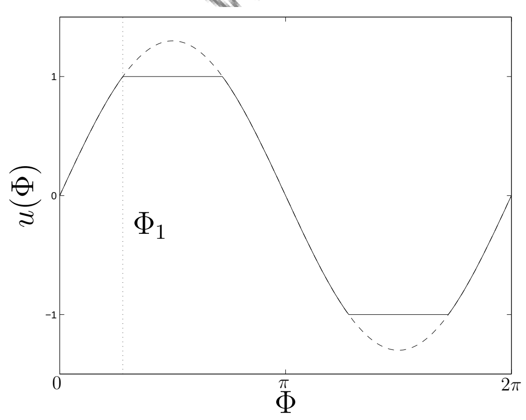
14 Oscillations and Describing Functions

14.1

The describing function of a saturation is given in the textbook, together with describing functions for some other nonlinearities.

We will derive it here anyway to demonstrate how to do it. Skip this part to begin with, it is mostly here for reference.

1. Apply the signal $e(t) = C \sin \Phi$, where $\Phi = \omega t$, to the input of the saturation. If $C \leq 1$ the output signal after the saturation will be $u(t) = e(t)$. If $C > 1$ the output signal $u(t)$ will be the solid curve in the figure below:



   Here Φ_1 is given by $C \sin \Phi_1 = 1$, i.e. $\Phi_1 = \arcsin(1/C)$.

2. Compute the Fourier coefficients a_1 and b_1 as

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} u(\Phi) \cos \Phi \, d\Phi \ , \quad b_1 = \frac{1}{\pi} \int_0^{2\pi} u(\Phi) \sin \Phi \, d\Phi$$

   As $u(\Phi)$ is an odd function and $\cos \Phi$ is even $a_1 = 0$. Utilizing symmetry we can write b_1 as

$$\begin{aligned} b_1 &= \frac{4}{\pi}\int_0^{\pi/2} u(\Phi)\sin\Phi \, d\Phi \\ &= \frac{4}{\pi}\left(\int_0^{\Phi_1} C\sin^2\Phi \, d\Phi + \int_{\Phi_1}^{\pi/2} \sin\Phi \, d\Phi\right) \\ &= \frac{4C}{\pi}\left(\frac{\Phi_1}{2} - \frac{\sin 2\Phi_1}{4} + \frac{\cos\Phi_1}{C}\right) \end{aligned}$$

As $\sin 2\Phi_1 = 2\sin\Phi_1\cos\Phi_1$, $\sin\Phi_1 = 1/C$ and $\cos\Phi_1 = \sqrt{C^2-1}/C$ we get

$$b_1 = \frac{2C}{\pi}\left(\arcsin\frac{1}{C} + \frac{\sqrt{C^2-1}}{C^2}\right)$$

3. The describing function is given by $Y_f(C) = (b_1 + ia_1)/C$

$$Y_f(C) = \frac{2}{\pi}\left(\arcsin\frac{1}{C} + \frac{\sqrt{C^2-1}}{C^2}\right)$$

This is valid for $C > 1$, $Y_f(C) = 1$ when $C \le 1$)

(a) The problem can be solved by a combination of theoretical computations and Matlab.

For $C \le 1$ the saturation does not affect the signal and consequently the describing function is $Y_f(C) = 1$. For larger amplitudes, the saturation clips the signal, and the describing function is

$$Y_f(C) = \frac{2}{\pi}\left(\arcsin\frac{1}{C} + \frac{\sqrt{C^2-1}}{C^2}\right)$$

The describing function is thus real, starts in 1 for $C \le 1$ and tends to zero when C tends to infinity. This means that $-1/Y_f(C)$ will start in -1 and tend to $-\infty$ when C grows. Hence, when drawn in the complex plane, it will cover the real line to the left of -1.

The transfer function of the linear part is

$$G(s) = \frac{10}{s(s+1)^2}$$

Since the system contains an integrator the argument of $G(i\omega)$ will start at $-90°$ for low frequencies, and due to the extra two poles, the argument will tend to $-270°$ for increasing ω (each stable pole leads to $90°$ phase loss asymptotically). This implies that $G(i\omega)$ must cross the real axis at some point and there is a possibility that it will cross $-1/Y_f$. Using the fact that

$$\arg G(i\omega) = \arg 10 - \arg(i\omega) - \arg((i\omega+1))^2 = 0 - 90° - 2\arctan\omega$$

we find that $\arg G(i\omega) = -180°$ (i.e. it crosses the negative real axis) when $\omega = 1$. Using also that

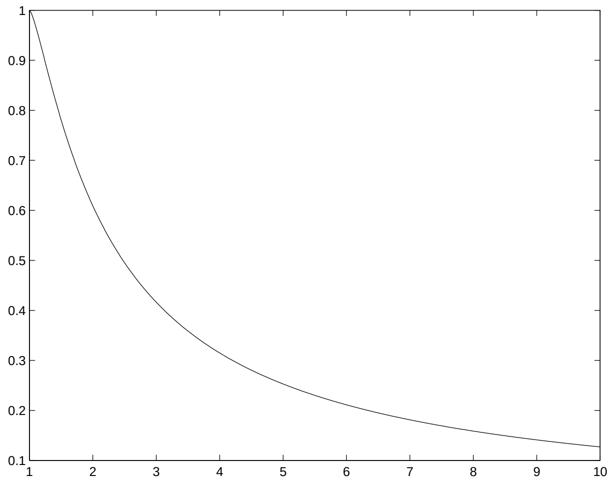
$$|G(i\omega)| = 10/(\omega(1+\omega^2))$$

we find that $\mid G(i1) \mid = 5$, i.e. $G(i\omega)$ crosses the negative real axis in the point -5, and there is an intersection with $-1/Y_f$. In order to find the corresponding value of C we need to solve the equation $-\frac{1}{Y_f(C)} = -5$

$$\frac{2}{\pi}\left(\arcsin\frac{1}{C} + \frac{\sqrt{C^2-1}}{C^2}\right) = 0.2$$

Given the complicated function, we solve this by plotting the describing function. The part for $C \geq 1$ is created with

```
>> C=1:0.01:10;
>> Yf=2/pi*(asin(1./C)+1./C.*sqrt(1-C.^(-2)));
>> plot(C,Yf)
```



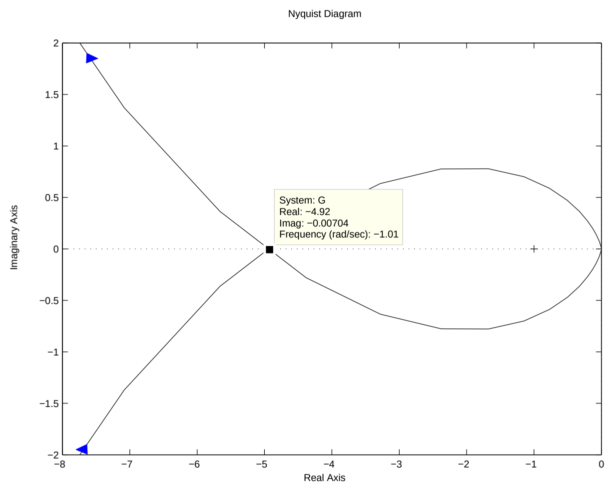
By zooming in one finds that $Y_f = 0.2$ for $C = 6.35$

For oscillations with $C < 6.35$ the curve $G(i\omega)$ will encircle $-1/Y_f$ and hence the amplitude of the oscillations will grow. Correspondingly, for oscillations with $C > 6.35$ the curve $G(i\omega)$ will not encircle $-1/Y_f$ and hence the amplitude of the oscillations will decay. Hence the describing function method predicts that there will be a limit cycle with angular frequency $\omega = 1$ and amplitude $C = 6.35$.

The Nyquist curve should of course be plotted in Matlab (normally one would start with a Nyquist plot in Matlab, to understand what one is looking for in the theoretical computations)

```
>> s=tf('s');
>> G=10/(s*(s+1)^2);
>> nyquist(G)
>> axis([-8 0 -2 2])
```

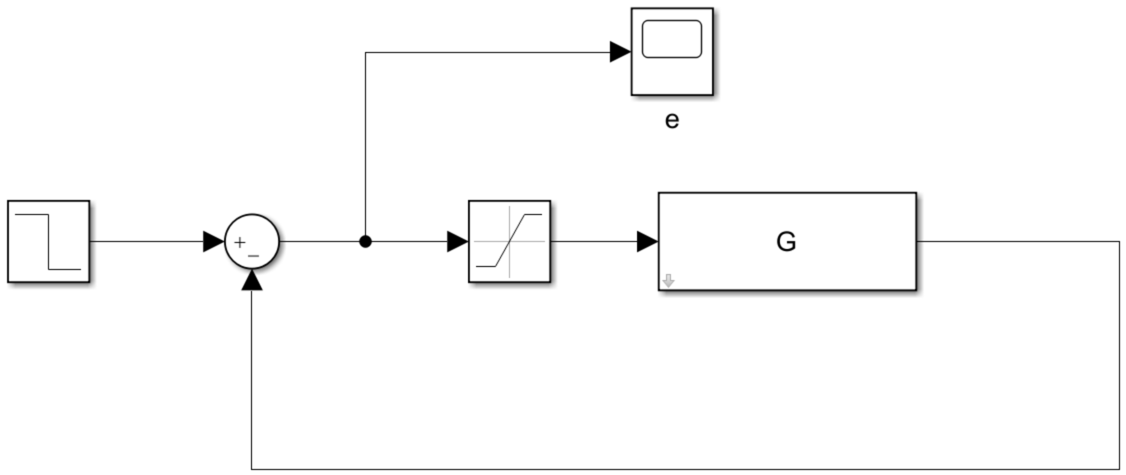
and by clicking in the plot one gets



Note: The curve does not pass exactly through -5, which is the correct value according to the analytical calculation, and this is caused by the automaticaly selection of frequency points in the Matlab function. A better plot is thus

```
>> w = logspace(-2,2,1000)
>> nyquist(G,w)
>> axis([-8 0 -2 2])
```

(b) An example of a Simulink model of the control system is shown in the figure below. A step with small amplitude which is turned off after a few seconds is sufficient to start the oscillations. The simulation results agree very well with the theoretical values from a).



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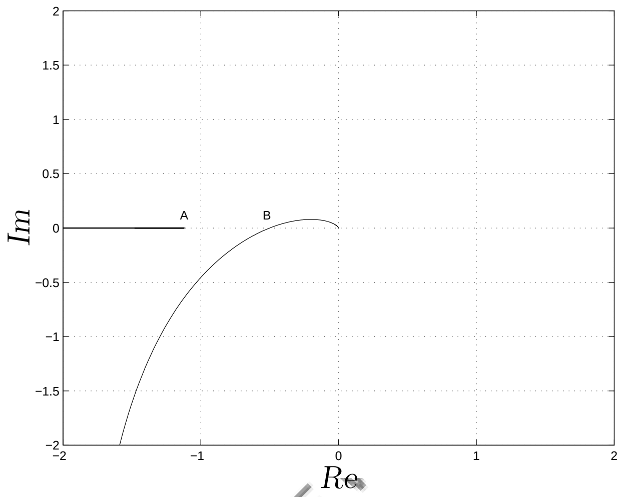
14.2

The Nyquist curve and the describing function are plotted below



The describing function changes direction in a point A, which corresponds to when $Y_f(C)$ takes it maximum value. The corresponding value of C can be found by differentiating $Y_f(C)$ with respect to C. Differentiation of

$$Y_f(C) = \frac{4H}{\pi C}\sqrt{1 - D^2/C^2}$$

with respect to C gives that the derivative is zero for $C = \sqrt{2}D$ and that $A = -\frac{\pi D}{2H}$. A possible intersection occurs when the Nyquist curve crosses the negative real axis. We have that $\arg G(i\omega) = -\pi$ when $\omega = 1$, and $|G(i1)| = 1/2$. The point B is thus $-1/2$. That the oscillation barely can exist means that $B \approx A$. The amplitude of the oscillation is 2.5 yields $\sqrt{2}D = 2.5$. Hence, $D = 5 \cdot \sqrt{2}/4$ and $H = \pi \cdot 5 \cdot \sqrt{2}/4$. The frequency of the oscillation is $\omega = 1$.

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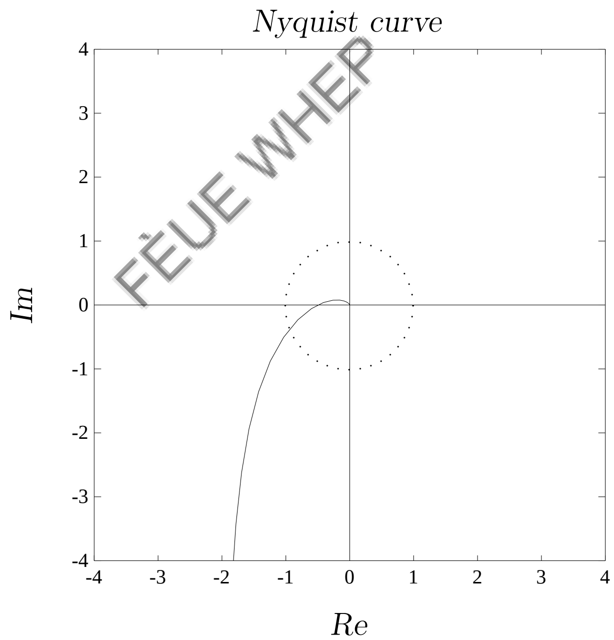
14.3

(a) The describing function of the relay is

$$Y_f(C) = 4/(\pi C), \quad \Rightarrow \quad -1/Y_f(C) = -\pi C/4$$

The curve $-1/Y_f(C)$ covers the entire negative real axis. The frequency response is

$$\begin{aligned} G(i\omega) &= \frac{K}{i\omega(i\omega+1)^2} = \frac{K(1-i\omega)^2 \cdot (-i\omega)}{\omega^2(1+\omega^2)^2} \\ &= \frac{K(1-\omega^2-2i\omega)(-i\omega)}{\omega^2(1+\omega^2)^2} = \frac{-2K\omega - iK(1-\omega^2)}{\omega(1+\omega^2)^2} \\ \arg G(i\omega) &= \arg(K) - \arg(i\omega) - 2\arg(1+i\omega) \\ &= 0 - \pi/2 - 2\mathrm{atan}(\omega) \end{aligned}$$



We will always have an oscillation as the Nyquist curve intersects with $-1/Y_f(C)$ for all values of K.

(b) At the intersection point we have that $\arg G(i\omega) = -\pi$, or alternatively that the imaginary part is 0. This occurs for $\omega = 1$. As $|G(i1)| = K/2$ the amplitude of the oscillation is given by

$$-\frac{K}{2} = -\frac{\pi C}{4}$$

The requirement that $C < 0.1$ results in $K < \pi/20$.

(c) With a possibly dynamic feedback $L(s)$, the phase of the linear loop-gain will be $\arg L(i\omega)G(i\omega) = \arg L(i\omega) + \arg G(i\omega)$. A controller yielding a phase lead (positive phase, rotating the Nyquist curve counter-clockwise) at $\omega \geq 1$ will thus allow us to use an increased gain K. One such controller is a PD-controller $1 + T_D s$ which will have the phase $\operatorname{atan}(T_D)$ at $\omega = 1$. Note that the phase of $L(i\omega)G(i\omega)$ now asymptotically tends to $-\pi$ instead of $-3\pi/2$ when $w \to \infty$, and for sufficiently large T_D the Nyquist curve of the loop gain does not even cross the real axis (not a general phenomena but occuring in combination with this particular system $G(s)$).

```
>> s = tf('s');
>> G = 1/(s*(1+s)^2);
>> L1 = 1;L2 = 1 + 0.1*s; L3 = 1 + 0.25*s;L4 = 1+2*s;
>> nyquist(L1*G,L2*G,L3*G,L4*G);
>> axis([-1 0 -1 1]);
>> figure
>> bode(L1*G,L2*G,L3*G,L4*G);
```

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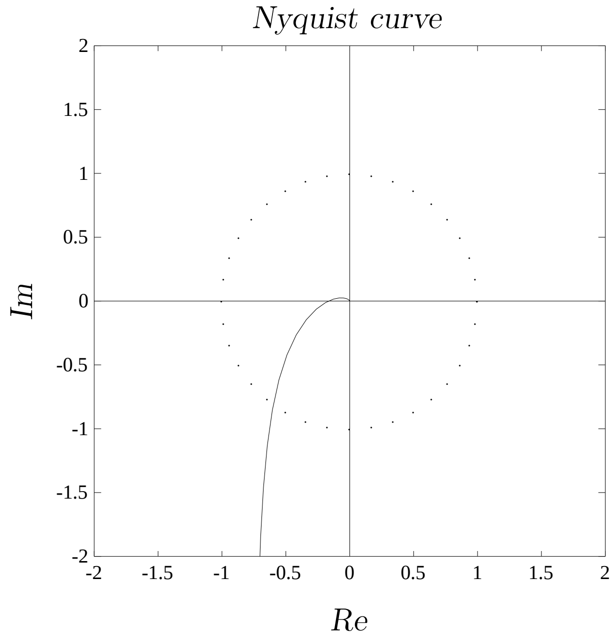
14.4

The describing function of an ideal relay:

$$Y_f(C) = \frac{4}{\pi \cdot C} \quad \Rightarrow \quad -\frac{1}{Y_f(C)} = -\frac{\pi}{4} \cdot C$$

(a) Plot the Nyquist curve of $G(s)H(s) = G(s)$

$$\begin{aligned} G(i\omega) &= \frac{1}{i\omega(i\omega+1)(i\omega+2)} = \frac{-i(1-i\omega)(2-i\omega)}{\omega(\omega^2+1)(\omega^2+4)} \\ &= -\frac{3}{(\omega^2+1)(\omega^2+4)} - i\frac{2-\omega^2}{\omega(\omega^2+1)(\omega^2+4)} \end{aligned}$$



If the point $-1/Y_N(C)$ is encircled by the Nyquist curve the amplitude of the oscillation will increase and otherwise it will decrease. This results in a stable oscillation. The frequency and amplitude can be determined from the intersection of the curves which occurs when $\operatorname{Im} G(i\omega) = 0$, i.e. when $\omega = \sqrt{2}$. As $\operatorname{Re} G(i\sqrt{2}) = -1/6$, we get

$$-1/6 = -\frac{\pi C}{4} \quad \Rightarrow \quad C = \frac{2}{3\pi}$$

Hence, the oscillation has the amplitude $2/(3\pi)$ and the frequency $\omega = \sqrt{2}$.

(b) Study $G(i\omega)H(i\omega)$

$$\begin{aligned} G(i\omega) \cdot H(i\omega) &= \frac{-i(1-i\omega)(2-i\omega)(1+Ki\omega)}{\omega(\omega^2+1)(\omega^2+4)} \\ &= \frac{-3+2K-K\omega^2}{(\omega^2+1)(\omega^2+4)} + i\frac{-2+\omega^2-3K\omega^2}{\omega(\omega^2+1)(\omega^2+4)} \end{aligned}$$

According to (a), we will avoid oscillations if $\operatorname{Im} G(i\omega)H(i\omega) < 0, \ \forall \omega$.

$$\begin{aligned} &-2+\omega^2-3K\omega^2 < 0 \quad \Rightarrow \\ &K > \frac{\omega^2-2}{3\omega^2} \end{aligned}$$

As $(\omega^2-2)/(3\omega^2) < 1/3, \ \forall \omega$ we can choose any $K > 1/3$.

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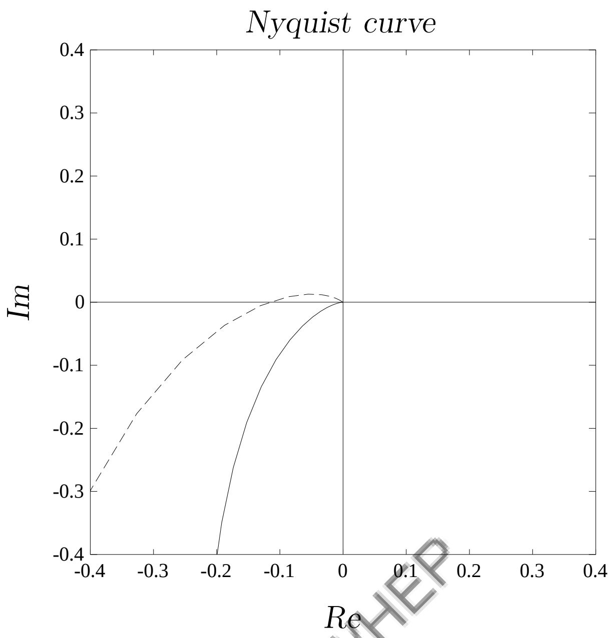


14.5

(a) Alternative (i): The describing function of a relay with hysteresis is given by

$$\begin{aligned} Y_f(C) &= \frac{4}{\pi C}\left(\sqrt{1-1/(2C)^2} - i/(2C)\right), \quad C \geq 0.5 \\ -1/Y_f(C) &= -\frac{\pi C}{4}\sqrt{1-1/(2C)^2} - i\frac{\pi}{8} \end{aligned}$$

which means that the imaginary part of $-1/Y_f$ will be $-\pi/8$ independent of C and the real part will start at zero and tend to $-\infty$.

The transfer function of the linear part is

$$G(s) = \frac{1}{s(s+1)}$$

Since the transfer function contains an integrator the argument will start at $-90°$ and since the relative degree is two the argument will tend to $-180°$. This indicates that there will be an intersection between $G(i\omega)$ and $-1/Y_f$. From the transfer function we have

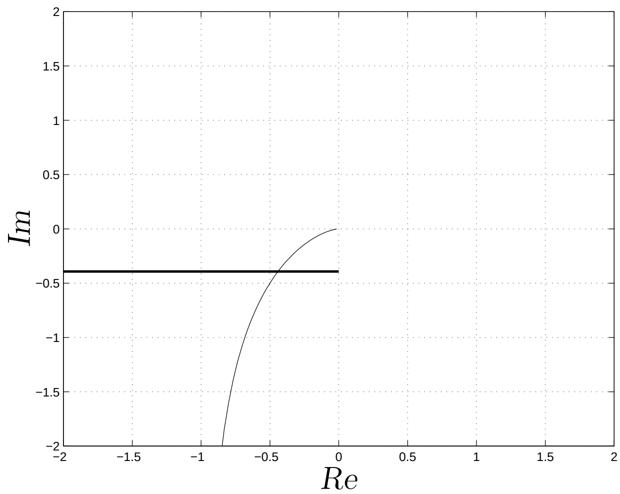
$$G(i\omega) = \frac{1}{i\omega(1+i\omega)} = \frac{-w-i}{\omega(1+\omega^2)}$$

Putting the real and imaginary parts of $G(i\omega)$ and $-1/Y_f$ equal to each other gives

$$\begin{aligned} \frac{1}{\omega(1+\omega^2)} &= \frac{\pi}{8} \\ \frac{\pi C}{4}\sqrt{1-1/(2C)^2} &= \frac{1}{1+\omega^2} \end{aligned}$$

The first equation has the approximate solution $\omega = 1.125$, which inserted in the second equation implies the solution $C = 0.75$.

Plot the Nyquist curve and the describing function.



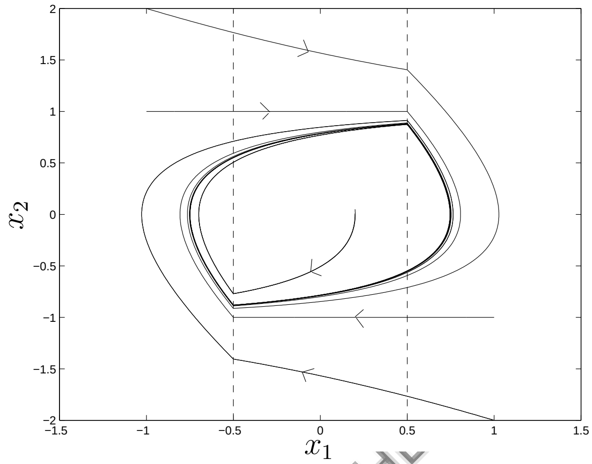
The curves intersect when $\omega = 1.235, C = 0.75$. This result can be found by looking at the plot or by solving the system of equations

C small $\Rightarrow$ $-1/Y_f(C)$ is encircled $\Rightarrow$ the amplitude of the oscillation increases
C large $\Rightarrow$ $-1/Y_f(C)$ is not encircled $\Rightarrow$ the amplitude of the oscillation decreases.
Thus, the oscillation is stable.

(b) Build a model in Simulink and verify the result.

(c) $x_1 = \theta,\ x_2 = \dot{\theta}$ yield

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u \end{aligned}, \qquad u = \begin{cases} 1, & x_1 < -0.5 \\ -1, & x_1 > 0.5 \end{cases}$$



Go back

14.6

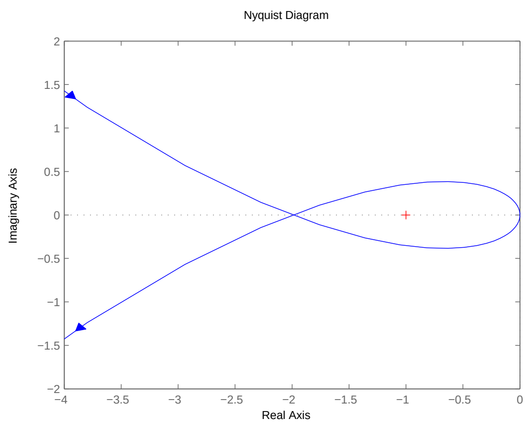
Inserting the numerical values for the PID coefficients gives the transfer function

$$G(s) = \frac{s^2 + 2s + 1}{s^3}$$

for the controller together with the motor. Evaluating G for $s = i\omega$ gives

$$G(i\omega) = \frac{-2\omega + i(1 - \omega^2)}{\omega^3}$$

It follows that G crosses the negative real axis at $\omega = \pm 1$ with $G(i) = -2$. A plot of the Nyquist curve is given below.

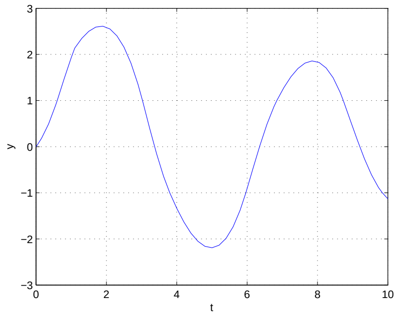


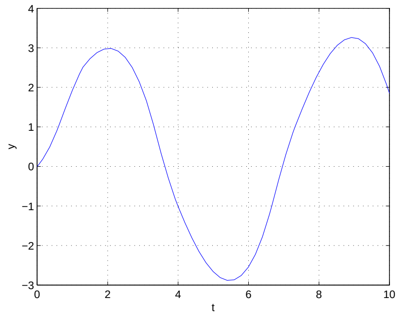
(a) Since the point -1 is not encircled by the Nyquist curve the closed loop system is asymptoticaldly stable when the amplifier is linear.

(b) The describing function for the saturation is

$$Y_f = \frac{2}{\pi}\left(\arcsin\frac{1}{C} + \frac{1}{C}\sqrt{1 - \frac{1}{C^2}}\right)$$

The condition $GY_f = -1$ gives $Y_f = 0.5$ which in turn gives $C \approx 2.5$. For values of C less than ≈ 2.5 the point $-1/Y_f(C)$ is not encircled so the amplitude ought to decrease, while for values of C greater than ≈ 2.5 the point $-1/Y_f(C)$ is encircled which indicates an increasing amplitude. The oscillation with $\omega = 1$ and $C \approx 2.5$ therefore probably has un unstable amplitude. This is confirmed by simulation. Below the output of the linear part is plotted for different initial amplitudes, showing a decreasing and an increasing oscillation.





It is clear that the control system will work well as long as there is no disturbance large enough to start an oscillation with an amplitude above the critical limit. (The growing oscillations that are created by large disturbances can be seen as a *windup* phenomenon of the integrator part of the regulator. When controlling a double integrator using a PID controller it is therefore very important to have some form of *anti-windup* compensation of the integral part.)

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14.7

The describing function is real. The Bode diagram shows that $\arg G_O(i\omega) = -180°$ and $\mid G_O(i\omega) \mid = 2$ at $\omega = 2$. This implies that the Nyquist curve crosses the negative real axis in the point -2 for $\omega = 2$. We hence have to solve the equation

$$-2 = \frac{-1}{Y_f(C)}$$

which implies

$$Y_f(C) = \frac{4}{\pi C}\sqrt{1 - \frac{1}{C^2}} = \frac{1}{2}$$

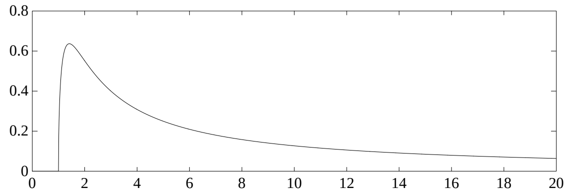
and

$$C^4 - \frac{64}{\pi^2}C^2 + \frac{64}{\pi^2} = 0$$

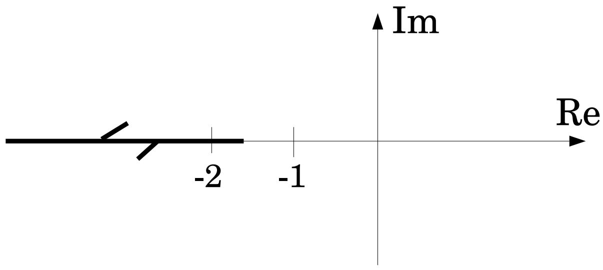
This gives

$$C = \overset{+}{(-)}\, 2.29 \quad \text{resp} \quad \overset{+}{(-)}\, 1.11$$

By inserting some values of C one realizes that $Y_f(C)$ looks like the figure

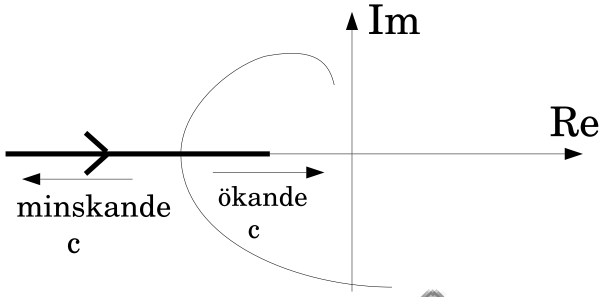


The function $-1/Y_f(C)$ thus moves along the real axis from $-\infty$ towards 0 when C increases, but stops at rougly $-1/.6$ and starts moving back towards $-\infty$. Hence, the curve $-1/Y_f(C)$ will intersect the Nyquist curve twice, as the computations indicate.



Analysis of the two candidate solutions gives

(I). C=1.11. For fixed amplitudes smaller than this value, when we think of the nonlinearity as a static gain with gain $Y_f(C)$, the point $-1/Y_f(C)$ will act as the point -1 in linear stability analysis, and tells us that the closed-loop system in a linear analysis would be asymptotically stable since it is not encircled. That means that any oscillation would decay, and C would not be constant as assumed. Instead it must decrease, and a new thought fixed value of C would once again indicate asymptotic stability. Hence, if initial oscillations are small, we suspect they will die out. (The relay here has a dead-zone which zeroes out everything between -1 and 1 so the result is reasonable, as a sinusoidal with amplitude 1.1 will almost completely be zeroed out and almost no energy enters the system. If the open-loop system G_0 is stable it is reasonable that the output will go to zero if the input almost always is zero)



(II). C=2.29. For fixed amplitudes larger than this value, when we think of the nonlinearity as a static gain with gain $Y_f(C)$, the point $-1/Y_f(C)$ will act as the point -1 in linear stability analysis, and tells us that the closed-loop system in a linear analysis would be asymptotically stable. That means that any oscillation would decay, and C would not be constant as assumed. Instead it would decrease, and a new thought fixed value of C would once again indicate asymptotic stability and C would decrease. However, if it decreases below 2.29, the point $-1/Y_f(C)$ is encircled by the Nyquist curve, and linear analysis tells us the system would be unstable and C would have to increase. At 2.29, we reach a stationary case were we neither increase nor decrease C according to linear theory, and we should suspect we will have oscillations with this amplitude. The limit cycle will have amplitude $C = 2.29$ and angular frequency $\omega = 2$.

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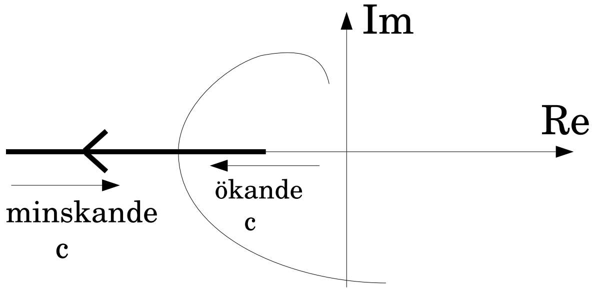

17 To Compensate Exactly for Nonlinearities

17.1

If we let

$$u = r - \cos x_1$$

we get a linear closed-loop system.

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17.2

The control signal

$$u = -y^4 + y^2 + r = -x_1^4 + x_1^2 + r$$

results in an exact feedback linearization.

Go back

17.3

The system is defined by

$$(*) \qquad \begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases}$$

Make the change of variables:

$$\begin{cases} z_1 = y \\ z_2 = \dot{y} \end{cases}$$

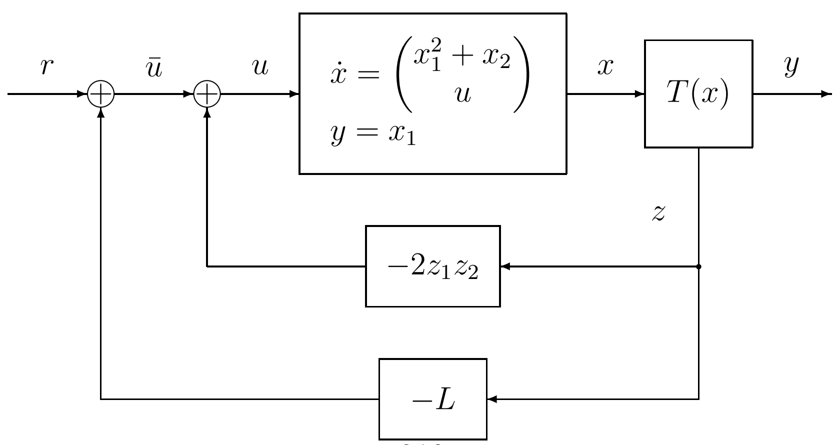
This results in

$$\dot{z}_1 = \dot{y} = z_2$$

$$\begin{aligned}
\dot{z}_2 &= \ddot{y} = \frac{d}{dt}\left(\dot{x}_1\right) \\
&= \frac{d}{dt}\left(x_1^2 + x_2\right) \\
&= 2x_1\dot{x}_1 + \dot{x}_2 = [\text{according to } (*)] \\
&= 2x_1\left(x_1^2 + x_2\right) + u \\
&= \begin{bmatrix} (*) \Rightarrow x_2 = \dot{x}_1 - x_1^2 \\ x_1 = y = z_1 \\ \dot{x}_1 = \dot{y} = z_2 \end{bmatrix} \\
&= 2z_1\left(z_1^2 + z_2 - z_1^2\right) + u \\
&= 2z_1 z_2 + u = \alpha(z) + \beta(z)u
\end{aligned}$$

An exact feedback linearization results from

$$u = \frac{-\alpha(z) + \bar{u}}{\beta(z)} = -2z_1 z_2 + \bar{u}.$$



Go back

17.4

As $\dot{x}_1$ depends on u we cannot choose y to be x_1. Hence, choose $y = x_2$.

$$\begin{aligned}
\dot{y} &= \dot{x}_2 = \sqrt{1+x_1} - \sqrt{1+x_2} \\
\ddot{y} &= \ddot{x}_2 = \frac{d}{dt}\left(\sqrt{1+x_1} - \sqrt{1+x_2}\right) = \\
&= \frac{1}{2\sqrt{1+x_1}}\dot{x}_1 - \frac{1}{2\sqrt{1+x_2}}\dot{x}_2 = \cdots = \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{1+x_1}} - \frac{\sqrt{1+x_1}}{\sqrt{1+x_2}}\right) + \frac{u}{2\sqrt{1+x_1}}
\end{aligned}$$

Thus, the relative degree is 2. Now, do the change of variables $z_1 = y, z_2 = \dot{y} \Rightarrow$

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \text{/from above/} = \frac{1}{2}\left(\frac{1}{\sqrt{1+x_1}} - \frac{\sqrt{1+x_1}}{\sqrt{1+x_2}}\right) + \frac{u}{2\sqrt{1+x_1}} = \\
&= \frac{1}{2}\left(\frac{1}{z_2+\sqrt{1+z_1}} - \frac{z_2+\sqrt{1+z_1}}{\sqrt{1+z_1}}\right) + \frac{1}{2}\frac{1}{z_2+\sqrt{1+z_1}}u = \\
&= \alpha(z) + \beta(z)u
\end{aligned}$$

Choose $u = \frac{1}{\beta(z)}\left(\bar{u} - \alpha(z)\right)$ to get an exact feedback linearization. What are the poles of the system?

Go back

17.5

(a) The force is

$$m\ddot{y} = F - k(y) - d(\dot{y})$$

Do the change of variables $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = F$ which results in the state-space form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m}(-k(x_1) - d(x_2) + x_3) \\
\dot{x}_3 &= -x_3 + u \\
y &= x_1
\end{aligned}$$

(b) Relative degree ν? Differentiate y with respect to time

$$\begin{aligned}
\dot{y} &= \dot{x}_1 = x_2 \\
\ddot{y} &= \dot{x}_2 = \frac{1}{m}(-k(x_1) - d(x_2) + x_3) \\
y^{(3)} &= \frac{1}{m}(-k'(x_1)\dot{x}_1 - d'(x_2)\dot{x}_2 + \dot{x}_3) \\
&= \frac{1}{m}(-k'(x_1)\dot{x}_1 - d'(x_2)\dot{x}_2 - x_3 + u)
\end{aligned}$$

As $\nu = n = 3$ we can make an exact feedback linearization. Make the change of variables

$$\begin{cases} z_1 = y \\ z_2 = \dot{y} \\ z_3 = \ddot{y} \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_1 = z_1 \\ x_2 = z_2 \\ x_3 = k(z_1) + d(z_2) + mz_3 \end{cases}$$

which results in the state-space form

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= \frac{1}{m}(-k'(z_1)z_2 - d'(z_2)z_3 - k(z_1) - d(z_2) - mz_3 + u) \\
y &= z_1
\end{aligned}$$

The control signal

$$u = m\tilde{u} + k'(z_1)z_2 + d'(z_2)z_3 + k(z_1) + d(z_2) + mz_3$$

results in a linear system from $\tilde{u}$ y.

Go back

Selected chapters from draft of

An Introduction to Game Theory
by
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Contents

Preface

Game theoretic reasoning pervades economic theory and is used widely in other social and behavioral sciences. This book presents the main ideas of game theory and shows how they can be used to understand economic, social, political, and biological phenomena. It assumes no knowledge of economics, political science, or any other social or behavioral science. It emphasizes the ideas behind the theory rather than their mathematical expression, and assumes no specific mathematical knowledge beyond that typically taught in US and Canadian high schools. (Chapter 17 reviews the mathematical concepts used in the book.) In particular, calculus is not used, except in the appendix of Chapter 9 (Section 9.7). Nevertheless, all concepts are defined precisely, and logical reasoning is used extensively. The more comfortable you are with tight logical analysis, the easier you will find the arguments. In brief, my aim is to explain the main ideas of game theory as simply as possible while maintaining complete precision.

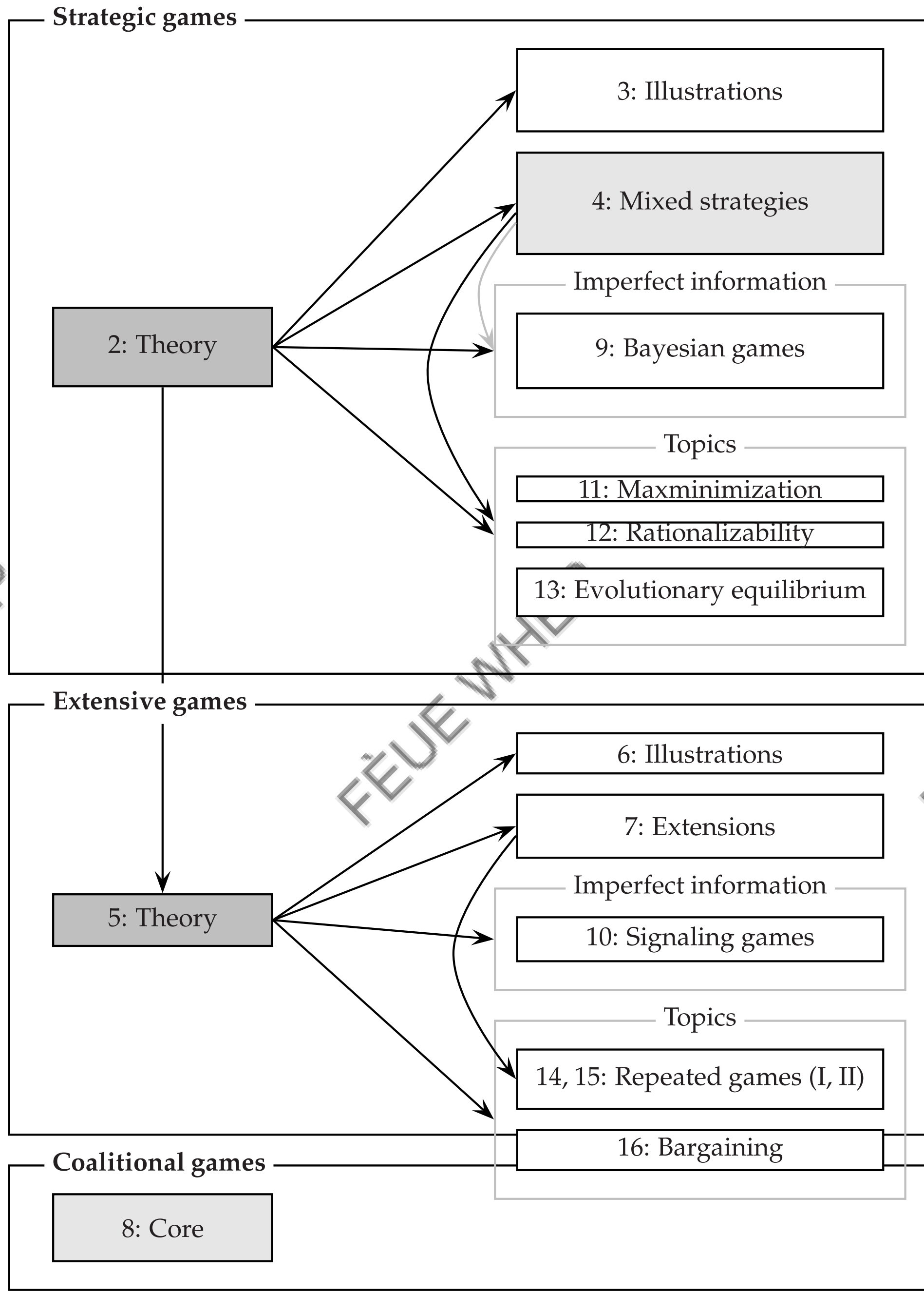
The only way to appreciate the theory is to see it in action, or better still to put it into action. So the book includes a wide variety of illustrations from the social and behavioral sciences, and over 200 exercises.

The structure of the book is illustrated in the figure on the next page. The gray boxes indicate core chapters (the darker gray, the more important). An black arrow from Chapter i to Chapter j means that Chapter j depends on Chapter i. The gray arrow from Chapter 4 to Chapter 9 means that the latter depends weakly on the former; for all but Section 9.8 only an understanding of expected payoffs (Section 4.1.3) is required, not a knowledge of mixed strategy Nash equilibrium. (Two chapters are not included in this figure: Chapter 1 reviews the theory of a single rational decision-maker, and Chapter 17 reviews the mathematical concepts used in the book.)

Each topic is presented with the aid of "Examples", which highlight theoretical points, and "Illustrations", which demonstrate how the theory may be used to understand social, economic, political, and biological phenomena. The "Illustrations" for the key models of strategic and extensive games are grouped in separate chapters (3 and 6), whereas those for the other models occupy the same chapters as the theory. The "Illustrations" introduce no new theoretical points, and any or all of them may be skipped without loss of continuity.

The limited dependencies between chapters mean that several routes may be taken through the book.

- At a minimum, you should study Chapters 2 (Nash Equilibrium: Theory) and 5 (Extensive Games with Perfect Information: Theory).
- Optionally you may sample some sections of Chapters 3 (Nash Equilibrium:



xivFigure 0.1 The structure of the book. The area of each box is proportional to the length of the chapter the box represents. The boxes corresponding to the core chapters are shaded gray; the ones shaded dark gray are more central that the ones shaded light gray. An arrow from Chapter i to Chapter j means that Chapter i is a prerequisite for Chapter j. The gray arrow from Chapter 4 to Chapter 9 means that the latter depends only weakly on the former.

Illustrations) and 6 (Extensive Games with Perfect Information: Illustrations).

- You may add to this plan any combination of Chapters 4 (Mixed Strategy Equilibrium), 9 (Bayesian Games, except Section 9.8), 7 (Extensive Games with Perfect Information: Extensions and Discussion), 8 (Coalitional Games and the Core), and 16 (Bargaining).

- If you read Chapter 4 (Mixed Strategy Equilibrium) then you may in addition study any combination of the remaining chapters covering strategic games, and if you study Chapter 7 (Extensive Games with Perfect Information: Extensions and Discussion) then you are ready to tackle Chapters 14 and 15 (Repeated Games).

All the material should be accessible to undergraduate students. A one-semester course for third or fourth year North American economics majors (who have been exposed to a few of the main ideas in first and second year courses) could cover up to about half the material in the book in moderate detail.

Personal pronouns

The lack of a sex-neutral third person singular pronoun in English has led many writers of formal English to use "he" for this purpose. Such usage conflicts with that of everyday speech. People may say "when an airplane pilot is working, he needs to concentrate", but they do not usually say "when a flight attendant is working, he needs to concentrate" or "when a secretary is working, he needs to concetrate". The use of "he" only for roles in which men traditionally predominate in Western societies suggests that women may not play such roles; I find this insinuation unacceptable.

To quote the *New Oxford Dictionary of English*, "[the use of *he* to refer to refer to a person of unspecified sex] has become . . . a hallmark of old-fashioned language or sexism in language." Writers have become sensitive to this issue in the last half century, but the lack of a sex-neutral pronoun "has been felt since at least as far back as Middle English" (*Webster's Dictionary of English Usage*, Merriam-Webster Inc., 1989, p. 499). A common solution has been to use "they", a usage that the *New Oxford Dictionary of English* endorses (and employs). This solution can create ambiguity when the pronoun follows references to more than one person; it also does not always sound natural. I choose a different solution: I use "she" exclusively. Obviously this usage, like that of "he", is not sex-neutral; but its use may do something to counterbalance the widespread use of "he", and does not seem likely to do any harm.

Acknowledgements

I owe a huge debt to Ariel Rubinstein. I have learned, and continue to learn, vastly from him about game theory. His influence on this book will be clear to anyone

familiar with our jointly-authored book *A course in game theory*. Had we not written that book and our previous book *Bargaining and markets*, I doubt that I would have embarked on this project.

Discussions over the years with Jean-Pierre Benoît, Vijay Krishna, Michael Peters, and Carolyn Pitchik have improved my understanding of many game theoretic topics.

Many people have generously commented on all or parts of drafts of the book. I am particularly grateful to Jeffrey Banks, Nikolaos Benos, Ted Bergstrom, Tilman Börgers, Randy Calvert, Vu Cao, Rachel Croson, Eddie Dekel, Marina De Vos, Laurie Duke, Patrick Elias, Mukesh Eswaran, Xinhua Gu, Costas Halatsis, Joe Harrington, Hiroyuki Kawakatsu, Lewis Kornhauser, Jack Leach, Simon Link, Bart Lipman, Kin Chung Lo, Massimo Marinacci, Peter McCabe, Barry O'Neill, Robin G. Osborne, Marco Ottaviani, Marie Rekkas, Bob Rosenthal, Al Roth, Matthew Shum, Giora Slutzki, Michael Smart, Nick Vriend, and Chuck Wilson.

I thank also the anonymous reviewers consulted by Oxford University Press and several other presses; the suggestions in their reviews greatly improved the book.

The book has its origins in a course I taught at Columbia University in the early 1980s. My experience in that course, and in courses at McMaster University, where I taught from early drafts, and at the University of Toronto, brought the book to its current form. The Kyoto Institute of Economic Research at Kyoto University provided me with a splendid environment in which to work on the book during two months in 1999.

References

The "Notes" section at the end of each chapter attempts to assign credit for the ideas in the chapter. Several cases present difficulties. In some cases, ideas evolved over a long period of time, with contributions by many people, making their origins hard to summarize in a sentence or two. In a few cases, my research has led to a conclusion about the origins of an idea different from the standard one. In all cases, I cite the relevant papers without regard to their difficulty.

Over the years, I have taken exercises from many sources. I have attempted to remember where I got them from, and have given credit, but I have probably missed some.

Examples addressing economic, political, and biological issues

The following tables list examples that address economic, political, and biological issues. [SO FAR CHECKED ONLY THROUGH CHAPTER 7.]

Games related to economic issues (THROUGH CHAPTER 7)

Exercise 31.1, Section 2.8.4, Exercise 42.1	Provision of a public good
Section 2.9.4	Collective decision-making
Section 3.1, Exercise 133.1	Cournot's model of oligopoly
Section 3.1.5	Common property
Section 3.2, Exercise 133.2, Exercise 143.2, Exercise 189.1, Exercise 210.1	Bertrand's model of oligopoly
Exercise 75.1	Competition in product characteristics
Section 3.5	Auctions with perfect information
Section 3.6	Accident law
Section 4.6	Expert diagnosis
Exercise 125.2, Exercise 208.1	Price competition between sellers
Section 4.8	Reporting a crime (private provision of a public good)
Example 141.1	All-pay auction with perfect information
Exercise 172.2	Entry into an industry by a financially-constrained challenger
Exercise 175.1	The "rotten kid theorem"
Section 6.2.2	The holdup game
Section 6.3	Stackelberg's model of duopoly
Exercise 207.2	A market game
Section 7.2	Entry into a monopolized industry
Section 7.5	Exit from a declining industry
Example 227.1	Chain-store game

Games related to political issues (THROUGH CHAPTER 7)

Exercise 32.2	Voter participation
Section 2.9.3	Voting
Exercise 47.3	Approval voting
Section 2.9.4	Collective decision-making
Section 3.3, Exercise 193.3, Exercise 193.4, Section 7.3	Hotelling's model of electoral competition

Exercise 73.1	Electoral competition between policy-motivated candidates
Exercise 73.2	Electoral competition between citizen-candidates
Exercise 88.3	Lobbying as an auction
Exercise 115.3	Voter participation
Exercise 139.1	Allocating resources in election campaigns
Section 6.4	Buying votes in a legislature
Section 7.4	Committee decision-making
Exercise 224.1	Cohesion of governing coalitions

Games related to biological issues (THROUGH CHAPTER 7)

Exercise 16.1	Hermaphroditic fish
Section 3.4	War of attrition

Typographic conventions, numbering, and nomenclature

In formal definitions, the terms being defined are set in **boldface**. Terms are set in *italics* when they are defined informally.

Definitions, propositions, examples, and exercises are numbered according to the page on which they appear. If the first such object on page z is an exercise, for example, it is called Exercise $z.1$; if the next object on that page is a definition, it is called Definition $z.2$. For example, the definition of a strategic game with ordinal preferences on page 11 is Definition 11.1. This scheme allows numbered items to found rapidly, and also facilitates precise index entries.

Symbol/term	Meaning
?	Exercise
??	Hard exercise
▶	Definition
■	Proposition
◆	Example: a game that illustrates a game-theoretic point
Illustration	A game, or family of games, that shows how the theory can illuminate observed phenomena

I maintain a website for the book. The current URL is
`http://www.economics.utoronto.ca/osborne/igt/`.

Draft chapter from *An introduction to game theory* by Martin J. Osborne
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1 Introduction

1.1 What is game theory?

GAME THEORY aims to help us understand situations in which decision-makers interact. A game in the everyday sense—"a competitive activity ... in which players contend with each other according to a set of rules", in the words of my dictionary—is an example of such a situation, but the scope of game theory is vastly larger. Indeed, I devote very little space to games in the everyday sense; my main focus is the use of game theory to illuminate economic, political, and biological phenomena.

A list of some of the applications I discuss will give you an idea of the range of situations to which game theory can be applied: firms competing for business, political candidates competing for votes, jury members deciding on a verdict, animals fighting over prey, bidders competing in an auction, the evolution of siblings' behavior towards each other, competing experts' incentives to provide correct diagnoses, legislators' voting behavior under pressure from interest groups, and the role of threats and punishment in long-term relationships.

Like other sciences, game theory consists of a collection of models. A model is an abstraction we use to understand our observations and experiences. What "understanding" entails is not clear-cut. Partly, at least, it entails our perceiving relationships between situations, isolating principles that apply to a range of problems, so that we can fit into our thinking new situations that we encounter. For example, we may fit our observation of the path taken by a lobbed tennis ball into a model that assumes the ball moves forward at a constant velocity and is pulled towards the ground by the constant force of "gravity". This model enhances our understanding because it fits well no matter how hard or in which direction the ball is hit, and applies also to the paths taken by baseballs, cricket balls, and a wide variety of other missiles, launched in any direction.

A model is unlikely to help us understand a phenomenon if its assumptions are wildly at odds with our observations. At the same time, a model derives power from its simplicity; the assumptions upon which it rests should capture the essence of the situation, not irrelevant details. For example, when considering the path taken by a lobbed tennis ball we should ignore the dependence of the force of gravity on the distance of the ball from the surface of the earth.

Models cannot be judged by an absolute criterion: they are neither "right" nor "wrong". Whether a model is useful or not depends, in part, on the purpose for which we use it. For example, when I determine the shortest route from Florence to Venice, I do not worry about the projection of the map I am using; I work under the assumption that the earth is flat. When I determine the shortest route from Beijing to Havana, however, I pay close attention to the projection—I assume that the earth is spherical. And were I to climb the Matterhorn I would assume that the earth is neither flat nor spherical!

One reason for improving our understanding of the world is to enhance our ability to mold it to our desires. The understanding that game theoretic models give is particularly relevant in the social, political, and economic arenas. Studying game theoretic models (or other models that apply to human interaction) may also suggest ways in which our behavior may be modified to improve our own welfare. By analyzing the incentives faced by negotiators locked in battle, for example, we may see the advantages and disadvantages of various strategies.

The models of game theory are precise expressions of ideas that can be presented verbally. However, verbal descriptions tend to be long and imprecise; in the interest of conciseness and precision, I frequently use mathematical symbols when describing models. Although I use the language of mathematics, I use few of its concepts; the ones I use are described in Chapter 17. My aim is to take advantage of the precision and conciseness of a mathematical formulation without losing sight of the underlying ideas.

Game-theoretic modeling starts with an idea related to some aspect of the interaction of decision-makers. We express this idea precisely in a model, incorporating features of the situation that appear to be relevant. This step is an art. We wish to put enough ingredients into the model to obtain nontrivial insights, but not so many that we are lead into irrelevant complications; we wish to lay bare the underlying structure of the situation as opposed to describe its every detail. The next step is to analyze the model—to discover its implications. At this stage we need to adhere to the rigors of logic; we must not introduce extraneous considerations absent from the model. Our analysis may yield results that confirm our idea, or that suggest it is wrong. If it is wrong, the analysis should help us to understand why it is wrong. We may see that an assumption is inappropriate, or that an important element is missing from the model; we may conclude that our idea is invalid, or that we need to investigate it further by studying a different model. Thus, the interaction between our ideas and models designed to shed light on them runs in two directions: the implications of models help us determine whether our ideas make sense, and these ideas, in the light of the implications of the models, may show us how the assumptions of our models are inappropriate. In either case, the process of formulating and analyzing a model should improve our understanding of the situation we are considering.

AN OUTLINE OF THE HISTORY OF GAME THEORY

Some game-theoretic ideas can be traced to the 18th century, but the major development of the theory began in the 1920s with the work of the mathematician Emile Borel (1871–1956) and the polymath John von Neumann (1903–57). A decisive event in the development of the theory was the publication in 1944 of the book *Theory of games and economic behavior* by von Neumann and Oskar Morgenstern. In the 1950s game-theoretic models began to be used in economic theory and political science, and psychologists began studying how human subjects behave in experimental games. In the 1970s game theory was first used as a tool in evolutionary biology. Subsequently, game theoretic methods have come to dominate microeconomic theory and are used also in many other fields of economics and a wide range of other social and behavioral sciences. The 1994 Nobel prize in economics was awarded to the game theorists John C. Harsanyi (1920–2000), John F. Nash (1928–), and Reinhard Selten (1930–).

JOHN VON NEUMANN

John von Neumann, the most important figure in the early development of game theory, was born in Budapest, Hungary, in 1903. He displayed exceptional mathematical ability as a child (he had mastered calculus by the age of 8), but his father, concerned about his son's financial prospects, did not want him to become a mathematician. As a compromise he enrolled in mathematics at the University of Budapest in 1921, but immediately left to study chemistry, first at the University of Berlin and subsequently at the Swiss Federal Institute of Technology in Zurich, from which he earned a degree in chemical engineering in 1925. During his time in Germany and Switzerland he returned to Budapest to write examinations, and in 1926 obtained a PhD in mathematics from the University of Budapest. He taught in Berlin and Hamburg, and, from 1930 to 1933, at Princeton University. In 1933 he became the youngest of the first six professors of the School of Mathematics at the Institute for Advanced Study in Princeton (Einstein was another).

Von Neumann's first published scientific paper appeared in 1922, when he was 19 years old. In 1928 he published a paper that establishes a key result on strictly competitive games (a result that had eluded Borel). He made many major contributions in pure and applied mathematics and in physics—enough, according to Halmos (1973), "for about three ordinary careers, in pure mathematics alone". While at the Institute for Advanced Study he collaborated with the Princeton economist Oskar Morgenstern in writing *Theory of games and economic behavior*, the book that established game theory as a field. In the 1940s he became increasingly involved in applied work. In 1943 he became a consultant to the Manhattan project, which was developing an atomic bomb. In 1944 he became involved with the development of the first electronic computer, to which he made major contributions. He stayed at Princeton until 1954, when he became a member of the US Atomic Energy Commission. He died in 1957.

1.2 The theory of rational choice

The theory of rational choice is a component of many models in game theory. Briefly, this theory is that a decision-maker chooses the best action according to her preferences, among all the actions available to her. No qualitative restriction is placed on the decision-maker's preferences; her "rationality" lies in the consistency of her decisions when faced with different sets of available actions, not in the nature of her likes and dislikes.

1.2.1 Actions

The theory is based on a model with two components: a set A consisting of all the actions that, under some circumstances, are available to the decision-maker, and a specification of the decision-maker's preferences. In any given situation the decision-maker is faced with a subset[1] of A, from which she must choose a single element. The decision-maker knows this subset of available choices, and takes it as given; in particular, the subset is not influenced by the decision-maker's preferences. The set A could, for example, be the set of bundles of goods that the decision-maker can possibly consume; given her income at any time, she is restricted to choose from the subset of A containing the bundles she can afford.

1.2.2 Preferences and payoff functions

As to preferences, we assume that the decision-maker, when presented with any pair of actions, knows which of the pair she prefers, or knows that she regards both actions as equally desirable (is "indifferent between the actions"). We assume further that these preferences are consistent in the sense that if the decision-maker prefers the action a to the action b, and the action b to the action c, then she prefers the action a to the action c. No other restriction is imposed on preferences. In particular, we do not rule out the possibility that a person's preferences are altruistic in the sense that how much she likes an outcome depends on some other person's welfare. Theories that use the model of rational choice aim to derive implications that do not depend on any qualitative characteristic of preferences.

How can we describe a decision-maker's preferences? One way is to specify, for each possible pair of actions, the action the decision-maker prefers, or to note that the decision-maker is indifferent between the actions. Alternatively we can "represent" the preferences by a *payoff function*, which associates a number with each action in such a way that actions with higher numbers are preferred. More

[1] See Chapter 17 for a description of mathematical terminology.

precisely, the payoff function u represents a decision-maker's preferences if, for any actions a in A and b in A,

$$u(a) > u(b) \text{ if and only if the decision-maker prefers } a \text{ to } b. \tag{5.1}$$

(A better name than payoff function might be "preference indicator function"; in economic theory a payoff function that represents a consumer's preferences is often referred to as a "utility function".)

◆ EXAMPLE 5.2 (Payoff function representing preferences) A person is faced with the choice of three vacation packages, to Havana, Paris, and Venice. She prefers the package to Havana to the other two, which she regards as equivalent. Her preferences between the three packages are represented by any payoff function that assigns the same number to both Paris and Venice and a higher number to Havana. For example, we can set $u(\text{Havana}) = 1$ and $u(\text{Paris}) = u(\text{Venice}) = 0$, or $u(\text{Havana}) = 10$ and $u(\text{Paris}) = u(\text{Venice}) = 1$, or $u(\text{Havana}) = 0$ and $u(\text{Paris}) = u(\text{Venice}) = -2$.

? EXERCISE 5.3 (Altruistic preferences) Person 1 cares both about her income and about person 2's income. Precisely, the value she attaches to each unit of her own income is the same as the value she attaches to any two units of person 2's income. How do her preferences order the outcomes $(1,4)$, $(2,1)$, and $(3,0)$, where the first component in each case is person 1's income and the second component is person 2's income? Give a payoff function consistent with these preferences.

A decision-maker's preferences, in the sense used here, convey only *ordinal* information. They may tell us that the decision-maker prefers the action a to the action b to the action c, for example, but they do not tell us "how much" she prefers a to b, or whether she prefers a to b "more" than she prefers b to c. Consequently a payoff function that represents a decision-maker's preferences also conveys only ordinal information. It may be tempting to think that the payoff numbers attached to actions by a payoff function convey intensity of preference—that if, for example, a decision-maker's preferences are represented by a payoff function u for which $u(a) = 0$, $u(b) = 1$, and $u(c) = 100$, then the decision-maker likes c a lot more than b but finds little difference between a and b. *But a payoff function contains no such information!* The *only* conclusion we can draw from the fact that $u(a) = 0$, $u(b) = 1$, and $u(c) = 100$ is that the decision-maker prefers c to b to a; her preferences are represented equally well by the payoff function v for which $v(a) = 0$, $v(b) = 100$, and $v(c) = 101$, for example, or any other function w for which $w(a) < w(b) < w(c)$.

From this discussion we see that a decision-maker's preferences are represented by many different payoff functions. Looking at the condition (5.1) under which the payoff function u represents a decision-maker's preferences, we see that if u represents a decision-maker's preferences and the payoff function v assigns a higher number to the action a than to the action b if and only if the payoff function u does so, then v also represents these preferences. Stated more compactly, if u represents a decision-maker's preferences and v is another payoff function for which

$$v(a) > v(b) \text{ if and only if } u(a) > u(b)$$

then v also represents the decision-maker's preferences. Or, more succinctly, if u represents a decision-maker's preferences then any increasing function of u also represents these preferences.

? EXERCISE 6.1 (Alternative representations of preferences) A decision-maker's preferences over the set $A = \{a, b, c\}$ are represented by the payoff function u for which $u(a) = 0$, $u(b) = 1$, and $u(c) = 4$. Are they also represented by the function v for which $v(a) = -1$, $v(b) = 0$, and $v(c) = 2$? How about the function w for which $w(a) = w(b) = 0$ and $w(c) = 8$?

Sometimes it is natural to formulate a model in terms of preferences and then find payoff functions that represent these preferences. In other cases it is natural to start with payoff functions, even if the analysis depends only on the underlying preferences, not on the specific representation we choose.

1.2.3 The theory of rational choice

The theory of rational choice is that in any given situation the decision-maker chooses the member of the available subset of A that is best according to her preferences. Allowing for the possibility that there are several equally attractive best actions, **the theory of rational choice** is:

> *the action chosen by a decision-maker is at least as good, according to her preferences, as every other available action.*

For any action, we can design preferences with the property that no other action is preferred. Thus if we have no information about a decision-maker's preferences, and make no assumptions about their character, any *single* action is consistent with the theory. However, if we assume that a decision-maker who is indifferent between two actions sometimes chooses one action and sometimes the other, not every *collection* of choices for different sets of available actions is consistent with the theory. Suppose, for example, we observe that a decision-maker chooses a whenever she faces the set $\{a, b\}$, but sometimes chooses b when facing the set $\{a, b, c\}$. The fact that she always chooses a when faced with $\{a, b\}$ means that she prefers a to b (if she were indifferent then she would sometimes choose b). But then when she faces the set $\{a, b, c\}$ she must choose either a or c, never b. Thus her choices are inconsistent with the theory. (More concretely, if you choose the same dish from the menu of your favorite lunch spot whenever there are no specials then, regardless of your preferences, it is inconsistent for you to choose some other item *from the menu* on a day when there is an off-menu special.)

If you have studied the standard economic theories of the consumer and the firm, you have encountered the theory of rational choice before. In the economic theory of the consumer, for example, the set of available actions is the set of all bundles of goods that the consumer can afford. In the theory of the firm, the set of available actions is the set of all input-output vectors, and the action a is preferred to the action b if and only if a yields a higher profit than does b.

1.2.4 Discussion

The theory of rational choice is enormously successful; it is a component of countless models that enhance our understanding of social phenomena. It pervades economic theory to such an extent that arguments are classified as "economic" as much because they apply the theory of rational choice as because they involve particularly "economic" variables.

Nevertheless, under some circumstances its implications are at variance with observations of human decision-making. To take a small example, adding an undesirable action to a set of actions sometimes significantly changes the action chosen (see Rabin 1998, 38). The significance of such discordance with the theory depends upon the phenomenon being studied. If we are considering how the markup of price over cost in an industry depends on the number of firms, for example, this sort of weakness in the theory may be unimportant. But if we are studying how advertising, designed specifically to influence peoples' preferences, affects consumers' choices, then the inadequacies of the model of rational choice may be crucial.

No general theory currently challenges the supremacy of rational choice theory. But you should bear in mind as you read this book that the model of choice that underlies most of the theories has its limits; some of the phenomena that you may think of explaining using a game theoretic model may lie beyond these limits. As always, the proof of the pudding is in the eating: if a model enhances our understanding of the world, then it serves its purpose.

1.3 Coming attractions

Part I presents the main models in game theory: a strategic game, an extensive game, and a coalitional game. These models differ in two dimensions. A strategic game and an extensive game focus on the actions of individuals, whereas a coalitional game focuses on the outcomes that can be achieved by groups of individuals; a strategic game and a coalitional game consider situations in which actions are chosen once and for all, whereas an extensive game allows for the possibility that plans may be revised as they are carried out.

The model, consisting of actions and preferences, to which rational choice theory is applied is tailor-made for the theory; if we want to develop another theory, we need to add elements to the model in addition to actions and preferences. The same is not true of most models in game theory: strategic interaction is sufficiently complex that even a relatively simple model can admit more than one theory of the outcome. We refer to a theory that specifies a set of outcomes for a model as a "solution". Chapter 2 describes the model of a strategic game and the solution of Nash equilibrium for such games. The theory of Nash equilibrium in a strategic game has been applied to a vast variety of situations; a handful of some of the most significant applications are discussed in Chapter 3.

Chapter 4 extends the notion of Nash equilibrium in a strategic game to allow for the possibility that a decision-maker, when indifferent between actions, may not always choose the same action, or, alternatively, identical decision-makers facing the same set of actions may choose different actions if more than one is best.

The model of an extensive game, which adds a temporal dimension to the description of strategic interaction captured by a strategic game, is studied in Chapters 5, 6, and 7. Part I concludes with Chapter 8, which discusses the model of a coalitional game and a solution concept for such a game, the core.

Part II extends the models of a strategic game and an extensive game to situations in which the players do not know the other players' characteristics or past actions. Chapter 9 extends the model of a strategic game, and Chapter 10 extends the model of an extensive game.

The chapters in Part III cover topics outside the basic theory. Chapters 11 and 12 examine two theories of the outcome in a strategic game that are alternatives to the theory of Nash equilibrium. Chapter 13 discusses how a variant of the notion of Nash equilibrium in a strategic game can be used to model behavior that is the outcome of evolutionary pressure rather than conscious choice. Chapters 14 and 15 use the model of an extensive game to study long-term relationships, in which the same group of players repeatedly interact. Finally, Chapter 16 uses strategic, extensive, and coalitional models to gain an understanding of the outcome of bargaining.

Notes

Von Neumann and Morgenstern (1944) established game theory as a field. The information about John von Neumann in the box on page 3 is drawn from Ulam (1958), Halmos (1973), Thompson (1987), Poundstone (1992), and Leonard (1995). Aumann (1985), on which I draw in the opening section, contains a very readable discussion of the aims and achievements of game theory. Two papers that discuss the limitations of rational choice theory are Rabin (1998) and Elster (1998).

2 Nash Equilibrium: Theory

Prerequisite: Chapter 1.

2.1 Strategic games

A STRATEGIC GAME is a model of interacting decision-makers. In recognition of the interaction, we refer to the decision-makers as *players*. Each player has a set of possible *actions*. The model captures interaction between the players by allowing each player to be affected by the actions of *all* players, not only her own action. Specifically, each player has *preferences* about the action *profile*—the list of all the players' actions. (See Section 17.5, in the mathematical appendix, for a discussion of profiles.)

More precisely, a strategic game is defined as follows. (The qualification "with ordinal preferences" distinguishes this notion of a strategic game from a more general notion studied in Chapter 4.)

▶ DEFINITION 11.1 (*Strategic game with ordinal preferences*) A **strategic game** (with ordinal preferences) consists of

- a set of **players**
- for each player, a set of **actions**
- for each player, **preferences** over the set of action profiles.

A very wide range of situations may be modeled as strategic games. For example, the players may be firms, the actions prices, and the preferences a reflection of the firms' profits. Or the players may be candidates for political office, the actions campaign expenditures, and the preferences a reflection of the candidates' probabilities of winning. Or the players may be animals fighting over some prey, the actions concession times, and the preferences a reflection of whether an animal wins or loses. In this chapter I describe some simple games designed to capture fundamental conflicts present in a variety of situations. The next chapter is devoted to more detailed applications to specific phenomena.

As in the model of rational choice by a single decision-maker (Section 1.2), it is frequently convenient to specify the players' preferences by giving *payoff functions* that represent them. Bear in mind that these payoffs have only *ordinal* significance. If a player's payoffs to the action profiles a, b, and c are 1, 2, and 10, for example, the only conclusion we can draw is that the player prefers c to b and b to a; the numbers do *not* imply that the player's preference between c and b is stronger than her preference between a and b.

Time is absent from the model. The idea is that each player chooses her action once and for all, and the players choose their actions "simultaneously" in the sense that no player is informed, when she chooses her action, of the action chosen by any other player. (For this reason, a strategic game is sometimes referred to as a "simultaneous move game".) Nevertheless, an action may involve activities that extend over time, and may take into account an unlimited number of contingencies. An action might specify, for example, "if company X's stock falls below \$10, buy 100 shares; otherwise, do not buy any shares". (For this reason, an action is sometimes called a "strategy".) However, the fact that time is absent from the model means that when analyzing a situation as a strategic game, we abstract from the complications that may arise if a player is allowed to change her plan as events unfold: we assume that actions are chosen once and for all.

2.2 Example: the Prisoner's Dilemma

One of the most well-known strategic games is the *Prisoner's Dilemma*. Its name comes from a story involving suspects in a crime; its importance comes from the huge variety of situations in which the participants face incentives similar to those faced by the suspects in the story.

◆ EXAMPLE 12.1 (Prisoner's Dilemma) Two suspects in a major crime are held in separate cells. There is enough evidence to convict each of them of a minor offense, but not enough evidence to convict either of them of the major crime unless one of them acts as an informer against the other (finks). If they both stay quiet, each will be convicted of the minor offense and spend one year in prison. If one and only one of them finks, she will be freed and used as a witness against the other, who will spend four years in prison. If they both fink, each will spend three years in prison.

This situation may be modeled as a strategic game:

Players The two suspects.

Actions Each player's set of actions is $\{Quiet, Fink\}$.

Preferences Suspect 1's ordering of the action profiles, from best to worst, is $(Fink, Quiet)$ (she finks and suspect 2 remains quiet, so she is freed), $(Quiet, Quiet)$ (she gets one year in prison), $(Fink, Fink)$ (she gets three years in prison), $(Quiet, Fink)$ (she gets four years in prison). Suspect 2's ordering is $(Quiet, Fink)$, $(Quiet, Quiet)$, $(Fink, Fink)$, $(Fink, Quiet)$.

We can represent the game compactly in a table. First choose payoff functions that represent the suspects' preference orderings. For suspect 1 we need a function u_1 for which

$$u_1(Fink, Quiet) > u_1(Quiet, Quiet) > u_1(Fink, Fink) > u_1(Quiet, Fink).$$

A simple specification is $u_1(Fink, Quiet) = 3$, $u_1(Quiet, Quiet) = 2$, $u_1(Fink, Fink) = 1$, and $u_1(Quiet, Fink) = 0$. For suspect 2 we can similarly choose the function u_2 for which $u_2(Quiet, Fink) = 3$, $u_2(Quiet, Quiet) = 2$, $u_2(Fink, Fink) = 1$, and $u_2(Fink, Quiet) = 0$. Using these representations, the game is illustrated in Figure 13.1. In this figure the two rows correspond to the two possible actions of player 1, the two columns correspond to the two possible actions of player 2, and the numbers in each box are the players' payoffs to the action profile to which the box corresponds, with player 1's payoff listed first.

		Suspect 2	
		Quiet	*Fink*
Suspect 1	*Quiet*	$2, 2$	$0, 3$
	Fink	$3, 0$	$1, 1$

Figure 13.1 The *Prisoner's Dilemma* (Example 12.1).

The *Prisoner's Dilemma* models a situation in which there are gains from cooperation (each player prefers that both players choose *Quiet* than they both choose *Fink*) but each player has an incentive to "free ride" (choose *Fink*) whatever the other player does. The game is important not because we are interested in understanding the incentives for prisoners to confess, but because many other situations have similar structures. Whenever each of two players has two actions, say C (corresponding to *Quiet*) and D (corresponding to *Fink*), player 1 prefers (D, C) to (C, C) to (D, D) to (C, D), and player 2 prefers (C, D) to (C, C) to (D, D) to (D, C), the *Prisoner's Dilemma* models the situation that the players face. Some examples follow.

2.2.1 *Working on a joint project*

You are working with a friend on a joint project. Each of you can either work hard or goof off. If your friend works hard then you prefer to goof off (the outcome of the project would be better if you worked hard too, but the increment in its value to you is not worth the extra effort). You prefer the outcome of your both working hard to the outcome of your both goofing off (in which case nothing gets accomplished), and the worst outcome for you is that you work hard and your friend goofs off (you hate to be "exploited"). If your friend has the same preferences then the game that models the situation you face is given in Figure 14.1, which, as you can see, differs from the *Prisoner's Dilemma* only in the names of the actions.

	Work hard	*Goof off*
Work hard	$2,2$	$0,3$
Goof off	$3,0$	$1,1$

Figure 14.1 Working on a joint project.

I am *not* claiming that a situation in which two people pursue a joint project *necessarily* has the structure of the *Prisoner's Dilemma*, only that the players' preferences in such a situation *may* be the same as in the *Prisoner's Dilemma*! If, for example, each person prefers to work hard than to goof off when the other person works hard, then the *Prisoner's Dilemma* does *not* model the situation: the players' preferences are different from those given in Figure 14.1.

? EXERCISE 14.1 (Working on a joint project) Formulate a strategic game that models a situation in which two people work on a joint project in the case that their preferences are the same as those in the game in Figure 14.1 except that each person prefers to work hard than to goof off when the other person works hard. Present your game in a table like the one in Figure 14.1.

2.2.2 Duopoly

In a simple model of a duopoly, two firms produce the same good, for which each firm charges either a low price or a high price. Each firm wants to achieve the highest possible profit. If both firms choose *High* then each earns a profit of $1000. If one firm chooses *High* and the other chooses *Low* then the firm choosing *High* obtains no customers and makes a loss of $200, whereas the firm choosing *Low* earns a profit of $1200 (its unit profit is low, but its volume is high). If both firms choose *Low* then each earns a profit of $600. Each firm cares only about its profit, so we can represent its preferences by the profit it obtains, yielding the game in Figure 14.2.

	High	*Low*
High	$1000, 1000$	$-200, 1200$
Low	$1200, -200$	$600, 600$

Figure 14.2 A simple model of a price-setting duopoly.

Bearing in mind that what matters are the players' preferences, not the particular payoff functions that we use to represent them, we see that this game, like the previous one, differs from the *Prisoner's Dilemma* only in the names of the actions.

The action *High* plays the role of *Quiet*, and the action *Low* plays the role of *Fink*; firm 1 prefers $(Low, High)$ to $(High, High)$ to (Low, Low) to $(High, Low)$, and firm 2 prefers $(High, Low)$ to $(High, High)$ to (Low, Low) to $(Low, High)$.

As in the previous example, I do not claim that the incentives in a duopoly are *necessarily* those in the *Prisoner's Dilemma*; different assumptions about the relative sizes of the profits in the four cases generate a different game. Further, in this case one of the abstractions incorporated into the model—that each firm has only two prices to choose between—may not be harmless; if the firms may choose among many prices then the structure of the interaction may change. (A richer model is studied in Section 3.2.)

2.2.3 The arms race

Under some assumptions about the countries' preferences, an arms race can be modeled as the *Prisoner's Dilemma*. (Because the *Prisoner's Dilemma* was first studied in the early 1950s, when the USA and USSR were involved in a nuclear arms race, you might suspect that US nuclear strategy was influenced by game theory; the evidence suggests that it was not.) Assume that each country can build an arsenal of nuclear bombs, or can refrain from doing so. Assume also that each country's favorite outcome is that it has bombs and the other country does not; the next best outcome is that neither country has any bombs; the next best outcome is that both countries have bombs (what matters is relative strength, and bombs are costly to build); and the worst outcome is that only the other country has bombs. In this case the situation is modeled by the *Prisoner's Dilemma*, in which the action *Don't build bombs* corresponds to *Quiet* in Figure 13.1 and the action *Build bombs* corresponds to *Fink*. However, once again the assumptions about preferences necessary for the *Prisoner's Dilemma* to model the situation may not be satisfied: a country may prefer *not* to build bombs if the other country does not, for example (bomb-building may be very costly), in which case the situation is modeled by a different game.

2.2.4 Common property

Two farmers are deciding how much to allow their sheep to graze on the village common. Each farmer prefers that her sheep graze a lot than a little, regardless of the other farmer's action, but prefers that both farmers' sheep graze a little than both farmers' sheep graze a lot (in which case the common is ruined for future use). Under these assumptions the game is the *Prisoner's Dilemma*. (A richer model is studied in Section 3.1.5.)

2.2.5 Other situations modeled as the Prisoner's Dilemma

A huge number of other situations have been modeled as the *Prisoner's Dilemma*, from mating hermaphroditic fish to tariff wars between countries.

? EXERCISE 16.1 (Hermaphroditic fish) Members of some species of hermaphroditic fish choose, in each mating encounter, whether to play the role of a male or a female. Each fish has a preferred role, which uses up fewer resources and hence allows more future mating. A fish obtains a payoff of H if it mates in its preferred role and L if it mates in the other role, where $H > L$. (Payoffs are measured in terms of number of offspring, which fish are evolved to maximize.) Consider an encounter between two fish whose preferred roles are the same. Each fish has two possible actions: mate in either role, and insist on its preferred role. If both fish offer to mate in either role, the roles are assigned randomly, and each fish's payoff is $\frac{1}{2}(H+L)$ (the average of H and L). If each fish insists on its preferred role, the fish do not mate; each goes off in search of another partner, and obtains the payoff S. The higher the chance of meeting another partner, the larger is S. Formulate this situation as a strategic game and determine the range of values of S, for any given values of H and L, for which the game differs from the *Prisoner's Dilemma* only in the names of the actions.

2.3 Example: Bach or Stravinsky?

In the *Prisoner's Dilemma* the main issue is whether or not the players will cooperate (choose *Quiet*). In the following game the players agree that it is better to cooperate than not to cooperate, but disagree about the best outcome.

◆ EXAMPLE 16.2 (Bach or Stravinsky?) Two people wish to go out together. Two concerts are available: one of music by Bach, and one of music by Stravinsky. One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer.

We can model this situation as the two-player strategic game in Figure 16.1, in which the person who prefers Bach chooses a row and the person who prefers Stravinsky chooses a column.

	Bach	*Stravinsky*
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

Figure 16.1 *Bach or Stravinsky?* (*BoS*) (Example 16.2).

This game is also referred to as the "Battle of the Sexes" (though the conflict it models surely occurs no more frequently between people of the opposite sex than it does between people of the same sex). I refer to the games as *BoS*, an acronym that fits both names. (I assume that each player is indifferent between listening to Bach and listening to Stravinsky when she is alone only for consistency with the standard specification of the game. As we shall see, the analysis of the game remains the same in the absence of this assumption.)

Like the *Prisoner's Dilemma*, *BoS* models a wide variety of situations. Consider, for example, two officials of a political party deciding the stand to take on an issue.

Suppose that they disagree about the best stand, but are both better off if they take the same stand than if they take different stands; both cases in which they take different stands, in which case voters do not know what to think, are equally bad. Then *BoS* captures the situation they face. Or consider two merging firms that currently use different computer technologies. As two divisions of a single firm they will both be better off if they both use the same technology; each firm prefers that the common technology be the one it used in the past. *BoS* models the choices the firms face.

2.4 Example: Matching Pennies

Aspects of both conflict and cooperation are present in both the *Prisoner's Dilemma* and *BoS*. The next game is purely conflictual.

◆ EXAMPLE 17.1 (Matching Pennies) Two people choose, simultaneously, whether to show the Head or the Tail of a coin. If they show the same side, person 2 pays person 1 a dollar; if they show different sides, person 1 pays person 2 a dollar. Each person cares only about the amount of money she receives, and (naturally!) prefers to receive more than less. A strategic game that models this situation is shown in Figure 17.1. (In this representation of the players' preferences, the payoffs are equal to the amounts of money involved. We could equally well work with another representation—for example, 2 could replace each 1, and 1 could replace each -1.)

	Head	*Tail*
Head	$1, -1$	$-1, \ \ 1$
Tail	$-1, \ \ 1$	$1, -1$

Figure 17.1 *Matching Pennies* (Example 17.1).

In this game the players' interests are diametrically opposed (such a game is called "strictly competitive"): player 1 wants to take the same action as the other player, whereas player 2 wants to take the opposite action.

This game may, for example, model the choices of appearances for new products by an established producer and a new firm in a market of fixed size. Suppose that each firm can choose one of two different appearances for the product. The established producer prefers the newcomer's product to look different from its own (so that its customers will not be tempted to buy the newcomer's product), whereas the newcomer prefers that the products look alike. Or the game could model a relationship between two people in which one person wants to be like the other, whereas the other wants to be different.

? EXERCISE 17.2 (Games without conflict) Give some examples of two-player strategic games in which each player has two actions and the players have the same preferences, so that there is no conflict between their interests. (Present your games as tables like the one in Figure 17.1.)

2.5 Example: the Stag Hunt

A sentence in *Discourse on the origin and foundations of inequality among men* (1755) by the philosopher Jean-Jacques Rousseau discusses a group of hunters who wish to catch a stag. They will succeed if they all remain sufficiently attentive, but each is tempted to desert her post and catch a hare. One interpretation of the sentence is that the interaction between the hunters may be modeled as the following strategic game.

◆ EXAMPLE 18.1 (Stag Hunt) Each of a group of hunters has two options: she may remain attentive to the pursuit of a stag, or catch a hare. If all hunters pursue the stag, they catch it and share it equally; if any hunter devotes her energy to catching a hare, the stag escapes, and the hare belongs to the defecting hunter alone. Each hunter prefers a share of the stag to a hare.

The strategic game that corresponds to this specification is:

Players The hunters.

Actions Each player's set of actions is $\{Stag, Hare\}$.

Preferences For each player, the action profile in which all players choose *Stag* (resulting in her obtaining a share of the stag) is ranked highest, followed by any profile in which she chooses *Hare* (resulting in her obtaining a hare), followed by any profile in which she chooses *Stag* and one or more of the other players chooses *Hare* (resulting in her leaving empty-handed).

Like other games with many players, this game cannot easily be presented in a table like that in Figure 17.1. For the case in which there are two hunters, the game is shown in Figure 18.1.

	Stag	*Hare*
Stag	2, 2	0, 1
Hare	1, 0	1, 1

Figure 18.1 The *Stag Hunt* (Example 18.1) for the case of two hunters.

The variant of the two-player *Stag Hunt* shown in Figure 19.1 has been suggested as an alternative to the *Prisoner's Dilemma* as a model of an arms race, or, more generally, of the "security dilemma" faced by a pair of countries. The game differs from the *Prisoner's Dilemma* in that a country prefers the outcome in which both countries refrain from arming themselves to the one in which it alone arms itself: the cost of arming outweighs the benefit if the other country does not arm itself.

	Refrain	*Arm*
Refrain	$3,3$	$0,2$
Arm	$2,0$	$1,1$

Figure 19.1 A variant of the two-player *Stag Hunt* that models the "security dilemma".

2.6 Nash equilibrium

What actions will be chosen by the players in a strategic game? We wish to assume, as in the theory of a rational decision-maker (Section 1.2), that each player chooses the best available action. In a game, the best action for any given player depends, in general, on the other players' actions. So when choosing an action a player must have in mind the actions the other players will choose. That is, she must form a *belief* about the other players' actions.

On what basis can such a belief be formed? The assumption underlying the analysis in this chapter and the next two chapters is that each player's belief is derived from her past experience playing the game, and that this experience is sufficiently extensive that she *knows* how her opponents will behave. No one tells her the actions her opponents will choose, but her previous involvement in the game leads her to be sure of these actions. (The question of *how* a player's experience can lead her to the correct beliefs about the other players' actions is addressed briefly in Section 4.9.)

Although we assume that each player has experience playing the game, we assume that she views each play of the game in isolation. She does not become familiar with the behavior of specific opponents and consequently does not condition her action on the opponent she faces; nor does she expect her current action to affect the other players' future behavior.

It is helpful to think of the following idealized circumstances. For each player in the game there is a population of many decision-makers who may, on any occasion, take that player's role. In each play of the game, players are selected randomly, one from each population. Thus each player engages in the game repeatedly, against ever-varying opponents. Her experience leads her to beliefs about the actions of "typical" opponents, not any specific set of opponents.

As an example, think of the interaction between buyers and sellers. Buyers and sellers repeatedly interact, but to a first approximation many of the pairings may be modeled as random. In many cases a buyer transacts only once with any given seller, or interacts repeatedly but anonymously (when the seller is a large store, for example).

In summary, the solution theory we study has two components. First, each player chooses her action according to the model of rational choice, given her belief about the other players' actions. Second, every player's belief about the other players' actions is correct. These two components are embodied in the following definition.

JOHN F. NASH, JR.

A few of the ideas of John F. Nash Jr., developed while he was a graduate student at Princeton from 1948 to 1950, transformed game theory. Nash was born in 1928 in Bluefield, West Virginia, USA, where he grew up. He was an undergraduate mathematics major at Carnegie Institute of Technology from 1945 to 1948. In 1948 he obtained both a B.S. and an M.S., and began graduate work in the Department of Mathematics at Princeton University. (One of his letters of recommendation, from a professor at Carnegie Institute of Technology, was a single sentence: "This man is a genius" (Kuhn et al. 1995, 282).) A paper containing the main result of his thesis was submitted to the *Proceedings of the National Academy of Sciences* in November 1949, fourteen months after he started his graduate work. ("A fine goal to set … graduate students", to quote Kuhn! (See Kuhn et al. 1995, 282.)) He completed his PhD the following year, graduating on his 22nd birthday. His thesis, 28 pages in length, introduces the equilibrium notion now known as "Nash equilibrium" and delineates a class of strategic games that have Nash equilibria (Proposition 116.1 in this book). The notion of Nash equilibrium vastly expanded the scope of game theory, which had previously focussed on two-player "strictly competitive" games (in which the players' interests are directly opposed). While a graduate student at Princeton, Nash also wrote the seminal paper in bargaining theory, Nash (1950b) (the ideas of which originated in an elective class in international economics he took as an undergraduate). He went on to take an academic position in the Department of Mathematics at MIT, where he produced "a remarkable series of papers" (Milnor 1995, 15); he has been described as "one of the most original mathematical minds of [the twentieth] century" (Kuhn 1996). He shared the 1994 Nobel prize in economics with the game theorists John C. Harsanyi and Reinhard Selten.

> A *Nash equilibrium* is an action profile a^* with the property that no player i can do better by choosing an action different from a_i^*, given that every other player j adheres to a_j^*.

In the idealized setting in which the players in any given play of the game are drawn randomly from a collection of populations, a Nash equilibrium corresponds to a *steady state*. If, whenever the game is played, the action profile is the same Nash equilibrium a^*, then no player has a reason to choose any action different from her component of a^*; there is no pressure on the action profile to change. Expressed differently, a Nash equilibrium embodies a stable "social norm": if everyone else adheres to it, no individual wishes to deviate from it.

The second component of the theory of Nash equilibrium—that the players' beliefs about each other's actions are correct—implies, in particular, that two players' beliefs about a third player's action are the same. For this reason, the condition is sometimes said to be that the players' "expectations are coordinated".

The situations to which we wish to apply the theory of Nash equilibrium do not in general correspond exactly to the idealized setting described above. For example, in some cases the players do not have much experience with the game; in others they do not view each play of the game in isolation. Whether or not the notion of Nash equilibrium is appropriate in any given situation is a matter of judgment. In some cases, a poor fit with the idealized setting may be mitigated by other considerations. For example, inexperienced players may be able to draw conclusions about their opponents' likely actions from their experience in other situations, or from other sources. (One aspect of such reasoning is discussed in the box on page 30). Ultimately, the test of the appropriateness of the notion of Nash equilibrium is whether it gives us insights into the problem at hand.

With the aid of an additional piece of notation, we can state the definition of a Nash equilibrium precisely. Let a be an action profile, in which the action of each player i is a_i. Let a'_i be any action of player i (either equal to a_i, or different from it). Then (a'_i, a_{-i}) denotes the action profile in which every player j *except* i chooses her action a_j as specified by a, whereas player i chooses a'_i. (The $-i$ subscript on a stands for "except i".) That is, (a'_i, a_{-i}) is the action profile in which all the players other than i adhere to a while i "deviates" to a'_i. (If $a'_i = a_i$ then of course $(a'_i, a_{-i}) = (a_i, a_{-i}) = a$.) If there are three players, for example, then (a'_2, a_{-2}) is the action profile in which players 1 and 3 adhere to a (player 1 chooses a_1, player 3 chooses a_3) and player 2 deviates to a'_2.

Using this notation, we can restate the condition for an action profile a^* to be a Nash equilibrium: no player i has any action a_i for which she prefers (a_i, a^*_{-i}) to a^*. Equivalently, for every player i and every action a_i of player i, the action profile a^* is at least as good for player i as the action profile (a_i, a^*_{-i}).

▶ DEFINITION 21.1 (*Nash equilibrium of strategic game with ordinal preferences*) The action profile a^* in a strategic game with ordinal preferences is a **Nash equilibrium** if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player j chooses a^*_j. Equivalently, for every player i,

$$u_i(a^*) \geq u_i(a_i, a^*_{-i}) \text{ for every action } a_i \text{ of player } i, \qquad (21.2)$$

where u_i is a payoff function that represents player i's preferences.

This definition implies neither that a strategic game necessarily has a Nash equilibrium, nor that it has at most one. Examples in the next section show that some games have a single Nash equilibrium, some possess no Nash equilibrium, and others have many Nash equilibria.

The definition of a Nash equilibrium is designed to model a steady state among experienced players. An alternative approach to understanding players' actions in strategic games assumes that the players know each others' preferences, and considers what each player can deduce about the other players' actions from their rationality and their knowledge of each other's rationality. This approach is studied in Chapter 12. For many games, it leads to a conclusion different from that of

Nash equilibrium. For games in which the conclusion is the same the approach offers us an alternative interpretation of a Nash equilibrium, as the outcome of rational calculations by players who do not necessarily have any experience playing the game.

Studying Nash equilibrium experimentally

The theory of strategic games lends itself to experimental study: arranging for subjects to play games and observing their choices is relatively straightforward. A few years after game theory was launched by von Neumann and Morgenstern's (1944) book, reports of laboratory experiments began to appear. Subsequently a huge number of experiments have been conducted, illuminating many issues relevant to the theory. I discuss selected experimental evidence throughout the book.

The theory of Nash equilibrium, as we have seen, has two components: the players act in accordance with the theory of rational choice, given their beliefs about the other players' actions, and these beliefs are correct. If every subject understands the game she is playing and faces incentives that correspond to the preferences of the player whose role she is taking, then a divergence between the observed outcome and a Nash equilibrium can be blamed on a failure of one or both of these two components. Experimental evidence has the potential of indicating the types of games for which the theory works well and, for those in which the theory does not work well, of pointing to the faulty component and giving us hints about the characteristics of a better theory. In designing an experiment that cleanly tests the theory, however, we need to confront several issues.

The model of rational choice takes preferences as given. Thus to test the theory of Nash equilibrium experimentally, we need to ensure that each subject's preferences are those of the player whose role she is taking in the game we are examining. The standard way of inducing the appropriate preferences is to pay each subject an amount of money directly related to the payoff given by a payoff function that represents the preferences of the player whose role the subject is taking. Such remuneration works if each subject likes money and cares only about the amount of money she receives, ignoring the amounts received by her opponents. The assumption that people like receiving money is reasonable in many cultures, but the assumption that people care only about their own monetary rewards—are "selfish"—may, in some contexts at least, not be reasonable. Unless we check whether our subjects are selfish in the context of our experiment, we will jointly test two hypotheses: that humans are selfish—a hypothesis not part of game theory—and that the notion of Nash equilibrium models their behavior. In some cases we may indeed wish to test these hypotheses jointly. But in order to test the theory of Nash equilibrium alone we need to ensure that we induce the preferences we wish to study.

Assuming that better decisions require more effort, we need also to ensure that each subject finds it worthwhile to put in the extra effort required to obtain a higher payoff. If we rely on monetary payments to provide incentives, the amount of money a subject can obtain must be sufficiently sensitive to the quality of her decisions to compensate her for the effort she expends (paying a flat fee, for example, is inappropriate). In some cases, monetary payments may not be necessary: under some circumstances, subjects drawn from a highly competitive culture like that of the USA may be sufficiently motivated by the possibility of obtaining a high score, even if that score does not translate into a monetary payoff.

The notion of Nash equilibrium models action profiles compatible with steady states. Thus to study the theory experimentally we need to collect observations of subjects' behavior when they have experience playing the game. But they should not have obtained that experience while knowingly facing the same opponents repeatedly, for the theory assumes that the players consider each play of the game in isolation, not as part of an ongoing relationship. One option is to have each subject play the game against many different opponents, gaining experience about how the other subjects on average play the game, but not about the choices of any other given player. Another option is to describe the game in terms that relate to a situation in which the subjects already have experience. A difficulty with this second approach is that the description we give may connote more than simply the payoff numbers of our game. If we describe the *Prisoner's Dilemma* in terms of cooperation on a joint project, for example, a subject may be biased toward choosing the action she has found appropriate when involved in joint projects, even if the structures of those interactions were significantly different from that of the *Prisoner's Dilemma*. As she plays the experimental game repeatedly she may come to appreciate how it differs from the games in which she has been involved previously, but her biases may disappear only slowly.

Whatever route we take to collect data on the choices of subjects experienced in playing the game, we confront a difficult issue: how do we know when the outcome has converged? Nash's theory concerns only equilibria; it has nothing to say about the path players' choices will take on the way to an equilibrium, and so gives us no guide as to whether 10, 100, or 1,000 plays of the game are enough to give a chance for the subjects' expectations to become coordinated.

Finally, we can expect the theory of Nash equilibrium to correspond to reality only approximately: like all useful theories, it definitely is not *exactly* correct. How do we tell whether the data are close enough to the theory to support it? One possibility is to compare the theory of Nash equilibrium with some other theory. But for many games there is no obvious alternative theory—and certainly not one with the generality of Nash equilibrium. Statistical tests can sometimes aid in deciding whether the data is consistent with the theory, though ultimately we remain the judge of whether or not our observations persuade us that the theory enhances our understanding of human behavior in the game.

2.7 Examples of Nash equilibrium

2.7.1 Prisoner's Dilemma

By examining the four possible pairs of actions in the *Prisoner's Dilemma* (reproduced in Figure 24.1), we see that (*Fink*, *Fink*) is the unique Nash equilibrium.

	Quiet	*Fink*
Quiet	2, 2	0, 3
Fink	3, 0	1, 1

Figure 24.1 The *Prisoner's Dilemma*.

The action pair (*Fink*, *Fink*) is a Nash equilibrium because (*i*) given that player 2 chooses *Fink*, player 1 is better off choosing *Fink* than *Quiet* (looking at the right column of the table we see that *Fink* yields player 1 a payoff of 1 whereas *Quiet* yields her a payoff of 0), and (*ii*) given that player 1 chooses *Fink*, player 2 is better off choosing *Fink* than *Quiet* (looking at the bottom row of the table we see that *Fink* yields player 2 a payoff of 1 whereas *Quiet* yields her a payoff of 0).

No other action profile is a Nash equilibrium:

- (*Quiet*, *Quiet*) does not satisfy (21.2) because when player 2 chooses *Quiet*, player 1's payoff to *Fink* exceeds her payoff to *Quiet* (look at the first components of the entries in the left column of the table). (Further, when player 1 chooses *Quiet*, player 2's payoff to *Fink* exceeds her payoff to *Quiet*: player 2, as well as player 1, wants to deviate. To show that a pair of actions is not a Nash equilibrium, however, it is not necessary to study player 2's decision once we have established that player 1 wants to deviate: it is enough to show that *one* player wishes to deviate to show that a pair of actions is not a Nash equilibrium.)

- (*Fink*, *Quiet*) does not satisfy (21.2) because when player 1 chooses *Fink*, player 2's payoff to *Fink* exceeds her payoff to *Quiet* (look at the second components of the entries in the bottom row of the table).

- (*Quiet*, *Fink*) does not satisfy (21.2) because when player 2 chooses *Fink*, player 1's payoff to *Fink* exceeds her payoff to *Quiet* (look at the first components of the entries in the right column of the table).

In summary, in the only Nash equilibrium of the *Prisoner's Dilemma* both players choose *Fink*. In particular, the incentive to free ride eliminates the possibility that the mutually desirable outcome (*Quiet*, *Quiet*) occurs. In the other situations discussed in Section 2.2 that may be modeled as the *Prisoner's Dilemma*, the outcomes predicted by the notion of Nash equilibrium are thus as follows: both people goof off when working on a joint project; both duopolists charge a low price; both countries build bombs; both farmers graze their sheep a lot. (The overgrazing of a common thus predicted is sometimes called the "tragedy of the commons". The intuition that some of these dismal outcomes may be avoided if the same pair of people play the game repeatedly is explored in Chapter 14.)

In the *Prisoner's Dilemma*, the Nash equilibrium action of each player (*Fink*) is the best action for each player not only if the other player chooses her equilibrium action (*Fink*), but also if she chooses her other action (*Quiet*). The action pair (*Fink*, *Fink*) is a Nash equilibrium because if a player believes that her opponent will choose *Fink* then it is optimal for her to choose *Fink*. But in fact it is optimal for a player to choose *Fink regardless* of the action she expects her opponent to choose. In most of the games we study, a player's Nash equilibrium action does not satisfy this condition: the action is optimal if the other players choose their Nash equilibrium actions, but some other action is optimal if the other players choose non-equilibrium actions.

? EXERCISE 25.1 (Altruistic players in the *Prisoner's Dilemma*) Each of two players has two possible actions, *Quiet* and *Fink*; each action pair results in the players' receiving amounts of *money* equal to the numbers corresponding to that action pair in Figure 24.1. (For example, if player 1 chooses *Quiet* and player 2 chooses *Fink*, then player 1 receives nothing, whereas player 2 receives \$3.) The players are not "selfish"; rather, the preferences of each player i are represented by the payoff function $m_i(a) + \alpha m_j(a)$, where $m_i(a)$ is the amount of money received by player i when the action profile is a, j is the other player, and α is a given nonnegative number. Player 1's payoff to the action pair $(Quiet, Quiet)$, for example, is $2 + 2\alpha$.

a. Formulate a strategic game that models this situation in the case $\alpha = 1$. Is this game the *Prisoner's Dilemma*?

b. Find the range of values of α for which the resulting game is the *Prisoner's Dilemma*. For values of α for which the game is not the *Prisoner's Dilemma*, find its Nash equilibria.

? EXERCISE 25.2 (Selfish and altruistic social behavior) Two people enter a bus. Two adjacent cramped seats are free. Each person must decide whether to sit or stand. Sitting alone is more comfortable than sitting next to the other person, which is more comfortable than standing.

a. Suppose that each person cares only about her own comfort. Model the situation as a strategic game. Is this game the *Prisoner's Dilemma*? Find its Nash equilibrium (equilibria?).

b. Suppose that each person is altruistic, ranking the outcomes according to the *other* person's comfort, and, out of politeness, prefers to stand than to sit if the other person stands. Model the situation as a strategic game. Is this game the *Prisoner's Dilemma*? Find its Nash equilibrium (equilibria?).

c. Compare the people's comfort in the equilibria of the two games.

EXPERIMENTAL EVIDENCE ON THE *Prisoner's Dilemma*

The *Prisoner's Dilemma* has attracted a great deal of attention by economists, psychologists, sociologists, and biologists. A huge number of experiments have been conducted with the aim of discovering how people behave when playing the game. Almost all these experiments involve each subject's playing the game repeatedly against an unchanging opponent, a situation that calls for an analysis significantly different from the one in this chapter (see Chapter 14).

The evidence on the outcome of isolated plays of the game is inconclusive. No experiment of which I am aware carefully induces the appropriate preferences and is specifically designed to elicit a steady state action profile (see the box on page 22). Thus in each case the choice of *Quiet* by a player could indicate that she is not "selfish" or that she is not experienced in playing the game, rather than providing evidence against the notion of Nash equilibrium.

In two experiments with very low payoffs, each subject played the game a small number of times against different opponents; between 50% and 94% of subjects chose *Fink*, depending on the relative sizes of the payoffs and some details of the design (Rapoport, Guyer, and Gordon 1976, 135–137, 211–213, and 223-226). A more recent experiment finds that in the last 10 of 20 rounds of play against different opponents, 78% of subjects choose *Fink* (Cooper, DeJong, Forsythe, and Ross 1996). In face-to-face games in which communication is allowed, the incidence of the choice of *Fink* tends to be lower: from 29% to 70% depending on the nature of the communication allowed (Deutsch 1958, and Frank, Gilovich, and Regan 1993, 163–167). (In all these experiments, the subjects were college students in the USA or Canada.)

One source of the variation in the results seems to be that some designs induce preferences that differ from those of the *Prisoner's Dilemma*; no clear answer emerges to the question of whether the notion of Nash equilibrium is relevant to the *Prisoner's Dilemma*. If, nevertheless, one interprets the evidence as showing that some subjects in the *Prisoner's Dilemma* systematically choose *Quiet* rather than *Fink*, one must fault the rational choice component of Nash equilibrium, not the coordinated expectations component. Why? Because, as noted in the text, *Fink* is optimal *no matter* what a player thinks her opponent will choose, so that any model in which the players act according to the model of rational choice, whether or not their expectations are coordinated, predicts that each player chooses *Fink*.

2.7.2 BoS

To find the Nash equilibria of *BoS* (Figure 16.1), we can examine each pair of actions in turn:

- $(Bach, Bach)$: If player 1 switches to *Stravinsky* then her payoff decreases from 2 to 0; if player 2 switches to *Stravinsky* then her payoff decreases from 1 to 0.

Thus a deviation by either player decreases her payoff. Thus (*Bach*, *Bach*) is a Nash equilibrium.

- (*Bach*, *Stravinsky*): If player 1 switches to *Stravinsky* then her payoff increases from 0 to 1. Thus (*Bach*, *Stravinsky*) is not a Nash equilibrium. (Player 2 can increase her payoff by deviating, too, but to show the pair is not a Nash equilibrium it suffices to show that one player can increase her payoff by deviating.)
- (*Stravinsky*, *Bach*): If player 1 switches to *Bach* then her payoff increases from 0 to 2. Thus (*Stravinsky*, *Bach*) is not a Nash equilibrium.
- (*Stravinsky*, *Stravinsky*): If player 1 switches to *Bach* then her payoff decreases from 1 to 0; if player 2 switches to *Bach* then her payoff decreases from 2 to 0. Thus a deviation by either player decreases her payoff. Thus (*Stravinsky*, *Stravinsky*) is a Nash equilibrium.

We conclude that the game has two Nash equilibria: (*Bach*, *Bach*) and (*Stravinsky*, *Stravinsky*). That is, both of these outcomes are compatible with a steady state; both outcomes are stable social norms. If, in every encounter, both players choose *Bach*, then no player has an incentive to deviate; if, in every encounter, both players choose *Stravinsky*, then no player has an incentive to deviate. If we use the game to model the choices of men when matched with women, for example, then the notion of Nash equilibrium shows that two social norms are stable: both players choose the action associated with the outcome preferred by women, and both players choose the action associated with the outcome preferred by men.

2.7.3 Matching Pennies

By checking each of the four pairs of actions in *Matching Pennies* (Figure 17.1) we see that the game has no Nash equilibrium. For the pairs of actions (*Head*, *Head*) and (*Tail*, *Tail*), player 2 is better off deviating; for the pairs of actions (*Head*, *Tail*) and (*Tail*, *Head*), player 1 is better off deviating. Thus for this game the notion of Nash equilibrium isolates no steady state. In Chapter 4 we return to this game; an extension of the notion of a Nash equilibrium gives us an understanding of the likely outcome.

2.7.4 The Stag Hunt

Inspection of Figure 18.1 shows that the two-player *Stag Hunt* has two Nash equilibria: (*Stag*, *Stag*) and (*Hare*, *Hare*). If one player remains attentive to the pursuit of the stag, then the other player prefers to remain attentive; if one player chases a hare, the other one prefers to chase a hare (she cannot catch a stag alone). (The equilibria of the variant of the game in Figure 19.1 are analogous: (*Refrain*, *Refrain*) and (*Arm*, *Arm*).)

Unlike the Nash equilibria of *BoS*, one of these equilibria is better for both players than the other: each player prefers (*Stag*, *Stag*) to (*Hare*, *Hare*). This fact has no bearing on the equilibrium status of (*Hare*, *Hare*), since the condition for an equilibrium is that a *single* player cannot gain by deviating, *given* the other player's behavior. Put differently, an equilibrium is immune to any *unilateral* deviation; coordinated deviations by groups of players are not contemplated. However, the existence of two equilibria raises the possibility that one equilibrium might more likely be the outcome of the game than the other. I return to this issue in Section 2.7.6.

I argue that the many-player *Stag Hunt* (Example 18.1) also has two Nash equilibria: the action profile (*Stag*, . . . , *Stag*) in which every players joins in the pursuit of the stag, and the profile (*Hare*, . . . , *Hare*) in which every player catches a hare.

- (*Stag*, . . . , *Stag*) is a Nash equilibrium because each player prefers this profile to that in which she alone chooses *Hare*. (A player is better off remaining attentive to the pursuit of the stag than running after a hare if all the other players remain attentive.)
- (*Hare*, . . . , *Hare*) is a Nash equilibrium because each player prefers this profile to that in which she alone pursues the stag. (A player is better off catching a hare than pursuing the stag if no one else pursues the stag.)
- No other profile is a Nash equilibrium, because in any other profile at least one player chooses *Stag* and at least one player chooses *Hare*, so that any player choosing *Stag* is better off switching to *Hare*. (A player is better off catching a hare than pursing the stag if at least one other person chases a hare, since the stag can be caught only if everyone pursues it.)

? EXERCISE 28.1 (Variants of the *Stag Hunt*) Consider two variants of the n-hunter *Stag Hunt* in which only m hunters, with $2 \leq m < n$, need to pursue the stag in order to catch it. (Continue to assume that there is a single stag.) Assume that a captured stag is shared only by the hunters that catch it.

a. Assume, as before, that each hunter prefers the fraction $1/n$ of the stag to a hare. Find the Nash equilibria of the strategic game that models this situation.

b. Assume that each hunter prefers the fraction $1/k$ of the stag to a hare, but prefers the hare to any smaller fraction of the stag, where k is an integer with $m \leq k \leq n$. Find the Nash equilibria of the strategic game that models this situation.

The following more difficult exercise enriches the hunters' choices in the *Stag Hunt*. This extended game has been proposed as a model that captures Keynes' basic insight about the possibility of multiple economic equilibria, some undesirable (Bryant 1983, 1994).

?? EXERCISE 28.2 (Extension of the *Stag hunt*) Extend the n-hunter *Stag Hunt* by giving each hunter K (a positive integer) units of effort, which she can allocate between pursuing the stag and catching hares. Denote the effort hunter i devotes to pursuing the stag by e_i, a nonnegative integer equal to at most K. The chance that the stag is caught depends on the smallest of all the hunters' efforts, denoted $\min_j e_j$. ("A chain is as strong as its weakest link.") Hunter i's payoff to the action profile $(e_1, \ldots, e_n)$ is $2\min_j e_j - e_i$. (She is better off the more likely the stag is caught, and worse off the more effort she devotes to pursuing the stag, which means she catches fewer hares.) Is the action profile $(e, \ldots, e)$, in which every hunter devotes the same effort to pursuing the stag, a Nash equilibrium for any value of e? (What is a player's payoff to this profile? What is her payoff if she deviates to a lower or higher effort level?) Is any action profile in which not all the players' effort levels are the same a Nash equilibrium? (Consider a player whose effort exceeds the minimum effort level of all players. What happens to her payoff if she reduces her effort level to the minimum?)

2.7.5 Hawk–Dove

The game in the next exercise captures a basic feature of animal conflict.

? EXERCISE 29.1 (Hawk–Dove) Two animals are fighting over some prey. Each can be passive or aggressive. Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome when its opponent is passive to that in which its opponent is aggressive. Formulate this situation as a strategic game and find its Nash equilibria.

2.7.6 A coordination game

Consider two people who wish to go out together, but who, unlike the dissidents in *BoS*, agree on the more desirable concert—say they both prefer *Bach*. A strategic game that models this situation is shown in Figure 29.1; it is an example of a *coordination game*. By examining the four action pairs, we see that the game has two Nash equilibria: (*Bach*, *Bach*) and (*Stravinsky*, *Stravinsky*). In particular, the action pair (*Stravinsky*, *Stravinsky*) in which both people choose their less-preferred concert is a Nash equilibrium.

	Bach	*Stravinsky*
Bach	2, 2	0, 0
Stravinsky	0, 0	1, 1

Figure 29.1 *A coordination game.*

Is the equilibrium in which both people choose *Stravinsky* plausible? People who argue that the technology of Apple computers originally dominated that of IBM computers, and that the Beta format for video recording is better than VHS, would say "yes". In both cases users had a strong interest in adopting the same standard, and one standard was better than the other; in the steady state that emerged in each case, the inferior technology was adopted by a large majority of users.

FOCAL POINTS

In games with many Nash equilibria, the theory isolates more than one pattern of behavior compatible with a steady state. In some games, some of these equilibria seem more likely to attract the players' attentions than others. To use the terminology of Schelling (1960), some equilibria are *focal*. In the coordination game in Figure 29.1, where the players agree on the more desirable Nash equilibrium and obtain the same payoff to every nonequilibrium action pair, the preferable equilibrium seems more likely to be focal (though two examples are given in the text of steady states involving the inferior equilibrium). In the variant of this game in which the two equilibria are equally good (i.e. $(2, 2)$ is replaced by $(1, 1)$), nothing in the structure of the game gives any clue as to which steady state might occur. In such a game, the names or nature of the actions, or other information, may predispose the players to one equilibrium rather than the other.

Consider, for example, voters in an election. Pre-election polls may give them information about each other's intended actions, pointing them to one of many Nash equilibria. Or consider a situation in which two players independently divide \$100 into two piles, each receiving \$10 if they choose the same divisions and nothing otherwise. The strategic game that models this situation has many Nash equilibria, in each of which both players choose the same division. But the equilibrium in which both players choose the $(\$50, \$50)$ division seems likely to command the players' attentions, possibly for esthetic reasons (it is an appealing division), and possibly because it is a steady state in an unrelated game in which the chosen division determines the players' payoffs.

The theory of Nash equilibrium is neutral about the equilibrium that will occur in a game with many equilibria. If features of the situation not modeled by the notion of a strategic game make some equilibria focal then those equilibria may be more likely to emerge as steady states, and the rate at which a steady state is reached may be higher than it otherwise would have been.

If two people played this game in a laboratory it seems likely that the outcome would be $(\textit{Bach}, \textit{Bach})$. Nevertheless, $(\textit{Stravinsky}, \textit{Stravinsky})$ also corresponds to a steady state: if either action pair is reached, there is no reason for either player to deviate from it.

2.7.7 Provision of a public good

The model in the next exercise captures an aspect of the provision of a "public good", like a park or a swimming pool, whose use by one person does not diminish its value to another person (at least, not until it is overcrowded). (Other aspects of public good provision are studied in Section 2.8.4.)

? EXERCISE 31.1 (Contributing to a public good) Each of n people chooses whether or not to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute, where $2 \leq k \leq n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows: (*i*) any outcome in which the good is provided and she does not contribute, (*ii*) any outcome in which the good is provided and she contributes, (*iii*) any outcome in which the good is not provided and she does not contribute, (*iv*) any outcome in which the good is not provided and she contributes. Formulate this situation as a strategic game and find its Nash equilibria. (Is there a Nash equilibrium in which more than k people contribute? One in which k people contribute? One in which fewer than k people contribute? (Be careful!))

2.7.8 *Strict and nonstrict equilibria*

In all the Nash equilibria of the games we have studied so far a deviation by a player leads to an outcome *worse* for that player than the equilibrium outcome. The definition of Nash equilibrium (21.1), however, requires only that the outcome of a deviation be *no better* for the deviant than the equilibrium outcome. And, indeed, some games have equilibria in which a player is indifferent between her equilibrium action and some other action, given the other players' actions.

Consider the game in Figure 31.1. This game has a unique Nash equilibrium, namely (T, L). (For every other pair of actions, one of the players is better off changing her action.) When player 2 chooses L, as she does in this equilibrium, player 1 is equally happy choosing T or B; if she deviates to B then she is no worse off than she is in the equilibrium. We say that the Nash equilibrium (T, L) is not a *strict equilibrium*.

	L	M	R
T	$1,1$	$1,0$	$0,1$
B	$1,0$	$0,1$	$1,0$

Figure 31.1 A game with a unique Nash equilibrium, which is not a strict equilibrium.

For a general game, an equilibrium is strict if each player's equilibrium action is *better* than all her other actions, given the other players' actions. Precisely, an action profile a^* is a **strict Nash equilibrium** if for every player i we have $u_i(a^*) > u_i(a_i, a^*_{-i})$ for every action $a_i \neq a^*_i$ of player i. (Contrast the strict inequality in this definition with the weak inequality in (21.2).)

2.7.9 *Additional examples*

The following exercises are more difficult than most of the previous ones. In the first two, the number of actions of each player is arbitrary, so you cannot mechanically examine each action profile individually, as we did for games in which each player has two actions. Instead, you can consider groups of action profiles that have features in common, and show that all action profiles in any given group are or are not equilibria. Deciding how best to group the profiles into types calls for some intuition about the character of a likely equilibrium; the exercises contain suggestions on how to proceed.

EXERCISE 32.1 (Guessing two-thirds of the average) Each of three people announces an integer from 1 to K. If the three integers are different, the person whose integer is closest to $\frac{2}{3}$ of the average of the three integers wins \$1. If two or more integers are the same, \$1 is split equally between the people whose integer is closest to $\frac{2}{3}$ of the average integer. Is there any integer k such that the action profile (k, k, k), in which every person announces the same integer k, is a Nash equilibrium? (If $k \geq 2$, what happens if a person announces a smaller number?) Is any other action profile a Nash equilibrium? (What is the payoff of a person whose number is the highest of the three? Can she increase this payoff by announcing a different number?)

Game theory is used widely in political science, especially in the study of elections. The game in the following exercise explores citizens' costly decisions to vote.

EXERCISE 32.2 (Voter participation) Two candidates, A and B, compete in an election. Of the n citizens, k support candidate A and m $(= n - k)$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$, and $-c$ in these three cases, where $0 < c < 1$.

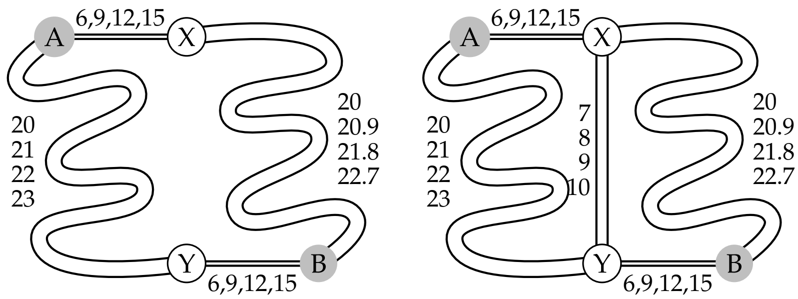
a. For $k = m = 1$, is the game the same (except for the names of the actions) as any considered so far in this chapter?

b. For $k = m$, find the set of Nash equilibria. (Is the action profile in which everyone votes a Nash equilibrium? Is there any Nash equilibrium in which the candidates tie and not everyone votes? Is there any Nash equilibrium in which one of the candidates wins by one vote? Is there any Nash equilibrium in which one of the candidates wins by two or more votes?)

c. What is the set of Nash equilibria for $k < m$?

If, when sitting in a traffic jam, you have ever thought about the time you might save if another road were built, the next exercise may lead you to think again.

EXERCISE 32.3 (Choosing a route) Four people must drive from A to B at the same time. Two routes are available, one via X and one via Y. (Refer to the left panel of Figure 33.1.) The roads from A to X, and from Y to B are both short and narrow; in each case, one car takes 6 minutes, and each additional car increases the travel time *per car* by 3 minutes. (If two cars drive from A to X, for example, *each car* takes 9 minutes.) The roads from A to Y, and from X to B are long and wide; on A to Y one car takes 20 minutes, and each additional car increases the travel time *per car* by 1 minute; on X to B one car takes 20 minutes, and each additional car increases the travel time *per car* by 0.9 minutes. Formulate this situation as a strategic game and find the Nash equilibria. (If all four people take one of the routes, can any of them do better by taking the other route? What if three take one route and one takes the other route, or if two take each route?)



Original network. Network with new road from X to Y.

Figure 33.1 Getting from A to B: the road networks in Exercise 32.3. The numbers beside each road are the travel times *per car* when 1, 2, 3, or 4 cars take that road.

Now suppose that a relatively short, wide road is built from X to Y, giving each person four options for travel from A to B: A–X–B, A–Y–B, A–X–Y–B, and A–Y–X–B. Assume that a person who takes A–X–Y–B travels the A–X portion at the same time as someone who takes A–X–B, and the Y–B portion at the same time as someone who takes A–Y–B. (Think of there being constant flows of traffic.) On the road between X and Y, one car takes 7 minutes and each additional car increases the travel time *per car* by 1 minute. Find the Nash equilibria in this new situation. Compare each person's travel time with her travel time in the equilibrium before the road from X to Y was built.

2.8 Best response functions

2.8.1 Definition

We can find the Nash equilibria of a game in which each player has only a few actions by examining each action profile in turn to see if it satisfies the conditions for equilibrium. In more complicated games, it is often better to work with the players' "best response functions".

Consider a player, say player i. For any given actions of the players other than i, player i's actions yield her various payoffs. We are interested in the best actions—those that yield her the highest payoff. In *BoS*, for example, *Bach* is the best action for player 1 if player 2 chooses *Bach*; *Stravinsky* is the best action for player 1 if player 2 chooses *Stravinsky*. In particular, in *BoS*, player 1 has a single best action for each action of player 2. By contrast, in the game in Figure 31.1, both T and B are best actions for player 1 if player 2 chooses L: they both yield the payoff of 1, and player 1 has no action that yields a higher payoff (in fact, she has no other action).

We denote the set of player i's best actions when the list of the other players' actions is a_{-i} by $B_i(a_{-i})$. Thus in *BoS* we have $B_1(Bach) = \{Bach\}$ and $B_1(Stravinsky) = \{Stravinsky\}$; in the game in Figure 31.1 we have $B_1(L) = \{T, B\}$.

Precisely, we define the function B_i by

$$B_i(a_{-i}) = \{a_i \text{ in } A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_i' \text{ in } A_i\} :$$

any action in $B_i(a_{-i})$ is at least as good for player i as every other action of player i when the other players' actions are given by a_{-i}. We call B_i the **best response function** of player i.

The function B_i is *set-valued*: it associates a set of actions with any list of the other players' actions. Every member of the set $B_i(a_{-i})$ is a **best response** of player i to a_{-i}: if each of the other players adheres to a_{-i} then player i can do no better than choose a member of $B_i(a_{-i})$. In some games, like *BoS*, the set $B_i(a_{-i})$ consists of a single action for every list a_{-i} of actions of the other players: no matter what the other players do, player i has a *single* optimal action. In other games, like the one in Figure 31.1, $B_i(a_{-i})$ contains more than one action for some lists a_{-i} of actions of the other players.

2.8.2 Using best response functions to define Nash equilibrium

A Nash equilibrium is an action profile with the property that no player can do better by changing her action, given the other players' actions. Using the terminology just developed, we can alternatively define a Nash equilibrium to be an action profile for which every player's action is a best response to the other players' actions. That is, we have the following result.

■ PROPOSITION 34.1 *The action profile a^* is a Nash equilibrium of a strategic game with ordinal preferences if and only if every player's action is a best response to the other players' actions:*

$$a_i^* \text{ is in } B_i(a_{-i}^*) \text{ for every player } i. \tag{34.2}$$

If each player i has a single best response to each list a_{-i} of the other players' actions, we can write the conditions in (34.2) as equations. In this case, for each player i and each list a_{-i} of the other players' actions, denote the single member of $B_i(a_{-i})$ by $b_i(a_{-i})$ (that is, $B_i(a_{-i}) = \{b_i(a_{-i})\}$). Then (34.2) is equivalent to

$$a_i^* = b_i(a_{-i}^*) \text{ for every player } i, \tag{34.3}$$

a collection of n equations in the n unknowns a_i^*, where n is the number of players in the game. For example, in a game with two players, say 1 and 2, these equations are

$$\begin{aligned} a_1^* &= b_1(a_2^*) \\ a_2^* &= b_2(a_1^*). \end{aligned}$$

That is, in a two-player game in which each player has a single best response to every action of the other player, (a_1^*, a_2^*) is a Nash equilibrium if and only if player 1's action a_1^* is her best response to player 2's action a_2^*, and player 2's action a_2^* is her best response to player 1's action a_1^*.

2.8.3 *Using best response functions to find Nash equilibria*

The definition of a Nash equilibrium in terms of best response functions suggests a method for finding Nash equilibria:

- find the best response function of each player
- find the action profiles that satisfy (34.2) (which reduces to (34.3) if each player has a single best response to each list of the other players' actions).

To illustrate this method, consider the game in Figure 35.1. First find the best response of player 1 to each action of player 2. If player 2 chooses L, then player 1's best response is M (2 is the highest payoff for player 1 in this column); indicate the best response by attaching a star to player 1's payoff to (M, L). If player 2 chooses C, then player 1's best response is T, indicated by the star attached to player 1's payoff to (T, C). And if player 2 chooses R, then both T and B are best responses for player 1; both are indicated by stars. Second, find the best response of player 2 to each action of player 1 (for each row, find highest payoff of player 2); these best responses are indicated by attaching stars to player 2's payoffs. Finally, find the boxes in which both players' payoffs are starred. Each such box is a Nash equilibrium: the star on player 1's payoff means that player 1's action is a best response to player 2's action, and the star on player 2's payoff means that player 2's action is a best response to player 1's action. Thus we conclude that the game has two Nash equilibria: (M, L) and (B, R).

	L	C	R
T	$1\,, 2^*$	$2^*, 1$	$1^*, 0$
M	$2^*, 1^*$	$0\,, 1^*$	$0\,, 0$
B	$0\,, 1$	$0\,, 0$	$1^*, 2^*$

Figure 35.1 Using best response functions to find Nash equilibria in a two-player game in which each player has three actions.

? EXERCISE 35.1 (Finding Nash equilibria using best response functions)

a. Find the players' best response functions in the *Prisoner's Dilemma* (Figure 13.1), *BoS* (Figure 16.1), *Matching Pennies* (Figure 17.1), and the two-player *Stag Hunt* (Figure 18.1) (and verify the Nash equilibria of these games).

b. Find the Nash equilibria of the game in Figure 36.1 by finding the players' best response functions.

	L	C	R
T	$2,2$	$1,3$	$0,1$
M	$3,1$	$0,0$	$0,0$
B	$1,0$	$0,0$	$0,0$

Figure 36.1 The game in Exercise 35.1b.

The players' best response functions for the game in Figure 35.1 are presented in a different format in Figure 36.2. In this figure, player 1's actions are on the horizontal axis and player 2's are on the vertical axis. (Thus the columns correspond to choices of player 1, and the rows correspond to choices of player 2, whereas the reverse is true in Figure 35.1. I choose this orientation for Figure 36.2 for consistency with the convention for figures of this type.) Player 1's best responses are indicated by circles, and player 2's by dots. Thus the circle at (T, C) reflects the fact that T is player 1's best response to player 2's choice of C, and the circles at (T, R) and (B, R) reflect the fact that T and B are both best responses of player 1 to player 2's choice of R. Any action pair marked by both a circle and a dot is a Nash equilibrium: the circle means that player 1's action is a best response to player 2's action, and the dot indicates that player 2's action is a best response to player 1's action.

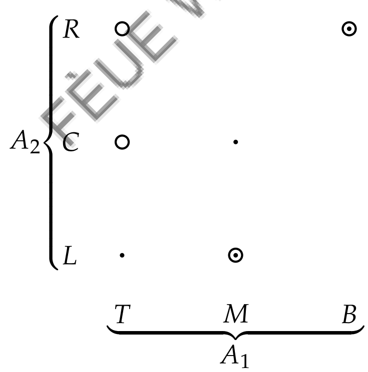


Figure 36.2 The players' best response functions for the game in Figure 35.1. Player 1's best responses are indicated by circles, and player 2's by dots. The action pairs for which there is both a circle and a dot are the Nash equilibria.

? EXERCISE 36.1 (Constructing best response functions) Draw the analogue of Figure 36.2 for the game in Exercise 35.1b.

? EXERCISE 36.2 (Dividing money) Two people have \$10 to divide between themselves. They use the following process to divide the money. Each person names a number of dollars (a nonnegative integer), at most equal to 10. If the sum of the amounts that the people name is at most 10 then each person receives the amount of money she names (and the remainder is destroyed). If the sum of the amounts that the people name exceeds 10 and the amounts named are different then the person who names the smaller amount receives that amount and the other person receives the remaining money. If the sum of the amounts that the people name exceeds 10 and the amounts named are the same then each person receives \$5. Determine the best response of each player to each of the other player's actions, plot them in a diagram like Figure 36.2, and thus find the Nash equilibria of the game.

A diagram like Figure 36.2 is a convenient representation of the players' best response functions also in a game in which each player's set of actions is an interval of numbers, as the next example illustrates.

◆ EXAMPLE 37.1 (A synergistic relationship) Two individuals are involved in a synergistic relationship. If both individuals devote more effort to the relationship, they are both better off. For any given effort of individual j, the return to individual i's effort first increases, then decreases. Specifically, an effort level is a nonnegative number, and individual i's preferences (for $i = 1, 2$) are represented by the payoff function $a_i(c + a_j - a_i)$, where a_i is i's effort level, a_j is the other individual's effort level, and $c > 0$ is a constant.

The following strategic game models this situation.

Players The two individuals.

Actions Each player's set of actions is the set of effort levels (nonnegative numbers).

Preferences Player i's preferences are represented by the payoff function $a_i(c + a_j - a_i)$, for $i = 1, 2$.

In particular, each player has infinitely many actions, so that we cannot present the game in a table like those used previously (Figure 36.1, for example).

To find the Nash equilibria of the game, we can construct and analyze the players' best response functions. Given a_j, individual i's payoff is a quadratic function of a_i that is zero when $a_i = 0$ and when $a_i = c + a_j$, and reaches a maximum in between. The symmetry of quadratic functions (see Section 17.4) implies that the best response of each individual i to a_j is

$$b_i(a_j) = \tfrac{1}{2}(c + a_j).$$

(If you know calculus, you can reach the same conclusion by setting the derivative of player i's payoff with respect to a_i equal to zero.)

The best response functions are shown in Figure 38.1. Player 1's actions are plotted on the horizontal axis and player 2's actions are plotted on the vertical axis. Player 1's best response function associates an action for player 1 with every action for player 2. Thus to interpret the function b_1 in the diagram, take a point a_2 on the vertical axis, and go across to the line labeled b_1 (the steeper of the two lines), then read down to the horizontal axis. The point on the horizontal axis that you reach is $b_1(a_2)$, the best action for player 1 when player 2 chooses a_2. Player 2's best response function, on the other hand, associates an action for player 2 with every action of player 1. Thus to interpret this function, take a point a_1 on the horizontal axis, and go up to b_2, then across to the vertical axis. The point on the vertical axis that you reach is $b_2(a_1)$, the best action for player 2 when player 1 chooses a_1.

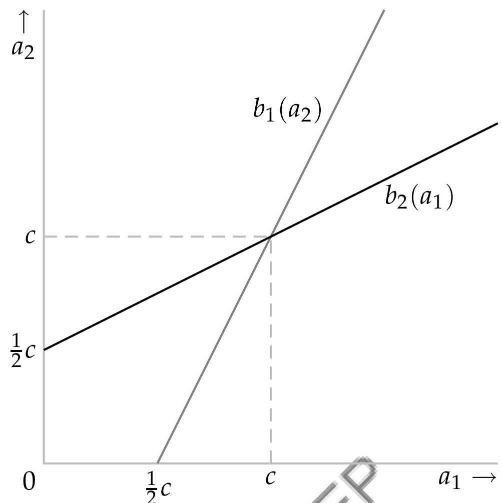


Figure 38.1 The players' best response functions for the game in Example 37.1. The game has a unique Nash equilibrium, $(a_1^*, a_2^*) = (c, c)$.

At a point (a_1, a_2) where the best response functions intersect in the figure, we have $a_1 = b_1(a_2)$, because (a_1, a_2) is on the graph of b_1, player 1's best response function, and $a_2 = b_2(a_1)$, because (a_1, a_2) is on the graph of b_2, player 1's best response function. Thus any such point (a_1, a_2) is a Nash equilibrium. In this game the best response functions intersect at a single point, so there is one Nash equilibrium. In general, they may intersect more than once; every point at which they intersect is a Nash equilibrium.

To find the point of intersection of the best response functions precisely, we can solve the two equations in (34.3):

$$\begin{aligned} a_1 &= \tfrac{1}{2}(c + a_2) \\ a_2 &= \tfrac{1}{2}(c + a_1). \end{aligned}$$

Substituting the second equation in the first, we get $a_1 = \frac{1}{2}(c + \frac{1}{2}(c + a_1)) = \frac{3}{4}c + \frac{1}{4}a_1$, so that $a_1 = c$. Substituting this value of a_1 into the second equation, we get $a_2 = c$. We conclude that the game has a unique Nash equilibrium $(a_1, a_2) = (c, c)$. (To reach this conclusion, it suffices to solve the two equations; we do not have to draw Figure 38.1. However, the diagram shows us at once that the game has a unique equilibrium, in which both players' actions exceed $\frac{1}{2}c$, facts that serve to check the results of our algebra.)

In the game in this example, each player has a unique best response to every action of the other player, so that the best response functions are lines. If a player has many best responses to some of the other players' actions, then her best response function is "thick" at some points; several examples in the next chapter have this property (see, for example, Figure 64.1). Example 37.1 is special also because the game has a unique Nash equilibrium—the best response functions cross once. As we have seen, some games have more than one equilibrium, and others have no equilibrium. A pair of best response functions that illustrates some of the possibilities is shown in Figure 39.1. In this figure the shaded area of player 1's best response function indicates that for a_2 between $\overline{a}_2$ and $\underline{a}_2$, player 1 has a range of best responses. For example, all actions of player 1 from a_1^{**} to a_1^{***} are best responses to the action a_2^{***} of player 2. For a game with these best response functions, the set of Nash equilibria consists of the pair of actions (a_1^*, a_2^*) and all the pairs of actions on player 2's best response function between (a_1^{**}, a_2^{**}) and (a_1^{***}, a_2^{***}).

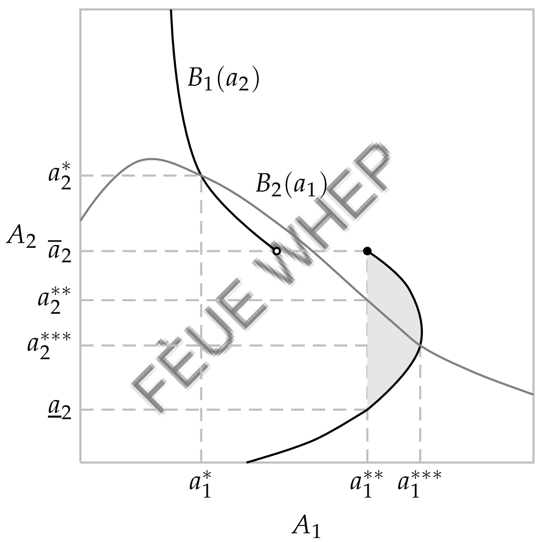


Figure 39.1 An example of the best response functions of a two-player game in which each player's set of actions is an interval of numbers. The set of Nash equilibria of the game consists of the pair of actions (a_1^*, a_2^*) and all the pairs of actions on player 2's best response function between (a_1^{**}, a_2^{**}) and (a_1^{***}, a_2^{***}).

? EXERCISE 39.1 (Strict and nonstrict Nash equilibria) Which of the Nash equilibria of the game whose best response functions are given in Figure 39.1 are strict (see the definition on page 31)?

Another feature that differentiates the best response functions in Figure 39.1 from those in Figure 38.1 is that the best response function b_1 of player 1 is not continuous. When player 2's action is $\overline{a}_2$, player 1's best response is a_1^{**} (indicated by the small disk at $(a_1^{**}, \overline{a}_2)$), but when player 2's action is slightly greater than $\overline{a}_2$, player 1's best response is significantly less than a_1^{**}. (The small circle indicates a point excluded from the best response function.) Again, several examples in the next chapter have this feature. From Figure 39.1 we see that if a player's best response function is discontinuous, then depending on where the discontinuity occurs, the best response functions may not intersect at all—the game may, like *Matching Pennies*, have no Nash equilibrium.

? EXERCISE 40.1 (Finding Nash equilibria using best response functions) Find the Nash equilibria of the two-player strategic game in which each player's set of actions is the set of nonnegative numbers and the players' payoff functions are $u_1(a_1, a_2) = a_1(a_2 - a_1)$ and $u_2(a_1, a_2) = a_2(1 - a_1 - a_2)$.

? EXERCISE 40.2 (A joint project) Two people are engaged in a joint project. If each person i puts in the effort x_i, a nonnegative number equal to at most 1, which costs her $c(x_i)$, the outcome of the project is worth $f(x_1, x_2)$. The worth of the project is split equally between the two people, regardless of their effort levels. Formulate this situation as a strategic game. Find the Nash equilibria of the game when (*a*) $f(x_1, x_2) = 3x_1x_2$ and $c(x_i) = x_i^2$ for $i = 1, 2$, and (*b*) $f(x_1, x_2) = 4x_1x_2$ and $c(x_i) = x_i$ for $i = 1, 2$. In each case, is there a pair of effort levels that yields both players higher payoffs than the Nash equilibrium effort levels?

2.8.4 Illustration: contributing to a public good

Exercise 31.1 models decisions on whether to contribute to the provision of a "public good". We now study a model in which two people decide not only whether to contribute, but also *how much* to contribute.

Denote person i's wealth by w_i, and the amount she contributes to the public good by c_i ($0 \le c_i \le w_i$); she spends her remaining wealth $w_i - c_i$ on "private goods" (like clothes and food, whose consumption by one person precludes their consumption by anyone else). The amount of the public good is equal to the sum of the contributions. Each person cares both about the amount of the public good and her consumption of private goods.

Suppose that person i's preferences are represented by the payoff function $v_i(c_1 + c_2) + w_i - c_i$. Because w_i is a constant, person i's preferences are alternatively represented by the payoff function

$$u_i(c_1, c_2) = v_i(c_1 + c_2) - c_i.$$

This situation is modeled by the following strategic game.

Players The two people.

Actions Player i's set of actions is the set of her possible contributions (nonnegative numbers less than or equal to w_i), for $i = 1, 2$.

Preferences Player i's preferences are represented by the payoff function $u_i(c_1, c_2) = v_i(c_1 + c_2) - c_i$, for $i = 1, 2$.

To find the Nash equilibria of this strategic game, consider the players' best response functions. Player 1's best response to the contribution c_2 of player 2 is the value of c_1 that maximizes $v_1(c_1 + c_2) - c_1$. Without specifying the form of the function v_1 we cannot explicitly calculate this optimal value. However, we can determine how it varies with c_2.

First consider player 1's best response to $c_2 = 0$. Suppose that the form of the function v_1 is such that the function $u_1(c_1, 0)$ increases up to its maximum, then decreases (as in Figure 41.1). Then player 1's best response to $c_2 = 0$, which I denote $b_1(0)$, is unique. This best response is the value of c_1 that maximizes $u_1(c_1, 0) = v_1(c_1) - c_1$ subject to $0 \leq c_1 \leq w_1$. Assume that $0 < b_1(0) < w_1$: player 1's optimal contribution to the public good when player 2 makes no contribution is positive and less than her entire wealth.

Now consider player 1's best response to $c_2 = k > 0$. This best response is the value of c_1 that maximizes $u_1(c_1, k) = v_1(c_1 + k) - c_1$. Now, we have

$$u_1(c_1, k) = u_1(c_1 + k, 0) + k.$$

That is, the graph of $u_1(c_1, k)$ as a function of c_1 is the translation to the left k units and up k units of the graph of $u_1(c_1, 0)$ as a function of c_1 (refer to Figure 41.1). Thus if $k \leq b_1(0)$ then $b_1(k) = b_1(0) - k$: if player 2's contribution increases from 0 to k then player 1's best response decreases by k. If $k > b_1(0)$ then, given the form of $u_1(c_1, 0)$, we have $b_1(k) = 0$.

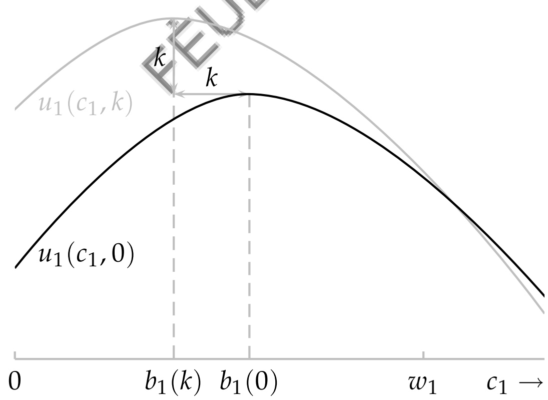


Figure 41.1 The relation between player 1's best responses $b_1(0)$ and $b_1(k)$ to $c_2 = 0$ and $c_2 = k$ in the game of contributing to a public good.

We conclude that if player 2 increases her contribution by k then player 1's best response is to reduce her contribution by k (or to zero, if k is larger than player 1's original contribution)!

The same analysis applies to player 2: for every unit more that player 1 contributes, player 2 contributes a unit less, so long as her contribution is nonnegative. The function v_2 may be different from the function v_1, so that player 1's best contribution $b_1(0)$ when $c_2 = 0$ may be different from player 2's best contribution $b_2(0)$

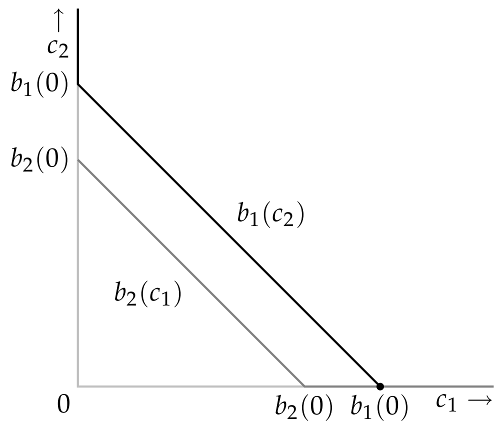


Figure 42.1 The best response functions for the game of contributing to a public good in Section 2.8.4 in a case in which $b_1(0) > b_2(0)$. The best response function of player 1 is the black line; that of player 2 is the gray line.

when $c_1 = 0$. But both best response functions have the same character: the slope of each function is -1 where the value of the function is positive. They are shown in Figure 42.1 for a case in which $b_1(0) > b_2(0)$.

We deduce that if $b_1(0) > b_2(0)$ then the game has a unique Nash equilibrium, $(b_1(0), 0)$: player 2 contributes nothing. Similarly, if $b_1(0) < b_2(0)$ then the unique Nash equilibrium is $(0, b_2(0))$: player 1 contributes nothing. That is, the person who contributes more when the other person contributes nothing is the only one to make a contribution in a Nash equilibrium. Only if $b_1(0) = b_2(0)$, which is not likely if the functions v_1 and v_2 differ, is there an equilibrium in which both people contribute. In this case the downward-sloping parts of the best response functions coincide, so that any pair of contributions (c_1, c_2) with $c_1 + c_2 = b_1(0)$ and $c_i \geq 0$ for $i = 1, 2$ is a Nash equilibrium.

In summary, the notion of Nash equilibrium predicts that, except in unusual circumstances, only one person contributes to the provision of the public good when each person's payoff function takes the form $v_i(c_1 + c_2) + w_i - c_i$, each function $v_i(c_i) - c_i$ increases to a maximum, then decreases, and each person optimally contributes less than her entire wealth when the other person does not contribute. The person who contributes is the one who wishes to contribute more when the other person does not contribute. In particular, the identity of the person who contributes does not depend on the distribution of wealth; any distribution in which each person optimally contributes less than her entire wealth when the other person does not contribute leads to the same outcome.

The next exercise asks you to consider a case in which the amount of the public good affects each person's enjoyment of the private good. (The public good might be clean air, which improves each person's enjoyment of her free time.)

? EXERCISE 42.1 (Contributing to a public good) Consider the model in this section when $u_i(c_1, c_2)$ is the sum of three parts: the amount $c_1 + c_2$ of the public good provided, the amount $w_i - c_i$ person i spends on private goods, and a term $(w_i - c_i)(c_1 + c_2)$ that reflects an interaction between the amount of the public good and her private consumption—the greater the amount of the public good, the more she values her private consumption. In summary, suppose that person i's payoff is $c_1 + c_2 + w_i - c_i + (w_i - c_i)(c_1 + c_2)$, or

$$w_i + c_j + (w_i - c_i)(c_1 + c_2),$$

where j is the other person. Assume that $w_1 = w_2 = w$, and that each player i's contribution c_i may be any number (positive or negative, possibly larger than w). Find the Nash equilibrium of the game that models this situation. (You can calculate the best responses explicitly. Imposing the sensible restriction that c_i lie between 0 and w complicates the analysis, but does not change the answer.) Show that in the Nash equilibrium both players are worse off than they are when they both contribute one half of their wealth to the public good. If you can, extend the analysis to the case of n people. As the number of people increases, how does the total amount contributed in a Nash equilibrium change? Compare the players' equilibrium payoffs with their payoffs when each contributes half her wealth to the public good, as n increases without bound. (The game is studied further in Exercise 358.3.)

2.9 Dominated actions

2.9.1 *Strict domination*

You drive up to a red traffic light. The left lane is free; in the right lane there is a car that may turn right when the light changes to green, in which case it will have to wait for a pedestrian to cross the side street. Assuming you wish to progress as quickly as possible, the action of pulling up in the left lane "strictly dominates" that of pulling up in the right lane. If the car in the right lane turns right then you are much better off in the left lane, where your progress will not be impeded; and even if the car in the right lane does not turn right, you are still better off in the left lane, rather than behind the other car.

In any game, a player's action "strictly dominates" another action if it is superior, *no matter what the other players do*.

▶ DEFINITION 43.1 (*Strict domination*) In a strategic game with ordinal preferences, player i's action a_i'' **strictly dominates** her action a_i' if

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function that represents player i's preferences.

In the *Prisoner's Dilemma*, for example, the action *Fink* strictly dominates the action *Quiet*: regardless of her opponent's action, a player prefers the outcome when she chooses *Fink* to the outcome when she chooses *Quiet*. In *BoS*, on the other hand, neither action strictly dominates the other: *Bach* is better than *Stravinsky* if the other player chooses *Bach*, but is worse than *Stravinsky* if the other player chooses *Stravinsky*.

If an action strictly dominates the action a_i, we say that a_i is **strictly dominated**. A strictly dominated action is not a best response to any actions of the other players: whatever the other players do, some other action is better. Since a player's Nash equilibrium action is a best response to the other players' Nash equilibrium actions,

a strictly dominated action is not used in any Nash equilibrium.

When looking for the Nash equilibria of a game, we can thus eliminate from consideration all strictly dominated actions. For example, we can eliminate *Quiet* for each player in the *Prisoner's Dilemma*, leaving (*Fink*, *Fink*) as the only candidate for a Nash equilibrium. (As we know, this action pair is indeed a Nash equilibrium.)

The fact that the action a_i'' strictly dominates the action a_i' of course does *not* imply that a_i'' strictly dominates *all* actions. Indeed, a_i'' may itself be strictly dominated. In the left-hand game in Figure 44.1, for example, M strictly dominates T, but B is better than M if player 2 chooses R. (I give only the payoffs of player 1 in the figure, because those of player 2 are not relevant.) Since T is strictly dominated, the game has no Nash equilibrium in which player 1 uses it; but the game may also not have any equilibrium in which player 1 uses M. In the right-hand game, M strictly dominates T, but is itself strictly dominated by B. In this case, in any Nash equilibrium player 1's action is B (her only action that is not strictly dominated).

	L	R
T	1	0
M	2	1
B	1	3

	L	R
T	1	0
M	2	1
B	3	2

Figure 44.1 Two games in which player 1's action T is strictly dominated by M. (Only player 1's payoffs are given.) In the left-hand game, B is better than M if player 2 chooses R; in the right-hand game, M itself is strictly dominated, by B.

A strictly dominated action is incompatible not only with a steady state, but also with rational behavior by a player who confronts a game for the first time. This fact is the first step in a theory different from Nash equilibrium, explored in Chapter 12.

2.9.2 Weak domination

As you approach the red light in the situation at the start of the previous section, there is a car in *each* lane. The car in the right lane may, or may not, be turning right; if it is, it may be delayed by a pedestrian crossing the side street. The car in the left lane cannot turn right. In this case your pulling up in the left lane "weakly dominates", though does not strictly dominate, your pulling up in the right lane. If the car in the right lane does not turn right, then both lanes are equally good; if it does, then the left lane is better.

In any game, a player's action "weakly dominates" another action if the first action is at least as good as the second action, no matter what the other players do, and is better than the second action for some actions of the other players.

▶ DEFINITION 45.1 (*Weak domination*) In a strategic game with ordinal preferences, player i's action a_i'' **weakly dominates** her action a_i' if

$$u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions}$$

and

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}) \text{ for some list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function that represents player i's preferences.

For example, in the game in Figure 45.1 (in which, once again, only player 1's payoffs are given), M weakly dominates T, and B weakly dominates M; B strictly dominates T.

	L	R
T	1	0
M	2	0
B	2	1

Figure 45.1 A game illustrating weak domination. (Only player 1's payoffs are given.) The action M weakly dominates T; B weakly dominates M. The action B strictly dominates T.

In a *strict* Nash equilibrium (Section 2.7.8) no player's equilibrium action is weakly dominated: every non-equilibrium action for a player yields her a payoff less than does her equilibrium action, and hence does not weakly dominate the equilibrium action.

Can an action be weakly dominated in a nonstrict Nash equilibrium? Definitely. Consider the games in Figure 46.1. In both games B weakly (but not strictly) dominates C for both players. But in both games (C, C) is a Nash equilibrium: *given* that player 2 chooses C, player 1 cannot do better than choose C, and *given* that player 1 chooses C, player 2 cannot do better than choose C. Both games also have a Nash equilibrium, (B, B), in which neither player's action is weakly dominated. In the left-hand game this equilibrium is better for both players than the equilibrium (C, C) in which both players' actions are weakly dominated, whereas in the right-hand game it is worse for both players than (C, C).

? EXERCISE 45.2 (Strict equilibria and dominated actions) For the game in Figure 46.2, determine, for each player, whether any action is strictly dominated or weakly dominated. Find the Nash equilibria of the game; determine whether any equilibrium is strict.

	B	C
B	$1,1$	$0,0$
C	$0,0$	$0,0$

	B	C
B	$1,1$	$2,0$
C	$0,2$	$2,2$

Figure 46.1 Two strategic games with a Nash equilibrium (C,C) in which both players' actions are weakly dominated.

	L	C	R
T	$0,0$	$1,0$	$1,1$
M	$1,1$	$1,1$	$3,0$
B	$1,1$	$2,1$	$2,2$

Figure 46.2 The game in Exercise 45.2.

? EXERCISE 46.1 (Nash equilibrium and weakly dominated actions) Give an example of a two-player strategic game in which each player has finitely many actions and in the only Nash equilibrium both players' actions are weakly dominated.

2.9.3 *Illustration: voting*

Two candidates, A and B, vie for office. Each of an odd number of citizens may vote for either candidate. (Abstention is not possible.) The candidate who obtains the most votes wins. (Because the number of citizens is odd, a tie is impossible.) A majority of citizens prefer A to win than B to win.

The following strategic game models the citizens' voting decisions in this situation.

Players The citizens.

Actions Each player's set of actions consists of voting for A and voting for B.

Preferences All players are indifferent between all action profiles in which a majority of players vote for A and between all action profiles in which a majority of players vote for B. Some players (a majority) prefer an action profile of the first type to one of the second type, and the others have the reverse preference.

I claim that a citizen's voting for her less preferred candidate is weakly dominated by her voting for her favorite candidate. Suppose that citizen i prefers candidate A; fix the votes of all citizens other than i. If citizen i switches from voting for B to voting for A then, depending on the other citizens' votes, either the outcome does not change, or A wins rather than B; such a switch cannot cause the winner to change from A to B. That is, citizen i's switching from voting for B to voting for A either has no effect on the outcome, or makes her better off; it cannot make her worse off.

The game has Nash equilibria in which some, or all, citizens' actions are weakly dominated. For example, the action profile in which all citizens vote for B is a Nash equilibrium (no citizen's switching her vote has any effect on the outcome).

? EXERCISE 47.1 (Voting) Find all the Nash equilibria of the game. (First consider action profiles in which the winner obtains one more vote than the loser and at least one citizen who votes for the winner prefers the loser to the winner, then profiles in which the winner obtains one more vote than the loser and all citizens who vote for the winner prefer the winner to the loser, and finally profiles in which the winner obtains three or more votes more than the loser.) Is there any equilibrium in which no player uses a weakly dominated action?

Consider a variant of the game in which the number of candidates is greater than two. A variant of the argument above shows that a citizen's action of voting for her least preferred candidate is weakly dominated by all her other actions. The next exercise asks you to show that no other action is weakly dominated.

? EXERCISE 47.2 (Voting between three candidates) Suppose there are three candidates, A, B, and C. A tie for first place is possible in this case; assume that a citizen who prefers a win by x to a win by y ranks a tie between x and y between an outright win for x and an outright win for y. Show that a citizen's only weakly dominated action is a vote for her least preferred candidate. Find a Nash equilibrium in which some citizen does not vote for her favorite candidate, but the action she takes is not weakly dominated.

? EXERCISE 47.3 (Approval voting) In the system of "approval voting", a citizen may vote for as many candidates as she wishes. If there are two candidates, say A and B, for example, a citizen may vote for neither candidate, for A, for B, or for both A and B. As before, the candidate who obtains the most votes wins. Show that any action that includes a vote for a citizen's least preferred candidate is weakly dominated, as is any action that does not include a vote for her most preferred candidate. More difficult: show that if there are k candidates then for a citizen who prefers candidate 1 to candidate 2 to $\ldots$ to candidate k the action that consists of votes for candidates 1 and $k-1$ is *not* weakly dominated.

2.9.4 Illustration: collective decision-making

The members of a group of people are affected by a policy, modeled as a number. Each person i has a favorite policy, denoted x_i^*; she prefers the policy y to the policy z if and only if y is closer to x_i^* than is z. The number n of people is odd. The following mechanism is used to choose a policy: each person names a policy, and the policy chosen is the median of those named. (That is, the policies named are put in order, and the one in the middle is chosen. If, for example, there are five people, and they name the policies -2, 0, 0.6, 5, and 10, then the policy 0.6 is chosen.)

What outcome does this mechanism induce? Does anyone have an incentive to name her favorite policy, or are people induced to distort their preferences? We can answer these questions by studying the following strategic game.

Players The n people.

Actions Each person's set of actions is the set of policies (numbers).

Preferences Each person i prefers the action profile a to the action profile a' if and only if the median policy named in a is closer to x_i^* than is the median policy named in a'.

I claim that for each player i, the action of naming her favorite policy x_i^* weakly dominates *all* her other actions. The reason is that relative to the situation in which she names x_i^*, she can change the median only by naming a policy *further* from her favorite policy than the current median; no change in the policy she names moves the median closer to her favorite policy.

Precisely, I show that for each action $x_i \neq x_i^*$ of player i, (a) for *all* actions of the other players, player i is at least as well off naming x_i^* as she is naming x_i, and (b) for *some* actions of the other players she is better off naming x_i^* than she is naming x_i. Take $x_i > x_i^*$.

a. For any list of actions of the players *other than* player i, denote the value of the $\frac{1}{2}(n-1)$th highest action by $\underline{a}$ and the value of the $\frac{1}{2}(n+1)$th highest action by $\overline{a}$ (so that half of the remaining players' actions are at most $\underline{a}$ and half of them are at least $\overline{a}$).

- If $\overline{a} \leq x_i^*$ or $\underline{a} \geq x_i$ then the median policy is the same whether player i names x_i^* or x_i.
- If $\overline{a} > x_i^*$ and $\underline{a} < x_i$ then when player i names x_i^* the median policy is at most the greater of x_i^* and $\underline{a}$ and when player i names x_i the median policy is at least the lesser of x_i and $\overline{a}$. Thus player i is worse off naming x_i than she is naming x_i^*.

b. Suppose that half of the remaining players name policies less than x_i^* and half of them name policies greater than x_i. Then the outcome is x_i^* if player i names x_i^*, and x_i if she names x_i. Thus she is better off naming x_i^* than she is naming x_i.

A symmetric argument applies when $x_i < x_i^*$.

If we think of the mechanism as asking the players to name their favorite policies, then the result is that telling the truth weakly dominates all other actions.

An implication of the fact that player i's naming her favorite policy x_i^* weakly dominates *all* her other actions is that the action profile in which every player names her favorite policy is a Nash equilibrium. That is, truth-telling is a Nash equilibrium, in the interpretation of the previous paragraph.

? EXERCISE 49.1 (Other Nash equilibria of the game modeling collective decision-making) Find two Nash equilibria in which the outcome is the median favorite policy, and one in which it is not.

? EXERCISE 49.2 (Another mechanism for collective decision-making) Consider the variant of the mechanism for collective decision-making described above in which the policy chosen is the *mean*, rather than the median, of the policies named by the players. Does a player's action of naming her favorite policy weakly dominate all her other actions?

2.10 Equilibrium in a single population: symmetric games and symmetric equilibria

A Nash equilibrium of a strategic game corresponds to a steady state of an interaction between the members of several populations, one for each player in the game, each play of the game involving one member of each population. Sometimes we want to model a situation in which the members of a *single* homogeneous population are involved anonymously in a symmetric interaction. Consider, for example, pedestrians approaching each other on a sidewalk or car drivers arriving simultaneously at an intersection from different directions. In each case, the members of each encounter are drawn from the same population: pairs from a single population of pedestrians meet each other, and groups from a single population of car drivers simultaneously approach intersections. And in each case, every participant's role is the same.

I restrict attention here to cases in which each interaction involves two participants. Define a two-player game to be "symmetric" if each player has the same set of actions and each player's evaluation of an outcome depends only on her action and that of her opponent, not on whether she is player 1 or player 2. That is, player 1 feels the same way about the outcome (a_1, a_2), in which her action is a_1 and her opponent's action is a_2, as player 2 feels about the outcome (a_2, a_1), in which *her* action is a_1 and her opponent's action is a_2. In particular, the players' preferences may be represented by payoff functions in which both players' payoffs are the same whenever the players choose the same action: $u_1(a, a) = u_2(a, a)$ for every action a.

▶ DEFINITION 49.3 (*Symmetric two-player strategic game with ordinal preferences*) A two-player strategic game with ordinal preferences is **symmetric** if the players' sets of actions are the same and the players' preferences are represented by payoff functions u_1 and u_2 for which $u_1(a_1, a_2) = u_2(a_2, a_1)$ for every action pair (a_1, a_2).

A two-player game in which each player has two actions is symmetric if the players' preferences are represented by payoff functions that take the form shown in Figure 50.1, where w, x, y, and z are arbitrary numbers. Several of the two-player games we have considered are symmetric, including the *Prisoner's Dilemma*, the two-player *Stag Hunt* (given again in Figure 50.2), and the game in Exercise 36.2. *BoS* (Figure 16.1) and *Matching Pennies* (Figure 17.1) are not symmetric.

	A	B
A	w, w	x, y
B	y, x	z, z

Figure 50.1 A two-player symmetric game.

	Quiet	*Fink*
Quiet	2, 2	0, 3
Fink	3, 0	1, 1

	Stag	*Hare*
Stag	2, 2	0, 1
Hare	1, 0	1, 1

Figure 50.2 Two symmetric games: the *Prisoner's Dilemma* (left) and the two-player *Stag Hunt* (right).

? EXERCISE 50.1 (Symmetric strategic games) Which of the games in Exercises 29.1 and 40.1, Example 37.1, Section 2.8.4, and Figure 46.1 are symmetric?

When the players in a symmetric two-player game are drawn from a single population, nothing distinguishes one of the players in any given encounter from the other. We may call them "player 1" and "player 2", but these labels are only for our convenience. There is only one role in the game, so that a steady state is characterized by a *single* action used by every participant whenever playing the game. An action a^* corresponds to such a steady state if no player can do better by using any other action, given that all the other players use a^*. An action a^* has this property if and only if (a^*, a^*) is a Nash equilibrium of the game. In other words, the solution that corresponds to a steady state of pairwise interactions between the members of a single population is "symmetric Nash equilibrium": a Nash equilibrium in which both players take the same action. The idea of this notion of equilibrium does not depend on the game's having only two players, so I give a definition for a game with any number of players.

▶ DEFINITION 50.2 (*Symmetric Nash equilibrium*) An action profile a^* in a strategic game with ordinal preferences in which each player has the same set of actions is a **symmetric Nash equilibrium** if it is a Nash equilibrium and a_i^* is the same for every player i.

As an example, consider a model of approaching pedestrians. Each participant in any given encounter has two possible actions—to step to the right, and to step to the left—and is better off when participants both step in the same direction than when they step in different directions (in which case a collision occurs). The resulting symmetric strategic game is given in Figure 51.1. The game has two symmetric Nash equilibria, namely $(Left, Left)$ and $(Right, Right)$. That is, there are two steady states, in one of which every pedestrian steps to the left as she

	Left	*Right*
Left	1, 1	0, 0
Right	0, 0	1, 1

Figure 51.1 Approaching pedestrians.

approaches another pedestrian, and in another of which both participants step to the right. (The latter steady state seems to prevail in the USA and Canada.)

A symmetric game may have no symmetric Nash equilibrium. Consider, for example, the game in Figure 51.2. This game has two Nash equilibria, (X, Y) and (Y, X), neither of which is symmetric. You may wonder if, in such a situation, there is a steady state in which each player does not always take the same action in every interaction. This question is addressed in Section 4.7.

	X	Y
X	0, 0	1, 1
Y	1, 1	0, 0

Figure 51.2 A symmetric game with no symmetric Nash equilibrium.

? EXERCISE 51.1 (Equilibrium for pairwise interactions in a single population) Find all the Nash equilibria of the game in Figure 51.3. Which of the equilibria, if any, correspond to a steady state if the game models pairwise interactions between the members of a single population?

	A	B	C
A	1, 1	2, 1	4, 1
B	1, 2	5, 5	3, 6
C	1, 4	6, 3	0, 0

Figure 51.3 The game in Exercise 51.1.

Notes

The notion of a strategic game originated in the work of Borel (1921) and von Neumann (1928). The notion of Nash equilibrium (and its interpretation) is due to Nash (1950a). (The idea that underlies it goes back at least to Cournot (1838, Ch. 7).)

The *Prisoner's Dilemma* appears to have first been considered by Melvin Dresher and Merrill Flood, who used it in an experiment at the RAND Corporation in January 1950 (Flood 1958/59, 11–17); it is an example in Nash's PhD thesis, submitted in May 1950. The story associated with it is due to Tucker (1950) (see Straffin 1980). O'Neill (1994, 1010–1013) argues that there is no evidence that game theory (and in particular the *Prisoner's Dilemma*) influenced US nuclear strategists in the 1950s. The idea that a common property will be overused is very old (in Western thought, it goes back at least to Aristotle (Ostrom 1990, 2)); a precise modern analysis was initiated by Gordon (1954). Hardin (1968) coined the phrase "tragedy of the commons".

BoS, like the *Prisoner's Dilemma*, is an example in Nash's PhD thesis; Luce and Raiffa (1957, 90–91) name it and associate a story with it. *Matching Pennies* was first considered by von Neumann (1928). Rousseau's sentence about hunting stags is interpreted as a description of a game by Ullmann-Margalit (1977, 121) and Jervis (1977/78), following discussion by Waltz (1959, 167–169) and Lewis (1969, 7, 47).

The information about John Nash in the box on p. 20 comes from Leonard (1994), Kuhn et al. (1995), Kuhn (1996), Myerson (1996), Nasar (1998), and Nash (1995). *Hawk–Dove* is known also as "Chicken" (two drivers approach each other on a narrow road; the one who pulls over first is "chicken"). It was first suggested (in a more complicated form) as a model of animal conflict by Maynard Smith and Price (1973). The discussion of focal points in the box on p. 30 draws on Schelling (1960, 54–58).

Games modeling voluntary contributions to a public good were first considered by Olson (1965, Section I.D). The game in Exercise 31.1 is studied in detail by Palfrey and Rosenthal (1984). The result in Section 2.8.4 is due to Warr (1983) and Bergstrom, Blume, and Varian (1986).

Game theory was first used to study voting behavior by Farquharson (1969) (whose book was completed in 1958). The system of "approval voting" in Exercise 47.3 was first studied formally by Brams and Fishburn (1978, 1983).

Exercise 16.1 is based on Leonard (1990). Exercise 25.2 is based on Ullmann-Margalit (1977, 48). The game in Exercise 28.2 is taken from Van Huyck, Battalio, and Beil (1990). The game in Exercise 32.1 is taken from Moulin (1986, 72). The game in Exercise 32.2 was first studied by Palfrey and Rosenthal (1983). Exercise 32.3 is based on Braess (1968); see also Murchland (1970). The game in Exercise 36.2 is taken from Brams (1993).

3 Nash Equilibrium: Illustrations

Prerequisite: Chapter 2.

In this chapter I discuss in detail a few key models that use the notion of Nash equilibrium to study economic, political, and biological phenomena. The discussion shows how the notion of Nash equilibrium improves our understanding of a wide variety of phenomena. It also illustrates some of the many forms strategic games and their Nash equilibria can take. The models in Sections 3.1 and 3.2 are related to each other, whereas those in each of the other sections are independent of each other.

3.1 Cournot's model of oligopoly

3.1.1 Introduction

How does the outcome of competition among the firms in an industry depend on the characteristics of the demand for the firms' output, the nature of the firms' cost functions, and the number of firms? Will the benefits of technological improvements be passed on to consumers? Will a reduction in the number of firms generate a less desirable outcome? To answer these questions we need a model of the interaction between firms competing for the business of consumers. In this section and the next I analyze two such models. Economists refer to them as models of "oligopoly" (competition between a small number of sellers), though they involve no restriction on the number of firms; the label reflects the strategic interaction they capture. Both models were studied first in the nineteenth century, before the notion of Nash equilibrium was formalized for a general strategic game. The first is due to the economist Cournot (1838).

3.1.2 General model

A single good is produced by n firms. The cost to firm i of producing q_i units of the good is $C_i(q_i)$, where C_i is an increasing function (more output is more costly to produce). All the output is sold at a single price, determined by the demand for the good and the firms' total output. Specifically, if the firms' total output is Q then the market price is $P(Q)$; P is called the "inverse demand function". Assume that P is a decreasing function when it is positive: if the firms' total output increases, then the price decreases (unless it is already zero). If the output of each firm i is q_i, then the price is $P(q_1 + \cdots + q_n)$, so that firm i's revenue is $q_i P(q_1 + \cdots + q_n)$. Thus firm i's profit, equal to its revenue minus its cost, is

$$\pi_i(q_1, \ldots, q_n) = q_i P(q_1 + \cdots + q_n) - C_i(q_i). \tag{54.1}$$

Cournot suggested that the industry be modeled as the following strategic game, which I refer to as **Cournot's oligopoly game**.

Players The firms.

Actions Each firm's set of actions is the set of its possible outputs (nonnegative numbers).

Preferences Each firm's preferences are represented by its profit, given in (54.1).

3.1.3 Example: duopoly with constant unit cost and linear inverse demand function

For specific forms of the functions C_i and P we can compute a Nash equilibrium of Cournot's game. Suppose there are two firms (the industry is a "duopoly"), each firm's cost function is the same, given by $C_i(q_i) = cq_i$ for all q_i ("unit cost" is constant, equal to c), and the inverse demand function is linear where it is positive, given by

$$P(Q) = \begin{cases} \alpha - Q & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha, \end{cases} \tag{54.2}$$

where $\alpha > 0$ and $c \geq 0$ are constants. This inverse demand function is shown in Figure 55.1. (Note that the price $P(Q)$ cannot be equal to $\alpha - Q$ for *all* values of Q, for then it would be negative for $Q > \alpha$.) Assume that $c < \alpha$, so that there is some value of total output Q for which the market price $P(Q)$ is greater than the firms' common unit cost c. (If c were to exceed α, there would be no output for the firms at which they could make any profit, because the market price never exceeds α.)

To find the Nash equilibria in this example, we can use the procedure based on the firms' best response functions (Section 2.8.3). First we need to find the firms' payoffs (profits). If the firms' outputs are q_1 and q_2 then the market price $P(q_1 + q_2)$ is $\alpha - q_1 - q_2$ if $q_1 + q_2 \leq \alpha$ and zero if $q_1 + q_2 > \alpha$. Thus firm 1's profit is

$$\begin{aligned} \pi_1(q_1, q_2) &= q_1(P(q_1 + q_2) - c) \\ &= \begin{cases} q_1(\alpha - c - q_1 - q_2) & \text{if } q_1 + q_2 \leq \alpha \\ -cq_1 & \text{if } q_1 + q_2 > \alpha. \end{cases} \end{aligned}$$

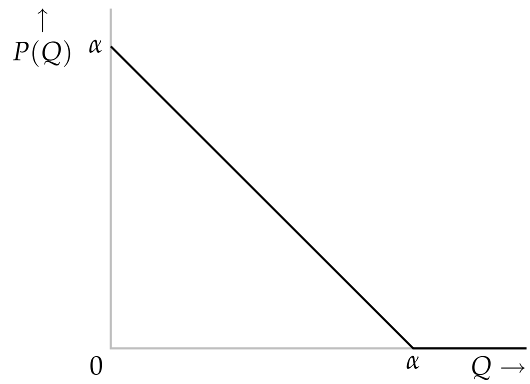


Figure 55.1 The inverse demand function in the example of Cournot's game studied in Section 3.1.3.

To find firm 1's best response to any given output q_2 of firm 2, we need to study firm 1's profit as a function of its output q_1 for given values of q_2. If $q_2 = 0$ then firm 1's profit is $\pi_1(q_1, 0) = q_1(\alpha - c - q_1)$ for $q_1 \leq \alpha$, a quadratic function that is zero when $q_1 = 0$ and when $q_1 = \alpha - c$. This function is the black curve in Figure 56.1. Given the symmetry of quadratic functions (Section 17.4), the output q_1 of firm 1 that maximizes its profit is $q_1 = \frac{1}{2}(\alpha - c)$. (If you know calculus, you can reach the same conclusion by setting the derivative of firm 1's profit with respect to q_1 equal to zero and solving for q_1.) Thus firm 1's best response to an output of zero for firm 2 is $b_1(0) = \frac{1}{2}(\alpha - c)$.

As the output q_2 of firm 2 increases, the profit firm 1 can obtain at any given output decreases, because more output of firm 2 means a lower price. The gray curve in Figure 56.1 is an example of $\pi_1(q_1, q_2)$ for $q_2 > 0$ and $q_2 < \alpha - c$. Again this function is a quadratic up to the output $q_1 = \alpha - q_2$ that leads to a price of zero. Specifically, the quadratic is $\pi_1(q_1, q_2) = q_1(\alpha - c - q_2 - q_1)$, which is zero when $q_1 = 0$ and when $q_1 = \alpha - c - q_2$. From the symmetry of quadratic functions (or some calculus) we conclude that the output that maximizes $\pi_1(q_1, q_2)$ is $q_1 = \frac{1}{2}(\alpha - c - q_2)$. (When $q_2 = 0$, this is equal to $\frac{1}{2}(\alpha - c)$, the best response to an output of zero that we found in the previous paragraph.)

When $q_2 > \alpha - c$, the value of $\alpha - c - q_2$ is negative. Thus for such a value of q_2, we have $q_1(\alpha - c - q_2 - q_1) < 0$ for all positive values of q_1: firm 1's profit is negative for any positive output, so that its best response is to produce the output of zero.

We conclude that the best response of firm 1 to the output q_2 of firm 2 depends on the value of q_2: if $q_2 \leq \alpha - c$ then firm 1's best response is $\frac{1}{2}(\alpha - c - q_2)$, whereas if $q_2 > \alpha - c$ then firm 1's best response is 0. Or, more compactly,

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c - q_2) & \text{if } q_2 \leq \alpha - c \\ 0 & \text{if } q_2 > \alpha - c. \end{cases}$$

Because firm 2's cost function is the same as firm 1's, its best response function b_2 is also the same: for any number q, we have $b_2(q) = b_1(q)$. Of course, firm 2's

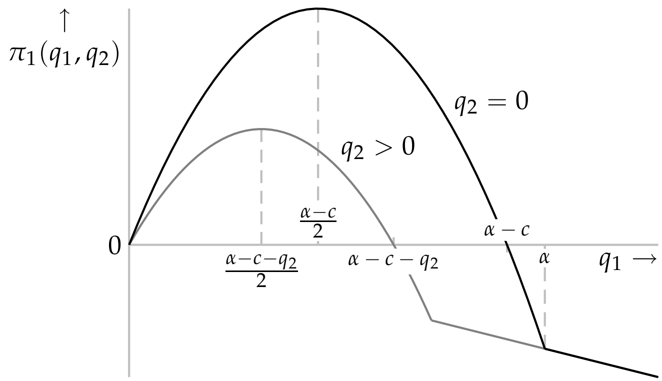


Figure 56.1 Firm 1's profit as a function of its output, given firm 2's output. The black curve shows the case $q_2 = 0$, whereas the gray curve shows a case in which $q_2 > 0$.

best response function associates a value of firm 2's output with every output of firm 1, whereas firm 1's best response function associates a value of firm 1's output with every output of firm 2, so we plot them relative to different axes. They are shown in Figure 56.2 (b_1 is black; b_2 is gray). As for a general game (see Section 2.8.3), b_1 associates each point on the vertical axis with a point on the horizontal axis, and b_2 associates each point on the horizontal axis with a point on the vertical axis.

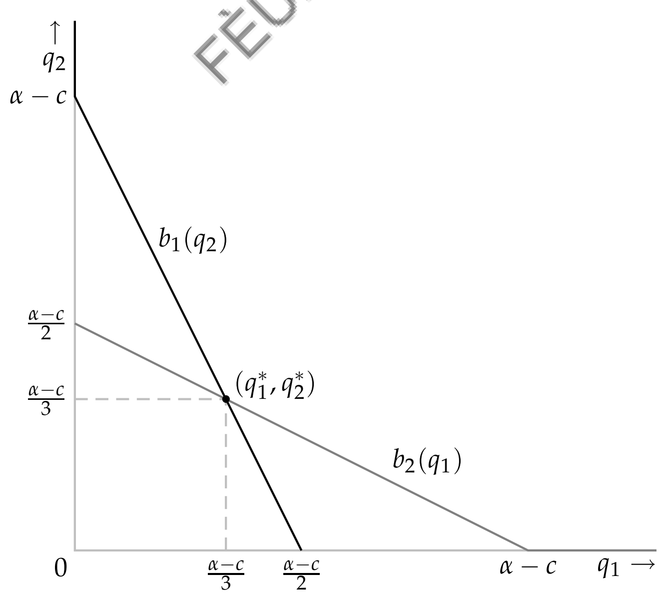


Figure 56.2 The best response functions in Cournot's duopoly game when the inverse demand function is given by (54.2) and the cost function of each firm is cq. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$.

A Nash equilibrium is a pair (q_1^*, q_2^*) of outputs for which q_1^* is a best response to q_2^*, and q_2^* is a best response to q_1^*:

$$q_1^* = b_1(q_2^*) \quad \text{and} \quad q_2^* = b_2(q_1^*)$$

(see (34.3)). The set of such pairs is the set of points at which the best response functions in Figure 56.2 intersect. From the figure we see that there is exactly one such point, which is given by the solution of the two equations

$$\begin{aligned} q_1 &= \tfrac{1}{2}(\alpha - c - q_2) \\ q_2 &= \tfrac{1}{2}(\alpha - c - q_1). \end{aligned}$$

Solving these two equations (by substituting the second into the first and then isolating q_1, for example) we find that $q_1^* = q_2^* = \frac{1}{3}(\alpha - c)$.

In summary, when there are two firms, the inverse demand function is given by $P(Q) = \alpha - Q$ for $Q \leq \alpha$, and the cost function of each firm is $C_i(q_i) = cq_i$, Cournot's oligopoly game has a unique Nash equilibrium $(q_1^*, q_2^*) = (\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$. The total output in this equilibrium is $\frac{2}{3}(\alpha - c)$, so that the price at which output is sold is $P(\frac{2}{3}(\alpha - c)) = \frac{1}{3}(\alpha + 2c)$. As α increases (meaning that consumers are willing to pay more for the good), the equilibrium price and the output of each firm increases. As c (the unit cost of production) increases, the output of each firm falls and the price rises; each unit increase in c leads to a two-thirds of a unit increase in the price.

? EXERCISE 57.1 (Cournot's duopoly game with linear inverse demand and different unit costs) Find the Nash equilibrium of Cournot's game when there are two firms, the inverse demand function is given by (54.2), the cost function of each firm i is $C_i(q_i) = c_i q_i$, where $c_1 > c_2$, and $c_1 < \alpha$. (There are two cases, depending on the size of c_1 relative to c_2.) Which firm produces more output in an equilibrium? What is the effect of technical change that lowers firm 2's unit cost c_2 (while not affecting firm 1's unit cost c_1) on the firms' equilibrium outputs, the total output, and the price?

? EXERCISE 57.2 (Cournot's duopoly game with linear inverse demand and a quadratic cost function) Find the Nash equilibrium of Cournot's game when there are two firms, the inverse demand function is given by (54.2), and the cost function of each firm i is $C_i(q_i) = q_i^2$.

In the next exercise each firm's cost function has a component that is independent of output. You will find in this case that Cournot's game may have more than one Nash equilibrium.

? EXERCISE 57.3 (Cournot's duopoly game with linear inverse demand and a fixed cost) Find the Nash equilibria of Cournot's game when there are two firms, the inverse demand function is given by (54.2), and the cost function of each firm i is given by

$$C_i(q_i) = \begin{cases} 0 & \text{if } q_i = 0 \\ f + cq_i & \text{if } q_i > 0, \end{cases}$$

where $c \geq 0$, $f > 0$, and $c < \alpha$. (Note that the fixed cost f affects only the firm's decision of whether or not to operate; it does not affect the output a firm wishes to produce *if it wishes to operate*.)

So far we have assumed that each firm's objective is to maximize its profit. The next exercise asks you to consider a case in which one firm's objective is to maximize its market share.

? EXERCISE 58.1 (Variant of Cournot's game, with market-share maximizing firms) Find the Nash equilibrium (equilibria?) of a variant of the example of Cournot's duopoly game that differs from the one in this section (linear inverse demand, constant unit cost) only in that one of the two firms chooses its output to maximize its market share subject to not making a loss, rather than to maximize its profit. What happens if *each* firm maximizes its market share?

3.1.4 Properties of Nash equilibrium

Two economically interesting properties of a Nash equilibrium of Cournot's game concern the relation between the firms' equilibrium profits and the profits they could obtain if they acted collusively, and the character of an equilibrium when the number of firms is large.

Comparison of Nash equilibrium with collusive outcomes In Cournot's game with two firms, is there any pair of outputs at which *both* firms' profits exceed their levels in a Nash equilibrium? The next exercise asks you to show that the answer is "yes" in the example considered in the previous section. Specifically, both firms can increase their profits relative to their equilibrium levels by reducing their outputs.

? EXERCISE 58.2 (Nash equilibrium of Cournot's duopoly game and collusive outcomes) Find the total output (call it Q^*) that maximizes the firms' *total* profit in Cournot's game when there are two firms and the inverse demand function and cost functions take the forms assumed Section 3.1.3. Compare $\frac{1}{2}Q^*$ with each firm's output in the Nash equilibrium, and show that each firm's equilibrium profit is less than its profit in the "collusive" outcome in which each firm produces $\frac{1}{2}Q^*$. Why is this collusive outcome not a Nash equilibrium?

The same is true more generally. For nonlinear inverse demand functions and cost functions, the shapes of the firms' best response functions differ, in general, from those in the example studied in the previous section. But for many inverse demand functions and cost functions the game has a Nash equilibrium and, for any equilibrium, there are pairs of outputs in which each firm's output is less than its equilibrium level and each firm's profit exceeds its equilibrium level.

To see why, suppose that (q_1^*, q_2^*) is a Nash equilibrium and consider the set of pairs (q_1, q_2) of outputs at which firm 1's profit is at least its equilibrium profit. The assumption that P is decreasing (higher total output leads to a lower price) implies that if (q_1, q_2) is in this set and $q_2' < q_2$ then (q_1, q_2') is also in the set. (We have $q_1 + q_2' < q_1 + q_2$, and hence $P(q_1 + q_2') > P(q_1 + q_2)$, so that firm 1's profit at (q_1, q_2') exceeds its profit at (q_1, q_2).) Thus in Figure 59.1 the set of pairs of outputs at which firm 1's profit is at least its equilibrium profit lies on or below the line $q_2 = q_2^*$; an example of such a set is shaded light gray. Similarly, the set of pairs of outputs at which firm 2's profit is at least its equilibrium profit lies on or to the left of the line $q_1 = q_1^*$, and an example is shaded light gray.

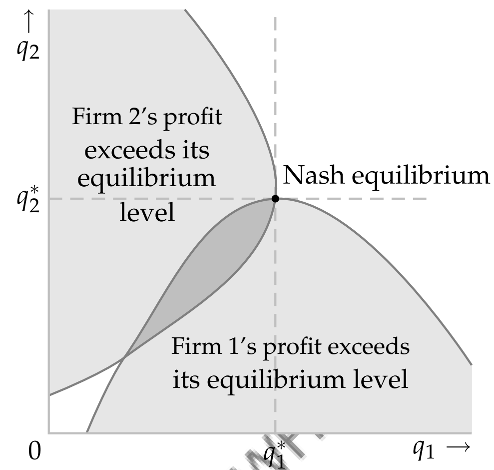


Figure 59.1 The pair (q_1^*, q_2^*) is a Nash equilibrium; along each gray curve one of the firm's profits is constant, equal to its profit at the equilibrium. The area shaded dark gray is the set of pairs of outputs at which both firms' profits exceed their equilibrium levels.

We see that if the parts of the boundaries of these sets indicated by the gray lines in the figure are smooth then the two sets must intersect; in the figure the intersection is shaded dark gray. At every pair of outputs in this area each firm's output is less than its equilibrium level ($q_i < q_i^*$ for $i = 1, 2$) and each firm's profit is higher than its equilibrium profit. That is, *both* firms are better off by restricting their outputs.

Dependence of Nash equilibrium on number of firms How does the equilibrium outcome in Cournot's game depend on the number of firms? If each firm's cost function has the same constant unit cost c, the best outcome for consumers compatible with no firm's making a loss has a price of c and a total output of $\alpha - c$. The next exercise asks you to show that if, for this cost function, the inverse demand function is linear (as in Section 3.1.3), then the price in the Nash equilibrium of Cournot's game decreases as the number of firms increases, approaching c. That is, from the viewpoint of consumers, the outcome is better the larger the number of firms, and when the number of firms is very large, the outcome is close to the best one compatible with nonnegative profits for the firms.

? EXERCISE 59.1 (Cournot's game with many firms) Consider Cournot's game in the case of an arbitrary number n of firms; retain the assumptions that the inverse demand function takes the form (54.2) and the cost function of each firm i is $C_i(q_i) = cq_i$ for all q_i, with $c < \alpha$. Find the best response function of each firm and set up the conditions for $(q_1^*, \ldots, q_n^*)$ to be a Nash equilibrium (see (34.3)), assuming that there is a Nash equilibrium in which all firms' outputs are positive. Solve these equations to find the Nash equilibrium. (For $n = 2$ your answer should be $(\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$, the equilibrium found in the previous section. First show that in an equilibrium all firms produce the same output, then solve for that output. If you cannot show that all firms produce the same output, simply assume that they do.) Find the price at which output is sold in a Nash equilibrium and show that this price decreases as n increases, approaching c as the number of firms increases without bound.

The main idea behind this result does not depend on the assumptions on the inverse demand function and the firms' cost functions. Suppose, more generally, that the inverse demand function is any decreasing function, that each firm's cost function is the same, denoted by C, and that there is a single output, say $\underline{q}$, at which the average cost of production $C(q)/q$ is minimal. In this case, any given total output is produced most efficiently by each firm's producing $\underline{q}$, and the lowest price compatible with the firms' not making losses is the minimal value of the average cost. The next exercise asks you to show that in a Nash equilibrium of Cournot's game in which the firms' total output is large relative to $\underline{q}$, this is the price at which the output is sold.

⁇ EXERCISE 60.1 (Nash equilibrium of Cournot's game with small firms) Suppose that there are infinitely many firms, all of which have the same cost function C. Assume that $C(0) = 0$, and for $q > 0$ the function $C(q)/q$ has a unique minimizer $\underline{q}$; denote the minimum of $C(q)/q$ by $\underline{p}$. Assume that the inverse demand function P is decreasing. Show that in any Nash equilibrium the firms' total output Q^* satisfies

$$P(Q^* + \underline{q}) \leq \underline{p} \leq P(Q^*).$$

(That is, the price is at least the minimal value $\underline{p}$ of the average cost, but is close enough to this minimum that increasing the total output of the firms by $\underline{q}$ would reduce the price to at most $\underline{p}$.) To establish these inequalities, show that if $P(Q^*) < \underline{p}$ or $P(Q^* + \underline{q}) > \underline{p}$ then Q^* is not the total output of the firms in a Nash equilibrium, because in each case at least one firm can deviate and increase its profit.

3.1.5 A generalization of Cournot's game: using common property

In Cournot's game, the payoff function of each firm i is $q_i P(q_1 + \cdots + q_n) - C_i(q_i)$. In particular, each firm's payoff depends only on its output and the sum of all the firm's outputs, not on the distribution of the total output among the firms, and decreases when this sum increases (given that P is decreasing). That is, the payoff of each firm i may be written as $f_i(q_i, q_1 + \cdots + q_n)$, where the function f_i is decreasing in its second argument (given the value of its first argument, q_i).

This general payoff function captures many situations in which players compete in using a piece of common property whose value to any one player diminishes as total use increases. The property might be a village green, for example; the higher the total number of sheep grazed there, the less valuable the green is to any given farmer.

The first property of a Nash equilibrium in Cournot's model discussed in the previous section applies to this general model: common property is "overused" in a Nash equilibrium in the sense that every player's payoff increases when every player reduces her use of the property from its equilibrium level. For example, all farmers' payoffs increase if each farmer reduces her use of the village green from its equilibrium level: in an equilibrium the green is "overgrazed". The argument is the same as the one illustrated in Figure 59.1 in the case of two players, because this argument depends only on the fact that each player's payoff function is smooth and is decreasing in the other player's action. (In Cournot's model, the "common property" that is overused is the demand for the good.)

? EXERCISE 61.1 (Interaction among resource-users) A group of n firms uses a common resource (a river or a forest, for example) to produce output. As more of the resource is used, any given firm can produce less output. Denote by x_i the amount of the resource used by firm i $(= 1, \ldots, n)$. Assume specifically that firm i's output is $x_i(1 - (x_1 + \cdots + x_n))$ if $x_1 + \cdots + x_n \leq 1$, and zero otherwise. Each firm i chooses x_i to maximize its output. Formulate this situation as a strategic game. Find values of α and c such that the game is the same as the one studied in Exercise 59.1, and hence find its Nash equilibria. Find an action profile $(x_1, \ldots, x_n)$ at which each firm's output is higher than it is at the Nash equilibrium.

3.2 Bertrand's model of oligopoly

3.2.1 General model

In Cournot's game, each firm chooses an output; the price is determined by the demand for the good in relation to the total output produced. In an alternative model of oligopoly, associated with a review of Cournot's book by Bertrand (1883), each firm chooses a price, and produces enough output to meet the demand it faces, given the prices chosen by all the firms. The model is designed to shed light on the same questions that Cournot's game addresses; as we shall see, some of the answers it gives are different.

The economic setting for the model is similar to that for Cournot's game. A single good is produced by n firms; each firm can produce q_i units of the good at a cost of $C_i(q_i)$. It is convenient to specify demand by giving a "demand function" D, rather than an inverse demand function as we did for Cournot's game. The interpretation of D is that if the good is available at the price p then the total amount demanded is $D(p)$.

Assume that if the firms set different prices then all consumers purchase the good from the firm with the lowest price, which produces enough output to meet this demand. If more than one firm sets the lowest price, all the firms doing so share the demand at that price equally. A firm whose price is not the lowest price receives no demand and produces no output. (Note that a firm does not choose its output strategically; it simply produces enough to satisfy all the demand it faces, given the prices, even if its price is below its unit cost, in which case it makes a loss. This assumption can be modified at the price of complicating the model.)

In summary, **Bertrand's oligopoly game** is the following strategic game.

Players The firms.

Actions Each firm's set of actions is the set of possible prices (nonnegative numbers).

Preferences Firm i's preferences are represented by its profit, equal to $p_i D(p_i)/m - C_i(D(p_i)/m)$ if firm i is one of m firms setting the lowest price ($m = 1$ if firm i's price p_i is lower than every other price), and equal to zero if some firm's price is lower than p_i.

3.2.2 *Example: duopoly with constant unit cost and linear demand function*

Suppose, as in Section 3.1.3, that there are two firms, each of whose cost functions has constant unit cost c (that is, $C_i(q_i) = cq_i$ for $i = 1, 2$). Assume that the demand function is $D(p) = \alpha - p$ for $p \leq \alpha$ and $D(p) = 0$ for $p > \alpha$, and that $c < \alpha$.

Because the cost of producing each unit is the same, equal to c, firm i makes the profit of $p_i - c$ on every unit it sells. Thus its profit is

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j, \end{cases}$$

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$).

As before, we can find the Nash equilibria of the game by finding the firms' best response functions. If firm j charges p_j, what is the best price for firm i to charge? We can reason informally as follows. If firm i charges p_j, it shares the market with firm j; if it charges slightly less, it sells to the entire market. Thus if p_j exceeds c, so that firm i makes a positive profit selling the good at a price slightly below p_j, firm i is definitely better off serving all the market at such a price than serving half of the market at the price p_j. If p_j is very high, however, firm i may be able to do even better: by reducing its price significantly below p_j it may increase its profit, because the extra demand engendered by the lower price may more than compensate for the lower revenue per unit sold. Finally, if p_j is less than c, then firm i's profit is negative if it charges a price less than or equal to p_j, whereas this profit is zero if it charges a higher price. Thus in this case firm i would like to charge any price greater than p_j, to make sure that it gets no customers. (Remember that if customers arrive at its door it is obliged to serve them, whether or not it makes a profit by so doing.)

We can make these arguments precise by studying firm i's payoff as a function of its price p_i for various values of the price p_j of firm j. Denote by p^m the value of p (price) that maximizes $(p-c)(\alpha-p)$. This price would be charged by a firm with a monopoly of the market (because $(p-c)(\alpha-p)$ is the profit of such a firm). Three cross-sections of firm i's payoff function, for different values of p_j, are shown in black in Figure 63.1. (The gray dashed line is the function $(p_i-c)(\alpha-p_i)$.)

- If $p_j < c$ (firm j's price is below the unit cost) then firm i's profit is negative if $p_i \leq p_j$ and zero if $p_i > p_j$ (see the left panel of Figure 63.1). Thus *any* price greater than p_j is a best response to p_j. That is, the set of firm i's best responses is $B_i(p_j) = \{p_i\colon p_i > p_j\}$.

- If $p_j = c$ then the analysis is similar to that of the previous case except that p_j, as well as any price greater than p_j, yields a profit of zero, and hence is a best response to p_j: $B_i(p_j) = \{p_i\colon p_i \geq p_j\}$.

- If $c < p_j \leq p^m$ then firm i's profit increases as p_i increases to p_j, then drops abruptly at p_j (see the middle panel of Figure 63.1). Thus there is no best response: firm i wants to choose a price less than p_j, but is better off the closer that price is to p_j. For any price less than p_j there is a higher price that is also less than p_j, so there is no best price. (I have assumed that a firm can choose *any* number as its price; in particular, it is not restricted to charge an integral number of cents.) Thus $B_i(p_j)$ is empty (has no members).

- If $p_j > p^m$ then p^m is the unique best response of firm i (see the right panel of Figure 63.1): $B_i(p_j) = \{p^m\}$.

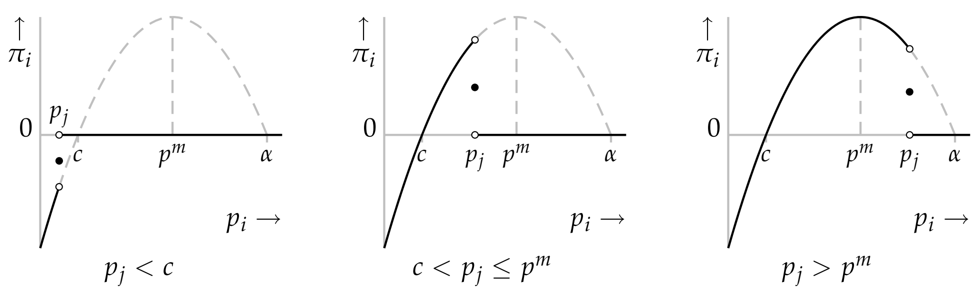


Figure 63.1 Three cross-sections (in black) of firm i's payoff function in Bertrand's duopoly game. Where the payoff function jumps, its value is given by the small disk; the small circles indicate points that are excluded as values of the functions.

In summary, firm i's best response function is given by

$$B_i(p_j) = \begin{cases} \{p_i\colon p_i > p_j\} & \text{if } p_j < c \\ \{p_i\colon p_i \geq p_j\} & \text{if } p_j = c \\ \varnothing & \text{if } c < p_j \leq p^m \\ \{p^m\} & \text{if } p^m < p_j, \end{cases}$$

where ∅ denotes the set with no members (the "empty set"). Note the respects in which this best response function differs qualitatively from a firm's best response function in Cournot's game: for some actions of its opponent, a firm has no best response, and for some actions it has multiple best responses.

The fact that firm i has *no* best response when $c < p_j < p^m$ is an artifact of modeling price as a continuous variable (a firm can choose its price to be any nonnegative number). If instead we assume that each firm's price must be a multiple of some indivisible unit ϵ (e.g. price must be an integral number of cents) then firm i's optimal response to a price p_j with $c < p_j < p^m$ is $p_j - \epsilon$. I model price as a continuous variable because doing so simplifies some of the analysis; in Exercise 65.2 you are asked to study the case of discrete prices.

When $p_j < c$, firm i's set of best responses is the set of all prices greater than p_j. In particular, prices between p_j and c are best responses. You may object that setting a price less than c is not very sensible. Such a price exposes firm i to the risk of making a loss (if firm j chooses a higher price) and has no advantage over the price of c, regardless of firm j's price. That is, such a price is *weakly dominated* (Definition 45.1) by the price c. Nevertheless, such a price *is* a best response! That is, it is optimal for firm i to choose such a price, *given* firm j's price: there is no price that yields firm i a higher profit, *given* firm j's price. The point is that when asking if a player's action is a best response to her opponent's action, we do not consider the "risk" that the opponent will take some other action.

Figure 64.1 shows the firms' best response functions (firm 1's on the left, firm 2's on the right). The shaded gray area in the left panel indicates that for a price p_2 less than c, *any* price greater than p_2 is a best response for firm 1. The absence of a black line along the sloping left boundary of this area indicates that only prices p_1 *greater than* (not equal to) p_2 are included. The black line along the top of the area indicates that for $p_2 = c$ any price greater than *or equal to* c is a best response. As before, the dot indicates a point that is included, whereas the small circle indicates a point that is excluded. Firm 2's best response function has a similar interpretation.

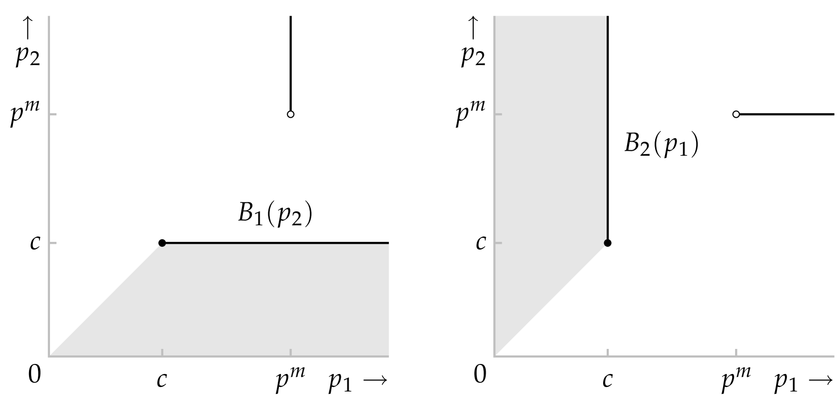


Figure 64.1 The firms' best response functions in Bertrand's duopoly game. Firm 1's best response function is in the left panel; firm 2's is in the right panel.

A Nash equilibrium is a pair (p_1^*, p_2^*) of prices such that p_1^* is a best response to p_2^*, and p_2^* is a best response to p_1^*—that is, p_1^* is in $B_1(p_2^*)$ and p_2^* is in $B_2(p_1^*)$ (see (34.2)). If we superimpose the two best response functions, any such pair is in the intersection of their graphs. If you do so, you will see that the graphs have a single point of intersection, namely $(p_1^*, p_2^*) = (c, c)$. That is, the game has a single Nash equilibrium, in which each firm charges the price c.

The method of finding the Nash equilibria of a game by constructing the players' best response functions is systematic. So long as these functions may be computed, the method straightforwardly leads to the set of Nash equilibria. However, in some games we can make a direct argument that avoids the need to construct the entire best response functions. Using a combination of intuition and trial and error we find the action profiles that seem to be equilibria, then we show precisely that any such profile is an equilibrium and every other profile is not an equilibrium. To show that a pair of actions is not a Nash equilibrium we need only find a *better* response for one of the players—not necessarily the *best* response.

In Bertrand's game we can argue as follows. (*i*) First we show that $(p_1, p_2) = (c, c)$ is a Nash equilibrium. If one firm charges the price c then the other firm can do no better than charge the price c also, because if it raises its price it sells no output, and if it lowers its price it makes a loss. (*ii*) Next we show that no other pair (p_1, p_2) is a Nash equilibrium, as follows.

- If $p_i < c$ for either $i = 1$ or $i = 2$ then the profit of the firm whose price is lowest (or the profit of both firms, if the prices are the same) is negative, and this firm can increase its profit (to zero) by raising its price to c.
- If $p_i = c$ and $p_j > c$ then firm i is better off increasing its price slightly, making its profit positive rather than zero.
- If $p_i > c$ and $p_j > c$, suppose that $p_i \geq p_j$. Then firm i can increase its profit by lowering p_i to slightly below p_j if $D(p_j) > 0$ (i.e. if $p_j < \alpha$) and to p^m if $D(p_j) = 0$ (i.e. if $p_j \geq \alpha$).

In conclusion, both arguments show that when the unit cost of production is a constant c, the same for both firms, and demand is linear, Bertrand's game has a unique Nash equilibrium, in which each firm's price is equal to c.

? EXERCISE 65.1 (Bertrand's duopoly game with constant unit cost) Consider the extent to which the analysis depends upon the demand function D taking the specific form $D(p) = \alpha - p$. Suppose that D is any function for which $D(p) \geq 0$ for all p and there exists $\overline{p} > c$ such that $D(p) > 0$ for all $p \leq \overline{p}$. Is (c, c) still a Nash equilibrium? Is it still the only Nash equilibrium?

? EXERCISE 65.2 (Bertrand's duopoly game with discrete prices) Consider the variant of the example of Bertrand's duopoly game in this section in which each firm is restricted to choose a price that is an integral number of cents. Assume that c is an integral number of cents and that $\alpha > c + 1$. Is (c, c) a Nash equilibrium of this game? Is there any other Nash equilibrium?

3.2.3 Discussion

For a duopoly in which both firms have the same constant unit cost and the demand function is linear, the Nash equilibria of Cournot's and Bertrand's games generate different economic outcomes. The equilibrium price in Bertrand's game is equal to the common unit cost c, whereas the price associated with the equilibrium of Cournot's game is $\frac{1}{3}(\alpha + 2c)$, which exceeds c because $c < \alpha$. In particular, the equilibrium price in Bertrand's game is the lowest price compatible with the firms' not making losses, whereas the price at the equilibrium of Cournot's game is higher. In Cournot's game, the price decreases towards c as the number of firms increases (Exercise 59.1), whereas in Bertrand's game it is c even if there are only two firms. In the next exercise you are asked to show that as the number of firms increases in Bertrand's game, the price remains c.

? EXERCISE 66.1 (Bertrand's oligopoly game) Consider Bertrand's oligopoly game when the cost and demand functions satisfy the conditions in Section 3.2.2 and there are n firms, with $n \geq 3$. Show that the set of Nash equilibria is the set of profiles $(p_1, \ldots, p_n)$ of prices for which $p_i \geq c$ for all i and at least two prices are equal to c. (Show that any such profile is a Nash equilibrium, and that every other profile is not a Nash equilibrium.)

What accounts for the difference between the Nash equilibria of Cournot's and Bertrand's games? The key point is that different strategic variables (output in Cournot's game, price in Bertrand's game) imply different strategic reasoning by the firms. In Cournot's game a firm changes its behavior if it can increase its profit by changing its output, on the assumption that the other firms' outputs will remain the same and the price will adjust to clear the market. In Bertrand's game a firm changes its behavior if it can increase its profit by changing its price, on the assumption that the other firms' prices will remain the same and their outputs will adjust to clear the market. Which assumption makes more sense depends on the context. For example, the wholesale market for agricultural produce may fit Cournot's game better, whereas the retail market for food may fit Bertrand's game better.

Under some variants of the assumptions in the previous section, Bertrand's game has no Nash equilibrium. In one case the firms' cost functions have constant unit costs, and these costs are different; in another case the cost functions have a fixed component. In both these cases, as well as in some other cases, an equilibrium is restored if we modify the way in which consumers are divided between the firms when the prices are the same, as the following exercises show. (We can think of the division of consumers between firms charging the same price as being determined as part of the equilibrium. Note that we retain the assumption that if the firms charge different prices then the one charging the lower price receives all the demand.)

? EXERCISE 66.2 (Bertrand's duopoly game with different unit costs) Consider Bertrand's duopoly game under a variant of the assumptions of Section 3.2.2 in which the firms' unit costs are different, equal to c_1 and c_2, where $c_1 < c_2$. Denote by p_1^m the price that maximizes $(p - c_1)(\alpha - p)$, and assume that $c_2 < p_1^m$ and that the function $(p - c_1)(\alpha - p)$ is increasing in p up to p_1^m.

a. Suppose that the rule for splitting up consumers when the prices are equal assigns all consumers to firm 1 when both firms charge the price c_2. Show that $(p_1, p_2) = (c_2, c_2)$ is a Nash equilibrium and that no other pair of prices is a Nash equilibrium.

b. Show that no Nash equilibrium exists if the rule for splitting up consumers when the prices are equal assigns some consumers to firm 2 when both firms charge c_2.

⁇ EXERCISE 67.1 (Bertrand's duopoly game with fixed costs) Consider Bertrand's game under a variant of the assumptions of Section 3.2.2 in which the cost function of each firm i is given by $C_i(q_i) = f + cq_i$ for $q_i > 0$, and $C_i(0) = 0$, where f is positive and less than the maximum of $(p - c)(\alpha - p)$ with respect to p. Denote by $\overline{p}$ the price p that satisfies $(p - c)(\alpha - p) = f$ and is less than the maximizer of $(p - c)(\alpha - p)$ (see Figure 67.1). Show that if firm 1 gets all the demand when both firms charge the same price then $(\overline{p}, \overline{p})$ is a Nash equilibrium. Show also that no other pair of prices is a Nash equilibrium. (First consider cases in which the firms charge the same price, then cases in which they charge different prices.)

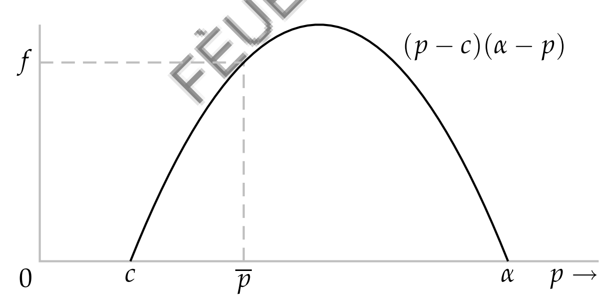


Figure 67.1 The determination of the price $\overline{p}$ in Exercise 67.1.

COURNOT, BERTRAND, AND NASH: SOME HISTORICAL NOTES

Associating the names of Cournot and Bertrand with the strategic games in Sections 3.1 and 3.2 invites two conclusions. First, that Cournot, writing in the first half of the nineteenth century, developed the concept of Nash equilibrium in the context of a model of oligopoly. Second, that Bertrand, dissatisfied with Cournot's game, proposed an alternative model in which price rather than output is the strategic variable. On both points the history is much less straightforward.

Cournot presented his "equilibrium" as the outcome of a dynamic adjustment process in which, in the case of two firms, the firms alternately choose best responses to each other's outputs. During such an adjustment process, each firm, when choosing an output, acts on the assumption that the other firm's output will remain the same, an assumption shown to be incorrect when the other firm subsequently adjusts its output. The fact that the adjustment process rests on the firms' acting on assumptions constantly shown to be false was the subject of criticism in a leading presentation of Cournot's model (Fellner 1949) available at the time Nash was developing his idea.

Certainly Nash did not literally generalize Cournot's idea: the evidence suggests that he was completely unaware of Cournot's work when developing the notion of Nash equilibrium (Leonard 1994, 502–503). In fact, only gradually, as Nash's work was absorbed into mainstream economic theory, was Cournot's solution interpreted as a Nash equilibrium (Leonard 1994, 507–509).

The association of the price-setting model with Bertrand (a mathematician) rests on a paragraph in a review of Cournot's book written by Bertrand in 1883. (Cournot's book, published in 1838, had previously been largely ignored.) The review is confused. Bertrand is under the impression that in *Cournot's* model the firms compete in prices, undercutting each other to attract more business! He argues that there is "no solution" because there is no limit to the fall in prices, a result he says that Cournot's formulation conceals (Bertrand 1883, 503). In brief, Bertrand's understanding of Cournot's work is flawed; he sees that price competition leads each firm to undercut the other, but his conclusion about the outcome is incorrect.

Through the lens of modern game theory we see that the models associated with Cournot and Bertrand are strategic games that differ only in the strategic variable, the solution in both cases being a Nash equilibrium. Until Nash's work, the picture was much murkier.

3.3 Electoral competition

What factors determine the number of political parties and the policies they propose? How is the outcome of an election affected by the electoral system and the voters' preferences among policies? A model that is the foundation for many theories of political phenomena addresses these questions. In the model, each of several candidates chooses a policy; each citizen has preferences over policies and votes for one of the candidates.

A simple version of this model is a strategic game in which the players are the candidates and a policy is a number, referred to as a "position". (The compression of all policy differences into one dimension is a major abstraction, though political positions are often categorized on a left–right axis.) After the candidates have chosen positions, each of a set of citizens votes (nonstrategically) for the candidate whose position she likes best. The candidate who obtains the most votes wins. Each candidate cares only about winning; no candidate has an ideological attachment to any position. Specifically, each candidate prefers to win than to tie for first place (in which case perhaps the winner is determined randomly) than to lose, and if she ties for first place she prefers to do so with as few other candidates as possible.

There is a continuum of voters, each with a favorite position. The distribution of these favorite positions over the set of all possible positions is arbitrary. In particular, this distribution may not be uniform: a large fraction of the voters may have favorite positions close to one point, while few voters have favorite positions close to some other point. A position that turns out to have special significance is the *median* favorite position: the position m with the property that exactly half of the voters' favorite positions are at most m, and half of the voters' favorite positions are at least m. (I assume that there is only one such position.)

Each voter's distaste for any position is given by the distance between that position and her favorite position. In particular, for any value of k, a voter whose favorite position is x^* is indifferent between the positions $x^* - k$ and $x^* + k$. (Refer to Figure 69.1.)

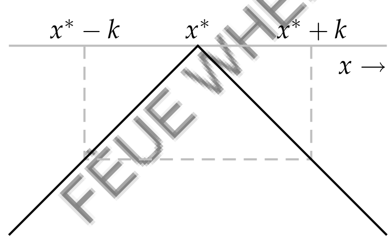


Figure 69.1 The payoff of a voter whose favorite position is x^*, as a function of the winning position, x.

Under this assumption, each candidate attracts the votes of all citizens whose favorite positions are closer to her position than to the position of any other candidate. An example is shown in Figure 70.1. In this example there are three candidates, with positions x_1, x_2, and x_3. Candidate 1 attracts the votes of every citizen whose favorite position is in the interval, labeled "votes for 1", up to the midpoint $\frac{1}{2}(x_1 + x_2)$ of the line segment from x_1 to x_2; candidate 2 attracts the votes of every citizen whose favorite position is in the interval from $\frac{1}{2}(x_1 + x_2)$ to $\frac{1}{2}(x_2 + x_3)$; and candidate 3 attracts the remaining votes. I assume that citizens whose favorite position is $\frac{1}{2}(x_1 + x_2)$ divide their votes equally between candidates 1 and 2, and those whose favorite position is $\frac{1}{2}(x_2 + x_3)$ divide their votes equally between candidates 2 and 3. If two or more candidates take the same position then they share equally the votes that the position attracts.

In summary, I consider the following strategic game, which, in honor of its originator, I call **Hotelling's model of electoral competition**.

Players The candidates.

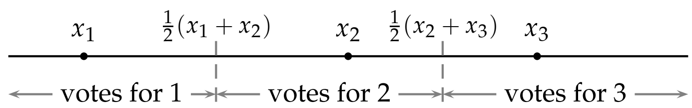


Figure 70.1 The allocation of votes between three candidates, with positions x_1, x_2, and x_3.

Actions Each candidate's set of actions is the set of positions (numbers).

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins outright, k to every terminal history in which she ties for first place with $n - k$ other candidates (for $1 \leq k \leq n - 1$), and 0 to every terminal history in which she loses, where positions attract votes in the way described in the previous paragraph.

Suppose there are two candidates. We can find a Nash equilibrium of the game by studying the players' best response functions. Fix the position x_2 of candidate 2 and consider the best position for candidate 1. First suppose that $x_2 < m$. If candidate 1 takes a position to the left of x_2 then candidate 2 attracts the votes of all citizens whose favorite positions are to the right of $\frac{1}{2}(x_1 + x_2)$, a set that includes the 50% of citizens whose favorite positions are to the right of m, and more. Thus candidate 2 wins, and candidate 1 loses. If candidate 1 takes a position to the right of x_2 then she wins so long as the dividing line between her supporters and those of candidate 2 is less than m (see Figure 70.2). If she is so far to the right that this dividing line lies to the right of m then she loses. She prefers to win than to lose, and is indifferent between all the outcomes in which she wins, so her set of best responses to x_2 is the set of positions that causes the midpoint $\frac{1}{2}(x_1 + x_2)$ of the line segment from x_2 to x_1 to be less than m. (If this midpoint is *equal* to m then the candidates tie.) The condition $\frac{1}{2}(x_1 + x_2) < m$ is equivalent to $x_1 < 2m - x_2$, so candidate 1's set of best responses to x_2 is the set of all positions between x_2 and $2m - x_2$ (excluding the points x_2 and $2m - x_2$).

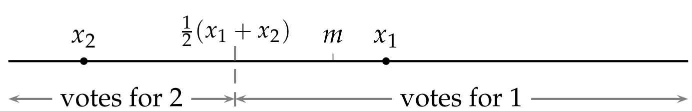


Figure 70.2 An action profile (x_1, x_2) for which candidate 1 wins.

A symmetric argument applies to the case in which $x_2 > m$. In this case candidate 1's set of best responses to x_2 is the set of all positions between $2m - x_2$ and x_2.

Finally consider the case in which $x_2 = m$. In this case candidate 1's unique best response is to choose the *same* position, m! If she chooses any other position then she loses, whereas if she chooses m then she ties for first place.

In summary, candidate 1's best response function is defined by

$$B_1(x_2) = \begin{cases} \{x_1 : x_2 < x_1 < 2m - x_2\} & \text{if } x_2 < m \\ \{m\} & \text{if } x_2 = m \\ \{x_1 : 2m - x_2 < x_1 < x_2\} & \text{if } x_2 > m. \end{cases}$$

Candidate 2 faces exactly the same incentives as candidate 1, and hence has the same best response function. The candidates' best response functions are shown in Figure 71.1.

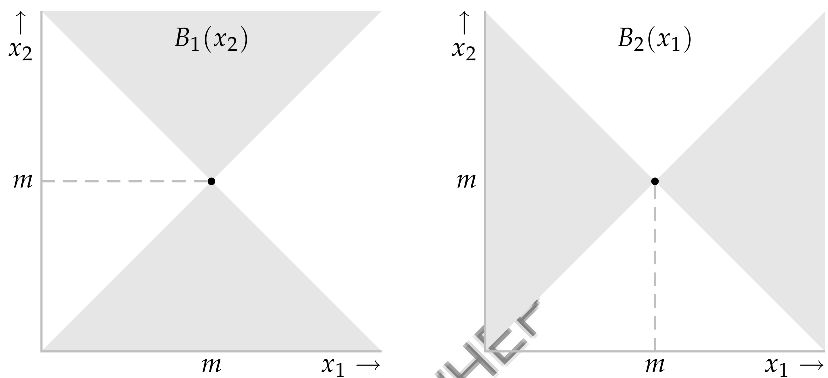


Figure 71.1 The candidates' best response functions in Hotelling's model of electoral competition with two candidates. Candidate 1's best response function is in the left panel; candidate 2's is in the right panel. (The edges of the shaded areas are excluded.)

If you superimpose the two best response functions, you see that the game has a unique Nash equilibrium, in which both candidates choose the position m, the voters' median favorite position. (Remember that the edges of the shaded area, which correspond to pairs of positions that result in ties, are excluded from the best response functions.) The outcome is that the election is a tie.

As in the case of Bertrand's duopoly game in the previous section, we can make a direct argument that (m, m) is the unique Nash equilibrium of the game, without constructing the best response functions. First, (m, m) is an equilibrium: it results in a tie, and if either candidate chooses a position different from m then she loses. Second, no other pair of positions is a Nash equilibrium, by the following argument.

- If one candidate loses then she can do better by moving to m, where she either wins outright (if her opponent's position is different from m) or ties for first place (if her opponent's position is m).
- If the candidates tie (because their positions are either the same or symmetric about m), then either candidate can do better by moving to m, where she wins outright.

Our conclusion is that the competition between the candidates to secure a majority of the votes drives them to select the same position, equal to the median of the citizens' favorite positions. Hotelling (1929, 54), the originator of the model, writes that this outcome is "strikingly exemplified." He continues, "The competition for votes between the Republican and Democratic parties [in the USA] does not lead to a clear drawing of issues, an adoption of two strongly contrasted positions between which the voter may choose. Instead, each party strives to make its platform as much like the other's as possible."

? EXERCISE 72.1 (Electoral competition with asymmetric voters' preferences) Consider a variant of Hotelling's model in which voters's preferences are asymmetric. Specifically, suppose that each voter cares twice as much about policy differences to the left of her favorite position than about policy differences to the right of her favorite position. How does this affect the Nash equilibrium?

In the model considered so far, no candidate has the option of staying out of the race. Suppose that we give each candidate this option; assume that it is better than losing and worse than tying for first place. Then the Nash equilibrium remains as before: both players enter the race and choose the position m. The direct argument differs from the one before only in that in addition we need to check that there is no equilibrium in which one or both of the candidates stays out of the race. If one candidate stays out then, given the other candidate's position, she can enter and either win outright or tie for first place. If both candidates stay out, then either candidate can enter and win outright.

The next exercise asks you to consider the Nash equilibria of this variant of the model when there are three candidates.

? EXERCISE 72.2 (Electoral competition with three candidates) Consider a variant of Hotelling's model in which there are three candidates and each candidate has the option of staying out of the race, which she regards as better than losing and worse than tying for first place. Use the following arguments to show that the game has no Nash equilibrium. First, show that there is no Nash equilibrium in which a single candidate enters the race. Second, show that in any Nash equilibrium in which more than one candidate enters, all candidates that enter tie for first place. Third, show that there is no Nash equilibrium in which two candidates enter the race. Fourth, show that there is no Nash equilibrium in which all three candidates enter the race and choose the same position. Finally, show that there is no Nash equilibrium in which all three candidates enter the race, and do not all choose the same position.

?? EXERCISE 72.3 (Electoral competition in two districts) Consider a variant of Hotelling's model that captures features of a US presidential election. Voters are divided between two districts. District 1 is worth more electoral college votes than is district 2. The winner is the candidate who obtains the most electoral college votes. Denote by m_i the median favorite position among the citizens of district i, for $i = 1$, 2; assume that $m_2 < m_1$. Each of two candidates chooses a single position. Each citizen votes (nonstrategically) for the candidate whose position in closest to her favorite position. The candidate who wins a majority of the votes in a district obtains all the electoral college votes of that district; if the candidates obtain the same number of votes in a district, they each obtain half of the electoral college votes of that district. Find the Nash equilibrium (equilibria?) of the strategic game that models this situation.

So far we have assumed that the candidates care only about winning; they are not at all concerned with the winner's position. The next exercise asks you to consider the case in which each candidate cares *only* about the winner's position, and not at all about winning. (You may be surprised by the equilibrium.)

?? EXERCISE 73.1 (Electoral competition between candidates who care only about the winning position) Consider the variant of Hotelling's model in which the candidates (like the citizens) care about the winner's position, and not at all about winning *per se*. There are two candidates. Each candidate has a favorite position; her dislike for other positions increases with their distance from her favorite position. Assume that the favorite position of one candidate is less than m and the favorite position of the other candidate is greater than m. Assume also that if the candidates tie when they take the positions x_1 and x_2 then the outcome is the compromise policy $\frac{1}{2}(x_1 + x_2)$. Find the set of Nash equilibria of the strategic game that models this situation. (First consider pairs (x_1, x_2) of positions for which either $x_1 < m$ and $x_2 < m$, or $x_1 > m$ and $x_2 > m$. Next consider pairs (x_1, x_2) for which either $x_1 < m < x_2$, or $x_2 < m < x_1$, then those for which $x_1 = m$ and $x_2 \neq m$, or $x_1 \neq m$ and $x_2 = m$. Finally consider the pair (m, m).)

The set of candidates in Hotelling's model is given. The next exercise asks you to analyze a model in which the set of candidates is generated as part of an equilibrium.

?? EXERCISE 73.2 (Citizen-candidates) Consider a game in which the players are the citizens. Any citizen may, at some cost $c > 0$, become a candidate. Assume that the only position a citizen can espouse is her favorite position, so that a citizen's only decision is whether to stand as a candidate. After all citizens have (simultaneously) decided whether to become candidates, each citizen votes for her favorite candidate, as in Hotelling's model. Citizens care about the position of the winning candidate; a citizen whose favorite position is x loses $|x - x^*|$ if the winning candidate's position is x^*. (For any number z, $|z|$ denotes the absolute value of z: $|z| = z$ if $z > 0$ and $|z| = -z$ if $z < 0$.) Winning confers the benefit b. Thus a citizen who becomes a candidate and ties with $k - 1$ other candidates for first place obtains the payoff $b/k - c$; a citizen with favorite position x who becomes a candidate and is not one of the candidates tied for first place obtains the payoff $-|x - x^*| - c$, where x^* is the winner's position; and a citizen with favorite position x who does not become a candidate obtains the payoff $-|x - x^*|$, where x^* is the winner's position. Assume that for every position x there is a citizen for whom x is the favorite position. Show that if $b \leq 2c$ then the game has a Nash equilibrium in which one citizen becomes a candidate. Is there an equilibrium (for any values of b and c) in which two citizens, each with favorite position m, become candidates? Is there an equilibrium in which two citizens with favorite positions different from m become candidates?

Hotelling's model assumes a basic agreement among the voters about the ordering of the positions. For example, if one voter prefers x to y to z and another voter prefers y to z to x, no voter prefers z to x to y. The next exercise asks you to study a model that does not so restrict the voters' preferences.

? EXERCISE 74.1 (Electoral competition for more general preferences) There is a finite number of positions and a finite, odd, number of voters. For any positions x and y, each voter either prefers x to y or prefers y to x. (No voter regards any two positions as equally desirable.) We say that a position x^* is a *Condorcet winner* if for every position y different from x^*, a majority of voters prefer x^* to y.

a. Show that for any configuration of preferences there is at most one Condorcet winner.

b. Give an example in which no Condorcet winner exists. (Suppose there are three positions (x, y, and z) and three voters. Assume that voter 1 prefers x to y to z. Construct preferences for the other two voters such that one voter prefers x to y and the other prefers y to x, one prefers x to z and the other prefers z to x, and one prefers y to z and the other prefers z to y. The preferences you construct must, of course, satisfy the condition that a voter who prefers a to b and b to c also prefers a to c, where a, b, and c are any positions.)

c. Consider the strategic game in which two candidates simultaneously choose positions, as in Hotelling's model. If the candidates choose different positions, each voter endorses the candidate whose position she prefers, and the candidate who receives the most votes wins. If the candidates choose the same position, they tie. Show that this game has a unique Nash equilibrium if the voters' preferences are such that there is a Condorcet winner, and has no Nash equilibrium if the voters' preferences are such that there is no Condorcet winner.

A variant of Hotelling's model of electoral competition can be used to analyze the choices of product characteristics by competing firms in situations in which price is not a significant variable. (Think of radio stations that offer different styles of music, for example.) The set of positions is the range of possible characteristics for the product, and the citizens are consumers rather than voters. Consumers' tastes differ; each consumer buys (at a fixed price, possibly zero) one unit of the product she likes best. The model differs substantially from Hotelling's model of electoral competition in that each firm's objective is to maximize its market share, rather than to obtain a market share larger than that of any other firm. In the next exercise you are asked to show that the Nash equilibria of this game in the case of two or three firms are the same as those in Hotelling's model of electoral competition.

? EXERCISE 75.1 (Competition in product characteristics) In the variant of Hotelling's model that captures competing firms' choices of product characteristics, show that when there are two firms the unique Nash equilibrium is (m, m) (both firms offer the consumers' median favorite product) and when there are three firms there is no Nash equilibrium. (Start by arguing that when there are two firms whose products differ, either firm is better off making its product more similar to that of its rival.)

3.4 The War of Attrition

The game known as the *War of Attrition* elaborates on the ideas captured by the game *Hawk–Dove* (Exercise 29.1). It was originally posed as a model of a conflict between two animals fighting over prey. Each animal chooses the time at which it intends to give up. When an animal gives up, its opponent obtains all the prey (and the time at which the winner intended to give up is irrelevant). If both animals give up at the same time then they each have an equal chance of obtaining the prey. Fighting is costly: each animal prefers as short a fight as possible.

The game models not only such a conflict between animals, but also many other disputes. The "prey" can be any indivisible object, and "fighting" can be any costly activity—for example, simply waiting.

To define the game precisely, let time be a continuous variable that starts at 0 and runs indefinitely. Assume that the value party i attaches to the object in dispute is $v_i > 0$ and the value it attaches to a 50% chance of obtaining the object is $v_i/2$. Each unit of time that passes before the dispute is settled (i.e. one of the parties concedes) costs each party one unit of payoff. Thus if player i concedes first, at time t_i, her payoff is $-t_i$ (she spends t_i units of time and does not obtain the object). If the other player concedes first, at time t_j, player i's payoff is $v_i - t_j$ (she obtains the object after t_j units of time). If both players concede at the same time, player i's payoff is $\frac{1}{2}v_i - t_i$, where t_i is the common concession time. The **War of Attrition** is the following strategic game.

Players The two parties to a dispute.

Actions Each player's set of actions is the set of possible concession times (nonnegative numbers).

Preferences Player i's preferences are represented by the payoff function

$$u_i(t_1, t_2) = \begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{1}{2}v_i - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j, \end{cases}$$

where j is the other player.

To find the Nash equilibria of this game, we start, as before, by finding the players' best response functions. Intuitively, if player j's intended concession time is early enough (t_j is small) then it is optimal for player i to wait for player j to concede. That is, in this case player i should choose a concession time later than t_j; any such time is equally good. By contrast, if player j intends to hold out for a long time (t_j is large) then player i should concede immediately. Because player i values the object at v_i, the length of time it is worth her waiting is v_i.

To make these ideas precise, we can study player i's payoff function for various fixed values of t_j, the concession time of player j. The three cases that the intuitive argument suggests are qualitatively different are shown in Figure 76.1: $t_j < v_i$ in the left panel, $t_j = v_i$ in the middle panel, and $t_j > v_i$ in the right panel. Player i's best responses in each case are her actions for which her payoff is highest: the set of times after t_j if $t_j < v_i$, 0 and the set of times after t_j if $t_j = v_i$, and 0 if $t_j > v_i$.

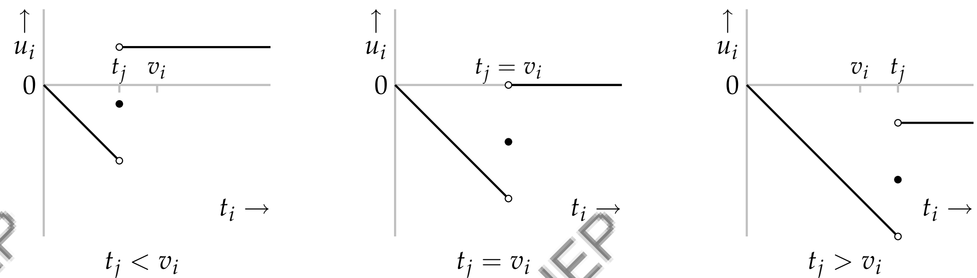


Figure 76.1 Three cross-sections of player i's payoff function in the *War of Attrition*.

In summary, player i's best response function is given by

$$B_i(t_j) = \begin{cases} \{t_i \colon t_i > t_j\} & \text{if } t_j < v_i \\ \{t_i \colon t_i = 0 \text{ or } t_i > t_j\} & \text{if } t_j = v_i \\ \{0\} & \text{if } t_j > v_i. \end{cases}$$

For a case in which $v_1 > v_2$, this function is shown in the left panel of Figure 77.1 for $i = 1$ and $j = 2$ (player 1's best response function), and in the right panel for $i = 2$ and $j = 1$ (player 2's best response function).

Superimposing the players' best response functions, we see that there are two areas of intersection: the vertical axis at and above v_1 and the horizontal axis at and to the right of v_2. Thus (t_1, t_2) is a Nash equilibrium of the game if and only if either

$$t_1 = 0 \text{ and } t_2 \geq v_1$$

or

$$t_2 = 0 \text{ and } t_1 \geq v_2.$$

In words, in every equilibrium either player 1 concedes immediately and player 2 concedes at time v_1 or later, or player 2 concedes immediately and player 1 concedes at time v_2 or later.

? EXERCISE 76.1 (Direct argument for Nash equilibria of *War of Attrition*) Give a direct argument, not using information about the entire best response functions, for the set of Nash equilibria of the *War of Attrition*. (Argue that if $t_1 = t_2$, $0 <$

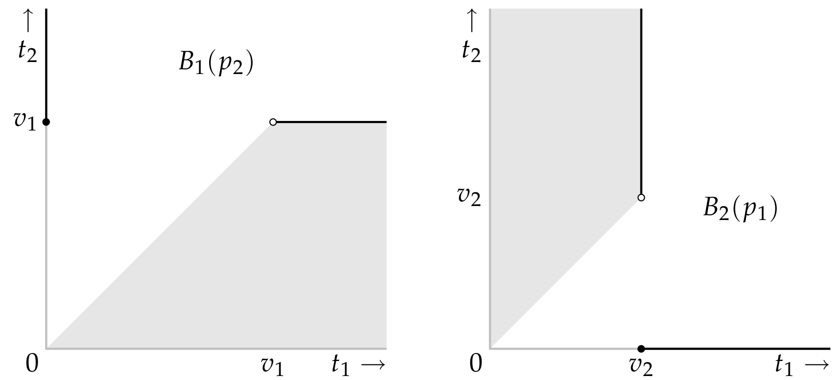


Figure 77.1 The players' best response functions in the *War of Attrition* (for a case in which $v_1 > v_2$). Player 1's best response function is in the left panel; player 2's is in the right panel. (The sloping edges are excluded.)

$t_i < t_j$, or $0 = t_i < t_j < v_i$ (for $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$) then the pair (t_1, t_2) is not a Nash equilibrium. Then argue that any remaining pair is a Nash equilibrium.)

Three features of the equilibria are notable. First, in no equilibrium is there any fight: one player always concedes immediately. Second, either player may concede first, regardless of the players' valuations. In particular, there are always equilibria in which the player who values the object more highly concedes first. Third, the equilibria are *asymmetric* (the players' actions are different), even when $v_1 = v_2$, in which case the game is symmetric—the players' sets of actions are the same and player 1's payoff to (t_1, t_2) is the same as player 2's payoff to (t_2, t_1) (Definition 49.3). Given this asymmetry, the populations from which the two players are drawn must be distinct in order to interpret the Nash equilibria as action profiles compatible with steady states. One player might be the current owner of the object in dispute, and the other a challenger, for example. In this case the equilibria correspond to the two conventions that a challenger always gives up immediately, and that an owner always does so. (Some evidence is discussed in the box on page 379.) If all players—those in the role of player 1 as well as those in the role of player 2—are drawn from a single population, then only symmetric equilibria are relevant (see Section 2.10). The *War of Attrition* has no such equilibria, so the notion of Nash equilibrium makes no prediction about the outcome in such a situation. (A solution that does make a prediction is studied in Example 376.1.)

? EXERCISE 77.1 (Variant of *War of Attrition*) Consider the variant of the *War of Attrition* in which each player attaches no value to the time spent waiting for the other player to concede, but the object in dispute loses value as time passes. (Think of a rotting animal carcass or a melting ice cream cone.) Assume that the value of the object to each player i after t units of time is $v_i - t$ (and the value of a 50% chance of obtaining the object is $\frac{1}{2}(v_i - t)$). Specify the strategic game that models this situation (take care with the payoff functions). Construct the analogue of Figure 76.1, find the players' best response functions, and hence find the Nash equilibria of the game.

The *War of Attrition* is an example of a "game of timing", in which each player's action is a number and each player's payoff depends sensitively on whether her action is greater or less than the other player's action. In many such games, each player's strategic variable is the time at which to act, hence the name "game of timing". The next two exercises are further examples of such games. (In the first the strategic variable is time, whereas in the second it is not.)

? EXERCISE 78.1 (Timing product release) Two firms are developing competing products for a market of fixed size. The longer a firm spends on development, the better its product. But the first firm to release its product has an advantage: the customers it obtains will not subsequently switch to its rival. (Once a person starts using a product, the cost of switching to an alternative, even one significantly better, is too high to make a switch worthwhile.) A firm that releases its product first, at time t, captures the share $h(t)$ of the market, where h is a function that increases from time 0 to time T, with $h(0) = 0$ and $h(T) = 1$. The remaining market share is left for the other firm. If the firms release their products at the same time, each obtains half of the market. Each firm wishes to obtain the highest possible market share. Model this situation as a strategic game and find its Nash equilibrium (equilibria?). (When finding firm i's best response to firm j's release time t_j, there are three cases: that in which $h(t_j) < \frac{1}{2}$ (firm j gets less than half of the market if it is the first to release its product), that in which $h(t_j) = \frac{1}{2}$, and that in which $h(t_j) > \frac{1}{2}$.)

? EXERCISE 78.2 (A fight) Each of two people has one unit of a resource. Each person chooses how much of the resource to use in fighting the other individual and how much to use productively. If each person i devotes y_i to fighting then the total output is $f(y_1, y_2) \geq 0$ and person i obtains the fraction $p_i(y_1, y_2)$ of the output, where

$$p_i(y_1, y_2) = \begin{cases} 1 & \text{if } y_i > y_j \\ \frac{1}{2} & \text{if } y_i = y_j \\ 0 & \text{if } y_i < y_j. \end{cases}$$

The function f is continuous (small changes in y_1 and y_2 cause small changes in $f(y_1, y_2)$), is decreasing in both y_1 and y_2 (the more each player devotes to fighting, the less output is produced), and satisfies $f(1, 1) = 0$ (if each player devotes all her resource to fighting then no output is produced). (If you prefer to deal with a specific function f, take $f(y_1, y_2) = 2 - y_1 - y_2$.) Each person cares only about the amount of output she receives, and prefers to receive as much as possible. Specify this situation as a strategic game and find its Nash equilibrium (equilibria?). (Use a direct argument: first consider pairs (y_1, y_2) with $y_1 \neq y_2$, then those with $y_1 = y_2 < 1$, then those with $y_1 = y_2 = 1$.)

3.5 Auctions

3.5.1 Introduction

In an "auction", a good is sold to the party who submits the highest bid. Auctions, broadly defined, are used to allocate significant economic resources, from works of art to short-term government bonds to offshore tracts for oil and gas exploration to the radio spectrum. They take many forms. For example, bids may be called out sequentially (as in auctions for works of art) or may be submitted in sealed envelopes; the price paid may be the highest bid, or some other price; if more than one unit of a good is being sold, bids may be taken on all units simultaneously, or the units may be sold sequentially. A game-theoretic analysis helps us to understand the consequences of various auction designs; it suggests, for example, the design likely to be the most effective at allocating resources, and the one likely to raise the most revenue. In this section I discuss auctions in which every buyer knows her own valuation and every other buyer's valuation of the item being sold. Chapter 9 develops tools that allow us to study, in Section 9.7, auctions in which buyers are not perfectly informed of each other's valuations.

AUCTIONS FROM BABYLONIA TO EBAY

Auctioning has a very long history. Herodotus, a Greek writer of the fifth century BC who, together with Thucydides, created the intellectual field of history, describes auctions in Babylonia. He writes that the Babylonians' "most sensible" custom was an annual auction in each village of the women of marriageable age. The women most attractive to the men were sold first; they commanded positive prices, whereas men were paid to be matched with the least desirable women. In each auction, bids appear to have been called out sequentially, the man who bid the most winning and paying the price he bid.

Auctions were also used in Athens in the fifth and fourth centuries BC to sell the rights to collect taxes, to dispose of confiscated property, and to lease land and mines. The evidence on the nature of the auctions is slim, but some interesting accounts survive. For example, the Athenian politician Andocides (c. 440–391 BC) reports collusive behavior in an auction of tax-collection rights (see Langdon 1994, 260).

Auctions were frequent in ancient Rome, and continued to be used in medieval Europe after the end of the Roman empire (tax-collection rights were annually auctioned by the towns of the medieval and early modern Low Countries, for example). The earliest use of the English word "auction" given by the *Oxford English Dictionary* dates from 1595, and concerns an auction "when will be sold Slaves, household goods, etc.". Rules surviving from the auctions of this era show that in some cases, at least, bids were called out sequentially, with the bidder remaining at the end obtaining the object at the price she bid (Cassady 1967, 30–31). A variant of this mechanism, in which a time limit is imposed on the bids, is reported by the English diarist and naval administrator Samuel Pepys (1633–1703). The auctioneer lit a short candle, and bids were valid only if made before the flame went out. Pepys reports that a flurry of bidding occurred at the last moment. At an auction on September 3, 1662, a bidder "cunninger than the rest" told him that just as the flame goes out, "the smoke descends", signaling the moment at which one should bid, an observation Pepys found "very pretty" (Pepys 1970, 185–186).

The auction houses of Sotheby's and Christie's were founded in the mid-18th century. At the beginning of the twenty-first century, they are being eclipsed, at least in the value of the goods they sell, by online auction companies. For example, eBay, founded in September 1995, sold US$1.3 billion of merchandise in 62 million auctions during the second quarter of 2000, roughly double the numbers for the second quarter of the previous year; Sotheby's and Christie's together sell around US$1 billion of art and antiques each quarter.

The mechanism used by eBay shares a feature with the ones Pepys observed: all bids must be received before some fixed time. The way in which the price is determined differs. In an eBay auction, a bidder submits a "proxy bid" that is not revealed; the prevailing price is a small increment above the second-highest proxy bid. As in the 17th century auctions Pepys observed, many bidders on eBay act at the last moment—a practice known as "sniping" in the argot of cyberspace. Other online auction houses use different termination rules. For example, Amazon waits ten minutes after a bid before closing an auction. The fact that last-minute bidding is much less common in Amazon auctions than it is in eBay auctions has attracted the attention of game theorists, who have begun to explore models that explain it in terms of the difference in the auctions' termination rules (see, for example, Ockenfels and Roth 2000).

In recent years, many countries have auctioned the rights to the radio spectrum, used for wireless communication. These auctions have been much studied by game theorists; they are discussed in the box on page 298.

3.5.2 Second-price sealed-bid auctions

In a common form of auction, people sequentially submit increasing bids for an object. (The word "auction" comes from the Latin *augere*, meaning "to increase".) When no one wishes to submit a bid higher than the current bid, the person making the current bid obtains the object at the price she bid.

Given that every person is certain of her valuation of the object before the bidding begins, during the bidding no one can learn anything relevant to her actions. Thus we can model the auction by assuming that each person decides, before bidding begins, the most she is willing to bid—her "maximal bid". When the players carry out their plans, the winner is the person whose maximal bid is highest. How much does she need to bid? Eventually only she and the person with the second highest maximal bid will be left competing against each other. In order to win, she therefore needs to bid slightly more than the *second highest* maximal bid. If the bidding increment is small, we can take the price the winner pays to be *equal* to the second highest maximal bid.

Thus we can model such an auction as a strategic game in which each player chooses an amount of money, interpreted as the *maximal* amount she is willing to bid, and the player who chooses the highest amount obtains the object and pays a price equal to the second highest amount.

This game models also a situation in which the people simultaneously put bids in sealed envelopes, and the person who submits the highest bid wins and pays a price equal to the *second* highest bid. For this reason the game is called a *second-price sealed-bid* auction.

To define the game precisely, denote by v_i the value player i attaches to the object; if she obtains the object at the price p her payoff is $v_i - p$. Assume that the players' valuations of the object are all different and all positive; number the players 1 through n in such a way that $v_1 > v_2 > \cdots > v_n > 0$. Each player i submits a (sealed) bid b_i. If player i's bid is higher than every other bid, she obtains the object at a price equal to the second-highest bid, say b_j, and hence receives the payoff $v_i - b_j$. If some other bid is higher than player i's bid, player i does not obtain the object, and receives the payoff of zero. If player i is in a tie for the highest bid, her payoff depends on the way in which ties are broken. A simple (though arbitrary) assumption is that the winner is the player among those submitting the highest bid whose number is smallest (i.e. whose valuation of the object is highest). (If the highest bid is submitted by players 2, 5, and 7, for example, the winner is player 2.) Under this assumption, player i's payoff when she bids b_i and is in a tie for the highest bid is $v_i - b_i$ if her number is lower than that of any other player submitting the bid b_i, and 0 otherwise.

In summary, a **second-price sealed-bid auction** (with perfect information) is the following strategic game.

Players The n bidders, where $n \geq 2$.

Actions The set of actions of each player is the set of possible bids (nonnegative numbers).

Preferences The payoff of any player i is $v_i - b_j$, where b_j is the highest bid submitted by a player other than i if either b_i is higher than every other bid, or b_i is at least as high as every other bid and the number of every other player who bids b_i is greater than i. Otherwise player i's payoff is 0.

This game has many Nash equilibria. One equilibrium is $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$: each player's bid is equal to her valuation of the object. Because $v_1 > v_2 > \cdots > v_n$, the outcome is that player 1 obtains the object at the price b_2; her payoff is $v_1 - b_2$ and every other player's payoff is zero. This profile is a Nash equilibrium by the following argument.

- If player 1 changes her bid to some other price at least equal to b_2 then the outcome does not change (recall that she pays the *second* highest bid, not the highest bid). If she changes her bid to a price less than b_2 then she loses and obtains the payoff of zero.

- If some other player lowers her bid or raises it to some price at most equal to b_1 then she remains a loser; if she raises her bid above b_1 then she wins but, in paying the price b_1, makes a loss (because her valuation is less than b_1).

Another equilibrium is $(b_1, \ldots, b_n) = (v_1, 0, \ldots, 0)$. In this equilibrium, player 1 obtains the object and pays the price of zero. The profile is an equilibrium because if player 1 changes her bid then the outcome remains the same, and if any of the remaining players raises her bid then either the outcome remains the same (if her new bid is at most v_1) or causes her to obtain the object at a price that exceeds her valuation (if her bid exceeds v_1). (The auctioneer obviously has an incentive for the price to be bid up, but she is not a player in the game!)

In both of these equilibria, player 1 obtains the object. But there are also equilibria in which player 1 does not obtain the object. Consider, for example, the action profile $(v_2, v_1, 0, \ldots, 0)$, in which player 2 obtains the object at the price v_2 and every player (including player 2) receives the payoff of zero. This action profile is a Nash equilibrium by the following argument.

- If player 1 raises her bid to v_1 or more, she wins the object but her payoff remains zero (she pays the price v_1, bid by player 2). Any other change in her bid has no effect on the outcome.
- If player 2 changes her bid to some other price greater than v_2, the outcome does not change. If she changes her bid to v_2 or less she loses, and her payoff remains zero.
- If any other player raises her bid to at most v_1, the outcome does not change. If she raises her bid above v_1 then she wins, but in paying the price v_1 (bid by player 2) she obtains a negative payoff.

? EXERCISE 82.1 (Nash equilibrium of second-price sealed-bid auction) Find a Nash equilibrium of a second-price sealed-bid auction in which player n obtains the object.

Player 2's bid in this equilibrium exceeds her valuation, and thus may seem a little rash: if player 1 were to increase her bid to any value less than v_1, player 2's payoff would be negative (she would obtain the object at a price greater than her valuation). This property of the action profile does not affect its status as an equilibrium, because in a Nash equilibrium a player does not consider the "risk" that another player will take an action different from her equilibrium action; each player simply chooses an action that is optimal, *given* the other players' actions. But the property does suggest that the equilibrium is less plausible as the outcome of the auction than the equilibrium in which every player bids her valuation.

The same point takes a different form when we interpret the strategic game as a model of events that unfold over time. Under this interpretation, player 2's action

v_1 means that she will continue bidding until the price reaches v_1. If player 1 is *sure* that player 2 will continue bidding until the price is v_1, then player 1 rationally stops bidding when the price reaches v_2 (or, indeed, when it reaches any other level at most equal to v_1). But there is little reason for player 1 to believe that player 2 will in fact stay in the bidding if the price exceeds v_2: player 2's action is not credible, because if the bidding were to go above v_2, player 2 would rationally withdraw.

The weakness of the equilibrium is reflected in the fact that player 2's bid v_1 is weakly dominated by the bid v_2. More generally,

> *in a second-price sealed-bid auction (with perfect information), a player's bid equal to her valuation weakly dominates all her other bids.*

That is, for any bid $b_i \neq v_i$, player i's bid v_i is at least as good as b_i, *no matter what the other players bid*, and is better than b_i for some actions of the other players. (See Definition 45.1.) A player who bids less than her valuation stands not to win in some cases in which she could profit by winning (when the highest of the other bids is between her bid and her valuation), and never stands to gain relative to the situation in which she bids her valuation; a player who bids more than her valuation stands to win in some cases in which she obtains a negative payoff by doing so (when the highest of the remaining bids is between her valuation and her bid), and never stands to gain relative to the situation in which she bids her valuation. The key point is that in a second-price auction, a player who changes her bid does not lower the price she pays, but only possibly changes her status from that of a winner into that of a loser, or vice versa.

A precise argument is shown in Figure 84.1, which compares player i's payoffs to the bid v_i with her payoffs to a bid $b_i < v_i$ (top table), and to a bid $b_i < v_i$ (bottom table), as a function of the highest of the other players' bids, denoted $\overline{b}$. In each case, for all bids of the other players, player i's payoffs to v_i are at least as large as her payoffs to the other bid, and for bids of the other players such that $\overline{b}$ is in the middle column of each table, player i's payoffs to v_i are greater than her payoffs to the other bid. Thus player i's bid v_i weakly dominates all her other bids.

In summary, a second-price auction has many Nash equilibria, but the equilibrium $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$ in which every player's bid is equal to her valuation of the object is distinguished by the fact that every player's action weakly dominates all her other actions.

? EXERCISE 83.1 (Second-price sealed-bid auction with two bidders) Find *all* the Nash equilibria of a second-price sealed-bid auction with two bidders. (Construct the players' best response functions. Apart from a difference in the tie-breaking rule, the game is the same as the one in Exercise 77.1.)

		Highest of other players' bids, $\overline{b}$		
		$\overline{b} < b_i$ or $\overline{b} = b_i$ & b_i wins	$b_i < \overline{b} < v_i$ or $\overline{b} = b_i$ & b_i loses	$\overline{b} > v_i$
i's bid	$b_i < v_i$	$v_i - \overline{b}$	0	0
	v_i	$v_i - \overline{b}$	$v_i - \overline{b}$	0

		$\overline{b} \leq v_i$	$v_i < \overline{b} < b_i$ or $\overline{b} = b_i$ & b_i wins	$\overline{b} > b_i$ or $\overline{b} = b_i$ & b_i loses
i's bid	v_i	$v_i - \overline{b}$	0	0
	$b_i > v_i$	$v_i - \overline{b}$	$v_i - \overline{b}\ (< 0)$	0

Figure 84.1 Player i's payoffs in a second-price sealed-bid auction, as a function of the highest of the other player's bids, denoted $\overline{b}$. The top table gives her payoffs to the bids $b_i < v_i$ and v_i, and the bottom table gives her payoffs to the bids v_i and $b_i > v_i$.

3.5.3 *First-price sealed-bid auctions*

A first-price auction differs from a second-price auction only in that the winner pays the price she bids, not the second highest bid. Precisely, a **first-price sealed-bid auction** (with perfect information) is defined as follows.

Players The n bidders, where $n \geq 2$.

Actions The set of actions of each player is the set of possible bids (nonnegative numbers).

Preferences The payoff of any player i is $v_i - b_i$ if either b_i is higher than every other bid, or b_i is at least as high as every other bid and the number of every other player who bids b_i is greater than i. Otherwise player i's payoff is 0.

This game models an auction in which people submit sealed bids and the highest bid wins. (You conduct such an auction when you solicit offers for a car you wish to sell, or, as a buyer, get estimates from contractors to fix your leaky basement, assuming in both cases that you do not inform potential bidders of existing bids.) The game models also a dynamic auction in which the auctioneer begins by announcing a high price, which she gradually lowers until someone indicates her willingness to buy the object. (Flowers in the Netherlands are sold in this way.) A bid in the strategic game is interpreted as the price at which the bidder will indicate her willingness to buy the object in the dynamic auction.

One Nash equilibrium of a first-price sealed-bid auction is $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$, in which player 1's bid is player 2's valuation v_2 and every other player's bid is her own valuation. The outcome of this equilibrium is that player 1 obtains the object at the price v_2.

? EXERCISE 84.1 (Nash equilibrium of first-price sealed-bid auction) Show that $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$ is a Nash equilibrium of a first-price sealed-bid auction.

A first-price sealed-bid auction has many other equilibria, but in all equilibria the winner is the player who values the object most highly (player 1), by the following argument. In any action profile $(b_1, \ldots, b_n)$ in which some player $i \neq 1$ wins, we have $b_i > b_1$. If $b_i > v_2$ then i's payoff is negative, so that she can do better by reducing her bid to 0; if $b_i \leq v_2$ then player 1 can increase her payoff from 0 to $v_1 - b_i$ by bidding b_i, in which case she wins. Thus no such action profile is a Nash equilibrium.

? EXERCISE 85.1 (First-price sealed-bid auction) Show that in a Nash equilibrium of a first-price sealed-bid auction the two highest bids are the same, one of these bids is submitted by player 1, and the highest bid is at least v_2 and at most v_1. Show also that any action profile satisfying these conditions is a Nash equilibrium.

In any equilibrium in which the winning bid exceeds v_2, at least one player's bid exceeds her valuation. As in a second-price sealed-bid auction, such a bid seems "risky", because it would yield the bidder a negative payoff if it were to win. In the equilibrium there is no risk, because the bid does not win; but, as before, the fact that the bid has this property reduces the plausibility of the equilibrium.

As in a second-price sealed-bid auction, the potential "riskiness" to player i of a bid $b_i > v_i$ is reflected in the fact that it is weakly dominated by the bid v_i, as shown by the following argument.

- If the other players' bids are such that player i loses when she bids b_i, then the outcome is the same whether she bids b_i or v_i.

- If the other players' bids are such that player i wins when she bids b_i, then her payoff is negative when she bids b_i and zero when she bids v_i (whether or not this bid wins).

However, in a first-price auction, unlike a second-price auction, a bid $b_i < v_i$ of player i is *not* weakly dominated by the bid v_i. If fact, such a bid is not weakly dominated by *any* bid. It is not weakly dominated by a bid $b'_i < b_i$, because if the other players' highest bid is between b'_i and b_i then b'_i loses whereas b_i wins and yields player i a positive payoff. And it is not weakly dominated by a bid $b'_i > b_i$, because if the other players' highest bid is less than b_i then both b_i and b'_i win and b_i yields a lower price.

Further, even though the bid v_i weakly dominates higher bids, this bid is itself weakly dominated, by a lower bid! If player i bids v_i her payoff is 0 regardless of the other players' bids, whereas if she bids less than v_i her payoff is either 0 (if she loses) or positive (if she wins).

In summary,

> *in a first-price sealed-bid auction (with perfect information), a player's bid of at least her valuation is weakly dominated, and a bid of less than her valuation is not weakly dominated.*

An implication of this result is that in *every* Nash equilibrium of a first-price sealed-bid auction at least one player's action is weakly dominated. However, this property of the equilibria depends on the assumption that a bid may be any number. In the variant of the game in which bids and valuations are restricted to be multiples of some discrete monetary unit ϵ (e.g. a cent), an action profile $(v_2 - \epsilon, v_2 - \epsilon, b_3, \ldots, b_n)$ for any $b_j \leq v_j - \epsilon$ for $j = 3, \ldots, n$ is a Nash equilibrium in which no player's bid is weakly dominated. Further, every equilibrium in which no player's bid is weakly dominated takes this form. When ϵ is small, each such equilibrium is close to an equilibrium $(v_2, v_2, b_3, \ldots, b_n)$ (with $b_j \leq v_j$ for $j = 3, \ldots, n$) of the game with unrestricted bids. On this (somewhat *ad hoc*) basis, I select action profiles $(v_2, v_2, b_3, \ldots, b_n)$ with $b_j \leq v_j$ for $j = 3, \ldots, n$ as "distinguished" equilibria of a first-price sealed-bid auction.

One conclusion of this analysis is that while both second-price and first-price auctions have many Nash equilibria, yielding a variety of outcomes, their distinguished equilibria yield the *same* outcome. (Recall that the distinguished equilibrium of a second-price sealed-bid auction is the action profile in which every player bids her valuation.) In every distinguished equilibrium of each game, the object is sold to player 1 at the price v_2. In particular, the auctioneer's revenue is the same in both cases. Thus if we restrict attention to the distinguished equilibria, the two auction forms are "revenue equivalent". The rules are different, but the players' equilibrium bids adjust to the difference and lead to the same outcome:

> *the single Nash equilibrium in which no player's bid is weakly dominated in a second-price auction yields the same outcome as the distinguished equilibria of a first-price auction.*

? EXERCISE 86.1 (Third-price auction) Consider a *third*-price sealed-bid auction, which differs from a first- and a second-price auction only in that the winner (the person who submits the highest bid) pays the third highest price. (Assume that there are at least three bidders.)

a. Show that for any player i the bid of v_i weakly dominates any lower bid, but does not weakly dominate any higher bid. (To show the latter, for any bid $b_i > v_i$ find bids for the other players such that player i is better off bidding b_i than bidding v_i.)

b. Show that the action profile in which each player bids her valuation is not a Nash equilibrium.

c. Find a Nash equilibrium. (There are ones in which every player submits the same bid.)

3.5.4 Variants

Uncertain valuations One respect in which the models in this section depart from reality is in the assumption that each bidder is certain of both her own valuation and every other bidder's valuation. In most, if not all, actual auctions, information is surely less perfect. The case in which the players are uncertain about each other's valuations has been thoroughly explored, and is discussed in Section 9.7. The result that a player's bidding her valuation weakly dominates all her other actions in a second-price auction survives when players are uncertain about each other's valuations, as does the revenue-equivalence of first- and second-price auctions under some conditions on the players' preferences.

Common valuations In some auctions the main difference between the bidders is not that the value the object differently but that they have different information about its value. For example, the bidders for an oil tract may put similar values on any given amount of oil, but have different information about how much oil is in the tract. Such auctions involve informational considerations that do not arise in the model we have studied in this section; they are studied in Section 9.7.3.

Multi-unit auctions In some auctions, like those for Treasury Bills (short-term government bonds) in the USA, many units of an object are available, and each bidder may value positively more than one unit. In each of the types of auction described below, each bidder submits a bid for each unit of the good. That is, an action is a list of bids $(b^1, \ldots, b^k)$, where b^1 is the player's bid for the first unit of the good, b^2 is her bid for the second unit, and so on. The player who submits the highest bid for any given unit obtains that unit. The auctions differ in the prices paid by the winners. (The first type of auction generalizes a first-price auction, whereas the next two generalize a second-price auction.)

Discriminatory auction The price paid for each unit is the winning bid for that unit.

Uniform-price auction The price paid for each unit is the same, equal to the highest rejected bid among all the bids for all units.

Vickrey auction A bidder who wins k objects pays the sum of the k highest rejected bids submitted by the *other* bidders.

The next exercise asks you to study these auctions when two units of an object are available.

?? EXERCISE 87.1 (Multi-unit auctions) Two units of an object are available. There are n bidders. Bidder i values the first unit that she obtains at v_i and the second unit at w_i, where $v_i > w_i > 0$. Each bidder submits two bids; the two highest bids win. Retain the tie-breaking rule in the text. Show that in discriminatory and uniform-price auctions, player i's action of bidding v_i and w_i does not dominate all her other actions, whereas in a Vickrey auction it does. (In the case of a Vickrey auction, consider separately the cases in which the other players' bids are such that player i wins no units, one unit, and two units when her bids are v_i and w_i.)

Goods for which the demand exceeds the supply at the going price are sometimes sold to the people who are willing to wait longest in line. We can model such situations as multi-unit auctions in which each person's bid is the amount of time she is willing to wait.

?? EXERCISE 88.1 (Waiting in line) Two hundred people are willing to wait in line to see a movie at a theater whose capacity is one hundred. Denote person i's valuation of the movie in excess of the price of admission, expressed in terms of the amount of time she is willing to wait, by v_i. That is, person i's payoff if she waits for t_i units of time is $v_i - t_i$. Each person attaches no value to a second ticket, and cannot buy tickets for other people. Assume that $v_1 > v_2 > \cdots > v_{200}$. Each person chooses an arrival time. If several people arrive at the same time then their order in line is determined by their index (lower-numbered people go first). If a person arrives to find 100 or more people already in line, her payoff is zero. Model the situation as a variant of a discriminatory multi-unit auction, in which each person submits a bid for only one unit, and find its Nash equilibria. (Look at your answer to Exercise 85.1 before seeking the Nash equilibria.) Arrival times for people at movies do not in general seem to conform with a Nash equilibrium. What feature missing from the model could explain the pattern of arrivals?

The next exercise is another application of a multi-unit auction. As in the previous exercise each person wants to buy only one unit, but in this case the price paid by the winners is the highest losing bid.

? EXERCISE 88.2 (Internet pricing) A proposal to deal with congestion on electronic message pathways is that each message should include a field stating an amount of money the sender is willing to pay for the message to be sent. Suppose that during some time interval, each of n people wants to send one message and the capacity of the pathway is k messages, with $k < n$. The k messages whose bids are highest are the ones sent, and each of the persons sending these messages pays a price equal to the $(k+1)$st highest bid. Model this situation as a multi-unit auction. (Use the same tie-breaking rule as the one in the text.) Does a person's action of bidding the value of her message weakly dominate all her other actions? (Note that the auction differs from those considered in Exercise 87.1 because each person submits only one bid. Look at the argument in the text that in a second-price sealed-bid auction a player's action of bidding her value weakly dominates all her other actions.)

Lobbying as an auction Variants of the models in this section can be used to understand some situations that are not explicitly auctions. An example, illustrated in the next exercise, is the competition between groups pressuring a government to follow policies they favor. This exercise shows also that the outcome of an auction may depend significantly (and perhaps counterintuitively) on the form the auction takes.

? EXERCISE 88.3 (Lobbying as an auction) A government can pursue three policies, x, y, and z. The monetary values attached to these policies by two interest groups, A and B, are given in Figure 89.1. The government chooses a policy in response to the payments the interest groups make to it. Consider the following two mechanisms.

First-price auction Each interest group chooses a policy and an amount of money it is willing to pay. The government chooses the policy proposed by the group willing to pay the most. This group makes its payment to the government, and the losing group makes no payment.

Menu auction Each interest group states, for each policy, the amount it is willing to pay to have the government implement that policy. The government chooses the policy for which the sum of the payments the groups are willing to make is the highest, and *each* group pays the government the amount of money it is willing to pay for that policy.

In each case each interest group's payoff is the value it attaches to the policy implemented minus the payment it makes. Assume that a tie is broken by the government's choosing the policy, among those tied, whose name is first in the alphabet.

	x	y	z
Interest group A	0	3	-100
Interest group B	0	-100	3

Figure 89.1 The values of the interest groups for the policies x, y, and z in Exercise 88.3.

Show that the first-price auction has a Nash equilibrium in which lobby A says it will pay 103 for y, lobby B says it will pay 103 for z, and the government's revenue is 103. Show that the menu auction has a Nash equilibrium in which lobby A announces that it will pay 3 for x, 6 for y, and 0 for z, and lobby B announces that it will pay 3 for x, 0 for y, and 6 for z, and the government chooses x, obtaining a revenue of 6. (In each case the pair of actions given is in fact the unique equilibrium.)

3.6 Accident law

3.6.1 Introduction

In some situations, laws influence the participants' payoffs and hence their actions. For example, a law may provide for the victim of an accident to be compensated by a party who was at fault, and the size of the compensation may affect the care that each party takes. What laws can we expect to produce socially desirable outcomes? A game theoretic analysis is useful in addressing this question.

3.6.2 The game

Consider the interaction between an *injurer* (player 1) and a *victim* (player 2). The victim suffers a loss that depends on the amounts of care taken by both her and the injurer. (How badly you hurt yourself when you fall down on the sidewalk in front of my house depends on both how well I have cleared the ice and how carefully you tread.) Denote by a_i the amount of care player i takes, measured in monetary terms, and by $L(a_1, a_2)$ the loss, also measured in monetary terms, suffered by the victim, as a function of the amounts of care. (In many cases the victim does not suffer a loss with certainty, but only with probability less than one. In such cases we can interpret $L(a_1, a_2)$ as the expected loss—the average loss suffered over many occurrences.) Assume that $L(a_1, a_2) > 0$ for all values of (a_1, a_2), and that more care taken by either player reduces the loss: L is decreasing in a_1 for any fixed value of a_2, and decreasing in a_2 for any fixed value of a_1.

A legal rule determines the fraction of the loss borne by the injurer, as a function of the amounts of care taken. Denote this fraction by $\rho(a_1, a_2)$. If $\rho(a_1, a_2) = 0$ for all (a_1, a_2), for example, the victim bears the entire loss, regardless of how much care she takes or how little care the injurer takes. At the other extreme, $\rho(a_1, a_2) = 1$ for all (a_1, a_2) means that the victim is fully compensated by the injurer no matter how careless she is or how careful the injurer is.

If the amounts of care are (a_1, a_2) then the injurer bears the cost a_1 of taking care and suffers the loss of $L(a_1, a_2)$, of which she bears the fraction $\rho(a_1, a_2)$. Thus the injurer's payoff is

$$-a_1 - \rho(a_1, a_2)L(a_1, a_2).$$

Similarly, the victim's payoff is

$$-a_2 - (1 - \rho(a_1, a_2))L(a_1, a_2).$$

For any given legal rule, embodied in ρ, we can model the interaction between the injurer and victim as the following strategic game.

Players The injurer and the victim.

Actions The set of actions of each player is the set of possible levels of care (nonnegative numbers).

Preferences The injurer's preferences are represented by the payoff function $-a_1 - \rho(a_1, a_2)L(a_1, a_2)$ and the victim's preferences are represented by the payoff function $-a_2 - (1 - \rho(a_1, a_2))L(a_1, a_2)$, where a_1 is the injurer's level of care and a_2 is the victim's level of care.

How do the equilibria of this game depend upon the legal rule? Do any legal rules lead to socially desirable equilibrium outcomes?

I restrict attention to a class of legal rules known as *negligence with contributory negligence*. (This class was established in the USA in the mid-nineteenth century, and prevailed until the mid-1970s.) Each rule in this class requires the injurer to compensate the victim for a loss if and only if *both* the victim is sufficiently careful *and* the injurer is sufficiently careless; the required compensation is the total loss. Rules in the class differ in the standards of care they specify for each party. The rule that specifies the standards of care X_1 for the injurer and X_2 for the victim requires the injurer to pay the victim the entire loss $L(a_1, a_2)$ when $a_1 < X_1$ (the injurer is insufficiently careful) and $a_2 \geq X_2$ (the victim is sufficiently careful), and nothing otherwise. That is, under this rule the fraction $\rho(a_1, a_2)$ of the loss borne by the injurer is

$$\rho(a_1, a_2) = \begin{cases} 1 & \text{if } a_1 < X_1 \text{ and } a_2 \geq X_2 \\ 0 & \text{if } a_1 \geq X_1 \text{ or } a_2 < X_2. \end{cases}$$

Included in this class of rules are those for which X_1 is a positive finite number and $X_2 = 0$ (the injurer has to pay if she is not sufficiently careful, even if the victim takes no care at all), known as rules of *pure negligence*, and that for which X_1 is infinite and $X_2 = 0$ (the injurer has to pay regardless of how careful she is and how careless the victim is), known as the rule of *strict liability*.

3.6.3 Nash equilibrium

Suppose we decide that the pair $(\hat{a}_1, \hat{a}_2)$ of actions is socially desirable. We wish to answer the question: are there values of X_1 and X_2 such that the game generated by the rule of negligence with contributory negligence for (X_1, X_2) has $(\hat{a}_1, \hat{a}_2)$ as its unique Nash equilibrium? If the answer is affirmative, then, assuming the solution concept of Nash equilibrium is appropriate for the situation we are considering, we have found a legal rule that induces the socially desirable outcome.

Specifically, suppose that we select as socially desirable the pair $(\hat{a}_1, \hat{a}_2)$ of actions that maximizes the sum of the players' payoffs. That is,

$$(\hat{a}_1, \hat{a}_2) \text{ maximizes } -a_1 - a_2 - L(a_1, a_2).$$

(For some functions L, this pair $(\hat{a}_1, \hat{a}_2)$ may be a reasonable candidate for a socially desirable outcome; in other cases it may induce a very inequitable distribution of payoff between the players, and thus be an unlikely candidate.)

I claim that the unique Nash equilibrium of the game induced by the legal rule of negligence with contributory negligence for $(X_1, X_2) = (\hat{a}_1, \hat{a}_2)$ is $(\hat{a}_1, \hat{a}_2)$. That is, if the standards of care are equal to their socially desirable levels, then these are the levels chosen by an injurer and a victim in the only equilibrium of the game. The outcome is that the injurer pays no compensation: her level of care is $\hat{a}_1$, just high enough that $\rho(a_1, a_2) = 0$. At the same time the victim's level of care is $\hat{a}_2$, high enough that if the injurer reduces her level of care even slightly then she has to pay full compensation.

I first argue that $(\hat{a}_1, \hat{a}_2)$ is *a* Nash equilibrium of the game, then show that it is the *only* equilibrium. To show that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium, I need to show that the injurer's action $\hat{a}_1$ is a best response to the victim's action $\hat{a}_2$ and *vice versa*.

Injurer's action Given that the victim's action is $\hat{a}_2$, the injurer has to pay compensation if and only if $a_1 < \hat{a}_1$. Thus the injurer's payoff is

$$u_1(a_1, \hat{a}_2) = \begin{cases} -a_1 - L(a_1, \hat{a}_2) & \text{if } a_1 < \hat{a}_1 \\ -a_1 & \text{if } a_1 \geq \hat{a}_1. \end{cases} \tag{91.1}$$

For $a_1 = \hat{a}_1$, this payoff is $-\hat{a}_1$. If she takes more care than $\hat{a}_1$, she is worse off, because care is costly and, beyond $\hat{a}_1$, does not reduce her liability for compensation. If she takes less care, then, given the victim's level of care, she has to pay compensation, and we need to compare the money saved by taking less care with the size of the compensation. The argument is a little tricky. First, by definition,

$$(\hat{a}_1, \hat{a}_2) \text{ maximizes } -a_1 - a_2 - L(a_1, a_2).$$

Hence

$$\hat{a}_1 \text{ maximizes } -a_1 - \hat{a}_2 - L(a_1, \hat{a}_2)$$

(given $\hat{a}_2$). Because $\hat{a}_2$ is a constant, it follows that

$$\hat{a}_1 \text{ maximizes } -a_1 - L(a_1, \hat{a}_2).$$

But from (91.1) we see that $-a_1 - L(a_1, \hat{a}_2)$ is the injurer's payoff $u_1(a_1, \hat{a}_2)$ when her action is $a_1 < \hat{a}_1$ and the victim's action is $\hat{a}_2$. We conclude that the injurer's payoff takes a form like that in the left panel of Figure 92.1. In particular, $\hat{a}_1$ maximizes $u_1(a_1, \hat{a}_2)$, so that $\hat{a}_1$ is a best response to $\hat{a}_2$.

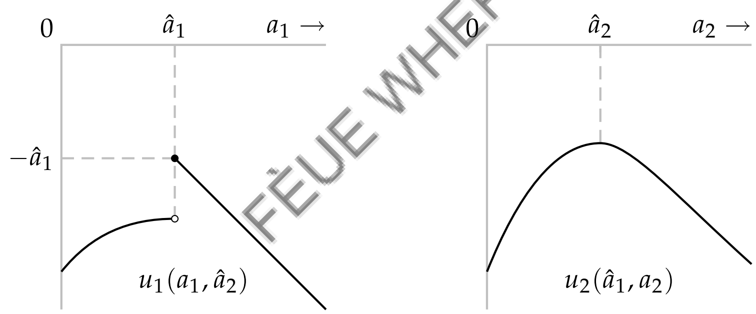


Figure 92.1 Left panel: the injurer's payoff as a function of her level of care a_1 when the victim's level of care is $a_2 = \hat{a}_2$ (see (91.1)). Right panel: the victim's payoff as a function of her level of care a_2 when the injurer's level of care is $a_1 = \hat{a}_1$ (see (92.1)).

Victim's action Given that the injurer's action is $\hat{a}_1$, the victim never receives compensation. Thus her payoff is

$$u_2(\hat{a}_1, a_2) = -a_2 - L(\hat{a}_1, a_2). \tag{92.1}$$

We can argue as we did for the injurer. By definition, $(\hat{a}_1, \hat{a}_2)$ maximizes $-a_1 - a_2 - L(a_1, a_2)$, so

$$\hat{a}_2 \text{ maximizes } -\hat{a}_1 - a_2 - L(\hat{a}_1, a_2)$$

(given $\hat{a}_1$). Because $\hat{a}_1$ is a constant, it follows that

$$\hat{a}_2 \text{ maximizes } -a_2 - L(\hat{a}_1, a_2), \tag{92.2}$$

which is the victim's payoff (see (92.1) and the right panel of Figure 92.1). That is, $\hat{a}_2$ maximizes $u_2(\hat{a}_1, a_2)$, so that $\hat{a}_2$ is a best response to $\hat{a}_1$.

We conclude that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium of the game induced by the legal rule of negligence with contributory negligence when the standards of care are $\hat{a}_1$ for the injurer and $\hat{a}_2$ for the victim.

To show that $(\hat{a}_1, \hat{a}_2)$ is the *only* Nash equilibrium of the game, first consider the injurer's best response function. Her payoff function is

$$u_1(a_1, a_2) = \begin{cases} -a_1 - L(a_1, a_2) & \text{if } a_1 < \hat{a}_1 \text{ and } a_2 \geq \hat{a}_2 \\ -a_1 & \text{if } a_1 \geq \hat{a}_1 \text{ or } a_2 < \hat{a}_2. \end{cases}$$

We can split the analysis into three cases, according to the victim's level of care.

$a_2 < \hat{a}_2$: In this case the injurer does not have to pay any compensation, regardless of her level of care; her payoff is $-a_1$, so that her best response is $a_1 = 0$.

$a_2 = \hat{a}_2$: In this case the injurer's best response is $\hat{a}_1$, as argued when showing that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium.

$a_2 > \hat{a}_2$: In this case the injurer's best response is at most $\hat{a}_1$, because her payoff for larger values of a_1 is equal to $-a_1$, a decreasing function of a_1.

We conclude that the injurer's best response function takes a form like that shown in the left panel of Figure 93.1.

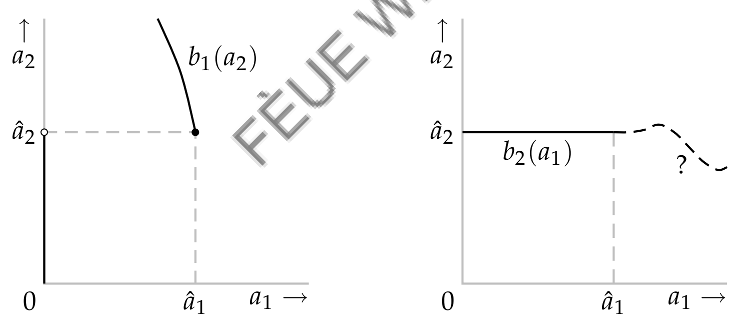


Figure 93.1 The players' best response functions under the rule of negligence with contributory negligence when $(X_1, X_2) = (\hat{a}_1, \hat{a}_2)$. Left panel: the injurer's best response function b_1. Right panel: the victim's best response function b_2. (The position of the victim's best response function for $a_1 > \hat{a}_1$ is not significant, and is not determined in the text.)

Now, given that the injurer's best response to any value of a_2 is never greater than $\hat{a}_1$, in any equilibrium we have $a_1 \leq \hat{a}_1$: any point (a_1, a_2) at which the victim's best response function crosses the injurer's best response function must have $a_1 \leq \hat{a}_1$. (Draw a few possible best response functions for the victim in the left panel of Figure 93.1.) We know that the victim's best response to $\hat{a}_1$ is $\hat{a}_2$ (because $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium), so we need to worry only about the victim's best responses to values of a_1 with $a_1 < \hat{a}_1$ (i.e. for cases in which the injurer takes insufficient care).

Let $a_1 < \hat{a}_1$. Then if the victim takes insufficient care she bears the loss; otherwise she is compensated for the loss, and hence bears only the cost a_2 of her taking care. Thus the victim's payoff is

$$u_2(a_1, a_2) = \begin{cases} -a_2 - L(a_1, a_2) & \text{if } a_2 < \hat{a}_2 \\ -a_2 & \text{if } a_2 \geq \hat{a}_2. \end{cases} \tag{94.1}$$

Now, by (92.2) the level of care $\hat{a}_2$ maximizes $-a_2 - L(\hat{a}_1, a_2)$, so that

$$-a_2 - L(\hat{a}_1, a_2) \leq -\hat{a}_2 - L(\hat{a}_1, \hat{a}_2) \text{ for all } a_2.$$

Further, the loss is nonnegative, so $-\hat{a}_2 - L(\hat{a}_1, \hat{a}_2) \leq -\hat{a}_2$. We conclude that

$$-a_2 - L(\hat{a}_1, a_2) \leq -\hat{a}_2 \text{ for all } a_2. \tag{94.2}$$

Finally, the loss increases as the injurer takes less care, so that given $a_1 < \hat{a}_1$ we have $L(a_1, a_2) > L(\hat{a}_1, a_2)$ for all a_2. Thus $-a_2 - L(a_1, a_2) < -a_2 - L(\hat{a}_1, a_2)$ for all a_2, and hence, using (94.2),

$$-a_2 - L(a_1, a_2) < -\hat{a}_2 \text{ for all } a_2.$$

From (94.1) it follows that the victim's best response to any $a_1 < \hat{a}_1$ is $\hat{a}_2$, as shown in the right panel of Figure 93.1.

Combining the two best response functions we see that $(\hat{a}_1, \hat{a}_2)$, the pair of levels of care that maximizes the sum of the players' payoffs, is the unique Nash equilibrium of the game. That is, the rule of negligence with contributory negligence for standards of care equal to $\hat{a}_1$ and $\hat{a}_2$ induces the players to choose these levels of care. If legislators can determine the values of $\hat{a}_1$ and $\hat{a}_2$ then by writing these levels into law they will induce a game that has as its unique Nash equilibrium the socially optimal actions.

Other standards also induce a pair of levels of care equal to $(\hat{a}_1, \hat{a}_2)$, as you are asked to show in the following exercise.

EXERCISE 94.3 (Alternative standards of care under negligence with contributory negligence) Show that $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium for the rule of negligence with contributory negligence for any value of (X_1, X_2) for which *either* $X_1 = \hat{a}_1$ and $X_2 \leq \hat{a}_2$ (including the pure negligence case of $X_2 = 0$), *or* $X_1 \geq M$ and $X_2 = \hat{a}_2$ for sufficiently large M. (Use the lines of argument in the text.)

EXERCISE 94.4 (Equilibrium under strict liability) Study the Nash equilibrium (equilibria?) of the game studied in the text under the rule of strict liability, in which X_1 is infinite and $X_2 = 0$ (i.e. the injurer is liable for the loss no matter how careful she is and how careless the victim is). How are the equilibrium actions related to $\hat{a}_1$ and $\hat{a}_2$?

Notes

The model in Section 3.1 was developed by Cournot (1838). The model in Section 3.2 is widely credited to Bertrand (1883). The box on p. 67 is based on Leonard (1994) and Magnan de Bornier (1992). The models are discussed in more detail by Shapiro (1989).

The model in Section 3.3 is due to Hotelling (1929) (though the focus of his paper is a model in which the players are firms that choose not only locations, but also prices). Downs (1957, especially Ch. 8) popularized Hotelling's model, using it to gain insights about electoral competition. Shepsle (1991) and Osborne (1995) survey work in the field.

The *War of Attrition* studied in Section 3.4 is due to Maynard Smith (1974); it is a variant of the *Dollar Auction* presented by Shubik (1971).

Vickrey (1961) initiated the formal modeling of auctions, as studied in Section 3.5. The literature is surveyed by Wilson (1992). The box on page 79 draws on Herodotus' *Histories* (Book 1, paragraph 196; see for example Herodotus 1998, 86), Langdon (1994), Cassady (1967, Ch. 3), Shubik (1983), Andreau (1999, 38–39), the website `www.eBay.com`, Ockenfels and Roth (2000), and personal correspondence with Robin G. Osborne (on ancient Greece and Rome) and John H. Munro (on medieval Europe).

The model of accident law discussed in Section 94.3 originated with Brown (1973) and Diamond (1974); the result about negligence with contributory negligence is due to Brown (1973, 340–341). The literature is surveyed by Benoît and Kornhauser (1995).

Novshek and Sonnenschein (1978) study, in a general setting, the issue addressed in Exercise 60.1. A brief summary of the early work on common property is given in the Notes to Chapter 2. The idea of the tie-breaking rule being determined by the equilibrium, used in Exercises 66.2 and 67.1, is due to Simon and Zame (1990). The result in Exercise 73.1 is due to Wittman (1977). Exercise 73.2 is based on Osborne and Slivinski (1996). The notion of a Condorcet winner defined in Exercise 74.1 is associated with Marie-Jean-Antoine-Nicolas de Caritat, marquis de Condorcet (1743–1794), an early student of voting procedures. The game in Exercise 78.1 is a variant of a game studied by Blackwell and Girschick (1954, Example 5 in Ch. 2). It is an example of a *noisy duel* (which models the situation of duelists, each of whom chooses when to fire a single bullet, which her opponent hears, as she gradually approaches her rival). Duels were first modeled as games in the late 1940s by members of the RAND Corporation in the USA; see Karlin (1959b, Ch. 5). Exercise 88.3 is based on Boylan (1997). The situation considered in Exercise 88.1, in which people decide when to join a queue, is studied by Holt and Sherman (1982). Exercise 88.2 is based on MacKie-Mason and Varian (1995).

4 Mixed Strategy Equilibrium

4.1 Introduction

4.1.1 Stochastic steady states

A NASH EQUILIBRIUM of a strategic game is an action profile in which every player's action is optimal given every other player's action (Definition 21.1). Such an action profile corresponds to a steady state of the idealized situation in which for each player in the game there is a population of individuals, and whenever the game is played, one player is drawn randomly from each population (see Section 2.6). In a steady state, every player's behavior is the same whenever she plays the game, and no player wishes to change her behavior, knowing (from her experience) the other players' behavior. In a steady state in which each player's "behavior" is simply an action and within each population all players choose the same action, the outcome of every play of the game is the same Nash equilibrium.

More general notions of a steady state allow the players' choices to vary, as long as the pattern of choices remains constant. For example, different members of a given population may choose different actions, each player choosing the same action whenever she plays the game. Or each individual may, on each occasion she plays the game, choose her action probabilistically according to the same, unchanging distribution. These two more general notions of a steady state are equivalent: a steady state of the first type in which the fraction p of the population representing player i chooses the action a corresponds to a steady state of the second type in which each member of the population representing player i chooses a with probability p. In both cases, in each play of the game the probability that the individual in the role of player i chooses a is p. Both these notions of steady state are modeled by a mixed strategy Nash equilibrium, a generalization of the notion of Nash equilibrium. For expository convenience, in most of this chapter I interpret such an equilibrium as a model of the second type of steady state, in which each player chooses her actions probabilistically; such a steady state is called *stochastic* ("involving probability").

4.1.2 Example: Matching Pennies

An analysis of the game *Matching Pennies* (Example 17.1) illustrates the idea of a stochastic steady state. My discussion focuses on the outcomes of this game, given in Figure 98.1, rather than payoffs that represent the players' preferences, as before.

	Head	*Tail*
Head	$\$1, -\$1$	$-\$1, \$1$
Tail	$-\$1, \$1$	$\$1, -\$1$

Figure 98.1 The outcomes of *Matching Pennies*.

As we saw previously, this game has no Nash equilibrium: no pair of actions is compatible with a steady state in which each player's action is the same whenever the game is played. I claim, however, that the game has a *stochastic* steady state in which each player chooses each of her actions with probability $\frac{1}{2}$. To establish this result, I need to argue that if player 2 chooses each of her actions with probability $\frac{1}{2}$, then player 1 optimally chooses each of her actions with probability $\frac{1}{2}$, and vice versa.

Suppose that player 2 chooses each of her actions with probability $\frac{1}{2}$. If player 1 chooses *Head* with probability p and *Tail* with probability $1-p$ then each outcome $(Head, Head)$ and $(Head, Tail)$ occurs with probability $\frac{1}{2}p$, and each outcome $(Tail, Head)$ and $(Tail, Tail)$ occurs with probability $\frac{1}{2}(1-p)$. Thus player 1 gains \$1 with probability $\frac{1}{2}p + \frac{1}{2}(1-p)$, which is equal to $\frac{1}{2}$, and loses \$1 with probability $\frac{1}{2}$. In particular, the probability distribution over outcomes is independent of p! Thus *every* value of p is optimal. In particular, player 1 can do no better than choose *Head* with probability $\frac{1}{2}$ and *Tail* with probability $\frac{1}{2}$. A similar analysis shows that player 2 optimally chooses each action with probability $\frac{1}{2}$ when player 1 does so. We conclude that the game has a stochastic steady state in which each player chooses each action with probability $\frac{1}{2}$.

I further claim that, under a reasonable assumption on the players' preferences, the game has no other steady state. This assumption is that each player wants the probability of her gaining \$1 to be as large as possible. More precisely, if $p > q$ then each player prefers to gain \$1 with probability p and lose \$1 with probability $1-p$ than to gain \$1 with probability q and lose \$1 with probability $1-q$.

To show that under this assumption there is no steady state in which the probability of each player's choosing *Head* is different from $\frac{1}{2}$, denote the probability with which player 2 chooses *Head* by q (so that she chooses *Tail* with probability $1-q$). If player 1 chooses *Head* with probability p then she gains \$1 with probability $pq + (1-p)(1-q)$ (the probability that the outcome is either $(Head, Head)$

or (*Tail*, *Tail*)) and loses \$1 with probability $(1-p)q + p(1-q)$. The first probability is equal to $1 - q + p(2q-1)$ and the second is equal to $q + p(1-2q)$. Thus if $q < \frac{1}{2}$ (player 2 chooses *Head* with probability less than $\frac{1}{2}$), the first probability is decreasing in p and the second is increasing in p, so that the lower is p, the better is the outcome for player 1; the value of p that induces the best probability distribution over outcomes for player 1 is 0. That is, if player 2 chooses *Head* with probability less than $\frac{1}{2}$, then the uniquely best policy for player 1 is to choose *Tail* with certainty. A similar argument shows that if player 2 chooses *Head* with probability greater than $\frac{1}{2}$, the uniquely best policy for player 1 is to choose *Head* with certainty.

Now, if player 1 chooses one of her actions with certainty, an analysis like that in the previous paragraph leads to the conclusion that the optimal policy of player 2 is to choose one of her actions with certainty (*Head* if player 1 chooses *Tail* and *Tail* if player 1 chooses *Head*).

We conclude that there is no steady state in which the probability that player 2 chooses *Head* is different from $\frac{1}{2}$. A symmetric argument leads to the conclusion that there is no steady state in which the probability that player 1 chooses *Head* is different from $\frac{1}{2}$. Thus the only stochastic steady state is that in which each player chooses each of her actions with probability $\frac{1}{2}$.

As discussed in the first section, the stable pattern of behavior we have found can be alternatively interpreted as a steady state in which no player randomizes. Instead, half the players in the population of individuals who take the role of player 1 in the game choose *Head* whenever they play the game and half of them choose *Tail* whenever they play the game; similarly half of those who take the role of player 2 choose *Head* and half choose *Tail*. Given that the individuals involved in any given play of the game are chosen randomly from the populations, in each play of the game each individual faces with probability $\frac{1}{2}$ an opponent who chooses *Head*, and with probability $\frac{1}{2}$ an opponent who chooses *Tail*.

? EXERCISE 99.1 (Variant of Matching Pennies) Find the steady state(s) of the game that differs from *Matching Pennies* only in that the outcomes of (*Head*,*Head*) and of (*Tail*,*Tail*) are that player 1 gains \$2 and player 2 loses \$1.

4.1.3 Generalizing the analysis: expected payoffs

The fact that *Matching Pennies* has only two outcomes for each player (gain \$1, lose \$1) makes the analysis of a stochastic steady state particularly simple, because it allows us to deduce, under a weak assumption, the players' preferences regarding *lotteries* (probability distributions) over outcomes from their preferences regarding deterministic outcomes (outcomes that occur with certainty). If a player prefers the deterministic outcome a to the deterministic outcome b, it is very plausible that if $p > q$ then she prefers the lottery in which a occurs with probability p (and b occurs with probability $1-p$) to the lottery in which a occurs with probability q (and b occurs with probability $1-q$).

In a game with more than two outcomes for some player, we cannot extrapolate in this way from preferences regarding deterministic outcomes to preferences regarding lotteries over outcomes. Suppose, for example, that a game has three possible outcomes, a, b, and c, and that a player prefers a to b to c. Does she prefer the deterministic outcome b to the lottery in which a and c each occur with probability $\frac{1}{2}$, or vice versa? The information about her preferences over deterministic outcomes gives us no clue about the answer to this question. She may prefer b to the lottery in which a and c each occur with probability $\frac{1}{2}$, or she may prefer this lottery to b; both preferences are consistent with her preferring a to b to c. In order to study her behavior when she is faced with choices between lotteries, we need to add to the model a description of her preferences regarding lotteries over outcomes.

A standard assumption in game theory restricts attention to preferences regarding lotteries over outcomes that may be represented by the expected value of a payoff function over deterministic outcomes. (See Section 17.7.3 if you are unfamiliar with the notion of "expected value".) That is, for every player i there is a payoff function u_i with the property that player i prefers one lottery over outcomes to another if and only if, according to u_i, the expected value of the first lottery exceeds the expected value of the second lottery.

For example, suppose that there are three outcomes, a, b, and c, and lottery P yields a with probability p_a, b with probability p_b, and c with probability p_c, whereas lottery Q yields these three outcomes with probabilities q_a, q_b, and q_c. Then the assumption is that for each player i there are numbers $u_i(a)$, $u_i(b)$, and $u_i(c)$ such that player i prefers lottery P to lottery Q if and only if $p_a u_i(a) + p_b u_i(b) + p_c u_i(c) > q_a u_i(a) + q_b u_i(b) + q_c u_i(c)$. (I discuss the representation of preferences by the expected value of a payoff function in more detail in Section 4.12, an appendix to this chapter.)

The first systematic investigation of preferences regarding lotteries represented by the expected value of a payoff function over deterministic outcomes was undertaken by von Neumann and Morgenstern (1944). Accordingly such preferences are called **vNM preferences**. A payoff function over deterministic outcomes (u_i in the previous paragraph) whose expected value represents such preferences is called a **Bernoulli payoff function** (in honor of Daniel Bernoulli (1700–1782), who appears to have been one of the first persons to use such a function to represent preferences).

The restrictions on preferences regarding deterministic outcomes required for them to be represented by a payoff function are relatively innocuous (see Section 1.2.2). The same is not true of the restrictions on preferences regarding lotteries over outcomes required for them to be represented by the expected value of a payoff function. (I do not discuss these restrictions, but the box at the end of this section gives an example of preferences that violate them.) Nevertheless, we obtain many insights from models that assume preferences take this form; following standard game theory (and standard economic theory), I maintain the assumption throughout the book.

The assumption that a player's preferences be represented by the expected value of a payoff function does not restrict her attitudes to risk: a person whose preferences are represented by such a function may have an arbitrarily strong like or dislike for risk. Suppose, for example, that a, b, and c are three outcomes, and a person prefers a to b to c. A person who is very averse to risky outcomes prefers to obtain b for sure rather than to face the lottery in which a occurs with probability p and c occurs with probability $1-p$, even if p is relatively large. Such preferences may be represented by the expected value of a payoff function u for which $u(a)$ is close to $u(b)$, which is much larger than $u(c)$. A person who is not at all averse to risky outcomes prefers the lottery to the certain outcome b, even if p is relatively small. Such preferences are represented by the expected value of a payoff function u for which $u(a)$ is much larger than $u(b)$, which is close to $u(c)$. If $u(a) = 10$, $u(b) = 9$, and $u(c) = 0$, for example, then the person prefers the certain outcome b to any lottery between a and c that yields a with probability less than $\frac{9}{10}$. But if $u(a) = 10$, $u(b) = 1$, and $u(c) = 0$, she prefers any lottery between a and c that yields a with probability greater than $\frac{1}{10}$ to the certain outcome b.

Suppose that the outcomes are amounts of money and a person's preferences are represented by the expected value of a payoff function in which the payoff of each outcome is equal to the amount of money involved. Then we say the person is *risk neutral*. Such a person compares lotteries according to the expected amount of money involved. (For example, she is indifferent between receiving \$100 for sure and the lottery that yields \$0 with probability $\frac{9}{10}$ and \$1000 with probability $\frac{1}{10}$.) On the one hand, the fact that people buy insurance suggests that in some circumstances preferences are *risk averse*: people prefer to obtain \$$z$ with certainty than to receive the outcome of a lottery that yields \$$z$ on average. On the other hand, the fact that people buy lottery tickets that pay, on average, much less than their purchase price, suggests that in other circumstances preferences are *risk preferring*. In both cases, preferences over lotteries are not represented by expected *monetary* values, though they still may be represented by the expected value of a *payoff* function (in which the payoffs to outcome are different from the monetary values of the outcomes).

Any given preferences over deterministic outcomes are represented by many different payoff functions (see Section 1.2.2). The same is true of preferences over lotteries; the relation between payoff functions whose expected values represent the same preferences is discussed in Section 4.12.2 in the appendix to this chapter. In particular, we may choose arbitrary payoffs for the outcomes that are best and worst according to the preferences, as long as the payoff to the best outcome exceeds the payoff to the worst outcome. For example, suppose there are three outcomes, a, b, and c, and a person prefers a to b to c, and is indifferent between b and the lottery that yields a with probability $\frac{1}{2}$ and c with probability $\frac{1}{2}$. Then we may choose $u(a) = 3$ and $u(c) = 1$, in which case $u(b) = 2$; or, for example, we may choose $u(a) = 10$ and $u(c) = 0$, in which case $u(b) = 5$.

SOME EVIDENCE ON EXPECTED PAYOFF FUNCTIONS

Consider the following two lotteries (the first of which is, in fact, deterministic):

Lottery 1 You receive \$2 million with certainty

Lottery 2 You receive \$10 million with probability 0.1, \$2 million with probability 0.89, and nothing with probability 0.01.

Which do you prefer? Now consider two more lotteries:

Lottery 3 You receive \$2 million with probability 0.11 and nothing with probability 0.89

Lottery 4 You receive \$10 million with probability 0.1 and nothing with probability 0.9.

Which do you prefer? A significant fraction of experimental subjects say they prefer lottery 1 to lottery 2, and lottery 4 to lottery 3. (See, for example, Conlisk (1989) and Camerer (1995, 622–623).)

These preferences cannot be represented by an expected payoff function! If they could be, there would exist a payoff function u for which the expected payoff of lottery 1 exceeds that of lottery 2:

$$u(2) > 0.1u(10) + 0.89u(2) + 0.01u(0),$$

where the amounts of money are expressed in millions. Subtracting $0.89u(2)$ and adding $0.89u(0)$ to each side we obtain

$$0.11u(2) + 0.89u(0) > 0.1u(10) + 0.9u(0).$$

But this inequality says that the expected payoff of lottery 3 exceeds that of lottery 4! Thus preferences represented by an expected payoff function that yield a preference for lottery 1 over lottery 2 must also yield a preference for lottery 3 over lottery 4.

Preferences represented by the expected value of a payoff function *are*, however, consistent with a person's being indifferent between lotteries 1 and 2, and between lotteries 3 and 4. Suppose we assume that when a person is almost indifferent between two lotteries, she may make a "mistake". Then a person's expressed preference for lottery 1 over lottery 2 and for lottery 4 over lottery 3 is not directly inconsistent with her preferences being represented by the expected value of a payoff function in which she is almost indifferent between lotteries 1 and 2 and between lotteries 3 and 4. If, however, we add the assumption that mistakes are distributed symmetrically, then the frequency with which people express a preference for lottery 2 over lottery 1 and for lottery 4 over lottery 3 (also inconsistent with preferences represented by the expected value of a payoff function) should be similar to that with which people express a preference for lottery 1 over lottery 2 and for lottery 3 over lottery 4. In fact, however, the second pattern is significantly more common than the first (Conlisk 1989), so that a more significant modification of the theory is needed to explain the observations.

A limitation of the evidence is that it is based on the preferences expressed by people faced with *hypothetical* choices; understandably (given the amounts of money involved), no experiment has been run in which subjects were paid according to the lotteries they chose! Experiments with stakes consistent with normal research budgets show few choices inconsistent with preferences represented by the expected value of a payoff function (Conlisk 1989). This evidence, however, does not contradict the evidence based on hypothetical choices with large stakes: with larger stakes subjects might make choices in line with the preferences they express when asked about hypothetical choices.

In summary, the evidence for an inconsistency with preferences compatible with an expected payoff function is, at a minimum, suggestive. It has spurred the development of alternative theories. Nevertheless, the vast majority of models in game theory (and also in economics) that involve choice under uncertainty currently assume that each decision-maker's preferences are represented by the expected value of a payoff function. I maintain this assumption throughout the book, although many of the ideas I discuss appear not to depend on it.

4.2 Strategic games in which players may randomize

To study stochastic steady states, we extend the notion of a strategic game given in Definition 11.1 by endowing each player with vNM preferences about lotteries over the set of action profiles.

▶ DEFINITION 103.1 A **strategic game** (with vNM preferences) consists of

- a set of **players**
- for each player, a set of **actions**
- for each player, **preferences** regarding lotteries over action profiles that may be represented by the expected value of a ("Bernoulli") payoff function over action profiles.

A two-player strategic game with vNM preferences in which each player has finitely many actions may be presented in a table like those in Chapter 2. Such a table looks exactly the same as it did before, though the interpretation of the numbers in the boxes is different. In Chapter 2 these numbers are values of payoff functions that represent the players' preferences over deterministic outcomes; here they are the values of (Bernoulli) payoff functions whose expected values represent the players' preferences over lotteries.

Given the change in the interpretation of the payoffs, two tables that represent the same strategic game with ordinal preferences no longer necessarily represent the same strategic game with vNM preferences. For example, the two tables in Figure 104.1 represent the same game with ordinal preferences—namely the *Prisoner's Dilemma* (Section 2.2). In both cases the best outcome for each player is that in which she chooses F and the other player chooses Q, the next best outcome is (Q, Q), then comes (F, F), and the worst outcome is that in which she chooses Q and the other player chooses F. However, the tables represent *different* strategic games with vNM preferences. For example, in the left table player 1's payoff to (Q, Q) is the *same* as her expected payoff to the lottery that yields (F, Q) with probability $\frac{1}{2}$ and (F, F) with probability $\frac{1}{2}$ $(2 = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 1)$, whereas in the right table her payoff to (Q, Q) is *greater than* her expected payoff to this lottery $(3 > \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 1)$. Thus the left table represents a situation in which player 1 is indifferent between the deterministic outcome (Q, Q) and the lottery in which (F, Q) occurs with probability $\frac{1}{2}$ and (F, F) occurs with probability $\frac{1}{2}$. In the right table, however, she prefers the deterministic outcome (Q, Q) to the lottery.

	Q	F
Q	$2, 2$	$0, 3$
F	$3, 0$	$1, 1$

	Q	F
Q	$3, 3$	$0, 4$
F	$4, 0$	$1, 1$

Figure 104.1 Two tables that represent the same strategic game with ordinal preferences but different strategic games with vNM preferences.

To show, as in this example, that two tables represent different strategic games with vNM preferences we need only find a pair of lotteries whose expected payoffs are ordered differently by the two tables. To show that they represent the *same* strategic game with vNM preferences is more difficult; see Section 4.12.2.

? EXERCISE 104.1 (Extensions of *BoS* with vNM preferences) Construct a table of payoffs for a strategic game with vNM preferences in which the players' preferences over deterministic outcomes are the same as they are in *BoS* (Example 16.2), and their preferences over lotteries satisfy the following condition: each player is indifferent between going to her less preferred concert in the company of the other player and the lottery in which with probability $\frac{1}{2}$ she and the other player go to different concerts and with probability $\frac{1}{2}$ they both go to her more preferred concert. Do the same in the case that each player is indifferent between going to her less preferred concert in the company of the other player and the lottery in which with probability $\frac{3}{4}$ she and the other player go to different concerts and with probability $\frac{1}{4}$ they both go to her more preferred concert. (In each case set each player's payoff to the outcome that she least prefers equal to 0 and her payoff to the outcome that she most prefers equal to 2.)

Despite the importance of saying how the numbers in a payoff table should be interpreted, users of game theory sometimes fail to make the interpretation clear. When interpreting discussions of Nash equilibrium in the literature, a reasonably safe assumption is that if the players are not allowed to choose their actions randomly then the numbers in payoff tables are payoffs that represent the players' ordinal preferences, whereas if the players are allowed to randomize then the numbers are payoffs whose expected values represent the players' preferences regarding lotteries over outcomes.

4.3 Mixed strategy Nash equilibrium

4.3.1 Mixed strategies

In the generalization of the notion of Nash equilibrium that models a stochastic steady state of a strategic game with vNM preferences, we allow each player to choose a probability distribution over her set of actions rather than restricting her to choose a single deterministic action. We refer to such a probability distribution as a **mixed strategy**.

I usually use α to denote a profile of mixed strategies; $\alpha_i(a_i)$ is the probability assigned by player i's mixed strategy α_i to her action a_i. To specify a mixed strategy of player i we need to give the probability it assigns to each of player i's actions. For example, the strategy of player 1 in *Matching Pennies* that assigns probability $\frac{1}{2}$ to each action is the strategy α_1 for which $\alpha_1(Head) = \frac{1}{2}$ and $\alpha_1(Tail) = \frac{1}{2}$. Because this way of describing a mixed strategy is cumbersome, I often use a shorthand for a game that is presented in a table like those in Figure 104.1: I write a mixed strategy as a list of probabilities, one for each action, *in the order the actions are given in the table*. For example, the mixed strategy $(\frac{1}{3}, \frac{2}{3})$ for player 1 in either of the games in Figure 104.1 assigns probability $\frac{1}{3}$ to Q and probability $\frac{2}{3}$ to F.

A mixed strategy may assign probability 1 to a single action: by *allowing* a player to choose probability distributions, we do not prohibit her from choosing deterministic actions. We refer to such a mixed strategy as a **pure strategy**. Player i's choosing the pure strategy that assigns probability 1 to the action a_i is equivalent to her simply choosing the action a_i, and I denote this strategy simply by a_i.

4.3.2 Equilibrium

The notion of equilibrium that we study is called "mixed strategy Nash equilibrium". The idea behind it is the same as the idea behind the notion of Nash equilibrium for a game with ordinal preferences: a mixed strategy Nash equilibrium is a mixed strategy profile α^* with the property that no player i has a mixed strategy α_i such that she prefers the lottery over outcomes generated by the strategy profile $(\alpha_i, \alpha^*_{-i})$ to the lottery over outcomes generated by the strategy profile α^*. The following definition gives this condition using payoff functions whose expected values represent the players' preferences.

▶ DEFINITION 105.1 (*Mixed strategy Nash equilibrium of strategic game with vNM preferences*) The mixed strategy profile α^* in a strategic game with vNM preferences is a **(mixed strategy) Nash equilibrium** if, for each player i and every mixed strategy α_i of player i, the expected payoff to player i of α^* is at least as large as the expected payoff to player i of $(\alpha_i, \alpha^*_{-i})$ according to a payoff function whose expected value represents player i's preferences over lotteries. Equivalently, for each player i,

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i}) \text{ for every mixed strategy } \alpha_i \text{ of player } i, \tag{106.1}$$

where $U_i(\alpha)$ is player i's expected payoff to the mixed strategy profile α.

4.3.3 Best response functions

When studying mixed strategy Nash equilibria, as when studying Nash equilibria of strategic games with ordinal preferences, the players' best response functions (Section 2.8) are often useful. As before, I denote player i's best response function by B_i. For a strategic game with ordinal preferences, $B_i(a_{-i})$ is the set of player i's best actions when the list of the other players' actions is α_{-i}. For a strategic game with vNM preferences, $B_i(\alpha_{-i})$ is the set of player i's best mixed strategies when the list of the other players' mixed strategies is α_{-i}. From the definition of a mixed strategy equilibrium, a profile α^* of mixed strategies is a mixed strategy Nash equilibrium if and only if every player's mixed strategy is a best response to the other players' mixed strategies (cf. Proposition 34.1):

> the mixed strategy profile α^* is a mixed strategy Nash equilibrium if and only if α^*_i is in $B_i(\alpha^*_{-i})$ for every player i.

4.3.4 Best response functions in two-player two-action games

The analysis of *Matching Pennies* in Section 4.1.2 shows that each player's set of best responses to the other player's mixed strategy is either a single pure strategy or the set of *all* mixed strategies. (For example, if player 2's mixed strategy assigns probability less than $\frac{1}{2}$ to *Head* then player 1's unique best response is the pure strategy *Tail*, if player 2's mixed strategy assigns probability greater than $\frac{1}{2}$ to *Head* then player 1's unique best response is the pure strategy *Head*, and if player 2's mixed strategy assigns probability $\frac{1}{2}$ to *Head* then all of player 1's mixed strategies are best responses.)

In any two-player game in which each player has two actions, the set of each player's best responses has a similar character: it consists either of a single pure strategy, or of all mixed strategies. The reason lies in the form of the payoff functions.

Consider a two-player game in which each player has two actions, T and B for player 1 and L and R for player 2. Denote by u_i, for $i = 1, 2$, a Bernoulli payoff function for player i. (That is, u_i is a payoff function over action pairs whose expected value represents player i's preferences regarding lotteries over action pairs.) Player 1's mixed strategy α_1 assigns probability $\alpha_1(T)$ to her action T and probability $\alpha_1(B)$ to her action B (with $\alpha_1(T) + \alpha_1(B) = 1$). For convenience, let $p = \alpha_1(T)$, so that $\alpha_1(B) = 1 - p$. Similarly, denote the probability $\alpha_2(L)$ that player 2's mixed strategy assigns to L by q, so that $\alpha_2(R) = 1 - q$.

We take the players' choices to be independent, so that when the players use the mixed strategies α_1 and α_2, the probability of any action pair (a_1, a_2) is the product of the probability player 1's mixed strategy assigns to a_1 and the probability player 2's mixed strategy assigns to a_2. (See Section 17.7.2 in the mathematical appendix if you are not familiar with the idea of independence.) Thus the probability distribution generated by the mixed strategy pair (α_1, α_2) over the four possible outcomes of the game has the form given in Figure 107.1: (T, L) occurs with probability pq, (T, R) occurs with probability $p(1-q)$, (B, L) occurs with probability $(1-p)q$, and (B, R) occurs with probability $(1-p)(1-q)$.

	$L\ (q)$	$R\ (1-q)$
$T\ (p)$	pq	$p(1-q)$
$B\ (1-p)$	$(1-p)q$	$(1-p)(1-q)$

Figure 107.1 The probabilities of the four outcomes in a two-player two-action strategic game when player 1's mixed strategy is $(p, 1-p)$ and player 2's mixed strategy is $(q, 1-q)$.

From this probability distribution we see that player 1's expected payoff to the mixed strategy pair (α_1, α_2) is

$$pq \cdot u_1(T, L) + p(1-q) \cdot u_1(T, R) + (1-p)q \cdot u_1(B, L) + (1-p)(1-q) \cdot u_1(B, R),$$

which we can alternatively write as

$$p[q \cdot u_1(T, L) + (1-q) \cdot u_1(T, R)] + (1-p)[q \cdot u_1(B, L) + (1-q) \cdot u_1(B, R)].$$

The first term in square brackets is player 1's expected payoff when she uses a *pure* strategy that assigns probability 1 to T and player 2 uses her mixed strategy α_2; the second term in square brackets is player 1's expected payoff when she uses a *pure* strategy that assigns probability 1 to B and player 2 uses her mixed strategy α_2. Denote these two expected payoffs $E_1(T, \alpha_2)$ and $E_1(B, \alpha_2)$. Then player 1's expected payoff to the mixed strategy pair (α_1, α_2) is

$$pE_1(T, \alpha_2) + (1-p)E_1(B, \alpha_2).$$

That is, player 1's expected payoff to the mixed strategy pair (α_1, α_2) is a weighted average of her expected payoffs to T and B when player 2 uses the mixed strategy α_2, with weights equal to the probabilities assigned to T and B by α_1.

In particular, player 1's expected payoff, given player 2's mixed strategy, is a *linear* function of p—when plotted in a graph, it is a straight line. A case in which $E_1(T, \alpha_2) > E_1(B, \alpha_2)$ is illustrated in Figure 108.1.

? EXERCISE 107.1 (Expected payoffs) Construct diagrams like Figure 108.1 for *BoS* (Figure 16.1) and the game in Figure 19.1 (in each case treating the numbers in the tables as Bernoulli payoffs). In each diagram, plot player 1's expected payoff as a function of the probability p that she assigns to her top action in three cases: when the probability q that player 2 assigns to her left action is 0, $\frac{1}{2}$, and 1.

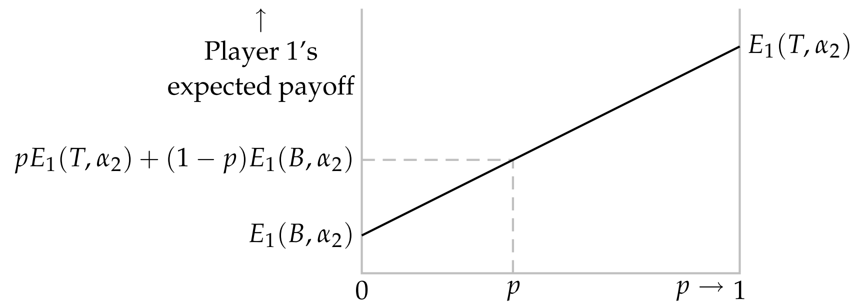


Figure 108.1 Player 1's expected payoff as a function of the probability p she assigns to T in the game in which her actions are T and B, when player 2's mixed strategy is α_2 and $E_1(T, \alpha_2) > E_1(B, \alpha_2)$.

A significant implication of the linearity of player 1's expected payoff is that there are three possibilities for her best response to a given mixed strategy of player 2:

- player 1's unique best response is the pure strategy T (if $E_1(T, \alpha_2) > E_1(B, \alpha_2)$, as in Figure 108.1)
- player 1's unique best response is the pure strategy B (if $E_1(B, \alpha_2) > E_1(T, \alpha_2)$, in which case the line representing player 1's expected payoff as a function of p in the analogue of Figure 108.1 slopes down)
- all mixed strategies of player 1 yield the same expected payoff, and hence all are best responses (if $E_1(T, \alpha_2) = E_1(B, \alpha_2)$, in which case the line representing player 1's expected payoff as a function of p in the analogue of Figure 108.1 is horizontal).

In particular, a mixed strategy $(p, 1-p)$ for which $0 < p < 1$ is never the *unique* best response; either it is not a best response, or *all* mixed strategies are best responses.

? EXERCISE 108.1 (Best responses) For each game and each value of q in Exercise 107.1, use the graphs you drew in that exercise to find player 1's set of best responses.

4.3.5 Example: Matching Pennies

The argument in Section 4.1.2 establishes that *Matching Pennies* has a unique mixed strategy Nash equilibrium, in which each player's mixed strategy assigns probability $\frac{1}{2}$ to *Head* and probability $\frac{1}{2}$ to *Tail*. I now describe an alternative route to this conclusion that uses the method described in Section 2.8.3, which involves explicitly constructing the players' best response functions; this method may be used in other games.

Represent each player's preferences by the expected value of a payoff function that assigns the payoff 1 to a gain of \$1 and the payoff -1 to a loss of \$1. The resulting strategic game with vNM preferences is shown in Figure 109.1.

	Head	*Tail*
Head	$1,-1$	$-1,\ \ 1$
Tail	$-1,\ \ 1$	$1,-1$

Figure 109.1 *Matching Pennies.*

Denote by p the probability that player 1's mixed strategy assigns to *Head*, and by q the probability that player 2's mixed strategy assigns to *Head*. Then, given player 2's mixed strategy, player 1's expected payoff to the pure strategy *Head* is

$$q \cdot 1 + (1-q) \cdot (-1) = 2q - 1$$

and her expected payoff to *Tail* is

$$q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q.$$

Thus if $q < \frac{1}{2}$ then player 1's expected payoff to *Tail* exceeds her expected payoff to *Head*, and hence exceeds also her expected payoff to every mixed strategy that assigns a positive probability to *Head*. Similarly, if $q > \frac{1}{2}$ then her expected payoff to *Head* exceeds her expected payoff to *Tail*, and hence exceeds her expected payoff to every mixed strategy that assigns a positive probability to *Tail*. If $q = \frac{1}{2}$ then both *Head* and *Tail*, and hence all her mixed strategies, yield the same expected payoff. We conclude that player 1's best responses to player 2's strategy are her mixed strategy that assigns probability 0 to *Head* if $q < \frac{1}{2}$, her mixed strategy that assigns probability 1 to *Head* if $q > \frac{1}{2}$, and all her mixed strategies if $q = \frac{1}{2}$. That is, denoting by $B_1(q)$ the set of probabilities player 1 assigns to *Head* in best responses to q, we have

$$B_1(q) = \begin{cases} \{0\} & \text{if } q < \frac{1}{2} \\ \{p\colon 0 \le p \le 1\} & \text{if } q = \frac{1}{2} \\ \{1\} & \text{if } q > \frac{1}{2}. \end{cases}$$

The best response function of player 2 is similar: $B_2(p) = \{1\}$ if $p < \frac{1}{2}$, $B_2(p) = \{q\colon 0 \le q \le 1\}$ if $p = \frac{1}{2}$, and $B_2(p) = \{0\}$ if $p > \frac{1}{2}$. Both best response functions are illustrated in Figure 110.1.

The set of mixed strategy Nash equilibria of the game corresponds (as before) to the set of intersections of the best response functions in this figure; we see that there is one intersection, corresponding to the equilibrium we found previously, in which each player assigns probability $\frac{1}{2}$ to *Head*.

Matching Pennies has no Nash equilibrium if the players are not allowed to randomize. If a game has a Nash equilibrium when randomization is not allowed, is it possible that it has additional equilibria when randomization is allowed? The following example shows that the answer is positive.

4.3.6 Example: BoS

Consider the two-player game with vNM preferences in which the players' preferences over deterministic action profiles are the same as in *BoS* and their prefer-

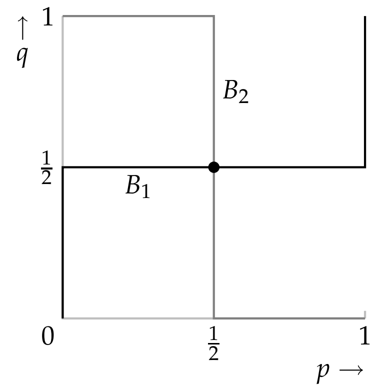


Figure 110.1 The players' best response functions in *Matching Pennies* (Figure 109.1) when randomization is allowed. The probabilities assigned by players 1 and 2 to *Head* are p and q respectively. The best response function of player 1 is black and that of player 2 is gray. The disk indicates the unique Nash equilibrium.

ences over lotteries are represented by the expected value of the payoff functions specified in Figure 110.2. What are the mixed strategy equilibria of this game?

	B	S
B	$2,1$	$0,0$
S	$0,0$	$1,2$

Figure 110.2 A version of the game *Bach or Stravinsky?* with vNM preferences.

First construct player 1's best response function. Suppose that player 2 assigns probability q to B. Then player 1's expected payoff to B is $2 \cdot q + 0 \cdot (1-q) = 2q$ and her expected payoff to S is $0 \cdot q + 1 \cdot (1-q) = 1-q$. Thus if $2q > 1-q$, or $q > \frac{1}{3}$, then her unique best response is B, while if $q < \frac{1}{3}$ then her unique best response is S. If $q = \frac{1}{3}$ then both B and S, and hence all player 1's mixed strategies, yield the same expected payoffs, so that every mixed strategy is a best response. In summary, player 1's best response function is

$$B_1(q) = \begin{cases} \{0\} & \text{if } q < \frac{1}{3} \\ \{p : 0 \leq p \leq 1\} & \text{if } q = \frac{1}{3} \\ \{1\} & \text{if } q > \frac{1}{3}. \end{cases}$$

Similarly we can find player 2's best response function. The best response functions of both players are shown in Figure 111.1.

We see that the game has three mixed strategy Nash equilibria, in which $(p,q) = (0,0)$, $(\frac{2}{3}, \frac{1}{3})$, and $(1,1)$. The first and third equilibria correspond to the Nash equilibria of the ordinal version of the game when the players were not allowed to randomize (Section 2.7.2). The second equilibrium is new. In this equilibrium each player chooses both B and S with positive probability (so that each of the four outcomes (B,B), (B,S), (S,B), and (S,S) occurs with positive probability).

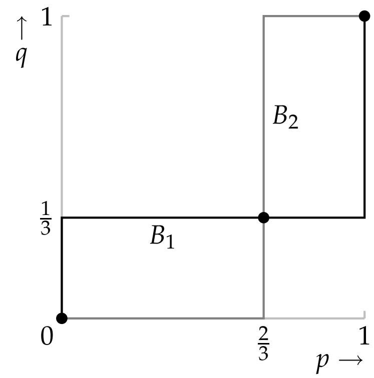


Figure 111.1 The players' best response functions in *BoS* (Figure 110.2) when randomization is allowed. The probabilities assigned by players 1 and 2 to B are p and q respectively. The best response function of player 1 is black and that of player 2 is gray. The disks indicate the Nash equilibria (two pure, one mixed).

? EXERCISE 111.1 (Mixed strategy equilibria of *Hawk–Dove*) Consider the two-player game with vNM preferences in which the players' preferences over deterministic action profiles are the same as in *Hawk–Dove* (Exercise 29.1) and their preferences over lotteries satisfy the following two conditions. Each player is indifferent between the outcome $(Passive, Passive)$ and the lottery that assigns probability $\frac{1}{2}$ to $(Aggressive, Aggressive)$ and probability $\frac{1}{2}$ to the outcome in which she is aggressive and the other player is passive, and between the outcome in which she is passive and the other player is aggressive and the lottery that assigns probability $\frac{2}{3}$ to the outcome $(Aggressive, Aggressive)$ and probability $\frac{1}{3}$ to the outcome $(Passive, Passive)$. Find payoffs whose expected values represent these preferences (take each player's payoff to $(Aggressive, Aggressive)$ to be 0 and each player's payoff to the outcome in which she is passive and the other player is aggressive to be 1). Find the mixed strategy Nash equilibrium of the resulting strategic game.

Both *Matching Pennies* and *BoS* have finitely many mixed strategy Nash equilibria: the players' best response functions intersect at a finite number of points (one for *Matching Pennies*, three for *BoS*). One of the games in the next exercise has a continuum of mixed strategy Nash equilibria because segments of the players' best response functions coincide.

? EXERCISE 111.2 (Games with mixed strategy equilibria) Find all the mixed strategy Nash equilibria of the strategic games in Figure 111.2.

	L	R
T	$6, 0$	$0, 6$
B	$3, 2$	$6, 0$

	L	R
T	$0, 1$	$0, 2$
B	$2, 2$	$0, 1$

Figure 111.2 Two strategic games with vNM preferences.

? EXERCISE 112.1 (A coordination game) Two people can perform a task if, and only if, they both exert effort. They are both better off if they both exert effort and perform the task than if neither exerts effort (and nothing is accomplished); the worst outcome for each person is that she exerts effort and the other does not (in which case again nothing is accomplished). Specifically, the players' preferences are represented by the expected value of the payoff functions in Figure 112.1, where c is a positive number less than 1 that can be interpreted as the cost of exerting effort. Find all the mixed strategy Nash equilibria of this game. How do the equilibria change as c increases? Explain the reasons for the changes.

	No effort	*Effort*
No effort	$0, 0$	$0, -c$
Effort	$-c, 0$	$1-c, 1-c$

Figure 112.1 The coordination game in Exercise 112.1.

?? EXERCISE 112.2 (Swimming with sharks) You and a friend are spending two days at the beach and would like to go for a swim. Each of you believes that with probability π the water is infested with sharks. If sharks are present, anyone who goes swimming today will surely be attacked. You each have preferences represented by the expected value of a payoff function that assigns $-c$ to being attacked by a shark, 0 to sitting on the beach, and 1 to a day's worth of undisturbed swimming. If one of you is attacked by sharks on the first day then you both deduce that a swimmer will surely be attacked the next day, and hence do not go swimming the next day. If no one is attacked on the first day then you both retain the belief that the probability of the water's being infested is π, and hence swim on the second day only if $-\pi c + 1 - \pi \geq 0$. Model this situation as a strategic game in which you and your friend each decides whether to go swimming on your first day at the beach. If, for example, you go swimming on the first day, you (and your friend, if she goes swimming) are attacked with probability π, in which case you stay out of the water on the second day; you (and your friend, if she goes swimming) swim undisturbed with probability $1 - \pi$, in which case you swim on the second day. Thus your expected payoff if you swim on the first day is $\pi(-c + 0) + (1 - \pi)(1 + 1) = -\pi c + 2(1 - \pi)$, independent of your friend's action. Find the mixed strategy Nash equilibria of the game (depending on c and π). Does the existence of a friend make it more or less likely that you decide to go swimming on the first day? (Penguins diving into water where seals may lurk are sometimes said to face the same dilemma, though Court (1996) argues that they do not.)

4.3.7 A useful characterization of mixed strategy Nash equilibrium

The method we have used so far to study the set of mixed strategy Nash equilibria of a game involves constructing the players' best response functions. Other methods are sometimes useful. I now present a characterization of mixed strategy Nash equilibrium that gives us an easy way to check whether a mixed strategy profile is an equilibrium, and is the basis of a procedure (described in Section 4.10) for finding all equilibria of a game.

The key point is an observation made in Section 4.3.4 for two-player two-action games: a player's expected payoff to a mixed strategy profile is a weighted average of her expected payoffs to her pure strategies, where the weight attached to each pure strategy is the probability assigned to that strategy by the player's mixed strategy. This property holds for any game (with any number of players) in which each player has finitely many actions. We can state it more precisely as follows.

> A player's expected payoff to the mixed strategy profile α is a weighted average of her expected payoffs to all mixed strategy profiles of the type (a_i, α_{-i}), where the weight attached to (a_i, α_{-i}) is the probability $\alpha_i(a_i)$ assigned to a_i by player i's mixed strategy α_i. (113.1)

Symbolically we have

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) U_i(a_i, \alpha_{-i}),$$

where A_i is player i's set of actions (pure strategies) and $U_i(a_i, \alpha_{-i})$ is her expected payoff when she uses the pure strategy that assigns probability 1 to a_i and every other player j uses her mixed strategy α_j. (See the end of Section 17.3 in the appendix on mathematics for an explanation of the $\sum$ notation.)

This property leads to a useful characterization of mixed strategy Nash equilibrium. Let α^* be a mixed strategy Nash equilibrium and denote by E_i^* player i's expected payoff in the equilibrium (i.e. $E_i^* = U_i(\alpha^*)$). Because α^* is an equilibrium, player i's expected payoff, given α_{-i}^*, to each of her pure strategies is at most E_i^*. Now, by (113.1), E_i^* is a weighted average of player i's expected payoffs to the pure strategies to which α_i^* assigns positive probability. Thus player i's expected payoffs to these pure strategies are all equal to E_i^*. (If any were smaller then the weighted average would be smaller.) We conclude that the expected payoff to each action to which α_i^* assigns positive probability is E_i^* and the expected payoff to every other action is at most E_i^*. Conversely, if these conditions are satisfied for every player i then α^* is a mixed strategy Nash equilibrium: the expected payoff to α_i^* is E_i^*, and the expected payoff to any other mixed strategy is at most E_i^*, because by (113.1) it is a weighted average of E_i^* and numbers that are at most E_i^*.

This argument establishes the following result.

■ PROPOSITION 113.2 (Characterization of mixed strategy Nash equilibrium of finite game) *A mixed strategy profile α^* in a strategic game with vNM preferences in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player i,*

- *the expected payoff, given α_{-i}^*, to every action to which α_i^* assigns positive probability is the same*

- *the expected payoff, given α^*_{-i}, to every action to which α^*_i assigns zero probability is at most the expected payoff to any action to which α^*_i assigns positive probability.*

Each player's expected payoff in an equilibrium is her expected payoff to any of her actions that she uses with positive probability.

The significance of this result is that it gives conditions for a mixed strategy Nash equilibrium in terms of each player's expected payoffs only to her *pure* strategies. For games in which each player has finitely many actions, it allows us easily to check whether a mixed strategy profile is an equilibrium. For example, in *BoS* (Section 4.3.6) the strategy pair $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$ is a mixed strategy Nash equilibrium because given player 2's strategy $(\frac{1}{3}, \frac{2}{3})$, player 1's expected payoffs to B and S are both equal to $\frac{2}{3}$, and given player 1's strategy $(\frac{2}{3}, \frac{1}{3})$, player 2's expected payoffs to B and S are both equal to $\frac{2}{3}$.

The next example is slightly more complicated.

◆ EXAMPLE 114.1 (Checking whether a mixed strategy profile is a mixed strategy Nash equilibrium) I claim that for the game in Figure 114.1 (in which the dots indicate irrelevant payoffs), the indicated pair of strategies, $(\frac{3}{4}, 0, \frac{1}{4})$ for player 1 and $(0, \frac{1}{3}, \frac{2}{3})$ for player 2, is a mixed strategy Nash equilibrium. To verify this claim, it suffices, by Proposition 113.2, to study each player's expected payoffs to her three pure strategies. For player 1 these payoffs are

T: $\frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$
M: $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$
B: $\frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 0 = \frac{5}{3}$.

Player 1's mixed strategy assigns positive probability to T and B and probability zero to M, so the two conditions in Proposition 113.2 are satisfied for player 1. The expected payoff to each of player 2's pure strategies is $\frac{5}{2}$ $(\frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 4 = \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 1 = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 7 = \frac{5}{2})$, so the two conditions in Proposition 113.2 are satisfied also for her.

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$

Figure 114.1 A partially-specified strategic game, illustrating a method of checking whether a mixed strategy profile is a mixed strategy Nash equilibrium. The dots indicate irrelevant payoffs.

Note that the expected payoff to player 2's action L, which she uses with probability zero, is the *same* as the expected payoff to her other two actions. This equality is consistent with Proposition 113.2, the second part of which requires only that the expected payoffs to actions used with probability zero be *no greater than* the expected payoffs to actions used with positive probability (not that they necessarily be less). Note also that the fact that player 2's expected payoff to L is the same as her expected payoffs to C and R does *not* imply that the game has a mixed strategy Nash equilibrium in which player 2 uses L with positive probability—it may, or it may not, depending on the unspecified payoffs.

? EXERCISE 115.1 (Choosing numbers) Players 1 and 2 each choose a positive integer up to K. If the players choose the same number then player 2 pays \$1 to player 1; otherwise no payment is made. Each player's preferences are represented by her expected monetary payoff.

a. Show that the game has a mixed strategy Nash equilibrium in which each player chooses each positive integer up to K with probability $1/K$.

b. (More difficult.) Show that the game has no other mixed strategy Nash equilibria. (Deduce from the fact that player 1 assigns positive probability to some action k that player 2 must do so; then look at the implied restriction on player 1's equilibrium strategy.)

? EXERCISE 115.2 (Silverman's game) Each of two players chooses a positive integer. If player i's integer is greater than player j's integer and less than three times this integer then player j pays \$1 to player i. If player i's integer is at least three times player j's integer then player i pays \$1 to player j. If the integers are equal, no payment is made. Each player's preferences are represented by her expected monetary payoff. Show that the game has no Nash equilibrium in pure strategies, and that the pair of mixed strategies in which each player chooses 1, 2, and 5 each with probability $\frac{1}{3}$ is a mixed strategy Nash equilibrium. (In fact, this pair of mixed strategies is the unique mixed strategy Nash equilibrium.)

?? EXERCISE 115.3 (Voter participation) Consider the game of voter participation in Exercise 32.2. Assume that $k \leq m$ and that each player's preferences are represented by the expectation of her payoffs given in Exercise 32.2. Show that there is a value of p between 0 and 1 such that the game has a mixed strategy Nash equilibrium in which every supporter of candidate A votes with probability p, k supporters of candidate B vote with certainty, and the remaining $m - k$ supporters of candidate B abstain. How do the probability p that a supporter of candidate A votes and the expected number of voters ("turnout") depend upon c? (Note that if every supporter of candidate A votes with probability p then the probability that exactly $k - 1$ of them vote is $kp^{k-1}(1 - p)$.)

?? EXERCISE 115.4 (Defending territory) General A is defending territory accessible by two mountain passes against an attack by general B. General A has three divisions at her disposal, and general B has two divisions. Each general allocates her divisions between the two passes. General A wins the battle at a pass if and only if she assigns at least as many divisions to the pass as does general B; she successfully defends her territory if and only if she wins the battle at both passes. Formulate this situation as a strategic game and find all its mixed strategy equilibria. (First argue that in every equilibrium B assigns probability zero to the action of allocating one division to each pass. Then argue that in any equilibrium she assigns probability $\frac{1}{2}$ to each of her other actions. Finally, find A's equilibrium strategies.) In an equilibrium do the generals concentrate all their forces at one pass, or spread them out?

An implication of Proposition 113.2 is that a nondegenerate mixed strategy equilibrium (a mixed strategy equilibrium that is not also a pure strategy equilibrium) is never a *strict* Nash equilibrium: every player whose mixed strategy assigns positive probability to more than one action is indifferent between her equilibrium mixed strategy and every action to which this mixed strategy assigns positive probability.

Any equilibrium that is not strict, whether in mixed strategies or not, has less appeal than a strict equilibrium because some (or all) of the players lack a positive incentive to choose their equilibrium strategies, given the other players' behavior. There is no reason for them *not* to choose their equilibrium strategies, but at the same time there is no reason for them not to choose another strategy that is equally good. Many pure strategy equilibria—especially in complex games—are also not strict, but among mixed strategy equilibria the problem is pervasive.

Given that in a mixed strategy equilibrium no player has a positive incentive to choose her equilibrium strategy, what determines how she randomizes in equilibrium? From the examples above we see that a player's equilibrium mixed strategy in a two-player game keeps the *other* player indifferent between a set of her actions, so that *she* is willing to randomize. In the mixed strategy equilibrium of *BoS*, for example, player 1 chooses B with probability $\frac{2}{3}$ so that player 2 is indifferent between B and S, and hence is willing to choose each with positive probability. Note, however, that the theory is *not* that the players consciously choose their strategies with this goal in mind! Rather, the conditions for equilibrium are designed to ensure that it is consistent with a steady state. In *BoS*, for example, if player 1 chooses B with probability $\frac{2}{3}$ and player 2 chooses B with probability $\frac{1}{3}$ then neither player has any reason to change her action. We have not yet studied how a steady state might come about, but have rather simply looked for strategy profiles consistent with steady states. In Section 4.9 I briefly discuss some theories of how a steady state might be reached.

4.3.8 Existence of equilibrium in finite games

Every game we have examined has at least one mixed strategy Nash equilibrium. In fact, every game in which each player has *finitely* many actions has at least one such equilibrium.

■ PROPOSITION 116.1 (Existence of mixed strategy Nash equilibrium in finite games) *Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy Nash equilibrium.*

This result is of no help in *finding* equilibria. But it is a useful fact to know: your quest for an equilibrium of a game in which each player has finitely many actions in principle may succeed! Note that the finiteness of the number of actions of each player is only *sufficient* for the existence of an equilibrium, not *necessary*; many games in which the players have infinitely many actions possess mixed strategy Nash equilibria. Note also that a player's mixed strategy in a mixed strategy Nash equilibrium may assign probability 1 to a single action; if every player's strategy does so then the equilibrium corresponds to a ("pure strategy") equilibrium of the associated game with ordinal preferences. Relatively advanced mathematical tools are needed to prove the result; see, for example, Osborne and Rubinstein (1994, 19–20).

4.4 Dominated actions

In a strategic game with ordinal preferences, one action of a player strictly dominates another action if it is superior, no matter what the other players do (see Definition 43.1). In a game with vNM preferences in which players may randomize, we extend this definition to allow an action to be dominated by a *mixed strategy*.

DEFINITION 117.1 (*Strict domination*) In a strategic game with vNM preferences, player i's mixed strategy α_i **strictly dominates** her action a_i' if

$$U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function whose expected value represents player i's preferences over lotteries and $U_i(\alpha_i, a_{-i})$ is player i's expected payoff under u_i when she uses the mixed strategy α_i and the actions chosen by the other players are given by a_{-i}.

As before, if a mixed strategy strictly dominates an action, we say that the action is **strictly dominated**. Figure 117.1 (in which only player 1's payoffs are given) shows that an action that is not strictly dominated by any pure strategy (i.e. is not strictly dominated in the sense of Definition 43.1) may be strictly dominated by a mixed strategy. The action T of player 1 is not strictly (or weakly) dominated by either M or B, but it is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B, because if player 2 chooses L then the mixed strategy yields player 1 the payoff of 2, whereas the action T yields her the payoff of 1, and if player 2 chooses R then the mixed strategy yields player 1 the payoff of $\frac{3}{2}$, whereas the action T yields her the payoff of 1.

	L	R
T	1	1
M	4	0
B	0	3

Figure 117.1 Player 1's payoffs in a strategic game with vNM preferences. The action T of player 1 is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B.

? EXERCISE 118.1 (Strictly dominated actions) In Figure 117.1, the mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B is not the only mixed strategy that strictly dominates T. Find all the mixed strategies that do so.

In a Nash equilibrium of a strategic game with ordinal preferences no player uses a strictly dominated action (Section 2.9.1). I now argue that the same is true of a mixed strategy Nash equilibrium of a strategic game with vNM preferences. In fact, I argue that a strictly dominated action is not a best response to any collection of mixed strategies of the other players. Suppose that player i's action a_i' is strictly dominated by her mixed strategy α_i, and the other players' mixed strategies are given by α_{-i}. Player i's expected payoff $U_i(\alpha_i, \alpha_{-i})$ when she uses the mixed strategy α_i and the other players use the mixed strategies α_{-i} is a weighted average of her payoffs $U_i(\alpha_i, a_{-i})$ as a_{-i} varies over all the collections of actions for the other players, with the weight on each a_{-i} equal to the probability with which it occurs when the other players' mixed strategies are α_{-i}. Player i's expected payoff when she uses the action a_i' and the other players use the mixed strategies α_{-i} is a similar weighted average; the weights are the same, but the terms take the form $u_i(a_i', a_{-i})$ rather than $U_i(\alpha_i, a_{-i})$. The fact that a_i' is strictly dominated by α_i means that $U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i})$ for *every* collection a_{-i} of the other players' actions. Hence player i's expected payoff when she uses the mixed strategy α_i exceeds her expected payoff when she uses the action a_i', given α_{-i}. Consequently,

> *a strictly dominated action is not used with positive probability in any mixed strategy equilibrium.*

Thus when looking for mixed strategy equilibria we can eliminate from consideration every strictly dominated action.

As before, we can define the notion of weak domination (see Definition 45.1).

▸ DEFINITION 118.2 (*Weak domination*) In a strategic game with vNM preferences, player i's mixed strategy α_i **weakly dominates** her action a_i' if

$$U_i(\alpha_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions}$$

and

$$U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function whose expected value represents player i's preferences over lotteries and $U_i(\alpha_i, a_{-i})$ is player i's expected payoff under u_i when she uses the mixed strategy α_i and the actions chosen by the other players are given by a_{-i}.

We saw that a weakly dominated action may be used in a Nash equilibrium (see Figure 46.1). Thus a weakly dominated action may be used with positive probability in a mixed strategy equilibrium, so that we *cannot* eliminate *weakly* dominated actions from consideration when finding mixed strategy equilibria!

? EXERCISE 119.1 (Eliminating dominated actions when finding equilibria) Find all the mixed strategy Nash equilibria of the game in Figure 119.1 by first eliminating any strictly dominated actions and then constructing the players' best response functions.

	L	M	R
T	$2,2$	$0,3$	$1,2$
B	$3,1$	$1,0$	$0,2$

Figure 119.1 The strategic game with vNM preferences in Exercise 119.1.

The fact that a player's strategy in a mixed strategy Nash equilibrium may be weakly dominated raises the question of whether a game necessarily has a mixed strategy Nash equilibrium in which no player's strategy is weakly dominated. The following result (which is not easy to prove) shows that the answer is affirmative for a finite game.

■ PROPOSITION 119.2 (Existence of mixed strategy Nash equilibrium with no weakly dominated strategies in finite games) *Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy Nash equilibrium in which no player's strategy is weakly dominated.*

4.5 Pure equilibria when randomization is allowed

The analysis in Section 4.3.6 shows that the mixed strategy Nash equilibria of *BoS* in which each player's strategy is pure correspond precisely to the Nash equilibria of the version of the game (considered in Section 2.3) in which the players are not allowed to randomize. The same is true for a general game: equilibria when the players are not allowed to randomize remain equilibria when they are allowed to randomize, and any pure equilibria that exist when they are allowed to randomize are equilibria when they are not allowed to randomize.

To establish this claim, let N be a set of players and let A_i, for each player i, be a set of actions. Consider the following two games.

G: the strategic game with ordinal preferences in which the set of players is N, the set of actions of each player i is A_i, and the preferences of each player i are represented by the payoff function u_i

G': the strategic game with vNM preferences in which the set of players is N, the set of actions of each player i is A_i, and the preferences of each player i are represented by the expected value of u_i.

First I argue that any Nash equilibrium of G corresponds to a mixed strategy Nash equilibrium (in which each player's strategy is pure) of G'. Let a^* be a Nash equilibrium of G, and for each player i let α_i^* be the mixed strategy that assigns probability 1 to a_i^*. Since a^* is a Nash equilibrium of G we know that in G' no player i has an action that yields her a payoff higher than does a_i^* when all the other players adhere to α_{-i}^*. Thus α^* satisfies the two conditions in Proposition 113.2, so that it is a mixed strategy equilibrium of G', establishing the following result.

■ PROPOSITION 120.1 (Pure strategy equilibria survive when randomization is allowed) *Let a^* be a Nash equilibrium of G and for each player i let α_i^* be the mixed strategy of player i that assigns probability one to the action a_i^*. Then α^* is a mixed strategy Nash equilibrium of G'.*

Next I argue that any mixed strategy Nash equilibrium of G' in which each player's strategy is pure corresponds to a Nash equilibrium of G. Let α^* be a mixed strategy Nash equilibrium of G' in which every player's mixed strategy is pure; for each player i, denote by a_i^* the action to which α_i assigns probability one. Then no mixed strategy of player i yields her a payoff higher than does α_i^* when the other players' mixed strategies are given by α_{-i}^*. Hence, in particular, no *pure* strategy of player i yields her a payoff higher than does α_i^*. Thus a^* is a Nash equilibrium of G. In words, if a pure strategy is optimal for a player when she is allowed to randomize then it remains optimal when she is prohibited from randomizing. (More generally, prohibiting a decision-maker from taking an action that is not optimal does not change the set of actions that are optimal.)

■ PROPOSITION 120.2 (Pure strategy equilibria survive when randomization is prohibited) *Let α^* be a mixed strategy Nash equilibrium of G' in which the mixed strategy of each player i assigns probability one to the single action a_i^*. Then a^* is a Nash equilibrium of G.*

4.6 Illustration: expert diagnosis

I seem to confront the following predicament all too frequently. Something about which I am relatively ill-informed (my car, my computer, my body) stops working properly. I consult an expert, who makes a diagnosis and recommends an action. I am not sure if the diagnosis is correct—the expert, after all, has an interest in selling her services. I have to decide whether to follow the expert's advice or to try to fix the problem myself, put up with it, or consult another expert.

4.6.1 Model

A simple model that captures the main features of this situation starts with the assumption that there are two types of problem, *major* and *minor*. Denote the fraction of problems that are major by r, and assume that $0 < r < 1$. An expert knows, on seeing a problem, whether it is *major* or *minor*; a consumer knows only the probability r. (The diagnosis is costly neither to the expert nor to the consumer.) An expert may recommend either a major or a minor repair (regardless of the true nature of the problem), and a consumer may either accept the expert's recommendation or seek another remedy. A major repair fixes both a major problem and a minor one.

Assume that a consumer always accepts an expert's advice to obtain a minor repair—there is no reason for her to doubt such a diagnosis—but may either accept or reject advice to obtain a major repair. Further assume that an expert always recommends a major repair for a major problem—a minor repair does not fix a major problem, so there is no point in an expert's recommending one for a major problem—but may recommend either repair for a minor problem. Suppose that an expert obtains the same profit $\pi > 0$ (per unit of time) from selling a minor repair to a consumer with a minor problem as she does from selling a major repair to a consumer with a major problem, but obtains the profit $\pi' > \pi$ from selling a major repair to a consumer with a minor problem. (The rationale is that in the last case the expert does not in fact perform a major repair, at least not in its entirety.) A consumer pays an expert E for a major repair and $I < E$ for a minor one; the cost she effectively bears if she chooses some other remedy is $E' > E$ if her problem is major and $I' > I$ if it is minor. (Perhaps she consults other experts before proceeding, or works on the problem herself, in either case spending valuable time.) I assume throughout that $E > I'$.

Under these assumptions we can model the situation as a strategic game in which the expert has two actions (recommend a minor repair for a minor problem; recommend a major repair for a minor problem), and the consumer has two actions (accept the recommendation of a major repair; reject the recommendation of a major repair). I name the actions as follows.

Expert *Honest* (recommend a minor repair for a minor problem and a major repair for a major problem) and *Dishonest* (recommend a major repair for both types of problem).

Consumer *Accept* (buy whatever repair the expert recommends) and *Reject* (buy a minor repair but seek some other remedy if a major repair is recommended)

Assume that each player's preferences are represented by her expected monetary payoff. Then the players' payoffs to the four action pairs are as follows; the strategic game is given in Figure 122.1.

(H, A): With probability r the consumer's problem is major, so she pays E, and with probability $1 - r$ it is minor, so she pays I. Thus her expected payoff is $-rE - (1 - r)I$. The expert's profit is π.

(D, A): The consumer's payoff is $-E$. The consumer's problem is major with probability r, yielding the expert π, and minor with probability $1 - r$, yielding the expert π', so that the expert's expected payoff is $r\pi + (1 - r)\pi'$.

(H, R): The consumer's cost is E' if her problem is major (in which case she rejects the expert's advice to get a major repair) and I if her problem is minor, so that her expected payoff is $-rE' - (1-r)I$. The expert obtains a payoff only if the consumer's problem is minor, in which case she gets π; thus her expected payoff is $(1-r)\pi$.

(D, R): The consumer never accepts the expert's advice, and thus obtains the expected payoff $-rE' - (1-r)I'$. The expert does not get any business, and thus obtains the payoff of 0.

		Consumer	
		Accept (q)	*Reject* $(1-q)$
Expert	*Honest* (p)	$\pi, -rE - (1-r)I$	$(1-r)\pi, -rE' - (1-r)I$
	Dishonest $(1-p)$	$r\pi + (1-r)\pi', -E$	$0, -rE' - (1-r)I'$

Figure 122.1 A game between an expert and a consumer with a problem.

4.6.2 *Nash equilibrium*

To find the Nash equilibria of the game we can construct the best response functions, as before. Denote by p the probability the expert assigns to H and by q the probability the consumer assigns to A.

Expert's best response function If $q = 0$ (i.e. the consumer chooses R with probability one) then the expert's best response is $p = 1$ (since $(1-r)\pi > 0$). If $q = 1$ (i.e. the consumer chooses A with probability one) then the expert's best response is $p = 0$ (since $\pi' > \pi$, so that $r\pi + (1-r)\pi' > \pi$). For what value of q is the expert indifferent between H and D? Given q, the expert's expected payoff to H is $q\pi + (1-q)(1-r)\pi$ and her expected payoff to D is $q[r\pi + (1-r)\pi']$, so she is indifferent between the two actions if

$$q\pi + (1-q)(1-r)\pi = q[r\pi + (1-r)\pi'].$$

Upon simplification, this yields $q = \pi/\pi'$. We conclude that the expert's best response function takes the form shown in both panels of Figure 123.1.

Consumer's best response function If $p = 0$ (i.e. the expert chooses D with probability one) then the consumer's best response depends on the relative sizes of E and $rE' + (1-r)I'$. If $E < rE' + (1-r)I'$ then the consumer's best response is $q = 1$, whereas if $E > rE' + (1-r)I'$ then her best response is $q = 0$; if $E = rE' + (1-r)I'$ then she is indifferent between R and A.

If $p = 1$ (i.e. the expert chooses H with probability one) then the consumer's best response is $q = 1$ (given $E < E'$).

We conclude that if $E < rE' + (1-r)I'$ then the consumer's best response to every value of p is $q = 1$, as shown in the left panel of Figure 123.1. If $E > rE' + (1-r)I'$ then the consumer is indifferent between A and R if

$$p[rE + (1-r)I] + (1-p)E = p[rE' + (1-r)I] + (1-p)[rE' + (1-r)I'],$$

which reduces to

$$p = \frac{E - [rE' + (1-r)I']}{(1-r)(E-I')}.$$

In this case the consumer's best response function takes the form shown in the right panel of Figure 123.1.

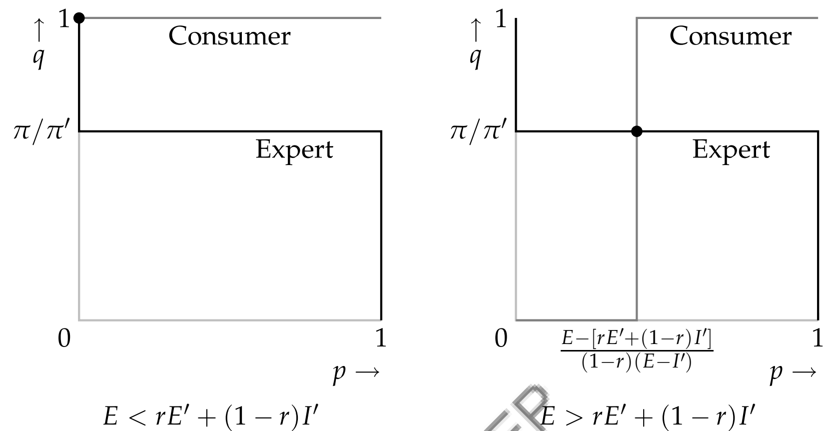


Figure 123.1 The players' best response functions in the game of expert diagnosis. The probability assigned by the expert to H is p and the probability assigned by the consumer to A is q.

Equilibrium Given the best response functions, if $E < rE' + (1-r)I'$ then the pair of pure strategies (D, A) is the unique Nash equilibrium. The condition $E < rE' + (1-r)I'$ says that the cost of a major repair by an expert is less than the *expected* cost of an alternative remedy; the only equilibrium yields the dismal outcome for the consumer in which the expert is always dishonest and the consumer always accepts her advice.

If $E > rE' + (1-r)I'$ then the unique equilibrium of the game is in mixed strategies, with $(p, q) = (p^*, q^*)$, where

$$p^* = \frac{E - [rE' + (1-r)I']}{(1-r)(E-I')} \quad \text{and} \quad q^* = \frac{\pi}{\pi'}.$$

In this equilibrium the expert is sometimes honest, sometimes dishonest, and the consumer sometimes accepts her advice to obtain a major repair, and sometimes ignores such advice.

As discussed in the introduction to the chapter, a mixed strategy equilibrium can be given more than one interpretation as a steady state. In the game we are studying, and the games studied earlier in the chapter, I have focused on the interpretation in which each player chooses her action randomly, with probabilities given by her equilibrium mixed strategy, every time she plays the game. In the game of expert diagnosis a different interpretation fits well: among the population of individuals who may play the role of each given player, every individual chooses the same action whenever she plays the game, but different individuals choose different actions; the fraction of individuals who choose each action is equal to the equilibrium probability that that action is used in a mixed strategy equilibrium. Specifically, if $E > rE' + (1-r)I'$ then the fraction p^* of experts is honest (recommending minor repairs for minor problems) and the fraction $1 - p^*$ is dishonest (recommending major repairs for minor problems), while the fraction q^* of consumers is credulous (accepting any recommendation) and the fraction $1 - q^*$ is wary (accepting only a recommendation of a minor repair). Honest and dishonest experts obtain the same expected payoff, as do credulous and wary consumers.

? EXERCISE 124.1 (Equilibrium in the expert diagnosis game) Find the set of mixed strategy Nash equilibria of the game when $E = rE' + (1-r)I'$.

4.6.3 Properties of the mixed strategy Nash equilibrium

Studying how the equilibrium is affected by changes in the parameters of the model helps us understand the nature of the strategic interaction between the players. I consider the effects of three changes.

Suppose that major problems become less common (cars become more reliable, more resources are devoted to preventive healthcare). If we rearrange the expression for p^* to

$$p^* = 1 - \frac{r(E' - E)}{(1-r)(E - I')},$$

we see that p^* increases as r decreases (the numerator of the fraction decreases and the denominator increases). Thus in a mixed strategy equilibrium, the experts are more honest when major problems are less common. Intuitively, if a major problem is less likely then a consumer has less to lose from ignoring an expert's advice, so that the probability of an expert's being honest has to rise in order that her advice be heeded. The value of q^* is not affected by the change in r: the probability of a consumer's accepting an expert's advice remains the same when major problems become less common. *Given* the expert's behavior, a decrease in r increases the consumer's payoff to rejecting the expert's advice more than it increases her payoff to accepting this advice, so that she prefers to reject the advice. But this partial analysis is misleading: in the equilibrium that exists after r decreases, the consumer is exactly as likely to accept the expert's advice as she was before the change.

Now suppose that major repairs become less expensive relative to minor ones (technological advances reduce the cost of complex equipment). We see that p^* decreases as E decreases (with E' and I' constant): when major repairs are less costly, experts are less honest. As major repairs become less costly, a consumer has more potentially to lose from ignoring an expert's advice, so that she heeds the advice even if experts are less likely to be honest.

Finally, suppose that the profit π' from an expert's fixing a minor problem with an alleged major repair falls (the government requires experts to return replaced parts to the consumer, making it more difficult for an expert to fraudulently claim to have performed a major repair). Then q^* increases—consumers become less wary. Experts have less to gain from acting dishonestly, so that consumers can be more confident of their advice.

EXERCISE 125.1 (Incompetent experts) Consider a (realistic?) variant of the model, in which the experts are not entirely competent. Assume that each expert always correctly recognizes a major problem but correctly recognizes a minor problem with probability $s < 1$: with probability $1 - s$ she mistakenly thinks that a minor problem is major, and, if the consumer accepts her advice, performs a major repair and obtains the profit π. Maintain the assumption that each consumer believes (correctly) that the probability her problem is major is r. As before, a consumer who does not give the job of fixing her problem to an expert bears the cost E' if it is major and I' if it is minor.

Suppose, for example, that an expert is honest and a consumer rejects advice to obtain a major repair. With probability r the consumer's problem is major, so that the expert recommends a major repair, which the consumer rejects; the consumer bears the cost E'. With probability $1 - r$ the consumer's problem is minor. In this case with probability s the expert correctly diagnoses it as minor, and the consumer accepts her advice and pays I; with probability $1 - s$ the expert diagnoses it as major, and the consumer rejects her advice and bears the cost I'. Thus the consumer's expected payoff in this case is $-rE' - (1 - r)[sI + (1 - s)I']$.

Construct the payoffs for every pair of actions and find the mixed strategy equilibrium in the case $E > rE' + (1 - r)I'$. Does incompetence breed dishonesty? More wary consumers?

EXERCISE 125.2 (Choosing a seller) Each of two sellers has available one indivisible unit of a good. Seller 1 posts the price p_1 and seller 2 posts the price p_2. Each of two buyers would like to obtain one unit of the good; they simultaneously decide which seller to approach. If both buyers approach the same seller, each trades with probability $\frac{1}{2}$; the disappointed buyer does not subsequently have the option to trade with the other seller. (This assumption models the risk faced by a buyer that a good is sold out when she patronizes a seller with a low price.) Each buyer's preferences are represented by the expected value of a payoff function that assigns the payoff 0 to not trading and the payoff $1 - p$ to purchasing one unit of the good at the price p. (Neither buyer values more than one unit.) For any pair (p_1, p_2) of prices with $0 \leq p_i \leq 1$ for $i = 1, 2$, find the Nash equilibria (in pure and in mixed strategies) of the strategic game that models this situation. (There are three main cases: $p_2 < 2p_1 - 1$, $2p_1 - 1 < p_2 < \frac{1}{2}(1 + p_1)$, and $p_2 > \frac{1}{2}(1 + p_1)$.)

4.7 Equilibrium in a single population

In Section 2.10 I discussed deterministic steady states in situations in which the members of a single population interact. I now discuss stochastic steady states in such situations.

First extend the definitions of a symmetric strategic game and a symmetric Nash equilibrium (Definitions 49.3 and 50.2) to a game with vNM preferences. Recall that a two-player strategic game with ordinal preferences is symmetric if each player has the same set of actions and each player's evaluation of an outcome depends only on her action and that of her opponent, not on whether she is player 1 or player 2. A symmetric game with vNM preferences satisfies the same conditions; its definition differs from Definition 49.3 only because a player's evaluation of an outcome is given by her expected payoff rather than her ordinal preferences.

▶ DEFINITION 126.1 (*Symmetric two-player strategic game with vNM preferences*) A two-player strategic game with vNM preferences is **symmetric** if the players' sets of actions are the same and the players' preferences are represented by the expected values of payoff functions u_1 and u_2 for which $u_1(a_1, a_2) = u_2(a_2, a_1)$ for every action pair (a_1, a_2).

A Nash equilibrium of a strategic game with ordinal preferences in which every player's set of actions is the same is symmetric if all players take the same action. This notion of equilibrium extends naturally to strategic games with vNM preferences. (As before, it does not depend on the game's having only two players, so I define it for a game with any number of players.)

▶ DEFINITION 126.2 (*Symmetric mixed strategy Nash equilibrium*) A profile α^* of mixed strategies in a strategic game with vNM preferences in which each player has the same set of actions is a **symmetric mixed strategy Nash equilibrium** if it is a mixed strategy Nash equilibrium and α_i^* is the same for every player i.

Now consider again the game of approaching pedestrians (Figure 51.1, reproduced in Figure 126.1), interpreting the payoff numbers as Bernoulli payoffs whose expected values represent the players' preferences over lotteries. We found that this game has two deterministic steady states, corresponding to the two symmetric Nash equilibria in pure strategies, $(Left, Left)$ and $(Right, Right)$. The game also has a symmetric mixed strategy Nash equilibrium, in which each player assigns probability $\frac{1}{2}$ to *Left* and probability $\frac{1}{2}$ to *Right*. This equilibrium corresponds to a steady state in which half of all encounters result in collisions! (With probability $\frac{1}{4}$ player 1 chooses *Left* and player 2 chooses *Right*, and with probability $\frac{1}{4}$ player 1 chooses *Right* and player 2 chooses *Left*.)

	Left	*Right*
Left	1,1	0,0
Right	0,0	1,1

Figure 126.1 Approaching pedestrians.

In this example not only is the game symmetric, but the players' interests coincide. The game in Figure 127.1 is symmetric, but the players prefer to take different actions rather than the same actions. This game has no pure symmetric equilibrium, but has a symmetric mixed strategy equilibrium, in which each player chooses each action with probability $\frac{1}{2}$.

	X	Y
X	$0,0$	$1,1$
Y	$1,1$	$0,0$

Figure 127.1 A symmetric game.

These two examples show that a symmetric game may have no symmetric *pure* strategy equilibrium. But both games have a symmetric mixed strategy Nash equilibrium, as does any symmetric game in which each player has finitely many actions, by the following result. (Relatively advanced mathematical tools are needed to prove the result.)

■ PROPOSITION 127.1 (Existence of symmetric mixed strategy Nash equilibrium in symmetric finite games) *Every strategic game with vNM preferences in which each player has the same finite set of actions has a symmetric mixed strategy Nash equilibrium.*

? EXERCISE 127.2 (Approaching cars) Members of a single population of car drivers are randomly matched in pairs when they simultaneously approach intersections from different directions. In each interaction, each driver can either stop or continue. The drivers' preferences are represented by the expected value of the payoff functions given in Figure 127.2; the parameter ϵ, with $0 < \epsilon < 1$, reflects the fact that each driver dislikes being the only one to stop. Find the symmetric Nash equilibrium (equilibria?) of the game (find both the equilibrium strategies and the equilibrium payoffs).

	Stop	*Continue*
Stop	$1,1$	$1-\epsilon,2$
Continue	$2,1-\epsilon$	$0,0$

Figure 127.2 The game in Exercise 127.2.

Now suppose that drivers are (re)educated to feel guilty about choosing *Continue*, with the consequence that their payoffs when choosing *Continue* fall by $\delta > 0$. That is, the entry $(2, 1-\epsilon)$ in Figure 127.2 is replaced by $(2-\delta, 1-\epsilon)$, the entry $(1-\epsilon, 2)$ is replaced by $(1-\epsilon, 2-\delta)$, and the entry $(0,0)$ is replaced by $(-\delta, -\delta)$. Show that all drivers are *better off* in the symmetric equilibrium of this game than they are in the symmetric equilibrium of the original game. Why is the society better off if everyone feels guilty about being aggressive? (The equilibrium of this game, like that of the equilibrium of the game of expert diagnosis in Section 4.6, may attractively be interpreted as representing a steady state in which some members of the population always choose one action, and other members always choose the other action.)

? EXERCISE 128.1 (Bargaining) Pairs of players from a single population bargain over the division of a pie of size 10. The members of a pair simultaneously make demands; the possible demands are the nonnegative *even* integers up to 10. If the demands sum to 10 then each player receives her demand; if the demands sum to less than 10 then each player receives her demand plus half of the pie that remains after both demands have been satisfied; if the demands sum to more than 10 then neither player receives any payoff. Find all the symmetric mixed strategy Nash equilibria in which each player assigns positive probability to at most two demands. (Many situations in which each player assigns positive probability to two actions, say a' and a'', can be ruled out as equilibria because when one player uses such a strategy, some action a''' yields the other player a payoff higher than does a' and/or a''.)

4.8 Illustration: reporting a crime

A crime is observed by a group of n people. Each person would like the police to be informed, but prefers that someone else make the phone call. Specifically, suppose that each person attaches the value v to the police being informed and bears the cost c if she makes the phone call, where $v > c > 0$. Then the situation is modeled by the following strategic game with vNM preferences.

Players The n people.

Actions Each player's set of actions is $\{Call, Don't\ call\}$.

Preferences Each player's preferences are represented by the expected value of a payoff function that assigns 0 to the profile in which no one calls, $v - c$ to any profile in which she calls, and v to any profile in which at least one person calls, but she does not.

This game is a variant of the one in Exercise 31.1, with $k = 1$. It has n pure Nash equilibria, in each of which exactly one person calls. (If that person switches to not calling, her payoff falls from $v - c$ to 0; if any other person switches to calling, her payoff falls from v to $v - c$.) If the members of the group differ in some respect, then these asymmetric equilibria may be compelling as steady states. For example, the social norm in which the oldest person in the group makes the phone call is stable.

If the members of the group either do not differ significantly or are not aware of any differences among themselves—if they are drawn from a single homogeneous population—then there is no way for them to coordinate, and a symmetric equilibrium, in which every player uses the same strategy, is more compelling.

The game has no symmetric pure Nash equilibrium. (If everyone calls, then any person is better off switching to not calling. If no one calls, then any person is better off switching to calling.)

However, it has a symmetric mixed strategy equilibrium in which each person calls with positive probability less than one. In any such equilibrium, each person's expected payoff to calling is equal to her expected payoff to not calling. Each person's payoff to calling is $v - c$, and her payoff to not calling is 0 if no one else calls and v if at least one other person calls, so the equilibrium condition is

$$v - c = 0 \cdot \Pr\{\text{no one else calls}\} + v \cdot \Pr\{\text{at least one other person calls}\},$$

or

$$v - c = v \cdot (1 - \Pr\{\text{no one else calls}\}),$$

or

$$c/v = \Pr\{\text{no one else calls}\}. \tag{129.1}$$

Denote by p the probability with which each person calls. The probability that no one else calls is the probability that every one of the other $n-1$ people does not call, namely $(1-p)^{n-1}$. Thus the equilibrium condition is $c/v = (1-p)^{n-1}$, or

$$p = 1 - (c/v)^{1/(n-1)}.$$

This number p is between 0 and 1, so we conclude that the game has a unique symmetric mixed strategy equilibrium, in which each person calls with probability $1 - (c/v)^{1/(n-1)}$. That is, there is a steady state in which whenever a person is in a group of n people facing the situation modeled by the game, she calls with probability $1 - (c/v)^{1/(n-1)}$.

How does this equilibrium change as the size of the group increases? We see that as n increases, the probability p that any given person calls decreases. (As n increases, $1/(n-1)$ decreases, so that $(c/v)^{1/(n-1)}$ increases.) What about the probability that *at least* one person calls? Fix any player i. Then the event "no one calls" is the same as the event "i does not call and no one *other than i* calls". Thus

$$\Pr\{\text{no one calls}\} = \Pr\{i \text{ does not call}\}\Pr\{\text{no one else calls}\}. \tag{129.2}$$

Now, the probability that any given person calls decreases as n increases, or equivalently the probability that she does not call increases as n increases. Further, from the equilibrium condition (129.1), $\Pr\{\text{no one else calls}\}$ is equal to c/v, *independent of n*. We conclude that the probability that no one calls *increases* as n increases. That is, the larger the group, the *less* likely the police are informed of the crime!

The condition defining a mixed strategy equilibrium is responsible for this result. For any given person to be indifferent between calling and not calling this condition requires that the probability that no one else calls be independent of the size of the group. Thus each person's probability of not calling is larger in a larger group, and hence, by the laws of probability reflected in (129.2), the probability that no one calls is larger in a larger group.

The result that the larger the group, the less likely any given person calls is not surprising. The result that the larger the group, the less likely at least one person calls is a more subtle implication of the notion of equilibrium. In a larger group no individual is any less concerned that the police should be called, but in a steady state the behavior of the group drives down the chance that the police are notified of the crime.

? EXERCISE 130.1 (Contributing to a public good) Consider an extension of the analysis above to the game in Exercise 31.1 for $k \geq 2$. (In this case a player may contribute even though the good is not provided; the player's payoff in this case is $-c$.) Denote by $Q_{n-1,m}(p)$ the probability that exactly m of a group of $n-1$ players contribute when each player contributes with probability p. What condition must be satisfied by $Q_{n-1,k-1}(p)$ in a symmetric mixed strategy equilibrium (in which each player contributes with the same probability)? (When does a player's contribution make a difference to the outcome?) For the case $v = 1$, $n = 4$, $k = 2$, and $c = \frac{3}{8}$ find the equilibria explicitly. (You need to use the fact that $Q_{3,1}(p) = 3p(1-p)^2$, and do a bit of algebra.)

REPORTING A CRIME: SOCIAL PSYCHOLOGY AND GAME THEORY

Thirty-eight people witnessed the brutal murder of Catherine ("Kitty") Genovese over a period of half an hour in New York City in March 1964. During this period, none of them significantly responded to her screams for help; none even called the police. Journalists, psychiatrists, sociologists, and others subsequently struggled to understand the witnesses' inaction. Some ascribed it to apathy engendered by life in a large city: "Indifference to one's neighbor and his troubles is a conditioned reflex of life in New York as it is in other big cities" (Rosenthal 1964, 81–82).

The event particularly interested social psychologists. It led them to try to understand the circumstances under which a bystander would help someone in trouble. Experiments quickly suggested that, contrary to the popular theory, people—even those living in large cities—are not in general apathetic to others' plights. An experimental subject who is the lone witness of a person in distress is very likely to try to help. But as the size of the group of witnesses increases, there is a decline not only in the probability that any given one of them offers assistance, but also in the probability that at least one of them offers assistance. Social psychologists hypothesize that three factors explain these experimental findings. First, "diffusion of responsibility": the larger the group, the lower the psychological cost of not helping. Second, "audience inhibition": the larger the group, the greater the embarrassment suffered by a helper in case the event turns out to be one in which help is inappropriate (because, for example, it is not in fact an emergency). Third, "social influence": a person infers the appropriateness of helping from others' behavior, so that in a large group everyone else's lack of intervention leads any given person to think intervention is less likely to be appropriate.

In terms of the model in Section 4.8, these three factors raise the expected cost and/or reduce the expected benefit of a person's intervening. They all seem plausible. However, they are not needed to explain the phenomenon: our game-theoretic analysis shows that even if the cost and benefit are *independent* of group size, a decrease in the probability that at least one person intervenes is an implication of equilibrium. This game-theoretic analysis has an advantage over the socio-psychological one: it derives the conclusion from the same principles that underlie all the other models studied so far (oligopoly, auctions, voting, and elections, for example), rather than positing special features of the specific environment in which a group of bystanders may come to the aid of a person in distress.

The critical element missing from the socio-psychological analysis is the notion of an *equilibrium*. Whether any given person intervenes depends on the probability she assigns to some other person's intervening. In an equilibrium each person must be indifferent between intervening and not intervening, and as we have seen this condition leads inexorably to the conclusion that an increase in group size reduces the probability that at least one person intervenes.

4.9 The formation of players' beliefs

In a Nash equilibrium, each player chooses a strategy that maximizes her expected payoff, *knowing* the other players' strategies. So far we have not considered how players may acquire such information. Informally, the idea underlying the previous analysis is that the players have learned each other's strategies from their experience playing the game. In the idealized situation to which the analysis corresponds, for each player in the game there is a large population of individuals who may take the role of that player; in any play of the game, one participant is drawn randomly from each population. In this situation, a new individual who joins a population that is in a steady state (i.e. is using a Nash equilibrium strategy profile) can learn the other players' strategies by observing their actions over many plays of the game. As long as the turnover in players is small enough, existing players' encounters with neophytes (who may use nonequilibrium strategies) will be sufficiently rare that their beliefs about the steady state will not be disturbed, so that a new player's problem is simply to learn the other players' actions.

This analysis leaves open the question of what might happen if new players simultaneously join more than one population in sufficient numbers that they have a significant chance of facing opponents who are themselves new. In particular, can we expect a steady state to be reached when no one has experience playing the game?

4.9.1 Eliminating dominated actions

In some games the players may reasonably be expected to choose their Nash equilibrium actions from an introspective analysis of the game. At an extreme, each player's best action may be independent of the other players' actions, as in the *Prisoner's Dilemma* (Example 12.1). In such a game no player needs to worry about the other players' actions. In a less extreme case, some player's best action may depend on the other players' actions, but the actions the other players will choose may be clear because each of these players has an action that strictly dominates all others. For example, in the game in Figure 132.1, player 2's action R strictly dominates L, so that no matter what player 1 thinks player 1 will do, she should choose R. Consequently, player 1, who can deduce by this argument that player 2 will choose R, may reason that she should choose B. That is, even inexperienced players may be led to the unique Nash equilibrium (B, R) in this game.

	L	R
T	$1, 2$	$0, 3$
B	$0, 0$	$1, 1$

Figure 132.1 A game in which player 2 has a strictly dominant action whereas player 1 does not.

This line of argument may be extended. For example, in the game in Figure 132.2 player 1's action T is strictly dominated, so player 1 may reason that player 2 will deduce that player 1 will not choose T. Consequently player 1 may deduce that player 2 will choose R, and hence herself may choose B rather than M.

	L	R
T	$0, 2$	$0, 0$
M	$2, 1$	$1, 2$
B	$1, 1$	$2, 2$

Figure 132.2 A game in which player 1 may reason that she should choose B because player 2 will reason that player 1 will not choose T, so that player 2 will choose R.

The set of action profiles that remain at the end of such a reasoning process contains all Nash equilibria; for many games (unlike these examples) it contains many other action profiles. In fact, in many games it does not eliminate any action profile, because no player has a strictly dominated action. Nevertheless, in some classes of games the process is powerful; its logical consequences are explored in Chapter 12.

4.9.2 Learning

Another approach to the question of how a steady state might be reached assumes that each player starts with an unexplained "prior" belief about the other players' actions, and changes these beliefs—"learns"—in response to information she receives. She may learn, for example, from observing the fortunes of other players like herself, from discussing the game with such players, or from her own experience playing the game. Here I briefly discuss two theories in which the same set of participants repeatedly play a game, each participant changing her beliefs about the others' strategies in response to her observations of their actions.

Best response dynamics A particularly simple theory assumes that in each period after the first, each player believes that the other players will choose the actions they chose in the previous period. In the first period, each player chooses a best response to an arbitrary deterministic belief about the other players' actions. In every subsequent period, each player chooses a best response to the other players' actions in the *previous* period. This process is known as *best response dynamics*. An action profile that remains the same from period to period is a pure Nash equilibrium of the game. Further, a pure Nash equilibrium in which each player's action is her only best response to the other players' actions is an action profile that remains the same from period to period.

In some games the sequence of action profiles generated best response dynamics converges to a pure Nash equilibrium, regardless of the players' initial beliefs. The example of Cournot's duopoly game studied in Section 3.1.3 is such a game. Looking at the best response functions in Figure 56.2, you can convince yourself that from arbitrary initial actions, the players' actions approach the Nash equilibrium (q_1^*, q_2^*).

? EXERCISE 133.1 (Best response dynamics in Cournot's duopoly game) Find the sequence of pairs of outputs chosen by the firms in Cournot's duopoly game under the assumptions of Section 3.1.3 if they both initially choose 0. (If you know how to solve a first-order difference equation, find a formula for the outputs in each period; if not, find the outputs in the first few periods.)

? EXERCISE 133.2 (Best response dynamics in Bertrand's duopoly game) Consider Bertrand's duopoly game in which the set of possible prices is discrete, under the assumptions of Exercise 65.2. Does the sequences of prices under best response dynamics converge to a Nash equilibrium when both prices initially exceed $c + 1$? What happens when both prices are initially equal to c?

For other games there are initial beliefs for which the sequence of action profiles generated by the process does not converge. In *BoS* (Example 16.2), for example, if player 1 initially believes that player 2 will choose *Stravinsky* and player 2 initially believes that player 1 will choose *Bach*, then the players' choices will subsequently alternate indefinitely between the action pairs (*Bach*, *Stravinsky*) and (*Stravinsky*, *Bach*). This example highlights the limited extent to which a player is assumed to reason in the model, which does not consider the possibility that she cottons on to the fact that her opponent's action is always a best response to her own previous action.

Fictitious play Under best response dynamics, the players' beliefs are continually revealed to be incorrect unless the starting point is a Nash equilibrium: the players' actions change from period to period. Further, each player believes that every other player is using a pure strategy: a player's belief does not admit the possibility that her opponents' actions are realizations of mixed strategies.

Another theory, known as *fictitious play*, assumes that players consider actions in all the previous periods when forming a belief about their opponents' strategies. They treat these actions as realizations of mixed strategies. Consider a two-player game. Each player begins with an arbitrary probabilistic belief about the other player's action. In the first play of the game she chooses a best response to this belief and observes the other player's action, say A. She then changes her belief to one that assigns probability one to A; in the second period, she chooses a best response to this belief and observes the other player's action, say B. She then changes her belief to one that assigns probability $\frac{1}{2}$ to both A and B, and chooses a best response to this belief. She continues to change her belief each period; in any period she adopts the belief that her opponent is using a mixed strategy in which the probability of each action is proportional to the frequency with which her opponent chose that action in the previous periods. (If, for example, in the first six periods player 2 chooses A twice, B three times, and C once, player 1's belief in period 7 assigns probability $\frac{1}{3}$ to A, probability $\frac{1}{2}$ to B, and probability $\frac{1}{6}$ to C.)

In the game *Matching Pennies* (Example 17.1), reproduced in Figure 134.1, this process works as follows. Suppose that player 1 begins with the belief that player 2's action will be *Tail*, and player 2 begins with the belief that player 1's action will be *Head*. Then in period 1 both players choose *Tail*. Thus in period 2 both players believe that their opponent will choose *Tail*, so that player 1 chooses *Tail* and player 2 chooses *Head*. Consequently in period 3, player 1's belief is that player 2 will choose *Head* with probability $\frac{1}{2}$ and *Tail* with probability $\frac{1}{2}$, and player 2's belief is that player 1 will definitely choose *Tail*. Thus in period 3, both *Head* and *Tail* are best responses of player 1 to her belief, so that she may take either action; the unique best response of player 2 is *Head*. The process continues similarly in subsequent periods.

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Figure 134.1 *Matching Pennies.*

In two-player games like *Matching Pennies*, in which the players' interests are directly opposed, and in any two-player game in which each player has two actions, this process converges to a mixed strategy Nash equilibrium from any initial beliefs. That is, after a sufficiently large number of periods, the frequencies with which each player chooses her actions are close to the frequencies induced by her mixed strategy in the Nash equilibrium. For other games there are initial beliefs for which the process does not converge. (The simplest example is too complicated to present compactly.)

People involved in an interaction that we model as a game may form beliefs about their opponents' strategies from an analysis of the structure of the players' payoffs, from their observations of their opponents' actions, and from information they obtain from other people involved in similar interactions. The models I have outlined allow us to explore the logical implications of two ways in which players may draw inferences from their opponents' actions. Models that assume the players to be more sophisticated may give more insights into the types of situation in which a Nash equilibrium is likely to be attained; this topic is an active area of current research.

4.10 Extension: Finding all mixed strategy Nash equilibria

We can find all the mixed strategy Nash equilibria of a two-player game in which each player has two actions by constructing the players' best response functions, as we have seen. In more complicated games, this method is usually not practical.

The following systematic method of finding all mixed strategy Nash equilibria of a game is suggested by the characterization of an equilibrium in Proposition 113.2.

- For each player i, choose a subset S_i of her set A_i of actions.
- Check whether there exists a mixed strategy profile α such that (*i*) the set of actions to which each strategy α_i assigns positive probability is S_i and (*ii*) α satisfies the conditions in Proposition 113.2.
- Repeat the analysis for every collection of subsets of the players' sets of actions.

The following example illustrates this method for a two-player game in which each player has two actions.

◆ EXAMPLE 135.1 (Finding all mixed strategy equilibria of a two-player game in which each player has two actions) Consider a two-player game in which each player has two actions. Denote the actions and payoffs as in Figure 136.1. Each player's set of actions has three nonempty subsets: two each consisting of a single action, and one consisting of both actions. Thus there are nine (3×3) pairs of subsets of the players' action sets. For each pair (S_1, S_2), we check if there is a pair (α_1, α_2) of mixed strategies such that each strategy α_i assigns positive probability only to actions in S_i and the conditions in Proposition 113.2 are satisfied.

- Checking the four pairs of subsets in which each player's subset consists of a single action amounts to checking whether any of the four pairs of actions is a pure strategy equilibrium. (For each player, the first condition in Proposition 113.2 is automatically satisfied, because there is only one action in each subset.)
- Consider the pair of subsets $\{T, B\}$ for player 1 and $\{L\}$ for player 2. The second condition in Proposition 113.2 is automatically satisfied for player 1, who has no actions to which she assigns probability 0, and the first condition is automatically satisfied for player 2, because she assigns positive probability to only one action. Thus for there to be a mixed strategy equilibrium in which player 1's probability of using T is p we need $u_{11} = u_{21}$ (player 1's payoffs to her two actions must be equal) and

$$pv_{11} + (1 - p)v_{21} \geq pv_{12} + (1 - p)v_{22}$$

(L must be at least as good as R, given player 1's mixed strategy). If $u_{11} \neq u_{21}$, or if there is no probability p satisfying the inequality, then there is no equilibrium of this type. A similar argument applies to the three other pairs of subsets in which one player's subset consists of both her actions and the other player's subset consists of a single action.

- To check whether there is a mixed strategy equilibrium in which the subsets are $\{T, B\}$ for player 1 and $\{L, R\}$ for player 2, we need to find a pair of mixed strategies that satisfies the first condition in Proposition 113.2 (the second condition is automatically satisfied because both players assign positive probability to both their actions). That is, we need to find probabilities p and q (if any such exist) for which

$$qu_{11} + (1-q)u_{12} = qu_{21} + (1-q)u_{22} \quad \text{and} \quad pv_{11} + (1-p)v_{21} = pv_{12} + (1-p)v_{22}.$$

	L	R
T	u_{11}, v_{11}	u_{12}, v_{12}
B	u_{21}, v_{21}	u_{22}, v_{22}

Figure 136.1 A two-player strategic game.

For example, in *BoS* we find the two pure equilibria when we check pairs of subsets in which each subset consists of a single action, we find no equilibria when we check pairs in which one subset consists of a single action and the other consists of both actions, and we find the mixed strategy equilibrium when we check the pair $(\{B, S\}, \{B, S\})$.

? EXERCISE 136.1 (Finding all mixed strategy equilibria of two-player games) Use the method described above to find all the mixed strategy equilibria of the games in Figure 111.2.

In a game in which each player has two actions, for any subset of any player's set of actions at most one of the two conditions in Proposition 113.2 is relevant (the first if the subset contains both actions and the second if it contains only one action). When a player has three or more actions and we consider a subset of her set of actions that contains two actions, both conditions are relevant, as the next example illustrates.

◆ EXAMPLE 136.2 (Finding all mixed strategy equilibria of a variant of *BoS*) Consider the variant of *BoS* given in Figure 137.1. First, by inspection we see that the game has two pure strategy Nash equilibria, namely (B, B) and (S, S).

Now consider the possibility of an equilibrium in which player 1's strategy is pure whereas player 2's strategy assigns positive probability to two or more actions. If player 1's strategy is B then player 2's payoffs to her three actions (2, 0, and 1) are all different, so the first condition in Proposition 113.2 is not satisfied. Thus

	B	S	X
B	$4,2$	$0,0$	$0,1$
S	$0,0$	$2,4$	$1,3$

Figure 137.1 A variant of the game *BoS*.

there is no equilibrium of this type. Similar reasoning rules out an equilibrium in which player 1's strategy is S and player 2's strategy assigns positive probability to more than one action, and also an equilibrium in which player 2's strategy is pure and player 1's strategy assigns positive probability to both of her actions.

Next consider the possibility of an equilibrium in which player 1's strategy assigns positive probability to both her actions and player 2's strategy assigns positive probability to two of her three actions. Denote by p the probability player 1's strategy assigns to B. There are three possibilities for the pair of player 2's actions that have positive probability.

B and S: For the conditions in Proposition 113.2 to be satisfied we need player 2's expected payoff to B to be equal to her expected payoff to S and at least her expected payoff to X. That is, we need

$$2p = 4(1-p) \geq p + 3(1-p).$$

The equation implies that $p = \frac{2}{3}$, which does not satisfy the inequality. (That is, if p is such that B and S yield the same expected payoff, then X yields a higher expected payoff.) Thus there is no equilibrium of this type.

B and X: For the conditions in Proposition 113.2 to be satisfied we need player 2's expected payoff to B to be equal to her expected payoff to X and at least her expected payoff to S. That is, we need

$$2p = p + 3(1-p) \geq 4(1-p).$$

The equation implies that $p = \frac{3}{4}$, which satisfies the inequality. For the first condition in Proposition 113.2 to be satisfied for player 1 we need player 1's expected payoffs to B and S to be equal: $4q = 1 - q$, where q is the probability player 2 assigns to B, or $q = \frac{1}{5}$. Thus the pair of mixed strategies $((\frac{3}{4}, \frac{1}{4}), (\frac{1}{5}, 0, \frac{4}{5}))$ is a mixed strategy equilibrium.

S and X: For every strategy of player 2 that assigns positive probability only to S and X, player 1's expected payoff to S exceeds her expected payoff to B. Thus there is no equilibrium of this sort.

The final possibility is that there is an equilibrium in which player 1's strategy assigns positive probability to both her actions and player 2's strategy assigns positive probability to all three of her actions. Let p be the probability player 1's strategy assigns to B. Then for player 2's expected payoffs to her three actions to be equal we need

$$2p = 4(1-p) = p + 3(1-p).$$

For the first equality we need $p = \frac{2}{3}$, violating the second equality. That is, there is no value of p for which player 2's expected payoffs to her three actions are equal, and thus no equilibrium in which she chooses each action with positive probability.

We conclude that the game has three mixed strategy equilibria: $((1,0),(1,0,0))$ (i.e. the pure strategy equilibrium (B,B)), $((0,1),(0,1,0))$ (i.e. the pure strategy equilibrium (S,S)), and $((\frac{3}{4},\frac{1}{4}),(\frac{1}{5},0,\frac{4}{5}))$.

? EXERCISE 138.1 (Finding all mixed strategy equilibria of a two-player game) Use the method described above to find all the mixed strategy Nash equilibria of the strategic game in Figure 138.1.

	L	M	R
T	$2,2$	$0,3$	$1,3$
B	$3,2$	$1,1$	$0,2$

Figure 138.1 The strategic game with vNM preferences in Exercise 138.1.

As you can see from the examples, this method has the disadvantage that for games in which each player has several strategies, or in which there are several players, the number of possibilities to examine is huge. Even in a two-player game in which each player has three actions, each player's set of actions has seven nonempty subsets (three each consisting of a single action, three consisting of two actions, and the entire set of actions), so that there are 49 (7×7) possible collections of subsets to check. In a symmetric game, like the one in the next exercise, many cases involve the same argument, reducing the number of distinct cases to be checked.

? EXERCISE 138.2 (Rock, paper, scissors) Each of two players simultaneously announces either *Rock*, or *Paper*, or *Scissors*. *Paper* beats (wraps) *Rock*, *Rock* beats (blunts) *Scissors*, and *Scissors* beats (cuts) *Paper*. The player who names the winning object receives \$1 from her opponent; if both players make the same choice then no payment is made. Each player's preferences are represented by the expected amount of money she receives. (An example of the variant of Hotelling's model of electoral competition considered in Exercise 74.1 has the same payoff structure. Suppose there are three possible positions, A, B, and C, and three citizens, one of whom prefers A to B to C, one of whom prefers B to C to A, and one of whom prefers C to A to B. Two candidates simultaneously choose positions. If the candidates choose different positions each citizen votes for the candidate whose position she prefers; if both candidates choose the same position they tie for first place.)

a. Formulate this situation as a strategic game and find all its mixed strategy equilibria (give both the equilibrium strategies and the equilibrium payoffs).

b. Find all the mixed strategy equilibria of the modified game in which player 1 is prohibited from announcing *Scissors*.

?? EXERCISE 139.1 (Election campaigns) A new political party, A, is challenging an established party, B. The race involves three localities of different sizes. Party A can wage a strong campaign in only one locality; B must commit resources to defend its position in one of the localities, without knowing which locality A has targeted. If A targets district i and B devotes its resources to some other district then A gains a_i votes at the expense of B; let $a_1 > a_2 > a_3 > 0$. If B devotes resources to the district that A targets then A gains no votes. Each party's preferences are represented by the expected number of votes it gains. (Perhaps seats in a legislature are allocated proportionally to vote shares.) Formulate this situation as a strategic game and find its mixed strategy equilibria.

Although games with many players cannot in general be conveniently represented in tables like those we use for two-player games, three-player games can be accommodated. We construct one table for each of player 3's actions; player 1 chooses a row, player 2 chooses a column, and player 3 chooses a *table*. The next exercise is an example of such a game.

? EXERCISE 139.2 (A three-player game) Find the mixed strategy Nash equilibria of the three-player game in Figure 139.1, in which each player has two actions.

	A	B
A	$1,1,1$	$0,0,0$
B	$0,0,0$	$0,0,0$

A

	A	B
A	$0,0,0$	$0,0,0$
B	$0,0,0$	$4,4,4$

B

Figure 139.1 The three-player game in Exercise 139.2.

4.11 Extension: Mixed strategy Nash equilibria of games in which each player has a continuum of actions

In all the games studied so far in this chapter each player has finitely many actions. In the previous chapter we saw that many situations may conveniently be modeled as games in which each player has a continuum of actions. (For example, in Cournot's model the set of possible outputs for a firm is the set of nonnegative numbers, and in Hotelling's model the set of possible positions for a candidate is the set of nonnegative numbers.) The principles involved in finding mixed strategy equilibria of such games are the same as those involved in finding mixed strategy equilibria of games in which each player has finitely many actions, though the techniques are different.

Proposition 113.2 says that a strategy profile in a game in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player, (a) every action to which her strategy assigns positive probability yields the same expected payoff, and (b) no action yields a higher expected payoff. Now, a mixed strategy of a player who has a continuum of actions is determined by the probabilities it assigns to sets of actions, not by the probabilities it assigns to single actions (all of which may be zero, for example). Thus (a) does not fit such a game. However, the following restatement of the result, equivalent to Proposition 113.2 for a game in which each player has finitely many actions, does fit.

■ PROPOSITION 140.1 (Characterization of mixed strategy Nash equilibrium) *A mixed strategy profile α^* in a strategic game with vNM preferences is a mixed strategy Nash equilibrium if and only if, for each player i,*

- *α_i^* assigns probability zero to the set of actions a_i for which the action profile (a_i, α_{-i}^*) yields player i an expected payoff less than her expected payoff to α^**
- *for no action a_i does the action profile (a_i, α_{-i}^*) yield player i an expected payoff greater than her expected payoff to α^*.*

A significant class of games in which each player has a continuum of actions consists of games in which each player's set of actions is a one-dimensional interval of numbers. Consider such a game with two players; let player i's set of actions be the interval from $\underline{a}_i$ to $\overline{a}_i$, for $i = 1, 2$. Identify each player's mixed strategy with a *cumulative* probability distribution on this interval. (See Section 17.7.4 in the appendix on mathematics if you are not familiar with this notion.) That is, the mixed strategy of each player i is a nondecreasing function F_i for which $0 \leq F_i(a_i) \leq 1$ for every action a_i; the number $F_i(a_i)$ is the probability that player i's action is at most a_i.

The form of a mixed strategy Nash equilibrium in such a game may be very complex. Some such games, however, have equilibria of a particularly simple form, in which each player's equilibrium mixed strategy assigns probability zero except in an interval. Specifically, consider a pair (F_1, F_2) of mixed strategies that satisfies the following conditions for $i = 1, 2$.

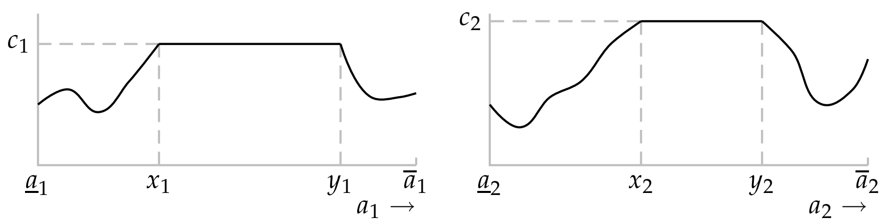
- There are numbers x_i and y_i such that player i's mixed strategy F_i assigns probability zero except in the interval from x_i to y_i: $F_i(z) = 0$ for $z < x_i$, and $F(z) = 1$ for $z \geq y_i$.
- Player i's expected payoff when her action is a_i and the other player uses her mixed strategy F_j takes the form

$$\begin{cases} = c_i & \text{for } x_i \leq a_i \leq y_i \\ \leq c_i & \text{for } a_i < x_i \text{ and } a_i > y_i \end{cases}$$

  where c_i is a constant.

(The second condition is illustrated in Figure 141.1.) By Proposition 140.1, such a pair of mixed strategies, if it exists, is a mixed strategy Nash equilibrium of the game, in which player i's expected payoff is c_i, for $i = 1, 2$.

The next example illustrates how a mixed strategy equilibrium of such a game may be found. The example is designed to be very simple; be warned that in most such games an analysis of the equilibria is, at a minimum, somewhat more



Player 1's expected payoff given F_2 Player 2's expected payoff given F_1

Figure 141.1 If (*i*) F_1 assigns positive probability only to actions in the interval from x_1 to y_1, (*ii*) F_2 assigns positive probability only to the actions in the interval from x_2 to y_2, (*iii*) given player 2's mixed strategy F_2, player 1's expected payoff takes the form shown in the left panel, and (*iv*) given player 1's mixed strategy F_1, player 2's expected payoff takes the form shown in the right panel, then (F_1, F_2) is a mixed strategy equilibrium.

complex. Further, my analysis is not complete: I merely find an equilibrium, rather than studying all equilibria. (In fact, the game has no other equilibria.)

EXAMPLE 141.1 (All-pay auction) Two people submit sealed bids for an object worth \$$K$ to each of them. Each person's bid may be any nonnegative number up to \$$K$. The winner is the person whose bid is higher; in the event of a tie each person receives half of the object, which she values at \$$K/2$. Each person pays her bid, *whether or not she wins*, and has preferences represented by the expected amount of money she receives.

This situation may be modeled by the following strategic game, known as an **all-pay auction**.

Players The two bidders.

Actions Each player's set of actions is the set of possible bids (nonnegative numbers up to K)

Payoff functions Each player i's preferences are represented by the expected value of the payoff function given by

$$u_i(a_1, a_2) = \begin{cases} -a_i & \text{if } a_i < a_j \\ K/2 - a_i & \text{if } a_i = a_j \\ K - a_i & \text{if } a_i > a_j, \end{cases}$$

where j is the other player.

One situation that may be modeled as such an auction is a lobbying process in which each of two interest groups spends resources to persuade a government to carry out the policy it prefers, and the group that spends the most wins. Another situation that may be modeled as such an auction is the competition between two firms to develop a new product by some deadline, where the firm that spends the most develops a better product, which captures the entire market.

An all-pay auction has no pure strategy Nash equilibrium, by the following argument.

- No pair of actions (x, x) with $x < K$ is a Nash equilibrium, because either player can increase her payoff by slightly increasing her bid.
- (K, K) is not a Nash equilibrium, because either player can increase her payoff from $-K/2$ to 0 by reducing her bid to 0.
- No pair of actions (a_1, a_2) with $a_1 \neq a_2$ is a Nash equilibrium because the player whose bid is higher can increase her payoff by reducing her bid (and the player whose bid is lower can, if her bid is positive, increase her payoff by reducing her bid to 0).

Consider the possibility that the game has a mixed strategy Nash equilibrium. Denote by F_i the mixed strategy (i.e. cumulative probability distribution over the interval of possible bids) of player i. I look for an equilibrium in which neither mixed strategy assigns positive probability to any *single* bid. (Remember that there are infinitely many possible bids.) In this case $F_i(a_i)$ is both the probability that player i bids at most a_i and the probability that she bids less than a_i. I further restrict attention to strategy pairs (F_1, F_2) for which, for $i = 1, 2$, there are numbers x_i and y_i such that F_i assigns positive probability only to the interval from x_i to y_i.

To investigate the possibility of such an equilibrium, consider player 1's expected payoff when she uses the action a_1, given player 2's mixed strategy F_2.

- If $a_1 < x_2$ then a_1 is less than player 2's bid with probability one, so that player 1's payoff is $-a_1$.
- If $a_1 > y_2$ then a_1 exceeds player 2's bid with probability one, so that player 1's payoff is $K - a_1$.
- If $x_2 \leq a_1 \leq y_2$ then player 1's expected payoff is calculated as follows. With probability $F_2(a_1)$ player 2's bid is less than a_1, in which case player 1's payoff is $K - a_1$; with probability $1 - F_2(a_1)$ player 2's bid exceeds a_1, in which case player 1's payoff is $-a_1$; and, by assumption, the probability that player 2's bid is exactly equal to a_1 is zero. Thus player 1's expected payoff is

$$(K - a_1)F_2(a_1) + (-a_1)(1 - F_2(a_1)) = KF_2(a_1) - a_1.$$

We need to find values of x_2 and y_2 and a strategy F_2 such that player 1's expected payoff satisfies the condition illustrated in the left panel of Figure 141.1: it is constant on the interval from x_1 to y_1, and less than this constant for $a_1 < x_1$ and $a_1 > y_1$. The constancy of the payoff on the interval from x_1 to y_1 requires that $KF_2(a_1) - a_1 = c_1$ for $x_1 \leq a_1 \leq y_1$, for some constant c_1. We also need $F_2(x_2) = 0$ and $F_2(y_2) = 1$ (because I am restricting attention to equilibria in which neither player's strategy assigns positive probability to any single action), and F_2 must be nondecreasing (so that it is a cumulative probability distribution). Analogous conditions must be satisfied by x_2, y_2, and F_1.

We see that if $x_1 = x_2 = 0$, $y_1 = y_2 = K$, and $F_1(z) = F_2(z) = z/K$ for all z with $0 \leq z \leq K$ then all these conditions are satisfied. Each player's expected payoff is constant, equal to 0 for all her actions a_1.

Thus the game has a mixed strategy Nash equilibrium in which each player randomizes "uniformly" over all her actions. In this equilibrium each player's expected payoff is 0: on average, the amount a player spends is exactly equal to the value of the object. (A more involved argument shows that this equilibrium is the *only* mixed strategy Nash equilibrium of the game.)

EXERCISE 143.1 (All-pay auction with many bidders) Consider the generalization of the game considered in the previous example in which there are $n \geq 2$ bidders. Find a mixed strategy Nash equilibrium in which each player uses the same mixed strategy. (If you know how, find each player's mean bid in the equilibrium.)

EXERCISE 143.2 (Bertrand's duopoly game) Consider Bertrand's oligopoly game (Section 3.2) when there are two firms. Assume that each firm's preferences are represented by its expected profit. Show that if the function $(p - c)D(p)$ is increasing in p, and increases without bound as p increases without bound, then for every $\underline{p} > c$, the game has a mixed strategy Nash equilibrium in which each firm uses the same mixed strategy F, with $F(\underline{p}) = 0$ and $F(p) > 0$ for $p > \underline{p}$.

In the games in the example and exercises each player's payoff depends only on her action and whether this action is greater than, equal to, or less than the other players' actions. The limited dependence of each player's payoff on the other players' actions makes the calculation of a player's expected payoff straightforward. In many games, each player's payoff is affected more substantially by the other players' actions, making the calculation of expected payoff more complex; more sophisticated mathematical tools are required to analyze such games.

4.12 Appendix: Representing preferences over lotteries by the expected value of a payoff function

4.12.1 Expected payoffs

Suppose that a decision-maker has preferences over a set of deterministic outcomes, and that each of her actions results in a *lottery* (probability distribution) over these outcomes. In order to determine the action she chooses, we need to know her preferences over these lotteries. As argued in Section 4.1.3, we cannot *derive* these preferences from her preferences over deterministic outcomes, but have to specify them as part of the model.

So assume that we are given the decision-maker's preferences over lotteries. As in the case of preferences over deterministic outcomes, under some fairly weak assumptions we can represent these preferences by a payoff function. (Refer to Section 1.2.2.) That is, when there are K deterministic outcomes we can find a function, say U, over lotteries such that

$$U(p_1, \ldots, p_K) > U(p'_1, \ldots, p'_K)$$

if and only if the decision-maker prefers the lottery $(p_1, \ldots, p_K)$ to the lottery $(p'_1, \ldots, p'_K)$ (where $(p_1, \ldots, p_K)$ is the lottery in which outcome 1 occurs with probability p_1, outcome 2 occurs with probability p_2, and so on).

For many purposes, however, we need more structure: we cannot get very far without restricting to preferences for which there is a more specific representation. The standard approach, developed by von Neumann and Morgenstern (1944), is to impose an additional assumption—the "independence axiom"—that allows us to conclude that the decision-maker's preferences can be represented by an *expected payoff function*. More precisely, the independence axiom (which I do not describe) allows us to conclude that there is a payoff function u over *deterministic* outcomes such that the decision-maker's preference relation over lotteries is represented by the function $U(p_1, \ldots, p_K) = \sum_{k=1}^{K} p_k u(a_k)$, where a_k is the kth outcome of the lottery:

$$\sum_{k=1}^{K} p_k u(a_k) > \sum_{k=1}^{K} p'_k u(a_k) \tag{144.1}$$

if and only if the decision-maker prefers the lottery $(p_1, \ldots, p_K)$ to the lottery $(p'_1, \ldots, p'_K)$. That is, the decision-maker evaluates a lottery by its *expected payoff* according to the function u, which is known as the decision-maker's *Bernoulli payoff function*.

Suppose, for example, that there are three possible deterministic outcomes: the decision-maker may receive \$0, \$1, or \$5, and naturally prefers \$5 to \$1 to \$0. Suppose that she prefers the lottery $(\frac{1}{2}, 0, \frac{1}{2})$ to the lottery $(0, \frac{3}{4}, \frac{1}{4})$ (where the first number in each list is the probability of \$0, the second number is the probability of \$1, and the third number is the probability of \$5). This preference is consistent with preferences represented by the expected value of a payoff function u for which $u(0) = 0$, $u(1) = 1$, and $u(5) = 4$, because

$$\tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 4 > \tfrac{3}{4} \cdot 1 + \tfrac{1}{4} \cdot 4.$$

(Many other payoff functions are consistent with a preference for $(\frac{1}{2}, 0, \frac{1}{2})$ over $(0, \frac{3}{4}, \frac{1}{4})$. Among those in which $u(0) = 0$ and $u(5) = 4$, for example, any function for which $u(1) < \frac{4}{3}$ does the job.) Suppose, on the other hand, that the decision-maker prefers the lottery $(0, \frac{3}{4}, \frac{1}{4})$ to the lottery $(\frac{1}{2}, 0, \frac{1}{2})$. This preference is consistent with preferences represented by the expected value of a payoff function u for which $u(0) = 0$, $u(1) = 3$, and $u(5) = 4$, because

$$\tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 4 < \tfrac{3}{4} \cdot 3 + \tfrac{1}{4} \cdot 4.$$

? EXERCISE 144.2 (Preferences over lotteries) There are three possible outcomes; in the outcome a_i a decision-maker gains \$$a_i$, where $a_1 < a_2 < a_3$. The decision-maker prefers a_3 to a_2 to a_1 and she prefers the lottery $(0.3, 0, 0.7)$ to $(0.1, 0.4, 0.5)$ to $(0.3, 0.2, 0.5)$ to $(0.45, 0, 0.55)$. Is this information consistent with the decision-maker's preferences being represented by the expected value of a payoff function? If so, find a payoff function consistent with the information. If not, show why not. Answer the same questions when, alternatively, the decision-maker prefers the lottery $(0.4, 0, 0.6)$ to $(0, 0.5, 0.5)$ to $(0.3, 0.2, 0.5)$ to $(0.45, 0, 0.55)$.

Preferences represented by the expected value of a (Bernoulli) payoff function have the great advantage that they are completely specified by that payoff function. Once we know $u(a_k)$ for each possible outcome a_k we know the decision-maker's preferences among all lotteries. This significant advantage does, however, carry with it a small price: it is very easy to confuse a Bernoulli payoff function with a payoff function that represents the decision-maker's preferences over deterministic outcomes.

To describe the relation between the two, suppose that a decision-maker's preferences over lotteries are represented by the expected value of the Bernoulli payoff function u. Then certainly u is a payoff function that represents the decision-maker's preferences over deterministic outcomes (which are special cases of lotteries, in which a single outcome is assigned probability 1). However, the converse is *not* true: if the decision-maker's preferences over deterministic outcomes are represented by the payoff function u (i.e. the decision-maker prefers a to a' if and only if $u(a) > u(a')$), then u is *not* necessarily a Bernoulli payoff function whose expected value represents the decision-maker's preferences over lotteries. For instance, suppose that the decision-maker prefers \$5 to \$1 to \$0, and prefers the lottery $(\frac{1}{2}, 0, \frac{1}{2})$ to the lottery $(0, \frac{3}{4}, \frac{1}{4})$. Then her preferences over deterministic outcomes are consistent with the payoff function u for which $u(0) = 0$, $u(1) = 3$, and $u(5) = 4$. However, her preferences over lotteries are *not* consistent with the expected value of this function (since $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 4 < \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 4$). The moral is that you should be careful to determine the type of payoff function you are dealing with.

4.12.2 Equivalent Bernoulli payoff functions

If a decision-maker's preferences in a deterministic environment are represented by the payoff function u then they are represented also by any payoff function that is an increasing function of u (see Section 1.2.2). The analogous property is not satisfied by Bernoulli payoff functions. Consider the example discussed above. A Bernoulli payoff function u for which $u(0) = 0$, $u(1) = 1$, and $u(5) = 4$ is consistent with a preference for the lottery $(\frac{1}{2}, 0, \frac{1}{2})$ over $(0, \frac{3}{4}, \frac{1}{4})$, but the function $\sqrt{u}$, for which $u(0) = 0$, $u(1) = 1$, and $u(5) = 2$, is not consistent with such a preference ($\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2 < \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 2$), though the square root function is increasing (larger numbers have larger square roots).

Under what circumstances do the expected values of two Bernoulli payoff functions represent the same preferences? The next result shows that they do so if and only if one payoff function is an increasing *linear* function of the other.

■ LEMMA 145.1 (Equivalence of Bernoulli payoff functions) *Suppose there are at least three possible outcomes. The expected values of the Bernoulli payoff functions u and v represent the same preferences over lotteries if and only if there exist numbers η and θ with $\theta > 0$ such that $u(x) = \eta + \theta v(x)$ for all x.*

If the expected value of u represents a decision-maker's preferences over lotteries then so, for example, do the expected values of $2u$, $1+u$, and $-1+4u$; but the expected values of u^2 and of $\sqrt{u}$ do not.

Part of the lemma is easy to establish. Let u be a Bernoulli payoff function whose expected value represents a decision-maker's preferences, and let $v(x) = \eta + \theta u(x)$ for all x, where η and θ are constants with $\theta > 0$. I argue that the expected values of u and of v represent the same preferences. Suppose that the decision-maker prefers the lottery $(p_1, \ldots, p_K)$ to the lottery $(p'_1, \ldots, p'_K)$. Then her expected payoff to $(p_1, \ldots, p_K)$ exceeds her expected payoff to $(p'_1, \ldots, p'_K)$, or

$$\sum_{k=1}^{K} p_k u(a_k) > \sum_{k=1}^{K} p'_k u(a_k) \tag{146.1}$$

(see (144.1)). Now,

$$\sum_{k=1}^{K} p_k v(a_k) = \sum_{k=1}^{K} p_k \eta + \sum_{k=1}^{K} p_k \theta u(a_k) = \eta + \theta \sum_{k=1}^{K} p_k u(a_k),$$

using the fact that the sum of the probabilities p_k is 1. Similarly,

$$\sum_{k=1}^{K} p'_k v(a_k) = \eta + \theta \sum_{k=1}^{K} p'_k u(a_k).$$

Substituting for u in (146.1) we obtain

$$\left(\sum_{k=1}^{K} p_k v(a_k) - \eta \right) /\theta > \left(\sum_{k=1}^{K} p'_k v(a_k) - \eta \right) /\theta,$$

which, given $\theta > 0$, is equivalent to

$$\sum_{k=1}^{K} p_k v(a_k) > \sum_{k=1}^{K} p'_k v(a_k) :$$

according to v, the expected payoff of $(p_1, \ldots, p_K)$ exceeds the expected payoff of $(p'_1, \ldots, p'_K)$. We conclude that if u represents the decision-maker's preferences then so does the function v defined by $v(x) = \eta + \theta u(x)$.

I omit the more difficult argument that if the expected values of the Bernoulli payoff functions u and v represent the same preferences over lotteries then $v(x) = \eta + \theta u(x)$ for some constants η and $\theta > 0$.

? EXERCISE 146.2 (Normalized Bernoulli payoff functions) Suppose that a decision-maker's preferences can be represented by the expected value of the Bernoulli payoff function u. Find a Bernoulli payoff function whose expected value represents the decision-maker's preferences and that assigns a payoff of 1 to the best outcome and a payoff of 0 to the worst outcome.

4.12.3 Equivalent strategic games with vNM preferences

Turning to games, consider the three payoff tables in Figure 147.1. All three tables represent the same strategic game with deterministic preferences: in each case, player 1 prefers (B, B) to (S, S) to (B, S), which she regards as indifferent to (S, B), and player 2 prefers (S, S) to (B, B) to (B, S), which she regards as indifferent to (S, B). However, only the left and middle tables represent the same strategic game with vNM preferences. The reason is that the payoff functions in the middle table are linear functions of the payoff functions in the left table, whereas the payoff functions in the right table are not. Specifically, denote the Bernoulli payoff functions of player i in the three games by u_i, v_i, and w_i. Then

$$v_1(a) = 2u_1(a) \text{ and } v_2(a) = -3 + 3u_2(a),$$

so that the left and middle tables represent the same strategic game with vNM preferences. However, w_1 is not a linear function of u_1. If it were, there would exist constants η and $\theta > 0$ such that $w_1(a) = \eta + \theta u_1(a)$ for each action pair a, or

$$\begin{aligned} 0 &= \eta + \theta \cdot 0 \\ 1 &= \eta + \theta \cdot 1 \\ 3 &= \eta + \theta \cdot 2, \end{aligned}$$

but these three equations have no solution. Thus the left and right tables represent different strategic games with vNM preferences. (As you can check, w_2 is not a linear function of u_2 either; but for the games not to be equivalent it is sufficient that *one* player's preferences be different.) Another way to see that player 1's vNM preferences in the left and right games are different is to note that in the left table player 1 is indifferent between the certain outcome (S, S) and the lottery in which (B, B) occurs with probability $\frac{1}{2}$ and (S, B) occurs with probability $\frac{1}{2}$ (each yields an expected payoff of 1), whereas in the right table she prefers the latter (since it yields an expected payoff of 1.5).

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

	B	S
B	$4, \ 0$	$0, -3$
S	$0, -3$	$2, \ 3$

	B	S
B	$3, 2$	$0, 1$
S	$0, 1$	$1, 4$

Figure 147.1 All three tables represent the same strategic game with ordinal preferences, but only the left and middle games, not the right one, represent the same strategic game with vNM preferences.

? EXERCISE 147.1 (Games equivalent to the *Prisoner's Dilemma*) Which of the tables in Figure 148.1 represents the same strategic game with vNM preferences as the *Prisoner's Dilemma* as specified in the left panel of Figure 104.1, when the numbers are interpreted as Bernoulli payoffs?

	C	D
C	$3,3$	$0,4$
D	$4,0$	$2,2$

	C	D
C	$6,\ 0$	$0,\ 2$
D	$9,-4$	$3,-2$

Figure 148.1 The payoff tables for Exercise 147.1.

Notes

The ideas behind mixed strategies and preferences represented by expected payoffs date back in Western thought at least to the eighteenth century (see Guilbaud (1961) and Kuhn (1968), and Bernoulli (1738), respectively). The modern formulation of a mixed strategy is due to Borel (1921; 1924, 204–221; 1927); the model of the representation of preferences by an expected payoff function is due to von Neumann and Morgenstern (1944). The model of a mixed strategy Nash equilibrium and Proposition 116.1 on the existence of a mixed strategy Nash equilibrium in a finite game are due to Nash (1950a, 1951). Proposition 119.2 is an implication of the existence of a "trembling hand perfect equilibrium", due to Selten (1975, Theorem 5).

The example in the box on page 102 is taken from Allais (1953). Conlisk (1989) discusses some of the evidence on the theory of expected payoffs; Machina (1987) and Hey (1997) survey the subject. (The purchasing power of the largest prize in Allais' example was roughly US\$6.6m in 1989 (the date of Conlisk's paper, in which the prize is US\$5m) and roughly US\$8m in 1999.) The model in Section 4.6 is due to Pitchik and Schotter (1987). The model in Section 4.8 is a special case of the one in Palfrey and Rosenthal (1984); the interpretation and analysis that I describe is taken from an unpublished 1984 paper of William F. Samuelson. The box on page 130 draws upon Rosenthal (1964), Latané and Nida (1981), Brown (1986), and Aronson (1995). Best response dynamics were first studied by Cournot (1838, Ch. VII), in the context of his duopoly game. Fictitious play was suggested by Brown (1951). Robinson (1951) shows that the process converges to a mixed strategy Nash equilibrium in any two-player game in which the players' interests are opposed; Shapley (1964, Section 5) exhibits a game outside this class in which the process does not converge. Recent work on learning in games is surveyed by Fudenberg and Levine (1998).

The game in Exercise 115.2 is due to David L. Silverman (see Silverman 1981–82 and Heuer 1995). Exercise 115.3 is based on Palfrey and Rosenthal (1983). Exercise 115.4 is taken from Shubik (1982, 226) (who finds only one of the continuum of equilibria of the game).

The model in Exercise 125.2 is taken from Peters (1984). Exercise 127.2 is a variant of an exercise of Moulin (1986, pp. 167, 185). Exercise 130.1 is based on Palfrey and Rosenthal (1984). The game *Rock-Paper-Scissors* (Exercise 138.2) was first studied by Borel (1924) and von Neumann (1928). Exercise 139.1 is based on Karlin (1959a, 92–94), who attributes the game to an unpublished paper by Dresher.

Exercise 143.1 is based on a result in Baye, Kovenock, and de Vries (1996). The mixed strategy Nash equilibria of Bertrand's model of duopoly (Exercise 143.2) are studied in detail by Baye and Morgan (1996).

The method of finding all mixed strategy equilibrium described in Section 4.10 is computationally very intense in all but the simplest games. Some computationally more efficient methods are implemented in the computer program `GAMBIT`, located at `http://www.hss.caltech.edu/\symbol{126}gambit/Gambit.html`.

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5 Extensive Games with Perfect Information: Theory

Prerequisite: Chapters 1 and 2.

5.1 Introduction

The model of a strategic game suppresses the sequential structure of decision-making. When applying the model to situations in which decision-makers move sequentially, we assume that each decision-maker chooses her plan of action once and for all; she is committed to this plan, which she cannot modify as events unfold. The model of an extensive game, by contrast, describes the sequential structure of decision-making explicitly, allowing us to study situations in which each decision-maker is free to change her mind as events unfold.

In this chapter and the next two we study a model in which each decision-maker is always fully informed about all previous actions. In Chapter 10 we study a more general model, which allows each decision-maker, when taking an action, to be imperfectly informed about previous actions.

5.2 Extensive games with perfect information

5.2.1 Definition

To describe an extensive game with perfect information, we need to specify the set of players and their preferences, as for a strategic game (Definition 11.1). In addition, we need to specify the order of the players' moves and the actions each player may take at each point. We do so by specifying the set of all sequences of actions that can possibly occur, together with the player who moves at each point in each sequence. We refer to each possible sequence of actions as a *terminal history* and to the function that gives the player who moves at each point in each terminal history as the *player function*. That is, an extensive game has four components:

- players
- terminal histories

- player function
- preferences for the players.

Before giving precise definitions of these components, I give an example that illustrates them informally.

◆ EXAMPLE 152.1 (Entry game) An incumbent faces the possibility of entry by a challenger. (The challenger may, for example, be a firm considering entry into an industry currently occupied by a monopolist, a politician competing for the leadership of a party, or an animal considering competing for the right to mate with a congener of the opposite sex.) The challenger may enter or not. If it enters, the incumbent may either acquiesce or fight.

We may model this situation as an extensive game with perfect information in which the terminal histories are $(In, Acquiesce)$, $(In, Fight)$, and *Out*, and the player function assigns the challenger to the start of the game and the incumbent to the history *In*.

At the start of an extensive game, and after any sequence of events, a player chooses an action. The sets of actions available to the players are not, however, given explicitly in the description of the game. Instead, the description of the game specifies the set of terminal histories and the player function, from which we can deduce the available sets of actions.

In the entry game, for example, the actions available to the challenger at the start of the game are *In* and *Out*, because these actions (and no others) begin terminal histories, and the actions available to the incumbent are *Acquiesce* and *Fight*, because these actions (and no others) follow *In* in terminal histories. More generally, suppose that (C, D) and (C, E) are terminal histories and the player function assigns player 1 to the start of the game and player 2 to the history C. Then two of the actions available to player 2 after player 1 chooses C at the start of the game are D and E.

The terminal histories of a game are specified as a set of sequences. But not every set of sequences is a legitimate set of terminal histories. If (C, D) is a terminal history, for example, there is no sense in specifying C as a terminal history: the fact that (C, D) is terminal implies that after C is chosen at the start of the game, some player may choose D, so that the action C does not end the game. More generally, a sequence that is a *proper subhistory* of a terminal history cannot itself be a terminal history. This restriction is the only one we need to impose on a set of sequences in order that the set be interpretable as a set of terminal histories.

To state the restriction precisely, define the **subhistories** of a finite sequence $(a^1, a^2, \ldots, a^k)$ of actions to be the empty sequence consisting of no actions, denoted $\varnothing$ (representing the start of the game), and all sequences of the form $(a^1, a^2, \ldots, a^m)$ where $1 \leq m \leq k$. (In particular, the entire sequence is a subhistory of itself.) Similarly, define the **subhistories** of an infinite sequence $(a^1, a^2, \ldots)$ of actions to be the empty sequence $\varnothing$, every sequence of the form $(a^1, a^2, \ldots, a^m)$ where m is a positive integer, and the entire sequence $(a^1, a^2, \ldots)$. A subhistory not equal to the entire sequence is called a **proper subhistory**. A sequence of actions that is a subhistory of some terminal history is called simply a **history**.

In the entry game in Example 152.1, the subhistories of $(In, Acquiesce)$ are the empty history $\varnothing$ and the sequences In and $(In, Acquiesce)$; the proper subhistories are the empty history and the sequence In.

▶ DEFINITION 153.1 (*Extensive game with perfect information*) An **extensive game with perfect information** consists of

- a set of **players**
- a set of sequences (**terminal histories**) with the property that no sequence is a proper subhistory of any other sequence
- a function (the **player function**) that assigns a player to every sequence that is a proper subhistory of some terminal history
- for each player, **preferences** over the set of terminal histories.

The set of terminal histories is the set of all sequences of actions that may occur; the player assigned by the player function to any history h is the player who takes an action after h.

As for a strategic game, we may specify a player's preferences by giving a payoff function that represents them (see Section 1.2.2). In some situations an outcome is associated with each terminal history, and the players' preferences are naturally defined over these outcomes, rather than directly over the terminal histories. For example, if we are modeling firms choosing prices then we may think in terms of each firm's caring about its profit—the outcome of a profile of prices—rather than directly about the profile of prices. However, any preferences over outcomes (e.g. profits) may be translated into preferences over terminal histories (e.g. sequences of prices). In the general definition, outcomes are conveniently identified with terminal histories and preferences are defined directly over these histories, avoiding the need for an additional element in the specification of the game.

◆ EXAMPLE 153.2 (Entry game) In the situation described in Example 152.1, suppose that the best outcome for the challenger is that it enters and the incumbent acquiesces, and the worst outcome is that it enters and the incumbent fights, whereas the best outcome for the incumbent is that the challenger stays out, and the worst outcome is that it enters and there is a fight. Then the situation may be modeled as the following extensive game with perfect information.

Players The challenger and the incumbent.

Terminal histories $(In, Acquiesce)$, $(In, Fight)$, and Out.

Player function $P(\varnothing) = Challenger$ and $P(In) = Incumbent$.

Preferences The challenger's preferences are represented by the payoff function u_1 for which $u_1(In, Acquiesce) = 2$, $u_1(Out) = 1$, and $u_1(In, Fight) = 0$, and the incumbent's preferences are represented by the payoff function u_2 for which $u_2(Out) = 2$, $u_2(In, Acquiesce) = 1$, and $u_2(In, Fight) = 0$.

This game is readily illustrated in a diagram. The small circle at the top of Figure 154.1 represents the empty history (the start of the game). The label above this circle indicates that the challenger chooses an action at the start of the game ($P(\varnothing) = \textit{Challenger}$). The two branches labeled *In* and *Out* represent the challenger's choices. The segment labeled *In* leads to a small disk, where it is the incumbent's turn to choose an action ($P(\textit{In}) = \textit{Incumbent}$) and her choices are *Acquiesce* and *Fight*. The pair of numbers beneath each terminal history gives the players' payoffs to that history, with the challenger's payoff listed first. (The players' payoffs may be given in any order. For games like this one, in which the players move in a well-defined order, I generally list the payoffs in that order. For games in which the players' names are 1, 2, 3, and so on, I list the payoffs in the order of their names.)

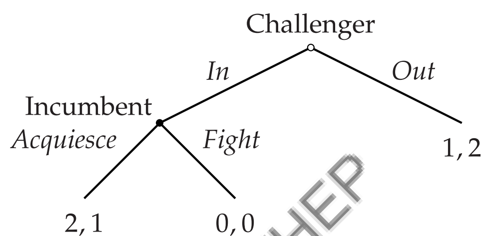

Figure 154.1 The entry game of Example 153.2. The challenger's payoff is the first number in each pair.

Definition 153.1 does not directly specify the sets of actions available to the players at their various moves. As I discussed briefly before the definition, we can deduce these sets from the set of terminal histories and the player function. If, for some nonterminal history h, the sequence (h, a) is a history, then a is one of the actions available to the player who moves after h. Thus the set of all actions available to the player who moves after h is

$$A(h) = \{a\colon (h, a) \text{ is a history}\}. \tag{154.1}$$

For example, for the game in Figure 154.1, the histories are $\varnothing$, *In*, *Out*, $(\textit{In}, \textit{Acquiesce})$, and $(\textit{In}, \textit{Fight})$. Thus the set of actions available to the player who moves at the start of the game, namely the challenger, is $A(\varnothing) = \{\textit{In}, \textit{Out}\}$, and the set of actions available to the player who moves after the history *In*, namely the incumbent, is $A(\textit{In}) = \{\textit{Acquiesce}, \textit{Fight}\}$.

? EXERCISE 154.2 (Examples of extensive games with perfect information)

a. Represent in a diagram like Figure 154.1 the two-player extensive game with perfect information in which the terminal histories are (C, E), (C, F), (D, G), and (D, H), the player function is given by $P(\varnothing) = 1$ and $P(C) = P(D) = 2$, player 1 prefers (C, F) to (D, G) to (C, E) to (D, H), and player 2 prefers (D, G) to (C, F) to (D, H) to (C, E).

b. Write down the set of players, set of terminal histories, player function, and players' preferences for the game in Figure 158.1.

c. The political figures Rosa and Ernesto each has to take a position on an issue. The options are Berlin (B) or Havana (H). They choose sequentially. A third person, Karl, determines who chooses first. Both Rosa and Ernesto care only about the actions they choose, not about who chooses first. Rosa prefers the outcome in which both she and Ernesto choose B to that in which they both choose H, and prefers this outcome to either of the ones in which she and Ernesto choose different actions; she is indifferent between these last two outcomes. Ernesto's preferences differ from Rosa's in that the roles of B and H are reversed. Karl's preferences are the same as Ernesto's. Model this situation as an extensive game with perfect information. (Specify the components of the game and represent the game in a diagram.)

Definition 153.1 allows terminal histories to be infinitely long. Thus we can use the model of an extensive game to study situations in which the participants do not consider any particular fixed horizon when making decisions. If the length of the longest terminal history is in fact finite, we say that the game has a **finite horizon**.

Even a game with a finite horizon may have infinitely many terminal histories, because some player has infinitely many actions after some history. If a game has a finite horizon *and* finitely many terminal histories we say it is **finite**. Note that a game that is not finite cannot be represented in a diagram like Figure 154.1, because such a figure allows for only finitely many branches.

An extensive game with perfect information models a situation in which each player, when choosing an action, knows all actions chosen previously (has *perfect information*), and always moves alone (rather than simultaneously with other players). Some economic and political situations that the model encompasses are discussed in the next chapter. The competition between interest groups courting legislators is one example. This situation may be modeled as an extensive game in which the groups sequentially offer payments to induce the legislators to vote for their favorite version of a bill (Section 6.4). A race (between firms developing a new technology, or between directors making competing movies, for instance), is another example. This situation is modeled as an extensive game in which the parties alternately decide how much effort to expend (Section 6.5). Parlor games such as chess, ticktacktoe, and go, in which there are no random events, the players move sequentially, and each player always knows all actions taken previously, may also be modeled as extensive games with perfect information (see the box on page 176).

In Section 7.1 I discuss a more general notion of an extensive game in which players may move simultaneously, though each player, when choosing an action, still knows all previous actions. In Chapter 10 I discuss a much more general notion that allows arbitrary patterns of information. In each case I sometimes refer to the object under consideration simply as an "extensive game".

5.2.2 Solutions

In the entry game in Figure 154.1, it seems clear that the challenger will enter and the incumbent will subsequently acquiesce. The challenger can reason that if it enters then the incumbent will acquiesce, because doing so is better for the incumbent than fighting. Given that the incumbent will respond to entry in this way, the challenger is better off entering.

This line of argument is called *backward induction*. Whenever a player has to move, she deduces, for each of her possible actions, the actions that the players (including herself) will subsequently rationally take, and chooses the action that yields the terminal history she most prefers.

While backward induction may be applied to the game in Figure 154.1, it cannot be applied to every extensive game with perfect information. Consider, for example, the variant of this game shown in Figure 156.1, in which the incumbent's payoff to the terminal history $(In, Fight)$ is 1 rather than 0. If, in the modified game, the challenger enters, the incumbent is indifferent between acquiescing and fighting. Backward induction does not tell the challenger what the incumbent will do in this case, and thus leaves open the question of which action the challenger should choose. Games with infinitely long histories present another difficulty for backward induction: they have no end from which to start the induction. The generalization of an extensive game with perfect information that allows for simultaneous moves (studied in Chapter 7) poses yet another problem: when players move simultaneously we cannot in general straightforwardly deduce each player's optimal action. (As in a strategic game, each player's best action depends on the other players' actions.)

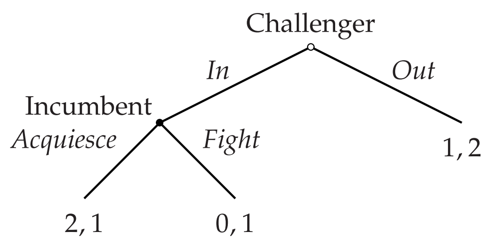

Figure 156.1 A variant of the entry game of Figure 154.1. The challenger's payoff is the first number in each pair.

Another approach to defining equilibrium takes off from the notion of Nash equilibrium. It seeks to model patterns of behavior that can persist in a steady state. The resulting notion of equilibrium applies to all extensive games with perfect information. Because the idea of backward induction is more limited, and the principles behind the notion of Nash equilibrium have been established in previous chapters, I begin by discussing the steady state approach. In games in which backward induction is well-defined, this approach turns out to lead to the backward induction outcome, so that there is no conflict between the two ideas.

5.3 Strategies and outcomes

5.3.1 Strategies

A key concept in the study of extensive games is that of a *strategy*. A player's strategy specifies the action the player chooses for *every* history after which it is her turn to move.

▶ DEFINITION 157.1 (*Strategy*) A **strategy** of player i in an extensive game with perfect information is a function that assigns to each history h after which it is player i's turn to move (i.e. $P(h) = i$, where P is the player function) an action in $A(h)$ (the set of actions available after h).

Consider the game in Figure 157.1.

- Player 1 moves only at the start of the game (i.e. after the empty history), when the actions available to her are C and D. Thus she has two strategies: one that assigns C to the empty history, and one that assigns D to the empty history.
- Player 2 moves after both the history C and the history D. After the history C the actions available to her are E and F, and after the history D the actions available to her are G and H. Thus a strategy of player 2 is a function that assigns either E or F to the history C, and either G or H to the history D. That is, player 2 has *four* strategies, which are shown in Figure 157.2.

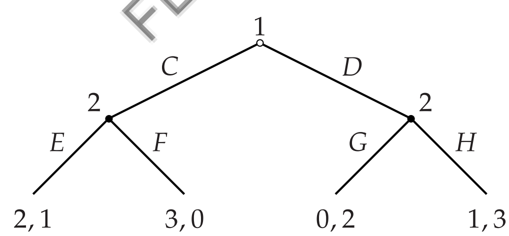

Figure 157.1 An extensive game with perfect information.

	Action assigned to history C	Action assigned to history D
Strategy 1	E	G
Strategy 2	E	H
Strategy 3	F	G
Strategy 4	F	H

Figure 157.2 The four strategies of player 2 in the game in Figure 157.1.

I refer to the strategies of player 1 in this game simply as C and D, and to the strategies of player 2 simply as EG, EH, FG, and FH. For many other finite games I use a similar shorthand: I write a player's strategy as a list of actions, one for each history after which it is the player's turn to move. In general I write the actions in the order in which they occur in the game, and, if they are available at the same "stage", from left to right as they appear in the diagram of the game. When the meaning of a list of actions is unclear, I explicitly give the history after which each action is taken.

Each of player 2's strategies in the game in Figure 157.1 may be interpreted as a plan of action or contingency plan: it specifies what player 2 does *if* player 1 chooses C, *and* what she does *if* player 1 chooses D. In every game, a player's strategy provides sufficient information to determine her *plan of action*: the actions she intends to take, *whatever* the other players do. In particular, if a player appoints an agent to play the game for her, and tells the agent her strategy, then the agent has enough information to carry out her wishes, *whatever* actions the other players take.

In some games some players' strategies are *more* than plans of action. Consider the game in Figure 158.1. Player 1 moves both at the start of the game and after the history (C, E). In each case she has two actions, so she has *four* strategies: CG (i.e. choose C at the start of the game and G after the history (C, E)), CH, DG, and DH. In particular, each strategy specifies an action after the history (C, E) *even if it specifies the action D at the beginning of the game*, in which case the history (C, E) does not occur! The point is that Definition 157.1 requires that a strategy of any player i specify an action for *every* history after which it is player i's turn to move, *even for histories that, if the strategy is followed, do not occur*.

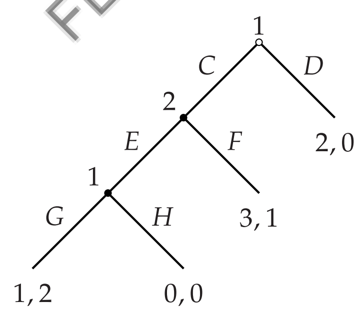

Figure 158.1 An extensive game in which player 1 moves both before and after player 2.

In view of this point and the fact that "strategy" is a synonym for "plan of action" in everyday language, you may regard the word "strategy" as inappropriate for the concept in Definition 157.1. You are right. You may also wonder why we cannot restrict attention to plans of action.

For the purposes of the notion of Nash equilibrium (discussed in the next section), we *could* in fact work with plans of action rather than strategies. But, as we shall see, the notion of Nash equilibrium for an extensive game is not satisfactory; the concept we adopt depends on the players' full strategies. When discussing this concept (in Section 5.5.4) I elaborate on the interpretation of a strategy. At the moment, you may think of a player's strategy as a plan of what to do, whatever the other players do, both if the player carries out her intended actions, and also if she makes mistakes. For example, we can interpret the strategy DG of player 1 in the game in Figure 158.1 to mean "I intend to choose D, but if I make a mistake and choose C instead then I will subsequently choose G". (Because the notion of Nash equilibrium depends only on plans of action, I could delay the definition of a strategy to the start of Section 5.5. I do not do so because the notion of a strategy is central to the study of extensive games, and its precise definition is much simpler than that of a plan of action.)

? EXERCISE 159.1 (Strategies in extensive games) What are the strategies of the players in the entry game (Example 153.2)? What are Rosa's strategies in the game in Exercise 154.2c?

5.3.2 Outcomes

A strategy profile determines the terminal history that occurs. Denote the strategy profile by s and the player function by P. At the start of the game player $P(\varnothing)$ moves. Her strategy is $s_{P(\varnothing)}$, and she chooses the action $s_{P(\varnothing)}(\varnothing)$. Denote this action by a^1. If the history a^1 is not terminal, player $P(a^1)$ moves next. Her strategy is $s_{P(a^1)}$, and she chooses the action $s_{P(a^1)}(a^1)$. Denote this action by a^2. If the history (a^1, a^2) is not terminal, then again the player function specifies whose turn it is to move, and that player's strategy specifies the action she chooses. The process continues until a terminal history is constructed. We refer to this terminal history as the **outcome of** s, and denote it $O(s)$.

In the game in Figure 158.1, for example, the outcome of the strategy pair (DG, E) is the terminal history D, and the outcome of (CH, E) is the terminal history (C, E, H).

Note that the outcome $O(s)$ of the strategy profile s depends only on the players' plans of action, not their full strategies. That is, to determine $O(s)$ we do *not* need to refer to any component of any player's strategy that specifies her actions after histories precluded by that strategy.

5.4 Nash equilibrium

As for strategic games, we are interested in notions of equilibrium that model the players' behavior in a steady state. That is, we look for patterns of behavior with the property that if every player knows every other player's behavior, she has no reason to change her own behavior. I start by defining a Nash equilibrium: a strategy profile from which no player wishes to deviate, given the other players' strategies. The definition is an adaptation of that of a Nash equilibrium in a strategic game (21.1).

► DEFINITION 159.2 (*Nash equilibrium of extensive game with perfect information*) The strategy profile s^* in an extensive game with perfect information is a **Nash equi-**

librium if, for every player i and every strategy r_i of player i, the terminal history $O(s^*)$ generated by s^* is at least as good according to player i's preferences as the terminal history $O(r_i, s^*_{-i})$ generated by the strategy profile (r_i, s^*_{-i}) in which player i chooses r_i while every other player j chooses s^*_j. Equivalently, for each player i,

$$u_i(O(s^*)) \geq u_i(O(r_i, s^*_{-i})) \text{ for every strategy } r_i \text{ of player } i,$$

where u_i is a payoff function that represents player i's preferences and O is the outcome function of the game.

One way to find the Nash equilibria of an extensive game in which each player has finitely many strategies is to list each player's strategies, find the outcome of each strategy profile, and analyze this information as for a strategic game. That is, we construct the following strategic game, known as the **strategic form** of the extensive game.

Players The set of players in the extensive game.

Actions Each player's set of actions is her set of strategies in the extensive game.

Preferences Each player's payoff to each action profile is her payoff to the terminal history generated by that action profile in the extensive game.

From Definition 159.2 we see that

> the set of Nash equilibria of any extensive game with perfect information is the set of Nash equilibria of its strategic form.

◆ EXAMPLE 160.1 (Nash equilibria of the entry game) In the entry game in Figure 154.1, the challenger has two strategies, *In* and *Out*, and the incumbent has two strategies, *Acquiesce* and *Fight*. The strategic form of the game is shown in Figure 160.1. We see that it has two Nash equilibria: (*In*, *Acquiesce*) and (*Out*, *Fight*). The first equilibrium is the pattern of behavior isolated by backward induction, discussed at the start of Section 5.2.2.

		Incumbent	
		Acquiesce	*Fight*
Challenger	*In*	2, 1	0, 0
	Out	1, 2	1, 2

Figure 160.1 The strategic form of the entry game in Figure 154.1.

In the second equilibrium the challenger always chooses *Out*. This strategy is optimal given the incumbent's strategy to fight in the event of entry. Further, the incumbent's strategy *Fight* is optimal given the challenger's strategy: the challenger chooses *Out*, so whether the incumbent plans to choose *Acquiesce* or *Fight* makes no difference to its payoff. Thus neither player can increase its payoff by choosing a different strategy, *given the other player's strategy*.

Thinking about the extensive game in this example raises a question about the Nash equilibrium $(Out, Fight)$ that does not arise when thinking about the strategic form: how does the challenger know that the incumbent will choose *Fight* if it enters? We interpret the strategic game to model a situation in which, whenever the challenger plays the game, it observes the incumbent's action, even if it chooses *Out*. By contrast, we interpret the extensive game to model a situation in which a challenger that always chooses *Out* never observes the incumbent's action, because the incumbent never moves. In a strategic game, the rationale for the Nash equilibrium condition that each player's strategy be optimal given the other players' strategies is that in a steady state, each player's experience playing the game leads her belief about the other players' actions to be correct. This rationale does not apply to the Nash equilibrium $(Out, Fight)$ of the (extensive) entry game, because a challenger who always chooses *Out* never observes the incumbent's action after the history *In*.

We can escape from this difficulty in interpreting a Nash equilibrium of an extensive game by considering a slightly perturbed steady state in which, on rare occasions, nonequilibrium actions are taken (perhaps players make mistakes, or deliberately experiment), and the perturbations allow each player eventually to observe every other player's action after *every* history. Given such perturbations, each player eventually learns the other players' entire strategies.

Interpreting the Nash equilibrium $(Out, Fight)$ as such a perturbed steady state, however, we run into another problem. On those (rare) occasions when the challenger enters, the subsequent behavior of the incumbent to fight is not a steady state in the remainder of the game: if the challenger enters, the incumbent is better off acquiescing than fighting. That is, the Nash equilibrium $(Out, Fight)$ does not correspond to a *robust* steady state of the extensive game.

Note that the extensive game embodies the assumption that the incumbent cannot commit, at the beginning of the game, to fight if the challenger enters; it is free to choose either *Acquiesce* or *Fight* in this event. If the incumbent *could* commit to fight in the event of entry then the analysis would be different. Such a commitment would induce the challenger to stay out, an outcome that the incumbent prefers. In the absence of the possibility of the incumbent's making a commitment, we might think of the its *announcing* at the start of the game that it intends to fight; but such a *threat* is not credible, because after the challenger enters the incumbent's only incentive is to acquiesce.

? EXERCISE 161.1 (Nash equilibria of extensive games) Find the Nash equilibria of the games in Exercise 154.2*a* and Figure 158.1. (When constructing the strategic form of each game, be sure to include *all* the strategies of each player.)

? EXERCISE 161.2 (Voting by alternating veto) Two people select a policy that affects them both by alternately vetoing policies until only one remains. First person 1 vetoes a policy. If more than one policy remains, person 2 then vetoes a policy. If more than one policy still remains, person 1 then vetoes another policy. The process continues until only one policy has not been vetoed. Suppose there are three possible policies, X, Y, and Z, person 1 prefers X to Y to Z, and person 2 prefers Z to Y to X. Model this situation as an extensive game and find its Nash equilibria.

5.5 Subgame perfect equilibrium

5.5.1 Definition

The notion of Nash equilibrium ignores the sequential structure of an extensive game; it treats strategies as choices made once and for all before play begins. Consequently, as we saw in the previous section, the steady state to which a Nash equilibrium corresponds may not be robust.

I now define a notion of equilibrium that models a robust steady state. This notion requires each player's strategy to be optimal, given the other players' strategies, not only at the start of the game, but after every possible history.

To define this concept, I first define the notion of a subgame. For any nonterminal history h, the *subgame* following h is the part of the game that remains after h has occurred. For example, the subgame following the history *In* in the entry game (Example 152.1) is the game in which the incumbent is the only player, and there are two terminal histories, *Acquiesce* and *Fight*.

▶ DEFINITION 162.1 (*Subgame*) Let Γ be an extensive game with perfect information, with player function P. For any nonterminal history h of Γ, the **subgame** $\Gamma(h)$ **following the history** h is the following extensive game.

Players The players in Γ.

Terminal histories The set of all sequences h' of actions such that (h, h') is a terminal history of Γ.

Player function The player $P(h, h')$ is assigned to each proper subhistory h' of a terminal history.

Preferences Each player prefers h' to h'' if and only if she prefers (h, h') to (h, h'') in Γ.

Note that the subgame following the initial history $\varnothing$ is the entire game. Every other subgame is called a *proper subgame*. Because there is a subgame for every nonterminal history, the number of subgames is equal to the number of nonterminal histories.

As an example, the game in Figure 157.1 has three nonterminal histories (the initial history, C, and D), and hence three subgames: the whole game (the part of the game following the initial history), the game following the history C, and the game following the history D. The two proper subgames are shown in Figure 163.1.

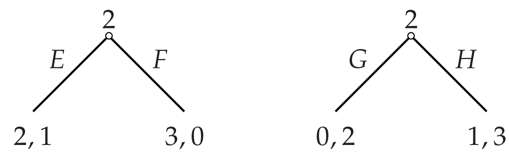

Figure 163.1 The two proper subgames of the extensive game in Figure 157.1.

The game in Figure 158.1 also has three nonterminal histories, and hence three subgames: the whole game, the game following the history C, and the game following the history (C, E). The two proper subgames are shown in Figure 163.2.

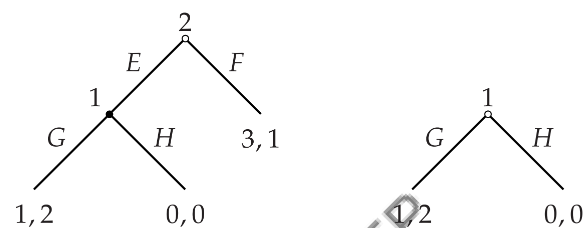

Figure 163.2 The two proper subgames of the extensive game in Figure 158.1.

? EXERCISE 163.1 (Subgames) Find all the subgames of the game in Exercise 154.2*c*.

In an equilibrium that corresponds to a perturbed steady state in which *every* history sometimes occurs, the players' behavior must correspond to a steady state in *every subgame*, not only in the whole game. Interpreting the actions specified by a player's strategy in a subgame to give the player's behavior if, possibly after a series of mistakes, that subgame is reached, this condition is embodied in the following informal definition.

> A *subgame perfect equilibrium* is a strategy profile s^* with the property that in no subgame can any player i do better by choosing a strategy different from s_i^*, given that every other player j adheres to s_j^*.

(Compare this definition with that of a Nash equilibrium of a strategic game, on page 19.)

For example, the Nash equilibrium $(Out, Fight)$ of the entry game (Example 152.1) is not a subgame perfect equilibrium because in the subgame following the history *In*, the strategy *Fight* is not optimal for the incumbent: *in this subgame*, the incumbent is better off choosing *Acquiesce* than it is choosing *Fight*. The Nash equilibrium $(In, Acquiesce)$ *is* a subgame perfect equilibrium: each player's strategy is optimal, given the other player's strategy, both in the whole game, and in the subgame following the history *In*.

To define the notion of subgame perfect equilibrium precisely, we need a new piece of notation. Let h be a history and s a strategy profile. Suppose that h occurs (even though it is not necessarily consistent with s), and *afterwards* the players adhere to the strategy profile s. Denote the resulting terminal history by $O_h(s)$. That is, $O_h(s)$ is the terminal history consisting of h followed by the outcome generated in the subgame following h by the strategy profile induced by s in the subgame. Note that for any strategy profile s, we have $O_\varnothing(s) = O(s)$ (where $\varnothing$, as always, denotes the initial history).

As an example, consider again the entry game. Let s be the strategy profile $(Out, Fight)$ and let h be the history In. If h occurs, and *afterwards* the players adhere to s, the resulting terminal history is $O_h(s) = (In, Fight)$.

▶ DEFINITION 164.1 (*Subgame perfect equilibrium*) The strategy profile s^* in an extensive game with perfect information is a **subgame perfect equilibrium** if, for every player i, every history h after which it is player i's turn to move (i.e. $P(h) = i$), and every strategy r_i of player i, the terminal history $O_h(s^*)$ generated by s^* after the history h is at least as good according to player i's preferences as the terminal history $O_h(r_i, s^*_{-i})$ generated by the strategy profile (r_i, s^*_{-i}) in which player i chooses r_i while every other player j chooses s^*_j. Equivalently, for every player i and every history h after which it is player i's turn to move,

$$u_i(O_h(s^*)) \geq u_i(O_h(r_i, s^*_{-i})) \text{ for every strategy } r_i \text{ of player } i,$$

where u_i is a payoff function that represents player i's preferences and $O_h(s)$ is the terminal history consisting of h followed by the sequence of actions generated by s after h.

The important point in this definition is that each player's strategy is required to be optimal for *every* history after which it is the player's turn to move, not only at the start of the game as in the definition of a Nash equilibrium (159.2).

5.5.2 Subgame perfect equilibrium and Nash equilibrium

In a subgame perfect equilibrium every player's strategy is optimal, in particular, after the initial history (put $h = \varnothing$ in the definition, and remember that $O_\varnothing(s) = O(s)$). Thus:

> Every subgame perfect equilibrium is a Nash equilibrium.

In fact, a subgame perfect equilibrium generates a Nash equilibrium in every subgame: if s^* is a subgame perfect equilibrium then, for any history h and player i, the strategy induced by s^*_i in the subgame following h is optimal given the strategies induced by s^*_{-i} in the subgame. Further, any strategy profile that generates a Nash equilibrium in every subgame is a subgame perfect equilibrium, so that we can give the following alternative definition.

> A *subgame perfect equilibrium* is a strategy profile that induces a Nash equilibrium in every subgame.

In a Nash equilibrium every player's strategy is optimal, given the other players' strategies, in the whole game. As we have seen, it may *not* be optimal in some subgames. I claim, however, that it *is* optimal in any subgame that is reached when the players follow their strategies. Given this claim, the significance of the requirement in the definition of a subgame perfect equilibrium that each player's strategy be optimal after every history, relative to the requirement in the definition of a Nash equilibrium, is that each player's strategy be optimal after histories that do not occur if the players follow their strategies (like the history *In* when the challenger's action is *Out* at the beginning of the entry game).

To show my claim, suppose that s^* is a Nash equilibrium of a game in which you are player i. Then your strategy s_i^* is optimal given the other players' strategies s_{-i}^*. When the other players follow their strategies, there comes a point (possibly the start of the game) when you have to move for the first time. Suppose that at this point you follow your strategy s_i^*; denote the action you choose by C. Now, after having chosen C, should you change your strategy in the rest of the game, given that the other players will continue to adhere to their strategies? No! If you could do better by changing your strategy after choosing C—say by switching to the strategy s_i' in the subgame—then you could have done better at the start of the game by choosing the strategy that chooses C and then follows s_i'. That is, if your plan is optimal, given the other players' strategies, at the start of the game, and you stick to it, then you never want to change your mind after play begins, as long as the other players stick to their strategies. (The general principle is known as the *Principle of Optimality* in dynamic programming.)

5.5.3 Examples

◆ EXAMPLE 165.1 (Entry game) Consider again the entry game of Example 152.1, which has two Nash equilibria, $(In, Acquiesce)$ and $(Out, Fight)$. The fact that the Nash equilibrium $(Out, Fight)$ is not a subgame perfect equilibrium follows from the formal definition as follows. For $s^* = (Out, Fight)$, $i =$ Incumbent, $r_i = Acquiesce$, and $h = In$, we have $O_h(s^*) = (In, Fight)$ and $O_h(r_i, s_{-i}^*) = (In, Acquiesce)$, so that the inequality in the definition is violated: $u_i(O_h(s^*)) = 0$ and $u_i(O_h(r_i, s_{-i}^*)) = 1$.

The Nash equilibrium $(In, Acquiesce)$ *is* a subgame perfect equilibrium because (*a*) it is a Nash equilibrium, so that at the start of the game the challenger's strategy *In* is optimal, given the incumbent's strategy *Acquiesce*, and (*b*) after the history *In*, the incumbent's strategy *Acquiesce* in the subgame is optimal. In the language of the formal definition, let $s^* = (In, Acquiesce)$.

- The challenger moves after one history, namely $h = \varnothing$. We have $O_h(s^*) = (In, Acquiesce)$ and hence for $i =$ challenger we have $u_i(O_h(s^*)) = 2$, whereas for the only other strategy of the challenger, $r_i = Out$, we have $u_i(O_h(r_i, s_{-i}^*)) = 1$.

- The incumbent moves after one history, namely $h = In$. We have $O_h(s^*) = (In, Acquiesce)$ and hence for $i =$ incumbent we have $u_i(O_h(s^*)) = 1$, whereas for the only other strategy of the incumbent, $r_i = Fight$, we have $u_i(O_h(r_i, s^*_{-i})) = 0$.

Every subgame perfect equilibrium is a Nash equilibrium, so we conclude that the game has a unique subgame perfect equilibrium, $(In, Acquiesce)$.

◆ EXAMPLE 166.1 (Variant of entry game) Consider the variant of the entry game in which the incumbent is indifferent between fighting and acquiescing if the challenger enters (see Figure 156.1). This game, like the original game, has two Nash equilibria, $(In, Acquiesce)$ and $(Out, Fight)$. But now *both* of these equilibria are subgame perfect equilibria, because after the history *In* both *Fight* and *Acquiesce* are optimal for the incumbent.

In particular, the game has a steady state in which every challenger always chooses *In* and every incumbent always chooses *Acquiesce*. If you, as the challenger, were playing the game for the first time, you would probably regard the action *In* as "risky", because after the history *In* the incumbent is indifferent between *Acquiesce* and *Fight*, and you prefer the terminal history *Out* to the terminal history $(In, Fight)$. Indeed, as discussed in Section 5.2.2, backward induction does not yield a clear solution of this game. But the subgame perfect equilibrium $(In, Acquiesce)$ corresponds to a perfectly reasonable steady state. If you had played the game hundreds of times against opponents drawn from the same population, and on every occasion your opponent had chosen *Acquiesce*, you could reasonably expect your next opponent to choose *Acquiesce*, and thus optimally choose *In*.

? EXERCISE 166.2 (Checking for subgame perfect equilibria) Which of the Nash equilibria of the game in Figure 158.1 are subgame perfect?

5.5.4 *Interpretation*

A Nash equilibrium of a strategic game corresponds to a steady state in an idealized setting in which the participants in each play of the game are drawn randomly from a collection of populations (see Section 2.6). The idea is that each player's long experience playing the game leads her to correct beliefs about the other players' actions; given these beliefs her equilibrium action is optimal.

A subgame perfect equilibrium of an extensive game corresponds to a slightly perturbed steady state, in which all players, on rare occasions, take nonequilibrium actions, so that after long experience each player forms correct beliefs about the other players' entire strategies, and thus knows how the other players will behave in every subgame. Given these beliefs, no player wishes to deviate from her strategy either at the start of the game or after *any* history.

This interpretation of a subgame perfect equilibrium, like the interpretation of a Nash equilibrium as a steady state, does not require a player to know the other players' preferences, or to think about the other players' rationality. It entails interpreting a strategy as a plan specifying a player's actions not only after histories consistent with the strategy, but also after histories that result when the player chooses arbitrary alternative actions, perhaps because she makes mistakes or deliberately experiments.

The subgame perfect equilibria of some extensive game can be given other interpretations. In some cases, one alternative interpretation is particularly attractive. Consider an extensive game with perfect information in which each player has a unique best action at every history after which it is her turn to move, and the horizon is finite. In such a game, a player who knows the other players' preferences and knows that the other players are rational can use backward induction to deduce her optimal strategy, as discussed in Section 5.2.2. Thus we can interpret a subgame perfect equilibrium as the outcome of the players' rational calculations about each other's strategies.

This interpretation of a subgame perfect equilibrium entails an interpretation of a strategy different from the one that fits the steady state interpretation. Consider, for example, the game in Figure 158.1. When analyzing this game, player 1 must consider the consequences of choosing C. Thus she must think about player 2's action after the history C, and hence must form a belief about what player 2 thinks she (player 1) will do after the history (C, E). The component of her strategy that specifies her action after this history reflects this belief. For instance, the strategy DG means that player 1 chooses D at the start of the game and believes that were she to choose C, player 2 would believe that after the history (C, E) she would choose G. In an arbitrary game, the interpretation of a subgame perfect equilibrium as the outcome of the players' rational calculations about each other's strategies entails interpreting the components of a player's strategy that assign actions to histories inconsistent with other parts of the strategy as specifying the player's belief about the other players' beliefs about what the player will do if one of these histories occurs.

This interpretation of a subgame perfect equilibrium is not free of difficulties, which are discussed in Section 7.7. Further, the interpretation is not tenable in games in which some player has more than one optimal action after some history, or in the more general extensive games considered in Section 7.1 and Chapter 10. Nevertheless, in some of the games studied in this chapter and the next it is an appealing alternative to the steady state interpretation. Further, an extension of the procedure of backward induction can be used to find all subgame perfect equilibria of finite horizon games, as we shall see in the next section. (This extension cannot be given an appealing behavioral interpretation in games in which some player has more than one optimal action after some history.)

5.6 Finding subgame perfect equilibria of finite horizon games: backward induction

We found the subgame perfect equilibria of the games in Examples 165.1 and 166.1 by finding the Nash equilibria of the games and checking whether each of these equilibria is subgame perfect. In a game with a finite horizon the set of subgame perfect equilibria may be found more directly by using an extension of the procedure of backward induction discussed briefly in Section 5.2.2.

Define the *length of a subgame* to be the length of the longest history in the subgame. (The lengths of the subgames in Figure 163.2, for example, are 2 and 1.) The procedure of backward induction works as follows. We start by finding the optimal actions of the players who move in the subgames of length 1 (the "last" subgames). Then, taking these actions as given, we find the optimal actions of the players who move first in the subgames of length 2. We continue working back to the beginning of the game, at each stage k finding the optimal actions of the players who move at the start of the subgames of length k, given the optimal actions we have found in all shorter subgames.

At each stage k of this procedure, the optimal actions of the players who move at the start of the subgames of length k are easy to determine: they are simply the actions that yield the players the highest payoffs, given the optimal actions in all shorter subgames.

Consider, for example, the game in Figure 168.1.

- First consider subgames of length 1. The game has two such subgames, in both of which player 2 moves. In the subgame following the history C, player 2's optimal action is E, and in the subgame following the history D, her optimal action is H.
- Now consider subgames of length 2. The game has one such subgame, namely the entire game, at the start of which player 1 moves. Given the optimal actions in the subgames of length 1, player 1's choosing C at the start of the game yields her a payoff of 2, whereas her choosing D yields her a payoff of 1. Thus player 1's optimal action at the start of the game is C.

The game has no subgame of length greater than 2, so the procedure of backward induction yields the strategy pair (C, EH).

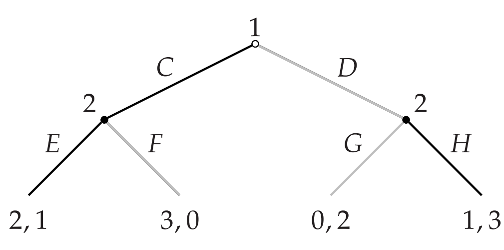

Figure 168.1 A game illustrating the procedure of backward induction. The actions selected by backward induction are indicated in black.

As another example, consider the game in Figure 158.1. We first deduce that in the subgame of length 1 following the history (C, E), player 1 chooses G; then that at the start of the subgame of length 2 following the history C, player 2 chooses E;

then that at the start of the whole game, player 1 chooses D. Thus the procedure of backward induction in this game yields the strategy pair (DG, E).

In any game in which this procedure selects a single action for the player who moves at the start of each subgame, the strategy profile thus selected is the unique subgame perfect equilibrium of the game. (You should find this result very plausible, though a complete proof is not trivial.)

What happens in a game in which at the start of some subgames more than one action is optimal? In such a game an extension of the procedure of backward induction locates all subgame perfect equilibrium. This extension traces back *separately* the implications for behavior in the longer subgames of *every combination* of optimal actions in the shorter subgames.

Consider, for example, the game in Figure 170.1.

- The game has three subgames of length one, in each of which player 2 moves. In the subgames following the histories C and D, player 2 is indifferent between her two actions. In the subgame following the history E, player 2's unique optimal action is K. Thus there are *four* combinations of player 2's optimal actions in the subgames of length 1: FHK, FIK, GHK, and GIK (where the first component in each case is player 2's action after the history C, the second component is her action after the history D, and the third component is her action after the history E).
- The game has a single subgame of length two, namely the whole game, in which player 1 moves first. We now consider player 1's optimal action in this game for *every combination* of the optimal actions of player 2 in the subgames of length 1.
  - For the combinations FHK and FIK of optimal actions of player 2, player 1's optimal action at the start of the game is C.
  - For the combination GHK of optimal actions of player 2, the actions C, D, and E are all optimal for player 1.
  - For the combination GIK of optimal actions of player, player 1's optimal action at the start of the game is D.

Thus the strategy pairs isolated by the procedure are (C, FHK), (C, FIK), (C, GHK), (D, GHK), (E, GHK), and (D, GIK).

The procedure, which for simplicity I refer to simply as **backward induction**, may be described compactly for an arbitrary game as follows.

- Find, for each subgame of length 1, the set of optimal actions of the player who moves first. Index the subgames by j, and denote by $S_j^*(1)$ the set of optimal actions in subgame j. (If the player who moves first in subgame j has a unique optimal action, then $S_j^*(1)$ contains a single action.)
- For each combination of actions consisting of one from each set $S_j^*(1)$, find, for each subgame of length two, the set of optimal actions of the player who

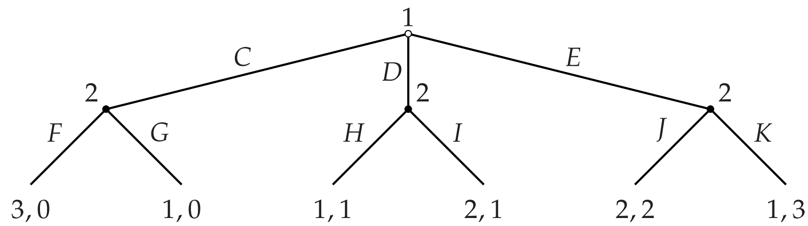

Figure 170.1 A game in which the first-mover in some subgames has multiple optimal actions.

moves first. The result is a set of strategy profiles for each subgame of length two. Denote by $S^*_\ell(2)$ the set of strategy profiles in subgame ℓ.

- Continue by examining successively longer subgames until reaching the start of the game. At each stage k, for each combination of strategy profiles consisting of one from each set $S^*_p(k-1)$ constructed in the previous stage, find, for each subgame of length k, the set of optimal actions of the player who moves first, and hence a set of strategy profiles for each subgame of length k.

The set of strategy profiles that this procedure yields for the whole game is the set of subgame perfect equilibria of the game.

■ PROPOSITION 170.1 (Subgame perfect equilibrium of finite horizon games and backward induction) *The set of subgame perfect equilibria of a finite horizon extensive game with perfect information is equal to the set of strategy profiles isolated by the procedure of backward induction.*

You should find this result, like my claim for games in which the player who moves at the start of every subgame has a single optimal action, very plausible, though again a complete proof is not trivial.

In the terminology of my description of the general procedure, the analysis for the game in Figure 170.1 is as follows. Number the subgames of length one from left to right. Then we have $S^*_1(1) = \{F, G\}$, $S^*_2(1) = \{H, I\}$, and $S^*_3(1) = \{K\}$. There are four lists of actions consisting of one action from each set: *FHK*, *FIK*, *GHK*, and *GIK*. For *FHK* and *FIK*, the action C of player 1 is optimal at the start of the game; for *GHK* the actions C, D, and E are all optimal; and for *GIK* the action D is optimal. Thus the set $S^*(2)$ of strategy profiles consists of (C, FHK), (C, FIK), (C, GHK), (D, GHK), (E, GHK), and (D, GIK). There are no longer subgames, so this set of strategy profiles is the set of subgame perfect equilibria of the game.

Each example I have presented so far in this section is a finite game—that is, a game that not only has a finite horizon, but also a finite number of terminal histories. In such a game, the player who moves first in any subgame has finitely many actions; at least one action is optimal. Thus in such a game the procedure of backward induction isolates at least one strategy profile. Using Proposition 170.1, we conclude that every finite game has a subgame perfect equilibrium.

■ PROPOSITION 171.1 (Existence of subgame perfect equilibrium) *Every finite extensive game with perfect information has a subgame perfect equilibrium.*

Note that this result does *not* claim that a finite extensive game has a *single* subgame perfect equilibrium. (As we have seen, the game in Figure 170.1, for example, has more than one subgame perfect equilibrium.)

A finite horizon game in which some player does not have finitely many actions after some history may or may not possess a subgame perfect equilibrium. A simple example of a game that does not have a subgame perfect equilibrium is the trivial game in which a single player chooses a number *less than* 1 and receives a payoff equal to the number she chooses. There is no greatest number less than one, so the single player has no optimal action, and thus the game has no subgame perfect equilibrium.

? EXERCISE 171.2 (Finding subgame perfect equilibria) Find the subgame perfect equilibria of the games in parts *a* and *c* of Exercise 154.2, and in Figure 171.1.

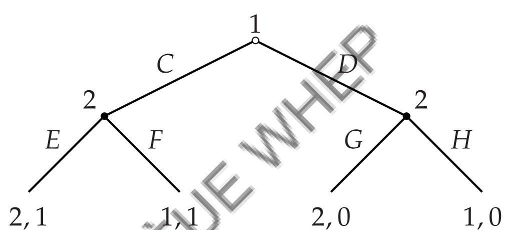

Figure 171.1 One of the games for Exercise 171.2.

? EXERCISE 171.3 (Voting by alternating veto) Find the subgame perfect equilibria of the game in Exercise 161.2. Does the game have any Nash equilibrium that is not a subgame perfect equilibrium? Is any outcome generated by a Nash equilibrium not generated by any subgame perfect equilibrium? Consider variants of the game in which player 2's preferences may be different from those specified in Exercise 161.2. Are there any preferences for which the outcome in a subgame perfect equilibrium of the game in which player 1 moves first differs from the outcome in a subgame perfect equilibrium of the game in which player 2 moves first?

? EXERCISE 171.4 (Burning a bridge) Army 1, of country 1, must decide whether to attack army 2, of country 2, which is occupying an island between the two countries. In the event of an attack, army 2 may fight, or retreat over a bridge to its mainland. Each army prefers to occupy the island than not to occupy it; a fight is the worst outcome for both armies. Model this situation as an extensive game with perfect information and show that army 2 can increase its subgame perfect equilibrium payoff (and reduce army 1's payoff) by burning the bridge to its mainland, eliminating its option to retreat if attacked.

? EXERCISE 172.1 (Sharing heterogeneous objects) A group of n people have to share k objects that the value differently. Each person assigns values to the objects; no one assigns the same value to two different objects. Each person evaluates a set of objects according to the sum of the values she assigns to the objects in the set. The following procedure is used to share the objects. The players are ordered 1 through n. Person 1 chooses an object, then person 2 does so, and so on; if $k > n$, then after person n chooses an object, person 1 chooses a second object, then person 2 chooses a second object, and so on. Objects are chosen until none remain. (In Canada and the USA professional sports teams use a similar procedure to choose new players.) Denote by $G(n,k)$ the extensive game that models this procedure. If $k \leq n$ then obviously $G(n,k)$ has a subgame perfect equilibrium in which each player's strategy is to choose her favorite object among those remaining when her turn comes. Show that if $k > n$ then $G(n,k)$ may have no subgame perfect equilibrium in which person 1 chooses her favorite object on the first round. (You can give an example in which $n = 2$ and $k = 3$.) Now fix $n = 2$. Define x_k to be the object least preferred by the person who does *not* choose at stage k (i.e. who does not choose the last object); define x_{k-1} to be the object, among all those except x_k, least preferred by the person who does *not* choose at stage $k-1$. Similarly, for any j with $2 \leq j \leq k$, given $x_j, \ldots, x_k$, define x_{j-1} to be the object, among all those excluding $\{x_j, \ldots, x_k\}$, least preferred by the person who does *not* choose at stage $j-1$. Show that the game $G(2,3)$ has a subgame perfect equilibrium in which for every $j = 1, \ldots, k$ the object x_j is chosen at stage j. (This result is true for $G(2,k)$ for all values of k.) If $n \geq 3$ then interestingly a person may be better off in all subgame perfect equilibria of $G(n,k)$ when she comes later in the ordering of players. (An example, however, is difficult to construct; one is given in Brams and Straffin (1979).)

The next exercise shows how backward induction can cause a relatively minor change in the way in which a game ends to reverberate to the start of the game, leading to a very different action for the first-mover.

?? EXERCISE 172.2 (An entry game with a financially-constrained firm) An incumbent in an industry faces the possibility of entry by a challenger. First the challenger chooses whether or not to enter. If it does not enter, neither firm has any further action; the incumbent's payoff is TM (it obtains the profit M in each of the following $T \geq 1$ periods) and the challenger's payoff is 0. If the challenger enters, it pays the entry cost $f > 0$, and in each of T periods the incumbent first commits to fight or cooperate with the challenger in that period, then the challenger chooses whether to stay in the industry or to exit. (Note that the order of the firms' moves within a period differs from that in the game in Example 152.1.) If, in any period, the challenger stays in, each firm obtains in that period the profit $-F < 0$ if the incumbent fights and $C > \max\{F, f\}$ if it cooperates. If, in any period, the challenger exits, both firms obtain the profit zero in that period (regardless of the incumbent's action); the incumbent obtains the profit $M > 2C$ and the challenger the profit

0 in every subsequent period. Once the challenger exits, it cannot subsequently re-enter. Each firm cares about the sum of its profits.

a. Find the subgame perfect equilibria of the extensive game that models this situation.

b. Consider a variant of the situation, in which the challenger is constrained by its financial war chest, which allows it to survive at most $T-2$ fights. Specifically, consider the game that differs from the one in part *a* only in that the history in which the challenger enters, in each of the following $T-2$ periods the incumbent fights and the challenger stays in, and in period $T-1$ the incumbent fights, is a terminal history (the challenger has to exit), in which the incumbent's payoff is M (it is the only firm in the industry in the last period) and the challenger's payoff is $-f$. Find the subgame perfect equilibria of this game.

◆ EXAMPLE 173.1 (Dollar auction) Consider an auction in which an object is sold to the highest bidder, but *both* the highest bidder *and* the second highest bidder pay their bids to the auctioneer. When such an auction is conducted and the object is a dollar, the outcome is sometimes that the object is sold at a price *greater* than a dollar. (Shubik writes that "A total of payments between three and five dollars is not uncommon" (1971, 110).) Obviously such an outcome is inconsistent with a subgame perfect equilibrium of an extensive game that models the auction: every participant has the option of not bidding, so that in no subgame perfect equilibrium can anyone's payoff be negative.

Why, then, do such outcomes occur? Suppose that there are two participants, and that both start bidding. If the player making the lower bid thinks that making a bid above the other player's bid will induce the other player to quit, she may be better off doing so than stopping bidding. For example, if the bids are currently \$0.50 and \$0.51, the player bidding \$0.50 is better off bidding \$0.52 *if* doing so induces the other bidder to quit, because she then wins the dollar and obtains a payoff of \$0.48, rather than losing \$0.50. The same logic applies even if the bids are greater than \$1.00, as long as they do not differ by more than \$1.00. If, for example, they are currently \$2.00 and \$2.01, then the player bidding \$2.00 loses only \$1.02 if a bid of \$2.02 induces her opponent to quit, whereas she loses \$2.00 if she herself quits. That is, in subgames in which bids have been made, the player making the second highest bid may optimally beat a bid that exceeds \$1.00, depending on the other players' strategies and the difference between the top two bids. (When discussing outcomes in which the total payment to the auctioneer exceeds \$1, Shubik remarks that "In playing this game, a large crowd is desirable ... the best time is during a party when spirits are high and the propensity to calculate does not settle in until at least two bids have been made" (1971, 109).)

In the next exercise you are asked to find the subgame perfect equilibria of an extensive game that models a simple example of such an auction.

? EXERCISE 173.2 (Dollar auction) An object that two people each value at v (a positive integer) is sold in an auction. In the auction, the people alternately have the opportunity to bid; a bid must be a positive integer greater than the previous bid. (In the situation that gives the game its name, v is 100 cents.) On her turn, a player may pass rather than bid, in which case the game ends and the other player receives the object; *both* players pay their last bids (if any). (If player 1 passes initially, for example, player 2 receives the object and makes no payment; if player 1 bids 1, player 2 bids 3, and then player 1 passes, player 2 obtains the object and pays 3, and player 1 pays 1.) Each person's wealth is w, which exceeds v; neither player may bid more than her wealth. For $v = 2$ and $w = 3$ model the auction as an extensive game and find its subgame perfect equilibria. (A much more ambitious project is to find all subgame perfect equilibria for arbitrary values of v and w.)

In all the extensive games studied so far in this chapter, each player has available finitely many actions whenever she moves. The next example shows how the procedure of backward induction may be used to find the subgame perfect equilibria of games in which a continuum of actions is available after some histories.

◆ EXAMPLE 174.1 (A synergistic relationship) Consider a variant of the situation in Example 37.1, in which two individuals are involved in a synergistic relationship. Suppose that the players choose their effort levels sequentially, rather than simultaneously. First individual 1 chooses her effort level a_1, then individual 2 chooses her effort level a_2. An effort level is a nonnegative number, and individual i's preferences (for $i = 1, 2$) are represented by the payoff function $a_i(c + a_j - a_i)$, where j is the other individual and $c > 0$ is a constant.

To find the subgame perfect equilibria, we first consider the subgames of length 1, in which individual 2 chooses a value of a_2. Individual 2's optimal action after the history a_1 is her best response to a_1, which we found to be $\frac{1}{2}(c + a_1)$ in Example 37.1. Thus individual 2's strategy in any subgame perfect equilibrium is the function that associates with each history a_1 the action $\frac{1}{2}(c + a_1)$.

Now consider individual 1's action at the start of the game. Given individual 2's strategy, individual 1's payoff if she chooses a_1 is $a_1(c + \frac{1}{2}(c + a_1) - a_1)$, or $\frac{1}{2}a_1(3c - a_1)$. This function is a quadratic that is zero when $a_1 = 0$ and when $a_1 = 3c$, and reaches a maximum in between. Thus individual 1's optimal action at the start of the game is $a_1 = \frac{3}{2}c$.

We conclude that the game has a unique subgame perfect equilibrium, in which individual 1's strategy is $a_1 = \frac{3}{2}c$ and individual 2's strategy is the function that associates with each history a_1 the action $\frac{1}{2}(c + a_1)$. The outcome of the equilibrium is that individual 1 chooses $a_1 = \frac{3}{2}c$ and individual 2 chooses $a_2 = \frac{5}{4}c$.

? EXERCISE 174.2 (Firm–union bargaining) A firm's output is $L(100 - L)$ when it uses $L \leq 50$ units of labor, and 2500 when it uses $L > 50$ units of labor. The price of output is 1. A union that represents workers presents a wage demand (a nonnegative number w), which the firm either accepts or rejects. If the firm accepts the demand, it chooses the number L of workers to employ (which you should take to be a continuous variable, not an integer); if it rejects the demand, no production takes place ($L = 0$). The firm's preferences are represented by its profit; the union's preferences are represented by the value of wL.

a. Formulate this situation as an extensive game with perfect information.

b. Find the subgame perfect equilibrium (equilibria?) of the game.

c. Is there an outcome of the game that both parties prefer to any subgame perfect equilibrium outcome?

d. Find a Nash equilibrium for which the outcome differs from any subgame perfect equilibrium outcome.

EXERCISE 175.1 (The "rotten kid theorem") A child's action a (a number) affects both her own private income $c(a)$ and her parent's income $p(a)$; for all values of a we have $c(a) < p(a)$. The child is selfish: she cares only about the amount of money she has. Her loving parent cares both about how much money she has and how much her child has. Specifically, her preferences are represented by a payoff equal to the smaller of the amount of money she has and the amount of money her child has. The parent may transfer money to the child. First the child takes an action, then the parent decides how much money to transfer. Model this situation as an extensive game and show that in a subgame perfect equilibrium the child takes an action that maximizes the sum of her private income and her parent's income. (In particular, the child's action does not maximize her own private income. The result is not limited to the specific form of the parent's preferences, but holds for any preferences with the property that a parent who is allocating a fixed amount x of money between herself and her child wishes to give more to the child when x is larger.)

EXERCISE 175.2 (Comparing simultaneous and sequential games) The set of actions available to player 1 is A_1; the set available to player 2 is A_2. Player 1's preferences over pairs (a_1, a_2) are represented by the payoff $u_1(a_1, a_2)$, and player 2's preferences are represented by the payoff $u_2(a_1, a_2)$. Compare the Nash equilibria (in pure strategies) of the strategic game in which the players choose actions simultaneously with the subgame perfect equilibria of the extensive game in which player 1 chooses an action, then player 2 does so. (For each history a_1 in the extensive game, the set of actions available to player 2 is A_2.)

a. Show that if, for every value of a_1, there is a unique member of A_2 that maximizes $u_2(a_1, a_2)$, then in every subgame perfect equilibrium of the extensive game, player 1's payoff is at least equal to her highest payoff in any Nash equilibrium of the strategic game.

b. Show that player 2's payoff in every subgame perfect equilibrium of the extensive game may be higher than her highest payoff in any Nash equilibrium of the strategic game.

c. Show that if for some values of a_1 more than one member of A_2 maximizes $u_2(a_1, a_2)$, then the extensive game may have a subgame perfect equilibrium in which player 1's payoff is less than her payoff in all Nash equilibria of the strategic game.

(For parts *b* and *c* you can give examples in which both A_1 and A_2 contain two actions.)

TICKTACKTOE, CHESS, AND RELATED GAMES

Ticktacktoe, chess, and related games may be modeled as extensive games with perfect information. (A history is a sequence of moves and each player prefers to win than to tie than to lose.) Both ticktacktoe and chess may be modeled as finite games, so by Proposition 171.1 each game has a subgame perfect equilibrium. (The official rules of chess allow indefinitely long sequences of moves, but the game seems to be well modeled by an extensive game in which a draw is declared automatically if a position is repeated three times, rather than a player having the option of declaring a draw in this case, as in the official rules.) The subgame perfect equilibria of ticktacktoe are of course known, whereas those of chess are not (yet).

Ticktacktoe and chess are "strictly competitive" games (Definition 339.1): in every outcome, either one player loses and the other wins, or the players draw. A result in a later chapter implies that for such a game all Nash equilibria yield the same outcome (Corollary 342.1). Further, a player's Nash equilibrium strategy yields *at least* her equilibrium payoff, regardless of the other players' strategies (Proposition 341.1a). (The same is definitely not true for an arbitrary game that is not strictly competitive: look, for example, at the game in Figure 29.1.) Because any subgame perfect equilibrium is a Nash equilibrium, the same is true for subgame perfect equilibrium strategies.

We conclude that in ticktacktoe and chess, either (*a*) one of the players has a strategy that guarantees she wins, or (*b*) each player has a strategy that guarantees at worst a draw.

In ticktacktoe, of course, we know that (*b*) is true. Chess is more subtle. In particular, it is not known whether White has a strategy that guarantees it wins, or Black has a strategy that guarantees it wins, or each player has a strategy that guarantees at worst a draw. The empirical evidence suggests that Black does not have a winning strategy, but this result has not been proved. When will a subgame perfect equilibrium of chess be found? (The answer "never" underestimates human ingenuity!)

? EXERCISE 176.1 (Subgame perfect equilibria of ticktacktoe) Ticktacktoe has subgame perfect equilibria in which the first player puts her first X in a corner. The second player's move is the same in all these equilibria. What is it?

? EXERCISE 176.2 (Toetacktick) Toetacktick is a variant of ticktacktoe in which a player who puts three marks in a line *loses* (rather than wins). Find a strategy of the first-mover that guarantees that she does not lose. (If fact, in all subgame perfect equilibria the game is a draw.)

? EXERCISE 177.1 (Three Men's Morris, or Mill) The ancient game of "Three Men's Morris" is played on a ticktacktoe board. Each player has three counters. The players move alternately. On each of her first three turns, a player places a counter on an unoccupied square. On each subsequent move, a player may move a counter to an adjacent square (vertically or horizontally, but not diagonally). The first player whose counters are in a row (vertically, horizontally, or diagonally) wins. Find a subgame perfect equilibrium strategy of player 1, and the equilibrium outcome.

Notes

The notion of an extensive game is due to von Neumann and Morgenstern (1944). Kuhn (1950, 1953) suggested the formulation described in this chapter. The description of an extensive game in terms of histories was suggested by Ariel Rubinstein. The notion of subgame perfect equilibrium is due to Selten (1965). Proposition 171.1 is due to Kuhn (1953). The interpretation of a strategy when a subgame perfect equilibrium is interpreted as the outcome of the players' reasoning about each others' rational actions is due to Rubinstein (1991). The principle of optimality in dynamic programming is discussed by Bellman (1957, 83), for example.

The procedure in Exercises 161.2 and 171.3 was first studied by Mueller (1978) and Moulin (1981). The idea in Exercise 171.4 goes back at least to Sun-tzu, who, in *The art of warfare* (probably written between 500BC and 300BC), advises "in surrounding the enemy, leave him a way out; do not press an enemy that is cornered" (end of Ch. 7; see, for example, Sun-tzu (1993, 132)). (That is, if no bridge exists in the situation described in the exercise, army 1 should build one.) Schelling (1966, 45) quotes Sun-tzu and gives examples of the strategy's being used in antiquity. My formulation of the exercise comes from Tirole (1988, 316). The model in Exercise 172.1 is studied by Kohler and Chandrasekaran (1971) and Brams and Straffin (1979). The game in Exercise 172.2 is based on Benoît (1984, Section 1). The dollar auction (Exercise 173.2) was introduced into the literature by Shubik (1971). Some of its subgame perfect equilibria, for arbitrary values of v and w, are studied by O'Neill (1986) and Leininger (1989); see also Taylor (1995, Chs. 1 and 6). Poundstone (1992, 257–272) writes informally about the game and its possible applications. The result in Exercise 175.1 is due to Becker (1974); see also Bergstrom (1989). The first formal study of chess is Zermelo (1913); see Schwalbe and Walker (2000) for a discussion of this paper and related work. Exercises 176.1, 176.2, and 177.1 are taken from Gardner (1959, Ch. 4), which includes several other intriguing examples.

6 Extensive Games with Perfect Information: Illustrations

Prerequisite: Chapter 5.

6.1 Introduction

THE first three sections of this chapter illustrate the notion of subgame perfect equilibrium in games in which the longest history has length two or three. The last section studies a game with an arbitrary finite horizon. Games with infinite horizons are studied in Chapters 16 and 14.

6.2 The ultimatum game and the holdup game

6.2.1 The ultimatum game

Bargaining over the division of a pie may naturally be modeled as an extensive game. Chapter 16 studies several such models. Here I analyze a very simple game that is the basis of one of the richer models studied in the later chapter. The game is so simple, in fact, that you may not initially think of it as a model of "bargaining".

Two people use the following procedure to split $\$c$. Person 1 offers person 2 an amount of money up to $\$c$. If 2 accepts this offer then 1 receives the remainder of the $\$c$. If 2 rejects the offer then *neither* person receives any payoff. Each person cares *only* about the amount of money she receives, and (naturally!) prefers to receive as much as possible.

Assume that the amount person 1 offers can be any number, not necessarily an integral number of cents. Then the following extensive game, known as the **ultimatum game**, models the procedure.

Players The two people.

Terminal histories The set of sequences (x, Z), where x is a number with $0 \le x \le c$ (the amount of money that person 1 offers to person 2) and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function $P(\varnothing) = 1$ and $P(x) = 2$ for all x.

Preferences Each person's preferences are represented by payoffs equal to the amounts of money she receives. For the terminal history (x, Y) person 1 receives $c - x$ and person 2 receives x; for the terminal history (x, N) each person receives 0.

This game has a finite horizon, so we can use backward induction to find its subgame perfect equilibria. First consider the subgames of length 1, in which person 2 either accepts or rejects an offer of person 1. For every possible offer of person 1, there is such a subgame. In the subgame that follows an offer x of person 1 for which $x > 0$, person 2's optimal action is to accept (if she rejects, she gets nothing). In the subgame that follows the offer $x = 0$, person 2 is indifferent between accepting and rejecting. Thus in a subgame perfect equilibrium person 2's strategy either accepts all offers (including 0), or accepts all offers $x > 0$ and rejects the offer $x = 0$.

Now consider the whole game. For each possible subgame perfect equilibrium strategy of person 2, we need to find the optimal strategy of person 1.

- If person 2 accepts all offers (including 0), then person 1's optimal offer is 0 (which yields her the payoff \$$c$).
- If person 2 accepts all offers except zero, then *no* offer of person 1 is optimal! No offer $x > 0$ is optimal, because the offer $x/2$ (for example) is better, given that person 2 accept both offers. And an offer of 0 is not optimal because person 2 rejects it, leading to a payoff of 0 for person 1, who is thus better off offering any positive amount less than \$$c$.

We conclude that the only subgame perfect equilibrium of the game is the strategy pair in which person 1 offers 0 and person 2 accepts all offers. In this equilibrium, person 1's payoff is \$$c$ and person 2's payoff is zero.

This one-sided outcome is a consequence of the one-sided structure of the game. If we allow person 2 to make a counteroffer after rejecting person 1's opening offer (and possibly allow further responses by both players), so that the model corresponds more closely to a "bargaining" situation, then under some circumstances the outcome is less one-sided. (An extension of this type is explored in Chapter 16.)

? EXERCISE 180.1 (Nash equilibria of the ultimatum game) Find the values of x for which there is a Nash equilibrium of the ultimatum game in which person 1 offers x.

? EXERCISE 180.2 (Subgame perfect equilibria of the ultimatum game with indivisible units) Find the subgame perfect equilibria of the variant of the ultimatum game in which the amount of money is available only in multiples of a cent.

? EXERCISE 180.3 (Dictator game and impunity game) The "dictator game" differs from the ultimatum game only in that person 2 does not have the option to reject person 1's offer (and thus has no strategic role in the game). The "impunity game" differs from the ultimatum game only in that person 1's payoff when person 2 rejects any offer x is $c - x$, rather than 0. (The game is named for the fact that person 2 is unable to "punish" person 1 for making a low offer.) Find the subgame perfect equilibria of each game.

?? EXERCISE 181.1 (Variants of ultimatum game and impunity game with equity-conscious players) Consider variants of the ultimatum game and impunity game in which each person cares not only about the amount of money she receives, but also about the equity of the allocation. Specifically, suppose that person i's preferences are represented by the payoff function given by $u_i(x_1, x_2) = x_i - \beta_i|x_1 - x_2|$, where x_i is the amount of money person i receives, $\beta_i > 0$, and, for any number z, $|z|$ denotes the absolute value of z (i.e. $|z| = z$ if $z > 0$ and $|z| = -z$ if $z < 0$). Find the set of subgame perfect equilibria of each game and compare them. Are there any values of β_1 and β_2 for which an offer is rejected in equilibrium? (An interesting further variant of the ultimatum game in which person 1 is uncertain about the value of β_2 is considered in Exercise 222.2.)

EXPERIMENTS ON THE ULTIMATUM GAME

The sharp prediction of the notion of subgame perfect equilibrium in the ultimatum game lends itself to experimental testing. The first test was conducted in the late 1970s among graduate students of economics in a class at the University of Cologne (in what was then West Germany). The amount c available varied among the games played; it ranged from 4 DM to 10 DM (around US\$2 to US\$5 at the time). A group of 42 students was split into two groups and seated on different sides of a room. Each member of one subgroup played the role of player 1 in an ultimatum game. She wrote down on a form the amount (up to c) that she demanded. Her form was then given to a randomly determined member of the other group, who, playing the role of player 2, either accepted what remained of the amount c or rejected it (in which case neither player received any payoff). Each player had 10 minutes to make her decision. The entire experiment was repeated a week later. (Güth, Schmittberger, and Schwarze 1982.)

In the first experiment the average demand by people playing the role of player 1 was $0.65c$, and in the second experiment it was $0.69c$, much less than the amount c or $c - 0.01$ predicted by the notion of subgame perfect equilibrium (0.01DM was the smallest monetary unit; see Exercise 180.2). Almost 20% of offers were rejected over the two experiments, including one of 3DM (out of a pie of 7DM) and five of around 1DM (out of pies of between 4DM and 6DM). Many other experiments, including one in which the amount of money to be divided was much larger (Hoffman, McCabe, and Smith 1996), have produced similar results. In brief, the results do not accord well with the predictions of subgame perfect equilibrium.

Or do they? Each player in the ultimatum game cares only about the amount of money she receives. But an experimental subject may care also about the amount of money her opponent receives. Further, a variant of the ultimatum game in which the players are equity-conscious has subgame perfect equilibria in which offers are significant (as you will have discovered if you did Exercise 181.1).

However, if people are equity-conscious in the strategic environment of the ultimatum game, they should be equity-conscious also in related environments; an explanation of the experimental results in the ultimatum game based on the nature of preferences is not convincing if it applies only to that environment. Several related games have been studied, among them the dictator game and the impunity game (Exercise 180.3). In the subgame perfect equilibria of these games, player 1 offers 0; in a variant in which the players are equity-conscious, player 1's offers are no higher than they are in the analogous variant of the ultimatum game, and, for moderate degrees of equity-conscience, are lower (see Exercise 181.1). These features of the equilibria are broadly consistent with the experimental evidence on dictator, impunity, and ultimatum games (see, for example, Forsythe, Horowitz, Savin, and Sefton 1994, Bolton and Zwick 1995, and Güth and Huck 1997).

One feature of the experimental results is inconsistent with subgame perfect equilibrium even when players are equity-conscious (at least given the form of the payoff functions in Exercise 181.1): positive offers are sometimes rejected. The equilibrium strategy of an equity-conscious player 2 in the ultimatum game rejects inequitable offers, but, knowing this, player 1 does not, in equilibrium, make such an offer. To generate rejections in equilibrium we need to further modify the model by assuming that people differ in their degree of equity-conscience, and that player 1 does not know the degree of equity-conscience of player 2 (see Exercise 222.2).

An alternative explanation of the experimental results focuses on player 2's behavior. The evidence is consistent with player 1's significant offers in the ultimatum game being driven by a fear that player 2 will reject small offers—a fear that is rational, because small offers are often rejected. Why does player 2 behave in this way? One argument is that in our daily lives, we use "rules of thumb" that work well in the situations in which we are typically involved; we do not calculate our rational actions in each situation. Further, we are not typically involved in one-shot situations with the structure of the ultimatum game. Instead, we usually engage in repeated interactions, where it is advantageous to "punish" a player who makes a paltry offer, and to build a reputation for not accepting such offers. Experimental subjects may apply such rules of thumb rather than carefully thinking through the logic of the game, and thus reject low offers in an ultimatum game, but accept them in an impunity game, where rejection does not affect the proposer. The experimental evidence so far collected is broadly consistent with both this explanation and the explanation based on the nature of players' preferences.

? EXERCISE 183.1 (Bargaining over two indivisible objects) Consider a variant of the ultimatum game, with indivisible units. Two people use the following procedure to allocate two desirable identical indivisible objects. One person proposes an allocation (both objects go to person 1, both go to person 2, one goes to each person), which the other person then either accepts or rejects. In the event of rejection, neither person receives either object. Each person cares only about the number of objects she obtains. Construct an extensive game that models this situation and find its subgame perfect equilibria. Does the game have any Nash equilibrium that is not a subgame perfect equilibrium? Is there any outcome that is generated by a Nash equilibrium but not by any subgame perfect equilibrium?

?? EXERCISE 183.2 (Dividing a cake fairly) Two players use the following procedure to divide a cake. Player 1 divides the cake into two pieces, and then player 2 chooses one of the pieces; player 1 obtains the remaining piece. The cake is continuously divisible (no lumps!), and each player likes all parts of it.

a. Suppose that the cake is perfectly homogeneous, so that each player cares only about the size of the piece of cake she obtains. How is the cake divided in a subgame perfect equilibrium?

b. Suppose that the cake is not homogeneous: the players evaluate different parts of it differently. Represent the cake by the set C, so that a piece of the cake is a subset P of C. Assume that if P is a subset of P' not equal to P' (smaller than P') then each player prefers P' to P. Assume also that the players' preferences are continuous: if player i prefers P to P' then there is a subset of P not equal to P that player i also prefers to P'. Let (P_1, P_2) (where P_1 and P_2 together constitute the whole cake C) be the division chosen by player 1 in a subgame perfect equilibrium of the divide-and-choose game, P_2 being the piece chosen by player 2. Show that player 2 is indifferent between P_1 and P_2, and player 1 likes P_1 at least as much as P_2. Give an example in which player 1 prefers P_1 to P_2.

6.2.2 The holdup game

Before engaging in an ultimatum game in which she may accept or reject an offer of person 1, person 2 takes an action that affects the size c of the pie to be divided. She may exert little effort, resulting in a small pie, of size c_L, or great effort, resulting in a large pie, of size c_H. She dislikes exerting effort. Specifically, assume that her payoff is $x - E$ if her share of the pie is x, where $E = L$ if she exerts little effort and $E = H > L$ if she exerts great effort. The extensive game that models this situation is known as the **holdup game**.

? EXERCISE 183.3 (Holdup game) Formulate the holdup game precisely. (Write down the set of players, set of terminal histories, player function, and the players' preferences.)

What is the subgame perfect equilibrium of the holdup game? Each subgame that follows person 2's choice of effort is an ultimatum game, and thus has a unique subgame perfect equilibrium, in which person 1 offers 0 and person 2 accepts all offers. Now consider person 2's choice of effort at the start of the game. If she chooses L then her payoff, given the outcome in the following subgame, is $-L$, whereas if she chooses H then her payoff is $-H$. Consequently she chooses L. Thus the game has a unique subgame perfect equilibrium, in which person 2 exerts little effort and person 1 obtains all of the resulting small pie.

This equilibrium does not depend on the values of c_L, c_H, L, and H (given that $H > L$). In particular, even if c_H is much larger than c_L, but H is only slightly larger than L, person 2 exerts little effort in the equilibrium, although both players could be much better off if person 2 were to exert great effort (which, in this case, is not very great) and person 2 were to obtain some of the extra pie. No such superior outcome is sustainable in an equilibrium because person 2, having exerted great effort, may be "held up" for the entire pie by person 1.

This result does not depend sensitively on the extreme subgame perfect equilibrium outcome of the ultimatum game. In Section 16.3 I analyze a model in which a similar result may emerge when the bargaining following person 2's choice of effort generates a more equal division of the pie.

6.3 Stackelberg's model of duopoly

6.3.1 General model

In the models of oligopoly studied in Sections 3.1 and 3.2, each firm chooses its action not knowing the other firms' actions. How do the conclusions change when the firms move sequentially? Is a firm better off moving before or after the other firms?

In this section I consider a market in which there are two firms, both producing the same good. Firm i's cost of producing q_i units of the good is $C_i(q_i)$; the price at which output is sold when the total output is Q is $P_d(Q)$. (In Section 3.1 I denote this function P; here I add a d subscript to avoid a conflict with the player function of the extensive game.) Each firm's strategic variable is output, as in Cournot's model (Section 3.1), but the firms make their decisions sequentially, rather than simultaneously: one firm chooses its output, then the other firm does so, knowing the output chosen by the first firm.

We can model this situation by the following extensive game, known as **Stackelberg's duopoly game** (after its originator).

Players The two firms.

Terminal histories The set of all sequences (q_1, q_2) of outputs for the firms (where each q_i, the output of firm i, is a nonnegative number).

Player function $P(\varnothing) = 1$ and $P(q_1) = 2$ for all q_1.

Preferences The payoff of firm i to the terminal history (q_1, q_2) is its profit $q_i P(q_1 + q_2) - C_i(q_i)$, for $i = 1, 2$.

Firm 1 moves at the start of the game. Thus a strategy of firm 1 is simply an output. Firm 2 moves after every history in which firm 1 chooses an output. Thus a strategy of firm 2 is a *function* that associates an output for firm 2 with each possible output of firm 1.

The game has a finite horizon, so we may use backward induction to find its subgame perfect equilibria.

- First, for any output of firm 1, we find the outputs of firm 2 that maximize its profit. Suppose that for each output q_1 of firm 1 there is one such output of firm 2; denote it $b_2(q_1)$. Then in any subgame perfect equilibrium, firm 2's strategy is b_2.

- Next, we find the outputs of firm 1 that maximize its profit, *given the strategy of firm 2*. When firm 1 chooses the output q_1, firm 2 chooses the output $b_2(q_1)$, resulting in a total output of $q_1 + b_2(q_1)$, and hence a price of $P_d(q_1 + b_2(q_1))$. Thus firm 1's output in a subgame perfect equilibrium is a value of q_1 that maximizes
$$q_1 P_d(q_1 + b_2(q_1)) - C_1(q_1). \tag{185.1}$$
Suppose that there is one such value of q_1; denote it q_1^*.

We conclude that if firm 2 has a unique best response $b_2(q_1)$ to each output q_1 of firm 1, and firm 1 has a unique best action q_1^*, given firm 2's best responses, then the subgame perfect equilibrium of the game is (q_1^*, b_2): firm 1's equilibrium strategy is q_1^* and firm 2's equilibrium strategy is the function b_2. The output chosen by firm 2, given firm 1's equilibrium strategy, is $b_2(q_1^*)$; denote this output q_2^*.

When firm 1 chooses any output q_1, the outcome, given that firm 2 uses its equilibrium strategy, is the pair of outputs $(q_1, b_2(q_1))$. That is, as firm 1 varies its output, the outcome varies along firm 2's best response function b_2. Thus we can characterize the subgame perfect equilibrium outcome (q_1^*, q_2^*) as the point on firm 2's best response function that maximizes firm 1's profit.

6.3.2 Example: constant unit cost and linear inverse demand

Suppose that $C_i(q_i) = cq_i$ for $i = 1, 2$, and
$$P_d(Q) = \begin{cases} \alpha - Q & \text{if } Q \le \alpha \\ 0 & \text{if } Q > \alpha, \end{cases} \tag{185.2}$$
where $c > 0$ and $c < \alpha$ (as in the example of Cournot's duopoly game in Section 3.1.3). We found that under these assumptions firm 2 has a unique best response to each output q_1 of firm 1, given by
$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c - q_1) & \text{if } q_1 \le \alpha - c \\ 0 & \text{if } q_1 > \alpha - c. \end{cases}$$

Thus in a subgame perfect equilibrium of Stackelberg's game firm 2's strategy is this function b_2 and firm 1's strategy is the output q_1 that maximizes

$$q_1(\alpha - c - (q_1 + \tfrac{1}{2}(\alpha - c - q_1))) = \tfrac{1}{2}q_1(\alpha - c - q_1)$$

(refer to (185.1)). This function is a quadratic in q_1 that is zero when $q_1 = 0$ and when $q_1 = \alpha - c$. Thus its maximizer is $q_1 = \frac{1}{2}(\alpha - c)$.

We conclude that the game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output $\frac{1}{2}(\alpha - c)$ and firm 2's strategy is b_2. The outcome of the equilibrium is that firm 1 produces the output $q_1^* = \frac{1}{2}(\alpha - c)$ and firm 2 produces the output $q_2^* = b_2(q_1^*) = b_2(\frac{1}{2}(\alpha - c)) = \frac{1}{2}(\alpha - c - \frac{1}{2}(\alpha - c)) = \frac{1}{4}(\alpha - c)$. Firm 1's profit is $q_1^*(P(q_1^* + q_2^*) - c) = \frac{1}{8}(\alpha - c)^2$, and firm 2's profit is $q_1^*(P(q_1^* + q_2^*) - c) = \frac{1}{16}(\alpha - c)^2$. By contrast, in the unique Nash equilibrium of Cournot's (simultaneous-move) game under the same assumptions, each firm produces $\frac{1}{3}(\alpha - c)$ units of output and obtains the profit $\frac{1}{9}(\alpha - c)^2$. Thus under our assumptions firm 1 produces more output and obtains more profit in the subgame perfect equilibrium of the sequential game in which it moves first than it does in the Nash equilibrium of Cournot's game, and firm 2 produces less output and obtains less profit.

? EXERCISE 186.1 (Stackelberg's duopoly game with quadratic costs) Find the subgame perfect equilibrium of Stackelberg's duopoly game when $C_i(q_i) = q_i^2$ for $i = 1, 2$, and $P_d(Q) = \alpha - Q$ for all $Q \leq \alpha$ (with $P_d(Q) = 0$ for $Q > \alpha$). Compare the equilibrium outcome with the Nash equilibrium of Cournot's game under the same assumptions (Exercise 57.2).

6.3.3 *Properties of subgame perfect equilibrium*

First-mover's equilibrium profit In the example just studied, the first-mover is better off in the subgame perfect equilibrium of Stackelberg's game than it is in the Nash equilibrium of Cournot's game. A weak version of this result holds under very general conditions: for any cost and inverse demand functions for which firm 2 has a unique best response to each output of firm 1, firm 1 is at least as well off in any subgame perfect equilibrium of Stackelberg's game as it is in any Nash equilibrium of Cournot's game. This result follows from the general result in Exercise 175.2*a*. The argument is simple. One of firm 1's options in Stackelberg's game is to choose its output in some Nash equilibrium of Cournot's game. If it chooses such an output then firm 2's best action is to choose its output in the same Nash equilibrium, given the assumption that it has a unique best response to each output of firm 1. Thus by choosing such an output, firm 1 obtains its profit at a Nash equilibrium of Cournot's game; by choosing a different output it may possibly obtain a higher payoff.

Equilibrium outputs In the example in the previous section, firm 1 produces more output in the subgame perfect equilibrium of Stackelberg's game than it does in the Nash equilibrium of Cournot's game, and firm 2 produces less. A weak form of this result holds whenever firm 2's best response function is decreasing where it is positive (i.e. a higher output for firm 1 implies a lower optimal output for firm 2).

The argument is illustrated in Figure 187.1. The firms' best response functions are the curves labeled b_1 (dashed) and b_2. The Nash equilibrium of Cournot's game is the intersection $(\overline{q}_1, \overline{q}_2)$ of these curves. Along each gray curve, firm 1's profit is constant; the *lower* curve corresponds to a *higher* profit. (For any given value of firm 1's output, a reduction in the output of firm 2 increases the price and thus increases firm 1's profit.) Each constant-profit curve of firm 1 is horizontal where it crosses firm 1's best response function, because the best response is precisely the output that maximizes firm 1's profit, given firm 2's output. (Cf. Figure 59.1.) Thus the subgame perfect equilibrium outcome—the point on firm 2's best response function that yields the highest profit for firm 1—is the point (q_1^*, q_2^*) in the figure. In particular, given that the best response function of firm 2 is downward-sloping, firm 1 produces at least as much, and firm 2 produces at most as much, in the subgame perfect equilibrium of Stackelberg's game as in the Nash equilibrium of Cournot's game.

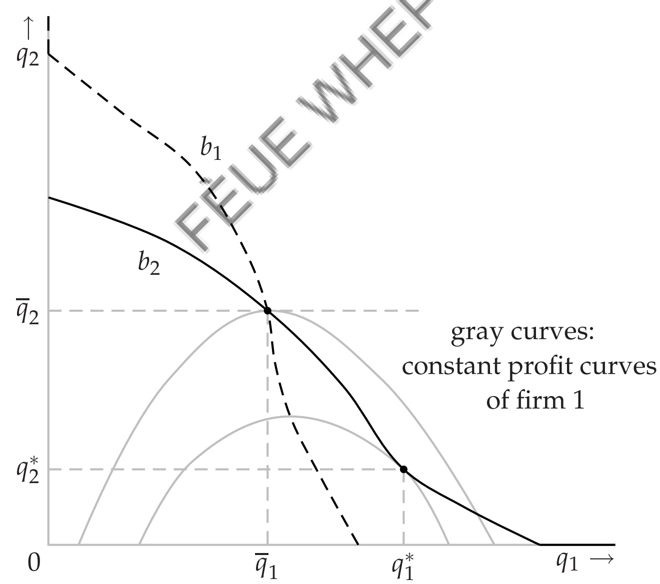

Figure 187.1 The subgame perfect equilibrium outcome (q_1^*, q_2^*) of Stackelberg's game and the Nash equilibrium $(\overline{q}_1, \overline{q}_2)$ of Cournot's game. Along each gray curve, firm 1's profit is constant; the lower curve corresponds to higher profit than does the upper curve. Each curve has a slope of zero where it crosses firm 1's best response function b_1.

For some cost and demand functions, firm 2's output in a subgame perfect equilibrium of Stackelberg's game is zero. An example is shown in Figure 188.1. The discontinuity in firm 2's best response function at q_1^* in this example may arise because firm 2 incurs a "fixed" cost—a cost independent of its output—when it produces a positive output (see Exercise 57.3). When firm 1's output is q_1^*, firm 2's maximal profit is zero, which it obtains both when it produces no output (and does not pay the fixed cost) and when it produces the output $\hat{q}_2$. When firm 1 produces less than q_1^*, firm 2's maximal profit is positive, and firm 2 optimally produces a positive output; when firm 1 produces more than q_1^*, firm 2 optimally produces no output. Given this form of firm 2's best response function and the form of firm 1's constant profit curves shown in the figure, the point on firm 2's best response function that yields firm 1 the highest profit is $(q_1^*, 0)$.

I claim that this example has a unique subgame perfect equilibrium, in which firm 1 produces q_1^* and firm 2's strategy coincides with its best response function except at q_1^*, where the strategy specifies the output 0. The output firm 2's equilibrium strategy specifies after each history must be a best response to firm 1's output, so the only question regarding firm 2's strategy is whether it specifies an output of 0 or $\hat{q}_2$ when firm 1's output is q_1^*. The argument that there is no subgame perfect equilibrium in which firm 2's strategy specifies the output $\hat{q}_2$ is similar to the argument that there is no subgame perfect equilibrium in the ultimatum game in which person 2 rejects the offer 0. If firm 2 produces the output $\hat{q}_2$ in response to firm 1's output q_1^* then firm 1 has no optimal output: it would like to produce a little more than q_1^*, inducing firm 2 to produce zero, but is better off the closer its output is to q_1^*. Because there is no smallest output greater than q_1^*, no output is *optimal* for firm 1 in this case. Thus the game has no subgame perfect equilibrium in which firm 2's strategy specifies the output $\hat{q}_2$ in response to firm 1's output q_1^*.

Note that if firm 2 were entirely absent from the market, firm 1 would produce $\hat{q}_1$, less than q_1^*. Thus firm 2's presence affects the outcome, even though it produces no output.

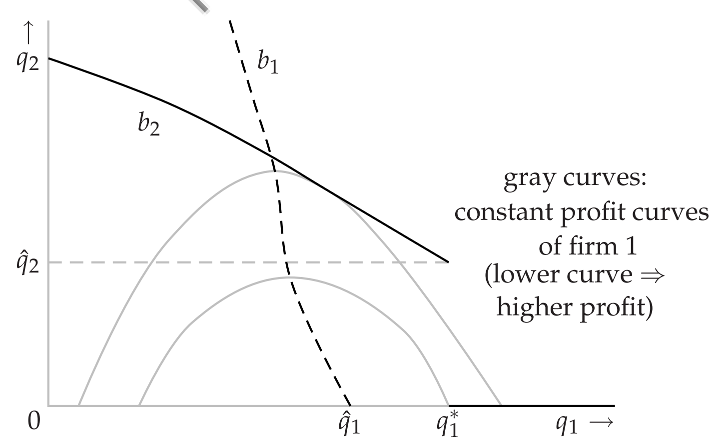

Figure 188.1 The subgame perfect equilibrium output q_1^* of firm 1 in Stackelberg's sequential game when firm 2 incurs a fixed cost. Along each gray curve, firm 1's profit is constant; the lower curve corresponds to higher profit than does the upper curve.

? EXERCISE 188.1 (Stackelberg's duopoly game with fixed costs) Suppose that the inverse demand function is given by (185.2) and the cost function of each firm i is given by

$$C_i(q_i) = \begin{cases} 0 & \text{if } q_i = 0 \\ f + cq_i & \text{if } q_i > 0, \end{cases}$$

where $c \geq 0$, $f > 0$, and $c < \alpha$, as in Exercise 57.3. Show that if $c = 0$, $\alpha = 12$, and $f = 4$, Stackelberg's game has a unique subgame perfect equilibrium, in which firm 1's output is 8 and firm 2's output is zero. (Use your results from Exercise 57.3).

The value of commitment Firm 1's output in a subgame perfect equilibrium of Stackelberg's game is *not* in general a best response to firm 2's output: if firm 1 could adjust its output after firm 2 has chosen its output, then it would do so! (In the case shown in Figure 187.1, it would reduce its output.) However, if firm 1 had this opportunity, and firm 2 knew that it had the opportunity, then firm 2 would choose a different output. Indeed, if we simply add a third stage to the game, in which firm 1 chooses an output, then the first stage is irrelevant, and *firm 2* is effectively the first-mover; in the subgame perfect equilibrium firm 1 is worse off than it is in the Nash equilibrium of the simultaneous-move game. (In the example in the previous section, the unique subgame perfect equilibrium has firm 2 choose the output $(\alpha - c)/2$ and firm 1 choose the output $(\alpha - c)/4$.) In summary, even though firm 1 can increase its profit by changing its output after firm 2 has chosen its output, in the game in which it has this opportunity it is worse off than it is in the game in which it must choose its output before firm 2 and cannot subsequently modify this output. That is, firm 1 prefers to be *committed* not to change its mind.

? EXERCISE 189.1 (Sequential variant of Bertrand's duopoly game) Consider the variant of Bertrand's duopoly game (Section 3.2) in which first firm 1 chooses a price, then firm 2 chooses a price. Assume that each firm is restricted to choose a price that is an integral number of cents (as in Exercise 65.2), that each firm's unit cost is constant, equal to c (an integral number of cents), and that the monopoly profit is positive.

a. Specify an extensive game with perfect information that models this situation.

b. Give an example of a strategy of firm 1 and an example of a strategy of firm 2.

c. Find the subgame perfect equilibria of the game.

6.4 Buying votes

A legislature has k members, where k is an odd number. Two rival bills, X and Y, are being considered. The bill that attracts the votes of a majority of legislators will pass. Interest group X favors bill X, whereas interest group Y favors bill Y. Each group wishes to entice a majority of legislators to vote for its favorite bill. First interest group X gives an amount of money (possibly zero) to each legislator, then interest group Y does so. Each interest group wishes to spend as little as possible. Group X values the passing of bill X at $\$V_X > 0$ and the passing of bill Y

at zero, and group Y values the passing of bill Y at $\$V_Y > 0$ and the passing of bill X at zero. (For example, group X is indifferent between an outcome in which it spends V_X and bill X is passed and one in which it spends nothing and bill Y is passed.) Each legislator votes for the favored bill of the interest group that offers her the most money; a legislator to whom both groups offer the same amount of money votes for bill Y (an arbitrary assumption that simplifies the analysis without qualitatively changing the outcome). For example, if $k = 3$, the amounts offered to the legislators by group X are $x = (100, 50, 0)$, and the amounts offered by group Y are $y = (100, 0, 50)$, then legislators 1 and 3 vote for Y and legislator 2 votes for X, so that Y passes. (In some legislatures the inducements offered to legislators are more subtle than cash transfers.)

We can model this situation as the following extensive game.

Players The two interest groups, X and Y.

Terminal histories The set of all sequences (x, y), where x is a list of payments to legislators made by interest group X and y is a list of payments to legislators made by interest group Y. (That is, both x and y are lists of k nonnegative integers.)

Player function $P(\varnothing) = X$ and $P(x) = Y$ for all x.

Preferences The preferences of interest group X are represented by the payoff function

$$\begin{cases} V_X - (x_1 + \cdots + x_k) & \text{if bill } X \text{ passes} \\ -(x_1 + \cdots + x_k) & \text{if bill } Y \text{ passes,} \end{cases}$$

where bill Y passes after the terminal history (x, y) if and only if the number of components of y that are at least equal to the corresponding components of x is at least $\frac{1}{2}(k+1)$ (a bare majority of the k legislators). The preferences of interest group Y are represented by the analogous function (where V_Y replaces V_X, y replaces x, and Y replaces X).

Before studying the subgame perfect equilibria of this game for arbitrary values of the parameters, consider two examples. First suppose that $k = 3$ and $V_X = V_Y = 300$. Under these assumptions, the most group X is willing to pay to get bill X passed is 300. For any payments it makes to the three legislators that sum to at most 300, two of the payments sum to at most 200, so that if group Y matches these payments it spends less than V_Y $(= 300)$ and gets bill Y passed. Thus in any subgame perfect equilibrium group X makes no payments, group Y makes no payments, and (given the tie-breaking rule) bill Y is passed.

Now suppose that $k = 3$, $V_X = 300$, and $V_Y = 100$. In this case by paying each legislator more than 50, group X makes matching payments by group Y unprofitable: only by spending more than V_Y $(= 100)$ can group Y cause bill Y to be passed. However, there is no subgame perfect equilibrium in which group X pays each legislator more than 50, because it can always pay a little less (as long as the payments still exceed 50) and still prevent group Y from profitably matching. In the only subgame perfect equilibrium group X pays each legislator exactly 50, and group Y makes no payments. Given group X's action, group Y is indifferent between matching X's payments (so that bill Y is passed), and making no payments. However, there is no subgame perfect equilibrium in which group Y matches group X's payments, because if this were group Y's response then group X could increase its payments a little, making matching payments by group Y unprofitable.

For arbitrary values of the parameters the subgame perfect equilibrium outcome takes one of the forms in these two examples: either no payments are made and bill Y is passed, or group X makes payments that group Y does not wish to match, group Y makes no payments, and bill X is passed.

To find the subgame perfect equilibria in general, we may use backward induction. First consider group Y's best response to an arbitrary strategy x of group X. Let $\mu = \frac{1}{2}(k+1)$, a bare majority of k legislators, and denote by m_x the sum of the smallest μ components of x—the total payments Y needs to make to buy off a bare majority of legislators.

- If $m_x < V_Y$ then group Y can buy off a bare majority of legislators for less than V_Y, so that its best response to x is to match group X's payments to the μ legislators to whom group X's payments are smallest; the outcome is that bill Y is passed.
- If $m_x > V_Y$ then the cost to group Y of buying off any majority of legislators exceeds V_Y, so that group Y's best response to x is to make no payments; the outcome is that bill X is passed.
- If $m_x = V_Y$ then both the actions in the previous two cases are best responses by group Y to x.

We conclude that group Y's strategy in a subgame perfect equilibrium has the following properties.

- After a history x for which $m_x < V_Y$, group Y matches group X's payments to the μ legislators to whom X's payments are smallest.
- After a history x for which $m_x > V_Y$, group Y makes no payments.
- After a history x for which $m_x = V_Y$, group Y either makes no payments or matches group X's payments to the μ legislators to whom X's payments are smallest.

Given that group Y's subgame perfect equilibrium strategy has these properties, what should group X do? If it chooses a list of payments x for which $m_x < V_Y$ then group Y matches its payments to a bare majority of legislators, and bill Y passes. If it reduces all its payments, the same bill is passed. Thus the only list of payments x with $m_x < V_Y$ that may be optimal is $(0, \ldots, 0)$. If it chooses a list of payments x with $m_x > V_Y$ then group Y makes no payments, and bill X passes. If it reduces all its payments a little (keeping the payments to every bare majority greater than V_Y), the outcome is the same. Thus no list of payments x for which $m_x > V_Y$ is optimal.

We conclude that in any subgame perfect equilibrium we have either $x = (0, \ldots, 0)$ (group X makes no payments) or $m_x = V_Y$ (the smallest sum of group X's payments to a bare majority of legislators is V_Y). Under what conditions does each case occur? If group X needs to spend more than V_X to deter group Y from matching its payments to a bare majority of legislators, then its best strategy is to make no payments ($x = (0, \ldots, 0)$). How much does it need to spend to deter group Y? It needs to pay more than V_Y to every bare majority of legislators, so it needs to pay each legislator more than V_Y/μ, in which case its total payment is more than kV_Y/μ. Thus if $V_X < kV_Y/\mu$, group X is better off making no payments than getting bill X passed by making payments large enough to deter group Y from matching its payments to a bare majority of legislators.

If $V_X > kV_Y/\mu$, on the other hand, group X can afford to make payments large enough to deter group Y from matching. In this case its best strategy is to pay each legislator V_Y/μ, so that its total payment to every bare majority of legislators is V_Y. Given this strategy, group Y is indifferent between matching group X's payments to a bare majority of legislators and making no payments. I claim that the game has no subgame perfect equilibrium in which group Y matches. The argument is similar to the argument that the ultimatum game has no subgame perfect equilibrium in which person 2 rejects the offer 0. Suppose that group Y matches. Then group X can increase its payoff by increasing its payments a little (keeping the total less than V_X), thereby deterring group Y from matching, and ensuring that bill X passes. Thus in any subgame perfect equilibrium group Y makes no payments in response to group X's strategy.

In conclusion, if $V_X \neq kV_Y/\mu$ then the game has a unique subgame perfect equilibrium, in which group Y's strategy is

- match group X's payments to the μ legislators to whom X's payments are smallest after a history x for which $m_x < V_Y$
- make no payments after a history x for which $m_x \geq V_Y$

and group X's strategy depends on the relative sizes of V_X and V_Y:

- if $V_X < kV_Y/\mu$ then group X makes no payments;
- if $V_X > kV_Y/\mu$ then group X pays each legislator V_Y/μ.

If $V_X < kV_Y/\mu$ then the outcome is that neither group makes any payment, and bill Y is passed; if $V_X > kV_Y/\mu$ then the outcome is that group X pays each legislator V_Y/μ, group Y makes no payments, and bill X is passed. (If $V_X = kV_Y/\mu$ then the analysis is more complex.)

Three features of the subgame perfect equilibrium are significant. First, the outcome favors the second-mover in the game (group Y): only if $V_X > kV_Y/\mu$, which is close to $2V_Y$ when k is large, does group X manage to get bill X passed. Second, group Y never makes any payments! According to its equilibrium strategy it is prepared to make payments in response to certain strategies of group X, but given group X's *equilibrium* strategy it spends not a cent. Third, if group X makes any payments (as it does in the equilibrium for $V_X > kV_Y/\mu$) then it makes a payment to *every* legislator. If there were no competing interest group but nonetheless each legislator would vote for bill X only if she were paid at least some amount, then group X would make payments to only a bare majority of legislators; if it were to act in this way in the presence of group Y it would supply group Y with almost a majority of legislators who could be induced to vote for bill Y at no cost.

? EXERCISE 193.1 (Three interest groups buying votes) Consider a variant of the model in which there are *three* bills, X, Y, and Z, and *three* interest groups, X, Y, and Z, who choose lists of payments sequentially. Ties are broken in favor of the group moving later. Find the bill that is passed in any subgame perfect equilibrium when $k = 3$ and (*a*) $V_X = V_Y = V_Z = 300$, (*b*) $V_X = 300$, $V_Y = V_Z = 100$, and (*c*) $V_X = 300$, $V_Y = 202$, $V_Z = 100$. (You may assume that in each case a subgame perfect equilibrium exists; note that you are not asked to find the subgame perfect equilibria themselves.)

? EXERCISE 193.2 (Interest groups buying votes under supermajority rule) Consider an alternative variant of the model in which a supermajority is required to pass a bill. There are two bills, X and Y, and a "default outcome". A bill passes if and only if it receives at least $k^* > \frac{1}{2}(k+1)$ votes; if neither bill passes the default outcome occurs. There are two interest groups. Both groups attach value 0 to the default outcome. Find the bill that is passed in any subgame perfect equilibrium when $k = 7$, $k^* = 5$, and (*a*) $V_X = V_Y = 700$ and (*b*) $V_X = 750$, $V_Y = 400$. In each case, would the legislators be better off or worse off if a simple majority of votes were required to pass a bill?

? EXERCISE 193.3 (Sequential positioning by two political candidates) Consider the variant of Hotelling's model of electoral competition in Section 3.3 in which the n candidates choose their positions sequentially, rather than simultaneously. Model this situation as an extensive game. Find the subgame perfect equilibrium (equilibria?) when $n = 2$.

?? EXERCISE 193.4 (Sequential positioning by three political candidates) Consider a further variant of Hotelling's model of electoral competition in which the n candidates choose their positions sequentially and each candidate has the option of staying out of the race. Assume that each candidate prefers to stay out than to enter and lose, prefers to enter and tie with any number of candidates than to stay out, and prefers to tie with as few other candidates as possible. Model the situation as an extensive game and find the subgame perfect equilibrium outcomes when $n = 2$ (easy) and when $n = 3$ and the voters' favorite positions are distributed uniformly from 0 to 1 (i.e. the fraction of the voters' favorite positions less than x is x) (hard).