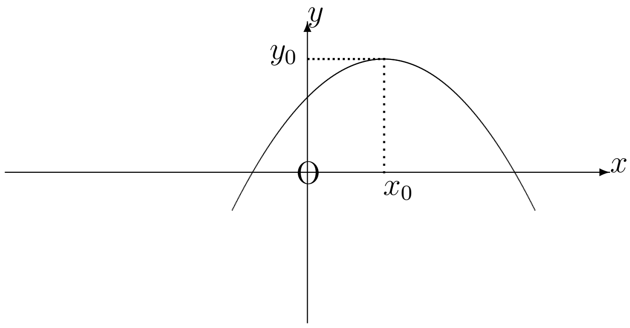
11.2.2 LOCAL MAXIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local maximum"** if y_0 is greater than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



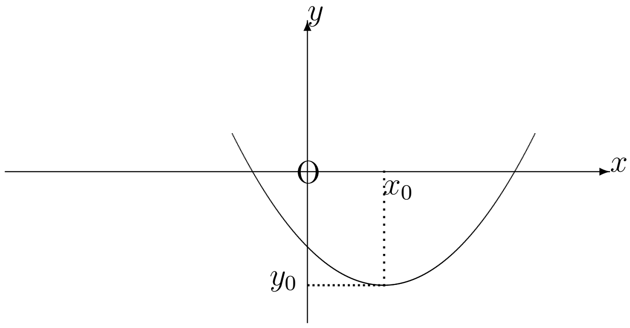
Note:
It may well happen that, for points on the curve which are some distance away from (x_0, y_0), their y co-ordinates are greater than y_0; hence, the definition of a local maximum point must refer to the behaviour of y in the immediate neighbourhood of the point.

11.2.3 LOCAL MINIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local minimum"** if y_0 is less than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



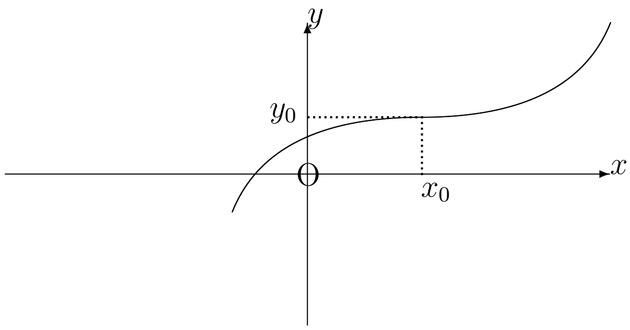
Note:
It may well happen that, for points on the curve which are some distance away from (x_0, y_0), their y co-ordinates are less than y_0; hence, the definition of a local minimum point must refer to the behaviour of y in the immediate neighbourhood of the point.

11.2.4 POINTS OF INFLEXION

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"point of inflexion"** if the curve exhibits a change in the direction bending there.



11.2.5 THE LOCATION OF STATIONARY POINTS AND THEIR NATURE

In order to determine the location of any stationary points on the curve whose equation is

$$y = f(x),$$

we simply obtain an expression for the derivate of y with respect to x, then equate it to zero. That is, we solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Having located a stationary point (x_0, y_0), we may then determine whether it is a local maximum, a local minimum, or a point of inflexion using two alternative methods. These methods will be illustrated by examples:

METHOD 1. - The "First Derivative" Method

Suppose ϵ denotes a number which is relatively small compared with x_0.

If we examine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$, first at $x = x_0 - \epsilon$ and then at $x = x_0 + \epsilon$, the following conclusions may be drawn:

(a) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from positive to negative, there is a local maximum at (x_0, y_0).

(b) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from negative to positive, there is a local minimum at (x_0, y_0).

(c) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not change, there is a point of inflexion at (x_0, y_0).

EXAMPLES

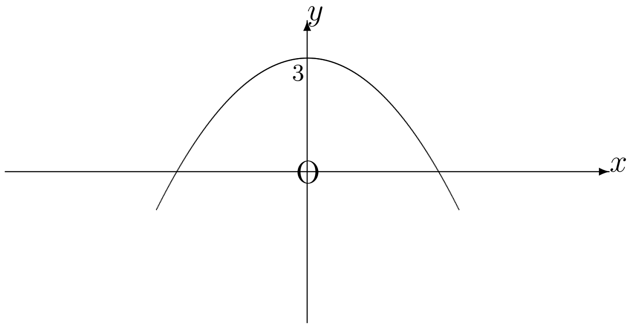
1. Determine the stationary point on the graph whose equation is

$$y = 3 - x^2.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x,$$

which is equal to zero at the point where $x = 0$ and hence, $y = 3$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and
If $x = 0 + \epsilon$, (for example $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.
Hence, there is a local maximum at the point $(0, 3)$.



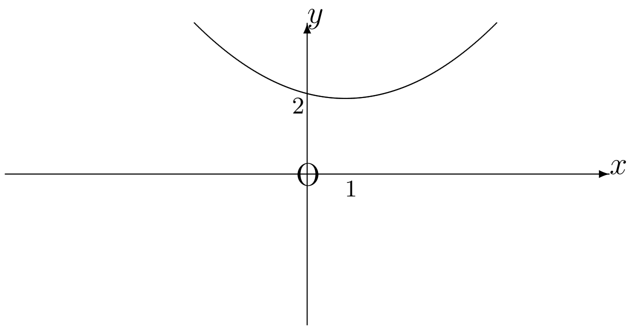
2. Determine the stationary point on the graph whose equation is

$$y = x^2 - 2x + 3.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2,$$

which is equal to zero at the point where $x = 1$ and hence, $y = 2$.
If $x = 1 - \epsilon$, (for example, $x = 1 - 0.01 = 0.99$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ and
If $x = 1 + \epsilon$, (for example, $x = 1 + 0.01) = 1.01$, then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a local minimum at the point $(1, 2)$.



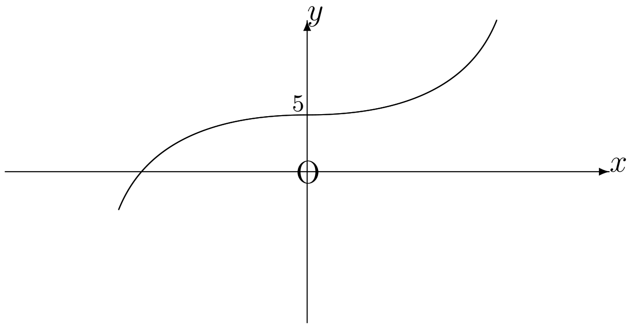
3. Determine the stationary point on the graph whose equation is

$$y = 5 + x^3.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

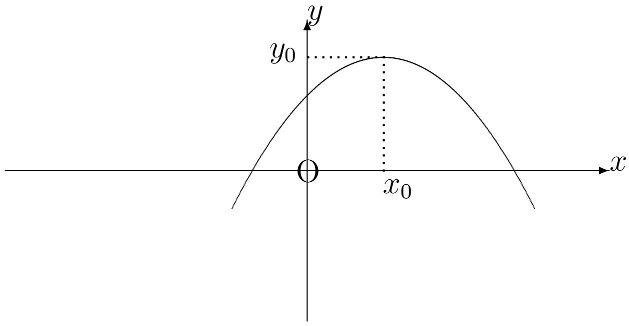
which is equal to zero at the point where $x = 0$ and hence, $y = 5$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and
If $x = 0 + \epsilon$, (for example, $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a point of inflexion at $(0, 5)$.



METHOD 2. - The "Second Derivative" Method

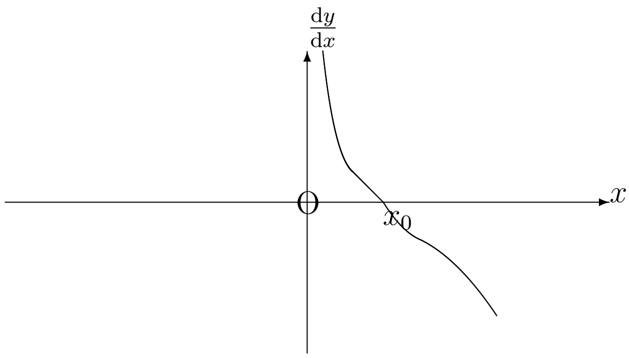
This method considers the general appearance of the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x, which is called the **"first derived curve"**. The properties of the first derived curve in the neighbourhood of a stationary point (x_0, y_0) may be used to predict the nature of this point.

(a) Local Maxima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ steadily decrease from large positive values to large negative values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going downwards"** tendency at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is **negative**. In other words,

$$\frac{d^2y}{dx^2} < 0 \quad \text{at} \quad x = x_0.$$

This is the second derivative test for a local maximum.

EXAMPLE

For the curve whose equation is

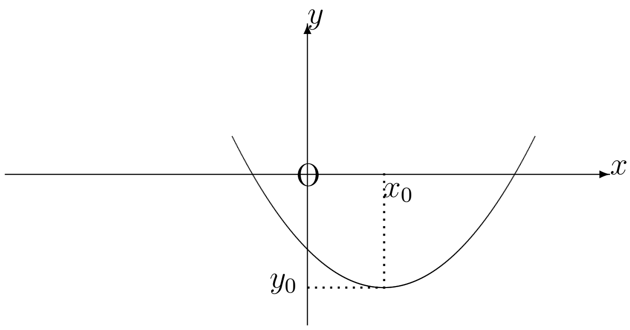
$$y = 3 - x^2,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2.$$

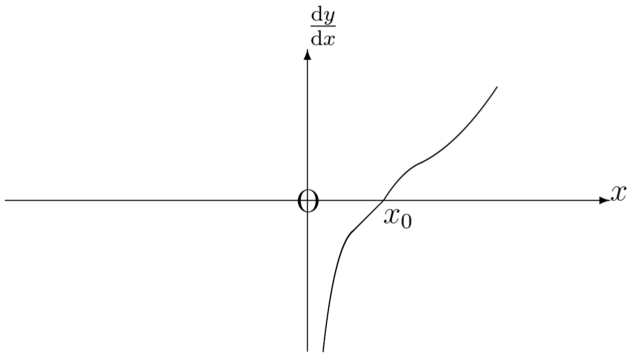
The second derivative is negative <u>everywhere</u>, so it is certainly negative at the stationary point $(0, 3)$ obtained in the previous method. Hence, $(0, 3)$ is a local maximum.

(b) Local Minima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ steadily increase from large negative values to large positive values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going upwards"** tendency at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is **positive**. In other words,

$$\frac{d^2y}{dx^2} > 0 \quad \text{at} \quad x = x_0.$$

This is the second derivative test for a local minimum.

EXAMPLE

For the curve whose equation is

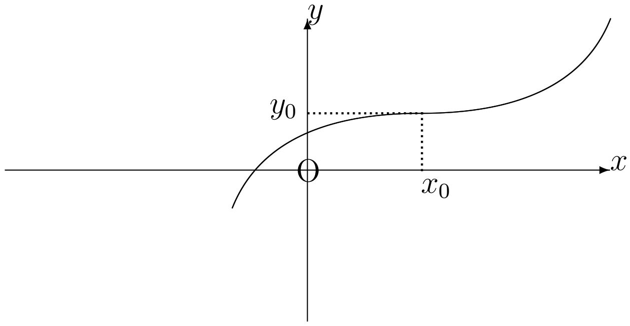
$$y = x^2 - 2x + 3,$$

we have

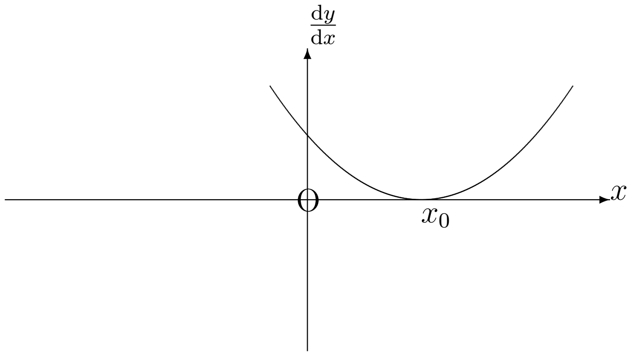
$$\frac{dy}{dx} = 2x - 2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 2.$$

The second derivative is positive everywhere, so it is certainly positive at the stationary point $(1, 2)$ obtained in the previous method. Hence, $(1, 2)$ is a local minimum.

(c) Points of inflexion



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ appear to reach either a minimum or a maximum value at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is zero and changes sign as x passes through the value x_0.

$$\frac{d^2y}{dx^2} = 0 \quad \text{at} \quad x = x_0 \quad \text{and} \quad \text{changes} \quad \text{sign.}$$

This is the second derivative test for a point of inflexion.

EXAMPLE

For the curve whose equation is

$$y = 5 + x^3,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

The second derivative is zero when $x = 0$ and changes sign as x passes through the value zero.

Hence, the stationary point $(0, 5)$ found previously is a point of inflexion.

Notes:
(i) For a stationary point of inflexion, it is not enough that

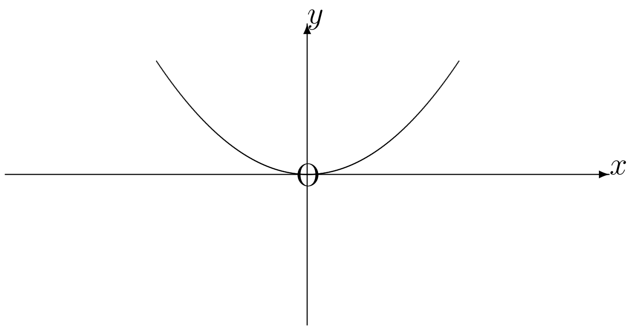
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

without also the change of sign.

For example, the curve whose equation is

$$y = x^4$$

is easily shown (by Method 1) to have a local minimum at the point $(0, 0)$; and yet, for this curve, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$.



(ii) Some curves contain what are called **"ordinary points of inflexion"**. They are not stationary points and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$; but the rest of the condition for a point of inflexion still holds. That is,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \text{and} \quad \text{changes} \quad \text{sign.}$$

EXAMPLE

For the curve whose equation is

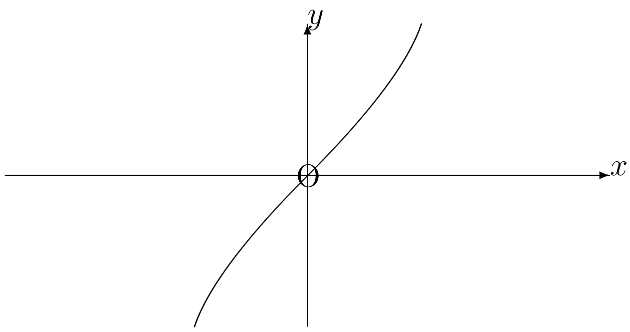
$$y = x^3 + x,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

Hence, there are no stationary points at all; but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$ and changes sign as x passes through $x = 0$.

That is, there is an ordinary point of inflexion at $(0, 0)$.



Notes:
(i) In any interval of the x-axis, the greatest value of a function of x will be either the greatest maximum or possibly the value at one end of the interval. Similarly, the least value of the function will be either the smallest minimum or possibly the value at one end of the interval.

(ii) In sketching a curve whose maxima, minima and points of inflexion are known, it may also be necessary to determine, from the equation of the curve, its points of intersection with the axes of reference.

11.2.6 EXERCISES

1. Determine the local maxima, local minima and points of inflexion (including ordinary points of inflexion) on the curves whose equations are given in the following:

   (a)
   $$y = x^3 - 6x^2 + 9x + 6;$$

   (b)
   $$y = x + \frac{1}{x}.$$

   In each case, give also a sketch of the curve.

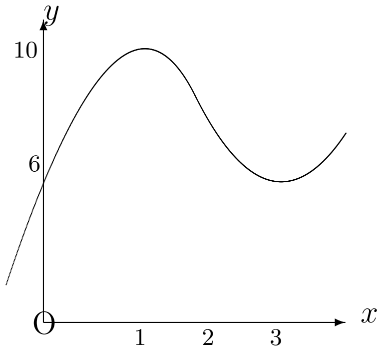
2. Show that the curve whose equation is
   $$y = \frac{1}{2x+1} + \ln(2x+1)$$
   has a local minimum at a point on the y-axis.

3. The horse-power, P, transmitted by a belt is given by
   $$P = k\left[Tv - \frac{wv^3}{g}\right],$$
   where k is a constant, v is the speed of the belt, T is the tension on the driving side and w is the weight per unit length of the belt. Determine the speed for which the horse-power is a maximum.

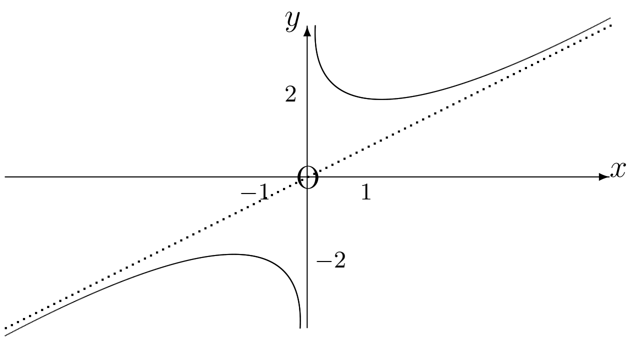
4. For x lying in the interval $-3 \leq x \leq 5$, determine the least and greatest values of the function
   $$x^3 - 12x + 20$$

11.2.7 ANSWERS TO EXERCISES

1. (a) Local maximum at $(1, 10)$, local minumum at $(3, 6)$, ordinary point of inflexion at $(2, 8)$;



   (b) Local maximum at $(-1, -2)$, local minimum at $(1, 2)$.



2. Local minimum at the point $(0, 1)$.

3. The horse-power is maximum when

$$v = \sqrt{\frac{gT}{2w}}.$$

4. The greatest value is 85 at $(5, 85)$; the least value is 4 at $(2, 4)$.

"JUST THE MATHS"

UNIT NUMBER

11.3

DIFFERENTIATION APPLICATIONS 3
(Curvature)

by

A.J.Hobson

UNIT 11.3 - DIFFERENTIATION APPLICATIONS 3

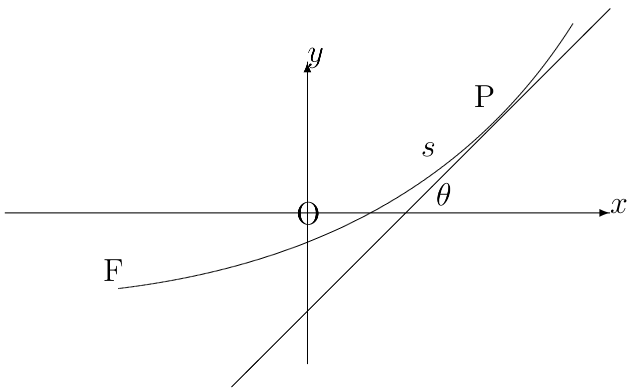
CURVATURE

11.3.1 INTRODUCTION

In the discussion which follows, consideration will be given to a method of measuring the **"tightness of bends"** on a curve. This measure will be called **"curvature"** and its definition will imply that very tight bends have large curvature.

We shall also need to distinguish between curves which are **"concave upwards"** ($\cup$), having positive curvature, and curves which are **"concave downwards"** ($\cap$), having negative curvature.

DEFINITION
Suppose we are given a curve whose equation is $y = f(x)$; and suppose that θ is the angle made with the positive x-axis by the tangent to the curve at a point, P(x, y), on it. If s is the distance to P, measured along the curve from some fixed point, F, on it then the curvature, κ, at P, is defined as the rate of increase of θ with respect to s.

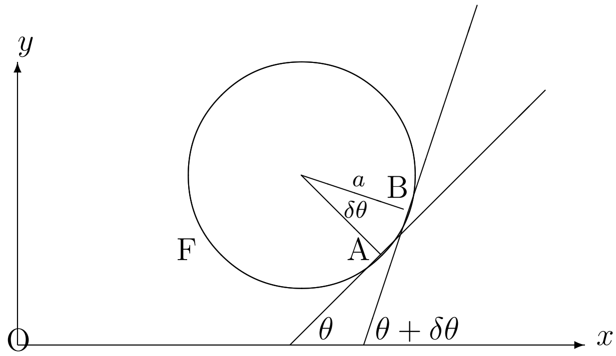


$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

EXAMPLE

Determine the curvature at any point of a circle with radius a.

Solution



We shall let A be a point on the circle at which the tangent is inclined to the positive x-axis at an angle, θ, and let B be a point (close to A) at which the tangent is inclined to the positive x-axis at an angle, $\theta + \delta\theta$. The length of the arc, AB, will be called δs, where we shall assume that distances, s, are measured along the circle in a counter-clockwise sense from the fixed point, F.

The diagram shows that $\delta\theta$ is both the angle between the two tangents **and** the angle subtended at the centre of the circle by the arc, AB.

Thus, $\delta s = a\delta\theta$ which can be written

$$\frac{\delta\theta}{\delta s} = \frac{1}{a}.$$

Allowing $\delta\theta$, and hence δs, to approach zero, we conclude that

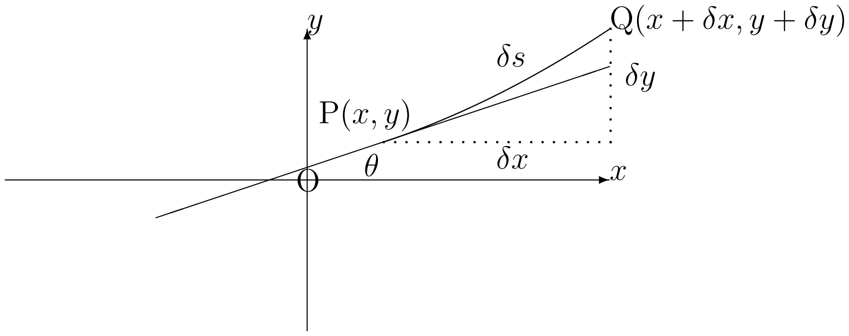
$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{a}.$$

We note, however, that, for the lower half of the circle, θ **increases** as s increases, while, in the upper half of the circle, θ **decreases** as s increases. The curvature will therefore be positive for the lower half (which is concave upwards) and negative for the upper half (which is concave downwards).

Summary
The curvature at any point of a circle is numerically equal to the reciprocal of the radius.

11.3.2 CURVATURE IN CARTESIAN CO-ORDINATES

Given a curve whose equation is $y = f(x)$, suppose $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ are two neighbouring points on it which are separated by a distance of δs along the curve.



In this diagram, we may observe that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \tan\theta$$

and also that

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lim_{\delta s \to 0} \frac{\delta x}{\delta s} = \cos\theta.$$

The curvature may therefore be evaluated as follows:

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\cos\theta.$$

But,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{1}{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Finally,

$$\cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}};$$

and so,

$$\kappa = \pm\frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}.$$

Notes:

(i) For a curve which is concave upwards at a particular point, the gradient, $\frac{\mathrm{d}y}{\mathrm{d}x}$, will **increase** as x increases through the point. Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ will be positive at the point.

(ii) For a curve which is concave downwards at a particular point, the gradient, $\frac{\mathrm{d}y}{\mathrm{d}x}$, will **decrease** as x increases through the point. Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ will be negative at the point.

(ii) In future, therefore, we may allow the value of the curvature to take the same sign as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, giving the formula

$$\kappa = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1+\frac{\mathrm{d}y}{\mathrm{d}x}^2\right]^{\frac{3}{2}}}.$$

EXAMPLE

Use the cartesian formula to determine the curvature at any point on the circle, centre $(0,0)$ with radius a.

Solution

The equation of the circle is

$$x^2 + y^2 = a^2,$$

which means that, for the upper half,

$$y = \sqrt{a^2 - x^2}$$

and, for the lower half,

$$y = -\sqrt{a^2 - x^2}.$$

Considering, firstly, the upper half,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{a^2 - x^2}}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2-x^2}}}{a^2 - x^2} = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Therefore,

$$\kappa = \frac{-\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}}{\left(1 + \frac{x^2}{a^2-x^2}\right)^{\frac{3}{2}}} = -\frac{a^2}{a^3} = -\frac{1}{a}.$$

For the lower half of the circle,

$$\kappa = \frac{1}{a}.$$

11.3.3 EXERCISES

In the following questions, state your answer in decimals correct to three places of decimals:

1. Calculate the curvature at the point $(-1, 3)$ on the curve whose equation is
$$y = x + 3x^2 - x^3.$$

2. Calculate the curvature at the origin on the curve whose equation is
$$y = \frac{x - x^2}{1 + x^2}.$$

3. Calculate the curvature at the point $(1, 1)$ on the curve whose equation is
$$x^3 - 2xy + y^3 = 0.$$

4. Calculate the curvature at the point for which $\theta = 30°$ on the curve whose parametric equations are
$$x = 1 + \sin\theta \quad \text{and} \quad y = \sin\theta - \frac{1}{2}\cos 2\theta.$$

11.3.4 ANSWERS TO EXERCISES

1. $\kappa = 0.023$
2. $\kappa = -0.707$
3. $\kappa = -5.650$
4. $\kappa = 0.179$

"JUST THE MATHS"

UNIT NUMBER

11.4

DIFFERENTIATION APPLICATIONS 4
(Circle, radius & centre of curvature)

by

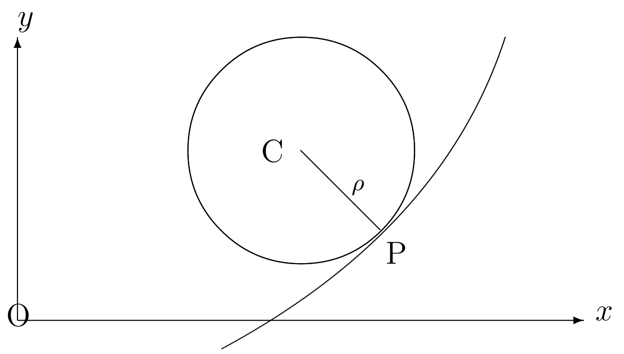
A.J.Hobson

UNIT 11.4 DIFFERENTIATION APPLICATIONS 4

CIRCLE, RADIUS AND CENTRE OF CURVATURE

11.4.1 INTRODUCTION

At a point, P, on a given curve, suppose we were to draw a circle which **just touches** the curve and has the same value of the curvature (including its sign). This circle is called the **"circle of curvature at P"**. Its radius, ρ, is called the **"radius of curvature at P"** and its centre is called the **"centre of curvature at P"**.



11.4.2 RADIUS OF CURVATURE

Using the earlier examples on the circle (Unit 11.3), we conclude that, if the curvature at P is κ, then $\rho = \frac{1}{\kappa}$ and, hence,

$$\rho = \frac{\mathrm{d}s}{\mathrm{d}\theta}.$$

Furthermore, in cartesian co-ordinates,

$$\rho = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}.$$

Note:
If we are interested in the radius of curvature simply as a length, then, for curves with

negative curvature, we would use only the **numerical** value obtained in the above formula. However, in a later discussion, it is necessary to use the appropriate sign for the radius of curvature.

EXAMPLE

Calculate the radius of curvature at the point $(0.5, -1)$ of the curve whose equation is

$$y^2 = 2x.$$

Solution

Differentiating implicitly,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}.$$

Also

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^2}.\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y^3}.$$

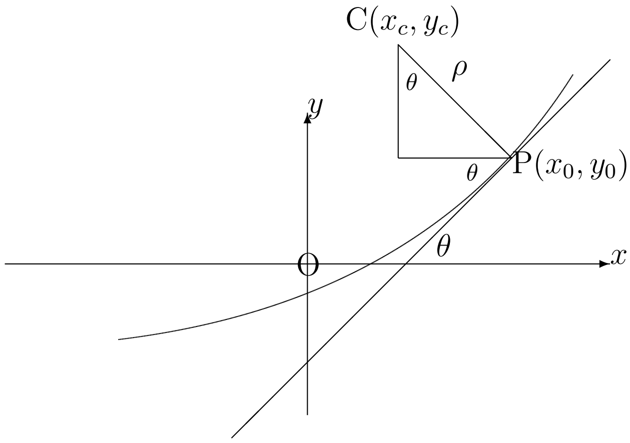
Hence, at the point $(0.5, -1)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$.

We conclude that

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{1} = 2\sqrt{2}.$$

11.4.3 CENTRE OF CURVATURE

We shall consider a point, (x_0, y_0), on an arc of a curve whose equation is $y = f(x)$ and for which the curvature is positive, the arc lying in the first quadrant. But it may be shown that the formulae obtained for the co-ordinates, (x_c, y_c), of the centre of curvature apply in any situation, provided that the curvature is associated with its appropriate sign.



From the diagram,

$$
\begin{aligned}
x_c &= x_0 - \rho \sin\theta, \\
y_c &= y_0 + \rho \cos\theta.
\end{aligned}
$$

Note:
Although the formulae apply in any situation, it is a good idea to sketch the curve in order estimate, roughly, where the centre of curvature is going to be. This is especially important where there is uncertainty about the precise value of the angle θ.

EXAMPLE

Determine the centre of curvature at the point $(0.5, -1)$ of the curve whose equation is

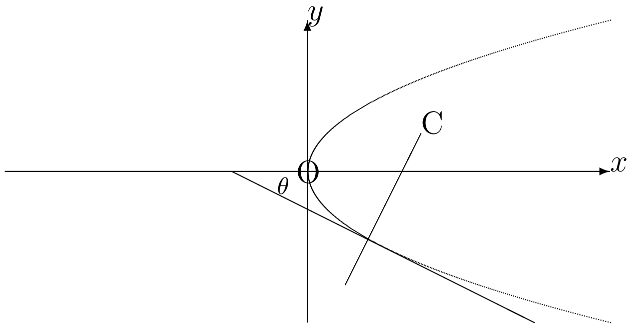
$$y^2 = 2x.$$

Solution

From the earlier example on calculating radius of curvature,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y} \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^3},$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$ and $\rho = 2\sqrt{2}$ at the point $(0.5, -1)$.



The diagram shows that the co-ordinates, (x_c, y_c), of the centre of curvature will be such that $x_c > 0.5$ and $y_c > -1$. This will be so provided that the angle, θ, is a negative acute angle; (that is, its cosine will be positive and its sine will be negative).

In fact,

$$\theta = \tan^{-1}(-1) = -45^\circ.$$

Hence,

$$\begin{aligned} x_c &= 0.5 - 2\sqrt{2}\sin(-45^\circ), \\ y_c &= -1 + 2\sqrt{2}\cos(-45^\circ). \end{aligned}$$

That is,

$$x_c = 2.5 \quad \text{and} \quad y_c = 1.$$

11.4.4 EXERCISES

In the following questions, state your results in decimals correct to three places of decimals:

1. Calculate the radius of curvature at the point $(-1, 3)$ on the curve whose equation is
$$y = x + 3x^2 - x^3$$
and hence obtain the co-ordinates of the centre of curvature.

2. Calculate the radius of curvature at the origin on the curve whose equation is
$$y = \frac{x - x^2}{1 + x^2}$$
and hence obtain the co-ordinates of the centre of curvature.

3. Calculate the radius of curvature at the point $(1, 1)$ on the curve whose equation is
$$x^3 - 2xy + y^3 = 0$$
and hence obtain the co-ordinates of the centre of curvature.

4. Calculate the radius of curvature at the point for which $\theta = 30^\circ$ on the curve whose parametric equations are
$$x = 1 + \sin\theta \quad \text{and} \quad y = \sin\theta - \frac{1}{2}\cos 2\theta$$
and hence obtain the co-ordinates of the centre of curvature.

11.4.5 ANSWERS TO EXERCISES

1. $\rho = 43.6705, \quad (x_c, y_c) = (42.333, 8.417)$.
2. $\rho = -1.414 \quad (x_c, y_c) = (1, -1)$.
3. $\rho = -0.177 \quad (x_c, y_c) = (0.875, 0.875)$.
4. $\rho = 0.590 \quad (x_c, y_c = (-3.500, 2.750)$.

"JUST THE MATHS"

UNIT NUMBER

11.5

DIFFERENTIATION APPLICATIONS 5
(Maclaurin's and Taylor's series)

by

A.J.Hobson

UNIT 11.5 - DIFFERENTIATION APPLICATIONS 5

MACLAURIN'S AND TAYLOR'S SERIES

11.5.1 MACLAURIN'S SERIES

One of the simplest kinds of function to deal with, in either algebra or calculus, is a polynomial (see Unit 1.8). Polynomials are easy to substitute numerical values into and they are easy to differentiate.

One useful application of the present section is to approximate, to a polynomial, functions which are not already in polynomial form.

THE GENERAL THEORY

Suppose $f(x)$ is a given function of x which is not in the form of a polynomial, and let us assume that it may be expressed in the form of an infinite series of ascending powers of x; that is, a **"power series"**, (see Unit 2.4).

More specifically, we assume that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ \ldots$$

This assumption cannot be justified unless there is a way of determining the **"coefficients"**, a_0, a_1, a_2, a_3, a_4, etc.; but this is possible as an application of differentiation as we now show:

(a) Firstly, if we substitute $x = 0$ into the assumed formula for $f(x)$, we obtain $f(0) = a_0$; in other words,

$$a_0 = f(0).$$

(b) Secondly, if we differentiate the assumed formula for $f(x)$ once with respect to x, we obtain

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$$

which, on substituting $x = 0$, gives $f'(0) = a_1$; in other words,

$$a_1 = f'(0).$$

(c) Differentiating a second time leads to the result that

$$f''(x) = 2a_2 + (3 \times 2)a_3 x + (4 \times 3)a_4 x^2 + \ldots$$

which, on substituting $x = 0$ gives $f''(0) = 2a_2$; in other words,

$$a_2 = \frac{1}{2} f''(0).$$

(d) Differentiating yet again leads to the result that

$$f'''(x) = (3 \times 2)a_3 + (4 \times 3 \times 2)a_4 x + \ldots$$

which, on substituting $x = 0$ gives $f'''(0) = (3 \times 2)a_3$; in other words,

$$a_3 = \frac{1}{3!} f'''(0).$$

(e) Continuing this process with further differentiation will lead to the general formula

$$a_n = \frac{1}{n!} f^{(n)}(0),$$

where $f^{(n)}(0)$ means the value, at $x = 0$ of the n-th derivative of $f(x)$.

Summary

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

This is called the **"Maclaurin's series for** $f(x)$**"**.

Notes:

(i) We must assume, ofcourse, that all of the derivatives of $f(x)$ exist at $x = 0$ in the first place; otherwise the above result is invalid.
It is also necessary to examine, for convergence or divergence, the Maclaurin's series obtained

for a particular function. The result may not be used when the series diverges; (see Units 2.3 and 2.4).

(b) If x is small and it is possible to neglect powers of x after the n-th power, then Maclaurin's series approximates $f(x)$ to a polynomial of degree n.

11.5.2 STANDARD SERIES

Here, we determine the Maclaurin's series for some of the functions which occur frequently in the applications of mathematics to science and engineering. The ranges of values of x for which the results are valid will be stated without proof.

1. **The Exponential Series**

   (i) $f(x) \equiv e^x$; hence, $f(0) = e^0 = 1$.

   (ii) $f'(x) = e^x$; hence, $f'(0) = e^0 = 1$.

   (iii) $f''(x) = e^x$; hence, $f''(0) = e^0 = 1$.

   (iv) $f'''(x) = e^x$; hence, $f'''(0) = e^0 = 1$.

   (v) $f^{(iv)}(x) = e^x$; hence, $f^{(iv)}(0) = e^0 = 1$.

   Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots$$

   and it may be shown that this series is valid for all values of x.

2. **The Sine Series**

   (i) $f(x) \equiv \sin x$; hence, $f(0) = \sin 0 = 0$.

   (ii) $f'(x) = \cos x$; hence, $f'(0) = \cos 0 = 1$.

   (iii) $f''(x) = -\sin x$; hence, $f''(0) = -\sin 0 = 0$.

   (iv) $f'''(x) = -\cos x$; hence, $f'''(0) = -\cos 0 = -1$.

   (v) $f^{(iv)}(x) = \sin x$; hence, $f^{(iv)}(0) = \sin 0 = 0$.

   (vi) $f^{(v)}(x) = \cos x$; hence, $f^{(v)}(0) = \cos 0 = 1$.

   Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots$$

   and it may be shown that this series is valid for all values of x.

3. The Cosine Series

(i) $f(x) \equiv \cos x$; hence, $f(0) = \cos 0 = 1$.

(ii) $f'(x) = -\sin x$; hence, $f'(0) = -\sin 0 = 0$.

(iii) $f''(x) = -\cos x$; hence, $f''(0) = -\cos 0 = -1$.

(iv) $f'''(x) = \sin x$; hence, $f'''(0) = \sin 0 = 0$.

(v) $f^{(iv)}(x) = \cos x$; hence, $f^{(iv)}(0) = \cos 0 = 1$.

Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ \ldots$$

and it may be shown that this series is valid for all values of x.

4. The Logarithmic Series

It is not possible to find a Maclaurin's series for the function $\ln x$, since neither the function nor its derivatives exist at $x = 0$.

As an alternative, we may consider the function $\ln(1 + x)$ instead.

(i) $f(x) \equiv \ln(1 + x)$; hence, $f(0) = \ln 1 = 0$.

(ii) $f'(x) = \frac{1}{1+x}$; hence, $f'(0) = 1$.

(iii) $f''(x) = -\frac{1}{(1+x)^2}$; hence, $f''(0) = 1$.

(iv) $f'''(x) = \frac{2}{(1+x)^3}$; hence, $f'''(0) = 2$.

(v) $f^{(iv)}(x) = -\frac{2\times 3}{(1+x)^4}$; hence, $f^{(iv)}(0) = -(2 \times 3)$.

Thus,

$$\ln(1 + x) = x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - (2 \times 3)\frac{x^4}{4!} + \ \ldots$$

which simplifies to

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots$$

and it may be shown that this series is valid for $-1 < x \leq 1$.

5. The Binomial Series

The statement of the Binomial Formula has already appeared in Unit 2.2; and it was seen there that

(a) When n is a positive integer, the expansion of $(1 + x)^n$ in ascending powers of x is a **finite** series;

(b) When n is a negative integer or a fraction, the expansion of $(1+x)^n$ in ascending powers of x is an **infinite** series.

Here, we examine the proof of the Binomial Formula.

(i) $f(x) \equiv (1+x)^n$**;** hence, $f(0) = 1$**.**

(ii) $f'(x) = n(1+x)^{n-1}$**;** hence, $f'(0) = n$**.**

(iii) $f''(x) = n(n-1)(1+x)^{n-2}$**;** hence, $f''(0) = n(n-1)$**.**

(iv) $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$**;** hence, $f'''(0) = n(n-1)(n-2)$**.**

(v) $f^{(iv)}(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$**;** hence, $f^{(iv)}(0) = n(n-1)(n-2)(n-3)$**.**

Thus,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ \ldots$$

If n is a positive integer, all of the derivatives of $(1+x)^n$ after the n-th derivative are identically equal to zero; so the series is a finite series ending with the term in x^n.

In all other cases, the series is an infinite series and it may be shown that it is valid whenever $-1 < x \leq 1$.

EXAMPLES

1. Use the Maclaurin's series for $\sin x$ to evaluate

$$\lim_{x \to 0} \frac{x + \sin x}{x(x+1)}.$$

Solution

Substituting the series for $\sin x$ gives

$$\lim_{x \to 0} \frac{x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots}{x^2 + x}$$

$$= \lim_{x \to 0} \frac{2x - \frac{x^3}{6} + \frac{x^5}{120} - \ \ldots}{x^2 + x}$$

$$= \lim_{x \to 0} \frac{2 - \frac{x^2}{6} + \frac{x^4}{120} - \ \ldots}{x + 1} = 2.$$

2. Use a Maclaurin's series to evaluate $\sqrt{1.01}$ correct to six places of decimals.

   Solution

   We shall consider the expansion of the function $(1+x)^{\frac{1}{2}}$ and then substitute $x = 0.01$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ \ldots$$

   That is,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ \ldots$$

   Substituting $x = 0.01$ gives

$$\sqrt{1.01} = 1 + \frac{1}{2} \times 0.01 - \frac{1}{8} \times 0.0001 + \frac{1}{16} \times 0.000001 - \ \ldots$$

$$= 1 + 0.005 - 0.0000125 + 0.0000000625 - \ \ldots$$

   The fourth term will not affect the sixth decimal place in the result given by the first three terms; and this is equal to 1.004988 correct to six places of decimals.

3. Assuming the Maclaurin's series for e^x and $\sin x$ and assuming that they may be multiplied together term-by-term, obtain the expansion of $e^x \sin x$ in ascending powers of x as far as the term in x^5.

   Solution

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{120} + \ldots\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24} + \ \ldots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \ \ldots$$

11.5.3 TAYLOR'S SERIES

A useful consequence of Maclaurin's series is known as Taylor's series and one form of it may be stated as follows:

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ \ldots$$

Proof:

To obtain this result from Maclaurin's series, we simply let $f(x+h) \equiv F(x)$. Then,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \frac{x^3}{3!}F'''(0) + \ ..$$

But, $F(0) = f(h)$, $F'(0) = f'(h)$, $F''(0) = f''(h)$, $F'''(0) = f'''(h)$,. . . which proves the result.

Note: An alternative form of Taylor's series, often used for approximations, may be obtained by interchanging the symbols x and h to give

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ \ldots$$

EXAMPLE

Given that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, use Taylor's series to evaluate $\sin(x+h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{4}$ and $h = 0.01$.

Solution

Using the sequence of derivatives as in the Maclaurin's series for $\sin x$, we have

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + ...$$

Subsituting $x = \frac{\pi}{4}$ and $h = 0.01$, we obtain

$$\sin\left(\frac{\pi}{4} + 0.01\right) = \frac{1}{\sqrt{2}}\left(1 + 0.01 - \frac{(0.01)^2}{2!} - \frac{(0.01)^3}{3!} + \ \ldots\right)$$

$$= \frac{1}{\sqrt{2}}(1 + 0.01 - 0.00005 - 0.000000017 + ...)$$

The fourth term does not affect the fifth decimal place in the sum of the first three terms; and so

$$\sin\left(\frac{\pi}{4} + 0.01\right) \simeq \frac{1}{\sqrt{2}} \times 1.00995 \simeq 0.71414$$

11.5.4 EXERCISES

1. Determine the first three non-vanishing terms of the Maclaurin's series for the function $\sec x$.
2. Determine the Maclaurin's series for the function $\tan x$ as far as the term in x^5.
3. Determine the Maclaurin's series for the function $\ln(1 + e^x)$ as far as the term in x^4.
4. Use the Maclaurin's series for the function e^x to deduce the expansion, in ascending powers of x of the function e^{-x} and then use these two series to obtain the expansion, in ascending powers of x, of the functions

   (a)
   $$\frac{e^x + e^{-x}}{2}(\equiv \cosh x);$$

   (b)
   $$\frac{e^x - e^{-x}}{2}(\equiv \sinh x).$$

5. Use the Maclaurin's series for the function $\cos x$ and the Binomial Series for the function $\frac{1}{1+x}$ to obtain the expansion of the function

   $$\frac{\cos x}{1 + x}$$

   in ascending powers of x as far as the term in x^4.
6. From the Maclaurin's series for the function $\cos x$, deduce the expansions of the functions $\cos 2x$ and $\sin^2 x$ as far as the term in x^4.

7. Use appropriate Maclaurin's series to evaluate the following limits:

   (a)
$$\lim_{x\to 0}\left[\frac{e^x+e^{-x}-2}{2\cos 2x-2}\right];$$

   (b)
$$\lim_{x\to 0}\left[\frac{\sin^2 x-x^2\cos x}{x^4}\right].$$

8. Use a Maclaurin's series to evaluate $\sqrt[3]{1.05}$ correct to four places of decimals.

9. Expand $\cos(x+h)$ as a series of ascending powers of h.

   Given that $\sin\frac{\pi}{6}=\frac{1}{2}$ and $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, evaluate $\cos(x+h)$, correct to five places of decimals, in the case when $x=\frac{\pi}{6}$ and $h=-0.05$.

11.5.5 ANSWERS TO EXERCISES

1.
$$1+\frac{x^2}{2}+\frac{x^4}{8}+\ ..$$

2.
$$x+\frac{x^3}{3}+\frac{2x^5}{15}+\ \ldots$$

3.
$$\ln 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ \ldots$$

4. (a)
$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ \ldots;$$

   (b)
$$\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ ..$$

5.

$$1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{13x^4}{24} - \ \dots$$

6.

$$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} - \ \dots$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \ \dots$$

7. (a) $-\frac{1}{4}$, (b) $\frac{1}{6}$

8. 1.0164

9. 0.74156

"JUST THE MATHS"

UNIT NUMBER

11.6

DIFFERENTIATION APPLICATIONS 6
(Small increments and small errors)

by

A.J.Hobson

UNIT 11.6 - DIFFERENTIATION APPLICATIONS 6

SMALL INCREMENTS AND SMALL ERRORS

11.6.1 SMALL INCREMENTS

Given that a dependent variable, y, and an independent variable, x are related by means of the formula

$$y = f(x),$$

suppose that x is subject to a small **"increment"**, δx,

In the present context we use the term "increment" to mean that δx is positive when x is **increased**, but negative when x is **decreased**.

The exact value of the corresponding increment, δy, in y is given by

$$\delta y = f(x + \delta x) - f(x),$$

but this can often be a cumbersome expression to evaluate.

However, since δx is small, we may recall, from the definition of a derivative (Unit 10.2), that

$$\frac{f(x + \delta x) - f(x)}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x}.$$

That is,

$$\frac{\delta y}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x};$$

and we may conclude that

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

For a diagramatic approach to this approximation for the increment in y, let us consider the graph of y against x in the neighbourhood of the two points $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ on the curve whose equation is $y = f(x)$.

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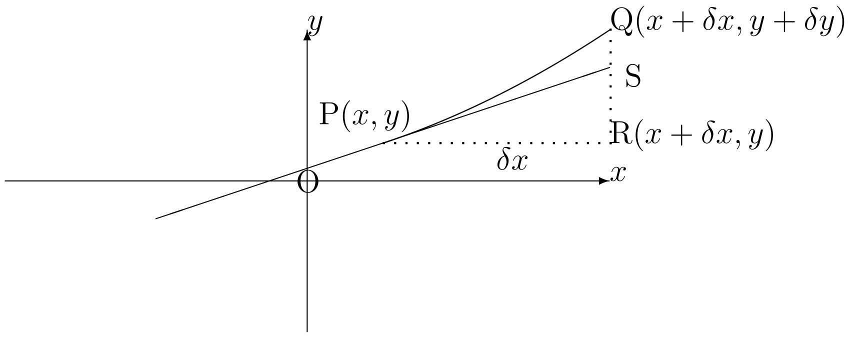


In the diagram, PR $= \delta x$, QR $= \delta y$ and the gradient of the line PS is given by the value of $\frac{dy}{dx}$ at P.

Taking SR as an approximation to QR, we obtain

$$\frac{\mathrm{SR}}{\mathrm{PR}} = \left[\frac{dy}{dx}\right]_{\mathrm{P}}.$$

In other words,

$$\frac{\mathrm{SR}}{\delta x} = \left[\frac{dy}{dx}\right]_{\mathrm{P}}.$$

Hence,

$$\delta y \simeq \left[\frac{dy}{dx}\right]_{\mathrm{P}} \delta x,$$

which is the same result as before.

Notes:

(i) The quantity $\frac{dy}{dx}\delta x$ is known as the **"total differential of y"** (or simply the "differential of y"). It provides an approximation **(including the appropriate sign)** for the increment, δy, in y subject to an increment of δx in x.

(ii) It is important **not** to use the word "differential" when referring to a "derivative". Rather, the correct alternative to "derivative" is "differential coefficient".

(iii) A more rigorous approach to the calculation of δy is to use the result known as "Taylor's Theorem" (see Unit 11.5) which, in this context, would give the formula

$$f(x+\delta x) = f(x) + f'(x)\delta x + \frac{f''(x)}{2!}(\delta x)^2 + \frac{f'''(x)}{3!}(\delta x)^3 + \ldots.$$

Hence, if δx is small enough for powers of two and above to be neglected, then

$$f(x+\delta x) - f(x) \simeq f'(x)\delta x$$

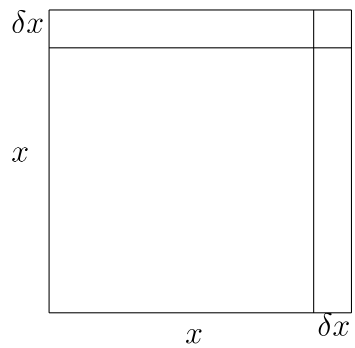
to the first order of approximation.

EXAMPLES

1. If a square has side $x$cms., determine both the exact and the approximate values of the increment in the area Acms^2. when x is increased by δx.

   Solution

   (a) Exact Method



The area is given by the formula

$$A = x^2.$$

If x increases by δx, then the increase, δA, in A may be obtained from the formula

$$A + \delta A = (x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2.$$

That is,

$$\delta A = 2x\delta x + (\delta x)^2.$$

(b) Approximate Method

Here, we use

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x$$

to give

$$\delta A \simeq 2x\delta x;$$

and we observe from the diagram that the two results differ only by the area of the small square, with side δx.

2. If

$$y = xe^{-x},$$

calculate, approximately, the change in y when x increases from 5 to 5.03.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}(1 - x),$$

so that

$$\delta y \simeq e^{-x}(1 - x)\delta x,$$

where $x = 5$ and $\delta x = 0.3$.

Hence,

$$\delta y \simeq e^{-5}.(1 - 5).(0.3) \simeq -0.00809,$$

showing a **decrease** of 0.00809 in y.

We may compare this with the exact value which is given by

$$\delta y = 5.3e^{-5.3} - 5e^{-5} \simeq -0.00723$$

3. If

$$y = xe^{-x},$$

determine, in terms of x, the percentage change in y when x is increased by 2%.

Solution

Once again, we have

$$\delta y = e^{-x}(1-x)\delta x;$$

but, this time, $\delta x = 0.02x$, so that

$$\delta y = e^{-x}(1-x) \times 0.02x.$$

The **percentage** change in y is given by

$$\frac{\delta y}{y} \times 100 = \frac{e^{-x}(1-x) \times 0.02x}{xe^{-x}} \times 100 = 2(1-x).$$

That is, y increases by $2(1-x)$%, which will be positive when $x < 1$ and negative when $x > 1$.

Note:

It is usually more meaningful to discuss increments in the form of a percentage, since this gives a better idea of how much a variable has changed in porportion to its original value.

11.6.2 SMALL ERRORS

In the functional relationship

$$y = f(x),$$

let us suppose that x is known to be subject to an error in measurement; then we consider what error will be likely in the calculated value of y.

In particular, suppose x is known to be **too large** by a small amount, δx, in which case the correct value of x could be obtained if we **decreased** it by δx; or, what amounts to the same thing, if we **increased** it by $-\delta x$.

Correspondingly, the value of y will **increase** by approximately $-\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; that is, y will **decrease** by approximately $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$.

Summary

We conclude that, if x is too large by an amount δx, then y is too large by approximately $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; though, ofcourse, if $\frac{\mathrm{d}y}{\mathrm{d}x}$ itself is negative, y will be too small when x is too large and vice versa.

EXAMPLES

1. If
$$y = x^2 \sin x,$$
calculate, approximately, the error in y when x is measured as 3, but this measurement is subsequently discovered to be too large by 0.06.
Solution
We have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \cos x + 2x \sin x$$
and, hence,
$$\delta y \simeq (x^2 \cos x + 2x \sin x)\delta x,$$
where $x = 3$ and $\delta x = 0.06$.
The error in y is therefore given approximately by
$$\delta y \simeq (3^2 \cos 3 + 6 \sin 3) \times 0.06 \simeq -0.4838$$
That is, y is too small by approximately 0.4838.

2. If
$$y = \frac{x}{1+x},$$
determine approximately, in terms of x, the percentage error in y when x is subject to an error of 5%.
Solution
We have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2},$$
so that

$$\delta y \simeq \frac{1}{(1+x)^2}\delta x,$$

where $\delta x = 0.05x$.

The **percentage** error in y is thus given by

$$\frac{\delta y}{y} \times 100 \simeq \frac{1}{(1+x)^2} \times 0.05x \times \frac{x+1}{x} \times 100 = \frac{5}{1+x}.$$

Hence, y is too large by approximately $\frac{5}{1+x}$% which will be positive when $x > -1$ and negative when $x < -1$.

11.6.3 EXERCISES

1. If
$$y = \frac{e^{2x}}{x},$$
calculate, approximately, the change in y when x is increased from 1 to 1.0025.
State your answer correct to three significant figures.

2. If
$$y = (2x+1)^5,$$
determine approximately, in terms of x, the percentage change in y when x increases by 0.1%.

3. If
$$y = x^3 \ln x,$$
calculate approximately, correct to the nearest integer, the error in y when x is measured as 4, but this measurement is subsequently discovered to be too small by 0.12.

4. If
$$y = \cos(3x^2+2),$$
determine approximately, in terms of x, the percentage error in y if x is too large by 2%.
You may assume that $3x^2+2$ lies between π and $\frac{3\pi}{2}$.

11.6.4 ANSWERS TO EXERCISES

1. y increases by approximately 0.0185.
2. y increases by approximately $\frac{x}{(2x+1)}\%$
3. y is too small by approximately 10.
4. y is too small by approximately $-12x^2 \tan(3x^2 + 2)$.

"JUST THE MATHS"

UNIT NUMBER

12.1

INTEGRATION 1
(Elementary indefinite integrals)

by

A.J.Hobson

UNIT 12.1 - INTEGRATION 1 - ELEMENTARY INDEFINITE INTEGRALS

12.1.1 THE DEFINITION OF AN INTEGRAL

In Differential Calculus, we are given functions of x and asked to obtain their derivatives; but, in Integral Calculus, we are given functions of x and asked what they are the derivatives of. The process of answering this question is called **"integration"**.

In other words **integration is the reverse of differentiation**.

DEFINITION
Given a function $f(x)$, another function, z, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x)$$

is called an integral of $f(x)$ with respect to x.

Notes:
(i) The above definition refers to **an** integral of $f(x)$ rather than **the** integral of $f(x)$. This is because, having found a possible function, z, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x),$$

$z + C$ is also an integral for any constant value, C.

(ii) We call $z + C$ the **"indefinite integral of $f(x)$ with respect to x"** and we write

$$\int f(x)\mathrm{d}x = z + C.$$

(iii) C is an **arbitrary constant** called the **"constant of integration"**.

(iv) The symbol $\mathrm{d}x$ does not denote a number; it is to be taken as a label indicating the variable with respect to which we are integrating. It may seem obvious that this will be x, but it could happen, for instance, that x is already dependent upon some other variable, t, in which case it would be vital to indicate the variable with respect to which we are integrating.

(v) In any integration problem, the function being integrated is called the **"integrand"**.

Result:
Two functions z_1 and z_2 are both integrals of the same function $f(x)$ if and only if they differ by a constant.

Proof:
(a) Suppose, firstly, that

$$z_1 - z_2 = C,$$

where C is a constant.

Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}[z_1 - z_2] = 0.$$

That is,

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} - \frac{\mathrm{d}z_2}{\mathrm{d}x} = 0,$$

or

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} = \frac{\mathrm{d}z_2}{\mathrm{d}x}.$$

From our definition, this shows that both z_1 and z_2 are integrals of the same function.

(b) Secondly, suppose that z_1 and z_2 are integrals of the same function. Then

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} = \frac{\mathrm{d}z_2}{\mathrm{d}x}.$$

That is,

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} - \frac{\mathrm{d}z_2}{\mathrm{d}x} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}[z_1 - z_2] = 0.$$

Hence,

$$z_1 - z_2 = C,$$

where C may be any constant.

ILLUSTRATIONS
Any result so far encountered in differentiation could be re-stated in reverse as a result on integration as shown in the following illustrations:

1.

$$\int 3x^2\mathrm{d}x = x^3 + C.$$

2.

$$\int x^2\mathrm{d}x = \frac{x^3}{3} + C.$$

3.

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad \text{Provided} \quad n \neq -1.$$

4.

$$\int \frac{1}{x} \mathrm{d}x \quad \text{i.e.} \quad \int x^{-1} \mathrm{d}x = \ln x + C.$$

5.

$$\int e^x \mathrm{d}x = e^x + C.$$

6.

$$\int \cos x \mathrm{d}x = \sin x + C.$$

7.

$$\int \sin x \mathrm{d}x = -\cos x + C.$$

Note:
Basic integrals of the above kinds may simply be quoted from a table of standard integrals in a suitable formula booklet. More advanced integrals are obtainable using the rules to be discussed below.

12.1.2 ELEMENTARY TECHNIQUES OF INTEGRATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants. Then,

$$\int [Af(x) + Bg(x)] \mathrm{d}x = A \int f(x) \mathrm{d}x + B \int g(x) \mathrm{d}x.$$

The proof follows from the fact that differentiation is already linear and hence the derivative of the right hand side is the integrand of the left hand side. The result itself is easily extended to linear combinations of three or more functions.

ILLUSTRATIONS

1.

$$\int (x^2 + 3x - 7) \mathrm{d}x = \frac{x^3}{3} + 3\frac{x^2}{2} - 7x + C.$$

2.

$$\int (3\cos x + 4\sec^2 x)\mathrm{d}x = 3\sin x + 4\tan x + C.$$

(b) Functions of a Linear Function

(i) Inspection Method
Provided the method of **differentiating** functions of a linear function has been fully understood, the fastest method of **integrating** such functions is to examine, by inspection, what needs to be differentiated in order to arrive at them.

EXAMPLES

1. Determine the indefinite integral

$$\int (2x+3)^{12}\mathrm{d}x.$$

Solution
In order to arrive at the function $(2x+3)^{12}$ by a differentiation process, we would have to begin with a function related to $(2x+3)^{13}$.
In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+3)^{13}\right] = 13(2x+3)^{12}.2 = 26(2x+3)^{12}.$$

Since this is 26 times the function we are trying to integrate, we may say that

$$\int (2x+3)^{12} = \frac{(2x+3)^{13}}{26} + C.$$

2. Determine the indefinite integral

$$\int \cos(3-5x)\mathrm{d}x.$$

Solution
In order to arrive at the function $\cos(3-5x)$ by a differentiation process, we would have to begin with a function related to $\sin(3-5x)$.
In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(3-5x)] = \cos(3-5x).-5 = -5\cos(3-5x).$$

Since this is -5 times the function we are trying to integrate, we may say that

$$\int \cos(3-5x) = -\frac{\sin(3-5x)}{5} + C.$$

3. Determine the indefinite integral
$$\int e^{4x+1} \mathrm{d}x.$$

Solution

In order to arrive at the function e^{4x+1} by a differentiation process, we would have to begin with a function related to e^{4x+1} itself because the derivative of a power of e always contains the **same** power of e.

In fact
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{4x+1}\right] = e^{4x+1}.4.$$

Since this is 4 times the function we are trying to integrate, we may say that
$$\int e^{4x+1} \mathrm{d}x = \frac{e^{4x+1}}{4} + C.$$

4. Determine the indefinite integral
$$\int \frac{1}{7x+3} \mathrm{d}x.$$

Solution

In order to arrive at the function $\frac{1}{7x+3}$ by a differentiation process, we would have to begin with a function related to $\ln(7x+3)$.

In fact,
$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(7x+3)] = \frac{1}{7x+3}.7 = \frac{7}{7x+3}$$

Since this is 7 times the function we are trying to integrate, we may say that
$$\int \frac{1}{7x+3} \mathrm{d}x = \frac{\ln(7x+3)}{7} + C.$$

Note:

In each of these examples, we are essentially treating the linear function as if it were a single x, then dividing the result by the coefficient of x in that linear function.

(ii) Substitution Method

The method to be discussed here will eventually be applied to functions other than functions of a linear function; but the latter serve as a useful way of introducing the technique of **"Integration by Substitution"**.

In the integral of the form $\int f(ax+b)\mathrm{d}x$, we may substitute $u = ax+b$ proceeding as follows:

Suppose

$$z = \int f(ax+b)\mathrm{d}x.$$

Then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(ax+b).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(u).$$

But,

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{\mathrm{d}z}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}u} = f(u).\frac{\mathrm{d}x}{\mathrm{d}u}.$$

Hence,

$$z = \int f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u.$$

Note:
The secret of this integration by substitution formula is that, apart from putting $u = ax + b$ into $f(ax + b)$, we replace the symbol $\mathrm{d}x$ with $\frac{\mathrm{d}x}{\mathrm{d}u}.\mathrm{d}u$; almost as if we had divided $\mathrm{d}x$ by $\mathrm{d}u$ then immediately multiplied by it again, though, strictly, this would not be allowed since $\mathrm{d}x$ and $\mathrm{d}u$ are not numbers.

EXAMPLES

1. Determine the indefinite integral

$$z = \int (2x+3)^{12}\mathrm{d}x.$$

Solution
Putting $u = 2x + 3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}$.
Thus,

$$z = \int u^{12}.\frac{1}{2}\mathrm{d}u = \frac{u^{13}}{13} \times \frac{1}{2} + C.$$

That is,

$$z = \frac{(2x+3)^{13}}{26} + C,$$

as before.

2. Determine the indefinite integral

$$z = \int \cos(3 - 5x)\mathrm{d}x.$$

Solution
Putting $u = 3 - 5x$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = -5$ and hence $\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{5}$.
Thus,

$$z = \int \cos u. - \frac{1}{5}\mathrm{d}u = -\frac{1}{5}\sin u + C.$$

That is,

$$z = -\frac{1}{5}\sin(3 - 5x) + C,$$

as before.

3. Determine the indefinite integral

$$z = \int e^{4x+1}\mathrm{d}x.$$

Solution
Putting $u = 4x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 4$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{4}$.
Thus,

$$z = \int e^u.\frac{1}{4}\mathrm{d}u = \frac{e^u}{4} + C.$$

That is,

$$z = \frac{e^{4x+1}}{4} + C,$$

as before.

4. Determine the indefinite integral

$$z = \int \frac{1}{7x + 3}\mathrm{d}x.$$

Solution
Putting $u = 7x + 3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 7$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{7}$
Thus,

$$z = \int \frac{1}{u}.\frac{1}{7}\mathrm{d}u = \frac{1}{7}\ln u + C.$$

That is,

$$z = \frac{1}{7}\ln(7x + 3) + C,$$

as before.

12.1.3 EXERCISES

1. Integrate the following functions with respect to x:

   (a)
   $$x^5;$$

   (b)
   $$x^{\frac{3}{2}};$$

   (c)
   $$\frac{1}{x^6};$$

   (d)
   $$2x^2 - x + 5;$$

   (e)
   $$x^3 - 7x^2 + x + 1.$$

2. Use a substitution of the form $u = ax + b$ in order to determine the following integrals:

   (a)
   $$\int \sin(5x - 6)\mathrm{d}x;$$

   (b)
   $$\int e^{2x+11}\mathrm{d}x;$$

   (c)
   $$\int (3x + 2)^6\mathrm{d}x.$$

3. Write down, by inspection, the indefinite integrals with respect to x of the following functions:

   (a)
   $$(1 + 2x)^{10};$$

(b)

$$e^{12x+4};$$

(c)

$$\frac{1}{3x-1};$$

(d)

$$\sin(3-5x);$$

(e)

$$\frac{9}{(4-x)^5};$$

(f)

$$\text{cosec}^2(7x+1).$$

12.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{x^6}{6}+C;$$

(b)

$$\frac{2}{5}x^{\frac{5}{2}}+C;$$

(c)

$$-\frac{1}{5x^5}+C;$$

(d)

$$\frac{2}{3}x^3-\frac{1}{2}x^2+5x+C;$$

(e)

$$\frac{1}{4}x^4-\frac{7}{3}x^3+\frac{1}{2}x^2+x+C.$$

2. (a)

$$-\frac{1}{5}\cos(5x-6)+C;$$

(b)

$$\frac{1}{2}e^{2x+11}+C;$$

(c)

$$\frac{1}{21}(3x+2)^7+C.$$

3. (a)

$$\frac{1}{22}(1+2x)^{11}+C;$$

(b)

$$\frac{1}{12}e^{12x+4}+C;$$

(c)

$$\frac{1}{3}\ln(3x-1)+C;$$

(d)

$$\frac{1}{5}\cos(3-5x)+C;$$

(e)

$$\frac{9}{4(4-x)^4}+C;$$

(f)

$$-\frac{1}{7}\cot(7x+1)+C.$$

APPENDIX - A Table of Standard Integrals

$f(x)$	$\int\ f(x)\ \mathrm{d}x$
a (const.)	ax
x^n	$x^{n+1}/(n+1)\ n \neq -1$
$1/x$	$\ln x$
$\sin ax$	$-(1/a)\cos ax$
$\cos ax$	$(1/a)\sin ax$
$\sec^2 ax$	$(1/a)\tan ax$
$\mathrm{cosec}^2\ ax$	$-(1/a)\cot ax$
$\sec\ ax.\tan ax$	$(1/a)\mathrm{sec}ax$
$\mathrm{cosec}\ ax.\cot ax$	$-(1/a)\mathrm{cosec}ax$
e^{ax}	$(1/a)e^{ax}$
a^x	$a^x/\ln a$
$\sinh ax$	$(1/a)\cosh ax$
$\cosh ax$	$(1/a)\sinh ax$
$\mathrm{sech}^2\ ax$	$(1/a)\tanh ax$
$\mathrm{sech}\ ax.\tanh ax$	$-(1/a)\mathrm{sech}ax$
$\mathrm{cosech}\ ax.\coth ax$	$-(1/a)\mathrm{cosech}ax$
$\cot ax$	$(1/a)\ln(\sin ax)$
$\tan ax$	$-(1/a)\ln(\cos ax)$
$\tanh ax$	$(1/a)\ln(\cosh ax)$
$\coth ax$	$(1/a)\ln(\sinh ax)$
$1/\sqrt{}(a^2-x^2)$	$\sin^{-1}(x/a)$
$1/(a^2+x^2)$	$(1/a)\tan^{-1}(x/a)$
$1/\sqrt{}(x^2+a^2)$	$\sinh^{-1}(x/a)$ or $\ln(x+\sqrt{x^2+a^2})$
$1/\sqrt{}(x^2-a^2)$	$\cosh^{-1}(x/a)$ or $\ln(x+\sqrt{x^2-a^2})$
$1/(a^2-x^2)$	$(1/a)\tanh^{-1}(x/a)$ or $\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ when $\lvert x\rvert < a$, $\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right)$ when $\lvert x\rvert > a$.

"JUST THE MATHS"

UNIT NUMBER

12.2

INTEGRATION 2
(Introduction to definite integrals)

by

A.J.Hobson

UNIT 12.2 - INTEGRATION 2
INTRODUCTION TO DEFINITE INTEGRALS

12.2.1 DEFINITION AND EXAMPLES

So far, all the integrals considered have been **"indefinite integrals"** since each result has contained an arbitrary constant which cannot be assigned a value without further information.

In practical applications of integration, however, a different kind of integral, called a **"definite integral"**, is encountered and is represented by a numerical value rather than a function plus an arbitrary constant.

Suppose that

$$\int f(x)\mathrm{d}x = g(x) + C.$$

Then the symbol

$$\int_a^b f(x)\mathrm{d}x$$

is used to mean

(Value of $g(x) + C$ at $x = b$) minus (Value of $g(x) + C$ at $x = a$).

In other words, since C will cancel out,

$$\int_a^b f(x)\mathrm{d}x = g(b) - g(a).$$

The right hand side of this statement can also be written

$$[g(x)]_a^b\,,$$

a notation which is used as the middle stage of a definite integral calculation.

The values a and b are known as the **"lower limit"** and **"upper limit"**, respectively, of the definite integral (even when a is larger than b).

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos x\mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x = [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

2. Evaluate the definite integral

$$\int_1^3 (2x+1)^2 \mathrm{d}x.$$

Solution

Using the method of integrating a function of a linear function, we obtain

$$\int_1^3 (2x+1)^2 \mathrm{d}x = \left[\frac{(2x+1)^3}{6}\right]_1^3 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

Notes:

(i) If we had decided to multiply out the integrand $(2x+1)^2$ before integrating, giving

$$4x^2 + 4x + 1,$$

the integration process would have yielded the expression

$$4\frac{x^3}{3} + 2x^2 + x,$$

which differs only from the previous result by the constant value $\frac{1}{6}$; students may like to check this. Hence the numerical result for the definite integral will be the same.

(ii) An alternative method of evaluating the definite integral would be to make the substitution

$$u = 2x + 1.$$

But, whenever substitution is used for definite integrals, it is not necessary to return to the original variable at the end as long as the limits of integration are changed to the appropriate values for u.

Replacing $\mathrm{d}x$ by $\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ (that is, $\frac{1}{2}\mathrm{d}u$) and the limits $x = 1$ and $x = 3$ by $u = 2 \times 1 + 1 = 3$ and $u = 2 \times 3 + 1 = 7$, respectively, we obtain

$$\int_3^7 u^2 \frac{1}{2} \mathrm{d}u = \left[\frac{u^3}{6}\right]_3^7 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

12.2.2 EXERCISES

Evaluate the following definite integrals:

1.
$$\int_0^{\frac{\pi}{3}} \sin 4x \mathrm{d}x.$$

2.
$$\int_{-1}^{1} (x+1)^7 \mathrm{d}x.$$

3.
$$\int_0^{\frac{\pi}{4}} \sec^2(x+\pi)\ \mathrm{d}x.$$

4.
$$\int_1^2 \frac{1}{4+3x}\ \mathrm{d}x.$$

5.
$$\int_0^1 e^{1-7x}\ \mathrm{d}x.$$

12.2.3 ANSWERS TO EXERCISES

1.
$$0.375$$

2.
$$32.$$

3.
$$1.$$

4.
$$\frac{1}{3}\ln\frac{10}{7} \quad \text{or} \quad 0.119 \quad \text{approximately}$$

5.
$$0.388 \text{ approximately}$$

"JUST THE MATHS"

UNIT NUMBER

12.3

INTEGRATION 3
(The method of completing the square)

by

A.J.Hobson

UNIT 12.3 - INTEGRATION 3
THE METHOD OF COMPLETING THE SQUARE

12.3.1 INTRODUCTION AND EXAMPLES

A substitution such as $u = \alpha x + \beta$ may also be used with integrals of the form

$$\int \frac{1}{px^2 + qx + r} \mathrm{d}x \quad \text{and} \quad \int \frac{1}{\sqrt{px^2 + qx + r}} \mathrm{d}x,$$

where, in the first of these, we assume that the quadratic will not factorise into simple linear factors; otherwise the method of partial fractions would be used to integrate it (see Unit 12.6).

Note:
The two types of integral here are often written, for convenience, as

$$\int \frac{\mathrm{d}x}{px^2 + qx + r} \quad \text{and} \quad \int \frac{\mathrm{d}x}{\sqrt{px^2 + qx + r}}.$$

In order to deal with such functions, we shall need to quote standard results which may be deduced from previous ones developed in the differentiation of inverse trigonometric and hyperbolic functions.

They are as follows:

1.
$$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

2.
$$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \sin^{-1}\frac{x}{a} + C.$$

3.
$$\int \frac{1}{\sqrt{a^2 + x^2}} \, \mathrm{d}x = \sinh^{-1}\frac{x}{a} + C \quad \text{or} \quad \ln(x + \sqrt{x^2 + a^2}) + C.$$

4.
$$\int \frac{1}{\sqrt{x^2 - a^2}} \, \mathrm{d}x = \cosh^{-1}\frac{x}{a} + C \quad \text{or} \quad \ln(x + \sqrt{x^2 - a^2}) + C.$$

5.

$$\int \frac{1}{a^2 - x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C;$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

EXAMPLES

1. Determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{\sqrt{x^2 + 2x - 3}}.$$

Solution

Completing the square in the quadratic expression gives

$$x^2 + 2x - 3 \equiv (x+1)^2 - 4 \equiv (x+1)^2 - 2^2.$$

Hence,

$$z = \int \frac{\mathrm{d}x}{\sqrt{(x+1)^2 - 2^2}}.$$

Putting $u = x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.
Thus,

$$z = \int \frac{\mathrm{d}u}{\sqrt{u^2 - 2^2}},$$

giving

$$z = \ln\left[u + \sqrt{u^2 - 2^2}\right] + C.$$

Returning to the variable, x, we have

$$z = \ln\left[x + 1 + \sqrt{x^2 + 2x - 3}\right] + C.$$

2. Evaluate the definite integral

$$z = \int_3^7 \frac{\mathrm{d}x}{x^2 - 6x + 25}.$$

Solution

Completing the square in the quadratic expression gives

$$x^2 - 6x + 25 \equiv (x-3)^2 + 16.$$

Hence,

$$z = \int_3^7 \frac{\mathrm{d}x}{(x-3)^2 + 16}.$$

Putting $u = x - 3$, we obtain $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.
Thus,

$$z = \int_0^4 \frac{\mathrm{d}u}{u^2 + 16},$$

giving

$$z = \left[\frac{1}{4}\tan^{-1}\frac{u}{4}\right]_0^4 = \frac{\pi}{16}.$$

Alternatively, without changing the original limits of integration,

$$z = \left[\frac{1}{4}\tan^{-1}\frac{x-3}{4}\right]_3^7.$$

Note:
In cases like the two examples discussed above, when $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and therefore $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$, it seems pointless to go through the laborious process of actually **making** the substitution in detail. All we need to do is to treat the linear expression within the completed square as if it were a single x, then write the result straight down !

12.3.2 EXERCISES

1. Use a table of standard integrals to write down the indefinite integrals of the following functions:

   (a)

   $$\frac{1}{\sqrt{4 - x^2}};$$

   (b)

   $$\frac{1}{9 + x^2};$$

(c)

$$\frac{1}{\sqrt{x^2-7}}.$$

2. By completing the square, evaluate the following definite integrals:

(a)

$$\int_{-1}^{\sqrt{3}-1} \frac{\mathrm{d}x}{x^2+2x+2};$$

(b)

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}}.$$

12.3.3 ANSWERS TO EXERCISES

1. (a)

$$\sin^{-1}\frac{x}{2}+C;$$

(b)

$$\frac{1}{3}\tan^{-1}\frac{x}{3}+C;$$

(c)

$$\ln(x+\sqrt{x^2-7})+C.$$

2. (a)

$$\left[\tan^{-1}(x+1)\right]_{-1}^{\sqrt{3}-1}=\frac{\pi}{3};$$

(b)

$$\left[\sin^{-1}\frac{x+1}{2}\right]_0^1=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}.$$

"JUST THE MATHS"

UNIT NUMBER

12.4

INTEGRATION 4
(Integration by substitution in general)

by

A.J.Hobson

UNIT 12.4 - INTEGRATION 4
INTEGRATION BY SUBSTITUTION IN GENERAL

12.4.1 EXAMPLES USING THE STANDARD FORMULA

With any integral

$$\int f(x)\mathrm{d}x,$$

it may be convenient to make some kind of substitution relating the variable, x, to a new variable, u. In such cases, we may use the formula discussed in Unit 12.1, namely

$$\int f(x)\mathrm{d}x = \int f(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u,$$

where it is assumed that, on the right hand side, the integrand has been expressed wholly in terms of u.

For this Unit, substitutions other than linear ones will be given in the problems to be solved.

EXAMPLES

1. Use the substitution $x = a\sin u$ to show that

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$$

Solution

To be precise, we shall assume for simplicity that u is the **acute** angle for which $x = a\sin u$. In effect, we shall be making the substitution $u = \sin^{-1}\frac{x}{a}$ using the principal value of the inverse function; we can certainly do this because the expression $\sqrt{a^2 - x^2}$ requires that $-a < x < a$.

If $x = a\sin u$, then $\frac{\mathrm{d}x}{\mathrm{d}u} = a\cos u$, so that the integral becomes

$$\int \frac{a\cos u}{\sqrt{a^2 - a^2\sin^2 u}}\mathrm{d}u.$$

But, from trigonometric identities,

$$\sqrt{a^2 - a^2\sin^2 u} \equiv a\cos u,$$

both sides being positive when u is an acute angle.

We are thus left with

$$\int 1\mathrm{d}u = u + C = \sin^{-1}\frac{x}{a} + C.$$

2. Use the substitution $u = \frac{1}{x}$ to determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{x\sqrt{1+x^2}}.$$

Solution

Converting the substitution to the form

$$x = \frac{1}{u},$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2}.$$

Hence,

$$z = \int \frac{1}{\frac{1}{u}\sqrt{1+\frac{1}{u^2}}}. - \frac{1}{u^2}\mathrm{d}u$$

That is,

$$z = \int -\frac{1}{\sqrt{u^2+1}} = -\ln(u + \sqrt{u^2+1}) + C.$$

Returning to the original variable, x, we have

$$z = -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) + C.$$

Note:

This example is somewhat harder than would be expected under examination conditions.

12.4.2 INTEGRALS INVOLVING A FUNCTION AND ITS DERIVATIVE

The method of integration by substitution provides two useful results applicable to a wide range of problems. They are as follows:

(a)

$$\int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C$$

provided $n \neq -1$.

(b)

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x) + C.$$

These two results are readily established by means of the substitution

$$u = f(x).$$

In both cases $\frac{du}{dx} = f'(x)$ and hence $\frac{dx}{du} = \frac{1}{f'(x)}$. This converts the integrals, respectively, into

(a)

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

and (b)

$$\int \frac{1}{u} du = \ln u + C.$$

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \sin^3 x.\cos x \; dx.$$

Solution

In this example we can consider $\sin x$ to be $f(x)$ and $\cos x$ to be $f'(x)$.

Thus, by quoting result (a), we obtain

$$\int_0^{\frac{\pi}{3}} \sin^3 x.\cos x dx = \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{3}} = \frac{9}{64},$$

using $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

2. Integrate the function

$$\frac{2x+1}{x^2+x-11}$$

with respect to x.

Solution

Here, we can identify $x^2 + x - 11$ with $f(x)$ and $2x + 1$ with $f'(x)$.

Thus, by quoting result (b), we obtain

$$\int \frac{2x+1}{x^2+x-11} dx = \ln(x^2 + x - 11) + C.$$

12.4.3 EXERCISES

1. Use the substitution $u = x + 3$ in order to determine the indefinite integral
$$\int x\sqrt{3+x}\ \mathrm{d}x.$$

2. Use the substitution $u = x^2 - 1$ in order to evaluate the definite integral
$$\int_1^5 x\sqrt{x^2-1}\ \mathrm{d}x.$$

3. Integrate the following functions with respect to x:

   (a)
$$\sin^7 x.\cos x;$$

   (b)
$$\cos^5 x.\sin x;$$

   (c)
$$\frac{4x-3}{2x^2-3x+13};$$

   (d)
$$\cot x.$$

12.4.4 ANSWERS TO EXERCISES

1.
$$\frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.$$

2.
$$\left[\frac{1}{3}(x^2-1)^{\frac{3}{2}}\right]_1^5 = \frac{1}{3}24^{\frac{3}{2}} \simeq 39.192$$

3. (a)

$$\frac{\sin^8 x}{8} + C;$$

(b)

$$-\frac{\cos^6 x}{6} + C;$$

(c)

$$\ln(2x^2 - 3x + 13) + C;$$

(d)

$$\ln \sin x + C.$$

"JUST THE MATHS"

UNIT NUMBER

12.5

INTEGRATION 5
(Integration by parts)

by

A.J.Hobson

UNIT 12.5 - INTEGRATION 5

INTEGRATION BY PARTS

12.5.1 THE STANDARD FORMULA

The technique to be discussed here provides a convenient method for integrating the product of two functions. However, it is possible to develop a suitable formula by considering, instead, the **derivative** of the product of two functions.

We consider, first, the following comparison:

$\frac{\mathrm{d}}{\mathrm{d}x}[x\sin x] = x\cos x + \sin x$	$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
$x\cos x = \frac{\mathrm{d}}{\mathrm{d}x}[x\sin x] - \sin x$	$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[uv] - v\frac{\mathrm{d}u}{\mathrm{d}x}$
$\int x\cos x \ \mathrm{d}x = x\sin x - \int \sin x \ \mathrm{d}x$	$\int u\frac{\mathrm{d}v}{\mathrm{d}x} \ \mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x} \ \mathrm{d}x$
$= x\sin x + \cos x + C$	

We see that, by labelling the product of two given functions as $u\frac{\mathrm{d}v}{\mathrm{d}x}$, we may express the integral of this product in terms of another integral which, it is anticipated, will be simpler than the original.

To summarise, the formula for **"integration by parts"** is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x} \ \mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x} \ \mathrm{d}x.$$

EXAMPLES

1. Determine

$$I = \int x^2 e^{3x} \ \mathrm{d}x.$$

Solution

In theory, it does not matter which element of the product x^2e^{3x} is labelled as u and which is labelled as $\frac{\mathrm{d}v}{\mathrm{d}x}$; but the integral obtained on the right-hand-side of the integration by parts formula must be simpler than the original.

In this case we shall take

$$u = x^2 \ \text{ and } \ \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Hence,

$$I = x^2 \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.2x \, \mathrm{d}x.$$

That is,

$$I = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int xe^{3x} \, \mathrm{d}x.$$

The integral on the right-hand-side still contains the product of two functions and so we must use integration by parts a second time, setting

$$u = x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Thus,

$$I = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.1 \, \mathrm{d}x\right].$$

The integration may now be completed to obtain

$$I = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C,$$

or

$$I = \frac{e^{3x}}{27}\left[9x^2 - 6x + 2\right] + C.$$

2. Determine

$$I = \int x \ln x \, \mathrm{d}x.$$

Solution

In this case, we cannot effectively choose $\frac{\mathrm{d}v}{\mathrm{d}x} = \ln x$ since we would need to know the integral of $\ln x$ in order to find v. Hence, we choose

$$u = \ln x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = x,$$

obtaining

$$I = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}.\frac{1}{x}\ \mathrm{d}x.$$

That is,

$$I = \frac{1}{2}x^2 \ln x - \int \frac{x}{2}\ \mathrm{d}x,$$

giving

$$I = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C.$$

3. Determine

$$I = \int \ln x\ \mathrm{d}x.$$

Solution
It is possible to regard this as an integration by parts problem if we set

$$u = \ln x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = x\ln x - \int x.\frac{1}{x}\ \mathrm{d}x,$$

giving

$$I = x\ln x - x + C.$$

4. Evaluate

$$I = \int_0^1 \sin^{-1}x\ \mathrm{d}x.$$

Solution
In a similar way to the previous example, it is possible to regard this as an integration by parts problem if we set

$$u = \sin^{-1}x \ \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = [x\sin^{-1}x]_0^1 - \int_0^1 x.\frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x.$$

That is,

$$I = [x\sin^{-1}x + \sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1.$$

5. Determine

$$I = \int e^{2x}\cos x\ \mathrm{d}x.$$

Solution
In this example, it makes little difference whether we choose e^{2x} or $\cos x$ to be u; but we shall set

$$u = e^{2x} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x.$$

Hence,

$$I = e^{2x}\sin x - \int (\sin x).2e^{2x}\ \mathrm{d}x.$$

That is,

$$I = e^{2x}\sin x - 2\int e^{2x}\sin x\ \mathrm{d}x.$$

Now we need to integrate by parts again, setting

$$u = e^{2x} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x.$$

Therefore,

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x - \int (-\cos x).2e^{2x}\ \mathrm{d}x\right].$$

In other words, the original integral has appeared again on the right hand side to give

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x + 2I\right].$$

On simplification,

$$5I = e^{2x}\sin x + 2e^{2x}\cos x,$$

so that

$$I = \frac{1}{5}e^{2x}[\sin x + 2\cos x] + C.$$

Note:

The above examples suggest a priority order for choosing u in a typical integration by parts problem. For example, if the product to be integrated contains a logarithm or an inverse function, then we must choose the logarithm or the inverse function as u; but if there are powers of x without logarithms or inverse functions, then we choose the power of x to be u.

The order of priorities is as follows:

1. LOGARITHMS or INVERSE FUNCTIONS;
2. POWERS OF x;
3. POWERS OF e.

12.5.2 EXERCISES

1. Use integration by parts to evaluate the definite integral

$$\int_0^1 x^3 e^{2x}\ \mathrm{d}x.$$

2. Use integration by parts to integrate the following functions with respect to x:

   (a)

   $$x^2\cos 2x;$$

   (b)

   $$x^5\ln x;$$

   (c)

   $$\tan^{-1}x;$$

(d)

$$x\tan^{-1}x.$$

3. Use integration by parts to evaluate the definite integral

$$\int_0^{\pi} e^{-2x}\sin 3x \; \mathrm{d}x.$$

12.5.3 ANSWERS TO EXERCISES

1.

$$\left[\frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3)\right]_0^1 = \frac{1}{8}\left(e^2 + 3\right) \simeq 1.299$$

2. (a)

$$\frac{1}{4}[2x^2\sin 2x + 2x\cos 2x - \sin 2x] + C;$$

(b)

$$\frac{x^6}{36}[6\ln x - 1] + C;$$

(c)

$$x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C;$$

(d)

$$\frac{1}{2}[x^2\tan^{-1}x - x + \tan^{-1}x] + C.$$

3.

$$\left[\frac{e^{-2x}}{13}(3\cos 3x - 2\sin 3x)\right]_0^{\pi} = -\frac{3}{13}(e^{-2\pi} + 1) \simeq -0.231$$

"JUST THE MATHS"

UNIT NUMBER

12.6

INTEGRATION 6
(Integration by partial fractions)

by

A.J.Hobson

UNIT 12.6 - INTEGRATION 6
INTEGRATION BY PARTIAL FRACTIONS

12.6.1 INTRODUCTION AND ILLUSTRATIONS

If the ratio of two polynomials, whose denominator has been factorised, is expressed as a sum of partial fractions, each partial fraction will be of a type whose integral can be determined by the methods of preceeding sections of this chapter.

The following summary of results will cover most elementary problems involving partial fractions:

RESULTS

1.

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln(ax+b) + C.$$

2.

$$\int \frac{1}{(ax+b)^n}\,\mathrm{d}x = \frac{1}{a}.\frac{(ax+b)^{-n+1}}{-n+1} + C \quad \text{provided} \quad n \neq 1.$$

3.

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

4.

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C,$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

5.

$$\int \frac{2ax+b}{ax^2+bx+c}\,\mathrm{d}x = \ln(ax^2+bx+c) + C.$$

ILLUSTRATIONS

We use some of the results of examples on partial fractions in Unit 1.8

1.

$$\int \frac{7x+8}{(2x+3)(x-1)}\,\mathrm{d}x = \int \left[\frac{1}{2x+3} + \frac{3}{x-1}\right]\,\mathrm{d}x$$

$$= \frac{1}{2}\ln(2x+3) + 3\ln(x-1) + C.$$

2.

$$\int_6^8 \frac{3x^2+9}{(x-5)(x^2+2x+7)}\,\mathrm{d}x = \int_6^8 \left[\frac{2}{x-5} + \frac{x+1}{x^2+2x+7}\right]\,\mathrm{d}x$$

$$= \left[2\ln(x-5) + \frac{1}{2}\ln(x^2+2x+7)\right]_6^8 \simeq 2.427$$

3.

$$\int \frac{9}{(x+1)^2(x-2)} = \int \left[\frac{-1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}\right]\,\mathrm{d}x$$

$$= -\ln(x+1) + \frac{3}{x+1} + \ln(x-2) + C.$$

4.

$$\int \frac{4x^2+x+6}{(x-4)(x^2+4x+5)}\,\mathrm{d}x = \int \left[\frac{2}{x-4} + \frac{2x+1}{x^2+4x+5}\right]\mathrm{d}x$$

$$= 2\ln(x-4) + \ln(x^2+4x+5) - 3\tan^{-1}(x+2) + C.$$

Note:
In the last example above, the second partial fraction has a numerator of $2x+1$ which is not the derivative of x^2+4x+5. But we simply rearrange the numerator as $(2x+4)-3$ to give a third integral which requires the technique of completing the square (discussed in Unit 12.3).

12.6.2 EXERCISES

Integrate the following functions with respect to x:

1. (a)

$$\frac{3x+5}{(x+1)(x+2)};$$

(b)

$$\frac{17x+11}{(x+1)(x-2)(x+3)};$$

(c)

$$\frac{3x^2-8}{(x-1)(x^2+x-7)}.$$

(d)

$$\frac{2x+1}{(x+2)^2(x-3)};$$

(e)

$$\frac{9+11x-x^2}{(x+1)^2(x+2)};$$

(f)

$$\frac{x^5}{(x+2)(x-4)}.$$

2. Evaluate the following definite integrals

(a)

$$\int_2^5 \frac{7x^2+11x+47}{(x-1)(x^2+2x+10)}\,\mathrm{d}x;$$

(b)

$$\int_1^3 \frac{4x^2+1}{x(2x-1)^2}\,\mathrm{d}x.$$

12.6.3 ANSWERS TO EXERCISES

1. (a)

$$2\ln(x+1)+\ln(x+2)+C;$$

(b)

$$\ln(x+1)+3\ln(x-2)-4\ln(x+3)+C;$$

(c)

$$\ln(x-1)+\ln(x^2+x-7)+C;$$

(d)

$$-\frac{3}{5(x+2)}-\frac{7}{25}\ln(x+2)+\frac{7}{25}\ln(x-3)+C;$$

(e)

$$-\frac{3}{(x+1)^2}+\frac{16}{x+1}-\frac{17}{x+2}$$

$$\frac{3}{x+1}+16\ln(x+1)-17\ln(x+2)+C;$$

(f)

$$\frac{x^4}{4}+\frac{2x^3}{3}+6x^2+40x+\frac{16}{3}\ln(x+2)+\frac{512}{3}\ln(x-4)+C.$$

2. (a)

$$\left[5\ln(x-1)+\ln(x^2+2x+10)+\frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_2^5\simeq -2.726;$$

(b)

$$\left[\ln x-\frac{2}{2x-1}\right]_1^3\simeq 2.699$$

"JUST THE MATHS"

UNIT NUMBER

12.7

INTEGRATION 7
(Further trigonometric functions)

by

A.J.Hobson

UNIT 12.7 - INTEGRATION 7 - FURTHER TRIGONOMETRIC FUNCTIONS

12.7.1 PRODUCTS OF SINES AND COSINES

In order to integrate the product of a sine and a cosine, or two cosines, or two sines, we may use one of the following trigonometric identities:

$$\text{sinAcosB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B})+\sin(\text{A}-\text{B})\right];$$

$$\text{cosAsinB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B})-\sin(\text{A}-\text{B})\right];$$

$$\text{cosAcosB} \equiv \frac{1}{2}\left[\cos(\text{A}+\text{B})+\cos(\text{A}-\text{B})\right];$$

$$\text{sinAsinB} \equiv \frac{1}{2}\left[\cos(\text{A}-\text{B})-\cos(\text{A}+\text{B})\right].$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin 2x \cos 5x \ \mathrm{d}x.$$

Solution

$$\int \sin 2x \cos 5x \ \mathrm{d}x = \frac{1}{2}\int \left[\sin 7x - \sin 3x\right] \mathrm{d}x$$

$$= -\frac{\cos 7x}{14} + \frac{\cos 3x}{6} + C.$$

2. Determine the indefinite integral

$$\int \sin 3x \sin x \ \mathrm{d}x.$$

Solution

$$\int \sin 3x \sin x \ \mathrm{d}x = \frac{1}{2}\int \left[\cos 2x - \cos 4x\right] \mathrm{d}x$$

$$= \frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C.$$

12.7.2 POWERS OF SINES AND COSINES

In this section, we consider the two integrals,

$$\int \sin^n x \, dx \quad \text{and} \quad \int \cos^n \, dx,$$

where n is a positive integer.

(a) The Complex Number Method

A single method which will cover both of the above integrals requires us to use the methods of Unit 6.5 in order to express $\cos^n x$ and $\sin^n x$ as a sum of whole multiples of sines or cosines of whole multiples of x.

EXAMPLE

Determine the indefinite integral

$$\int \sin^4 x \, dx.$$

Solution

By the complex number method,

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

The Working:

$$j^4 2^4 \sin^4 x \equiv \left(z - \frac{1}{z}\right)^4,$$

where $z \equiv \cos x + j \sin x$.

That is,

$$16\sin^4 x \equiv z^4 - 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 - 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4;$$

or, after cancelling common factors,

$$16\sin^4 x \equiv z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \equiv z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6,$$

which gives

$$16\sin^4 x \equiv 2\cos 4x - 8\cos 2x + 6,$$

or

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

Hence,

$$\int \sin^4 x \ \mathrm{d}x = \frac{1}{8}\left[\frac{\sin 4x}{4} - 4\frac{\sin 2x}{2} + 3x\right] + C$$

$$= \frac{1}{32}[\sin 4x - 8\sin 2x + 12x] + C.$$

(b) Odd Powers of Sines and Cosines

The following method uses the facts that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = \sin x.$$

We illustrate with examples in which use is made of the trigonometric identity

$$\cos^2 A + \sin^2 A \equiv 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^3 x \ \mathrm{d}x.$$

Solution

$$\int \sin^3 x \, dx = \int \sin^2 x . \sin x \, dx.$$

That is,

$$\int \sin^3 x dx = \int \left(1 - \cos^2 x\right) \sin x \, dx$$

$$= \int \left(\sin x - \cos^2 x . \sin x\right) \, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + C.$$

2. Determine the indefinite integral

$$\int \cos^7 x \, dx.$$

Solution

$$\int \cos^7 x \, dx = \int \cos^6 x . \cos x \, dx.$$

That is,

$$\int \cos^7 x \, dx = \int \left(1 - \sin^2 x\right)^3 . \cos x \, dx$$

$$= \int \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right) . \cos x \, dx$$

$$= \sin x - \sin^3 x + 3.\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

(c) Even Powers of Sines and Cosines

The method illustrated here becomes tedious if the even power is higher than 4. In such cases, it is best to use the complex number method in paragraph (a) above.

In the examples which follow, we shall need the trigonometric identity

$$\cos 2A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^2 x \; \mathrm{d}x.$$

Solution

$$\int \sin^2 x \; \mathrm{d}x = \int \frac{1}{2}(1 - \cos 2x) \; \mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \; \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^2 x \; \mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \; \mathrm{d}x$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

3. Determine the indefinite integral

$$\int \cos^4 x \; \mathrm{d}x.$$

Solution

$$\int \cos^4 x \; \mathrm{d}x = \int \left[\cos^2 x\right]^2 \; \mathrm{d}x = \int \left[\frac{1}{2}(1 + \cos 2x)\right]^2 \; \mathrm{d}x.$$

That is,

$$\int \cos^4 x \; \mathrm{d}x = \int \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) \; \mathrm{d}x$$

$$= \int \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2}[1 + \cos 4x]\right) \; \mathrm{d}x$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

12.7.3 EXERCISES

1. Determine the indefinite integral

$$\int \cos x \cos 3x \; \mathrm{d}x.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \cos 4x \sin 2x \; \mathrm{d}x.$$

3. Determine the following indefinite integrals:

(a)

$$\int \sin^5 x \; \mathrm{d}x;$$

(b)

$$\int \cos^3 x \; \mathrm{d}x.$$

4. Evaluate the following definite integrals:

(a)

$$\int_0^{\frac{\pi}{8}} \sin^4 x \; \mathrm{d}x;$$

(b)

$$\int_0^{\frac{\pi}{2}} \cos^6 x \; \mathrm{d}x.$$

12.7.4 ANSWERS TO EXERCISES

1.

$$\frac{\sin 4x}{8} + \frac{\sin 2x}{4} + C.$$

2.

$$-\frac{\sqrt{3}}{4} \simeq -0.433$$

3. (a)

$$-\frac{\cos^5 x}{5} + 2\frac{\cos^3 x}{3} - \cos x + C,$$

or

$$\frac{1}{16}\left[-\frac{\cos 5x}{5} + \frac{5\cos 3x}{3} - 10\cos x\right] + C \quad \text{by complex numbers;}$$

(b)

$$\sin x - \frac{\sin^3 x}{3} + C,$$

or

$$\frac{1}{4}\left[\frac{\sin 3x}{3} - 3\sin x\right] + C \quad \text{by complex numbers.}$$

4. (a)

$$1.735 \times 10^{-3} \text{ approx;}$$

(b)

$$-1.$$

"JUST THE MATHS"

UNIT NUMBER

12.8

INTEGRATION 8
(The tangent substitutions)

by

A.J.Hobson

UNIT 12.8 - INTEGRATION 8

THE TANGENT SUBSTITUTIONS

There are two types of integral, involving sines and cosines, which require a special substitution using a tangent function. They are described as follows:

12.8.1 THE SUBSTITUTION $t = \tan x$

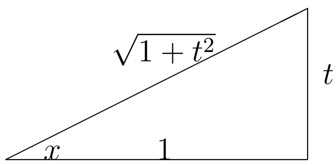
This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin^2 x + c\cos^2 x} \, \mathrm{d}x,$$

where a, b and c are constants; though, in most exercises, at least one of these three constants will be zero.

A simple right-angled triangle will show that, if $t = \tan x$, then

$$\sin x \equiv \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos x \equiv \frac{1}{\sqrt{1+t^2}}.$$



Furthermore,

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \sec^2 x \equiv 1 + t^2 \quad \text{so that} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{1}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{4 - 3\sin^2 x} \, \mathrm{d}x.$$

Solution

$$\int \frac{1}{4 - 3\sin^2 x}\,\mathrm{d}x$$

$$= \int \frac{1}{4 - \frac{3t^2}{1+t^2}}.\frac{1}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{4+t^2}\,\mathrm{d}t$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{2}\tan^{-1}\left[\frac{\tan x}{2}\right] + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{\sin^2 x + 9\cos^2 x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{\sin^2 x + 9\cos^2 x}\,\mathrm{d}x$$

$$= \int \frac{1}{\frac{t^2}{1+t^2} + \frac{9}{1+t^2}}.\frac{1}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{t^2+9}\,\mathrm{d}t$$

$$= \frac{1}{3}\tan^{-1}\frac{t}{3} + C = \frac{1}{3}\tan^{-1}\left[\frac{\tan x}{3}\right] + C.$$

12.8.2 THE SUBSTITUTION $t = \tan(x/2)$

This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin x + c\cos x}\,\mathrm{d}x,$$

where a, b and c are constants; though, in most exercises, one or more of these constants will be zero.

In order to make the substitution, we make the following observations:

(i)

$$\sin x \equiv 2\sin(x/2).\cos(x/2) \equiv 2\tan(x/2).\cos^2(x/2) \equiv \frac{2\tan(x/2)}{\sec^2(x/2)} \equiv \frac{2\tan(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\sin x \equiv \frac{2t}{1+t^2}.$$

(ii)

$$\cos x \equiv \cos^2(x/2) - \sin^2(x/2) \equiv \cos^2(x/2)\left[1-\tan^2(x/2)\right] \equiv \frac{1-\tan^2(x/2)}{\sec^2(x/2)} \equiv \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\cos x \equiv \frac{1-t^2}{1+t^2}.$$

(iii)

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \frac{1}{2}\sec^2(x/2) \equiv \frac{1}{2}\left[1+\tan^2(x/2)\right] \equiv \frac{1}{2}[1+t^2].$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{2}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

Solution

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{1+t^2+2t}\,\mathrm{d}t$$

$$= \int \frac{2}{(1+t)^2}\,\mathrm{d}t$$

$$= -\frac{2}{1+t} + C = -\frac{2}{1+\tan(x/2)} + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{4\frac{1-t^2}{1+t^2} - \frac{6t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{4-4t^2-6t}\,\mathrm{d}t = \int -\frac{1}{2t^2+3t-2}\,\mathrm{d}t$$

$$= \int -\frac{1}{(2t-1)(t+2)}\,\mathrm{d}t$$

$$= \int \frac{1}{5}\left[\frac{1}{t+2} - \frac{2}{2t-1}\right]\mathrm{d}t$$

$$= \frac{1}{5}[\ln(t+2) - \ln(2t-1)] + C = \frac{1}{5}\ln\left[\frac{\tan(x/2)+2}{2\tan(x/2)-1}\right] + C.$$

12.8.3 EXERCISES

1. Determine the indefinite integral

$$\int \frac{1}{4 + 12\cos^2 x}\,\mathrm{d}x.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{4}} \frac{1}{5\cos^2 x + 3\sin^2 x}\,\mathrm{d}x.$$

3. Determine the indefinite integral

$$\int \frac{1}{5 + 3\cos x}\,\mathrm{d}x.$$

4. Evaluate the definite integral

$$\int_3^{3.1} \frac{1}{12\sin x + 5\cos x}\,\mathrm{d}x.$$

12.8.4 ANSWERS TO EXERCISES

1.

$$\frac{1}{4}\tan^{-1}\left[\frac{\tan x}{2}\right] + C.$$

2.

$$\left[\frac{1}{\sqrt{15}}\tan^{-1}\left(\sqrt{\frac{3}{5}}\tan x\right)\right]_0^{\frac{\pi}{4}} \simeq 0.1702$$

3.

$$\frac{1}{2}\tan^{-1}\left[\frac{\tan(x/2)}{2}\right] + C.$$

4.

$$\left[\frac{1}{13}[5\ln(5\tan(x/2) + 1) - \ln(\tan(x/2) - 5)\right]_3^{3.1} \simeq 0.348$$

"JUST THE MATHS"

UNIT NUMBER

12.9

INTEGRATION 9
(Reduction formulae)

by

A.J.Hobson

UNIT 12.9 - INTEGRATION 9

REDUCTION FORMULAE

INTRODUCTION

For certain integrals, both definite and indefinite, the function being integrated (that is, the "integrand") consists of a product of two functions, one of which involves an unspecified integer, say n. Using the method of integration by parts, it is sometimes possible to express such an integral in terms of a similar integral where n has been replaced by $(n-1)$, or sometimes $(n-2)$. The relationship between the two integrals is called a **"reduction formula"** and, by repeated application of this formula, the original integral may be determined in terms of n.

12.9.1 INDEFINITE INTEGRALS

The method will be illustrated by examples.

EXAMPLES

1. Obtain a reduction formula for the indefinite integral

$$I_n = \int x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^x$, we obtain

$$I_n = x^n e^x - \int e^x.nx^{n-1} \, \mathrm{d}x.$$

That is,

$$I_n = x^n e^x - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = x^3 e^x - 3I_2,$$

where

$$I_2 = x^2 e^x - 2I_1$$

and

$$I_1 = xe^x - I_0.$$

But

$$I_0 = \int e^x \, \mathrm{d}x = e^x + \text{constant},$$

which leads us to the conclusion that

$$I_3 = x^3e^x - 3\left[x^2e^x - 2\left(xe^x - e^x\right)\right] + \text{constant}.$$

In other words,

$$I_3 = e^x\left[x^3 - 3x^2 + 6x - 6\right] + C,$$

where C is an arbitrary constant.

2. Obtain a reduction formula for the indefinite integral

$$I_n = \int x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = x^n \sin x - \int \sin x . nx^{n-1} \, \mathrm{d}x = x^n \sin x - n \int x^{n-1} \sin x \, \mathrm{d}x.$$

Using integration by parts in this last integral, with $u = x^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = x^n \sin x - n\left\{-x^{n-1}\cos x + \int \cos x.(n-1)x^{n-2} \, \mathrm{d}x\right\}.$$

That is,

$$I_n = x^n \sin x + nx^{n-1}\cos x - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = x^2 \sin x + 2x \cos x - 2I_0,$$

where

$$I_0 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x + C,$$

where C is an arbitrary constant.
Also, substituting $n = 3$,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 3.2.I_1,$$

where

$$I_1 = \int x \cos x \, \mathrm{d}x = x \sin x + \cos x + \text{constant}.$$

Therefore,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 6x \sin x - 6 \cos x + D,$$

where D is an arbitrary constant.

12.9.2 DEFINITE INTEGRALS

Integrals of the type encountered in the previous section may also include upper and lower limits of integration. The process of finding a reduction formula is virtually the same, except that the limits of integration are inserted where appropriate. Again, the method is illustrated by examples.

EXAMPLES

1. Obtain a reduction formula for the definite integral

$$I_n = \int_0^1 x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution
From the first example in section 12.9.1,

$$I_n = [x^n e^x]_0^1 - nI_{n-1} = e - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = e - 3I_2,$$

where

$$I_2 = e - 2I_1$$

and

$$I_1 = e - I_0.$$

But

$$I_0 = \int_0^1 e^x \ \mathrm{d}x = e - 1,$$

which leads us to the conclusion that

$$I_3 = e - 3e + 6e - 6e + 6 = 6 - 2e.$$

2. Obtain a reduction formula for the definite integral

$$I_n = \int_0^{\pi} x^n \cos x \ \mathrm{d}x$$

and, hence, determine I_2 and I_3.
Solution
From the second example in section 12.9.1,

$$I_n = \left[x^n \sin x + nx^{n-1} \cos x\right]_0^{\pi} - n(n-1)I_{n-2} = -n\pi^{n-1} - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = -2\pi - 2I_0,$$

where

$$I_0 = \int_0^{\pi} \cos x \ \mathrm{d}x = [\sin x]_0^{\pi} = 0.$$

Hence,

$$I_2 = -2\pi.$$

Also, substituting $n = 3$,

$$I_3 = -3\pi^2 - 3.2.I_1,$$

where

$$I_1 = \int_0^{\pi} x \cos x \, \mathrm{d}x = [x \sin x + \cos x]_0^{\pi} = -2.$$

Therefore,

$$I_3 = -3\pi^2 + 12.$$

12.9.3 EXERCISES

1. Obtain a reduction formula for

$$I_n = \int x^n e^{2x} \, \mathrm{d}x$$

when $n \geq 1$ and, hence, determine I_3.

2. Obtain a reduction formula for

$$I_n = \int_0^1 x^n e^{2x} \mathrm{d}x$$

when $n \geq 1$ and, hence, evaluate I_4.

3. Obtain a reduction formula for

$$I_n = \int x^n \sin x \, \mathrm{d}x$$

when $n \geq 1$ and, hence, determine I_4.

4. Obtain a reduction formula for

$$I_n = \int_0^{\pi} x^n \sin x \, \mathrm{d}x$$

when $n \geq 1$ and, hence, evaluate I_3.

5. If

$$I_n = \int (\ln x)^n \, \mathrm{d}x,$$

where $n \geq 1$, show that

$$I_n = x(\ln x)^n - nI_{n-1}$$

and, hence, determine I_3.

6. If

$$I_n = \int \left(x^2 + a^2\right)^n \, \mathrm{d}x,$$

show that

$$I_n = \frac{1}{2n+1}\left[x\left(x^2 + a^2\right)^n + 2na^2 I_{n-1}\right].$$

Hint: Write $(x^2 + a^2)^n$ as $1.(x^2 + a^2)^n$.

12.9.4 ANSWERS TO EXERCISES

1.

$$I_n = \frac{1}{2}\left[x^n e^{2x} - nI_{n-1}\right],$$

giving

$$I_3 = \frac{e^{2x}}{8}\left[4x^3 - 6x^2 + 6x - 3\right] + C.$$

2.

$$I_n = \frac{1}{2}\left[e^2 - nI_{n-1}\right],$$

giving

$$I_4 = \frac{1}{4}\left[e^2 - 3\right].$$

3.

$$I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2},$$

giving

$$I_4 = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.$$

4.

$$I_n = \pi^n - n(n-1)I_{n-2},$$

giving

$$I_3 = \pi^3 - 6\pi.$$

5.

$$I_3 = x \left[\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6 \right] + C.$$

"JUST THE MATHS"

UNIT NUMBER

12.10

INTEGRATION 10
(Further reduction formulae)

by

A.J.Hobson

UNIT 12.10 - INTEGRATION 10

FURTHER REDUCTION FORMULAE

INTRODUCTION

As an extension to the idea of reduction formulae, there are two particular definite integrals which are worthy of special consideration. They are

$$\int_0^{\frac{\pi}{2}} \sin^n x \ \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \ \mathrm{d}x.$$

But, first, we shall establish the reduction formulae for the equivalent indefinite integrals.

12.10.1 INTEGER POWERS OF A SINE

Suppose that

$$I_n = \int \sin^n x \ \mathrm{d}x;$$

then, by writing the integrand as the product of two functions, we have

$$I_n = \int \sin^{n-1} x \sin x \ \mathrm{d}x.$$

Using integration by parts, with $u = \sin^{n-1} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = \sin^{n-1} x(-\cos x) + \int (n-1)\sin^{n-2} x \cos^2 x \ \mathrm{d}x.$$

But, since $\cos^2 x \equiv 1 - \sin^2 x$, this becomes

$$I_n = -\sin^{n-1} x \cos x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n}\left[-\sin^{n-1} x \cos x + (n-1)I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \sin^6 x \, dx.$$

Solution

$$I_6 = \frac{1}{6}\left[-\sin^5 x \cos x + 5I_4\right],$$

where

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x + 3I_2\right], \quad I_2 = \frac{1}{2}\left[-\sin x \cos x + I_0\right]$$

and

$$I_0 = \int dx = x + \text{constant}.$$

Hence,

$$I_2 = \frac{1}{2}\left[-\sin x \cos x + x + \text{constant}\right];$$

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x - \frac{3}{2}\sin x \cos x + \frac{3}{2}x + \text{constant}\right];$$

$$I_6 = \frac{1}{6}\left[-\sin^5 x \cos x - \frac{5}{4}\sin^3 x \cos x - \frac{15}{8}\sin x \cos x + \frac{15}{8}x + \text{constant}\right].$$

$$\text{Thus,} \quad \int \sin^6 x \, dx = -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5x}{16} + C,$$

where C is an arbitrary constant.

12.10.2 INTEGER POWERS OF A COSINE

Suppose that

$$I_n = \int \cos^n x \, \mathrm{d}x;$$

then, by writing the integrand as the product of two functions, we have

$$I_n = \int \cos^{n-1} x \cos x \, \mathrm{d}x.$$

Using integration by parts, with $u = \cos^{n-1} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x \, \mathrm{d}x.$$

But, since $\sin^2 x \equiv 1 - \cos^2 x$, this becomes

$$I_n = \cos^{n-1} x \sin x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n}\left[\cos^{n-1} x \sin x + (n-1)I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \cos^5 x \, \mathrm{d}x.$$

Solution

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + 4I_3\right],$$

where

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2I_1\right]$$

and

$$I_1 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2\sin x + \text{constant}\right];$$

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + \frac{4}{3}\cos^2 x \sin x + \frac{8}{3}\sin x + \text{constant}\right];$$

We conclude that

$$\int \cos^5 x \, \mathrm{d}x = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + C,$$

where C is an arbitrary constant.

12.10.3 WALLIS'S FORMULAE

Here, we consider the definite integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x.$$

Denoting either of these integrals by I_n, the reduction formula reduces to

$$I_n = \frac{n-1}{n} I_{n-2}$$

in both cases, from the previous two sections.

Convenient results may be obtained from this formula according as n is an odd number or an even number, as follows:

(a) n is an odd number

Repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}. \ldots \frac{6}{7}.\frac{4}{5}.\frac{2}{3}.I_1.$$

But

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, \mathrm{d}x \quad \text{or} \quad I_1 = \int_0^{\frac{\pi}{2}} \cos x \, \mathrm{d}x,$$

both of which have a value of 1.

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5). \ldots 6.4.2}{n(n-2)(n-4). \ldots 7.5.3},$$

which is **the first of "Wallis's formulae"**.

(b) n is an even number

This time, repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}. \ldots \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.I_0.$$

But

$$I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}.$$

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5). \ldots 5.3.1}{n(n-2)(n-4). \ldots 6.4.2} \frac{\pi}{2},$$

which is **the second of "Wallis's formulae"**.

EXAMPLES

1. Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x.$$

   Solution
$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x = \frac{4.2}{5.3} = \frac{8}{15}.$$

2. Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x.$$

   Solution
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x = \frac{3.1}{4.2}\frac{\pi}{2} = \frac{3\pi}{16}.$$

12.10.4 COMBINATIONS OF SINES AND COSINES

Another type of problem to which Wallis's formulae may be applied is of the form

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, \mathrm{d}x,$$

where either m or n (or both) is an even number. We simply use $\sin^2 x \equiv 1 - \cos^2 x$ or $\cos^2 x \equiv 1 - \sin^2 x$ in order to convert the problem to several integrals of the types already discussed.

EXAMPLE

Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^5 x \left(1 - \cos^2 x\right) \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\cos^5 x - \cos^7 x\right) \, \mathrm{d}x,$$

which may be interpreted as

$$I_5 - I_7 = \frac{4.2}{5.3} - \frac{5.4.3}{6.4.2} = \frac{8}{15} - \frac{16}{35} = \frac{8}{105}.$$

12.10.5 EXERCISES

1. Determine the indefinite integral

$$\int \cos^4 x \ \mathrm{d}x.$$

2. Determine the indefinite integral

$$\int \sin^7 x \ \mathrm{d}x.$$

3. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^6 x \ \mathrm{d}x.$$

4. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^9 x \ \mathrm{d}x.$$

5. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \sin^6 x \ \mathrm{d}x.$$

6. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x \ \mathrm{d}x.$$

12.10.6 ANSWERS TO EXERCISES

1.

$$\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3x}{8} + C.$$

2.

$$-\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x - \frac{24}{105}\sin^2 x \cos x - \frac{16}{35}\cos x + C.$$

3.

$$\frac{5\pi}{32}.$$

4.

$$\frac{128}{315}.$$

5.

$$\frac{5\pi}{32}.$$

6.

$$-\frac{4}{105}.$$

"JUST THE MATHS"

UNIT NUMBER

13.1

INTEGRATION APPLICATIONS 1
(The area under a curve)

by

A.J.Hobson

UNIT 13.1 - INTEGRATION APPLICATIONS 1

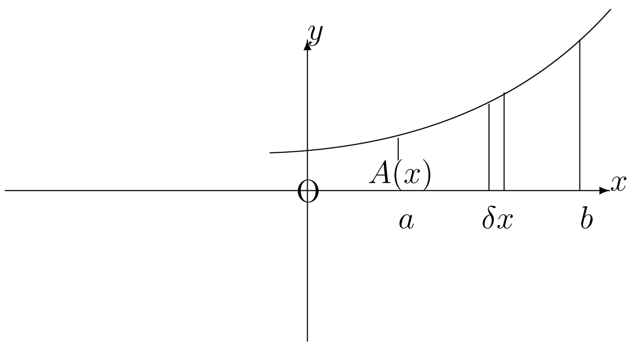
THE AREA UNDER A CURVE

13.1.1 THE ELEMENTARY FORMULA

We shall consider, here, a method of calculating the area contained between the x-axis of a cartesian co-ordinate system and the arc, from $x = a$ to $x = b$, of the curve whose equation is

$$y = f(x).$$

Suppose that $A(x)$ represents the area contained between the curve, the x-axis, the y-axis and the ordinate at some arbitrary value of x.



A small increase of δx in x will lead to a corresponding increase of δA in A approximating in area to that of a narrow rectangle whose width is δx and whose height is $f(x)$.

Thus,

$$\delta A \simeq f(x)\delta x,$$

which may be written

$$\frac{\delta A}{\delta x} \simeq f(x).$$

By allowing δx to tend to zero, the approximation disappears to give

$$\frac{\mathrm{d}A}{\mathrm{d}x} = f(x).$$

Hence, on integrating both sides with respect to x,

$$A(x) = \int f(x)\ \mathrm{d}x.$$

The constant of integration would need to be such that $A = 0$ when $x = 0$; but, in fact, we do not need to know the value of this constant because the required area, from $x = a$ to $x = b$, is given by

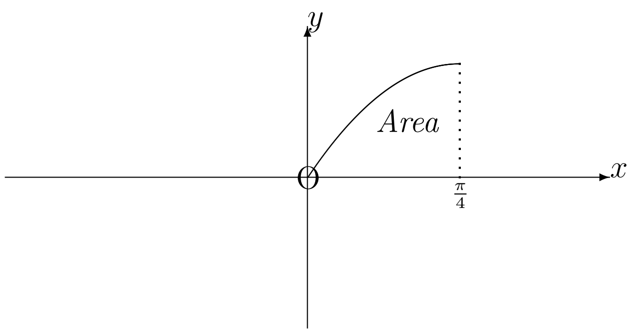
$$A(b) - A(a) = \int_a^b f(x)\ \mathrm{d}x.$$

EXAMPLES

1. Determine the area contained between the x-axis and the curve whose equation is $y = \sin 2x$, from $x = 0$ to $x = \frac{\pi}{4}$.

   Solution

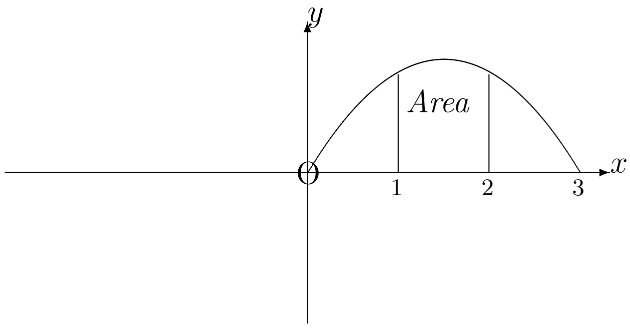
$$\int_0^{\frac{\pi}{4}} \sin 2x\ \mathrm{d}x = \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$



2. Determine the area contained between the x-axis and the curve whose equation is $y = 3x - x^2$, from $x = 1$ to $x = 2$.

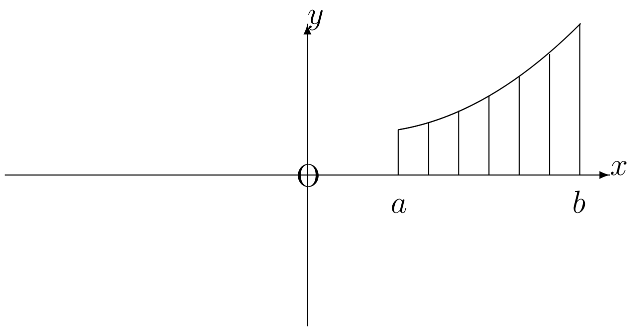
Solution

$$\int_1^2 (3x - x^2)\,\mathrm{d}x = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^2 = \left(6 - \frac{8}{3}\right) - \left(\frac{3}{2} - \frac{1}{3}\right) = \frac{13}{6}.$$



13.1.2 DEFINITE INTEGRATION AS A SUMMATION

Consider, now, the same area as in the previous section, but regarded (approximately) as the sum of a large number of narrow rectangles with typical width δx and typical height $f(x)$. The narrower the strips, the better will be the approximation.



Hence, we may state an alternative expression for the area from $x = a$ to $x = b$ in the form

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x.$$

Since this new expression represents the same area as before, we may conclude that

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x = \int_a^b f(x)\delta x.$$

Notes:

(i) The above result shows that an area which lies wholly **below** the x-axis will be **negative** and so care must be taken with curves which cross the x-axis between $x = a$ and $x = b$.

(ii) If c is any value of x between $x = a$ and $x = b$, the above result shows that

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x.$$

(iii) To calculate the TOTAL area contained between the x-axis and a curve which crosses the x-axis between $x = a$ and $x = b$, account must be taken of any parts of the area which are negative.

(iv) It is usually a good idea to sketch the area under consideration before evaluating the appropriate definite integrals.

(v) It will be seen shortly that the formula obtained for definite integration as a summation has a wider field of application than simply the calculation of areas.

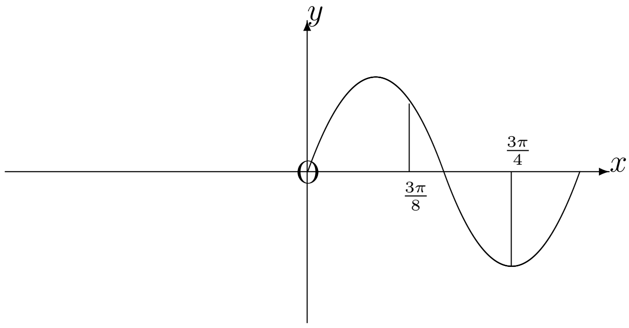
EXAMPLES

1. Determine the total area between the x-axis and the curve whose equation is $y = \sin 2x$, from $x = \frac{3\pi}{8}$ and $x = \frac{3\pi}{4}$.
   Solution

$$\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sin 2x\ \mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2x\ \mathrm{d}x.$$

   That is,

$$\left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}} - \left[-\frac{\cos 2x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right) - \left(0 - \frac{1}{2}\right) = 1 - \frac{1}{2\sqrt{2}}.$$



2. Evaluate the definite integral,

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \, dx.$$

Solution

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \, dx = \left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} = -\frac{1}{2\sqrt{2}}.$$

13.1.3 EXERCISES

1. Determine the areas bounded by the following curves and the x-axis between the ordinates $x = 1$ and $x = 3$:

   (a)
   $$y = 2x^2 + x + 1;$$
   (b)
   $$y = (1 - x)^2;$$
   (c)
   $$y = 2\sqrt{x}.$$

2. Sketch the curve whose equation is
   $$y = (1 - x)(2 + x)$$
   and determine the area contained between the x-axis and the portion of the curve above the x-axis.

3. To the nearest whole number, determine the area bounded between $x = 1$ and $x = 2$ by the curves whose equations are
$$y = 3e^{2x} \text{ and } y = 3e^{-x}.$$

4. Determine the area bounded between $x = 0$ and $x = \frac{\pi}{3}$ by the curves whose equations are
$$y = \sin x \text{ and } y = \sin 2x.$$

5. Determine the total area, from $x = 0$ to $x = \frac{3\pi}{10}$, contained between the x-axis and the curve whose equation is
$$y = \cos 5x.$$

13.1.4 ANSWERS TO EXERCISES

1. (a)
$$\frac{70}{3};$$
(b)
$$\frac{8}{3};$$
(c)
$$4\sqrt{3} - \frac{4}{3}.$$

2.
$$\frac{9}{2}.$$

3.
$$70.$$

4.
$$0.25$$

5.
$$\frac{2\sqrt{2} - 1}{5\sqrt{2}} - \simeq 0.259$$

"JUST THE MATHS"

UNIT NUMBER

13.2

INTEGRATION APPLICATIONS 2
(Mean values)
&
(Root mean square values)

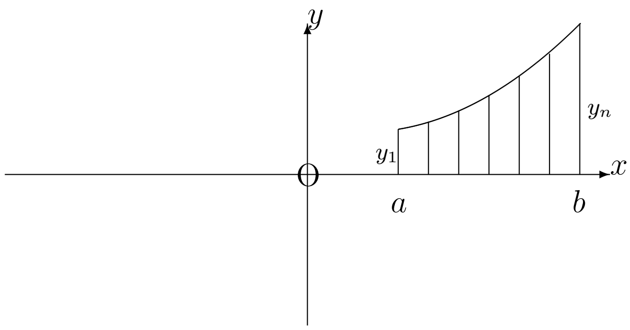
by

A.J.Hobson

UNIT 13.2 - INTEGRATION APPLICATIONS 2

MEAN AND ROOT MEAN SQUARE VALUES

13.2.1 MEAN VALUES



On the curve whose equation is

$$y = f(x),$$

suppose that $y_1, y_2, y_3, ..., y_n$ are the y-coordinates which correspond to n different x-coordinates, $a = x_1, x_2, x_3, ..., x_n = b$.

The average (that is, the arithmetic mean) of these n y-coordinates is

$$\frac{y_1 + y_2 + y_3 + ... + y_n}{n}.$$

But now suppose that we wished to determine the average (arithmetic mean) of **all** the y-coordinates, from $x = a$ to $x = b$ on the curve whose equation is $y = f(x)$.

We could make a reasonable approximation by taking a very **large** number, n, of y-coordinates separated in the x-direction by very **small** distances. If these distances are typically represented by δx, then the required mean value could be written

$$\frac{y_1\delta x + y_2\delta x + y_3\delta x + ... + y_n\delta x}{n\delta x},$$

in which the denominator is equivalent to $(b - a + \delta x)$, since there are only $n - 1$ spaces between the n y-coordinates.

Allowing the number of y-coordinates to increase indefinitely, δx will to tend to zero and we obtain the formula for the **"Mean Value"** in the form

$$\text{M.V.} = \frac{1}{b-a} \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x.$$

That is,

$$\text{M.V.} = \frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x.$$

Note:
In cases where the definite integral in this formula represents the area between the curve and the x-axis, the Mean Value provides the height of a rectangle, with base $b - a$, having the same area as that represented by the definite integral.

EXAMPLE

Determine the Mean Value of the function

$$f(x) \equiv x^2 - 5x$$

from $x = 1$ to $x = 4$.

Solution

The Mean Value is given by

$$\text{M.V.} = \frac{1}{4-1} \int_1^4 (x^2 - 5x)\ \mathrm{d}x = \frac{1}{3}\left[\frac{x^3}{3} - \frac{5x^2}{2}\right]_1^4 =$$

$$\frac{1}{3}\left[\left(\frac{64}{3} - 40\right) - \left(\frac{1}{3} - \frac{5}{2}\right)\right] = -\frac{33}{2}.$$

13.2.2 ROOT MEAN SQUARE VALUES

It is sometimes convenient to use an alternative kind of average for the values of a function, $f(x)$, between $x = a$ and $x = b$.

The **"Root Mean Square Value"** provides a measure of "central tendency" for the **numerical** values of $f(x)$ and is defined to be the square root of the Mean Value of $f(x)$ from $x = a$ to $x = b$.

Hence,

$$\text{R.M.S.V.} = \sqrt{\frac{1}{b-a}\int_a^b \ [f(x)]^2 \ \mathrm{d}x}.$$

EXAMPLE

Determine the Root Mean Square Value of the function, $f(x) \equiv x^2 - 5$, from $x = 1$ to $x = 3$.

Solution

The Root Mean Square Value is given by

$$\text{R.M.S.V.} = \sqrt{\frac{1}{3-1}\int_1^3 \ (x^2-5)^2 \ \mathrm{d}x}.$$

Temporarily ignoring the square root, we obtain the **"Mean Square Value"**,

$$\text{M.S.V.} = \frac{1}{2}\int_1^3 \ (x^4 - 10x^2 + 25) \ \mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{10x^3}{3} + 25x\right]_1^3 = \frac{1}{2}\left[\left(\frac{243}{5} - \frac{270}{3} + 75\right) - \left(\frac{1}{5} - \frac{10}{3} + 25\right)\right] = \frac{176}{30}.$$

Thus,

$$\text{R.M.S.V.} = \sqrt{\frac{176}{30}} \simeq 2.422$$

13.2.3 EXERCISES

1. (a) Determine the Mean Value of the function, $(x-1)(x-2)$, from $x = 1$ to $x = 2$;
   (b) Determine, correct to three significant figures, the Mean Value of the function, $\frac{1}{2x+5}$, from $x = 3$ to $x = 5$;
   (c) Determine the Mean Value of the function, $\sin 2t$, from $t = 0$ to $t = \frac{\pi}{2}$;
   (d) Determine, correct to three places of decimals, the Mean Value of the function, e^{-x}, from $x = 1$ to $x = 5$;
   (e) Determine, correct to three significant figures, the mean value of the function, xe^{-2x}, from $x = 0$ to $x = 2$.
2. (a) Determine the Root Mean Square Value of the function, $3x + 1$, from $x = -2$ to $x = 2$;
   (b) Determine the Root Mean Square Value, of the function, e^{x}, from $x = 0$ to $x = 1$, correct to three decimal places;
   (c) Determine the Root Mean Square Value of the function, $\cos x$, from $x = \frac{\pi}{2}$ to $x = \pi$;
   (d) Determine the Root Mean Square Value of the function, $(4x-5)^{\frac{3}{2}}$, from $x = 1.25$ to $x = 1.5$.

13.2.4 ANSWERS TO EXERCISES

1. (a) $-\frac{1}{6}$;
   (b) 0.0775;
   (c) $\frac{2}{\pi}$;
   (d) -0.076;
   (e) 0.114
2. (a) $\sqrt{13} \simeq= 3.606$;
   (b) 1.787;
   (c) $\frac{1}{\sqrt{2}}$;
   (d) $\frac{1}{2}$.

"JUST THE MATHS"

UNIT NUMBER

13.3

INTEGRATION APPLICATIONS 3
(Volumes of revolution)

by

A.J.Hobson

UNIT 13.3 - INTEGRATION APLICATIONS 3

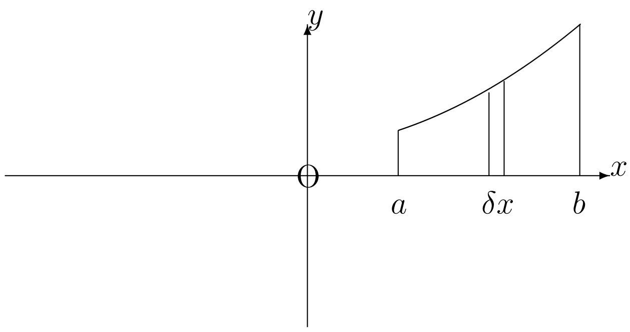
VOLUMES OF REVOLUTION

13.3.1 VOLUMES OF REVOLUTION ABOUT THE X-AXIS

Suppose that the area between a curve whose equation is

$$y = f(x)$$

and the x-axis, from $x = a$ to $x = b$, lies wholly above the x-axis; suppose, also, that this area is rotated through 2π radians about the x-axis. Then a solid figure is obtained whose volume may be determined as an application of definite integration.



When a narrow strip of width, δx, and height, y, is rotated through 2π radians about the x-axis, we obtain a disc whose volume, δV, is given approximately by

$$\delta V \simeq \pi y^2 \delta x.$$

Thus, the total volume, V, obtained is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x.$$

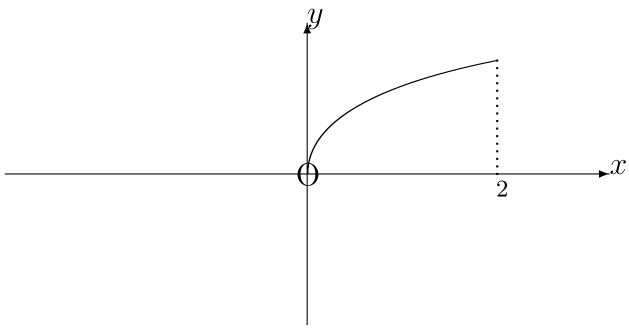
That is,

$$V = \int_a^b \pi y^2 \, \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line, $x = 2$, and the parabola, $y^2 = 8x$, is rotated through 2π radians about the x-axis.

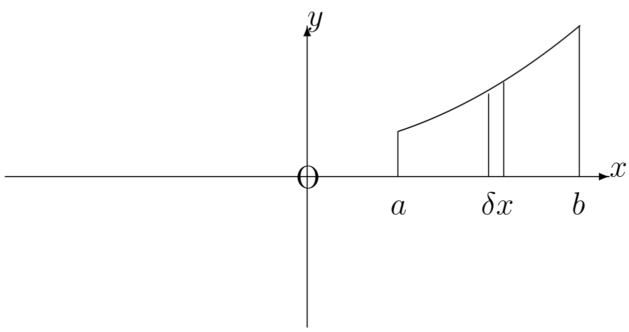
Solution



$$V = \int_0^2 \pi \times 8x \, \mathrm{d}x = [4\pi x^2]_0^2 = 16\pi.$$

13.3.2 VOLUMES OF REVOLUTION ABOUT THE Y-AXIS

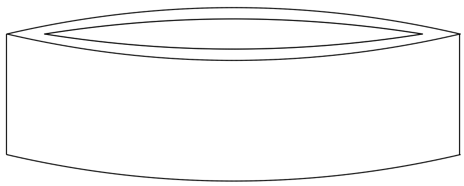
First we consider the same diagram as in the previous section:



This time, if the narrow strip of width, δx, is rotated through 2π radians about the y-axis, we obtain, approximately, a cylindrical shell of internal radius, x, external radius, $x + \delta x$ and height, y.

The volume, δV, of the shell is thus given by

$$\delta V \simeq 2\pi xy\delta x.$$



The total volume is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi xy\delta x.$$

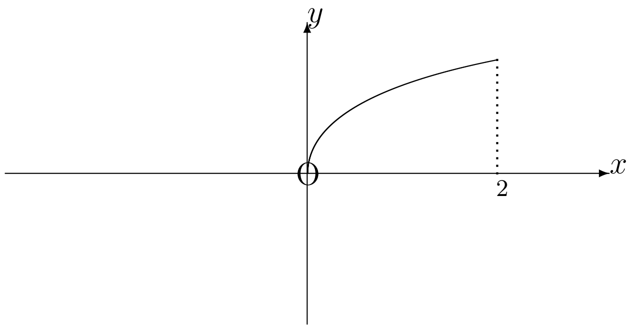
That is,

$$V = \int_a^b 2\pi xy \, \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line $x = 2$ and the parabola $y^2 = 8x$ is rotated through 2π radians about the y-axis.

Solution



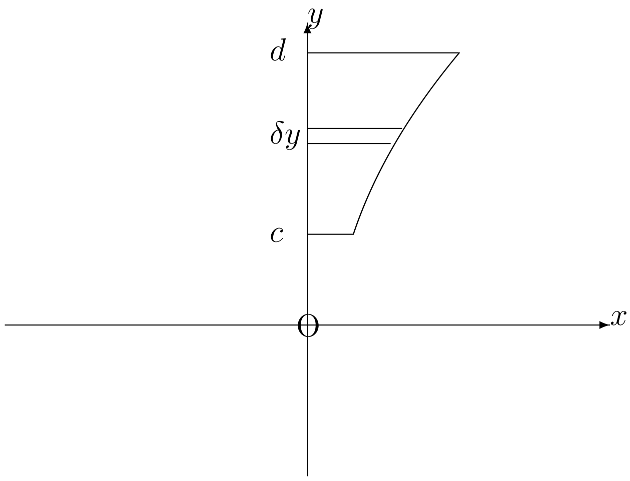
$$V = \int_0^2 \; 2\pi x \times \sqrt{8x} \; \mathrm{d}x.$$

In other words,

$$V = \pi 4\sqrt{2} \int_0^2 \; x^{\frac{3}{2}} \mathrm{d}x = \pi 4\sqrt{2} \left[\frac{2x^{\frac{5}{2}}}{5} \right]_0^2 = \frac{64\pi}{5}.$$

Note:

It may be required to find the volume of revolution about the y-axis of an area which is contained between a curve and the y-axis from $y = c$ to $y = d$.



But here we simply interchange the roles of x and y in the original formula for rotation about the x-axis; that is

$$V = \int_c^d \pi x^2 \text{ d}y.$$

Similarly, the volume of rotation of the above area about the x-axis is given by

$$V = \int_c^d 2\pi yx \text{ d}y.$$

13.3.3 EXERCISES

1. By using a straight line through the origin, obtain a formula for the volume, V, of a solid right-circular cone with height, h, and base radius, r.
2. Determine the volume obtained when the segment straight line
$$y = 5 - 4x,$$
lying between $x = 0$ and $x = 1$, is rotated through 2π radians about (a) the x-axis and (b) the y-axis.
3. Determine the volume obtained when the part of the curve
$$y = \cos 3x,$$
lying between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$, is rotated through 2π radians about the x-axis.

4. Determine the volume obtained when the part of the curve
$$y = \frac{1}{x\sqrt{2+x}},$$
lying between $x = 2$ and $x = 7$, is rotated through 2π radians about the x-axis.

5. Determine the volume obtained when the part of the curve
$$y = \frac{1}{(x-1)(x-5)},$$
lying between $x = 6$ and $x = 8$, is rotated through 2π radians about the y-axis.

6. Determine the volume obtained when the part of the curve
$$x = ye^{-y},$$
lying between $y = 0$ and $y = 1$, is rotated through 2π radians about the y-axis.

7. Determine the volume obtained when the part of the curve
$$y = \sin 2x,$$
lying between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$, is rotated through 2π radians about the y-axis.

8. Determine the volume obtained when the part of the curve
$$y = x(1-x^3)^{\frac{1}{4}},$$
lying between $x = 0$ and $x = 1$, is rotated through 2π radians about the x-axis.

9. Determine the volume obtained when the part of the curve
$$x = (4-y^2)^2,$$
lying between $y = 1$ and $y = 2$, is rotated through 2π radians about the x-axis.

10. Determine the volume obtained when the part of the curve
$$y = x\sec(x^3),$$
lying between $x = 0$ and $x = 0.5$, is rotated through 2π radians about the x-axis.

11. Determine the volume obtained when the part of the curve
$$y = \frac{1}{x^2-1},$$
lying between $x = 2$ and $x = 3$ is rotated through 2π radians about the y-axis.

13.3.4 ANSWERS TO EXERCISES

1.

$$V = \frac{1}{3}\pi r^2 h.$$

2.

$$\text{(a)} \quad \frac{\pi}{3} \simeq 1.047 \quad \text{(b)} \quad \frac{7\pi}{3} \simeq 7.330$$

3.

$$\frac{\pi^2}{12} \simeq 0.822$$

4.

0.214 approximately.

5.

8.010 approximately.

6.

0.254 approximately.

7.

3.364 approximately.

8.

$$\frac{2\pi}{9} \simeq 0.698$$

9.

$$9\pi \simeq 28.274$$

10.

0.132 approximately.

11.

3.081 approximately.

"JUST THE MATHS"

UNIT NUMBER

13.4

INTEGRATION APPLICATIONS 4
(Lengths of curves)

by

A.J.Hobson

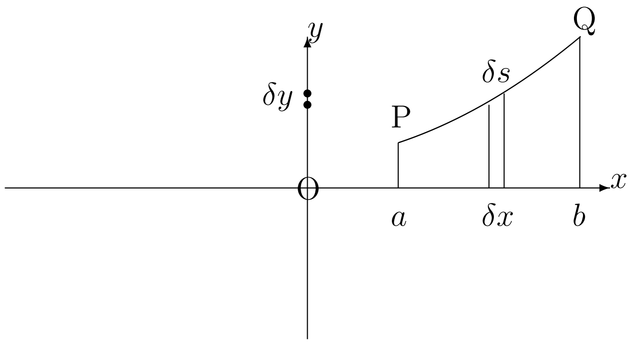
UNIT 13.4 - INTEGRATION APPLICATIONS 4 - LENGTHS OF CURVES

13.4.1 THE STANDARD FORMULAE

The problem, in this unit, is to calculate the length of the arc of the curve with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

The total length, s, of arc is thus given by

$$s = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

That is,

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Notes:
(i) If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$s = \pm \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(ii) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y, so that the length of the arc is given by

$$s = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$

EXAMPLES

1. A curve has equation

$$9y^2 = 16x^3.$$

Determine the length of the arc of the curve between the point $\left(1, \frac{4}{3}\right)$ and the point $\left(4, \frac{32}{3}\right)$.

Solution

We may write the equation of the curve in the form

$$y = \frac{4x^{\frac{3}{2}}}{3};$$

and so,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^{\frac{1}{2}}.$$

Hence,

$$s = \int_1^4 \sqrt{1 + 4x}\ \mathrm{d}x = \left[\frac{(1 + 4x)^{\frac{3}{2}}}{6}\right]_1^4 = \frac{17^{\frac{3}{2}}}{6} - \frac{5^{\frac{3}{2}}}{6} \simeq 13.55$$

2. A curve is given parametrically by

$$x = t^2 - 1, \quad y = t^3 + 1.$$

Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 1$.

Solution

Since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2,$$

we have

$$s = \int_0^1 \sqrt{4t^2 + 6t^4}\ \mathrm{d}t = \int_0^1 t\sqrt{4 + 6t^2}\ \mathrm{d}t = \left[\frac{1}{18}\left(4 + 6t^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{1}{18}\left(10^{\frac{3}{2}} - 8\right) \simeq 1.31$$

13.4.2 EXERCISES

1. A straight line has equation

$$y = 3x + 2.$$

Use (a) elementary trigonometry and (b) definite integration to determine the length of the line segment joining the point where $x = 3$ and the point where $x = 7$.

2. A curve has equation

$$y = \frac{1}{2}x^2 - \frac{1}{4}\ln x.$$

Determine the length of the arc of the curve between $x = 1$ and $x = e$.

3. A curve has equation

$$x = 2(y + 3)^{\frac{3}{2}}.$$

Determine the length of the arc of the curve between $y = -2$ and $y = 1$, stating your answer in decimals correct to four significant figures.

4. A curve is given parametrically by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 2\pi$.

5. A curve is given parametrically by

$$x = 4(\cos\theta + \theta\sin\theta), \quad y = 4(\sin\theta - \theta\cos\theta).$$

Determine the length of the arc of the curve between the point where $\theta = 0$ and the point where $\theta = \frac{\pi}{4}$.

6. A curve is given parametrically by

$$x = e^u \sin u, \quad y = e^u \cos u.$$

Determine the length of the arc of the curve between the point where $u = 0$ and the point where $u = 1$.

13.4.3 ANSWERS TO EXERCISES

1.

$$4\sqrt{10} \simeq 12.65$$

2.

$$\frac{2e^2 - 1}{4} \simeq 3.44$$

3.

$$14.33$$

4.

$$8.$$

5.

$$\frac{\pi^2}{8}.$$

6.

$$\sqrt{2}(e - 1) \simeq 2.43$$

"JUST THE MATHS"

UNIT NUMBER

13.5

INTEGRATION APPLICATIONS 5
(Surfaces of revolution)

by

A.J.Hobson

UNIT 13.5 - INTEGRATION APPLICATIONS 5

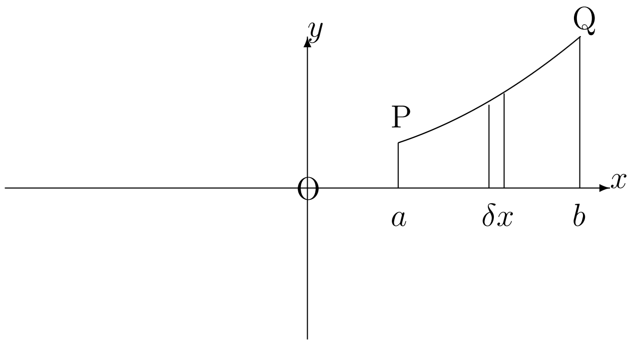
SURFACES OF REVOLUTION

13.5.1 SURFACES OF REVOLUTION ABOUT THE X-AXIS

The problem, in this unit, is to calculate the surface area obtained when the arc of the curve, with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$, is rotated through 2π radians about the x-axis or the y-axis.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis, respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

When the arc, of length δs, is rotated through 2π radians about the x-axis, it generates a thin band whose area is, approximately,

$$2\pi y \delta s = 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\, \delta x.$$

The total surface area, S, is thus given by

$$S = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\, \delta x.$$

That is,

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi y \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$S = \pm\int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

EXAMPLES

1. A curve has equation

$$y^2 = 2x.$$

Determine the surface area obtained when the arc of the curve between the point $(2, 2))$ and the point $(8, 4)$ is rotated through 2π radians about the x-axis.

Solution

We may write the equation of the arc of the curve in the form

$$y = \sqrt{2x} = \sqrt{2}x^{\frac{1}{2}};$$

and so,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{2x}}.$$

Hence,

$$S = \int_2^8 2\pi\sqrt{2x}\sqrt{1+\frac{1}{2x}}\ \mathrm{d}x = \int_2^8 \sqrt{2x+1}\ \mathrm{d}x = \left[\frac{(2x+1)^{\frac{3}{2}}}{3}\right]_2^8.$$

Thus,

$$S = \frac{17^{\frac{3}{2}}}{3} - \frac{5^{\frac{3}{2}}}{3} \simeq 19.64$$

2. A curve is given parametrically by

$$x = \sqrt{2}\cos\theta, \quad y = \sqrt{2}\sin\theta.$$

Determine the surface area obtained when the arc of the curve between the point $(0, \sqrt{2})$ and the point $(1, 1)$ is rotated through 2π radians about the x-axis.

Solution

The parameters of the two points are $\frac{\pi}{2}$ and $\frac{\pi}{4}$, respectively; and, since

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sqrt{2}\sin\theta \;\text{ and }\; \frac{\mathrm{d}y}{\mathrm{d}\theta} = \sqrt{2}\cos\theta,$$

we have

$$S = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\sqrt{2}\pi\sin\theta\sqrt{2\sin^2\theta + 2\cos^2\theta}\;\mathrm{d}\theta = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\pi\sin\theta\;\mathrm{d}\theta.$$

Thus,

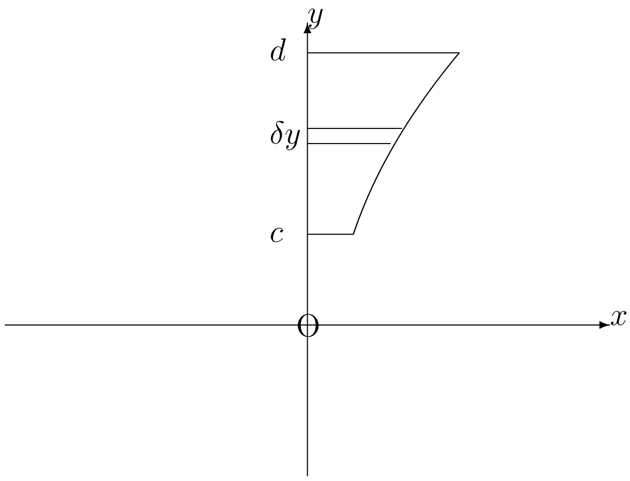
$$S = -\left[-4\pi\cos\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{4\pi}{\sqrt{2}} \simeq 8.89$$

13.5.2 SURFACES OF REVOLUTION ABOUT THE Y-AXIS

For a curve whose equation is of the form $x = g(y)$, the surface of revolution about the y-axis of an arc joining the two points at which $y = c$ and $y = d$ is given by

$$S = \int_a^b 2\pi x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\;\mathrm{d}y.$$

We simply reverse the roles of x and y in the previous section.



Alternatively, if the curve is given parametrically,

$$S = \pm \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

EXAMPLE

If the arc of the parabola, with equation

$$x^2 = 2y,$$

joining the two points $(2, 2)$ and $(4, 8)$, is rotated through 2π radians about the y-axis, determine the surface area obtained.

Solution

Using the result from the previous section, the surface area obtained is given by

$$S = \int_2^8 2\pi\sqrt{2y}\sqrt{1 + \frac{1}{2y}}\ \mathrm{d}y \simeq 19.64$$

13.5.3 EXERCISES

1. Use a straight line through the origin to determine the surface area of a right-circular cone with height, h, and base radius, r.

2. Determine the surface area obtained when the arc of the curve $x = y^3$, between $y = 0$ and $y = 1$, is rotated through 2π radians about the y-axis.

3. A curve is given parametrically by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

   Determine the surface area obtained when the arc of the curve between the point where $t = 0$ and the point where $t = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

   State your answer correct to three places of decimals.

4. A curve is given parametrically by

$$x = 4(\cos\theta + \theta\sin\theta), \quad y = 4(\sin\theta - \theta\cos\theta).$$

   Determine the surface area obtained when the arc of the curve between the point where $\theta = 0$ and the point where $\theta = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

5. A curve is given parametrically by

$$x = e^u \cos u, \quad y = e^u \sin u.$$

   Determine the surface area obtained when the arc of the curve between the point where $u = 0$ and the point where $u = \frac{\pi}{4}$ is rotated through 2π radians about the y-axis.

   State your answer correct to three places of decimals.

13.5.4 ANSWERS TO EXERCISES

1.
$$\pi r\sqrt{r^2+h^2}.$$

2.
$$\frac{\pi(10\sqrt{10}-1)}{27} \simeq 3.56$$

3.
$$3.891$$

4.
$$32\pi\left(3-\left(\frac{\pi}{2}\right)^2\right) \simeq 53.54$$

5.
$$1.037$$

"JUST THE MATHS"

UNIT NUMBER

13.6

INTEGRATION APPLICATIONS 6
(First moments of an arc)

by

A.J.Hobson

UNIT 13.6 - INTEGRATION APPLICATIONS 6

FIRST MOMENTS OF AN ARC

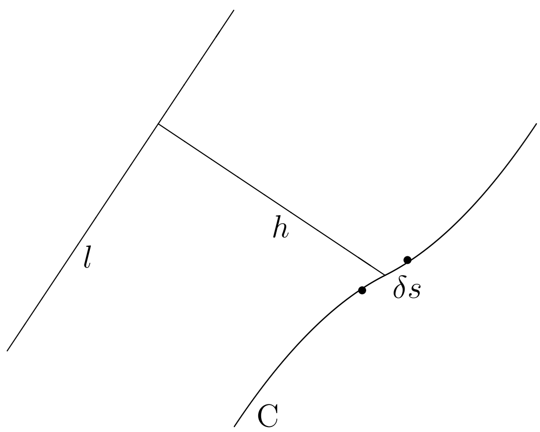
13.6.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates; and suppose that δs is the length of a small element of this arc.

Then the **"first moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} h\delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

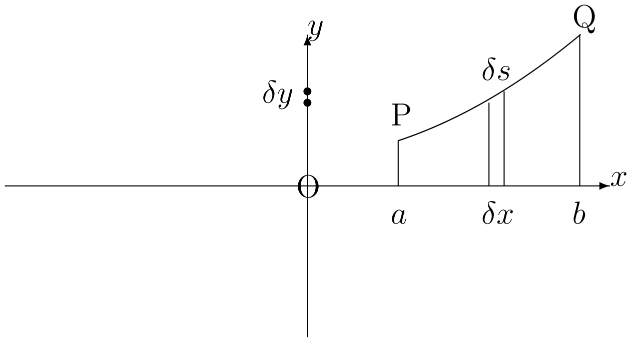


13.6.2 FIRST MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve, whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The first moment of each element about the y-axis is x times the length of the element; that is $x\delta s$, implying that the total first moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} x\delta s.$$

But, from Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

so that the first moment of the arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the y-axis is given by

$$\pm \int_{t_1}^{t_2} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.6.3 FIRST MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the first moment about the x-axis will be

$$\int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the first moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

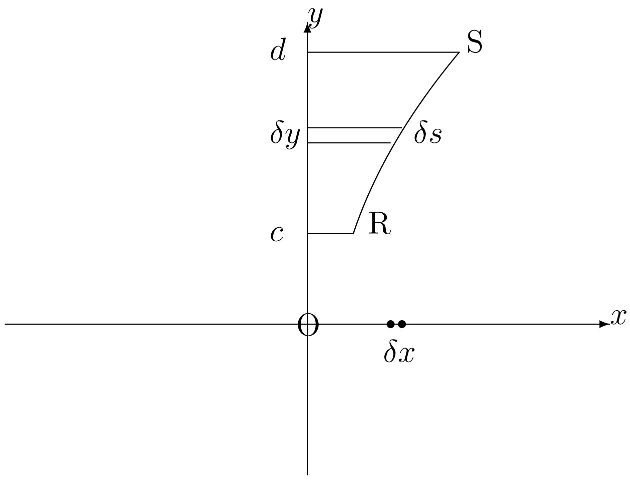
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.6.2 so that the first moment of the arc about the x-axis is given by

$$\int_c^d y\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the x-axis is given by

$$\pm\int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

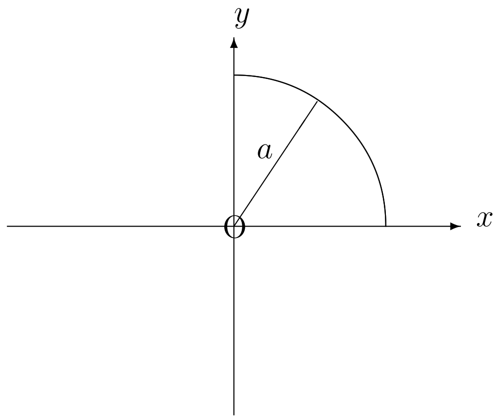
EXAMPLES

1. Determine the first moments about the x-axis and the y-axis of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The first moment of the arc about the y-axis is therefore given by

$$\int_0^a x\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a \frac{x}{y}\sqrt{x^2 + y^2}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$ and $y = \sqrt{a^2 - x^2}$.

Hence,

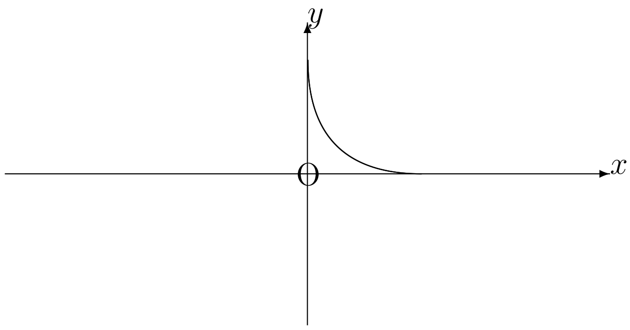
$$\text{first moment} = \int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\ \mathrm{d}x = \left[-a\sqrt{(a^2 - x^2)}\right]_0^a = a^2.$$

By symmetry, the first moment of the arc about the x-axis will also be a^2.

2. Determine the first moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \;\text{ and }\; \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} y\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\ \mathrm{d}\theta$$

$$= 3a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5}.$$

Similarly, the first moment of the arc about the y-axis is given by

$$\int_0^{\frac{\pi}{2}} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} a\cos^3\theta.(3a\cos\theta\sin\theta)\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\ \mathrm{d}\theta = 3a^2\left[-\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5},$$

though, again, this second result could be deduced, by symmetry, from the first.

13.6.4 THE CENTROID OF AN ARC

Having calculated the first moments of an arc about both the x-axis and the y-axis it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $s\overline{x}$, where s is the total length of the arc;

and

(b) The first moment about the x-axis is given by $s\overline{y}$, where s is the total length of the arc.

The point is called the **"centroid"** or the **"geometric centre"** of the arc and, for an arc of the curve with equation $y = f(x)$, between $x = a$ and $x = b$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}.$$

Notes:

(i) The first moment of an arc about an axis through its centroid will, by definition, be zero. In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δs, to the y-axis, the first moment about the given axis will be

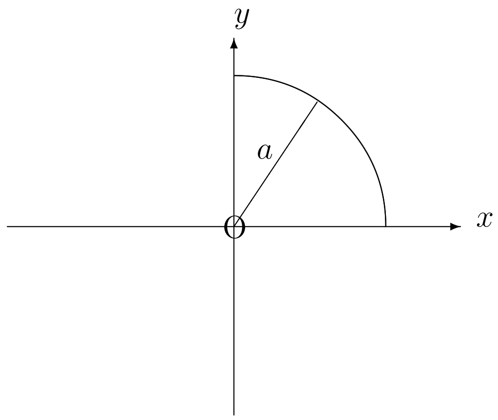
$$\sum_{\mathrm{C}} (x-\overline{x})\delta s = \sum_{\mathrm{C}} x\delta s - \overline{x}\sum_{\mathrm{C}} \delta s = s\overline{x} - s\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole arc at a single point for the purposes of considering first moments. In practice, it corresponds, for example, to the position of the centre of mass of a thin wire with uniform density.

EXAMPLES

1. Determine the cartesian co-ordinates of the centroid of the arc of the circle, with equation
$$x^2 + y^2 = a^2,$$
lying in the first quadrant.

Solution

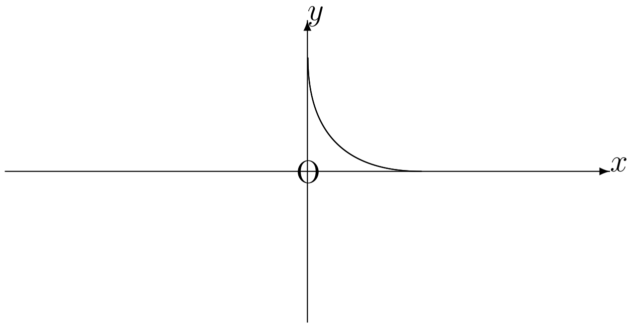


From an earlier example in this unit, we know that the first moments of the arc about the x-axis and the y-axis are both equal to a^2.

Also, the length of the arc is $\frac{\pi a}{2}$, which implies that
$$\overline{x} = \frac{2a}{\pi} \text{ and } \overline{y} = \frac{2a}{\pi}.$$

2. Determine the cartesian co-ordinates of the centroid of the first quadrant arc of the curve with parametric equations
$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From an earlier example in this unit, we know that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

and that the first moments of the arc about the x-axis and the y-axis are both equal to $\frac{3a^2}{5}$.
Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus,

$$\overline{x} = \frac{2a}{5} \text{ and } \overline{y} = \frac{2a}{5}.$$

13.6.5 EXERCISES

1. Determine the first moment about the y-axis of the arc of the curve with equation
$$y = x^2,$$
lying between $x = 0$ and $x = 1$.

2. Determine the first moment about the x-axis of the arc of the curve with equation
$$x = 5y^2,$$
lying between $y = 0.1$ and $y = 0.5$.

3. Determine the first moment about the x-axis of the arc of the curve with equation
$$y = 2\sqrt{x},$$
lying between $x = 3$ and $x = 24$.

4. Verify, using integration, that the centroid of the straight line segment, defined by the equation
$$y = 3x + 2,$$
from $x = 0$ to $x = 1$, lies at its centre point.

5. Determine the cartesian co-ordinates of the centroid of the arc of the circle given parametrically by
$$x = 5\cos\theta, \quad y = 5\sin\theta,$$
from $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$.

6. For the curve whose equation is
$$9y^2 = x(3 - x)^2,$$
show that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$
Hence show that the centroid of the first quadrant arch of this curve lies at the point $\left(\frac{7}{5}, \frac{\sqrt{3}}{4}\right)$.

13.6.6 ANSWERS TO EXERCISES

1.

$$\frac{5\sqrt{5}-1}{12} \simeq 0.85$$

2.

$$\frac{13\sqrt{26}-\sqrt{2}}{150} \simeq 0.43$$

3.

$$156.$$

4.

$$\overline{x} = \frac{1}{2} \quad \text{and} \quad \overline{y} = \frac{7}{2}.$$

5.

$$\overline{x} = \frac{15}{\pi} \simeq 4.77, \quad \overline{y} = 0.$$

"JUST THE MATHS"

UNIT NUMBER

13.7

INTEGRATION APPLICATIONS 7
(First moments of an area)

by

A.J.Hobson

UNIT 13.7 - INTEGRATION APPLICATIONS 7

FIRST MOMENTS OF AN AREA

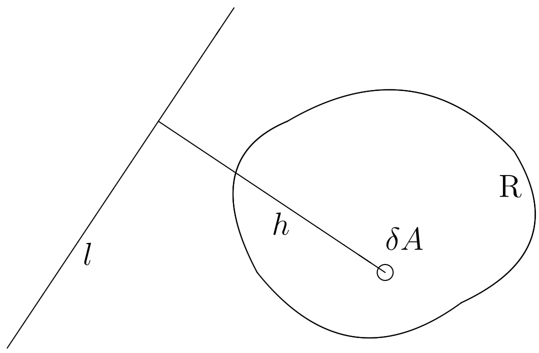
13.7.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy-plane of cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the **"first moment"** of R about a fixed line, l, in the plane of R is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h\delta A,$$

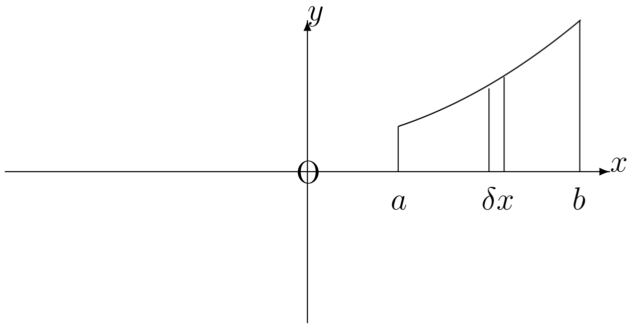
where h is the perpendicular distance, from l, of the element with area, δA.



13.7.2 FIRST MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may divided up into small elements by using a network, consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

But all of the elements in a narrow 'strip' of width δx and height y (parallel to the y-axis) have the same perpendicular distance, x, from the y-axis.

Hence the first moment of this strip about the y-axis is x times the area of the strip; that is, $x(y\delta x)$, implying that the total first moment of the region about the y-axis is given by

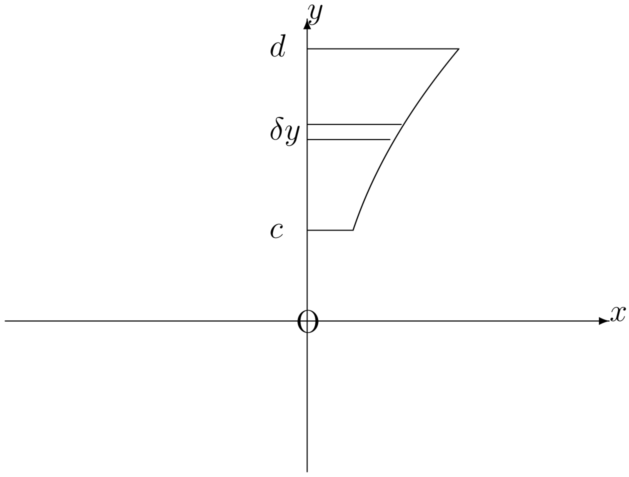
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy \, \mathrm{d}x.$$

Note:
First moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment about the x-axis is given by

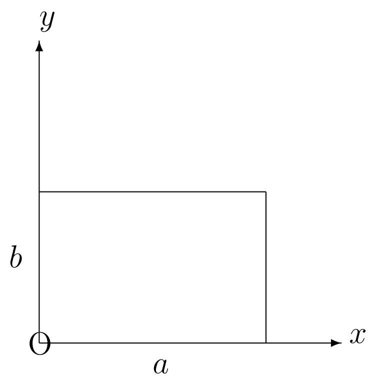
$$\int_c^d yx \, \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a rectangular region, with sides of lengths a and b about the side of length b.

   Solution



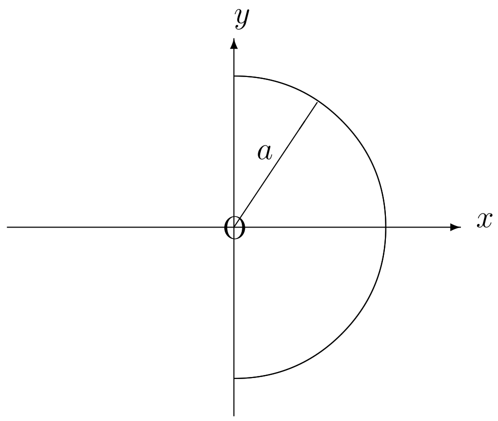
The first moment about the y-axis is given by

$$\int_0^a xb \, \mathrm{d}x = \left[\frac{x^2 b}{2}\right]_0^a = \frac{1}{2}a^2 b.$$

2. Determine the first moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the first moment about the y-axis is given by

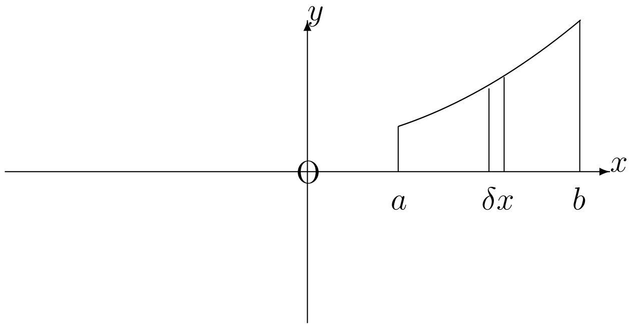
$$2\int_0^a x\sqrt{a^2 - x^2}\,\mathrm{d}x = \left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}a^3.$$

Note:
Although first moments about the x-axis will be discussed mainly in the next section of this Unit, we note that the symmetry of the above region shows that its first moment about the x-axis would be zero; this is because, for each $y(x\delta y)$, there will be a corresponding $-y(x\delta y)$ in calculating the first moments of the strips parallel to the x-axis.

13.7.3 FIRST MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the first moment of a rectangular region about one of its sides. This result may now be used to determine the first moment about the x-axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow strip, of width δx and height y, is regarded as approximately a rectangle, its first moment about the x-axis is $\frac{1}{2}y^2\delta x$. Hence, the first moment of the whole region about the x-axis is given by

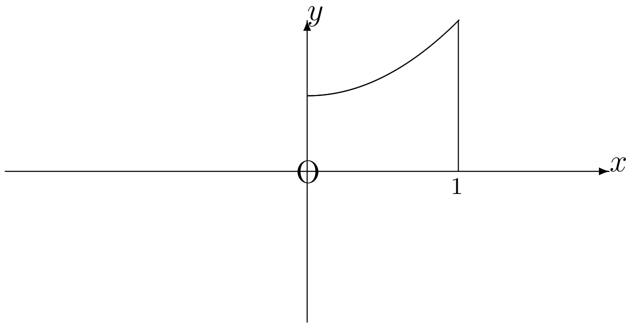
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{2}y^2\delta x = \int_a^b \frac{1}{2}y^2 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the first moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution

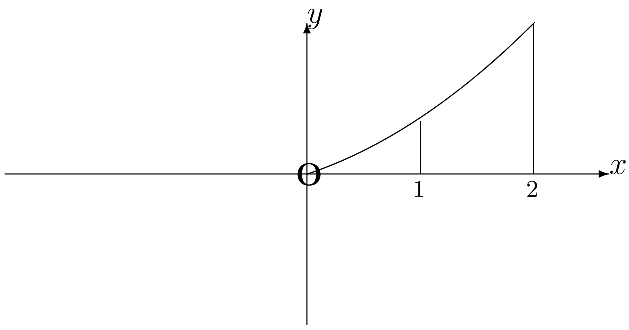


$$\text{First moment } = \int_0^1 \frac{1}{2}(x^2+1)^2 \, \mathrm{d}x = \frac{1}{2}\int_0^1 (x^4+2x^2+1) \, \mathrm{d}x = \frac{1}{2}\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^1 = \frac{28}{15}.$$

2. Determine the first moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the lines $x = 1$, $x = 2$ and the curve

$$y = xe^x.$$

Solution



$$\text{First moment } = \int_1^2 \frac{1}{2}x^2e^{2x} \, \mathrm{d}x$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \int_1^2 xe^{2x}\ \mathrm{d}x\right)$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \left[x\frac{e^{2x}}{2}\right]_1^2 + \int_1^2 \frac{e^{2x}}{2}\ \mathrm{d}x\right).$$

That is,

$$\frac{1}{2}\left[x^2\frac{e^{2x}}{2} - x\frac{e^{2x}}{2} + \frac{e^{2x}}{4}\right]_1^2 = \frac{5e^4 - e^2}{8} \simeq 33.20$$

13.7.4 THE CENTROID OF AN AREA

Having calculated the first moments of a two dimensional region about both the x-axis and the y-axis, it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $A\overline{x}$, where A is the total area of the region;

and

(b) The first moment about the x-axis is given by $A\overline{y}$, where A is the toal area of the region.

The point is called the **"centroid"** or the **"geometric centre"** of the region and, in the case of a region bounded, in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b\ xy\ \mathrm{d}x}{\int_a^b\ y\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b\ \frac{1}{2}y^2\ \mathrm{d}x}{\int_a^b\ y\ \mathrm{d}x}.$$

Notes:

(i) The first moment of an area, about an axis through its centroid will, by definition, be zero. In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δA, to the y-axis, the first moment about the given axis will be

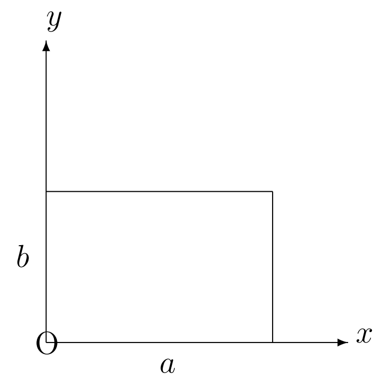
$$\sum_{\mathrm{R}} (x - \overline{x})\delta A = \sum_{\mathrm{R}} x\delta A - \overline{x}\sum_{\mathrm{R}} \delta A = A\overline{x} - A\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole area at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass for a thin plate with uniform density, whose shape is that of the region which we have been considering.

EXAMPLES

1. Determine the position of the centroid of a rectangular region with sides of lengths, a and b.

   Solution



The area of the rectangle is ab and, from Example 1 in section 13.7.2, the first moments about the y-axis and the x-axis are $\frac{1}{2}a^2b$ and $\frac{1}{2}b^2a$, respectively.

Hence,

$$\overline{x} = \frac{\frac{1}{2}a^2b}{ab} = \frac{1}{2}a$$

and

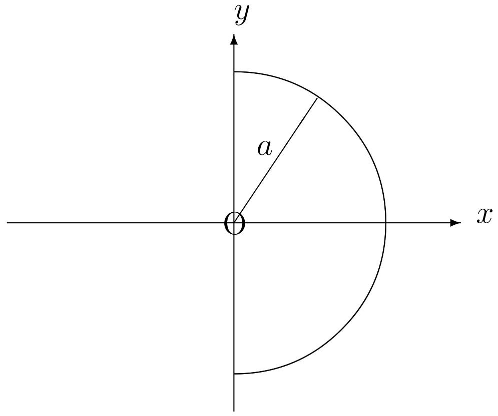
$$\overline{y} = \frac{\frac{1}{2}b^2a}{ab} = \frac{1}{2}b,$$

as we would expect for a rectangle.

2. Determine the position of the centroid of the semi-circular region bounded, in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



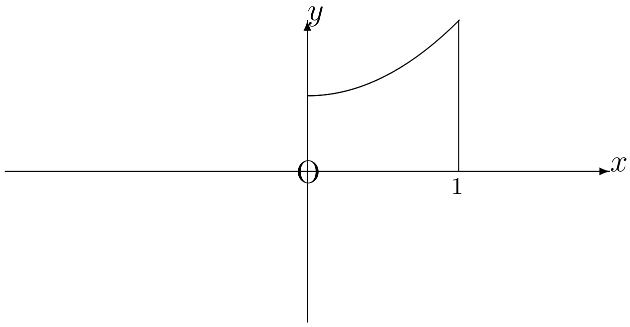
The area of the semi-circular region is $\frac{1}{2}\pi a^2$ and so, from Example 2, in section 13.7.2,

$$\overline{x} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \quad \text{and} \quad \overline{y} = 0.$$

3. Determine the position of the centroid of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



The first moment about the y-axis is given by

$$\int_0^1 x(x^2+1)\,\mathrm{d}x = \left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^1 = \frac{3}{4}.$$

The area is given by

$$\int_0^1 (x^2+1)\,\mathrm{d}x = \left[\frac{x^3}{3}+x\right]_0^1 = \frac{4}{3}.$$

Hence,

$$\overline{x} = \frac{3}{4} \div \frac{4}{3} = 1.$$

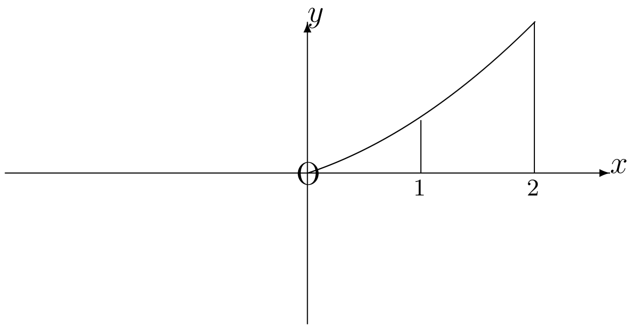
The first moment about the x-axis is $\frac{28}{15}$, from Example 1 in section 13.7.3; and, therefore,

$$\overline{y} = \frac{28}{15} \div \frac{4}{3} = \frac{7}{5}.$$

4. Determine the position of the centroid of the region bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = xe^x.$$

Solution



The first moment about the y-axis is given by

$$\int_1^2 x^2e^x \, \mathrm{d}x = \left[x^2e^x - 2xe^x + 2e^x\right]_1^2 \simeq 12.06,$$

using integration by parts (twice).
The area is given by

$$\int_1^2 xe^x \, \mathrm{d}x = [xe^x - e^x]_1^2 \simeq 7.39$$

using integration by parts (once).
Hence,

$$\overline{x} \simeq 12.06 \div 7.39 \simeq 1.63$$

The first moment about the x-axis is approximately 33.20, from Example 2 in section 13.7.3; and so,

$$\overline{y} \simeq 33.20 \div 7.39 \simeq 4.47$$

13.7.5 EXERCISES

Determine the position of the centroid of each of the following regions of the xy-plane:

1. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$x^2 + y^2 = a^2.$$

2. Bounded by the line $x = 1$ and the semi-circle whose equation is

$$(x-1)^2 + y^2 = 4, \ \ x > 1.$$

3. Bounded in the fourth quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 2x^2 - 1.$$

4. Bounded in the first quadrant by the x-axis and the curve whose equation is

$$y = \sin x.$$

5. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = xe^{-2x}.$$

13.7.6 ANSWERS TO EXERCISES

1.

$$\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right).$$

2.

$$\left(\frac{11}{3\pi}, 0\right).$$

3.

$$\left(\frac{3\sqrt{2}}{16}, -\frac{13}{20}\right).$$

4.

$$\left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

5.

$$(0.28, 0.04).$$

"JUST THE MATHS"

UNIT NUMBER

13.8

INTEGRATION APPLICATIONS 8 (First moments of a volume)

by

A.J.Hobson

UNIT 13.8 - INTEGRATION APPLICATIONS 8

FIRST MOMENTS OF A VOLUME

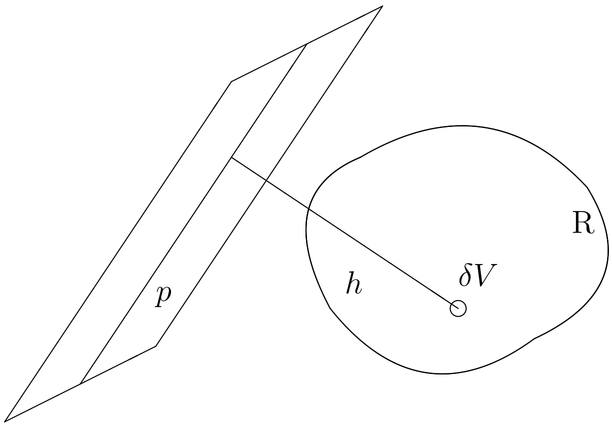
13.8.1 INTRODUCTION

Suppose that R denotes a region of space (with volume V) and suppose that δV is the volume of a small element of this region.

Then the **"first moment"** of R about a fixed plane, p, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h\delta V,$$

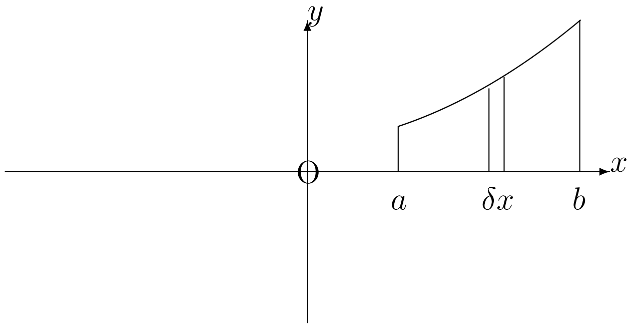
where h is the perpendicular distance, from p, of the element with volume, δV.



13.8.2 FIRST MOMENT OF A VOLUME OF REVOLUTION ABOUT A PLANE THROUGH THE ORIGIN, PERPENDICULAR TO THE X-AXIS.

Let us consider the volume of revolution about the x-axis of a region, in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



For a narrow 'strip' of width, δx, and height, y, parallel to the y-axis, the volume of revolution will be a thin disc with volume $\pi y^2 \delta x$ and all the elements of volume within it have the same perpendicular distance, x, from the plane about which moments are being taken.

Hence the first moment of this disc about the given plane is x times the volume of the disc; that is, $x(\pi y^2 \delta x)$, implying that the total first moment is given by

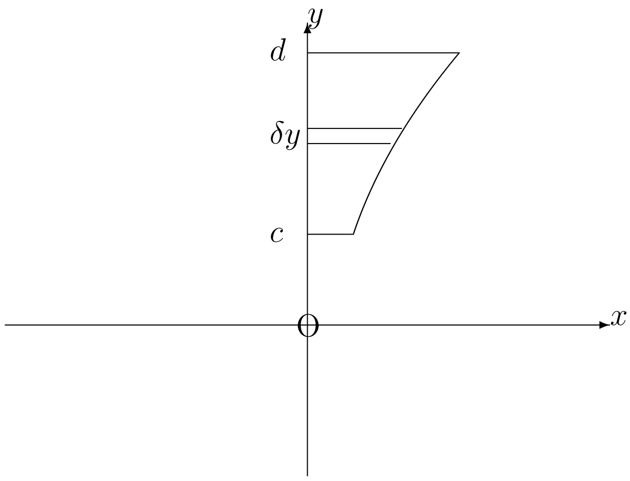
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi x y^2 \delta x = \int_a^b \pi x y^2 \, \mathrm{d}x.$$

Note:
For the volume of revolution about the y-axis of a region in the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment of the volume about a plane through the origin, perpendicular to the y-axis, is given by

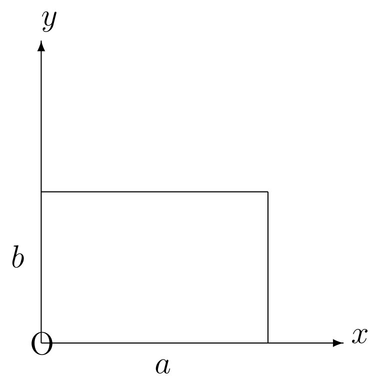
$$\int_c^d \pi y x^2 \, \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a solid right-circular cylinder with height, a and radius b, about one end.

   Solution



Let us consider the volume of revolution about the x-axis of the region, bounded in the first quadrant of the xy-plane, by the x-axis, the y-axis and the lines $x = a$, $y = b$.

The first moment of the volume about a plane through the origin, perpendicular to the x-axis, is given by

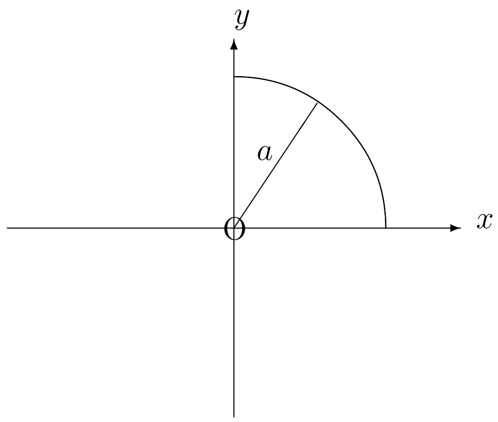
$$\int_0^a \pi x b^2 \ \mathrm{d}x = \left[\frac{\pi x^2 b^2}{2}\right]_0^a = \frac{\pi a^2 b^2}{2}.$$

2. Determine the first moment of volume, about its plane base, of a solid hemisphere with radius a.

   Solution

   Let us consider the volume of revolution about the x-axis of the region, bounded in the first quadrant, by the x-axis, y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$



The first moment of the volume about a plane through the origin, perpendicular to the x-axis is given by

$$\int_0^a \pi x(a^2 - x^2) \ \mathrm{d}x = \left[\pi(\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)\right]_0^a = \pi\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{\pi a^4}{4}.$$

Note:
The symmetry of the solid figures in the above two examples shows that their first moments about a plane through the origin, perpendicular to the y-axis would be zero. This is because, for each $y\delta V$ in the calculation of the total first moment, there will be a corresponding $-y\delta V$.

In much the same way, the first moments of volume about the xy-plane (or indeed any plane of symmetry) would also be zero.

13.8.3 THE CENTROID OF A VOLUME

Suppose R denotes a volume of revolution about the x-axis of a region of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$

Having calculated the first moment of R about a plane through the origin, perpendicular to the x-axis (assuming that this is not a plane of symmetry), it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $V\overline{x}$, where V is the total volume of revolution about the x-axis.

The point is called the **"centroid"** or the **"geometric centre"** of the volume, and $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b \ \pi x y^2 \ \mathrm{d}x}{\int_a^b \ \pi y^2 \mathrm{d}x} = \frac{\int_a^b \ x y^2 \ \mathrm{d}x}{\int_a^b \ y^2 \mathrm{d}x}.$$

Notes:

(i) The centroid effectively tries to concentrate the whole volume at a single point for the purposes of considering first moments. It will always lie on the line of intersection of any two planes of symmetry.

(ii) In practice, the centroid corresponds to the position of the centre of mass for a solid with uniform density, whose shape is that of the volume of revolution which we have been considering.

(iii) For a volume of revolution about the y-axis, from $y = c$ to $y = d$, the centroid will lie on the y-axis, and its distance, $\overline{y}$, from the origin will be given by

$$\overline{y} = \frac{\int_c^d \ \pi y x^2 \ \mathrm{d}y}{\int_c^d \ \pi x^2 \mathrm{d}y} = \frac{\int_c^d \ y x^2 \ \mathrm{d}y}{\int_c^d \ x^2 \mathrm{d}y}.$$

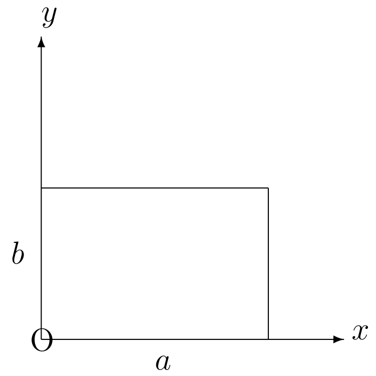
(iv) The first moment of a volume about a plane through its centroid will, by definition, be zero. In particular, if we take the plane through the y-axis, perpendicular to the x-axis to be parallel to the plane through the centroid, with x as the perpendicular distance from an element, δV, to the plane through the y-axis, the first moment about the plane through the centroid will be

$$\sum_{\mathrm{R}} (x - \overline{x})\delta V = \sum_{\mathrm{R}} x\delta V - \overline{x}\sum_{\mathrm{R}} \delta V = V\overline{x} - V\overline{x} = 0.$$

EXAMPLES

1. Determine the position of the centroid of a solid right-circular cylinder with height, a, and radius, b.

   Solution



Using Example 1 in Section 13.8.2, the centroid will lie on the x-axis and the first moment about a plane through the origin, perpendicular to the x-axis is $\frac{\pi a^2 b^2}{2}$.

Also, the volume is $\pi b^2 a$.

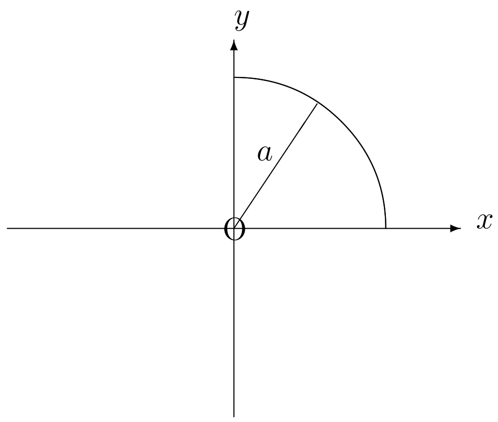
Hence,

$$\overline{x} = \frac{\frac{\pi a^2 b^2}{2}}{\pi b^2 a} = \frac{a}{2},$$

as we would expect for a cylinder.

2. Determine the position of the centroid of a solid hemisphere with base-radius, a.

Solution



Let us consider the volume of revolution about the x-axis of the region bounded in the first quadrant by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2$$

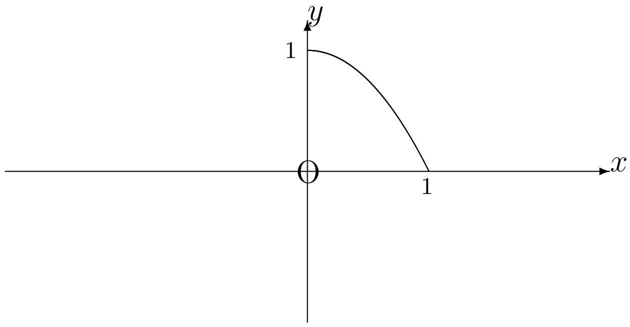
From Example 2 in Section 13.8.2, the centroid will lie on the x-axis and the first moment of volume about a plane through the origin, perpendicular to the x-axis is $\frac{\pi a^4}{4}$. Also, the volume of the hemisphere is $\frac{2}{3}\pi a^3$ and so,

$$\overline{x} = \frac{\frac{2}{3}\pi a^3}{\frac{\pi a^4}{4}} = \frac{3a}{8}.$$

3. Determine the position of the centroid of the volume of revolution about the y-axis of region bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - x^2.$$

Solution



Firstly, by symmetry, the centroid will lie on the y-axis.

Secondly, the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\int_0^1 \pi y(1-y)\ \mathrm{d}y = \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}.$$

Thirdly, the volume is given by

$$\int_0^1 \pi(1-y)\ \mathrm{d}y = \left[y - \frac{y^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

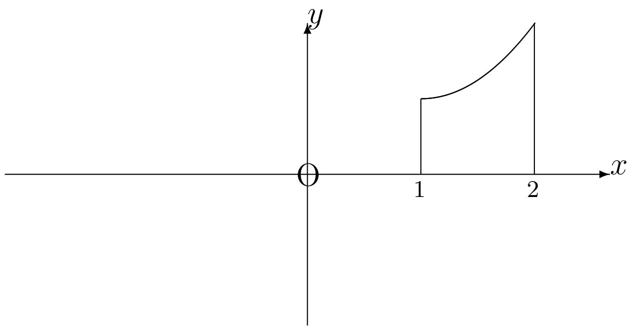
Hence,

$$\overline{y} = \frac{\pi}{6} \div \frac{\pi}{2} = \frac{1}{3}.$$

4. Determine the position of the centroid of the volume of revolution about the x-axis of the region, bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = e^x.$$

Solution



Firstly, by symmetry, the centroid will lie on the x axis.

Secondly, the First Moment about a plane through the origin, perpendicular to the x-axis is given by

$$\int_1^2 \pi x e^{2x}\ \mathrm{d}x = \pi\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]_1^2 \simeq 122.84,$$

using integration by parts.

The volume is given by

$$\int_1^2 \pi e^{2x}\ \mathrm{d}x = \pi\left[\frac{e^{2x}}{2}\right]_1^2 \simeq 74.15$$

Hence,

$$\overline{x} \simeq 122.84 \div 74.15 \simeq 1.66$$

13.8.4 EXERCISES

1. Determine the position of the centroid of the volume obtained when each of the following regions of the xy-plane is rotated through 2π radians about the x-axis:

   (a) Bounded in the first quadrant by the x-axis, the line $x = 1$ and the quarter-circle represented by

   $$(x-1)^2 + y^2 = 4,\ \ x > 1, y > 0;$$

(b) Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - 2x^2;$$

(c) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = \frac{\pi}{2}$ and the curve whose equation is

$$y = \sin x;$$

(d) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = \sqrt{x}e^{-x}.$$

2. A solid right-circular cone, whose vertex is at the origin, has, for its central axis, the part of the y-axis between $y = 0$ and $y = h$. Determine the position of the centroid of the cone.

13.8.5 ANSWERS TO EXERCISES

1. (a)

$$\overline{x} = 1.75;$$

(b)

$$\overline{x} \simeq 0.22;$$

(c)

$$\overline{x} \simeq 1.10;$$

(d)

$$\overline{x} \simeq 0.36$$

2.

$$\overline{y} = \frac{3h}{4}.$$

"JUST THE MATHS"

UNIT NUMBER

13.9

INTEGRATION APPLICATIONS 9
(First moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.9 - INTEGRATION APPLICATIONS 9

FIRST MOMENTS OF A SURFACE OF REVOLUTION

13.9.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"first moment"** about a plane through the origin, perpendicular to the x-axis, is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} 2\pi x y \delta s,$$

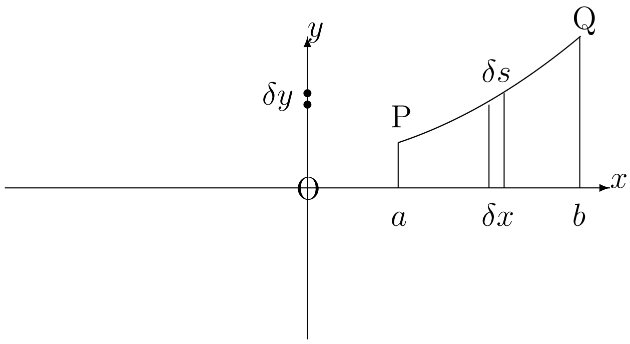
where x is the perpendicular distance, from the plane of moments, of the thin band, with surface area $2\pi y \delta s$, so generated.

13.9.2 INTEGRATION FORMULAE FOR FIRST MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x;$$

so that, for the surface of revolution of the arc about the x-axis, the first moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi xy\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b 2\pi xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the first moment about the plane through the origin, perpendicular to the x-axis is given by

$$\text{First Moment} = \pm\int_{t_1}^{t_2} 2\pi xy\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

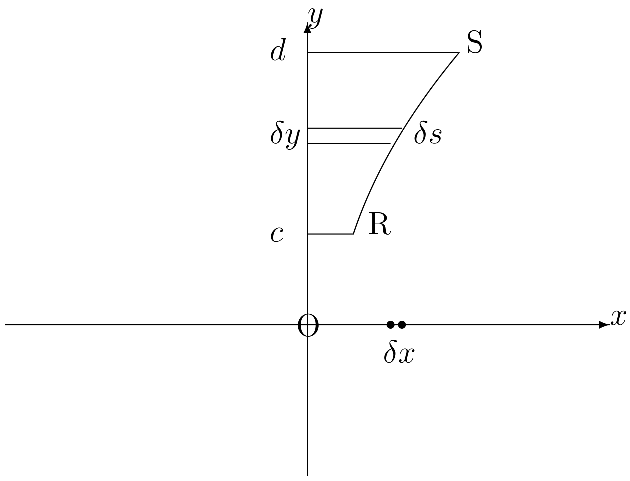
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\int_c^d 2\pi yx\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the first moment about a plane through the origin, perpendicular to the y-axis, is given by

$$\text{First moment} = \pm \int_{t_1}^{t_2} 2\pi y x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

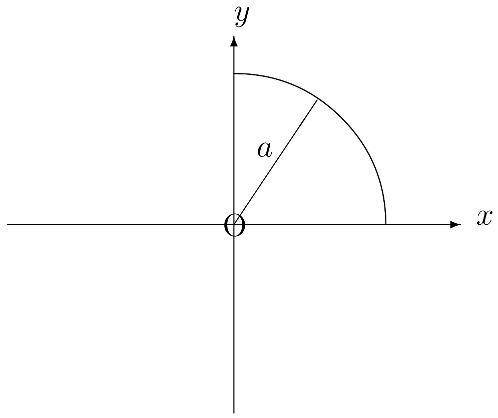
EXAMPLES

1. Determine the first moment about a plane through the origin, perpendicular to the x-axis, for the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.
The first moment about the specified plane is therefore given by

$$\int_0^a 2\pi xy\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi xy\sqrt{\frac{x^2 + y^2}{y^2}}\ \mathrm{d}x.$$

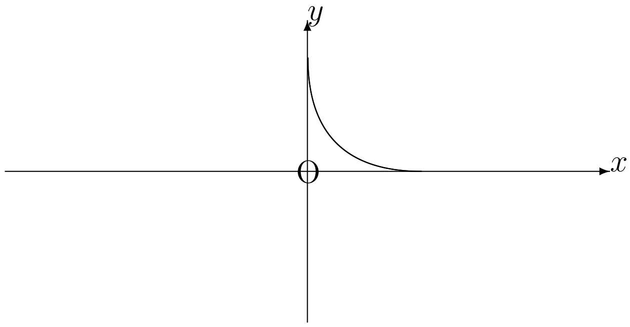
But $x^2 + y^2 = a^2$, and so the first moment becomes

$$\int_0^a 2\pi ax\ \mathrm{d}x = [\pi ax^2]_0^a = \pi a^3.$$

2. Determine the first moments about planes through the origin, (a) perpendicular to the x-axis and (b) perpendicular to the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \;\text{ and }\; \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi xy\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^2\cos^3\theta\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 6\pi a^3\cos^4\theta\sin^4\theta\ \mathrm{d}\theta.$$

Using $2\sin\theta\cos\theta \equiv \sin 2\theta$, the integral reduces to

$$\frac{3\pi a^3}{8}\int_0^{\frac{\pi}{2}} \sin^4 2\theta\ \mathrm{d}\theta,$$

which, by the methods of Unit 12.7, gives

$$\frac{3\pi a^3}{32}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right)\ \mathrm{d}\theta = \frac{3\pi a^3}{32}\left[\frac{3\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16}\right]_0^{\frac{\pi}{2}} = \frac{9\pi a^3}{128}.$$

By symmetry, or by direct integration, the first moment about a plane through the origin, perpendicular to the y-axis is also $\frac{9\pi a^3}{128}$.

13.9.3 THE CENTROID OF A SURFACE OF REVOLUTION

Having calculated the first moment of a surface of revolution about a plane through the origin, perpendicular to the x-axis, it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $S\overline{x}$, where S is the total surface area.

The point is called the **"centroid"** or the **"geometric centre"** of the surface of revolution and, for the surface of revolution of the arc of the curve whose equation is $y = f(x)$, between $x = a$ and $x = b$, the value of $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b 2\pi xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}{\int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} = \frac{\int_a^b xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}.$$

Note:
The centroid effectively tries to concentrate the whole surface at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass of a thin sheet, for example, with uniform density.

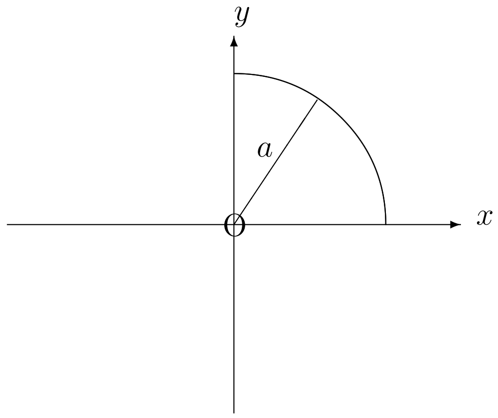
EXAMPLES

1. Determine the position of the centroid of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 of Section 13.9.2, we know that the first moment of the surface about a plane through the origin, perpendicular to the the x-axis is equal to πa^3.

Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2,$$

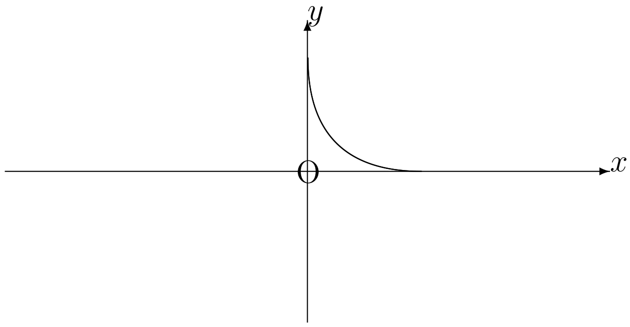
which implies that

$$\overline{x} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}.$$

2. Determine the position of the centroid of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 of Section 13.9.2 that that the first moment of the surface about a plane through the origin, perpendicular to the x-axis is equal to $\frac{9\pi a^3}{128}$.
Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a \sin^3\theta .3a\cos\theta\sin\theta \, \mathrm{d}\theta = \int_{0}^{\frac{\pi}{2}} 3a^2\sin^4\theta\cos\theta \, \mathrm{d}\theta = 3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$\overline{x} = \frac{15a}{128}.$$

13.9.4 EXERCISES

1. Determine the first moment, about a plane through the origin, perpendicular to the x-axis, of the surface of revolution (about the x-axis) of the straight-line segment joining the origin to the point $(3, 4)$.

2. Determine the first moment about a plane through the origin, perpendicular to the x-axis, of the surface of revolution (about the x-axis) of the arc of the curve whose equation is

$$y^2 = 4x,$$

lying between $x = 0$ and $x = 1$.

3. Determine the first moment about a plane through the origin, perpendicular to the y-axis, of the surface of revolution (about the y-axis) of the arc of the curve whose equation is

$$y^2 = 4(x - 1),$$

lying between $y = 2$ and $y = 4$.

4. Determine the first moment, about a plane through the origin, perpendicular to the y-axis, of the surface of revolution (about the y-axis) of the arc of the curve whose parametricc equations are

$$x = 2\cos t, \quad y = 3\sin t,$$

joining the point $(2, 0)$ to the point $(0, 3)$.

5. Determine the position of the centroid of a hollow right-circular cone with height h.

6. For the curve whose equation is

$$9y^2 = x(3 - x)^2,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$

Hence, show that the centroid of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x-axis lies at the point $\left(\frac{5}{4}, 0\right)$.

13.9.5 ANSWERS TO EXERCISES

1.

$$40\pi.$$

2.

$$4\pi\left[\frac{12\sqrt{2}}{5}-\frac{4}{15}\right]\simeq 39.3$$

3.

$$\left[\frac{8\pi}{5}\left(1+\frac{y^2}{4}\right)^{\frac{5}{2}}\right]_2^4\simeq 41.98$$

4.

$$\left[-\frac{4\pi}{5}\left(4+5\cos^2 t\right)^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}}\simeq 47.75$$

5. Along the central axis, at a distance of $\frac{2h}{3}$ from the vertex.
6.

$$\text{First moment}=\frac{15\pi}{4}\quad\text{Surface Area}=3\pi.$$

"JUST THE MATHS"

UNIT NUMBER

13.10

INTEGRATION APPLICATIONS 10
(Second moments of an arc)

by

A.J.Hobson

UNIT 13.10 - INTEGRATION APPLICATIONS 10

SECOND MOMENTS OF AN ARC

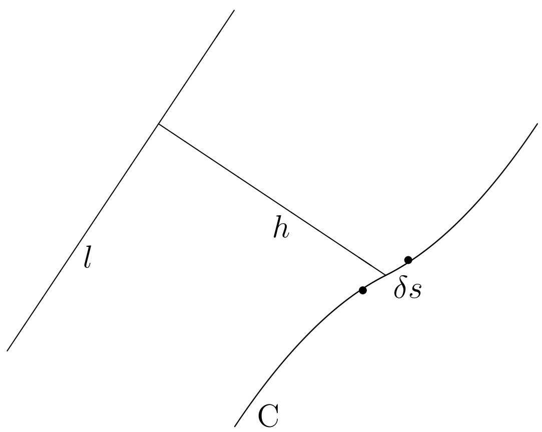
13.10.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then the **"second moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} h^2 \delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

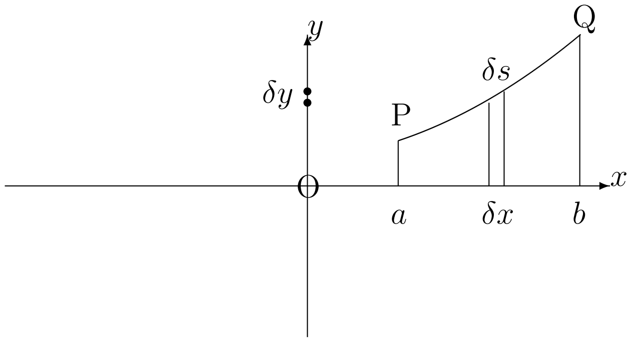


13.10.2 THE SECOND MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The second moment of each element about the y-axis is x^2 times the length of the element; that is, $x^2\delta s$, implying that the total second moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} x^2 \delta s.$$

But, by Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

so that the second moment of arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b x^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the second moment of the arc about the y-axis is given by

$$\pm \int_{t_1}^{t_2} x^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.10.3 THE SECOND MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the second moment about the x-axis will be

$$\int_a^b y^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the second moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

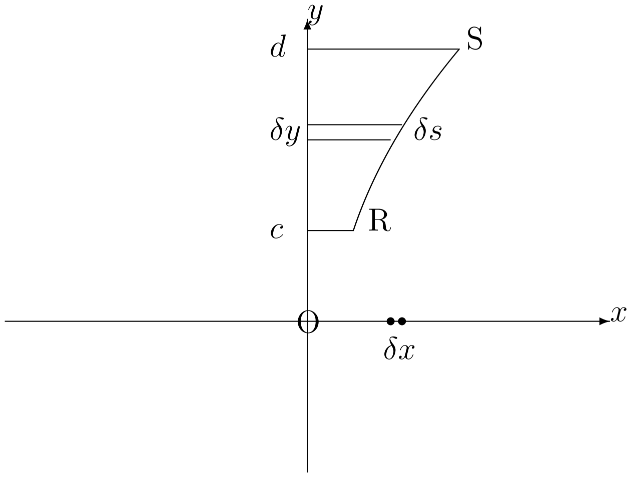
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.10.2 so that the second moment about the x-axis is given by

$$\int_c^d y^2 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \, \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the second moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

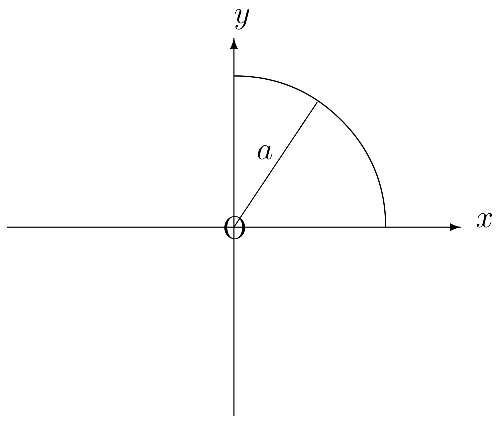
EXAMPLES

1. Determine the second moments about the x-axis and the y-axis of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The second moment about the y-axis is therefore given by

$$\int_0^a x^2\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a \frac{x^2}{y}\sqrt{x^2 + y^2}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$ and, hence,

$$\text{second moment} = \int_0^a \frac{ax^2}{y}\mathrm{d}x.$$

Making the substitution $x = a\sin u$ gives

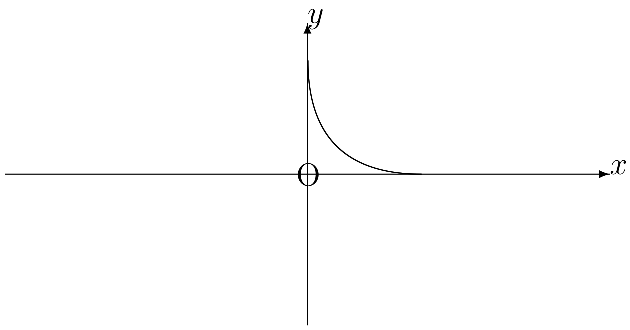
$$\text{second moment} = \int_0^{\frac{\pi}{2}} a^3\sin^2 u \; \mathrm{d}u = a^3 \int_0^{\frac{\pi}{2}} \frac{1-\cos 2u}{2} \; \mathrm{d}u = a^3 \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^3}{4}.$$

By symmetry, the second moment about the x-axis will also be $\frac{\pi a^3}{4}$.

2. Determine the second moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta \sin\theta \;\text{ and }\; \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the second moment about the y-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} x^2 \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta} \; \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a^2\cos^6\theta.3a\cos\theta\sin\theta \; \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \cos^7\theta \sin\theta \, \mathrm{d}\theta$$

$$= 3a^2 \left[-\frac{\cos^8\theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8}.$$

Similarly, the second moment about the x-axis is given by

$$\int_0^{\frac{\pi}{2}} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} a^2 \sin^6\theta.(3a\cos\theta\sin\theta) \, \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \sin^7\theta \cos\theta \, \mathrm{d}\theta = 3a^3 \left[\frac{\sin^8\theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8},$$

though, again, this second result could be deduced, by symmetry, from the first.

13.10.4 THE RADIUS OF GYRATION OF AN ARC

Having calculated the second moment of an arc about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by sk^2, where s is the total length of the arc.

We simply divide the value of the second moment by s in order to obtain the value of k^2 and, hence, the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole arc at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

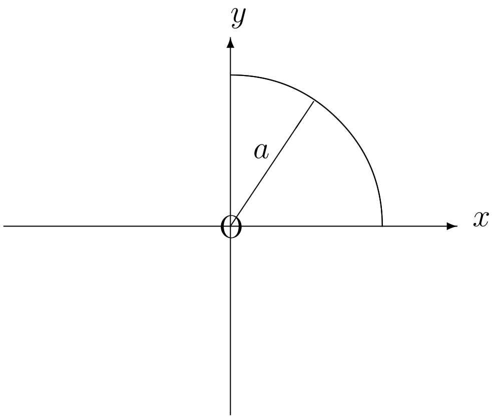
EXAMPLES

1. Determine the radius of gyration, about the y-axis, of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 in Section 13.10.3, we know that the Second Moment of the arc about the y-axis is equal to $\frac{\pi a^3}{4}$.

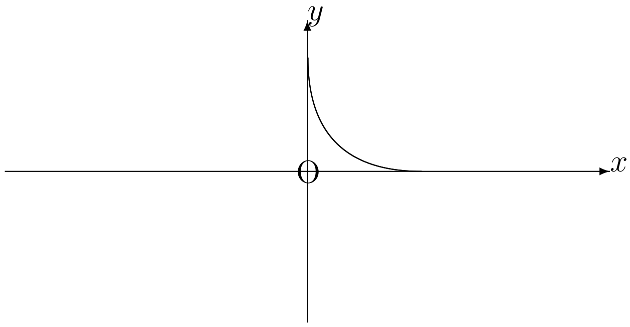
Also, the length of the arc is $\frac{\pi a}{2}$, which implies that the radius of gyration is

$$\sqrt{\frac{\pi a^3}{4} \times \frac{2}{\pi a}} = \frac{a}{\sqrt{2}}.$$

2. Determine the radius of gyration, about the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in Section 13.10.3, we know that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

and that the second moment of the arc about the y-axis is equal to $\frac{3a^3}{8}$. Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus, the radius of gyration is

$$\sqrt{\frac{3a^3}{8}\times\frac{2}{3a}} = \frac{a}{2}.$$

13.10.5 EXERCISES

1. Determine the second moments about (a) the x-axis and (b) the y-axis of the straight line segment with equation

$$y = 2x + 1,$$

lying between $x = 0$ and $x = 3$.

2. Determine the second moment about the y-axis of the first-quadrant arc of the curve whose equation is

$$25y^2 = 4x^5,$$

lying between $x = 0$ and $x = 2$.

3. Determine, correct to two places of decimals, the second moment, about the x-axis, of the arc of the curve whose equation is

$$y = e^x,$$

lying between $x = 0.1$ and $x = 0.5$.

4. Given that

$$\int \sqrt{x^2 - 1}\ \mathrm{d}x = \frac{1}{2}\left(x\sqrt{x^2 - 1} - \ln(x + \sqrt{x^2 - 1})\right) + C,$$

determine, correct to two places of decimals, the second moment, about the x-axis, of the arc of the curve whose equation is

$$y^2 = 8x,$$

lying between $x = 0$ and $x = 1$.

5. Verify, using integration, that the radius of gyration, about the y-axis, of the straight line segment defined by the equation

$$y = 3x + 2,$$

from $x = 0$ to $x = 1$ is $\frac{1}{\sqrt{3}}$.

6. Determine the radius of gyration about the x-axis of the arc of the circle given parametrically by

$$x = 5\cos\theta, \quad y = 5\sin\theta,$$

from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

7. For the curve whose equation is

$$9y^2 = x(3-x)^2,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-x}{2\sqrt{x}}.$$

Hence, determine, correct to three significant figures, the radius of gyration, about the y-axis, of the first quadrant arch of this curve.

13.10.6 ANSWERS TO EXERCISES

1.

$$\text{(a)} \ \frac{9\sqrt{5}}{2} \quad \text{(b)} \ 12\sqrt{5}.$$

2.

$$\frac{52}{9} \simeq 5.78$$

3.

$$1.29$$

4.

$$8.59$$

5.

$$\text{Second moment} = \frac{\sqrt{10}}{3} \quad \text{Length} = \sqrt{10}.$$

6.

$$k = \sqrt{\frac{125(\pi-2)}{10\pi}} \simeq 2.13$$

7.

$$k \simeq 1.68$$

"JUST THE MATHS"

UNIT NUMBER

13.11

INTEGRATION APPLICATIONS 11
(Second moments of an area (A))

by

A.J.Hobson

UNIT 13.11 - INTEGRATION APPLICATIONS 11

SECOND MOMENTS OF AN AREA (A)

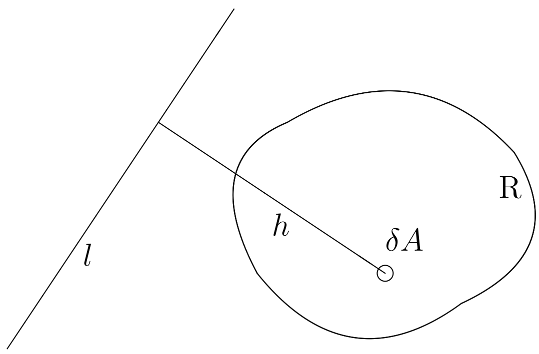
13.11.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy-plane in cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, **not necessarily in the plane of** R, is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h^2 \delta A,$$

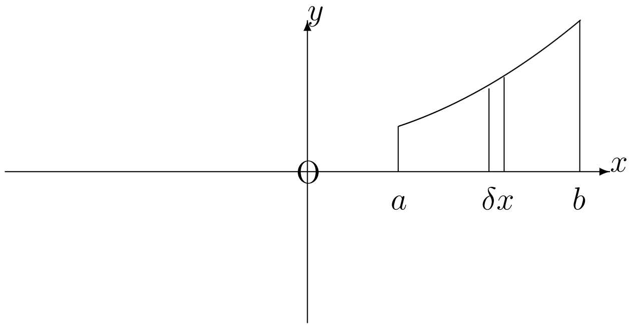
where h is the perpendicular distance from l of the element with area, δA.



13.11.2 THE SECOND MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may be divided up into small elements by using a network consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

But all of the elements in a narrow 'strip', of width δx and height y (parallel to the y-axis), have the same perpendicular distance, x, from the y-axis.

Hence the second moment of this strip about the y-axis is x^2 times the area of the strip; that is, $x^2(y\delta x)$, implying that the total second moment of the region about the y-axis is given by

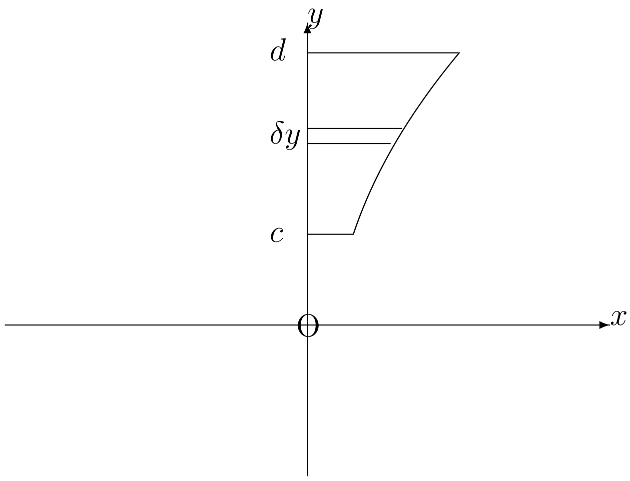
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 y \delta x = \int_a^b x^2 y \ \mathrm{d}x.$$

Note:
Second moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

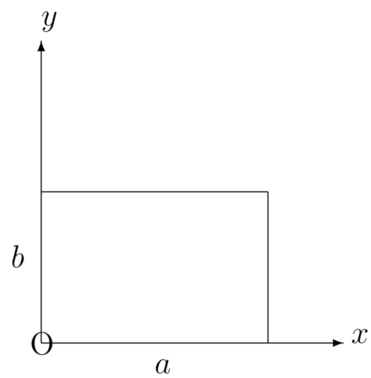
$$\int_c^d y^2 x \ \mathrm{d}y.$$



EXAMPLES

1. Determine the second moment of a rectangular region with sides of lengths, a and b, about the side of length b.

   Solution



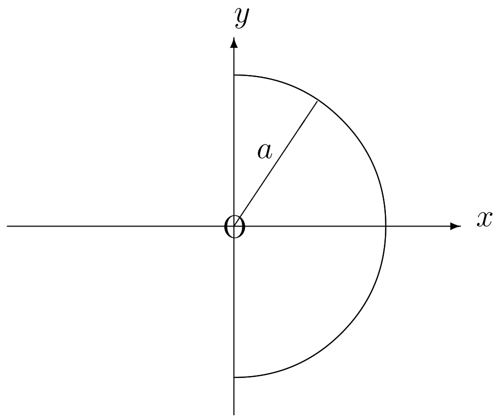
The second moment about the y-axis is given by

$$\int_0^a x^2 b \, \mathrm{d}x = \left[\frac{x^3 b}{3}\right]_0^a = \frac{1}{3}a^3 b.$$

2. Determine the second moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the second moment about the y-axis is given by

$$2\int_0^a x^2\sqrt{a^2 - x^2}\ \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} a^2\sin^2\theta . a\cos\theta . a\cos\theta \mathrm{d}\theta,$$

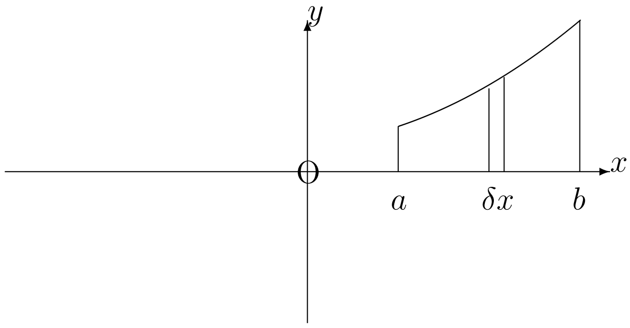
if we substitute $x = a\sin\theta$.
This simplifies to

$$2a^4\int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4}\ \mathrm{d}\theta = \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2}\ \mathrm{d}\theta$$

$$= \frac{a^4}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{8}.$$

13.11.3 THE SECOND MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the second moment of a rectangular region about one of its sides. This result may now be used to determine the second moment about the x-axis, of a region enclosed, in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is $y = f(x)$.



If a narrow strip of width δx and height y is regarded, approximately, as a rectangle, its second moment about the x-axis is $\frac{1}{3}y^3\delta x$. Hence the second moment of the whole region about the x-axis is given by

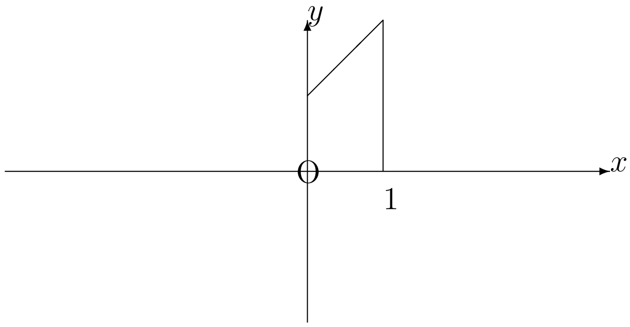
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{3}y^3\delta x = \int_a^b \frac{1}{3}y^3 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the second moment about the x-axis of the region bounded, in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment } = \int_0^1 \frac{1}{3}(x+1)^3 \, \mathrm{d}x$$

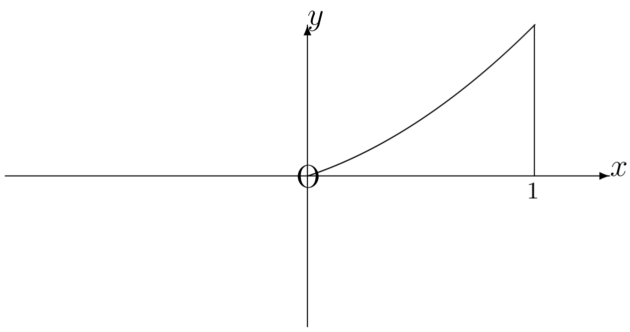
$$= \frac{1}{3}\int_0^1 (x^3 + 3x^2 + 3x + 1) \, \mathrm{d}x = \frac{1}{3}\left[\frac{x^4}{4} + x^3 + + \frac{3x^2}{2} + x\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{4} + 1 + \frac{3}{2} + 1\right) = \frac{5}{4}.$$

2. Determine the second moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve

$$y = xe^x.$$

Solution



$$\text{Second moment } = \int_0^1 \frac{1}{3}x^3 e^{3x}\,\mathrm{d}x$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \int_0^1 x^2 e^{3x}\,\mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \int_0^1 2x\frac{e^{3x}}{3}\,\mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \frac{2xe^{3x}}{9} - \frac{2}{3}\int_0^1 \frac{e^{3x}}{3}\,\mathrm{d}x\right).$$

That is,

$$\frac{1}{3}\left[x^3\frac{e^{3x}}{3} - x^2\frac{e^{3x}}{3} + \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27}\right]_0^1 = \frac{4e^3+2}{81} \simeq 1.02$$

Note:
The Second Moment of an area about a certain axis is closely related to its **"moment of inertia"** about that axis. In fact, for a thin plate with uniform density, ρ, the moment of inertia is ρ times the second moment of area, since multiplication by ρ, of elements of area, converts them into elements of mass.

13.11.7 EXERCISES

Determine the second moment of each of the following regions of the xy-plane about the axis specified:

1. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
$$y = 1 - 2x^2.$$
Axis: The y-axis.
2. Bounded in the first quadrant by the x-axis and the curve whose equation is
$$y = \sin x.$$
Axis: The x-axis.
3. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is
$$y = e^{-2x}.$$
Axis: The x-axis
4. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is
$$y = e^{-2x}.$$
Axis: The y-axis.

13.11.8 ANSWERS TO EXERCISES

1. $$\frac{\sqrt{2}}{30}.$$
2. $$\frac{4}{9}.$$
3. $$0.055, \text{ approximately.}$$
4. $$0.083, \text{ approximately.}$$

"JUST THE MATHS"

UNIT NUMBER

13.12

INTEGRATION APPLICATIONS 12
(Second moments of an area (B))

by

A.J.Hobson

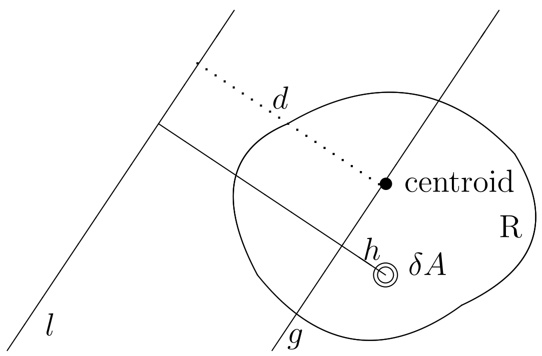
UNIT 13.12 - INTEGRATION APPLICATIONS 12

SECOND MOMENTS OF AN AREA (B)

13.12.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis, in the same plane as R and having a perpendicular distance of d from the first axis.



We have

$$M_l = \sum_{\mathrm{R}} (h+d)^2 \delta A = \sum_{\mathrm{R}} (h^2 + 2hd + d^2).$$

That is,

$$M_l = \sum_{\mathrm{R}} h^2 \delta A + 2d \sum_{\mathrm{R}} h\delta A + d^2 \sum_{\mathrm{R}} \delta A = M_g + Ad^2,$$

since the summation, $\sum_{\mathrm{R}} h\delta A$, is the first moment about the an axis through the centroid and therefore zero; (see Unit 13.7, section 13.7.4).

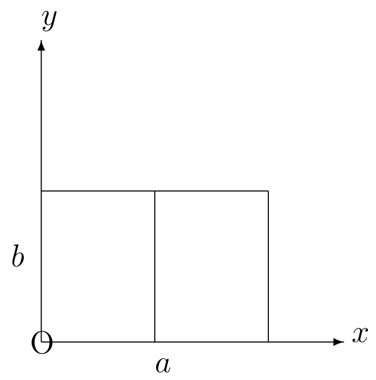
The Parallel Axis Theorem states that

$$M_l = M_g + Ad^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region about an axis through its centroid, parallel to one side.

   Solution



   For a rectangular region with sides of length a and b, the second moment about the side of length b is $\frac{a^3b}{3}$ from Example 1 in the previous Unit, section 13.11.2.

   The perpendicular distance between the two axes is then $\frac{a}{2}$, so that the required second moment, M_g is given by

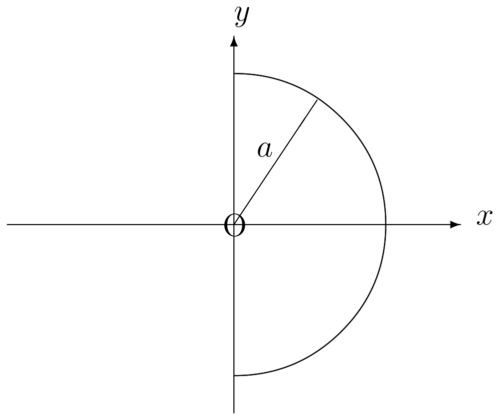
$$\frac{a^3b}{3} = M_g + ab\left(\frac{a}{2}\right)^2 = M_g + \frac{a^3b}{4}$$

   Hence,

$$M_g = \frac{a^3b}{12}.$$

2. Determine the second moment of a semi-circular region about an axis through its centroid, parallel to its diameter.

Solution



The second moment of the semi-circular region about its diameter is $\frac{\pi a^4}{8}$, from Example 2 in the previous Unit, section 13.11.2.

Also the position of the centroid, from Example 2 in Unit 13.7, section 13.7.4, is a distance of $\frac{4a}{3\pi}$ from the diameter, along the radius which perpendicular to it.

Hence,

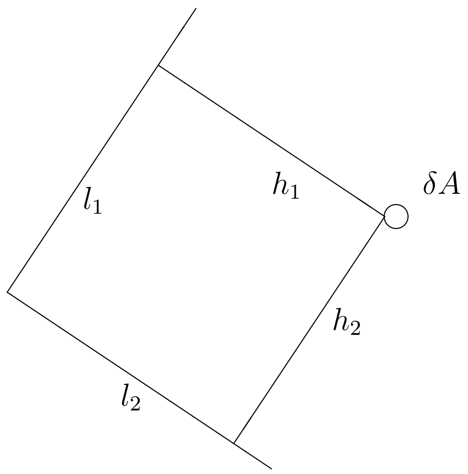
$$\frac{\pi a^4}{8} = M_g + \frac{\pi a^2}{2}.\left(\frac{4a}{3\pi}\right)^2 = M_g + \frac{8a^4}{9\pi^2}.$$

That is,

$$M_g = \frac{\pi a^4}{8} - \frac{8a^4}{9\pi^2}.$$

13.12.2 THE PERPENDICULAR AXIS THEOREM

Suppose l_1 and l_2 are two straight lines, at right-angles to each other, in the plane of a region R with area A and suppose h_1 and h_2 are the perpendicular distances from these two lines, respectively, of an element δA in R.



The second moment about l_1 is given by

$$M_1 = \sum_R h_1^2 \delta A$$

and the second moment about l_2 is given by

$$M_2 = \sum_R h_2^2 \delta A.$$

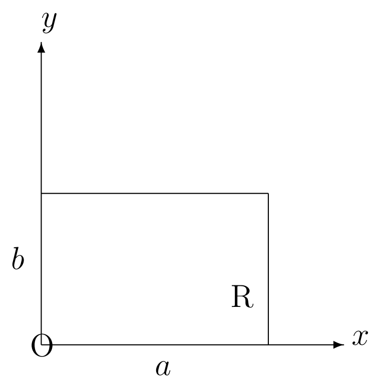
Adding these two together gives the second moment about an axis, perpendicular to the plane of R and passing through the point of intersection of l_1 and l_2. This is because the square of the perpendicular distance, h_3, of δA from this new axis is given, from Pythagoras's Theorem, by

$$h_3^2 = h_1^2 + h_2^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region, R, with sides of length a and b, about an axis through one corner, perpendicular to the plane of R.

Solution

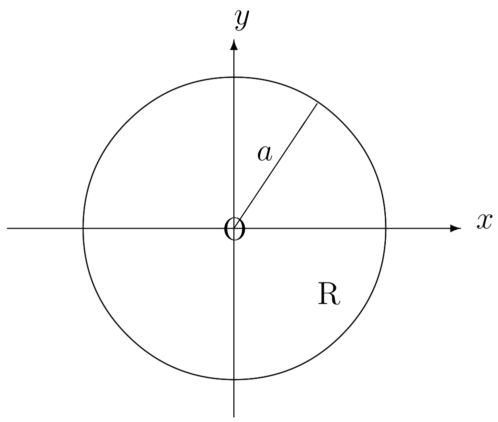


Using Example 1 in the previous Unit, section 13.11.2, the required second moment is

$$\frac{1}{3}a^3b + \frac{1}{3}b^3a = \frac{1}{3}ab(a^2 + b^2).$$

2. Determine the second moment of a circular region, R, with radius a, about an axis through its centre, perpendicular to the plane of R.

Solution



The second moment of R about a diameter is, from Example 2 in the previous Unit, section 13.11.2, equal to $\frac{\pi a^4}{4}$; that is, twice the value of the second moment of a semi-circular region about its diameter.

The required second moment is thus

$$\frac{\pi a^4}{4} + \frac{\pi a^4}{4} = \frac{\pi a^4}{2}.$$

13.12.3 THE RADIUS OF GYRATION OF AN AREA

Having calculated the second moment of a two dimensional region about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Ak^2, where A is the total area of the region.

We simply divide the value of the second moment by A in order to obtain the value of k^2 and hence the value of k.

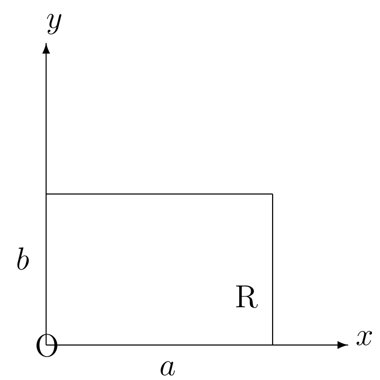
The value of k is called the **"radius of gyration"** of the given region about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole area at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a rectangular region, R, with sides of lengths a and b about an axis through one corner, perpendicular to the plane of R.

   Solution



Using Example 1 from the previous section, the second moment is

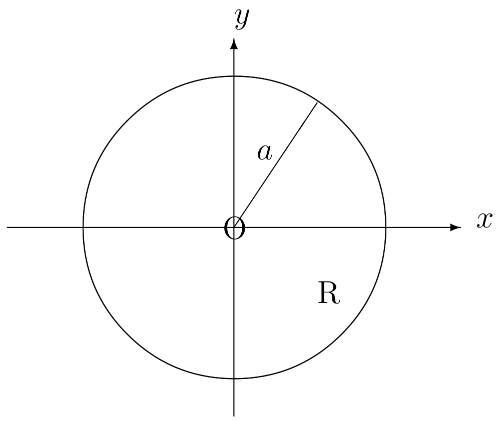
$$\frac{1}{3}ab(a^2+b^2)$$

and, since the area itself is ab, we obtain

$$k=\sqrt{a^2+b^2}.$$

2. Determine the radius of gyration of a circular region, R, about an axis through its centre, perpendicular to the plane of R.

Solution



From Example 2 in the previous section, the second moment about the given axis is $\frac{\pi a^4}{2}$ and, since the area itself is πa^2, we obtain

$$k=\frac{a}{\sqrt{2}}.$$

13.12.4 EXERCISES

Determine the radius of gyration of each of the following regions of the xy-plane about the axis specified:

1. Bounded in the first quadrant by the x-axis, the y-axis and the lines $x = a$, $y = b$. Axis: Through the point $\left(\frac{a}{2}, \frac{b}{2}\right)$, perpendicular to the xy-plane.
2. Bounded in the first quadrant by the x-axis, the y-axis and the lines $x = a$, $y = b$. Axis: The line $x = 4a$.
3. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
$$x^2 + y^2 = a^2.$$
Axis: Through the origin, perpendicular to the xy-plane.
4. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
$$x^2 + y^2 = a^2.$$
Axis: The line $x = a$.

13.12.5 ANSWERS TO EXERCISES

1.
$$\frac{1}{12}\left(a^2 + b^2\right).$$

2.
$$\frac{7a}{\sqrt{3}}.$$

3.
$$\frac{a}{\sqrt{2}}.$$

4.
$$\frac{a\sqrt{5}}{2}.$$

"JUST THE MATHS"

UNIT NUMBER

13.13

INTEGRATION APPLICATIONS 13
(Second moments of a volume (A))

by

A.J.Hobson

UNIT 13.13 - INTEGRATION APPLICATIONS 13

SECOND MOMENTS OF A VOLUME (A)

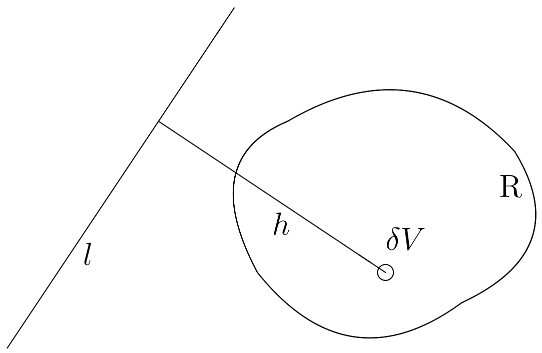
13.13.1 INTRODUCTION

Suppose that R denotes a region (with volume V) in space and suppose that δV is the volume of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h^2 \delta V,$$

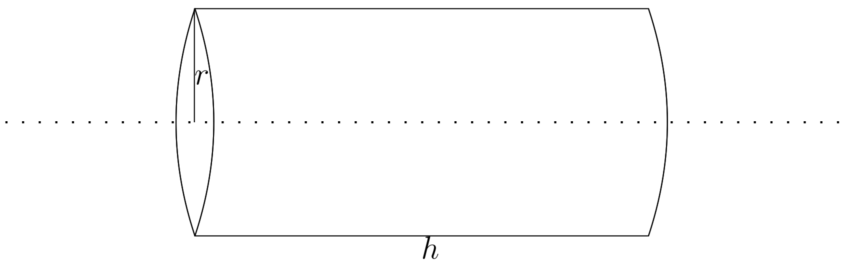
where h is the perpendicular distance from l of the element with volume δV.



EXAMPLE

Determine the second moment, about its own axis, of a solid right-circular cylinder with height, h, and radius, a.

Solution



In a thin cylindrical shell with internal radius, r, and thickness, δr, all of the elements of volume have the same perpendicular distance, r, from the axis of moments.

Hence the second moment of this shell will be the product of its volume and r^2; that is, $r^2(2\pi r h \delta r)$.

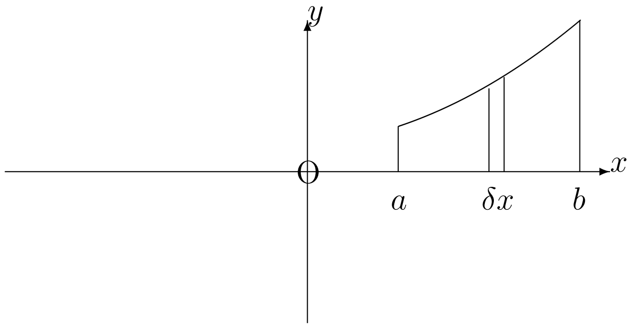
The total second moment is therefore given by

$$\lim_{\delta r \to 0} \sum_{r=0}^{r=a} r^2(2\pi r h \delta r) = \int_0^a 2\pi h r^3 \, \mathrm{d}r = \frac{\pi a^4 h}{2}.$$

13.13.2 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution of a narrow 'strip', of width, δx, and height, y, (parallel to the y-axis), is a cylindrical 'shell', of internal radius x, height, y, and thickness, δx.

Hence, from the example in the previous section, its second moment about the y-axis is $2\pi x^3 y \delta x$.

Thus, the total second moment about the y-axis is given by

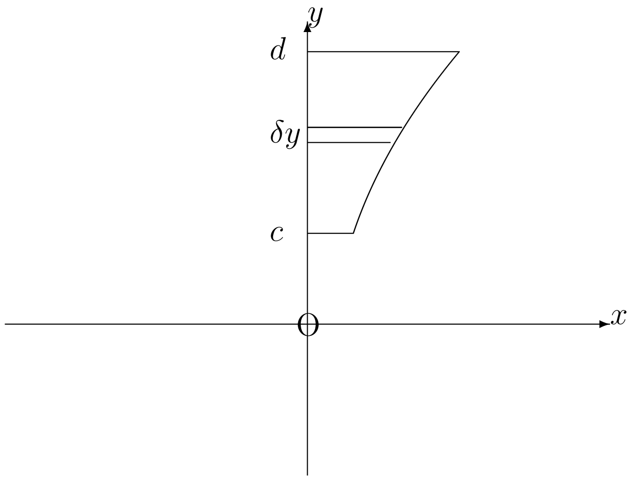
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x^3 y \delta x = \int_a^b 2\pi x^3 y \, \mathrm{d}x.$$

Note:
Second moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for the volume of revolution, about the x-axis, of a region in the first quadrant bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

$$\int_c^d 2\pi y^3 x \, \mathrm{d}y.$$



EXAMPLE

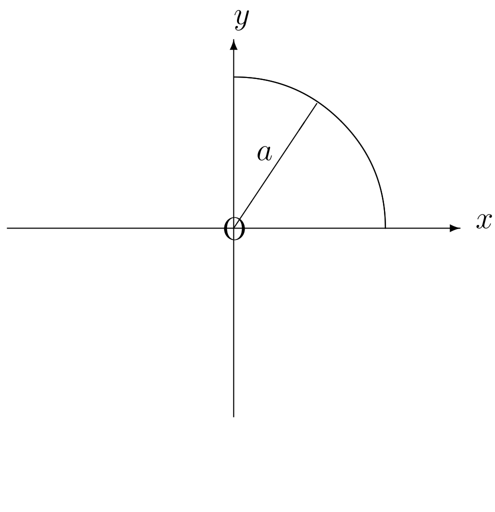
Determine the second moment, about a diameter, of a solid sphere with radius a.

Solution

We may consider, first, the volume of revolution about the y-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2,$$

then double the result obtained.



The total second moment is given by

$$2\int_0^a 2\pi x^3\sqrt{a^2-x^2}\ \mathrm{d}x = 4\pi\int_0^{\frac{\pi}{2}} a^3\sin^3\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

This simplifies to

$$4\pi a^5\int_0^{\frac{\pi}{2}} \sin^3\theta\ \cos^2\theta\mathrm{d}\theta = 4\pi\int_0^{\frac{\pi}{2}} \left(\cos^2\theta - \cos^4\theta\right)\sin\theta\mathrm{d}\theta,$$

if we make use of the trigonometric identity

$$\sin^2\theta \equiv 1 - \cos^2\theta.$$

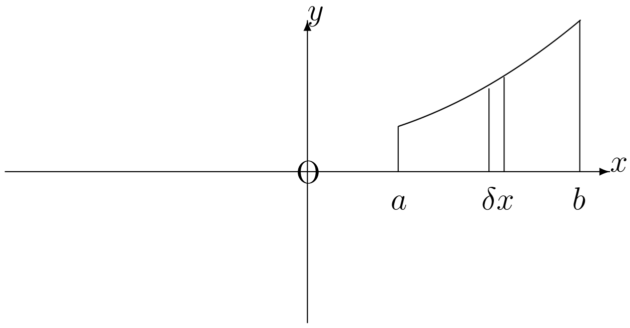
The total second moment is now given by

$$4\pi a^5\left[-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}} = 4\pi a^5\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8\pi a^5}{15}.$$

13.13.3 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE X-AXIS

In the introduction to this Unit, a formula was established for the second moment of a solid right-circular cylinder about its own axis. This result may now be used to determine the second moment about the x-axis for the volume of revolution about this axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution about the x-axis of a narrow strip, of width δx and height y, is a cylindrical 'disc' whose second moment about the x-axis is $\frac{\pi y^4 \delta x}{2}$. Hence the second moment of the whole region about the x-axis is given by

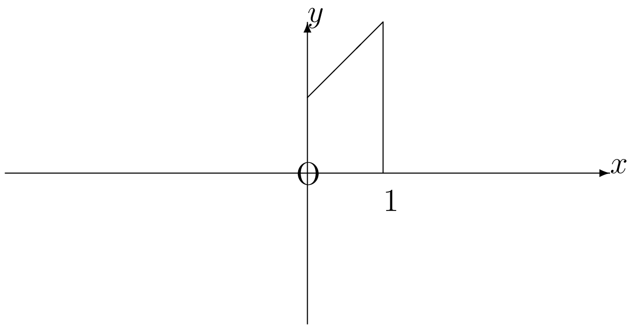
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{\pi y^4}{2} \delta x = \int_a^b \frac{\pi y^4}{2} \, \mathrm{d}x.$$

EXAMPLE

Determine the second moment about the x-axis for the volume of revolution about this axis of the region bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{\pi(x+1)^4}{2}\,\mathrm{d}x = \left[\pi\frac{(x+1)^4}{10}\right]_0^1 = \frac{31\pi}{10}.$$

Note:
The second moment of a volume about a certain axis is closely related to its **"moment of inertia"** about that axis. In fact, for a solid, with uniform density, ρ, the Moment of Inertia is ρ times the second moment of volume, since multiplication by ρ of elements of volume converts them into elements of mass.

13.13.4 EXERCISES

1. Determine the second moment about a diameter of a circular disc with small thickness, t, and radius, a.
2. Determine the second moment, about the axis specified, for the volume of revolution of each of the following regions of the xy-plane about this axis:
   (a) Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$y = 1 - 2x^2.$$
   Axis: The y-axis.
   (b) Bounded in the first quadrant by the x-axis and the curve whose equation is
   $$y^2 = \sin x.$$
   Axis: The x-axis.

(c) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The x-axis

(d) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The y-axis.

13.13.5 ANSWERS TO EXERCISES

1.

$$\frac{\pi a^4 t}{4}.$$

2. (a)

$$\frac{\pi}{24}.$$

(b)

$$\frac{\pi^2}{4}.$$

(c)

$$0.196, \text{ approximately.}$$

(d)

$$0.337, \text{ approximately.}$$

"JUST THE MATHS"

UNIT NUMBER

13.14

INTEGRATION APPLICATIONS 14
(Second moments of a volume (B))

by

A.J.Hobson

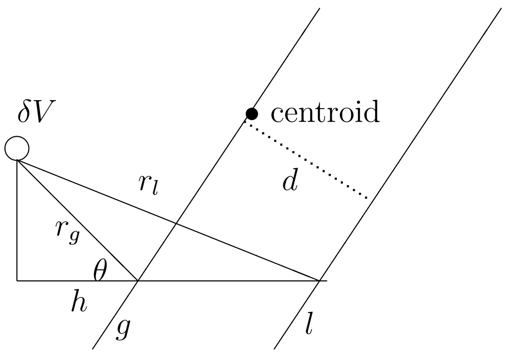
UNIT 13.14 - INTEGRATION APPLICATIONS 14

SECOND MOMENTS OF A VOLUME (B)

13.14.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis and has a perpendicular distance of d from the first axis.



In the above **three-dimensional** diagram, we have

$$M_l = \sum_{\rm R} r_l^2 \delta V \text{ and } M_g = \sum_{\rm R} r_g^2 \delta V.$$

But, from the Cosine Rule,

$$r_l^2 = r_g^2 + d^2 - 2r_g d\cos(180^\circ - \theta) = r_g^2 + d^2 + 2r_g d\cos\theta.$$

Hence,

$$r_l^2 = r_g^2 + d^2 + 2dh$$

and so

$$\sum_R r_l^2 \delta V = \sum_R r_g^2 \delta V + \sum_R d^2 \delta V + 2d \sum_R h\delta V.$$

Finally, the expression

$$\sum_R h\delta V$$

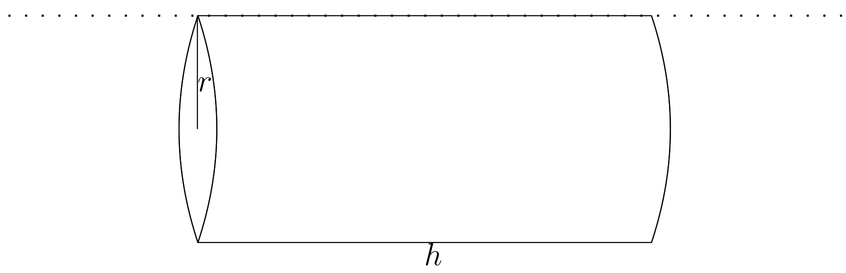
represents the first moment of R about a plane through the centroid, which is perpendicular to the plane containing l and g. Such first moment will be zero and hence,

$$M_l = M_g + Vd^2.$$

EXAMPLE

Determine the second moment of a solid right-circular cylinder about one of its generators (that is, a line in the surface, parallel to the central axis).

Solution



The second moment of the cylinder about the central axis was shown, in Unit 13.13, section 13.13.2, to be $\frac{\pi a^4 h}{2}$; and, since this axis and the generator are a distance a apart, the required second moment is given by

$$\frac{\pi a^4 h}{2} + (\pi a^2 h)a^2 = \frac{3\pi a^4 h}{2}.$$

13.14.2 THE RADIUS OF GYRATION OF A VOLUME

Having calculated the second moment of a three-dimensional region about a certain axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Vk^2, where V is the total volume of the region.

We simply divide the value of the second moment by V in order to obtain the value of k^2 and hence, the value of k.

The value of k is called the **"radius of gyration"** of the given region about the given axis.

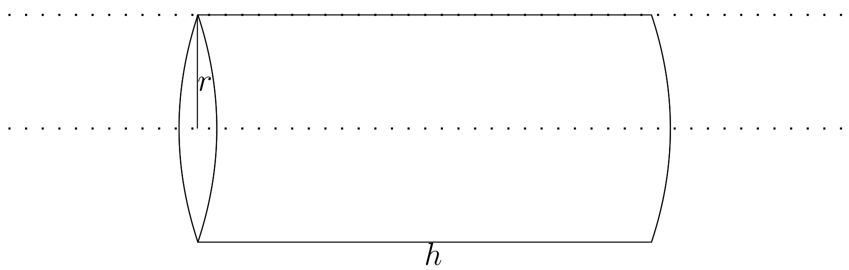
Note:

The radius of gyration effectively tries to concentrate the whole volume at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a solid right-circular cylinder with height, h, and radius, a, about (a) its own axis and (b) one of its generators.

   Solution

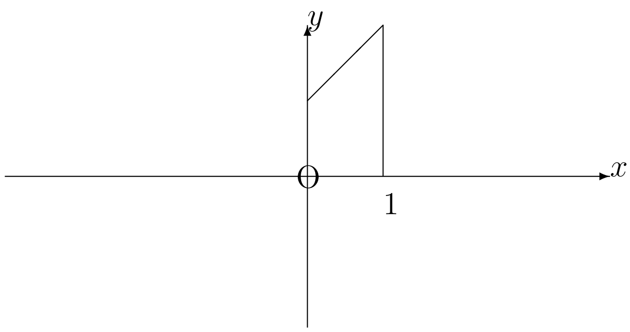
   

   Using earlier examples, together with the volume, $V = \pi a^2 h$, the required radii of gyration are (a) $\sqrt{\frac{\pi a^4 h}{2} \div \pi a^2 h} = \frac{a}{\sqrt{2}}$ and (b) $\sqrt{\frac{3\pi a^4 h}{2} \div \pi a^2 h} = a\sqrt{\frac{3}{2}}$.

2. Determine the radius of gyration of the volume of revolution about the x-axis of the region, bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



From Unit 13.13, section 13.13.3, the second moment about the given axis is $\frac{31\pi}{10}$. The volume itself is given by

$$\int_0^1 \pi(x+1)^2 \, \mathrm{d}x = \left[\pi\frac{(x+1)^3}{3}\right]_0^1 = \frac{7\pi}{3}.$$

Hence,

$$k^2 = \frac{31\pi}{10} \times \frac{3}{7\pi} = \frac{93}{70}.$$

That is,

$$k = \sqrt{\frac{93}{70}} \simeq 1.15$$

13.14.3 EXERCISES

1. Determine the radius of gyration of a hollow cylinder with internal radius, a, and external radius, b, about
   (a) its central axis;
   (b) a generator lying in its outer surface.
2. Determine the radius of gyration of a solid hemisphere, with radius a, about
   (a) its base-diameter;
   (b) an axis through its centroid, parallel to its base-diameter.
3. For a solid right-circular cylinder with height, h, and radius, a, determine the radius of gyration about
   (a) a diameter of one end;
   (b) an axis through the centroid, perpendicular to the axis of the cylinder.
4. For a solid right-circular cone with height, h, and base-radius, a, determine the radius of gyration about
   (a) the axis of the cone;
   (b) a line through the vertex, perpendicular to the axis of the cone;
   (c) a line through the centroid, perpendicular to the axis of the cone.

13.14.4 ANSWERS TO EXERCISES

1. (a)
$$\sqrt{\frac{a^2+b^2}{2}};$$
   (b)
$$\sqrt{\frac{3b^2+a^2}{2}}.$$
2. (a)
$$a\sqrt{\frac{2}{5}};$$
   (b)
$$a\sqrt{\frac{173}{320}}.$$
3. (a)
$$\sqrt{\frac{3a^2+4h^2}{12}};$$
   (b)
$$\sqrt{\frac{3a^2+7h^2}{12}}.$$
4. (a)
$$a\sqrt{\frac{3}{10}};$$
   (b)
$$\sqrt{\frac{3a^2}{20}+\frac{3h^2}{5}};$$
   (c)
$$\sqrt{\frac{3a^2}{20}+\frac{3h^2}{80}}.$$

"JUST THE MATHS"

UNIT NUMBER

13.15

INTEGRATION APPLICATIONS 15
(Second moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.15 - INTEGRATION APPLICATIONS 15

SECOND MOMENTS OF A SURFACE OF REVOLUTION

13.15.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"second moment"** about the x-axis, is given by

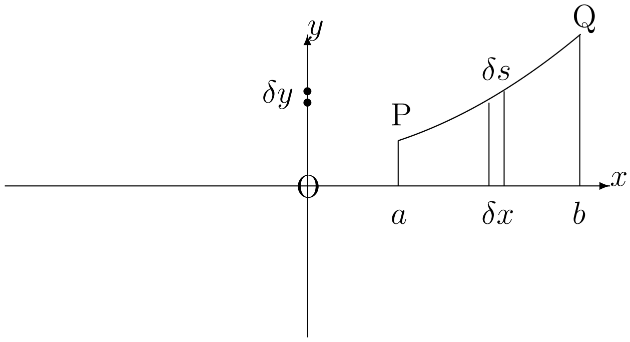
$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} y^2.2\pi y \delta s.$$

13.15.2 INTEGRATION FORMULAE FOR SECOND MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

so that, for the surface of revolution of the arc about the x-axis, the second moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y^3 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{t_1}^{t_2} 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the second moment about the plane through the origin, perpendicular to the x-axis, is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi y^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$$

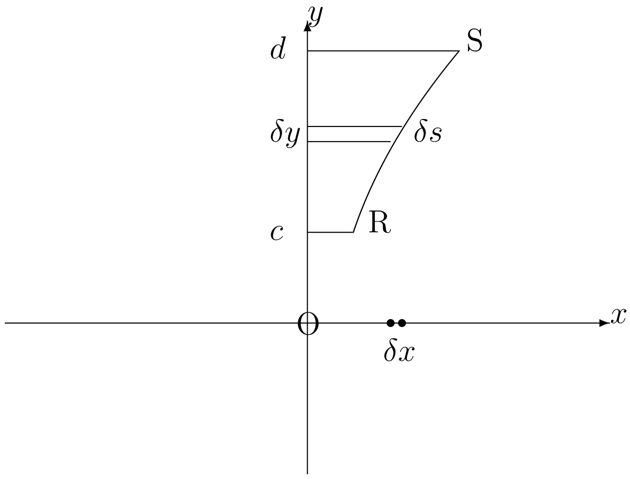
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the second moment about the y-axis is given by

$$\int_c^d 2\pi x^3 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the second moment about the y-axis is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi x^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

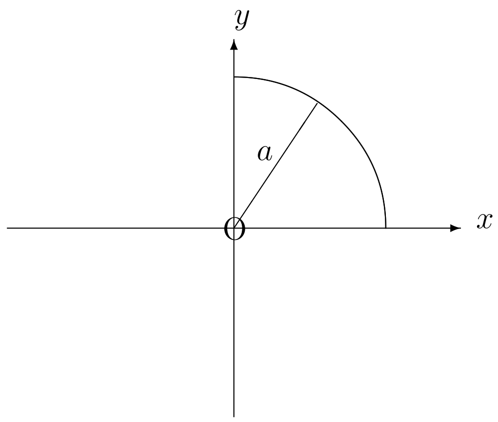
EXAMPLES

1. Determine the second moment about the x-axis of the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and, hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The second moment about the x-axis is therefore given by

$$\int_0^a 2\pi y^3\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi y^3\sqrt{\frac{x^2+y^2}{y^2}}\ \mathrm{d}x.$$

But $x^2+y^2=a^2$, and so the second moment becomes

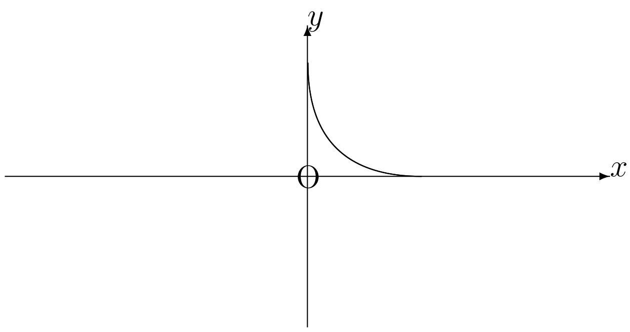
$$\int_0^a 2\pi a(a^2-x^2)\ \mathrm{d}x = 2\pi a\left[a^2x-\frac{x^3}{3}\right]_0^a = \frac{4\pi a^4}{3}.$$

2. Determine the second moment about the axis of revolution, when the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$

is rotated through 2π radians about (a) the x-axis and (b) the y-axis.

Solution



(a) Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the second moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^0 2\pi y^3\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta+\sin^2\theta\equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^3 \sin^{27}\theta . 3a\cos\theta\sin\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 6\pi a^4 \sin^{28}\theta\cos\theta \, \mathrm{d}\theta$$

$$= 6\pi a^4 \int_0^{\frac{\pi}{2}} \sin^{28}\theta\cos\theta \, \mathrm{d}\theta,$$

which, by the methods of Unit 12.7 gives

$$6\pi a^4 \left[\frac{\sin^{29}\theta}{29}\right]_0^{\frac{\pi}{2}} = \frac{6\pi a^4}{29}.$$

(b) By symmetry, or by direct integration, the second moment about the y-axis is also $\frac{6\pi a^4}{29}$.

13.15.3 THE RADIUS OF GYRATION OF A SURFACE OF REVOLUTION

Having calculated the second moment of a surface of revolution about a specified axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Sk^2, where S is the total surface area of revolution.

We simply divide the value of the second moment by S in order to obtain the value of k^2 and hence the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole surface at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

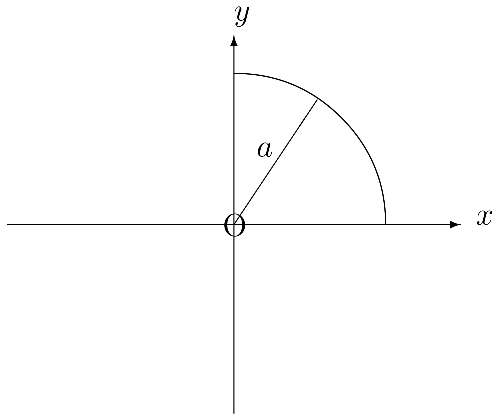
EXAMPLES

1. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From an Example 1 in section 13.15.2, we know that the second moment of the surface about the x-axis is equal to $\frac{4\pi a^4}{3}$.
Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2,$$

which implies that

$$k^2 = \frac{4\pi a^4}{3} \times \frac{1}{2\pi a^2} \quad = \frac{2a^2}{3}.$$

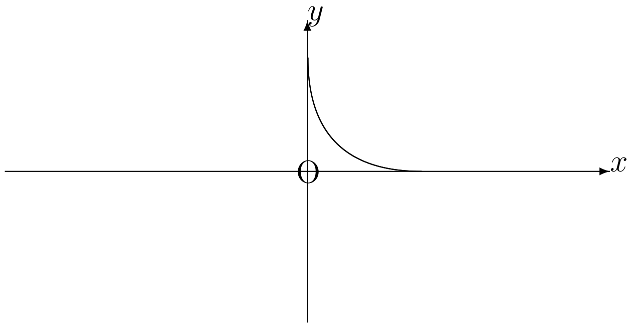
The radius of gyration is thus given by

$$k = a\sqrt{\frac{2}{3}}.$$

2. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 in section 13.15.2, that that the second moment of the surface about the x-axis is equal to $\frac{6\pi a^4}{29}$.
Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta = \int_{0}^{\frac{\pi}{2}} 3a^2\sin^4\theta\cos\theta\ \mathrm{d}\theta = 3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$k^2 = \frac{6\pi a^4}{29}\times\frac{5}{3\pi a^2} = \frac{10a^2}{29}.$$

13.15.4 EXERCISES

1. Determine the second moment, about the x-axis, of the surface of revolution (about the x-axis) of the straight-line segment joining the origin to the point $(2, 3)$.

2. Determine the second moment about the x-axis, of the surface of revolution (about the x-axis) of the first quadrant arc of the curve whose equation is $y^2 = 4x$, lying between $x = 0$ and $x = 1$.

3. Determine, correct to two places of decimals, the second moment about the y-axis, of the surface of revolution (about the y-axis) of the first quadrant arc of the curve whose equation is $3y = x^3$, lying between $x = 1$ and $x = 2$.

4. Determine, correct to two places of decimals, the second moment, about the y-axis, of the surface of revolution (about the y-axis) of the arc of the circle given parametrically by
$$x = 2\cos t, \quad y = 2\sin t,$$
joining the point $(\sqrt{2}, \sqrt{2})$ to the point $(0, 2)$.

5. Determine the radius of gyration of a hollow right-circular cone with maximum radius, a, about its central axis.

6. For the curve whose equation is $9y^2 = x(3-x)^2$, show that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-x}{2\sqrt{x}}.$$
Hence, show that the radius of gyration about the y-axis of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x-axis is 4, correct to the nearest whole number.

13.15.5 ANSWERS TO EXERCISES

1.
$$\frac{\pi 27\sqrt{13}}{2}.$$

2.
$$\frac{32\pi}{5}[4-\sqrt{2}] \simeq 51.99$$

3.
$$70.44$$

4.
$$0.73$$

5.
$$k = \frac{a}{\sqrt{2}}.$$

6.
$$\text{Second moment} \simeq 139.92, \quad \text{surface area} \simeq 9.42$$

"JUST THE MATHS"

UNIT NUMBER

13.16

INTEGRATION APPLICATIONS 16
(Centres of pressure)

by

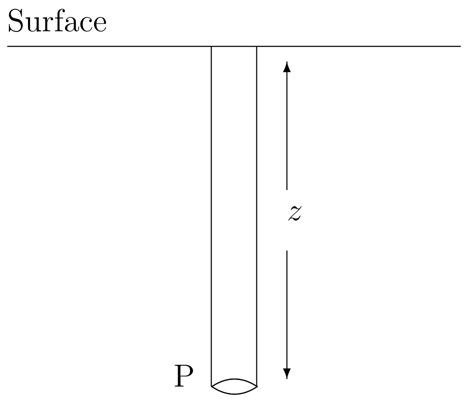
A.J.Hobson

UNIT 13.16 - INTEGRATION APPLICATIONS 16

CENTRES OF PRESSURE

13.16.1 THE PRESSURE AT A POINT IN A LIQUID

In the following diagram, we consider the pressure in a liquid at a point, P, whose depth below the surface of the liquid is z.



Ignoring atmospheric pressure, the pressure, p, at P is measured as the thrust acting upon unit area and is due to the weight of the column of liquid with height z above it.

Hence,

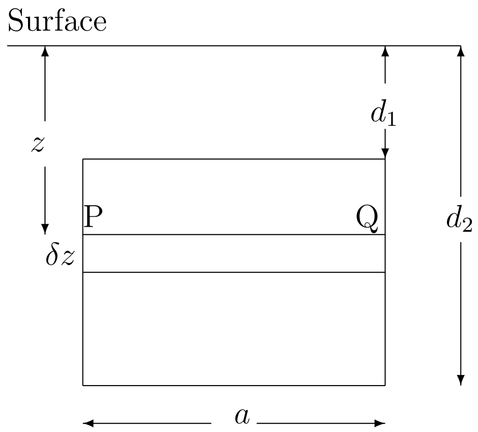
$$p = wz$$

where w is the weight, per unit volume, of the liquid.

Note:
The pressure at P is directly proportional to the depth of P below the surface; and we shall assume that the pressure acts equally in all directions at P.

13.16.2 THE PRESSURE ON AN IMMERSED PLATE

We now consider a rectangular plate, with dimensions a and $(d_2 - d_1)$, immersed vertically in a liquid as shown below.



For a thin strip, PQ, of width, δz, at a depth, z, below the surface of the liquid, the thrust on PQ will be the pressure at P multiplied by the area of the strip; that is, $wz \times a\delta z$.

The total thrust on the whole plate will therefore be

$$\sum_{z=d_1}^{z=d_2} waz\delta z.$$

Allowing δz to tend to zero, the total thrust becomes

$$\int_{d_1}^{d_2} waz \ \mathrm{d}z = \left[\frac{waz^2}{2}\right]_{d_1}^{d_2} = \frac{wa}{2}(d_2^2 - d_1^2).$$

This may be written

$$wa(d_2 - d_1)\left(\frac{d_2 + d_1}{2}\right),$$

where, in this form, $a(d_2 - d_1)$ is the area of the plate and $\frac{d_2+d_1}{2}$ is the depth of the centroid of the plate.

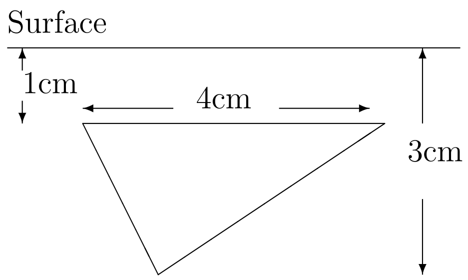
We conclude that

$$\text{Total Thrust} = \text{Area of Plate} \times \text{Pressure at the Centroid.}$$

Note:
It may be shown that this result holds whatever the shape of the plate is; and even when the plate is not vertical.

EXAMPLES

1. A triangular plate is immersed vertically in a liquid for which the weight per unit volume is w. The dimensions of the plate and its position in the liquid is shown in the following diagram:



Determine the total thrust on the plate as a multiple of w.

Solution

The area of the plate is given by

$$\text{Area} = \frac{1}{2} \times 4 \times 2 = 4\text{cm}^2$$

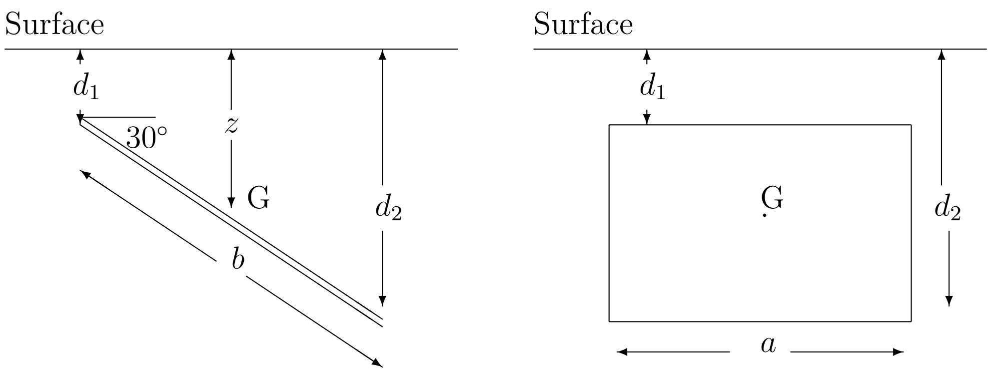
The centroid of the plate, which is at a distance from its horizontal side equal to one third of its perpendicular height, will lie at a depth of

$$\left(1 + \frac{1}{3} \times 2\right) \text{cms.} = \frac{5}{3}\text{cms.}$$

Hence, the pressure at the centroid is $\frac{5w}{3}$ and we conclude that

$$\text{Total thrust} = 4 \times \frac{5w}{3} = \frac{20w}{3}.$$

2. The following diagram shows a rectangular plate immersed in a liquid for which the weight per unit volume is w; and the plate is inclined at 30° to the horizontal:



Determine the total thrust on the plate as a multiple of w.

Solution

The depth, z, of the centroid, G, of the plate is given by

$$z = d_1 + \frac{b}{2}\sin 30^\circ = d_1 + \frac{b}{4}.$$

Hence, the pressure, p, at G is given by

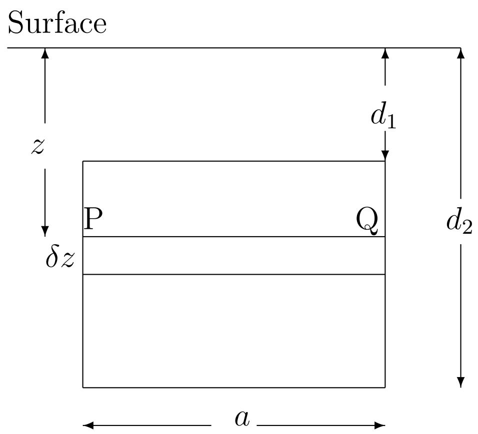
$$p = \left(d_1 + \frac{b}{4}\right) w;$$

and, since the area of the plate is ab, we obtain

$$\text{Total thrust} = ab\left(d_1 + \frac{b}{4}\right) w.$$

13.16.3 THE DEPTH OF THE CENTRE OF PRESSURE

In this section, we consider again an earlier diagram for a rectangular plate, immersed vertically in a liquid whose weight per unit volume is w.



We have already seen that the total thrust on the plate is

$$\int_{d_1}^{d_2} waz \ \mathrm{d}z = w\int_{d_1}^{d_2} az \ \mathrm{d}z$$

and is the resultant of varying thrusts, acting according to depth, at each level of the plate.

But, by taking first moments of these thrusts about the line in which the plane of the plate intersects the surface of the liquid, we may determine a particular depth at which the total thrust may be considered to act.

This depth is called **"the depth of the centre of pressure"**.

The Calculation

In the diagram, the thrust on the strip PQ is $waz\delta z$ and its first moment about the line in the surface is $waz^2\delta z$ so that the sum of the first moments on all such strips is given by

$$\sum_{z=d_1}^{z=d_2} waz^2\delta z = w\int_{d_1}^{d_2} az^2 \ \mathrm{d}z$$

where the definite integral is, in fact, the <u>second</u> moment of the plate about the line in the surface.

Next, we define the depth, C_p, of the centre of pressure to be such that

$$\text{Total thrust} \times C_p = \text{sum of first moments of strips like PQ.}$$

That is,

$$w\int_{d_1}^{d_2} az\ \mathrm{d}z \times C_p = w\int_{d_1}^{d_2} az^2\ \mathrm{d}z$$

and, hence,

$$C_p = \frac{\int_{d_1}^{d_2} az^2\ \mathrm{d}z}{\int_{d_1}^{d_2} az\ \mathrm{d}z},$$

which may be interpreted as

$$C_p = \frac{Ak^2}{A\overline{z}} = \frac{k^2}{\overline{z}},$$

where A is the area of the plate, k is the radius of gyration of the plate about the line in the surface of the liquid and $\overline{z}$ is the depth of the centroid of the plate.

Notes:

(i) It may be shown that the formula

$$C_p = \frac{k^2}{\overline{z}}$$

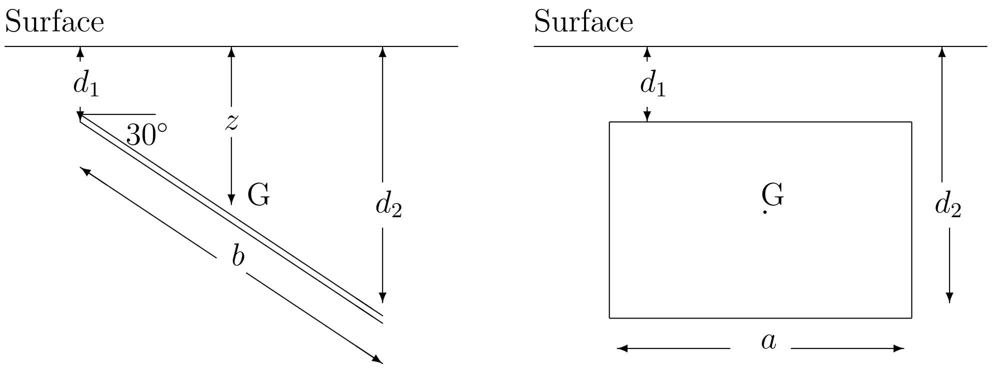
holds for any shape of plate immersed at any angle.

(ii) The phrase, "centre of pressure" suggests a particular point at which the total thrust is considered to act; but this is simply for convenience. The calculation is only for the depth of the centre of pressure.

EXAMPLE

Determine the depth of the centre of pressure for the second example of the previous section.

Solution



The depth of the centroid is

$$d_1 + \frac{b}{4}$$

and the square of the radius of gyration of the plate about an axis through the centroid, parallel to the side with length a is $\frac{a^2}{12}$.

Furthermore, the perpendicular distance between this axis and the line of intersection of the plane of the plate with the surface of the liquid is

$$\frac{b}{2} + \frac{d_1}{\sin 30°} = \frac{b}{2} + 2d_1.$$

Hence, the square of the radius of gyration of the plate about the line in the surface is

$$\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2,$$

using the Theorem of Parallel Axes.

Finally, the depth of the centre of pressure is given by

$$C_p = \frac{\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2}{d_1 + \frac{b}{4}}.$$

13.16.4 EXERCISES

1. A thin equilateral triangular plate is immersed vertically in a liquid for which the weight per unit volume is w, with one edge on the surface. If the length of each side is a, determine the total thrust on the plate.

2. A thin plate is bounded by the arc of a parabola and a straight line segment of length 1.2m perpendicular to the axis of symmetry of the parabola, this axis being of length 0.4m.

   If the plate is immersed vertically in a liquid with the straight edge on the surface, determine the total thrust on the plate in the form lw, where w is the weight per unit volume of the liquid and l is in decimals, correct to two places.

3. A thin rectangular plate, with sides of length 10cm and 20cm is immersed in a liquid so that the sides of length 10cm are horizontal and the sides of length 20cm are incline at 55° to the horizontal. If the uppermost side of the plate is at a depth of 13cm, determine the total thrust on then plate in the form lw, where w is the mass per unit volume of the liquid.

4. A thin circular plate, with diameter 0.5m is immersed vertically in a tank of liquid so that the uppermost point on its circumference is 2m below the surface. Determine the depth of the centre of pressure. correct to two places of decimals.

5. A thin plate is in the form of a trapezium with parallel sides of length 1m and 2.5m, a distance of 0.75m apart, and the remaining two sides inclined equally to either one of the parallel sides.

   If the plate is immersed vertically in water with the side of length 2.5m on the surface, calculate the depth of the centre of pressure, correct to two places of decimals.

13.16.5 ANSWERS TO EXERCISES

1.

$$\text{Total thrust} = \frac{wa^3}{8}.$$

2.

$$\text{Total Thrust} = 5.12w.$$

3.

$$C_p \simeq 2.26\text{m}.$$

4.

$$C_p \simeq 0.46\text{m}.$$

"JUST THE MATHS"

UNIT NUMBER

14.1

PARTIAL DIFFERENTIATION 1
(Partial derivatives of the first order)

by

A.J.Hobson

UNIT 14.1 - PARTIAL DIFFERENTIATION 1 - PARTIAL DERIVATIVES OF THE FIRST ORDER

14.1.1 FUNCTIONS OF SEVERAL VARIABLES

In most scientific problems, it is likely that a variable quantity under investigation will depend (for its values), not only on **one** other variable quantity, but on **several** other variable quantities.

The type of notation used may be indicated by examples such as the following:

1.
$$z = f(x, y),$$
which means that the variable, z, depends (for its values) on two variables, x and y.

2.
$$w = F(x, y, z),$$
which means that the variable, w, depends (for its values) on three variables, x, y and z.

Normally, the variables on the right-hand side of examples like those above may be chosen independently of one another and, as such, are called the **"independent variables"**. By contrast, the variable on the left-hand side is called the **"dependent variable"**.

Notes:

(i) Some relationships between several variables are not stated as an **explicit** formula for one of the variables in terms of the others.

An illustration of this type would be $x^2 + y^2 + z^2 = 16$.

In such cases, it may be necessary to specify separately which is the dependent variable.

(ii) The variables on the right-hand side of an explicit formula, giving a dependent variable in terms of them, may not actually be independent of one another. This would occur if those variables were already, themselves, dependent on a quantity not specifically mentioned in the formula.

For example, in the formula

$$z = xy^2 + \sin(x - y),$$

suppose it is also known that $x = t - 1$ and $y = 3t + 2$.

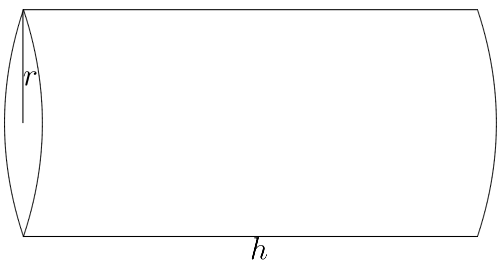
Then the variables x and y are not independent of each other. In fact, by eliminating t, we obtain

$$y = 3(x + 1) + 2 = 3x + 5.$$

14.1.2 THE DEFINITION OF A PARTIAL DERIVATIVE

ILLUSTRATION

Consider the formulae for the volume, V, and the surface area, S, of a solid right-circular cylinder with radius, r, and height, h.



The relevant formulae are

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r^2 + 2\pi r h,$$

so that both V and S are functions of the two variables, r and h.

But suppose it were possible for r to be held constant while h is allowed to vary. Then the corresponding rates of increase of V and S with respect to h are given by

$$\left[\frac{\mathrm{d}V}{\mathrm{d}h}\right]_{r \text{ const.}} = \pi r^2$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}h}\right]_{r \text{ const.}} = 2\pi r.$$

These two expressions are called the **"partial derivatives of V and S with respect to h"**.

Similarly, suppose it were possible for h to be held constant while r is allowed to vary. Then the corresponding rates of increase of V and S with respect to r are given by

$$\left[\frac{\mathrm{d}V}{\mathrm{d}r}\right]_{h \text{ const.}} = 2\pi rh$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}r}\right]_{h \text{ const.}} = 4\pi r + 2\pi h.$$

These two expressions are called the **"partial derivatives of V and S with respect to r"**.

THE NOTATION FOR PARTIAL DERIVATIVES

In the defining illustration above, the notation used for the partial derivatives of V and S was an adaptation of the notation for what will, in future, be referred to as **ordinary** derivatives.

It was, however, rather cumbersome; and the more standard notation which uses the symbol ∂ rather than d is indicated by restating the earlier results as

$$\frac{\partial V}{\partial h} = \pi r^2, \quad \frac{\partial S}{\partial h} = 2\pi r$$

and

$$\frac{\partial V}{\partial r} = 2\pi rh, \quad \frac{\partial S}{\partial r} = 4\pi r + 2\pi h.$$

In this notation, it is understood that each independent variable (except the one with respect to which we are differentiating) is held constant.

EXAMPLES

1. Determine the partial derivatives of the following functions with respect to each of the independent variables:

(a)

$$z = (x^2 + 3y)^5;$$

Solution

$$\frac{\partial z}{\partial x} = 5(x^2 + 3y)^4.2x = 10x(x^2 + 3y)^4$$

and

$$\frac{\partial z}{\partial y} = 5(x^2 + 3y)^4.3 = 15(x^2 + 3y)^4.$$

(b)

$$w = ze^{3x-7y};$$

Solution

$$\frac{\partial w}{\partial x} = 3ze^{3x-7y},$$

$$\frac{\partial w}{\partial y} = -7ze^{3x-7y},$$

and

$$\frac{\partial w}{\partial z} = e^{3x-7y}.$$

(c)

$$z = x\sin(2x^2 + 5y).$$

Solution

$$\frac{\partial z}{\partial x} = \sin(2x^2 + 5y) + 4x^2\cos(2x^2 + 5y)$$

and

$$\frac{\partial z}{\partial y} = 5x\cos(2x^2 + 5y).$$

2. If

$$z = f(x^2 + y^2),$$

show that

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

Solution

$$\frac{\partial z}{\partial x} = 2xf'(x^2 + y^2)$$

and

$$\frac{\partial z}{\partial y} = 2yf'(x^2 + y^2).$$

Hence,

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

3. Given the formula

$$\cos(x + 2z) + 3y^2 + 2xyz = 0$$

as an implicit relationship between two independent variables x and y and a dependent variable z, determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x, y and z.

Solution

Differentiating the formula partially with respect to x and y in turn, we obtain

$$-\sin(x + 2z).\left(1 + 2\frac{\partial z}{\partial x}\right) + 2y\left(x\frac{\partial z}{\partial x} + y\right) = 0$$

and

$$-\sin(x + 2z).2\frac{\partial z}{\partial y} + 6y + 2x\left(y\frac{\partial z}{\partial y} + z\right) = 0,$$

respectively.

Thus,

$$\frac{\partial z}{\partial x} = \frac{\sin(x + 2z) - 2y^2}{2yx - 2\sin(x + 2z)}$$

and

$$\frac{\partial z}{\partial y} = \frac{2xz + 6y}{2\sin(x + 2z) - 2xy} = \frac{xz + 3y}{\sin(x + 2z) - xy}.$$

14.1.3 EXERCISES

1. Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in the following cases:

   (a)
   $$z = 2x^2 - 4xy + y^3;$$

   (b)
   $$z = \cos(5x - 3y);$$

   (c)
   $$z = e^{x^2+2y^2};$$

   (d)
   $$z = x\sin(y - x).$$

2. If
   $$z = (x + y)\ln\left(\frac{x}{y}\right),$$
   show that
   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

3. Determine $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ in the following cases:

   (a)
   $$w = x^5 + 3xyz + z^2;$$

   (b)
   $$w = ze^{2x-3y};$$

   (c)
   $$w = \sin(x^2 - yz).$$

14.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\partial z}{\partial x} = 4x - 4y \quad \text{and} \quad \frac{\partial z}{\partial y} = -4x + 3y^2;$$

(b)

$$\frac{\partial z}{\partial x} = -5\sin(5x - 3y) \quad \text{and} \quad \frac{\partial z}{\partial y} = 3\sin(5x - 3y);$$

(c)

$$\frac{\partial z}{\partial x} = 2xe^{x^2+2y^2} \quad \text{and} \quad \frac{\partial z}{\partial y} = 4ye^{x^2+2y^2};$$

(d)

$$\frac{\partial z}{\partial x} = \sin(y - x) - x\cos(y - x) \quad \text{and} \quad \frac{\partial z}{\partial y} = x\cos(y - x).$$

2.

$$\frac{\partial z}{\partial x} = \ln\left(\frac{x}{y}\right) + \frac{x+y}{x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \ln\left(\frac{x}{y}\right) - \frac{x+y}{y}.$$

3. (a)

$$\frac{\partial w}{\partial x} = 5x^4 + 3yz, \quad \frac{\partial w}{\partial y} = 3xz, \quad \frac{\partial w}{\partial z} = 3xy + 2z;$$

(b)

$$\frac{\partial w}{\partial x} = 2ze^{2x-3y}, \quad \frac{\partial w}{\partial y} = -3ze^{2x-3y}, \frac{\partial w}{\partial z} = e^{2x-3y};$$

(c)

$$\frac{\partial w}{\partial x} = 2x\cos(x^2 - yz), \quad \frac{\partial w}{\partial y} = -z\cos(x^2 - yz), \quad \frac{\partial w}{\partial z} = -y\cos(x^2 - yz).$$

"JUST THE MATHS"

UNIT NUMBER

14.2

PARTIAL DIFFERENTIATION 2
(Partial derivatives of order higher than one)

by

A.J.Hobson

UNIT 14.2 - PARTIAL DIFFERENTIATION 2

PARTIAL DERIVATIVES OF THE SECOND AND HIGHER ORDERS

14.2.1 STANDARD NOTATIONS AND THEIR MEANINGS

In Unit 14.1, the partial derivatives encountered are known as partial derivatives of the **first order**; that is, the dependent variable was differentiated only **once** with respect to each independent variable.

But a partial derivative will, in general contain **all** of the independent variables, suggesting that we may need to differentiate again with respect to **any** of those variables.

For example, in the case where a variable, z, is a function of two independent variables, x and y, the possible partial derivatives of the second order are

(i)

$$\frac{\partial^2 z}{\partial x^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right);$$

(ii)

$$\frac{\partial^2 z}{\partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right);$$

(iii)

$$\frac{\partial^2 z}{\partial x \partial y}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right);$$

(iv)

$$\frac{\partial^2 z}{\partial y \partial x}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

The last two can be shown to give the same result for all elementary functions likely to be encountered in science and engineering.

Note:

Occasionally, it may be necessary to use partial derivatives of order higher than two, as illustrated, for example, by

$$\frac{\partial^3 z}{\partial x \partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]$$

and

$$\frac{\partial^4 z}{\partial x^2 \partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]\right).$$

EXAMPLES

Determine all the first and second order partial derivatives of the following functions:

1.

$$z = 7x^3 - 5x^2y + 6y^3.$$

Solution

$$\frac{\partial z}{\partial x} = 21x^2 - 10xy; \qquad \frac{\partial z}{\partial y} = -5x^2 + 18y^2;$$

$$\frac{\partial^2 z}{\partial x^2} = 42x - 10y; \qquad \frac{\partial^2 z}{\partial y^2} = 36y;$$

$$\frac{\partial^2 z}{\partial y \partial x} = -10x; \qquad \frac{\partial^2 z}{\partial x \partial y} = -10x.$$

2.

$$z = y \sin x + x \cos y.$$

Solution

$$\frac{\partial z}{\partial x} = y \cos x + \cos y; \qquad \frac{\partial z}{\partial y} = \sin x - x \sin y;$$

$$\frac{\partial^2 z}{\partial x^2} = -y \sin x; \qquad \frac{\partial^2 z}{\partial y^2} = -x \cos y;$$

$$\frac{\partial^2 z}{\partial y \partial x} = \cos x - \sin y; \qquad \frac{\partial^2 z}{\partial x \partial y} = \cos x - \sin y.$$

3.

$$z = e^{xy}(2x - y).$$

Solution

$$\begin{aligned}\frac{\partial z}{\partial x} &= e^{xy}[y(2x-y)+2] \\ &= e^{xy}[2xy - y^2 + 2];\end{aligned} \qquad \begin{aligned}\frac{\partial z}{\partial y} &= e^{xy}[x(2x-y)-1] \\ &= e^{xy}[2x^2 - xy - 1];\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= e^{xy}[y(2xy - y^2 + 2) + 2y] \\ &= e^{xy}[2xy^2 - y^3 + 4y];\end{aligned} \qquad \begin{aligned}\frac{\partial^2 z}{\partial y^2} &= e^{xy}[x(2x^2 - xy - 1) - x] \\ &= e^{xy}[2x^3 - x^2y - 2x];\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y \partial x} &= e^{xy}[x(2xy - y^2 + 2) + 2x - 2y] \\ &= e^{xy}[2x^2y - xy^2 + 4x - 2y];\end{aligned} \qquad \begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= e^{xy}[y(2x^2 - xy - 1) + 4x - y] \\ &= e^{xy}[2x^2y - xy^2 + 4x - 2y].\end{aligned}$$

14.2.2 EXERCISES

1. Determine all the first and second order partial derivatives of the following functions:

   (a)

   $$z = 5x^2y^3 - 7x^3y^5;$$

   (b)

   $$z = x^4 \sin 3y.$$

2. Determine all the first and second order partial derivatives of the function

   $$w \equiv z^2 e^{xy} + x\cos(y^2 z).$$

3. If

   $$z = (x+y)\ln\left(\frac{x}{y}\right),$$

   show that

   $$x^2 \frac{\partial^2 z}{\partial x^2} = y^2 \frac{\partial^2 z}{\partial y^2}.$$

4. If

$$z = f(x + ay) + F(x - ay),$$

show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial y^2}.$$

14.2.3 ANSWERS TO EXERCISES

1. (a) The required partial derivatives are as follows:

$$\frac{\partial z}{\partial x} = 10xy^3 - 21x^2y^5; \qquad \frac{\partial z}{\partial y} = 15x^2y^2 - 35x^3y^4;$$

$$\frac{\partial^2 z}{\partial x^2} = 10y^3 - 42xy^5; \qquad \frac{\partial^2 z}{\partial y^2} = 30x^2y - 140x^3y^3;$$

$$\frac{\partial^2 z}{\partial y \partial x} = 30xy^2 - 105x^2y^4; \qquad \frac{\partial^2 z}{\partial x \partial y} = 30xy^2 - 105x^2y^4.$$

(b) The required partial derivatives are as follows:

$$\frac{\partial z}{\partial x} = 4x^3 \sin 3y; \qquad \frac{\partial z}{\partial y} = 3x^4 \cos 3y;$$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 \sin 3y; \qquad \frac{\partial^2 z}{\partial y^2} = -9x^4 \sin 3y;$$

$$\frac{\partial^2 z}{\partial y \partial x} = 12x^3 \cos 3y; \qquad \frac{\partial^2 z}{\partial x \partial y} = 12x^3 \cos 3y.$$

2. The required partial derivatives are as follows:

$$\frac{\partial w}{\partial x} = yz^2e^{xy} + \cos(y^2z); \quad \frac{\partial w}{\partial y} = z^2xe^{xy} - 2xyz\sin(y^2z); \quad \frac{\partial w}{\partial z} = 2ze^{xy} - xy^2\sin(y^2z);$$

$$\frac{\partial^2 w}{\partial x^2} = y^2z^2e^{xy}; \quad \frac{\partial^2 w}{\partial y^2} = z^2x^2e^{xy} - 2xz\sin(y^2z) + 4xy^2z^2\cos(y^2z); \quad \frac{\partial^2 w}{\partial z^2} = 2e^{xy} - xy^4\cos(y^2z);$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x} = z^2 e^{xy} + z^2 xye^{xy} - 2yz\sin(y^2 z);$$

$$\frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 w}{\partial z \partial y} = 2zxe^{xy} - 2xy\sin(y^2 z) - 2xy^3 z\cos(y^2 z);$$

$$\frac{\partial^2 w}{\partial z \partial x} = \frac{\partial^2 w}{\partial x \partial z} = 2zye^{xy} - y^2\sin(y^2 z).$$

3.

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x} - \frac{y}{x^2} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = -\frac{1}{y} + \frac{x}{y^2}.$$

4.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + F''(x-ay) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 F''(x-ay).$$