathbf{b}}$ to the space curve C at a particular point P.

since the first term is zero by (10.10), and the second is zero because it is the vector product of two parallel (in this case identical) vectors. Integrating, we obtain the required result

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{c}, \qquad (10.11)$$

where $\mathbf{c}$ is a constant vector.

As a further point of interest we may note that in an infinitesimal time dt the change in the position vector of the small mass is $d\mathbf{r}$ and the element of area swept out by the position vector of the particle is simply $dA = \frac{1}{2}|\mathbf{r} \times d\mathbf{r}|$. Dividing both sides of this equation by dt, we conclude that

$$\frac{dA}{dt} = \frac{1}{2}\left|\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right| = \frac{|\mathbf{c}|}{2},$$

and that the physical interpretation of the above result (10.11) is that the position vector $\mathbf{r}$ of the small mass sweeps out equal areas in equal times. This result is in fact valid for motion under any force that acts along the line joining the two particles. ◀

10.3 Space curves

In the previous section we mentioned that the velocity vector of a particle is a tangent to the curve in space along which the particle moves. We now give a more complete discussion of curves in space and also a discussion of the geometrical interpretation of the vector derivative.

A curve C in space can be described by the vector $\mathbf{r}(u)$ joining the origin O of a coordinate system to a point on the curve (see figure 10.3). As the parameter u varies, the end-point of the vector moves along the curve. In Cartesian coordinates,

$$\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k},$$

where $x = x(u)$, $y = y(u)$ and $z = z(u)$ are the *parametric* equations of the curve.

This parametric representation can be very useful, particularly in mechanics when the parameter may be the time t. We can, however, also represent a space curve by $y = f(x)$, $z = g(x)$, which can be easily converted into the above parametric form by setting $u = x$, so that

$$\mathbf{r}(u) = u\mathbf{i} + f(u)\mathbf{j} + g(u)\mathbf{k}.$$

Alternatively, a space curve can be represented in the form $F(x, y, z) = 0$, $G(x, y, z) = 0$, where each equation represents a surface and the curve is the intersection of the two surfaces.

A curve may sometimes be described in parametric form by the vector $\mathbf{r}(s)$, where the parameter s is the *arc length* along the curve measured from a fixed point. Even when the curve is expressed in terms of some other parameter, it is straightforward to find the arc length between any two points on the curve. For the curve described by $\mathbf{r}(u)$, let us consider an infinitesimal vector displacement

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

along the curve. The square of the infinitesimal distance moved is then given by

$$(ds)^2 = d\mathbf{r}\cdot d\mathbf{r} = (dx)^2 + (dy)^2 + (dz)^2,$$

from which it can be shown that

$$\left(\frac{ds}{du}\right)^2 = \frac{d\mathbf{r}}{du}\cdot\frac{d\mathbf{r}}{du}.$$

Therefore, the arc length between two points on the curve $\mathbf{r}(u)$, given by $u = u_1$ and $u = u_2$, is

$$s = \int_{u_1}^{u_2} \sqrt{\frac{d\mathbf{r}}{du}\cdot\frac{d\mathbf{r}}{du}}\,du. \qquad (10.12)$$

► *A curve lying in the xy-plane is given by $y = y(x)$, $z = 0$. Using (10.12), show that the arc length along the curve between $x = a$ and $x = b$ is given by $s = \int_a^b \sqrt{1 + y'^2}\,dx$, where $y' = dy/dx$.*

Let us first represent the curve in parametric form by setting $u = x$, so that

$$\mathbf{r}(u) = u\mathbf{i} + y(u)\mathbf{j}.$$

Differentiating with respect to u, we find

$$\frac{d\mathbf{r}}{du} = \mathbf{i} + \frac{dy}{du}\mathbf{j},$$

from which we obtain

$$\frac{d\mathbf{r}}{du}\cdot\frac{d\mathbf{r}}{du} = 1 + \left(\frac{dy}{du}\right)^2.$$

Therefore, remembering that $u = x$, from (10.12) the arc length between $x = a$ and $x = b$ is given by

$$s = \int_a^b \sqrt{\frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}}\, du = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

This result was derived using more elementary methods in chapter 2. ◄

If a curve C is described by $\mathbf{r}(u)$ then, by considering figures 10.1 and 10.3, we see that, at any given point on the curve, $d\mathbf{r}/du$ is a vector tangent to C at that point, in the direction of increasing u. In the special case where the parameter u is the arc length s along the curve then $d\mathbf{r}/ds$ is a *unit* tangent vector to C and is denoted by $\hat{\mathbf{t}}$.

The rate at which the unit tangent $\hat{\mathbf{t}}$ changes with respect to s is given by $d\hat{\mathbf{t}}/ds$, and its magnitude is defined as the *curvature* κ of the curve C at a given point,

$$\kappa = \left|\frac{d\hat{\mathbf{t}}}{ds}\right| = \left|\frac{d^2\hat{\mathbf{r}}}{ds^2}\right|.$$

We can also define the quantity $\rho = 1/\kappa$, which is called the *radius of curvature*.

Since $\hat{\mathbf{t}}$ is of constant (unit) magnitude, it follows from (10.8) that it is perpendicular to $d\hat{\mathbf{t}}/ds$. The unit vector in the direction perpendicular to $\hat{\mathbf{t}}$ is denoted by $\hat{\mathbf{n}}$ and is called the *principal normal* at the point. We therefore have

$$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\,\hat{\mathbf{n}}. \tag{10.13}$$

The unit vector $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$, which is perpendicular to the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$, is called the *binormal* to C. The vectors $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ form a right-handed rectangular cooordinate system (or *triad*) at any given point on C (see figure 10.3). As s changes so that the point of interest moves along C, the triad of vectors also changes.

The rate at which $\hat{\mathbf{b}}$ changes with respect to s is given by $d\hat{\mathbf{b}}/ds$ and is a measure of the *torsion* τ of the curve at any given point. Since $\hat{\mathbf{b}}$ is of constant magnitude, from (10.8) it is perpendicular to $d\hat{\mathbf{b}}/ds$. We may further show that $d\hat{\mathbf{b}}/ds$ is also perpendicular to $\hat{\mathbf{t}}$, as follows. By definition $\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0$, which on differentiating yields

$$\begin{aligned}
0 = \frac{d}{ds}\left(\hat{\mathbf{b}} \cdot \hat{\mathbf{t}}\right) &= \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds} \\
&= \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \kappa\,\hat{\mathbf{n}} \\
&= \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}},
\end{aligned}$$

where we have used the fact that $\hat{\mathbf{b}} \cdot \hat{\mathbf{n}} = 0$. Hence, since $d\hat{\mathbf{b}}/ds$ is perpendicular to both $\hat{\mathbf{b}}$ and $\hat{\mathbf{t}}$, we must have $d\hat{\mathbf{b}}/ds \propto \hat{\mathbf{n}}$. The constant of proportionality is $-\tau$, so we finally obtain

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau\,\hat{\mathbf{n}}. \tag{10.14}$$

Taking the dot product of each side with $\hat{\mathbf{n}}$, we see that the torsion of a curve is given by

$$\tau = -\,\hat{\mathbf{n}} \cdot \frac{d\hat{\mathbf{b}}}{ds}.$$

We may also define the quantity $\sigma = 1/\tau$, which is called the *radius of torsion*.

Finally, we consider the derivative $d\hat{\mathbf{n}}/ds$. Since $\hat{\mathbf{n}} = \hat{\mathbf{b}} \times \hat{\mathbf{t}}$ we have

$$\begin{aligned}
\frac{d\hat{\mathbf{n}}}{ds} &= \frac{d\hat{\mathbf{b}}}{ds} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \frac{d\hat{\mathbf{t}}}{ds} \\
&= -\tau\,\hat{\mathbf{n}} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \kappa\,\hat{\mathbf{n}} \\
&= \tau\,\hat{\mathbf{b}} - \kappa\,\hat{\mathbf{t}}.
\end{aligned} \tag{10.15}$$

In summary, $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ and their derivatives with respect to s are related to one another by the relations (10.13), (10.14) and (10.15), the *Frenet–Serret formulae*,

$$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\,\hat{\mathbf{n}}, \qquad \frac{d\hat{\mathbf{n}}}{ds} = \tau\,\hat{\mathbf{b}} - \kappa\,\hat{\mathbf{t}}, \qquad \frac{d\hat{\mathbf{b}}}{ds} = -\tau\,\hat{\mathbf{n}}. \tag{10.16}$$

►*Show that the acceleration of a particle travelling along a trajectory* $\mathbf{r}(t)$ *is given by*

$$\mathbf{a}(t) = \frac{dv}{dt}\,\hat{\mathbf{t}} + \frac{v^2}{\rho}\,\hat{\mathbf{n}},$$

where v is the speed of the particle, $\hat{\mathbf{t}}$ is the unit tangent to the trajectory, $\hat{\mathbf{n}}$ is its principal normal and ρ is its radius of curvature.

The velocity of the particle is given by

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\,\hat{\mathbf{t}},$$

where ds/dt is the speed of the particle, which we denote by v, and $\hat{\mathbf{t}}$ is the unit vector tangent to the trajectory. Writing the velocity as $\mathbf{v} = v\,\hat{\mathbf{t}}$, and differentiating once more with respect to time t, we obtain

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\,\hat{\mathbf{t}} + v\frac{d\hat{\mathbf{t}}}{dt};$$

but we note that

$$\frac{d\hat{\mathbf{t}}}{dt} = \frac{ds}{dt}\frac{d\hat{\mathbf{t}}}{ds} = v\kappa\,\hat{\mathbf{n}} = \frac{v}{\rho}\,\hat{\mathbf{n}}.$$

Therefore, we have

$$\mathbf{a}(t) = \frac{dv}{dt}\,\hat{\mathbf{t}} + \frac{v^2}{\rho}\,\hat{\mathbf{n}}.$$

This shows that in addition to an acceleration dv/dt along the tangent to the particle's trajectory, there is also an acceleration v^2/ρ in the direction of the principal normal. The latter is often called the *centripetal* acceleration. ◄

Finally, we note that a curve $\mathbf{r}(u)$ representing the trajectory of a particle may sometimes be given in terms of some parameter u that is not necessarily equal to the time t but is functionally related to it in some way. In this case the velocity of the particle is given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{du}\frac{du}{dt}.$$

Differentiating again with respect to time gives the acceleration as

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{du}\frac{du}{dt}\right) = \frac{d^2\mathbf{r}}{du^2}\left(\frac{du}{dt}\right)^2 + \frac{d\mathbf{r}}{du}\frac{d^2u}{dt^2}.$$

10.4 Vector functions of several arguments

The concept of the derivative of a vector is easily extended to cases where the vectors (or scalars) are functions of more than one independent scalar variable, $u_1, u_2, \ldots, u_n$. In this case, the results of subsection 10.1.1 are still valid, except that the derivatives become partial derivatives $\partial\mathbf{a}/\partial u_i$ defined as in ordinary differential calculus. For example, in Cartesian coordinates,

$$\frac{\partial\mathbf{a}}{\partial u} = \frac{\partial a_x}{\partial u}\mathbf{i} + \frac{\partial a_y}{\partial u}\mathbf{j} + \frac{\partial a_z}{\partial u}\mathbf{k}.$$

In particular, (10.7) generalises to the chain rule of partial differentiation discussed in section 5.5. If $\mathbf{a} = \mathbf{a}(u_1, u_2, \ldots, u_n)$ and each of the u_i is also a function $u_i(v_1, v_2, \ldots, v_n)$ of the variables v_i then, generalising (5.17),

$$\frac{\partial\mathbf{a}}{\partial v_i} = \frac{\partial\mathbf{a}}{\partial u_1}\frac{\partial u_1}{\partial v_i} + \frac{\partial\mathbf{a}}{\partial u_2}\frac{\partial u_2}{\partial v_i} + \cdots + \frac{\partial\mathbf{a}}{\partial u_n}\frac{\partial u_n}{\partial v_i} = \sum_{j=1}^{n}\frac{\partial\mathbf{a}}{\partial u_j}\frac{\partial u_j}{\partial v_i}. \quad (10.17)$$

A special case of this rule arises when $\mathbf{a}$ is an explicit function of some variable v, as well as of scalars $u_1, u_2, \ldots, u_n$ that are themselves functions of v; then we have

$$\frac{d\mathbf{a}}{dv} = \frac{\partial\mathbf{a}}{\partial v} + \sum_{j=1}^{n}\frac{\partial\mathbf{a}}{\partial u_j}\frac{\partial u_j}{\partial v}. \quad (10.18)$$

We may also extend the concept of the differential of a vector given in (10.9) to vectors dependent on several variables $u_1, u_2, \ldots, u_n$:

$$d\mathbf{a} = \frac{\partial\mathbf{a}}{\partial u_1}du_1 + \frac{\partial\mathbf{a}}{\partial u_2}du_2 + \cdots + \frac{\partial\mathbf{a}}{\partial u_n}du_n = \sum_{j=1}^{n}\frac{\partial\mathbf{a}}{\partial u_j}du_j. \quad (10.19)$$

As an example, the infinitesimal change in an electric field $\mathbf{E}$ in moving from a position $\mathbf{r}$ to a neighbouring one $\mathbf{r} + d\mathbf{r}$ is given by

$$d\mathbf{E} = \frac{\partial\mathbf{E}}{\partial x}dx + \frac{\partial\mathbf{E}}{\partial y}dy + \frac{\partial\mathbf{E}}{\partial z}dz. \quad (10.20)$$

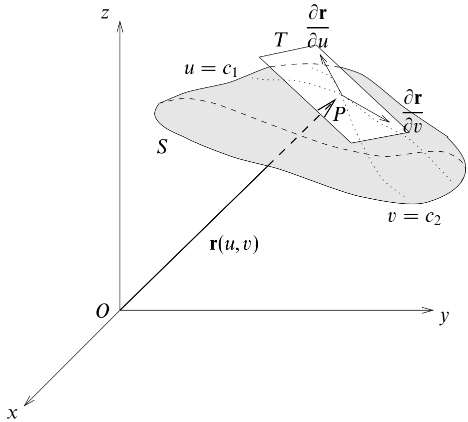


Figure 10.4 The tangent plane T to a surface S at a particular point P; $u = c_1$ and $v = c_2$ are the coordinate curves, shown by dotted lines, that pass through P. The broken line shows some particular parametric curve $\mathbf{r} = \mathbf{r}(\lambda)$ lying in the surface.

10.5 Surfaces

A surface S in space can be described by the vector $\mathbf{r}(u, v)$ joining the origin O of a coordinate system to a point on the surface (see figure 10.4). As the parameters u and v vary, the end-point of the vector moves over the surface. This is very similar to the parametric representation $\mathbf{r}(u)$ of a curve, discussed in section 10.3, but with the important difference that we require *two* parameters to describe a surface, whereas we need only one to describe a curve.

In Cartesian coordinates the surface is given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

where $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ are the parametric equations of the surface. We can also represent a surface by $z = f(x, y)$ or $g(x, y, z) = 0$. Either of these representations can be converted into the parametric form in a similar manner to that used for equations of curves. For example, if $z = f(x, y)$ then by setting $u = x$ and $v = y$ the surface can be represented in parametric form by

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}.$$

Any curve $\mathbf{r}(\lambda)$, where λ is a parameter, on the surface S can be represented by a pair of equations relating the parameters u and v, for example $u = f(\lambda)$ and $v = g(\lambda)$. A parametric representation of the curve can easily be found by straightforward substitution, i.e. $\mathbf{r}(\lambda) = \mathbf{r}(u(\lambda), v(\lambda))$. Using (10.17) for the case where the vector is a function of a single variable λ so that the LHS becomes a total derivative, the tangent to the curve $\mathbf{r}(\lambda)$ at any point is given by

$$\frac{d\mathbf{r}}{d\lambda} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{d\lambda} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{d\lambda}. \tag{10.21}$$

The two curves $u =$ constant and $v =$ constant passing through any point P on S are called *coordinate curves*. For the curve $u =$ constant, for example, we have $du/d\lambda = 0$, and so from (10.21) its tangent vector is in the direction $\partial\mathbf{r}/\partial v$. Similarly, the tangent vector to the curve $v =$ constant is in the direction $\partial\mathbf{r}/\partial u$.

If the surface is smooth then at any point P on S the vectors $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ are linearly independent and define the *tangent plane* T at the point P (see figure 10.4). A vector normal to the surface at P is given by

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}. \tag{10.22}$$

In the neighbourhood of P, an infinitesimal vector displacement $d\mathbf{r}$ is written

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\, du + \frac{\partial \mathbf{r}}{\partial v}\, dv.$$

The *element of area* at P, an infinitesimal parallelogram whose sides are the coordinate curves, has magnitude

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u}\, du \times \frac{\partial \mathbf{r}}{\partial v}\, dv \right| = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv = |\mathbf{n}|\, du\, dv. \tag{10.23}$$

Thus the total area of the surface is

$$A = \iint_R \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv = \iint_R |\mathbf{n}|\, du\, dv, \tag{10.24}$$

where R is the region in the uv-plane corresponding to the range of parameter values that define the surface.

► *Find the element of area on the surface of a sphere of radius a, and hence calculate the total surface area of the sphere.*

We can represent a point $\mathbf{r}$ on the surface of the sphere in terms of the two parameters θ and ϕ:

$$\mathbf{r}(\theta, \phi) = a \sin\theta \cos\phi\, \mathbf{i} + a \sin\theta \sin\phi\, \mathbf{j} + a \cos\theta\, \mathbf{k},$$

where θ and ϕ are the polar and azimuthal angles respectively. At any point P, vectors tangent to the coordinate curves $\theta =$ constant and $\phi =$ constant are

$$\frac{\partial \mathbf{r}}{\partial \theta} = a \cos\theta \cos\phi\, \mathbf{i} + a \cos\theta \sin\phi\, \mathbf{j} - a \sin\theta\, \mathbf{k},$$
$$\frac{\partial \mathbf{r}}{\partial \phi} = -a \sin\theta \sin\phi\, \mathbf{i} + a \sin\theta \cos\phi\, \mathbf{j}.$$

A normal **n** to the surface at this point is then given by

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= a^2\sin\theta(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}),$$

which has a magnitude of $a^2 \sin\theta$. Therefore, the element of area at P is, from (10.23),

$$dS = a^2 \sin\theta\, d\theta\, d\phi,$$

and the total surface area of the sphere is given by

$$A = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\; a^2 \sin\theta = 4\pi a^2.$$

This familiar result can, of course, be proved by much simpler methods! ◄

10.6 Scalar and vector fields

We now turn to the case where a particular scalar or vector quantity is defined not just at a point in space but continuously as a *field* throughout some region of space R (which is often the whole space). Although the concept of a field is valid for spaces with an arbitrary number of dimensions, in the remainder of this chapter we will restrict our attention to the familiar three-dimensional case. A *scalar field* $\phi(x, y, z)$ associates a scalar with each point in R, while a *vector field* $\mathbf{a}(x, y, z)$ associates a vector with each point. In what follows, we will assume that the variation in the scalar or vector field from point to point is both continuous and differentiable in R.

Simple examples of scalar fields include the pressure at each point in a fluid and the electrostatic potential at each point in space in the presence of an electric charge. Vector fields relating to the same physical systems are the velocity vector in a fluid (giving the local speed and direction of the flow) and the electric field.

With the study of continuously varying scalar and vector fields there arises the need to consider their derivatives and also the integration of field quantities along lines, over surfaces and throughout volumes in the field. We defer the discussion of line, surface and volume integrals until the next chapter, and in the remainder of this chapter we concentrate on the definition of vector differential operators and their properties.

10.7 Vector operators

Certain differential operations may be performed on scalar and vector fields and have wide-ranging applications in the physical sciences. The most important operations are those of finding the *gradient* of a scalar field and the *divergence* and *curl* of a vector field. It is usual to define these operators from a strictly mathematical point of view, as we do below. In the following chapter, however, we will discuss their geometrical definitions, which rely on the concept of integrating vector quantities along lines and over surfaces.

Central to all these differential operations is the vector operator ∇, which is called *del* (or sometimes *nabla*) and in Cartesian coordinates is defined by

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{10.25}$$

The form of this operator in non-Cartesian coordinate systems is discussed in sections 10.9 and 10.10.

10.7.1 Gradient of a scalar field

The *gradient* of a scalar field $\phi(x, y, z)$ is defined by

$$\text{grad } \phi = \nabla\phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}. \tag{10.26}$$

Clearly, $\nabla\phi$ is a vector field whose x-, y- and z- components are the first partial derivatives of $\phi(x, y, z)$ with respect to x, y and z respectively. Also note that the vector field $\nabla\phi$ should not be confused with the vector operator $\phi\nabla$, which has components $(\phi\, \partial/\partial x, \phi\, \partial/\partial y, \phi\, \partial/\partial z)$.

►*Find the gradient of the scalar field* $\phi = xy^2z^3$.

From (10.26) the gradient of ϕ is given by

$$\nabla\phi = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}. \blacktriangleleft$$

The gradient of a scalar field ϕ has some interesting geometrical properties. Let us first consider the problem of *calculating the rate of change of* ϕ *in some particular direction*. For an infinitesimal vector displacement $d\mathbf{r}$, forming its scalar product with $\nabla\phi$ we obtain

$$\begin{aligned}
\nabla\phi \cdot d\mathbf{r} &= \left(\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}\right) \cdot (\mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dx), \\
&= \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy + \frac{\partial \phi}{\partial z}\, dz, \\
&= d\phi,
\end{aligned} \tag{10.27}$$

which is the infinitesimal change in ϕ in going from position $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$. In particular, if $\mathbf{r}$ depends on some parameter u such that $\mathbf{r}(u)$ defines a space curve

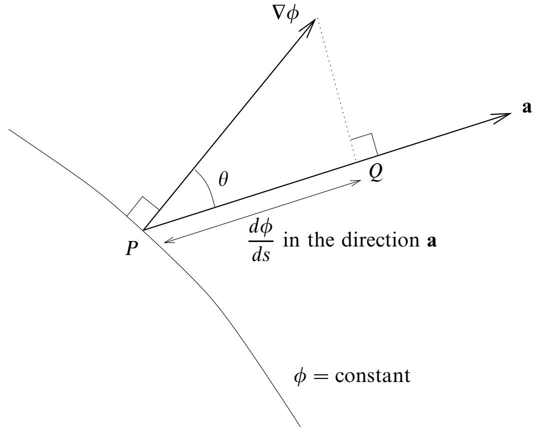


Figure 10.5 Geometrical properties of $\nabla\phi$. PQ gives the value of $d\phi/ds$ in the direction **a**.

then the total derivative of ϕ with respect to u along the curve is simply

$$\frac{d\phi}{du} = \nabla\phi \cdot \frac{d\mathbf{r}}{du}. \tag{10.28}$$

In the particular case where the parameter u is the arc length s along the curve, the total derivative of ϕ with respect to s along the curve is given by

$$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{\mathbf{t}}, \tag{10.29}$$

where $\hat{\mathbf{t}}$ is the unit tangent to the curve at the given point, as discussed in section 10.3.

In general, the rate of change of ϕ with respect to the distance s in a particular direction **a** is given by

$$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{\mathbf{a}} \tag{10.30}$$

and is called the directional derivative. Since $\hat{\mathbf{a}}$ is a unit vector we have

$$\frac{d\phi}{ds} = |\nabla\phi| \cos\theta$$

where θ is the angle between $\hat{\mathbf{a}}$ and $\nabla\phi$ as shown in figure 10.5. Clearly $\nabla\phi$ lies in the direction of the fastest increase in ϕ, and $|\nabla\phi|$ is the largest possible value of $d\phi/ds$. Similarly, the largest rate of decrease of ϕ is $d\phi/ds = -|\nabla\phi|$ in the direction of $-\nabla\phi$.

►*For the function $\phi = x^2y + yz$ at the point $(1, 2, -1)$, find its rate of change with distance in the direction $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. At this same point, what is the greatest possible rate of change with distance and in which direction does it occur?*

The gradient of ϕ is given by (10.26):

$$\nabla\phi = 2xy\mathbf{i} + (x^2 + z)\mathbf{j} + y\mathbf{k},$$
$$= 4\mathbf{i} + 2\mathbf{k} \qquad \text{at the point } (1, 2, -1).$$

The unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, so the rate of change of ϕ with distance s in this direction is, using (10.30),

$$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{\mathbf{a}} = \frac{1}{\sqrt{14}}(4 + 6) = \frac{10}{\sqrt{14}}.$$

From the above discussion, at the point $(1, 2, -1)$ $d\phi/ds$ will be greatest in the direction of $\nabla\phi = 4\mathbf{i} + 2\mathbf{k}$ and has the value $|\nabla\phi| = \sqrt{20}$ in this direction. ◄

We can extend the above analysis to find the rate of change of a vector field (rather than a scalar field as above) in a particular direction. The scalar differential operator $\hat{\mathbf{a}} \cdot \nabla$ can be shown to give the rate of change with distance in the direction $\hat{\mathbf{a}}$ of the quantity (vector or scalar) on which it acts. In Cartesian coordinates it may be written as

$$\hat{\mathbf{a}} \cdot \nabla = a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}. \tag{10.31}$$

Thus we can write the infinitesimal change in an electric field in moving from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ given in (10.20) as $d\mathbf{E} = (d\mathbf{r} \cdot \nabla)\mathbf{E}$.

A second interesting geometrical property of $\nabla\phi$ may be found by considering the surface defined by $\phi(x, y, z) = c$, where c is some constant. If $\hat{\mathbf{t}}$ is a unit tangent to this surface at some point then clearly $d\phi/ds = 0$ in this direction and from (10.29) we have $\nabla\phi \cdot \hat{\mathbf{t}} = 0$. In other words, *$\nabla\phi$ is a vector normal to the surface $\phi(x, y, z) = c$ at every point*, as shown in figure 10.5. If $\hat{\mathbf{n}}$ is a unit normal to the surface in the direction of increasing $\phi(x, y, z)$, then the gradient is sometimes written

$$\nabla\phi \equiv \frac{\partial\phi}{\partial n}\hat{\mathbf{n}}, \tag{10.32}$$

where $\partial\phi/\partial n \equiv |\nabla\phi|$ is the rate of change of ϕ in the direction $\hat{\mathbf{n}}$ and is called the *normal derivative*.

►*Find expressions for the equations of the tangent plane and the line normal to the surface $\phi(x, y, z) = c$ at the point P with coordinates x_0, y_0, z_0. Use the results to find the equations of the tangent plane and the line normal to the surface of the sphere $\phi = x^2 + y^2 + z^2 = a^2$ at the point $(0, 0, a)$.*

A vector normal to the surface $\phi(x, y, z) = c$ at the point P is simply $\nabla\phi$ evaluated at that point; we denote it by $\mathbf{n}_0$. If $\mathbf{r}_0$ is the position vector of the point P relative to the origin,

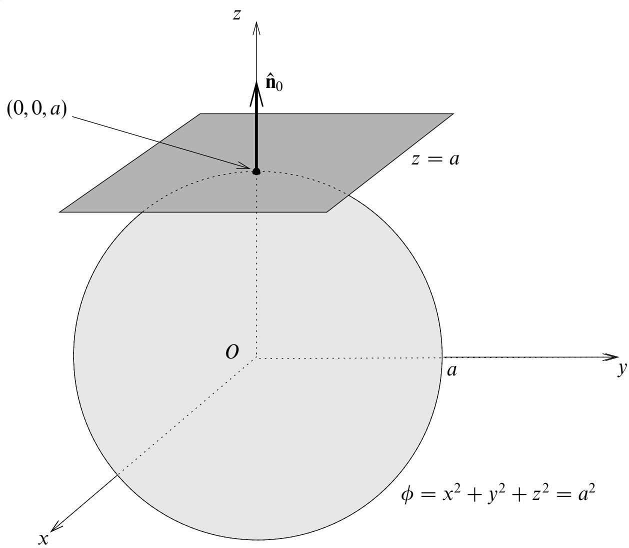


Figure 10.6 The tangent plane and the normal to the surface of the sphere $\phi = x^2 + y^2 + z^2 = a^2$ at the point $\mathbf{r}_0$ with coordinates $(0, 0, a)$.

and $\mathbf{r}$ is the position vector of any point on the tangent plane, then the vector equation of the tangent plane is, from (7.41),

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n}_0 = 0.$$

Similarly, if $\mathbf{r}$ is the position vector of any point on the straight line passing through P (with position vector $\mathbf{r}_0$) in the direction of the normal $\mathbf{n}_0$ then the vector equation of this line is, from subsection 7.7.1,

$$(\mathbf{r} - \mathbf{r}_0) \times \mathbf{n}_0 = \mathbf{0}.$$

For the surface of the sphere $\phi = x^2 + y^2 + z^2 = a^2$,

$$\begin{aligned}\nabla\phi &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ &= 2a\mathbf{k} \qquad \text{at the point } (0, 0, a).\end{aligned}$$

Therefore the equation of the tangent plane to the sphere at this point is

$$(\mathbf{r} - \mathbf{r}_0) \cdot 2a\mathbf{k} = 0.$$

This gives $2a(z - a) = 0$ or $z = a$, as expected. The equation of the line normal to the sphere at the point $(0, 0, a)$ is

$$(\mathbf{r} - \mathbf{r}_0) \times 2a\mathbf{k} = \mathbf{0},$$

which gives $2ay\mathbf{i} - 2ax\mathbf{j} = \mathbf{0}$ or $x = y = 0$, i.e. the z-axis, as expected. The tangent plane and normal to the surface of the sphere at this point are shown in figure 10.6. ◄

Further properties of the gradient operation, which are analogous to those of the ordinary derivative, are listed in subsection 10.8.1 and may be easily proved.

In addition to these, we note that the gradient operation also obeys the chain rule as in ordinary differential calculus, i.e. if ϕ and ψ are scalar fields in some region R then

$$\nabla\,[\phi(\psi)] = \frac{\partial \phi}{\partial \psi}\nabla\psi.$$

10.7.2 Divergence of a vector field

The *divergence* of a vector field $\mathbf{a}(x, y, z)$ is defined by

$$\operatorname{div}\mathbf{a} = \nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}, \tag{10.33}$$

where a_x, a_y and a_z are the x-, y- and z- components of $\mathbf{a}$. Clearly, $\nabla \cdot \mathbf{a}$ is a scalar field. Any vector field $\mathbf{a}$ for which $\nabla \cdot \mathbf{a} = 0$ is said to be *solenoidal.*

►*Find the divergence of the vector field* $\mathbf{a} = x^2y^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}$.

From (10.33) the divergence of $\mathbf{a}$ is given by

$$\nabla \cdot \mathbf{a} = 2xy^2 + 2yz^2 + 2x^2z = 2(xy^2 + yz^2 + x^2z). \blacktriangleleft$$

We will discuss fully the geometric definition of divergence and its physical meaning in the next chapter. For the moment, we merely note that the divergence can be considered as a quantitative measure of how much a vector field diverges (spreads out) or converges at any given point. For example, if we consider the vector field $\mathbf{v}(x, y, z)$ describing the local velocity at any point in a fluid then $\nabla \cdot \mathbf{v}$ is equal to the net rate of outflow of fluid per unit volume, evaluated at a point (by letting a small volume at that point tend to zero).

Now if some vector field $\mathbf{a}$ is itself derived from a scalar field via $\mathbf{a} = \nabla\phi$ then $\nabla \cdot \mathbf{a}$ has the form $\nabla \cdot \nabla\phi$ or, as it is usually written, $\nabla^2\phi$, where ∇^2 (del squared) is the scalar differential operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{10.34}$$

$\nabla^2\phi$ is called the *Laplacian* of ϕ and appears in several important partial differential equations of mathematical physics, discussed in chapters 20 and 21.

►*Find the Laplacian of the scalar field* $\phi = xy^2z^3$.

From (10.34) the Laplacian of ϕ is given by

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 2xz^3 + 6xy^2z. \blacktriangleleft$$

10.7.3 Curl of a vector field

The *curl* of a vector field $\mathbf{a}(x, y, z)$ is defined by

$$\operatorname{curl}\mathbf{a} = \nabla \times \mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k},$$

where a_x, a_y and a_z are the x-, y- and z- components of $\mathbf{a}$. The RHS can be written in a more memorable form as a determinant:

$$\nabla \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}, \tag{10.35}$$

where it is understood that, on expanding the determinant, the partial derivatives in the second row act on the components of $\mathbf{a}$ in the third row. Clearly, $\nabla \times \mathbf{a}$ is itself a vector field. Any vector field $\mathbf{a}$ for which $\nabla \times \mathbf{a} = \mathbf{0}$ is said to be *irrotational*.

► *Find the curl of the vector field* $\mathbf{a} = x^2y^2z^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}$.

The curl of $\mathbf{a}$ is given by

$$\nabla\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y^2z^2 & y^2z^2 & x^2z^2 \end{vmatrix} = -2\left[y^2z\mathbf{i} + (xz^2 - x^2y^2z)\mathbf{j} + x^2yz^2\mathbf{k}\right]. \blacktriangleleft$$

For a vector field $\mathbf{v}(x, y, z)$ describing the local velocity at any point in a fluid, $\nabla \times \mathbf{v}$ is a measure of the angular velocity of the fluid in the neighbourhood of that point. If a small paddle wheel were placed at various points in the fluid then it would tend to rotate in regions where $\nabla \times \mathbf{v} \neq \mathbf{0}$, while it would not rotate in regions where $\nabla \times \mathbf{v} = \mathbf{0}$.

Another insight into the physical interpretation of the curl operator is gained by considering the vector field $\mathbf{v}$ describing the velocity at any point in a rigid body rotating about some axis with angular velocity $\boldsymbol{\omega}$. If $\mathbf{r}$ is the position vector of the point with respect to some origin on the axis of rotation then the velocity of the point is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Without any loss of generality, we may take $\boldsymbol{\omega}$ to lie along the z-axis of our coordinate system, so that $\boldsymbol{\omega} = \omega\,\mathbf{k}$. The velocity field is then $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$. The curl of this vector field is easily found to be

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\boldsymbol{\omega}. \tag{10.36}$$

$$
\begin{aligned}
\nabla(\phi + \psi) &= \nabla\phi + \nabla\psi \\
\nabla \cdot (\mathbf{a} + \mathbf{b}) &= \nabla \cdot \mathbf{a} + \nabla \cdot \mathbf{b} \\
\nabla \times (\mathbf{a} + \mathbf{b}) &= \nabla \times \mathbf{a} + \nabla \times \mathbf{b} \\
\nabla(\phi\psi) &= \phi\nabla\psi + \psi\nabla\phi \\
\nabla(\mathbf{a} \cdot \mathbf{b}) &= \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} \\
\nabla \cdot (\phi\mathbf{a}) &= \phi\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\phi \\
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \\
\nabla \times (\phi\mathbf{a}) &= \nabla\phi \times \mathbf{a} + \phi\nabla \times \mathbf{a} \\
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}
\end{aligned}
$$

Table 10.1 Vector operators acting on sums and products. The operator ∇ is defined in (10.25); ϕ and ψ are scalar fields, **a** and **b** are vector fields.

Therefore the curl of the velocity field is a vector equal to twice the angular velocity vector of the rigid body about its axis of rotation. We give a full geometrical discussion of the curl of a vector in the next chapter.

10.8 Vector operator formulae

In the same way as for ordinary vectors (chapter 7), for vector operators certain identities exist. In addition, we must consider various relations involving the action of vector operators on sums and products of scalar and vector fields. Some of these relations have been mentioned earlier, but we list all the most important ones here for convenience. The validity of these relations may be easily verified by direct calculation (a quick method of deriving them using tensor notation is given in chapter 26).

Although some of the following vector relations are expressed in Cartesian coordinates, it may be proved that they are all independent of the choice of coordinate system. This is to be expected since grad, div and curl all have clear geometrical definitions, which are discussed more fully in the next chapter and which do not rely on any particular choice of coordinate system.

10.8.1 Vector operators acting on sums and products

Let ϕ and ψ be scalar fields and **a** and **b** be vector fields. Assuming these fields are differentiable, the action of grad, div and curl on various sums and products of them is presented in table 10.1.

These relations can be proved by direct calculation.

►*Show that*
$$\nabla \times (\phi \mathbf{a}) = \nabla\phi \times \mathbf{a} + \phi \nabla \times \mathbf{a}.$$

The x-component of the LHS is

$$\begin{aligned}
\frac{\partial}{\partial y}(\phi a_z) - \frac{\partial}{\partial z}(\phi a_y) &= \phi \frac{\partial a_z}{\partial y} + \frac{\partial \phi}{\partial y} a_z - \phi \frac{\partial a_y}{\partial z} - \frac{\partial \phi}{\partial z} a_y, \\
&= \phi \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + \left(\frac{\partial \phi}{\partial y} a_z - \frac{\partial \phi}{\partial z} a_y \right), \\
&= \phi(\nabla \times \mathbf{a})_x + (\nabla\phi \times \mathbf{a})_x,
\end{aligned}$$

where, for example, $(\nabla\phi \times \mathbf{a})_x$ denotes the x-component of the vector $\nabla\phi \times \mathbf{a}$. Incorporating the y- and z- components, which can be similarly found, we obtain the stated result. ◄

Some useful special cases of the relations in table 10.1 are worth noting. If $\mathbf{r}$ is the position vector relative to some origin and $r = |\mathbf{r}|$, then

$$\begin{aligned}
\nabla\phi(r) &= \frac{d\phi}{dr}\hat{\mathbf{r}}, \\
\nabla \cdot [\phi(r)\mathbf{r}] &= 3\phi(r) + r\frac{d\phi(r)}{dr}, \\
\nabla^2\phi(r) &= \frac{d^2\phi(r)}{dr^2} + \frac{2}{r}\frac{d\phi(r)}{dr}, \\
\nabla \times [\phi(r)\mathbf{r}] &= \mathbf{0}.
\end{aligned}$$

These results may be proved straightforwardly using Cartesian coordinates but far more simply using spherical polar coordinates, which are discussed in subsection 10.9.2. Particular cases of these results are

$$\nabla r = \hat{\mathbf{r}}, \qquad \nabla \cdot \mathbf{r} = 3, \qquad \nabla \times \mathbf{r} = \mathbf{0},$$

together with

$$\begin{aligned}
\nabla\left(\frac{1}{r}\right) &= -\frac{\hat{\mathbf{r}}}{r^2}, \\
\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) &= -\nabla^2\left(\frac{1}{r}\right) = 4\pi\delta(\mathbf{r}),
\end{aligned}$$

where $\delta(\mathbf{r})$ is the Dirac delta function, discussed in chapter 13. The last equation is important in the solution of certain partial differential equations and is discussed further in chapter 20.

10.8.2 Combinations of grad, div and curl

We now consider the action of two vector operators in succession on a scalar or vector field. We can immediately discard four of the nine obvious combinations of grad, div and curl, since they clearly do not make sense. If ϕ is a scalar field and

a is a vector field, these four combinations are grad(grad ϕ), div(div **a**), curl(div **a**) and grad(curl **a**). In each case the second (outer) vector operator is acting on the wrong type of field, i.e. scalar instead of vector or vice versa. In grad(grad ϕ), for example, grad acts on grad ϕ, which is a vector field, but we know that grad only acts on scalar fields (although in fact we will see in chapter 26 that we can form the *outer product* of the del operator with a vector to give a tensor, but that need not concern us here).

Of the five valid combinations of grad, div and curl, two are identically zero, namely

$$\text{curl grad } \phi = \nabla \times \nabla\phi = \mathbf{0}, \tag{10.37}$$

$$\text{div curl } \mathbf{a} = \nabla \cdot (\nabla \times \mathbf{a}) = 0. \tag{10.38}$$

From (10.37), we see that if **a** is derived from the gradient of some scalar function such that $\mathbf{a} = \nabla\phi$ then it is necessarily irrotational ($\nabla \times \mathbf{a} = 0$). We also note that if **a** is an irrotational vector field then another irrotational vector field is $\mathbf{a} + \nabla\phi + \mathbf{c}$, where ϕ is any scalar field and **c** is a constant vector. This follows since

$$\nabla \times (\mathbf{a} + \nabla\phi + \mathbf{c}) = \nabla \times \mathbf{a} + \nabla \times \nabla\phi = \mathbf{0}.$$

Similarly, from (10.38) we may infer that if **b** is the curl of some vector field **a** such that $\mathbf{b} = \nabla \times \mathbf{a}$ then **b** is solenoidal ($\nabla \cdot \mathbf{b} = 0$). Obviously, if **b** is solenoidal and **c** is any constant vector then $\mathbf{b} + \mathbf{c}$ is also solenoidal.

The three remaining combinations of grad, div and curl are

$$\text{div grad } \phi = \nabla \cdot \nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}, \tag{10.39}$$

$$\begin{aligned} \text{grad div } \mathbf{a} &= \nabla(\nabla \cdot \mathbf{a}), \\ &= \left(\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_y}{\partial x \partial y} + \frac{\partial^2 a_z}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 a_x}{\partial y \partial x} + \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_z}{\partial y \partial z}\right)\mathbf{j} \\ &\quad + \left(\frac{\partial^2 a_x}{\partial z \partial x} + \frac{\partial^2 a_y}{\partial z \partial y} + \frac{\partial^2 a_z}{\partial z^2}\right)\mathbf{k}, \end{aligned} \tag{10.40}$$

$$\text{curl curl } \mathbf{a} = \nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}, \tag{10.41}$$

where (10.39) and (10.40) are expressed in Cartesian coordinates. In (10.41), the term $\nabla^2\mathbf{a}$ has the linear differential operator ∇^2 acting on a vector (as opposed to a scalar as in (10.39)), which of course consists of a sum of unit vectors multiplied by components. Two cases arise.

(i) If the unit vectors are constants (i.e. they are independent of the values of the coordinates) then the differential operator gives a non-zero contribution only when acting upon the components, the unit vectors being merely multipliers.

(ii) If the unit vectors vary as the values of the coordinates change (i.e. are not constant in direction throughout the whole space) then the derivatives of these vectors appear as contributions to $\nabla^2 \mathbf{a}$.

Cartesian coordinates are an example of the first case in which each component satisfies $(\nabla^2 \mathbf{a})_i = \nabla^2 a_i$. In this case (10.41) can be applied to each component separately:

$$[\nabla \times (\nabla \times \mathbf{a})]_i = [\nabla(\nabla \cdot \mathbf{a})]_i - \nabla^2 a_i. \tag{10.42}$$

However, cylindrical and spherical polar coordinates come in the second class. For them (10.41) is still true, but the further step to (10.42) cannot be made.

More complicated vector operator relations may be proved using the relations given above.

►*Show that*

$$\nabla \cdot (\nabla\phi \times \nabla\psi) = 0,$$

where ϕ and ψ are scalar fields.

From the previous section we have

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

If we let $\mathbf{a} = \nabla\phi$ and $\mathbf{b} = \nabla\psi$ then we obtain

$$\nabla \cdot (\nabla\phi \times \nabla\psi) = \nabla\psi \cdot (\nabla \times \nabla\phi) - \nabla\phi \cdot (\nabla \times \nabla\psi) = 0, \tag{10.43}$$

since $\nabla \times \nabla\phi = 0 = \nabla \times \nabla\psi$, from (10.37). ◄

10.9 Cylindrical and spherical polar coordinates

The operators we have discussed in this chapter, i.e. grad, div, curl and ∇^2, have all been defined in terms of Cartesian coordinates, but for many physical situations other coordinate systems are more natural. For example, many systems, such as an isolated charge in space, have spherical symmetry and spherical polar coordinates would be the obvious choice. For axisymmetric systems, such as fluid flow in a pipe, cylindrical polar coordinates are the natural choice. The physical laws governing the behaviour of the systems are often expressed in terms of the vector operators we have been discussing, and so it is necessary to be able to express these operators in these other, non-Cartesian, coordinates. We first consider the two most common non-Cartesian coordinate systems, i.e. cylindrical and spherical polars, and go on to discuss general curvilinear coordinates in the next section.

10.9.1 Cylindrical polar coordinates

As shown in figure 10.7, the position of a point in space P having Cartesian coordinates x, y, z may be expressed in terms of cylindrical polar coordinates

ρ, ϕ, z, where

$$x = \rho \cos\phi, \qquad y = \rho \sin\phi, \qquad z = z, \tag{10.44}$$

and $\rho \geq 0$, $0 \leq \phi < 2\pi$ and $-\infty < z < \infty$. The position vector of P may therefore be written

$$\mathbf{r} = \rho \cos\phi\, \mathbf{i} + \rho \sin\phi\, \mathbf{j} + z\, \mathbf{k}. \tag{10.45}$$

If we take the partial derivatives of $\mathbf{r}$ with respect to ρ, ϕ and z respectively then we obtain the three vectors

$$\mathbf{e}_\rho = \frac{\partial \mathbf{r}}{\partial \rho} = \cos\phi\, \mathbf{i} + \sin\phi\, \mathbf{j}, \tag{10.46}$$

$$\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = -\rho \sin\phi\, \mathbf{i} + \rho \cos\phi\, \mathbf{j}, \tag{10.47}$$

$$\mathbf{e}_z = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}. \tag{10.48}$$

These vectors lie in the directions of increasing ρ, ϕ and z respectively but are not all of unit length. Although $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ form a useful set of basis vectors in their own right (we will see in section 10.10 that such a basis is sometimes the *most* useful), it is usual to work with the corresponding *unit* vectors, which are obtained by dividing each vector by its modulus to give

$$\hat{\mathbf{e}}_\rho = \mathbf{e}_\rho = \cos\phi\, \mathbf{i} + \sin\phi\, \mathbf{j}, \tag{10.49}$$

$$\hat{\mathbf{e}}_\phi = \frac{1}{\rho}\, \mathbf{e}_\phi = -\sin\phi\, \mathbf{i} + \cos\phi\, \mathbf{j}, \tag{10.50}$$

$$\hat{\mathbf{e}}_z = \mathbf{e}_z = \mathbf{k}. \tag{10.51}$$

These three unit vectors, like the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, form an orthonormal triad at each point in space, i.e. the basis vectors are mutually orthogonal and of unit length (see figure 10.7). Unlike the fixed vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, however, $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ change direction as P moves.

The expression for a general infinitesimal vector displacement $d\mathbf{r}$ in the position of P is given, from (10.19), by

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial \rho}\, d\rho + \frac{\partial \mathbf{r}}{\partial \phi}\, d\phi + \frac{\partial \mathbf{r}}{\partial z}\, dz \\ &= d\rho\, \mathbf{e}_\rho + d\phi\, \mathbf{e}_\phi + dz\, \mathbf{e}_z \\ &= d\rho\, \hat{\mathbf{e}}_\rho + \rho\, d\phi\, \hat{\mathbf{e}}_\phi + dz\, \hat{\mathbf{e}}_z. \end{aligned} \tag{10.52}$$

This expression illustrates an important difference between Cartesian and cylindrical polar coordinates (or non-Cartesian coordinates in general). In Cartesian coordinates, the distance moved in going from x to $x + dx$, with y and z held constant, is simply $ds = dx$. However, in cylindrical polars, if ϕ changes by $d\phi$, with ρ and z held constant, then the distance moved is *not* $d\phi$, but $ds = \rho\, d\phi$.

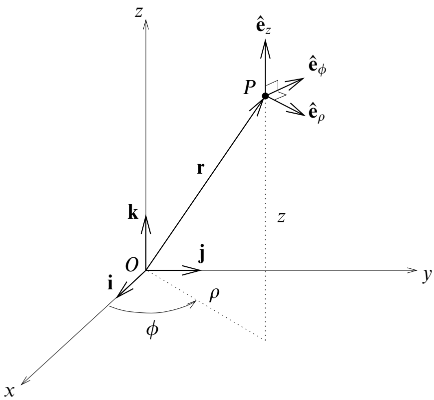


Figure 10.7 Cylindrical polar coordinates ρ, ϕ, z.

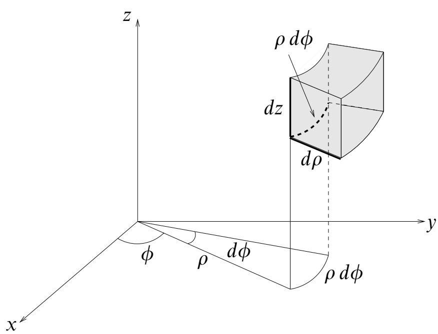


Figure 10.8 The element of volume in cylindrical polar coordinates is given by $\rho\, d\rho\, d\phi\, dz$.

Factors, such as the ρ in $\rho\, d\phi$, that multiply the coordinate differentials to give distances are known as *scale factors*. From (10.52), the scale factors for the ρ-, ϕ- and z- coordinates are therefore 1, ρ and 1 respectively.

The magnitude ds of the displacement $d\mathbf{r}$ is given in cylindrical polar coordinates by

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2,$$

where in the second equality we have used the fact that the basis vectors are orthonormal. We can also find the volume element in a cylindrical polar system (see figure 10.8) by calculating the volume of the infinitesimal parallelepiped

$$
\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix} \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}
$$

Table 10.2 Vector operators in cylindrical polar coordinates; Φ is a scalar field and $\mathbf{a}$ is a vector field.

defined by the vectors $d\rho\,\hat{\mathbf{e}}_\rho$, $\rho\,d\phi\,\hat{\mathbf{e}}_\phi$ and $dz\,\hat{\mathbf{e}}_z$:

$$dV = |d\rho\,\hat{\mathbf{e}}_\rho\cdot(\rho\,d\phi\,\hat{\mathbf{e}}_\phi\ \times\ dz\,\hat{\mathbf{e}}_z)| = \rho\,d\rho\,d\phi\,dz,$$

which again uses the fact that the basis vectors are orthonormal. For a simple coordinate system such as cylindrical polars the expressions for $(ds)^2$ and dV are obvious from the geometry.

We will now express the vector operators discussed in this chapter in terms of cylindrical polar coordinates. Let us consider a vector field $\mathbf{a}(\rho,\phi,z)$ and a scalar field $\Phi(\rho,\phi,z)$, where we use Φ for the scalar field to avoid confusion with the azimuthal angle ϕ. We must first write the vector field in terms of the basis vectors of the cylindrical polar coordinate system, i.e.

$$\mathbf{a} = a_\rho\,\hat{\mathbf{e}}_\rho + a_\phi\,\hat{\mathbf{e}}_\phi + a_z\,\hat{\mathbf{e}}_z,$$

where a_ρ, a_ϕ and a_z are the components of $\mathbf{a}$ in the ρ-, ϕ- and z- directions respectively. The expressions for grad, div, curl and ∇^2 can then be calculated and are given in table 10.2. Since the derivations of these expressions are rather complicated we leave them until our discussion of general curvilinear coordinates in the next section; the reader could well postpone examination of these formal proofs until some experience of using the expressions has been gained.

►*Express the vector field* $\mathbf{a} = yz\,\mathbf{i} - y\,\mathbf{j} + xz^2\,\mathbf{k}$ *in cylindrical polar coordinates, and hence calculate its divergence. Show that the same result is obtained by evaluating the divergence in Cartesian coordinates.*

The basis vectors of the cylindrical polar coordinate system are given in (10.49)–(10.51). Solving these equations simultaneously for $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ we obtain

$$
\begin{aligned}
\mathbf{i} &= \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{j} &= \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{k} &= \hat{\mathbf{e}}_z.
\end{aligned}
$$

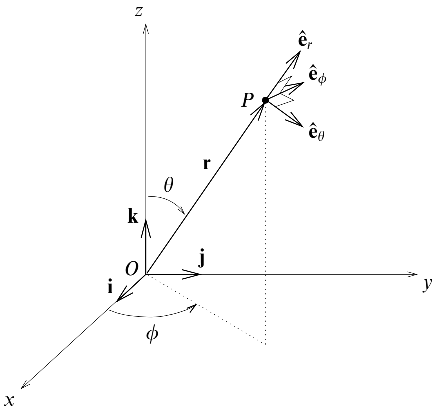


Figure 10.9 Spherical polar coordinates r, θ, ϕ.

Substituting these relations and (10.44) into the expression for $\mathbf{a}$ we find

$$\begin{aligned}\mathbf{a} &= z\rho\sin\phi\,(\cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi) - \rho\sin\phi\,(\sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi) + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z \\ &= (z\rho\sin\phi\cos\phi - \rho\sin^2\phi)\,\hat{\mathbf{e}}_\rho - (z\rho\sin^2\phi + \rho\sin\phi\cos\phi)\,\hat{\mathbf{e}}_\phi + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z.\end{aligned}$$

Substituting into the expression for $\nabla\cdot\mathbf{a}$ given in table 10.2,

$$\begin{aligned}\nabla\cdot\mathbf{a} &= 2z\sin\phi\cos\phi - 2\sin^2\phi - 2z\sin\phi\cos\phi - \cos^2\phi + \sin^2\phi + 2z\rho\cos\phi \\ &= 2z\rho\cos\phi - 1.\end{aligned}$$

Alternatively, and much more quickly in this case, we can calculate the divergence directly in Cartesian coordinates. We obtain

$$\nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 2zx - 1,$$

which on substituting $x = \rho\cos\phi$ yields the same result as the calculation in cylindrical polars. ◄

Finally, we note that similar results can be obtained for (two-dimensional) polar coordinates in a plane by omitting the z-dependence. For example, $(ds)^2 = (d\rho)^2 + \rho^2(d\phi)^2$, while the element of volume is replaced by the element of area $dA = \rho\,d\rho\,d\phi$.

10.9.2 Spherical polar coordinates

As shown in figure 10.9, the position of a point in space P, with Cartesian coordinates x, y, z, may be expressed in terms of spherical polar coordinates r, θ, ϕ, where

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta, \tag{10.53}$$

and $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The position vector of P may therefore be written as

$$\mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}.$$

If, in a similar manner to that used in the previous section for cylindrical polars, we find the partial derivatives of $\mathbf{r}$ with respect to r, θ and ϕ respectively and divide each of the resulting vectors by its modulus then we obtain the unit basis vectors

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},\\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},\\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.
\end{aligned}$$

These unit vectors are in the directions of increasing r, θ and ϕ respectively and are the orthonormal basis set for spherical polar coordinates, as shown in figure 10.9.

A general infinitesimal vector displacement in spherical polars is, from (10.19),

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi; \tag{10.54}$$

thus the scale factors for the r-, θ- and ϕ- coordinates are 1, r and $r\sin\theta$ respectively. The magnitude ds of the displacement $d\mathbf{r}$ is given by

$$(ds)^2 = d\mathbf{r}\cdot d\mathbf{r} = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2,$$

since the basis vectors form an orthonormal set. The element of volume in spherical polar coordinates (see figure 10.10) is the volume of the infinitesimal parallelepiped defined by the vectors $dr\,\hat{\mathbf{e}}_r$, $r\,d\theta\,\hat{\mathbf{e}}_\theta$ and $r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi$ and is given by

$$dV = |dr\,\hat{\mathbf{e}}_r\cdot(r\,d\theta\,\hat{\mathbf{e}}_\theta \;\times\; r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi)| = r^2\sin\theta\,dr\,d\theta\,d\phi,$$

where again we use the fact that the basis vectors are orthonormal. The expressions for $(ds)^2$ and dV in spherical polars can be obtained from the geometry of this coordinate system.

We will now express the standard vector operators in spherical polar coordinates, using the same techniques as for cylindrical polar coordinates. We consider a scalar field $\Phi(r,\theta,\phi)$ and a vector field $\mathbf{a}(r,\theta,\phi)$. The latter may be written in terms of the basis vectors of the spherical polar coordinate system as

$$\mathbf{a} = a_r\,\hat{\mathbf{e}}_r + a_\theta\,\hat{\mathbf{e}}_\theta + a_\phi\,\hat{\mathbf{e}}_\phi,$$

where a_r, a_θ and a_ϕ are the components of $\mathbf{a}$ in the r-, θ- and ϕ- directions respectively. The expressions for grad, div, curl and ∇^2 are given in table 10.3. The derivations of these results are given in the next section.

As a final note, we mention that, in the expression for $\nabla^2\Phi$ given in table 10.3,

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi$$

$$\nabla\cdot\mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ a_r & ra_\theta & r\sin\theta\ a_\phi \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Table 10.3 Vector operators in spherical polar coordinates; Φ is a scalar field and **a** is a vector field.

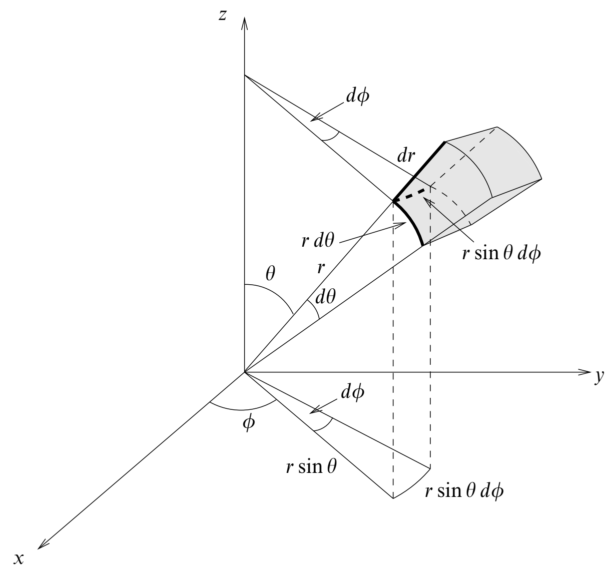


Figure 10.10 The element of volume in spherical polar coordinates is given by $r^2\sin\theta\,dr\,d\theta\,d\phi$.

we can rewrite the first term on the RHS as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi),$$

which can often be useful in shortening calculations.

10.10 General curvilinear coordinates

As indicated earlier, the contents of this section are more formal and technically complicated than hitherto. The section could be omitted until the reader has had some experience of using its results.

Cylindrical and spherical polars are just two examples of what are called *general curvilinear coordinates.* In the general case, the position of a point P having Cartesian coordinates x, y, z may be expressed in terms of the three curvilinear coordinates u_1, u_2, u_3, where

$$x = x(u_1, u_2, u_3), \qquad y = y(u_1, u_2, u_3), \qquad z = z(u_1, u_2, u_3),$$

and similarly

$$u_1 = u_1(x, y, z), \qquad u_2 = u_2(x, y, z), \qquad u_3 = u_3(x, y, z).$$

We assume that all these functions are continuous, differentiable and have a single-valued inverse, except perhaps at or on certain isolated points or lines, so that there is a one-to-one correspondence between the x, y, z and u_1, u_2, u_3 systems. The u_1-, u_2- and u_3- coordinate curves of a general curvilinear system are analogous to the x-, y- and z- axes of Cartesian coordinates. The surfaces $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$, where c_1, c_2, c_3 are constants, are called the *coordinate surfaces* and each pair of these surfaces has its intersection in a curve called a *coordinate curve* or *line* (see figure 10.11).

If at each point in space the three coordinate surfaces passing through the point meet at right angles then the curvilinear coordinate system is called *orthogonal.* For example, in spherical polars $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$ and the three coordinate surfaces passing through the point (R, Θ, Φ) are the sphere $r = R$, the circular cone $\theta = \Theta$ and the plane $\phi = \Phi$, which intersect at right angles at that point. Therefore spherical polars form an orthogonal coordinate system (as do cylindrical polars) .

If $\mathbf{r}(u_1, u_2, u_3)$ is the position vector of the point P then $\mathbf{e}_1 = \partial\mathbf{r}/\partial u_1$ is a vector tangent to the u_1-curve at P (for which u_2 and u_3 are constants) in the direction of increasing u_1. Similarly, $\mathbf{e}_2 = \partial\mathbf{r}/\partial u_2$ and $\mathbf{e}_3 = \partial\mathbf{r}/\partial u_3$ are vectors tangent to the u_2- and u_3- curves at P in the direction of increasing u_2 and u_3 respectively. Denoting the lengths of these vectors by h_1, h_2 and h_3, the *unit* vectors in each of these directions are given by

$$\hat{\mathbf{e}}_1 = \frac{1}{h_1}\frac{\partial \mathbf{r}}{\partial u_1}, \quad \hat{\mathbf{e}}_2 = \frac{1}{h_2}\frac{\partial \mathbf{r}}{\partial u_2}, \quad \hat{\mathbf{e}}_3 = \frac{1}{h_3}\frac{\partial \mathbf{r}}{\partial u_3},$$

where $h_1 = |\partial\mathbf{r}/\partial u_1|$, $h_2 = |\partial\mathbf{r}/\partial u_2|$ and $h_3 = |\partial\mathbf{r}/\partial u_3|$.

The quantities h_1, h_2, h_3 are the scale factors of the curvilinear coordinate system. The element of distance associated with an infinitesimal change du_i in one of the coordinates is $h_i\, du_i$. In the previous section we found that the scale

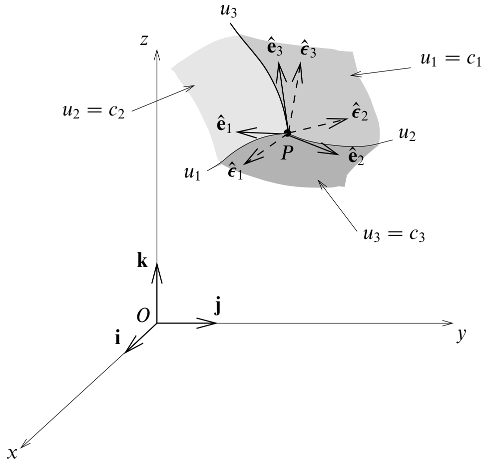


Figure 10.11 General curvilinear coordinates.

factors for cylindrical and spherical polar coordinates were

$$\begin{array}{llll} \text{for cylindrical polars} & h_\rho = 1, & h_\phi = \rho, & h_z = 1, \\ \text{for spherical polars} & h_r = 1, & h_\theta = r, & h_\phi = r\sin\theta. \end{array}$$

Although the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form a perfectly good basis for the curvilinear coordinate system, it is usual to work with the corresponding unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$. For an orthogonal curvilinear coordinate system these unit vectors form an orthonormal basis.

An infinitesimal vector displacement in general curvilinear coordinates is given by, from (10.19),

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}\,du_1 + \frac{\partial \mathbf{r}}{\partial u_2}\,du_2 + \frac{\partial \mathbf{r}}{\partial u_3}\,du_3 \qquad (10.55)$$

$$= du_1\,\mathbf{e}_1 + du_2\,\mathbf{e}_2 + du_3\,\mathbf{e}_3 \qquad (10.56)$$

$$= h_1\,du_1\,\hat{\mathbf{e}}_1 + h_2\,du_2\,\hat{\mathbf{e}}_2 + h_3\,du_3\,\hat{\mathbf{e}}_3. \qquad (10.57)$$

In the case of *orthogonal* curvilinear coordinates, where the $\hat{\mathbf{e}}_i$ are mutually perpendicular, the element of arc length is given by

$$(ds)^2 = d\mathbf{r}\cdot d\mathbf{r} = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2. \qquad (10.58)$$

The volume element for the coordinate system is the volume of the infinitesimal parallelepiped defined by the vectors $(\partial\mathbf{r}/\partial u_i)\,du_i = du_i\,\mathbf{e}_i = h_i\,du_i\,\hat{\mathbf{e}}_i$, for $i = 1, 2, 3$.

For orthogonal coordinates this is given by

$$\begin{aligned} dV &= |du_1\,\mathbf{e}_1 \cdot (du_2\,\mathbf{e}_2 \ \times\ du_3\,\mathbf{e}_3)| \\ &= |h_1\,\hat{\mathbf{e}}_1 \cdot (h_2\,\hat{\mathbf{e}}_2 \ \times\ h_3\,\hat{\mathbf{e}}_3)|\,du_1\,du_2\,du_3 \\ &= h_1 h_2 h_3\,du_1\,du_2\,du_3. \end{aligned}$$

Now, in addition to the set $\{\hat{\mathbf{e}}_i\}$, $i = 1, 2, 3$, there exists another useful set of three unit basis vectors at P. Since ∇u_1 is a vector normal to the surface $u_1 = c_1$, a unit vector in this direction is $\hat{\boldsymbol{\epsilon}}_1 = \nabla u_1/|\nabla u_1|$. Similarly, $\hat{\boldsymbol{\epsilon}}_2 = \nabla u_2/|\nabla u_2|$ and $\hat{\boldsymbol{\epsilon}}_3 = \nabla u_3/|\nabla u_3|$ are unit vectors normal to the surfaces $u_2 = c_2$ and $u_3 = c_3$ respectively.

Therefore at each point P in a curvilinear coordinate system, there exist, in general, two sets of unit vectors: $\{\hat{\mathbf{e}}_i\}$, tangent to the coordinate curves, and $\{\hat{\boldsymbol{\epsilon}}_i\}$, normal to the coordinate surfaces. A vector $\mathbf{a}$ can be written in terms of either set of unit vectors:

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3 = A_1\hat{\boldsymbol{\epsilon}}_1 + A_2\hat{\boldsymbol{\epsilon}}_2 + A_3\hat{\boldsymbol{\epsilon}}_3,$$

where a_1, a_2, a_3 and A_1, A_2, A_3 are the components of $\mathbf{a}$ in the two systems. It may be shown that the two bases become identical if the coordinate system is orthogonal.

Instead of the *unit* vectors discussed above, we could instead work directly with the two sets of vectors $\{\mathbf{e}_i = \partial\mathbf{r}/\partial u_i\}$ and $\{\boldsymbol{\epsilon}_i = \nabla u_i\}$, which are not, in general, of unit length. We can then write a vector $\mathbf{a}$ as

$$\mathbf{a} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 = \beta_1\boldsymbol{\epsilon}_1 + \beta_2\boldsymbol{\epsilon}_2 + \beta_3\boldsymbol{\epsilon}_3,$$

or more explicitly as

$$\mathbf{a} = \alpha_1\frac{\partial\mathbf{r}}{\partial u_1} + \alpha_2\frac{\partial\mathbf{r}}{\partial u_2} + \alpha_3\frac{\partial\mathbf{r}}{\partial u_3} = \beta_1\nabla u_1 + \beta_2\nabla u_2 + \beta_3\nabla u_3,$$

where α_1, α_2, α_3 and β_1, β_2, β_3 are called the *contravariant* and *covariant* components of $\mathbf{a}$ respectively. A more detailed discussion of these components, in the context of tensor analysis, is given in chapter 26. The (in general) non-unit bases $\{\mathbf{e}_i\}$ and $\{\boldsymbol{\epsilon}_i\}$ are often the most natural bases in which to express vector quantities.

►*Show that* $\{\mathbf{e}_i\}$ *and* $\{\boldsymbol{\epsilon}_i\}$ *are reciprocal systems of vectors.*

Let us consider the scalar product $\mathbf{e}_i \cdot \boldsymbol{\epsilon}_j$; using the Cartesian expressions for $\mathbf{r}$ and ∇, we obtain

$$\begin{aligned} \mathbf{e}_i \cdot \boldsymbol{\epsilon}_j &= \frac{\partial\mathbf{r}}{\partial u_i} \cdot \nabla u_j \\ &= \left(\frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k}\right) \cdot \left(\frac{\partial u_j}{\partial x}\mathbf{i} + \frac{\partial u_j}{\partial y}\mathbf{j} + \frac{\partial u_j}{\partial z}\mathbf{k}\right) \\ &= \frac{\partial x}{\partial u_i}\frac{\partial u_j}{\partial x} + \frac{\partial y}{\partial u_i}\frac{\partial u_j}{\partial y} + \frac{\partial z}{\partial u_i}\frac{\partial u_j}{\partial z} = \frac{\partial u_j}{\partial u_i}. \end{aligned}$$

In the last step we have used the chain rule for partial differentiation. Therefore $\mathbf{e}_i \cdot \boldsymbol{\epsilon}_j = 1$ if $i = j$, and $\mathbf{e}_i \cdot \boldsymbol{\epsilon}_j = 0$ otherwise. Hence $\{\mathbf{e}_i\}$ and $\{\boldsymbol{\epsilon}_j\}$ are reciprocal systems of vectors. ◄

We now derive expressions for the standard vector operators in *orthogonal* curvilinear coordinates. Despite the useful properties of the non-unit bases discussed above, the remainder of our discussion in this section will be in terms of the unit basis vectors $\{\hat{\mathbf{e}}_i\}$. The expressions for the vector operators in cylindrical and spherical polar coordinates given in tables 10.2 and 10.3 respectively can be found from those derived below by inserting the appropriate scale factors.

Gradient

The change $d\Phi$ in a scalar field Φ resulting from changes du_1, du_2, du_3 in the coordinates u_1, u_2, u_3 is given by, from (5.5),

$$d\Phi = \frac{\partial \Phi}{\partial u_1}\, du_1 + \frac{\partial \Phi}{\partial u_2}\, du_2 + \frac{\partial \Phi}{\partial u_3}\, du_3.$$

For orthogonal curvilinear coordinates u_1, u_2, u_3 we find from (10.57), and comparison with (10.27), that we can write this as

$$d\Phi = \nabla\Phi \cdot d\mathbf{r}, \tag{10.59}$$

where $\nabla\Phi$ is given by

$$\nabla\Phi = \frac{1}{h_1}\frac{\partial \Phi}{\partial u_1}\,\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial \Phi}{\partial u_2}\,\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial \Phi}{\partial u_3}\,\hat{\mathbf{e}}_3. \tag{10.60}$$

This implies that the del operator can be written

$$\nabla = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}.$$

►*Show that for orthogonal curvilinear coordinates $\nabla u_i = \hat{\mathbf{e}}_i/h_i$. Hence show that the two sets of vectors $\{\hat{\mathbf{e}}_i\}$ and $\{\hat{\boldsymbol{\epsilon}}_i\}$ are identical in this case.*

Letting $\Phi = u_i$ in (10.60) we find immediately that $\nabla u_i = \hat{\mathbf{e}}_i/h_i$. Therefore $|\nabla u_i| = 1/h_i$, and so $\hat{\boldsymbol{\epsilon}}_i = \nabla u_i/|\nabla u_i| = h_i \nabla u_i = \hat{\mathbf{e}}_i$. ◄

Divergence

In order to derive the expression for the divergence of a vector field in orthogonal curvilinear coordinates, we must first write the vector field in terms of the basis vectors of the coordinate system:

$$\mathbf{a} = a_1\,\hat{\mathbf{e}}_1 + a_2\,\hat{\mathbf{e}}_2 + a_3\,\hat{\mathbf{e}}_3.$$

The divergence is then given by

$$\nabla \cdot \mathbf{a} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 a_1) + \frac{\partial}{\partial u_2}(h_3 h_1 a_2) + \frac{\partial}{\partial u_3}(h_1 h_2 a_3)\right]. \tag{10.61}$$

►*Prove the expression for* $\nabla \cdot \mathbf{a}$ *in orthogonal curvilinear coordinates.*

Let us consider the sub-expression $\nabla \cdot (a_1\hat{\mathbf{e}}_1)$. Now $\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = h_2\nabla u_2 \times h_3\nabla u_3$. Therefore

$$\begin{aligned}\nabla \cdot (a_1\hat{\mathbf{e}}_1) &= \nabla \cdot (a_1h_2h_3\nabla u_2 \times \nabla u_3),\\ &= \nabla(a_1h_2h_3) \cdot (\nabla u_2 \times \nabla u_3) + a_1h_2h_3\nabla \cdot (\nabla u_2 \times \nabla u_3).\end{aligned}$$

However, $\nabla \cdot (\nabla u_2 \times \nabla u_3) = 0$, from (10.43), so we obtain

$$\nabla \cdot (a_1\hat{\mathbf{e}}_1) = \nabla(a_1h_2h_3) \cdot \left(\frac{\hat{\mathbf{e}}_2}{h_2} \times \frac{\hat{\mathbf{e}}_3}{h_3}\right) = \nabla(a_1h_2h_3) \cdot \frac{\hat{\mathbf{e}}_1}{h_2h_3};$$

letting $\Phi = a_1h_2h_3$ in (10.60) and substituting into the above equation, we find

$$\nabla \cdot (a_1\hat{\mathbf{e}}_1) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(a_1h_2h_3).$$

Repeating the analysis for $\nabla \cdot (a_2\hat{\mathbf{e}}_2)$ and $\nabla \cdot (a_3\hat{\mathbf{e}}_3)$, and adding the results we obtain (10.61), as required. ◄

Laplacian

In the expression for the divergence (10.61), let

$$\mathbf{a} = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\hat{\mathbf{e}}_3,$$

where we have used (10.60). We then obtain

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right],$$

which is the expression for the Laplacian in orthogonal curvilinear coordinates.

Curl

The curl of a vector field $\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$ in orthogonal curvilinear coordinates is given by

$$\nabla \times \mathbf{a} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix}. \qquad (10.62)$$

►*Prove the expression for* $\nabla \times \mathbf{a}$ *in orthogonal curvilinear coordinates.*

Let us consider the sub-expression $\nabla \times (a_1\hat{\mathbf{e}}_1)$. Since $\hat{\mathbf{e}}_1 = h_1\nabla u_1$ we have

$$\begin{aligned}\nabla \times (a_1\hat{\mathbf{e}}_1) &= \nabla \times (a_1h_1\nabla u_1),\\ &= \nabla(a_1h_1) \times \nabla u_1 \quad + \quad a_1h_1\nabla \times \nabla u_1.\end{aligned}$$

But $\nabla \times \nabla u_1 = 0$, so we obtain

$$\nabla \times (a_1\hat{\mathbf{e}}_1) = \nabla(a_1h_1) \times \frac{\hat{\mathbf{e}}_1}{h_1}.$$

$$\nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\hat{\mathbf{e}}_3$$

$$\nabla\cdot\mathbf{a} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3a_1) + \frac{\partial}{\partial u_2}(h_3h_1a_2) + \frac{\partial}{\partial u_3}(h_1h_2a_3)\right]$$

$$\nabla\times\mathbf{a} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

Table 10.4 Vector operators in orthogonal curvilinear coordinates u_1, u_2, u_3. Φ is a scalar field and $\mathbf{a}$ is a vector field.

Letting $\Phi = a_1h_1$ in (10.60) and substituting into the above equation, we find

$$\nabla\times(a_1\hat{\mathbf{e}}_1) = \frac{\hat{\mathbf{e}}_2}{h_3h_1}\frac{\partial}{\partial u_3}(a_1h_1) - \frac{\hat{\mathbf{e}}_3}{h_1h_2}\frac{\partial}{\partial u_2}(a_1h_1).$$

The corresponding analysis of $\nabla\times(a_2\,\hat{\mathbf{e}}_2)$ produces terms in $\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_1$, whilst that of $\nabla\times(a_3\,\hat{\mathbf{e}}_3)$ produces terms in $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. When the three results are added together, the coefficients multiplying $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ are the same as those obtained by writing out (10.62) explicitly, thus proving the stated result. ◄

The general expressions for the vector operators in orthogonal curvilinear coordinates are shown for reference in table 10.4. The explicit results for cylindrical and spherical polar coordinates, given in tables 10.2 and 10.3 respectively, are obtained by substituting the appropriate set of scale factors in each case.

A discussion of the expressions for vector operators in tensor form, which are valid even for non-orthogonal curvilinear coordinate systems, is given in chapter 26.

10.11 Exercises

10.1 Evaluate the integral

$$\int \left[\mathbf{a}(\dot{\mathbf{b}}\cdot\mathbf{a} + \mathbf{b}\cdot\dot{\mathbf{a}}) + \dot{\mathbf{a}}(\mathbf{b}\cdot\mathbf{a}) - 2(\dot{\mathbf{a}}\cdot\mathbf{a})\mathbf{b} - \dot{\mathbf{b}}|\mathbf{a}|^2\right]\,dt$$

in which $\dot{\mathbf{a}}$, $\dot{\mathbf{b}}$ are the derivatives of $\mathbf{a}$, $\mathbf{b}$ with respect to t.

10.2 At time $t = 0$, the vectors $\mathbf{E}$ and $\mathbf{B}$ are given by $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{B} = \mathbf{B}_0$, where the unit vectors, $\mathbf{E}_0$ and $\mathbf{B}_0$ are fixed and orthogonal. The equations of motion are

$$\frac{d\mathbf{E}}{dt} = \mathbf{E}_0 + \mathbf{B}\times\mathbf{E}_0,$$
$$\frac{d\mathbf{B}}{dt} = \mathbf{B}_0 + \mathbf{E}\times\mathbf{B}_0.$$

Find $\mathbf{E}$ and $\mathbf{B}$ at a general time t, showing that after a long time the directions of $\mathbf{E}$ and $\mathbf{B}$ have almost interchanged.

10.3 The general equation of motion of a (non-relativistic) particle of mass m and charge q when it is placed in a region where there is a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ is

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B});$$

here $\mathbf{r}$ is the position of the particle at time t and $\dot{\mathbf{r}} = d\mathbf{r}/dt$, etc. Write this as three separate equations in terms of the Cartesian components of the vectors involved.

For the simple case of crossed uniform fields $\mathbf{E} = E\mathbf{i}$, $\mathbf{B} = B\mathbf{j}$, in which the particle starts from the origin at $t = 0$ with $\dot{\mathbf{r}} = v_0\mathbf{k}$, find the equations of motion and show the following:

(a) if $v_0 = E/B$ then the particle continues its initial motion;
(b) if $v_0 = 0$ then the particle follows the space curve given in terms of the parameter ξ by

$$x = \frac{mE}{B^2q}(1 - \cos\xi), \qquad y = 0, \qquad z = \frac{mE}{B^2q}(\xi - \sin\xi).$$

Interpret this curve geometrically and relate ξ to t. Show that the total distance travelled by the particle after time t is given by

$$\frac{2E}{B}\int_0^t \left|\sin\frac{Bqt'}{2m}\right| dt'.$$

10.4 Use vector methods to find the maximum angle to the horizontal at which a stone may be thrown so as to ensure that it is always moving away from the thrower.

10.5 If two systems of coordinates with a common origin O are rotating with respect to each other, the measured accelerations differ in the two systems. Denoting by $\mathbf{r}$ and $\mathbf{r}'$ position vectors in frames $OXYZ$ and $OX'Y'Z'$, respectively, the connection between the two is

$$\ddot{\mathbf{r}}' = \ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rotation of $OXYZ$ with respect to $OX'Y'Z'$ (taken as fixed). The third term on the RHS is known as the Coriolis acceleration, whilst the final term gives rise to a centrifugal force.

Consider the application of this result to the firing of a shell of mass m from a stationary ship on the steadily rotating earth, working to the first order in ω ($= 7.3 \times 10^{-5}$ rad s^{-1}). If the shell is fired with velocity $\mathbf{v}$ at time $t = 0$ and only reaches a height that is small compared with the radius of the earth, show that its acceleration, as recorded on the ship, is given approximately by

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times (\mathbf{v} + \mathbf{g}t),$$

where $m\mathbf{g}$ is the weight of the shell measured on the ship's deck.

The shell is fired at another stationary ship (a distance $\mathbf{s}$ away) and $\mathbf{v}$ is such that the shell would have hit its target had there been no Coriolis effect.

(a) Show that without the Coriolis effect the time of flight of the shell would have been $\tau = -2\mathbf{g} \cdot \mathbf{v}/g^2$.
(b) Show further that when the shell actually hits the sea it is off-target by approximately

$$\frac{2\tau}{g^2}[(\mathbf{g} \times \boldsymbol{\omega}) \cdot \mathbf{v}](\mathbf{g}\tau + \mathbf{v}) - (\boldsymbol{\omega} \times \mathbf{v})\tau^2 - \frac{1}{3}(\boldsymbol{\omega} \times \mathbf{g})\tau^3.$$

(c) Estimate the order of magnitude Δ of this miss for a shell for which the initial speed v is 300 m s^{-1}, firing close to its maximum range ($\mathbf{v}$ makes an angle of $\pi/4$ with the vertical) in a northerly direction, whilst the ship is stationed at latitude 45° North.

10.6 Prove that for a space curve $\mathbf{r} = \mathbf{r}(s)$, where s is the arc length measured along the curve from a fixed point, the triple scalar product

$$\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^3\mathbf{r}}{ds^3}$$

at any point on the curve has the value $\kappa^2\tau$, where κ is the curvature and τ the torsion at that point.

10.7 For the twisted space curve $y^3 + 27axz - 81a^2y = 0$, given parametrically by

$$x = au(3 - u^2), \qquad y = 3au^2, \qquad z = au(3 + u^2),$$

show that the following hold:

(a) $ds/du = 3\sqrt{2}a(1 + u^2)$, where s is the distance along the curve measured from the origin;
(b) the length of the curve from the origin to the Cartesian point $(2a, 3a, 4a)$ is $4\sqrt{2}a$;
(c) the radius of curvature at the point with parameter u is $3a(1 + u^2)^2$;
(d) the torsion τ and curvature κ at a general point are equal;
(e) any of the Frenet–Serret formulae that you have not already used directly are satisfied.

10.8 The shape of the curving slip road joining two motorways, that cross at right angles and are at vertical heights $z = 0$ and $z = h$, can be approximated by the space curve

$$\mathbf{r} = \frac{\sqrt{2}h}{\pi} \ln\cos\left(\frac{z\pi}{2h}\right)\mathbf{i} + \frac{\sqrt{2}h}{\pi} \ln\sin\left(\frac{z\pi}{2h}\right)\mathbf{j} + z\mathbf{k}.$$

Show that the radius of curvature ρ of the slip road is $(2h/\pi)\operatorname{cosec}(z\pi/h)$ at height z and that the torsion $\tau = -1/\rho$. To shorten the algebra, set $z = 2h\theta/\pi$ and use θ as the parameter.

10.9 In a magnetic field, field lines are curves to which the magnetic induction $\mathbf{B}$ is everywhere tangential. By evaluating $d\mathbf{B}/ds$, where s is the distance measured along a field line, prove that the radius of curvature at any point on a line is given by

$$\rho = \frac{B^3}{|\mathbf{B} \times (\mathbf{B} \cdot \nabla)\mathbf{B}|}.$$

10.10 Find the areas of the given surfaces using parametric coordinates.

(a) Using the parameterisation $x = u\cos\phi$, $y = u\sin\phi$, $z = u\cot\Omega$, find the sloping surface area of a right circular cone of semi-angle Ω whose base has radius a. Verify that it is equal to $\frac{1}{2}\times$perimeter of the base $\times$slope height.
(b) Using the same parameterization as in (a) for x and y, and an appropriate choice for z, find the surface area between the planes $z = 0$ and $z = Z$ of the paraboloid of revolution $z = \alpha(x^2 + y^2)$.

10.11 Parameterising the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

by $x = a\cos\theta\sec\phi$, $y = b\sin\theta\sec\phi$, $z = c\tan\phi$, show that an area element on its surface is

$$dS = \sec^2\phi\left[c^2\sec^2\phi\left(b^2\cos^2\theta + a^2\sin^2\theta\right) + a^2b^2\tan^2\phi\right]^{1/2} d\theta\, d\phi.$$

Use this formula to show that the area of the curved surface $x^2 + y^2 - z^2 = a^2$ between the planes $z = 0$ and $z = 2a$ is

$$\pi a^2 \left(6 + \frac{1}{\sqrt{2}} \sinh^{-1} 2\sqrt{2} \right).$$

10.12 For the function

$$z(x, y) = (x^2 - y^2)e^{-x^2-y^2},$$

find the location(s) at which the steepest gradient occurs. What are the magnitude and direction of that gradient? The algebra involved is easier if plane polar coordinates are used.

10.13 Verify by direct calculation that

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

10.14 In the following exercises, **a**, **b** and **c** are vector fields.

(a) Simplify

$$\nabla \times \mathbf{a}(\nabla \cdot \mathbf{a}) \quad + \quad \mathbf{a} \times [\nabla \times (\nabla \times \mathbf{a})] \quad + \quad \mathbf{a} \times \nabla^2 \mathbf{a}.$$

(b) By explicitly writing out the terms in Cartesian coordinates, prove that

$$[\mathbf{c} \cdot (\mathbf{b} \cdot \nabla) - \mathbf{b} \cdot (\mathbf{c} \cdot \nabla)]\, \mathbf{a} = (\nabla \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c}).$$

(c) Prove that $\mathbf{a} \times (\nabla \times \mathbf{a}) = \nabla(\frac{1}{2}a^2) - (\mathbf{a} \cdot \nabla)\mathbf{a}$.

10.15 Evaluate the Laplacian of the function

$$\psi(x, y, z) = \frac{zx^2}{x^2 + y^2 + z^2}$$

(a) directly in Cartesian coordinates, and (b) after changing to a spherical polar coordinate system. Verify that, as they must, the two methods give the same result.

10.16 Verify that (10.42) is valid for each component separately when **a** is the Cartesian vector $x^2 y\, \mathbf{i} + xyz\, \mathbf{j} + z^2 y\, \mathbf{k}$, by showing that each side of the equation is equal to $z\, \mathbf{i} + (2x + 2z)\, \mathbf{j} + x\, \mathbf{k}$.

10.17 The (Maxwell) relationship between a time-independent magnetic field **B** and the current density **J** (measured in SI units in $\mathrm{A\,m^{-2}}$) producing it,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

can be applied to a long cylinder of conducting ionised gas which, in cylindrical polar coordinates, occupies the region $\rho < a$.

(a) Show that a uniform current density $(0, C, 0)$ and a magnetic field $(0, 0, B)$, with B constant $(= B_0)$ for $\rho > a$ and $B = B(\rho)$ for $\rho < a$, are consistent with this equation. Given that $B(0) = 0$ and that **B** is continuous at $\rho = a$, obtain expressions for C and $B(\rho)$ in terms of B_0 and a.

(b) The magnetic field can be expressed as $\mathbf{B} = \nabla \times \mathbf{A}$, where **A** is known as the vector potential. Show that a suitable **A** that has only one non-vanishing component, $A_\phi(\rho)$, can be found, and obtain explicit expressions for $A_\phi(\rho)$ for both $\rho < a$ and $\rho > a$. Like **B**, the vector potential is continuous at $\rho = a$.

(c) The gas pressure $p(\rho)$ satisfies the hydrostatic equation $\nabla p = \mathbf{J} \times \mathbf{B}$ and vanishes at the outer wall of the cylinder. Find a general expression for p.

10.18 Evaluate the Laplacian of a vector field using two different coordinate systems as follows.

(a) For cylindrical polar coordinates ρ, ϕ, z, evaluate the derivatives of the three unit vectors with respect to each of the coordinates, showing that only $\partial \hat{\mathbf{e}}_\rho / \partial \phi$ and $\partial \hat{\mathbf{e}}_\phi / \partial \phi$ are non-zero.

(i) Hence evaluate $\nabla^2 \mathbf{a}$ when $\mathbf{a}$ is the vector $\hat{\mathbf{e}}_\rho$, i.e. a vector of unit magnitude everywhere directed radially outwards and expressed by $a_\rho = 1$, $a_\phi = a_z = 0$.

(ii) Note that it is trivially obvious that $\nabla \times \mathbf{a} = \mathbf{0}$ and hence that equation (10.41) requires that $\nabla(\nabla \cdot \mathbf{a}) = \nabla^2 \mathbf{a}$.

(iii) Evaluate $\nabla(\nabla \cdot \mathbf{a})$ and show that the latter equation holds, but that

$$[\nabla(\nabla \cdot \mathbf{a})]_\rho \neq \nabla^2 a_\rho.$$

(b) Rework the same problem in Cartesian coordinates (where, as it happens, the algebra is more complicated).

10.19 Maxwell's equations for electromagnetism in free space (i.e. in the absence of charges, currents and dielectric or magnetic media) can be written

$$\text{(i)}\ \ \nabla \cdot \mathbf{B} = 0, \qquad \text{(ii)}\ \ \nabla \cdot \mathbf{E} = 0,$$
$$\text{(iii)}\ \ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \quad \text{(iv)}\ \ \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}.$$

A vector $\mathbf{A}$ is defined by $\mathbf{B} = \nabla \times \mathbf{A}$, and a scalar ϕ by $\mathbf{E} = -\nabla\phi - \partial \mathbf{A}/\partial t$. Show that if the condition

$$\text{(v)}\ \ \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

is imposed (this is known as choosing the Lorentz gauge), then $\mathbf{A}$ and ϕ satisfy wave equations as follows:

$$\text{(vi)}\ \ \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0,$$
$$\text{(vii)}\ \ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{0}.$$

The reader is invited to proceed as follows.

(a) Verify that the expressions for $\mathbf{B}$ and $\mathbf{E}$ in terms of $\mathbf{A}$ and ϕ are consistent with (i) and (iii).
(b) Substitute for $\mathbf{E}$ in (ii) and use the derivative with respect to time of (v) to eliminate $\mathbf{A}$ from the resulting expression. Hence obtain (vi).
(c) Substitute for $\mathbf{B}$ and $\mathbf{E}$ in (iv) in terms of $\mathbf{A}$ and ϕ. Then use the gradient of (v) to simplify the resulting equation and so obtain (vii).

10.20 In a description of the flow of a very viscous fluid that uses spherical polar coordinates with axial symmetry, the components of the velocity field $\mathbf{u}$ are given in terms of the *stream function* ψ by

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = \frac{-1}{r\sin\theta}\frac{\partial \psi}{\partial r}.$$

Find an explicit expression for the differential operator E defined by

$$E\psi = -(r\sin\theta)(\nabla \times \mathbf{u})_\phi.$$

The stream function satisfies the equation of motion $E^2\psi = 0$ and, for the flow of a fluid past a sphere, takes the form $\psi(r,\theta) = f(r)\sin^2\theta$. Show that $f(r)$ satisfies the (ordinary) differential equation

$$r^4 f^{(4)} - 4r^2 f'' + 8rf' - 8f = 0.$$

10.21 Paraboloidal coordinates u, v, ϕ are defined in terms of Cartesian coordinates by

$$x = uv\cos\phi, \qquad y = uv\sin\phi, \qquad z = \tfrac{1}{2}(u^2 - v^2).$$

Identify the coordinate surfaces in the u, v, ϕ system. Verify that each coordinate surface ($u =$ constant, say) intersects every coordinate surface on which one of the other two coordinates (v, say) is constant. Show further that the system of coordinates is an orthogonal one and determine its scale factors. Prove that the u-component of $\nabla \times \mathbf{a}$ is given by

$$\frac{1}{(u^2+v^2)^{1/2}}\left(\frac{a_\phi}{v} + \frac{\partial a_\phi}{\partial v}\right) - \frac{1}{uv}\frac{\partial a_v}{\partial \phi}.$$

10.22 Non-orthogonal curvilinear coordinates are difficult to work with and should be avoided if at all possible, but the following example is provided to illustrate the content of section 10.10.

In a new coordinate system for the region of space in which the Cartesian coordinate z satisfies $z \geq 0$, the position of a point $\mathbf{r}$ is given by (α_1, α_2, R), where α_1 and α_2 are respectively the cosines of the angles made by $\mathbf{r}$ with the x- and y-coordinate axes of a Cartesian system and $R = |\mathbf{r}|$. The ranges are $-1 \leq \alpha_i \leq 1$, $0 \leq R < \infty$.

(a) Express $\mathbf{r}$ in terms of α_1, α_2, R and the unit Cartesian vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.
(b) Obtain expressions for the vectors $\mathbf{e}_i$ $(= \partial\mathbf{r}/\partial\alpha_1, \ldots)$ and hence show that the scale factors h_i are given by

$$h_1 = \frac{R(1-\alpha_2^2)^{1/2}}{(1-\alpha_1^2-\alpha_2^2)^{1/2}}, \qquad h_2 = \frac{R(1-\alpha_1^2)^{1/2}}{(1-\alpha_1^2-\alpha_2^2)^{1/2}}, \qquad h_3 = 1.$$

(c) Verify formally that the system is not an orthogonal one.
(d) Show that the volume element of the coordinate system is

$$dV = \frac{R^2\, d\alpha_1\, d\alpha_2\, dR}{(1-\alpha_1^2-\alpha_2^2)^{1/2}},$$

and demonstrate that this is always less than or equal to the corresponding expression for an orthogonal curvilinear system.
(e) Calculate the expression for $(ds)^2$ for the system, and show that it differs from that for the corresponding orthogonal system by

$$\frac{2\alpha_1\alpha_2 R^2}{1-\alpha_1^2-\alpha_2^2}\, d\alpha_1 d\alpha_2.$$

10.23 Hyperbolic coordinates u, v, ϕ are defined in terms of Cartesian coordinates by

$$x = \cosh u \cos v \cos\phi, \qquad y = \cosh u \cos v \sin\phi, \qquad z = \sinh u \sin v.$$

Sketch the coordinate curves in the $\phi = 0$ plane, showing that far from the origin they become concentric circles and radial lines. In particular, identify the curves $u = 0$, $v = 0$, $v = \pi/2$ and $v = \pi$. Calculate the tangent vectors at a general point, show that they are mutually orthogonal and deduce that the appropriate scale factors are

$$h_u = h_v = (\cosh^2 u - \cos^2 v)^{1/2}, \qquad h_\phi = \cosh u \cos v.$$

Find the most general function $\psi(u)$ of u only that satisfies Laplace's equation $\nabla^2\psi = 0$.

10.24 In a Cartesian system, A and B are the points $(0, 0, -1)$ and $(0, 0, 1)$ respectively. In a new coordinate system a general point P is given by (u_1, u_2, u_3) with $u_1 = \frac{1}{2}(r_1 + r_2)$, $u_2 = \frac{1}{2}(r_1 - r_2)$, $u_3 = \phi$; here r_1 and r_2 are the distances AP and BP and ϕ is the angle between the plane ABP and $y = 0$.

(a) Express z and the perpendicular distance ρ from P to the z-axis in terms of u_1, u_2, u_3.
(b) Evaluate $\partial x/\partial u_i$, $\partial y/\partial u_i$, $\partial z/\partial u_i$, for $i = 1, 2, 3$.
(c) Find the Cartesian components of $\hat{\mathbf{u}}_j$ and hence show that the new coordinates are mutually orthogonal. Evaluate the scale factors and the infinitesimal volume element in the new coordinate system.
(d) Determine and sketch the forms of the surfaces u_i = constant.
(e) Find the most general function f of u_1 only that satisfies $\nabla^2 f = 0$.

10.12 Hints and answers

10.1 Group the term so that they form the total derivatives of compound vector expressions. The integral has the value $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{h}$.

10.3 For crossed uniform fields, $\ddot{x} + (Bq/m)^2 x = q(E - Bv_0)/m$, $\ddot{y} = 0$, $m\dot{z} = qBx + mv_0$; (b) $\xi = Bqt/m$; the path is a cycloid in the plane $y = 0$; $ds = [(dx/dt)^2 + (dz/dt)^2]^{1/2}\, dt$.

10.5 $\mathbf{g} = \ddot{\mathbf{r}}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, where $\ddot{\mathbf{r}}'$ is the shell's acceleration measured by an observer fixed in space. To first order in ω, the direction of $\mathbf{g}$ is radial, i.e. parallel to $\ddot{\mathbf{r}}'$.

(a) Note that $\mathbf{s}$ is orthogonal to $\mathbf{g}$.
(b) If the actual time of flight is T, use $(\mathbf{s} + \Delta) \cdot \mathbf{g} = 0$ to show that

$$T \approx \tau(1 + 2g^{-2}(\mathbf{g} \times \boldsymbol{\omega}) \cdot \mathbf{v} + \cdots).$$

In the Coriolis terms, it is sufficient to put $T \approx \tau$.
(c) For this situation $(\mathbf{g} \times \boldsymbol{\omega}) \cdot \mathbf{v} = 0$ and $\boldsymbol{\omega} \times \mathbf{v} = \mathbf{0}$; $\tau \approx 43$ s and $\Delta = 10$–15 m to the East.

10.7 (a) Evaluate $(d\mathbf{r}/du) \cdot (d\mathbf{r}/du)$.
(b) Integrate the previous result between $u = 0$ and $u = 1$.
(c) $\hat{\mathbf{t}} = [\sqrt{2}(1 + u^2)]^{-1}[(1 - u^2)\mathbf{i} + 2u\mathbf{j} + (1 + u^2)\mathbf{k}]$. Use $d\hat{\mathbf{t}}/ds = (d\hat{\mathbf{t}}/du)/(ds/du)$; $\rho^{-1} = |d\hat{\mathbf{t}}/ds|$.
(d) $\hat{\mathbf{n}} = (1+u^2)^{-1}[-2u\mathbf{i} + (1-u^2)\mathbf{j}]$. $\hat{\mathbf{b}} = [\sqrt{2}(1+u^2)]^{-1}[(u^2-1)\mathbf{i} - 2u\mathbf{j} + (1+u^2)\mathbf{k}]$. Use $d\hat{\mathbf{b}}/ds = (d\hat{\mathbf{b}}/du)/(ds/du)$ and show that this equals $-[3a(1 + u^2)^2]^{-1}\hat{\mathbf{n}}$.
(e) Show that $d\hat{\mathbf{n}}/ds = \tau(\hat{\mathbf{b}} - \hat{\mathbf{t}}) = -2[3\sqrt{2}a(1 + u^2)^3]^{-1}[(1 - u^2)\mathbf{i} + 2u\mathbf{j}]$.

10.9 Note that $d\mathbf{B} = (d\mathbf{r} \cdot \nabla)\mathbf{B}$ and that $\mathbf{B} = B\hat{\mathbf{t}}$, with $\hat{\mathbf{t}} = d\mathbf{r}/ds$. Obtain $(\mathbf{B} \cdot \nabla)\mathbf{B}/B = \hat{\mathbf{t}}(dB/ds) + \hat{\mathbf{n}}(B/\rho)$ and then take the vector product of $\hat{\mathbf{t}}$ with this equation.

10.11 To integrate $\sec^2\phi(\sec^2\phi + \tan^2\phi)^{1/2}\, d\phi$ put $\tan\phi = 2^{-1/2}\sinh\psi$.

10.13 Work in Cartesian coordinates, regrouping the terms obtained by evaluating the divergence on the LHS.

10.15 (a) $2z(x^2+y^2+z^2)^{-3}[(y^2+z^2)(y^2+z^2-3x^2)-4x^4]$; (b) $2r^{-1}\cos\theta\,(1-5\sin^2\theta\cos^2\phi)$; both are equal to $2zr^{-4}(r^2 - 5x^2)$.

10.17 Use the formulae given in table 10.2.

(a) $C = -B_0/(\mu_0 a)$; $B(\rho) = B_0\rho/a$.
(b) $B_0\rho^2/(3a)$ for $\rho < a$, and $B_0[\rho/2 - a^2/(6\rho)]$ for $\rho > a$.
(c) $[B_0^2/(2\mu_0)][1 - (\rho/a)^2]$.

10.19 Recall that $\nabla \times \nabla\phi = \mathbf{0}$ for any scalar ϕ and that $\partial/\partial t$ and ∇ act on different variables.

10.21 Two sets of paraboloids of revolution about the z-axis and the sheaf of planes containing the z-axis. For constant u, $-\infty < z < u^2/2$; for constant v, $-v^2/2 < z < \infty$. The scale factors are $h_u = h_v = (u^2 + v^2)^{1/2}$, $h_\phi = uv$.

10.23 The tangent vectors are as follows: for $u = 0$, the line joining $(1, 0, 0)$ and $(-1, 0, 0)$; for $v = 0$, the line joining $(1, 0, 0)$ and $(\infty, 0, 0)$; for $v = \pi/2$, the line $(0, 0, z)$; for $v = \pi$, the line joining $(-1, 0, 0)$ and $(-\infty, 0, 0)$.
$\psi(u) = 2\tan^{-1} e^u + c$, derived from $\partial[\cosh u(\partial\psi/\partial u)]/\partial u = 0$.

11

Line, surface and volume integrals

In the previous chapter we encountered continuously varying scalar and vector fields and discussed the action of various differential operators on them. In addition to these differential operations, the need often arises to consider the integration of field quantities along lines, over surfaces and throughout volumes. In general the integrand may be scalar or vector in nature, but the evaluation of such integrals involves their reduction to one or more scalar integrals, which are then evaluated. In the case of surface and volume integrals this requires the evaluation of double and triple integrals (see chapter 6).

11.1 Line integrals

In this section we discuss *line* or *path integrals*, in which some quantity related to the field is integrated between two given points in space, A and B, along a prescribed curve C that joins them. In general, we may encounter line integrals of the forms

$$\int_C \phi \, d\mathbf{r}, \qquad \int_C \mathbf{a} \cdot d\mathbf{r}, \qquad \int_C \mathbf{a} \times d\mathbf{r}, \tag{11.1}$$

where ϕ is a scalar field and $\mathbf{a}$ is a vector field. The three integrals themselves are respectively vector, scalar and vector in nature. As we will see below, in physical applications line integrals of the second type are by far the most common.

The formal definition of a line integral closely follows that of ordinary integrals and can be considered as the limit of a sum. We may divide the path C joining the points A and B into N small line elements $\Delta\mathbf{r}_p$, $p = 1, \ldots, N$. If (x_p, y_p, z_p) is any point on the line element $\Delta\mathbf{r}_p$ then the second type of line integral in (11.1), for example, is defined as

$$\int_C \mathbf{a} \cdot d\mathbf{r} = \lim_{N \to \infty} \sum_{p=1}^{N} \mathbf{a}(x_p, y_p, z_p) \cdot \Delta\mathbf{r}_p,$$

where it is assumed that all $|\Delta\mathbf{r}_p| \to 0$ as $N \to \infty$.

Each of the line integrals in (11.1) is evaluated over some curve C that may be either open (A and B being distinct points) or closed (the curve C forms a loop, so that A and B are coincident). In the case where C is closed, the line integral is written $\oint_C$ to indicate this. The curve may be given either parametrically by $\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$ or by means of simultaneous equations relating x, y, z for the given path (in Cartesian coordinates). A full discussion of the different representations of space curves was given in section 10.3.

In general, the value of the line integral depends not only on the end-points A and B but also on the path C joining them. For a closed curve we must also specify the direction around the loop in which the integral is taken. It is usually taken to be such that a person walking around the loop C in this direction always has the region R on his/her left; this is equivalent to traversing C in the anticlockwise direction (as viewed from above).

11.1.1 Evaluating line integrals

The method of evaluating a line integral is to reduce it to a set of scalar integrals. It is usual to work in Cartesian coordinates, in which case $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$. The first type of line integral in (11.1) then becomes simply

$$\int_C \phi\, d\mathbf{r} = \mathbf{i}\int_C \phi(x, y, z)\, dx + \mathbf{j}\int_C \phi(x, y, z)\, dy + \mathbf{k}\int_C \phi(x, y, z)\, dz.$$

The three integrals on the RHS are ordinary scalar integrals that can be evaluated in the usual way once the path of integration C has been specified. Note that in the above we have used relations of the form

$$\int \phi\, \mathbf{i}\, dx = \mathbf{i}\int \phi\, dx,$$

which is allowable since the Cartesian unit vectors are of constant magnitude and direction and hence may be taken out of the integral. If we had been using a different coordinate system, such as spherical polars, then, as we saw in the previous chapter, the unit basis vectors would not be constant. In that case the basis vectors could not be factorised out of the integral.

The second and third line integrals in (11.1) can also be reduced to a set of scalar integrals by writing the vector field $\mathbf{a}$ in terms of its Cartesian components as $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, where a_x, a_y, a_z are each (in general) functions of x, y, z. The second line integral in (11.1), for example, can then be written as

$$\begin{aligned}\int_C \mathbf{a}\cdot d\mathbf{r} &= \int_C (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})\\ &= \int_C (a_x\, dx + a_y\, dy + a_z\, dz)\\ &= \int_C a_x\, dx + \int_C a_y\, dy + \int_C a_z\, dz. \qquad (11.2)\end{aligned}$$

A similar procedure may be followed for the third type of line integral in (11.1), which involves a cross product.

Line integrals have properties that are analogous to those of ordinary integrals. In particular, the following are useful properties (which we illustrate using the second form of line integral in (11.1) but which are valid for all three types).

(i) Reversing the path of integration changes the sign of the integral. If the path C along which the line integrals are evaluated has A and B as its end-points then

$$\int_A^B \mathbf{a} \cdot d\mathbf{r} = -\int_B^A \mathbf{a} \cdot d\mathbf{r}.$$

This implies that if the path C is a loop then integrating around the loop in the opposite direction changes the sign of the integral.

(ii) If the path of integration is subdivided into smaller segments then the sum of the separate line integrals along each segment is equal to the line integral along the whole path. So, if P is any point on the path of integration that lies between the path's end-points A and B then

$$\int_A^B \mathbf{a} \cdot d\mathbf{r} = \int_A^P \mathbf{a} \cdot d\mathbf{r} + \int_P^B \mathbf{a} \cdot d\mathbf{r}.$$

►*Evaluate the line integral* $I = \int_C \mathbf{a} \cdot d\mathbf{r}$, *where* $\mathbf{a} = (x+y)\mathbf{i} + (y-x)\mathbf{j}$, *along each of the paths in the xy-plane shown in figure 11.1, namely*

(i) *the parabola* $y^2 = x$ *from* $(1,1)$ *to* $(4,2)$,
(ii) *the curve* $x = 2u^2 + u + 1$, $y = 1 + u^2$ *from* $(1,1)$ *to* $(4,2)$,
(iii) *the line* $y = 1$ *from* $(1,1)$ *to* $(4,1)$, *followed by the line* $x = 4$ *from* $(4,1)$ *to* $(4,2)$.

Since each of the paths lies entirely in the xy-plane, we have $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$. We can therefore write the line integral as

$$I = \int_C \mathbf{a} \cdot d\mathbf{r} = \int_C [(x+y)\,dx + (y-x)\,dy]. \tag{11.3}$$

We must now evaluate this line integral along each of the prescribed paths.

Case (i). Along the parabola $y^2 = x$ we have $2y\,dy = dx$. Substituting for x in (11.3) and using just the limits on y, we obtain

$$I = \int_{(1,1)}^{(4,2)} [(x+y)\,dx + (y-x)\,dy] = \int_1^2 [(y^2+y)2y + (y-y^2)]\,dy = 11\tfrac{1}{3}.$$

Note that we could just as easily have substituted for y and obtained an integral in x, which would have given the same result.

Case (ii). The second path is given in terms of a parameter u. We could eliminate u between the two equations to obtain a relationship between x and y directly and proceed as above, but it is usually quicker to write the line integral in terms of the parameter u. Along the curve $x = 2u^2 + u + 1$, $y = 1 + u^2$ we have $dx = (4u+1)\,du$ and $dy = 2u\,du$.

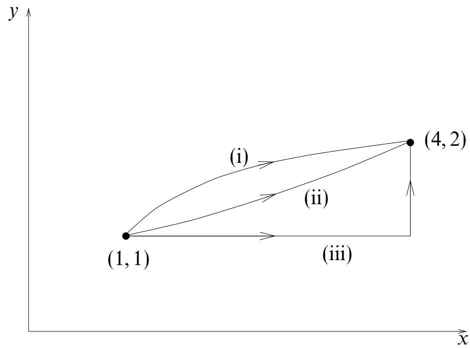


Figure 11.1 Different possible paths between the points (1, 1) and (4, 2).

Substituting for x and y in (11.3) and writing the correct limits on u, we obtain

$$I = \int_{(1,1)}^{(4,2)} [(x+y)\,dx + (y-x)\,dy]$$
$$= \int_0^1 [(3u^2+u+2)(4u+1) - (u^2+u)2u]\,du = 10\tfrac{2}{3}.$$

Case (iii). For the third path the line integral must be evaluated along the two line segments separately and the results added together. First, along the line $y = 1$ we have $dy = 0$. Substituting this into (11.3) and using just the limits on x for this segment, we obtain

$$\int_{(1,1)}^{(4,1)} [(x+y)\,dx + (y-x)\,dy] = \int_1^4 (x+1)\,dx = 10\tfrac{1}{2}.$$

Next, along the line $x = 4$ we have $dx = 0$. Substituting this into (11.3) and using just the limits on y for this segment, we obtain

$$\int_{(4,1)}^{(4,2)} [(x+y)\,dx + (y-x)\,dy] = \int_1^2 (y-4)\,dy = -2\tfrac{1}{2}.$$

The value of the line integral along the whole path is just the sum of the values of the line integrals along each segment, and is given by $I = 10\frac{1}{2} - 2\frac{1}{2} = 8$. ◄

When calculating a line integral along some curve C, which is given in terms of x, y and z, we are sometimes faced with the problem that the curve C is such that x, y and z are not single-valued functions of one another over the entire length of the curve. This is a particular problem for closed loops in the xy-plane (and also for some open curves). In such cases the path may be subdivided into shorter line segments along which one coordinate is a single-valued function of the other two. The sum of the line integrals along these segments is then equal to the line integral along the entire curve C. A better solution, however, is to represent the curve in a parametric form $\mathbf{r}(u)$ that is valid for its entire length.

►*Evaluate the line integral* $I = \oint_C x\,dy$, *where* C *is the circle in the xy-plane defined by* $x^2 + y^2 = a^2$, $z = 0$.

Adopting the usual convention mentioned above, the circle C is to be traversed in the anticlockwise direction. Taking the circle as a whole means x is not a single-valued function of y. We must therefore divide the path into two parts with $x = +\sqrt{a^2 - y^2}$ for the semicircle lying to the right of $x = 0$, and $x = -\sqrt{a^2 - y^2}$ for the semicircle lying to the left of $x = 0$. The required line integral is then the sum of the integrals along the two semicircles. Substituting for x, it is given by

$$I = \oint_C x\,dy = \int_{-a}^{a} \sqrt{a^2 - y^2}\,dy + \int_{a}^{-a} \left(-\sqrt{a^2 - y^2}\right)\,dy$$
$$= 4\int_0^a \sqrt{a^2 - y^2}\,dy = \pi a^2.$$

Alternatively, we can represent the entire circle parametrically, in terms of the azimuthal angle ϕ, so that $x = a\cos\phi$ and $y = a\sin\phi$ with ϕ running from 0 to 2π. The integral can therefore be evaluated over the whole circle at once. Noting that $dy = a\cos\phi\,d\phi$, we can rewrite the line integral completely in terms of the parameter ϕ and obtain

$$I = \oint_C x\,dy = a^2 \int_0^{2\pi} \cos^2\phi\,d\phi = \pi a^2. \blacktriangleleft$$

11.1.2 Physical examples of line integrals

There are many physical examples of line integrals, but perhaps the most common is the expression for the total work done by a force $\mathbf{F}$ when it moves its point of application from a point A to a point B along a given curve C. We allow the magnitude and direction of $\mathbf{F}$ to vary along the curve. Let the force act at a point $\mathbf{r}$ and consider a small displacement $d\mathbf{r}$ along the curve; then the small amount of work done is $dW = \mathbf{F}\cdot d\mathbf{r}$, as discussed in subsection 7.6.1 (note that dW can be either positive or negative). Therefore, the total work done in traversing the path C is

$$W_C = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

Naturally, other physical quantities can be expressed in such a way. For example, the electrostatic potential energy gained by moving a charge q along a path C in an electric field $\mathbf{E}$ is $-q\int_C \mathbf{E}\cdot d\mathbf{r}$. We may also note that Ampère's law concerning the magnetic field $\mathbf{B}$ associated with a current-carrying wire can be written as

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 I,$$

where I is the current enclosed by a closed path C traversed in a right-handed sense with respect to the current direction.

Magnetostatics also provides a physical example of the third type of line integral in (11.1). If a loop of wire C carrying a current I is placed in a magnetic field $\mathbf{B}$ then the force $d\mathbf{F}$ on a small length $d\mathbf{r}$ of the wire is given by $d\mathbf{F} = I\,d\mathbf{r} \times \mathbf{B}$, and so the total (vector) force on the loop is

$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B}.$$

11.1.3 Line integrals with respect to a scalar

In addition to those listed in (11.1), we can form other types of line integral, which depend on a particular curve C but for which we integrate with respect to a scalar du, rather than the vector differential $d\mathbf{r}$. This distinction is somewhat arbitrary, however, since we can always rewrite line integrals containing the vector differential $d\mathbf{r}$ as a line integral with respect to some scalar parameter. If the path C along which the integral is taken is described parametrically by $\mathbf{r}(u)$ then

$$d\mathbf{r} = \frac{d\mathbf{r}}{du}\,du,$$

and the second type of line integral in (11.1), for example, can be written as

$$\int_C \mathbf{a} \cdot d\mathbf{r} = \int_C \mathbf{a} \cdot \frac{d\mathbf{r}}{du}\,du.$$

A similar procedure can be followed for the other types of line integral in (11.1).

Commonly occurring special cases of line integrals with respect to a scalar are

$$\int_C \phi\,ds, \qquad \int_C \mathbf{a}\,ds,$$

where s is the arc length along the curve C. We can always represent C parametrically by $\mathbf{r}(u)$, and from section 10.3 we have

$$ds = \sqrt{\frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}}\,du.$$

The line integrals can therefore be expressed entirely in terms of the parameter u and thence evaluated.

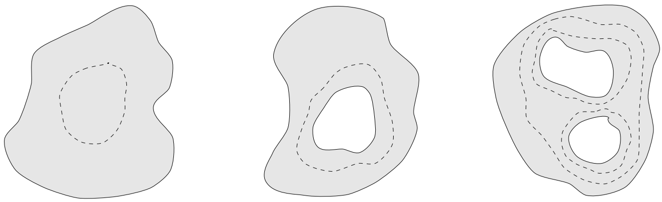
►*Evaluate the line integral $I = \int_C (x-y)^2\,ds$, where C is the semicircle of radius a running from $A = (a,0)$ to $B = (-a,0)$ and for which $y \geq 0$.*

The semicircular path from A to B can be described in terms of the azimuthal angle ϕ (measured from the x-axis) by

$$\mathbf{r}(\phi) = a\cos\phi\,\mathbf{i} + a\sin\phi\,\mathbf{j},$$

where ϕ runs from 0 to π. Therefore the element of arc length is given, from section 10.3, by

$$ds = \sqrt{\frac{d\mathbf{r}}{d\phi} \cdot \frac{d\mathbf{r}}{d\phi}}\,d\phi = a(\cos^2\phi + \sin^2\phi)\,d\phi = a\,d\phi.$$



(a) (b) (c)

Figure 11.2 (a) A simply connected region; (b) a doubly connected region; (c) a triply connected region.

Since $(x-y)^2 = a^2(1-\sin 2\phi)$, the line integral becomes

$$I = \int_C (x-y)^2\, ds = \int_0^{\pi} a^3(1-\sin 2\phi)\, d\phi = \pi a^3. \blacktriangleleft$$

As discussed in the previous chapter, the expression (10.58) for the square of the element of arc length in three-dimensional orthogonal curvilinear coordinates u_1, u_2, u_3 is

$$(ds)^2 = h_1^2\,(du_1)^2 + h_2^2\,(du_2)^2 + h_3^2\,(du_3)^2,$$

where h_1, h_2, h_3 are the scale factors of the coordinate system. If a curve C in three dimensions is given parametrically by the equations $u_i = u_i(\lambda)$ for $i = 1, 2, 3$ then the element of arc length along the curve is

$$ds = \sqrt{h_1^2\left(\frac{du_1}{d\lambda}\right)^2 + h_2^2\left(\frac{du_2}{d\lambda}\right)^2 + h_3^2\left(\frac{du_3}{d\lambda}\right)^2}\; d\lambda.$$

11.2 Connectivity of regions

In physical systems it is usual to define a scalar or vector field in some region R. In the next and some later sections we will need the concept of the *connectivity* of such a region in both two and three dimensions.

We begin by discussing planar regions. A plane region R is said to be *simply connected* if every simple closed curve within R can be continuously shrunk to a point without leaving the region (see figure 11.2(a)). If, however, the region R contains a hole then there exist simple closed curves that cannot be shrunk to a point without leaving R (see figure 11.2(b)). Such a region is said to be doubly connected, since its boundary has two distinct parts. Similarly, a region with $n-1$ holes is said to be *n-fold connected*, or *multiply connected* (the region in figure 11.2(c) is triply connected).

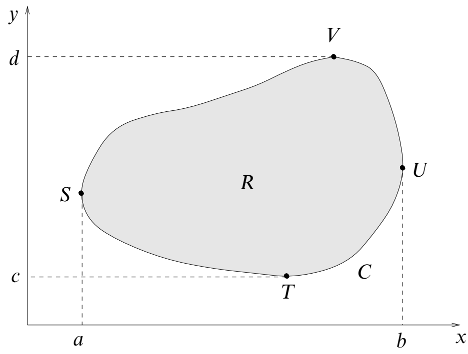


Figure 11.3 A simply connected region R bounded by the curve C.

These ideas can be extended to regions that are not planar, such as general three-dimensional surfaces and volumes. The same criteria concerning the shrinking of closed curves to a point also apply when deciding the connectivity of such regions. In these cases, however, the curves must lie in the surface or volume in question. For example, the interior of a torus is not simply connected, since there exist closed curves in the interior that cannot be shrunk to a point without leaving the torus. The region between two concentric spheres of different radii is simply connected.

11.3 Green's theorem in a plane

In subsection 11.1.1 we considered (amongst other things) the evaluation of line integrals for which the path C is closed and lies entirely in the xy-plane. Since the path is closed it will enclose a region R of the plane. We now discuss how to express the line integral around the loop as a double integral over the enclosed region R.

Suppose the functions $P(x, y)$, $Q(x, y)$ and their partial derivatives are single-valued, finite and continuous inside and on the boundary C of some simply connected region R in the xy-plane. *Green's theorem in a plane* (sometimes called the divergence theorem in two dimensions) then states

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy, \tag{11.4}$$

and so relates the line integral around C to a double integral over the enclosed region R. This theorem may be proved straightforwardly in the following way. Consider the simply connected region R in figure 11.3, and let $y = y_1(x)$ and

$y = y_2(x)$ be the equations of the curves STU and SVU respectively. We then write

$$\begin{aligned}
\iint_R \frac{\partial P}{\partial y}\,dx\,dy &= \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy\,\frac{\partial P}{\partial y} = \int_a^b dx\Big[P(x,y)\Big]_{y=y_1(x)}^{y=y_2(x)} \\
&= \int_a^b \Big[P(x,y_2(x)) - P(x,y_1(x))\Big]\,dx \\
&= -\int_a^b P(x,y_1(x))\,dx - \int_b^a P(x,y_2(x))\,dx = -\oint_C P\,dx.
\end{aligned}$$

If we now let $x = x_1(y)$ and $x = x_2(y)$ be the equations of the curves TSV and TUV respectively, we can similarly show that

$$\begin{aligned}
\iint_R \frac{\partial Q}{\partial x}\,dx\,dy &= \int_c^d dy \int_{x_1(y)}^{x_2(y)} dx\,\frac{\partial Q}{\partial x} = \int_c^d dy\Big[Q(x,y)\Big]_{x=x_1(y)}^{x=x_2(y)} \\
&= \int_c^d \Big[Q(x_2(y),y) - Q(x_1(y),y)\Big]\,dy \\
&= \int_d^c Q(x_1,y)\,dy + \int_c^d Q(x_2,y)\,dy = \oint_C Q\,dy.
\end{aligned}$$

Subtracting these two results gives Green's theorem in a plane.

►*Show that the area of a region R enclosed by a simple closed curve C is given by $A = \frac{1}{2}\oint_C(x\,dy - y\,dx) = \oint_C x\,dy = -\oint_C y\,dx$. Hence calculate the area of the ellipse $x = a\cos\phi$, $y = b\sin\phi$.*

In Green's theorem (11.4) put $P = -y$ and $Q = x$; then

$$\oint_C (x\,dy - y\,dx) = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy = 2A.$$

Therefore the area of the region is $A = \frac{1}{2}\oint_C(x\,dy - y\,dx)$. Alternatively, we could put $P = 0$ and $Q = x$ and obtain $A = \oint_C x\,dy$, or put $P = -y$ and $Q = 0$, which gives $A = -\oint_C y\,dx$.

The area of the ellipse $x = a\cos\phi$, $y = b\sin\phi$ is given by

$$\begin{aligned}
A = \frac{1}{2}\oint_C (x\,dy - y\,dx) &= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\phi + \sin^2\phi)\,d\phi \\
&= \frac{ab}{2}\int_0^{2\pi} d\phi = \pi ab. ◄
\end{aligned}$$

It may further be shown that Green's theorem in a plane is also valid for multiply connected regions. In this case, the line integral must be taken over all the distinct boundaries of the region. Furthermore, each boundary must be traversed in the positive direction, so that a person travelling along it in this direction always has the region R on their left. In order to apply Green's theorem

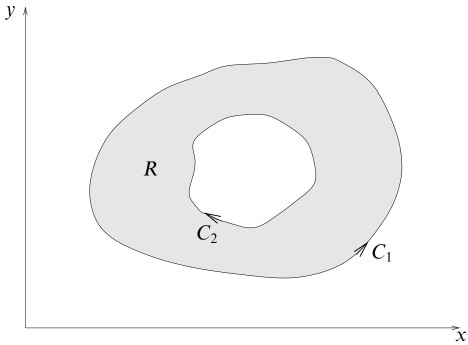


Figure 11.4 A doubly connected region R bounded by the curves C_1 and C_2.

to the region R shown in figure 11.4, the line integrals must be taken over both boundaries, C_1 and C_2, in the directions indicated, and the results added together.

We may also use Green's theorem in a plane to investigate the path independence (or not) of line integrals when the paths lie in the xy-plane. Let us consider the line integral

$$I = \int_A^B (P\,dx + Q\,dy).$$

For the line integral from A to B to be independent of the path taken, it must have the same value along any two arbitrary paths C_1 and C_2 joining the points. Moreover, if we consider as the path the closed loop C formed by $C_1 - C_2$ then the line integral around this loop must be zero. From Green's theorem in a plane, (11.4), we see that a *sufficient* condition for $I = 0$ is that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \tag{11.5}$$

throughout some simply connected region R containing the loop, where we assume that these partial derivatives are continuous in R.

It may be shown that (11.5) is also a *necessary* condition for $I = 0$ and is equivalent to requiring $P\,dx + Q\,dy$ to be an exact differential of some function $\phi(x, y)$ such that $P\,dx + Q\,dy = d\phi$. It follows that $\int_A^B (P\,dx + Q\,dy) = \phi(B) - \phi(A)$ and that $\oint_C (P\,dx + Q\,dy)$ around any closed loop C in the region R is identically zero. These results are special cases of the general results for paths in three dimensions, which are discussed in the next section.

►*Evaluate the line integral*

$$I = \oint_C [(e^x y + \cos x \sin y)\, dx + (e^x + \sin x \cos y)\, dy]\,,$$

around the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Clearly, it is not straightforward to calculate this line integral directly. However, if we let

$$P = e^x y + \cos x \sin y \qquad \text{and} \qquad Q = e^x + \sin x \cos y,$$

then $\partial P/\partial y = e^x + \cos x \cos y = \partial Q/\partial x$, and so $P\,dx + Q\,dy$ is an exact differential (it is actually the differential of the function $f(x, y) = e^x y + \sin x \sin y$). From the above discussion, we can conclude immediately that $I = 0$. ◄

11.4 Conservative fields and potentials

So far we have made the point that, in general, the value of a line integral between two points A and B depends on the path C taken from A to B. In the previous section, however, we saw that, for paths in the xy-plane, line integrals whose integrands have certain properties are independent of the path taken. We now extend that discussion to the full three-dimensional case.

For line integrals of the form $\int_C \mathbf{a} \cdot d\mathbf{r}$, there exists a class of vector fields for which the line integral between two points is *independent* of the path taken. Such vector fields are called *conservative*. A vector field $\mathbf{a}$ that has continuous partial derivatives in a simply connected region R is conservative if, and only if, any of the following is true.

(i) The integral $\int_A^B \mathbf{a} \cdot d\mathbf{r}$, where A and B lie in the region R, is independent of the path from A to B. Hence the integral $\oint_C \mathbf{a} \cdot d\mathbf{r}$ around any closed loop in R is zero.
(ii) There exists a single-valued function ϕ of position such that $\mathbf{a} = \nabla\phi$.
(iii) $\nabla \times \mathbf{a} = \mathbf{0}$.
(iv) $\mathbf{a} \cdot d\mathbf{r}$ is an exact differential.

The validity or otherwise of any of these statements implies the same for the other three, as we will now show.

First, let us assume that (i) above is true. If the line integral from A to B is independent of the path taken between the points then its value must be a function only of the positions of A and B. We may therefore write

$$\int_A^B \mathbf{a} \cdot d\mathbf{r} = \phi(B) - \phi(A), \tag{11.6}$$

which defines a single-valued scalar function of position ϕ. If the points A and B are separated by an infinitesimal displacement $d\mathbf{r}$ then (11.6) becomes

$$\mathbf{a} \cdot d\mathbf{r} = d\phi,$$

which shows that we require $\mathbf{a} \cdot d\mathbf{r}$ to be an exact differential: condition (iv). From (10.27) we can write $d\phi = \nabla\phi \cdot d\mathbf{r}$, and so we have

$$(\mathbf{a} - \nabla\phi) \cdot d\mathbf{r} = 0.$$

Since $d\mathbf{r}$ is arbitrary, we find that $\mathbf{a} = \nabla\phi$; this immediately implies $\nabla \times \mathbf{a} = \mathbf{0}$, condition (iii) (see (10.37)).

Alternatively, if we suppose that there exists a single-valued function of position ϕ such that $\mathbf{a} = \nabla\phi$ then $\nabla \times \mathbf{a} = \mathbf{0}$ follows as before. The line integral around a closed loop then becomes

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \oint_C \nabla\phi \cdot d\mathbf{r} = \oint d\phi.$$

Since we defined ϕ to be single-valued, this integral is zero as required.

Now suppose $\nabla \times \mathbf{a} = \mathbf{0}$. From Stoke's theorem, which is discussed in section 11.9, we immediately obtain $\oint_C \mathbf{a} \cdot d\mathbf{r} = 0$; then $\mathbf{a} = \nabla\phi$ and $\mathbf{a} \cdot d\mathbf{r} = d\phi$ follow as above.

Finally, let us suppose $\mathbf{a} \cdot d\mathbf{r} = d\phi$. Then immediately we have $\mathbf{a} = \nabla\phi$, and the other results follow as above.

►*Evaluate the line integral $I = \int_A^B \mathbf{a} \cdot d\mathbf{r}$, where $\mathbf{a} = (xy^2 + z)\mathbf{i} + (x^2 y + 2)\mathbf{j} + x\mathbf{k}$, A is the point (c, c, h) and B is the point $(2c, c/2, h)$, along the different paths*

(i) *C_1, given by $x = cu$, $y = c/u$, $z = h$,*
(ii) *C_2, given by $2y = 3c - x$, $z = h$.*

Show that the vector field $\mathbf{a}$ is in fact conservative, and find ϕ such that $\mathbf{a} = \nabla\phi$.

Expanding out the integrand, we have

$$I = \int_{(c,c,h)}^{(2c,c/2,h)} \left[(xy^2 + z)\,dx + (x^2 y + 2)\,dy + x\,dz\right], \tag{11.7}$$

which we must evaluate along each of the paths C_1 and C_2.

(i) Along C_1 we have $dx = c\,du$, $dy = -(c/u^2)\,du$, $dz = 0$, and on substituting in (11.7) and finding the limits on u, we obtain

$$I = \int_1^2 c\left(h - \frac{2}{u^2}\right)\,du = c(h-1).$$

(ii) Along C_2 we have $2\,dy = -dx$, $dz = 0$ and, on substituting in (11.7) and using the limits on x, we obtain

$$I = \int_c^{2c} \left(\tfrac{1}{2}x^3 - \tfrac{9}{4}cx^2 + \tfrac{9}{4}c^2 x + h - 1\right)\,dx = c(h-1).$$

Hence the line integral has the same value along paths C_1 and C_2. Taking the curl of $\mathbf{a}$, we have

$$\nabla \times \mathbf{a} = (0-0)\mathbf{i} + (1-1)\mathbf{j} + (2xy - 2xy)\mathbf{k} = \mathbf{0},$$

so $\mathbf{a}$ is a conservative vector field, and the line integral between two points must be independent of the path taken. Since $\mathbf{a}$ is conservative, we can write $\mathbf{a} = \nabla\phi$. Therefore, ϕ must satisfy

$$\frac{\partial \phi}{\partial x} = xy^2 + z,$$

which implies that $\phi = \frac{1}{2}x^2y^2 + zx + f(y,z)$ for some function f. Secondly, we require

$$\frac{\partial \phi}{\partial y} = x^2y + \frac{\partial f}{\partial y} = x^2y + 2,$$

which implies $f = 2y + g(z)$. Finally, since

$$\frac{\partial \phi}{\partial z} = x + \frac{\partial g}{\partial z} = x,$$

we have $g = \text{constant} = k$. It can be seen that we have explicitly constructed the function $\phi = \frac{1}{2}x^2y^2 + zx + 2y + k$. ◄

The quantity ϕ that figures so prominently in this section is called the *scalar potential function* of the conservative vector field $\mathbf{a}$ (which satisfies $\nabla \times \mathbf{a} = \mathbf{0}$), and is unique up to an arbitrary additive constant. Scalar potentials that are multi-valued functions of position (but in simple ways) are also of value in describing some physical situations, the most obvious example being the scalar magnetic potential associated with a current-carrying wire. When the integral of a field quantity around a closed loop is considered, provided the loop does not enclose a net current, the potential is single-valued and all the above results still hold. If the loop does enclose a net current, however, our analysis is no longer valid and extra care must be taken.

If, instead of being conservative, a vector field $\mathbf{b}$ satisfies $\nabla \cdot \mathbf{b} = 0$ (i.e. $\mathbf{b}$ is solenoidal) then it is both possible and useful, for example in the theory of electromagnetism, to define a *vector field* $\mathbf{a}$ such that $\mathbf{b} = \nabla \times \mathbf{a}$. It may be shown that such a vector field $\mathbf{a}$ always exists. Further, if $\mathbf{a}$ is one such vector field then $\mathbf{a}' = \mathbf{a} + \nabla\psi + \mathbf{c}$, where ψ is any scalar function and $\mathbf{c}$ is any constant vector, also satisfies the above relationship, i.e. $\mathbf{b} = \nabla \times \mathbf{a}'$. This was discussed more fully in subsection 10.8.2.

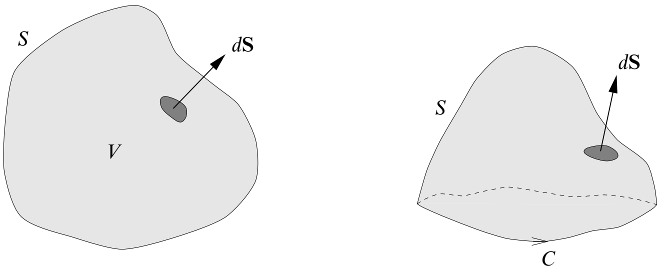
11.5 Surface integrals

As with line integrals, integrals over surfaces can involve vector and scalar fields and, equally, can result in either a vector or a scalar. The simplest case involves entirely scalars and is of the form

$$\int_S \phi \, dS. \tag{11.8}$$

As analogues of the line integrals listed in (11.1), we may also encounter surface integrals involving vectors, namely

$$\int_S \phi \, d\mathbf{S}, \qquad \int_S \mathbf{a} \cdot d\mathbf{S}, \qquad \int_S \mathbf{a} \times d\mathbf{S}. \tag{11.9}$$



(a) (b)

Figure 11.5 (a) A closed surface and (b) an open surface. In each case a normal to the surface is shown: $d\mathbf{S} = \hat{\mathbf{n}}\,dS$.

All the above integrals are taken over some surface S, which may be either open or closed, and are therefore, in general, double integrals. Following the notation for line integrals, for surface integrals over a closed surface $\int_S$ is replaced by $\oint_S$.

The vector differential $d\mathbf{S}$ in (11.9) represents a vector area element of the surface S. It may also be written $d\mathbf{S} = \hat{\mathbf{n}}\,dS$, where $\hat{\mathbf{n}}$ is a unit normal to the surface at the position of the element and dS is the scalar area of the element used in (11.8). The convention for the direction of the normal $\hat{\mathbf{n}}$ to a surface depends on whether the surface is open or closed. A closed surface, see figure 11.5(a), does not have to be simply connected (for example, the surface of a torus is not), but it does have to enclose a volume V, which may be of infinite extent. The direction of $\hat{\mathbf{n}}$ is taken to point outwards from the enclosed volume as shown. An open surface, see figure 11.5(b), spans some perimeter curve C. The direction of $\hat{\mathbf{n}}$ is then given by the right-hand sense with respect to the direction in which the perimeter is traversed, i.e. follows the right-hand screw rule discussed in subsection 7.6.2. An open surface does not have to be simply connected but for our purposes it must be two-sided (a Möbius strip is an example of a one-sided surface).

The formal definition of a surface integral is very similar to that of a line integral. We divide the surface S into N elements of area ΔS_p, $p = 1, 2, \ldots, N$, each with a unit normal $\hat{\mathbf{n}}_p$. If (x_p, y_p, z_p) is any point in ΔS_p then the second type of surface integral in (11.9), for example, is defined as

$$\int_S \mathbf{a} \cdot d\mathbf{S} = \lim_{N\to\infty} \sum_{p=1}^{N} \mathbf{a}(x_p, y_p, z_p) \cdot \hat{\mathbf{n}}_p \Delta S_p,$$

where it is required that all $\Delta S_p \to 0$ as $N \to \infty$.

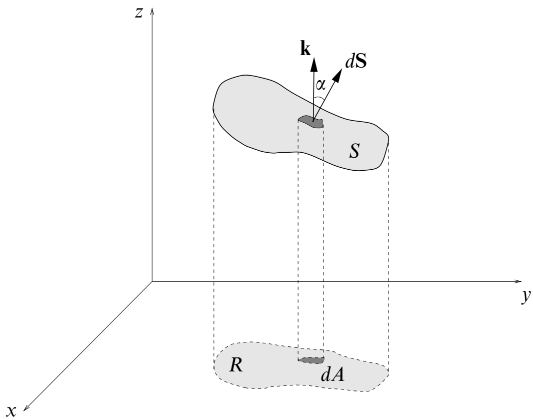


Figure 11.6 A surface S (or part thereof) projected onto a region R in the xy-plane; $d\mathbf{S}$ is a surface element.

11.5.1 Evaluating surface integrals

We now consider how to evaluate surface integrals over some general surface. This involves writing the scalar area element dS in terms of the coordinate differentials of our chosen coordinate system. In some particularly simple cases this is very straightforward. For example, if S is the surface of a sphere of radius a (or some part thereof) then using spherical polar coordinates θ, ϕ on the sphere we have $dS = a^2 \sin\theta\, d\theta\, d\phi$. For a general surface, however, it is not usually possible to represent the surface in a simple way in any particular coordinate system. In such cases, it is usual to work in Cartesian coordinates and consider the projections of the surface onto the coordinate planes.

Consider a surface (or part of a surface) S as in figure 11.6. The surface S is projected onto a region R of the xy-plane, so that an element of surface area dS projects onto the area element dA. From the figure, we see that $dA = |\cos\alpha|\, dS$, where α is the angle between the unit vector $\mathbf{k}$ in the z-direction and the unit normal $\hat{\mathbf{n}}$ to the surface at P. So, at any given point of S, we have simply

$$dS = \frac{dA}{|\cos\alpha|} = \frac{dA}{|\hat{\mathbf{n}}\cdot\mathbf{k}|}.$$

Now, if the surface S is given by the equation $f(x, y, z) = 0$ then, as shown in subsection 10.7.1, the unit normal at any point of the surface is given by $\hat{\mathbf{n}} = \nabla f/|\nabla f|$ evaluated at that point, cf. (10.32). The scalar element of surface area then becomes

$$dS = \frac{dA}{|\hat{\mathbf{n}}\cdot\mathbf{k}|} = \frac{|\nabla f|\, dA}{\nabla f\cdot\mathbf{k}} = \frac{|\nabla f|\, dA}{\partial f/\partial z}, \tag{11.10}$$

where $|\nabla f|$ and $\partial f/\partial z$ are evaluated on the surface S. We can therefore express any surface integral over S as a double integral over the region R in the xy-plane.

►*Evaluate the surface integral $I = \int_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a} = x\mathbf{i}$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$.*

The surface of the hemisphere is shown in figure 11.7. In this case dS may be easily expressed in spherical polar coordinates as $dS = a^2 \sin\theta \, d\theta \, d\phi$, and the unit normal to the surface at any point is simply $\hat{\mathbf{r}}$. On the surface of the hemisphere we have $x = a \sin\theta \cos\phi$ and so

$$\mathbf{a} \cdot d\mathbf{S} = x\,(\mathbf{i} \cdot \hat{\mathbf{r}})\; dS = (a \sin\theta \cos\phi)(\sin\theta \cos\phi)(a^2 \sin\theta \, d\theta \, d\phi).$$

Therefore, inserting the correct limits on θ and ϕ, we have

$$I = \int_S \mathbf{a} \cdot d\mathbf{S} = a^3 \int_0^{\pi/2} d\theta \, \sin^3\theta \int_0^{2\pi} d\phi \, \cos^2\phi = \frac{2\pi a^3}{3}.$$

We could, however, follow the general prescription above and project the hemisphere S onto the region R in the xy-plane that is a circle of radius a centred at the origin. Writing the equation of the surface of the hemisphere as $f(x, y) = x^2 + y^2 + z^2 - a^2 = 0$ and using (11.10), we have

$$I = \int_S \mathbf{a} \cdot d\mathbf{S} = \int_S x\,(\mathbf{i} \cdot \hat{\mathbf{r}})\; dS = \int_R x\,(\mathbf{i} \cdot \hat{\mathbf{r}})\, \frac{|\nabla f|\, dA}{\partial f/\partial z}.$$

Now $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{r}$, so on the surface S we have $|\nabla f| = 2|\mathbf{r}| = 2a$. On S we also have $\partial f/\partial z = 2z = 2\sqrt{a^2 - x^2 - y^2}$ and $\mathbf{i} \cdot \hat{\mathbf{r}} = x/a$. Therefore, the integral becomes

$$I = \iint_R \frac{x^2}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy.$$

Although this integral may be evaluated directly, it is quicker to transform to plane polar coordinates:

$$\begin{aligned} I &= \iint_{R'} \frac{\rho^2 \cos^2\phi}{\sqrt{a^2 - \rho^2}}\, \rho \, d\rho \, d\phi \\ &= \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^a \frac{\rho^3 \, d\rho}{\sqrt{a^2 - \rho^2}}. \end{aligned}$$

Making the substitution $\rho = a \sin u$, we finally obtain

$$I = \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^{\pi/2} a^3 \sin^3 u \, du = \frac{2\pi a^3}{3}. \blacktriangleleft$$

In the above discussion we assumed that any line parallel to the z-axis intersects S only once. If this is not the case, we must split up the surface into smaller surfaces S_1, S_2 etc. that are of this type. The surface integral over S is then the sum of the surface integrals over S_1, S_2 and so on. This is always necessary for closed surfaces.

Sometimes we may need to project a surface S (or some part of it) onto the zx- or yz-plane, rather than the xy-plane; for such cases, the above analysis is easily modified.

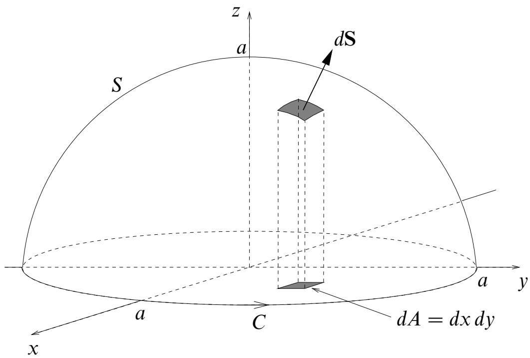


Figure 11.7 The surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

11.5.2 Vector areas of surfaces

The vector area of a surface S is defined as

$$\mathbf{S} = \int_S d\mathbf{S},$$

where the surface integral may be evaluated as above.

► *Find the vector area of the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$.*

As in the previous example, $d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$ in spherical polar coordinates. Therefore the vector area is given by

$$\mathbf{S} = \iint_S a^2 \sin\theta \; \hat{\mathbf{r}} \, d\theta \, d\phi.$$

Now, since $\hat{\mathbf{r}}$ varies over the surface S, it also must be integrated. This is most easily achieved by writing $\hat{\mathbf{r}}$ in terms of the constant Cartesian basis vectors. On S we have

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},$$

so the expression for the vector area becomes

$$\begin{aligned}\mathbf{S} &= \mathbf{i}\left(a^2\int_0^{2\pi}\cos\phi\,d\phi\int_0^{\pi/2}\sin^2\theta\,d\theta\right) + \mathbf{j}\left(a^2\int_0^{2\pi}\sin\phi\,d\phi\int_0^{\pi/2}\sin^2\theta\,d\theta\right)\\ &\quad + \mathbf{k}\left(a^2\int_0^{2\pi}d\phi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right)\\ &= \mathbf{0} + \mathbf{0} + \pi a^2\mathbf{k} = \pi a^2\mathbf{k}.\end{aligned}$$

Note that the magnitude of $\mathbf{S}$ is the projected area, of the hemisphere onto the xy-plane, and not the surface area of the hemisphere. ◄

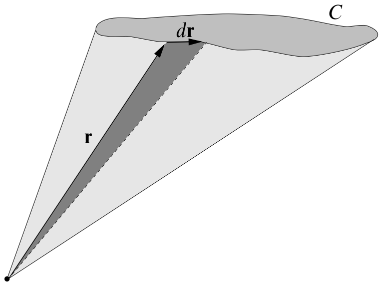


Figure 11.8 The conical surface spanning the perimeter C and having its vertex at the origin.

The hemispherical shell discussed above is an example of an open surface. For a closed surface, however, the vector area is always zero. This may be seen by projecting the surface down onto each Cartesian coordinate plane in turn. For each projection, every positive element of area on the upper surface is cancelled by the corresponding negative element on the lower surface. Therefore, each component of $\mathbf{S} = \oint_S d\mathbf{S}$ vanishes.

An important corollary of this result is that the vector area of an open surface depends only on its perimeter, or boundary curve, C. This may be proved as follows. If surfaces S_1 and S_2 have the same perimeter then $S_1 - S_2$ is a closed surface, for which

$$\oint d\mathbf{S} = \int_{S_1} d\mathbf{S} - \int_{S_2} d\mathbf{S} = \mathbf{0}.$$

Hence $\mathbf{S}_1 = \mathbf{S}_2$. Moreover, we may derive an expression for the vector area of an open surface S solely in terms of a line integral around its perimeter C. Since we may choose any surface with perimeter C, we will consider a cone with its vertex at the origin (see figure 11.8). The vector area of the elementary triangular region shown in the figure is $d\mathbf{S} = \frac{1}{2}\mathbf{r} \times d\mathbf{r}$. Therefore, the vector area of the cone, and hence of *any* open surface with perimeter C, is given by the line integral

$$\mathbf{S} = \frac{1}{2}\oint_C \mathbf{r} \times d\mathbf{r}.$$

For a surface confined to the xy-plane, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$, and we obtain for this special case that the area of the surface is given by $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$, as we found in section 11.3.

►*Find the vector area of the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, by evaluating the line integral $\mathbf{S} = \frac{1}{2}\oint_C \mathbf{r} \times d\mathbf{r}$ around its perimeter.*

The perimeter C of the hemisphere is the circle $x^2 + y^2 = a^2$, on which we have

$$\mathbf{r} = a\cos\phi\,\mathbf{i} + a\sin\phi\,\mathbf{j}, \qquad d\mathbf{r} = -a\sin\phi\,d\phi\,\mathbf{i} + a\cos\phi\,d\phi\,\mathbf{j}.$$

Therefore the cross product $\mathbf{r} \times d\mathbf{r}$ is given by

$$\mathbf{r} \times d\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi & a\sin\phi & 0 \\ -a\sin\phi\,d\phi & a\cos\phi\,d\phi & 0 \end{vmatrix} = a^2(\cos^2\phi + \sin^2\phi)\,d\phi\,\mathbf{k} = a^2\,d\phi\,\mathbf{k},$$

and the vector area becomes

$$\mathbf{S} = \tfrac{1}{2}a^2\mathbf{k}\int_0^{2\pi} d\phi = \pi a^2\,\mathbf{k}. ◄$$

11.5.3 Physical examples of surface integrals

There are many examples of surface integrals in the physical sciences. Surface integrals of the form (11.8) occur in computing the total electric charge on a surface or the mass of a shell, $\int_S \rho(\mathbf{r})\,dS$, given the charge or mass density $\rho(\mathbf{r})$. For surface integrals involving vectors, the second form in (11.9) is the most common. For a vector field $\mathbf{a}$, the surface integral $\int_S \mathbf{a}\cdot d\mathbf{S}$ is called the *flux* of $\mathbf{a}$ through S. Examples of physically important flux integrals are numerous. For example, let us consider a surface S in a fluid with density $\rho(\mathbf{r})$ that has a velocity field $\mathbf{v}(\mathbf{r})$. The mass of fluid crossing an element of surface area $d\mathbf{S}$ in time dt is $dM = \rho\mathbf{v}\cdot d\mathbf{S}\,dt$. Therefore the *net* total mass flux of fluid crossing S is $M = \int_S \rho(\mathbf{r})\mathbf{v}(\mathbf{r})\cdot d\mathbf{S}$. As a another example, the electromagnetic flux of energy out of a given volume V bounded by a surface S is $\oint_S(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}$.

The solid angle, to be defined below, subtended at a point O by a surface (closed or otherwise) can also be represented by an integral of this form, although it is not strictly a flux integral (unless we imagine isotropic rays radiating from O). The integral

$$\Omega = \int_S \frac{\mathbf{r}\cdot d\mathbf{S}}{r^3} = \int_S \frac{\hat{\mathbf{r}}\cdot d\mathbf{S}}{r^2}, \tag{11.11}$$

gives the *solid angle* Ω *subtended at* O *by a surface* S if $\mathbf{r}$ is the position vector measured from O of an element of the surface. A little thought will show that (11.11) takes account of all three relevant factors: the size of the element of surface, its inclination to the line joining the element to O and the distance from O. Such a general expression is often useful for computing solid angles when the three-dimensional geometry is complicated. Note that (11.11) remains valid when the surface S is not convex and when a single ray from O in certain directions would cut S in more than one place (but we exclude multiply connected regions).

In particular, when the surface is closed $\Omega = 0$ if O is outside S and $\Omega = 4\pi$ if O is an interior point.

Surface integrals resulting in vectors occur less frequently. An example is afforded, however, by the total resultant force experienced by a body immersed in a stationary fluid in which the hydrostatic pressure is given by $p(\mathbf{r})$. The pressure is everywhere inwardly directed and the resultant force is $\mathbf{F} = -\oint_S p\, d\mathbf{S}$, taken over the whole surface.

11.6 Volume integrals

Volume integrals are defined in an obvious way and are generally simpler than line or surface integrals since the element of volume dV is a scalar quantity. We may encounter volume integrals of the forms

$$\int_V \phi\, dV, \qquad \int_V \mathbf{a}\, dV. \tag{11.12}$$

Clearly, the first form results in a scalar, whereas the second form yields a vector. Two closely related physical examples, one of each kind, are provided by the total mass of a fluid contained in a volume V, given by $\int_V \rho(\mathbf{r})\, dV$, and the total linear momentum of that same fluid, given by $\int_V \rho(\mathbf{r})\mathbf{v}(\mathbf{r})\, dV$ where $\mathbf{v}(\mathbf{r})$ is the velocity field in the fluid. As a slightly more complicated example of a volume integral we may consider the following.

►*Find an expression for the angular momentum of a solid body rotating with angular velocity* $\boldsymbol{\omega}$ *about an axis through the origin.*

Consider a small volume element dV situated at position $\mathbf{r}$; its linear momentum is $\rho\, dV \dot{\mathbf{r}}$, where $\rho = \rho(\mathbf{r})$ is the density distribution, and its angular momentum about O is $\mathbf{r} \times \rho\dot{\mathbf{r}}\, dV$. Thus for the whole body the angular momentum $\mathbf{L}$ is

$$\mathbf{L} = \int_V (\mathbf{r} \times \dot{\mathbf{r}})\rho\, dV.$$

Putting $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$ yields

$$\mathbf{L} = \int_V [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]\, \rho\, dV = \int_V \boldsymbol{\omega} r^2 \rho\, dV - \int_V (\mathbf{r} \cdot \boldsymbol{\omega})\mathbf{r}\rho\, dV. \blacktriangleleft$$

The evaluation of the first type of volume integral in (11.12) has already been considered in our discussion of multiple integrals in chapter 6. The evaluation of the second type of volume integral follows directly since we can write

$$\int_V \mathbf{a}\, dV = \mathbf{i}\int_V a_x\, dV + \mathbf{j}\int_V a_y\, dV + \mathbf{k}\int_V a_z\, dV, \tag{11.13}$$

where a_x, a_y, a_z are the Cartesian components of $\mathbf{a}$. Of course, we could have written $\mathbf{a}$ in terms of the basis vectors of some other coordinate system (e.g. spherical polars) but, since such basis vectors are not, in general, constant, they

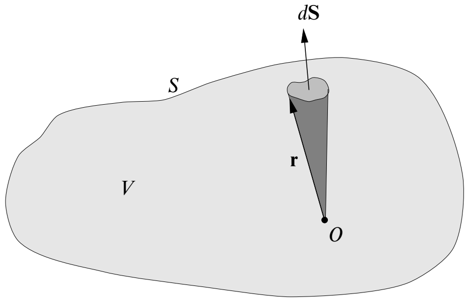


Figure 11.9 A general volume V containing the origin and bounded by the closed surface S.

cannot be taken out of the integral sign as in (11.13) and must be included as part of the integrand.

11.6.1 Volumes of three-dimensional regions

As discussed in chapter 6, the volume of a three-dimensional region V is simply $V = \int_V dV$, which may be evaluated directly once the limits of integration have been found. However, the volume of the region obviously depends only on the surface S that bounds it. We should therefore be able to express the volume V in terms of a surface integral over S. This is indeed possible, and the appropriate expression may derived as follows. Referring to figure 11.9, let us suppose that the origin O is contained within V. The volume of the small shaded cone is $dV = \frac{1}{3}\mathbf{r}\cdot d\mathbf{S}$; the total volume of the region is thus given by

$$V = \frac{1}{3}\oint_S \mathbf{r}\cdot d\mathbf{S}.$$

It may be shown that this expression is valid even when O is not contained in V. Although this surface integral form is available, in practice it is usually simpler to evaluate the volume integral directly.

►*Find the volume enclosed between a sphere of radius a centred on the origin and a circular cone of half-angle α with its vertex at the origin.*

The element of vector area $d\mathbf{S}$ on the surface of the sphere is given in spherical polar coordinates by $a^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$. Now taking the axis of the cone to lie along the z-axis (from which θ is measured) the required volume is given by

$$V = \frac{1}{3}\oint_S \mathbf{r}\cdot d\mathbf{S} = \frac{1}{3}\int_0^{2\pi} d\phi \int_0^{\alpha} a^2 \sin\theta\, \mathbf{r}\cdot\hat{\mathbf{r}}\, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi} d\phi \int_0^{\alpha} a^3 \sin\theta\, d\theta = \frac{2\pi a^3}{3}(1-\cos\alpha). \blacktriangleleft$$

11.7 Integral forms for grad, div and curl

In the previous chapter we defined the vector operators grad, div and curl in purely mathematical terms, which depended on the coordinate system in which they were expressed. An interesting application of line, surface and volume integrals is the expression of grad, div and curl in coordinate-free, geometrical terms. If ϕ is a scalar field and $\mathbf{a}$ is a vector field then it may be shown that at any point P

$$\nabla\phi = \lim_{V\to 0}\left(\frac{1}{V}\oint_S \phi\, d\mathbf{S}\right) \tag{11.14}$$

$$\nabla\cdot\mathbf{a} = \lim_{V\to 0}\left(\frac{1}{V}\oint_S \mathbf{a}\cdot d\mathbf{S}\right) \tag{11.15}$$

$$\nabla\times\mathbf{a} = \lim_{V\to 0}\left(\frac{1}{V}\oint_S d\mathbf{S}\times\mathbf{a}\right) \tag{11.16}$$

where V is a small volume enclosing P and S is its bounding surface. Indeed, we may consider these equations as the (geometrical) *definitions* of grad, div and curl. An alternative, but equivalent, geometrical definition of $\nabla\times\mathbf{a}$ at a point P, which is often easier to use than (11.16), is given by

$$(\nabla\times\mathbf{a})\cdot\hat{\mathbf{n}} = \lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{a}\cdot d\mathbf{r}\right), \tag{11.17}$$

where C is a plane contour of area A enclosing the point P and $\hat{\mathbf{n}}$ is the unit normal to the enclosed planar area.

It may be shown, *in any coordinate system*, that all the above equations are consistent with our definitions in the previous chapter, although the difficulty of proof depends on the chosen coordinate system. The most general coordinate system encountered in that chapter was one with orthogonal curvilinear coordinates u_1, u_2, u_3, of which Cartesians, cylindrical polars and spherical polars are all special cases. Although it may be shown that (11.14) leads to the usual expression for grad in curvilinear coordinates, the proof requires complicated manipulations of the derivatives of the basis vectors with respect to the coordinates and is not presented here. In Cartesian coordinates, however, the proof is quite simple.

►*Show that the geometrical definition of* grad *leads to the usual expression for* $\nabla\phi$ *in Cartesian coordinates.*

Consider the surface S of a small rectangular volume element $\Delta V = \Delta x\,\Delta y\,\Delta z$ that has its faces parallel to the x, y, and z coordinate surfaces; the point P (see above) is at one corner. We must calculate the surface integral (11.14) over each of its six faces. Remembering that the normal to the surface points outwards from the volume on each face, the two faces with $x =$ constant have areas $\Delta\mathbf{S} = -\mathbf{i}\,\Delta y\,\Delta z$ and $\Delta\mathbf{S} = \mathbf{i}\,\Delta y\,\Delta z$ respectively. Furthermore, over each small surface element, we may take ϕ to be constant, so that the net contribution to the surface integral from these two faces is then

$$[(\phi + \Delta\phi) - \phi]\,\Delta y\,\Delta z\,\mathbf{i} = \left(\phi + \frac{\partial \phi}{\partial x}\,\Delta x - \phi\right)\,\Delta y\,\Delta z\,\mathbf{i}$$
$$= \frac{\partial \phi}{\partial x}\,\Delta x\,\Delta y\,\Delta z\,\mathbf{i}.$$

The surface integral over the pairs of faces with $y =$ constant and $z =$ constant respectively may be found in a similar way, and we obtain

$$\oint_S \phi\, d\mathbf{S} = \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)\,\Delta x\,\Delta y\,\Delta z.$$

Therefore $\nabla\phi$ at the point P is given by

$$\nabla\phi = \lim_{\Delta x, \Delta y, \Delta z \to 0}\left[\frac{1}{\Delta x\,\Delta y\,\Delta z}\left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)\,\Delta x\,\Delta y\,\Delta z\right]$$
$$= \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}. \blacktriangleleft$$

We now turn to (11.15) and (11.17). These geometrical definitions may be shown straightforwardly to lead to the usual expressions for div and curl in orthogonal curvilinear coordinates.

►*By considering the infinitesimal volume element $dV = h_1h_2h_3\,\Delta u_1\,\Delta u_2\,\Delta u_3$ shown in figure 11.10, show that (11.15) leads to the usual expression for $\nabla\cdot\mathbf{a}$ in orthogonal curvilinear coordinates.*

Let us write the vector field in terms of its components with respect to the basis vectors of the curvilinear coordinate system as $\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$. We consider first the contribution to the RHS of (11.15) from the two faces with $u_1 =$ constant, i.e. $PQRS$ and the face opposite it (see figure 11.10). Now, the volume element is formed from the orthogonal vectors $h_1\,\Delta u_1\,\hat{\mathbf{e}}_1$, $h_2\,\Delta u_2\,\hat{\mathbf{e}}_2$ and $h_3\,\Delta u_3\,\hat{\mathbf{e}}_3$ at the point P and so for $PQRS$ we have

$$\Delta\mathbf{S} = h_2h_3\,\Delta u_2\,\Delta u_3\;\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2 = -h_2h_3\,\Delta u_2\,\Delta u_3\,\hat{\mathbf{e}}_1.$$

Reasoning along the same lines as in the previous example, we conclude that the contribution to the surface integral of $\mathbf{a}\cdot d\mathbf{S}$ over $PQRS$ and its opposite face taken together is given by

$$\frac{\partial}{\partial u_1}(\mathbf{a}\cdot\Delta\mathbf{S})\,\Delta u_1 = \frac{\partial}{\partial u_1}(a_1h_2h_3)\,\Delta u_1\,\Delta u_2\,\Delta u_3.$$

The surface integrals over the pairs of faces with $u_2 =$ constant and $u_3 =$ constant respectively may be found in a similar way, and we obtain

$$\oint_S \mathbf{a}\cdot d\mathbf{S} = \left[\frac{\partial}{\partial u_1}(a_1h_2h_3) + \frac{\partial}{\partial u_2}(a_2h_3h_1) + \frac{\partial}{\partial u_3}(a_3h_1h_2)\right]\,\Delta u_1\,\Delta u_2\,\Delta u_3.$$

Therefore $\nabla\cdot\mathbf{a}$ at the point P is given by

$$\nabla\cdot\mathbf{a} = \lim_{\Delta u_1, \Delta u_2, \Delta u_3 \to 0}\left[\frac{1}{h_1h_2h_3\,\Delta u_1\,\Delta u_2\,\Delta u_3}\oint_S \mathbf{a}\cdot d\mathbf{S}\right]$$
$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(a_1h_2h_3) + \frac{\partial}{\partial u_2}(a_2h_3h_1) + \frac{\partial}{\partial u_3}(a_3h_1h_2)\right]. \blacktriangleleft$$

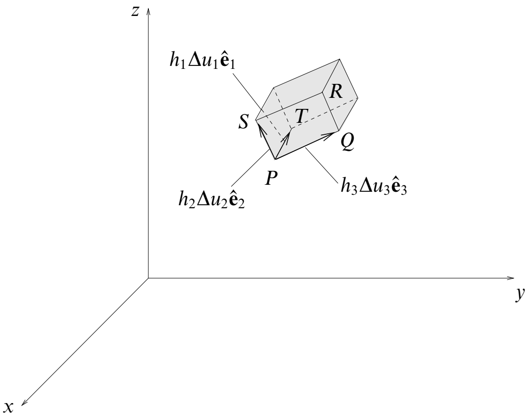


Figure 11.10 A general volume ΔV in orthogonal curvilinear coordinates u_1, u_2, u_3. PT gives the vector $h_1 \,\Delta u_1 \,\hat{\mathbf{e}}_1$, PS gives $h_2 \,\Delta u_2 \,\hat{\mathbf{e}}_2$ and PQ gives $h_3 \,\Delta u_3 \,\hat{\mathbf{e}}_3$.

►*By considering the infinitesimal planar surface element $PQRS$ in figure 11.10, show that (11.17) leads to the usual expression for $\nabla \times \mathbf{a}$ in orthogonal curvilinear coordinates.*

The planar surface $PQRS$ is defined by the orthogonal vectors $h_2 \,\Delta u_2 \,\hat{\mathbf{e}}_2$ and $h_3 \,\Delta u_3 \,\hat{\mathbf{e}}_3$ at the point P. If we traverse the loop in the direction $PSRQ$ then, by the right-hand convention, the unit normal to the plane is $\hat{\mathbf{e}}_1$. Writing $\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$, the line integral around the loop in this direction is given by

$$\begin{aligned}
\oint_{PSRQ} \mathbf{a} \cdot d\mathbf{r} &= a_2 h_2 \,\Delta u_2 + \left[a_3 h_3 + \frac{\partial}{\partial u_2}(a_3 h_3) \,\Delta u_2 \right] \Delta u_3 \\
&\quad - \left[a_2 h_2 + \frac{\partial}{\partial u_3}(a_2 h_2) \,\Delta u_3 \right] \Delta u_2 - a_3 h_3 \,\Delta u_3 \\
&= \left[\frac{\partial}{\partial u_2}(a_3 h_3) - \frac{\partial}{\partial u_3}(a_2 h_2) \right] \Delta u_2 \,\Delta u_3.
\end{aligned}$$

Therefore from (11.17) the component of $\nabla \times \mathbf{a}$ in the direction $\hat{\mathbf{e}}_1$ at P is given by

$$\begin{aligned}
(\nabla \times \mathbf{a})_1 &= \lim_{\Delta u_2, \Delta u_3 \to 0} \left[\frac{1}{h_2 h_3 \,\Delta u_2 \,\Delta u_3} \oint_{PSRQ} \mathbf{a} \cdot d\mathbf{r} \right] \\
&= \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial u_2}(h_3 a_3) - \frac{\partial}{\partial u_3}(h_2 a_2) \right].
\end{aligned}$$

The other two components are found by cyclically permuting the subscripts 1, 2, 3. ◄

Finally, we note that we can also write the ∇^2 operator as a surface integral by setting $\mathbf{a} = \nabla\phi$ in (11.15), to obtain

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \lim_{V \to 0} \left(\frac{1}{V} \oint_S \nabla \phi \cdot d\mathbf{S} \right).$$

11.8 Divergence theorem and related theorems

The divergence theorem relates the total flux of a vector field out of a closed surface S to the integral of the divergence of the vector field over the enclosed volume V; it follows almost immediately from our geometrical definition of divergence (11.15).

Imagine a volume V, in which a vector field $\mathbf{a}$ is continuous and differentiable, to be divided up into a large number of small volumes V_i. Using (11.15), we have for each small volume

$$(\nabla \cdot \mathbf{a})V_i \approx \oint_{S_i} \mathbf{a} \cdot d\mathbf{S},$$

where S_i is the surface of the small volume V_i. Summing over i we find that contributions from surface elements interior to S cancel since each surface element appears in two terms with opposite signs, the outward normals in the two terms being equal and opposite. Only contributions from surface elements that are also parts of S survive. If each V_i is allowed to tend to zero then we obtain the *divergence theorem*,

$$\int_V \nabla \cdot \mathbf{a}\, dV = \oint_S \mathbf{a} \cdot d\mathbf{S}. \tag{11.18}$$

We note that the divergence theorem holds for both simply and multiply connected surfaces, provided that they are closed and enclose some non-zero volume V. The divergence theorem may also be extended to tensor fields (see chapter 26).

The theorem finds most use as a tool in formal manipulations, but sometimes it is of value in transforming surface integrals of the form $\int_S \mathbf{a} \cdot d\mathbf{S}$ into volume integrals or vice versa. For example, setting $\mathbf{a} = \mathbf{r}$ we immediately obtain

$$\int_V \nabla \cdot \mathbf{r}\, dV = \int_V 3\, dV = 3V = \oint_S \mathbf{r} \cdot d\mathbf{S},$$

which gives the expression for the volume of a region found in subsection 11.6.1. The use of the divergence theorem is further illustrated in the following example.

►*Evaluate the surface integral $I = \int_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a} = (y - x)\,\mathbf{i} + x^2 z\,\mathbf{j} + (z + x^2)\,\mathbf{k}$ and S is the open surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.*

We could evaluate this surface integral directly, but the algebra is somewhat lengthy. We will therefore evaluate it by use of the divergence theorem. Since the latter only holds for closed surfaces enclosing a non-zero volume V, let us first consider the closed surface $S' = S + S_1$, where S_1 is the circular area in the xy-plane given by $x^2 + y^2 \leq a^2$, $z = 0$; S' then encloses a hemispherical volume V. By the divergence theorem we have

$$\int_V \nabla \cdot \mathbf{a}\, dV = \oint_{S'} \mathbf{a} \cdot d\mathbf{S} = \int_S \mathbf{a} \cdot d\mathbf{S} + \int_{S_1} \mathbf{a} \cdot d\mathbf{S}.$$

Now $\nabla \cdot \mathbf{a} = -1 + 0 + 1 = 0$, so we can write

$$\int_S \mathbf{a} \cdot d\mathbf{S} = -\int_{S_1} \mathbf{a} \cdot d\mathbf{S}.$$

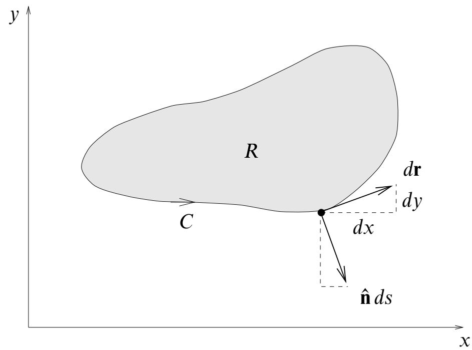


Figure 11.11 A closed curve C in the xy-plane bounding a region R. Vectors tangent and normal to the curve at a given point are also shown.

The surface integral over S_1 is easily evaluated. Remembering that the normal to the surface points outward from the volume, a surface element on S_1 is simply $d\mathbf{S} = -\mathbf{k}\,dx\,dy$. On S_1 we also have $\mathbf{a} = (y - x)\,\mathbf{i} + x^2\,\mathbf{k}$, so that

$$I = -\int_{S_1} \mathbf{a}\cdot d\mathbf{S} = \iint_R x^2\,dx\,dy,$$

where R is the circular region in the xy-plane given by $x^2 + y^2 \leq a^2$. Transforming to plane polar coordinates we have

$$I = \iint_{R'} \rho^2 \cos^2\phi\ \rho\,d\rho\,d\phi = \int_0^{2\pi} \cos^2\phi\,d\phi \int_0^a \rho^3\,d\rho = \frac{\pi a^4}{4}. \blacktriangleleft$$

It is also interesting to consider the two-dimensional version of the divergence theorem. As an example, let us consider a two-dimensional planar region R in the xy-plane bounded by some closed curve C (see figure 11.11). At any point on the curve the vector $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ is a tangent to the curve and the vector $\hat{\mathbf{n}}\,ds = dy\,\mathbf{i} - dx\,\mathbf{j}$ is a normal pointing out of the region R. If the vector field $\mathbf{a}$ is continuous and differentiable in R then the two-dimensional divergence theorem in Cartesian coordinates gives

$$\iint_R \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y}\right) dx\,dy = \oint \mathbf{a}\cdot\hat{\mathbf{n}}\,ds = \oint_C (a_x\,dy - a_y\,dx).$$

Letting $P = -a_y$ and $Q = a_x$, we recover Green's theorem in a plane, which was discussed in section 11.3.

11.8.1 Green's theorems

Consider two scalar functions ϕ and ψ that are continuous and differentiable in some volume V bounded by a surface S. Applying the divergence theorem to the vector field $\phi\nabla\psi$ we obtain

$$\begin{aligned}\oint_S \phi\nabla\psi \cdot d\mathbf{S} &= \int_V \nabla\cdot(\phi\nabla\psi)\,dV \\ &= \int_V \left[\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)\right]\,dV. \end{aligned} \tag{11.19}$$

Reversing the roles of ϕ and ψ in (11.19) and subtracting the two equations gives

$$\oint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S} = \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV. \tag{11.20}$$

Equation (11.19) is usually known as Green's first theorem and (11.20) as his second. Green's second theorem is useful in the development of the Green's functions used in the solution of partial differential equations (see chapter 21).

11.8.2 Other related integral theorems

There exist two other integral theorems which are closely related to the divergence theorem and which are of some use in physical applications. If ϕ is a scalar field and $\mathbf{b}$ is a vector field and both ϕ and $\mathbf{b}$ satisfy our usual differentiability conditions in some volume V bounded by a closed surface S then

$$\int_V \nabla\phi\,dV = \oint_S \phi\,d\mathbf{S}, \tag{11.21}$$

$$\int_V \nabla\times\mathbf{b}\,dV = \oint_S d\mathbf{S}\times\mathbf{b}. \tag{11.22}$$

►*Use the divergence theorem to prove (11.21).*

In the divergence theorem (11.18) let $\mathbf{a} = \phi\mathbf{c}$, where $\mathbf{c}$ is a constant vector. We then have

$$\int_V \nabla\cdot(\phi\mathbf{c})\,dV = \oint_S \phi\mathbf{c}\cdot d\mathbf{S}.$$

Expanding out the integrand on the LHS we have

$$\nabla\cdot(\phi\mathbf{c}) = \phi\nabla\cdot\mathbf{c} + \mathbf{c}\cdot\nabla\phi = \mathbf{c}\cdot\nabla\phi,$$

since $\mathbf{c}$ is constant. Also, $\phi\mathbf{c}\cdot d\mathbf{S} = \mathbf{c}\cdot\phi d\mathbf{S}$, so we obtain

$$\int_V \mathbf{c}\cdot(\nabla\phi)\,dV = \oint_S \mathbf{c}\cdot\phi\,d\mathbf{S}.$$

Since $\mathbf{c}$ is constant we may take it out of both integrals to give

$$\mathbf{c}\cdot\int_V \nabla\phi\,dV = \mathbf{c}\cdot\oint_S \phi\,d\mathbf{S},$$

and since $\mathbf{c}$ is arbitrary we obtain the stated result (11.21). ◄

Equation (11.22) may be proved in a similar way by letting $\mathbf{a} = \mathbf{b}\times\mathbf{c}$ in the divergence theorem, where $\mathbf{c}$ is again a constant vector.

11.8.3 Physical applications of the divergence theorem

The divergence theorem is useful in deriving many of the most important partial differential equations in physics (see chapter 20). The basic idea is to use the divergence theorem to convert an integral form, often derived from observation, into an equivalent differential form (used in theoretical statements).

►*For a compressible fluid with time-varying position-dependent density $\rho(\mathbf{r}, t)$ and velocity field $v(\mathbf{r}, t)$, in which fluid is neither being created nor destroyed, show that*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

For an arbitrary volume V in the fluid, the conservation of mass tells us that the rate of increase or decrease of the mass M of fluid in the volume must equal the net rate at which fluid is entering or leaving the volume, i.e.

$$\frac{dM}{dt} = -\oint_S \rho \mathbf{v} \cdot d\mathbf{S},$$

where S is the surface bounding V. But the mass of fluid in V is simply $M = \int_V \rho\, dV$, so we have

$$\frac{d}{dt}\int_V \rho\, dV + \oint_S \rho \mathbf{v} \cdot d\mathbf{S} = 0.$$

Taking the derivative inside the first integral on the RHS and using the divergence theorem to rewrite the second integral, we obtain

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_V \nabla \cdot (\rho \mathbf{v})\, dV = \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right] dV = 0.$$

Since the volume V is arbitrary, the integrand (which is assumed continuous) must be identically zero, so we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

This is known as the *continuity equation*. It can also be applied to other systems, for example those in which ρ is the density of electric charge or the heat content, etc. For the flow of an incompressible fluid, $\rho =$ constant and the continuity equation becomes simply $\nabla \cdot \mathbf{v} = 0$. ◄

In the previous example, we assumed that there were no sources or sinks in the volume V, i.e. that there was no part of V in which fluid was being created or destroyed. We now consider the case where a finite number of *point* sources and/or sinks are present in an incompressible fluid. Let us first consider the simple case where a single source is located at the origin, out of which a quantity of fluid flows radially at a rate Q ($\text{m}^3\ \text{s}^{-1}$). The velocity field is given by

$$\mathbf{v} = \frac{Q\mathbf{r}}{4\pi r^3} = \frac{Q\hat{\mathbf{r}}}{4\pi r^2}.$$

Now, for a sphere S_1 of radius r centred on the source, the flux across S_1 is

$$\oint_{S_1} \mathbf{v} \cdot d\mathbf{S} = |\mathbf{v}| 4\pi r^2 = Q.$$

Since $\mathbf{v}$ has a singularity at the origin it is not differentiable there, i.e. $\nabla\cdot\mathbf{v}$ is not defined there, but at all other points $\nabla\cdot\mathbf{v} = 0$, as required for an incompressible fluid. Therefore, from the divergence theorem, for any closed surface S_2 that does not enclose the origin we have

$$\oint_{S_2} \mathbf{v}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{v}\, dV = 0.$$

Thus we see that the surface integral $\oint_S \mathbf{v}\cdot d\mathbf{S}$ has value Q or zero depending on whether or not S encloses the source. In order that the divergence theorem is valid for *all* surfaces S, irrespective of whether they enclose the source, we write

$$\nabla\cdot\mathbf{v} = Q\delta(\mathbf{r}),$$

where $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function. The properties of this function are discussed fully in chapter 13, but for the moment we note that it is defined in such a way that

$$\delta(\mathbf{r}-\mathbf{a}) = 0 \qquad \text{for } \mathbf{r}\neq\mathbf{a},$$

$$\int_V f(\mathbf{r})\delta(\mathbf{r}-\mathbf{a})\, dV = \begin{cases} f(\mathbf{a}) & \text{if } \mathbf{a} \text{ lies in } V \\ 0 & \text{otherwise} \end{cases}$$

for any well-behaved function $f(\mathbf{r})$. Therefore, for any volume V containing the source at the origin, we have

$$\int_V \nabla\cdot\mathbf{v}\, dV = Q\int_V \delta(\mathbf{r})\, dV = Q,$$

which is consistent with $\oint_S \mathbf{v}\cdot d\mathbf{S} = Q$ for a closed surface enclosing the source. Hence, by introducing the Dirac delta function the divergence theorem can be made valid even for non-differentiable point sources.

The generalisation to several sources and sinks is straightforward. For example, if a source is located at $\mathbf{r} = \mathbf{a}$ and a sink at $\mathbf{r} = \mathbf{b}$ then the velocity field is

$$\mathbf{v} = \frac{(\mathbf{r}-\mathbf{a})Q}{4\pi|\mathbf{r}-\mathbf{a}|^3} - \frac{(\mathbf{r}-\mathbf{b})Q}{4\pi|\mathbf{r}-\mathbf{b}|^3}$$

and its divergence is given by

$$\nabla\cdot\mathbf{v} = Q\delta(\mathbf{r}-\mathbf{a}) - Q\delta(\mathbf{r}-\mathbf{b}).$$

Therefore, the integral $\oint_S \mathbf{v}\cdot d\mathbf{S}$ has the value Q if S encloses the source, $-Q$ if S encloses the sink and 0 if S encloses neither the source nor sink or encloses them both. This analysis also applies to other physical systems – for example, in electrostatics we can regard the sources and sinks as positive and negative point charges respectively and replace $\mathbf{v}$ by the electric field $\mathbf{E}$.

11.9 Stokes' theorem and related theorems

Stokes' theorem is the 'curl analogue' of the divergence theorem and relates the integral of the curl of a vector field over an open surface S to the line integral of the vector field around the perimeter C bounding the surface.

Following the same lines as for the derivation of the divergence theorem, we can divide the surface S into many small areas S_i with boundaries C_i and unit normals $\hat{\mathbf{n}}_i$. Using (11.17), we have for each small area

$$(\nabla \times \mathbf{a}) \cdot \hat{\mathbf{n}}_i \; S_i \approx \oint_{C_i} \mathbf{a} \cdot d\mathbf{r}.$$

Summing over i we find that on the RHS all parts of all interior boundaries that are not part of C are included twice, being traversed in opposite directions on each occasion and thus contributing nothing. Only contributions from line elements that are also parts of C survive. If each S_i is allowed to tend to zero then we obtain Stokes' theorem,

$$\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \oint_C \mathbf{a} \cdot d\mathbf{r}. \tag{11.23}$$

We note that Stokes' theorem holds for both simply and multiply connected open surfaces, provided that they are two-sided. Stokes' theorem may also be extended to tensor fields (see chapter 26).

Just as the divergence theorem (11.18) can be used to relate volume and surface integrals for certain types of integrand, Stokes' theorem can be used in evaluating surface integrals of the form $\oint_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S}$ as line integrals or vice versa.

►*Given the vector field* $\mathbf{a} = y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k}$, *verify Stokes' theorem for the hemispherical surface* $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Let us first evaluate the surface integral

$$\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S}$$

over the hemisphere. It is easily shown that $\nabla \times \mathbf{a} = -2\,\mathbf{k}$, and the surface element is $d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \; \hat{\mathbf{r}}$ in spherical polar coordinates. Therefore

$$\begin{aligned}
\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \left(-2a^2 \sin\theta\right) \hat{\mathbf{r}} \cdot \mathbf{k} \\
&= -2a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \left(\frac{z}{a}\right) d\theta \\
&= -2a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = -2\pi a^2.
\end{aligned}$$

We now evaluate the line integral around the perimeter curve C of the surface, which is the circle $x^2 + y^2 = a^2$ in the xy-plane. This is given by

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \oint_C (y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= \oint_C (y\,dx - x\,dy).$$

Using plane polar coordinates, on C we have $x = a\cos\phi$, $y = a\sin\phi$ so that $dx = -a\sin\phi\,d\phi$, $dy = a\cos\phi\,d\phi$, and the line integral becomes

$$\oint_C (y\,dx - x\,dy) = -a^2 \int_0^{2\pi} (\sin^2\phi + \cos^2\phi)\,d\phi = -a^2 \int_0^{2\pi} d\phi = -2\pi a^2.$$

Since the surface and line integrals have the same value, we have verified Stokes' theorem in this case. ◄

The two-dimensional version of Stokes' theorem also yields Green's theorem in a plane. Consider the region R in the xy-plane shown in figure 11.11, in which a vector field $\mathbf{a}$ is defined. Since $\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j}$, we have $\nabla \times \mathbf{a} = (\partial a_y/\partial x - \partial a_x/\partial y)\,\mathbf{k}$, and Stokes' theorem becomes

$$\iint_R \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) dx\,dy = \oint_C (a_x\,dx + a_y\,dy).$$

Letting $P = a_x$ and $Q = a_y$ we recover Green's theorem in a plane, (11.4).

11.9.1 Related integral theorems

As for the divergence theorem, there exist two other integral theorems that are closely related to Stokes' theorem. If ϕ is a scalar field and $\mathbf{b}$ is a vector field, and both ϕ and $\mathbf{b}$ satisfy our usual differentiability conditions on some two-sided open surface S bounded by a closed perimeter curve C, then

$$\int_S d\mathbf{S} \times \nabla\phi = \oint_C \phi\,d\mathbf{r}, \tag{11.24}$$

$$\int_S (d\mathbf{S} \times \nabla) \times \mathbf{b} = \oint_C d\mathbf{r} \times \mathbf{b}. \tag{11.25}$$

►*Use Stokes' theorem to prove (11.24).*

In Stokes' theorem, (11.23), let $\mathbf{a} = \phi\mathbf{c}$, where $\mathbf{c}$ is a constant vector. We then have

$$\int_S [\nabla \times (\phi\mathbf{c})] \cdot d\mathbf{S} = \oint_C \phi\mathbf{c} \cdot d\mathbf{r}. \tag{11.26}$$

Expanding out the integrand on the LHS we have

$$\nabla \times (\phi\mathbf{c}) = \nabla\phi \times \mathbf{c} + \phi\nabla \times \mathbf{c} = \nabla\phi \times \mathbf{c},$$

since $\mathbf{c}$ is constant, and the scalar triple product on the LHS of (11.26) can therefore be written

$$[\nabla \times (\phi\mathbf{c})] \cdot d\mathbf{S} = (\nabla\phi \times \mathbf{c}) \cdot d\mathbf{S} = \mathbf{c} \cdot (d\mathbf{S} \times \nabla\phi).$$

Substituting this into (11.26) and taking $\mathbf{c}$ out of both integrals because it is constant, we find

$$\mathbf{c}\cdot\int_S d\mathbf{S}\times\nabla\phi = \mathbf{c}\cdot\oint_C \phi\, d\mathbf{r}.$$

Since $\mathbf{c}$ is an arbitrary constant vector we therefore obtain the stated result (11.24). ◄

Equation (11.25) may be proved in a similar way, by letting $\mathbf{a} = \mathbf{b}\times\mathbf{c}$ in Stokes' theorem, where $\mathbf{c}$ is again a constant vector. We also note that by setting $\mathbf{b} = \mathbf{r}$ in (11.25) we find

$$\int_S (d\mathbf{S}\times\nabla)\times\mathbf{r} = \oint_C d\mathbf{r}\times\mathbf{r}.$$

Expanding out the integrand on the LHS gives

$$(d\mathbf{S}\times\nabla)\times\mathbf{r} = d\mathbf{S} - d\mathbf{S}(\nabla\cdot\mathbf{r}) = d\mathbf{S} - 3\,d\mathbf{S} = -2\,d\mathbf{S}.$$

Therefore, as we found in subsection 11.5.2, the vector area of an open surface S is given by

$$\mathbf{S} = \int_S d\mathbf{S} = \frac{1}{2}\oint_C \mathbf{r}\times d\mathbf{r}.$$

11.9.2 Physical applications of Stokes' theorem

Like the divergence theorem, Stokes' theorem is useful in converting integral equations into differential equations.

►*From Ampère's law, derive Maxwell's equation in the case where the currents are steady, i.e.* $\nabla\times\mathbf{B} - \mu_0\mathbf{J} = \mathbf{0}$.

Ampère's rule for a distributed current with current density $\mathbf{J}$ is

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0\int_S \mathbf{J}\cdot d\mathbf{S},$$

for any circuit C bounding a surface S. Using Stokes' theorem, the LHS can be transformed into $\int_S(\nabla\times\mathbf{B})\cdot d\mathbf{S}$; hence

$$\int_S (\nabla\times\mathbf{B} - \mu_0\mathbf{J})\cdot d\mathbf{S} = 0$$

for *any* surface S. This can only be so if $\nabla\times\mathbf{B} - \mu_0\mathbf{J} = \mathbf{0}$, which is the required relation. Similarly, from Faraday's law of electromagnetic induction we can derive Maxwell's equation $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$. ◄

In subsection 11.8.3 we discussed the flow of an incompressible fluid in the presence of several sources and sinks. Let us now consider *vortex* flow in an incompressible fluid with a velocity field

$$\mathbf{v} = \frac{1}{\rho}\hat{\mathbf{e}}_\phi,$$

in cylindrical polar coordinates ρ, ϕ, z. For this velocity field $\nabla\times\mathbf{v}$ equals zero everywhere except on the axis $\rho = 0$, where $\mathbf{v}$ has a singularity. Therefore $\oint_C \mathbf{v} \cdot d\mathbf{r}$ equals zero for any path C that does not enclose the vortex line on the axis and 2π if C does enclose the axis. In order for Stokes' theorem to be valid for all paths C, we therefore set

$$\nabla \times \mathbf{v} = 2\pi\delta(\rho),$$

where $\delta(\rho)$ is the Dirac delta function, to be discussed in subsection 13.1.3. Now, since $\nabla \times \mathbf{v} = \mathbf{0}$, except on the axis $\rho = 0$, there exists a scalar potential ψ such that $\mathbf{v} = \nabla\psi$. It may easily be shown that $\psi = \phi$, the polar angle. Therefore, if C does not enclose the axis then

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \oint d\phi = 0,$$

and if C does enclose the axis,

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \Delta\phi = 2\pi n,$$

where n is the number of times we traverse C. Thus ϕ is a multivalued potential.

Similar analyses are valid for other physical systems – for example, in magnetostatics we may replace the vortex lines by current-carrying wires and the velocity field $\mathbf{v}$ by the magnetic field $\mathbf{B}$.

11.10 Exercises

11.1 The vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = 2xz\mathbf{i} + 2yz^2\mathbf{j} + (x^2 + 2y^2z - 1)\mathbf{k}.$$

Calculate $\nabla \times \mathbf{F}$ and deduce that $\mathbf{F}$ can be written $F = \nabla\phi$. Determine the form of ϕ.

11.2 The vector field $\mathbf{Q}$ is defined by

$$\mathbf{Q} = \left[3x^2(y+z) + y^3 + z^3\right]\mathbf{i} + \left[3y^2(z+x) + z^3 + x^3\right]\mathbf{j} + \left[3z^2(x+y) + x^3 + y^3\right]\mathbf{k}.$$

Show that $\mathbf{Q}$ is a conservative field, construct its potential function and hence evaluate the integral $J = \int \mathbf{Q} \cdot d\mathbf{r}$ along any line connecting the point A at $(1, -1, 1)$ to B at $(2, 1, 2)$.

11.3 $\mathbf{F}$ is a vector field $xy^2\mathbf{i} + 2\mathbf{j} + x\mathbf{k}$, and L is a path parameterised by $x = ct$, $y = c/t$, $z = d$ for the range $1 \leq t \leq 2$. Evaluate (a) $\int_L \mathbf{F}\,dt$, (b) $\int_L \mathbf{F}\,dy$ and (c) $\int_L \mathbf{F} \cdot d\mathbf{r}$.

11.4 By making an appropriate choice for the functions $P(x, y)$ and $Q(x, y)$ that appear in Green's theorem in a plane, show that the integral of $x - y$ over the upper half of the unit circle centred on the origin has the value $-\frac{2}{3}$. Show the same result by direct integration in Cartesian coordinates.

11.5 Determine the point of intersection P, in the first quadrant, of the two ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Taking $b < a$, consider the contour L that bounds the area in the first quadrant that is common to the two ellipses. Show that the parts of L that lie along the coordinate axes contribute nothing to the line integral around L of $x\,dy - y\,dx$. Using a parameterisation of each ellipse similar to that employed in the example in section 11.3, evaluate the two remaining line integrals and hence find the total area common to the two ellipses.

11.6 By using parameterisations of the form $x = a\cos^n\theta$ and $y = a\sin^n\theta$ for suitable values of n, find the area bounded by the curves

$$x^{2/5} + y^{2/5} = a^{2/5} \quad \text{and} \quad x^{2/3} + y^{2/3} = a^{2/3}.$$

11.7 Evaluate the line integral

$$I = \oint_C \left[y(4x^2 + y^2)\, dx + x(2x^2 + 3y^2)\, dy \right]$$

around the ellipse $x^2/a^2 + y^2/b^2 = 1$.

11.8 Criticise the following 'proof' that $\pi = 0$.

(a) Apply Green's theorem in a plane to the functions $P(x, y) = \tan^{-1}(y/x)$ and $Q(x, y) = \tan^{-1}(x/y)$, taking the region R to be the unit circle centred on the origin.

(b) The RHS of the equality so produced is

$$\int\int_R \frac{y - x}{x^2 + y^2}\, dx\, dy,$$

which, either from symmetry considerations or by changing to plane polar coordinates, can be shown to have zero value.

(c) In the LHS of the equality, set $x = \cos\theta$ and $y = \sin\theta$, yielding $P(\theta) = \theta$ and $Q(\theta) = \pi/2 - \theta$. The line integral becomes

$$\int_0^{2\pi} \left[\left(\frac{\pi}{2} - \theta \right) \cos\theta - \theta \sin\theta \right] d\theta,$$

which has the value 2π.

(d) Thus $2\pi = 0$ and the stated result follows.

11.9 A single-turn coil C of arbitrary shape is placed in a magnetic field $\mathbf{B}$ and carries a current I. Show that the couple acting upon the coil can be written as

$$\mathbf{M} = I \int_C (\mathbf{B} \cdot \mathbf{r})\, d\mathbf{r} - I \int_C \mathbf{B}(\mathbf{r} \cdot d\mathbf{r}).$$

For a planar rectangular coil of sides $2a$ and $2b$ placed with its plane vertical and at an angle ϕ to a uniform horizontal field $\mathbf{B}$, show that $\mathbf{M}$ is, as expected, $4abBI \cos\phi\, \mathbf{k}$.

11.10 Find the vector area $\mathbf{S}$ of the part of the curved surface of the hyperboloid of revolution

$$\frac{x^2}{a^2} - \frac{y^2 + z^2}{b^2} = 1$$

that lies in the region $z \geq 0$ and $a \leq x \leq \lambda a$.

11.11 An axially symmetric solid body with its axis AB vertical is immersed in an incompressible fluid of density ρ_0. Use the following method to show that, whatever the shape of the body, for $\rho = \rho(z)$ in cylindrical polars the Archimedean upthrust is, as expected, $\rho_0 g V$, where V is the volume of the body.

Express the vertical component of the resultant force on the body, $-\int p\, d\mathbf{S}$, where p is the pressure, in terms of an integral; note that $p = -\rho_0 g z$ and that for an annular surface element of width dl, $\mathbf{n} \cdot \mathbf{n}_z\, dl = -d\rho$. Integrate by parts and use the fact that $\rho(z_A) = \rho(z_B) = 0$.

11.12 Show that the expression below is equal to the solid angle subtended by a rectangular aperture, of sides $2a$ and $2b$, at a point on the normal through its centre, and at a distance c from the aperture:

$$\Omega = 4\int_0^b \frac{ac}{(y^2+c^2)(y^2+c^2+a^2)^{1/2}}\,dy.$$

By setting $y = (a^2+c^2)^{1/2}\tan\phi$, change this integral into the form

$$\int_0^{\phi_1} \frac{4ac\cos\phi}{c^2+a^2\sin^2\phi}\,d\phi,$$

where $\tan\phi_1 = b/(a^2+c^2)^{1/2}$, and hence show that

$$\Omega = 4\tan^{-1}\left[\frac{ab}{c(a^2+b^2+c^2)^{1/2}}\right].$$

11.13 A vector field $\mathbf{a}$ is given by $-zxr^{-3}\mathbf{i}-zyr^{-3}\mathbf{j}+(x^2+y^2)r^{-3}\mathbf{k}$, where $r^2 = x^2+y^2+z^2$. Establish that the field is conservative (a) by showing that $\nabla\times\mathbf{a} = \mathbf{0}$, and (b) by constructing its potential function ϕ.

11.14 A vector field $\mathbf{a}$ is given by $(z^2+2xy)\,\mathbf{i}+(x^2+2yz)\,\mathbf{j}+(y^2+2zx)\,\mathbf{k}$. Show that $\mathbf{a}$ is conservative and that the line integral $\int \mathbf{a}\cdot d\mathbf{r}$ along any line joining $(1,1,1)$ and $(1,2,2)$ has the value 11.

11.15 A force $\mathbf{F}(\mathbf{r})$ acts on a particle at $\mathbf{r}$. In which of the following cases can $\mathbf{F}$ be represented in terms of a potential? Where it can, find the potential.

(a) $\mathbf{F} = F_0\left[\mathbf{i}-\mathbf{j}-\dfrac{2(x-y)}{a^2}\mathbf{r}\right]\exp\left(-\dfrac{r^2}{a^2}\right);$

(b) $\mathbf{F} = \dfrac{F_0}{a}\left[z\mathbf{k}+\dfrac{(x^2+y^2-a^2)}{a^2}\mathbf{r}\right]\exp\left(-\dfrac{r^2}{a^2}\right);$

(c) $\mathbf{F} = F_0\left[\mathbf{k}+\dfrac{a(\mathbf{r}\times\mathbf{k})}{r^2}\right].$

11.16 One of Maxwell's electromagnetic equations states that all magnetic fields $\mathbf{B}$ are solenoidal (i.e. $\nabla\cdot\mathbf{B} = 0$). Determine whether each of the following vectors could represent a real magnetic field; where it could, try to find a suitable vector potential $\mathbf{A}$, i.e. such that $\mathbf{B} = \nabla\times\mathbf{A}$. (Hint: seek a vector potential that is parallel to $\nabla\times\mathbf{B}$.):

(a) $\dfrac{B_0 b}{r^3}[(x-y)z\,\mathbf{i}+(x-y)z\,\mathbf{j}+(x^2-y^2)\,\mathbf{k}]$ in Cartesians with $r^2 = x^2+y^2+z^2$;

(b) $\dfrac{B_0 b^3}{r^3}[\cos\theta\ \cos\phi\,\hat{\mathbf{e}}_r-\sin\theta\ \cos\phi\,\hat{\mathbf{e}}_\theta+\sin 2\theta\ \sin\phi\,\hat{\mathbf{e}}_\phi]$ in spherical polars;

(c) $B_0 b^2\left[\dfrac{z\rho}{(b^2+z^2)^2}\,\hat{\mathbf{e}}_\rho+\dfrac{1}{b^2+z^2}\,\hat{\mathbf{e}}_z\right]$ in cylindrical polars.

11.17 The vector field $\mathbf{f}$ has components $y\mathbf{i}-x\mathbf{j}+\mathbf{k}$ and γ is a curve given parametrically by

$$\mathbf{r} = (a-c+c\cos\theta)\mathbf{i}+(b+c\sin\theta)\mathbf{j}+c^2\theta\mathbf{k}, \quad 0\le\theta\le 2\pi.$$

Describe the shape of the path γ and show that the line integral $\int_\gamma \mathbf{f}\cdot d\mathbf{r}$ vanishes. Does this result imply that $\mathbf{f}$ is a conservative field?

11.18 A vector field $\mathbf{a} = f(r)\mathbf{r}$ is spherically symmetric and everywhere directed away from the origin. Show that $\mathbf{a}$ is irrotational, but that it is also solenoidal only if $f(r)$ is of the form Ar^{-3}.

11.19 Evaluate the surface integral $\int \mathbf{r} \cdot d\mathbf{S}$, where $\mathbf{r}$ is the position vector, over that part of the surface $z = a^2 - x^2 - y^2$ for which $z \geq 0$, by each of the following methods.

(a) Parameterise the surface as $x = a \sin\theta \cos\phi$, $y = a \sin\theta \sin\phi$, $z = a^2 \cos^2\theta$, and show that

$$\mathbf{r} \cdot d\mathbf{S} = a^4(2\sin^3\theta \cos\theta + \cos^3\theta \sin\theta)\, d\theta\, d\phi.$$

(b) Apply the divergence theorem to the volume bounded by the surface and the plane $z = 0$.

11.20 Obtain an expression for the value ϕ_P at a point P of a scalar function ϕ that satisfies $\nabla^2\phi = 0$, in terms of its value and normal derivative on a surface S that encloses it, by proceeding as follows.

(a) In Green's second theorem, take ψ at any particular point Q as $1/r$, where r is the distance of Q from P. Show that $\nabla^2\psi = 0$, except at $r = 0$.
(b) Apply the result to the doubly connected region bounded by S and a small sphere Σ of radius δ centred on P.
(c) Apply the divergence theorem to show that the surface integral over Σ involving $1/\delta$ vanishes, and prove that the term involving $1/\delta^2$ has the value $4\pi\phi_P$.
(d) Conclude that

$$\phi_P = -\frac{1}{4\pi}\int_S \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial\phi}{\partial n}\, dS.$$

This important result shows that the value at a point P of a function ϕ that satisfies $\nabla^2\phi = 0$ everywhere within a closed surface S that encloses P may be expressed *entirely* in terms of its value and normal derivative on S. This matter is taken up more generally in connection with Green's functions in chapter 21 and in connection with functions of a complex variable in section 24.10.

11.21 Use result (11.21), together with an appropriately chosen scalar function ϕ, to prove that the position vector $\bar{\mathbf{r}}$ of the centre of mass of an arbitrarily shaped body of volume V and uniform density can be written

$$\bar{\mathbf{r}} = \frac{1}{V}\oint_S \tfrac{1}{2} r^2\, d\mathbf{S}.$$

11.22 A rigid body of volume V and surface S rotates with angular velocity $\boldsymbol{\omega}$. Show that

$$\boldsymbol{\omega} = -\frac{1}{2V}\oint_S \mathbf{u} \times d\mathbf{S},$$

where $\mathbf{u}(\mathbf{x})$ is the velocity of the point $\mathbf{x}$ on the surface S.

11.23 Demonstrate the validity of the divergence theorem:

(a) by calculating the flux of the vector

$$\mathbf{F} = \frac{\alpha\mathbf{r}}{(r^2 + a^2)^{3/2}}$$

through the spherical surface $|\mathbf{r}| = \sqrt{3}a$;
(b) by showing that

$$\nabla \cdot \mathbf{F} = \frac{3\alpha a^2}{(r^2 + a^2)^{5/2}}$$

and evaluating the volume integral of $\nabla \cdot \mathbf{F}$ over the interior of the sphere $|\mathbf{r}| = \sqrt{3}a$. The substitution $r = a\tan\theta$ will prove useful in carrying out the integration.

11.24 Prove equation (11.22) and, by taking $\mathbf{b} = zx^2\mathbf{i} + zy^2\mathbf{j} + (x^2 - y^2)\mathbf{k}$, show that the two integrals

$$I = \int x^2\, dV \quad \text{and} \quad J = \int \cos^2\theta \sin^3\theta \cos^2\phi\, d\theta\, d\phi,$$

both taken over the unit sphere, must have the same value. Evaluate both directly to show that the common value is $4\pi/15$.

11.25 In a uniform conducting medium with unit relative permittivity, charge density ρ, current density $\mathbf{J}$, electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, Maxwell's electromagnetic equations take the form (with $\mu_0\epsilon_0 = c^{-2}$)

$$\begin{array}{ll} \text{(i) } \nabla \cdot \mathbf{B} = 0, & \text{(ii) } \nabla \cdot \mathbf{E} = \rho/\epsilon_0, \\ \text{(iii) } \nabla \times \mathbf{E} + \dot{\mathbf{B}} = \mathbf{0}, & \text{(iv) } \nabla \times \mathbf{B} - (\dot{\mathbf{E}}/c^2) = \mu_0 \mathbf{J}. \end{array}$$

The density of stored energy in the medium is given by $\frac{1}{2}(\epsilon_0 E^2 + \mu_0^{-1} B^2)$. Show that the rate of change of the total stored energy in a volume V is equal to

$$-\int_V \mathbf{J}\cdot\mathbf{E}\, dV - \frac{1}{\mu_0}\oint_S (\mathbf{E}\times\mathbf{B})\cdot\, d\mathbf{S},$$

where S is the surface bounding V.

[The first integral gives the ohmic heating loss, whilst the second gives the electromagnetic energy flux out of the bounding surface. The vector $\mu_0^{-1}(\mathbf{E}\times\mathbf{B})$ is known as the Poynting vector.]

11.26 A vector field $\mathbf{F}$ is defined in cylindrical polar coordinates ρ, θ, z by

$$\mathbf{F} = F_0\left(\frac{x\cos\lambda z}{a}\mathbf{i} + \frac{y\cos\lambda z}{a}\mathbf{j} + (\sin\lambda z)\mathbf{k}\right) \equiv \frac{F_0\rho}{a}(\cos\lambda z)\mathbf{e}_\rho + F_0(\sin\lambda z)\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the Cartesian axes and $\mathbf{e}_\rho$ is the unit vector $(x/\rho)\mathbf{i} + (y/\rho)\mathbf{j}$.

(a) Calculate, as a surface integral, the flux of $\mathbf{F}$ through the closed surface bounded by the cylinders $\rho = a$ and $\rho = 2a$ and the planes $z = \pm a\pi/2$.
(b) Evaluate the same integral using the divergence theorem.

11.27 The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = (3x^2yz + y^3z + xe^{-x})\mathbf{i} + (3xy^2z + x^3z + ye^x)\mathbf{j} + (x^3y + y^3x + xy^2z^2)\mathbf{k}.$$

Calculate (a) directly, and (b) by using Stokes' theorem the value of the line integral $\int_L \mathbf{F}\cdot d\mathbf{r}$, where L is the (three-dimensional) closed contour $OABCDEO$ defined by the successive vertices $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(1,1,1)$, $(1,1,0)$, $(0,1,0)$, $(0,0,0)$.

11.28 A vector force field $\mathbf{F}$ is defined in Cartesian coordinates by

$$\mathbf{F} = F_0\left[\left(\frac{y^3}{3a^3} + \frac{y}{a}e^{xy/a^2} + 1\right)\mathbf{i} + \left(\frac{xy^2}{a^3} + \frac{x+y}{a}e^{xy/a^2}\right)\mathbf{j} + \frac{z}{a}e^{xy/a^2}\mathbf{k}\right].$$

Use Stokes' theorem to calculate

$$\oint_L \mathbf{F}\cdot d\mathbf{r},$$

where L is the perimeter of the rectangle $ABCD$ given by $A = (0,1,0)$, $B = (1,1,0)$, $C = (1,3,0)$ and $D = (0,3,0)$.

11.11 Hints and answers

11.1 Show that $\nabla \times \mathbf{F} = \mathbf{0}$. The potential $\phi_F(\mathbf{r}) = x^2 z + y^2 z^2 - z$.

11.3 (a) $c^3 \ln 2\,\mathbf{i} + 2\,\mathbf{j} + (3c/2)\mathbf{k}$; (b) $(-3c^4/8)\mathbf{i} - c\,\mathbf{j} - (c^2 \ln 2)\mathbf{k}$; (c) $c^4 \ln 2 - c$.

11.5 For P, $x = y = ab/(a^2 + b^2)^{1/2}$. The relevant limits are $0 \le \theta_1 \le \tan^{-1}(b/a)$ and $\tan^{-1}(a/b) \le \theta_2 \le \pi/2$. The total common area is $4ab\tan^{-1}(b/a)$.

11.7 Show that, in the notation of section 11.3, $\partial Q/\partial x - \partial P/\partial y = 2x^2$; $I = \pi a^3 b/2$.

11.9 $\mathbf{M} = I \int_C \mathbf{r} \times (d\mathbf{r} \times \mathbf{B})$. Show that the horizontal sides in the first term and the whole of the second term contribute nothing to the couple.

11.11 Note that, if $\hat{\mathbf{n}}$ is the outward normal to the surface, $\hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}\, dl$ is equal to $-d\rho$.

11.13 (b) $\phi = c + z/r$.

11.15 (a) Yes, $F_0(x - y)\exp(-r^2/a^2)$; (b) yes, $-F_0[(x^2 + y^2)/(2a)]\exp(-r^2/a^2)$; (c) no, $\nabla \times \mathbf{F} \neq \mathbf{0}$.

11.17 A spiral of radius c with its axis parallel to the z-direction and passing through (a, b). The pitch of the spiral is $2\pi c^2$. No, because (i) γ is not a closed loop and (ii) the line integral must be zero for *every* closed loop, not just for a particular one. In fact $\nabla \times \mathbf{f} = -2\mathbf{k} \neq \mathbf{0}$ shows that $\mathbf{f}$ is not conservative.

11.19 (a) $d\mathbf{S} = (2a^3 \cos\theta \sin^2\theta \cos\phi\,\mathbf{i} + 2a^3 \cos\theta \sin^2\theta \sin\phi\,\mathbf{j} + a^2 \cos\theta \sin\theta\,\mathbf{k})\, d\theta\, d\phi$.
(b) $\nabla \cdot \mathbf{r} = 3$; over the plane $z = 0$, $\mathbf{r} \cdot d\mathbf{S} = 0$.
The necessarily common value is $3\pi a^4/2$.

11.21 Write $\mathbf{r}$ as $\nabla(\frac{1}{2}r^2)$.

11.23 The answer is $3\sqrt{3}\pi\alpha/2$ in each case.

11.25 Identify the expression for $\nabla \cdot (\mathbf{E} \times \mathbf{B})$ and use the divergence theorem.

11.27 (a) The successive contributions to the integral are:
$1 - 2e^{-1}$, 0, $2 + \frac{1}{2}e$, $-\frac{7}{3}$, $-1 + 2e^{-1}$, $-\frac{1}{2}$.
(b) $\nabla \times \mathbf{F} = 2xyz^2\mathbf{i} - y^2z^2\mathbf{j} + ye^x\mathbf{k}$. Show that the contour is equivalent to the sum of two plane square contours in the planes $z = 0$ and $x = 1$, the latter being traversed in the negative sense. Integral $= \frac{1}{6}(3e - 5)$.

12

Fourier series

We have already discussed, in chapter 4, how complicated functions may be expressed as power series. However, this is not the only way in which a function may be represented as a series, and the subject of this chapter is the expression of functions as a sum of sine and cosine terms. Such a representation is called a *Fourier series*. Unlike Taylor series, a Fourier series can describe functions that are not everywhere continuous and/or differentiable. There are also other advantages in using trigonometric terms. They are easy to differentiate and integrate, their moduli are easily taken and each term contains only one characteristic frequency. This last point is important because, as we shall see later, Fourier series are often used to represent the response of a system to a periodic input, and this response often depends directly on the frequency content of the input. Fourier series are used in a wide variety of such physical situations, including the vibrations of a finite string, the scattering of light by a diffraction grating and the transmission of an input signal by an electronic circuit.

12.1 The Dirichlet conditions

We have already mentioned that Fourier series may be used to represent some functions for which a Taylor series expansion is not possible. The particular conditions that a function $f(x)$ must fulfil in order that it may be expanded as a Fourier series are known as the *Dirichlet conditions*, and may be summarised by the following four points:

(i) the function must be periodic;
(ii) it must be single-valued and continuous, except possibly at a finite number of finite discontinuities;
(iii) it must have only a finite number of maxima and minima within one period;
(iv) the integral over one period of $|f(x)|$ must converge.

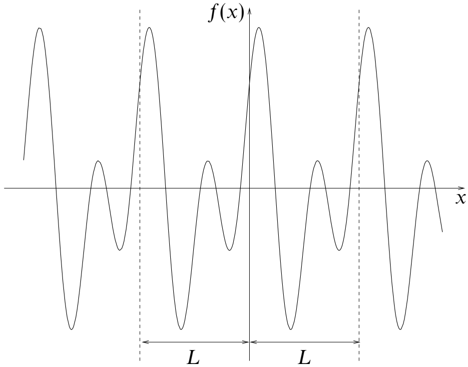


Figure 12.1 An example of a function that may be represented as a Fourier series without modification.

If the above conditions are satisfied then the Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous. The convergence of the Fourier series at points of discontinuity is discussed in section 12.4. The last three Dirichlet conditions are almost always met in real applications, but not all functions are periodic and hence do not fulfil the first condition. It may be possible, however, to represent a non-periodic function as a Fourier series by manipulation of the function into a periodic form. This is discussed in section 12.5. An example of a function that may, without modification, be represented as a Fourier series is shown in figure 12.1.

We have stated without proof that any function that satisfies the Dirichlet conditions may be represented as a Fourier series. Let us now show why this is a plausible statement. We require that any reasonable function (one that satisfies the Dirichlet conditions) can be expressed as a linear sum of sine and cosine terms. We first note that we cannot use just a sum of sine terms since sine, being an odd function (i.e. a function for which $f(-x) = -f(x)$), cannot represent even functions (i.e. functions for which $f(-x) = f(x)$). This is obvious when we try to express a function $f(x)$ that takes a non-zero value at $x = 0$. Clearly, since $\sin nx = 0$ for all values of n, we cannot represent $f(x)$ at $x = 0$ by a sine series. Similarly odd functions cannot be represented by a cosine series since cosine is an even function. Nevertheless, it is possible to represent *all* odd functions by a sine series and *all* even functions by a cosine series. Now, since all functions may be written as the sum of an odd and an even part,

$$\begin{aligned} f(x) &= \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)] \\ &= f_{\text{even}}(x) + f_{\text{odd}}(x), \end{aligned}$$

we can write any function as the sum of a sine series and a cosine series.

All the terms of a Fourier series are mutually orthogonal, i.e. the integrals, over one period, of the product of any two terms have the following properties:

$$\int_{x_0}^{x_0+L} \sin\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi px}{L}\right)dx = 0 \quad \text{for all } r \text{ and } p, \tag{12.1}$$

$$\int_{x_0}^{x_0+L} \cos\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi px}{L}\right)dx = \begin{cases} L & \text{for } r = p = 0, \\ \frac{1}{2}L & \text{for } r = p > 0, \\ 0 & \text{for } r \neq p, \end{cases} \tag{12.2}$$

$$\int_{x_0}^{x_0+L} \sin\left(\frac{2\pi rx}{L}\right)\sin\left(\frac{2\pi px}{L}\right)dx = \begin{cases} 0 & \text{for } r = p = 0, \\ \frac{1}{2}L & \text{for } r = p > 0, \\ 0 & \text{for } r \neq p, \end{cases} \tag{12.3}$$

where r and p are integers greater than or equal to zero; these formulae are easily derived. A full discussion of why it is possible to expand a function as a sum of mutually orthogonal functions is given in chapter 17.

The Fourier series expansion of the function $f(x)$ is conventionally written

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right)\right], \tag{12.4}$$

where a_0, a_r, b_r are constants called the *Fourier coefficients*. These coefficients are analogous to those in a power series expansion and the determination of their numerical values is the essential step in writing a function as a Fourier series.

This chapter continues with a discussion of how to find the Fourier coefficients for particular functions. We then discuss simplifications to the general Fourier series that may save considerable effort in calculations. This is followed by the alternative representation of a function as a complex Fourier series, and we conclude with a discussion of Parseval's theorem.

12.2 The Fourier coefficients

We have indicated that a series that satisfies the Dirichlet conditions may be written in the form (12.4). We now consider how to find the Fourier coefficients for any particular function. For a periodic function $f(x)$ of period L we will find that the Fourier coefficients are given by

$$a_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi rx}{L}\right)dx, \tag{12.5}$$

$$b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\sin\left(\frac{2\pi rx}{L}\right)dx, \tag{12.6}$$

where x_0 is arbitrary but is often taken as 0 or $-L/2$. The apparently arbitrary factor $\frac{1}{2}$ which appears in the a_0 term in (12.4) is included so that (12.5) may apply for $r = 0$ as well as $r > 0$. The relations (12.5) and (12.6) may be derived as follows.

Suppose the Fourier series expansion of $f(x)$ can be written as in (12.4),

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right) \right].$$

Then, multiplying by $\cos(2\pi px/L)$, integrating over one full period in x and changing the order of the summation and integration, we get

$$\begin{aligned} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi px}{L}\right) dx &= \frac{a_0}{2} \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi px}{L}\right) dx \\ &\quad + \sum_{r=1}^{\infty} a_r \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi rx}{L}\right) \cos\left(\frac{2\pi px}{L}\right) dx \\ &\quad + \sum_{r=1}^{\infty} b_r \int_{x_0}^{x_0+L} \sin\left(\frac{2\pi rx}{L}\right) \cos\left(\frac{2\pi px}{L}\right) dx. \end{aligned} \tag{12.7}$$

We can now find the Fourier coefficients by considering (12.7) as p takes different values. Using the orthogonality conditions (12.1)–(12.3) of the previous section, we find that when $p = 0$ (12.7) becomes

$$\int_{x_0}^{x_0+L} f(x) dx = \frac{a_0}{2} L.$$

When $p \neq 0$ the only non-vanishing term on the RHS of (12.7) occurs when $r = p$, and so

$$\int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi rx}{L}\right) dx = \frac{a_r}{2} L.$$

The other Fourier coefficients b_r may be found by repeating the above process but multiplying by $\sin(2\pi px/L)$ instead of $\cos(2\pi px/L)$ (see exercise 12.2).

►*Express the square-wave function illustrated in figure 12.2 as a Fourier series.*

Physically this might represent the input to an electrical circuit that switches between a high and a low state with time period T. The square wave may be represented by

$$f(t) = \begin{cases} -1 & \text{for } -\frac{1}{2}T \leq t < 0, \\ +1 & \text{for } 0 \leq t < \frac{1}{2}T. \end{cases}$$

In deriving the Fourier coefficients, we note firstly that the function is an odd function and so the series will contain only sine terms (this simplification is discussed further in the

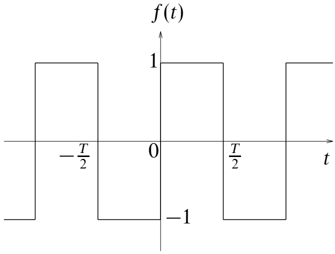


Figure 12.2 A square-wave function.

following section). To evaluate the coefficients in the sine series we use (12.6). Hence

$$\begin{aligned} b_r &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi rt}{T}\right)dt \\ &= \frac{4}{T}\int_0^{T/2} \sin\left(\frac{2\pi rt}{T}\right)dt \\ &= \frac{2}{\pi r}\,[1-(-1)^r]\,. \end{aligned}$$

Thus the sine coefficients are zero if r is even and equal to $4/(\pi r)$ if r is odd. Hence the Fourier series for the square-wave function may be written as

$$f(t) = \frac{4}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \cdots\right), \tag{12.8}$$

where $\omega = 2\pi/T$ is called the *angular frequency*. ◀

12.3 Symmetry considerations

The example in the previous section employed the useful property that since the function to be represented was odd, all the cosine terms of the Fourier series were absent. It is often the case that the function we wish to express as a Fourier series has a particular symmetry, which we can exploit to reduce the calculational labour of evaluating Fourier coefficients. Functions that are symmetric or antisymmetric about the origin (i.e. even and odd functions respectively) admit particularly useful simplifications. Functions that are odd in x have no cosine terms (see section 12.1) and all the a-coefficients are equal to zero. Similarly, functions that are even in x have no sine terms and all the b-coefficients are zero. Since the Fourier series of odd or even functions contain only half the coefficients required for a general periodic function, there is a considerable reduction in the algebra needed to find a Fourier series.

The consequences of symmetry or antisymmetry of the function about the quarter period (i.e. about $L/4$) are a little less obvious. Furthermore, the results are not used as often as those above and the remainder of this section can be omitted on a first reading without loss of continuity. The following argument gives the required results.

Suppose that $f(x)$ has even or odd symmetry about $L/4$, i.e. $f(L/4 - x) = \pm f(x - L/4)$. For convenience, we make the substitution $s = x - L/4$ and hence $f(-s) = \pm f(s)$. We can now see that

$$b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(s)\sin\left(\frac{2\pi rs}{L} + \frac{\pi r}{2}\right) ds,$$

where the limits of integration have been left unaltered since f is, of course, periodic in s as well as in x. If we use the expansion

$$\sin\left(\frac{2\pi rs}{L} + \frac{\pi r}{2}\right) = \sin\left(\frac{2\pi rs}{L}\right)\cos\left(\frac{\pi r}{2}\right) + \cos\left(\frac{2\pi rs}{L}\right)\sin\left(\frac{\pi r}{2}\right),$$

we can immediately see that the trigonometric part of the integrand is an odd function of s if r is even and an even function of s if r is odd. Hence if $f(s)$ is even and r is even then the integral is zero, and if $f(s)$ is odd and r is odd then the integral is zero. Similar results can be derived for the Fourier a-coefficients and we conclude that

(i) if $f(x)$ is even about $L/4$ then $a_{2r+1} = 0$ and $b_{2r} = 0$,
(ii) if $f(x)$ is odd about $L/4$ then $a_{2r} = 0$ and $b_{2r+1} = 0$.

All the above results follow automatically when the Fourier coefficients are evaluated in any particular case, but prior knowledge of them will often enable some coefficients to be set equal to zero on inspection and so substantially reduce the computational labour. As an example, the square-wave function shown in figure 12.2 is (i) an odd function of t, so that all $a_r = 0$, and (ii) even about the point $t = T/4$, so that $b_{2r} = 0$. Thus we can say immediately that only sine terms of odd harmonics will be present and therefore will need to be calculated; this is confirmed in the expansion (12.8).

12.4 Discontinuous functions

The Fourier series expansion usually works well for functions that are discontinuous in the required range. However, the series itself does not produce a discontinuous function and we state without proof that the value of the expanded $f(x)$ at a discontinuity will be half-way between the upper and lower values. Expressing this more mathematically, at a point of finite discontinuity, x_d, the Fourier series converges to

$$\tfrac{1}{2}\lim_{\epsilon\to 0}[f(x_d + \epsilon) + f(x_d - \epsilon)].$$

At a discontinuity, the Fourier series representation of the function will overshoot its value. Although as more terms are included the overshoot moves in position

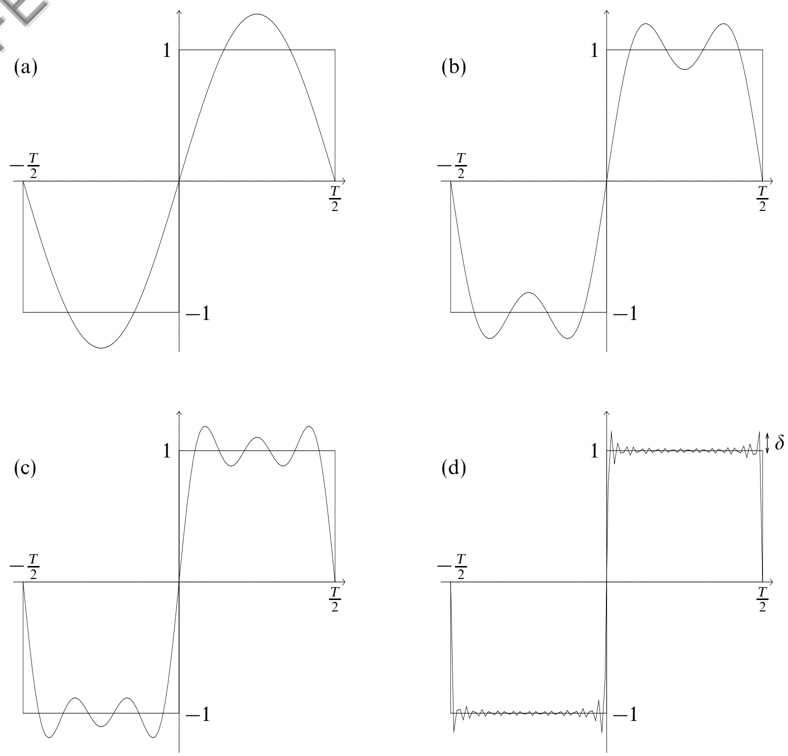


Figure 12.3 The convergence of a Fourier series expansion of a square-wave function, including (a) one term, (b) two terms, (c) three terms and (d) 20 terms. The overshoot δ is shown in (d).

arbitrarily close to the discontinuity, it never disappears even in the limit of an infinite number of terms. This behaviour is known as *Gibbs' phenomenon.* A full discussion is not pursued here but suffice it to say that the size of the overshoot is proportional to the magnitude of the discontinuity.

►*Find the value to which the Fourier series of the square-wave function discussed in section 12.2 converges at $t = 0$.*

It can be seen that the function is discontinuous at $t = 0$ and, by the above rule, we expect the series to converge to a value half-way between the upper and lower values, in other words to converge to zero in this case. Considering the Fourier series of this function, (12.8), we see that all the terms are zero and hence the Fourier series converges to zero as expected. The Gibbs phenomenon for the square-wave function is shown in figure 12.3. ◄

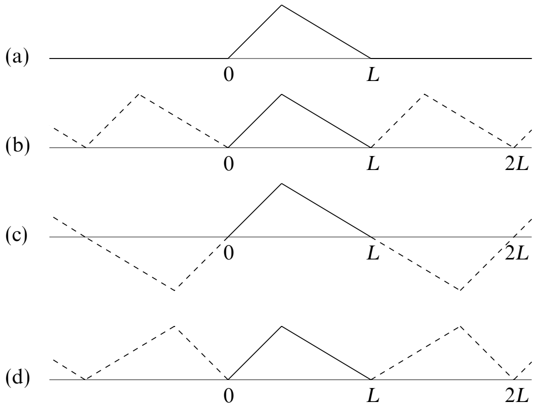


Figure 12.4   Possible periodic extensions of a function.

12.5 Non-periodic functions

We have already mentioned that a Fourier representation may sometimes be used for non-periodic functions. If we wish to find the Fourier series of a non-periodic function only within a fixed range then we may *continue* the function outside the range so as to make it periodic. The Fourier series of this periodic function would then correctly represent the non-periodic function in the desired range. Since we are often at liberty to extend the function in a number of ways, we can sometimes make it odd or even and so reduce the calculation required. Figure 12.4(b) shows the simplest extension to the function shown in figure 12.4(a). However, this extension has no particular symmetry. Figures 12.4(c), (d) show extensions as odd and even functions respectively with the benefit that only sine or cosine terms appear in the resulting Fourier series. We note that these last two extensions give a function of period $2L$.

In view of the result of section 12.4, it must be added that the continuation must not be discontinuous at the end-points of the interval of interest; if it is the series will not converge to the required value there. This requirement that the series converges appropriately may reduce the choice of continuations. This is discussed further at the end of the following example.

►*Find the Fourier series of* $f(x) = x^2$ *for* $0 < x \leq 2$.

We must first make the function periodic. We do this by extending the range of interest to $-2 < x \leq 2$ in such a way that $f(x) = f(-x)$ and then letting $f(x + 4k) = f(x)$, where k is any integer. This is shown in figure 12.5. Now we have an even function of period 4. The Fourier series will faithfully represent $f(x)$ in the range, $-2 < x \leq 2$, although not outside it. Firstly we note that since we have made the specified function even in x by extending

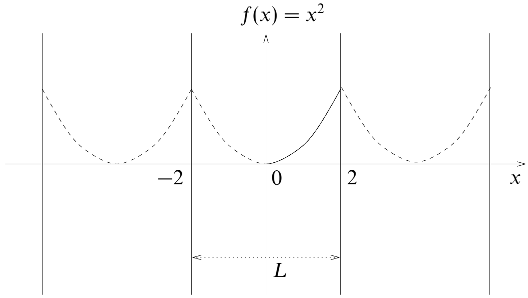


Figure 12.5 $f(x) = x^2$, $0 < x \leq 2$, with the range extended to give periodicity.

the range, all the coefficients b_r will be zero. Now we apply (12.5) and (12.6) with $L = 4$ to determine the remaining coefficients:

$$a_r = \frac{2}{4}\int_{-2}^{2} x^2 \cos\left(\frac{2\pi r x}{4}\right) dx = \frac{4}{4}\int_{0}^{2} x^2 \cos\left(\frac{\pi r x}{2}\right) dx,$$

where the second equality holds because the function is even in x. Thus

$$\begin{aligned} a_r &= \left[\frac{2}{\pi r}x^2 \sin\left(\frac{\pi r x}{2}\right)\right]_0^2 - \frac{4}{\pi r}\int_0^2 x \sin\left(\frac{\pi r x}{2}\right) dx \\ &= \frac{8}{\pi^2 r^2}\left[x \cos\left(\frac{\pi r x}{2}\right)\right]_0^2 - \frac{8}{\pi^2 r^2}\int_0^2 \cos\left(\frac{\pi r x}{2}\right) dx \\ &= \frac{16}{\pi^2 r^2}\cos \pi r \\ &= \frac{16}{\pi^2 r^2}(-1)^r. \end{aligned}$$

Since this expression for a_r has r^2 in its denominator, to evaluate a_0 we must return to the original definition,

$$a_r = \frac{2}{4}\int_{-2}^{2} f(x) \cos\left(\frac{\pi r x}{2}\right) dx.$$

From this we obtain

$$a_0 = \frac{2}{4}\int_{-2}^{2} x^2\, dx = \frac{4}{4}\int_{0}^{2} x^2\, dx = \frac{8}{3}.$$

The final expression for $f(x)$ is then

$$x^2 = \frac{4}{3} + 16\sum_{r=1}^{\infty} \frac{(-1)^r}{\pi^2 r^2}\cos\left(\frac{\pi r x}{2}\right) \qquad \text{for } 0 < x \leq 2. \blacktriangleleft$$

We note that in the above example we could have extended the range so as to make the function odd. In other words we could have set $f(x) = -f(-x)$ and then made $f(x)$ periodic in such a way that $f(x + 4) = f(x)$. In this case the resulting Fourier series would be a series of just sine terms. However, although this will faithfully represent the function inside the required range, it does not converge to the correct values of $f(x) = \pm 4$ at $x = \pm 2$; it converges, instead, to zero, the average of the values at the two ends of the range.

12.6 Integration and differentiation

It is sometimes possible to find the Fourier series of a function by integration or differentiation of another Fourier series. If the Fourier series of $f(x)$ is integrated term by term then the resulting Fourier series converges to the integral of $f(x)$. Clearly, when integrating in such a way there is a constant of integration that must be found. If $f(x)$ is a continuous function of x for all x and $f(x)$ is also periodic then the Fourier series that results from differentiating term by term converges to $f'(x)$, provided that $f'(x)$ itself satisfies the Dirichlet conditions. These properties of Fourier series may be useful in calculating complicated Fourier series, since simple Fourier series may easily be evaluated (or found from standard tables) and often the more complicated series can then be built up by integration and/or differentiation.

►*Find the Fourier series of* $f(x) = x^3$ *for* $0 < x \le 2$.

In the example discussed in the previous section we found the Fourier series for $f(x) = x^2$ in the required range. So, if we *integrate* this term by term, we obtain

$$\frac{x^3}{3} = \frac{4}{3}x + 32\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi^3 r^3}\sin\left(\frac{\pi r x}{2}\right) + c,$$

where c is, so far, an arbitrary constant. We have not yet found the Fourier series for x^3 because the term $\frac{4}{3}x$ appears in the expansion. However, by now *differentiating* the same initial expression for x^2 we obtain

$$2x = -8\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi r}\sin\left(\frac{\pi r x}{2}\right).$$

We can now write the full Fourier expansion of x^3 as

$$x^3 = -16\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi r}\sin\left(\frac{\pi r x}{2}\right) + 96\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi^3 r^3}\sin\left(\frac{\pi r x}{2}\right) + c.$$

Finally, we can find the constant, c, by considering $f(0)$. At $x = 0$, our Fourier expansion gives $x^3 = c$ since all the sine terms are zero, and hence $c = 0$. ◄

12.7 Complex Fourier series

As a Fourier series expansion in general contains both sine and cosine parts, it may be written more compactly using a complex exponential expansion. This simplification makes use of the property that $\exp(irx) = \cos rx + i\sin rx$. The complex Fourier series expansion is written

$$f(x) = \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi i r x}{L}\right), \qquad (12.9)$$

where the Fourier coefficients are given by

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\exp\left(-\frac{2\pi irx}{L}\right) dx. \tag{12.10}$$

This relation can be derived, in a similar manner to that of section 12.2, by multiplying (12.9) by $\exp(-2\pi ipx/L)$ before integrating and using the orthogonality relation

$$\int_{x_0}^{x_0+L} \exp\left(-\frac{2\pi ipx}{L}\right)\exp\left(\frac{2\pi irx}{L}\right) dx = \begin{cases} L & \text{for } r = p, \\ 0 & \text{for } r \neq p. \end{cases}$$

The complex Fourier coefficients in (12.9) have the following relations to the real Fourier coefficients:

$$\begin{aligned} c_r &= \tfrac{1}{2}(a_r - ib_r), \\ c_{-r} &= \tfrac{1}{2}(a_r + ib_r). \end{aligned} \tag{12.11}$$

Note that if $f(x)$ is real then $c_{-r} = c_r^*$, where the asterisk represents complex conjugation.

►*Find a complex Fourier series for $f(x) = x$ in the range $-2 < x < 2$.*

Using (12.10), for $r \neq 0$,

$$\begin{aligned} c_r &= \frac{1}{4}\int_{-2}^{2} x\exp\left(-\frac{\pi irx}{2}\right) dx \\ &= \left[-\frac{x}{2\pi ir}\exp\left(-\frac{\pi irx}{2}\right)\right]_{-2}^{2} + \int_{-2}^{2}\frac{1}{2\pi ir}\exp\left(-\frac{\pi irx}{2}\right) dx \\ &= -\frac{1}{\pi ir}[\exp(-\pi ir) + \exp(\pi ir)] + \left[\frac{1}{r^2\pi^2}\exp\left(-\frac{\pi irx}{2}\right)\right]_{-2}^{2} \\ &= \frac{2i}{\pi r}\cos\pi r - \frac{2i}{r^2\pi^2}\sin\pi r = \frac{2i}{\pi r}(-1)^r. \end{aligned} \tag{12.12}$$

For $r = 0$, we find $c_0 = 0$ and hence

$$x = \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \frac{2i(-1)^r}{r\pi}\exp\left(\frac{\pi irx}{2}\right).$$

We note that the Fourier series derived for x in section 12.6 gives $a_r = 0$ for all r and

$$b_r = -\frac{4(-1)^r}{\pi r},$$

and so, using (12.11), we confirm that c_r and c_{-r} have the forms derived above. It is also apparent that the relationship $c_r^* = c_{-r}$ holds, as we expect since $f(x)$ is real. ◄

12.8 Parseval's theorem

Parseval's theorem gives a useful way of relating the Fourier coefficients to the function that they describe. Essentially a conservation law, it states that

$$\frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 dx = \sum_{r=-\infty}^{\infty} |c_r|^2$$
$$= \left(\tfrac{1}{2}a_0\right)^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2). \qquad (12.13)$$

In a more memorable form, this says that the sum of the moduli squared of the complex Fourier coefficients is equal to the average value of $|f(x)|^2$ over one period. Parseval's theorem can be proved straightforwardly by writing $f(x)$ as a Fourier series and evaluating the required integral, but the algebra is messy. Therefore, we shall use an alternative method, for which the algebra is simple and which in fact leads to a more general form of the theorem.

Let us consider two functions $f(x)$ and $g(x)$, which are (or can be made) periodic with period L and which have Fourier series (expressed in complex form)

$$f(x) = \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi irx}{L}\right),$$
$$g(x) = \sum_{r=-\infty}^{\infty} \gamma_r \exp\left(\frac{2\pi irx}{L}\right),$$

where c_r and γ_r are the complex Fourier coefficients of $f(x)$ and $g(x)$ respectively. Thus

$$f(x)g^*(x) = \sum_{r=-\infty}^{\infty} c_r g^*(x) \exp\left(\frac{2\pi irx}{L}\right).$$

Integrating this equation with respect to x over the interval $(x_0, x_0 + L)$ and dividing by L, we find

$$\begin{aligned}\frac{1}{L}\int_{x_0}^{x_0+L} f(x)g^*(x)\,dx &= \sum_{r=-\infty}^{\infty} c_r \frac{1}{L}\int_{x_0}^{x_0+L} g^*(x)\exp\left(\frac{2\pi irx}{L}\right) dx \\ &= \sum_{r=-\infty}^{\infty} c_r \left[\frac{1}{L}\int_{x_0}^{x_0+L} g(x)\exp\left(\frac{-2\pi irx}{L}\right) dx\right]^* \\ &= \sum_{r=-\infty}^{\infty} c_r\gamma_r^*,\end{aligned}$$

where the last equality uses (12.10). Finally, if we let $g(x) = f(x)$ then we obtain Parseval's theorem (12.13). This result can be proved in a similar manner using the sine and cosine form of the Fourier series, but the algebra is slightly more complicated.

Parseval's theorem is sometimes used to sum series. However, if one is presented with a series to sum, it is not usually possible to decide which Fourier series should be used to evaluate it. Rather, useful summations are nearly always found serendipitously. The following example shows the evaluation of a sum by a Fourier series method.

►*Using Parseval's theorem and the Fourier series for $f(x) = x^2$ found in section 12.5, calculate the sum $\sum_{r=1}^{\infty} r^{-4}$.*

Firstly we find the average value of $[f(x)]^2$ over the interval $-2 < x \leq 2$:

$$\frac{1}{4}\int_{-2}^{2} x^4\,dx = \frac{16}{5}.$$

Now we evaluate the right-hand side of (12.13):

$$\left(\tfrac{1}{2}a_0\right)^2 + \tfrac{1}{2}\sum_{1}^{\infty} a_r^2 + \tfrac{1}{2}\sum_{1}^{\infty} b_n^2 = \left(\tfrac{4}{3}\right)^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}\frac{16^2}{\pi^4 r^4}.$$

Equating the two expression we find

$$\sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{\pi^4}{90}. \blacktriangleleft$$

12.9 Exercises

12.1 Prove the orthogonality relations stated in section 12.1.

12.2 Derive the Fourier coefficients b_r in a similar manner to the derivation of the a_r in section 12.2.

12.3 Which of the following functions of x could be represented by a Fourier series over the range indicated?

(a) $\tanh^{-1}(x)$, $\quad -\infty < x < \infty$;
(b) $\tan x$, $\quad -\infty < x < \infty$;
(c) $|\sin x|^{-1/2}$, $\quad -\infty < x < \infty$;
(d) $\cos^{-1}(\sin 2x)$, $\quad -\infty < x < \infty$;
(e) $x\sin(1/x)$, $\quad -\pi^{-1} < x \leq \pi^{-1}$, cyclically repeated.

12.4 By moving the origin of t to the centre of an interval in which $f(t) = +1$, i.e. by changing to a new independent variable $t' = t - \frac{1}{4}T$, express the square-wave function in the example in section 12.2 as a cosine series. Calculate the Fourier coefficients involved (a) directly and (b) by changing the variable in result (12.8).

12.5 Find the Fourier series of the function $f(x) = x$ in the range $-\pi < x \leq \pi$. Hence show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

12.6 For the function

$$f(x) = 1 - x, \qquad 0 \leq x \leq 1,$$

find (a) the Fourier sine series and (b) the Fourier cosine series. Which would be better for numerical evaluation? Relate your answer to the relevant periodic continuations.

12.7 For the continued functions used in exercise 12.6 and the derived corresponding series, consider (i) their derivatives and (ii) their integrals. Do they give meaningful equations? You will probably find it helpful to sketch all the functions involved.

12.8 The function $y(x) = x \sin x$ for $0 \le x \le \pi$ is to be represented by a Fourier series of period 2π that is either even or odd. By sketching the function and considering its derivative, determine which series will have the more rapid convergence. Find the full expression for the better of these two series, showing that the convergence $\sim n^{-3}$ and that alternate terms are missing.

12.9 Find the Fourier coefficients in the expansion of $f(x) = \exp x$ over the range $-1 < x < 1$. What value will the expansion have when $x = 2$?

12.10 By integrating term by term the Fourier series found in the previous question and using the Fourier series for $f(x) = x$ found in section 12.6, show that $\int \exp x\, dx = \exp x + c$. Why is it not possible to show that $d(\exp x)/dx = \exp x$ by differentiating the Fourier series of $f(x) = \exp x$ in a similar manner?

12.11 Consider the function $f(x) = \exp(-x^2)$ in the range $0 \le x \le 1$. Show how it should be continued to give as its Fourier series a series (the actual form is not wanted) (a) with only cosine terms, (b) with only sine terms, (c) with period 1 and (d) with period 2.

Would there be any difference between the values of the last two series at (i) $x = 0$, (ii) $x = 1$?

12.12 Find, without calculation, which terms will be present in the Fourier series for the periodic functions $f(t)$, of period T, that are given in the range $-T/2$ to $T/2$ by:

(a) $f(t) = 2$ for $0 \le |t| < T/4$, $f = 1$ for $T/4 \le |t| < T/2$;
(b) $f(t) = \exp[-(t - T/4)^2]$;
(c) $f(t) = -1$ for $-T/2 \le t < -3T/8$ and $3T/8 \le t < T/2, f(t) = 1$ for $-T/8 \le t < T/8$; the graph of f is completed by two straight lines in the remaining ranges so as to form a continuous function.

12.13 Consider the representation as a Fourier series of the displacement of a string lying in the interval $0 \le x \le L$ and fixed at its ends, when it is pulled aside by y_0 at the point $x = L/4$. Sketch the continuations for the region outside the interval that will

(a) produce a series of period L,
(b) produce a series that is antisymmetric about $x = 0$, and
(c) produce a series that will contain only cosine terms.
(d) What are (i) the periods of the series in (b) and (c) and (ii) the value of the 'a_0-term' in (c)?
(e) Show that a typical term of the series obtained in (b) is

$$\frac{32 y_0}{3n^2\pi^2} \sin\frac{n\pi}{4} \sin\frac{n\pi x}{L}.$$

12.14 Show that the Fourier series for the function $y(x) = |x|$ in the range $-\pi \le x < \pi$ is

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}.$$

By integrating this equation term by term from 0 to x, find the function $g(x)$ whose Fourier series is

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3}.$$

Deduce the value of the sum S of the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots.$$

12.15 Using the result of exercise 12.14, determine, as far as possible by inspection, the forms of the functions of which the following are the Fourier series:

(a)

$$\cos\theta + \frac{1}{9}\cos 3\theta + \frac{1}{25}\cos 5\theta + \cdots;$$

(b)

$$\sin\theta + \frac{1}{27}\sin 3\theta + \frac{1}{125}\sin 5\theta + \cdots;$$

(c)

$$\frac{L^2}{3} - \frac{4L^2}{\pi^2}\left[\cos\frac{\pi x}{L} - \frac{1}{4}\cos\frac{2\pi x}{L} + \frac{1}{9}\cos\frac{3\pi x}{L} - \cdots\right].$$

(You may find it helpful to first set $x = 0$ in the quoted result and so obtain values for $S_o = \sum(2m+1)^{-2}$ and other sums derivable from it.)

12.16 By finding a cosine Fourier series of period 2 for the function $f(t)$ that takes the form $f(t) = \cosh(t-1)$ in the range $0 \le t \le 1$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 + 1} = \frac{1}{e^2 - 1}.$$

Deduce values for the sums $\sum(n^2\pi^2 + 1)^{-1}$ over odd n and even n separately.

12.17 Find the (real) Fourier series of period 2 for $f(x) = \cosh x$ and $g(x) = x^2$ in the range $-1 \le x \le 1$. By integrating the series for $f(x)$ twice, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2(n^2\pi^2+1)} = \frac{1}{2}\left(\frac{1}{\sinh 1} - \frac{5}{6}\right).$$

12.18 Express the function $f(x) = x^2$ as a Fourier sine series in the range $0 < x \le 2$ and show that it converges to zero at $x = \pm 2$.

12.19 Demonstrate explicitly for the square-wave function discussed in section 12.2 that Parseval's theorem (12.13) is valid. You will need to use the relationship

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

Show that a filter that transmits frequencies only up to $8\pi/T$ will still transmit more than 90% of the power in such a square-wave voltage signal.

12.20 Show that the Fourier series for $|\sin\theta|$ in the range $-\pi \le \theta \le \pi$ is given by

$$|\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2 - 1}.$$

By setting $\theta = 0$ and $\theta = \pi/2$, deduce values for

$$\sum_{m=1}^{\infty} \frac{1}{4m^2-1} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{1}{16m^2-1}.$$

12.21 Find the complex Fourier series for the periodic function of period 2π defined in the range $-\pi \leq x \leq \pi$ by $y(x) = \cosh x$. By setting $x = 0$ prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{1}{2}\left(\frac{\pi}{\sinh \pi} - 1\right).$$

12.22 The repeating output from an electronic oscillator takes the form of a sine wave $f(t) = \sin t$ for $0 \leq t \leq \pi/2$; it then drops instantaneously to zero and starts again. The output is to be represented by a complex Fourier series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{4nti}.$$

Sketch the function and find an expression for c_n. Verify that $c_{-n} = c_n^*$. Demonstrate that setting $t = 0$ and $t = \pi/2$ produces differing values for the sum

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}.$$

Determine the correct value and check it using the result of exercise 12.20.

12.23 Apply Parseval's theorem to the series found in the previous exercise and so derive a value for the sum of the series

$$\frac{17}{(15)^2} + \frac{65}{(63)^2} + \frac{145}{(143)^2} + \cdots + \frac{16n^2+1}{(16n^2-1)^2} + \cdots.$$

12.24 A string, anchored at $x = \pm L/2$, has a fundamental vibration frequency of $2L/c$, where c is the speed of transverse waves on the string. It is pulled aside at its centre point by a distance y_0 and released at time $t = 0$. Its subsequent motion can be described by the series

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$

Find a general expression for a_n and show that only the odd harmonics of the fundamental frequency are present in the sound generated by the released string. By applying Parseval's theorem, find the sum S of the series $\sum_0^{\infty} (2m+1)^{-4}$.

12.25 Show that Parseval's theorem for two real functions whose Fourier expansions have cosine and sine coefficients a_n, b_n and α_n, β_n takes the form

$$\frac{1}{L}\int_0^L f(x)g^*(x)\,dx = \frac{1}{4}a_0\alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n).$$

(a) Demonstrate that for $g(x) = \sin mx$ or $\cos mx$ this reduces to the definition of the Fourier coefficients.

(b) Explicitly verify the above result for the case in which $f(x) = x$ and $g(x)$ is the square-wave function, both in the interval $-1 \leq x \leq 1$.

12.26 An odd function $f(x)$ of period 2π is to be approximated by a Fourier sine series having only m terms. The error in this approximation is measured by the square deviation

$$E_m = \int_{-\pi}^{\pi} \left[f(x) - \sum_{n=1}^{m} b_n \sin nx \right]^2 dx.$$

By differentiating E_m with respect to the coefficients b_n, find the values of b_n that minimise E_m.

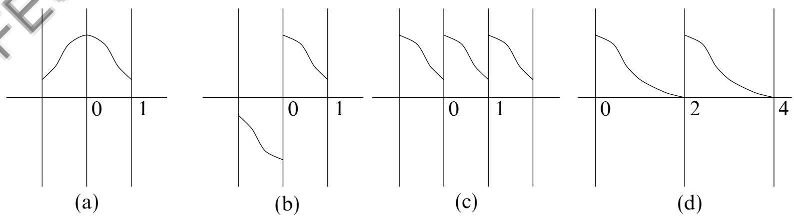


Figure 12.6 Continuations of $\exp(-x^2)$ in $0 \le x \le 1$ to give: (a) cosine terms only; (b) sine terms only; (c) period 1; (d) period 2.

Sketch the graph of the function $f(x)$, where

$$f(x) = \begin{cases} -x(\pi + x) & \text{for } -\pi \le x < 0, \\ x(x - \pi) & \text{for } 0 \le x < \pi. \end{cases}$$

If $f(x)$ is to be approximated by the first three terms of a Fourier sine series, what values should the coefficients have so as to minimise E_3? What is the resulting value of E_3?

12.10 Hints and answers

12.1 Note that the only integral of a sinusoid around a complete cycle of length L that is not zero is the integral of $\cos(2\pi nx/L)$ when $n = 0$.

12.3 Only (c). In terms of the Dirichlet conditions (section 12.1), the others fail as follows: (a) (i); (b) (ii); (d) (ii); (e) (iii).

12.5 $f(x) = 2\sum_1^\infty (-1)^{n+1} n^{-1} \sin nx$; set $x = \pi/2$.

12.7 (i) Series (a) from exercise 12.6 does not converge and cannot represent the function $y(x) = -1$. Series (b) reproduces the square-wave function of equation (12.8).
(ii) Series (a) gives the series for $y(x) = -x - \frac{1}{2}x^2 - \frac{1}{2}$ in the range $-1 \le x \le 0$ and for $y(x) = x - \frac{1}{2}x^2 - \frac{1}{2}$ in the range $0 \le x \le 1$. Series (b) gives the series for $y(x) = x + \frac{1}{2}x^2 + \frac{1}{2}$ in the range $-1 \le x \le 0$ and for $y(x) = x - \frac{1}{2}x^2 + \frac{1}{2}$ in the range $0 \le x \le 1$.

12.9 $f(x) = (\sinh 1)\left\{1 + 2\sum_1^\infty (-1)^n (1 + n^2\pi^2)^{-1}[\cos(n\pi x) - n\pi \sin(n\pi x)]\right\}$.
The series will converge to the same value as it does at $x = 0$, i.e. $f(0) = 1$.

12.11 See figure 12.6. (c) (i) $(1 + e^{-1})/2$, (ii) $(1 + e^{-1})/2$; (d) (i) $(1 + e^{-4})/2$, (ii) e^{-1}.

12.13 (d) (i) The periods are both $2L$; (ii) $y_0/2$.

12.15 $S_o = \pi^2/8$. If $S_e = \sum(2m)^{-2}$ then $S_e = \frac{1}{4}(S_e + S_o)$, yielding $S_o - S_e = \pi^2/12$ and $S_e + S_o = \pi^2/6$.
(a) $(\pi/4)(\pi/2 - |\theta|)$; (b) $(\pi\theta/4)(\pi/2 - |\theta|/2)$ from integrating (a). (c) Even function; average value $L^2/3$; $y(0) = 0$; $y(L) = L^2$; probably $y(x) = x^2$. Compare with the worked example in section 12.5.

12.17 $\cosh x = (\sinh 1)[1 + 2\sum_{n=1}^\infty (-1)^n (\cos n\pi x)/(n^2\pi^2 + 1)]$ and after integrating twice this form must be recovered. Use $x^2 = \frac{1}{3} + 4\sum(-1)^n(\cos n\pi x)/(n^2\pi^2)]$ to eliminate the quadratic term arising from the constants of integration; there is no linear term.

12.19 $C_{\pm(2m+1)} = \mp 2i/[(2m+1)\pi]$; $\sum |C_n|^2 = (4/\pi^2) \times 2 \times (\pi^2/8)$; the values $n = \pm 1$, ± 3 contribute $> 90\%$ of the total.

12.21 $c_n = [(-1)^n \sinh \pi]/[\pi(1+n^2)]$. Having set $x = 0$, separate out the $n = 0$ term and note that $(-1)^n = (-1)^{-n}$.

12.23 $(\pi^2 - 8)/16$.

12.25 (b) All a_n and α_n are zero; $b_n = 2(-1)^{n+1}/(n\pi)$ and $\beta_n = 4/(n\pi)$. You will need the result quoted in exercise 12.19.

13

Integral transforms

In the previous chapter we encountered the Fourier series representation of a periodic function in a fixed interval as a superposition of sinusoidal functions. It is often desirable, however, to obtain such a representation even for functions defined over an infinite interval and with no particular periodicity. Such a representation is called a *Fourier transform* and is one of a class of representations called *integral transforms*.

We begin by considering Fourier transforms as a generalisation of Fourier series. We then go on to discuss the properties of the Fourier transform and its applications. In the second part of the chapter we present an analogous discussion of the closely related *Laplace transform*.

13.1 Fourier transforms

The Fourier transform provides a representation of functions defined over an infinite interval and having no particular periodicity, in terms of a superposition of sinusoidal functions. It may thus be considered as a generalisation of the Fourier series representation of periodic functions. Since Fourier transforms are often used to represent time-varying functions, we shall present much of our discussion in terms of $f(t)$, rather than $f(x)$, although in some spatial examples $f(x)$ will be the more natural notation and we shall use it as appropriate. Our only requirement on $f(t)$ will be that $\int_{-\infty}^{\infty} |f(t)|\, dt$ is finite.

In order to develop the transition from Fourier series to Fourier transforms, we first recall that a function of period T may be represented as a complex Fourier series, cf. (12.9),

$$f(t) = \sum_{r=-\infty}^{\infty} c_r\, e^{2\pi i r t/T} = \sum_{r=-\infty}^{\infty} c_r\, e^{i\omega_r t}, \tag{13.1}$$

where $\omega_r = 2\pi r/T$. As the period T tends to infinity, the 'frequency quantum'

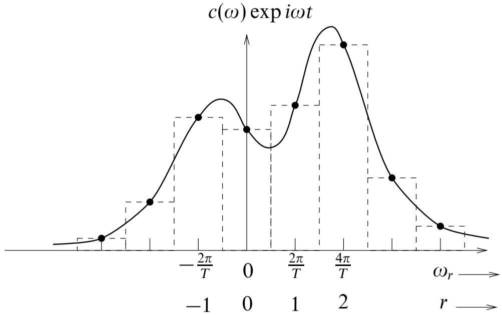


Figure 13.1 The relationship between the Fourier terms for a function of period T and the Fourier integral (the area below the solid line) of the function.

$\Delta\omega = 2\pi/T$ becomes vanishingly small and the spectrum of allowed frequencies ω_r becomes a continuum. Thus, the infinite sum of terms in the Fourier series becomes an integral, and the coefficients c_r become functions of the *continuous* variable ω, as follows.

We recall, cf. (12.10), that the coefficients c_r in (13.1) are given by

$$c_r = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-2\pi irt/T}\, dt = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_r t}\, dt, \tag{13.2}$$

where we have written the integral in two alternative forms and, for convenience, made one period run from $-T/2$ to $+T/2$ rather than from 0 to T. Substituting from (13.2) into (13.1) gives

$$f(t) = \sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(u)\, e^{-i\omega_r u}\, du\; e^{i\omega_r t}. \tag{13.3}$$

At this stage ω_r is still a discrete function of r equal to $2\pi r/T$.

The solid points in figure 13.1 are a plot of (say, the real part of) $c_r\, e^{i\omega_r t}$ as a function of r (or equivalently of ω_r) and it is clear that $(2\pi/T)c_r\, e^{i\omega_r t}$ gives the area of the rth broken-line rectangle. If T tends to ∞ then $\Delta\omega$ $(= 2\pi/T)$ becomes infinitesimal, the width of the rectangles tends to zero and, from the mathematical definition of an integral,

$$\sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_r)\, e^{i\omega_r t} \;\;\to\;\; \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega.$$

In this particular case

$$g(\omega_r) = \int_{-T/2}^{T/2} f(u)\, e^{-i\omega_r u}\, du,$$

and (13.3) becomes

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u}. \tag{13.4}$$

This result is known as *Fourier's inversion theorem.*

From it we may define the *Fourier transform* of $f(t)$ by

$$\widetilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt, \tag{13.5}$$

and its inverse by

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}(\omega)\, e^{i\omega t}\, d\omega. \tag{13.6}$$

Including the constant $1/\sqrt{2\pi}$ in the definition of $\widetilde{f}(\omega)$ (whose mathematical existence as $T \to \infty$ is assumed here without proof) is clearly arbitrary, the only requirement being that the product of the constants in (13.5) and (13.6) should equal $1/(2\pi)$. Our definition is chosen to be as symmetric as possible.

► *Find the Fourier transform of the exponential decay function $f(t) = 0$ for $t < 0$ and $f(t) = A\,e^{-\lambda t}$ for $t \geq 0$ ($\lambda > 0$).*

Using the definition (13.5) and separating the integral into two parts,

$$\begin{aligned}
\widetilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} (0)\, e^{-i\omega t}\, dt + \frac{A}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\lambda t}\, e^{-i\omega t}\, dt \\
&= 0 + \frac{A}{\sqrt{2\pi}}\left[-\frac{e^{-(\lambda+i\omega)t}}{\lambda + i\omega}\right]_0^{\infty} \\
&= \frac{A}{\sqrt{2\pi}(\lambda + i\omega)},
\end{aligned}$$

which is the required transform. It is clear that the multiplicative constant A does not affect the form of the transform, merely its amplitude. This transform may be verified by resubstitution of the above result into (13.6) to recover $f(t)$, but evaluation of the integral requires the use of complex-variable contour integration (chapter 24). ◄

13.1.1 The uncertainty principle

An important function that appears in many areas of physical science, either precisely or as an approximation to a physical situation, is the *Gaussian* or *normal* distribution. Its Fourier transform is of importance both in itself and also because, when interpreted statistically, it readily illustrates a form of *uncertainty principle*.

►*Find the Fourier transform of the normalised Gaussian distribution*

$$f(t) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right), \qquad -\infty < t < \infty.$$

This Gaussian distribution is centred on $t = 0$ and has a root mean square deviation $\Delta t = \tau$. (Any reader who is unfamiliar with this interpretation of the distribution should refer to chapter 30.)

Using the definition (13.5), the Fourier transform of $f(t)$ is given by

$$\begin{aligned}\widetilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right) \exp(-i\omega t)\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} \exp\left\{-\frac{1}{2\tau^2}\left[t^2 + 2\tau^2 i\omega t + (\tau^2 i\omega)^2 - (\tau^2 i\omega)^2\right]\right\} dt,\end{aligned}$$

where the quantity $-(\tau^2 i\omega)^2/(2\tau^2)$ has been both added and subtracted in the exponent in order to allow the factors involving the variable of integration t to be expressed as a complete square. Hence the expression can be written

$$\widetilde{f}(\omega) = \frac{\exp(-\frac{1}{2}\tau^2\omega^2)}{\sqrt{2\pi}} \left\{\frac{1}{\tau\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(t + i\tau^2\omega)^2}{2\tau^2}\right] dt\right\}.$$

The quantity inside the braces is the normalisation integral for the Gaussian and equals unity, although to show this strictly needs results from complex variable theory (chapter 24). That it is equal to unity can be made plausible by changing the variable to $s = t + i\tau^2\omega$ and assuming that the imaginary parts introduced into the integration path and limits (where the integrand goes rapidly to zero anyway) make no difference.

We are left with the result that

$$\widetilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\tau^2\omega^2}{2}\right), \tag{13.7}$$

which is another Gaussian distribution, centred on zero and with a root mean square deviation $\Delta\omega = 1/\tau$. It is interesting to note, and an important property, that the Fourier transform of a Gaussian is another Gaussian. ◄

In the above example the root mean square deviation in t was τ, and so it is seen that the deviations or 'spreads' in t and in ω are inversely related:

$$\Delta\omega\, \Delta t = 1,$$

independently of the value of τ. In physical terms, the narrower in time is, say, an electrical impulse the greater the spread of frequency components it must contain. Similar physical statements are valid for other pairs of Fourier-related variables, such as spatial position and wave number. In an obvious notation, $\Delta k \Delta x = 1$ for a Gaussian wave packet.

The uncertainty relations as usually expressed in quantum mechanics can be related to this if the de Broglie and Einstein relationships for momentum and energy are introduced; they are

$$p = \hbar k \qquad \text{and} \qquad E = \hbar\omega.$$

Here $\hbar$ is Planck's constant h divided by 2π. In a quantum mechanics setting $f(t)$

is a wavefunction and the distribution of the wave intensity in time is given by $|f|^2$ (also a Gaussian). Similarly, the intensity distribution in frequency is given by $|\widetilde{f}|^2$. These two distributions have respective root mean square deviations of $\tau/\sqrt{2}$ and $1/(\sqrt{2}\tau)$, giving, after incorporation of the above relations,

$$\Delta E\,\Delta t = \hbar/2 \qquad \text{and} \qquad \Delta p\,\Delta x = \hbar/2.$$

The factors of $1/2$ that appear are specific to the Gaussian form, but any distribution $f(t)$ produces for the product $\Delta E \Delta t$ a quantity $\lambda\hbar$ in which λ is strictly positive (in fact, the Gaussian value of $1/2$ is the minimum possible).

13.1.2 Fraunhofer diffraction

We take our final example of the Fourier transform from the field of optics. The pattern of transmitted light produced by a partially opaque (or phase-changing) object upon which a coherent beam of radiation falls is called a *diffraction pattern* and, in particular, when the cross-section of the object is small compared with the distance at which the light is observed the pattern is known as a *Fraunhofer* diffraction pattern.

We will consider only the case in which the light is monochromatic with wavelength λ. The direction of the incident beam of light can then be described by the *wave vector* $\mathbf{k}$; the magnitude of this vector is given by the *wave number* $k = 2\pi/\lambda$ of the light. The essential quantity in a Fraunhofer diffraction pattern is the dependence of the observed amplitude (and hence intensity) on the angle θ between the viewing direction $\mathbf{k}'$ and the direction $\mathbf{k}$ of the incident beam. This is entirely determined by the spatial distribution of the amplitude and phase of the light at the object, the transmitted intensity in a particular direction $\mathbf{k}'$ being determined by the corresponding Fourier component of this spatial distribution.

As an example, we take as an object a simple two-dimensional screen of width $2Y$ on which light of wave number k is incident normally; see figure 13.2. We suppose that at the position $(0, y)$ the amplitude of the transmitted light is $f(y)$ per unit length in the y-direction ($f(y)$ may be complex). The function $f(y)$ is called an *aperture function*. Both the screen and beam are assumed infinite in the z-direction.

Denoting the unit vectors in the x- and y- directions by $\mathbf{i}$ and $\mathbf{j}$ respectively, the total light amplitude at a position $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j}$, with $x_0 > 0$, will be the superposition of all the (Huyghens') wavelets originating from the various parts of the screen. For large r_0 $(= |\mathbf{r}_0|)$, these can be treated as plane waves to give[§]

$$A(\mathbf{r}_0) = \int_{-Y}^{Y} \frac{f(y)\exp[i\mathbf{k}' \cdot (\mathbf{r}_0 - y\mathbf{j})]}{|\mathbf{r}_0 - y\mathbf{j}|}\,dy. \tag{13.8}$$

[§] This is the approach first used by Fresnel. For simplicity we have omitted from the integral a multiplicative inclination factor that depends on angle θ and decreases as θ increases.

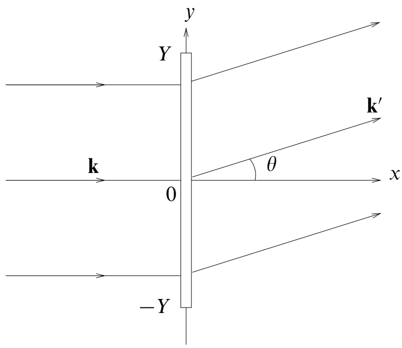


Figure 13.2 Diffraction grating of width $2Y$ with light of wavelength $2\pi/k$ being diffracted through an angle θ.

The factor $\exp[i\mathbf{k}' \cdot (\mathbf{r}_0 - y\mathbf{j})]$ represents the phase change undergone by the light in travelling from the point $y\mathbf{j}$ on the screen to the point $\mathbf{r}_0$, and the denominator represents the reduction in amplitude with distance. (Recall that the system is infinite in the z-direction and so the 'spreading' is effectively in two dimensions only.)

If the medium is the same on both sides of the screen then $\mathbf{k}' = k\cos\theta\,\mathbf{i}+k\sin\theta\,\mathbf{j}$, and if $r_0 \gg Y$ then expression (13.8) can be approximated by

$$A(\mathbf{r}_0) = \frac{\exp(i\mathbf{k}' \cdot \mathbf{r}_0)}{r_0} \int_{-\infty}^{\infty} f(y) \exp(-iky\sin\theta)\, dy. \tag{13.9}$$

We have used that $f(y) = 0$ for $|y| > Y$ to extend the integral to infinite limits. The intensity in the direction θ is then given by

$$I(\theta) = |A|^2 = \frac{2\pi}{{r_0}^2} |\widetilde{f}(q)|^2, \tag{13.10}$$

where $q = k\sin\theta$.

►*Evaluate $I(\theta)$ for an aperture consisting of two long slits each of width $2b$ whose centres are separated by a distance $2a$, $a > b$; the slits are illuminated by light of wavelength λ.*

The aperture function is plotted in figure 13.3. We first need to find $\widetilde{f}(q)$:

$$\begin{aligned}
\widetilde{f}(q) &= \frac{1}{\sqrt{2\pi}} \int_{-a-b}^{-a+b} e^{-iqx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{a-b}^{a+b} e^{-iqx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-iqx}}{iq} \right]_{-a-b}^{-a+b} + \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-iqx}}{iq} \right]_{a-b}^{a+b} \\
&= \frac{-1}{iq\sqrt{2\pi}} \left[e^{-iq(-a+b)} - e^{-iq(-a-b)} + e^{-iq(a+b)} - e^{-iq(a-b)} \right].
\end{aligned}$$

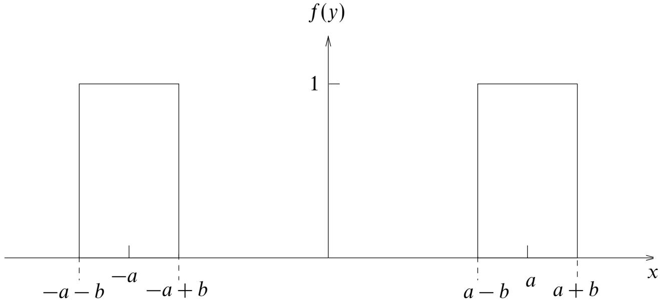


Figure 13.3 The aperture function $f(y)$ for two wide slits.

After some manipulation we obtain

$$\tilde{f}(q) = \frac{4\cos qa \sin qb}{q\sqrt{2\pi}}.$$

Now applying (13.10), and remembering that $q = (2\pi \sin\theta)/\lambda$, we find

$$I(\theta) = \frac{16\cos^2 qa \sin^2 qb}{q^2 {r_0}^2},$$

where r_0 is the distance from the centre of the aperture. ◄

13.1.3 The Dirac δ-function

Before going on to consider further properties of Fourier transforms we make a digression to discuss the Dirac δ-function and its relation to Fourier transforms. The δ-function is different from most functions encountered in the physical sciences but we will see that a rigorous mathematical definition exists; the utility of the δ-function will be demonstrated throughout the remainder of this chapter. It can be visualised as a very sharp narrow pulse (in space, time, density, etc.) which produces an integrated effect having a definite magnitude. The formal properties of the δ-function may be summarised as follows.

The Dirac δ-function has the property that

$$\delta(t) = 0 \qquad \text{for } t \neq 0, \tag{13.11}$$

but its fundamental defining property is

$$\int f(t)\delta(t-a)\,dt = f(a), \tag{13.12}$$

provided the range of integration includes the point $t = a$; otherwise the integral equals zero. This leads immediately to two further useful results:

$$\int_{-a}^{b} \delta(t)\,dt = 1 \qquad \text{for all } a, b > 0 \tag{13.13}$$

and

$$\int \delta(t-a)\,dt = 1, \tag{13.14}$$

provided the range of integration includes $t = a$.

Equation (13.12) can be used to derive further useful properties of the Dirac δ-function:

$$\delta(t) = \delta(-t), \tag{13.15}$$

$$\delta(at) = \frac{1}{|a|}\delta(t), \tag{13.16}$$

$$t\delta(t) = 0. \tag{13.17}$$

►*Prove that* $\delta(bt) = \delta(t)/|b|$.

Let us first consider the case where $b > 0$. It follows that

$$\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt = \int_{-\infty}^{\infty} f\left(\frac{t'}{b}\right)\delta(t')\frac{dt'}{b} = \frac{1}{b}f(0) = \frac{1}{b}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,$$

where we have made the substitution $t' = bt$. But $f(t)$ is arbitrary and so we immediately see that $\delta(bt) = \delta(t)/b = \delta(t)/|b|$ for $b > 0$.

Now consider the case where $b = -c < 0$. It follows that

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt &= \int_{\infty}^{-\infty} f\left(\frac{t'}{-c}\right)\delta(t')\left(\frac{dt'}{-c}\right) = \int_{-\infty}^{\infty} \frac{1}{c}f\left(\frac{t'}{-c}\right)\delta(t')\,dt' \\ &= \frac{1}{c}f(0) = \frac{1}{|b|}f(0) = \frac{1}{|b|}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,\end{aligned}$$

where we have made the substitution $t' = bt = -ct$. But $f(t)$ is arbitrary and so

$$\delta(bt) = \frac{1}{|b|}\delta(t),$$

for all b, which establishes the result. ◄

Furthermore, by considering an integral of the form

$$\int f(t)\delta(h(t))\,dt,$$

and making a change of variables to $z = h(t)$, we may show that

$$\delta(h(t)) = \sum_i \frac{\delta(t - t_i)}{|h'(t_i)|}, \tag{13.18}$$

where the t_i are those values of t for which $h(t) = 0$ and $h'(t)$ stands for dh/dt.

The derivative of the delta function, $\delta'(t)$, is defined by

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta'(t)\,dt &= \Big[f(t)\delta(t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\delta(t)\,dt \\ &= -f'(0), \end{aligned} \tag{13.19}$$

and similarly for higher derivatives.

For many practical purposes, effects that are not strictly described by a δ-function may be analysed as such, if they take place in an interval much shorter than the response interval of the system on which they act. For example, the idealised notion of an impulse of magnitude J applied at time t_0 can be represented by

$$j(t) = J\delta(t - t_0). \tag{13.20}$$

Many physical situations are described by a δ-function in space rather than in time. Moreover, we often require the δ-function to be defined in more than one dimension. For example, the charge density of a point charge q at a point $\mathbf{r}_0$ may be expressed as a three-dimensional δ-function

$$\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r_0}) = q\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \tag{13.21}$$

so that a discrete 'quantum' is expressed as if it were a continuous distribution. From (13.21) we see that (as expected) the total charge enclosed in a volume V is given by

$$\int_V \rho(\mathbf{r})\,dV = \int_V q\delta(\mathbf{r} - \mathbf{r}_0)\,dV = \begin{cases} q & \text{if } \mathbf{r}_0 \text{ lies in } V, \\ 0 & \text{otherwise.} \end{cases}$$

Closely related to the Dirac δ-function is the *Heaviside* or *unit step function* $H(t)$, for which

$$H(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{13.22}$$

This function is clearly discontinuous at $t = 0$ and it is usual to take $H(0) = 1/2$. The Heaviside function is related to the delta function by

$$H'(t) = \delta(t). \tag{13.23}$$

►*Prove relation (13.23).*

Considering the integral

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)H'(t)\,dt &= \Big[f(t)H(t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)H(t)\,dt \\ &= f(\infty) - \int_{0}^{\infty} f'(t)\,dt \\ &= f(\infty) - \Big[f(t)\Big]_{0}^{\infty} = f(0),\end{aligned}$$

and comparing it with (13.12) when $a = 0$ immediately shows that $H'(t) = \delta(t)$. ◄

13.1.4 Relation of the δ-function to Fourier transforms

In the previous section we introduced the Dirac δ-function as a way of representing very sharp narrow pulses, but in no way related it to Fourier transforms. We now show that the δ-function can equally well be defined in a way that more naturally relates it to the Fourier transform.

Referring back to the Fourier inversion theorem (13.4), we have

$$\begin{aligned}f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u} \\ &= \int_{-\infty}^{\infty} du\; f(u)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega\right\}.\end{aligned}$$

Comparison of this with (13.12) shows that we may write the δ-function as

$$\delta(t-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega. \tag{13.24}$$

Considered as a Fourier transform, this representation shows that a very narrow time peak at $t = u$ results from the superposition of a complete spectrum of harmonic waves, all frequencies having the same amplitude and all waves being in phase at $t = u$. This suggests that the δ-function may also be represented as the limit of the transform of a uniform distribution of unit height as the width of this distribution becomes infinite.

Consider the rectangular distribution of frequencies shown in figure 13.4(a). From (13.6), taking the inverse Fourier transform,

$$\begin{aligned}f_\Omega(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\Omega}^{\Omega} 1 \times e^{i\omega t}\, d\omega \\ &= \frac{2\Omega}{\sqrt{2\pi}}\frac{\sin\Omega t}{\Omega t}.\end{aligned} \tag{13.25}$$

This function is illustrated in figure 13.4(b) and it is apparent that, for large Ω, it becomes very large at $t = 0$ and also very narrow about $t = 0$, as we qualitatively

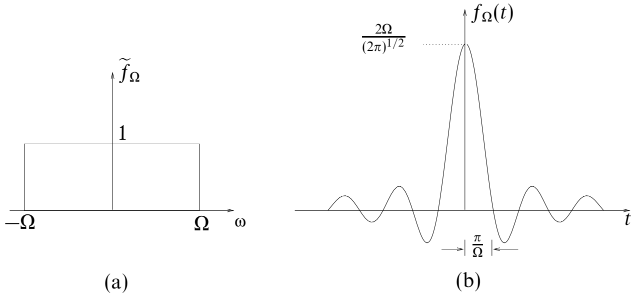


Figure 13.4 (a) A Fourier transform showing a rectangular distribution of frequencies between $\pm\Omega$; (b) the function of which it is the transform, which is proportional to $t^{-1}\sin\Omega t$.

expect and require. We also note that, in the limit $\Omega \to \infty$, $f_\Omega(t)$, as defined by the inverse Fourier transform, tends to $(2\pi)^{1/2}\delta(t)$ by virtue of (13.24). Hence we may conclude that the δ-function can also be represented by

$$\delta(t) = \lim_{\Omega\to\infty}\left(\frac{\sin\Omega t}{\pi t}\right). \tag{13.26}$$

Several other function representations are equally valid, e.g. the limiting cases of rectangular, triangular or Gaussian distributions; the only essential requirements are a knowledge of the area under such a curve and that undefined operations such as dividing by zero are not inadvertently carried out on the δ-function whilst some non-explicit representation is being employed.

We also note that the Fourier transform definition of the delta function, (13.24), shows that the latter is real since

$$\delta^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega = \delta(-t) = \delta(t).$$

Finally, the Fourier transform of a δ-function is simply

$$\widetilde{\delta}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t)\,e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}. \tag{13.27}$$

13.1.5 Properties of Fourier transforms

Having considered the Dirac δ-function, we now return to our discussion of the properties of Fourier transforms. As we would expect, Fourier transforms have many properties analogous to those of Fourier series in respect of the connection between the transforms of related functions. Here we list these properties without proof; they can be verified by working from the definition of the transform. As previously, we denote the Fourier transform of $f(t)$ by $\widetilde{f}(\omega)$ or $\mathscr{F}[f(t)]$.

(i) Differentiation:

$$\mathscr{F}\left[f'(t)\right] = i\omega\widetilde{f}(\omega). \tag{13.28}$$

This may be extended to higher derivatives, so that

$$\mathscr{F}\left[f''(t)\right] = i\omega\mathscr{F}\left[f'(t)\right] = -\omega^2\widetilde{f}(\omega),$$

and so on.

(ii) Integration:

$$\mathscr{F}\left[\int^t f(s)\,ds\right] = \frac{1}{i\omega}\widetilde{f}(\omega) + 2\pi c\delta(\omega), \tag{13.29}$$

where the term $2\pi c\delta(\omega)$ represents the Fourier transform of the constant of integration associated with the indefinite integral.

(iii) Scaling:

$$\mathscr{F}[f(at)] = \frac{1}{a}\widetilde{f}\left(\frac{\omega}{a}\right). \tag{13.30}$$

(iv) Translation:

$$\mathscr{F}[f(t+a)] = e^{ia\omega}\widetilde{f}(\omega). \tag{13.31}$$

(v) Exponential multiplication:

$$\mathscr{F}\left[e^{\alpha t}f(t)\right] = \widetilde{f}(\omega + i\alpha), \tag{13.32}$$

where α may be real, imaginary or complex.

►*Prove relation (13.28).*

Calculating the Fourier transform of $f'(t)$ directly, we obtain

$$\begin{aligned}\mathscr{F}\left[f'(t)\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)\,e^{-i\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega t}f(t)\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\omega\,e^{-i\omega t}f(t)\,dt \\ &= i\omega\widetilde{f}(\omega),\end{aligned}$$

if $f(t) \to 0$ at $t = \pm\infty$, as it must since $\int_{-\infty}^{\infty}|f(t)|\,dt$ is finite. ◄

To illustrate a use and also a proof of (13.32), let us consider an amplitude-modulated radio wave. Suppose a message to be broadcast is represented by $f(t)$. The message can be added electronically to a constant signal a of magnitude such that $a + f(t)$ is never negative, and then the sum can be used to modulate the amplitude of a carrier signal of frequency ω_c. Using a complex exponential notation, the transmitted amplitude is now

$$g(t) = A\,[a + f(t)]\,e^{i\omega_c t}. \tag{13.33}$$

Ignoring in the present context the effect of the term $Aa\exp(i\omega_c t)$, which gives a contribution to the transmitted spectrum only at $\omega = \omega_c$, we obtain for the new spectrum

$$\begin{aligned}\widetilde{g}(\omega) &= \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} f(t)\, e^{i\omega_c t}\, e^{-i\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} f(t)\, e^{-i(\omega-\omega_c)t}\, dt \\ &= A\widetilde{f}(\omega-\omega_c), \qquad (13.34)\end{aligned}$$

which is simply a shift of the whole spectrum by the carrier frequency. The use of different carrier frequencies enables signals to be separated.

13.1.6 Odd and even functions

If $f(t)$ is odd or even then we may derive alternative forms of Fourier's inversion theorem, which lead to the definition of different transform pairs. Let us first consider an odd function $f(t) = -f(-t)$, whose Fourier transform is given by

$$\begin{aligned}\widetilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)(\cos\omega t - i\sin\omega t)\, dt \\ &= \frac{-2i}{\sqrt{2\pi}} \int_{0}^{\infty} f(t)\sin\omega t\, dt,\end{aligned}$$

where in the last line we use the fact that $f(t)$ and $\sin\omega t$ are odd, whereas $\cos\omega t$ is even.

We note that $\widetilde{f}(-\omega) = -\widetilde{f}(\omega)$, i.e. $\widetilde{f}(\omega)$ is an odd function of ω. Hence

$$\begin{aligned}f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f}(\omega)\, e^{i\omega t}\, d\omega = \frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} \widetilde{f}(\omega)\sin\omega t\, d\omega \\ &= \frac{2}{\pi} \int_{0}^{\infty} d\omega\, \sin\omega t \left\{ \int_{0}^{\infty} f(u)\sin\omega u\, du \right\}.\end{aligned}$$

Thus we may define the *Fourier sine transform pair* for odd functions:

$$\widetilde{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t)\ \sin\omega t\, dt, \qquad (13.35)$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \widetilde{f}_s(\omega)\ \sin\omega t\, d\omega. \qquad (13.36)$$

Note that although the Fourier sine transform pair was derived by considering an odd function $f(t)$ defined over all t, the definitions (13.35) and (13.36) only require $f(t)$ and $\widetilde{f}_s(\omega)$ to be defined for positive t and ω respectively. For an

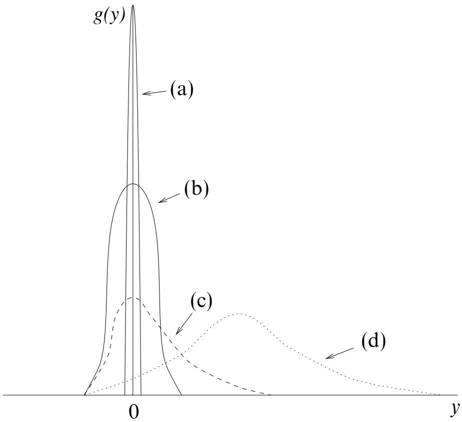


Figure 13.5 Resolution functions: (a) ideal δ-function; (b) typical unbiased resolution; (c) and (d) biases tending to shift observations to higher values than the true one.

even function, i.e. one for which $f(t) = f(-t)$, we can define the *Fourier cosine transform pair* in a similar way, but with $\sin \omega t$ replaced by $\cos \omega t$.

13.1.7 Convolution and deconvolution

It is apparent that any attempt to measure the value of a physical quantity is limited, to some extent, by the finite resolution of the measuring apparatus used. On the one hand, the physical quantity we wish to measure will be in general a function of an independent variable, x say, i.e. the true function to be measured takes the form $f(x)$. On the other hand, the apparatus we are using does not give the true output value of the function; a resolution function $g(y)$ is involved. By this we mean that the probability that an output value $y = 0$ will be recorded instead as being between y and $y+dy$ is given by $g(y)\,dy$. Some possible resolution functions of this sort are shown in figure 13.5. To obtain good results we wish the resolution function to be as close to a δ-function as possible (case (a)). A typical piece of apparatus has a resolution function of finite width, although if it is accurate the mean is centred on the true value (case (b)). However, some apparatus may show a bias that tends to shift observations to higher or lower values than the true ones (cases (c) and (d)), thereby exhibiting systematic error.

Given that the true distribution is $f(x)$ and the resolution function of our measuring apparatus is $g(y)$, we wish to calculate what the observed distribution $h(z)$ will be. The symbols x, y and z all refer to the same physical variable (e.g.

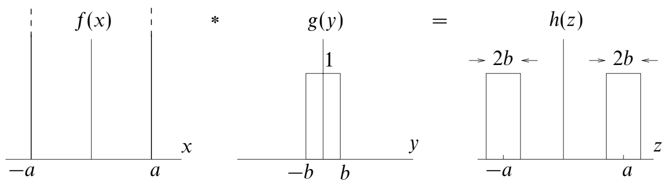


Figure 13.6 The convolution of two functions $f(x)$ and $g(y)$.

length or angle), but are denoted differently because the variable appears in the analysis in three different roles.

The probability that a true reading lying between x and $x + dx$, and so having probability $f(x)\,dx$ of being selected by the experiment, will be moved by the instrumental resolution by an amount $z - x$ into a small interval of width dz is $g(z - x)\,dz$. Hence the combined probability that the interval dx will give rise to an observation appearing in the interval dz is $f(x)\,dx\,g(z - x)\,dz$. Adding together the contributions from all values of x that can lead to an observation in the range z to $z + dz$, we find that the observed distribution is given by

$$h(z) = \int_{-\infty}^{\infty} f(x)g(z - x)\,dx. \tag{13.37}$$

The integral in (13.37) is called the *convolution* of the functions f and g and is often written $f * g$. The convolution defined above is commutative ($f * g = g * f$), associative and distributive. The observed distribution is thus the convolution of the true distribution and the experimental resolution function. The result will be that the observed distribution is broader and smoother than the true one and, if $g(y)$ has a bias, the maxima will normally be displaced from their true positions. It is also obvious from (13.37) that if the resolution is the ideal δ-function, $g(y) = \delta(y)$ then $h(z) = f(z)$ and the observed distribution is the true one.

It is interesting to note, and a very important property, that the convolution of any function $g(y)$ with a number of delta functions leaves a copy of $g(y)$ at the position of each of the delta functions.

►*Find the convolution of the function $f(x) = \delta(x + a) + \delta(x - a)$ with the function $g(y)$ plotted in figure 13.6.*

Using the convolution integral (13.37)

$$h(z) = \int_{-\infty}^{\infty} f(x)g(z - x)\,dx = \int_{-\infty}^{\infty} [\delta(x + a) + \delta(x - a)]g(z - x)\,dx$$
$$= g(z + a) + g(z - a).$$

This convolution $h(z)$ is plotted in figure 13.6. ◄

Let us now consider the Fourier transform of the convolution (13.37); this is given by

$$\begin{aligned}\widetilde{h}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-ikz} \left\{ \int_{-\infty}^{\infty} f(x)g(z-x)\, dx \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) \left\{ \int_{-\infty}^{\infty} g(z-x)\, e^{-ikz}\, dz \right\}.\end{aligned}$$

If we let $u = z - x$ in the second integral we have

$$\begin{aligned}\widetilde{h}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) \left\{ \int_{-\infty}^{\infty} g(u)\, e^{-ik(u+x)}\, du \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \int_{-\infty}^{\infty} g(u)\, e^{-iku}\, du \\ &= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi}\, \widetilde{f}(k) \times \sqrt{2\pi}\widetilde{g}(k) = \sqrt{2\pi}\, \widetilde{f}(k)\widetilde{g}(k). \qquad (13.38)\end{aligned}$$

Hence the Fourier transform of a convolution $f * g$ is equal to the product of the separate Fourier transforms multiplied by $\sqrt{2\pi}$; this result is called the *convolution theorem.*

It may be proved similarly that the converse is also true, namely that the Fourier transform of the product $f(x)g(x)$ is given by

$$\mathscr{F}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}} \widetilde{f}(k) * \widetilde{g}(k). \qquad (13.39)$$

►*Find the Fourier transform of the function in figure 13.3 representing two wide slits by considering the Fourier transforms of* (i) *two* δ*-functions, at* $x = \pm a$*,* (ii) *a rectangular function of height* 1 *and width* $2b$ *centred on* $x = 0$.

(i) The Fourier transform of the two δ-functions is given by

$$\begin{aligned}\widetilde{f}(q) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-a)\, e^{-iqx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x+a)\, e^{-iqx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left(e^{-iqa} + e^{iqa} \right) = \frac{2\cos qa}{\sqrt{2\pi}}.\end{aligned}$$

(ii) The Fourier transform of the broad slit is

$$\begin{aligned}\widetilde{g}(q) &= \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} e^{-iqx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iqx}}{-iq} \right]_{-b}^{b} \\ &= \frac{-1}{iq\sqrt{2\pi}} (e^{-iqb} - e^{iqb}) = \frac{2\sin qb}{q\sqrt{2\pi}}.\end{aligned}$$

We have already seen that the convolution of these functions is the required function representing two wide slits (see figure 13.6). So, using the convolution theorem, the Fourier transform of the convolution is $\sqrt{2\pi}$ times the product of the individual transforms, i.e. $4\cos qa\, \sin qb/(q\sqrt{2\pi})$. This is, of course, the same result as that obtained in the example in subsection 13.1.2. ◄

The inverse of convolution, called *deconvolution*, allows us to find a true distribution $f(x)$ given an observed distribution $h(z)$ and a resolution function $g(y)$.

►*An experimental quantity $f(x)$ is measured using apparatus with a known resolution function $g(y)$ to give an observed distribution $h(z)$. How may $f(x)$ be extracted from the measured distribution?*

From the convolution theorem (13.38), the Fourier transform of the measured distribution is

$$\widetilde{h}(k) = \sqrt{2\pi}\,\widetilde{f}(k)\widetilde{g}(k),$$

from which we obtain

$$\widetilde{f}(k) = \frac{1}{\sqrt{2\pi}}\frac{\widetilde{h}(k)}{\widetilde{g}(k)}.$$

Then on inverse Fourier transforming we find

$$f(x) = \frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}\left[\frac{\widetilde{h}(k)}{\widetilde{g}(k)}\right].$$

In words, to extract the true distribution, we divide the Fourier transform of the observed distribution by that of the resolution function for each value of k and then take the inverse Fourier transform of the function so generated. ◄

This explicit method of extracting true distributions is straightforward for exact functions but, in practice, because of experimental and statistical uncertainties in the experimental data or because data over only a limited range are available, it is often not very precise, involving as it does three (numerical) transforms each requiring in principle an integral over an infinite range.

13.1.8 Correlation functions and energy spectra

The *cross-correlation* of two functions f and g is defined by

$$C(z) = \int_{-\infty}^{\infty} f^*(x)g(x+z)\,dx. \tag{13.40}$$

Despite the formal similarity between (13.40) and the definition of the convolution in (13.37), the use and interpretation of the cross-correlation and of the convolution are very different; the cross-correlation provides a quantitative measure of the similarity of two functions f and g as one is displaced through a distance z relative to the other. The cross-correlation is often notated as $C = f \otimes g$, and, like convolution, it is both associative and distributive. Unlike convolution, however, it is *not* commutative, in fact

$$[f \otimes g](z) = [g \otimes f]^*(-z). \tag{13.41}$$

►*Prove the Wiener–Kinchin theorem,*

$$\widetilde{C}(k) = \sqrt{2\pi}\,[\widetilde{f}(k)]^*\widetilde{g}(k). \tag{13.42}$$

Following a method similar to that for the convolution of f and g, let us consider the Fourier transform of (13.40):

$$\begin{aligned}\widetilde{C}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, e^{-ikz}\left\{\int_{-\infty}^{\infty} f^*(x)g(x+z)\,dx\right\}\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f^*(x)\left\{\int_{-\infty}^{\infty} g(x+z)\,e^{-ikz}\,dz\right\}.\end{aligned}$$

Making the substitution $u = x + z$ in the second integral we obtain

$$\begin{aligned}\widetilde{C}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f^*(x)\left\{\int_{-\infty}^{\infty} g(u)\,e^{-ik(u-x)}\,du\right\}\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^*(x)\,e^{ikx}\,dx\int_{-\infty}^{\infty} g(u)\,e^{-iku}\,du\\ &= \frac{1}{\sqrt{2\pi}}\times\sqrt{2\pi}\,[\widetilde{f}(k)]^*\times\sqrt{2\pi}\,\widetilde{g}(k) = \sqrt{2\pi}\,[\widetilde{f}(k)]^*\widetilde{g}(k).\ ◄\end{aligned}$$

Thus the Fourier transform of the cross-correlation of f and g is equal to the product of $[\widetilde{f}(k)]^*$ and $\widetilde{g}(k)$ multiplied by $\sqrt{2\pi}$. This a statement of the *Wiener–Kinchin theorem.* Similarly we can derive the converse theorem

$$\mathscr{F}\left[f^*(x)g(x)\right] = \frac{1}{\sqrt{2\pi}}\widetilde{f}\otimes\widetilde{g}.$$

If we now consider the special case where g is taken to be equal to f in (13.40) then, writing the LHS as $a(z)$, we have

$$a(z) = \int_{-\infty}^{\infty} f^*(x)f(x+z)\,dx; \tag{13.43}$$

this is called the *auto-correlation function* of $f(x)$. Using the Wiener–Kinchin theorem (13.42) we see that

$$\begin{aligned}a(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{a}(k)\,e^{ikz}\,dk\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{2\pi}\,[\widetilde{f}(k)]^*\widetilde{f}(k)\,e^{ikz}\,dk,\end{aligned}$$

so that $a(z)$ is the inverse Fourier transform of $\sqrt{2\pi}\,|\widetilde{f}(k)|^2$, which is in turn called the *energy spectrum* of f.

13.1.9 Parseval's theorem

Using the results of the previous section we can immediately obtain *Parseval's theorem.* The most general form of this (also called the *multiplication theorem*) is obtained simply by noting from (13.42) that the cross-correlation (13.40) of two functions f and g can be written as

$$C(z) = \int_{-\infty}^{\infty} f^*(x)g(x+z)\,dx = \int_{-\infty}^{\infty} [\widetilde{f}(k)]^*\widetilde{g}(k)\,e^{ikz}\,dk. \tag{13.44}$$

Then, setting $z = 0$ gives the multiplication theorem

$$\int_{-\infty}^{\infty} f^*(x)g(x)\,dx = \int [\widetilde{f}(k)]^*\widetilde{g}(k)\,dk. \tag{13.45}$$

Specialising further, by letting $g = f$, we derive the most common form of Parseval's theorem,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\widetilde{f}(k)|^2\,dk. \tag{13.46}$$

When f is a physical amplitude these integrals relate to the total intensity involved in some physical process. We have already met a form of Parseval's theorem for Fourier series in chapter 12; it is in fact a special case of (13.46).

►*The displacement of a damped harmonic oscillator as a function of time is given by*

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ e^{-t/\tau}\sin\omega_0 t & \text{for } t \geq 0. \end{cases}$$

Find the Fourier transform of this function and so give a physical interpretation of Parseval's theorem.

Using the usual definition for the Fourier transform we find

$$\widetilde{f}(\omega) = \int_{-\infty}^{0} 0 \times e^{-i\omega t}\,dt + \int_{0}^{\infty} e^{-t/\tau}\sin\omega_0 t\, e^{-i\omega t}\,dt.$$

Writing $\sin\omega_0 t$ as $(e^{i\omega_0 t} - e^{-i\omega_0 t})/2i$ we obtain

$$\begin{aligned}\widetilde{f}(\omega) &= 0 + \frac{1}{2i}\int_0^{\infty} \left[e^{-it(\omega-\omega_0-i/\tau)} - e^{-it(\omega+\omega_0-i/\tau)}\right] dt \\ &= \frac{1}{2}\left[\frac{1}{\omega+\omega_0-i/\tau} - \frac{1}{\omega-\omega_0-i/\tau}\right],\end{aligned}$$

which is the required Fourier transform. The physical interpretation of $|\widetilde{f}(\omega)|^2$ is the energy content per unit frequency interval (i.e. the *energy spectrum*) whilst $|f(t)|^2$ is proportional to the sum of the kinetic and potential energies of the oscillator. Hence (to within a constant) Parseval's theorem shows the equivalence of these two alternative specifications for the total energy. ◄

13.1.10 Fourier transforms in higher dimensions

The concept of the Fourier transform can be extended naturally to more than one dimension. For instance we may wish to find the spatial Fourier transform of two- or three-dimensional functions of position. For example, in three dimensions we can define the Fourier transform of $f(x,y,z)$ as

$$\widetilde{f}(k_x,k_y,k_z) = \frac{1}{(2\pi)^{3/2}} \iiint f(x,y,z)\, e^{-ik_xx} e^{-ik_yy} e^{-ik_zz}\, dx\, dy\, dz, \tag{13.47}$$

and its inverse as

$$f(x,y,z) = \frac{1}{(2\pi)^{3/2}} \iiint \widetilde{f}(k_x,k_y,k_z)\, e^{ik_xx} e^{ik_yy} e^{ik_zz}\, dk_x\, dk_y\, dk_z. \tag{13.48}$$

Denoting the vector with components k_x, k_y, k_z by $\mathbf{k}$ and that with components x, y, z by $\mathbf{r}$, we can write the Fourier transform pair (13.47), (13.48) as

$$\widetilde{f}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{r}, \tag{13.49}$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int \widetilde{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k}. \tag{13.50}$$

From these relations we may deduce that the three-dimensional Dirac δ-function can be written as

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k}. \tag{13.51}$$

Similar relations to (13.49), (13.50) and (13.51) exist for spaces of other dimensionalities.

►*In three-dimensional space a function $f(\mathbf{r})$ possesses spherical symmetry, so that $f(\mathbf{r}) = f(r)$. Find the Fourier transform of $f(\mathbf{r})$ as a one-dimensional integral.*

Let us choose spherical polar coordinates in which the vector $\mathbf{k}$ of the Fourier transform lies along the polar axis ($\theta = 0$). This we can do since $f(\mathbf{r})$ is spherically symmetric. We then have

$$d^3\mathbf{r} = r^2 \sin\theta\, dr\, d\theta\, d\phi \qquad \text{and} \qquad \mathbf{k}\cdot\mathbf{r} = kr\cos\theta,$$

where $k = |\mathbf{k}|$. The Fourier transform is then given by

$$\begin{aligned}
\widetilde{f}(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{r} \\
&= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, f(r) r^2 \sin\theta\, e^{-ikr\cos\theta} \\
&= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr\, 2\pi f(r) r^2 \int_0^\pi d\theta\, \sin\theta\, e^{-ikr\cos\theta}.
\end{aligned}$$

The integral over θ may be straightforwardly evaluated by noting that

$$\frac{d}{d\theta}(e^{-ikr\cos\theta}) = ikr\sin\theta\, e^{-ikr\cos\theta}.$$

Therefore

$$\begin{aligned}
\widetilde{f}(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr\, 2\pi f(r) r^2 \left[\frac{e^{-ikr\cos\theta}}{ikr}\right]_{\theta=0}^{\theta=\pi} \\
&= \frac{1}{(2\pi)^{3/2}} \int_0^\infty 4\pi r^2 f(r) \left(\frac{\sin kr}{kr}\right) dr. \text{ ◄}
\end{aligned}$$

A similar result may be obtained for two-dimensional Fourier transforms in which $f(\mathbf{r}) = f(\rho)$, i.e. $f(\mathbf{r})$ is independent of azimuthal angle ϕ. In this case, using the integral representation of the Bessel function $J_0(x)$ given at the very end of subsection 18.5.3, we find

$$\widetilde{f}(\mathbf{k}) = \frac{1}{2\pi} \int_0^\infty 2\pi\rho f(\rho) J_0(k\rho)\, d\rho. \qquad (13.52)$$

13.2 Laplace transforms

Often we are interested in functions $f(t)$ for which the Fourier transform does not exist because $f \not\to 0$ as $t \to \infty$, and so the integral defining $\widetilde{f}$ does not converge. This would be the case for the function $f(t) = t$, which does not possess a Fourier transform. Furthermore, we might be interested in a given function only for $t > 0$, for example when we are given the value at $t = 0$ in an initial-value problem. This leads us to consider the Laplace transform, $\bar{f}(s)$ or $\mathscr{L}[f(t)]$, of $f(t)$, which is defined by

$$\bar{f}(s) \equiv \int_0^\infty f(t) e^{-st}\, dt, \qquad (13.53)$$

provided that the integral exists. We assume here that s is real, but complex values would have to be considered in a more detailed study. In practice, for a given function $f(t)$ there will be some real number s_0 such that the integral in (13.53) exists for $s > s_0$ but diverges for $s \leq s_0$.

Through (13.53) we define a *linear* transformation $\mathscr{L}$ that converts functions of the variable t to functions of a new variable s:

$$\mathscr{L}[af_1(t) + bf_2(t)] = a\mathscr{L}[f_1(t)] + b\mathscr{L}[f_2(t)] = a\bar{f}_1(s) + b\bar{f}_2(s). \qquad (13.54)$$

►*Find the Laplace transforms of the functions* (i) $f(t) = 1$, (ii) $f(t) = e^{at}$, (iii) $f(t) = t^n$, *for* $n = 0, 1, 2, \ldots$.

(i) By direct application of the definition of a Laplace transform (13.53), we find

$$\mathscr{L}[1] = \int_0^\infty e^{-st}\, dt = \left[\frac{-1}{s} e^{-st}\right]_0^\infty = \frac{1}{s}, \qquad \text{if } s > 0,$$

where the restriction $s > 0$ is required for the integral to exist.

(ii) Again using (13.53) directly, we find

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt \\ &= \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \frac{1}{s-a} \qquad \text{if } s > a.\end{aligned}$$

(iii) Once again using the definition (13.53) we have

$$\bar{f}_n(s) = \int_0^\infty t^n e^{-st}\, dt.$$

Integrating by parts we find

$$\begin{aligned} \bar{f}_n(s) &= \left[\frac{-t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\, dt \\ &= 0 + \frac{n}{s}\bar{f}_{n-1}(s), \qquad \text{if } s > 0. \end{aligned}$$

We now have a recursion relation between successive transforms and by calculating $\bar{f}_0$ we can infer $\bar{f}_1$, $\bar{f}_2$, etc. Since $t^0 = 1$, (i) above gives

$$\bar{f}_0 = \frac{1}{s}, \qquad \text{if } s > 0, \tag{13.55}$$

and

$$\bar{f}_1(s) = \frac{1}{s^2}, \qquad \bar{f}_2(s) = \frac{2!}{s^3}, \qquad \ldots, \qquad \bar{f}_n(s) = \frac{n!}{s^{n+1}} \qquad \text{if } s > 0.$$

Thus, in each case (i)–(iii), direct application of the definition of the Laplace transform (13.53) yields the required result. ◄

Unlike that for the Fourier transform, the inversion of the Laplace transform is not an easy operation to perform, since an explicit formula for $f(t)$, given $\bar{f}(s)$, is not straightforwardly obtained from (13.53). The general method for obtaining an inverse Laplace transform makes use of complex variable theory and is not discussed until chapter 25. However, progress can be made without having to find an *explicit* inverse, since we can prepare from (13.53) a 'dictionary' of the Laplace transforms of common functions and, when faced with an inversion to carry out, hope to find the given transform (together with its parent function) in the listing. Such a list is given in table 13.1.

When finding inverse Laplace transforms using table 13.1, it is useful to note that for all practical purposes the inverse Laplace transform is unique[§] and linear so that

$$\mathcal{L}^{-1}\left[a\bar{f}_1(s) + b\bar{f}_2(s)\right] = af_1(t) + bf_2(t). \tag{13.56}$$

In many practical problems the method of partial fractions can be useful in producing an expression from which the inverse Laplace transform can be found.

►*Using table 13.1 find $f(t)$ if*

$$\bar{f}(s) = \frac{s+3}{s(s+1)}.$$

Using partial fractions $\bar{f}(s)$ may be written

$$\bar{f}(s) = \frac{3}{s} - \frac{2}{s+1}.$$

§ This is not strictly true, since two functions can differ from one another at a finite number of isolated points but have the *same* Laplace transform.

$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/s^{n+1}$	0
$\sin bt$	$b/(s^2+b^2)$	0
$\cos bt$	$s/(s^2+b^2)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/(s^2-a^2)$	$\|a\|$
$\cosh at$	$s/(s^2-a^2)$	$\|a\|$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$	a
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$	a
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$	0
$t^{-1/2}$	$(\pi/s)^{1/2}$	0
$\delta(t-t_0)$	e^{-st_0}	0
$H(t-t_0) = \begin{cases} 1 & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$	e^{-st_0}/s	0

Table 13.1 Standard Laplace transforms. The transforms are valid for $s > s_0$.

Comparing this with the standard Laplace transforms in table 13.1, we find that the inverse transform of $3/s$ is 3 for $s > 0$ and the inverse transform of $2/(s+1)$ is $2e^{-t}$ for $s > -1$, and so

$$f(t) = 3 - 2e^{-t}, \qquad \text{if } s > 0. \blacktriangleleft$$

13.2.1 Laplace transforms of derivatives and integrals

One of the main uses of Laplace transforms is in solving differential equations. Differential equations are the subject of the next six chapters and we will return to the application of Laplace transforms to their solution in chapter 15. In the meantime we will derive the required results, i.e. the Laplace transforms of derivatives.

The Laplace transform of the first derivative of $f(t)$ is given by

$$\begin{aligned} \mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt} e^{-st}\,dt \\ &= \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\ &= -f(0) + s\bar{f}(s), \qquad \text{for } s > 0. \end{aligned} \tag{13.57}$$

The evaluation relies on integration by parts and higher-order derivatives may be found in a similar manner.

►*Find the Laplace transform of d^2f/dt^2.*

Using the definition of the Laplace transform and integrating by parts we obtain

$$\begin{aligned}\mathscr{L}\left[\frac{d^2f}{dt^2}\right] &= \int_0^\infty \frac{d^2f}{dt^2}e^{-st}\,dt \\ &= \left[\frac{df}{dt}e^{-st}\right]_0^\infty + s\int_0^\infty \frac{df}{dt}e^{-st}\,dt \\ &= -\frac{df}{dt}(0) + s[s\bar{f}(s) - f(0)], \qquad \text{for } s > 0,\end{aligned}$$

where (13.57) has been substituted for the integral. This can be written more neatly as

$$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2\bar{f}(s) - sf(0) - \frac{df}{dt}(0), \qquad \text{for } s > 0. \blacktriangleleft$$

In general the Laplace transform of the nth derivative is given by

$$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^n\bar{f} - s^{n-1}f(0) - s^{n-2}\frac{df}{dt}(0) - \cdots - \frac{d^{n-1}f}{dt^{n-1}}(0), \qquad \text{for } s > 0. \tag{13.58}$$

We now turn to integration, which is much more straightforward. From the definition (13.53),

$$\begin{aligned}\mathscr{L}\left[\int_0^t f(u)\,du\right] &= \int_0^\infty dt\,e^{-st}\int_0^t f(u)\,du \\ &= \left[-\frac{1}{s}e^{-st}\int_0^t f(u)\,du\right]_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}f(t)\,dt.\end{aligned}$$

The first term on the RHS vanishes at both limits, and so

$$\mathscr{L}\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\mathscr{L}\left[f\right]. \tag{13.59}$$

13.2.2 Other properties of Laplace transforms

From table 13.1 it will be apparent that multiplying a function $f(t)$ by e^{at} has the effect on its transform that s is replaced by $s-a$. This is easily proved generally:

$$\begin{aligned}\mathscr{L}\left[e^{at}f(t)\right] &= \int_0^\infty f(t)e^{at}e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= \bar{f}(s-a).\end{aligned} \tag{13.60}$$

As it were, multiplying $f(t)$ by e^{at} moves the origin of s by an amount a.

We may now consider the effect of multiplying the Laplace transform $\bar{f}(s)$ by e^{-bs} $(b > 0)$. From the definition (13.53),

$$\begin{aligned} e^{-bs}\bar{f}(s) &= \int_0^\infty e^{-s(t+b)} f(t)\,dt \\ &= \int_0^\infty e^{-sz} f(z-b)\,dz, \end{aligned}$$

on putting $t + b = z$. Thus $e^{-bs}\bar{f}(s)$ is the Laplace transform of a function $g(t)$ defined by

$$g(t) = \begin{cases} 0 & \text{for } 0 < t \le b, \\ f(t-b) & \text{for } t > b. \end{cases}$$

In other words, the function f has been translated to 'later' t (larger values of t) by an amount b.

Further properties of Laplace transforms can be proved in similar ways and are listed below.

(i) $$\mathcal{L}[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right), \tag{13.61}$$

(ii) $$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}, \qquad \text{for } n = 1, 2, 3, \ldots, \tag{13.62}$$

(iii) $$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(u)\,du, \tag{13.63}$$

provided $\lim_{t\to 0}[f(t)/t]$ exists.

Related results may be easily proved.

► *Find an expression for the Laplace transform of $t\,d^2f/dt^2$.*

From the definition of the Laplace transform we have

$$\begin{aligned} \mathcal{L}\left[t\frac{d^2f}{dt^2}\right] &= \int_0^\infty e^{-st}\,t\frac{d^2f}{dt^2}\,dt \\ &= -\frac{d}{ds}\int_0^\infty e^{-st}\frac{d^2f}{dt^2}\,dt \\ &= -\frac{d}{ds}[s^2\bar{f}(s) - sf(0) - f'(0)] \\ &= -s^2\frac{d\bar{f}}{ds} - 2s\bar{f} + f(0). \end{aligned}$$ ◄

Finally we mention the convolution theorem for Laplace transforms (which is analogous to that for Fourier transforms discussed in subsection 13.1.7). If the functions f and g have Laplace transforms $\bar{f}(s)$ and $\bar{g}(s)$ then

$$\mathcal{L}\left[\int_0^t f(u)g(t-u)\,du\right] = \bar{f}(s)\bar{g}(s), \tag{13.64}$$

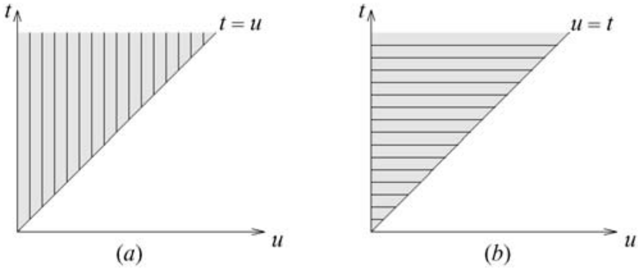


Figure 13.7 Two representations of the Laplace transform convolution (see text).

where the integral in the brackets on the LHS is the *convolution* of f and g, denoted by $f * g$. As in the case of Fourier transforms, the convolution defined above is commutative, i.e. $f * g = g * f$, and is associative and distributive. From (13.64) we also see that

$$\mathscr{L}^{-1}\left[\bar{f}(s)\bar{g}(s)\right] = \int_0^t f(u)g(t-u)\,du = f * g.$$

►*Prove the convolution theorem (13.64) for Laplace transforms.*

From the definition (13.64),

$$\begin{aligned}\bar{f}(s)\bar{g}(s) &= \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-sv} g(v)\,dv \\ &= \int_0^\infty du \int_0^\infty dv\, e^{-s(u+v)} f(u)g(v).\end{aligned}$$

Now letting $u + v = t$ changes the limits on the integrals, with the result that

$$\bar{f}(s)\bar{g}(s) = \int_0^\infty du\, f(u) \int_u^\infty dt\, g(t-u)\, e^{-st}.$$

As shown in figure 13.7(a) the shaded area of integration may be considered as the sum of vertical strips. However, we may instead integrate over this area by summing over horizontal strips as shown in figure 13.7(b). Then the integral can be written as

$$\begin{aligned}\bar{f}(s)\bar{g}(s) &= \int_0^t du\, f(u) \int_0^\infty dt\, g(t-u)\, e^{-st} \\ &= \int_0^\infty dt\, e^{-st} \left\{ \int_0^t f(u)g(t-u)\,du \right\} \\ &= \mathscr{L}\left[\int_0^t f(u)g(t-u)\,du \right]. \blacktriangleleft\end{aligned}$$

The properties of the Laplace transform derived in this section can sometimes be useful in finding the Laplace transforms of particular functions.

►*Find the Laplace transform of* $f(t) = t \sin bt$.

Although we could calculate the Laplace transform directly, we can use (13.62) to give

$$\bar{f}(s) = (-1)\frac{d}{ds}\mathscr{L}[\sin bt] = -\frac{d}{ds}\left(\frac{b}{s^2+b^2}\right) = \frac{2bs}{(s^2+b^2)^2}, \qquad \text{for } s > 0. \blacktriangleleft$$

13.3 Concluding remarks

In this chapter we have discussed Fourier and Laplace transforms in some detail. Both are examples of *integral transforms*, which can be considered in a more general context.

A general integral transform of a function $f(t)$ takes the form

$$F(\alpha) = \int_a^b K(\alpha, t) f(t)\, dt, \tag{13.65}$$

where $F(\alpha)$ is the transform of $f(t)$ with respect to the *kernel* $K(\alpha, t)$, and α is the transform variable. For example, in the Laplace transform case $K(s, t) = e^{-st}$, $a = 0$, $b = \infty$.

Very often the inverse transform can also be written straightforwardly and we obtain a transform pair similar to that encountered in Fourier transforms. Examples of such pairs are

(i) the Hankel transform

$$F(k) = \int_0^\infty f(x) J_n(kx) x\, dx,$$
$$f(x) = \int_0^\infty F(k) J_n(kx) k\, dk,$$

where the J_n are Bessel functions of order n, and

(ii) the Mellin transform

$$F(z) = \int_0^\infty t^{z-1} f(t)\, dt,$$
$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} t^{-z} F(z)\, dz.$$

Although we do not have the space to discuss their general properties, the reader should at least be aware of this wider class of integral transforms.

13.4 Exercises

13.1 Find the Fourier transform of the function $f(t) = \exp(-|t|)$.

(a) By applying Fourier's inversion theorem prove that

$$\frac{\pi}{2}\exp(-|t|) = \int_0^\infty \frac{\cos\omega t}{1+\omega^2}\,d\omega.$$

(b) By making the substitution $\omega = \tan\theta$, demonstrate the validity of Parseval's theorem for this function.

13.2 Use the general definition and properties of Fourier transforms to show the following.

(a) If $f(x)$ is periodic with period a then $\tilde{f}(k) = 0$, unless $ka = 2\pi n$ for integer n.
(b) The Fourier transform of $tf(t)$ is $id\tilde{f}(\omega)/d\omega$.
(c) The Fourier transform of $f(mt + c)$ is

$$\frac{e^{i\omega c/m}}{m}\tilde{f}\left(\frac{\omega}{m}\right).$$

13.3 Find the Fourier transform of $H(x-a)e^{-bx}$, where $H(x)$ is the Heaviside function.

13.4 Prove that the Fourier transform of the function $f(t)$ defined in the tf-plane by straight-line segments joining $(-T, 0)$ to $(0, 1)$ to $(T, 0)$, with $f(t) = 0$ outside $|t| < T$, is

$$\tilde{f}(\omega) = \frac{T}{\sqrt{2\pi}}\,\text{sinc}^2\left(\frac{\omega T}{2}\right),$$

where sinc x is defined as $(\sin x)/x$.

Use the general properties of Fourier transforms to determine the transforms of the following functions, graphically defined by straight-line segments and equal to zero outside the ranges specified:

(a) $(0, 0)$ to $(0.5, 1)$ to $(1, 0)$ to $(2, 2)$ to $(3, 0)$ to $(4.5, 3)$ to $(6, 0)$;
(b) $(-2, 0)$ to $(-1, 2)$ to $(1, 2)$ to $(2, 0)$;
(c) $(0, 0)$ to $(0, 1)$ to $(1, 2)$ to $(1, 0)$ to $(2, -1)$ to $(2, 0)$.

13.5 By taking the Fourier transform of the equation

$$\frac{d^2\phi}{dx^2} - K^2\phi = f(x),$$

show that its solution, $\phi(x)$, can be written as

$$\phi(x) = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ikx}\widetilde{f}(k)}{k^2 + K^2}\,dk,$$

where $\widetilde{f}(k)$ is the Fourier transform of $f(x)$.

13.6 By differentiating the definition of the Fourier sine transform $\tilde{f}_s(\omega)$ of the function $f(t) = t^{-1/2}$ with respect to ω, and then integrating the resulting expression by parts, find an elementary differential equation satisfied by $\tilde{f}_s(\omega)$. Hence show that this function is its own Fourier sine transform, i.e. $\tilde{f}_s(\omega) = Af(\omega)$, where A is a constant. Show that it is also its own Fourier cosine transform. Assume that the limit as $x \to \infty$ of $x^{1/2}\sin\alpha x$ can be taken as zero.

13.7 Find the Fourier transform of the unit rectangular distribution

$$f(t) = \begin{cases} 1 & |t| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the convolution of f with itself and, without further integration, deduce its transform. Deduce that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2}\, d\omega = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin^4 \omega}{\omega^4}\, d\omega = \frac{2\pi}{3}.$$

13.8 Calculate the Fraunhofer spectrum produced by a diffraction grating, uniformly illuminated by light of wavelength $2\pi/k$, as follows. Consider a grating with $4N$ equal strips each of width a and alternately opaque and transparent. The aperture function is then

$$f(y) = \begin{cases} A & \text{for } (2n+1)a \le y \le (2n+2)a, \qquad -N \le n < N, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show, for diffraction at angle θ to the normal to the grating, that the required Fourier transform can be written

$$\tilde{f}(q) = (2\pi)^{-1/2} \sum_{r=-N}^{N-1} \exp(-2iarq) \int_a^{2a} A \exp(-iqu)\, du,$$

where $q = k \sin\theta$.

(b) Evaluate the integral and sum to show that

$$\tilde{f}(q) = (2\pi)^{-1/2} \exp(-iqa/2) \frac{A \sin(2qaN)}{q \cos(qa/2)},$$

and hence that the intensity distribution $I(\theta)$ in the spectrum is proportional to

$$\frac{\sin^2(2qaN)}{q^2 \cos^2(qa/2)}.$$

(c) For large values of N, the numerator in the above expression has very closely spaced maxima and minima as a function of θ and effectively takes its mean value, 1/2, giving a low-intensity background. Much more significant peaks in $I(\theta)$ occur when $\theta = 0$ or the cosine term in the denominator vanishes. Show that the corresponding values of $|\tilde{f}(q)|$ are

$$\frac{2aNA}{(2\pi)^{1/2}} \qquad \text{and} \qquad \frac{4aNA}{(2\pi)^{1/2}(2m+1)\pi}, \qquad \text{with } m \text{ integral.}$$

Note that the constructive interference makes the maxima in $I(\theta) \propto N^2$, not N. Of course, observable maxima only occur for $0 \le \theta \le \pi/2$.

13.9 By finding the complex Fourier *series* for its LHS show that either side of the equation

$$\sum_{n=-\infty}^{\infty} \delta(t + nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{-2\pi nit/T}$$

can represent a periodic train of impulses. By expressing the function $f(t + nX)$, in which X is a constant, in terms of the Fourier *transform* $\tilde{f}(\omega)$ of $f(t)$, show that

$$\sum_{n=-\infty}^{\infty} f(t + nX) = \frac{\sqrt{2\pi}}{X} \sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2n\pi}{X}\right) e^{2\pi nit/X}.$$

This result is known as the *Poisson summation formula.*

13.10 In many applications in which the frequency spectrum of an analogue signal is required, the best that can be done is to sample the signal $f(t)$ a finite number of times at fixed intervals, and then use a *discrete Fourier transform* F_k to estimate discrete points on the (true) frequency spectrum $\tilde{f}(\omega)$.

(a) By an argument that is essentially the converse of that given in section 13.1, show that, if N samples f_n, beginning at $t = 0$ and spaced τ apart, are taken, then $\tilde{f}(2\pi k/(N\tau)) \approx F_k \tau$ where

$$F_k = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f_n e^{-2\pi nki/N}.$$

(b) For the function $f(t)$ defined by

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t < 1, \\ 0 & \text{otherwise,} \end{cases}$$

from which eight samples are drawn at intervals of $\tau = 0.25$, find a formula for $|F_k|$ and evaluate it for $k = 0, 1, \ldots, 7$.

(c) Find the exact frequency spectrum of $f(t)$ and compare the actual and estimated values of $\sqrt{2\pi}|\tilde{f}(\omega)|$ at $\omega = k\pi$ for $k = 0, 1, \ldots, 7$. Note the relatively good agreement for $k < 4$ and the lack of agreement for larger values of k.

13.11 For a function $f(t)$ that is non-zero only in the range $|t| < T/2$, the full frequency spectrum $\tilde{f}(\omega)$ can be constructed, in principle exactly, from values at discrete sample points $\omega = n(2\pi/T)$. Prove this as follows.

(a) Show that the coefficients of a complex Fourier *series* representation of $f(t)$ with period T can be written as

$$c_n = \frac{\sqrt{2\pi}}{T} \tilde{f}\left(\frac{2\pi n}{T}\right).$$

(b) Use this result to represent $f(t)$ as an infinite sum in the defining integral for $\tilde{f}(\omega)$, and hence show that

$$\tilde{f}(\omega) = \sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{T}\right) \operatorname{sinc}\left(n\pi - \frac{\omega T}{2}\right),$$

where sinc x is defined as $(\sin x)/x$.

13.12 A signal obtained by sampling a function $x(t)$ at regular intervals T is passed through an electronic filter, whose response $g(t)$ to a unit δ-function input is represented in a tg-plot by straight lines joining $(0,0)$ to $(T, 1/T)$ to $(2T, 0)$ and is zero for all other values of t. The output of the filter is the convolution of the input, $\sum_{-\infty}^{\infty} x(t)\delta(t - nT)$, with $g(t)$.
Using the convolution theorem, and the result given in exercise 13.4, show that the output of the filter can be written

$$y(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{\omega T}{2}\right) e^{-i\omega[(n+1)T - t]} \, d\omega.$$

13.13 Find the Fourier transform specified in part (a) and then use it to answer part (b).

(a) Find the Fourier transform of

$$f(\gamma, p, t) = \begin{cases} e^{-\gamma t} \sin pt & t > 0, \\ 0 & t < 0, \end{cases}$$

where γ (> 0) and p are constant parameters.

(b) The current $I(t)$ flowing through a certain system is related to the applied voltage $V(t)$ by the equation

$$I(t) = \int_{-\infty}^{\infty} K(t-u)V(u)\,du,$$

where

$$K(\tau) = a_1 f(\gamma_1, p_1, \tau) + a_2 f(\gamma_2, p_2, \tau).$$

The function $f(\gamma, p, t)$ is as given in (a) and all the a_i, γ_i (> 0) and p_i are fixed parameters. By considering the Fourier transform of $I(t)$, find the relationship that must hold between a_1 and a_2 if the total net charge Q passed through the system (over a very long time) is to be zero for an arbitrary applied voltage.

13.14 Prove the equality

$$\int_0^{\infty} e^{-2at} \sin^2 at\,dt = \frac{1}{\pi} \int_0^{\infty} \frac{a^2}{4a^4 + \omega^4}\,d\omega.$$

13.15 A linear amplifier produces an output that is the convolution of its input and its response function. The Fourier transform of the response function for a particular amplifier is

$$\tilde{K}(\omega) = \frac{i\omega}{\sqrt{2\pi}(\alpha + i\omega)^2}.$$

Determine the time variation of its output $g(t)$ when its input is the Heaviside step function. (Consider the Fourier transform of a decaying exponential function and the result of exercise 13.2(b).)

13.16 In quantum mechanics, two equal-mass particles having momenta $\mathbf{p}_j = \hbar\mathbf{k}_j$ and energies $E_j = \hbar\omega_j$ and represented by plane wavefunctions $\phi_j = \exp[i(\mathbf{k}_j \cdot \mathbf{r}_j - \omega_j t)]$, $j = 1, 2$, interact through a potential $V = V(|\mathbf{r}_1 - \mathbf{r}_2|)$. In first-order perturbation theory the probability of scattering to a state with momenta and energies $\mathbf{p}'_j, E'_j$ is determined by the modulus squared of the quantity

$$M = \iiint \psi_{\rm f}^* V \psi_{\rm i}\,d\mathbf{r}_1\,d\mathbf{r}_2\,dt.$$

The initial state, $\psi_{\rm i}$, is $\phi_1\phi_2$ and the final state, $\psi_{\rm f}$, is $\phi'_1\phi'_2$.

(a) By writing $\mathbf{r}_1 + \mathbf{r}_2 = 2\mathbf{R}$ and $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$ and assuming that $d\mathbf{r}_1\,d\mathbf{r}_2 = d\mathbf{R}\,d\mathbf{r}$, show that M can be written as the product of three one-dimensional integrals.

(b) From two of the integrals deduce energy and momentum conservation in the form of δ-functions.

(c) Show that M is proportional to the Fourier transform of V, i.e. to $\widetilde{V}(\mathbf{k})$ where $2\hbar\mathbf{k} = (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}'_2 - \mathbf{p}'_1)$ or, alternatively, $\hbar\mathbf{k} = \mathbf{p}'_1 - \mathbf{p}_1$.

13.17 For some ion–atom scattering processes, the potential V of the previous exercise may be approximated by $V = |\mathbf{r}_1 - \mathbf{r}_2|^{-1} \exp(-\mu|\mathbf{r}_1 - \mathbf{r}_2|)$. Show, using the result of the worked example in subsection 13.1.10, that the probability that the ion will scatter from, say, $\mathbf{p}_1$ to $\mathbf{p}'_1$ is proportional to $(\mu^2 + k^2)^{-2}$, where $k = |\mathbf{k}|$ and $\mathbf{k}$ is as given in part (c) of that exercise.

13.18 The equivalent duration and bandwidth, T_e and B_e, of a signal $x(t)$ are defined in terms of the latter and its Fourier transform $\tilde{x}(\omega)$ by

$$T_e = \frac{1}{x(0)} \int_{-\infty}^{\infty} x(t)\, dt,$$
$$B_e = \frac{1}{\tilde{x}(0)} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, d\omega,$$

where neither $x(0)$ nor $\tilde{x}(0)$ is zero. Show that the product $T_e B_e = 2\pi$ (this is a form of uncertainty principle), and find the equivalent bandwidth of the signal

$$x(t) = \exp(-|t|/T).$$

For this signal, determine the fraction of the total energy that lies in the frequency range $|\omega| < B_e/4$. You will need the indefinite integral with respect to x of $(a^2 + x^2)^{-2}$, which is

$$\frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3} \tan^{-1} \frac{x}{a}.$$

13.19 Calculate directly the auto-correlation function $a(z)$ for the product $f(t)$ of the exponential decay distribution and the Heaviside step function,

$$f(t) = \frac{1}{\lambda} e^{-\lambda t} H(t).$$

Use the Fourier transform and energy spectrum of $f(t)$ to deduce that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega z}}{\lambda^2 + \omega^2}\, d\omega = \frac{\pi}{\lambda} e^{-\lambda |z|}.$$

13.20 Prove that the cross-correlation $C(z)$ of the Gaussian and Lorentzian distributions

$$f(t) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right), \qquad g(t) = \left(\frac{a}{\pi}\right) \frac{1}{t^2 + a^2},$$

has as its Fourier transform the function

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tau^2\omega^2}{2}\right) \exp(-a|\omega|).$$

Hence show that

$$C(z) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(\frac{a^2 - z^2}{2\tau^2}\right) \cos\left(\frac{az}{\tau^2}\right).$$

13.21 Prove the expressions given in table 13.1 for the Laplace transforms of $t^{-1/2}$ and $t^{1/2}$, by setting $x^2 = ts$ in the result

$$\int_0^{\infty} \exp(-x^2)\, dx = \tfrac{1}{2}\sqrt{\pi}.$$

13.22 Find the functions $y(t)$ whose Laplace transforms are the following:

(a) $1/(s^2 - s - 2)$;
(b) $2s/[(s+1)(s^2+4)]$;
(c) $e^{-(\gamma+s)t_0}/[(s+\gamma)^2 + b^2]$.

13.23 Use the properties of Laplace transforms to prove the following without evaluating any Laplace integrals explicitly:

(a) $\mathscr{L}\left[t^{5/2}\right] = \frac{15}{8}\sqrt{\pi} s^{-7/2}$;
(b) $\mathscr{L}\left[(\sinh at)/t\right] = \frac{1}{2}\ln\left[(s+a)/(s-a)\right], \qquad s > |a|$;

(c) $\mathcal{L}\,[\sinh at \cos bt] = a(s^2 - a^2 + b^2)[(s-a)^2 + b^2]^{-1}[(s+a)^2 + b^2]^{-1}$.

13.24 Find the solution (the so-called *impulse response* or *Green's function*) of the equation

$$T\frac{dx}{dt} + x = \delta(t)$$

by proceeding as follows.

(a) Show by substitution that

$$x(t) = A(1 - e^{-t/T})H(t)$$

is a solution, for which $x(0) = 0$, of

$$T\frac{dx}{dt} + x = AH(t), \qquad (*)$$

where $H(t)$ is the Heaviside step function.

(b) Construct the solution when the RHS of $(*)$ is replaced by $AH(t-\tau)$, with $dx/dt = x = 0$ for $t < \tau$, and hence find the solution when the RHS is a rectangular pulse of duration τ.

(c) By setting $A = 1/\tau$ and taking the limit as $\tau \to 0$, show that the impulse response is $x(t) = T^{-1}e^{-t/T}$.

(d) Obtain the same result much more directly by taking the Laplace transform of each term in the original equation, solving the resulting algebraic equation and then using the entries in table 13.1.

13.25 This exercise is concerned with the limiting behaviour of Laplace transforms.

(a) If $f(t) = A + g(t)$, where A is a constant and the indefinite integral of $g(t)$ is bounded as its upper limit tends to ∞, show that

$$\lim_{s\to 0} s\bar{f}(s) = A.$$

(b) For $t > 0$, the function $y(t)$ obeys the differential equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = c\cos^2\omega t,$$

where a, b and c are positive constants. Find $\bar{y}(s)$ and show that $s\bar{y}(s) \to c/2b$ as $s \to 0$. Interpret the result in the t-domain.

13.26 By writing $f(x)$ as an integral involving the δ-function $\delta(\xi - x)$ and taking the Laplace transforms of both sides, show that the transform of the solution of the equation

$$\frac{d^4y}{dx^4} - y = f(x)$$

for which y and its first three derivatives vanish at $x = 0$ can be written as

$$\bar{y}(s) = \int_0^\infty f(\xi)\frac{e^{-s\xi}}{s^4 - 1}\,d\xi.$$

Use the properties of Laplace transforms and the entries in table 13.1 to show that

$$y(x) = \frac{1}{2}\int_0^x f(\xi)\,[\sinh(x-\xi) - \sin(x-\xi)]\,d\xi.$$

13.27 The function $f_a(x)$ is defined as unity for $0 < x < a$ and zero otherwise. Find its Laplace transform $\bar{f}_a(s)$ and deduce that the transform of $xf_a(x)$ is

$$\frac{1}{s^2}\left[1-(1+as)e^{-sa}\right].$$

Write $f_a(x)$ in terms of Heaviside functions and hence obtain an explicit expression for

$$g_a(x) = \int_0^x f_a(y)f_a(x-y)\,dy.$$

Use the expression to write $\bar{g}_a(s)$ in terms of the functions $\bar{f}_a(s)$ and $\bar{f}_{2a}(s)$, and their derivatives, and hence show that $\bar{g}_a(s)$ is equal to the square of $\bar{f}_a(s)$, in accordance with the convolution theorem.

13.28 Show that the Laplace transform of $f(t-a)H(t-a)$, where $a \geq 0$, is $e^{-as}\bar{f}(s)$ and that, if $g(t)$ is a periodic function of period T, $\bar{g}(s)$ can be written as

$$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}g(t)\,dt.$$

(a) Sketch the periodic function defined in $0 \leq t \leq T$ by

$$g(t) = \begin{cases} 2t/T & 0 \leq t < T/2, \\ 2(1-t/T) & T/2 \leq t \leq T, \end{cases}$$

and, using the previous result, find its Laplace transform.

(b) Show, by sketching it, that

$$\frac{2}{T}[tH(t) + 2\sum_{n=1}^{\infty}(-1)^n(t-\tfrac{1}{2}nT)H(t-\tfrac{1}{2}nT)]$$

is another representation of $g(t)$ and hence derive the relationship

$$\tanh x = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}.$$

13.5 Hints and answers

13.1 Note that the integrand has different analytic forms for $t < 0$ and $t \geq 0$. $(2/\pi)^{1/2}(1+\omega^2)^{-1}$.

13.3 $(1/\sqrt{2\pi})[(b-ik)/(b^2+k^2)]e^{-a(b+ik)}$.

13.5 Use or derive $\widetilde{\phi''}(k) = -k^2\tilde{\phi}(k)$ to obtain an algebraic equation for $\tilde{\phi}(k)$ and then use the Fourier inversion formula.

13.7 $(2/\sqrt{2\pi})(\sin\omega/\omega)$.
The convolution is $2-|t|$ for $|t| < 2$, zero otherwise. Use the convolution theorem.
$(4/\sqrt{2\pi})(\sin^2\omega/\omega^2)$.
Apply Parseval's theorem to f and to $f * f$.

13.9 The Fourier coefficient is T^{-1}, independent of n. Make the changes of variables $t \to \omega$, $n \to -n$ and $T \to 2\pi/X$ and apply the translation theorem.

13.11 (b) Recall that the infinite integral involved in defining $\tilde{f}(\omega)$ has a non-zero integrand only in $|t| < T/2$.

13.13 (a) $(1/\sqrt{2\pi})\{p/[(\gamma+i\omega)^2+p^2]\}$.
(b) Show that $Q = \sqrt{2\pi}\,\tilde{I}(0)$ and use the convolution theorem. The required relationship is $a_1p_1/(\gamma_1^2+p_1^2) + a_2p_2/(\gamma_2^2+p_2^2) = 0$.

13.15 $\tilde{g}(\omega) = 1/[\sqrt{2\pi}(\alpha+i\omega)^2]$, leading to $g(t) = te^{-\alpha t}$.

13.17 $\widetilde{V}(\mathbf{k}) \propto [-2\pi/(ik)] \int \{\exp[-(\mu - ik)r] - \exp[-(\mu + ik)r]\}\, dr$.

13.19 Note that the lower limit in the calculation of $a(z)$ is 0, for $z > 0$, and $|z|$, for $z < 0$. Auto-correlation $a(z) = [(1/(2\lambda^3)] \exp(-\lambda|z|)$.

13.21 Prove the result for $t^{1/2}$ by integrating that for $t^{-1/2}$ by parts.

13.23 (a) Use (13.62) with $n = 2$ on $\mathscr{L}\left[\sqrt{t}\right]$; (b) use (13.63);
(c) consider $\mathscr{L}\,[\exp(\pm at)\cos bt]$ and use the translation property, subsection 13.2.2.

13.25 (a) Note that $|\lim \int g(t)e^{-st}\, dt| \le |\lim \int g(t)\, dt|$.
(b) $(s^2 + as + b)\bar{y}(s) = \{c(s^2 + 2\omega^2)/[s(s^2 + 4\omega^2)]\} + (a + s)y(0) + y'(0)$.
For this damped system, at large t (corresponding to $s \to 0$) rates of change are negligible and the equation reduces to $by = c\cos^2 \omega t$. The average value of $\cos^2 \omega t$ is $\frac{1}{2}$.

13.27 $s^{-1}[1 - \exp(-sa)]$; $g_a(x) = x$ for $0 < x < a$, $g_a(x) = 2a - x$ for $a \le x \le 2a$, $g_a(x) = 0$ otherwise.

14

First-order ordinary differential equations

Differential equations are the group of equations that contain derivatives. Chapters 14–21 discuss a variety of differential equations, starting in this chapter and the next with those ordinary differential equations (ODEs) that have closed-form solutions. As its name suggests, an ODE contains only ordinary derivatives (no partial derivatives) and describes the relationship between these derivatives of the *dependent variable*, usually called y, with respect to the *independent variable*, usually called x. The solution to such an ODE is therefore a function of x and is written $y(x)$. For an ODE to have a closed-form solution, it must be possible to express $y(x)$ in terms of the standard elementary functions such as $\exp x$, $\ln x$, $\sin x$ etc. The solutions of some differential equations cannot, however, be written in closed form, but only as an infinite series; these are discussed in chapter 16.

Ordinary differential equations may be separated conveniently into different categories according to their general characteristics. The primary grouping adopted here is by the *order* of the equation. The order of an ODE is simply the order of the highest derivative it contains. Thus equations containing dy/dx, but no higher derivatives, are called first order, those containing d^2y/dx^2 are called second order and so on. In this chapter we consider first-order equations, and in the next, second- and higher-order equations.

Ordinary differential equations may be classified further according to *degree*. The degree of an ODE is the power to which the highest-order derivative is raised, after the equation has been rationalised to contain only integer powers of derivatives. Hence the ODE

$$\frac{d^3y}{dx^3} + x\left(\frac{dy}{dx}\right)^{3/2} + x^2 y = 0,$$

is of third order and second degree, since after rationalisation it contains the term $(d^3y/dx^3)^2$.

The *general solution* to an ODE is the most general function $y(x)$ that satisfies the equation; it will contain *constants of integration* which may be determined by the application of some suitable *boundary conditions*. For example, we may be told that for a certain first-order differential equation, the solution $y(x)$ is equal to zero when the parameter x is equal to unity; this allows us to determine the value of the constant of integration. The *general solutions* to nth-order ODEs, which are considered in detail in the next chapter, will contain n (essential) arbitrary constants of integration and therefore we will need n boundary conditions if these constants are to be determined (see section 14.1). When the boundary conditions have been applied, and the constants found, we are left with a *particular solution* to the ODE, which obeys the given boundary conditions. Some ODEs of degree greater than unity also possess *singular solutions*, which are solutions that contain no arbitrary constants and cannot be found from the general solution; singular solutions are discussed in more detail in section 14.3. When any solution to an ODE has been found, it is always possible to check its validity by substitution into the original equation and verification that any given boundary conditions are met.

In this chapter, firstly we discuss various types of first-degree ODE and then go on to examine those higher-degree equations that can be solved in closed form. At the outset, however, we discuss the general form of the solutions of ODEs; this discussion is relevant to both first- and higher-order ODEs.

14.1 General form of solution

It is helpful when considering the general form of the solution of an ODE to consider the inverse process, namely that of obtaining an ODE from a given group of functions, each one of which is a solution of the ODE. Suppose the members of the group can be written as

$$y = f(x, a_1, a_2, \ldots, a_n), \tag{14.1}$$

each member being specified by a different set of values of the parameters a_i. For example, consider the group of functions

$$y = a_1 \sin x + a_2 \cos x; \tag{14.2}$$

here $n = 2$.

Since an ODE is required for which *any* of the group is a solution, it clearly must not contain any of the a_i. As there are n of the a_i in expression (14.1), we must obtain $n+1$ equations involving them in order that, by elimination, we can obtain one final equation without them.

Initially we have only (14.1), but if this is differentiated n times, a total of $n+1$ equations is obtained from which (in principle) all the a_i can be eliminated, to give one ODE satisfied by all the group. As a result of the n differentiations, $d^n y/dx^n$ will be present in one of the $n+1$ equations and hence in the final equation, which will therefore be of nth order.

In the case of (14.2), we have

$$\frac{dy}{dx} = a_1 \cos x - a_2 \sin x,$$
$$\frac{d^2y}{dx^2} = -a_1 \sin x - a_2 \cos x.$$

Here the elimination of a_1 and a_2 is trivial (because of the similarity of the forms of y and d^2y/dx^2), resulting in

$$\frac{d^2y}{dx^2} + y = 0,$$

a second-order equation.

Thus, to summarise, a group of functions (14.1) with n parameters satisfies an nth-order ODE in general (although in some degenerate cases an ODE of less than nth order is obtained). The intuitive converse of this is that the general solution of an nth-order ODE contains n arbitrary parameters (constants); for our purposes, this will be assumed to be valid although a totally general proof is difficult.

As mentioned earlier, external factors affect a system described by an ODE, by fixing the values of the dependent variables for particular values of the independent ones. These externally imposed (or *boundary*) conditions on the solution are thus the means of determining the parameters and so of specifying precisely which function is the required solution. It is apparent that the number of boundary conditions should match the number of parameters and hence the order of the equation, if a unique solution is to be obtained. Fewer independent boundary conditions than this will lead to a number of undetermined parameters in the solution, whilst an excess will usually mean that no acceptable solution is possible.

For an nth-order equation the required n boundary conditions can take many forms, for example the value of y at n different values of x, or the value of any $n-1$ of the n derivatives dy/dx, d^2y/dx^2, $\dots$, d^ny/dx^n together with that of y, all for the same value of x, or many intermediate combinations.

14.2 First-degree first-order equations

First-degree first-order ODEs contain only dy/dx equated to some function of x and y, and can be written in either of two equivalent standard forms,

$$\frac{dy}{dx} = F(x, y), \qquad A(x, y)\,dx + B(x, y)\,dy = 0,$$

where $F(x, y) = -A(x, y)/B(x, y)$, and $F(x, y)$, $A(x, y)$ and $B(x, y)$ are in general functions of both x and y. Which of the two above forms is the more useful for finding a solution depends on the type of equation being considered. There are several different types of first-degree first-order ODEs that are of interest in the physical sciences. These equations and their respective solutions are discussed below.

14.2.1 Separable-variable equations

A separable-variable equation is one which may be written in the conventional form

$$\frac{dy}{dx} = f(x)g(y), \tag{14.3}$$

where $f(x)$ and $g(y)$ are functions of x and y respectively, including cases in which $f(x)$ or $g(y)$ is simply a constant. Rearranging this equation so that the terms depending on x and on y appear on opposite sides (i.e. are separated), and integrating, we obtain

$$\int \frac{dy}{g(y)} = \int f(x)\,dx.$$

Finding the solution $y(x)$ that satisfies (14.3) then depends only on the ease with which the integrals in the above equation can be evaluated. It is also worth noting that ODEs that at first sight do not appear to be of the form (14.3) can sometimes be made separable by an appropriate factorisation.

►*Solve*

$$\frac{dy}{dx} = x + xy.$$

Since the RHS of this equation can be factorised to give $x(1 + y)$, the equation becomes separable and we obtain

$$\int \frac{dy}{1+y} = \int x\,dx.$$

Now integrating both sides separately, we find

$$\ln(1 + y) = \frac{x^2}{2} + c,$$

and so

$$1 + y = \exp\left(\frac{x^2}{2} + c\right) = A\exp\left(\frac{x^2}{2}\right),$$

where c and hence A is an arbitrary constant. ◄

Solution method. *Factorise the equation so that it becomes separable. Rearrange it so that the terms depending on x and those depending on y appear on opposite sides and then integrate directly. Remember the constant of integration, which can be evaluated if further information is given.*

14.2.2 *Exact equations*

An *exact* first-degree first-order ODE is one of the form

$$A(x,y)\,dx + B(x,y)\,dy = 0 \qquad \text{and for which} \quad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{14.4}$$

In this case $A(x,y)\,dx + B(x,y)\,dy$ is an exact differential, $dU(x,y)$ say (see section 5.3). In other words

$$A\,dx + B\,dy = dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy,$$

from which we obtain

$$A(x,y) = \frac{\partial U}{\partial x}, \tag{14.5}$$

$$B(x,y) = \frac{\partial U}{\partial y}. \tag{14.6}$$

Since $\partial^2 U/\partial x\partial y = \partial^2 U/\partial y\partial x$ we therefore require

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{14.7}$$

If (14.7) holds then (14.4) can be written $dU(x,y) = 0$, which has the solution $U(x,y) = c$, where c is a constant and from (14.5) $U(x,y)$ is given by

$$U(x,y) = \int A(x,y)\,dx + F(y). \tag{14.8}$$

The function $F(y)$ can be found from (14.6) by differentiating (14.8) with respect to y and equating to $B(x,y)$.

►*Solve*

$$x\frac{dy}{dx} + 3x + y = 0.$$

Rearranging into the form (14.4) we have

$$(3x + y)\,dx + x\,dy = 0,$$

i.e. $A(x,y) = 3x + y$ and $B(x,y) = x$. Since $\partial A/\partial y = 1 = \partial B/\partial x$, the equation is exact, and by (14.8) the solution is given by

$$U(x,y) = \int (3x+y)\,dx + F(y) = c_1 \qquad \Rightarrow \qquad \frac{3x^2}{2} + yx + F(y) = c_1.$$

Differentiating $U(x,y)$ with respect to y and equating it to $B(x,y) = x$ we obtain $dF/dy = 0$, which integrates immediately to give $F(y) = c_2$. Therefore, letting $c = c_1 - c_2$, the solution to the original ODE is

$$\frac{3x^2}{2} + xy = c. \blacktriangleleft$$

Solution method. *Check that the equation is an exact differential using (14.7) then solve using (14.8). Find the function $F(y)$ by differentiating (14.8) with respect to y and using (14.6).*

14.2.3 Inexact equations: integrating factors

Equations that may be written in the form

$$A(x, y)\, dx + B(x, y)\, dy = 0 \qquad \text{but for which} \quad \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x} \tag{14.9}$$

are known as inexact equations. However, the differential $A\, dx + B\, dy$ can always be made exact by multiplying by an *integrating factor* $\mu(x, y)$, which obeys

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}. \tag{14.10}$$

For an integrating factor that is a function of both x and y, i.e. $\mu = \mu(x, y)$, there exists no general method for finding it; in such cases it may sometimes be found by inspection. If, however, an integrating factor exists that is a function of either x or y alone then (14.10) can be solved to find it. For example, if we assume that the integrating factor is a function of x alone, i.e. $\mu = \mu(x)$, then (14.10) reads

$$\mu \frac{\partial A}{\partial y} = \mu \frac{\partial B}{\partial x} + B \frac{d\mu}{dx}.$$

Rearranging this expression we find

$$\frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dx = f(x)\, dx,$$

where we require $f(x)$ also to be a function of x only; indeed this provides a general method of determining whether the integrating factor μ is a function of x alone. This integrating factor is then given by

$$\mu(x) = \exp\left\{\int f(x)\, dx\right\} \qquad \text{where} \qquad f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right). \tag{14.11}$$

Similarly, if $\mu = \mu(y)$ then

$$\mu(y) = \exp\left\{\int g(y)\, dy\right\} \qquad \text{where} \qquad g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right). \tag{14.12}$$

►*Solve*
$$\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}.$$

Rearranging into the form (14.9), we have
$$(4x + 3y^2)\,dx + 2xy\,dy = 0, \tag{14.13}$$
i.e. $A(x, y) = 4x + 3y^2$ and $B(x, y) = 2xy$. Now
$$\frac{\partial A}{\partial y} = 6y, \qquad \frac{\partial B}{\partial x} = 2y,$$
so the ODE is not exact in its present form. However, we see that
$$\frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = \frac{2}{x},$$
a function of x alone. Therefore an integrating factor exists that is also a function of x alone and, ignoring the arbitrary constant of integration, is given by
$$\mu(x) = \exp\left\{2\int \frac{dx}{x}\right\} = \exp(2\ln x) = x^2.$$
Multiplying (14.13) through by $\mu(x) = x^2$ we obtain
$$(4x^3 + 3x^2y^2)\,dx + 2x^3y\,dy = 4x^3\,dx + (3x^2y^2\,dx + 2x^3y\,dy) = 0.$$
By inspection this integrates immediately to give the solution $x^4 + y^2x^3 = c$, where c is a constant. ◄

Solution method. *Examine whether $f(x)$ and $g(y)$ are functions of only x or y respectively. If so, then the required integrating factor is a function of either x or y only, and is given by (14.11) or (14.12) respectively. If the integrating factor is a function of both x and y, then sometimes it may be found by inspection or by trial and error. In any case, the integrating factor μ must satisfy (14.10). Once the equation has been made exact, solve by the method of subsection 14.2.2.*

14.2.4 Linear equations

Linear first-order ODEs are a special case of inexact ODEs (discussed in the previous subsection) and can be written in the conventional form
$$\frac{dy}{dx} + P(x)y = Q(x). \tag{14.14}$$
Such equations can be made exact by multiplying through by an appropriate integrating factor in a similar manner to that discussed above. In this case, however, the integrating factor is always a function of x alone and may be expressed in a particularly simple form. An integrating factor $\mu(x)$ must be such that
$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}\left[\mu(x)y\right] = \mu(x)Q(x), \tag{14.15}$$

which may then be integrated directly to give

$$\mu(x)y = \int \mu(x)Q(x)\,dx. \tag{14.16}$$

The required integrating factor $\mu(x)$ is determined by the first equality in (14.15), i.e.

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu\frac{dy}{dx} + \mu P y,$$

which immediately gives the simple relation

$$\frac{d\mu}{dx} = \mu(x)P(x) \qquad \Rightarrow \qquad \mu(x) = \exp\left\{\int P(x)\,dx\right\}. \tag{14.17}$$

►*Solve*

$$\frac{dy}{dx} + 2xy = 4x.$$

The integrating factor is given immediately by

$$\mu(x) = \exp\left\{\int 2x\,dx\right\} = \exp x^2.$$

Multiplying through the ODE by $\mu(x) = \exp x^2$ and integrating, we have

$$y\exp x^2 = 4\int x\exp x^2\,dx = 2\exp x^2 + c.$$

The solution to the ODE is therefore given by $y = 2 + c\exp(-x^2)$. ◄

Solution method. *Rearrange the equation into the form (14.14) and multiply by the integrating factor $\mu(x)$ given by (14.17). The left- and right-hand sides can then be integrated directly, giving y from (14.16).*

14.2.5 Homogeneous equations

Homogeneous equation are ODEs that may be written in the form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right), \tag{14.18}$$

where $A(x,y)$ and $B(x,y)$ are homogeneous functions of the same degree. A function $f(x,y)$ is homogeneous of degree n if, for any λ, it obeys

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

For example, if $A = x^2y - xy^2$ and $B = x^3 + y^3$ then we see that A and B are both homogeneous functions of degree 3. In general, for functions of the form of A and B, we see that for both to be homogeneous, and of the same degree, we require the sum of the powers in x and y in each term of A and B to be the same (in this example equal to 3). The RHS of a homogeneous ODE can be written as a function of y/x. The equation may then be solved by making the substitution $y = vx$, so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = F(v).$$

This is now a separable equation and can be integrated directly to give

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}. \tag{14.19}$$

►*Solve*

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right).$$

Substituting $y = vx$ we obtain

$$v + x\frac{dv}{dx} = v + \tan v.$$

Cancelling v on both sides, rearranging and integrating gives

$$\int \cot v \, dv = \int \frac{dx}{x} = \ln x + c_1.$$

But

$$\int \cot v \, dv = \int \frac{\cos v}{\sin v} \, dv = \ln(\sin v) + c_2,$$

so the solution to the ODE is $y = x \sin^{-1} Ax$, where A is a constant. ◄

Solution method. *Check to see whether the equation is homogeneous. If so, make the substitution $y = vx$, separate variables as in (14.19) and then integrate directly. Finally replace v by y/x to obtain the solution.*

14.2.6 Isobaric equations

An isobaric ODE is a generalisation of the homogeneous ODE discussed in the previous section, and is of the form

$$\frac{dy}{dx} = \frac{A(x, y)}{B(x, y)}, \tag{14.20}$$

where the equation is dimensionally consistent if y and dy are each given a weight m relative to x and dx, i.e. if the substitution $y = vx^m$ makes it separable.

►*Solve*

$$\frac{dy}{dx} = \frac{-1}{2yx}\left(y^2 + \frac{2}{x}\right).$$

Rearranging we have

$$\left(y^2 + \frac{2}{x}\right) dx + 2yx\, dy = 0.$$

Giving y and dy the weight m and x and dx the weight 1, the sums of the powers in each term on the LHS are $2m+1$, 0 and $2m+1$ respectively. These are equal if $2m+1=0$, i.e. if $m=-\frac{1}{2}$. Substituting $y = vx^m = vx^{-1/2}$, with the result that $dy = x^{-1/2}\, dv - \frac{1}{2}vx^{-3/2}\, dx$, we obtain

$$v\, dv + \frac{dx}{x} = 0,$$

which is separable and may be integrated directly to give $\frac{1}{2}v^2 + \ln x = c$. Replacing v by $y\sqrt{x}$ we obtain the solution $\frac{1}{2}y^2 x + \ln x = c$. ◄

Solution method. *Write the equation in the form $A\,dx + B\,dy = 0$. Giving y and dy each a weight m and x and dx each a weight 1, write down the sum of powers in each term. Then, if a value of m that makes all these sums equal can be found, substitute $y = vx^m$ into the original equation to make it separable. Integrate the separated equation directly, and then replace v by yx^{-m} to obtain the solution.*

14.2.7 Bernoulli's equation

Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad \text{where } n \neq 0 \text{ or } 1. \tag{14.21}$$

This equation is very similar in form to the linear equation (14.14), but is in fact non-linear due to the extra y^n factor on the RHS. However, the equation can be made linear by substituting $v = y^{1-n}$ and correspondingly

$$\frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dv}{dx}.$$

Substituting this into (14.21) and dividing through by y^n, we find

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x),$$

which is a linear equation and may be solved by the method described in subsection 14.2.4.

►*Solve*

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3y^4.$$

If we let $v = y^{1-4} = y^{-3}$ then

$$\frac{dy}{dx} = -\frac{y^4}{3}\frac{dv}{dx}.$$

Substituting this into the ODE and rearranging, we obtain

$$\frac{dv}{dx} - \frac{3v}{x} = -6x^3,$$

which is linear and may be solved by multiplying through by the integrating factor (see subsection 14.2.4)

$$\exp\left\{-3\int \frac{dx}{x}\right\} = \exp(-3\ln x) = \frac{1}{x^3}.$$

This yields the solution

$$\frac{v}{x^3} = -6x + c.$$

Remembering that $v = y^{-3}$, we obtain $y^{-3} = -6x^4 + cx^3$. ◄

Solution method. *Rearrange the equation into the form (14.21) and make the substitution $v = y^{1-n}$. This leads to a linear equation in v, which can be solved by the method of subsection 14.2.4. Then replace v by y^{1-n} to obtain the solution.*

14.2.8 Miscellaneous equations

There are two further types of first-degree first-order equation that occur fairly regularly but do not fall into any of the above categories. They may be reduced to one of the above equations, however, by a suitable change of variable.

Firstly, we consider

$$\frac{dy}{dx} = F(ax + by + c), \tag{14.22}$$

where a, b and c are constants, i.e. x and y *only* appear on the RHS in the particular combination $ax + by + c$ and not in any other combination or by themselves. This equation can be solved by making the substitution $v = ax + by + c$, in which case

$$\frac{dv}{dx} = a + b\frac{dy}{dx} = a + bF(v), \tag{14.23}$$

which is separable and may be integrated directly.

►*Solve*

$$\frac{dy}{dx} = (x + y + 1)^2.$$

Making the substitution $v = x + y + 1$, we obtain, as in (14.23),

$$\frac{dv}{dx} = v^2 + 1,$$

which is separable and integrates directly to give

$$\int \frac{dv}{1 + v^2} = \int dx \quad \Rightarrow \quad \tan^{-1} v = x + c_1.$$

So the solution to the original ODE is $\tan^{-1}(x + y + 1) = x + c_1$, where c_1 is a constant of integration. ◄

Solution method. *In an equation such as (14.22), substitute $v = ax+by+c$ to obtain a separable equation that can be integrated directly. Then replace v by $ax + by + c$ to obtain the solution.*

Secondly, we discuss

$$\frac{dy}{dx} = \frac{ax + by + c}{ex + fy + g}, \tag{14.24}$$

where a, b, c, e, f and g are all constants. This equation may be solved by letting $x = X + \alpha$ and $y = Y + \beta$, where α and β are constants found from

$$a\alpha + b\beta + c = 0 \tag{14.25}$$

$$e\alpha + f\beta + g = 0. \tag{14.26}$$

Then (14.24) can be written as

$$\frac{dY}{dX} = \frac{aX + bY}{eX + fY},$$

which is homogeneous and can be solved by the method of subsection 14.2.5. Note, however, that if $a/e = b/f$ then (14.25) and (14.26) are not independent and so cannot be solved uniquely for α and β. However, in this case, (14.24) reduces to an equation of the form (14.22), which was discussed above.

►*Solve*

$$\frac{dy}{dx} = \frac{2x - 5y + 3}{2x + 4y - 6}.$$

Let $x = X + \alpha$ and $y = Y + \beta$, where α and β obey the relations

$$2\alpha - 5\beta + 3 = 0$$
$$2\alpha + 4\beta - 6 = 0,$$

which solve to give $\alpha = \beta = 1$. Making these substitutions we find

$$\frac{dY}{dX} = \frac{2X - 5Y}{2X + 4Y},$$

which is a homogeneous ODE and can be solved by substituting $Y = vX$ (see subsection 14.2.5) to obtain

$$\frac{dv}{dX} = \frac{2 - 7v - 4v^2}{X(2 + 4v)}.$$

This equation is separable, and using partial fractions we find

$$\int \frac{2 + 4v}{2 - 7v - 4v^2}\, dv = -\frac{4}{3}\int \frac{dv}{4v - 1} - \frac{2}{3}\int \frac{dv}{v + 2} = \int \frac{dX}{X},$$

which integrates to give

$$\ln X + \tfrac{1}{3}\ln(4v - 1) + \tfrac{2}{3}\ln(v + 2) = c_1,$$

or

$$X^3(4v - 1)(v + 2)^2 = \exp 3c_1.$$

Remembering that $Y = vX$, $x = X + 1$ and $y = Y + 1$, the solution to the original ODE is given by $(4y - x - 3)(y + 2x - 3)^2 = c_2$, where $c_2 = \exp 3c_1$. ◀

Solution method. *If in (14.24) $a/e \neq b/f$ then make the substitution $x = X + \alpha$, $y = Y + \beta$, where α and β are given by (14.25) and (14.26); the resulting equation is homogeneous and can be solved as in subsection 14.2.5. Substitute $v = Y/X$, $X = x - \alpha$ and $Y = y - \beta$ to obtain the solution. If $a/e = b/f$ then (14.24) is of the same form as (14.22) and may be solved accordingly.*

14.3 Higher-degree first-order equations

First-order equations of degree higher than the first do not occur often in the description of physical systems, since squared and higher powers of first-order derivatives usually arise from resistive or driving mechanisms, when an acceleration or other higher-order derivative is also present. They do sometimes appear in connection with geometrical problems, however.

Higher-degree first-order equations can be written as $F(x, y, dy/dx) = 0$. The most general standard form is

$$p^n + a_{n-1}(x, y)p^{n-1} + \cdots + a_1(x, y)p + a_0(x, y) = 0, \qquad (14.27)$$

where for ease of notation we write $p = dy/dx$. If the equation can be solved for one of x, y or p then either an explicit or a parametric solution can sometimes be obtained. We discuss the main types of such equations below, including Clairaut's equation, which is a special case of an equation explicitly soluble for y.

14.3.1 Equations soluble for p

Sometimes the LHS of (14.27) can be factorised into the form

$$(p - F_1)(p - F_2)\cdots(p - F_n) = 0, \qquad (14.28)$$

where $F_i = F_i(x, y)$. We are then left with solving the n first-degree equations $p = F_i(x, y)$. Writing the solutions to these first-degree equations as $G_i(x, y) = 0$, the general solution to (14.28) is given by the product

$$G_1(x, y)G_2(x, y) \cdots G_n(x, y) = 0. \tag{14.29}$$

►*Solve*

$$(x^3 + x^2 + x + 1)p^2 - (3x^2 + 2x + 1)yp + 2xy^2 = 0. \tag{14.30}$$

This equation may be factorised to give

$$[(x + 1)p - y][(x^2 + 1)p - 2xy] = 0.$$

Taking each bracket in turn we have

$$(x + 1)\frac{dy}{dx} - y = 0,$$
$$(x^2 + 1)\frac{dy}{dx} - 2xy = 0,$$

which have the solutions $y - c(x + 1) = 0$ and $y - c(x^2 + 1) = 0$ respectively (see section 14.2 on first-degree first-order equations). Note that the arbitrary constants in these two solutions can be taken to be the same, since only one is required for a first-order equation. The general solution to (14.30) is then given by

$$[y - c(x + 1)]\left[y - c(x^2 + 1)\right] = 0. \blacktriangleleft$$

Solution method. *If the equation can be factorised into the form (14.28) then solve the first-order ODE* $p - F_i = 0$ *for each factor and write the solution in the form* $G_i(x, y) = 0$. *The solution to the original equation is then given by the product (14.29).*

14.3.2 Equations soluble for x

Equations that can be solved for x, i.e. such that they may be written in the form

$$x = F(y, p), \tag{14.31}$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to y, so that

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p}\frac{dp}{dy}.$$

This results in an equation of the form $G(y, p) = 0$, which can be used together with (14.31) to eliminate p and give the general solution. Note that often a singular solution to the equation will be found at the same time (see the introduction to this chapter).

►*Solve*

$$6y^2p^2 + 3xp - y = 0. \tag{14.32}$$

This equation can be solved for x explicitly to give $3x = (y/p) - 6y^2p$. Differentiating both sides with respect to y, we find

$$3\frac{dx}{dy} = \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 6y^2\frac{dp}{dy} - 12yp,$$

which factorises to give

$$\left(1 + 6yp^2\right)\left(2p + y\frac{dp}{dy}\right) = 0. \tag{14.33}$$

Setting the factor containing dp/dy equal to zero gives a first-degree first-order equation in p, which may be solved to give $py^2 = c$. Substituting for p in (14.32) then yields the general solution of (14.32):

$$y^3 = 3cx + 6c^2. \tag{14.34}$$

If we now consider the first factor in (14.33), we find $6p^2y = -1$ as a possible solution. Substituting for p in (14.32) we find the singular solution

$$8y^3 + 3x^2 = 0.$$

Note that the singular solution contains no arbitrary constants and cannot be found from the general solution (14.34) by any choice of the constant c. ◄

Solution method. *Write the equation in the form (14.31) and differentiate both sides with respect to y. Rearrange the resulting equation into the form $G(y, p) = 0$, which can be used together with the original ODE to eliminate p and so give the general solution. If $G(y, p)$ can be factorised then the factor containing dp/dy should be used to eliminate p and give the general solution. Using the other factors in this fashion will instead lead to singular solutions.*

14.3.3 Equations soluble for y

Equations that can be solved for y, i.e. are such that they may be written in the form

$$y = F(x, p), \tag{14.35}$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to x, so that

$$\frac{dy}{dx} = p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{dp}{dx}.$$

This results in an equation of the form $G(x, p) = 0$, which can be used together with (14.35) to eliminate p and give the general solution. An additional (singular) solution to the equation is also often found.

►*Solve*

$$xp^2 + 2xp - y = 0. \tag{14.36}$$

This equation can be solved for y explicitly to give $y = xp^2 + 2xp$. Differentiating both sides with respect to x, we find

$$\frac{dy}{dx} = p = 2xp\frac{dp}{dx} + p^2 + 2x\frac{dp}{dx} + 2p,$$

which after factorising gives

$$(p+1)\left(p + 2x\frac{dp}{dx}\right) = 0. \tag{14.37}$$

To obtain the general solution of (14.36), we consider the factor containing dp/dx. This first-degree first-order equation in p has the solution $xp^2 = c$ (see subsection 14.3.1), which we then use to eliminate p from (14.36). Thus we find that the general solution to (14.36) is

$$(y - c)^2 = 4cx. \tag{14.38}$$

If instead, we set the other factor in (14.37) equal to zero, we obtain the very simple solution $p = -1$. Substituting this into (14.36) then gives

$$x + y = 0,$$

which is a singular solution to (14.36). ◄

Solution method. *Write the equation in the form (14.35) and differentiate both sides with respect to x. Rearrange the resulting equation into the form $G(x, p) = 0$, which can be used together with the original ODE to eliminate p and so give the general solution. If $G(x, p)$ can be factorised then the factor containing dp/dx should be used to eliminate p and give the general solution. Using the other factors in this fashion will instead lead to singular solutions.*

14.3.4 Clairaut's equation

Finally, we consider Clairaut's equation, which has the form

$$y = px + F(p) \tag{14.39}$$

and is therefore a special case of equations soluble for y, as in (14.35). It may be solved by a similar method to that given in subsection 14.3.3, but for Clairaut's equation the form of the general solution is particularly simple. Differentiating (14.39) with respect to x, we find

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{dF}{dp}\frac{dp}{dx} \quad \Rightarrow \quad \frac{dp}{dx}\left(\frac{dF}{dp} + x\right) = 0. \tag{14.40}$$

Considering first the factor containing dp/dx, we find

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad y = c_1x + c_2. \tag{14.41}$$

Since $p = dy/dx = c_1$, if we substitute (14.41) into (14.39) we find $c_1x + c_2 = c_1x + F(c_1)$. Therefore the constant c_2 is given by $F(c_1)$, and the general solution to (14.39) is

$$y = c_1x + F(c_1), \tag{14.42}$$

i.e. the general solution to Clairaut's equation can be obtained by replacing p in the ODE by the arbitrary constant c_1. Now, considering the second factor in (14.40), we also have

$$\frac{dF}{dp} + x = 0, \tag{14.43}$$

which has the form $G(x, p) = 0$. This relation may be used to eliminate p from (14.39) to give a singular solution.

►*Solve*

$$y = px + p^2. \tag{14.44}$$

From (14.42) the general solution is $y = cx + c^2$. But from (14.43) we also have $2p + x = 0 \Rightarrow p = -x/2$. Substituting this into (14.44) we find the singular solution $x^2 + 4y = 0$. ◄

Solution method. *Write the equation in the form (14.39), then the general solution is given by replacing p by some constant c, as shown in (14.42). Using the relation $dF/dp + x = 0$ to eliminate p from the original equation yields the singular solution.*

14.4 Exercises

14.1 A radioactive isotope decays in such a way that the number of atoms present at a given time, $N(t)$, obeys the equation

$$\frac{dN}{dt} = -\lambda N.$$

If there are initially N_0 atoms present, find $N(t)$ at later times.

14.2 Solve the following equations by separation of the variables:

(a) $y' - xy^3 = 0$;
(b) $y' \tan^{-1} x - y(1 + x^2)^{-1} = 0$;
(c) $x^2y' + xy^2 = 4y^2$.

14.3 Show that the following equations either are exact or can be made exact, and solve them:

(a) $y(2x^2y^2 + 1)y' + x(y^4 + 1) = 0$;
(b) $2xy' + 3x + y = 0$;
(c) $(\cos^2 x + y \sin 2x)y' + y^2 = 0$.

14.4 Find the values of α and β that make

$$dF(x, y) = \left(\frac{1}{x^2 + 2} + \frac{\alpha}{y}\right) dx + (xy^\beta + 1)\, dy$$

an exact differential. For these values solve $F(x, y) = 0$.

14.5 By finding suitable integrating factors, solve the following equations:

(a) $(1-x^2)y' + 2xy = (1-x^2)^{3/2}$;
(b) $y' - y\cot x + \operatorname{cosec} x = 0$;
(c) $(x + y^3)y' = y$ (treat y as the independent variable).

14.6 By finding an appropriate integrating factor, solve

$$\frac{dy}{dx} = -\frac{2x^2 + y^2 + x}{xy}.$$

14.7 Find, in the form of an integral, the solution of the equation

$$\alpha\frac{dy}{dt} + y = f(t)$$

for a general function $f(t)$. Find the specific solutions for

(a) $f(t) = H(t)$,
(b) $f(t) = \delta(t)$,
(c) $f(t) = \beta^{-1}e^{-t/\beta}H(t)$ with $\beta < \alpha$.

For case (c), what happens if $\beta \to 0$?

14.8 A series electric circuit contains a resistance R, a capacitance C and a battery supplying a time-varying electromotive force $V(t)$. The charge q on the capacitor therefore obeys the equation

$$R\frac{dq}{dt} + \frac{q}{C} = V(t).$$

Assuming that initially there is no charge on the capacitor, and given that $V(t) = V_0 \sin\omega t$, find the charge on the capacitor as a function of time.

14.9 Using tangential–polar coordinates (see exercise 2.20), consider a particle of mass m moving under the influence of a force f directed towards the origin O. By resolving forces along the instantaneous tangent and normal and making use of the result of exercise 2.20 for the instantaneous radius of curvature, prove that

$$f = -mv\frac{dv}{dr} \qquad \text{and} \qquad mv^2 = fp\frac{dr}{dp}.$$

Show further that $h = mpv$ is a constant of the motion and that the law of force can be deduced from

$$f = \frac{h^2}{mp^3}\frac{dp}{dr}.$$

14.10 Use the result of exercise 14.9 to find the law of force, acting towards the origin, under which a particle must move so as to describe the following trajectories:

(a) A circle of radius a that passes through the origin;
(b) An equiangular spiral, which is defined by the property that the angle α between the tangent and the radius vector is constant along the curve.

14.11 Solve

$$(y - x)\frac{dy}{dx} + 2x + 3y = 0.$$

14.12 A mass m is accelerated by a time-varying force $\alpha\exp(-\beta t)v^3$, where v is its velocity. It also experiences a resistive force ηv, where η is a constant, owing to its motion through the air. The equation of motion of the mass is therefore

$$m\frac{dv}{dt} = \alpha\exp(-\beta t)v^3 - \eta v.$$

Find an expression for the velocity v of the mass as a function of time, given that it has an initial velocity v_0.

14.13 Using the results about Laplace transforms given in chapter 13 for df/dt and $tf(t)$, show, for a function $y(t)$ that satisfies

$$t\frac{dy}{dt} + (t-1)y = 0 \qquad (*)$$

with $y(0)$ finite, that $\bar{y}(s) = C(1+s)^{-2}$ for some constant C.

Given that

$$y(t) = t + \sum_{n=2}^{\infty} a_n t^n,$$

determine C and show that $a_n = (-1)^{n-1}/(n-1)!$. Compare this result with that obtained by integrating $(*)$ directly.

14.14 Solve

$$\frac{dy}{dx} = \frac{1}{x+2y+1}.$$

14.15 Solve

$$\frac{dy}{dx} = -\frac{x+y}{3x+3y-4}.$$

14.16 If $u = 1 + \tan y$, calculate $d(\ln u)/dy$; hence find the general solution of

$$\frac{dy}{dx} = \tan x \cos y\ (\cos y + \sin y).$$

14.17 Solve

$$x(1-2x^2y)\frac{dy}{dx} + y = 3x^2y^2,$$

given that $y(1) = 1/2$.

14.18 A reflecting mirror is made in the shape of the surface of revolution generated by revolving the curve $y(x)$ about the x-axis. In order that light rays emitted from a point source at the origin are reflected back parallel to the x-axis, the curve $y(x)$ must obey

$$\frac{y}{x} = \frac{2p}{1-p^2},$$

where $p = dy/dx$. By solving this equation for x, find the curve $y(x)$.

14.19 Find the curve with the property that at each point on it the sum of the intercepts on the x- and y-axes of the tangent to the curve (taking account of sign) is equal to 1.

14.20 Find a parametric solution of

$$x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - y = 0$$

as follows.

(a) Write an equation for y in terms of $p = dy/dx$ and show that

$$p = p^2 + (2px+1)\frac{dp}{dx}.$$

(b) Using p as the independent variable, arrange this as a linear first-order equation for x.

(c) Find an appropriate integrating factor to obtain

$$x = \frac{\ln p - p + c}{(1-p)^2},$$

which, together with the expression for y obtained in (a), gives a parameterisation of the solution.

(d) Reverse the roles of x and y in steps (a) to (c), putting $dx/dy = p^{-1}$, and show that essentially the same parameterisation is obtained.

14.21 Using the substitutions $u = x^2$ and $v = y^2$, reduce the equation

$$xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2 - 1)\frac{dy}{dx} + xy = 0$$

to Clairaut's form. Hence show that the equation represents a family of conics and the four sides of a square.

14.22 The action of the control mechanism on a particular system for an input $f(t)$ is described, for $t \geq 0$, by the coupled first-order equations:

$$\dot{y} + 4z = f(t),$$
$$\dot{z} - 2z = \dot{y} + \tfrac{1}{2}y.$$

Use Laplace transforms to find the response $y(t)$ of the system to a unit step input, $f(t) = H(t)$, given that $y(0) = 1$ and $z(0) = 0$.

Questions 23 to 31 are intended to give the reader practice in choosing an appropriate method. The level of difficulty varies within the set; if necessary, the hints may be consulted for an indication of the most appropriate approach.

14.23 Find the general solutions of the following:

(a) $\dfrac{dy}{dx} + \dfrac{xy}{a^2 + x^2} = x$; (b) $\dfrac{dy}{dx} = \dfrac{4y^2}{x^2} - y^2$.

14.24 Solve the following first-order equations for the boundary conditions given:

(a) $y' - (y/x) = 1$, $y(1) = -1$;
(b) $y' - y\tan x = 1$, $y(\pi/4) = 3$;
(c) $y' - y^2/x^2 = 1/4$, $y(1) = 1$;
(d) $y' - y^2/x^2 = 1/4$, $y(1) = 1/2$.

14.25 An electronic system has two inputs, to each of which a constant unit signal is applied, but starting at different times. The equations governing the system thus take the form

$$\dot{x} + 2y = H(t),$$
$$\dot{y} - 2x = H(t-3).$$

Initially (at $t = 0$), $x = 1$ and $y = 0$; find $x(t)$ at later times.

14.26 Solve the differential equation

$$\sin x\frac{dy}{dx} + 2y\cos x = 1,$$

subject to the boundary condition $y(\pi/2) = 1$.

14.27 Find the complete solution of

$$\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx} + \frac{A}{x} = 0,$$

where A is a positive constant.

14.28 Find the solution of

$$(5x + y - 7)\frac{dy}{dx} = 3(x + y + 1).$$

14.29 Find the solution $y = y(x)$ of

$$x\frac{dy}{dx} + y - \frac{y^2}{x^{3/2}} = 0,$$

subject to $y(1) = 1$.

14.30 Find the solution of

$$(2\sin y - x)\frac{dy}{dx} = \tan y,$$

if (a) $y(0) = 0$, and (b) $y(0) = \pi/2$.

14.31 Find the family of solutions of

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = 0$$

that satisfy $y(0) = 0$.

14.5 Hints and answers

14.1 $N(t) = N_0 \exp(-\lambda t)$.

14.3 (a) exact, $x^2y^4 + x^2 + y^2 = c$; (b) IF $= x^{-1/2}$, $x^{1/2}(x + y) = c$; (c) IF $= \sec^2 x$, $y^2 \tan x + y = c$.

14.5 (a) IF $= (1 - x^2)^{-2}$, $y = (1 - x^2)(k + \sin^{-1} x)$; (b) IF $=$ cosec x, leading to $y = k\sin x + \cos x$; (c) exact equation is $y^{-1}(dx/dy) - xy^{-2} = y$, leading to $x = y(k + y^2/2)$.

14.7 $y(t) = e^{-t/\alpha}\int^t \alpha^{-1} e^{t'/\alpha} f(t')\,dt'$; (a) $y(t) = 1 - e^{-t/\alpha}$; (b) $y(t) = \alpha^{-1}e^{-t/\alpha}$; (c) $y(t) = (e^{-t/\alpha} - e^{-t/\beta})/(\alpha - \beta)$. It becomes case (b).

14.9 Note that, if the angle between the tangent and the radius vector is α, then $\cos\alpha = dr/ds$ and $\sin\alpha = p/r$.

14.11 Homogeneous equation, put $y = vx$ to obtain $(1 - v)(v^2 + 2v + 2)^{-1}\,dv = x^{-1}\,dx$; write $1 - v$ as $2 - (1 + v)$, and $v^2 + 2v + 2$ as $1 + (1 + v)^2$; $A[x^2 + (x + y)^2] = \exp\{4\tan^{-1}[(x + y)/x]\}$.

14.13 $(1 + s)(d\bar{y}/ds) + 2\bar{y} = 0$. $C = 1$; use separation of variables to show directly that $y(t) = te^{-t}$.

14.15 The equation is of the form of (14.22), set $v = x + y$; $x + 3y + 2\ln(x + y - 2) = A$.

14.17 The equation is isobaric with weight $y = -2$; setting $y = vx^{-2}$ gives $v^{-1}(1 - v)^{-1}(1 - 2v)\,dv = x^{-1}\,dx$; $4xy(1 - x^2y) = 1$.

14.19 The curve must satisfy $y = (1 - p^{-1})^{-1}(1 - x + px)$, which has solution $x = (p - 1)^{-2}$, leading to $y = (1 \pm \sqrt{x})^2$ or $x = (1 \pm \sqrt{y})^2$; the singular solution $p' = 0$ gives straight lines joining $(\theta, 0)$ and $(0, 1 - \theta)$ for any θ.

14.21 $v = qu + q/(q - 1)$, where $q = dv/du$. General solution $y^2 = cx^2 + c/(c - 1)$, hyperbolae for $c > 0$ and ellipses for $c < 0$. Singular solution $y = \pm(x \pm 1)$.

14.23 (a) Integrating factor is $(a^2 + x^2)^{1/2}$, $y = (a^2 + x^2)/3 + A(a^2 + x^2)^{-1/2}$; (b) separable, $y = x(x^2 + Ax + 4)^{-1}$.

14.25 Use Laplace transforms; $\bar{x}s(s^2 + 4) = s + s^2 - 2e^{-3s}$; $x(t) = \frac{1}{2}\sin 2t + \cos 2t - \frac{1}{2}H(t - 3) + \frac{1}{2}\cos(2t - 6)H(t - 3)$.

14.27 This is Clairaut's equation with $F(p) = A/p$. General solution $y = cx + A/c$; singular solution, $y = 2\sqrt{Ax}$.

14.29 Either Bernoulli's equation with $n = 2$ or an isobaric equation with $m = 3/2$; $y(x) = 5x^{3/2}/(2 + 3x^{5/2})$.

14.31 Show that $p = (Ce^x - 1)^{-1}$, where $p = dy/dx$; $y = \ln[C - e^{-x})/(C - 1)]$ or $\ln[D - (D - 1)e^{-x}]$ or $\ln(e^{-K} + 1 - e^{-x}) + K$.

15

Higher-order ordinary differential equations

Following on from the discussion of first-order ordinary differential equations (ODEs) given in the previous chapter, we now examine equations of second and higher order. Since a brief outline of the general properties of ODEs and their solutions was given at the beginning of the previous chapter, we will not repeat it here. Instead, we will begin with a discussion of various types of higher-order equation. This chapter is divided into three main parts. We first discuss linear equations with constant coefficients and then investigate linear equations with variable coefficients. Finally, we discuss a few methods that may be of use in solving general linear or non-linear ODEs. Let us start by considering some general points relating to *all* linear ODEs.

Linear equations are of paramount importance in the description of physical processes. Moreover, it is an empirical fact that, when put into mathematical form, many natural processes appear as higher-order linear ODEs, most often as second-order equations. Although we could restrict our attention to these second-order equations, the generalisation to nth-order equations requires little extra work, and so we will consider this more general case.

A linear ODE of general order n has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x). \qquad (15.1)$$

If $f(x) = 0$ then the equation is called *homogeneous*; otherwise it is *inhomogeneous*. The first-order linear equation studied in subsection 14.2.4 is a special case of (15.1). As discussed at the beginning of the previous chapter, the general solution to (15.1) will contain n arbitrary constants, which may be determined if n boundary conditions are also provided.

In order to solve any equation of the form (15.1), we must first find the general solution of the *complementary equation*, i.e. the equation formed by setting

$f(x) = 0$:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0. \qquad (15.2)$$

To determine the general solution of (15.2), we must find n linearly independent functions that satisfy it. Once we have found these solutions, the general solution is given by a linear superposition of these n functions. In other words, if the n solutions of (15.2) are $y_1(x)$, $y_2(x), \ldots, y_n(x)$, then the general solution is given by the linear superposition

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x), \qquad (15.3)$$

where the c_m are arbitrary constants that may be determined if n boundary conditions are provided. The linear combination $y_c(x)$ is called the *complementary function* of (15.1).

The question naturally arises how we establish that any n individual solutions to (15.2) are indeed linearly independent. For n functions to be linearly independent over an interval, there must not exist *any* set of constants c_1, $c_2, \ldots, c_n$ such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0 \qquad (15.4)$$

over the interval in question, except for the trivial case $c_1 = c_2 = \cdots = c_n = 0$.

A statement equivalent to (15.4), which is perhaps more useful for the practical determination of linear independence, can be found by repeatedly differentiating (15.4), $n - 1$ times in all, to obtain n simultaneous equations for c_1, $c_2, \ldots, c_n$:

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) &= 0 \\ c_1 y_1{}'(x) + c_2 y_2{}'(x) + \cdots + c_n y_n{}'(x) &= 0 \\ &\ \vdots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)} + \cdots + c_n y_n^{(n-1)}(x) &= 0, \end{aligned} \qquad (15.5)$$

where the primes denote differentiation with respect to x. Referring to the discussion of simultaneous linear equations given in chapter 8, if the determinant of the coefficients of c_1, $c_2, \ldots, c_n$ is non-zero then the only solution to equations (15.5) is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. In other words, the n functions $y_1(x)$, $y_2(x), \ldots, y_n(x)$ are linearly independent over an interval if

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1{}' & y_2{}' & & \vdots \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0 \qquad (15.6)$$

over that interval; $W(y_1, y_2, \ldots, y_n)$ is called the *Wronskian* of the set of functions. It should be noted, however, that the vanishing of the Wronskian does not guarantee that the functions are linearly dependent.

If the original equation (15.1) has $f(x) = 0$ (i.e. it is homogeneous) then of course the complementary function $y_c(x)$ in (15.3) is already the general solution. If, however, the equation has $f(x) \neq 0$ (i.e. it is inhomogeneous) then $y_c(x)$ is only one part of the solution. The general solution of (15.1) is then given by

$$y(x) = y_c(x) + y_p(x), \tag{15.7}$$

where $y_p(x)$ is the *particular integral*, which can be *any* function that satisfies (15.1) directly, provided it is linearly independent of $y_c(x)$. It should be emphasised for practical purposes that *any* such function, no matter how simple (or complicated), is equally valid in forming the general solution (15.7).

It is important to realise that the above method for finding the general solution to an ODE by superposing particular solutions assumes crucially that the ODE is linear. For non-linear equations, discussed in section 15.3, this method cannot be used, and indeed it is often impossible to find closed-form solutions to such equations.

15.1 Linear equations with constant coefficients

If the a_m in (15.1) are constants rather than functions of x then we have

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x). \tag{15.8}$$

Equations of this sort are very common throughout the physical sciences and engineering, and the method for their solution falls into two parts as discussed in the previous section, i.e. finding the complementary function $y_c(x)$ and finding the particular integral $y_p(x)$. If $f(x) = 0$ in (15.8) then we do not have to find a particular integral, and the complementary function is by itself the general solution.

15.1.1 Finding the complementary function $y_c(x)$

The complementary function must satisfy

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0 \tag{15.9}$$

and contain n arbitrary constants (see equation (15.3)). The standard method for finding $y_c(x)$ is to try a solution of the form $y = Ae^{\lambda x}$, substituting this into (15.9). After dividing the resulting equation through by $Ae^{\lambda x}$, we are left with a polynomial equation in λ of order n; this is the *auxiliary equation* and reads

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0. \tag{15.10}$$

In general the auxiliary equation has n roots, say $\lambda_1, \lambda_2, \ldots, \lambda_n$. In certain cases, some of these roots may be repeated and some may be complex. The three main cases are as follows.

(i) *All roots real and distinct*. In this case the n solutions to (15.9) are $\exp \lambda_m x$ for $m = 1$ to n. It is easily shown by calculating the Wronskian (15.6) of these functions that if all the λ_m are distinct then these solutions are linearly independent. We can therefore linearly superpose them, as in (15.3), to form the complementary function

$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}. \qquad (15.11)$$

(ii) *Some roots complex*. For the special (but usual) case that all the coefficients a_m in (15.9) are real, if one of the roots of the auxiliary equation (15.10) is complex, say $\alpha + i\beta$, then its complex conjugate $\alpha - i\beta$ is also a root. In this case we can write

$$\begin{aligned} c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} &= e^{\alpha x}(d_1 \cos\beta x + d_2 \sin\beta x) \\ &= A e^{\alpha x} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} (\beta x + \phi), \end{aligned} \qquad (15.12)$$

where A and ϕ are arbitrary constants.

(iii) *Some roots repeated*. If, for example, λ_1 occurs k times ($k > 1$) as a root of the auxiliary equation, then we have not found n linearly independent solutions of (15.9); formally the Wronskian (15.6) of these solutions, having two or more identical columns, is equal to zero. We must therefore find $k-1$ further solutions that are linearly independent of those already found and also of each other. By direct substitution into (15.9) we find that

$$x e^{\lambda_1 x}, \quad x^2 e^{\lambda_1 x}, \quad \ldots, \quad x^{k-1} e^{\lambda_1 x}$$

are also solutions, and by calculating the Wronskian it is easily shown that they, together with the solutions already found, form a linearly independent set of n functions. Therefore the complementary function is given by

$$y_c(x) = (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{\lambda_1 x} + c_{k+1} e^{\lambda_{k+1} x} + c_{k+2} e^{\lambda_{k+2} x} + \cdots + c_n e^{\lambda_n x}. \qquad (15.13)$$

If more than one root is repeated the above argument is easily extended. For example, suppose as before that λ_1 is a k-fold root of the auxiliary equation and, further, that λ_2 is an l-fold root (of course, $k > 1$ and $l > 1$). Then, from the above argument, the complementary function reads

$$\begin{aligned} y_c(x) = {} & (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{\lambda_1 x} \\ & + (c_{k+1} + c_{k+2} x + \cdots + c_{k+l} x^{l-1}) e^{\lambda_2 x} \\ & + c_{k+l+1} e^{\lambda_{k+l+1} x} + c_{k+l+2} e^{\lambda_{k+l+2} x} + \cdots + c_n e^{\lambda_n x}. \end{aligned} \qquad (15.14)$$

►*Find the complementary function of the equation*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x. \qquad (15.15)$$

Setting the RHS to zero, substituting $y = Ae^{\lambda x}$ and dividing through by $Ae^{\lambda x}$ we obtain the auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0.$$

The root $\lambda = 1$ occurs twice and so, although e^x is a solution to (15.15), we must find a further solution to the equation that is linearly independent of e^x. From the above discussion, we deduce that xe^x is such a solution, so that the full complementary function is given by the linear superposition

$$y_c(x) = (c_1 + c_2 x)e^x. \blacktriangleleft$$

Solution method. *Set the RHS of the ODE to zero (if it is not already so), and substitute $y = Ae^{\lambda x}$. After dividing through the resulting equation by $Ae^{\lambda x}$, obtain an nth-order polynomial equation in λ (the auxiliary equation, see (15.10)). Solve the auxiliary equation to find the n roots, $\lambda_1, \lambda_2, \ldots, \lambda_n$, say. If all these roots are real and distinct then $y_c(x)$ is given by (15.11). If, however, some of the roots are complex or repeated then $y_c(x)$ is given by (15.12) or (15.13), or the extension (15.14) of the latter, respectively.*

15.1.2 Finding the particular integral $y_p(x)$

There is no generally applicable method for finding the particular integral $y_p(x)$ but, for linear ODEs with constant coefficients and a simple RHS, $y_p(x)$ can often be found by inspection or by assuming a parameterised form similar to $f(x)$. The latter method is sometimes called the *method of undetermined coefficients*. If $f(x)$ contains only polynomial, exponential, or sine and cosine terms then, by assuming a trial function for $y_p(x)$ of similar form but one which contains a number of undetermined parameters and substituting this trial function into (15.9), the parameters can be found and $y_p(x)$ deduced. Standard trial functions are as follows.

(i) If $f(x) = ae^{rx}$ then try

$$y_p(x) = be^{rx}.$$

(ii) If $f(x) = a_1 \sin rx + a_2 \cos rx$ (a_1 or a_2 may be zero) then try

$$y_p(x) = b_1 \sin rx + b_2 \cos rx.$$

(iii) If $f(x) = a_0 + a_1 x + \cdots + a_N x^N$ (some a_m may be zero) then try

$$y_p(x) = b_0 + b_1 x + \cdots + b_N x^N.$$

(iv) If $f(x)$ is the sum or product of any of the above then try $y_p(x)$ as the sum or product of the corresponding individual trial functions.

It should be noted that this method fails if any term in the assumed trial function is also contained within the complementary function $y_c(x)$. In such a case the trial function should be multiplied by the smallest integer power of x such that it will then contain no term that already appears in the complementary function. The undetermined coefficients in the trial function can now be found by substitution into (15.8).

Three further methods that are useful in finding the particular integral $y_p(x)$ are those based on Green's functions, the variation of parameters, and a change in the dependent variable using knowledge of the complementary function. However, since these methods are also applicable to equations with variable coefficients, a discussion of them is postponed until section 15.2.

►*Find a particular integral of the equation*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x.$$

From the above discussion our first guess at a trial particular integral would be $y_p(x) = be^x$. However, since the complementary function of this equation is $y_c(x) = (c_1 + c_2x)e^x$ (as in the previous subsection), we see that e^x is already contained in it, as indeed is xe^x. Multiplying our first guess by the lowest integer power of x such that the result does not appear in $y_c(x)$, we therefore try $y_p(x) = bx^2e^x$. Substituting this into the ODE, we find that $b = 1/2$, so the particular integral is given by $y_p(x) = x^2e^x/2$. ◄

Solution method. *If the RHS of an ODE contains only functions mentioned at the start of this subsection then the appropriate trial function should be substituted into it, thereby fixing the undetermined parameters. If, however, the RHS of the equation is not of this form then one of the more general methods outlined in subsections 15.2.3–15.2.5 should be used; perhaps the most straightforward of these is the variation-of-parameters method.*

15.1.3 Constructing the general solution $y_c(x) + y_p(x)$

As stated earlier, the full solution to the ODE (15.8) is found by adding together the complementary function and any particular integral. In order to illustrate further the material discussed in the last two subsections, let us find the general solution to a new example, starting from the beginning.

►*Solve*

$$\frac{d^2y}{dx^2} + 4y = x^2 \sin 2x. \tag{15.16}$$

First we set the RHS to zero and assume the trial solution $y = Ae^{\lambda x}$. Substituting this into (15.16) leads to the auxiliary equation

$$\lambda^2 + 4 = 0 \qquad \Rightarrow \qquad \lambda = \pm 2i. \tag{15.17}$$

Therefore the complementary function is given by

$$y_c(x) = c_1 e^{2ix} + c_2 e^{-2ix} = d_1 \cos 2x + d_2 \sin 2x. \tag{15.18}$$

We must now turn our attention to the particular integral $y_p(x)$. Consulting the list of standard trial functions in the previous subsection, we find that a first guess at a suitable trial function for this case should be

$$(ax^2 + bx + c) \sin 2x + (dx^2 + ex + f) \cos 2x. \tag{15.19}$$

However, we see that this trial function contains terms in $\sin 2x$ and $\cos 2x$, both of which already appear in the complementary function (15.18). We must therefore multiply (15.19) by the smallest integer power of x which ensures that none of the resulting terms appears in $y_c(x)$. Since multiplying by x will suffice, we finally assume the trial function

$$(ax^3 + bx^2 + cx) \sin 2x + (dx^3 + ex^2 + fx) \cos 2x. \tag{15.20}$$

Substituting this into (15.16) to fix the constants appearing in (15.20), we find the particular integral to be

$$y_p(x) = -\frac{x^3}{12} \cos 2x + \frac{x^2}{16} \sin 2x + \frac{x}{32} \cos 2x. \tag{15.21}$$

The general solution to (15.16) then reads

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= d_1 \cos 2x + d_2 \sin 2x - \frac{x^3}{12} \cos 2x + \frac{x^2}{16} \sin 2x + \frac{x}{32} \cos 2x. \end{aligned}$$ ◄

15.1.4 Linear recurrence relations

Before continuing our discussion of higher-order ODEs, we take this opportunity to introduce the discrete analogues of differential equations, which are called *recurrence relations* (or sometimes *difference equations*). Whereas a differential equation gives a prescription, in terms of current values, for the new value of a dependent variable at a point only infinitesimally far away, a recurrence relation describes how the next in a sequence of values u_n, defined only at (non-negative) integer values of the 'independent variable' n, is to be calculated.

In its most general form a recurrence relation expresses the way in which u_{n+1} is to be calculated from all the preceding values $u_0, u_1, \ldots, u_n$. Just as the most general differential equations are intractable, so are the most general recurrence relations, and we will limit ourselves to analogues of the types of differential equations studied earlier in this chapter, namely those that are linear, have constant coefficients and possess simple functions on the RHS. Such equations occur over a broad range of engineering and statistical physics as well as in the realms of finance, business planning and gambling! They form the basis of many numerical methods, particularly those concerned with the numerical solution of ordinary and partial differential equations.

A general recurrence relation is exemplified by the formula

$$u_{n+1} = \sum_{r=0}^{N-1} a_r u_{n-r} + k, \tag{15.22}$$

where N and the a_r are fixed and k is a constant or a simple function of n. Such an equation, involving terms of the series whose indices differ by up to N (ranging from $n-N+1$ to n), is called an Nth-order recurrence relation. It is clear that, given values for $u_0, u_1, \ldots, u_{N-1}$, this is a definitive scheme for generating the series and therefore has a unique solution.

Parallelling the nomenclature of differential equations, if the term not involving any u_n is absent, i.e. $k = 0$, then the recurrence relation is called *homogeneous*. The parallel continues with the form of the general solution of (15.22). If v_n is the general solution of the homogeneous relation, and w_n is *any* solution of the full relation, then

$$u_n = v_n + w_n$$

is the most general solution of the complete recurrence relation. This is straightforwardly verified as follows:

$$\begin{aligned}
u_{n+1} &= v_{n+1} + w_{n+1} \\
&= \sum_{r=0}^{N-1} a_r v_{n-r} + \sum_{r=0}^{N-1} a_r w_{n-r} + k \\
&= \sum_{r=0}^{N-1} a_r (v_{n-r} + w_{n-r}) + k \\
&= \sum_{r=0}^{N-1} a_r u_{n-r} + k.
\end{aligned}$$

Of course, if $k = 0$ then $w_n = 0$ for all n is a trivial particular solution and the complementary solution, v_n, is itself the most general solution.

First-order recurrence relations

First-order relations, for which $N = 1$, are exemplified by

$$u_{n+1} = au_n + k, \tag{15.23}$$

with u_0 specified. The solution to the homogeneous relation is immediate,

$$u_n = Ca^n,$$

and, if k is a constant, the particular solution is equally straightforward: $w_n = K$ for all n, provided K is chosen to satisfy

$$K = aK + k,$$

i.e. $K = k(1-a)^{-1}$. This will be sufficient unless $a = 1$, in which case $u_n = u_0 + nk$ is obvious by inspection.

Thus the general solution of (15.23) is

$$u_n = \begin{cases} Ca^n + k/(1-a) & a \neq 1, \\ u_0 + nk & a = 1. \end{cases} \tag{15.24}$$

If u_0 is specified for the case of $a \neq 1$ then C must be chosen as $C = u_0 - k/(1-a)$, resulting in the equivalent form

$$u_n = u_0 a^n + k\frac{1-a^n}{1-a}. \tag{15.25}$$

We now illustrate this method with a worked example.

►*A house-buyer borrows capital B from a bank that charges a fixed annual rate of interest R%. If the loan is to be repaid over Y years, at what value should the fixed annual payments P, made at the end of each year, be set? For a loan over 25 years at 6%, what percentage of the first year's payment goes towards paying off the capital?*

Let u_n denote the outstanding debt at the end of year n, and write $R/100 = r$. Then the relevant recurrence relation is

$$u_{n+1} = u_n(1+r) - P$$

with $u_0 = B$. From (15.25) we have

$$u_n = B(1+r)^n - P\frac{1-(1+r)^n}{1-(1+r)}.$$

As the loan is to be repaid over Y years, $u_Y = 0$ and thus

$$P = \frac{Br(1+r)^Y}{(1+r)^Y - 1}.$$

The first year's interest is rB and so the fraction of the first year's payment going towards capital repayment is $(P - rB)/P$, which, using the above expression for P, is equal to $(1+r)^{-Y}$. With the given figures, this is (only) 23%. ◄

With only small modifications, the method just described can be adapted to handle recurrence relations in which the constant k in (15.23) is replaced by $k\alpha^n$, i.e. the relation is

$$u_{n+1} = au_n + k\alpha^n. \tag{15.26}$$

As for an inhomogeneous linear differential equation (see subsection 15.1.2), we may try as a potential particular solution a form which resembles the term that makes the equation inhomogeneous. Here, the presence of the term $k\alpha^n$ indicates that a particular solution of the form $u_n = A\alpha^n$ should be tried. Substituting this into (15.26) gives

$$A\alpha^{n+1} = aA\alpha^n + k\alpha^n,$$

from which it follows that $A = k/(\alpha - a)$ and that there is a particular solution having the form $u_n = k\alpha^n/(\alpha - a)$, provided $\alpha \neq a$. For the special case $\alpha = a$, the reader can readily verify that a particular solution of the form $u_n = An\alpha^n$ is appropriate. This mirrors the corresponding situation for linear differential equations when the RHS of the differential equation is contained in the complementary function of its LHS.

In summary, the general solution to (15.26) is

$$u_n = \begin{cases} C_1 a^n + k\alpha^n/(\alpha - a) & \alpha \neq a, \\ C_2 a^n + kn\alpha^{n-1} & \alpha = a, \end{cases} \tag{15.27}$$

with $C_1 = u_0 - k/(\alpha - a)$ and $C_2 = u_0$.

Second-order recurrence relations

We consider next recurrence relations that involve u_{n-1} in the prescription for u_{n+1} and treat the general case in which the intervening term, u_n, is also present. A typical equation is thus

$$u_{n+1} = au_n + bu_{n-1} + k. \tag{15.28}$$

As previously, the general solution of this is $u_n = v_n + w_n$, where v_n satisfies

$$v_{n+1} = av_n + bv_{n-1} \tag{15.29}$$

and w_n is *any* particular solution of (15.28); the proof follows the same lines as that given earlier.

We have already seen for a first-order recurrence relation that the solution to the homogeneous equation is given by terms forming a geometric series, and we consider a corresponding series of powers in the present case. Setting $v_n = A\lambda^n$ in (15.29) for some λ, as yet undetermined, gives the requirement that λ should satisfy

$$A\lambda^{n+1} = aA\lambda^n + bA\lambda^{n-1}.$$

Dividing through by $A\lambda^{n-1}$ (assumed non-zero) shows that λ could be either of the roots, λ_1 and λ_2, of

$$\lambda^2 - a\lambda - b = 0, \tag{15.30}$$

which is known as the *characteristic equation* of the recurrence relation.

That there are two possible series of terms of the form $A\lambda^n$ is consistent with the fact that two initial values (boundary conditions) have to be provided before the series can be calculated by repeated use of (15.28). These two values are sufficient to determine the appropriate coefficient A for each of the series. Since (15.29) is both linear and homogeneous, and is satisfied by both $v_n = A\lambda_1^n$ and $v_n = B\lambda_2^n$, its general solution is

$$v_n = A\lambda_1^n + B\lambda_2^n.$$

If the coefficients a and b are such that (15.30) has two equal roots, i.e. $a^2 = -4b$, then, as in the analogous case of repeated roots for differential equations (see subsection 15.1.1(iii)), the second term of the general solution is replaced by $Bn\lambda_1^n$ to give

$$v_n = (A + Bn)\lambda_1^n.$$

Finding a particular solution is straightforward if k is a constant: a trivial but adequate solution is $w_n = k(1 - a - b)^{-1}$ for all n. As with first-order equations, particular solutions can be found for other simple forms of k by trying functions similar to k itself. Thus particular solutions for the cases $k = Cn$ and $k = D\alpha^n$ can be found by trying $w_n = E + Fn$ and $w_n = G\alpha^n$ respectively.

►*Find the value of u_{16} if the series u_n satisfies*

$$u_{n+1} + 4u_n + 3u_{n-1} = n$$

for $n \geq 1$, with $u_0 = 1$ and $u_1 = -1$.

We first solve the characteristic equation,

$$\lambda^2 + 4\lambda + 3 = 0,$$

to obtain the roots $\lambda = -1$ and $\lambda = -3$. Thus the complementary function is

$$v_n = A(-1)^n + B(-3)^n.$$

In view of the form of the RHS of the original relation, we try

$$w_n = E + Fn$$

as a particular solution and obtain

$$E + F(n+1) + 4(E + Fn) + 3[E + F(n-1)] = n,$$

yielding $F = 1/8$ and $E = 1/32$.

Thus the complete general solution is

$$u_n = A(-1)^n + B(-3)^n + \frac{n}{8} + \frac{1}{32},$$

and now using the given values for u_0 and u_1 determines A as $7/8$ and B as $3/32$. Thus

$$u_n = \frac{1}{32}\left[28(-1)^n + 3(-3)^n + 4n + 1\right].$$

Finally, substituting $n = 16$ gives $u_{16} = 4\,035\,633$, a value the reader may (or may not) wish to verify by repeated application of the initial recurrence relation. ◄

Higher-order recurrence relations

It will be apparent that linear recurrence relations of order $N > 2$ do not present any additional difficulty in principle, though two obvious practical difficulties are (i) that the characteristic equation is of order N and in general will not have roots that can be written in closed form and (ii) that a correspondingly large number of given values is required to determine the N otherwise arbitrary constants in the solution. The algebraic labour needed to solve the set of simultaneous linear equations that determines them increases rapidly with N. We do not give specific examples here, but some are included in the exercises at the end of the chapter.

15.1.5 Laplace transform method

Having briefly discussed recurrence relations, we now return to the main topic of this chapter, i.e. methods for obtaining solutions to higher-order ODEs. One such method is that of Laplace transforms, which is very useful for solving linear ODEs with constant coefficients. Taking the Laplace transform of such an equation transforms it into a purely *algebraic* equation in terms of the Laplace transform of the required solution. Once the algebraic equation has been solved for this Laplace transform, the general solution to the original ODE can be obtained by performing an inverse Laplace transform. One advantage of this method is that, for given boundary conditions, it provides the solution in just one step, instead of having to find the complementary function and particular integral separately.

In order to apply the method we need only two results from Laplace transform theory (see section 13.2). First, the Laplace transform of a function $f(x)$ is defined by

$$\bar{f}(s) \equiv \int_0^\infty e^{-sx} f(x)\, dx, \tag{15.31}$$

from which we can derive the second useful relation. This concerns the Laplace transform of the nth derivative of $f(x)$:

$$\overline{f^{(n)}}(s) = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0), \tag{15.32}$$

where the primes and superscripts in parentheses denote differentiation with respect to x. Using these relations, along with table 13.1, on p. 455, which gives Laplace transforms of standard functions, we are in a position to solve a linear ODE with constant coefficients by this method.

►*Solve*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{-x}, \tag{15.33}$$

subject to the boundary conditions $y(0) = 2$, $y'(0) = 1$.

Taking the Laplace transform of (15.33) and using the table of standard results we obtain

$$s^2\bar{y}(s) - sy(0) - y'(0) - 3\,[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = \frac{2}{s+1},$$

which reduces to

$$(s^2 - 3s + 2)\bar{y}(s) - 2s + 5 = \frac{2}{s+1}. \tag{15.34}$$

Solving this algebraic equation for $\bar{y}(s)$, the Laplace transform of the required solution to (15.33), we obtain

$$\bar{y}(s) = \frac{2s^2 - 3s - 3}{(s+1)(s-1)(s-2)} = \frac{1}{3(s+1)} + \frac{2}{s-1} - \frac{1}{3(s-2)}, \tag{15.35}$$

where in the final step we have used partial fractions. Taking the inverse Laplace transform of (15.35), again using table 13.1, we find the specific solution to (15.33) to be

$$y(x) = \tfrac{1}{3}e^{-x} + 2e^{x} - \tfrac{1}{3}e^{2x}. \blacktriangleleft$$

Note that if the boundary conditions in a problem are given as symbols, rather than just numbers, then the step involving partial fractions can often involve a considerable amount of algebra. The Laplace transform method is also very convenient for solving sets of *simultaneous* linear ODEs with constant coefficients.

►*Two electrical circuits, both of negligible resistance, each consist of a coil having self-inductance L and a capacitor having capacitance C. The mutual inductance of the two circuits is M. There is no source of e.m.f. in either circuit. Initially the second capacitor is given a charge* CV_0*, the first capacitor being uncharged, and at time* $t = 0$ *a switch in the second circuit is closed to complete the circuit. Find the subsequent current in the first circuit.*

Subject to the initial conditions $q_1(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0$ and $q_2(0) = CV_0 = V_0/G$, say, we have to solve

$$L\ddot{q}_1 + M\ddot{q}_2 + Gq_1 = 0,$$
$$M\ddot{q}_1 + L\ddot{q}_2 + Gq_2 = 0.$$

On taking the Laplace transform of the above equations, we obtain

$$(Ls^2 + G)\bar{q}_1 + Ms^2\bar{q}_2 = sMV_0C,$$
$$Ms^2\bar{q}_1 + (Ls^2 + G)\bar{q}_2 = sLV_0C.$$

Eliminating $\bar{q}_2$ and rewriting as an equation for $\bar{q}_1$, we find

$$\bar{q}_1(s) = \frac{MV_0 s}{[(L+M)s^2 + G][(L-M)s^2 + G]}$$
$$= \frac{V_0}{2G}\left[\frac{(L+M)s}{(L+M)s^2 + G} - \frac{(L-M)s}{(L-M)s^2 + G}\right].$$

Using table 13.1,

$$q_1(t) = \tfrac{1}{2}V_0C(\cos\omega_1 t - \cos\omega_2 t),$$

where $\omega_1^2(L+M) = G$ and $\omega_2^2(L-M) = G$. Thus the current is given by

$$i_1(t) = \tfrac{1}{2}V_0C(\omega_2\sin\omega_2 t - \omega_1\sin\omega_1 t). \blacktriangleleft$$

Solution method. *Perform a Laplace transform, as defined in (15.31), on the entire equation, using (15.32) to calculate the transform of the derivatives. Then solve the resulting algebraic equation for $\bar{y}(s)$, the Laplace transform of the required solution to the ODE. By using the method of partial fractions and consulting a table of Laplace transforms of standard functions, calculate the inverse Laplace transform. The resulting function $y(x)$ is the solution of the ODE that obeys the given boundary conditions.*

15.2 Linear equations with variable coefficients

There is no generally applicable method of solving equations with coefficients that are functions of x. Nevertheless, there are certain cases in which a solution is possible. Some of the methods discussed in this section are also useful in finding the general solution or particular integral for equations with constant coefficients that have proved impenetrable by the techniques discussed above.

15.2.1 The Legendre and Euler linear equations

Legendre's linear equation has the form

$$a_n(\alpha x+\beta)^n\frac{d^n y}{dx^n} + \cdots + a_1(\alpha x+\beta)\frac{dy}{dx} + a_0 y = f(x), \tag{15.36}$$

where α, β and the a_n are constants and may be solved by making the substitution $\alpha x+\beta = e^t$. We then have

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{\alpha}{\alpha x+\beta}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\alpha^2}{(\alpha x+\beta)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

and so on for higher derivatives. Therefore we can write the terms of (15.36) as

$$\begin{aligned}
(\alpha x+\beta)\frac{dy}{dx} &= \alpha\frac{dy}{dt},\\
(\alpha x+\beta)^2\frac{d^2y}{dx^2} &= \alpha^2\frac{d}{dt}\left(\frac{d}{dt}-1\right)y,\\
&\;\;\vdots\\
(\alpha x+\beta)^n\frac{d^ny}{dx^n} &= \alpha^n\frac{d}{dt}\left(\frac{d}{dt}-1\right)\cdots\left(\frac{d}{dt}-n+1\right)y.
\end{aligned} \tag{15.37}$$

Substituting equations (15.37) into the original equation (15.36), the latter becomes a linear ODE with constant coefficients, i.e.

$$a_n\alpha^n \frac{d}{dt}\left(\frac{d}{dt}-1\right)\cdots\left(\frac{d}{dt}-n+1\right)y+\cdots+a_1\alpha\frac{dy}{dt}+a_0y=f\left(\frac{e^t-\beta}{\alpha}\right),$$

which can be solved by the methods of section 15.1.

A special case of Legendre's linear equation, for which $\alpha = 1$ and $\beta = 0$, is *Euler's equation*,

$$a_nx^n\frac{d^ny}{dx^n}+\cdots+a_1x\frac{dy}{dx}+a_0y=f(x); \qquad (15.38)$$

it may be solved in a similar manner to the above by substituting $x = e^t$. If $f(x) = 0$ in (15.38) then substituting $y = x^\lambda$ leads to a simple algebraic equation in λ, which can be solved to yield the solution to (15.38). In the event that the algebraic equation for λ has repeated roots, extra care is needed. If λ_1 is a k-fold root ($k > 1$) then the k linearly independent solutions corresponding to this root are x^{λ_1}, $x^{\lambda_1}\ln x, \ldots, x^{\lambda_1}(\ln x)^{k-1}$.

►*Solve*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0 \qquad (15.39)$$

by both of the methods discussed above.

First we make the substitution $x = e^t$, which, after cancelling e^t, gives an equation with constant coefficients, i.e.

$$\frac{d}{dt}\left(\frac{d}{dt}-1\right)y+\frac{dy}{dt}-4y=0 \qquad \Rightarrow \qquad \frac{d^2y}{dt^2}-4y=0. \qquad (15.40)$$

Using the methods of section 15.1, the general solution of (15.40), and therefore of (15.39), is given by

$$y=c_1e^{2t}+c_2e^{-2t}=c_1x^2+c_2x^{-2}.$$

Since the RHS of (15.39) is zero, we can reach the same solution by substituting $y = x^\lambda$ into (15.39). This gives

$$\lambda(\lambda-1)x^\lambda+\lambda x^\lambda-4x^\lambda=0,$$

which reduces to

$$(\lambda^2-4)x^\lambda=0.$$

This has the solutions $\lambda = \pm 2$, so we obtain again the general solution

$$y=c_1x^2+c_2x^{-2}. \blacktriangleleft$$

Solution method. *If the ODE is of the Legendre form (15.36) then substitute $\alpha x+\beta = e^t$. This results in an equation of the same order but with constant coefficients, which can be solved by the methods of section 15.1. If the ODE is of the Euler form (15.38) with a non-zero RHS then substitute $x = e^t$; this again leads to an equation of the same order but with constant coefficients. If, however, $f(x) = 0$ in the Euler equation (15.38) then the equation may also be solved by substituting*

$y = x^\lambda$. This leads to an algebraic equation whose solution gives the allowed values of λ; the general solution is then the linear superposition of these functions.

15.2.2 Exact equations

Sometimes an ODE may be merely the derivative of another ODE of one order lower. If this is the case then the ODE is called exact. The nth-order linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{15.41}$$

is exact if the LHS can be written as a simple derivative, i.e. if

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = \frac{d}{dx}\left[b_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_0(x)y\right]. \tag{15.42}$$

It may be shown that, for (15.42) to hold, we require

$$a_0(x) - a_1'(x) + a_2''(x) - \cdots + (-1)^n a_n^{(n)}(x) = 0, \tag{15.43}$$

where the prime again denotes differentiation with respect to x. If (15.43) is satisfied then straightforward integration leads to a new equation of one order lower. If this simpler equation can be solved then a solution to the original equation is obtained. Of course, if the above process leads to an equation that is itself exact then the analysis can be repeated to reduce the order still further.

►*Solve*

$$(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 1. \tag{15.44}$$

Comparing with (15.41), we have $a_2 = 1 - x^2$, $a_1 = -3x$ and $a_0 = -1$. It is easily shown that $a_0 - a_1' + a_2'' = 0$, so (15.44) is exact and can therefore be written in the form

$$\frac{d}{dx}\left[b_1(x)\frac{dy}{dx} + b_0(x)y\right] = 1. \tag{15.45}$$

Expanding the LHS of (15.45) we find

$$\frac{d}{dx}\left(b_1\frac{dy}{dx} + b_0 y\right) = b_1\frac{d^2y}{dx^2} + (b_1' + b_0)\frac{dy}{dx} + b_0' y. \tag{15.46}$$

Comparing (15.44) and (15.46) we find

$$b_1 = 1 - x^2, \qquad b_1' + b_0 = -3x, \qquad b_0' = -1.$$

These relations integrate consistently to give $b_1 = 1 - x^2$ and $b_0 = -x$, so (15.44) can be written as

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx} - xy\right] = 1. \tag{15.47}$$

Integrating (15.47) gives us directly the first-order linear ODE

$$\frac{dy}{dx} - \left(\frac{x}{1 - x^2}\right)y = \frac{x + c_1}{1 - x^2},$$

which can be solved by the method of subsection 14.2.4 and has the solution

$$y = \frac{c_1 \sin^{-1} x + c_2}{\sqrt{1 - x^2}} - 1. ◄$$

It is worth noting that, even if a higher-order ODE is not exact in its given form, it may sometimes be made exact by multiplying through by some suitable function, an *integrating factor*, cf. subsection 14.2.3. Unfortunately, no straightforward method for finding an integrating factor exists and one often has to rely on inspection or experience.

►*Solve*

$$x(1-x^2)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} - xy = x. \tag{15.48}$$

It is easily shown that (15.48) is not exact, but we also see immediately that by multiplying it through by $1/x$ we recover (15.44), which is exact and is solved above. ◄

Another important point is that an ODE need not be linear to be exact, although no simple rule such as (15.43) exists if it is not linear. Nevertheless, it is often worth exploring the possibility that a non-linear equation is exact, since it could then be reduced in order by one and may lead to a soluble equation. This is discussed further in subsection 15.3.3.

Solution method. *For a linear ODE of the form (15.41) check whether it is exact using equation (15.43). If it is not then attempt to find an integrating factor which when multiplying the equation makes it exact. Once the equation is exact write the LHS as a derivative as in (15.42) and, by expanding this derivative and comparing with the LHS of the ODE, determine the functions $b_m(x)$ in (15.42). Integrate the resulting equation to yield another ODE, of one order lower. This may be solved or simplified further if the new ODE is itself exact or can be made so.*

15.2.3 Partially known complementary function

Suppose we wish to solve the nth-order linear ODE

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{15.49}$$

and we happen to know that $u(x)$ is a solution of (15.49) when the RHS is set to zero, i.e. $u(x)$ is one part of the complementary function. By making the substitution $y(x) = u(x)v(x)$, we can transform (15.49) into an equation of order $n-1$ in dv/dx. This simpler equation may prove soluble.

In particular, if the original equation is of second order then we obtain a first-order equation in dv/dx, which may be soluble using the methods of section 14.2. In this way both the remaining term in the complementary function and the particular integral are found. This method therefore provides a useful way of calculating particular integrals for second-order equations with variable (or constant) coefficients.

►*Solve*

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x. \tag{15.50}$$

We see that the RHS does not fall into any of the categories listed in subsection 15.1.2, and so we are at an initial loss as to how to find the particular integral. However, the complementary function of (15.50) is

$$y_c(x) = c_1 \sin x + c_2 \cos x,$$

and so let us choose the solution $u(x) = \cos x$ (we could equally well choose $\sin x$) and make the substitution $y(x) = v(x)u(x) = v(x)\cos x$ into (15.50). This gives

$$\cos x \frac{d^2v}{dx^2} - 2\sin x \frac{dv}{dx} = \operatorname{cosec} x, \tag{15.51}$$

which is a first-order linear ODE in dv/dx and may be solved by multiplying through by a suitable integrating factor, as discussed in subsection 14.2.4. Writing (15.51) as

$$\frac{d^2v}{dx^2} - 2\tan x \frac{dv}{dx} = \frac{\operatorname{cosec} x}{\cos x}, \tag{15.52}$$

we see that the required integrating factor is given by

$$\exp\left\{-2\int \tan x\, dx\right\} = \exp\left[2\ln(\cos x)\right] = \cos^2 x.$$

Multiplying both sides of (15.52) by the integrating factor $\cos^2 x$ we obtain

$$\frac{d}{dx}\left(\cos^2 x \frac{dv}{dx}\right) = \cot x,$$

which integrates to give

$$\cos^2 x \frac{dv}{dx} = \ln(\sin x) + c_1.$$

After rearranging and integrating again, this becomes

$$\begin{aligned} v &= \int \sec^2 x \ln(\sin x)\, dx + c_1 \int \sec^2 x\, dx \\ &= \tan x \ln(\sin x) - x + c_1 \tan x + c_2. \end{aligned}$$

Therefore the general solution to (15.50) is given by $y = uv = v\cos x$, i.e.

$$y = c_1 \sin x + c_2 \cos x + \sin x \ln(\sin x) - x\cos x,$$

which contains the full complementary function and the particular integral. ◄

Solution method. *If $u(x)$ is a known solution of the nth-order equation (15.49) with $f(x) = 0$, then make the substitution $y(x) = u(x)v(x)$ in (15.49). This leads to an equation of order $n-1$ in dv/dx, which might be soluble.*

15.2.4 Variation of parameters

The method of variation of parameters proves useful in finding particular integrals for linear ODEs with variable (and constant) coefficients. However, it requires knowledge of the entire complementary function, not just of one part of it as in the previous subsection.

Suppose we wish to find a particular integral of the equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{15.53}$$

and the complementary function $y_c(x)$ (the general solution of (15.53) with $f(x) = 0$) is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where the functions $y_m(x)$ are known. We now assume that a particular integral of (15.53) can be expressed in a form similar to that of the complementary function, but with the constants c_m replaced by functions of x, i.e. we assume a particular integral of the form

$$y_p(x) = k_1(x)y_1(x) + k_2(x)y_2(x) + \cdots + k_n(x)y_n(x). \tag{15.54}$$

This will no longer satisfy the complementary equation (i.e. (15.53) with the RHS set to zero) but might, with suitable choices of the functions $k_i(x)$, be made equal to $f(x)$, thus producing not a complementary function but a particular integral.

Since we have n arbitrary functions $k_1(x), k_2(x), \ldots, k_n(x)$, but only one restriction on them (namely the ODE), we may impose a further $n-1$ constraints. We can choose these constraints to be as convenient as possible, and the simplest choice is given by

$$\begin{aligned}
k_1'(x)y_1(x) + k_2'(x)y_2(x) + \cdots + k_n'(x)y_n(x) &= 0\\
k_1'(x)y_1'(x) + k_2'(x)y_2'(x) + \cdots + k_n'(x)y_n'(x) &= 0\\
&\ \ \vdots\\
k_1'(x)y_1^{(n-2)}(x) + k_2'(x)y_2^{(n-2)}(x) + \cdots + k_n'(x)y_n^{(n-2)}(x) &= 0\\
k_1'(x)y_1^{(n-1)}(x) + k_2'(x)y_2^{(n-1)}(x) + \cdots + k_n'(x)y_n^{(n-1)}(x) &= \frac{f(x)}{a_n(x)},
\end{aligned} \tag{15.55}$$

where the primes denote differentiation with respect to x. The last of these equations is not a freely chosen constraint; given the previous $n-1$ constraints and the original ODE, it must be satisfied.

This choice of constraints is easily justified (although the algebra is quite messy). Differentiating (15.54) with respect to x, we obtain

$$y_p' = k_1 y_1' + k_2 y_2' + \cdots + k_n y_n' + [\,k_1' y_1 + k_2' y_2 + \cdots + k_n' y_n\,],$$

where, for the moment, we drop the explicit x-dependence of these functions. Since we are free to choose our constraints as we wish, let us define the expression in parentheses to be zero, giving the first equation in (15.55). Differentiating again we find

$$y_{\rm p}'' = k_1 y_1'' + k_2 y_2'' + \cdots + k_n y_n'' + [k_1' y_1' + k_2' y_2' + \cdots + k_n' y_n'].$$

Once more we can choose the expression in brackets to be zero, giving the second equation in (15.55). We can repeat this procedure, choosing the corresponding expression in each case to be zero. This yields the first $n-1$ equations in (15.55). The mth derivative of $y_{\rm p}$ for $m < n$ is then given by

$$y_{\rm p}^{(m)} = k_1 y_1^{(m)} + k_2 y_2^{(m)} + \cdots + k_n y_n^{(m)}.$$

Differentiating $y_{\rm p}$ once more we find that its nth derivative is given by

$$y_{\rm p}^{(n)} = k_1 y_1^{(n)} + k_2 y_2^{(n)} + \cdots + k_n y_n^{(n)} + [k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + \cdots + k_n' y_n^{(n-1)}].$$

Substituting the expressions for $y_{\rm p}^{(m)}$, $m = 0$ to n, into the original ODE (15.53), we obtain

$$\sum_{m=0}^{n} a_m [k_1 y_1^{(m)} + k_2 y_2^{(m)} + \cdots + k_n y_n^{(m)}] + a_n [k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + \cdots + k_n' y_n^{(n-1)}] = f(x),$$

i.e.

$$\sum_{m=0}^{n} a_m \sum_{j=1}^{n} k_j y_j^{(m)} + a_n [k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + \cdots + k_n' y_n^{(n-1)}] = f(x).$$

Rearranging the order of summation on the LHS, we find

$$\sum_{j=1}^{n} k_j [a_n y_j^{(n)} + \cdots + a_1 y_j' + a_0 y_j] + a_n [k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + \cdots + k_n' y_n^{(n-1)}] = f(x). \tag{15.56}$$

But since the functions y_j are solutions of the complementary equation of (15.53) we have (for all j)

$$a_n y_j^{(n)} + \cdots + a_1 y_j' + a_0 y_j = 0.$$

Therefore (15.56) becomes

$$a_n [k_1' y_1^{(n-1)} + k_2' y_2^{(n-1)} + \cdots + k_n' y_n^{(n-1)}] = f(x),$$

which is the final equation given in (15.55).

Considering (15.55) to be a set of simultaneous equations in the set of unknowns $k_1'(x), k_2', \ldots, k_n'(x)$, we see that the determinant of the coefficients of these functions is equal to the Wronskian $W(y_1, y_2, \ldots, y_n)$, which is non-zero since the solutions $y_m(x)$ are linearly independent; see equation (15.6). Therefore (15.55) can be solved for the functions $k_m'(x)$, which in turn can be integrated, setting all constants of integration equal to zero, to give $k_m(x)$. The general solution to (15.53) is then given by

$$y(x) = y_c(x) + y_p(x) = \sum_{m=1}^{n} [c_m + k_m(x)]\, y_m(x).$$

Note that if the constants of integration are included in the $k_m(x)$ then, as well as finding the particular integral, we redefine the arbitrary constants c_m in the complementary function.

►*Use the variation-of-parameters method to solve*

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x, \qquad (15.57)$$

subject to the boundary conditions $y(0) = y(\pi/2) = 0$.

The complementary function of (15.57) is again

$$y_c(x) = c_1 \sin x + c_2 \cos x.$$

We therefore assume a particular integral of the form

$$y_p(x) = k_1(x) \sin x + k_2(x) \cos x,$$

and impose the additional constraints of (15.55), i.e.

$$k_1'(x) \sin x + k_2'(x) \cos x = 0,$$
$$k_1'(x) \cos x - k_2'(x) \sin x = \operatorname{cosec} x.$$

Solving these equations for $k_1'(x)$ and $k_2'(x)$ gives

$$k_1'(x) = \cos x \operatorname{cosec} x = \cot x,$$
$$k_2'(x) = -\sin x \operatorname{cosec} x = -1.$$

Hence, ignoring the constants of integration, $k_1(x)$ and $k_2(x)$ are given by

$$k_1(x) = \ln(\sin x),$$
$$k_2(x) = -x.$$

The general solution to the ODE (15.57) is therefore

$$y(x) = [c_1 + \ln(\sin x)] \sin x + (c_2 - x) \cos x,$$

which is identical to the solution found in subsection 15.2.3. Applying the boundary conditions $y(0) = y(\pi/2) = 0$ we find $c_1 = c_2 = 0$ and so

$$y(x) = \ln(\sin x) \sin x - x \cos x. \blacktriangleleft$$

Solution method. *If the complementary function of (15.53) is known then assume a particular integral of the same form but with the constants replaced by functions of x. Impose the constraints in (15.55) and solve the resulting system of equations for the unknowns* $k_1'(x), k_2', \ldots, k_n'(x)$. *Integrate these functions, setting constants of integration equal to zero, to obtain* $k_1(x), k_2(x), \ldots, k_n(x)$ *and hence the particular integral.*

15.2.5 Green's functions

The Green's function method of solving linear ODEs bears a striking resemblance to the method of variation of parameters discussed in the previous subsection; it too requires knowledge of the entire complementary function in order to find the particular integral and therefore the general solution. The Green's function approach differs, however, since once the Green's function for a particular LHS of (15.1) and particular boundary conditions has been found, then the solution for *any* RHS (i.e. any $f(x)$) can be written down immediately, albeit in the form of an integral.

Although the Green's function method can be approached by considering the superposition of eigenfunctions of the equation (see chapter 17) and is also applicable to the solution of partial differential equations (see chapter 21), this section adopts a more utilitarian approach based on the properties of the Dirac delta function (see subsection 13.1.3) and deals only with the use of Green's functions in solving ODEs.

Let us again consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{15.58}$$

but for the sake of brevity we now denote the LHS by $\mathcal{L}y(x)$, i.e. as a linear differential operator acting on $y(x)$. Thus (15.58) now reads

$$\mathcal{L}y(x) = f(x). \tag{15.59}$$

Let us suppose that a function $G(x,z)$ (the *Green's function*) exists such that the general solution to (15.59), which obeys some set of imposed boundary conditions in the range $a \le x \le b$, is given by

$$y(x) = \int_a^b G(x,z)f(z)\,dz, \tag{15.60}$$

where z is the integration variable. If we apply the linear differential operator $\mathcal{L}$ to both sides of (15.60) and use (15.59) then we obtain

$$\mathcal{L}y(x) = \int_a^b [\mathcal{L}G(x,z)]\,f(z)\,dz = f(x). \tag{15.61}$$

Comparison of (15.61) with a standard property of the Dirac delta function (see subsection 13.1.3), namely

$$f(x) = \int_a^b \delta(x-z)f(z)\,dz,$$

for $a \le x \le b$, shows that for (15.61) to hold for any arbitrary function $f(x)$, we require (for $a \le x \le b$) that

$$\mathcal{L}G(x,z) = \delta(x-z), \tag{15.62}$$

i.e. the Green's function $G(x, z)$ must satisfy the original ODE with the RHS set equal to a delta function. $G(x, z)$ may be thought of physically as the response of a system to a unit impulse at $x = z$.

In addition to (15.62), we must impose two further sets of restrictions on $G(x, z)$. The first is the requirement that the general solution $y(x)$ in (15.60) obeys the boundary conditions. For *homogeneous* boundary conditions, in which $y(x)$ and/or its derivatives are required to be *zero* at specified points, this is most simply arranged by demanding that $G(x, z)$ itself obeys the boundary conditions when it is considered as a function of x alone; if, for example, we require $y(a) = y(b) = 0$ then we should also demand $G(a, z) = G(b, z) = 0$. Problems having inhomogeneous boundary conditions are discussed at the end of this subsection.

The second set of restrictions concerns the continuity or discontinuity of $G(x, z)$ and its derivatives at $x = z$ and can be found by integrating (15.62) with respect to x over the small interval $[z - \epsilon,\ z + \epsilon]$ and taking the limit as $\epsilon \to 0$. We then obtain

$$\lim_{\epsilon \to 0} \sum_{m=0}^{n} \int_{z-\epsilon}^{z+\epsilon} a_m(x) \frac{d^m G(x, z)}{dx^m}\, dx = \lim_{\epsilon \to 0} \int_{z-\epsilon}^{z+\epsilon} \delta(x - z)\, dx = 1. \tag{15.63}$$

Since $d^n G/dx^n$ exists at $x = z$ but with value infinity, the $(n-1)$th-order derivative must have a finite discontinuity there, whereas all the lower-order derivatives, $d^m G/dx^m$ for $m < n - 1$, must be continuous at this point. Therefore the terms containing these derivatives cannot contribute to the value of the integral on the LHS of (15.63). Noting that, apart from an arbitrary additive constant, $\int (d^m G/dx^m)\, dx = d^{m-1} G/dx^{m-1}$, and integrating the terms on the LHS of (15.63) by parts we find

$$\lim_{\epsilon \to 0} \int_{z-\epsilon}^{z+\epsilon} a_m(x) \frac{d^m G(x, z)}{dx^m}\, dx = 0 \tag{15.64}$$

for $m = 0$ to $n - 1$. Thus, since only the term containing $d^n G/dx^n$ contributes to the integral in (15.63), we conclude, after performing an integration by parts, that

$$\lim_{\epsilon \to 0} \left[a_n(x) \frac{d^{n-1} G(x, z)}{dx^{n-1}} \right]_{z-\epsilon}^{z+\epsilon} = 1. \tag{15.65}$$

Thus we have the further n constraints that $G(x, z)$ and its derivatives up to order $n - 2$ are continuous at $x = z$ but that $d^{n-1} G/dx^{n-1}$ has a discontinuity of $1/a_n(z)$ at $x = z$.

Thus the properties of the Green's function $G(x, z)$ for an nth-order linear ODE may be summarised by the following.

(i) $G(x, z)$ obeys the original ODE but with $f(x)$ on the RHS set equal to a delta function $\delta(x - z)$.

(ii) When considered as a function of x alone $G(x,z)$ obeys the specified (homogeneous) boundary conditions on $y(x)$.

(iii) The derivatives of $G(x,z)$ with respect to x up to order $n-2$ are continuous at $x=z$, but the $(n-1)$th-order derivative has a discontinuity of $1/a_n(z)$ at this point.

►*Use Green's functions to solve*

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x, \tag{15.66}$$

subject to the boundary conditions $y(0) = y(\pi/2) = 0$.

From (15.62) we see that the Green's function $G(x,z)$ must satisfy

$$\frac{d^2G(x,z)}{dx^2} + G(x,z) = \delta(x-z). \tag{15.67}$$

Now it is clear that for $x \neq z$ the RHS of (15.67) is zero, and we are left with the task of finding the general solution to the homogeneous equation, i.e. the complementary function. The complementary function of (15.67) consists of a linear superposition of $\sin x$ and $\cos x$ and *must* consist of different superpositions on either side of $x = z$, since its $(n-1)$th derivative (i.e. the first derivative in this case) is required to have a discontinuity there. Therefore we assume the form of the Green's function to be

$$G(x,z) = \begin{cases} A(z)\sin x + B(z)\cos x & \text{for } x < z, \\ C(z)\sin x + D(z)\cos x & \text{for } x > z. \end{cases}$$

Note that we have performed a similar (but not identical) operation to that used in the variation-of-parameters method, i.e. we have replaced the constants in the complementary function with functions (this time of z).

We must now impose the relevant restrictions on $G(x,z)$ in order to determine the functions $A(z), \ldots, D(z)$. The first of these is that $G(x,z)$ should itself obey the homogeneous boundary conditions $G(0,z) = G(\pi/2, z) = 0$. This leads to the conclusion that $B(z) = C(z) = 0$, so we now have

$$G(x,z) = \begin{cases} A(z)\sin x & \text{for } x < z, \\ D(z)\cos x & \text{for } x > z. \end{cases}$$

The second restriction is the continuity conditions given in equations (15.64), (15.65), namely that, for this second-order equation, $G(x,z)$ is continuous at $x = z$ and dG/dx has a discontinuity of $1/a_2(z) = 1$ at this point. Applying these two constraints we have

$$D(z)\cos z - A(z)\sin z = 0$$
$$-D(z)\sin z - A(z)\cos z = 1.$$

Solving these equations for $A(z)$ and $D(z)$, we find

$$A(z) = -\cos z, \qquad D(z) = -\sin z.$$

Thus we have

$$G(x,z) = \begin{cases} -\cos z \sin x & \text{for } x < z, \\ -\sin z \cos x & \text{for } x > z. \end{cases}$$

Therefore, from (15.60), the general solution to (15.66) that obeys the boundary conditions

$y(0) = y(\pi/2) = 0$ is given by

$$\begin{aligned}
y(x) &= \int_0^{\pi/2} G(x,z)\,\mathrm{cosec}\, z\, dz \\
&= -\cos x \int_0^x \sin z\, \mathrm{cosec}\, z\, dz - \sin x \int_x^{\pi/2} \cos z\, \mathrm{cosec}\, z\, dz \\
&= -x\cos x + \sin x \ln(\sin x),
\end{aligned}$$

which agrees with the result obtained in the previous subsections. ◀

As mentioned earlier, once a Green's function has been obtained for a given LHS and boundary conditions, it can be used to find a general solution for any RHS; thus, the solution of $d^2y/dx^2 + y = f(x)$, with $y(0) = y(\pi/2) = 0$, is given immediately by

$$\begin{aligned}
y(x) &= \int_0^{\pi/2} G(x,z) f(z)\, dz \\
&= -\cos x \int_0^x \sin z\; f(z)\, dz - \sin x \int_x^{\pi/2} \cos z\; f(z)\, dz. \qquad (15.68)
\end{aligned}$$

As an example, the reader may wish to verify that if $f(x) = \sin 2x$ then (15.68) gives $y(x) = (-\sin 2x)/3$, a solution easily verified by direct substitution. In general, analytic integration of (15.68) for arbitrary $f(x)$ will prove intractable; then the integrals must be evaluated numerically.

Another important point is that although the Green's function method above has provided a general solution, it is also useful for finding a particular integral if the complementary function is known. This is easily seen since in (15.68) the constant integration limits 0 and $\pi/2$ lead merely to constant values by which the factors $\sin x$ and $\cos x$ are multiplied; thus the complementary function is reconstructed. The rest of the general solution, i.e. the particular integral, comes from the variable integration limit x. Therefore by changing $\int_x^{\pi/2}$ to $-\int^x$, and so dropping the constant integration limits, we can find just the particular integral. For example, a particular integral of $d^2y/dx^2 + y = f(x)$ that satisfies the above boundary conditions is given by

$$y_{\mathrm{p}}(x) = -\cos x \int^x \sin z\; f(z)\, dz + \sin x \int^x \cos z\; f(z)\, dz.$$

A very important point to realise about the Green's function method is that a particular $G(x,z)$ applies to a given LHS of an ODE *and* the imposed boundary conditions, i.e. *the same equation with different boundary conditions will have a different Green's function*. To illustrate this point, let us consider again the ODE solved in (15.68), but with different boundary conditions.

►*Use Green's functions to solve*

$$\frac{d^2y}{dx^2} + y = f(x), \tag{15.69}$$

subject to the one-point boundary conditions $y(0) = y'(0) = 0$.

We again require (15.67) to hold and so again we assume a Green's function of the form

$$G(x,z) = \begin{cases} A(z)\sin x + B(z)\cos x & \text{for } x < z, \\ C(z)\sin x + D(z)\cos x & \text{for } x > z. \end{cases}$$

However, we now require $G(x,z)$ to obey the boundary conditions $G(0,z) = G'(0,z) = 0$, which imply $A(z) = B(z) = 0$. Therefore we have

$$G(x,z) = \begin{cases} 0 & \text{for } x < z, \\ C(z)\sin x + D(z)\cos x & \text{for } x > z. \end{cases}$$

Applying the continuity conditions on $G(x,z)$ as before now gives

$$C(z)\sin z + D(z)\cos z = 0,$$
$$C(z)\cos z - D(z)\sin z = 1,$$

which are solved to give

$$C(z) = \cos z, \qquad D(z) = -\sin z.$$

So finally the Green's function is given by

$$G(x,z) = \begin{cases} 0 & \text{for } x < z, \\ \sin(x-z) & \text{for } x > z, \end{cases}$$

and the general solution to (15.69) that obeys the boundary conditions $y(0) = y'(0) = 0$ is

$$y(x) = \int_0^\infty G(x,z)f(z)\,dz$$
$$= \int_0^x \sin(x-z)f(z)\,dz. \blacktriangleleft$$

Finally, we consider how to deal with inhomogeneous boundary conditions such as $y(a) = \alpha$, $y(b) = \beta$ or $y(0) = y'(0) = \gamma$, where α, β, γ are non-zero. The simplest method of solution in this case is to make a change of variable such that the boundary conditions in the new variable, u say, are homogeneous, i.e. $u(a) = u(b) = 0$ or $u(0) = u'(0) = 0$ etc. For nth-order equations we generally require n boundary conditions to fix the solution, but these n boundary conditions can be of various types: we could have the n-point boundary conditions $y(x_m) = y_m$ for $m = 1$ to n, or the one-point boundary conditions $y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = y_0$, or something in between. In all cases a suitable change of variable is

$$u = y - h(x),$$

where $h(x)$ is an $(n-1)$th-order polynomial that obeys the boundary conditions.

For example, if we consider the second-order case with boundary conditions $y(a) = \alpha$, $y(b) = \beta$ then a suitable change of variable is

$$u = y - (mx + c),$$

where $y = mx + c$ is the straight line through the points (a, α) and (b, β), for which $m = (\alpha - \beta)/(a - b)$ and $c = (\beta a - \alpha b)/(a - b)$. Alternatively, if the boundary conditions for our second-order equation are $y(0) = y'(0) = \gamma$ then we would make the same change of variable, but this time $y = mx + c$ would be the straight line through $(0, \gamma)$ with slope γ, i.e. $m = c = \gamma$.

Solution method. *Require that the Green's function $G(x, z)$ obeys the original ODE, but with the RHS set to a delta function $\delta(x - z)$. This is equivalent to assuming that $G(x, z)$ is given by the complementary function of the original ODE, with the constants replaced by functions of z; these functions are different for $x < z$ and $x > z$. Now require also that $G(x, z)$ obeys the given homogeneous boundary conditions and impose the continuity conditions given in (15.64) and (15.65). The general solution to the original ODE is then given by (15.60). For inhomogeneous boundary conditions, make the change of dependent variable $u = y - h(x)$, where $h(x)$ is a polynomial obeying the given boundary conditions.*

15.2.6 Canonical form for second-order equations

In this section we specialise from nth-order linear ODEs with variable coefficients to those of order 2. In particular we consider the equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{15.70}$$

which has been rearranged so that the coefficient of d^2y/dx^2 is unity. By making the substitution $y(x) = u(x)v(x)$ we obtain

$$v'' + \left(\frac{2u'}{u} + a_1\right)v' + \left(\frac{u'' + a_1u' + a_0u}{u}\right)v = \frac{f}{u}, \tag{15.71}$$

where the prime denotes differentiation with respect to x. Since (15.71) would be much simplified if there were no term in v', let us choose $u(x)$ such that the first factor in parentheses on the LHS of (15.71) is zero, i.e.

$$\frac{2u'}{u} + a_1 = 0 \quad \Rightarrow \quad u(x) = \exp\left\{-\tfrac{1}{2}\int a_1(z)\,dz\right\}. \tag{15.72}$$

We then obtain an equation of the form

$$\frac{d^2v}{dx^2} + g(x)v = h(x), \tag{15.73}$$

where

$$g(x) = a_0(x) - \tfrac{1}{4}[a_1(x)]^2 - \tfrac{1}{2}a_1'(x)$$
$$h(x) = f(x)\exp\left\{\tfrac{1}{2}\int a_1(z)\,dz\right\}.$$

Since (15.73) is of a simpler form than the original equation, (15.70), it may prove easier to solve.

►*Solve*

$$4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (x^2-1)y = 0. \qquad (15.74)$$

Dividing (15.74) through by $4x^2$, we see that it is of the form (15.70) with $a_1(x) = 1/x$, $a_0(x) = (x^2-1)/4x^2$ and $f(x) = 0$. Therefore, making the substitution

$$y = vu = v\exp\left(-\int \frac{1}{2x}\,dx\right) = \frac{Av}{\sqrt{x}},$$

we obtain

$$\frac{d^2v}{dx^2} + \frac{v}{4} = 0. \qquad (15.75)$$

Equation (15.75) is easily solved to give

$$v = c_1\sin\tfrac{1}{2}x + c_2\cos\tfrac{1}{2}x,$$

so the solution of (15.74) is

$$y = \frac{v}{\sqrt{x}} = \frac{c_1\sin\frac{1}{2}x + c_2\cos\frac{1}{2}x}{\sqrt{x}}. \blacktriangleleft$$

As an alternative to choosing $u(x)$ such that the coefficient of v' in (15.71) is zero, we could choose a different $u(x)$ such that the coefficient of v vanishes. For this to be the case, we see from (15.71) that we would require

$$u'' + a_1u' + a_0u = 0,$$

so $u(x)$ would have to be a solution of the original ODE with the RHS set to zero, i.e. part of the complementary function. If such a solution were known then the substitution $y = uv$ would yield an equation with no term in v, which could be solved by two straightforward integrations. This is a special (second-order) case of the method discussed in subsection 15.2.3.

Solution method. *Write the equation in the form (15.70), then substitute $y = uv$, where $u(x)$ is given by (15.72). This leads to an equation of the form (15.73), in which there is no term in dv/dx and which may be easier to solve. Alternatively, if part of the complementary function is known then follow the method of subsection 15.2.3.*

15.3 General ordinary differential equations

In this section, we discuss miscellaneous methods for simplifying general ODEs. These methods are applicable to both linear and non-linear equations and in some cases may lead to a solution. More often than not, however, finding a closed-form solution to a general non-linear ODE proves impossible.

15.3.1 Dependent variable absent

If an ODE does not contain the dependent variable y explicitly, but only its derivatives, then the change of variable $p = dy/dx$ leads to an equation of one order lower.

►*Solve*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4x \tag{15.76}$$

This is transformed by the substitution $p = dy/dx$ to the first-order equation

$$\frac{dp}{dx} + 2p = 4x. \tag{15.77}$$

The solution to (15.77) is then found by the method of subsection 14.2.4 and reads

$$p = \frac{dy}{dx} = ae^{-2x} + 2x - 1,$$

where a is a constant. Thus by direct integration the solution to the original equation, (15.76), is

$$y(x) = c_1e^{-2x} + x^2 - x + c_2. \blacktriangleleft$$

An extension to the above method is appropriate if an ODE contains only derivatives of y that are of order m and greater. Then the substitution $p = d^my/dx^m$ reduces the order of the ODE by m.

Solution method. *If the ODE contains only derivatives of y that are of order m and greater then the substitution $p = d^my/dx^m$ reduces the order of the equation by m.*

15.3.2 Independent variable absent

If an ODE does not contain the independent variable x explicitly, except in d/dx, d^2/dx^2 etc., then as in the previous subsection we make the substitution $p = dy/dx$

but also write

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dy}{dx}\frac{dp}{dy} = p\frac{dp}{dy}$$
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(p\frac{dp}{dy}\right) = \frac{dy}{dx}\frac{d}{dy}\left(p\frac{dp}{dy}\right) = p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \qquad (15.78)$$

and so on for higher-order derivatives. This leads to an equation of one order lower.

►*Solve*

$$1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0. \qquad (15.79)$$

Making the substitutions $dy/dx = p$ and $d^2y/dx^2 = p(dp/dy)$ we obtain the first-order ODE

$$1 + yp\frac{dp}{dy} + p^2 = 0,$$

which is separable and may be solved as in subsection 14.2.1 to obtain

$$(1 + p^2)y^2 = c_1.$$

Using $p = dy/dx$ we therefore have

$$p = \frac{dy}{dx} = \pm\sqrt{\frac{c_1^2 - y^2}{y^2}},$$

which may be integrated to give the general solution of (15.79); after squaring this reads

$$(x + c_2)^2 + y^2 = c_1^2. \blacktriangleleft$$

Solution method. *If the ODE does not contain x explicitly then substitute $p = dy/dx$, along with the relations for higher derivatives given in (15.78), to obtain an equation of one order lower, which may prove easier to solve.*

15.3.3 Non-linear exact equations

As discussed in subsection 15.2.2, an exact ODE is one that can be obtained by straightforward differentiation of an equation of one order lower. Moreover, the notion of exact equations is useful for both linear and non-linear equations, since an exact equation can be immediately integrated. It is possible, of course, that the resulting equation may itself be exact, so that the process can be repeated. In the non-linear case, however, there is no simple relation (such as (15.43) for the linear case) by which an equation can be shown to be exact. Nevertheless, a general procedure does exist and is illustrated in the following example.

►*Solve*

$$2y\frac{d^3y}{dx^3} + 6\frac{dy}{dx}\frac{d^2y}{dx^2} = x. \tag{15.80}$$

Directing our attention to the term on the LHS of (15.80) that contains the highest-order derivative, i.e. $2y\,d^3y/dx^3$, we see that it can be obtained by differentiating $2y\,d^2y/dx^2$ since

$$\frac{d}{dx}\left(2y\frac{d^2y}{dx^2}\right) = 2y\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\frac{d^2y}{dx^2}. \tag{15.81}$$

Rewriting the LHS of (15.80) using (15.81), we are left with $4(dy/dx)(d^2y/dy^2)$, which may itself be written as a derivative, i.e.

$$4\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[2\left(\frac{dy}{dx}\right)^2\right]. \tag{15.82}$$

Since, therefore, we can write the LHS of (15.80) as a sum of simple derivatives of other functions, (15.80) is exact. Integrating (15.80) with respect to x, and using (15.81) and (15.82), now gives

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = \int x\,dx = \frac{x^2}{2} + c_1. \tag{15.83}$$

Now we can repeat the process to find whether (15.83) is itself exact. Considering the term on the LHS of (15.83) that contains the highest-order derivative, i.e. $2y\,d^2y/dx^2$, we note that we obtain this by differentiating $2y\,dy/dx$, as follows:

$$\frac{d}{dx}\left(2y\frac{dy}{dx}\right) = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2.$$

The above expression already contains all the terms on the LHS of (15.83), so we can integrate (15.83) to give

$$2y\frac{dy}{dx} = \frac{x^3}{6} + c_1x + c_2.$$

Integrating once more we obtain the solution

$$y^2 = \frac{x^4}{24} + \frac{c_1x^2}{2} + c_2x + c_3. \blacktriangleleft$$

It is worth noting that both linear equations (as discussed in subsection 15.2.2) and non-linear equations may sometimes be made exact by multiplying through by an appropriate integrating factor. Although no general method exists for finding such a factor, one may sometimes be found by inspection or inspired guesswork.

Solution method. *Rearrange the equation so that all the terms containing y or its derivatives are on the LHS, then check to see whether the equation is exact by attempting to write the LHS as a simple derivative. If this is possible then the equation is exact and may be integrated directly to give an equation of one order lower. If the new equation is itself exact the process can be repeated.*

15.3.4 Isobaric or homogeneous equations

It is straightforward to generalise the discussion of first-order isobaric equations given in subsection 14.2.6 to equations of general order n. An nth-order isobaric equation is one in which every term can be made dimensionally consistent upon giving y and dy each a weight m, and x and dx each a weight 1. Then the nth derivative of y with respect to x, for example, would have dimensions m in y and $-n$ in x. In the special case $m = 1$, for which the equation is dimensionally consistent, the equation is called homogeneous (not to be confused with linear equations with a zero RHS). If an equation is isobaric or homogeneous then the change in dependent variable $y = vx^m$ ($y = vx$ in the homogeneous case) followed by the change in independent variable $x = e^t$ leads to an equation in which the new independent variable t is absent except in the form d/dt.

►*Solve*

$$x^3\frac{d^2y}{dx^2} - (x^2 + xy)\frac{dy}{dx} + (y^2 + xy) = 0. \tag{15.84}$$

Assigning y and dy the weight m, and x and dx the weight 1, the weights of the five terms on the LHS of (15.84) are, from left to right: $m + 1$, $m + 1$, $2m$, $2m$, $m + 1$. For these weights all to be equal we require $m = 1$; thus (15.84) is a homogeneous equation. Since it is homogeneous we now make the substitution $y = vx$, which, after dividing the resulting equation through by x^3, gives

$$x\frac{d^2v}{dx^2} + (1 - v)\frac{dv}{dx} = 0. \tag{15.85}$$

Now substituting $x = e^t$ into (15.85) we obtain (after some working)

$$\frac{d^2v}{dt^2} - v\frac{dv}{dt} = 0, \tag{15.86}$$

which can be integrated directly to give

$$\frac{dv}{dt} = \tfrac{1}{2}v^2 + c_1. \tag{15.87}$$

Equation (15.87) is separable, and integrates to give

$$\begin{aligned}\tfrac{1}{2}t + d_2 &= \int \frac{dv}{v^2 + d_1^2} \\ &= \frac{1}{d_1}\tan^{-1}\left(\frac{v}{d_1}\right).\end{aligned}$$

Rearranging and using $x = e^t$ and $y = vx$ we finally obtain the solution to (15.84) as

$$y = d_1 x \tan\left(\tfrac{1}{2}d_1 \ln x + d_1 d_2\right). \blacktriangleleft$$

Solution method. *Assume that y and dy have weight m, and x and dx weight 1, and write down the combined weights of each term in the ODE. If these weights can be made equal by assuming a particular value for m then the equation is isobaric (or homogeneous if $m = 1$). Making the substitution $y = vx^m$ followed by $x = e^t$ leads to an equation in which the new independent variable t is absent except in the form d/dt.*

15.3.5 Equations homogeneous in x or y alone

It will be seen that the intermediate equation (15.85) in the example of the previous subsection was simplified by the substitution $x = e^t$, in that this led to an equation in which the new independent variable t occurred only in the form d/dt; see (15.86). A closer examination of (15.85) reveals that it is dimensionally consistent in the independent variable x *taken alone*; this is equivalent to giving the dependent variable and its differential a weight $m = 0$. For any equation that is homogeneous in x alone, the substitution $x = e^t$ will lead to an equation that does not contain the new independent variable t except as d/dt. Note that the Euler equation of subsection 15.2.1 is a special, linear example of an equation homogeneous in x alone. Similarly, if an equation is homogeneous in y alone, then substituting $y = e^v$ leads to an equation in which the new dependent variable, v, occurs only in the form d/dv.

►*Solve*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{2}{y^3} = 0.$$

This equation is homogeneous in x alone, and on substituting $x = e^t$ we obtain

$$\frac{d^2y}{dt^2} + \frac{2}{y^3} = 0,$$

which does not contain the new independent variable t except as d/dt. Such equations may often be solved by the method of subsection 15.3.2, but in this case we can integrate directly to obtain

$$\frac{dy}{dt} = \sqrt{2(c_1 + 1/y^2)}.$$

This equation is separable, and we find

$$\int \frac{dy}{\sqrt{2(c_1 + 1/y^2)}} = t + c_2.$$

By multiplying the numerator and denominator of the integrand on the LHS by y, we find the solution

$$\frac{\sqrt{c_1y^2 + 1}}{\sqrt{2}c_1} = t + c_2.$$

Remembering that $t = \ln x$, we finally obtain

$$\frac{\sqrt{c_1y^2 + 1}}{\sqrt{2}c_1} = \ln x + c_2. \blacktriangleleft$$

Solution method. *If the weight of x taken alone is the same in every term in the ODE then the substitution $x = e^t$ leads to an equation in which the new independent variable t is absent except in the form d/dt. If the weight of y taken alone is the same in every term then the substitution $y = e^v$ leads to an equation in which the new dependent variable v is absent except in the form d/dv.*

15.3.6 Equations having $y = Ae^x$ as a solution

Finally, we note that if any general (linear or non-linear) nth-order ODE is satisfied identically by assuming that

$$y = \frac{dy}{dx} = \cdots = \frac{d^n y}{dx^n} \tag{15.88}$$

then $y = Ae^x$ is a solution of that equation. This must be so because $y = Ae^x$ is a non-zero function that satisfies (15.88).

►*Find a solution of*

$$(x^2 + x)\frac{dy}{dx}\frac{d^2 y}{dx^2} - x^2 y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 0. \tag{15.89}$$

Setting $y = dy/dx = d^2y/dx^2$ in (15.89), we obtain

$$(x^2 + x)y^2 - x^2y^2 - xy^2 = 0,$$

which is satisfied identically. Therefore $y = Ae^x$ is a solution of (15.89); this is easily verified by directly substituting $y = Ae^x$ into (15.89). ◄

Solution method. *If the equation is satisfied identically by making the substitutions $y = dy/dx = \cdots = d^n y/dx^n$ then $y = Ae^x$ is a solution.*

15.4 Exercises

15.1 A simple harmonic oscillator, of mass m and natural frequency ω_0, experiences an oscillating driving force $f(t) = ma\cos\omega t$. Therefore, its equation of motion is

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = a\cos\omega t,$$

where x is its position. Given that at $t = 0$ we have $x = dx/dt = 0$, find the function $x(t)$. Describe the solution if ω is approximately, but not exactly, equal to ω_0.

15.2 Find the roots of the auxiliary equation for the following. Hence solve them for the boundary conditions stated.

$$\text{(a)}\ \frac{d^2 f}{dt^2} + 2\frac{df}{dt} + 5f = 0, \qquad \text{with } f(0) = 1, f'(0) = 0.$$

$$\text{(b)}\ \frac{d^2 f}{dt^2} + 2\frac{df}{dt} + 5f = e^{-t}\cos 3t, \qquad \text{with } f(0) = 0, f'(0) = 0.$$

15.3 The theory of bent beams shows that at any point in the beam the 'bending moment' is given by K/ρ, where K is a constant (that depends upon the beam material and cross-sectional shape) and ρ is the radius of curvature at that point. Consider a light beam of length L whose ends, $x = 0$ and $x = L$, are supported at the same vertical height and which has a weight W suspended from its centre. Verify that at any point x ($0 \le x \le L/2$ for definiteness) the net magnitude of the bending moment (bending moment = force × perpendicular distance) due to the weight and support reactions, evaluated on either side of x, is $Wx/2$.

If the beam is only slightly bent, so that $(dy/dx)^2 \ll 1$, where $y = y(x)$ is the downward displacement of the beam at x, show that the beam profile satisfies the approximate equation

$$\frac{d^2y}{dx^2} = -\frac{Wx}{2K}.$$

By integrating this equation twice and using physically imposed conditions on your solution at $x = 0$ and $x = L/2$, show that the downward displacement at the centre of the beam is $WL^3/(48K)$.

15.4 Solve the differential equation

$$\frac{d^2f}{dt^2} + 6\frac{df}{dt} + 9f = e^{-t},$$

subject to the conditions $f = 0$ and $df/dt = \lambda$ at $t = 0$.

Find the equation satisfied by the positions of the turning points of $f(t)$ and hence, by drawing suitable sketch graphs, determine the number of turning points the solution has in the range $t > 0$ if (a) $\lambda = 1/4$, and (b) $\lambda = -1/4$.

15.5 The function $f(t)$ satisfies the differential equation

$$\frac{d^2f}{dt^2} + 8\frac{df}{dt} + 12f = 12e^{-4t}.$$

For the following sets of boundary conditions determine whether it has solutions, and, if so, find them:

(a) $f(0) = 0, \quad f'(0) = 0, \quad f(\ln\sqrt{2}) = 0;$
(b) $f(0) = 0, \quad f'(0) = -2, \quad f(\ln\sqrt{2}) = 0.$

15.6 Determine the values of α and β for which the following four functions are linearly dependent:

$$\begin{aligned} y_1(x) &= x\cosh x + \sinh x, \\ y_2(x) &= x\sinh x + \cosh x, \\ y_3(x) &= (x+\alpha)e^x, \\ y_4(x) &= (x+\beta)e^{-x}. \end{aligned}$$

You will find it convenient to work with those linear combinations of the $y_i(x)$ that can be written the most compactly.

15.7 A solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 4e^{-x}$$

takes the value 1 when $x = 0$ and the value e^{-1} when $x = 1$. What is its value when $x = 2$?

15.8 The two functions $x(t)$ and $y(t)$ satisfy the simultaneous equations

$$\begin{aligned} \frac{dx}{dt} - 2y &= -\sin t, \\ \frac{dy}{dt} + 2x &= 5\cos t. \end{aligned}$$

Find explicit expressions for $x(t)$ and $y(t)$, given that $x(0) = 3$ and $y(0) = 2$. Sketch the solution trajectory in the xy-plane for $0 \le t < 2\pi$, showing that the trajectory crosses itself at $(0, 1/2)$ and passes through the points $(0, -3)$ and $(0, -1)$ in the negative x-direction.

15.9 Find the general solutions of

(a) $\dfrac{d^3y}{dx^3} - 12\dfrac{dy}{dx} + 16y = 32x - 8,$

(b) $\dfrac{d}{dx}\left(\dfrac{1}{y}\dfrac{dy}{dx}\right) + (2a\coth 2ax)\left(\dfrac{1}{y}\dfrac{dy}{dx}\right) = 2a^2,$

where a is a constant.

15.10 Use the method of Laplace transforms to solve

(a) $\dfrac{d^2f}{dt^2} + 5\dfrac{df}{dt} + 6f = 0, \qquad f(0) = 1,\ f'(0) = -4,$

(b) $\dfrac{d^2f}{dt^2} + 2\dfrac{df}{dt} + 5f = 0, \qquad f(0) = 1,\ f'(0) = 0.$

15.11 The quantities $x(t)$, $y(t)$ satisfy the simultaneous equations

$$\ddot{x} + 2n\dot{x} + n^2x = 0,$$
$$\ddot{y} + 2n\dot{y} + n^2y = \mu\dot{x},$$

where $x(0) = y(0) = \dot{y}(0) = 0$ and $\dot{x}(0) = \lambda$. Show that

$$y(t) = \tfrac{1}{2}\mu\lambda t^2\left(1 - \tfrac{1}{3}nt\right)\exp(-nt).$$

15.12 Use Laplace transforms to solve, for $t \geq 0$, the differential equations

$$\ddot{x} + 2x + y = \cos t,$$
$$\ddot{y} + 2x + 3y = 2\cos t,$$

which describe a coupled system that starts from rest at the equilibrium position. Show that the subsequent motion takes place along a straight line in the xy-plane. Verify that the frequency at which the system is driven is equal to one of the resonance frequencies of the system; explain why there is *no* resonant behaviour in the solution you have obtained.

15.13 Two unstable isotopes A and B and a stable isotope C have the following decay rates per atom present: $A \to B$, $3\ \mathrm{s}^{-1}$; $A \to C$, $1\ \mathrm{s}^{-1}$; $B \to C$, $2\ \mathrm{s}^{-1}$. Initially a quantity x_0 of A is present, but there are no atoms of the other two types. Using Laplace transforms, find the amount of C present at a later time t.

15.14 For a lightly damped ($\gamma < \omega_0$) harmonic oscillator driven at its undamped resonance frequency ω_0, the displacement $x(t)$ at time t satisfies the equation

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = F\sin\omega_0 t.$$

Use Laplace transforms to find the displacement at a general time if the oscillator starts from rest at its equilibrium position.

(a) Show that ultimately the oscillation has amplitude $F/(2\omega_0\gamma)$, with a phase lag of $\pi/2$ relative to the driving force per unit mass F.

(b) By differentiating the original equation, conclude that if $x(t)$ is expanded as a power series in t for small t, then the first non-vanishing term is $F\omega_0 t^3/6$. Confirm this conclusion by expanding your explicit solution.

15.15 The 'golden mean', which is said to describe the most aesthetically pleasing proportions for the sides of a rectangle (e.g. the ideal picture frame), is given by the limiting value of the ratio of successive terms of the Fibonacci series u_n, which is generated by

$$u_{n+2} = u_{n+1} + u_n,$$

with $u_0 = 0$ and $u_1 = 1$. Find an expression for the general term of the series and verify that the golden mean is equal to the larger root of the recurrence relation's characteristic equation.

15.16 In a particular scheme for numerically modelling one-dimensional fluid flow, the successive values, u_n, of the solution are connected for $n \geq 1$ by the difference equation

$$c(u_{n+1} - u_{n-1}) = d(u_{n+1} - 2u_n + u_{n-1}),$$

where c and d are positive constants. The boundary conditions are $u_0 = 0$ and $u_M = 1$. Find the solution to the equation, and show that successive values of u_n will have alternating signs if $c > d$.

15.17 The first few terms of a series u_n, starting with u_0, are $1, 2, 2, 1, 6, -3$. The series is generated by a recurrence relation of the form

$$u_n = Pu_{n-2} + Qu_{n-4},$$

where P and Q are constants. Find an expression for the general term of the series and show that, in fact, the series consists of two interleaved series given by

$$u_{2m} = \tfrac{2}{3} + \tfrac{1}{3}4^m,$$
$$u_{2m+1} = \tfrac{7}{3} - \tfrac{1}{3}4^m,$$

for $m = 0, 1, 2, \ldots$.

15.18 Find an explicit expression for the u_n satisfying

$$u_{n+1} + 5u_n + 6u_{n-1} = 2^n,$$

given that $u_0 = u_1 = 1$. Deduce that $2^n - 26(-3)^n$ is divisible by 5 for all non-negative integers n.

15.19 Find the general expression for the u_n satisfying

$$u_{n+1} = 2u_{n-2} - u_n$$

with $u_0 = u_1 = 0$ and $u_2 = 1$, and show that they can be written in the form

$$u_n = \frac{1}{5} - \frac{2^{n/2}}{\sqrt{5}} \cos\left(\frac{3\pi n}{4} - \phi\right),$$

where $\tan\phi = 2$.

15.20 Consider the seventh-order recurrence relation

$$u_{n+7} - u_{n+6} - u_{n+5} + u_{n+4} - u_{n+3} + u_{n+2} + u_{n+1} - u_n = 0.$$

Find the most general form of its solution, and show that:

(a) if only the four initial values $u_0 = 0$, $u_1 = 2$, $u_2 = 6$ and $u_3 = 12$, are specified, then the relation has one solution that cycles repeatedly through this set of four numbers;
(b) but if, in addition, it is required that $u_4 = 20$, $u_5 = 30$ and $u_6 = 42$ then the solution is unique, with $u_n = n(n + 1)$.

15.21 Find the general solution of

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x,$$

given that $y(1) = 1$ and $y(e) = 2e$.

15.22 Find the general solution of

$$(x + 1)^2 \frac{d^2y}{dx^2} + 3(x + 1)\frac{dy}{dx} + y = x^2.$$

15.23 Prove that the general solution of

$$(x-2)\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + \frac{4y}{x^2} = 0$$

is given by

$$y(x) = \frac{1}{(x-2)^2}\left[k\left(\frac{2}{3x} - \frac{1}{2}\right) + cx^2\right].$$

15.24 Use the method of variation of parameters to find the general solutions of

(a) $\dfrac{d^2y}{dx^2} - y = x^n$, (b) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 2xe^x$.

15.25 Use the intermediate result of exercise 15.24(a) to find the Green's function that satisfies

$$\frac{d^2G(x,\xi)}{dx^2} - G(x,\xi) = \delta(x-\xi) \qquad \text{with} \qquad G(0,\xi) = G(1,\xi) = 0.$$

15.26 Consider the equation

$$F(x,y) = x(x+1)\frac{d^2y}{dx^2} + (2-x^2)\frac{dy}{dx} - (2+x)y = 0.$$

(a) Given that $y_1(x) = 1/x$ is one of its solutions, find a second linearly independent one,
   (i) by setting $y_2(x) = y_1(x)u(x)$, and
   (ii) by noting the sum of the coefficients in the equation.

(b) Hence, using the variation of parameters method, find the general solution of

$$F(x,y) = (x+1)^2.$$

15.27 Show generally that if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

with $y_1(0) = 0$ and $y_2(1) = 0$, then the Green's function $G(x,\xi)$ for the interval $0 \le x, \xi \le 1$ and with $G(0,\xi) = G(1,\xi) = 0$ can be written in the form

$$G(x,\xi) = \begin{cases} y_1(x)y_2(\xi)/W(\xi) & 0 < x < \xi, \\ y_2(x)y_1(\xi)/W(\xi) & \xi < x < 1, \end{cases}$$

where $W(x) = W[y_1(x), y_2(x)]$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

15.28 Use the result of the previous exercise to find the Green's function $G(x,\xi)$ that satisfies

$$\frac{d^2G}{dx^2} + 3\frac{dG}{dx} + 2G = \delta(x-x),$$

in the interval $0 \le x, \xi \le 1$, with $G(0,\xi) = G(1,\xi) = 0$. Hence obtain integral expressions for the solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \begin{cases} 0 & 0 < x < x_0, \\ 1 & x_0 < x < 1, \end{cases}$$

distinguishing between the cases (a) $x < x_0$, and (b) $x > x_0$.

15.29 The equation of motion for a driven damped harmonic oscillator can be written

$$\ddot{x} + 2\dot{x} + (1 + \kappa^2)x = f(t),$$

with $\kappa \neq 0$. If it starts from rest with $x(0) = 0$ and $\dot{x}(0) = 0$, find the corresponding Green's function $G(t, \tau)$ and verify that it can be written as a function of $t - \tau$ only. Find the explicit solution when the driving force is the unit step function, i.e. $f(t) = H(t)$. Confirm your solution by taking the Laplace transforms of both it and the original equation.

15.30 Show that the Green's function for the equation

$$\frac{d^2y}{dx^2} + \frac{y}{4} = f(x),$$

subject to the boundary conditions $y(0) = y(\pi) = 0$, is given by

$$G(x, z) = \begin{cases} -2\cos\frac{1}{2}x\sin\frac{1}{2}z & 0 \le z \le x, \\ -2\sin\frac{1}{2}x\cos\frac{1}{2}z & x \le z \le \pi. \end{cases}$$

15.31 Find the Green's function $x = G(t, t_0)$ that solves

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} = \delta(t - t_0)$$

under the initial conditions $x = dx/dt = 0$ at $t = 0$. Hence solve

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} = f(t),$$

where $f(t) = 0$ for $t < 0$.
Evaluate your answer explicitly for $f(t) = Ae^{-at}$ $(t > 0)$.

15.32 Consider the equation

$$\frac{d^2y}{dx^2} + f(y) = 0,$$

where $f(y)$ can be any function.

(a) By multiplying through by dy/dx, obtain the general solution relating x and y.
(b) A mass m, initially at rest at the point $x = 0$, is accelerated by a force

$$f(x) = A(x_0 - x)\left[1 + 2\ln\left(1 - \frac{x}{x_0}\right)\right].$$

Its equation of motion is $m\,d^2x/dt^2 = f(x)$. Find x as a function of time, and show that ultimately the particle has travelled a distance x_0.

15.33 Solve

$$2y\frac{d^3y}{dx^3} + 2\left(y + 3\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = \sin x.$$

15.34 Find the general solution of the equation

$$x\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = Ax.$$

15.35 Express the equation

$$\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (4x^2 + 6)y = e^{-x^2}\sin 2x$$

in canonical form and hence find its general solution.

15.36 Find the form of the solutions of the equation

$$\frac{dy}{dx}\frac{d^3y}{dx^3} - 2\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = 0$$

that have $y(0) = \infty$.

[You will need the result $\int^z \operatorname{cosech} u \, du = -\ln(\operatorname{cosech} z + \coth z)$.]

15.37 Consider the equation

$$x^p y'' + \frac{n+3-2p}{n-1} x^{p-1} y' + \left(\frac{p-2}{n-1}\right)^2 x^{p-2} y = y^n,$$

in which $p \neq 2$ and $n > -1$ but $n \neq 1$. For the boundary conditions $y(1) = 0$ and $y'(1) = \lambda$, show that the solution is $y(x) = v(x)x^{(p-2)/(n-1)}$, where $v(x)$ is given by

$$\int_0^{v(x)} \frac{dz}{\left[\lambda^2 + 2z^{n+1}/(n+1)\right]^{1/2}} = \ln x.$$

15.5 Hints and answers

15.1 The function is $a(\omega_0^2 - \omega^2)^{-1}(\cos\omega t - \cos\omega_0 t)$; for moderate t, $x(t)$ is a sine wave of linearly increasing amplitude $(t \sin\omega_0 t)/(2\omega_0)$; for large t it shows beats of maximum amplitude $2(\omega_0^2 - \omega^2)^{-1}$.

15.3 Ignore the term y'^2, compared with 1, in the expression for ρ. $y = 0$ at $x = 0$. From symmetry, $dy/dx = 0$ at $x = L/2$.

15.5 General solution $f(t) = Ae^{-6t} + Be^{-2t} - 3e^{-4t}$. (a) No solution, inconsistent boundary conditions; (b) $f(t) = 2e^{-6t} + e^{-2t} - 3e^{-4t}$.

15.7 The auxiliary equation has repeated roots and the RHS is contained in the complementary function. The solution is $y(x) = (A+Bx)e^{-x} + 2x^2e^{-x}$. $y(2) = 5e^{-2}$.

15.9 (a) The auxiliary equation has roots 2, 2, -4; $(A+Bx)\exp 2x + C\exp(-4x) + 2x + 1$; (b) multiply through by $\sinh 2ax$ and note that $\int \operatorname{cosech} 2ax \, dx = (2a)^{-1}\ln(|\tanh ax|)$; $y = B(\sinh 2ax)^{1/2}(|\tanh ax|)^A$.

15.11 Use Laplace transforms; write $s(s+n)^{-4}$ as $(s+n)^{-3} - n(s+n)^{-4}$.

15.13 $\mathscr{L}[C(t)] = x_0(s+8)/[s(s+2)(s+4)]$, yielding $C(t) = x_0[1 + \frac{1}{2}\exp(-4t) - \frac{3}{2}\exp(-2t)]$.

15.15 The characteristic equation is $\lambda^2 - \lambda - 1 = 0$. $u_n = [(1+\sqrt{5})^n - (1-\sqrt{5})^n]/(2^n\sqrt{5})$.

15.17 From u_4 and u_5, $P = 5, Q = -4$. $u_n = 3/2 - 5(-1)^n/6 + (-2)^n/4 + 2^n/12$.

15.19 The general solution is $A + B2^{n/2}\exp(i3\pi n/4) + C2^{n/2}\exp(i5\pi n/4)$. The initial values imply that $A = 1/5, B = (\sqrt{5}/10)\exp[i(\pi - \phi)]$ and $C = (\sqrt{5}/10)\exp[i(\pi + \phi)]$.

15.21 This is Euler's equation; setting $x = \exp t$ produces $d^2z/dt^2 - 2\,dz/dt + z = \exp t$, with complementary function $(A + Bt)\exp t$ and particular integral $t^2(\exp t)/2$; $y(x) = x + [x\ln x(1 + \ln x)]/2$.

15.23 After multiplication through by x^2 the coefficients are such that this is an exact equation. The resulting first-order equation, in standard form, needs an integrating factor $(x-2)^2/x^2$.

15.25 Given the boundary conditions, it is better to work with $\sinh x$ and $\sinh(1-x)$ than with $e^{\pm x}$; $G(x,\xi) = -[\sinh(1-\xi)\sinh x]/\sinh 1$ for $x < \xi$ and $-[\sinh(1-x)\sinh\xi]/\sinh 1$ for $x > \xi$.

15.27 Follow the method of subsection 15.2.5, but using general rather than specific functions.

15.29 $G(t,\tau) = 0$ for $t < \tau$ and $\kappa^{-1}e^{-(t-\tau)}\sin[\kappa(t-\tau)]$ for $t > \tau$. For a unit step input, $x(t) = (1+\kappa^2)^{-1}(1 - e^{-t}\cos\kappa t - \kappa^{-1}e^{-t}\sin\kappa t)$. Both transforms are equivalent to $s[(s+1)^2 + \kappa^2)]\bar{x} = 1$.

15.31 Use continuity and the step condition on $\partial G/\partial t$ at $t = t_0$ to show that $G(t, t_0) = \alpha^{-1}\{1 - \exp[\alpha(t_0 - t)]\}$ for $0 \le t_0 \le t$; $x(t) = A(\alpha - a)^{-1}\{a^{-1}[1 - \exp(-at)] - \alpha^{-1}[1 - \exp(-\alpha t)]\}$.

15.33 The LHS of the equation is exact for two stages of integration and then needs an integrating factor $\exp x$; $2y\, d^2y/dx^2 + 2y\, dy/dx + 2(dy/dx)^2$; $2y\, dy/dx + y^2 = d(y^2)/dx + y^2$; $y^2 = A\exp(-x) + Bx + C - (\sin x - \cos x)/2$.

15.35 Follow the method of subsection 15.2.6; $u(x) = e^{-x^2}$ and $v(x)$ satisfies $v'' + 4v = \sin 2x$, for which a particular integral is $(-x\cos 2x)/4$. The general solution is $y(x) = [A\sin 2x + (B - \frac{1}{4}x)\cos 2x]e^{-x^2}$.

15.37 The equation is isobaric, with y of weight m, where $m + p - 2 = mn$; $v(x)$ satisfies $x^2v'' + xv' = v^n$. Set $x = e^t$ and $v(x) = u(t)$, leading to $u'' = u^n$ with $u(0) = 0, u'(0) = \lambda$. Multiply both sides by u' to make the equation exact.

16

Series solutions of ordinary differential equations

In the previous chapter the solution of both homogeneous and non-homogeneous linear ordinary differential equations (ODEs) of order ≥ 2 was discussed. In particular we developed methods for solving some equations in which the coefficients were not constant but functions of the independent variable x. In each case we were able to write the solutions to such equations in terms of elementary functions, or as integrals. In general, however, the solutions of equations with variable coefficients cannot be written in this way, and we must consider alternative approaches.

In this chapter we discuss a method for obtaining solutions to linear ODEs in the form of convergent series. Such series can be evaluated numerically, and those occurring most commonly are named and tabulated. There is in fact no distinct borderline between this and the previous chapter, since solutions in terms of elementary functions may equally well be written as convergent series (i.e. the relevant Taylor series). Indeed, it is partly because some series occur so frequently that they are given special names such as $\sin x$, $\cos x$ or $\exp x$.

Since we shall be concerned principally with second-order linear ODEs in this chapter, we begin with a discussion of these equations, and obtain some general results that will prove useful when we come to discuss series solutions.

16.1 Second-order linear ordinary differential equations

Any homogeneous second-order linear ODE can be written in the form

$$y'' + p(x)y' + q(x)y = 0, \qquad (16.1)$$

where $y' = dy/dx$ and $p(x)$ and $q(x)$ are given functions of x. From the previous chapter, we recall that the most general form of the solution to (16.1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \qquad (16.2)$$

where $y_1(x)$ and $y_2(x)$ are *linearly independent* solutions of (16.1), and c_1 and c_2 are constants that are fixed by the boundary conditions (if supplied).

A full discussion of the linear independence of sets of functions was given at the beginning of the previous chapter, but for just two functions y_1 and y_2 to be linearly independent we simply require that y_2 is not a multiple of y_1. Equivalently, y_1 and y_2 must be such that the equation

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

is *only* satisfied for $c_1 = c_2 = 0$. Therefore the linear independence of $y_1(x)$ and $y_2(x)$ can usually be deduced by inspection but in any case can always be verified by the evaluation of the Wronskian of the two solutions,

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'. \tag{16.3}$$

If $W(x) \neq 0$ anywhere in a given interval then y_1 and y_2 are linearly independent in that interval.

An alternative expression for $W(x)$, of which we will make use later, may be derived by differentiating (16.3) with respect to x to give

$$W' = y_1 y_2'' + y_1' y_2' - y_2 y_1'' - y_2' y_1' = y_1 y_2'' - y_1'' y_2.$$

Since both y_1 and y_2 satisfy (16.1), we may substitute for y_1'' and y_2'' to obtain

$$W' = -y_1(py_2' + qy_2) + (py_1' + qy_1)y_2 = -p(y_1 y_2' - y_1' y_2) = -pW.$$

Integrating, we find

$$W(x) = C \exp\left\{ -\int^x p(u)\, du \right\}, \tag{16.4}$$

where C is a constant. We note further that in the special case $p(x) \equiv 0$ we obtain $W = \text{constant}$.

► *The functions $y_1 = \sin x$ and $y_2 = \cos x$ are both solutions of the equation $y'' + y = 0$. Evaluate the Wronskian of these two solutions, and hence show that they are linearly independent.*

The Wronskian of y_1 and y_2 is given by

$$W = y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1.$$

Since $W \neq 0$ the two solutions are linearly independent. We also note that $y'' + y = 0$ is a special case of (16.1) with $p(x) = 0$. We therefore expect, from (16.4), that W will be a constant, as is indeed the case. ◄

From the previous chapter we recall that, once we have obtained the general solution to the homogeneous second-order ODE (16.1) in the form (16.2), the general solution to the *inhomogeneous* equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{16.5}$$

can be written as the sum of the solution to the homogeneous equation $y_c(x)$ (the complementary function) and *any* function $y_p(x)$ (the particular integral) that satisfies (16.5) and is linearly independent of $y_c(x)$. We have therefore

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x). \tag{16.6}$$

General methods for obtaining y_p, that are applicable to equations with variable coefficients, such as the variation of parameters or Green's functions, were discussed in the previous chapter. An alternative description of the Green's function method for solving inhomogeneous equations is given in the next chapter. For the present, however, we will restrict our attention to the solutions of homogeneous ODEs in the form of convergent series.

16.1.1 Ordinary and singular points of an ODE

So far we have implicitly assumed that $y(x)$ is a *real* function of a *real* variable x. However, this is not always the case, and in the remainder of this chapter we broaden our discussion by generalising to a *complex* function $y(z)$ of a *complex* variable z.

Let us therefore consider the second-order linear homogeneous ODE

$$y'' + p(z)y' + q(z) = 0, \tag{16.7}$$

where now $y' = dy/dz$; this is a straightforward generalisation of (16.1). A full discussion of complex functions and differentiation with respect to a complex variable z is given in chapter 24, but for the purposes of the present chapter we need not concern ourselves with many of the subtleties that exist. In particular, we may treat differentiation with respect to z in a way analogous to ordinary differentiation with respect to a real variable x.

In (16.7), if, at some point $z = z_0$, the functions $p(z)$ and $q(z)$ are finite and can be expressed as complex power series (see section 4.5), i.e.

$$p(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^n, \qquad q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^n,$$

then $p(z)$ and $q(z)$ are said to be *analytic* at $z = z_0$, and this point is called an *ordinary point* of the ODE. If, however, $p(z)$ or $q(z)$, or both, diverge at $z = z_0$ then it is called a *singular point* of the ODE.

Even if an ODE is singular at a given point $z = z_0$, it may still possess a non-singular (finite) solution at that point. In fact the necessary and sufficient condition[§] for such a solution to exist is that $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ are both analytic at $z = z_0$. Singular points that have this property are called *regular*

[§] See, for example, H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics*, 3rd edn (Cambridge: Cambridge University Press, 1966), p. 479.

singular points, whereas any singular point not satisfying both these criteria is termed an *irregular* or *essential* singularity.

►*Legendre's equation has the form*

$$(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0, \tag{16.8}$$

where ℓ is a constant. Show that $z=0$ is an ordinary point and $z=\pm1$ are regular singular points of this equation.

Firstly, divide through by $1-z^2$ to put the equation into our standard form (16.7):

$$y'' - \frac{2z}{1-z^2}y' + \frac{\ell(\ell+1)}{1-z^2}y = 0.$$

Comparing this with (16.7), we identify $p(z)$ and $q(z)$ as

$$p(z) = \frac{-2z}{1-z^2} = \frac{-2z}{(1+z)(1-z)}, \qquad q(z) = \frac{\ell(\ell+1)}{1-z^2} = \frac{\ell(\ell+1)}{(1+z)(1-z)}.$$

By inspection, $p(z)$ and $q(z)$ are analytic at $z=0$, which is therefore an ordinary point, but both diverge for $z=\pm1$, which are thus singular points. However, at $z=1$ we see that both $(z-1)p(z)$ and $(z-1)^2q(z)$ are analytic and hence $z=1$ is a regular singular point. Similarly, at $z=-1$ both $(z+1)p(z)$ and $(z+1)^2q(z)$ are analytic, and it too is a regular singular point. ◄

So far we have assumed that z_0 is finite. However, we may sometimes wish to determine the nature of the point $|z|\to\infty$. This may be achieved straightforwardly by substituting $w=1/z$ into the equation and investigating the behaviour at $w=0$.

►*Show that Legendre's equation has a regular singularity at $|z|\to\infty$.*

Letting $w=1/z$, the derivatives with respect to z become

$$\frac{dy}{dz} = \frac{dy}{dw}\frac{dw}{dz} = -\frac{1}{z^2}\frac{dy}{dw} = -w^2\frac{dy}{dw},$$

$$\frac{d^2y}{dz^2} = \frac{dw}{dz}\frac{d}{dw}\left(\frac{dy}{dz}\right) = -w^2\left(-2w\frac{dy}{dw} - w^2\frac{d^2y}{dw^2}\right) = w^3\left(2\frac{dy}{dw} + w\frac{d^2y}{dw^2}\right).$$

If we substitute these derivatives into Legendre's equation (16.8) we obtain

$$\left(1-\frac{1}{w^2}\right)w^3\left(2\frac{dy}{dw} + w\frac{d^2y}{dw^2}\right) + 2\frac{1}{w}w^2\frac{dy}{dw} + \ell(\ell+1)y = 0,$$

which simplifies to give

$$w^2(w^2-1)\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw} + \ell(\ell+1)y = 0.$$

Dividing through by $w^2(w^2-1)$ to put the equation into standard form, and comparing with (16.7), we identify $p(w)$ and $q(w)$ as

$$p(w) = \frac{2w}{w^2-1}, \qquad q(w) = \frac{\ell(\ell+1)}{w^2(w^2-1)}.$$

At $w=0$, $p(w)$ is analytic but $q(w)$ diverges, and so the point $|z|\to\infty$ is a singular point of Legendre's equation. However, since wp and w^2q are both analytic at $w=0$, $|z|\to\infty$ is a regular singular point. ◄

Equation	Regular singularities	Essential singularities
Hypergeometric $z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0$	$0, 1, \infty$	—
Legendre $(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0$	$-1, 1, \infty$	—
Associated Legendre $(1-z^2)y'' - 2zy' + \left[\ell(\ell+1) - \dfrac{m^2}{1-z^2}\right] y = 0$	$-1, 1, \infty$	—
Chebyshev $(1-z^2)y'' - zy' + \nu^2 y = 0$	$-1, 1, \infty$	—
Confluent hypergeometric $zy'' + (c-z)y' - ay = 0$	0	∞
Bessel $z^2 y'' + zy' + (z^2 - \nu^2)y = 0$	0	∞
Laguerre $zy'' + (1-z)y' + \nu y = 0$	0	∞
Associated Laguerre $zy'' + (m+1-z)y' + (\nu - m)y = 0$	0	∞
Hermite $y'' - 2zy' + 2\nu y = 0$	—	∞
Simple harmonic oscillator $y'' + \omega^2 y = 0$	—	∞

Table 16.1 Important second-order linear ODEs in the physical sciences and engineering.

Table 16.1 lists the singular points of several second-order linear ODEs that play important roles in the analysis of many problems in physics and engineering. A full discussion of the solutions to each of the equations in table 16.1 and their properties is left until chapter 18. We now discuss the general methods by which series solutions may be obtained.

16.2 Series solutions about an ordinary point

If $z = z_0$ is an ordinary point of (16.7) then it may be shown that *every* solution $y(z)$ of the equation is also analytic at $z = z_0$. From now on we will take z_0 as the origin, i.e. $z_0 = 0$. If this is not already the case, then a substitution $Z = z - z_0$ will make it so. Since every solution is analytic, $y(z)$ can be represented by a power series of the form (see section 24.11)

$$y(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{16.9}$$

Moreover, it may be shown that such a power series converges for $|z| < R$, where R is the radius of convergence and is equal to the distance from $z = 0$ to the nearest singular point of the ODE (see chapter 24). At the radius of convergence, however, the series may or may not converge (as shown in section 4.5).

Since every solution of (16.7) is analytic at an ordinary point, it is always possible to obtain two *independent* solutions (from which the general solution (16.2) can be constructed) of the form (16.9). The derivatives of y with respect to z are given by

$$y' = \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n, \tag{16.10}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n. \tag{16.11}$$

Note that, in each case, in the first equality the sum can still start at $n = 0$ since the first term in (16.10) and the first two terms in (16.11) are automatically zero. The second equality in each case is obtained by shifting the summation index so that the sum can be written in terms of coefficients of z^n. By substituting (16.9)–(16.11) into the ODE (16.7), and requiring that the coefficients of each power of z sum to zero, we obtain a *recurrence relation* expressing each a_n in terms of the previous a_r $(0 \le r \le n-1)$.

►*Find the series solutions, about $z = 0$, of*

$$y'' + y = 0.$$

By inspection, $z = 0$ is an ordinary point of the equation, and so we may obtain two independent solutions by making the substitution $y = \sum_{n=0}^{\infty} a_n z^n$. Using (16.9) and (16.11) we find

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + \sum_{n=0}^{\infty} a_n z^n = 0,$$

which may be written as

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] z^n = 0.$$

For this equation to be satisfied we require that the coefficient of each power of z vanishes *separately*, and so we obtain the two-term recurrence relation

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad \text{for } n \ge 0.$$

Using this relation, we can calculate, say, the even coefficients a_2, a_4, a_6 and so on, for a given a_0. Alternatively, starting with a_1, we obtain the odd coefficients a_3, a_5, etc. Two independent solutions of the ODE can be obtained by setting either $a_0 = 0$ or $a_1 = 0$. Firstly, if we set $a_1 = 0$ and choose $a_0 = 1$ then we obtain the solution

$$y_1(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

Secondly, if we set $a_0 = 0$ and choose $a_1 = 1$ then we obtain a second, *independent*, solution

$$y_2(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

Recognising these two series as $\cos z$ and $\sin z$, we can write the general solution as

$$y(z) = c_1 \cos z + c_2 \sin z,$$

where c_1 and c_2 are arbitrary constants that are fixed by boundary conditions (if supplied). We note that both solutions converge for all z, as might be expected since the ODE possesses no singular points (except $|z| \to \infty$). ◄

Solving the above example was quite straightforward and the resulting series were easily recognised and written in *closed form* (i.e. in terms of elementary functions); *this is not usually the case.* Another simplifying feature of the previous example was that we obtained a two-term recurrence relation relating a_{n+2} and a_n, so that the odd- and even-numbered coefficients were independent of one another. In general, the recurrence relation expresses a_n in terms of any number of the previous a_r $(0 \le r \le n-1)$.

►*Find the series solutions, about $z = 0$, of*

$$y'' - \frac{2}{(1-z)^2} y = 0.$$

By inspection, $z = 0$ is an ordinary point, and therefore we may find two independent solutions by substituting $y = \sum_{n=0}^{\infty} a_n z^n$. Using (16.10) and (16.11), and multiplying through by $(1-z)^2$, we find

$$(1 - 2z + z^2) \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 2 \sum_{n=0}^{\infty} a_n z^n = 0,$$

which leads to

$$\sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 2 \sum_{n=0}^{\infty} n(n-1) a_n z^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n z^n - 2 \sum_{n=0}^{\infty} a_n z^n = 0.$$

In order to write all these series in terms of the coefficients of z^n, we must shift the summation index in the first two sums, obtaining

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n - 2 \sum_{n=0}^{\infty} (n+1) n a_{n+1} z^n + \sum_{n=0}^{\infty} (n^2 - n - 2) a_n z^n = 0,$$

which can be written as

$$\sum_{n=0}^{\infty} (n+1)[(n+2) a_{n+2} - 2n a_{n+1} + (n-2) a_n] z^n = 0.$$

By demanding that the coefficients of each power of z vanish separately, we obtain the three-term recurrence relation

$$(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n = 0 \qquad \text{for } n \geq 0,$$

which determines a_n for $n \geq 2$ in terms of a_0 and a_1. Three-term (or more) recurrence relations are a nuisance and, in general, can be difficult to solve. This particular recurrence relation, however, has two straightforward solutions. One solution is $a_n = a_0$ for all n, in which case (choosing $a_0 = 1$) we find

$$y_1(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}.$$

The other solution to the recurrence relation is $a_1 = -2a_0$, $a_2 = a_0$ and $a_n = 0$ for $n > 2$, so that (again choosing $a_0 = 1$) we obtain a *polynomial* solution to the ODE:

$$y_2(z) = 1 - 2z + z^2 = (1-z)^2.$$

The linear independence of y_1 and y_2 is obvious but can be checked by computing the Wronskian

$$W = y_1 y_2' - y_1' y_2 = \frac{1}{1-z}[-2(1-z)] - \frac{1}{(1-z)^2}(1-z)^2 = -3.$$

Since $W \neq 0$, the two solutions y_1 and y_2 are indeed linearly independent. The general solution of the ODE is therefore

$$y(z) = \frac{c_1}{1-z} + c_2(1-z)^2.$$

We observe that y_1 (and hence the general solution) is singular at $z = 1$, which is the singular point of the ODE nearest to $z = 0$, but the polynomial solution, y_2, is valid for all finite z. ◄

The above example illustrates the possibility that, in some cases, we may find that the recurrence relation leads to $a_n = 0$ for $n > N$, for one or both of the two solutions; we then obtain a *polynomial* solution to the equation. Polynomial solutions are discussed more fully in section 16.5, but one obvious property of such solutions is that they converge for all finite z. By contrast, as mentioned above, for solutions in the form of an infinite series the circle of convergence extends only as far as the singular point nearest to that about which the solution is being obtained.

16.3 Series solutions about a regular singular point

From table 16.1 we see that several of the most important second-order linear ODEs in physics and engineering have regular singular points in the finite complex plane. We must extend our discussion, therefore, to obtaining series solutions to ODEs about such points. In what follows we assume that the regular singular point about which the solution is required is at $z = 0$, since, as we have seen, if this is not already the case then a substitution of the form $Z = z - z_0$ will make it so.

If $z = 0$ is a regular singular point of the equation

$$y'' + p(z)y' + q(z)y = 0$$

then at least one of $p(z)$ and $q(z)$ is not analytic at $z = 0$, and in general we should not expect to find a power series solution of the form (16.9). We must therefore extend the method to include a more general form for the solution. In fact, it may be shown (Fuch's theorem) that there exists *at least one* solution to the above equation, of the form

$$y = z^\sigma \sum_{n=0}^{\infty} a_n z^n, \tag{16.12}$$

where the exponent σ is a number that may be real or complex and where $a_0 \neq 0$ (since, if it were otherwise, σ could be redefined as $\sigma + 1$ or $\sigma + 2$ or $\cdots$ so as to make $a_0 \neq 0$). Such a series is called a generalised power series or *Frobenius series*. As in the case of a simple power series solution, the radius of convergence of the Frobenius series is, in general, equal to the distance to the nearest singularity of the ODE.

Since $z = 0$ is a regular singularity of the ODE, it follows that $zp(z)$ and $z^2q(z)$ are analytic at $z = 0$, so that we may write

$$zp(z) \equiv s(z) = \sum_{n=0}^{\infty} s_n z^n,$$

$$z^2q(z) \equiv t(z) = \sum_{n=0}^{\infty} t_n z^n,$$

where we have defined the analytic functions $s(z)$ and $t(z)$ for later convenience. The original ODE therefore becomes

$$y'' + \frac{s(z)}{z} y' + \frac{t(z)}{z^2} y = 0.$$

Let us substitute the Frobenius series (16.12) into this equation. The derivatives of (16.12) with respect to z are given by

$$y' = \sum_{n=0}^{\infty} (n+\sigma) a_n z^{n+\sigma-1}, \tag{16.13}$$

$$y'' = \sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1) a_n z^{n+\sigma-2}, \tag{16.14}$$

and we obtain

$$\sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1) a_n z^{n+\sigma-2} + s(z) \sum_{n=0}^{\infty} (n+\sigma) a_n z^{n+\sigma-2} + t(z) \sum_{n=0}^{\infty} a_n z^{n+\sigma-2} = 0.$$

Dividing this equation through by $z^{\sigma-2}$, we find

$$\sum_{n=0}^{\infty} \left[(n+\sigma)(n+\sigma-1) + s(z)(n+\sigma) + t(z)\right] a_n z^n = 0. \tag{16.15}$$

Setting $z = 0$, all terms in the sum with $n > 0$ vanish, implying that

$$[\sigma(\sigma - 1) + s(0)\sigma + t(0)]a_0 = 0,$$

which, since we require $a_0 \neq 0$, yields the *indicial equation*

$$\sigma(\sigma - 1) + s(0)\sigma + t(0) = 0. \tag{16.16}$$

This equation is a quadratic in σ and in general has two roots, the nature of which determines the forms of possible series solutions.

The two roots of the indicial equation, σ_1 and σ_2, are called the *indices* of the regular singular point. By substituting each of these roots into (16.15) in turn and requiring that the coefficients of each power of z vanish separately, we obtain a recurrence relation (for each root) expressing each a_n as a function of the previous a_r $(0 \leq r \leq n-1)$. We will see that the larger root of the indicial equation always yields a solution to the ODE in the form of a Frobenius series (16.12). The form of the second solution depends, however, on the relationship between the two indices σ_1 and σ_2. There are three possible general cases: (i) distinct roots not differing by an integer; (ii) repeated roots; (iii) distinct roots differing by an integer (not equal to zero). Below, we discuss each of these in turn.

Before continuing, however, we note that, as was the case for solutions in the form of a simple power series, it is always worth investigating whether a Frobenius series found as a solution to a problem is summable in closed form or expressible in terms of known functions. We illustrate this point below, but the reader should avoid gaining the impression that this is always so or that, if one worked hard enough, a closed-form solution could always be found without using the series method. As mentioned earlier, this is *not* the case, and very often an infinite series solution is the best one can do.

16.3.1 Distinct roots not differing by an integer

If the roots of the indicial equation, σ_1 and σ_2, differ by an amount that is not an integer then the recurrence relations corresponding to each root lead to two linearly independent solutions of the ODE:

$$y_1(z) = z^{\sigma_1} \sum_{n=0}^{\infty} a_n z^n, \qquad y_2(z) = z^{\sigma_2} \sum_{n=0}^{\infty} b_n z^n,$$

with both solutions taking the form of a Frobenius series. The linear independence of these two solutions follows from the fact that y_2/y_1 is not a constant since $\sigma_1 - \sigma_2$ is not an integer. Because y_1 and y_2 are linearly independent, we may use them to construct the general solution $y = c_1 y_1 + c_2 y_2$.

We also note that this case includes complex conjugate roots where $\sigma_2 = \sigma_1^*$, since $\sigma_1 - \sigma_2 = \sigma_1 - \sigma_1^* = 2i \operatorname{Im} \sigma_1$ cannot be equal to a real integer.

► *Find the power series solutions about $z = 0$ of*

$$4zy'' + 2y' + y = 0.$$

Dividing through by $4z$ to put the equation into standard form, we obtain

$$y'' + \frac{1}{2z}y' + \frac{1}{4z}y = 0, \tag{16.17}$$

and on comparing with (16.7) we identify $p(z) = 1/(2z)$ and $q(z) = 1/(4z)$. Clearly $z = 0$ is a singular point of (16.17), but since $zp(z) = 1/2$ and $z^2q(z) = z/4$ are finite there, it is a regular singular point. We therefore substitute the Frobenius series $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$ into (16.17). Using (16.13) and (16.14), we obtain

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \frac{1}{2z}\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1} + \frac{1}{4z}\sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0,$$

which, on dividing through by $z^{\sigma-2}$, gives

$$\sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1) + \tfrac{1}{2}(n+\sigma) + \tfrac{1}{4}z\right] a_n z^n = 0. \tag{16.18}$$

If we set $z = 0$ then all terms in the sum with $n > 0$ vanish, and we obtain the indicial equation

$$\sigma(\sigma - 1) + \tfrac{1}{2}\sigma = 0,$$

which has roots $\sigma = 1/2$ and $\sigma = 0$. Since these roots do not differ by an integer, we expect to find two independent solutions to (16.17), in the form of Frobenius series.

Demanding that the coefficients of z^n vanish separately in (16.18), we obtain the recurrence relation

$$(n+\sigma)(n+\sigma-1)a_n + \tfrac{1}{2}(n+\sigma)a_n + \tfrac{1}{4}a_{n-1} = 0. \tag{16.19}$$

If we choose the larger root, $\sigma = 1/2$, of the indicial equation then (16.19) becomes

$$(4n^2 + 2n)a_n + a_{n-1} = 0 \qquad \Rightarrow \qquad a_n = \frac{-a_{n-1}}{2n(2n+1)}.$$

Setting $a_0 = 1$, we find $a_n = (-1)^n/(2n+1)!$, and so the solution to (16.17) is given by

$$\begin{aligned} y_1(z) &= \sqrt{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^n \\ &= \sqrt{z} - \frac{(\sqrt{z})^3}{3!} + \frac{(\sqrt{z})^5}{5!} - \cdots = \sin\sqrt{z}. \end{aligned}$$

To obtain the second solution we set $\sigma = 0$ (the smaller root of the indicial equation) in (16.19), which gives

$$(4n^2 - 2n)a_n + a_{n-1} = 0 \qquad \Rightarrow \qquad a_n = -\frac{a_{n-1}}{2n(2n-1)}.$$

Setting $a_0 = 1$ now gives $a_n = (-1)^n/(2n)!$, and so the second (independent) solution to (16.17) is

$$y_2(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^n = 1 - \frac{(\sqrt{z})^2}{2!} + \frac{(\sqrt{4})^4}{4!} - \cdots = \cos\sqrt{z}.$$

We may check that $y_1(z)$ and $y_2(z)$ are indeed linearly independent by computing the Wronskian as follows:

$$\begin{aligned} W &= y_1 y_2' - y_2 y_1' \\ &= \sin\sqrt{z}\left(-\frac{1}{2\sqrt{z}}\sin\sqrt{z}\right) - \cos\sqrt{z}\left(\frac{1}{2\sqrt{z}}\cos\sqrt{z}\right) \\ &= -\frac{1}{2\sqrt{z}}\left(\sin^2\sqrt{z} + \cos^2\sqrt{z}\right) = -\frac{1}{2\sqrt{z}} \neq 0. \end{aligned}$$

Since $W \neq 0$, the solutions $y_1(z)$ and $y_2(z)$ are linearly independent. Hence, the general solution to (16.17) is given by

$$y(z) = c_1 \sin\sqrt{z} + c_2 \cos\sqrt{z}. \blacktriangleleft$$

16.3.2 Repeated root of the indicial equation

If the indicial equation has a repeated root, so that $\sigma_1 = \sigma_2 = \sigma$, then obviously only one solution in the form of a Frobenius series (16.12) may be found as described above, i.e.

$$y_1(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n.$$

Methods for obtaining a second, linearly independent, solution are discussed in section 16.4.

16.3.3 Distinct roots differing by an integer

Whatever the roots of the indicial equation, the recurrence relation corresponding to the larger of the two always leads to a solution of the ODE. However, if the roots of the indicial equation differ by an integer then the recurrence relation corresponding to the smaller root may or may not lead to a second linearly independent solution, depending on the ODE under consideration. Note that for complex roots of the indicial equation, the 'larger' root is taken to be the one with the larger real part.

►*Find the power series solutions about $z = 0$ of*

$$z(z-1)y'' + 3zy' + y = 0. \tag{16.20}$$

Dividing through by $z(z-1)$ to put the equation into standard form, we obtain

$$y'' + \frac{3}{(z-1)}y' + \frac{1}{z(z-1)}y = 0, \tag{16.21}$$

and on comparing with (16.7) we identify $p(z) = 3/(z-1)$ and $q(z) = 1/[z(z-1)]$. We immediately see that $z = 0$ is a singular point of (16.21), but since $zp(z) = 3z/(z-1)$ and $z^2q(z) = z/(z-1)$ are finite there, it is a regular singular point and we expect to find at least one solution in the form of a Frobenius series. We therefore substitute $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$ into (16.21) and, using (16.13) and (16.14), we obtain

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \frac{3}{z-1}\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1} + \frac{1}{z(z-1)}\sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0,$$

which, on dividing through by $z^{\sigma-2}$, gives

$$\sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1) + \frac{3z}{z-1}(n+\sigma) + \frac{z}{z-1}\right] a_n z^n = 0.$$

Although we could use this expression to find the indicial equation and recurrence relations, the working is simpler if we now multiply through by $z-1$ to give

$$\sum_{n=0}^{\infty}\left[(z-1)(n+\sigma)(n+\sigma-1) + 3z(n+\sigma) + z\right] a_n z^n = 0. \tag{16.22}$$

If we set $z = 0$ then all terms in the sum with the exponent of z greater than zero vanish, and we obtain the indicial equation

$$\sigma(\sigma-1) = 0,$$

which has the roots $\sigma = 1$ and $\sigma = 0$. Since the roots differ by an integer (unity), it may not be possible to find two linearly independent solutions of (16.21) in the form of Frobenius series. We are guaranteed, however, to find one such solution corresponding to the larger root, $\sigma = 1$.

Demanding that the coefficients of z^n vanish separately in (16.22), we obtain the recurrence relation

$$(n-1+\sigma)(n-2+\sigma)a_{n-1} - (n+\sigma)(n+\sigma-1)a_n + 3(n-1+\sigma)a_{n-1} + a_{n-1} = 0,$$

which can be simplified to give

$$(n+\sigma-1)a_n = (n+\sigma)a_{n-1}. \tag{16.23}$$

On substituting $\sigma = 1$ into this expression, we obtain

$$a_n = \left(\frac{n+1}{n}\right) a_{n-1},$$

and on setting $a_0 = 1$ we find $a_n = n+1$; so one solution to (16.21) is given by

$$y_1(z) = z\sum_{n=0}^{\infty}(n+1)z^n = z(1 + 2z + 3z^2 + \cdots)$$
$$= \frac{z}{(1-z)^2}. \tag{16.24}$$

If we attempt to find a second solution (corresponding to the smaller root of the indicial equation) by setting $\sigma = 0$ in (16.23), we find

$$a_n = \left(\frac{n}{n-1}\right) a_{n-1}.$$

But we require $a_0 \neq 0$, so a_1 is formally infinite and the method fails. We discuss how to find a second linearly independent solution in the next section. ◄

One particular case is worth mentioning. If the point about which the solution is required, i.e. $z = 0$, is in fact an ordinary point of the ODE rather than a regular singular point, then substitution of the Frobenius series (16.12) leads to an indicial equation with roots $\sigma = 0$ and $\sigma = 1$. Although these roots differ by an integer (unity), the recurrence relations corresponding to the two roots yield two linearly independent power series solutions (one for each root), as expected from section 16.2.

16.4 Obtaining a second solution

Whilst attempting to construct solutions to an ODE in the form of Frobenius series about a regular singular point, we found in the previous section that when the indicial equation has a repeated root, or roots differing by an integer, we can (in general) find only one solution of this form. In order to construct the general solution to the ODE, however, we require two linearly independent solutions y_1 and y_2. We now consider several methods for obtaining a second solution in this case.

16.4.1 The Wronskian method

If y_1 and y_2 are two linearly independent solutions of the standard equation

$$y'' + p(z)y' + q(z)y = 0$$

then the Wronskian of these two solutions is given by $W(z) = y_1 y_2' - y_2 y_1'$. Dividing the Wronskian by y_1^2 we obtain

$$\frac{W}{y_1^2} = \frac{y_2'}{y_1} - \frac{y_1'}{y_1^2} y_2 = \frac{y_2'}{y_1} + \left[\frac{d}{dz}\left(\frac{1}{y_1}\right)\right] y_2 = \frac{d}{dz}\left(\frac{y_2}{y_1}\right),$$

which integrates to give

$$y_2(z) = y_1(z) \int^z \frac{W(u)}{y_1^2(u)}\, du.$$

Now using the alternative expression for $W(z)$ given in (16.4) with $C = 1$ (since we are not concerned with this normalising factor), we find

$$y_2(z) = y_1(z) \int^z \frac{1}{y_1^2(u)} \exp\left\{-\int^u p(v)\, dv\right\} du. \tag{16.25}$$

Hence, given y_1, we can in principle compute y_2. Note that the lower limits of integration have been omitted. If constant lower limits are included then they merely lead to a constant times the first solution.

▶*Find a second solution to (16.21) using the Wronskian method.*

For the ODE (16.21) we have $p(z) = 3/(z-1)$, and from (16.24) we see that one solution to (16.21) is $y_1 = z/(1-z)^2$. Substituting for p and y_1 in (16.25) we have

$$\begin{aligned} y_2(z) &= \frac{z}{(1-z)^2}\int^z \frac{(1-u)^4}{u^2}\exp\left(-\int^u \frac{3}{v-1}\,dv\right)du \\ &= \frac{z}{(1-z)^2}\int^z \frac{(1-u)^4}{u^2}\exp\left[-3\ln(u-1)\right]du \\ &= \frac{z}{(1-z)^2}\int^z \frac{u-1}{u^2}\,du \\ &= \frac{z}{(1-z)^2}\left(\ln z + \frac{1}{z}\right). \end{aligned}$$

By calculating the Wronskian of y_1 and y_2 it is easily shown that, as expected, the two solutions are linearly independent. In fact, as the Wronskian has already been evaluated as $W(u) = \exp[-3\ln(u-1)]$, i.e. $W(z) = (z-1)^{-3}$, no calculation is needed. ◄

An alternative (but equivalent) method of finding a second solution is simply to assume that the second solution has the form $y_2(z) = u(z)y_1(z)$ for some function $u(z)$ to be determined (this method was discussed more fully in subsection 15.2.3). From (16.25), we see that the second solution derived from the Wronskian is indeed of this form. Substituting $y_2(z) = u(z)y_1(z)$ into the ODE leads to a first-order ODE in which u' is the dependent variable; this may then be solved.

16.4.2 The derivative method

The derivative method of finding a second solution begins with the derivation of a recurrence relation for the coefficients a_n in a Frobenius series solution, as in the previous section. However, rather than putting $\sigma = \sigma_1$ in this recurrence relation to evaluate the first series solution, we now keep σ as a variable parameter. This means that the computed a_n are functions of σ and the computed solution is now a function of z and σ:

$$y(z,\sigma) = z^\sigma \sum_{n=0}^{\infty} a_n(\sigma)z^n. \tag{16.26}$$

Of course, if we put $\sigma = \sigma_1$ in this, we obtain immediately the first series solution, but for the moment we leave σ as a parameter.

For brevity let us denote the differential operator on the LHS of our standard ODE (16.7) by $\mathcal{L}$, so that

$$\mathcal{L} = \frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z),$$

and examine the effect of $\mathcal{L}$ on the series $y(z,\sigma)$ in (16.26). It is clear that the series $\mathcal{L}y(z,\sigma)$ will contain only a term in z^σ, since the recurrence relation defining the $a_n(\sigma)$ is such that these coefficients vanish for higher powers of z. But the coefficient of z^σ is simply the LHS of the indicial equation. Therefore, if the roots of the indicial equation are $\sigma = \sigma_1$ and $\sigma = \sigma_2$ then it follows that

$$\mathcal{L}y(z,\sigma) = a_0(\sigma - \sigma_1)(\sigma - \sigma_2)z^\sigma. \tag{16.27}$$

Therefore, as in the previous section, we see that for $y(z,\sigma)$ to be a solution of the ODE $\mathcal{L}y = 0$, σ must equal σ_1 or σ_2. For simplicity we shall set $a_0 = 1$ in the following discussion.

Let us first consider the case in which the two roots of the indicial equation are equal, i.e. $\sigma_2 = \sigma_1$. From (16.27) we then have

$$\mathcal{L}y(z,\sigma) = (\sigma - \sigma_1)^2 z^\sigma.$$

Differentiating this equation with respect to σ we obtain

$$\frac{\partial}{\partial\sigma}\left[\mathcal{L}y(z,\sigma)\right] = (\sigma - \sigma_1)^2 z^\sigma \ln z + 2(\sigma - \sigma_1)z^\sigma,$$

which equals zero if $\sigma = \sigma_1$. But since $\partial/\partial\sigma$ and $\mathcal{L}$ are operators that differentiate with respect to different variables, we can reverse their order, implying that

$$\mathcal{L}\left[\frac{\partial}{\partial\sigma}y(z,\sigma)\right] = 0 \qquad \text{at } \sigma = \sigma_1.$$

Hence, the function in square brackets, evaluated at $\sigma = \sigma_1$ and denoted by

$$\left[\frac{\partial}{\partial\sigma}y(z,\sigma)\right]_{\sigma=\sigma_1}, \tag{16.28}$$

is also a solution of the original ODE $\mathcal{L}y = 0$, and is in fact the second linearly independent solution that we were looking for.

The case in which the roots of the indicial equation differ by an integer is slightly more complicated but can be treated in a similar way. In (16.27), since $\mathcal{L}$ differentiates with respect to z we may multiply (16.27) by any function of σ, say $\sigma - \sigma_2$, and take this function inside the operator $\mathcal{L}$ on the LHS to obtain

$$\mathcal{L}\left[(\sigma - \sigma_2)y(z,\sigma)\right] = (\sigma - \sigma_1)(\sigma - \sigma_2)^2 z^\sigma. \tag{16.29}$$

Therefore the function

$$\left[(\sigma - \sigma_2)y(z,\sigma)\right]_{\sigma=\sigma_2}$$

is also a solution of the ODE $\mathcal{L}y = 0$. However, it can be proved[§] that this function is a simple multiple of the first solution $y(z,\sigma_1)$, showing that it is not linearly independent and that we must find another solution. To do this we differentiate (16.29) with respect to σ and find

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left\{\mathcal{L}\left[(\sigma - \sigma_2)y(z,\sigma)\right]\right\} = (\sigma - \sigma_2)^2 z^\sigma &+ 2(\sigma - \sigma_1)(\sigma - \sigma_2)z^\sigma \\ &+ (\sigma - \sigma_1)(\sigma - \sigma_2)^2 z^\sigma \ln z,\end{aligned}$$

[§] For a fuller discussion see, for example, K. F. Riley, *Mathematical Methods for the Physical Sciences* (Cambridge: Cambridge University Press, 1974), pp. 158–9.

which is equal to zero if $\sigma = \sigma_2$. As previously, since $\partial/\partial\sigma$ and $\mathcal{L}$ are operators that differentiate with respect to different variables, we can reverse their order to obtain

$$\mathcal{L}\left\{\frac{\partial}{\partial\sigma}\left[(\sigma-\sigma_2)y(z,\sigma)\right]\right\} = 0 \qquad \text{at } \sigma = \sigma_2,$$

and so the function

$$\left\{\frac{\partial}{\partial\sigma}\left[(\sigma-\sigma_2)y(z,\sigma)\right]\right\}_{\sigma=\sigma_2} \tag{16.30}$$

is also a solution of the original ODE $\mathcal{L}y = 0$, and is in fact the second linearly independent solution.

►*Find a second solution to (16.21) using the derivative method.*

From (16.23) the recurrence relation (with σ as a parameter) is given by

$$(n+\sigma-1)a_n = (n+\sigma)a_{n-1}.$$

Setting $a_0 = 1$ we find that the coefficients have the particularly simple form $a_n(\sigma) = (\sigma+n)/\sigma$. We therefore consider the function

$$y(z,\sigma) = z^\sigma \sum_{n=0}^{\infty} a_n(\sigma)z^n = z^\sigma \sum_{n=0}^{\infty} \frac{\sigma+n}{\sigma} z^n.$$

The smaller root of the indicial equation for (16.21) is $\sigma_2 = 0$, and so from (16.30) a second, linearly independent, solution to the ODE is given by

$$\left\{\frac{\partial}{\partial\sigma}\left[\sigma y(z,\sigma)\right]\right\}_{\sigma=0} = \left\{\frac{\partial}{\partial\sigma}\left[z^\sigma \sum_{n=0}^{\infty}(\sigma+n)z^n\right]\right\}_{\sigma=0}.$$

The derivative with respect to σ is given by

$$\frac{\partial}{\partial\sigma}\left[z^\sigma \sum_{n=0}^{\infty}(\sigma+n)z^n\right] = z^\sigma \ln z \sum_{n=0}^{\infty}(\sigma+n)z^n + z^\sigma \sum_{n=0}^{\infty} z^n,$$

which on setting $\sigma = 0$ gives the second solution

$$\begin{aligned} y_2(z) &= \ln z \sum_{n=0}^{\infty} nz^n + \sum_{n=0}^{\infty} z^n \\ &= \frac{z}{(1-z)^2}\ln z + \frac{1}{1-z} \\ &= \frac{z}{(1-z)^2}\left(\ln z + \frac{1}{z} - 1\right). \end{aligned}$$

This second solution is the same as that obtained by the Wronskian method in the previous subsection except for the addition of some of the first solution. ◄

16.4.3 Series form of the second solution

Using any of the methods discussed above, we can find the general form of the second solution to the ODE. This form is most easily found, however, using the derivative method. Let us first consider the case where the two solutions of the indicial equation are equal. In this case a second solution is given by (16.28), which may be written as

$$\begin{aligned} y_2(z) &= \left[\frac{\partial y(z,\sigma)}{\partial \sigma}\right]_{\sigma=\sigma_1} \\ &= (\ln z) z^{\sigma_1} \sum_{n=0}^{\infty} a_n(\sigma_1) z^n + z^{\sigma_1} \sum_{n=1}^{\infty} \left[\frac{da_n(\sigma)}{d\sigma}\right]_{\sigma=\sigma_1} z^n \\ &= y_1(z) \ln z + z^{\sigma_1} \sum_{n=1}^{\infty} b_n z^n, \end{aligned} \tag{16.31}$$

where $b_n = [da_n(\sigma)/d\sigma]_{\sigma=\sigma_1}$. One could equally obtain the coefficients b_n by direct substitution of the form (16.31) into the original ODE.

In the case where the roots of the indicial equation differ by an integer (not equal to zero), then from (16.30) a second solution is given by

$$\begin{aligned} y_2(z) &= \left\{\frac{\partial}{\partial \sigma}\left[(\sigma-\sigma_2) y(z,\sigma)\right]\right\}_{\sigma=\sigma_2} \\ &= \ln z \left[(\sigma-\sigma_2) z^{\sigma} \sum_{n=0}^{\infty} a_n(\sigma) z^n\right]_{\sigma=\sigma_2} + z^{\sigma_2} \sum_{n=0}^{\infty} \left[\frac{d}{d\sigma}(\sigma-\sigma_2) a_n(\sigma)\right]_{\sigma=\sigma_2} z^n. \end{aligned}$$

But, as we mentioned in the previous section, $[(\sigma-\sigma_2)y(z,\sigma)]$ at $\sigma = \sigma_2$ is just a multiple of the first solution $y(z,\sigma_1)$. Therefore the second solution is of the form

$$y_2(z) = c y_1(z) \ln z + z^{\sigma_2} \sum_{n=0}^{\infty} b_n z^n, \tag{16.32}$$

where c is a constant. In some cases, however, c might be zero, and so the second solution would not contain the term in $\ln z$ and could be written simply as a Frobenius series. Clearly this corresponds to the case in which the substitution of a Frobenius series into the original ODE yields two solutions automatically. In either case, the coefficients b_n may also be found by direct substitution of the form (16.32) into the original ODE.

16.5 Polynomial solutions

We have seen that the evaluation of successive terms of a series solution to a differential equation is carried out by means of a recurrence relation. The form of the relation for a_n depends upon n, the previous values of a_r ($r < n$) and the parameters of the equation. It may happen, as a result of this, that for some value of $n = N + 1$ the computed value a_{N+1} is zero and that all higher a_r also vanish. If this is so, and the corresponding solution of the indicial equation σ

is a positive integer or zero, then we are left with a finite polynomial of degree $N' = N + \sigma$ as a solution of the ODE:

$$y(z) = \sum_{n=0}^{N} a_n z^{n+\sigma}. \tag{16.33}$$

In many applications in theoretical physics (particularly in quantum mechanics) the termination of a potentially infinite series after a finite number of terms is of crucial importance in establishing physically acceptable descriptions and properties of systems. The condition under which such a termination occurs is therefore of considerable importance.

►*Find power series solutions about $z = 0$ of*

$$y'' - 2zy' + \lambda y = 0. \tag{16.34}$$

For what values of λ does the equation possess a polynomial solution? Find such a solution for $\lambda = 4$.

Clearly $z = 0$ is an ordinary point of (16.34) and so we look for solutions of the form $y = \sum_{n=0}^{\infty} a_n z^n$. Substituting this into the ODE and multiplying through by z^2 we find

$$\sum_{n=0}^{\infty} [n(n-1) - 2z^2 n + \lambda z^2] a_n z^n = 0.$$

By demanding that the coefficients of each power of z vanish separately we derive the recurrence relation

$$n(n-1)a_n - 2(n-2)a_{n-2} + \lambda a_{n-2} = 0,$$

which may be rearranged to give

$$a_n = \frac{2(n-2) - \lambda}{n(n-1)} a_{n-2} \qquad \text{for } n \geq 2. \tag{16.35}$$

The odd and even coefficients are therefore independent of one another, and two solutions to (16.34) may be derived. We either set $a_1 = 0$ and $a_0 = 1$ to obtain

$$y_1(z) = 1 - \lambda \frac{z^2}{2!} - \lambda(4-\lambda)\frac{z^4}{4!} - \lambda(4-\lambda)(8-\lambda)\frac{z^6}{6!} - \cdots \tag{16.36}$$

or set $a_0 = 0$ and $a_1 = 1$ to obtain

$$y_2(z) = z + (2-\lambda)\frac{z^3}{3!} + (2-\lambda)(6-\lambda)\frac{z^5}{5!} + (2-\lambda)(6-\lambda)(10-\lambda)\frac{z^7}{7!} + \cdots .$$

Now, from the recurrence relation (16.35) (or in this case from the expressions for y_1 and y_2 themselves) we see that for the ODE to possess a polynomial solution we require $\lambda = 2(n-2)$ for $n \geq 2$ or, more simply, $\lambda = 2n$ for $n \geq 0$, i.e. λ must be an even positive integer. If $\lambda = 4$ then from (16.36) the ODE has the polynomial solution

$$y_1(z) = 1 - \frac{4z^2}{2!} = 1 - 2z^2. \blacktriangleleft$$

A simpler method of obtaining finite polynomial solutions is to *assume* a solution of the form (16.33), where $a_N \neq 0$. Instead of starting with the lowest power of z, as we have done up to now, this time we start by considering the coefficient of the highest power z^N; such a power now exists because of our assumed form of solution.

►*By assuming a polynomial solution find the values of λ in (16.34) for which such a solution exists.*

We assume a polynomial solution to (16.34) of the form $y = \sum_{n=0}^{N} a_n z^n$. Substituting this form into (16.34) we find

$$\sum_{n=0}^{N} \left[n(n-1)a_n z^{n-2} - 2zna_n z^{n-1} + \lambda a_n z^n \right] = 0.$$

Now, instead of starting with the lowest power of z, we start with the highest. Thus, demanding that the coefficient of z^N vanishes, we require $-2N + \lambda = 0$, i.e. $\lambda = 2N$, as we found in the previous example. By demanding that the coefficient of a general power of z is zero, the same recurrence relation as above may be derived and the solutions found. ◄

16.6 Exercises

16.1 Find two power series solutions about $z = 0$ of the differential equation

$$(1 - z^2)y'' - 3zy' + \lambda y = 0.$$

Deduce that the value of λ for which the corresponding power series becomes an Nth-degree polynomial $U_N(z)$ is $N(N+2)$. Construct $U_2(z)$ and $U_3(z)$.

16.2 Find solutions, as power series in z, of the equation

$$4zy'' + 2(1 - z)y' - y = 0.$$

Identify one of the solutions and verify it by direct substitution.

16.3 Find power series solutions in z of the differential equation

$$zy'' - 2y' + 9z^5 y = 0.$$

Identify closed forms for the two series, calculate their Wronskian, and verify that they are linearly independent. Compare the Wronskian with that calculated from the differential equation.

16.4 Change the independent variable in the equation

$$\frac{d^2 f}{dz^2} + 2(z - a)\frac{df}{dz} + 4f = 0 \qquad (*)$$

from z to $x = z - \alpha$, and find two independent series solutions, expanded about $x = 0$, of the resulting equation. Deduce that the general solution of $(*)$ is

$$f(z, \alpha) = A(z - \alpha)e^{-(z-\alpha)^2} + B \sum_{m=0}^{\infty} \frac{(-4)^m m!}{(2m)!}(z - \alpha)^{2m},$$

with A and B arbitrary constants.

16.5 Investigate solutions of Legendre's equation at one of its singular points as follows.

(a) Verify that $z = 1$ is a regular singular point of Legendre's equation and that the indicial equation for a series solution in powers of $(z - 1)$ has roots 0 and 3.

(b) Obtain the corresponding recurrence relation and show that $\sigma = 0$ does not give a valid series solution.

(c) Determine the radius of convergence R of the $\sigma = 3$ series and relate it to the positions of the singularities of Legendre's equation.

16.6 Verify that $z = 0$ is a regular singular point of the equation

$$z^2 y'' - \tfrac{3}{2} z y' + (1 + z) y = 0,$$

and that the indicial equation has roots 2 and 1/2. Show that the general solution is given by

$$y(z) = 6a_0 z^2 \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) 2^{2n} z^n}{(2n+3)!}$$
$$+ b_0 \left(z^{1/2} + 2z^{3/2} - \frac{z^{1/2}}{4} \sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n} z^n}{n(n-1)(2n-3)!} \right).$$

16.7 Use the derivative method to obtain, as a second solution of Bessel's equation for the case when $\nu = 0$, the following expression:

$$J_0(z) \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{r=1}^{n} \frac{1}{r} \right) \left(\frac{z}{2} \right)^{2n},$$

given that the first solution is $J_0(z)$, as specified by (18.79).

16.8 Consider a series solution of the equation

$$zy'' - 2y' + yz = 0 \qquad (*)$$

about its regular singular point.

(a) Show that its indicial equation has roots that differ by an integer but that the two roots nevertheless generate linearly independent solutions

$$y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2n z^{2n+1}}{(2n+1)!},$$

$$y_2(z) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n-1) z^{2n}}{(2n)!}.$$

(b) Show that $y_1(z)$ is equal to $3a_0(\sin z - z\cos z)$ by expanding the sinusoidal functions. Then, using the Wronskian method, find an expression for $y_2(z)$ in terms of sinusoids. You will need to write z^2 as $(z/\sin z)(z \sin z)$ and integrate by parts to evaluate the integral involved.

(c) Confirm that the two solutions are linearly independent by showing that their Wronskian is equal to $-z^2$, as would be expected from the form of $(*)$.

16.9 Find series solutions of the equation $y'' - 2zy' - 2y = 0$. Identify one of the series as $y_1(z) = \exp z^2$ and verify this by direct substitution. By setting $y_2(z) = u(z) y_1(z)$ and solving the resulting equation for $u(z)$, find an explicit form for $y_2(z)$ and deduce that

$$\int_0^x e^{-v^2}\, dv = e^{-x^2} \sum_{n=0}^{\infty} \frac{n!}{2(2n+1)!} (2x)^{2n+1}.$$

16.10 Solve the equation

$$z(1-z)\frac{d^2 y}{dz^2} + (1-z)\frac{dy}{dz} + \lambda y = 0$$

as follows.

(a) Identify and classify its singular points and determine their indices.

(b) Find one series solution in powers of z. Give a formal expression for a second linearly independent solution.

(c) Deduce the values of λ for which there is a polynomial solution $P_N(z)$ of degree N. Evaluate the first four polynomials, normalised in such a way that $P_N(0) = 1$.

16.11 Find the general power series solution about $z = 0$ of the equation

$$z\frac{d^2y}{dz^2} + (2z-3)\frac{dy}{dz} + \frac{4}{z}y = 0.$$

16.12 Find the radius of convergence of a series solution about the origin for the equation $(z^2 + az + b)y'' + 2y = 0$ in the following cases:

$$\text{(a) } a = 5,\ b = 6; \qquad \text{(b) } a = 5,\ b = 7.$$

Show that if a and b are real and $4b > a^2$, then the radius of convergence is always given by $b^{1/2}$.

16.13 For the equation $y'' + z^{-3}y = 0$, show that the origin becomes a regular singular point if the independent variable is changed from z to $x = 1/z$. Hence find a series solution of the form $y_1(z) = \sum_0^\infty a_n z^{-n}$. By setting $y_2(z) = u(z)y_1(z)$ and expanding the resulting expression for du/dz in powers of z^{-1}, show that $y_2(z)$ has the asymptotic form

$$y_2(z) = c\left[z + \ln z - \tfrac{1}{2} + \mathrm{O}\left(\frac{\ln z}{z}\right)\right],$$

where c is an arbitrary constant.

16.14 Prove that the Laguerre equation,

$$z\frac{d^2y}{dz^2} + (1-z)\frac{dy}{dz} + \lambda y = 0,$$

has polynomial solutions $L_N(z)$ if λ is a non-negative integer N, and determine the recurrence relationship for the polynomial coefficients. Hence show that an expression for $L_N(z)$, normalised in such a way that $L_N(0) = N!$, is

$$L_N(z) = \sum_{n=0}^{N} \frac{(-1)^n (N!)^2}{(N-n)!(n!)^2} z^n.$$

Evaluate $L_3(z)$ explicitly.

16.15 The origin is an ordinary point of the Chebyshev equation,

$$(1-z^2)y'' - zy' + m^2 y = 0,$$

which therefore has series solutions of the form $z^\sigma \sum_0^\infty a_n z^n$ for $\sigma = 0$ and $\sigma = 1$.

(a) Find the recurrence relationships for the a_n in the two cases and show that there exist polynomial solutions $T_m(z)$:

(i) for $\sigma = 0$, when m is an even integer, the polynomial having $\frac{1}{2}(m+2)$ terms;

(ii) for $\sigma = 1$, when m is an odd integer, the polynomial having $\frac{1}{2}(m+1)$ terms.

(b) $T_m(z)$ is normalised so as to have $T_m(1) = 1$. Find explicit forms for $T_m(z)$ for $m = 0, 1, 2, 3$.

(c) Show that the corresponding non-terminating series solutions $S_m(z)$ have as their first few terms

$$S_0(z) = a_0\left(z + \frac{1}{3!}z^3 + \frac{9}{5!}z^5 + \cdots\right),$$

$$S_1(z) = a_0\left(1 - \frac{1}{2!}z^2 - \frac{3}{4!}z^4 - \cdots\right),$$

$$S_2(z) = a_0\left(z - \frac{3}{3!}z^3 - \frac{15}{5!}z^5 - \cdots\right),$$

$$S_3(z) = a_0\left(1 - \frac{9}{2!}z^2 + \frac{45}{4!}z^4 + \cdots\right).$$

16.16 Obtain the recurrence relations for the solution of Legendre's equation (18.1) in *inverse* powers of z, i.e. set $y(z) = \sum a_n z^{\sigma-n}$, with $a_0 \neq 0$. Deduce that, if ℓ is an integer, then the series with $\sigma = \ell$ will terminate and hence converge for all z, whilst the series with $\sigma = -(\ell+1)$ does not terminate and hence converges only for $|z| > 1$.

16.7 Hints and answers

16.1 Note that $z = 0$ is an ordinary point of the equation.
For $\sigma = 0$, $a_{n+2}/a_n = [n(n+2) - \lambda]/[(n+1)(n+2)]$ and, correspondingly, for $\sigma = 1$, $U_2(z) = a_0(1 - 4z^2)$ and $U_3(z) = a_0(z - 2z^3)$.

16.3 $\sigma = 0$ and 3; $a_{6m}/a_0 = (-1)^m/(2m)!$ and $a_{6m}/a_0 = (-1)^m/(2m+1)!$, respectively. $y_1(z) = a_0 \cos z^3$ and $y_2(z) = a_0 \sin z^3$. The Wronskian is $\pm 3a_0^2 z^2 \neq 0$.

16.5 (b) $a_{n+1}/a_n = [\ell(\ell+1) - n(n+1)]/[2(n+1)^2]$.
(c) $R = 2$, equal to the distance between $z = 1$ and the closest singularity at $z = -1$.

16.7 A typical term in the series for $y(\sigma, z)$ is $\dfrac{(-1)^n z^{2n}}{[(\sigma+2)(\sigma+4)\cdots(\sigma+2n)]^2}$.

16.9 The origin is an ordinary point. Determine the constant of integration by examining the behaviour of the related functions for small x.
$y_2(z) = (\exp z^2)\int_0^z \exp(-x^2)\,dx$.

16.11 Repeated roots $\sigma = 2$.

$$y(z) = az^2 - 4az^3 + 6bz^3 + \sum_{n=2}^{\infty} \frac{(n+1)(-2z)^{n+2}}{n!}\left\{\frac{a}{4} + b\left[\ln z + g(n)\right]\right\},$$

where

$$g(n) = \frac{1}{n+1} - \frac{1}{n} - \frac{1}{n-1} - \cdots - \frac{1}{2} - 2.$$

16.13 The transformed equation is $xy'' + 2y' + y = 0$; $a_n = (-1)^n(n+1)^{-1}(n!)^{-2}a_0$; $du/dz = A[y_1(z)]^{-2}$.

16.15 (a) (i) $a_{n+2} = [a_n(n^2 - m^2)]/[(n+2)(n+1)]$,
(ii) $a_{n+2} = \{a_n[(n+1)^2 - m^2]\}/[(n+3)(n+2)]$; (b) 1, z, $2z^2 - 1$, $4z^3 - 3z$.

17

Eigenfunction methods for differential equations

In the previous three chapters we dealt with the solution of differential equations of order n by two methods. In one method, we found n independent solutions of the equation and then combined them, weighted with coefficients determined by the boundary conditions; in the other we found solutions in terms of series whose coefficients were related by (in general) an n-term recurrence relation and thence fixed by the boundary conditions. For both approaches the linearity of the equation was an important or essential factor in the utility of the method, and in this chapter our aim will be to exploit the superposition properties of linear differential equations even further.

We will be concerned with the solution of equations of the inhomogeneous form

$$\mathcal{L}y(x) = f(x), \tag{17.1}$$

where $f(x)$ is a prescribed or general function *and* the boundary conditions to be satisfied by the solution $y = y(x)$, for example at the limits $x = a$ and $x = b$, are given. The expression $\mathcal{L}y(x)$ stands for a linear differential operator $\mathcal{L}$ acting upon the function $y(x)$.

In general, unless $f(x)$ is both known and simple, it will not be possible to find particular integrals of (17.1), even if complementary functions can be found that satisfy $\mathcal{L}y = 0$. The idea is therefore to exploit the linearity of $\mathcal{L}$ by building up the required solution $y(x)$ as a *superposition*, generally containing an infinite number of terms, of some set of functions $\{y_i(x)\}$ that each individually satisfy the boundary conditions. Clearly this brings in a quite considerable complication but since, within reason, we may select the set of functions to suit ourselves, we can obtain sizeable compensation for this complication. Indeed, if the set chosen is one containing functions that, when acted upon by $\mathcal{L}$, produce particularly simple results then we can 'show a profit' on the operation. In particular, if the set consists of those functions y_i for which

$$\mathcal{L}y_i(x) = \lambda_i y_i(x), \tag{17.2}$$

where λ_i is a constant (and which satisfy the boundary conditions), then a distinct advantage may be obtained from the manoeuvre because all the differentiation will have disappeared from (17.1).

Equation (17.2) is clearly reminiscent of the equation satisfied by the *eigenvectors* $\mathbf{x}^i$ of a linear operator $\mathcal{A}$, namely

$$\mathcal{A}\mathbf{x}^i = \lambda_i \mathbf{x}^i, \tag{17.3}$$

where λ_i is a constant and is called the *eigenvalue* associated with $\mathbf{x}^i$. By analogy, in the context of differential equations a function $y_i(x)$ satisfying (17.2) is called an *eigenfunction* of the operator $\mathcal{L}$ (under the imposed boundary conditions) and λ_i is then called the eigenvalue associated with the eigenfunction $y_i(x)$. Clearly, the eigenfunctions $y_i(x)$ of $\mathcal{L}$ are only determined up to an arbitrary scale factor by (17.2).

Probably the most familiar equation of the form (17.2) is that which describes a simple harmonic oscillator, i.e.

$$\mathcal{L}y \equiv -\frac{d^2y}{dt^2} = \omega^2 y, \qquad \text{where } \mathcal{L} \equiv -d^2/dt^2. \tag{17.4}$$

Imposing the boundary condition that the solution is periodic with period T, the eigenfunctions in this case are given by $y_n(t) = A_n e^{i\omega_n t}$, where $\omega_n = 2\pi n/T$, $n = 0, \pm 1, \pm 2, \ldots$ and the A_n are constants. The eigenvalues are $\omega_n^2 = n^2\omega_1^2 = n^2(2\pi/T)^2$. (Sometimes ω_n is referred to as the eigenvalue of this equation, but we will avoid such confusing terminology here.)

We may discuss a somewhat wider class of differential equations by considering a slightly more general form of (17.2), namely

$$\mathcal{L}y_i(x) = \lambda_i \rho(x) y_i(x), \tag{17.5}$$

where $\rho(x)$ is a *weight function*. In many applications $\rho(x)$ is unity for all x, in which case (17.2) is recovered; in general, though, it is a function determined by the choice of coordinate system used in describing a particular physical situation. The only requirement on $\rho(x)$ is that it is real and does not change sign in the range $a \leq x \leq b$, so that it can, without loss of generality, be taken to be non-negative throughout; of course, $\rho(x)$ must be the same function for all values of λ_i. A function $y_i(x)$ that satisfies (17.5) is called an eigenfunction of the operator $\mathcal{L}$ with respect to the weight function $\rho(x)$.

This chapter will not cover methods used to determine the eigenfunctions of (17.2) or (17.5), since we have discussed those in previous chapters, but, rather, will use the properties of the eigenfunctions to solve inhomogeneous equations of the form (17.1). We shall see later that the sets of eigenfunctions $y_i(x)$ of a particular class of operators called *Hermitian operators* (the operator in the simple harmonic oscillator equation is an example) have particularly useful properties and these will be studied in detail. It turns out that many of the interesting differential operators met within the physical sciences are Hermitian. Before continuing our discussion of the eigenfunctions of Hermitian operators, however, we will consider some properties of general sets of functions.

17.1 Sets of functions

In chapter 8 we discussed the definition of a vector space but concentrated on spaces of finite dimensionality. We consider now the *infinite*-dimensional space of all reasonably well-behaved functions $f(x)$, $g(x)$, $h(x)$, $\ldots$ on the interval $a \leq x \leq b$. That these functions form a linear vector space is shown by noting the following properties. The set is closed under

(i) addition, which is commutative and associative, i.e.

$$f(x) + g(x) = g(x) + f(x),$$
$$[f(x) + g(x)] + h(x) = f(x) + [g(x) + h(x)],$$

(ii) multiplication by a scalar, which is distributive and associative, i.e.

$$\lambda\,[f(x) + g(x)] = \lambda f(x) + \lambda g(x),$$
$$\lambda\,[\mu f(x)] = (\lambda\mu) f(x),$$
$$(\lambda + \mu) f(x) = \lambda f(x) + \mu f(x).$$

Furthermore, in such a space

(iii) there exists a ‘null vector’ 0 such that $f(x) + 0 = f(x)$,
(iv) multiplication by unity leaves any function unchanged, i.e. $1 \times f(x) = f(x)$,
(v) each function has an associated negative function $-f(x)$ that is such that $f(x) + [-f(x)] = 0$.

By analogy with finite-dimensional vector spaces we now introduce a set of linearly independent basis functions $y_n(x)$, $n = 0, 1, \ldots, \infty$, such that *any* ‘reasonable’ function in the interval $a \leq x \leq b$ (i.e. it obeys the Dirichlet conditions discussed in chapter 12) can be expressed as the linear sum of these functions:

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x).$$

Clearly if a different set of linearly independent basis functions $u_n(x)$ is chosen then the function can be expressed in terms of the new basis,

$$f(x) = \sum_{n=0}^{\infty} d_n u_n(x),$$

where the d_n are a different set of coefficients. In each case, provided the basis functions are linearly independent, the coefficients are unique.

We may also define an *inner product* on our function space by

$$\langle f|g\rangle = \int_a^b f^*(x)g(x)\rho(x)\,dx, \tag{17.6}$$

where $\rho(x)$ is the weight function, which we require to be real and non-negative in the interval $a \le x \le b$. As mentioned above, $\rho(x)$ is often unity for all x. Two functions are said to be *orthogonal* (with respect to the weight function $\rho(x)$) on the interval $[a, b]$ if

$$\langle f|g\rangle = \int_a^b f^*(x)g(x)\rho(x)\,dx = 0, \tag{17.7}$$

and the *norm* of a function is defined as

$$\|f\| = \langle f|f\rangle^{1/2} = \left[\int_a^b f^*(x)f(x)\rho(x)\,dx\right]^{1/2} = \left[\int_a^b |f(x)|^2\rho(x)\,dx\right]^{1/2}. \tag{17.8}$$

It is also common practice to define a *normalised* function by $\hat{f} = f/\|f\|$, which has unit norm.

An infinite-dimensional vector space of functions, for which an inner product is defined, is called a *Hilbert space*. Using the concept of the inner product, we can choose a basis of linearly independent functions $\hat{\phi}_n(x)$, $n = 0, 1, 2, \ldots$ that are orthonormal, i.e. such that

$$\langle \hat{\phi}_i|\hat{\phi}_j\rangle = \int_a^b \hat{\phi}_i^*(x)\hat{\phi}_j(x)\rho(x)\,dx = \delta_{ij}. \tag{17.9}$$

If $y_n(x)$, $n = 0, 1, 2, \ldots$, are a linearly independent, but not orthonormal, basis for the Hilbert space then an orthonormal set of basis functions $\hat{\phi}_n$ may be produced (in a similar manner to that used in the construction of a set of orthogonal eigenvectors of an Hermitian matrix; see chapter 8) by the following procedure:

$$\begin{aligned}
\phi_0 &= y_0,\\
\phi_1 &= y_1 - \hat{\phi}_0\langle\hat{\phi}_0|y_1\rangle,\\
\phi_2 &= y_2 - \hat{\phi}_1\langle\hat{\phi}_1|y_2\rangle - \hat{\phi}_0\langle\hat{\phi}_0|y_2\rangle,\\
&\ \vdots\\
\phi_n &= y_n - \hat{\phi}_{n-1}\langle\hat{\phi}_{n-1}|y_n\rangle - \cdots - \hat{\phi}_0\langle\hat{\phi}_0|y_n\rangle,\\
&\ \vdots
\end{aligned}$$

It is straightforward to check that each ϕ_n is orthogonal to all its predecessors ϕ_i, $i = 0, 1, 2, \ldots, n-1$. This method is called *Gram–Schmidt orthogonalisation*. Clearly the functions ϕ_n form an orthogonal set, but in general they do not have unit norms.

►*Starting from the linearly independent functions $y_n(x) = x^n$, $n = 0, 1, \ldots,$ construct three orthonormal functions over the range $-1 < x < 1$, assuming a weight function of unity.*

The first unnormalised function ϕ_0 is simply equal to the first of the original functions, i.e.

$$\phi_0 = 1.$$

The normalisation is carried out by dividing by

$$\langle \phi_0 | \phi_0 \rangle^{1/2} = \left(\int_{-1}^{1} 1 \times 1 \, du \right)^{1/2} = \sqrt{2},$$

with the result that the first normalised function $\hat{\phi}_0$ is given by

$$\hat{\phi}_0 = \frac{\phi_0}{\sqrt{2}} = \sqrt{\tfrac{1}{2}}.$$

The second unnormalised function is found by applying the above Gram–Schmidt orthogonalisation procedure, i.e.

$$\phi_1 = y_1 - \hat{\phi}_0 \langle \hat{\phi}_0 | y_1 \rangle.$$

It can easily be shown that $\langle \hat{\phi}_0 | y_1 \rangle = 0$, and so $\phi_1 = x$. Normalising then gives

$$\hat{\phi}_1 = \phi_1 \left(\int_{-1}^{1} u \times u \, du \right)^{-1/2} = \sqrt{\tfrac{3}{2}} x.$$

The third unnormalised function is similarly given by

$$\begin{aligned} \phi_2 &= y_2 - \hat{\phi}_1 \langle \hat{\phi}_1 | y_2 \rangle - \hat{\phi}_0 \langle \hat{\phi}_0 | y_2 \rangle \\ &= x^2 - 0 - \tfrac{1}{3}, \end{aligned}$$

which, on normalising, gives

$$\hat{\phi}_2 = \phi_2 \left(\int_{-1}^{1} \left(u^2 - \tfrac{1}{3}\right)^2 du \right)^{-1/2} = \tfrac{1}{2} \sqrt{\tfrac{5}{2}} (3x^2 - 1).$$

By comparing the functions $\hat{\phi}_0$, $\hat{\phi}_1$ and $\hat{\phi}_2$ with the list in subsection 18.1.1, we see that this procedure has generated (multiples of) the first three Legendre polynomials. ◄

If a function is expressed in terms of an *orthonormal* basis $\hat{\phi}_n(x)$ as

$$f(x) = \sum_{n=0}^{\infty} c_n \hat{\phi}_n(x) \tag{17.10}$$

then the coefficients c_n are given by

$$c_n = \langle \hat{\phi}_n | f \rangle = \int_a^b \hat{\phi}_n^*(x) f(x) \rho(x) \, dx. \tag{17.11}$$

Note that this is true only if the basis is orthonormal.

17.1.1 Some useful inequalities

Since for a Hilbert space $\langle f|f\rangle \geq 0$, the inequalities discussed in subsection 8.1.3 hold. The proofs are not repeated here, but the relationships are listed for completeness.

(i) The Schwarz inequality states that

$$|\langle f|g\rangle| \leq \langle f|f\rangle^{1/2}\langle g|g\rangle^{1/2}, \tag{17.12}$$

where the equality holds when $f(x)$ is a scalar multiple of $g(x)$, i.e. when they are linearly dependent.

(ii) The triangle inequality states that

$$\|f+g\| \leq \|f\| + \|g\|, \tag{17.13}$$

where again equality holds when $f(x)$ is a scalar multiple of $g(x)$.

(iii) Bessel's inequality requires the introduction of an *orthonormal* basis $\hat{\phi}_n(x)$ so that any function $f(x)$ can be written as

$$f(x) = \sum_{n=0}^{\infty} c_n \hat{\phi}_n(x),$$

where $c_n = \langle \hat{\phi}_n|f\rangle$. Bessel's inequality then states that

$$\langle f|f\rangle \geq \sum_n |c_n|^2. \tag{17.14}$$

The equality holds if the summation is over all the basis functions. If some values of n are omitted from the sum then the inequality results (unless, of course, the c_n happen to be zero for all values of n omitted, in which case the equality remains).

17.2 Adjoint, self-adjoint and Hermitian operators

Having discussed general sets of functions, we now return to the discussion of eigenfunctions of linear operators. We begin by introducing the *adjoint* of an operator $\mathcal{L}$, denoted by $\mathcal{L}^\dagger$, which is defined by

$$\int_a^b f^*(x)\,[\mathcal{L}g(x)]\;dx = \int_a^b [\mathcal{L}^\dagger f(x)]^* g(x)\,dx + \text{boundary terms}, \tag{17.15}$$

where the boundary terms are evaluated at the end-points of the interval $[a, b]$. Thus, for any given linear differential operator $\mathcal{L}$, the adjoint operator $\mathcal{L}^\dagger$ can be found by repeated integration by parts.

An operator is said to be *self-adjoint* if $\mathcal{L}^\dagger = \mathcal{L}$. If, in addition, certain boundary conditions are met by the functions f and g on which a self-adjoint operator acts,