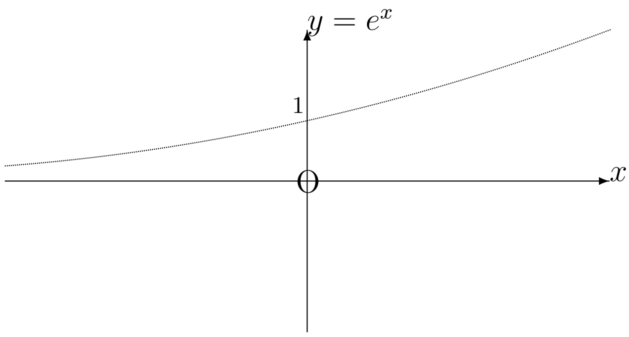
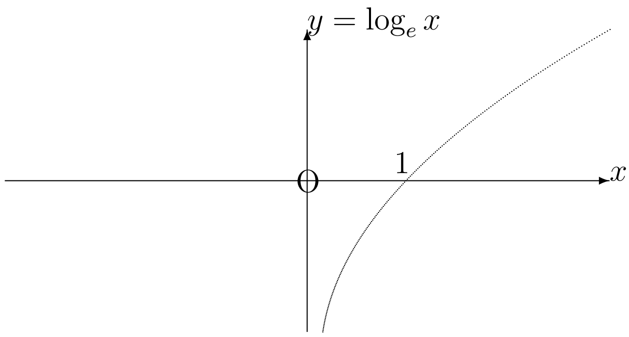
1.4.6 GRAPHS OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In the applications of mathematics to science and engineering, two commonly used "functions" are $y = e^x$ and $y = \log_e x$. Their graphs are as follows:





They are closely linked with each other by virtue of the two equivalent statements:

$$P = \log_e Q \quad \text{and} \quad Q = e^P$$

for any number P and any postive number Q.

Because of these statements, we would expect similarities in the graphs of the equations

$$y = \log_e x \quad \text{and} \quad y = e^x.$$

1.4.7 LOGARITHMIC SCALES

In a certain kind of graphical work (see Unit 5.3), some use is made of a linear scale along which numbers can be allocated according to their logarithmic distances from a chosen origin of measurement.

Considering firstly that 10 is the base of logarithms, the number 1 is always placed at the zero of measurement (since $\log_{10} 1 = 0$); 10 is placed at the first unit of measurement (since $\log_{10} 10 = 1$), 100 is placed at the second unit of measurement (since $\log_{10} 100 = 2$), and so on.

Negative powers of 10 such as $10^{-1} = 0.1$, $10^{-2} = 0.01$ etc. are placed at the points corresponding to -1 and -2 etc. respectively on an ordinary linear scale.

The logarithmic scale appears therefore in **"cycles"**, each cycle corresponding to a range of numbers between two consecutive powers of 10.

Intermediate numbers are placed at intervals which correspond to their logarithm values.

An extract from a typical logarithmic scale would be as follows:

0.1 0.2 0.3 0.4 1 2 3 4 10

Notes:
(i) A given set of numbers will determine how many cycles are required on the logarithmic scale. For example .3, .6, 5, 9, 23, 42, 166 will require **four** cycles.

(ii) Commercially printed logarithmic scales do not specify the base of logarithms; the change of base formula implies that logarithms to different bases are proportional to each other and hence their logarithmic scales will have the same relative shape.

1.4.8 EXERCISES

1. Without using tables or a calculator, evaluate

   (a) $\log_{10} 27 \div \log_{10} 3$;
   (b) $(\log_{10} 16 - \log_{10} 2) \div \log_{10} 2$.

2. Using properties of logarithms where possible, solve for x the following equations:

   (a) $\log_{10} \frac{7}{2} + 2\log_{10} \frac{4}{3} - \log_{10} \frac{7}{45} = 1 + \log_{10} x$;
   (b) $2\log_{10} 6 - (\log_{10} 4 + \log_{10} 9) = \log_{10} x$.
   (c) $10^x = 5(2^{10})$.

3. From the definition of a logarithm or the change of base formula, evaluate the following:

   (a) $\log_2 7$;
   (b) $\log_3 0.04$;
   (c) $\log_5 3$;
   (d) $3\log_3 2 - \log_3 4 + \log_3 \frac{1}{2}$.

4. Obtain y in terms of x for the following equations:

   (a) $2\ln y = 1 - x^2$;
   (b) $\ln x = 5 - 3\ln y$.

5. Rewrite the following statements without logarithms:

   (a) $\ln x = -\frac{1}{2}\ln(1 - 2v^3) + \ln C$;
   (b) $\ln(1 + y) = \frac{1}{2}x^2 + \ln 4$;
   (c) $\ln(4 + y^2) = 2\ln(x + 1) + \ln A$.

6. (a) If $\frac{I_0}{I} = 10^{ac}$, find c in terms of the other quantities in the formula.
   (b) If $y^p = Cx^q$, find q in terms of the other quantities in the formula.

1.4.9 ANSWERS TO EXERCISES

1. (a) 3; (b) 3.
2. (a) 4; (b) 1; (c) 3.70927
3. (a) 2.807; (b) -2.930; (c) 0.683; (d) 0
4. (a)

$$y = e^{\frac{1}{2}(1-x^2)};$$

(b)

$$y = \sqrt[3]{\frac{e^5}{x}}.$$

5. (a)

$$x = \frac{C}{\sqrt{1-2v^3}};$$

(b)

$$1 + y = 4e^{\frac{x^2}{2}};$$

(c)

$$4 + y^2 = A(x+1)^2.$$

6. (a)

$$c = \frac{1}{a}\log_{10}\frac{I_0}{I};$$

(b)

$$q = \frac{p\log y - \log C}{\log x},$$

using any base.

"JUST THE MATHS"

UNIT NUMBER

1.5

ALGEBRA 5
(Manipulation of algebraic expressions)

by

A.J.Hobson

UNIT 1.5 - ALGEBRA 5
MANIPULATION OF ALGEBRAIC EXPRESSIONS

1.5.1 SIMPLIFICATION OF EXPRESSIONS

An algebraic expression will, in general, contain a mixture of alphabetical symbols together with one or more numerical quantities; some of these symbols and numbers may be bracketted together.

Using the Language of Algebra and the Laws of Algebra discussed earlier, the method of simplification is to remove brackets and collect together any terms which have the same format

Some elementary illustrations are as follows:

1. $a + a + a + 3 + b + b + b + b + 8 \equiv 3a + 4b + 11.$
2. $11p^2 + 5q^7 - 8p^2 + q^7 \equiv 3p^2 + 6q^7.$
3. $a(2a - b) + b(a + 5b) - a^2 - 4b^2 \equiv 2a^2 - ab + ba + 5b^2 - a^2 - 4b^2 \equiv a^2 + b^2.$

More frequently, the expressions to be simplified will involve symbols which represent both the constants and variables encountered in scientific work. Typical examples in pure mathematics use symbols like x, y and z to represent the variable quantities.

Further illustrations use this kind of notation and, for simplicity, we shall omit the full-stop type of multiplication sign between symbols.

1. $x(2x + 5) + x^2(3 - x) \equiv 2x^2 + 5x + 3x^2 - x^3 \equiv 5x^2 + 5x - x^3.$
2. $x^{-1}(4x - x^2) - 6(1 - 3x) \equiv 4 - x - 6 + 18x \equiv 17x - 2.$

We need also to consider the kind of expression which involves two or more brackets multiplied together; but the routine is just an extension of what has already been discussed.

For example consider the expression

$$(a + b)(c + d).$$

Taking the first bracket as a single item for the moment, the Distributive Law gives

$$(a + b)c + (a + b)d.$$

Using the Distributive Law a second time gives

$$ac + bc + ad + bd.$$

In other words, each of the two terms in the first bracket are multiplied by each of the two terms in the second bracket, giving four terms in all.

Again, we illustrate with examples:

EXAMPLES

1. $(x + 3)(x - 5) \equiv x^2 + 3x - 5x - 15 \equiv x^2 - 2x - 15.$
2. $(x^3 - x)(x + 5) \equiv x^4 - x^2 + 5x^3 - 5x.$
3. $(x + a)^2 \equiv (x + a)(x + a) \equiv x^2 + ax + ax + a^2 \equiv x^2 + 2ax + a^2.$
4. $(x + a)(x - a) \equiv x^2 + ax - ax - a^2 \equiv x^2 - a^2.$

The last two illustrations above are significant for later work because they incorporate, respectively, the standard results for a **"perfect square"** and **"the difference between two squares"**.

1.5.2 FACTORISATION

Introduction
In an algebraic context, the word **"factor"** means the same as **"multiplier"**. Thus, to factorise an algebraic expression, is to write it as a product of separate multipliers or factors.

Some simple examples will serve to introduce the idea:

EXAMPLES

1. $$3x + 12 \equiv 3(x + 4).$$

2. $$8x^2 - 12x \equiv x(8x - 12) \equiv 4x(2x - 3).$$

3. $$5x^2 + 15x^3 \equiv x^2(5 + 15x) \equiv 5x^2(1 + 3x).$$

4.

$$6x + 3x^2 + 9xy \equiv x(6 + 3x + 9y) \equiv 3x(2 + x + 3y).$$

Note:
When none of the factors can be broken down into simpler factors, the original expression is said to have been factorised into **"irreducible factors"**.

Factorisation of quadratic expressions

A **"quadratic expression"** is an expression of the form

$$ax^2 + bx + c,$$

where, usually, a, b and c are fixed numbers (constants) while x is a variable number. The numbers a and b are called, respectively, the **"coefficients"** of x^2 and x while c is called the **"constant term"**; but, for brevity, we often say that the quadratic expression has coefficients a, b and c.

Note:
It is important that the coefficient a does not have the value zero otherwise the expression is not quadratic but **"linear"**.

The method of factorisation is illustrated by examples:

(a) When the coefficient of x^2 is 1

EXAMPLES

1.

$$x^2 + 5x + 6 \equiv (x + m)(x + n) \equiv x^2 + (m + n)x + mn.$$

This implies that $5 = m + n$ and $6 = mn$ which, by inspection gives $m = 2$ and $n = 3$. Hence

$$x^2 + 5x + 6 \equiv (x + 2)(x + 3).$$

2.

$$x^2 + 4x - 21 \equiv (x + m)(x + n) \equiv x^2 + (m + n)x + mn.$$

This implies that $4 = m + n$ and $-21 = mn$ which, by inspection, gives $m = -3$ and $n = 7$. Hence

$$x^2 + 4x - 21 \equiv (x - 3)(x + 7).$$

Notes:
(i) In general, for simple cases, it is better to try to carry out the factorisation entirely by inspection. This avoids the cumbersome use of m and n in the above two examples as follows:

$$x^2 + 2x - 8 \equiv (x+?)(x+?).$$

The two missing numbers must be such that their sum is 2 and their product is -8. The required values are therefore -2 and 4. Hence

$$x^2 + 2x - 8 \equiv (x - 2)(x + 4).$$

(ii) It is necessary, when factorising a quadratic expression, to be aware that either a perfect square or the difference of two squares might be involved. In these cases, the factorisation is a little simpler. For instance:

$$x^2 + 10x + 25 \equiv (x + 5)^2$$

and

$$x^2 - 64 \equiv (x - 8)(x + 8).$$

(iii) Some quadratic expressions will not conveniently factorise at all. For example, in the expression

$$x^2 - 13x + 2,$$

we cannot find two whole numbers whose sum is -13 while, at the same time, their product is 2.

(b) When the coefficient of x^2 is not 1

Quadratic expressions of this kind are usually more difficult to factorise than those in the previous paragraph. We first need to determine the possible pairs of factors of the coefficient of x^2 and the possible pairs of factors of the constant term; then we need to consider the possible combinations of these which provide the correct factors of the quadratic expression.

EXAMPLES

1. To factorise the expression

$$2x^2 + 11x + 12,$$

we observe that 2 is the product of 2 and 1 only, while 12 is the product of either 12 and 1, 6 and 2 or 4 and 3. All terms of the quadratic expression are positive and hence we may try $(2x+1)(x+12)$, $(2x+12)(x+1)$, $(2x+6)(x+2)$, $(2x+2)(x+6)$, $(2x+4)(x+3)$ and $(2x+3)(x+4)$. Only the last of these is correct and so

$$2x^2 + 11x + 12 \equiv (2x+3)(x+4).$$

2. To factorise the expression

$$6x^2 + 7x - 3,$$

we observe that 6 is the product of either 6 and 1 or 3 and 2 while 3 is the product of 3 and 1 only. A negative constant term implies that, in the final result, its two factors must have opposite signs. Hence we may try $(6x+3)(x-1)$, $(6x-3)(x+1)$, $(6x+1)(x-3)$, $(6x-1)(x+3)$, $(3x+3)(2x-1)$, $(3x-3)(2x+1)$, $(3x+1)(2x-3)$ and $(3x-1)(2x+3)$. Again, only the last of these is correct and so

$$6x^2 + 7x - 3 \equiv (3x-1)(2x+3).$$

Note:
The more factors there are in the coefficients considered, the more possibilities there are to try of the final factorisation.

1.5.3 COMPLETING THE SQUARE IN A QUADRATIC EXPRESSION

The following work is based on the standard expansions

$$(x+a)^2 \equiv x^2 + 2ax + a^2$$

and

$$(x-a)^2 \equiv x^2 - 2ax + a^2.$$

Both of these last expressions are called **"complete squares"** (or **"perfect squares"**) in which we observe that one half of the coefficient of x, when multiplied by itself, gives the constant term. That is

$$(\frac{1}{2} \times 2a)^2 = a^2.$$

ILLUSTRATIONS

1.

$$x^2 + 6x + 9 \equiv (x+3)^2.$$

2.

$$x^2 - 8x + 16 \equiv (x-4)^2.$$

3.

$$4x^2 - 4x + 1 \equiv 4\left[x^2 - x + \frac{1}{4}\right] \equiv 4\left(x - \frac{1}{2}\right)^2.$$

Of course it may happen that a given quadratic expression is NOT a complete square; but, by using one half of the coefficient of x, we may express it as the sum or difference of a complete square and a constant. This process is called **"completing the square"**, and the following examples illustrate it:

EXAMPLES

1.

$$x^2 + 6x + 11 \equiv (x+3)^2 + 2.$$

2.

$$x^2 - 8x + 7 \equiv (x-4)^2 - 9.$$

3.

$$\begin{aligned} 4x^2 - 4x + 5 &\equiv 4\left[x^2 - x + \frac{5}{4}\right] \\ &\equiv 4\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{5}{4}\right] \\ &\equiv 4\left[\left(x - \frac{1}{2}\right)^2 + 1\right] \\ &\equiv 4\left(x - \frac{1}{2}\right)^2 + 4. \end{aligned}$$

1.5.4 ALGEBRAIC FRACTIONS

We first recall the basic rules for combining fractions, namely

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

We also note that a single algebraic fraction may sometimes be simplified by the cancellation of common factors between the numerator and the denominator.

EXAMPLES

1.
$$\frac{5}{25+15x} \equiv \frac{1}{5+3x}, \quad \text{assuming that} \quad x \neq -\frac{5}{3}.$$

2.
$$\frac{4x}{3x^2+x} \equiv \frac{4}{3x+1}, \quad \text{assuming that} \quad x \neq 0 \quad \text{or} \quad -\frac{1}{3}.$$

3.
$$\frac{x+2}{x^2+3x+2} \equiv \frac{x+2}{(x+2)(x+1)} \equiv \frac{1}{x+1}, \quad \text{assuming that} \quad x \neq -1 \quad \text{or} \quad -2.$$

These elementary principles may now be used with more advanced combinations of algebraic fractions

EXAMPLES

1. Simplify the expression
$$\frac{3x+6}{x^2+3x+2} \times \frac{x+1}{2x+8}.$$

   Solution
   Using factorisation where possible, together with the rule for multiplying fractions, we obtain
$$\frac{3(x+2)(x+1)}{2(x+4)(x+1)(x+2)} \equiv \frac{3}{2(x+4)},$$
   assuming that $x \neq -1, -2$ or -4.

2. Simplify the expression
$$\frac{3}{x+2} \div \frac{x}{2x+4}.$$

   Solution
   Using factorisation where possible together with the rule for dividing fractions, we obtain
$$\frac{3}{x+2} \times \frac{2x+4}{x} \equiv \frac{3}{x+2} \times \frac{2(x+2)}{x} \equiv \frac{6}{x},$$
   assuming that $x \neq 0$ or -2.

3. Express
$$\frac{4}{x+y} - \frac{3}{y}$$
as a single fraction.

Solution

From the basic rule for adding and subtracting fractions, we obtain
$$\frac{4y - 3(x+y)}{(x+y)y} \equiv \frac{y-3x}{(x+y)y},$$
assuming that $y \neq 0$ and $x \neq -y$.

4. Express
$$\frac{x}{x+1} + \frac{4-x^2}{x^2-x-2}$$
as a single fraction.

Solution

This example could be tackled in the same way as the previous one but it is worth noticing that $x^2 - x - 2 \equiv (x+1)(x-2)$. Consequently, it is worth putting both fractions over the simplest common denominator, namely $(x+1)(x-2)$. Hence we obtain, if $x \neq 2$ or -1,
$$\frac{x(x-2)}{(x+1)(x-2)} + \frac{4-x^2}{(x+1)(x-2)} \equiv \frac{x^2-2x+4-x^2}{(x+1)(x-2)} \equiv \frac{2(2-x)}{(x+1)(x-2)} \equiv -\frac{2}{x+1}.$$

1.5.5 EXERCISES

1. Write down in their simplest forms
   (a) $5a - 2b - 3a + 6b$; (b) $11p + 5q - 2q + p$.

2. Simplify the following expressions:
   (a) $3x^2 - 2x + 5 - x^2 + 7x - 2$; (b) $x^3 + 5x^2 - 2x + 1 + x - x^2$.

3. Expand and simplify the following expressions:
   (a) $x(x^2 - 3x) + x^2(4x+7)$; (b) $(2x-1)(2x+1) - x^2 + 5x$;
   (c) $(x+3)(2x^2-5)$; (d) $2(3x+1)^2 + 5(x-7)^2$.

4. Factorise the following expressions:
   (a) $xy + 4x^2y$; (b) $2abc - 6ab^2$;
   (c) $\pi r^2 + 2\pi rh$; (d) $2xy^2z + 4x^2z$.

5. Factorise the following quadratic expressions:
   (a) $x^2 + 8x + 12$; (b) $x^2 + 11x + 18$; (c) $x^2 + 13x - 30$;
   (d) $3x^2 + 11x + 6$; (e) $4x^2 - 12x + 9$; (f) $9x^2 - 64$.
6. Complete the square in the following quadratic expressions:
   (a) $x^2 - 10x - 26$; (b) $x^2 - 5x + 4$; (c) $7x^2 - 2x + 1$.
7. Simplify the following:
   (a) $\frac{x^2+4x+4}{x^2+5x+6}$; (b) $\frac{x^2-1}{x^2+2x+1}$,
   assuming the values of x to be such that no denominators are zero.
8. Express each of the following as a single fraction:
   (a) $\frac{3}{x} + \frac{4}{y}$; (b) $\frac{4}{x} - \frac{6}{2x}$;
   (c) $\frac{1}{x+1} + \frac{1}{x+2}$; (d) $\frac{5x}{x^2+5x+4} - \frac{3}{x+4}$,
   assuming that the values of x and y are such that no denominators are zero.

1.5.6 ANSWERS TO EXERCISES

1. (a) $2a + 4b$; (b) $12p + 3q$.
2. (a) $2x^2 + 5x + 3$; (b) $x^3 + 4x^2 - x + 1$.
3. (a) $5x^3 + 4x^2$; (b) $3x^2 + 5x - 1$.
   (c) $2x^3 + 6x^2 - 5x - 15$; (d) $23x^2 - 58x + 247$.
4. (a) $xy(1 + 4x)$; (b) $2ab(c - 3b)$;
   (c) $\pi r(r + 2h)$; (d) $2xz(y^2 + 2x)$.
5. (a) $(x + 2)(x + 6)$; (b) $(x + 2)(x + 9)$; (c) $(x - 2)(x + 15$;
   (d) $(3x + 2)(x + 3)$; (e) $(2x - 3)^2$; (f) $(3x - 8)(3x + 8)$.
6. (a) $(x - 5)^2 - 51$; (b) $\left(x - \frac{5}{2}\right)^2 - \frac{9}{4}$; (c) $7\left[\left(x - \frac{1}{7}\right)^2 + \frac{6}{49}\right]$.
7. (a) $\frac{x+2}{x+3}$; (b) $\frac{x-1}{x+1}$.
8. (a) $\frac{3y+4x}{xy}$; (b) $\frac{1}{x}$;
   (c) $\frac{2x+3}{(x+1)(x+2)}$; (d) $\frac{2x-3}{x^2+5x+4}$.

"JUST THE MATHS"

UNIT NUMBER

1.6

ALGEBRA 6
(Formulae and algebraic equations)

by

A.J.Hobson

UNIT 1.6 - ALGEBRA 6 - FORMULAE AND ALGEBRAIC EQUATIONS

1.6.1 TRANSPOSITION OF FORMULAE

In dealing with technical formulae, it is often required to single out one of the quantities involved in terms of all the others. We are said to **"transpose the formula"** and make that quantity **"the subject of the equation"**.

In order to do this, steps of the following types may be carried out on both sides of a given formula:

(a) Addition or subtraction of the same value;

(b) Multiplication or division by the same value;

(c) The raising of both sides to equal powers;

(d) Taking logarithms of both sides.

EXAMPLES

1. Make x the subject of the formula

$$y = 3(x + 7).$$

Solution
Dividing both sides by 3 gives $\frac{y}{3} = x + 7$; then subtracting 7 gives $x = \frac{y}{3} - 7$.

2. Make y the subject of the formula

$$a = b + c\sqrt{x^2 - y^2}.$$

Solution
(i) Subtracting b gives $a - b = c\sqrt{x^2 - y^2}$;
(ii) Dividing by c gives $\frac{a-b}{c} = \sqrt{x^2 - y^2}$;
(iii) Squaring both sides gives $\left(\frac{a-b}{c}\right)^2 = x^2 - y^2$;
(iv) Subtracting x^2 gives $\left(\frac{a-b}{c}\right)^2 - x^2 = -y^2$;
(v) Multiplying throughout by -1 gives $x^2 - \left(\frac{a-b}{c}\right)^2 = y^2$;

(vi) Taking square roots of both sides gives

$$y = \pm\sqrt{x^2 - \left(\frac{a-b}{c}\right)^2}.$$

3. Make x the subject of the formula

$$e^{2x-1} = y^3.$$

Solution

Taking natural logarithms of both sides of the formula

$$2x - 1 = 3\ln y.$$

Hence

$$x = \frac{3\ln y + 1}{2}.$$

Note:

A genuine scientific formula will usually involve quantities which can assume only positive values; in which case we can ignore the negative value of a square root.

1.6.2 SOLUTION OF LINEAR EQUATIONS

A Linear Equation in a variable quantity x has the general form

$$ax + b = c.$$

Its solution is obtained by first subtracting b from both sides then dividing both sides by a. That is

$$x = \frac{c-b}{a}.$$

EXAMPLES

1. Solve the equation

$$5x + 11 = 20.$$

Solution

The solution is clearly $x = \frac{20-11}{5} = \frac{9}{5} = 1.8$

2. Solve the equation
$$3 - 7x = 12.$$

Solution
This time, the solution is $x = \frac{12-3}{-7} = \frac{9}{-7} \simeq -1.29$

1.6.3 SOLUTION OF QUADRATIC EQUATIONS

The standard form of a quadratic equation is

$$ax^2 + bx + c = 0,$$

where a, b and c are constants and $a \neq 0$.

We shall discuss three methods of solving such an equation related very closely to the previous discussion on quadratic expressions. The first two methods can be illustrated by examples.

(a) By Factorisation
This method depends on the ability to determine the factors of the left hand side of the given quadratic equation. This will usually be by trial and error.

EXAMPLES

1. Solve the quadratic equation
$$6x^2 + x - 2 = 0.$$

Solution
In factorised form, the equation can be written

$$(3x + 2)(2x - 1) = 0.$$

Hence, $x = -\frac{2}{3}$ or $x = \frac{1}{2}$.

2. Solve the quadratic equation
$$15x^2 - 17x - 4 = 0.$$

Solution
In factorised form, the equation can be written

$$(5x + 1)(3x - 4) = 0.$$

Hence, $x = -\frac{1}{5}$ or $x = \frac{4}{3}$

(b) By Completing the square

By looking at some numerical examples of this method, we shall be led naturally to a third method involving a **universal** formula for solving any quadratic equation.

EXAMPLES

1. Solve the quadratic equation
$$x^2 - 4x - 1 = 0.$$

   Solution
   On completing the square, the equation can be written
$$(x-2)^2 - 5 = 0.$$
   Thus,
$$x - 2 = \pm\sqrt{5}.$$
   That is,
$$x = 2 \pm \sqrt{5}.$$
   Left as it is, this is an answer in **"surd form"** but it could, of course, be expressed in decimals as 4.236 and −0.236.

2. Solve the quadratic equation
$$4x^2 + 4x - 2 = 0.$$

   Solution
   The equation may be written
$$4\left[x^2 + x - \frac{1}{2}\right] = 0$$
   and, on completing the square,
$$4\left[\left(x + \frac{1}{2}\right)^2 - \frac{3}{4}\right] = 0.$$
   Hence,
$$\left(x + \frac{1}{2}\right)^2 = \frac{3}{4},$$
   giving
$$x + \frac{1}{2} = \pm\sqrt{\frac{3}{4}}.$$

That is,

$$x = -\frac{1}{2} \pm \sqrt{\frac{3}{4}}$$

or

$$x = \frac{-1 \pm \sqrt{3}}{2}.$$

(c) By the Quadratic Formula

Starting now with an arbitrary quadratic equation

$$ax^2 + bx + c = 0,$$

we shall use the method of completing the square in order to establish the **general** solution.

The sequence of steps is as follows:

$$a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0;$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] = 0;$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a};$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}};$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}};$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note:
The quantity $b^2 - 4ac$ is called the **"discriminant"** of the equation and gives either two solutions, one solution or no solutions according as its value is positive, zero or negative.

The single solution case is usually interpreted as a pair of coincident solutions while the no solution case really means no **real** solutions. A more complete discussion of this case arises in the subject of **"complex numbers"** (see Unit 6.1).

EXAMPLES

Use the quadratic formula to solve the following:

1.

$$x^2 + 2x - 35 = 0.$$

Solution

$$x = \frac{-2 \pm \sqrt{4 + 140}}{2} = \frac{-2 \pm 12}{2} = 5 \quad \text{or} \quad -7.$$

2.

$$2x^2 - 3x - 7 = 0.$$

Solution

$$x = \frac{3 \pm \sqrt{9 + 56}}{4} = \frac{3 \pm \sqrt{65}}{4} = \frac{3 \pm 8.062}{4} \simeq 2.766 \quad \text{or} \quad -1.266$$

3.

$$9x^2 - 6x + 1 = 0.$$

Solution

$$x = \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{6}{18} = \frac{1}{3} \quad \text{only}.$$

4.

$$5x^2 + x + 1 = 0.$$

Solution

$$x = \frac{-1 \pm \sqrt{1 - 20}}{10}.$$

Hence, there are no real solutions

1.6.4 EXERCISES

1. Make the given symbol the subject of the following formulae
   (a) x: $a(x-a) = b(x+b)$;
   (b) b: $a = \frac{2-7b}{3+5b}$;
   (c) r: $n = \frac{1}{2L}\sqrt{\frac{r}{p}}$;
   (d) x: $ye^{x^2+1} = 5$.
2. Solve, for x, the following equations
   (a) $14x = 35$;
   (b) $3x - 4.7 = 2.8$;
   (c) $4(2x-5) = 3(2x+8)$.
3. Solve the following quadratic equations by factorisation:
   (a) $x^2 + 5x - 14 = 0$;
   (b) $8x^2 + 2x - 3 = 0$.
4. Where possible, solve the following quadratic equations by the formula:
   (a) $2x^2 - 3x + 1 = 0$; (b) $4x = 45 - x^2$;
   (c) $16x^2 - 24x + 9 = 0$; (d) $3x^2 + 2x + 11 = 0$.

1.6.5 ANSWERS TO EXERCISES

1. (a) $x = \frac{b^2+a^2}{a-b}$;
   (b) $b = \frac{2-3a}{7+5a}$;
   (c) $r = 4n^2L^2p$;
   (d) $x = \pm\sqrt{\ln 5 - \ln y - 1}$.
2. (a) 2.5; (b) 2.5; (c) 22.
3. (a) $x = -7$, $x = 2$;
   (b) $x = -\frac{3}{4}$, $x = \frac{1}{2}$.
4. (a) $x = 1$, $x = \frac{1}{2}$;
   (b) $x = 5$, $x = -9$;
   (c) $x = \frac{3}{4}$ only;
   (d) No solutions.

"JUST THE MATHS"

UNIT NUMBER

1.7

ALGEBRA 7
(Simultaneous linear equations)

by

A.J.Hobson

UNIT 1.7 - ALGEBRA 7 - SIMULTANEOUS LINEAR EQUATIONS

Introduction

When Mathematics is applied to scientific work, it is often necessary to consider several statements involving several variables which are required to have a common solution for those variables. We illustrate here the case of two simultaneous linear equations in two variables x and y and three simultaneous linear equations in three unknowns x, y and z.

1.7.1 TWO SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the simultaneous linear equations:

$$\begin{aligned} ax + by &= p, \\ cx + dy &= q. \end{aligned}$$

To obtain the solution we first eliminate one of the variables in order to calculate the other. For instance, to eliminate x, we try to make coefficient of x the same in both equations so that, subtracting one statement from the other, x disappears. In this case, we multiply the first equation by c and the second equation by a to give

$$\begin{aligned} cax + cby &= cp, \\ acx + ady &= aq. \end{aligned}$$

Subtracting the second equation from the first, we obtain

$$y(cb - ad) = cp - aq$$

which, in turn, means that

$$y = \frac{cp - aq}{cb - ad} \text{ provided } cb - ad \neq 0.$$

Having found the value of y, we could then either substitute back into one of the original equations to find x or begin again by eliminating y.

However, it is better not to think of the above explanation as providing a **formula** for solving two simultaneous linear equations. Rather, a numerical example should be dealt with from first principles with the numbers provided.

Note:
$cb - ad = 0$ relates to a degenerate case in which the left hand sides of the two equations are proportional to each other. Such cases will not be dealt with at this stage.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 6x - 2y &= 1, \qquad (1) \\ 4x + 7y &= 9. \qquad (2) \end{aligned}$$

Solution

Multiplying the first equation by 4 and the second equation by 6,

$$\begin{aligned} 24x - 8y &= 4, \qquad (4) \\ 24x + 42y &= 54. \qquad (5) \end{aligned}$$

Subtracting the second of these from the first, we obtain $-50y = -50$ and hence $y = 1$.

Substituting back into equation (1), $6x - 2 = 1$, giving $6x = 3$, and, hence, $x = \frac{1}{2}$.

Alternative Method

Multiplying the first equation by 7 and the second equation by -2, we obtain

$$\begin{aligned} 42x - 14y &= 7, \qquad (5) \\ -8x - 14y &= -18. \qquad (6) \end{aligned}$$

Subtracting equation (6) from equation (5) gives $50x = 25$ and hence, $x = \frac{1}{2}$.

Substituting into equation (1) gives $3 - 2y = 1$ and hence, $-2y = -2$; that is $y = 1$.

1.7.2 THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

Here, we consider three simultaneous equations of the general form

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3; \end{aligned}$$

but the method will be illustrated by a particular example.

The object of the method is to eliminate one of the variables from two different pairs of the three equations so that we are left with a pair of simultaneous equations from which to calculate the other two variables.

EXAMPLE

Solve, for x, y and z, the simultaneous linear equations

$$\begin{aligned} x - y + 2z &= 9, & (1) \\ 2x + y - z &= 1, & (2) \\ 3x - 2y + z &= 8. & (3) \end{aligned}$$

Solution

Firstly, we may eliminate z from equations (2) and (3) by adding them together. We obtain

$$5x - y = 9. \qquad (4)$$

Secondly, we may eliminate z from equations (1) and (2) by doubling equation (2) and adding it to equation (1). We obtain

$$5x + y = 11. \qquad (5)$$

If we now add equation (4) to equation (5), y will be eliminated to give

$$10x = 20 \quad \text{or} \quad x = 2.$$

Similarly, if we subtract equation (4) from equation (5), x will be eliminated to give

$$2y = 2 \quad \text{or} \quad y = 1.$$

Finally, if we substitute our values of x and y into one of the original equations [say equation (3)] we obtain

$$z = 8 - 3x + 2y = 8 - 6 + 2 = 4.$$

Thus,

$$x = 2, \quad y = 1 \quad \text{and} \quad z = 4.$$

1.7.3 ILL-CONDITIONED EQUATIONS

In the simultaneous linear equations of genuine scientific problems, the coefficients will often be decimal quantitities that have already been subjected to rounding errors; and the solving process will tend to amplify these errors. The result may be that such errors swamp the values of the variables being solved for; and we have what is called an **"ill-conditioned"** set of equations. The opposite of this is a **"well-conditioned"** set of equations and all of those so far discussed have been well-conditioned. But let us consider, now, the following example:

EXAMPLE

The simultaneous linear equations

$$\begin{aligned} x + y &= 1, \\ 1.001x + y &= 2 \end{aligned}$$

have the common solution $x = 1000$, $y = -999$.

However, suppose that the coefficient of x in the second equation is altered to 1.000, which is a mere 0.1%. Then the equations have no solution at all since $x + y$ cannot be equal to 1 and 2 at the same time.

Secondly, suppose that the coefficient of x in the second equation is altered to 0.999 which is still only a 0.2% alteration.

The solutions obtained are now $x = -1000$, $y = 1001$ and so a change of about 200% has occurred in original values of x and y.

1.7.4 EXERCISES

1. Solve, for x and y, the following pairs of simultaneous linear equations:

   (a)

$$\begin{aligned} x - 2y &= 5, \\ 3x + y &= 1; \end{aligned}$$

   (b)

$$\begin{aligned} 2x + 3y &= 42, \\ 5x - y &= 20. \end{aligned}$$

2. Solve, for x, y and z, the following sets of simultaneous equations:

   (a)

$$\begin{aligned} x + y + z &= 0, \\ 2x - y - 3z &= 4, \\ 3x + 3y &= 7; \end{aligned}$$

   (b)

$$\begin{aligned} x + y - 10 &= 0, \\ y + z - 3 &= 0, \\ x + z + 1 &= 0; \end{aligned}$$

   (c)

$$\begin{aligned} 2x - y - z &= 6, \\ x + 3y + 2z &= 1, \\ 3x - y - 5z &= 1; \end{aligned}$$

(d)

$$\begin{aligned} 2x - 5y + 2z &= 14, \\ 9x + 3y - 4z &= 13, \\ 7x + 3y - 2z &= 3; \end{aligned}$$

(e)

$$\begin{aligned} 4x - 7y + 6z &= -18, \\ 5x + y - 4z &= -9, \\ 3x - 2y + 3z &= 12. \end{aligned}$$

3. Solve the simultaneous linear equations

$$\begin{aligned} 1.985x - 1.358y &= 2.212, \\ 0.953x - 0.652y &= 1.062, \end{aligned}$$

and compare with the solutions obtained by changing the constant term, 1.062, of the second equation to 1.061.

1.7.5 ANSWERS TO EXERCISES

1. (a) $x = 1$, $y = -2$; (b) $x = 6$, $y = 10$.
2. (a) $x = -\frac{2}{9}$, $y = \frac{23}{9}$, $z = -\frac{7}{3}$;
   (b) $x = 3$, $y = 7$, $z = -4$;
   (c) $x = 3$, $y = -2$, $z = 2$;
   (d) $x = 1$, $y = -4$, $z = -4$;
   (e) $x = 3$, $y = 12$, $z = 9$.
3. $$x \simeq 0.6087, \quad y \simeq -0.7391$$

   compared with

   $$x \simeq 30.1304, \quad y \simeq 42.413$$

   a change of 4850% in x and 5839% in y.

"JUST THE MATHS"

UNIT NUMBER

1.8

ALGEBRA 8
(Polynomials)

by

A.J.Hobson

UNIT 1.8 - ALGEBRA 8 - POLYNOMIALS

Introduction

The work already covered in earlier units has frequently been concerned with mathematical expressions involving constant quantities together positive integer powers of a variable quantity (usually x). The general form of such expressions is

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$$

which is called a
"polynomial of degree n in x", having **"coefficients"** $a_0, a_1, a_2, a_3, ...a_n$, usually constant.

Note:
Polynomials of degree 1, 2 and 3 are called respectively **"linear"**, **"quadratic"** and **"cubic"** polynomials.

1.8.1 THE FACTOR THEOREM

If $P(x)$ denotes an algebraic polynomial which has the value zero when $x = \alpha$, then $x - \alpha$ is a factor of the polynomial and
$P(x) \equiv (x - \alpha)\times$ another polynomial, $Q(x)$, of one degree lower.

Notes:
(i) The statement of this Theorem includes some functional notation (i.e. $P(x), Q(x)$) which will be discussed fully as an introduction to the subject of Calculus in Unit 10.1.

(ii) $x = \alpha$ is called a **"root"** of the polynomial.

1.8.2 APPLICATION TO QUADRATIC AND CUBIC EXPRESSIONS

(a) Quadratic Expressions

Suppose we are given a quadratic expression where we suspect that at least one of the values of x making it zero is a whole number. We may systematically try

$$x = 0, 1, -1, 2, -2, 3, -3, 4, -4, ...$$

until such a value of x (say $x = \alpha$) is located. Then one of the factors of the quadratic expression is $x - \alpha$ enabling us to determine the other factor(s) easily.

EXAMPLES

1. By trial and error, the quadratic expression

$$x^2 + 2x - 3$$

has the value zero when $x = 1$; hence, $x - 1$ is a factor.

By further trial and error, the complete factorisation is

$$(x - 1)(x + 3).$$

2. By trial and error, the quadratic expression

$$3x^2 + 20x - 7$$

has the value zero when $x = -7$; hence, $(x + 7)$ is a factor.

By further trial and error, the complete factorisation is

$$(x + 7)(3x - 1).$$

(b) Cubic Expressions

The method just used for the factorisation of quadratic expressions may also be used for polynomials having powers of x higher than two. In particular, it may be used for cubic expressions whose standard form is

$$ax^3 + bx^2 + cx + d,$$

where a, b, c and d are constants.

EXAMPLES

1. Factorise the cubic expression

$$x^3 + 3x^2 - x - 3$$

assuming that there is at least one whole number value of x for which the expression has the value zero.

Solution

By trial and error, the cubic expression has the value zero when $x = 1$. Hence, by the Factor Theorem, $(x - 1)$ is a factor.

Thus,

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)Q(x)$$

where $Q(x)$ is some quadratic expression to be found and, if possible, factorised further. In other words,

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(px^2 + qx + r)$$

for some constants p, q and r.

Usually, it is fairly easy to determine $Q(x)$ using trial and error by comparing, initially, the highest and lowest powers of x on both sides of the above identity; then comparing any intermediate powers as necessary. We obtain

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x^2 + 4x + 3).$$

In this example, the quadratic expression does factorise even further giving

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x + 1)(x + 3).$$

2. Factorise the cubic expression

$$x^3 + 4x^2 + 4x + 1.$$

Solution

By trial and error, we discover that the cubic expression has value zero when $x = -1$, and so $x + 1$ must be a factor.
Hence,

$$x^3 + 4x^2 + 4x + 1 \equiv (x + 1)Q(x),$$

where $Q(x)$ is a quadratic expression to be found. In other words,

$$x^3 + 4x^2 + 4x + 1 \equiv (x + 1)(px^2 + qx + r)$$

for some constants p, q and r.

Comparing the relevant coefficients, we obtain

$$x^3 + 4x^2 + 4x + 1 \equiv (x + 1)(x^2 + 3x + 1)$$

but, this time, the quadratic part of the answer will not conveniently factorise further.

1.8.3 CUBIC EQUATIONS

There is no convenient general method of solving a cubic equation; hence, the discussion here will be limited to those equations where at least one of the solutions is known to be a fairly small whole number, that solution being obtainable by trial and error.

We illustrate with examples.

EXAMPLES

1. Solve the cubic equation
$$x^3 + 3x^2 - x - 3 = 0.$$

   Solution
   By trial and error, one solution is $x = 1$ and so $(x - 1)$ must be a factor of the left hand side. In fact, we obtain the new form of the equation as
$$(x - 1)(x^2 + 4x + 3) = 0.$$
   That is,
$$(x - 1)(x + 1)(x + 3) = 0.$$
   Hence, the solutions are $x = 1$, $x = -1$ and $x = -3$.

2. Solve the cubic equation
$$2x^3 - 7x^2 + 5x + 54 = 0.$$

   Solution
   By trial and error, one solution is $x = -2$ and so $(x + 2)$ must be a factor of the left hand side. In fact we obtain the new form of the equation as
$$(x + 2)(2x^2 - 11x + 27) = 0.$$
   The quadratic factor will not conveniently factorise, but we can use the quadratic formula to determine the values of x, if any, which make it equal to zero.They are
$$x = \frac{11 \pm \sqrt{121 - 216}}{4}.$$
   The discriminant here is negative so that the only (real) solution to the cubic equation is $x = -2$.

1.8.4 LONG DIVISION OF POLYNOMIALS

(a) Exact Division

Having used the Factor Theorem to find a linear factor of a polynomial expression, there will always be a remaining factor, $Q(x)$, which is some other polynomial whose degree is one lower than that of the original; but the determination of $Q(x)$ by trial and error is not the only method of doing so. An alternative method is to use the technique known as **"long division of polynomials"**; and the working is illustrated by the following examples:

EXAMPLES

1. Factorise the cubic expression
$$x^3 + 3x^2 - x - 3.$$

   Solution

   By trial and error, the cubic expression has the value zero when $x = 1$ so that $(x - 1)$ is a factor.

   Dividing the given cubic expression (called the **"dividend"**) by $(x - 1)$ (called the **"divisor"**), we have the following scheme:
$$
\begin{array}{rllll}
 & & x^2 & +4x & +3 \\ \cline{2-5}
x-1) & x^3 & +3x^2 & -x & -3 \\
 & \underline{x^3} & \underline{-x^2} & & \\
 & & 4x^2 & -x & -3 \\
 & & \underline{4x^2} & \underline{-4x} & \\
 & & & 3x & -3 \\
 & & & \underline{3x} & \underline{-3} \\
 & & & & 0
\end{array}
$$

   Note:
   At each stage, using positive powers of x or constants only, we examine what the highest power of x in the divisor would have to be multiplied by in order to give the highest power of x in the dividend. The results are written in the top line as the **"quotient"** so that, on multiplying down, we can subtract to find each **"remainder"**.
   The process stops when, to continue would need negative powers of x; the final remainder will be zero when the divisor is a factor of the original expression.

   We conclude here that
$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x^2 + 4x + 3).$$

   Further factorisation leads to the complete result,
$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x + 1)(x + 3).$$

2. Solve, completely, the cubic equation
$$x^3 + 4x^2 + 4x + 1 = 0.$$

   Solution

   By trial and error, one solution is $x = -1$ so that $(x + 1)$ is a factor of the left hand side.

Dividing the left hand side of the equation by $(x+1)$, we have the following scheme:

$$
\begin{array}{r@{}l}
 & x^2+3x+1 \\
\cline{2-2}
x+1) & x^3+4x^2+4x+1 \\
 & \underline{x^3+x^2} \\
 & \phantom{x^3+{}}3x^2+4x+1 \\
 & \phantom{x^3+{}}\underline{3x^2+3x} \\
 & \phantom{x^3+3x^2+{}}x+1 \\
 & \phantom{x^3+3x^2+{}}\underline{x+1} \\
 & \phantom{x^3+3x^2+{}x+{}}0
\end{array}
$$

Hence, the equation becomes

$$(x+1)(x^2+3x+1)=0,$$

giving $x=-1$ and $x=\frac{-3\pm\sqrt{9-4}}{2}\simeq -0.382 \quad \text{or} \quad -2.618$

(b) Non-exact Division

The technique for long division of polynomials may be used to divide any polynomial by another polynomial of lower or equal degree, even when this second polynomial is not a factor of the first.

The chief difference in method is that the remainder will not be zero; but otherwise we proceed as before.

EXAMPLES

1. Divide the polynomial $6x+5$ by the polynomial $3x-1$.

   Solution

$$
\begin{array}{r@{}l}
 & \phantom{6x+{}}2 \\
\cline{2-2}
3x-1) & 6x+5 \\
 & \underline{6x-2} \\
 & 7
\end{array}
$$

Hence,

$$\frac{6x+5}{3x-1}\equiv 2+\frac{7}{3x-1}.$$

2. Divide $3x^2 + 2x$ by $x + 1$.

   Solution

$$
\begin{array}{r}
3x - 1 \\
x + 1 \overline{\big) 3x^2 + 2x} \\
\underline{3x^2 + 3x} \\
-x \\
\underline{-x - 1} \\
1
\end{array}
$$

   Hence,

$$
\frac{3x^2 + 2x}{x + 1} \equiv 3x - 1 + \frac{1}{x + 1}.
$$

3. Divide $x^4 + 2x^3 - 2x^2 + 4x - 1$ by $x^2 + 2x - 3$.

   Solution

$$
\begin{array}{r}
x^2 \qquad + 1 \\
x^2 + 2x - 3 \overline{\big) x^4 + 2x^3 - 2x^2 + 4x - 1} \\
\underline{x^4 + 2x^3 - 3x^2} \qquad\quad \\
x^2 + 4x - 1 \\
\underline{x^2 + 2x - 3} \\
2x + 2
\end{array}
$$

   Hence,

$$
\frac{x^4 + 2x^3 - 2x^2 + 4x - 1}{x^2 + 2x - 3} \equiv x^2 + 1 + \frac{2x + 2}{x^2 + 2x - 3}.
$$

1.8.5 EXERCISES

1. Use the Factor Theorem to factorise the following quadratic expressions: assuming that at least one root is an integer:
   (a) $2x^2 + 7x + 3$;
   (b) $1 - 2x - 3x^2$.
2. Factorise the following cubic polynomials assuming that at least one root is an integer:
   (a) $x^3 + 2x^2 - x - 2$;
   (b) $x^3 - x^2 - 4$;
   (c) $x^3 - 4x^2 + 4x - 3$.

3. Solve completely the following cubic equations assuming that at least one solution is an integer:
   (a) $x^3 + 6x^2 + 11x + 6 = 0$;
   (b) $x^3 + 2x^2 - 31x + 28 = 0$.
4. (a) Divide $2x^3 - 11x^2 + 18x - 8$ by $x - 2$;
   (b) Divide $3x^3 + 12x^2 + 13x + 4$ by $x + 1$.
5. (a) Divide $x^2 - 2x + 1$ by $x^2 + 3x - 2$, expressing your answer in the form of a constant plus a ratio of two polynomials.
   (b) Divide x^5 by $x^2 - 2x - 8$, expressing your answer in the form of a polynomial plus a ratio of two other polynomials.

1.8.6 ANSWERS TO EXERCISES

1. (a) $(2x + 1)(x + 3)$; (b) $(x + 1)(1 - 3x)$.
2. (a) $(x - 1)(x + 1)(x + 2)$; (b) $(x - 2)(x^2 + x + 2)$; (c) $(x - 3)(x^2 - x + 1)$.
3. (a) $x = -1$, $x = -2$, $x = -3$; (b) $x = 1$, $x = 4$, $x = -7$.
4. (a) $2x^2 - 7x + 4$; (b) $3x^2 + 9x + 4$.
5. (a)

$$1 + \frac{3 - 5x}{x^2 + 3x - 2};$$

(b)

$$x^3 + 2x^2 + 12x + 40 + \frac{176x + 320}{x^2 - 2x - 8}.$$

"JUST THE MATHS"

UNIT NUMBER

1.9

ALGEBRA 9
(The theory of partial fractions)

by

A.J.Hobson

UNIT 1.9 - ALGEBRA 9 - THE THEORY OF PARTIAL FRACTIONS

1.9.1 INTRODUCTION

The theory of partial fractions applies chiefly to the ratio of two polynomials in which the degree of the numerator is strictly less than that of the denominator. Such a ratio is called a **"proper rational function"**.

For a rational function which is not proper, it is necessary first to use long division of polynomials in order to express it as the sum of a polynomial and a proper rational function.

RESULT

A proper rational function whose denominator has been factorised into its irreducible factors can be expressed as a sum of proper rational functions, called **"partial fractions"**; the denominators of the partial fractions are the irreducible factors of the denominator in the original rational function.

ILLUSTRATION

From previous work on fractions, it can be verified that

$$\frac{1}{2x+3} + \frac{3}{x-1} \equiv \frac{7x+8}{(2x+3)(x-1)}$$

and the expression on the left hand side may be interpretted as the decomposition into partial fractions of the expression on the right hand side.

1.9.2 STANDARD TYPES OF PARTIAL FRACTION PROBLEM

(a) Denominator of the given rational function has all linear factors.

EXAMPLE

Express the rational function

$$\frac{7x+8}{(2x+3)(x-1)}$$

in partial fractions.

Solution

There will be two partial fractions each of whose numerator must be of lower degree than 1; i.e. it must be a **constant**

We write

$$\frac{7x+8}{(2x+3)(x-1)} \equiv \frac{A}{2x+3} + \frac{B}{x-1}.$$

Multiplying throughout by $(2x+3)(x-1)$, we obtain

$$7x+8 \equiv A(x-1)+B(2x+3).$$

In order to determine A and B, any two suitable values of x may be substituted on both sides; and the most obvious values in this case are $x=1$ and $x=-\frac{3}{2}$.

It may, however, be argued that these two values of x must be disallowed since they cause denominators in the first identity above to take the value zero.

Nevertheless, we shall use these values in the second identity above since the arithmetic involved is negligibly different from taking values infinitesimally close to $x=1$ and $x=-\frac{3}{2}$.

Substituting $x=1$ gives

$$7+8=B(2+3).$$

Hence,

$$B=\frac{7+8}{2+3}=\frac{15}{5}=3.$$

Substituting $x=-\frac{3}{2}$ gives

$$7\times-\frac{3}{2}+8=A(-\frac{3}{2}-1).$$

Hence,

$$A=\frac{7\times-\frac{3}{2}+8}{-\frac{3}{2}-1}=\frac{-\frac{5}{2}}{-\frac{5}{2}}=1.$$

We conclude that

$$\frac{7x+8}{(2x+3)(x-1)}=\frac{1}{2x+3}+\frac{3}{x-1}.$$

The "Cover-up" Rule
A useful time-saver when the factors in the denominator of the given rational function are linear is to use the following routine which is equivalent to the method described above:

To obtain the constant numerator of the partial fraction for a particular linear factor, $ax+b$, in the original denominator, cover up $ax+b$ in the original rational function and then substitute $x=-\frac{b}{a}$ into what remains.

ILLUSTRATION
In the above example, we may simply cover up $x-1$, then substitute $x=1$ into the fraction

$$\frac{7x+8}{2x+3}.$$

Then we may cover up $2x + 3$ and substitute $x = -\frac{3}{2}$ into the fraction

$$\frac{7x + 8}{x - 1}.$$

Note:
We shall see later how the cover-up rule can also be brought into effective use when not all of the factors in the denominator of the given rational function are linear.

(b) Denominator of the given rational function contains one linear and one quadratic factor

EXAMPLE

Express the rational function

$$\frac{3x^2 + 9}{(x - 5)(x^2 + 2x + 7)}$$

in partial fractions.

Solution

We should observe firstly that the quadratic factor will not reduce conveniently into two linear factors. If it did, the method would be as in the previous paragraph. Hence we may write

$$\frac{3x^2 + 9}{(x - 5)(x^2 + 2x + 7)} \equiv \frac{A}{x - 5} + \frac{Bx + C}{x^2 + 2x + 7},$$

noticing that the second partial fraction may contain an x term in its numerator, yet still be a proper rational function.

Multiplying throughout by $(x - 5)(x^2 + 2x + 7)$, we obtain

$$3x^2 + 9 \equiv A(x^2 + 2x + 7) + (Bx + C)(x - 5).$$

A convenient value of x to substitute on both sides is $x = 5$ which gives

$$3 \times 5^2 + 9 = A(5^2 + 2 \times 5 + 7).$$

That is, $84 = 42A$ or $A = 2$.

No other convenient values of x may be substituted; but two polynomial expressions can be identical only if their corresponding coefficients are the same in value. We therefore equate suitable coefficients to find B and C; usually, the coefficients of the highest and lowest powers of x.

Equating coefficients of x^2, $3 = A + B$ and hence $B = 1$.

Equating constant terms (the coefficients of x^0), $9 = 7A - 5C = 14 - 5C$ and hence $C = 1$.

The result is therefore

$$\frac{3x^2 + 9}{(x-5)(x^2+2x+7)} \equiv \frac{2}{x-5} + \frac{x+1}{x^2+2x+7}.$$

Observations

It is easily verified that the value of A may be calculated by means of the cover-up rule, as in paragraph **(a)**; and, having found A, the values of B and C could be found by cross multiplying the numerators and denominators in the expression

$$\frac{2}{x-5} + \frac{?x+?}{x^2+2x+7}$$

in order to arrive at the numerator of the original rational function. This process essentially compares the coefficients of x^2 and x^0 as before.

(c) Denominator of the given rational function contains a repeated linear factor

In general, examples of this kind will not be more complicated than for a rational function with one repeated linear factor together with either a non-repeated linear factor or a quadratic factor.

EXAMPLE

Express the rational function

$$\frac{9}{(x+1)^2(x-2)}$$

in partial fractions.

Solution

First we observe that, from paragraph **(b)**, the partial fraction corresponding to the repeated linear factor would be of the form

$$\frac{Ax+B}{(x+1)^2};$$

but this may be written

$$\frac{A(x+1)+B-A}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{B-A}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2}.$$

Thus, a better form of statement for the problem as a whole, is

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{x-2}.$$

Eliminating fractions, we obtain

$$9 \equiv A(x+1)(x-2) + C(x-2) + D(x+1)^2.$$

Putting $x = -1$ gives $9 = -3C$ so that $C = -3$.

Putting $x = 2$ gives $9 = 9D$ so that $D = 1$.

Equating coefficients of x^2 gives $0 = A + D$ so that $A = -1$.

Therefore,

$$\frac{9}{(x+1)^2(x-2)} \equiv -\frac{1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}.$$

Notes:
(i) Similar partial fractions may be developed for higher repeated powers so that, for a repeated linear factor of power of n, there will be n corresponding partial fractions, each with a constant numerator. The labels for these numerators in future will be taken as A, B, C, etc. in sequence.

(ii) We observe that the numerator above the repeated factor itself (D in this case) could actually have been obtained by the cover-up rule; covering up $(x+1)^2$ in the original rational function, then substituting $x = -1$ into the rest.

(d) Keily's Method

A useful method for repeated linear factors is to use these factors one at a time, keeping the rest outside the expression as a factor.

EXAMPLE

Express the rational function

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{1}{x+1}\left[\frac{9}{(x+1)(x-2)}\right]$$

in partial fractions.

Solution

Using the cover-up rule inside the square brackets,

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{1}{x+1}\left[\frac{-3}{x+1}+\frac{3}{x-2}\right]$$

$$\equiv -\frac{3}{(x+1)^2}+\frac{3}{(x+1)(x-2)};$$

and, again by cover-up rule,

$$\equiv -\frac{3}{(x+1)^2}-\frac{1}{x+1}+\frac{1}{x-2}$$

as before.

Warning
Care must be taken with Keily's method when, even though the original rational function is proper, the resulting expression inside the square brackets is improper. This would have occurred, for instance, if the problem given had been

$$\frac{9x^2}{(x+1)^2(x-2)},$$

leading to

$$\frac{1}{x+1}\left[\frac{9x^2}{(x+1)(x-2)}\right].$$

In this case, long division would have to be used inside the square brackets before proceeding with Keily's method.

For such examples, it is probably better to use the method of paragraph **(c)**.

1.9.3 EXERCISES

Express the following rational functions in partial fractions:

1. $$\frac{3x+5}{(x+1)(x+2)}.$$

2.

$$\frac{17x+11}{(x+1)(x-2)(x+3)}.$$

3.

$$\frac{3x^2-8}{(x-1)(x^2+x-7)}.$$

4.

$$\frac{2x+1}{(x+2)^2(x-3)}.$$

5.

$$\frac{9+11x-x^2}{(x+1)^2(x+2)}.$$

6.

$$\frac{x^5}{(x+2)(x-4)}.$$

1.9.4 ANSWERS TO EXERCISES

1.

$$\frac{2}{x+1}+\frac{1}{x+2}.$$

2.

$$\frac{1}{x+1}+\frac{3}{x-2}-\frac{4}{x+3}.$$

3.

$$\frac{1}{x-1}+\frac{2x+1}{x^2+x-7}.$$

4.

$$\frac{3}{5(x+2)^2}-\frac{7}{25(x+2)}+\frac{7}{25(x-3)}.$$

5.

$$-\frac{3}{(x+1)^2}+\frac{16}{x+1}-\frac{17}{x+2}.$$

6.

$$x^3+2x^2+12x+40+\frac{16}{3(x+2)}+\frac{512}{3(x-4)}.$$

"JUST THE MATHS"

UNIT NUMBER

1.10

ALGEBRA 10
(Inequalities 1)

by

A.J.Hobson

UNIT 1.10 - ALGEBRA 10 - INEQUALITIES 1.

1.10.1 INTRODUCTION

If the symbols a and b denote numerical quantities, then the statement

$$a < b$$

is used to mean "a is less than b" while the statement

$$b > a$$

is used to mean "b is greater than a"

These are called **"strict inequalities"** because there is no allowance for the possibilty that a and b might be equal to each other. For example, if a is the number of days in a particular month and b is the number of hours in that month, then $b > a$.

Some inequalities do allow the possibility of a and b being equal to each other and are called **"weak inequalities"** written in one of the forms

$$a \leq b \quad b \leq a \quad a \geq b \quad b \geq a$$

For example, if a is the number of students who enrolled for a particular module in a university and b is the number of students who eventually passed that module, then $a \geq b$.

1.10.2 ALGEBRAIC RULES FOR INEQUALITIES

Given two different numbers, one of them must be strictly less than the other. Suppose a is the smaller of the two and b the larger; i.e.

$$a < b$$

Then
1. $a + c < b + c$ for any number c.

2. $ac < bc$ when c is positive but $ac > bc$ when c is negative.

3. $\frac{1}{a} > \frac{1}{b}$ provided a and b are **both positive.**

Note:
The only other situation in 3. which is consistent with $a < b$ will occur if a is negative and b is positive. In this case, $\frac{1}{a} < \frac{1}{b}$ because a negative number is always less than a positive number.

EXAMPLES

1. Simplify the inequality
$$2x + 3y > 5x - y + 7.$$

   Solution

   We simply deal with this in the same way as we would deal with an equation by adding appropriate quantities to both sides or subtracting appropriate quantities from both sides. We obtain
$$-3x + 4y > 7 \quad \text{or} \quad 3x - 4y + 7 < 0.$$

2. Solve the inequality
$$\frac{1}{x-1} < 2,$$
   assuming that $x \neq 1$.

   Solution

   Here we must be careful in case $x-1$ is negative; the argument is therefore in two parts:
   (a) If $x > 1$, i.e. $x - 1$ is positive, then the inequality can be rewritten as
$$1 < 2(x-1) \quad \text{or} \quad x - 1 > \frac{1}{2}.$$
   Hence,
$$x > \frac{3}{2}.$$
   (b) If $x < 1$, i.e. $x - 1$ is negative, then the inequality is automatically true since a negative number is bound to be less than a positive number.

   Conclusion:

   The given inequality is satisfied when $x < 1$ and when $x > \frac{3}{2}$.

3. Solve the inequality
$$2x - 7 \leq 3.$$

   Solution

   Adding 7 to both sides, then dividing both sides by 2 gives
$$x \leq 5.$$

4. Solve the inequality
$$\frac{x-1}{x-6} \geq 0.$$

Solution

We first observe that the fraction on the left of the inequality can equal zero only when $x = 1$.

Secondly, the only way in which a fraction can be positive is for both numerator and denominator to be positive or both numerator and denominator to be negative.

(a) Suppose $x - 1 > 0$ and $x - 6 > 0$; these two are covered by $x > 6$.

(b) Suppose $x - 1 < 0$ and $x - 6 < 0$; these two are covered by $x < 1$.

Note:
The value $x = 6$ is problematic because the given expression becomes infinite; in fact, as x passes through 6 from values below it to values above it, there is a sudden change from $-\infty$ to $+\infty$.

Conclusion:
The inequality is satisfied when either $x > 6$ or $x \leq 1$.

1.10.3 INTERVALS

In scientific calculations, a variable quantity x may be restricted to a certain range of values called an **"interval"** which may extend to ∞ or $-\infty$; but, in many cases, such intervals have an upper and a lower **"bound"**. The standard types of interval are as follows:

(a) $a < x < b$ denotes an **"open interval"** of all the values of x between a and b but excluding a and b themselves. The symbol (a, b) is also used to mean the same thing. For example, if x is a purely decimal quantity, it must lie in the open interval

$$-1 < x < 1.$$

(b) $a \leq x \leq b$ denotes a **"closed interval"** of all the values of x from a to b inclusive. The symbol $[a, b]$ is also used to mean the same thing. For example, the expression $\sqrt{1 - x^2}$ has real values only when

$$-1 \leq x \leq 1.$$

Note:
It is possible to encounter intervals which are closed at one end but open at the other; they may be called either **"half open"** or **"half closed"**. For example

$$a < x \leq b \quad \text{or} \quad a \leq x < b,$$

which can also be denoted respectively by $(a, b]$ and $[a, b)$.

(c) Intervals of the types

$$x > a \quad x \geq a \quad x < a \quad x \leq a$$

are called **"infinite intervals"**.

1.10.4 EXERCISES

1. Simplify the following inequalities:

   (a)

   $$x + y \leq 2x + y + 1;$$

   (b)

   $$2a - b > 1 + a - 2b - c.$$

2. Solve the following inequalities to find the range of values of x.

   (a)

   $$x + 3 < 6;$$

   (b)

   $$-2x \geq 10;$$

   (c)

   $$\frac{2}{x} > 18;$$

   (d)

   $$\frac{x+3}{2x-1} \leq 0.$$

3. Classify the following intervals as open, closed, or half-open/half-closed:

   (a)

   $$(5, 8);$$

   (b)

   $$(-3, -2);$$

(c)

$$[2,4);$$

(d)

$$[8,23];$$

(e)

$$(-\infty,\infty);$$

(f)

$$(0,\infty);$$

(g)

$$[0,\infty).$$

1.10.5 ANSWERS TO EXERCISES

1. (a) $x \geq -1$;
   (b) $a+b+c > 1$.
2. (a) $x < 3$;
   (b) $x \leq -5$;
   (c) $x < \frac{1}{9}$ since $x > 0$;
   (d) $-3 \leq x < \frac{1}{2}$.
3. (a) open;
   (b) open;
   (c) half-open/half-closed;
   (d) closed;
   (e) open;
   (f) open;
   (g) half-open/half-closed.

"JUST THE MATHS"

UNIT NUMBER

1.11

ALGEBRA 11
(Inequalities 2)

by

A.J.Hobson

UNIT 1.11 - ALGEBRA 11 - INEQUALITIES 2.

1.11.1 RECAP ON MODULUS, ABSOLUTE VALUE OR NUMERICAL VALUE

As seen in Unit 1.1, the Modulus of a numerical quantity ignores any negative signs if there are any. For example, the modulus of -3 is 3, but the modulus of 3 is also 3.

The modulus of an unspecified numerical quantity x is denoted by the symbol

$$\mid x \mid$$

and is defined by the two statements:

$$\mid x \mid = x \quad \text{if} \quad x \geq 0;$$

$$\mid x \mid = -x \quad \text{if} \quad x \leq 0.$$

Notes:
(i) An alternative, but less convenient formula for the modulus of x is

$$\mid x \mid = +\sqrt{x^2}.$$

(ii) It is possible to show that, for any two numbers a and b,

$$\mid a + b \mid \leq \mid a \mid + \mid b \mid .$$

This is called the **"triangle inequality"** and can be linked to the fact that the length of any side of a triangle is never greater than the sum of the lengths of the other two sides.

The proof is a little involved since it is necessary to consider all possible cases of a and b being positive, negative or zero together with a consideration of their relative sizes. It will not be included here.

1.11.2 INTERVAL INEQUALITIES

(a) Using the Modulus notation

In this section, we investigate the meaning of the inequality

$$\mid x - a \mid < k,$$

where a is any number and k is a positive number.

Case 1. $x - a > 0$.

The inequality can be rewritten as

$$x - a < k \quad \text{i.e.} \quad x < a + k.$$

Case 2. $x - a < 0$.

The inequality can be rewritten as

$$-(x - a) < k \quad \text{i.e.} \quad a - x < k \quad \text{i.e.} \quad x > a - k.$$

Combining the two cases, we conclude that

$$\mid x - a \mid < k \quad \text{means} \quad a - k < x < a + k,$$

which is an open interval having $x = a$ at the centre and extending to a distance of a either side of the centre. A similar interpretation could be given of $\mid x - a \mid \leq k$.

EXAMPLE

Obtain the closed interval represented by the statement

$$\mid x + 3 \mid \leq 10.$$

Solution

Using $a = -3$ and $k = 10$, we have

$$-3 - 10 \leq x \leq -3 + 10.$$

That is,

$$-13 \leq x \leq 7.$$

(b) Using Factorised Polynomials

Suppose a polynomial in x has been factorised into a number of linear factors corresponding to the degree of the polynomial. Then, if certain values of x are substituted in, the polynomial will be positive (or zero) as long as the number of individual factors which become negative is even. Similarly, the polynomial will be negative (or zero) as long as the number of individual factors which become negative is odd.

These observations enable us to find the ranges of values of x for which a factorised polynomial is positive or negative.

EXAMPLE

Find the range of values of x for which the polynomial

$$(x+3)(x-1)(x-2)$$

is strictly positive.

Solution

The first task is to find what are called the **"critical values"**. These are the values of x at which the polynomial becomes equal to zero. In our case, the critical values are $x = -3, 1, 2$.

Next, the critical values divide the x-line into separate intervals where, for the moment, we exclude the critical values themselves. In this case, we obtain

$$x < -3, \quad -3 < x < 1, \quad 1 < x < 2, \quad x > 2.$$

All that is now necessary is to select a value from each of these intervals and investigate how it affects the signs of the factors of the polynomial and hence the sign of the polynomial itself.

$x < -3$ gives (neg)(neg)(neg) and therefore < 0;

$-3 < x < 1$ gives (pos)(neg)(neg) and therefore > 0;

$1 < x < 2$ gives (pos)(pos)(neg) and therefore < 0;

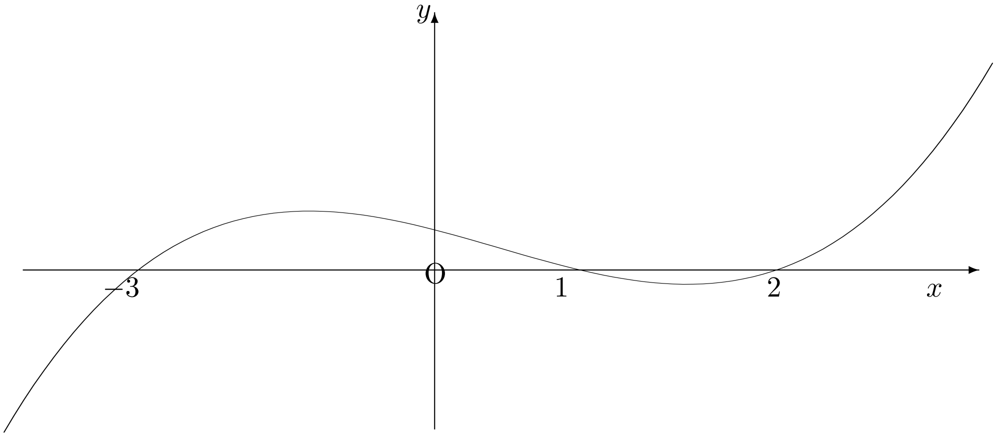
$x > 2$ gives (pos)(pos)(pos) and therefore > 0.

Clearly the critical values will not be included in the answer for this example because they cause the polynomial to have the value zero.

The required ranges are thus

$$-3 < x < 1 \quad \text{and} \quad x > 2.$$

Note:
An alternative method is to sketch the graph of the polynomial as a smooth curve passing through all the critical values on the x-axis. One point only between each critical value will make it clear whether the graph is on the positive side of the x-axis or the negative side. For the above example, the graph is as follows:



1.11.3 EXERCISES

1. Determine the **precise** ranges of values of x which satisfy the following inequalities:
   (a) $\mid x \mid < 2$;
   (b) $\mid x \mid > 3$;
   (c) $\mid x - 3 \mid < 1$;
   (d) $\mid x + 6 \mid < 4$;
   (e) $\mid x + 1 \mid \geq 2$;
   (f) $0 < \mid x + 3 \mid < 5$.
2. Determine the **precise** ranges of values of x which satisfy the following inequalities:
   (a) $(x - 4)(x + 2) \geq 0$;
   (b) $(x + 3)(x - 2)(x - 4) < 0$;
   (c) $(x + 1)^2(x - 3) > 0$;
   (d) $18x - 3x^2 > 0$.
3. By considering the expansion of $(a^2 - b^2)^2$, show that

$$a^4 + b^4 \geq 2a^2b^2.$$

1.11.4 ANSWERS TO EXERCISES

1. (a) $-2 < x < 2$;
   (b) $x < -3$ and $x > 3$;
   (c) $2 < x < 4$;
   (d) $-10 < x < -2$;
   (e) $x \geq 1$ and $x \leq -3$;
   (f) $-8 < x < -3$ and $-3 < x < 2$.
2. (a) $x \leq -2$ and $x \geq 4$;
   (b) $x < -3$ and $2 < x < 4$;
   (c) $x > 3$;
   (d) $0 < x < 6$.

"JUST THE MATHS"

UNIT NUMBER

2.1

SERIES 1
(Elementary progressions and series)

by

A.J.Hobson

UNIT 2.1 - SERIES 1 - ELEMENTARY PROGRESSIONS AND SERIES

2.1.1 ARITHMETIC PROGRESSIONS

The **"sequence"** of numbers,

$$a, a+d, a+2d, a+3d, ...$$

is said to form an **"arithmetic progression"**.

The symbol a represents the **"first term"**, the symbol d represents the **"common difference"** and the **"n-th term"** is given by the expression

$$a+(n-1)d$$

EXAMPLES

1. Determine the n-th term of the arithmetic progression 15, 12, 9, 6,...
   Solution
   The n-th term is
   $$15+(n-1)(-3) = 18-3n.$$
2. Determine the n-th term of the arithmetic progression
   $$8, 8.125, 8.25, 8.375, 8.5, ...$$
   Solution
   The n-th term is
   $$8+(n-1)(0.125) = 7.875+0.125n.$$
3. The 13th term of an arithmetic progression is 10 and the 25th term is 20; calculate
   (a) the common difference;
   (b) the first term;
   (c) the 17th term.
   Solution
   Letting a be the first term and d be the common difference, we have
   $$a+12d = 10$$
   and
   $$a+24d = 20.$$
   (a) Subtracting the first of these from the second gives $12d = 10$, so that the common difference $d = \frac{10}{12} = \frac{5}{6} \simeq 0.83$

(b) Substituting into the first of the relationships between a and d gives

$$a + 12 \times \frac{5}{6} = 10.$$

That is,

$$a + 10 = 10.$$

Hence, $a = 0$.

(c) The 17th term is

$$0 + 16 \times \frac{5}{6} = \frac{80}{6} = \frac{40}{3} \simeq 13.3$$

2.1.2 ARITHMETIC SERIES

If the terms of an arithmetic progression are added together, we obtain what is called an **"arithmetic series"**. The total sum of the first n terms of such a series can be denoted by S_n so that

$$S_n = a + [a + d] + [a + 2d] + ... + [a + (n - 2)d] + [a + (n - 1)d];$$

but this is not the most practical way of evaluating the sum of the n terms, especially when n is a very large number.

A trick which provides us with a more convenient formula for S_n is to write down the existing formula **backwards**. That is,

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + ... + [a + 2d] + [a + d] + a.$$

Adding the two statements now gives

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + ... + [2a + (n - 1)d] + [2a + (n - 1)d],$$

where, on the right hand side, there are n repetitions of the same expression.

Hence,

$$2S_n = n[2a + (n - 1)d]$$

or

$$S_n = \frac{n}{2}[2a + (n - 1)d].$$

This version of the formula is suitable if we know the values of a, n and d; but an alternative version can be used if we know only the first term, the last term and the number of terms. In this case,

$$S_n = \frac{n}{2}[\text{FIRST} \quad + \quad \text{LAST}]$$

which is simply n times the average of the first and last terms.

EXAMPLES

1. Determine the sum of the natural numbers from 1 to 100.
   Solution
   The sum is given by
   $$\frac{100}{2} \times [1 + 100] = 5050.$$
2. How many terms of the arithmetic series
   $$10 + 12 + 14 + ...$$
   must be taken so that the sum of the series is 252 ?
   Solution
   The first term is clearly 10 and the common difference is 2.
   Hence, letting n be the number of terms, we require that
   $$252 = \frac{n}{2}[20 + (n - 1) \times 2].$$
   That is,
   $$252 = \frac{n}{2}[2n + 18] = n(n + 9).$$
   By trial and error, $n = 12$ will balance this equation; but it is more conclusive to obtain n as the solution to the quadratic equation
   $$n^2 + 9n - 252 = 0$$
   or
   $$(n - 12)(n + 21) = 0,$$
   which gives $n = 12$ only, since the negative value $n = -21$ may be ignored.
3. A contractor agrees to sink a well 250 metres deep at a cost of £2.70 for the first metre, £2.85 for the second metre and an extra 15p for each additional metre. Find the cost of the last metre and the total cost.
   Solution
   In this problem we are dealing with an arithmetic series of 250 terms whose first term is 2.70 and whose common difference is 0.15. The cost of the last metre is the 250-th term of the series and therefore
   £$[2.70 + 249 \times 0.15]$ = £40.05
   The total cost will be
   £$\frac{250}{2} \times [2.70 + 40.05]$ = £5343.75

2.1.3 GEOMETRIC PROGRESSIONS

The sequence of numbers

$$a, ar, ar^2, ar^3, \ldots$$

is said to form a "geometric progression".

The symbol a represents the **"first term"**, the symbol r represents the **"common ratio"** and the **"n-th term"** is given by the expression

$$ar^{n-1}.$$

EXAMPLES

1. Determine the n-th term of the geometric progression

$$3, -12, 48, -192, \ldots$$

   Solution

   The progression has n-th term

$$3(-4)^{n-1}$$

   which will always be positive when n is an odd number and negative when n is an even number.

2. Determine the seventh term of the geometric progression

$$3, 6, 12, 24, \ldots$$

   Solution

   The seventh term is

$$3(2^6) = 192.$$

3. The third term of a geometric progression is 4.5 and the ninth term is 16.2. Determine the common ratio.

   Solution

   Firstly, we have

$$ar^2 = 4.5$$

   and

$$ar^8 = 16.2$$

   Dividing the second of these by the first gives

$$\frac{ar^8}{ar^2} = \frac{16.2}{4.5}.$$

Therefore,

$$r^6 = 3.6$$

and so

$$r \simeq 1.238$$

4. The expenses of a company are £200,000 a year. It is decided that each year they shall be reduced by 5% of those for the preceding year.
   What will be the expenses during the fourth year, the first reduction taking place at the end of the first year ?

   Solution
   In this problem, we use a geometric progression with first term 200,000 and common ratio 0.95.

   The expenses during the fourth year will thus be the fourth term of the progression; that is, £200,000 $\times(0.95)^3$ = £171475.

2.1.4 GEOMETRIC SERIES

If the terms of a geometric progression are added together, we obtain what is called a **"geometric series"** . The total sum of a geometric series with n terms may be denoted by S_n so that

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

but, as with arithmetic series, this is not the most practical formula for evaluating S_n.

This time, a trick to establish a convenient formula for S_n is to write down both S_n and rS_n, the latter giving

$$rS_n = \qquad ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ar^n.$$

Subtracting the second formula from the first gives

$$S_n - rS_n = a - ar^n,$$

so that

$$S_n = \frac{a(1-r^n)}{1-r}.$$

This is the version of the formula most commonly used since, in many practical applications, r will be less than one; but, for examples in which r is greater than one, it may be better to use the alternative version, namely

$$S_n = \frac{a(r^n-1)}{r-1}.$$

In fact, either version may be used whatever the value of r is.

EXAMPLES

1. Determine the sum of the geometric series

$$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$

Solution

The sum is given by

$$S_6 = \frac{4(1 - (\frac{1}{2})^6)}{1 - \frac{1}{2}} = \frac{4(1 - 0.0156)}{0.5} \simeq 7.875$$

2. A sum of money £C is invested for n years at an interest of $100r\%$, compounded annually. What will be the total interest earned by the end of the n-th year ?

Solution

At the end of year 1, the interest earned will be Cr.
At the end of year 2, the interest earned will be $(C + Cr)r = Cr(1 + r)$.
At the end of year 3, the interest earned will be $C(1 + r)r + C(1 + r)r^2 = Cr(1 + r)^2$.

This pattern reveals that,
at the end of year n, the interest earned will be $Cr(1 + r)^{n-1}$.

Thus the total interest earned by the end of year n will be

$$Cr + Cr(1 + r) + Cr(1 + r)^2 + ... + Cr(1 + r)^{n-1},$$

which is a geometric series of n terms with first term Cr and common ratio $1 + r$. Its sum is therefore

$$\frac{Cr((1 + r)^n - 1)}{r} = C((1 + r)^n - 1).$$

Note:

The same result can be obtained using only a geometric progression as follows:

At the end of year 1, the total amount will be $C + Cr = C(1 + r)$.
At the end of year 2, the total amount will be $C(1 + r) + C(1 + r)r = C(1 + r)^2$.
At the end of year 3, the total amount will be $C(1 + r)^2 + C(1 + r)^2 r = C(1 + r)^3$.
At the end of year n, the total amount will be $C(1 + r)^n$.

Thus the total interest earned will be $C(1 + r)^n - C = C((1 + r)^n - 1)$ as before.

The sum to infinity of a geometric series.

In a geometric series with n terms, suppose that the value of the common ratio, r, is numerically less than 1. Then the higher the value of n, the smaller the numerical value of r^n, to the extent that, as n approaches infinity, r^n approaches zero.

We conclude that, although it not possible to reach the end of a geometric series which has an infinite number of terms, its sum to infinity may be given by

$$S_\infty = \frac{a}{1-r}.$$

EXAMPLES

1. Determine the sum to infinity of the geometric series

$$5 - 1 + \frac{1}{5} - \ \ldots$$

   Solution
   The sum to infinity is

$$\frac{5}{1+\frac{1}{5}} = \frac{25}{6} \simeq 4.17$$

2. The yearly output of a silver mine is found to be decreasing by 25% of its previous year's output. If, in a certain year, its output was £25,000, what could be reckoned as its total future output ?

   Solution
   The total output, in pounds, for subsequent years will be given by

$$25000 \times 0.75 + 25000 \times (0.75)^2 + 25000 \times (0.75)^3 + ... = \frac{25000 \times 0.75}{1 - 0.75} = 75000.$$

2.1.5 MORE GENERAL PROGRESSIONS AND SERIES

Introduction

Not all progressions and series encountered in mathematics are either arithmetic or geometric. For instance:

$$1^2, 2^2, 3^2, 4^2, ..., n^2$$

has a clearly defined pattern but is not arithmetic or geometric.

An **arbitrary** progression of n numbers which conform to some regular pattern is often denoted by

$$u_1, u_2, u_3, u_4, ..., u_n,$$

and it may or may not be possible to find a simple formula for the sum S_n.

The Sigma Notation (Σ).

If the general term of a series with n terms is known, then the complete series can be written down in short notation as indicated by the following illustrations:

1.

$$a + (a + d) + (a + 2d) + ... + (a + [n - 1]d) = \sum_{r=1}^{n} (a + [r - 1]d).$$

2.

$$a + ar + ar^2 + ...ar^{n-1} = \sum_{k=1}^{n} ar^{k-1}.$$

3.

$$1^2 + 2^2 + 3^2 + ...n^2 = \sum_{r=1}^{n} r^2.$$

4.

$$-1^3 + 2^3 - 3^3 + 4^3 + ...(-1)^n n^3 = \sum_{r=1}^{n} (-1)^r r^3.$$

Notes:

(i) It is sometimes more convenient to count the terms of a series from zero rather than 1. For example:

$$a + (a + d) + (a + 2d) + ...a + [n - 1]d = \sum_{r=0}^{n-1} (a + rd)$$

and

$$a + ar + ar^2 + ar^3 + ...ar^{n-1} = \sum_{k=0}^{n-1} ar^k.$$

In general, for a series with n terms starting at u_0,

$$u_0 + u_1 + u_2 + u_3 + ... + u_{n-1} = \sum_{r=0}^{n-1} u_r.$$

(ii) We may also use the sigma notation for **"infinite series"** such as those we encountered in the sum to infinity of a geometric series. For example

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... = \sum_{r=1}^{\infty} \frac{1}{3^{r-1}} \text{ or } \sum_{r=0}^{\infty} \frac{1}{3^r}.$$

STANDARD RESULTS

It may be shown that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1),$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

and

$$\sum_{r=1}^{n} r^3 = \left[\frac{1}{2}n(n+1)\right]^2.$$

Outline Proofs:

The first of these is simply the formula for the sum of an arithmetic series with first term 1 and last term n.

The second is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1.$$

The third is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1.$$

EXAMPLE

Determine the sum to n terms of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 + \ldots$$

Solution

The series is

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r^3 + 3r^2 + 2r = \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r.$$

Using the three standard results, the summation becomes

$$\left[\frac{1}{2}n(n+1)\right]^2 + 3\left[\frac{1}{6}n(n+1)(2n+1)\right] + 2\left[\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{4}n(n+1)[n(n+1) + 4n + 2 + 4] = \frac{1}{4}n(n+1)[n^2 + 5n + 6].$$

This simplifies to

$$\frac{1}{4}n(n+1)(n+2)(n+3).$$

2.1.6 EXERCISES

1. Write down the next two terms and also the n-th term of the following sequences of numbers which are either arithmetic progressions or geometric progressions:

   (a) 40, 29, 18, 7, ...;

   (b) $\frac{13}{3}$, $\frac{17}{3}$, 7, ...;

   (c) 5, 15, 45, ...;

   (d) 10, 9.2, 8.4, ...;

   (e) 81, -54, 36, ...;

   (f) $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{16}$, ...

2. The third term of an arithmetic series is 34 and the 17th term is -8. Find the sum of the first 20 terms.

3. For the geometric series 1 + 1.2 + 1.44 + ..., find the 6th term and the sum of the first 10 terms.

4. A parent places in a savings bank £25 on his son's first birthday, £50 on his second, £75 on his third and so on, increasing the amount by £25 on each birthday. How much will be saved up (apart from any accrued interest) when the boy reaches his 16th birthday if the final amount is added on this day ?

5. Every year, a gardner takes 4 runners from each of his one year old strawberry plants in order to form 4 additional plants. If he starts with 5 plants, how many new plants will he take at the end of the 6th year and what will then be his total number of plants ?

6. A superball is dropped from a height of 10m. At each rebound, it rises to a height which is 90% of the height from which it has just fallen. What is the total distance through which the ball will have moved before it finally comes to rest ?

7. Express the series

   $$\frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \frac{8}{9} + ...n\text{terms}$$

   in both of the forms

   $$\sum_{r=1}^{n} u_r \quad \text{and} \quad \sum_{r=0}^{n-1} u_r.$$

   Hint:
   Find the pattern in the numerators and denominators separately

8. Determine the sum to n terms of the series

$$1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \ldots$$

2.1.7 ANSWERS TO EXERCISES

1. (a) -4, -15; $51 - 11n$;
   (b) $8\frac{1}{3}$, $9\frac{2}{3}$; $\frac{1}{3}(9 + 4n)$;
   (c) 135, 405; $5(3)^{n-1}$;

   (d) 7.6, 6.8; $10.8 - 0.8n$;
   (e) -24, 16; $(-1)^{n-1}2^{n-1}3^{5-n}$;
   (f) $\frac{9}{64}$, $\frac{27}{256}$; $\frac{3^{n-2}}{4^{n-1}}$.
2. $a = 40$, $d = -3$, $S_{20} = 230$.
3. 6th term $= (1.2)^5 \simeq 2.488$ and $S_{10} \simeq 25.96$
4. £3400.
5. Number at year 6 $= 5 \times 4^6 = 20480$; Total $= 27305$.
6. Total distance $= 10 + \frac{2\times10\times0.9}{1-0.9} = 190$m.
7.

$$\sum_{r=1}^{n} \frac{2r}{2r+1} \quad \text{and} \quad \sum_{r=0}^{n-1} \frac{2(r+1)}{2r+3}.$$

8.

$$\frac{1}{6}n(n+1)(4n+5).$$

"JUST THE MATHS"

UNIT NUMBER

2.2

SERIES 2
(Binomial series)

by

A.J.Hobson

UNIT 2.2 - SERIES 2 - BINOMIAL SERIES

INTRODUCTION

In this section, we shall be concerned with the methods of expanding (multiplying out) an expression of the form

$$(A + B)^n,$$

where A and B are either mathematical expressions or numerical values, and n is a given number which need not be a positive integer. However, we shall deal first with the case when n *is* a positive integer, since there is a useful aid to memory for obtaining the result.

2.2.1 PASCAL'S TRIANGLE

Initially, we consider some simple illustrations obtainable from very elementary algebraic techniques in earlier work:

1. $(A + B)^1 \equiv$
$$A + B.$$

2. $(A + B)^2 \equiv$
$$A^2 + 2AB + B^2.$$

3. $(A + B)^3 \equiv$
$$A^3 + 3A^2B + 3AB^2 + B^3.$$

4. $(A + B)^4 \equiv$
$$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4.$$

OBSERVATIONS

(i) We notice that, in each result, the expansion begins with the maximum possible power of A and ends with the maximum possible power of B.

(ii) In the sequence of terms from beginning to end, the powers of A **decrease** in steps of 1 while the powers of B **increase** in steps of 1.

(iii) The coefficients in the illustrated expansions follow the diagramatic pattern called **PASCAL'S TRIANGLE**:

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

and this suggests a general pattern where each line begins and ends with the number 1 and each of the other numbers is the sum of the two numbers above it in the previous line. For example, the next line would be

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

giving the result

5. $(A + B)^5 \equiv$

$$A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5.$$

(iv) The only difference which occurs in an expansion of the form

$$(A - B)^n$$

is that the terms are alternately positive and negative. For instance,

6. $(A - B)^6 \equiv$

$$A^6 - 6A^5B + 15A^4B^2 - 20A^3B^3 + 15A^2B^4 - 6AB^5 + B^6.$$

2.2.2 BINOMIAL FORMULAE

In $(A + B)^n$, if n is a large positive integer, then the method of Pascal's Triangle can become very tedious. If n is *not* a positive integer, then Pascal's Triangle cannot be used anyway.

A more general method which can be applied to any value of n is the binomial formula whose proof is best obtained as an application of differential calculus and hence will not be included here.

Before stating appropriate versions of the binomial formula, we need to introduce a standard notation called a **"factorial"** by means of the following definition:

DEFINITION
If n is a positive integer, the product

$$1.2.3.4.5..........n$$

is denoted by the symbol $n!$ and is called **"n factorial"**.

Note:
This definition could not be applied to the case when $n = 0$, but it is convenient to give a meaning to 0! We define it separately by the statement

$$0! = 1$$

and the logic behind this separate definition can be made plain in the applications of calculus. There is no meaning to $n!$ when n is a negative integer.

(a) Binomial formula for $(A + B)^n$ when n is a positive integer.

It can be shown that

$$(A + B)^n \equiv A^n + na^{n-1}B + \frac{n(n-1)}{2!}A^{n-2}B^2 + \frac{n(n-1)(n-2)}{3!}A^{n-3}B^3 + ... + B^n.$$

Notes:

(i) This is the same as the result which would be given by Pascal's Triangle.

(ii) The last term in the expansion is really

$$\frac{n(n-1)(n-2)(n-3).......3.2.1}{n!}A^{n-n}B^n = A^0B^n = B^n.$$

(iii) The coefficient of $A^{n-r}B^r$ in the expansion is

$$\frac{n(n-1)(n-2)(n-3).......(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

and this is sometimes denoted by the symbol $\binom{n}{r}$.

(iv) A commonly used version of the result is given by

$$(1 + x)^n \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... + x^n.$$

EXAMPLES

1. Expand fully the expression $(1+2x)^3$.
   Solution
   We first note that
   $$(A+B)^3 \equiv A^3 + 3A^2B + \frac{3.2}{2!}AB^2 + B^3 \equiv A^3 + 3A^2B + 3AB^2 + B^3.$$
   If we now replace A by 1 and B by $2x$, we obtain
   $$(1+2x)^3 \equiv 1 + 3(2x) + 3(2x)^2 + (2x)^3 \equiv 1 + 6x + 12x^2 + 8x^3.$$

2. Expand fully the expression $(2-x)^5$.
   Solution
   We first note that
   $$(A+B)^5 \equiv A^5 + 5A^4B + \frac{5.4}{2!}A^3B^2 + \frac{5.4.3}{3!}A^2B^3 + \frac{5.4.3.2}{4!}AB^4 + B^5.$$
   That is,
   $$(A+B)^5 \equiv A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5.$$
   We now replace A by 2 and B by $-x$ to obtain
   $$(2-x)^5 \equiv 2^5 + 5(2)^4(-x) + 10(2)^3(-x)^2 + 10(2)^2(-x)^3 + 5(2)(-x)^4 + (-x)^5.$$
   That is,
   $$(2-x)^5 \equiv 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5.$$

(b) Binomial formula for $(A+B)^n$ when n is negative or a fraction.

It turns out that the binomial formula for a positive integer index may still be used when the index is negative or a fraction, except that the series of terms will be an **infinite** series. That is, it will not terminate.

In order to state the most commonly used version of the more general result, we use the simplified form of the binomial formula in Note (iii) of the previous section:

RESULT

If n is negative or a fraction and x lies strictly between $x=-1$ and $x=1$, it can be shown that
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

EXAMPLES

1. Expand $(1+x)^{\frac{1}{2}}$ as far as the term in x^3.

   Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + ...$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - ......$$

   provided $-1 < x < 1$.

2. Expand $(2-x)^{-3}$ as far as the term in x^3 stating the values of x for which the series is valid.

   Solution

   We first convert the expression $(2-x)^{-3}$ to one in which the leading term in the bracket is 1. That is,

$$(2-x)^{-3} \equiv \left[2\left(1-\frac{x}{2}\right)\right]^{-3}$$

$$\equiv \frac{1}{8}\left(1+\left[-\frac{x}{2}\right]\right)^{-3}.$$

   The required binomial expansion is thus:

$$\frac{1}{8}\left[1+(-3)\left(-\frac{x}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{x}{2}\right)^3+......\right].$$

   That is,

$$\frac{1}{8}\left[1+\frac{3x}{2}+\frac{3x^2}{2}+\frac{5x^3}{4}+......\right].$$

   The expansion is valid provided that $-x/2$ lies strictly between -1 and 1. This will be so when x itself lies strictly between -2 and 2.

(c) Approximate Values

The Binomial Series may be used to calculate simple approximations, as illustrated by the following example:

EXAMPLE

Evaluate $\sqrt{1.02}$ correct to five places of decimals.

Solution

Using $1.02 = 1 + 0.02$, we may say that

$$\sqrt{1.02} = (1 + 0.02)^{\frac{1}{2}}.$$

That is,

$$\sqrt{1.02} = 1 + \frac{1}{2}(0.02) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1.2}(0.02)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2.3}(0.02)^3 + \ \ldots$$

$$= 1 + 0.01 - \frac{1}{8}(0.0004) + \frac{1}{16}(0.000008) - \ldots$$

$$= 1 + 0.01 - 0.00005 + 0.0000005 - \ \ldots$$

$$\simeq 1.010001 - 0.000050 = 1.009951$$

Hence $\sqrt{1.02} \simeq 1.00995$

2.2.3 EXERCISES

1. Expand the following, using Pascal's Triangle:

   (a)
   $$(1+x)^5;$$
   (b)
   $$(x+y)^6;$$
   (c)
   $$(x-y)^7;$$
   (d)
   $$(x-1)^8.$$

2. Use the result of question 1(a) to evaluate
   $$(1.01)^5$$
   without using a calculator.

3. Expand fully the following expressions:

   (a)
   $$(2x-1)^5;$$
   (b)
   $$\left(3+\frac{x}{2}\right)^4;$$
   (c)
   $$\left(x-\frac{2}{x}\right)^3.$$

4. Expand the following as far as the term in x^3, stating the values of x for which the expansions are valid:

   (a)
   $$(3+x)^{-1};$$

(b)

$$(1-2x)^{\frac{1}{2}};$$

(c)

$$(2+x)^{-4}.$$

5. Using the first four terms of the expansion for $(1+x)^n$, calculate an approximate value of $\sqrt{1.1}$, stating the result correct to five significant figures.

6. If x is small, show that

$$(1+x)^{-1}-(1-2x)^{\frac{1}{2}} \simeq \frac{3x^2}{2}.$$

2.2.4 ANSWERS TO EXERCISES

1. (a)

$$1+5x+10x^2+10x^3+5x^4+x^5;$$

(b)

$$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6;$$

(c)

$$x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7;$$

(d)

$$x^8-8x^7+28x^6-56x^5+70x^4-56x^3+28x^2-8x+1.$$

2. 1.0510100501 to ten places of decimals.

3. (a)

$$32x^5-80x^4+80x^3-40x^2+10x-1;$$

(b)

$$81+54x+\frac{27}{2}x^2+\frac{3}{2}x^3+\frac{1}{16}x^4;$$

(c)

$$x^3-6x+\frac{12}{x}-\frac{8}{x^3}.$$

4. (a)

$$\frac{1}{3}\left[1-\frac{x}{3}+\frac{x^2}{9}-\frac{x^3}{27}+......\right],$$

provided $-3 < x < 3$.

(b)

$$1-x-\frac{x^2}{2}-\frac{x^3}{2}-......,$$

provided $-\frac{1}{2} < x < \frac{1}{2}$.

(c)

$$\frac{1}{16}\left[1-2x+\frac{5x^2}{2}-\frac{5x^3}{2}+......\right],$$

provided $-2 < x < 2$.

5. 1.0488

6. Expand each bracket as far as the term in x^2.

"JUST THE MATHS"

UNIT NUMBER

2.3

SERIES 3
(Elementary convergence and divergence)

by

A.J.Hobson

UNIT 2.3 - SERIES 3 - ELEMENTARY CONVERGENCE AND DIVERGENCE

Introduction

In the examination of geometric series in Unit 2.1 and of binomial series in Unit 2.2, the idea was introduced of series which have a first term but no last term; that is, there are an infinite number of terms.

The general format of an infinite series may be specified by either

$$u_1 + u_2 + u_3 + \ldots = \sum_{r=1}^{\infty} u_r$$

or

$$u_0 + u_1 + u_2 + \ldots = \sum_{r=0}^{\infty} u_r.$$

In the first of these two forms, u_r is the r-th term while, in the second, u_r is the $(r+1)$-th term.

ILLUSTRATIONS

1.

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{r=1}^{\infty} \frac{1}{r} = \sum_{r=0}^{\infty} \frac{1}{r+1}.$$

2.

$$2 + 4 + 6 + 8 + \ldots = \sum_{r=1}^{\infty} 2r = \sum_{r=0}^{\infty} 2(r+1).$$

3.

$$1 + 3 + 5 + 7 + \ldots = \sum_{r=1}^{\infty} (2r-1) = \sum_{r=0}^{\infty} (2r+1).$$

2.3.1 THE DEFINITIONS OF CONVERGENCE AND DIVERGENCE

It has already been shown in Unit 2.1 (for geometric series) that an infinite series may have a **"sum to infinity"** even though it is not possible to reach the end of the series.

For example, the infinite geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{r=1}^{\infty} \frac{1}{2^r}$$

is such that the sum, S_n, of the first n terms is given by

$$S_n = \frac{\frac{1}{2}(1 - \frac{1}{2}^n)}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}.$$

As n becomes larger and larger, S_n approaches ever closer to the fixed value, 1.

We say that the **"limiting value"** of S_n as n **"tends to infinity"** is 1; and we write

$$\lim_{n \to \infty} S_n = 1.$$

Since this limiting value is a **finite** number, we say that the series **"converges"** to 1.

DEFINITION (A)
For the infinite series

$$\sum_{r=1}^{\infty} u_r,$$

the expression

$$u_1 + u_2 + u_3 + \ldots + u_n$$

is called its **"n-th partial sum"**.

DEFINITION (B)
If the n-th partial sum of an infinite series tends to a finite limit as n tends to infinity, the series is said to **"converge"**. In **all** other cases, the series is said to **"diverge"**.

ILLUSTRATIONS

1.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\;=\sum_{r=1}^{\infty}\frac{1}{2^r}\quad\text{converges.}$$

2.

$$1+2+3+4+\ldots.=\sum_{r=1}^{\infty}r\quad\text{diverges.}$$

3.

$$1-1+1-1+\ldots.=\sum_{r=1}^{\infty}(-1)^{n-1}\quad\text{diverges.}$$

Notes:

(i) The third illustration above shows that a series which diverges does not necessarily diverge to infinity.

(ii) Whether a series converges or diverges depends less on the starting terms than it does on the later terms. For example

$$7-15+2+39+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\;..$$

converges to $7-15+2+39+1=33+1=34$.

(iii) It will be seen, in the next section, that it is sometimes possible to test an infinite series for convergence or divergence without having to try and determine its sum to infinity.

2.3.2 TESTS FOR CONVERGENCE AND DIVERGENCE

In this section, the emphasis will be on the **use** of certain standard tests, rather than on their rigorous formal **proofs**. Only **outline** proofs will be suggested.

To begin with, we shall consider series of **positive** terms only.

TEST 1 - The r-th Term Test

An infinite series,

$$\sum_{r=1}^{\infty} u_r,$$

cannot converge unless its terms ultimately tend to zero; that is,

$$\lim_{r\to\infty} u_r = 0.$$

Outline Proof:

The series will converge only if the r-th partial sums, S_r, tend to a finite limit, L (say), as r tends to infinity; hence, if we observe that $u_r = S_r - S_{r-1}$, then u_r must tend to zero as r tends to infinity since S_r and S_{r-1} each tend to L.

ILLUSTRATIONS

1. The convergent series

$$\sum_{r=1}^{\infty} \frac{1}{2^r},$$

discussed earlier, is such that

$$\lim_{r\to\infty} \frac{1}{2^r} = 0.$$

2. The divergent series

$$\sum_{r=1}^{\infty} r,$$

discussed earlier, is such that

$$\lim_{r\to\infty} r \neq 0.$$

3. The series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is such that

$$\lim_{r\to\infty} \frac{1}{r} = 0,$$

but it will be shown later that this series is **divergent**.

That is, the converse of the r-th Term Test is not true. It does not imply that a series is convergent when its terms **do** tend to zero; merely that it is divergent when its terms **do not** tend to zero.

TEST 2 - The Comparison Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\sum_{r=1}^{\infty} v_r$$

is a second series which is known to **converge**.

Then the first series converges provided that $u_r \leq v_r$.

Similarly, if

$$\sum_{r=1}^{\infty} w_r$$

is a series which is known to **diverge**, then the first series diverges provided that $u_r \geq w_r$.

Note:
It may be necessary to ignore the first few values of r.

Outline Proof:

Suppose we think of u_r and v_r as the heights of two sets of rectangles, all with a common base-length of one unit.

If the series

$$\sum_{r=1}^{\infty} v_r$$

is **convergent** it represents a **finite** total area of an infinite number of rectangles.

The series

$$\sum_{r=1}^{\infty} u_r$$

represents a **smaller** area and, hence, is also finite.

A similar argument holds when

$$\sum_{r=1}^{\infty} w_r$$

is a **divergent** series and $u_r \geq w_r$.

A divergent series of **positive** terms can diverge only to $+\infty$ so that the set of rectangles determined by u_r generates an area that is greater than an area which is already infinite.

EXAMPLES

1. Show that the series

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ \ldots$$

is divergent.

Solution

The given series may be written as

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ \ldots,$$

a series whose terms are all greater than (or, for the second term, equal to) $\frac{1}{2}$. But the series

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ \ldots$$

is a divergent series and, hence, the series

$$\sum_{r=1}^{\infty}\frac{1}{r}$$

is divergent.

2. Given that

$$\sum_{r=1}^{\infty}\frac{1}{r^2}$$

is a convergent series, show that

$$\sum_{r=1}^{\infty}\frac{1}{r(r+1)}$$

is also a convergent series.

Solution

First, we observe that, for $r=1,2,3,4,\ldots,$

$$\frac{1}{r(r+1)}<\frac{1}{r.r}=\frac{1}{r^2}.$$

Hence, the terms of the series

$$\sum_{r=1}^{\infty}\frac{1}{r(r+1)}$$

are smaller in value than those of a known convergent series. It therefore converges also.

Note:
It may be shown that the series

$$\sum_{r=1}^{\infty} \frac{1}{r^p}$$

is convergent whenever $p > 1$ and divergent whenever $p \leq 1$. This result provides a useful standard tool to use with the Comparison Test.

TEST 3 - D'Alembert's Ratio Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = L;$$

Then the series converges if $L < 1$ and diverges if $L > 1$.

There is no conclusion if $L = 1$.

Outline Proof:

(i) If $L > 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be greater than 1 and so $u_{r+1} > u_r$ for a large enough value of r.

Hence, the terms cannot ultimately be decreasing; so Test 1 shows that the series diverges.

(ii) If $L < 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be less than 1 and so $u_{r+1} < u_r$ for a large enough value or r.

We will consider that this first occurs when $r = s$; and, from this value onwards, the terms steadily decrease in value.

Furthermore, we can certainly find a positive number, h, between L and 1 such that

$$\frac{u_{s+1}}{u_s} < h, \frac{u_{s+2}}{u_{s+1}} < h, \frac{u_{s+3}}{u_{s+2}} < h, \ \ldots$$

That is,

$$u_{s+1} < hu_s, \ u_{s+2} < hu_{s+1}, \ u_{s+3} < hu_{s+2}, \ \ldots,$$

which gives

$$u_{s+1} < hu_s, \ u_{s+2} < h^2u_s, \ u_{s+3} < h^3u_s, \ \ldots$$

But, since $L < h < 1$,

$$hu_s + h^2u_s + h^3u_s + \ \ldots$$

is a convergent geometric series; therefore, by the Comparison Test,

$$u_{s+1} + u_{s+2} + u_{s+3} + \ \ldots = \sum_{r=1}^{\infty} u_{s+r} \ \text{ converges,}$$

implying that the original series converges also.

(iii) If $L = 1$, there will be no conclusion since we have already encountered examples of both a convergent series **and** a divergent series which give $L = 1$.

In particular,

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent and gives

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = \lim_{r \to \infty} \frac{r}{r+1} = \lim_{r \to \infty} \frac{1}{1 + \frac{1}{r}} = 1.$$

Also,

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is convergent and gives

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r^2}{(r+1)^2} = \lim_{r\to\infty} \left(\frac{r}{r+1}\right)^2 = \lim_{r\to\infty} \left(\frac{1}{1+\frac{1}{r}}\right)^2 = 1.$$

Note:
A convenient way to calculate the limit as r tends to infinity of any ratio of two polynomials in r is first to divide the numerator and the denominator by the highest power of r.

For example,

$$\lim_{r\to\infty} \frac{3r^3+1}{2r^3+1} = \lim_{r\to\infty} \frac{3+\frac{1}{r^3}}{2+\frac{1}{r^3}} = \frac{3}{2}.$$

ILLUSTRATIONS

1. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} \frac{r}{2^r},$$

$$\frac{u_{r+1}}{u_r} = \frac{r+1}{2^{r+1}}.\frac{2^r}{r} = \frac{r+1}{2r}.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r+1}{2r} = \lim_{r\to\infty} \frac{1+\frac{1}{r}}{2} = \frac{1}{2}.$$

The limiting value is less than 1 so that the series converges.

2. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} 2^r,$$

$$\frac{u_{r+1}}{u_r} = \frac{2^{r+1}}{2^r} = 2.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} 2 = 2.$$

The limiting value is greater than 1 so that the series diverges.

2.3.3 EXERCISES

1. Use the "r-th Term Test" to show that the following series are divergent:

   (a)

$$\sum_{r=1}^{\infty} \frac{r}{r+2};$$

   (b)

$$\sum_{r=1}^{\infty} \frac{1+2r^2}{1+r^2}.$$

2. Use the "Comparison Test" to determine whether the following series are convergent or divergent:

   (a)

$$\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)};$$

   (b)

$$\sum_{r=1}^{\infty} \frac{r}{\sqrt{r^6+1}};$$

(c)

$$\sum_{r=1}^{\infty} \frac{r}{r^2+1}.$$

3. Use D'Alembert's Ratio Test to determine whether the following series are convergent or divergent:

   (a)

$$\sum_{r=1}^{\infty} \frac{2^r}{r^2};$$

   (b)

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)!};$$

   (c)

$$\sum_{r=1}^{\infty} \frac{r+1}{r!}.$$

4. Obtain an expression for the r-th term of the following infinite series and, hence, investigate them for convergence or divergence:

   (a)

$$\frac{1}{1\times 2} + \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} + \frac{1}{4\times 2^4} + \,..;$$

   (b)

$$1 + \frac{3}{2\times 4} + \frac{7}{4\times 9} + \frac{15}{8\times 16} + \frac{31}{16\times 25} + \,...;$$

   (c)

$$\frac{1}{\sqrt{3}-1} + \frac{1}{2-\sqrt{2}} + \frac{1}{\sqrt{5}-\sqrt{3}} + \frac{1}{\sqrt{6}-2} + \frac{1}{\sqrt{7}-\sqrt{5}} + \,..$$

2.3.4 ANSWERS TO EXERCISES

1. (a)

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = 1 \neq 0;$$

(b)

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = 2 \neq 0.$$

2. (a) Convergent;
   (b) Convergent;
   (c) Divergent.
3. (a) Divergent;
   (b) Convergent;
   (c) Convergent.
4. (a)

$$u_r = \frac{1}{r \times 2^r};$$

The series is convergent by D'Alembert's Ratio Test;

(b)

$$u_r = \frac{2^r - 1}{2^{r-1} r^2};$$

The series is convergent by Comparison Test;

(c)

$$u_r = \frac{1}{\sqrt{r+2} - \sqrt{r}};$$

The series is divergent by r-th Term Test.

Note:
For further discussion of limiting values, see Unit 10.1

"JUST THE MATHS"

UNIT NUMBER

2.4

SERIES 4
(Further convergence and divergence)

by

A.J.Hobson

UNIT 2.4 - SERIES 4- FURTHER CONVERGENCE AND DIVERGENCE

2.4.1 SERIES OF POSITIVE AND NEGATIVE TERMS

Introduction

In Units 2.2 and 2.3, most of the series considered have included only positive terms. But now we shall examine the concepts of convergence and divergence in cases where negative terms are present.

We note here, for example, that the r-th Term Test encountered in Unit 2.3 may be used for series whose terms are not necessarily all positive. This is because the formula

$$u_r = S_r - S_{r-1}$$

is valid for any series.

The series cannot converge unless the partial sums S_r and S_{r-1} both tend to the same finite limit as r tends to infinity which implies that u_r tends to zero as r tends to infinity.

A particularly simple kind of series with both positive and negative terms is one whose terms are alternately positive and negative. The following test is applicable to such series:

Test 4 - The Alternating Series Test

If

$$u_1 - u_2 + u_3 - u_4 + \ . \ . \ , \quad \text{where} \quad u_r > 0,$$

is such that

$$u_r > u_{r+1} \quad \textbf{and} \quad u_r \to 0 \quad \text{as} \quad r \to \infty,$$

then the series converges.

Outline Proof:

(a) Suppose we re-group the series as

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ . \ . \ .;$$

then, it may be considered in the form

$$\sum_{r=1}^{\infty} v_r,$$

where $v_1 = u_1 - u_2, v_2 = u_3 - u_4, v_3 = u_5 - u_6, \ \ldots .$

This means that v_r is positive, so that the corresponding r-th partial sums, $S_r = v_1 + v_2 + v_3 + \ \ldots \ + v_r$, steadily increase as r increases.

(b) Alternatively, suppose we re-group the series as

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ldots ;$$

then, it may be considered in the form

$$u_1 - \sum_{r=1}^{\infty} w_r,$$

where $w_1 = u_2 - u_3, w_2 = u_4 - u_5, w_3 = u_6 - u_7, \ \ldots .$

In this case, each partial sum, $S_r = u_1 - (w_1 + w_2 + w_3 + \ \ldots + w_r)$ is less than u_1 since positive quantities are being subtracted from it.

(c) We conclude that the partial sums of the original series are steadily increasing but are never greater than u_1. They must therefore tend to a finite limit as r tends to infinity; that is, the series converges.

ILUSTRATION

The series

$$\sum_{r=1}^{\infty} (-1)^{r-1}\frac{1}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots$$

is convergent since

$$\frac{1}{r} > \frac{1}{r+1} \ \ \textbf{and} \ \ \frac{1}{r} \to 0 \ \ \text{as} \ \ r \to \infty.$$

2.4.2 ABSOLUTE AND CONDITIONAL CONVERGENCE

In this section, a link is made between a series having both positive and negative terms and the corresponding series for which all of the terms are positive.

By making this link, we shall be able to make use of earlier tests for convergence and divergence.

DEFINITION (A)
If

$$\sum_{r=1}^{\infty} u_r$$

is a series with both positive and negative terms, it is said to be **"absolutely convergent"** if

$$\sum_{r=1}^{\infty} |u_r|$$

is convergent.

DEFINITION (B)
If

$$\sum_{r=1}^{\infty} u_r$$

is a convergent series of positive and negative terms, but

$$\sum_{r=1}^{\infty} |u_r|$$

is a divergent series, then the first of these two series is said to be **"conditionally convergent"**.

ILLUSTRATIONS

1. The series

   $$1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$

   converges absolutely since the series

   $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$

   converges.

2. The series
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ \ldots$$
is conditionally convergent since, although it converges (by the Alternating Series Test), the series
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ \ldots$$
is divergent.

Notes:

(i) It may be shown that any series of positive and negative terms which is **absolutely** convergent will also be convergent.

(ii) Any test for the convergence of a series of positive terms may be used as a test for the absolute convergence of a series of both positive and negative terms.

2.4.3 TESTS FOR ABSOLUTE CONVERGENCE

The Comparison Test

Given the series
$$\sum_{r=1}^{\infty} u_r,$$
suppose that $|u_r| \leq v_r$ where
$$\sum_{r=1}^{\infty} v_r$$
is a convergent series of positive terms. Then, the given series is absolutely convergent.

D'Alembert's Ratio Test

Given the series
$$\sum_{r=1}^{\infty} u_r,$$
suppose that
$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = L.$$

Then the given series is absolutely convergent if $L < 1$.

Note:

If $L > 1$, then $|u_{r+1}| > |u_r|$ for large enough values of r showing that the **numerical** values of the terms steadily increase. This implies that u_r does **not** tend to zero as r tends to infinity and, hence, by the r-th Term Test, the series diverges.

If $L = 1$, there is no conclusion.

EXAMPLES

1. Show that the series

$$\frac{1}{1 \times 2} - \frac{1}{2 \times 3} - \frac{1}{3 \times 4} + \frac{1}{4 \times 5} - \frac{1}{5 \times 6} - \frac{1}{6 \times 7} + \ ..$$

is absolutely convergent.

Solution

The r-th term of the series is numerically equal to

$$\frac{1}{r(r+1)},$$

which is always less than $\frac{1}{r^2}$, the r-th term of a known convergent series.

2. Show that the series

$$\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \ ..$$

is conditionally convergent.

Solution

The r-th term of the series is numerically equal to

$$\frac{r}{r^2+1},$$

which tends to zero as r tends to infinity.

Also,

$$\frac{r}{r^2+1} > \frac{r+1}{(r+1)^2+1}$$

since this may be reduced to the true statement $r^2 + r > 1$.
Hence, by the Alternating Series Test, the series converges.
However, it is also true that

$$\frac{r}{r^2+1} > \frac{r}{r^2+r} = \frac{1}{r+1};$$

and, hence, by the Comparison Test, the series of absolute values is divergent, since

$$\sum_{r=1}^{\infty} \frac{1}{r+1}$$

is divergent.

2.4.4 POWER SERIES

A series of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ .\ . = \sum_{r=0}^{\infty} a_r x^r \ \text{ or } \ \sum_{r=1}^{\infty} a_{r-1} x^{r-1},$$

where x is usually a variable quantity, is called a **"power Series in x with coefficients $a_0,\ a_1,\ a_2,\ a_3,\ .\ .\ .$"**.

Notes:

(i) In this kind of series, it is particularly useful to sum the series from $r = 0$ to infinity rather than from $r = 1$ to infinity so that the constant term at the beginning (if there is one) can be considered as the term in x^0.

But the various tests for convergence and divergence still apply in this alternative notation.

(ii) A power series will not necessarily be convergent (or divergent) for **all** values of x and it is usually required to determine the specific **range** of values of x for which the series converges. This can most frequently be done using D'Alembert's Ratio Test.

ILLUSTRATION

For the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ .\ . = \sum_{r=1}^{\infty} (-1)^{r-1}\frac{x^r}{r},$$

we have

$$\left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{(-1)^r x^{r+1}}{r+1} \cdot \frac{r}{(-1)^{r-1} x^r}\right| = \left|\frac{r}{r+1} x\right|,$$

which tends to $|x|$ as r tends to infinity.

Thus, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

If $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ . \ . \ ,$$

which converges by the Alternating Series Test; while, if $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ . \ . \ ,$$

which diverges.

The **precise** range of convergence for the given series is therefore $-1 < x \leq 1$.

2.4.5 EXERCISES

1. Show that the following alternating series are convergent:

   (a)

   $$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ . \ . \ .;$$

   (b)

   $$\frac{1}{3^2} - \frac{2}{3^3} + \frac{3}{3^4} - \frac{4}{3^5} + \ . \ .$$

2. Show that the following series are conditionally convergent:

   (a)

   $$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ . \ . \ ;$$

(b)

$$\frac{2}{1\times 3}-\frac{3}{2\times 4}+\frac{4}{3\times 5}-\frac{5}{4\times 6}+\ \ldots$$

3. Show that the following series are absolutely convergent:

   (a)

$$\frac{3}{2}+\frac{4}{3}\times\frac{1}{2}-\frac{5}{4}\times\frac{1}{2^2}-\frac{6}{5}\times\frac{1}{2^3}+\ \ldots;$$

   (b)

$$\frac{1}{3}+\frac{1\times 2}{3\times 5}-\frac{1\times 2\times 3}{3\times 5\times 7}-\frac{1\times 2\times 3\times 4}{3\times 5\times 7\times 9}+\ ..$$

4. Obtain the precise range of values of x for which each of the following power series is convergent:

   (a)

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ \ldots;$$

   (b)

$$\frac{x}{1\times 2}+\frac{x^2}{2\times 3}+\frac{x^3}{3\times 4}+\frac{x^4}{4\times 5}+\ \ldots;$$

   (c)

$$2x+\frac{3x^2}{2^3}+\frac{4x^3}{3^3}+\frac{5x^4}{4^3}+\ \ldots;$$

   (d)

$$1+\frac{2x}{5}+\frac{3x^2}{25}+\frac{4x^3}{125}+\ \ldots$$

2.4.6 ANSWERS TO EXERCISES

1. (a) Use $u_r = \frac{1}{2^{r-1}}$ (numerically);
   (b) Use $u_r = \frac{r}{3^{r+1}}$ (numerically).
2. (a) Use $u_r = \frac{1}{\sqrt{r}}$ (numerically);
   (b) Use $u_r = \frac{r+1}{r(r+2)}$ (numerically).
3. (a) Use $u_r = \frac{r+2}{r+1} \times \frac{1}{2^{r-1}}$ - (numerically);
   (b) Use $u_r = \frac{2^r(r!)^2}{(2r+1)!}$ (numerically).
4. (a) The power series converges for all values of x;
   (b) $-1 \leq x \leq 1$;
   (c) $-1 \leq x \leq 1$;
   (d) $-5 < x < 5$.

Note:
For further discussion of limiting values, see Unit 10.1

"JUST THE MATHS"

UNIT NUMBER

3.1

TRIGONOMETRY 1
(Angles & trigonometric functions)

by

A.J.Hobson

UNIT 3.1 - TRIGONOMETRY 1
ANGLES AND TRIGONOMETRIC FUNCTIONS

3.1.1 INTRODUCTION

The following results will be assumed without proof:

(i) The Circumference, C, and Diameter, D, of a circle are directly proportional to each other through the formula

$$C = \pi D$$

or, if the radius is r,

$$C = 2\pi r.$$

(ii) The area, A, of a circle is related to the radius, r, by means of the formula

$$A = \pi r^2.$$

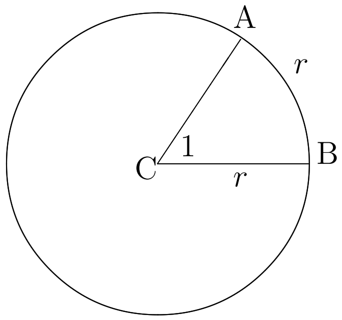
3.1.2 ANGULAR MEASURE

(a) Astronomical Units

The **"degree"** is a $\frac{1}{360}$th part of one complete revolution. It is based on the study of planetary motion where 360 is approximately the number of days in a year.

(b) Radian Measure

A **"radian"** is the angle subtended at the centre of a circle by an arc which is equal in length to the radius.



RESULTS

(i) Using the definition of a radian, together with the second formula for circumference on the previous page, we conclude that there are 2π radians in one complete revolution. That is, 2π radians is equivalent to $360°$ or, in other words π radians is equivalent to $180°$.

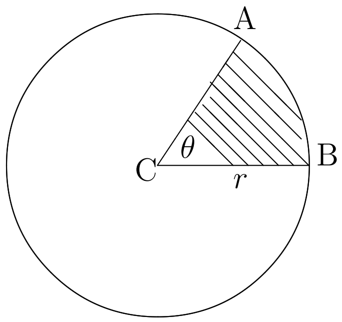
(ii) In the diagram overleaf, the arclength from A to B will be given by

$$\frac{\theta}{2\pi} \times 2\pi r = r\theta,$$

assuming that θ is measured in radians.

(iii) In the diagram below, the area of the sector ABC is given by

$$\frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$



(c) Standard Angles

The scaling factor for converting degrees to radians is

$$\frac{\pi}{180}$$

and the scaling factor for converting from radians to degrees is

$$\frac{180}{\pi}.$$

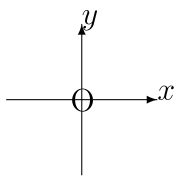
These scaling factors enable us to deal with any angle, but it is useful to list the expression, in radians, of some of the more well-known angles.

ILLUSTRATIONS

1. 15° is equivalent to $\frac{\pi}{180} \times 15 = \frac{\pi}{12}$.
2. 30° is equivalent to $\frac{\pi}{180} \times 30 = \frac{\pi}{6}$.
3. 45° is equivalent to $\frac{\pi}{180} \times 45 = \frac{\pi}{4}$.
4. 60° is equivalent to $\frac{\pi}{180} \times 60 = \frac{\pi}{3}$.
5. 75° is equivalent to $\frac{\pi}{180} \times 75 = \frac{5\pi}{12}$.
6. 90° is equivalent to $\frac{\pi}{180} \times 90 = \frac{\pi}{2}$.

(d) Positive and Negative Angles

For the measurement of angles in general, we consider the plane of the page to be divided into four quadrants by means of a cartesian reference system with axes Ox and Oy. The **"first quadrant"** is that for which x and y are both positive, and the other three quadrants are numbered from the first in an anticlockwise sense.



From the positive x-direction, we measure angles positively in the anticlockwise sense and negatively in the clockwise sense. Special names are given to the type of angles obtained as follows:

1. Angles in the range between $0°$ and $90°$ are called **"positive acute"** angles.

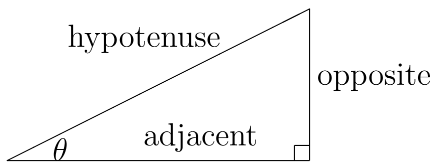
2. Angles in the range between $90°$ and $180°$ are called **"positive obtuse"** angles.

3. Angles in the range between $180°$ and $360°$ are called **"positive reflex"** angles.

4. Angles measured in the clockwise sense have similar names but preceded by the word **"negative"**.

3.1.3 TRIGONOMETRIC FUNCTIONS

We first consider a right-angled triangle in one corner of which is an angle θ other than the right-angle itself. The sides of the triangle are labelled in relation to this angle , θ, as **"opposite"**, **"adjacent"** and **"hypotenuse"** (see diagram below).



For future reference, we shall assume, without proof, the result known as **"Pythagoras' Theorem"**. This states that the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

DEFINITIONS

(a) The **"sine"** of the angle θ, denoted by $\sin\theta$, is defined by

$$\sin\theta \equiv \frac{\text{opposite}}{\text{hypotenuse}};$$

(b) The **"cosine"** of the angle θ, denoted by $\cos\theta$, is defined by

$$\cos\theta \equiv \frac{\text{adjacent}}{\text{hypotenuse}};$$

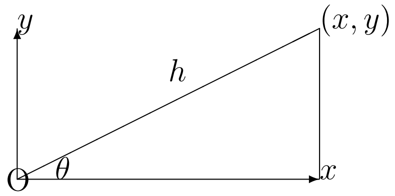
(c) The **"tangent"** of the angle θ, denoted by $\tan\theta$, is defined by

$$\tan\theta \equiv \frac{\text{opposite}}{\text{adjacent}}.$$

Notes:
(i) The traditional aid to remembering the above definitions is the abbreviation

S.O.H.C.A.H.T.O.A.

(ii) The definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be extended to angles of any size by regarding the end-points of the hypotenuse, with length h, to be, respectively, the origin and the point (x, y) in a cartesian system of reference.



For any values of x and y, positive, negative or zero, the three basic trigonometric functions are defined in general by the formulae

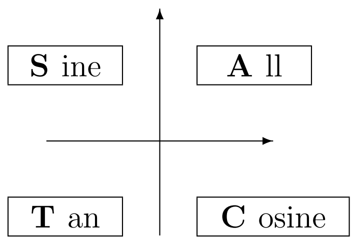
$$\sin\theta \equiv \frac{y}{h};$$

$$\cos\theta \equiv \frac{x}{h};$$

$$\tan\theta \equiv \frac{y}{x} \equiv \frac{\sin\theta}{\cos\theta}.$$

Clearly these reduce to the original definitions in the case when θ is a positive acute angle. Trigonometric functions can also be called **"trigonometric ratios"**.

(iii) It is useful to indicate diagramatically which of the three basic trigonometric functions have positive values in the various quadrants.



(iv) Three other trigonometric functions are sometimes used and are defined as the reciprocals of the three basic functions as follows:

"Secant"

$$\sec\theta \equiv \frac{1}{\cos\theta};$$

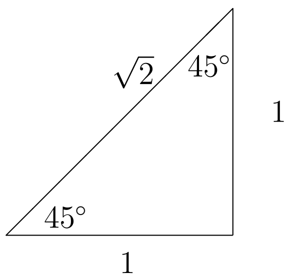
"Cosecant"

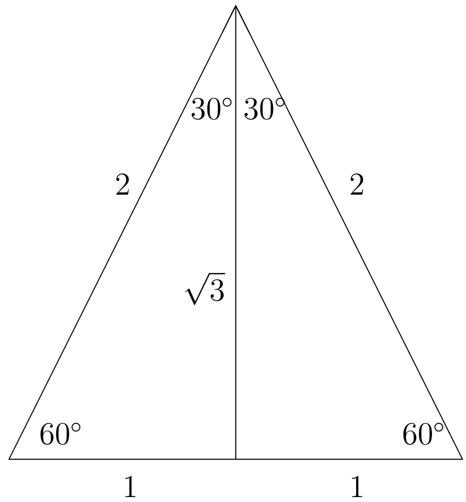
$$\mathrm{cosec}\theta \equiv \frac{1}{\sin\theta};$$

"Cotangent"

$$\cot\theta \equiv \frac{1}{\tan\theta}.$$

(v) The values of the functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the particluar angles 30°, 45° and 60° are easily obtained without calculator from the following diagrams:





The diagrams show that

(a) $\sin 45° = \frac{1}{\sqrt{2}}$; (b) $\cos 45° = \frac{1}{\sqrt{2}}$; (c) $\tan 45° = 1$;

(d) $\sin 30° = \frac{1}{2}$; (e) $\cos 30° = \frac{\sqrt{3}}{2}$ (f) $\tan 30° = \frac{1}{\sqrt{3}}$;

(g) $\sin 60° = \frac{\sqrt{3}}{2}$; (h) $\cos 60° = \frac{1}{2}$; (i) $\tan 60° = \sqrt{3}$.

3.1.4 EXERCISES

1. Express each of the following angles as a multiple of π
   (a) 65°; (b) 105°; (c) 72°; (d) 252°;
   (e) 20°; (f) −160°; (g) 9°; (h) 279°.
2. On a circle of radius 24cms., find the length of arc which subtends an angle at the centre of
   (a) $\frac{2}{3}$ radians.; (b) $\frac{3\pi}{5}$ radians.;
   (c) 75°; (d) 130°.
3. A wheel is turning at the rate of 48 revolutions per minute. Express this angular speed in
   (a) revolutions per second; (b) radians per minute; (c) radians per second.

4. A wheel, 4 metres in diameter, is rotating at 80 revolutions per minute. Determine the distance, in metres, travelled in one second by a point on the rim.

5. A chord AB of a circle, radius 5cms., subtends a right-angle at the centre of the circle. Calculate, correct to two places of decimals, the areas of the two segments into which AB divides the circle.

6. If $\tan\theta$ is positive and $\cos\theta = -\frac{4}{5}$, what is the value of $\sin\theta$?

7. Determine the length of the chord of a circle, radius 20cms., subtending an angle of 150° at the centre.

8. A ladder leans against the side of a vertical building with its foot 4 metres from the building. If the ladder is inclined at 70° to the ground, how far from the ground is the top of the ladder and how long is the ladder ?

3.1.5 ANSWERS TO EXERCISES

1. (a) $\frac{13\pi}{36}$; (b) $\frac{7\pi}{12}$; (c) $\frac{2\pi}{5}$; (d) $\frac{7\pi}{5}$; (e) $\frac{\pi}{9}$; (f) -$\frac{8\pi}{9}$; (g) $\frac{\pi}{20}$; (h) $\frac{31\pi}{20}$.
2. (a) 16 cms.; (b) $\frac{72\pi}{5}$ cms.; (c) 10π cms.; (d) $\frac{52\pi}{3}$ cms.
3. (a) $\frac{4}{5}$ revs. per sec.; (b) 96π rads. per min.; (c) $\frac{8\pi}{5}$ rads. per sec.
4. $\frac{16\pi}{3}$ metres.
5. 7.13 square cms. and 71.41 square cms.
6. $\sin\theta = -\frac{3}{5}$.
7. The chord has a length of 38.6cms. approximately.
8. The top of ladder is 11 metres from the ground and the length of the ladder is 11.7 metres.

"JUST THE MATHS"

UNIT NUMBER

3.2

TRIGONOMETRY 2
(Graphs of trigonometric functions)

by

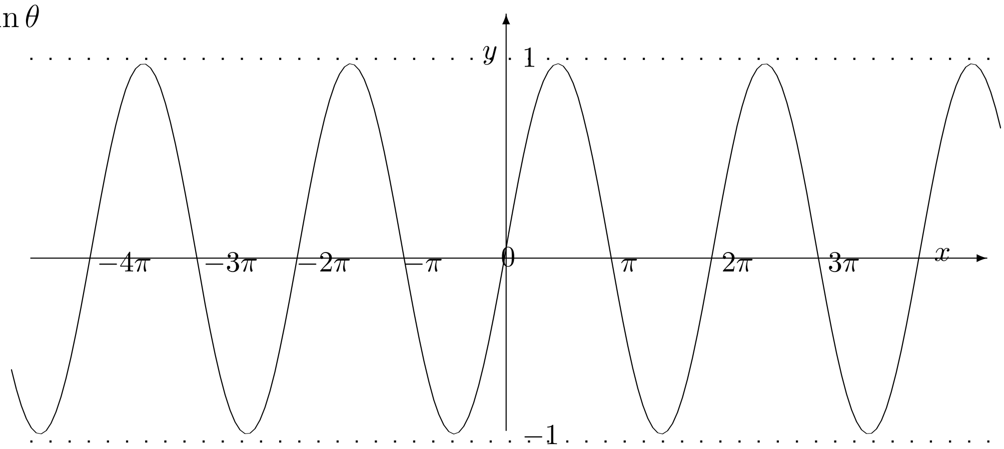
A.J.Hobson

UNIT 3.2 - TRIGONOMETRY 2.
GRAPHS OF TRIGONOMETRIC FUNCTIONS

3.2.1 GRAPHS OF ELEMENTARY TRIGONOMETRIC FUNCTIONS

The following diagrams illustrate the graphs of the basic trigonometric functions sinθ, cosθ and tanθ,

1. $y = \sin\theta$



The graph illustrates that

$$\sin(\theta + 2\pi) \equiv \sin\theta$$

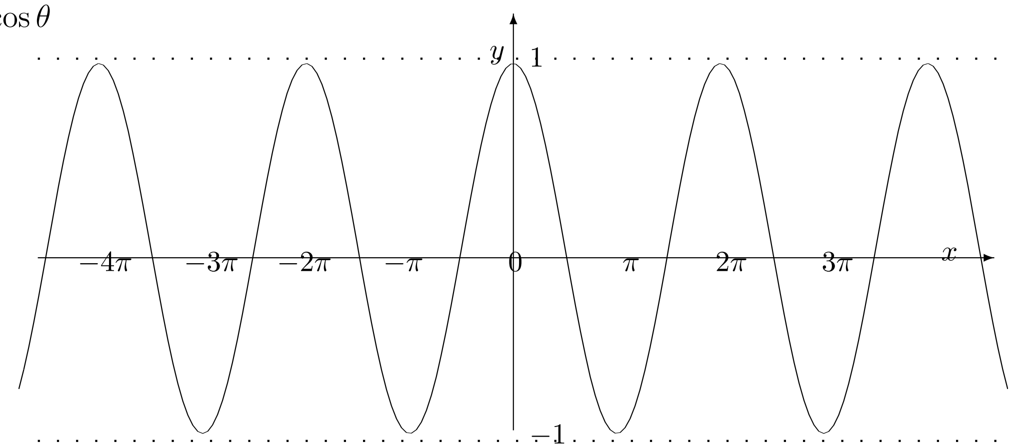
and we say that sinθ is a **"periodic function with period** 2π**"**.
Other numbers which can act as a period are $\pm 2n\pi$ where n is any integer; but 2π itself is the smallest positive period and, as such, is called the **"primitive period"** or sometimes the **"wavelength"**.

We may also observe that

$$\sin(-\theta) \equiv -\sin\theta$$

which makes sinθ what is called an **"odd function"**.

2. $y = \cos\theta$



The graph illustrates that

$$\cos(\theta + 2\pi) \equiv \cos\theta$$

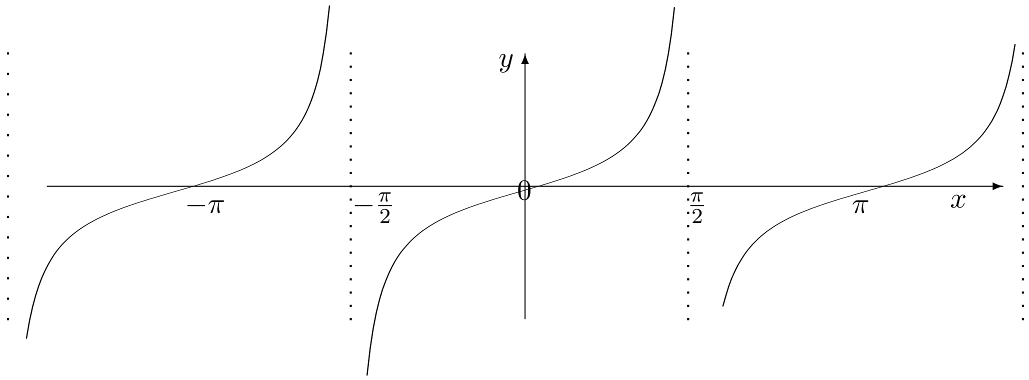
and so cosθ, like sinθ, is a periodic function with primitive period 2π

We may also observe that

$$\cos(-\theta) \equiv \cos\theta$$

which makes cosθ what is called an **"even function"**.

3. $y = \tan\theta$



This time, the graph illustrates that

$$\tan(\theta + \pi) \equiv \tan\theta$$

which implies that tanθ is a periodic function with primitive period π.

We may also observe that

$$\tan(-\theta) \equiv -\tan\theta$$

which makes tanθ an **"odd function"**.

3.2.2 GRAPHS OF MORE GENERAL TRIGONOMETRIC FUNCTIONS

In scientific work, it is possible to encounter functions of the form

$$\boxed{\text{A}\sin(\omega\theta + \alpha)} \text{ and } \boxed{\text{A}\cos(\omega\theta + \alpha)}$$

where ω and α are constants.

We may sketch their graphs by using the information in the previous examples 1. and 2.

EXAMPLES

1. Sketch the graph of

$$y = 5\cos(\theta - \pi).$$

Solution

The important observations to make first are that

(a) the graph will have the same shape as the basic cosine wave but will lie between $y = -5$ and $y = 5$ instead of between $y = -1$ and $y = 1$; we say that the graph has an **"amplitude"** of 5.

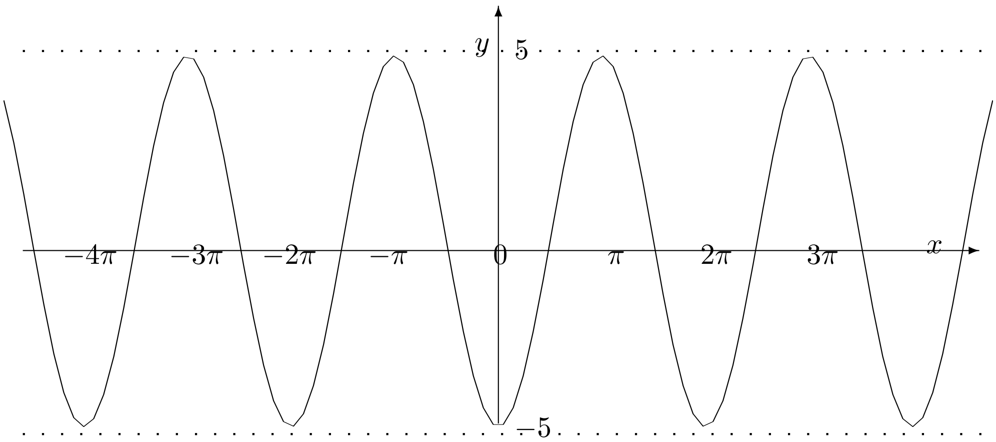
(b) the graph will cross the θ-axis at the points for which

$$\theta - \pi = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$$

that is

$$\theta = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$$

(c) The y-axis must be placed between the smallest **negative** intersection with the θ-axis and the smallest **positive** intersection with the θ - axis (in proportion to their values). In this case, the y-axis must be placed half way between $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$.



Of course, in this example, from earlier trigonometry results, we could have noticed that

$$5\cos(\theta - \pi) \equiv -5\cos\theta$$

so that graph consists of an "upsidedown" cosine wave with an amplitude of 5. However, not all examples can be solved in this way.

2. Sketch the graph of

$$y = 3\sin(2\theta + 1).$$

Solution

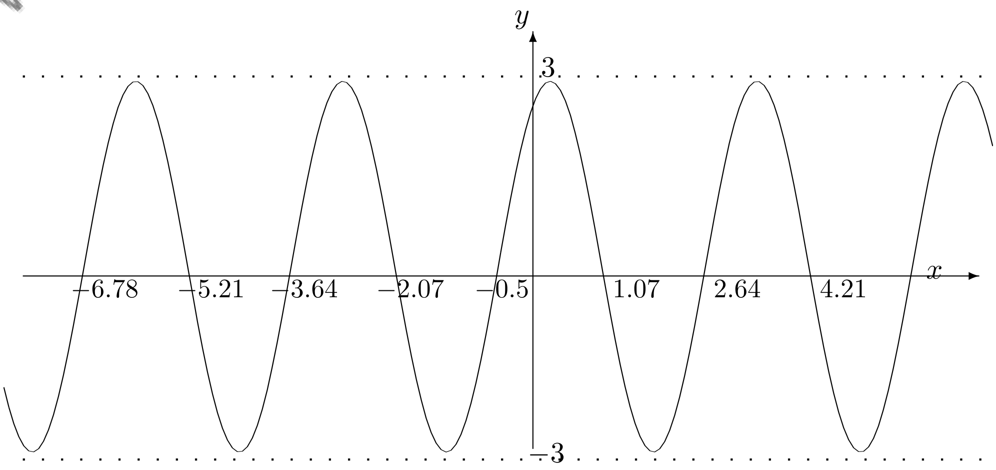
This time, the graph will have the same shape as the basic sine wave, but will have an amplitude of 3. It will cross the θ-axis at the points for which

$$2\theta + 1 = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, ...$$

and by solving for θ in each case, we obtain

$$\theta = ... - 6.78, -5.21, -3.64, -2.07, -0.5, 1.07, 2.64, 4.21, 5.78...$$

Hence, the y-axis must be placed between $\theta = -0.5$ and $\theta = 1.07$ but at about one third of the way from $\theta = -0.5$



3.2.3 EXERCISES

1. Make a table of values of θ and y, with θ in the range from 0 to 2π in steps of $\frac{\pi}{12}$, and hence, sketch the graphs of

   (a)
$$y = \sec\theta;$$

   (b)
$$y = \operatorname{cosec}\theta;$$

   (c)
$$y = \cot\theta.$$

2. Sketch the graphs of the following functions:

   (a)
$$y = 2\sin\left(\theta + \frac{\pi}{4}\right);$$

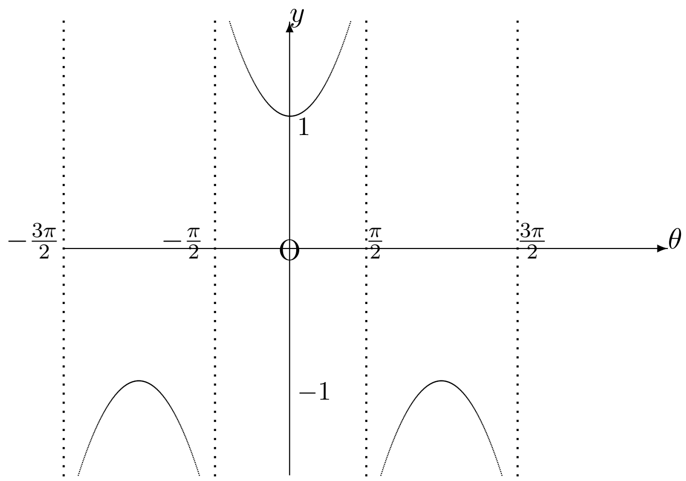
   (b)
$$y = 2\cos(3\theta - 1).$$

   (c)
$$y = 5\sin(7\theta + 2).$$

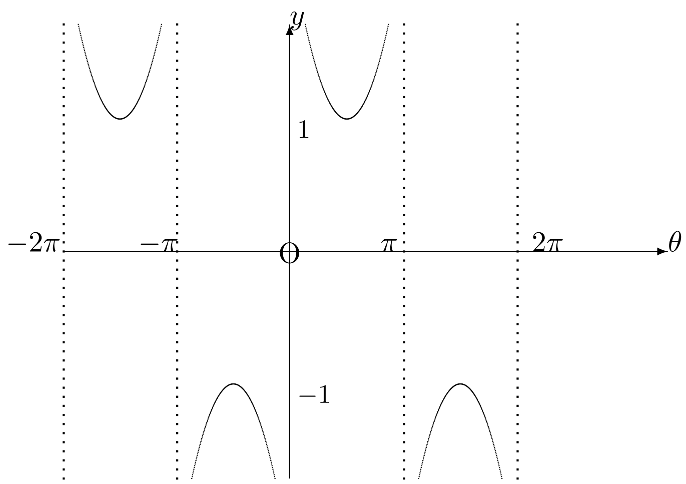
   (d)
$$y = -\cos\left(\theta - \frac{\pi}{3}\right).$$

3.2.4 ANSWERS TO EXERCISES

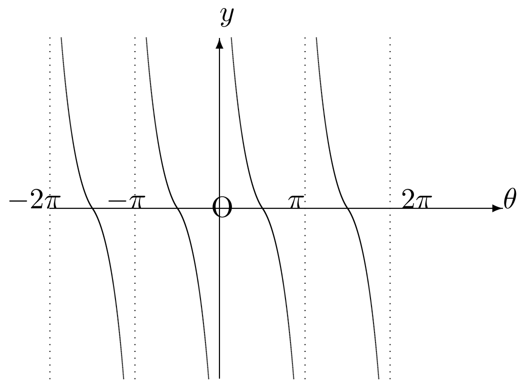
1. (a) The graph is



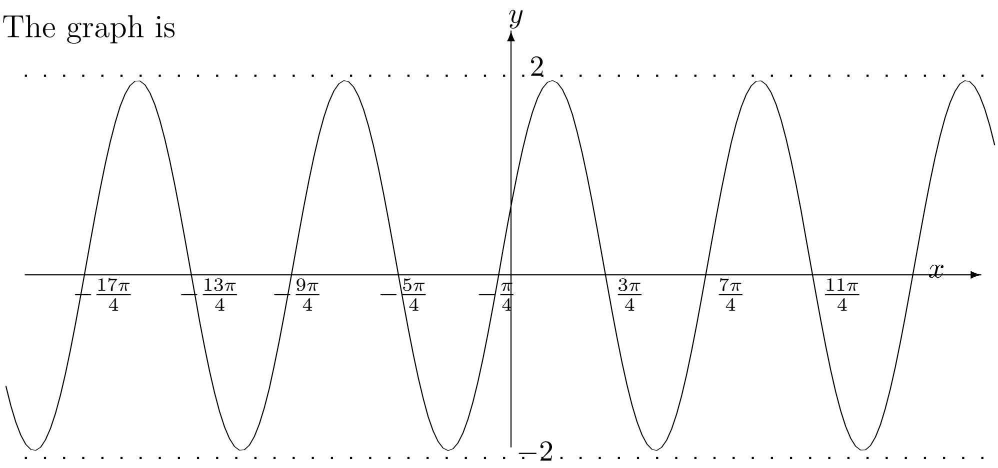
(b) The graph is



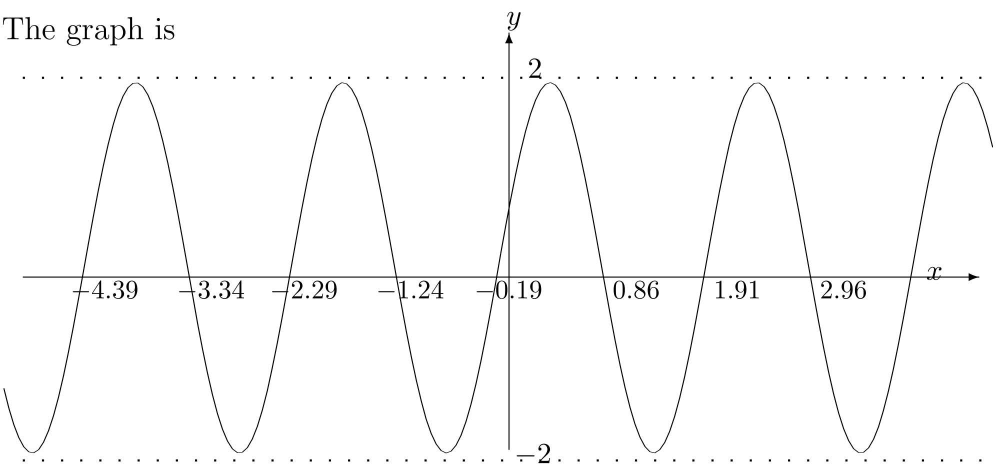
(c) The graph is



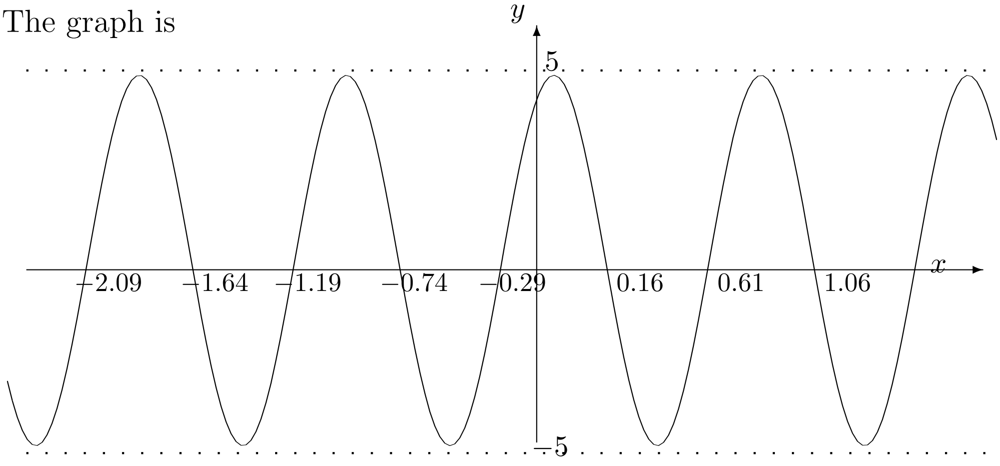
2. (a) The graph is



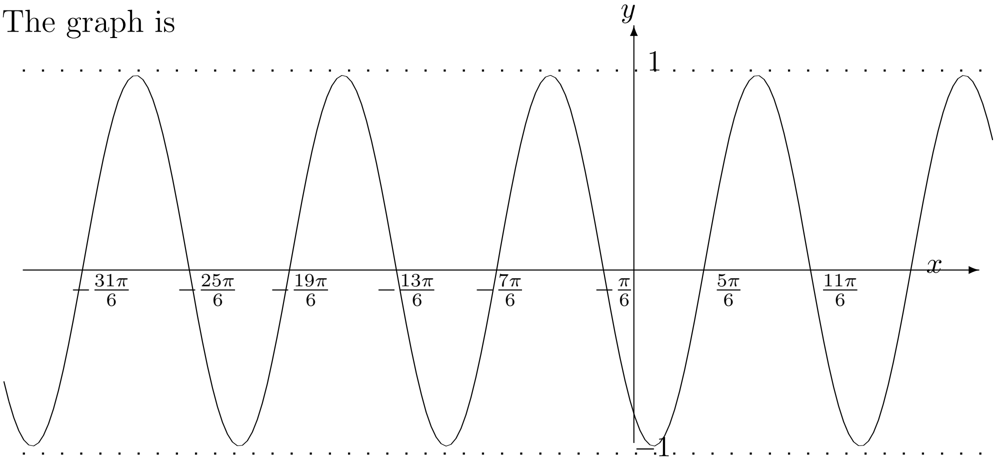
(b) The graph is



(c) The graph is



(d) The graph is



"JUST THE MATHS"

UNIT NUMBER

3.3

TRIGONOMETRY 3
(Approximations & inverse functions)

by

A.J.Hobson

UNIT 3.3 - TRIGONOMETRY APPROXIMATIONS AND INVERSE FUNCTIONS

3.3.1 APPROXIMATIONS FOR TRIGONOMETRIC FUNCTIONS

Three standard approximations for the functions $\sin\theta, \cos\theta$ and $\tan\theta$ respectively can be obtained from a set of results taken from the applications of Calculus. These are stated without proof as follows:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}.....$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}......$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + ...$$

These results apply **only if θ is in radians** but, if θ is small enough for θ^2 and higher powers of θ to be neglected, we conclude that

$$\sin\theta \simeq \theta,$$

$$\cos\theta \simeq 1,$$

$$\tan\theta \simeq \theta.$$

Better approximations are obtainable if more terms of the infinite series are used.

EXAMPLE

Approximate the function

$$5 + 2\cos\theta - 7\sin\theta$$

to a quartic polynomial in θ.

Solution

Using terms of the appropriate series up to and including the fourth power of θ, we deduce that

$$5 + 2\cos\theta - 7\sin\theta \simeq 5 + 2 - \theta^2 + \frac{\theta^4}{12} - 7\theta + 7\frac{\theta^3}{6}$$

$$= \frac{1}{12}\left[\theta^4 + 14\theta^3 - 12\theta^2 - 84\theta + 84\right].$$

3.3.2 INVERSE TRIGONOMETRIC FUNCTIONS

It is frequently necessary to determine possible angles for which the value of their sine, cosine or tangent is already specified. This is carried out using inverse trigonometric functions defined as follows:

(a) The symbol

$$\text{Sin}^{-1}x$$

denotes any angle whose sine value is the number x. It is necessary that $-1 \leq x \leq 1$ since the sine of an angle is always in this range.

(b) The symbol

$$\text{Cos}^{-1}x$$

denotes any angle whose cosine value is the number x. Again, $-1 \leq x \leq 1$.

(c) The symbol

$$\text{Tan}^{-1}x$$

denotes any angle whose tangent value is x. This time, x may be any value because the tangent function covers the range from $-\infty$ to ∞.

We note that because of the $\boxed{\textbf{A}\text{ ll}}$, $\boxed{\textbf{S}\text{ ine}}$, $\boxed{\textbf{T}\text{ angent}}$, $\boxed{\textbf{C}\text{ osine}}$ diagram, (see Unit 3.1), there will be two **basic** values of an inverse function from two different quadrants. But either of these two values may be increased or decreased by a whole multiple of 360° (2π) to yield other acceptable answers and hence an infinite number of possible answers.

EXAMPLES

1. Evaluate $\text{Sin}^{-1}(\frac{1}{2})$.

   Solution

   $\text{Sin}^{-1}(\frac{1}{2}) = 30^\circ \pm n360^\circ$ or $150^\circ \pm n360^\circ$.

2. Evaluate $\text{Tan}^{-1}(\sqrt{3})$.

   Solution

   $\text{Tan}^{-1}(\sqrt{3}) = 60^\circ \pm n360^\circ$ or $240^\circ \pm n360^\circ$.

   This result is in fact better written in the combined form
   $\text{Tan}^{-1}(\sqrt{3}) = 60^\circ \pm n180^\circ$
   THat is, angles in opposite quadrants have the same tangent.

   Another Type of Question

3. Obtain all of the solutions to the equation

   $$\cos 3x = -0.432$$

   which lie in the interval $-180^\circ \leq x \leq 180^\circ$.

   Solution

   This type of question is of a slightly different nature since we are asked for a specified **selection** of values rather than the general solution of the equation.

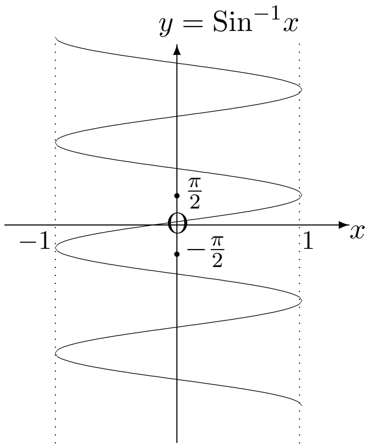
We require that $3x$ be any one of the angles (within an interval $-540^\circ \leq 3x \leq 540^\circ$) whose cosine is equal to -0.432. Using a calculator, the simplest angle which satisfies this condition is 115.59°; but the complete set is

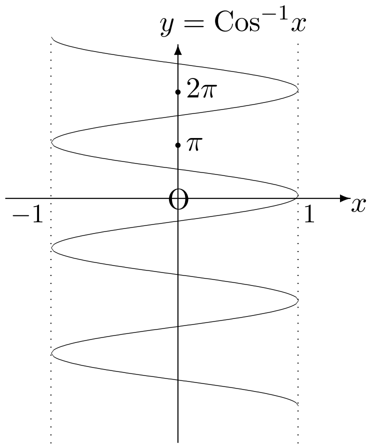
$$\pm 115.59^\circ \quad \pm 244.41^\circ \quad \pm 475.59^\circ$$

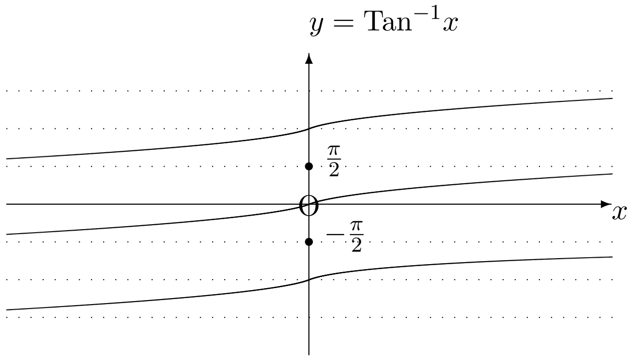
Thus, on dividing by 3, the possibilities for x are

$$\pm 38.5^\circ \quad \pm 81.5^\circ \quad \pm 158.5^\circ$$

Note: The graphs of inverse trigonometric functions are discussed fully in Unit 10.6, but we include them here for the sake of completeness







Of all the possible values obtained for an inverse trigonometric function, one particular one is called the **"Principal Value"**. It is the unique value which lies in a specified range described below, the explanation of which is best dealt with in connection with differential calculus.

To indicate such a principal value, we use the lower-case initial letter of each inverse function.

(a) $\theta = \sin^{-1}x$ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(b) $\theta = \cos^{-1}x$ lies in the range $0 \leq \theta \leq \pi$.

(c) $\theta = \tan^{-1}x$ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

EXAMPLES

1. Evaluate $\sin^{-1}(\frac{1}{2})$.

   Solution

   $\sin^{-1}(\frac{1}{2}) = 30^\circ$ or $\frac{\pi}{6}$.

2. Evaluate $\tan^{-1}(-\sqrt{3})$.

   Solution

   $\tan^{-1}(-\sqrt{3}) = -60^\circ$ or $-\frac{\pi}{3}$.

3. Write down a formula for u in terms of v in the case when

$$v = 5\cos(1 - 7u).$$

   Solution

   Dividing by 5 gives

$$\frac{v}{5} = \cos(1 - 7u).$$

   Taking the inverse cosine gives

$$\mathrm{Cos}^{-1}\left(\frac{v}{5}\right) = 1 - 7u.$$

   Subtracting 1 from both sides gives

$$\mathrm{Cos}^{-1}\left(\frac{v}{5}\right) - 1 = -7u.$$

   Dividing both sides by -7 gives

$$u = -\frac{1}{7}\left[\mathrm{Cos}^{-1}\left(\frac{v}{5}\right) - 1\right].$$

3.3.3 EXERCISES

1. If powers of θ higher then three can be neglected, find an approximation for the function

$$6\sin\theta + 2\cos\theta + 10\tan\theta$$

   in the form of a polynomial in θ.

2. If powers of θ higher than five can be neglected, find an approximation for the function

$$2\sin\theta - \theta\cos\theta$$

   in the form of a polynomial in θ.

3. If powers of θ higher than two can be neglected, show that the function
$$\frac{\theta \sin \theta}{1 - \cos \theta}$$
is approximately equal to 2.

4. Write down the principal values of the following:
   (a) $\mathrm{Sin}^{-1} 1$;
   (b) $\mathrm{Sin}^{-1} \left(-\frac{1}{2}\right)$;
   (c) $\mathrm{Cos}^{-1} \left(-\frac{\sqrt{3}}{2}\right)$;
   (d) $\mathrm{Tan}^{-1} 5$;
   (e) $\mathrm{Tan}^{-1}(-\sqrt{3})$;
   (f) $\mathrm{Cos}^{-1} \left(-\frac{1}{\sqrt{2}}\right)$.

5. Solve the following equations for x in the interval $0 \le x \le 360°$:
   (a)
   $$\tan x = 2.46$$
   (b)
   $$\cos x = 0.241$$
   (c)
   $$\sin x = -0.786$$
   (d)
   $$\tan x = -1.42$$
   (e)
   $$\cos x = -0.3478$$
   (f)
   $$\sin x = 0.987$$

   Give your answers correct to one decimal place.

6. Solve the following equations for the range given, stating your final answers in degrees correct to one decimal place:
   (a) $\sin 2x = -0.346$ for $0 \le x \le 360°$;
   (b) $\tan 3x = 1.86$ for $0 \le x \le 180°$;
   (c) $\cos 2x = -0.57$ for $-180° \le x \le 180°$;
   (d) $\cos 5x = 0.21$ for $0 \le x \le 45°$;
   (e) $\sin 4x = 0.78$ for $0 \le x \le 180°$.

7. Write down a formula for u in terms of v for the following:
   (a) $v = \sin u$;
   (b) $v = \cos 2u$;
   (c) $v = \tan(u + 1)$.

8. If x is positive, show diagramatically that

   (a)

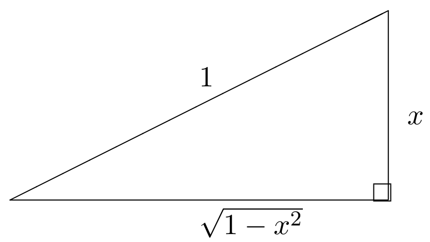
$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2};$$

   (b)

$$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}.$$

3.3.4 ANSWERS TO EXERCISES

1. $\frac{7\theta^3}{3} - \theta^2 + 16\theta + 2.$
2. $\theta + \frac{\theta^3}{6} - \frac{\theta^5}{40}.$
3. Substitute approximations for $\sin\theta$ and $\cos\theta$.
4. (a) $\frac{\pi}{2}$; (b) $-\frac{\pi}{6}$; (c) $\frac{5\pi}{6}$; (d) 1.373; (e) $-\frac{\pi}{3}$; (f) $\frac{3\pi}{4}$.
5. (a) 67.9° or 247.9°;
   (b) 76.1° or 283.9°;
   (c) 231.8° or 308.2°;
   (d) 125.2° or 305.32°;
   (e) 110.4° or 249.6°;
   (f) 80.8° or 99.2°
6. (a) 100.1°, 169.9°, 280.1°, 349.9°
   (b) 20.6°, 80.6°, 140.6°
   (c) ±62.4°, ±117.6°
   (d) 15.6°
   (e) 32.2°, 102.8°, 122.2°
7. (a) $u = \text{Sin}^{-1}v$; (b) $u = \frac{1}{2}\text{Cos}^{-1}v$; (c) $u = \text{Tan}^{-1}v - 1$.
8. A suitable diagram is



"JUST THE MATHS"

UNIT NUMBER

3.4

TRIGONOMETRY 4
(Solution of triangles)

by

A.J.Hobson

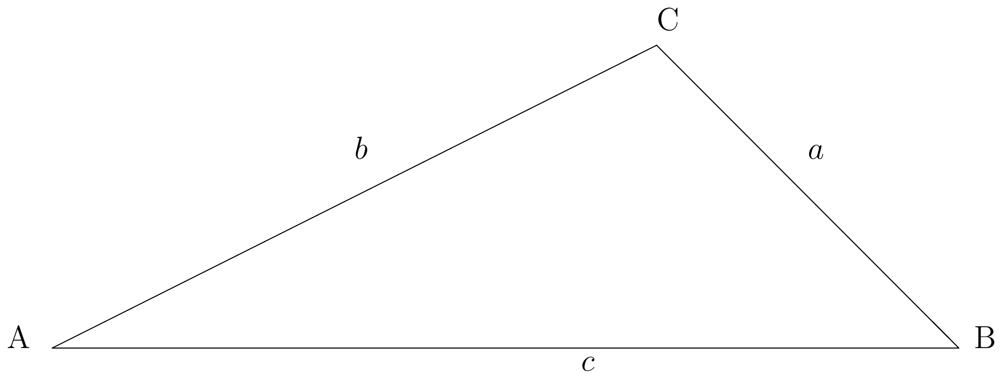
UNIT 3.4 - TRIGONOMETRY 4
SOLUTION OF TRIANGLES

3.4.1 INTRODUCTION

The **"solution of a triangle"** is defined to mean the complete set of data relating to the lengths of its three sides and the values of its three interior angles. It can be shown that these angles always add up to 180°.

If a sufficient amount of information is provided about **some** of this data, then it is usually possible to determine the remaining data.

We shall use a standardised type of diagram for an arbitrary triangle whose **"vertices"** (i.e. corners) are A,B and C and whose sides have lengths a, b and c. It is as follows:



The angles at A,B and C will be denoted by $\widehat{A}, \widehat{B}$ and $\widehat{C}$.

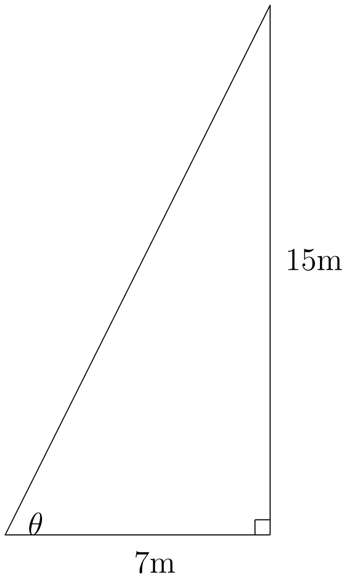
3.4.2 RIGHT-ANGLED TRIANGLES

Right-angled triangles are easier to solve than the more general kinds of triangle because all we need to use are the relationships between the lengths of the sides and the trigonometric ratios sine, cosine and tangent. An example will serve to illustrate the technique:

EXAMPLE

From the top of a vertical pylon, 15 meters high, a guide cable is to be secured into the (horizontal) ground at a distance of 7 meters from the base of the pylon.
What will be the length of the cable and what will be its inclination (in degrees) to the horizontal ?

Solution



From Pythagoras' Theorem, the length of the cable will be

$$\sqrt{7^2 + 15^2} \simeq 16.55\text{m}.$$

The angle of inclination to the horizontal will be θ, where

$$\tan\theta = \frac{15}{7}.$$

$$\text{Hence,} \quad \theta \simeq 65^\circ.$$

3.4.3 THE SINE AND COSINE RULES

Two powerful tools for the solution of triangles in general may be stated in relation to the earlier diagram as follows:

(a) The Sine Rule

$$\frac{a}{\sin\widehat{A}} = \frac{b}{\sin\widehat{B}} = \frac{c}{\sin\widehat{C}}.$$

(b) The Cosine Rule

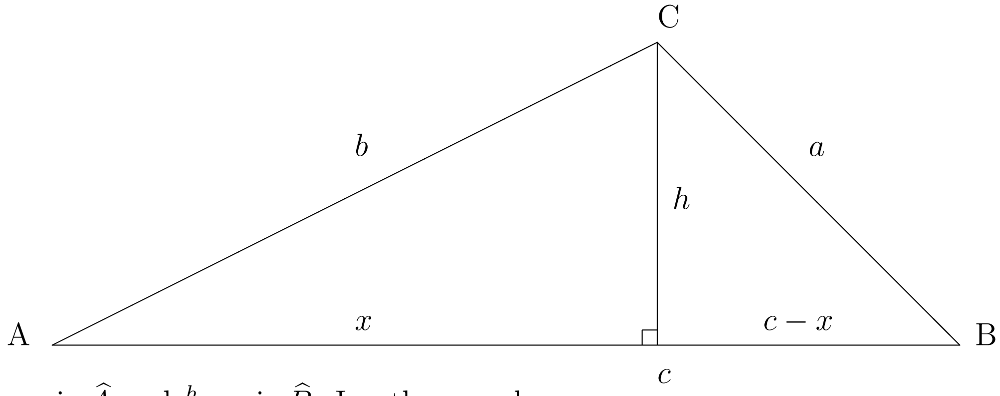
$$a^2 = b^2 + c^2 - 2bc\cos\widehat{A};$$
$$b^2 = c^2 + a^2 - 2ca\cos\widehat{B};$$
$$c^2 = a^2 + b^2 - 2ab\cos\widehat{C}.$$

Clearly, the last two of these are variations of the first.

We also observe that, whenever the angle on the right-hand-side is a right-angle, the Cosine Rule reduces to Pythagoras' Theorem.

The Proof of the Sine Rule

In the diagram encountered earlier, suppose we draw the perpendicular (of length h) from the vertex C onto the side AB.



Then $\frac{h}{b} = \sin \widehat{A}$ and $\frac{h}{a} = \sin \widehat{B}$. In other words,

$$b \sin \widehat{A} = a \sin \widehat{B}$$

or

$$\frac{b}{\sin \widehat{B}} = \frac{a}{\sin \widehat{A}}.$$

Clearly, the remainder of the Sine Rule can be obtained by considering the perpendicular drawn from a different vertex.

The Proof of the Cosine Rule

Using the same diagram as for the Sine Rule, we can assume that the side AB has lengths x and $c - x$ either side of the foot of the perpendicular drawn from C. Hence

$$h^2 = b^2 - x^2$$

and, at the same time,

$$h^2 = a^2 - (c - x)^2.$$

Expanding and equating the two expressions for h^2, we obtain

$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2$$

that is

$$a^2 = b^2 + c^2 - 2xc.$$

But $x = b \cos \widehat{A}$, and so

$$a^2 = b^2 + c^2 - 2bc \cos \widehat{A}.$$

EXAMPLES

1. Solve the triangle ABC in the case when $\widehat{A} = 20°, \widehat{B} = 30°$ and $c = 10$cm.

   Solution

   Firstly, the angle $\widehat{C} = 130°$ since the interior angles must add up to 180°.

   Thus, by the Sine Rule, we have

$$\frac{a}{\sin 20°} = \frac{b}{\sin 30°} = \frac{10}{\sin 130°}.$$

   That is,

$$\frac{a}{0.342} = \frac{b}{0.5} = \frac{10}{0.766}$$

   These give the results

$$a = \frac{10 \times 0.342}{0.766} \cong 4.47\text{cm}$$
$$b = \frac{10 \times 0.5}{0.766} \cong 6.53\text{cm}$$

2. Solve the triangle ABC in the case when $b = 9\text{cm}, c = 5\text{cm}$ and $\widehat{A} = 70°$.

   Solution

   In this case, the information prevents us from using the Sine Rule immediately, but the Cosine Rule **can** be applied as follows:

$$a^2 = 25 + 81 - 90\cos 70°$$

   giving

$$a^2 = 106 - 30.782 = 75.218$$

   Hence

$$a \simeq 8.673\text{cm} \simeq 8.67\text{cm}$$

   Now we can use the Sine Rule to complete the solution

$$\frac{8.673}{\sin 70°} = \frac{9}{\sin \widehat{B}} = \frac{5}{\sin \widehat{C}}.$$

   Thus,

$$\sin \widehat{B} = \frac{9 \times \sin 70°}{8.673} = \frac{9 \times 0.940}{8.673} \simeq 0.975$$

   This suggests that $\widehat{B} \simeq 77.19°$ in which case $\widehat{C} \simeq 180° - 70° - 77.19° \simeq 32.81°$ but, for the moment, we must also allow the possibility that $\widehat{B} \simeq 102.81°$ which would give $\widehat{C} \simeq 7.19°$

   However, we can show that the alternative solution is unacceptable because it is not consistent with the whole of the Sine Rule statement for this example. Thus the only solution is the one for which

$$a \simeq 8.67\text{cm}, \quad \widehat{B} \simeq 77.19°, \quad \widehat{C} \simeq 32.81°$$

Note: It is possible to encounter examples for which more than one solution **does** exist.

3.4.4 EXERCISES

Solve the triangle ABC in the following cases:

1. $c = 25\text{cm}$, $\widehat{A} = 35°$, $\widehat{B} = 68°$.
2. $c = 23\text{cm}$, $a = 30\text{cm}$, $\widehat{C} = 40°$.
3. $b = 4\text{cm}$, $c = 5\text{cm}$, $\widehat{A} = 60°$.
4. $a = 21\text{cm}$, $b = 23\text{cm}$, $c = 16\text{cm}$.

3.4.5 ANSWERS TO EXERCISES

1. $a \simeq 14.72\text{cm}$, $b \simeq 23.79\text{cm}$, $\widehat{C} \simeq 77°$.
2. $\widehat{A} \simeq 56.97°$, $\widehat{B} \simeq 83.03°$, $b = 35.52\text{cm}$;
   OR
   $\widehat{A} \simeq 123.03°$, $\widehat{B} \simeq 16.97°$, $b \simeq 10.44\text{cm}$.
3. $a \simeq 4.58\text{cm}$, $\widehat{B} \simeq 49.11°$, $\widehat{C} \simeq 70.89°$.
4. $\widehat{A} \simeq 62.13°$, $\widehat{B} \simeq 75.52°$, $\widehat{C} \simeq 42.35°$.

"JUST THE MATHS"

UNIT NUMBER

3.5

TRIGONOMETRY 5
(Trigonometric identities & wave-forms)

by

A.J.Hobson

UNIT 3.5 - TRIGONOMETRY 5
TRIGONOMETRIC IDENTITIES AND WAVE FORMS

3.5.1 TRIGONOMETRIC IDENTITIES

The standard trigonometric functions can be shown to satisfy a certain group of relationships for any value of the angle θ. They are called **"trigonometric identities"**.

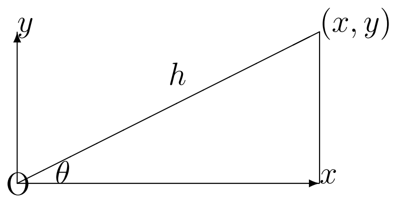
ILLUSTRATION

Prove that

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

Proof:

The following diagram was first encountered in Unit 3.1



From the diagram,

$$\cos\theta = \frac{x}{h} \quad \text{and} \quad \sin\theta = \frac{y}{h}.$$

But, by Pythagoras' Theorem,

$$x^2 + y^2 = h^2.$$

In other words,

$$\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h}\right)^2 = 1.$$

That is,

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

It is also worth noting various consequences of this identity:

(a) $\cos^2\theta \equiv 1 - \sin^2\theta$; (rearrangement).

(b) $\sin^2\theta \equiv 1 - \cos^2\theta$; (rearrangement).

(c) $\sec^2\theta \equiv 1 + \tan^2\theta$; (divide by $\cos^2\theta$).

(d) $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$; (divide by $\sin^2\theta$).

Other Trigonometric Identities in common use will not be **proved** here, but they are listed for reference. However, a booklet of Mathematical Formulae should be obtained.

$$\sec\theta \equiv \frac{1}{\cos\theta} \quad \operatorname{cosec}\theta \equiv \frac{1}{\sin\theta} \quad \cot\theta \equiv \frac{1}{\tan\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \quad 1 + \tan^2\theta \equiv \sec^2\theta \quad 1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin A \equiv 2\sin\frac{1}{2}A\cos\frac{1}{2}A$$

$$\cos A \equiv \cos^2\frac{1}{2}A - \sin^2\frac{1}{2}A \equiv 1 - 2\sin^2\frac{1}{2}A \equiv 2\cos^2\frac{1}{2}A - 1$$

$$\tan A \equiv \frac{2\tan\frac{1}{2}A}{1 - \tan^2\frac{1}{2}A}$$

$$\sin A + \sin B \equiv 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A\cos B \equiv \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\sin B \equiv \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos A\cos B \equiv \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A\sin B \equiv \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

EXAMPLES

1. Show that
$$\sin^2 2x \equiv \frac{1}{2}(1 - \cos 4x).$$
Solution
From the standard trigonometric identities, we have
$$\cos 4x \equiv 1 - 2\sin^2 2x$$
on replacing A by $2x$.
Rearranging this new identity, gives the required result.

2. Show that
$$\sin\left(\theta + \frac{\pi}{2}\right) \equiv \cos\theta.$$
Solution
The left hand side can be expanded as
$$\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2};$$
and the result follows, because $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

3. Simplify the expression
$$\frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha}.$$
Solution
Using separate trigonometric identities in the numerator and denominator, the expression becomes
$$\frac{2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\cos\left(\frac{2\alpha-3\alpha}{2}\right)}{-2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\sin\left(\frac{2\alpha-3\alpha}{2}\right)}$$
$$\equiv \frac{2\sin\left(\frac{5\alpha}{2}\right).\cos\left(\frac{-\alpha}{2}\right)}{-2\sin\left(\frac{5\alpha}{2}\right).\sin\left(\frac{-\alpha}{2}\right)}$$
$$\equiv \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$
$$\equiv \cot\left(\frac{\alpha}{2}\right).$$

4. Express $2\sin 3x\cos 7x$ as the difference of two sines.
Solution
$$2\sin 3x\cos 7x \equiv \sin(3x + 7x) + \sin(3x - 7x).$$
Hence,
$$2\sin 3x\cos 7x \equiv \sin 10x - \sin 4x.$$

3.5.2 AMPLITUDE, WAVE-LENGTH, FREQUENCY AND PHASE ANGLE

In the scientific applications of Mathematics, importance is attached to trigonometric functions of the form

$$A\sin(\omega t+\alpha) \quad \text{and} \quad A\cos(\omega t+\alpha),$$

where A, ω and α are constants and t is usually a time variable.

It is useful to note, from trigonometric identities, that the expanded forms of the above two functions are given by

$$A\sin(\omega t+\alpha) \equiv A\sin\omega t\cos\alpha + A\cos\omega t\sin\alpha$$

and

$$A\cos(\omega t+\alpha) \equiv A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha.$$

(a) The Amplitude

In view of the fact that the sine and the cosine of any angle always lies within the closed interval from -1 to $+1$ inclusive, the constant, A, represents the maximum value (numerically) which can be attained by each of the above trigonometric functions.

A is called the **"amplitude"** of each of the functions.

(b) The Wave Length (Or Period)

If the value, t, increases or decreases by a whole multiple of $\frac{2\pi}{\omega}$, then the value, $(\omega t+\alpha)$, increases or decreases by a whole multiple of 2π; and, hence, the functions remain unchanged in value.

A graph, against t, of either $A\sin(\omega t+\alpha)$ or $A\cos(\omega t+\alpha)$ would be repeated in shape at regular intervals of length $\frac{2\pi}{\omega}$.

The repeated shape of the graph is called the **"wave profile"** and $\frac{2\pi}{\omega}$ is called the **"wave-length"**, or **"period"** of each of the functions.

(c) The Frequency

If t is indeed a time variable, then the wave length (or period) represents the time taken to complete a single wave-profile. Consequently, the number of wave-profiles completed in one unit of time is given by $\frac{\omega}{2\pi}$.

$\frac{\omega}{2\pi}$ is called the **"frequency"** of each of the functions.

Note:
The constant ω itself is called the **"angular frequency"**; it represents the change in the quantity $(\omega t+\alpha)$ for every unit of change in the value of t.

(d) The Phase Angle

The constant, α, affects the starting value, at $t=0$, of the trigonometric functions $A\sin(\omega t+\alpha)$ and $A\cos(\omega t+\alpha)$. Each of these is said to be **"out of phase"**, by an amount, α, with the trigonometric functions $A\sin\omega t$ and $A\cos\omega t$ respectively.

α is called the **"phase angle"** of each of the two original trigonometric functions; but it can take infinitely many values differing only by a whole multiple of 360° (if working in degrees) or 2π (if working in radians).

EXAMPLES

1. Express $\sin t + \sqrt{3}\cos t$ in the form $A\sin(t+\alpha)$, with α in degrees, and hence solve the equation,

$$\sin t + \sqrt{3}\cos t = 1$$

for t in the range $0° \leq t \leq 360°$.

Solution

We require that

$$\sin t + \sqrt{3}\cos t \equiv A\sin t\cos\alpha + A\cos t\sin\alpha$$

Hence,

$$A\cos\alpha = 1 \quad \text{and} \quad A\sin\alpha = \sqrt{3},$$

which gives $A^2 = 4$ (using $\cos^2\alpha + \sin^2\alpha \equiv 1$) and also $\tan\alpha = \sqrt{3}$.

Thus,

$$A = 2 \quad \text{and} \quad \alpha = 60° \text{ (principal value)}.$$

To solve the given equation, we may now use

$$2\sin(t + 60°) = 1,$$

so that

$$t + 60° = \text{Sin}^{-1}\frac{1}{2} = 30° + k360° \quad \text{or} \quad 150° + k360°,$$

where k may be any integer.

For the range $0° \leq t \leq 360°$, we conclude that

$$t = 330° \quad \text{or} \quad 90°.$$

2. Determine the amplitude and phase-angle which will express the trigonometric function $a\sin\omega t + b\cos\omega t$ in the form $A\sin(\omega t + \alpha)$.

Apply the result to the expression $3\sin 5t - 4\cos 5t$ stating α in degrees, correct to one decimal place, and lying in the interval from $-180°$ to $180°$.

Solution

We require that

$$A\sin(\omega t + \alpha) \equiv a\sin\omega t + b\cos\omega t;$$

and, hence, from trigonometric identities,

$$A\sin\alpha = b \quad \text{and} \quad A\cos\alpha = a.$$

Squaring each of these and adding the results together gives

$$A^2 = a^2 + b^2 \quad \text{that is} \quad A = \sqrt{a^2+b^2}.$$

Also,

$$\frac{A\sin\alpha}{A\cos\alpha} = \frac{b}{a},$$

which gives

$$\alpha = \tan^{-1}\frac{b}{a};$$

but the particular angle chosen must ensure that $\sin\alpha = \frac{b}{A}$ and $\cos\alpha = \frac{a}{A}$ have the correct sign.

Applying the results to the expression $3\sin 5t - 4\cos 5t$, we have

$$A = \sqrt{3^2+4^2}$$

and

$$\alpha = \tan^{-1}\left(-\frac{4}{3}\right).$$

But $\sin\alpha \left(= -\frac{4}{5}\right)$ is negative and $\cos\alpha \left(= \frac{3}{5}\right)$ is positive so that α may be taken as an angle between zero and $-90°$; that is $\alpha = -53.1°$.

We conclude that

$$3\sin 5t - 4\cos 5t \equiv 5\sin(5t - 53.1°).$$

3. Solve the equation

$$4\sin 2t + 3\cos 2t = 1$$

for t in the interval from $-180°$ to $180°$.

Solution

Expressing the left hand side of the equation in the form $A\sin(2t+\alpha)$, we require

$$A = \sqrt{4^2+3^2} = 5 \quad \text{and} \quad \alpha = \tan^{-1}\frac{3}{4}.$$

Also $\sin\alpha \left(= \frac{3}{5}\right)$ is positive and $\cos\alpha \left(= \frac{4}{5}\right)$ is positive so that α may be taken as an angle in the interval from zero to $90°$.

Hence, $\alpha = 36.87°$ and the equation to be solved becomes

$$5\sin(2t + 36.87°) = 1.$$

Its solutions are obtained by making t the **"subject"** of the equation to give

$$t = \frac{1}{2}\left[\mathrm{Sin}^{-1}\frac{1}{5} - 36.87°\right].$$

The possible values of $\mathrm{Sin}^{-1}\frac{1}{5}$ are $11.53° + k360°$ and $168.46° + k360°$, where k may be any integer. But, to give values of t which are numerically less than $180°$, we may use only $k = 0$ and $k = 1$ in the first of these and $k = 0$ and $k = -1$ in the second.

The results obtained are

$$t = -12.67°,\ 65.80°,\ 167.33° \quad \text{and} \quad -114.21°$$

3.5.3 EXERCISES

1. Simplify the following expressions:

   (a)
   $$(1 + \cos x)(1 - \cos x);$$

   (b)
   $$(1 + \sin x)^2 - 2\sin x(1 + \sin x).$$

2. Show that
   $$\cos\left(\theta - \frac{\pi}{2}\right) \equiv \sin\theta$$

3. Express $2\sin 4x \sin 5x$ as the difference of two cosines.

4. Use the table of trigonometric identities to show that

   (a)
   $$\frac{\sin 5x + \sin x}{\cos 5x + \cos x} \equiv \tan 3x;$$

   (b)
   $$\frac{1 - \cos 2x}{1 + \cos 2x} \equiv \tan^2 x;$$

   (c)
   $$\tan x.\tan 2x + 2 \equiv \tan 2x.\cot x;$$

   (d)
   $$\cot(x + y) \equiv \frac{\cot x.\cot y - 1}{\cot y + \cot x}.$$

5. Solve the following equations by writing the trigonometric expression on the left-hand-side in the form suggested, being careful to see whether the phase angle is required in degrees or radians and ensuring that your final answers are in the range given:

   (a) $\cos t + 7\sin t = 5, \quad 0° \leq t \leq 360°$, (transposed to the form $A\cos(t - \alpha)$, with α in degrees.
   (b) $\sqrt{2}\cos t - \sin t = 1, \quad 0° \leq t \leq 360°$, (transposed to the form $A\cos(t + \alpha)$, with α in degrees.
   (c) $2\sin t - \cos t = 1, \quad 0 \leq t \leq 2\pi$, (transposed to the form $A\sin(t - \alpha)$, with α in radians.
   (d) $3\sin t - 4\cos t = 0.6, \quad 0° \leq t \leq 360°$, (transposed to the form $A\sin(t - \alpha)$, with α in degrees.

6. Determine the amplitude and phase-angle which will express the trigonometric function $a\cos\omega t + b\sin\omega t$ in the form $A\cos(\omega t + \alpha)$.

   Apply the result to the expression $4\cos 5t - 4\sqrt{3}\sin 5t$ stating α in degrees and lying in the interval from $-180°$ to $180°$.

7. Solve the equation

$$2 \cos 3t + 5 \sin 3t = 4$$

for t in the interval from zero to 360°, expressing t in decimals correct to two decimal places.

3.5.4 ANSWERS TO EXERCISES

1. (a) $\sin^2 x$; (b) $\cos^2 x$.
2. Use the cos(A - B) formula to expand left hand side.
3. $\cos x - \cos 9x$.
4. (a) Use the formulae for sinA + sinB and cosA + cosB;
   (b) Use the formulae for $\cos 2x$ to make the 1's cancel;
   (c) Both sides are identically equal to $\frac{2}{1-\tan^2 x}$;
   (d) Invert the formula for $\tan(x + y)$.
5. (a) 36.9°, 126.9°;
   (b) 19.5°, 270°;
   (c) 0, 3.14;
   (d) 226.24°
6.
$$A = \sqrt{a^2 + b^2}, \quad \text{and} \quad \alpha = \tan^{-1}\left(-\frac{b}{a}\right);$$

$$4 \cos 5t - 4\sqrt{3} \sin 5t \equiv 8 \cos(5t + 60°).$$

7.
$$\sqrt{29} \cos(3t - 68.20°) = 4 \quad \text{or} \quad \sqrt{29} \sin(3t + 21.80°) = 4$$

give

$$t = 8.72°, \ 36.74°, \ 156.74° \quad \text{and} \quad 276.74°$$

"JUST THE MATHS"

UNIT NUMBER

4.1

HYPERBOLIC FUNCTIONS 1
(Definitions, graphs and identities)

by

A.J.Hobson

UNIT 4.1 - HYPERBOLIC FUNCTIONS DEFINITIONS, GRAPHS AND IDENTITIES

4.1.1 INTRODUCTION

In this section, we introduce a new group of mathematical functions, based on the functions

$$e^x \quad \text{and} \quad e^{-x}$$

whose properties resemble, very closely, those of the standard trigonometric functions. But, whereas trigonometric functions can be related to the geometry of a circle (and are sometimes called the **"circular functions"**), it can be shown that the new group of functions are related to the geometry of a hyperbola (see unit 5.7). Because of this, they are called **"hyperbolic functions"**.

4.1.2 DEFINITIONS

(a) Hyperbolic Cosine

The hyperbolic cosine of a number, x, is denoted by $\cosh x$ and is defined by

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}.$$

Note:
The name of the function is pronounced **"cosh"**.

(b) Hyperbolic Sine

The hyperbolic sine of a number, x, is denoted by $\sinh x$ and is defined by

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}.$$

Note:
The name of the function is pronounced **"shine"**

(c) Hyperbolic Tangent

The hyperbolic tangent of a number, x, is denoted by $\tanh x$ and is defined by

$$\tanh x \equiv \frac{\sinh x}{\cosh x}.$$

Notes:
(i) The name of the function is pronounced **"than"**.

(ii) In terms of exponentials, it is easily shown that

$$\tanh x \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}.$$

(d) Other Hyperbolic Functions

Other, less commonly used, hyperbolic functions are defined as follows:

(i) Hyperbolic secant, pronounced **"shek"**, is defined by

$$\text{sech} x \equiv \frac{1}{\cosh x}.$$

(ii) Hyperbolic cosecant, pronounced **"coshek"** is defined by

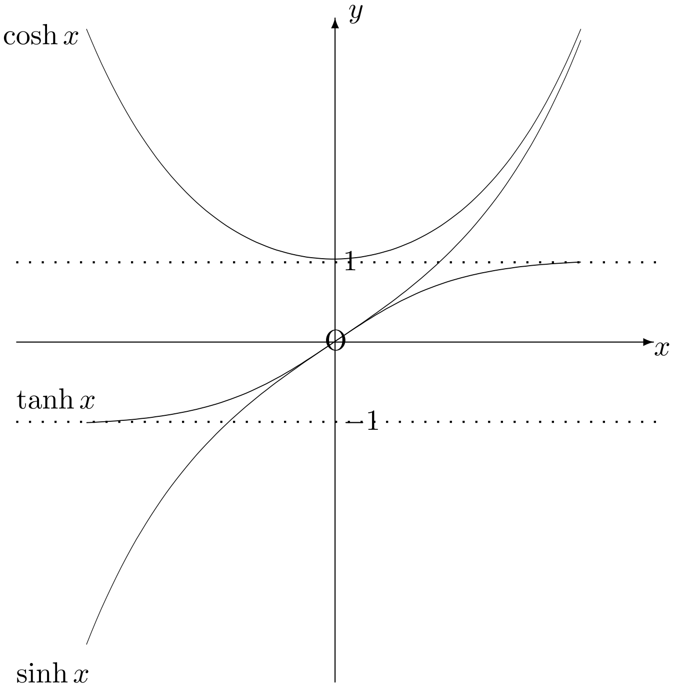
$$\text{cosech} x \equiv \frac{1}{\sinh x}.$$

(iii) Hyperbolic cotangent, pronounced **"coth"** is defined by

$$\text{coth} x \equiv \frac{1}{\tanh x} \equiv \frac{\cosh x}{\sinh x}.$$

4.1.3 GRAPHS OF HYPERBOLIC FUNCTIONS

It is useful to see the graphs of the functions $\cosh x$, $\sinh x$ and $\tanh x$ drawn with reference to the same set of axes. It can be shown that they are as follows:



Note:
We observe that the graph of $\cosh x$ exists only for y greater than or equal to 1; and that graph of $\tanh x$ exists only for y lying between -1 and $+1$. The graph of $\sinh x$, however, covers the whole range of x and y values from $-\infty$ to $+\infty$.

4.1.4 HYPERBOLIC IDENTITIES

It is possible to show that, to every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions though, in some cases, the comparison is more direct than in other cases.

ILLUSTRATIONS

1.

$$e^x \equiv \cosh x + \sinh x.$$

Proof

This follows directly from the definitions of $\cosh x$ and $\sinh x$.

2.

$$e^{-x} \equiv \cosh x - \sinh x.$$

Proof

Again, this follows directly from the definitions of $\cosh x$ and $\sinh x$.

3.

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

Proof

This follows if we multiply together the results of the previous two illustrations since $e^x.e^{-x} = 1$ and $(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv \cosh^2 x - \sinh^2 x$.

Notes:

(i) Dividing throughout by $\cosh^2 x$ gives the identity

$$1 - \tanh^2 x \equiv \text{sech}^2 x.$$

(ii) Dividing throughout by $\sinh^2 x$ gives the identity

$$\coth^2 x - 1 \equiv \text{cosech}^2 x.$$

4.

$$\sinh(x+y) \equiv \sinh x \cosh y + \cosh x \sinh y.$$

Proof:

The right hand side may be expressed in the form

$$\frac{e^x - e^{-x}}{2}.\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}.\frac{e^y - e^{-y}}{2},$$

which expands out to

$$\frac{e^{(x+y)} + e^{(x-y)} - e^{(-x+y)} - e^{(-x-y)}}{4} + \frac{e^{(x+y)} - e^{(x-y)} + e^{(-x+y)} - e^{(-x-y)}}{4};$$

and this simplifies to

$$\frac{2e^{(x+y)} - 2e^{(-x-y)}}{4} \equiv \frac{e^{(x+y)} - e^{-(x+y)}}{2} \equiv \sinh(x+y).$$

5.

$$\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y.$$

Proof

The proof is similar to the previous illustration.

6.

$$\tanh(x + y) \equiv \frac{\tanh x + \tanh y}{1 - \tanh x \tanh y}.$$

Proof

The proof again is similar to that in Illustration No. 4.

4.1.5 OSBORN'S RULE

Many other results, similar to those previously encountered in the standard list of trigonometric identities can be proved in the same way as for Illustration No. 4 above; that is, we substitute the definitions of the appropriate hyperbolic functions.

However, if we merely wish to **write down** a hyperbolic identity without proving it, we may use the following observation due to Osborn:

Starting with any trigonometric identity, change cos to cosh and sin to sinh. Then, if the trigonometric identity contains (or implies) two sine functions multiplied together, change the sign in front of the relevant term from + to − or vice versa.

ILLUSTRATIONS

1.

$$\cos^2 x + \sin^2 x \equiv 1,$$

which leads to the hyperbolic identity

$$\cosh^2 x - \sinh^2 x \equiv 1$$

since the trigonometric identity contains two sine functions multiplied together.

2.

$$\sin(x - y) \equiv \sin x \cos y - \cos x \sin y,$$

which leads to the hyperbolic identity

$$\sinh(x - y) \equiv \sinh x \cosh y - \cosh x \sinh y$$

in which no changes of sign are required.

3.

$$\sec^2 x \equiv 1 + \tan^2 x,$$

which leads to the hyperbolic identity

$$\text{sech}^2 x \equiv 1 - \tanh^2 x$$

since $\tan^2 x$ in the trigonometric identity implies that two sine functions are multiplied together; that is,

$$\tan^2 x \equiv \frac{\sin^2 x}{\cos^2 x}.$$

4.1.6 EXERCISES

1. If
$$L = 2C \sinh \frac{H}{2C},$$
determine the value of L when $H = 63$ and $C = 50$

2. If
$$v^2 = 1.8L \tanh \frac{6.3d}{L},$$
determine the value of v when $d = 40$ and $L = 315$.

3. Use Osborn's Rule to write down hyperbolic identities for
   (a)
   $$\sinh 2A;$$
   (b)
   $$\cosh 2A.$$

4. Use the results of the previous question to simplify the expression
$$\frac{1 + \sinh 2A + \cosh 2A}{1 - \sinh 2A - \cosh 2A}.$$

5. Use Osborn's rule to write down the hyperbolic identity which corresponds to the trigonometric identity
$$2 \sin x \sin y \equiv \cos(x - y) - \cos(x + y)$$
and prove your result.

6. If
$$a = c \cosh x \quad \text{and} \quad b = c \sinh x,$$
show that
$$(a + b)^2 e^{-2x} \equiv a^2 - b^2 \equiv c^2.$$

4.1.7 ANSWERS TO EXERCISES

1. 67.25

2. 19.40

3. (a)

$$\sinh 2A \equiv 2\sinh A\cosh A;$$

(b)

$$\cosh 2A \equiv \cosh^2 A + \sinh^2 A \equiv 2\cosh^2 A - 1 \equiv 1 + 2\sinh^2 A.$$

4.

$$-\coth A.$$

5.

$$-2\sinh x\sinh y \equiv \cosh(x-y) - \cosh(x+y).$$

"JUST THE MATHS"

UNIT NUMBER

4.2

HYPERBOLIC FUNCTIONS 2
(Inverse hyperbolic functions)

by

A.J.Hobson

UNIT 4.2 - HYPERBOLIC FUNCTIONS 2
INVERSE HYPERBOLIC FUNCTIONS

4.2.1 - INTRODUCTION

The three basic inverse hyperbolic functions are $\text{Cosh}^{-1}x$, $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$.

It may be shown that they are given by the following formulae:

(a)

$$\text{Cosh}^{-1}x = \pm \ln(x + \sqrt{x^2 - 1});$$

(b)

$$\text{Sinh}^{-1}x = \ln(x + \sqrt{x^2 + 1});$$

(c)

$$\text{Tanh}^{-1}x = \frac{1}{2} \ln \frac{1+x}{1-x}.$$

Notes:

(i) The positive value of $\text{Cosh}^{-1}x$ is called the **'principal value"** and is denoted by $\cosh^{-1}x$ (using a lower-case c).

(ii) $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$ have only **one** value but, for uniformity, we customarily denote them by $\sinh^{-1}x$ and $\tanh^{-1}x$.

4.2.2 THE PROOFS OF THE STANDARD FORMULAE

(a) Inverse Hyperbolic Cosine

If we let $y = \text{Cosh}^{-1}x$, then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Hence,

$$2x = e^y + e^{-y}.$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0,$$

which is a quadratic equation in e^y having solutions, from the quadratic formula, given by

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Taking natural logarithms of both sides gives

$$y = \ln(x \pm \sqrt{x^2 - 1}).$$

However, the two values $x + \sqrt{x^2 - 1}$ and $x - \sqrt{x^2 - 1}$ are reciprocals of each other, since their product is the value, 1; and so

$$y = \pm \ln(x + \sqrt{x^2 - 1}).$$

(b) Inverse Hyperbolic Sine

If we let $y = \text{Sinh}^{-1}x$, then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}.$$

Hence,

$$2x = e^y - e^{-y},$$

or

$$(e^y)^2 - 2xe^y - 1 = 0,$$

which is a quadratic equation in e^y having solutions, from the quadratic formula, given by

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

However, $x - \sqrt{x^2+1}$ has a negative value and cannot, therefore, be equated to a power of e, which is positive. Hence, this part of the expression for e^y must be ignored.

Taking natural logarithms of both sides gives

$$y = \ln(x + \sqrt{x^2+1}).$$

(c) Inverse Hyperbolic Tangent

If we let $y = \text{Tanh}^{-1}x$, then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}.$$

Hence,

$$x\left(e^{2y}+1\right) = e^{2y} - 1,$$

giving

$$e^{2y} = \frac{1+x}{1-x}.$$

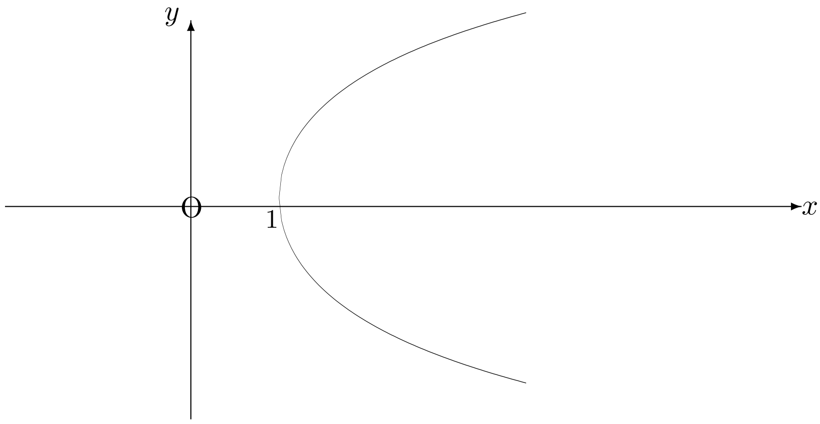
Taking natural logarithms of both sides,

$$y = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

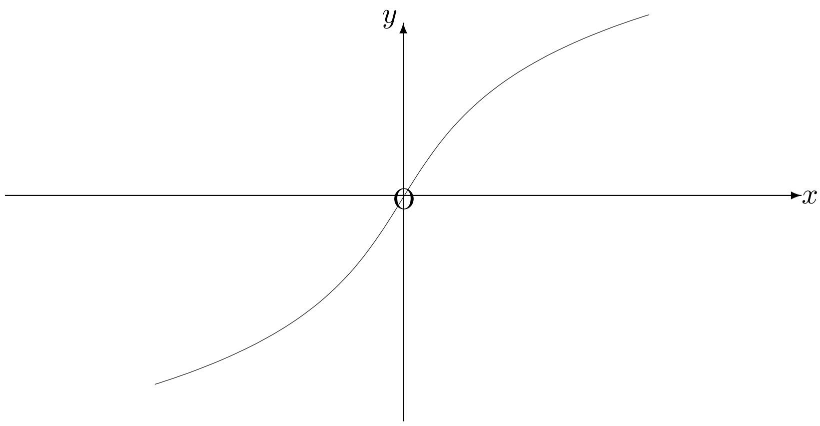
Note:
The graphs of inverse hyperbolic functions are discussed fully in Unit 10.7, but we include them here for the sake of completeness:

The graphs are as follows:

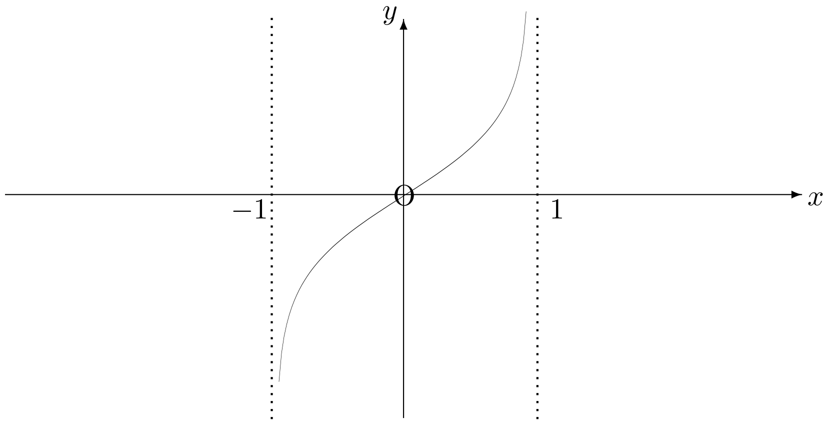
(a) $y = \text{Cosh}^{-1}x$



(b) $y = \text{Sinh}^{-1}x$



(c) $y = \text{Tanh}^{-1}x$



4.2.3 EXERCISES

1. Use the standard formulae to evaluate (a) $\sinh^{-1}7$ and (b) $\cosh^{-1}9$.

2. Express $\cosh 2x$ and $\sinh 2x$ in terms of exponentials and hence solve, for x, the equation

$$2\cosh 2x - \sinh 2x = 2.$$

3. Obtain a formula which equates $\text{cosech}^{-1}x$ to the natural logarithm of an expression in x, distinguishing between the two cases $x > 0$ and $x < 0$.

4. If $t = \tanh(x/2)$, prove that

   (a)

$$\sinh x = \frac{2t}{1-t^2}$$

   and

   (b)

$$\cosh x = \frac{1+t^2}{1-t^2}.$$

   Hence solve, for x, the equation

$$7\sinh x + 20\cosh x = 24.$$

4.2.4 ANSWERS TO EXERCISES

1. (a)

$$\ln(7 + \sqrt{49 + 1}) \simeq 2.644;$$

(b)

$$\ln(9 + \sqrt{81 - 1}) \simeq 2.887$$

2.

$$\left(e^{2x}\right)^2 - 4e^{2x} + 3 = 0,$$

which gives $e^{2x} = 1$ or 3 and hence $x = 0$ or $\frac{1}{2}\ln 3 \simeq 0.549$.

3. If $x > 0$, then

$$\text{cosech}^{-1}x = \ln \frac{1 + \sqrt{1 + x^2}}{x}.$$

If $x < 0$, then

$$\text{cosech}^{-1}x = \ln \frac{1 - \sqrt{1 + x^2}}{x}.$$

4. Use

$$\sinh x \equiv \frac{2\tanh(x/2)}{\text{sech}^2(x/2)}$$

and

$$\cosh x \equiv \frac{\left(1 + \tanh^2(x/2)\right)}{\text{sech}^2(x/2)}.$$

This gives $t = -\frac{1}{2}$ or $t = \frac{2}{11}$ and hence $x \simeq -1.099$ or 0.368 which agrees with the solution obtained using exponentials.

"JUST THE MATHS"

UNIT NUMBER

5.1

GEOMETRY 1
(Co-ordinates, distance & gradient)

by

A.J.Hobson

UNIT 5.1 - GEOMETRY 1

CO-ORDINATES, DISTANCE AND GRADIENT

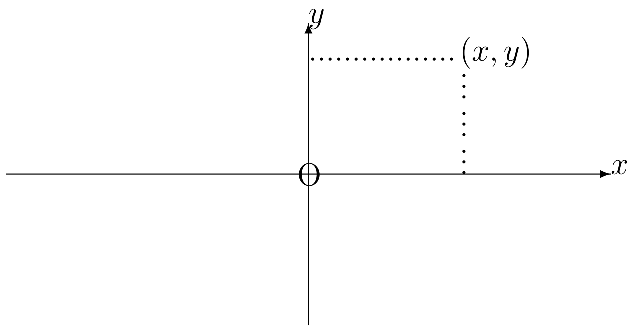
5.1.1 CO-ORDINATES

(a) Cartesian Co-ordinates

The position of a point, P, in a plane may be specified completely if we know its perpendicular distances from two chosen fixed straight lines, where we distinguish between positive distances on one side of each line and negative distances on the other side of each line.

It is not essential that the two chosen fixed lines should be at right-angles to each other, but we usually take them to be so for the sake of convenience.

Consider the following diagram:



The horizontal directed line, Ox, is called the **"x-axis"** and distances to the right of the origin (point O) are taken as positive.

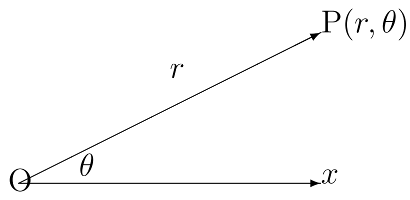
The vertical directed line, Oy, is called the **"y-axis"** and distances above the origin (point O) are taken as positive.

The notation (x, y) denotes a point whose perpendicular distances from Oy and Ox are x and y respectively, these being called the **"cartesian co-ordinates"** of the point.

(b) Polar Co-ordinates

An alternative method of fixing the position of a point P in a plane is to choose first a point, O, called the **"pole"** and directed line , Ox, emanating from the pole in one direction only and called the **"initial line"**.

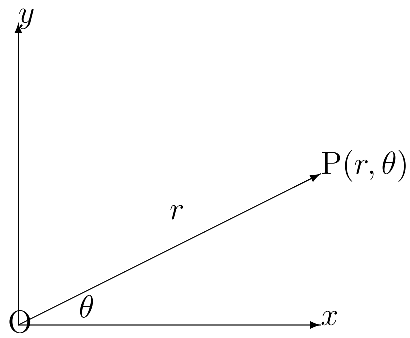
Consider the following diagram:



The position of P is determined by its distance r from the pole and the angle, θ which the line OP makes with the initial line, measuring this angle positively in a counter-clockwise sense or negatively in a clockwise sense from the initial line. The notation (r, θ) denotes the **"polar co-ordinates"** of the point.

5.1.2 THE RELATIONSHIP BETWEEN POLAR AND CARTESIAN CO-ORDINATES

It is convenient to superimpose the diagram for Polar Co-ordinates onto the diagram for Cartesian Co-ordinates as follows:



The trigonometry of the combined diagram shows that

(a) $x = r.\cos\theta$ and $y = r.\sin\theta$;

(b) $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

EXAMPLES

1. Express the equation
$$2x + 3y = 1$$
in polar co-ordinates.

   Solution

   Substituting for x and y separately, we obtain
$$2r\cos\theta + 3r\sin\theta = 1$$
   That is
$$r = \frac{1}{2\cos\theta + 3\sin\theta}$$

2. Express the equation
$$r = \sin\theta$$
in cartesian co-ordinates.

Solution

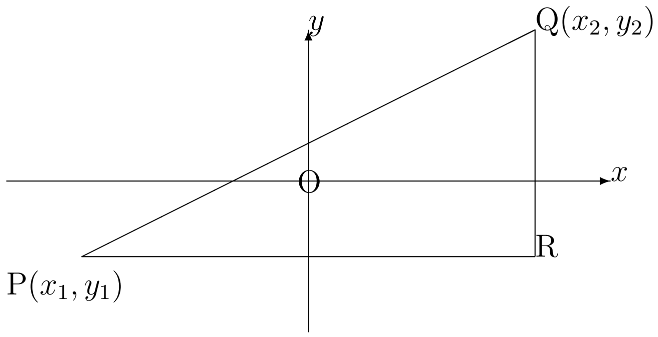
We could try substituting for r and θ separately, but it is easier, in this case, to rewrite the equation as
$$r^2 = r\sin\theta$$
which gives
$$x^2 + y^2 = y$$

5.1.3 THE DISTANCE BETWEEN TWO POINTS

Given two points (x_1, y_1) and (x_2, y_2), the quantity $\mid x_2 - x_1 \mid$ is called the **"horizontal separation"** of the two points and the quantity $\mid y_2 - y_1 \mid$ is called the **"vertical separation"** of the two points, assuming, of course, that the x-axis is horizontal.

The expressions for the horizontal and vertical separations remain valid even when one or more of the co-ordinates is negative. For example, the horizontal separation of the points $(5, 7)$ and $(-3, 2)$ is given by $\mid -3 - 5 \mid = 8$ which agrees with the fact that the two points are on opposite sides of the y-axis.

The actual distance between (x_1, y_1) and (x_2, y_2) is easily calculated from Pythagoras' Theorem, using the horizontal and vertical separations of the points.



In the diagram,
$$PQ^2 = PR^2 + RQ^2.$$
That is,
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$
giving
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note:
We do not need to include the modulus signs of the horizontal and vertical separations

because we are squaring them and therefore, any negative signs will disappear. For the same reason, it does not matter which way round the points are labelled.

EXAMPLE
Calculate the distance, d, between the two points $(5, -3)$ and $(-11, -7)$.

Solution

Using the formula, we obtain

$$d = \sqrt{(5+11)^2 + (-3+7)^2}.$$

That is,

$$d = \sqrt{256+16} = \sqrt{272} \cong 16.5$$

5.1.4 GRADIENT

The gradient of the straight-line segment, PQ, joining two points P and Q in a plane is defined to be the tangent of the angle which PQ makes with the positive x-direction.

In practice, when the co-ordinates of the two points are P(x_1, y_1) and Q(x_2, y_2), the gradient, m, is given by either

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

or

$$m = \frac{y_1 - y_2}{x_1 - x_2},$$

both giving the same result.

This is not quite the same as the ratio of the horizontal and vertical separations since we distinguish between positive gradient and negative gradient.

EXAMPLE
Determine the gradient of the straight-line segment joining the two points $(8, -13)$ and $(-2, 5)$ and hence calculate the angle which the segment makes with the positive x-direction.

Solution

$$m = \frac{5+13}{-2-8} = \frac{-13-5}{8+2} = -1.8$$

Hence, the angle, θ, which the segment makes with the positive x-direction is given by

$$\tan\theta = -1.8$$

Thus,

$$\theta = \tan^{-1}(-1.8) \simeq 119^\circ.$$

5.1.5 EXERCISES

1. A square, side d, has vertices O,A,B,C (labelled counter-clockwise) where O is the pole of a system of polar co-ordinates. Determine the polar co-ordinates of A,B and C when

(a) OA is the initial line;

(b) OB is the initial line.

2. Express the following cartesian equations in polar co-ordinates:

(a)

$$x^2 + y^2 - 2y = 0;$$

(b)

$$y^2 = 4a(a - x).$$

3. Express the following polar equations in cartesian co-ordinates:

(a)

$$r^2 \sin 2\theta = 3;$$

(b)

$$r = 1 + \cos\theta.$$

4. Determine the length of the line segment joining the following pairs of points given in cartesian co-ordinates:

(a) $(0, 0)$ and $(3, 4)$;

(b) $(-2, -3)$ and $(1, 1)$;

(c) $(-4, -6)$ and $(-1, -2)$;

(d) $(2, 4)$ and $(-3, 16)$;

(e) $(-1, 3)$ and $(11, -2)$.

5. Determine the gradient of the straight-line segment joining the two points $(-5, -0.5)$ and $(4.5, -1)$.

5.1.6 ANSWERS TO EXERCISES

1. (a) A$(d, 0)$, B$\left(d\sqrt{2}, \frac{\pi}{4}\right)$, C$\left(d, \frac{\pi}{2}\right)$;

   (b) A$\left(d, -\frac{\pi}{4}\right)$, B$(d\sqrt{2}, 0)$, C$\left(d, \frac{\pi}{4}\right)$.

2. (a) $r = 2\sin\theta$;

   (b) $r^2\sin^2\theta = 4a(a - r\cos\theta)$.

3. (a) $xy = \frac{3}{2}$;

   (b) $x^4 + y^4 + 2x^2y^2 - 2x^3 - 2xy^2 - y^2 = 0$.

4. (a) 5; (b) 5; (c) 5; (d) 13; (e) 13.

5. $m = -\frac{1}{19}$.

"JUST THE MATHS"

UNIT NUMBER

5.2

GEOMETRY 2
(The straight line)

by

A.J.Hobson

UNIT 5.2 - GEOMETRY 2

THE STRAIGHT LINE

5.2.1 PREAMBLE

It is not possible to give a satisfactory diagramatic definition of a straight line since the attempt is likely to assume a knowledge of linear measurement which, itself, depends on the concept of a straight line. For example, it is no use defining a straight line as "the shortest path between two points" since the word "shortest" assumes a knowledge of linear measurement.

In fact, the straight line is defined algebraically as follows:

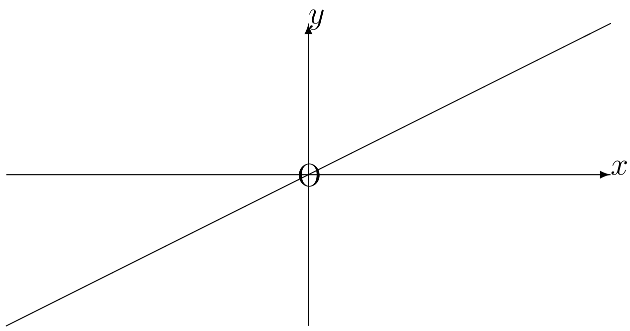
DEFINITION
A straight line is a set of points, (x, y), satisfying an equation of the form

$$ax + by + c = 0$$

where a, b and c are constants. This equation is called a **"linear equation"** and the symbol (x, y) itself, rather than a dot on the page, represents an arbitrary point of the line.

5.2.2 STANDARD EQUATIONS OF A STRAIGHT LINE

(a) Having a given gradient and passing through the origin



Let the gradient be m; then, from the diagram, all points (x, y) on the straight line **(but no others)** satisfy the relationship,

$$\frac{y}{x} = m.$$

That is,

$$\boxed{y = mx}$$

which is the equation of this straight line.

EXAMPLE
Determine, in degrees, the angle, θ, which the straight line,

$$\sqrt{3}y = x,$$

makes with the positive x-direction.

Solution

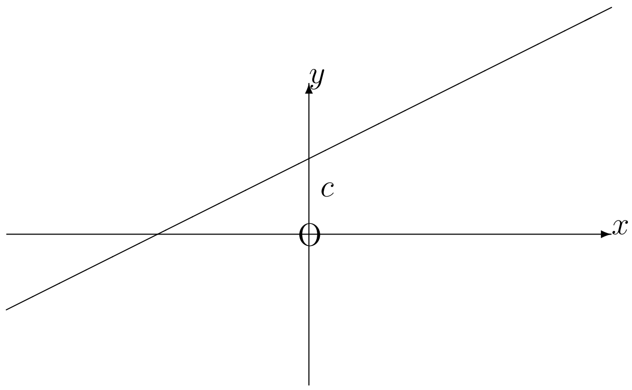
The gradient of the straight line is given by

$$\tan\theta = \frac{1}{\sqrt{3}}.$$

Hence,

$$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ.$$

(b) Having a given gradient, and a given intercept on the vertical axis



Let the gradient be m and let the intercept be c; then, in this case we can imagine that the relationship between x and y in the previous section is altered only by adding the number c to all of the y co-ordinates. Hence the equation of the straight line is

$$\boxed{y = mx + c.}$$

EXAMPLE
Determine the gradient, m, and intercept c on the y-axis of the straight line whose equation is

$$7x - 5y - 3 = 0.$$

Solution

On rearranging the equation, we have

$$y = \frac{7}{5}x - \frac{3}{5}.$$

Hence,

$$m = \frac{7}{5}$$

and

$$c = -\frac{3}{5}.$$

This straight line will intersect the y-axis **below** the origin because the intercept is negative.

(c) Having a given gradient and passing through a given point

Let the gradient be m and let the given point be (x_1, y_1). Then,

$$y = mx + c,$$

where

$$y_1 = mx_1 + c.$$

Hence, on subtracting the second of these from the first, we obtain

$$\boxed{y - y_1 = m(x - x_1).}$$

EXAMPLE
Determine the equation of the straight line having gradient $\frac{3}{8}$ and passing through the point $(-7, 2)$.

Solution

From the formula,

$$y - 2 = \frac{3}{8}(x + 7).$$

That is

$$8y - 16 = 3x + 21,$$

giving

$$8y = 3x + 37.$$

(d) Passing through two given points

Let the two given points be (x_1, y_1) and (x_2, y_2). Then, the gradient is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence, from the previous section, the equation of the straight line is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1);$$

but this is more usually written

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Note:
The same result is obtained no matter which way round the given points are taken as (x_1, y_1) and (x_2, y_2).

EXAMPLE
Determine the equation of the straight line joining the two points $(-5, 3)$ and $(2, -7)$, stating the values of its gradient and its intercept on the y-axis.

Solution (Method 1).

$$\frac{y-3}{-7-3} = \frac{x+5}{2+5},$$

giving

$$7(y-3) = -10(x+5).$$

That is,

$$10x + 7y + 29 = 0.$$

Solution (Method 2).

$$\frac{y+7}{3+7} = \frac{x-2}{-5-2},$$

giving

$$-7(y+7) = 10(x-2).$$

That is,

$$10x + 7y + 29 = 0,$$

as before.

By rewriting the equation of the line as

$$y = -\frac{10}{7}x - \frac{29}{7}$$

we see that the gradient is $-\frac{10}{7}$ and the intercept on the y-axis is $-\frac{29}{7}$.

(e) The parametric equations of a straight line

In the previous section, the common value of the two fractions

$$\frac{y-y_1}{y_2-y_1} \quad \text{and} \quad \frac{x-x_1}{x_2-x_1}$$

is called the **"parameter"** of the point (x, y) and is usually denoted by t.

By equating each fraction separately to t, we obtain

$$x = x_1 + (x_2 - x_1)t \quad \text{and} \quad y = y_1 + (y_2 - y_1)t.$$

These are called the **"parametric equations"** of the straight line while (x_1, y_1) and (x_2, y_2) are known as the **"base points"** of the parametric representation of the line.

Notes:
(i) In the above parametric representation, (x_1, y_1) has parameter $t = 0$ and (x_2, y_2) has parameter $t = 1$.

(ii) Other parametric representations of the same line can be found by using the given base points in the opposite order, or by using a different pair of points on the line as base points.

EXAMPLES

1. Use parametric equations to find two other points on the line joining $(3, -6)$ and $(-1, 4)$.

   Solution

   One possible parametric representation of the line is

   $$x = 3 - 4t \quad y = -6 + 10t.$$

   To find another two points, we simply substitute any two values of t other than 0 or 1. For example, with $t = 2$ and $t = 3$,

   $$x = -5, y = 14 \quad \text{and} \quad x = -9, y = 24.$$

   A pair of suitable points is therefore $(-5, 14)$ and $(-9, 24)$.

2. The co-ordinates, x and y, of a moving particle are given, at time t, by the equations

   $$x = 3 - 4t \quad \text{and} \quad y = 5 + 2t$$

   Determine the gradient of the straight line along which the particle moves.

   Solution

   Eliminating t, we have

   $$\frac{x-3}{-4} = \frac{y-5}{2}.$$

   That is,

   $$2(x-3) = -4(y-5),$$

   giving

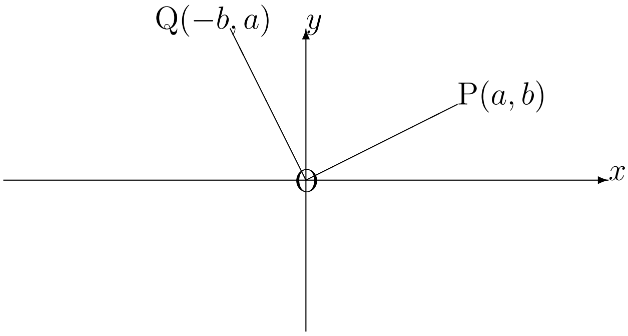
   $$y = -\frac{2}{4}x + \frac{26}{4}.$$

   Hence, the gradient of the line is

   $$-\frac{2}{4} = -\frac{1}{2}.$$

5.2.3 PERPENDICULAR STRAIGHT LINES

The perpendicularity of two straight lines is not dependent on either their length or their precise position in the plane. Hence, without loss of generality, we may consider two straight line segments of equal length passing through the origin. The following diagram indicates appropriate co-ordinates and angles to demonstrate perpendicularity:



In the diagram, the gradient of OP $= \frac{b}{a}$ and the gradient of OQ $= \frac{a}{-b}$.

Hence the **product of the gradients is equal to** -1 or, in other words, **each gradient is minus the reciprocal of the other gradient**.

EXAMPLE
Determine the equation of the straight line which passes through the point $(-2, 6)$ and is perpendicular to the straight line,

$$3x + 5y + 11 = 0.$$

Solution

The gradient of the given line is $-\frac{3}{5}$ which implies that the gradient of a perpendicular line is $\frac{5}{3}$. Hence, the required line has equation

$$y - 6 = \frac{5}{3}(x + 2),$$

giving

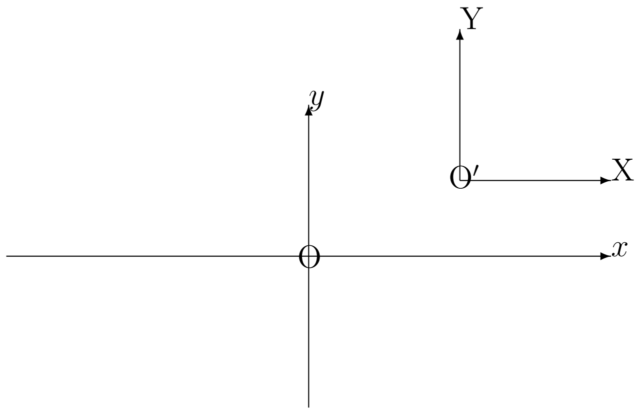
$$3y - 18 = 5x + 10.$$

That is,

$$3y = 5x + 28.$$

5.2.4 CHANGE OF ORIGIN

Given a cartesian system of reference with axes Ox and Oy, it may sometimes be convenient to consider a new set of axes O$'X$ parallel to Ox and O$'Y$ parallel to Oy with new origin at O$'$ whose co-ordinates are (h, k) referred to the original set of axes.



In the above diagram, everything is drawn in the first quadrant, but the relationships obtained between the old and new co-ordinates will apply in all situations. They are

$$X = x - h \quad \text{and} \quad Y = y - k$$

or

$$x = X + h \quad \text{and} \quad y = Y + k.$$

EXAMPLE
Given the straight line,

$$y = 3x + 11,$$

determine its equation referred to new axes with new origin at the point $(-2, 5)$.

Solution

Using

$$x = X - 2 \quad \text{and} \quad y = Y + 5,$$

we obtain

$$Y + 5 = 3(X - 2) + 11.$$

That is,

$$Y = 3X,$$

which is a straight line through the new origin with gradient 3.

Note:
If we had spotted that the point $(-2, 5)$ was **on** the original line, the new line would be bound to pass through the new origin; and its gradient would not alter in the change of origin.

5.2.5 EXERCISES

1. Determine the equations of the following straight lines:
   (a) having gradient 4 and intercept -7 on the y-axis;
   (b) having gradient $\frac{1}{3}$ and passing through the point $(-2, 5)$;
   (c) passing through the two points $(1, 6)$ and $(5, 9)$.
2. Determine the equation of the straight line passing through the point $(1, -5)$ which is perpendicular to the straight line whose cartesian equation is
$$x + 2y = 3.$$
3. Given the straight line
$$y = 4x + 2,$$
referred to axes Ox and Oy, determine its equation referred to new axes O$'X$ and O$'Y$ with new origin at the point where $x = 7$ and $y = -3$ (assuming that Ox is parallel to O$'X$ and Oy is parallel to O$'Y$).
4. Use the parametric equations of the straight line joining the two points $(-3, 4)$ and $(7, -1)$ in order to find its point of intersection with the straight line whose cartesian equation is
$$x - y + 4 = 0.$$

5.2.6 ANSWERS TO EXERCISES

1. (a)
$$y = 4x - 7;$$
   (b)
$$3y = x + 17;$$
   (c)
$$4y = 3x + 21.$$
2.
$$y = 2x - 7.$$
3.
$$Y = 4X + 33.$$
4.
$$x = -3 + 10t \quad y = 4 - 5t,$$
giving the point of intersection (at $t = \frac{1}{5}$) as $(-1, 3)$.

"JUST THE MATHS"

UNIT NUMBER

5.3

GEOMETRY 3
(Straight line laws)

by

A.J.Hobson

UNIT 5.3 - GEOMETRY 3

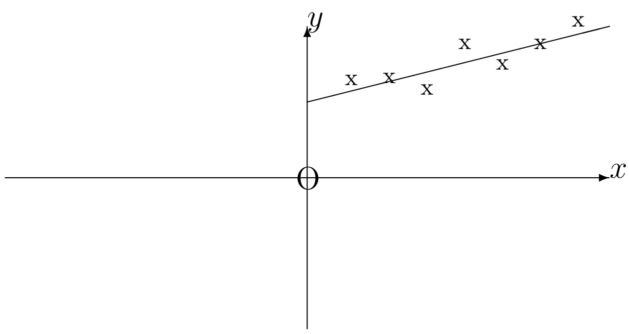
STRAIGHT LINE LAWS

5.3.1 INTRODUCTION

In practical work, the theory of an experiment may show that two variables, x and y, are connected by a straight line equation (or **"straight line law"**) of the form

$$y = mx + c.$$

In order to estimate the values of m and c, we could use the experimental data to plot a graph of y against x and obtain the **"best straight line"** passing through (or near) the plotted points to average out any experimental errors. Points which are obviously out of character with the rest are usually ignored.



It would seem logical, having obtained the best straight line, to measure the gradient, m, and the intercept, c, on the y-axis. However, this not always the wisest way of proceeding and should be avoided in general. The reasons for this are as follows:

(i) Economical use of graph paper may make it impossible to read the intercept, since this part of the graph may be "off the page".

(ii) The use of symbols other than x or y in scientific work may leave doubts as to which is the equivalent of the y-axis and which is the equivalent of the x-axis. Consequently, the gradient may be incorrectly calculated from the graph.

The safest way of finding m and c is to take two sets of readings, (x_1, y_1) and (x_2, y_2), from the best straight line drawn then solve the simultaneous linear equations

$$
\begin{aligned}
y_1 &= mx_1 + c, \\
y_2 &= mx_2 + c.
\end{aligned}
$$

It is a good idea if the two points chosen are as far apart as possible, since this will reduce errors in calculation due to the use of small quantities.

5.3.2 LAWS REDUCIBLE TO LINEAR FORM

Other experimental laws which are not linear can sometimes be reduced to linear form by using the experimental data to plot variables other than x or y, but related to them.

EXAMPLES

1. $y = ax^2 + b$.

   Method

   We let $X = x^2$, so that $y = aX + b$ and hence we may obtain a straight line by plotting y against X.

2. $y = ax^2 + bx$.

   Method

   Here, we need to consider the equation in the equivalent form $\frac{y}{x} = ax + b$ so that, by letting $Y = \frac{y}{x}$, giving $Y = ax + b$, a straight line will be obtained if we plot Y against x.

   Note:
   If one of the sets of readings taken in the experiment happens to be
   $(x, y) = (0, 0)$, we must ignore it in this example.

3. $xy = ax + b$.

   Method

   Two alternatives are available here as follows:
   (a) Letting $xy = Y$, giving $Y = ax + b$, we could plot a graph of Y against x.

   (b) Writing the equation as $y = a + \frac{b}{x}$, we could let $\frac{1}{x} = X$, giving $y = a + bX$, and plot a graph of y against X.

4. $y = ax^b$.

   Method

   This kind of law brings in the properties of logarithms since, if we take logarithms of both sides (base 10 will do here), we obtain

   $$\log_{10} y = \log_{10} a + b \log_{10} x.$$

   Letting $\log_{10} y = Y$ and $\log_{10} x = X$, we have

   $$Y = \log_{10} a + bX,$$

   so that a straight line will be obtained by plotting Y against X.

5. $y = ab^x$.

   Method

   Here again, logarithms may be used to give

   $$\log_{10} y = \log_{10} a + x \log_{10} b$$

   Letting $\log_{10} y = Y$, we have

   $$Y = \log_{10} a + x \log_{10} b,$$

   which will give a straight line if we plot Y against x.

6. $y = ae^{bx}$.

   Method

   In this case, it makes sense to take **natural** logarithms of both sides to give

   $$\log_e y = \log_e a + bx,$$

   which may also be written

   $$\ln y = \ln a + bx$$

   Hence, letting $\ln y = Y$, we can obtain a straight line by plotting a graph of Y against x.

Note:

In all six of the above examples, it is even more important **not** to try to read off the gradient and the intercept from the graph drawn. As before, we should take two sets of readings for x (or X) and y (or Y), substitute them in the straight-line form of the equation and solve two simultaneous linear equations for the constants required.

5.3.3 THE USE OF LOGARITHMIC GRAPH PAPER

In Examples 4,5 and 6 in the previous section, it can be very tedious looking up on a calculator the logarithms of large sets of numbers. We may use, instead, a special kind of graph paper on which there is printed a logarithmic scale (see Unit 1.4) along one or both of the axis directions.

0.1 0.2 0.3 0.4 1 2 3 4 10

Effectively, the logarithmic scale has already looked up the logarithms of the numbers assigned to it provided these numbers are allocated to each **"cycle"** of the scale in successive powers of 10.

Data which includes numbers spread over several different successive powers of ten will need graph paper which has at least that number of cycles in the appropriate axis direction.

For example, the numbers 0.03, 0.09, 0.17, 0.33, 1.82, 4.65, 12, 16, 20, 50 will need **four** cycles on the logarithmic scale.

Accepting these restrictions, which make logarithmic graph paper less economical to use than ordinary graph paper, all we need to do is to plot the **actual** values of the variables whose logarithms we would otherwise have needed to look up. This will give the straight line graph from which we take the usual two sets of readings; these are then substituted into the form of the experimental equation which occurs immediately after taking logarithms of both sides.

It will not matter which base of logarithms is being used since logarithms to two different bases are proportional to each other anyway. The logarithmic graph paper does not, therefore, specify a base.

EXAMPLES

1. $y = ax^b$.

   Method

   (i) Taking logarithms (base 10), $\log_{10} y = \log_{10} a + b \log_{10} x$.

   (ii) Plot a graph of y against x, both on logarithmic scales.

   (iii) Estimate the position of the "best straight line".

   (iv) Read off from the graph two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned}\log_{10} y_1 &= \log_{10} a + b \log_{10} x_1, \\ \log_{10} y_2 &= \log_{10} a + b \log_{10} x_2.\end{aligned}$$

If it is possible to choose readings which are powers of 10, so much the better, but this is not essential.

2. $y = ab^x$.

Method

(i) Taking logarithms (base 10), $\log_{10} y = \log_{10} a + x \log_{10} b$.

(ii) Plot a graph of y against x with y on a logarithmic scale and x on a linear scale.

(iii) Estimate the position of the best straight line.

(iv) Read off from the graph two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned}\log_{10} y_1 &= \log_{10} a + x_1 \log_{10} b, \\ \log_{10} y_2 &= \log_{10} a + x_2 \log_{10} b.\end{aligned}$$

If it is possible to choose zero for the x_1 value, so much the better, but this is not essential.

3. $y = ae^{bx}$.

Method

(i) Taking natural logarithms, $\ln y = \ln a + bx$.

(ii) Plot a graph of y against x with y on a logarithmic scale and x on a linear scale.

(iii) Estimate the position of the best straight line.

(iv) Read off two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned}\ln y_1 &= \ln a + bx_1, \\ \ln y_2 &= \ln a + bx_2.\end{aligned}$$

If it possible to choose zero for the x_1 value, so much the better, but this is not essential.

5.3.5 EXERCISES

In these exercises, use logarithmic graph paper where possible.

1. The following values of x and y can be represented approximately by the law $y = a+bx^2$:

x	0	2	4	6	8	10
y	7.76	11.8	24.4	43.6	71.2	107.0

   Use a straight line graph to find approximately the values of a and b.

2. The following values of x and y are assumed to follow the law $y = ab^x$:

x	0.2	0.4	0.6	0.8	1.4	1.8
y	0.508	0.645	0.819	1.040	2.130	3.420

   Use a straight line graph to find approximately the values of a and b.

3. The following values of x and y are assumed to follow the law $y = ae^{kx}$:

x	0.2	0.5	0.7	1.1	1.3
y	1.223	1.430	1.571	1.921	2.127

   Use a straight line graph to find approximately the values of a and k.

4. The table below gives the pressure, P, and the volume, V, of a certain quantity of steam at maximum density:

P	12.27	17.62	24.92	34.77	47.87	65.06
V	3,390	2,406	1,732	1,264	934.6	699.0

   Assuming that $PV^n = C$, use a straight line graph to find approximately the values of n and C.

5. The coefficient of self induction, L, of a coil, and the number of turns, N, of wire are related by the formula $L = aN^b$, where a and b are constants.
   For the following pairs of observed values, use a straight line graph to calculate approximate values of a and b:

N	25	35	50	75	150	200	250
L	1.09	2.21	5.72	9.60	44.3	76.0	156.0

6. Measurements taken, when a certain gas undergoes compression, give the following values of pressure, p, and temperature, T:

p	10	15	20	25	35	50
T	270	289	303	315	333	353

   Assuming a law of the form $T = ap^n$, use a straight line graph to calculate approximately the values of a and n. Hence estimate the value of T when $p = 32$.

5.3.6 ANSWERS TO EXERCISES

The following answers are approximate; check only that the order of your results are correct. Any slight variations in the position of your straight line could affect the result considerably.

1. $a \simeq 8.0$, $b \simeq 0.99$
2. $a \simeq 0.4$, $b \simeq 3.3$
3. $a \simeq 1.1$, $k \simeq 0.5$
4. $n \simeq 1.06$, $C \simeq 65887$
5. $a \simeq 1.38 \times 10^{-3}$, $b \simeq 2.08$
6. $a \simeq 183.95$, $n \simeq 0.17$

"JUST THE MATHS"

UNIT NUMBER

5.4

GEOMETRY 4
(Elementary linear programming)

by

A.J.Hobson

UNIT 5.4 - GEOMETRY 4

ELEMENTARY LINEAR PROGRAMMING

5.4.1 FEASIBLE REGIONS

(i) The equation, $y = mx + c$, of a straight line is satisfied only by points which lie on the line. But it is useful to investigate the conditions under which a point with co-ordinates (x, y) may lie on one side of the line or the other.

(ii) For example, the inequality $y < mx + c$ is satisfied by points which lie **below** the line and the inequality $y > mx + c$ is satisfied by points which lie **above** the line.

(iii) Linear inequalities of the form $Ax+By+C < 0$ or $Ax+By+C > 0$ may be interpretted in the same way by converting, if necessary, to one of the forms in (ii).

(iv) Weak inequalities of the form $Ax + By + C \leq 0$ or $Ax + By + C \geq 0$ include the points which lie on the line itself as well as those lying on one side of it.

(v) Several simultaneous linear inequalties may be used to determine a region of the xy-plane throughout which all of the inequalities are satisfied. The region is called the **"feasible region"**.

EXAMPLES

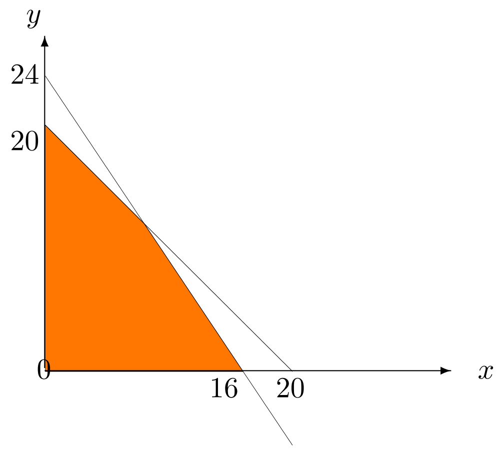
1. Determine the feasible region for the simultaneous inequalities

$$x \geq 0, \ y \geq 0, \ x + y \leq 20, \ \text{and } 3x + 2y \leq 48$$

Solution

We require the points of the first quadrant which lie on or below the straight line $y = 20 - x$ and on or below the straight line $y = -\frac{3}{2}x + 16$.

The feasible region is shown as the shaded area in the following diagram:



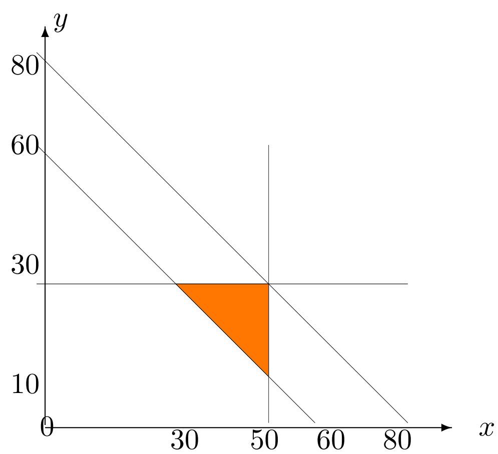
2. Determine the feasible region for the following simultaneous inequalities:

$$0 \le x \le 50,\ 0 \le y \le 30,\ x + y \le 80,\ x + y \ge 60$$

Solution

We require the points which lie on or to the left of the straight line $x = 50$, on or below the straight line $y = 30$, on or below the straight line $y = 80 - x$ and on or above the straight line $y = 60 - x$.

The feasible region is shown as the shaded area in the following diagram:



5.4.2 OBJECTIVE FUNCTIONS

An important application of the feasible region discussed in the previous section is that of maximising (or minimising) a linear function of the form $px + qy$ subject to a set of simultaneous linear inequalities. Such a function is known as an **"objective function"**

Essentially, it is required that a straight line with gradient $-\frac{p}{q}$ is moved across the appropriate feasible region until it reaches the highest possible point of that region for a maximum value or the lowest possible point for a minimum value. This will imply that the straight line $px + qy = r$ is such that r is the optimum value required.

However, for convenience, it may be shown that the optimum value of the objective function always occurs at one of the corners of the feasible region so that we simply evaluate it at each corner and choose the maximum (or minimum) value.

EXAMPLES

1. A farmer wishes to buy a number of cows and sheep. Cows cost £18 each and sheep cost £12 each.

   The farmer has accommodation for not more than 20 animals, and cannot afford to pay more than £288.

   If he can reasonably expect to make a profit of £11 per cow and £9 per sheep, how many of each should he buy in order to make his total profit as large as possible ?

   Solution

   Suppose he needs to buy x cows and y sheep; then, his profit is the objective function $P \equiv 11x + 9y$.

   Also,

   $$x \geq 0,\ \ y \geq 0,\ \ x + y \leq 20,\ \text{ and } 18x + 12y \leq 288 \text{ or } 3x + 2y \leq 48.$$

   Thus, we require to maximize $P \equiv 11x + 9y$ in the feasible region for the first example of the previous section.

   The corners of the region are the points $(0, 0)$, $(16, 0)$, $(0, 20)$ and $(8, 12)$, the last of these being the point of intersection of the two straight lines $x+y = 20$ and $3x+2y = 48$.

   The maximum value occurs at the point $(8, 12)$ and is equal to $88 + 108 = 196$. Hence, the farmer should buy 8 cows and 12 sheep.

2. A cement manufacturer has two depots, D_1 and D_2, which contain current stocks of 80 tons and 20 tons of cement respectively.

   Two customers C_1 and C_2 place orders for 50 and 30 tons respectively.

   The transport cost is £1 per ton, per mile and the distances, in miles, between D_1, D_2, C_1 and C_2 are given by the following table:

	C_1	C_2
D_1	40	30
D_2	10	20

   From which depots should the orders be dispatched in order to minimise the transport costs ?

   Solution

   Suppose that D_1 distributes x tons to C_1 and y tons to C_2; then D_2 must distribute $50 - x$ tons to C_1 and $30 - y$ tons to C_2.

   All quantities are positive and the following inequalties must be satisfied:

$$x \leq 50,\ \ y \leq 30,\ \ x + y \leq 80,\ \ 80 - (x + y) \leq 20 \text{ or } x + y \geq 60$$

   The total transport costs, T, are made up of $40x,\ 30y,\ 10(50 - x)$ and $20(30 - y)$.

   That is,

$$T \equiv 30x + 10y + 1100,$$

   and this is the objective function to be minimised.

   From the diagram in the second example of the previous section, we need to evaluate the objective function at the points $(30, 30),\ (50, 30)$ and $(50, 10)$.

   The minimum occurs, in fact, at the point $(30, 30)$ so that D_1 should send 30 tons to C_1 and 30 tons to C_2 while D_2 should send 20 tons to C_1 but 0 tons to C_2.

5.4.3 EXERCISES

1. Sketch, on separate diagrams, the regions of the xy-plane which correspond to the following inequalities (assuming that $x \geq 0$ and $y \geq 0$):

   (a)
   $$x + y \leq 6;$$
   (b)
   $$x + y \geq 4;$$
   (c)
   $$3x + y \geq 6;$$
   (d)
   $$x + 3y \geq 6.$$

2. Sketch the feasible region for which all the inequalities in question 1 are satisfied.

3. Maximise the objective function $5x + 7y$ subject to the simultaneous linear inequalities
   $$x \geq 0,\ y \geq 0,\ 3x + 2y \geq 6 \text{ and } x + y \leq 4.$$

4. A mine manager has contracts to supply, weekly,

   100 tons of grade 1 coal,
   700 tons of grade 2 coal,
   2000 tons of grade 3 coal,
   4500 tons of grade 4 coal.

   Two seams, A and B, are being worked at a cost of £4000 and £10,000, respectively, per shift, and the yield, in tons per shift, from each seam is given by the following table:

	Grade 1	Grade 2	Grade 3	Grade 4
A	200	100	200	400
B	100	100	500	1500

   How many shifts per week should each seam be worked, in order to fulfill the contracts most economically ?

5. A manufacturer employs 5 skilled and 10 semi-skilled workers to make an article in two qualities, standard and deluxe.

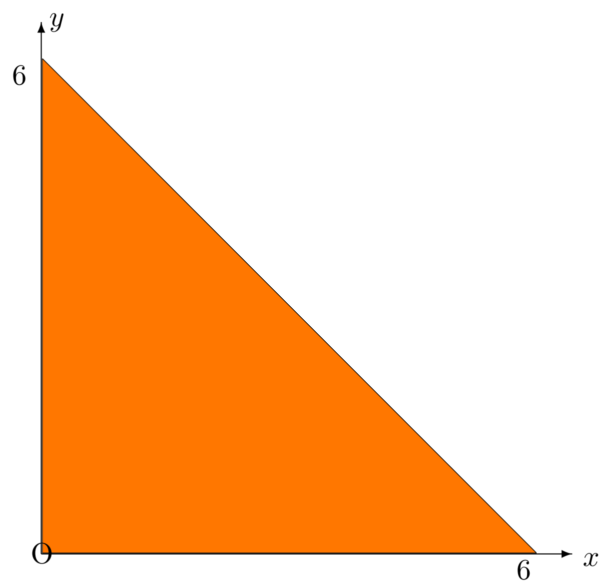
   The deluxe model requires 2 hour's work by skilled workers; the standard model requires 1 hour's work by skilled workers and 3 hour's work by semi-skilled workers.

   No worker works more than 8 hours per day and profit is £10 on the deluxe model and £8 on the standard model.

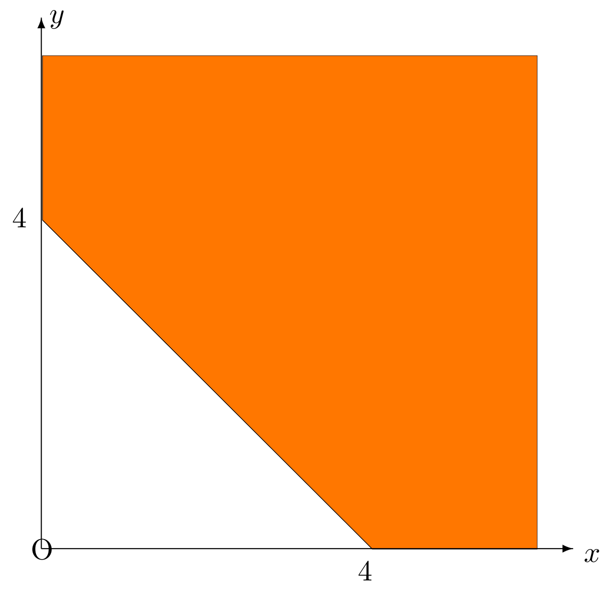
   How many of each type, per day, should be made in order to maximise profits ?

5.4.4 ANSWERS TO EXERCISES

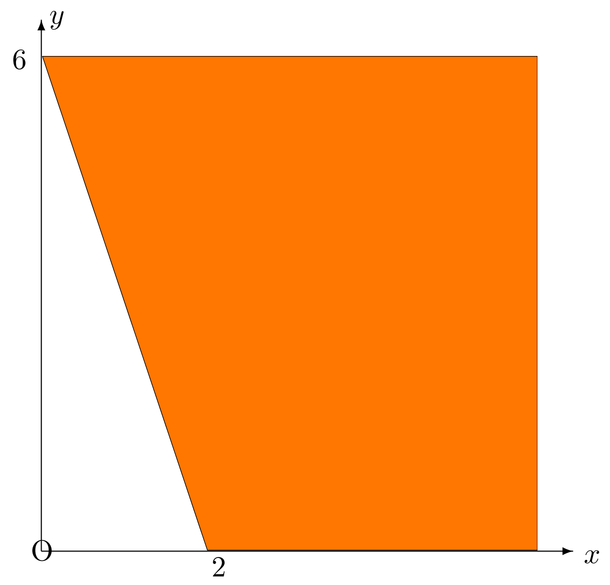
1. (a) The region is as follows:



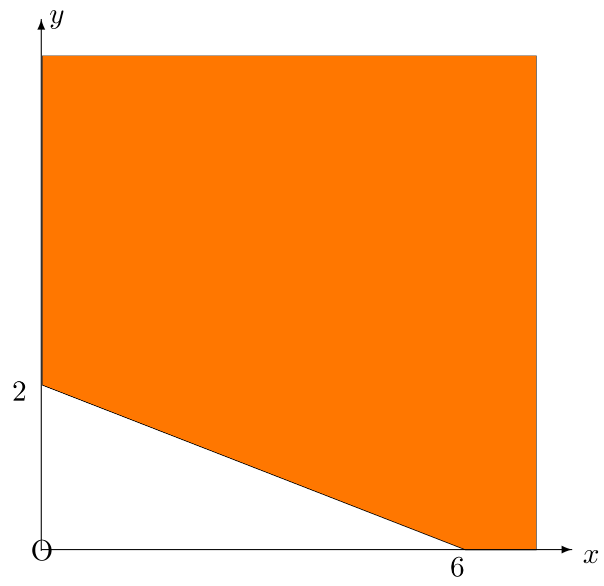
(b) The region is as follows:



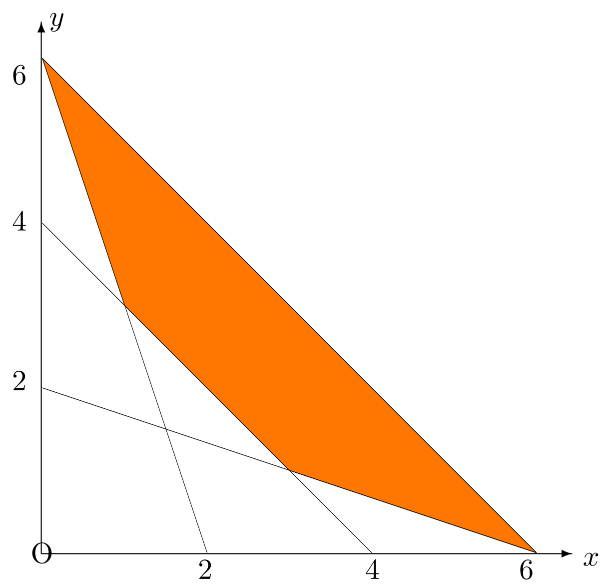
(c) The region is as follows:



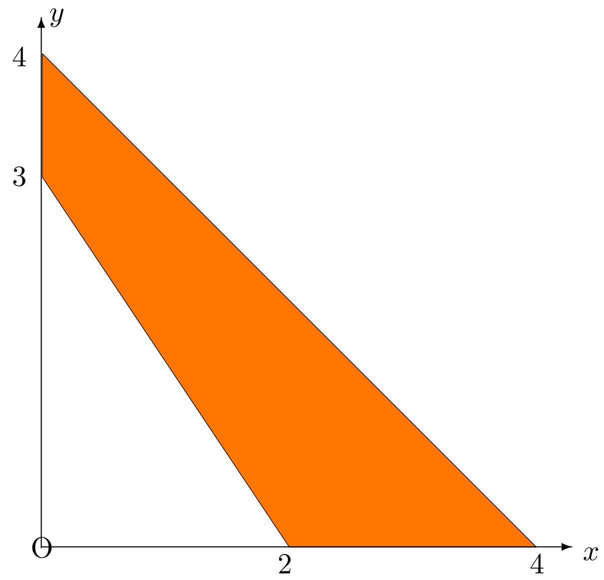
(d) The region is as follows:



2. The feasible region is as follows:



3. The feasible region is as follows:



The maximum value of $5x + 7y$ occurs at the point $(0, 4)$ and is equal to 28.

4. Subject to the simultaneous inequalities

$$x \geq 0,\ y \geq 0,\ 2x + y \geq 10,\ x + y \geq 7,\ 2x + 5y \geq 20 \text{ and } 4x + 5y \geq 45,$$

the function $2x + 5y$ has minimum value 20 at any point on the line $2x + 5y = 20$.

5. Subject to the simultaneous inequalities

$$x \geq 0,\ y \geq 0,\ x + 2y \leq 0 \text{ and } 3x + 2y \leq 80,$$

the objective function $P \equiv 8x + 10y$ has maximum value 260 at the point $(20, 10)$.

“JUST THE MATHS”

UNIT NUMBER

5.5

GEOMETRY 5
(Conic sections - the circle)

by

A.J.Hobson

UNIT 5.5 - GEOMETRY 5

CONIC SECTIONS - THE CIRCLE

5.5.1 INTRODUCTION

In this and the following three units, we shall investigate some of the geometry of four standard curves likely to be encountered in the scientific applications of Mathematics. They are the Circle, the Parabola, the Ellipse and the Hyperbola.

These curves could be generated, if desired, by considering plane sections through a cone; and, because of this, they are often called **"conic sections"** or even just **"conics"**. We shall not discuss this interpretation further, but rather use a more analytical approach.

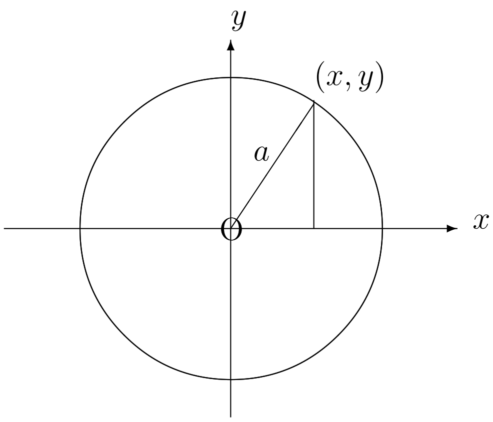
The properties of the four standard conics to be included here will be restricted to those required for simple applications work and, therefore, these notes will not provide an extensive course on elementary co-ordinate geometry.

Useful results from previous work which will be used in these units include the Change of Origin technique (Unit 5.2) and the method of Completing the Square (Unit 1.5). These results should be reviewed, if necessary, by the student.

DEFINITION
A circle is the path traced out by (or **"locus"** of) a point which moves at a fixed distance, called the **"radius"**, from a fixed point, called the **"centre"**.

5.5.2 STANDARD EQUATIONS FOR A CIRCLE

(a) Circle with centre at the origin and having radius a.



Using Pythagoras's Theorem in the diagram, the equation which is satisfied by every point (x, y) on the circle, but no other points in the plane of the axes, is

$$x^2 + y^2 = a^2.$$

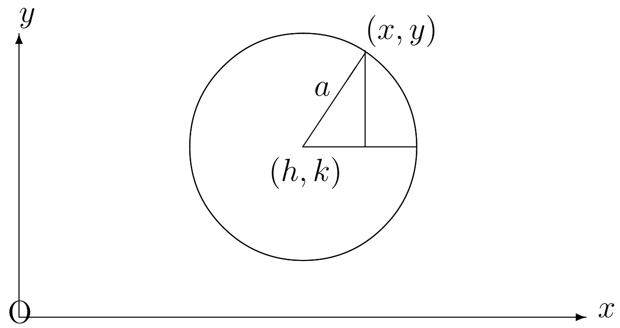
This is therefore the cartesian equation of a circle with centre $(0, 0)$ and radius a.

Note:
The angle θ in the diagram could be used as a parameter for the point (x, y) to give the parametric equations

$$x = a\cos\theta, \quad y = a\sin\theta.$$

Each point on the curve has infinitely many possible parameter values, all differing by a multiple of 2π; but it is usually most convenient to choose the value which lies in the interval $-\pi < \theta \leq \pi$.

(b) Circle with centre (h, k) having radius a.



If we were to consider a temporary change of origin to the point (h, k) with X-axis and Y-axis, the circle would have equation

$$X^2 + Y^2 = a^2,$$

with reference to the new axes. But, from previous work,

$$X = x - h \text{ and } Y = y - k.$$

Hence, with reference to the original axes, the circle has equation

$$(x - h)^2 + (y - k)^2 = a^2;$$

or, in its expanded form,

$$x^2 + y^2 - 2hx - 2ky + c = 0,$$

where

$$c = h^2 + k^2 - a^2.$$

Notes:
(i) The parametric equations of this circle with reference to the temporary new axes through the point (h, k) would be

$$X = a\cos\theta, \quad Y = a\sin\theta.$$

Hence, the parametric equations of the circle with reference to the original axes are

$$x = h + a\cos\theta, \quad y = k + a\sin\theta.$$

(ii) If the equation of a circle is encountered in the form

$$(x - h)^2 + (y - k)^2 = a^2,$$

it is very easy to identify the centre, (h, k) and the radius, a. If the equation is encountered in its expanded form, the best way to identify the centre and radius is to **complete the square in the x and y terms** in order to return to the first form.

EXAMPLES

1. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$x^2 + y^2 + 4x + 6y + 4 = 0.$$

Solution

Completing the square in the x terms,

$$x^2 + 4x \equiv (x + 2)^2 - 4.$$

Completing the square in the y terms,

$$y^2 + 6y \equiv (y + 3)^2 - 9.$$

The equation of the circle therefore becomes

$$(x + 2)^2 + (y + 3)^2 = 9.$$

Hence the centre is the point $(-2, -3)$ and the radius is 3.

2. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$5x^2 + 5y^2 - 10x + 15y + 1 = 0.$$

Solution

Here it is best to divide throughout by the coefficient of the x^2 and y^2 terms, even if some of the new coefficients become fractions. We obtain

$$x^2 + y^2 - 2x + 3y + \frac{1}{5} = 0.$$

Completing the square in the x terms,

$$x^2 - 2x \equiv (x - 1)^2 - 1.$$

Completing the square in the y terms,

$$y^2 + 3y \equiv \left(y + \frac{3}{2}\right)^2 - \frac{9}{4}.$$

The equation of the circle therefore becomes

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{61}{20}.$$

Hence the centre is the point $\left(1, -\frac{3}{2}\right)$ and the radius is $\sqrt{\frac{61}{20}} \cong 1.75$

Note:
Not every equation of the form

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

represents a circle because, for some combinations of h, k and c, the radius would not be a real number. In fact,

$$a = \sqrt{h^2 + k^2 - c},$$

which could easily turn out to be unreal.

5.5.3 EXERCISES

1. Write down the equation of the circle with centre $(4, -3)$ and radius 2.
2. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is
$$x^2 + y^2 - 2x + 4y - 11 = 0.$$
3. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is
$$36x^2 + 36y^2 - 36x - 24y - 131 = 0.$$
4. Determine the equation of the circle passing through the point $(4, -3)$ and having centre $(2, 1)$. What is the radius of the circle and what are its parametric equations ?
5. Use the parametric equations of the straight line joining the two points $(2, 4)$ and $-4, 2)$ to find its points of intersection with the circle whose equation is
$$x^2 + y^2 + 4x - 2y = 0.$$

   Hint:
   Substitute the parametric equations of the straight line into the equation of the circle and find two solutions for the parameter.

5.5.4 ANSWERS TO EXERCISES

1. The equation of the circle is either

$$(x-4)^2+(y+3)^2=4,$$

or

$$x^2+y^2-8x+6y+21=0.$$

2. The centre is $(1,-2)$ and the radius is 4.

3. The centre is $\left(\frac{1}{2},\frac{1}{3}\right)$ and radius is 2.

4. The equation is

$$(x-2)^2+(y-1)^2=20$$

and the radius is $\sqrt{20}$.
The parametric equations are

$$x=2+\sqrt{20}\cos\theta, \quad y=1+\sqrt{20}\sin\theta.$$

5. From

$$x=2-6t, \quad \text{and} \quad y=4-2t,$$

we obtain

$$40t^2-60t+20=0,$$

giving $t=\frac{1}{2}$ and $t=1$.
The points of intersection are $(-1,3)$ and $(-4,2)$.

"JUST THE MATHS"

UNIT NUMBER

5.6

GEOMETRY 6
(Conic sections - the parabola)

by

A.J.Hobson

UNIT 5.6 - GEOMETRY 6

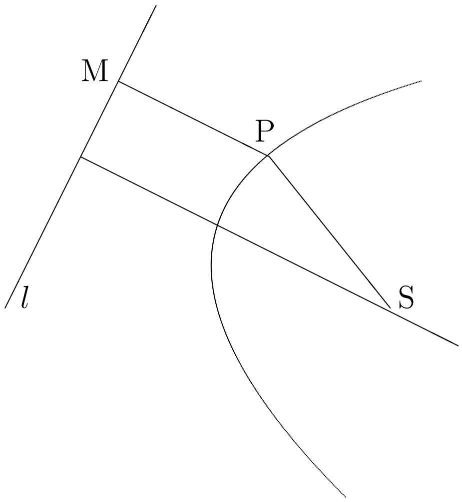
CONIC SECTIONS - THE PARABOLA

5.5.1 INTRODUCTION

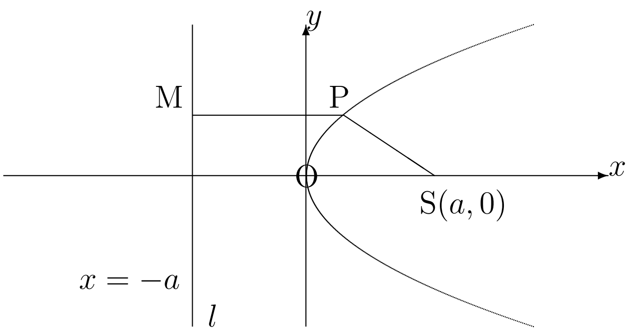
The Standard Form for the equation of a Parabola

DEFINITION

A parabola is the path traced out by (or **"locus"** of) a point, P, whose distance, SP, from a fixed point, S, called the **"focus"**, is equal to its perpendicular distance, PM, from a fixed line, l, called the **"directrix"**.



For convenience, we may take the directrix, l, to be a vertical line - with the perpendicular onto it from the focus, S, being the x-axis. We could take the y-axis to be the directrix itself, but the equation of the parabola turns out to be simpler if we use a different line; namely the line parallel to the directrix passing through the mid-point of the perpendicular from the focus onto the directrix. This point is one of the points on the parabola so that we make the curve pass through the origin.



Letting the focus be the point $(a, 0)$ (since it lies on the x-axis) the definition of the parabola implies that SP = PM. That is,

$$\sqrt{(x-a)^2 + y^2} = x + a.$$

Squaring both sides gives

$$(x-a)^2 + y^2 = x^2 + 2ax + a^2,$$

or

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2.$$

This reduces to

$$y^2 = 4ax$$

and is the standard equation of a parabola with **"vertex"** at the origin, focus at $(a, 0)$ and axis of symmetry along the x-axis. All other versions of the equation of a parabola which we shall consider will be based on this version.

Notes:
(i) If a is negative, the bowl of the parabola faces in the opposite direction towards negative x values.

(ii) Any equation of the form $y^2 = kx$, where k is a constant, represents a parabola with vertex at the origin and axis of symmetry along the x-axis. Its focus will lie at the point $\left(\frac{k}{4}, 0\right)$; it is worth noting this observation for future reference.

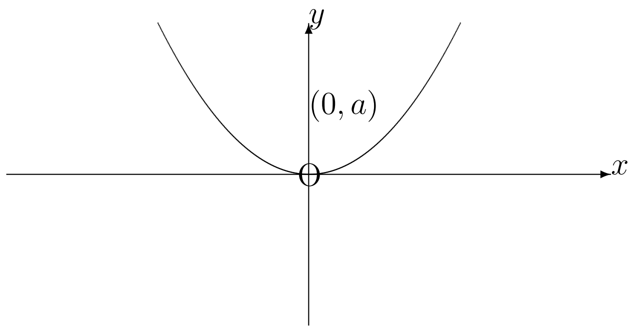
(iii) The parabola $y^2 = 4ax$ may be represented parametrically by the pair of equations

$$x = at^2, \quad y = 2at;$$

but the parameter, t, has no significance in the diagram such as was the case for the circle.

5.6.2 OTHER FORMS OF THE EQUATION OF A PARABOLA

(a) Vertex at $(0, 0)$ with focus at $(0, a)$



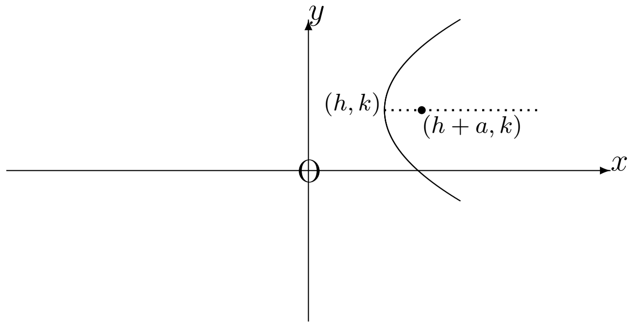
This parabola is effectively the same as the standard one except that the roles of x and y have been reversed. We may assume, therefore that the curve has equation

$$x^2 = 4ay$$

with associated parametric equations

$$x = 2at, \quad y = at^2$$

(b) Vertex at (h, k) with focus at $(h + a, k)$



If we were to consider a temporary change of origin to the point (h, k), with X-axis and Y-axis, the parabola would have equation

$$Y^2 = 4aX.$$

With reference to the original axes, therefore, the parabola has equation

$$(y - k)^2 = 4a(x - h).$$

Notes:
(i) Such a parabola will often be encountered in the expanded form of this equation, containing quadratic terms in y and linear terms in x. Conversion to the stated form by completing the square in the y terms will make it possible to identify the vertex and focus.

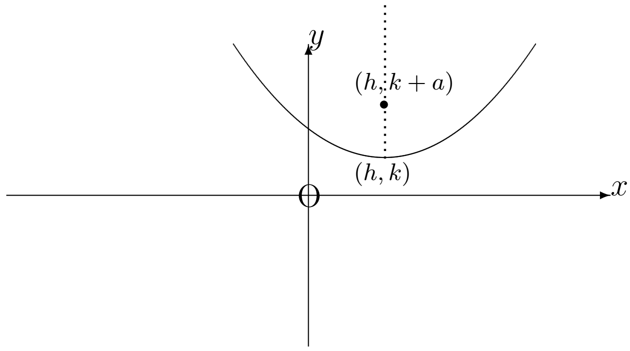
(ii) With reference to the new axes with origin at the point (h, k), the parametric equations of the parabola would be

$$X = at^2, \quad Y = 2at.$$

Hence, with reference to the original axes, the parametric equations are

$$x = h + at^2, \quad y = k + 2at.$$

(c) Vertex at (h, k) with focus at $(h, k + a)$



If we were to consider a temporary change of origin to the point (h, k) with X-axis and Y-axis, the parabola would have equation

$$X^2 = 4aY.$$

With reference to the original axes, therefore, the parabola has equation

$$(x - h)^2 = 4a(y - k).$$

Notes:
(i) Such a parabola will often be encountered in the expanded form of this equation, containing quadratic terms in x and linear terms in y. Conversion to the stated form by completing the square in the x terms will make it possible to identify the vertex and focus.

(ii) With reference to the new axes with origin at the point (h, k), the parametric equations of the parabola would be

$$X = 2at, \quad Y = at^2.$$

Hence, with reference to the original axes, the parametric equations are

$$x = h + 2at, \quad y = k + at^2.$$

EXAMPLES

1. Give a sketch of the parabola whose cartesian equation is

$$y^2 - 6y + 3x = 10,$$

showing the co-ordinates of the vertex, focus and intesections with the x-axis and y-axis.

Solution

First, we must complete the square in the y terms obtaining

$$y^2 - 6y \equiv (y - 3)^2 - 9.$$

Hence, the equation becomes

$$(y-3)^2 - 9 + 3x = 10.$$

That is,

$$(y-3)^2 = 19 - 3x,$$

or

$$(y-3)^2 = 4.\left(-\frac{3}{4}\right)\left(x - \frac{19}{3}\right).$$

Thus, the vertex lies at the point $\left(\frac{19}{3}, 3\right)$ and the focus lies at the point $\left(\frac{19}{3} - \frac{3}{4}, 3\right)$; that is, $\left(\frac{67}{12}, 3\right)$.

The parabola intersects the x-axis where $y = 0$, i.e.

$$3x = 10,$$

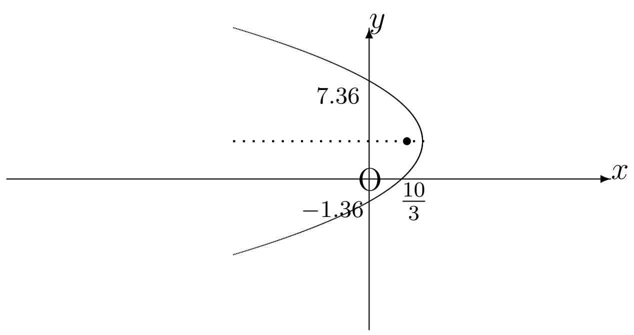
giving $x = \frac{10}{3}$.

The parabola intersects the y-axis where $x = 0$; that is,

$$y^2 - 6y - 10 = 0,$$

which is a quadratic equation with solutions

$$y = \frac{6 \pm \sqrt{36+40}}{2} \cong 7.36 \ \text{ or } \ -1.36$$



2. Use the parametric equations of the parabola

$$x^2 = 8y$$

to determine its points of intesection with the straight line

$$y = x + 6.$$

Solution

The parametric equations are $x = 4t$, $y = 2t^2$.

Substituting these into the equation of the straight line, we have

$$2t^2 = 4t + 6.$$

That is,

$$t^2 - 2t - 3 = 0,$$

or

$$(t-3)(t+1) = 0,$$

which is a quadratic equation in t having solutions $t = 3$ and $t = -1$.

The points of intersection are therefore $(12, 18)$ and $(-4, 2)$.

5.6.3 EXERCISES

1. For the following parabolae, determine the co-ordinates of the vertex, the focus and the points of intersection with the x- axis and y-axis where appropriate:

   (a)
   $$(y-1)^2 = 4(x-2);$$

   (b)
   $$(x+1)^2 = 8(y-3);$$

   (c)
   $$2x = y^2 + 4y + 6;$$

   (d)
   $$x^2 + 4x - 4y + 6 = 0.$$

2. Use the parametric equations of the parabola
   $$y^2 = 12x$$
   to determine its points of intersection with the straight line
   $$6x + 5y - 12 = 0.$$

5.6.4 ANSWERS TO EXERCISES

1. (a) Vertex $(2, 1)$, Focus $(3, 1)$, Intersection $(\frac{9}{4}, 0)$ with the x-axis;
   (b) Vertex $(-1, 3)$, Focus $(-1, 5)$, Intersection $(0, \frac{25}{8})$ with the y-axis;
   (c) Vertex $(1, -2)$, Focus $(\frac{3}{2}, -2)$, Intersection $(3, 0)$ with the x-axis;
   (d) Vertex $(-2, \frac{1}{2})$, Focus $(-2, \frac{3}{2})$, Intersection $(0, \frac{3}{2})$ with the y-axis.

2. $x = 3t^2$ and $y = 6t$ give $18t^2 + 30t - 12 = 0$ with solutions $t = \frac{1}{3}$ and $t = -2$. Hence the points of intersection are $(\frac{1}{3}, 2)$ and $(12, -12)$.

"JUST THE MATHS"

UNIT NUMBER

5.7

GEOMETRY 7
(Conic sections - the ellipse)

by

A.J.Hobson

UNIT 5.7 - GEOMETRY 7

CONIC SECTIONS - THE ELLIPSE

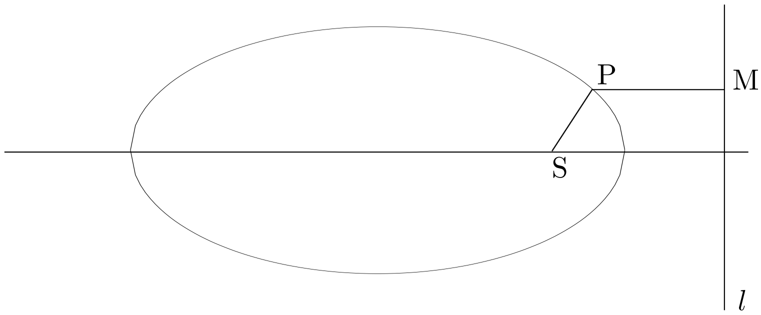
5.7.1 INTRODUCTION

The Standard Form for the equation of an Ellipse

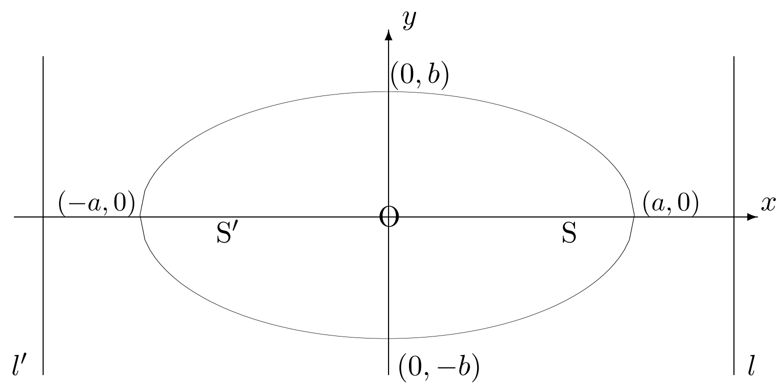
DEFINITION
The Ellipse is the path traced out by (or **"locus** of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

where $\epsilon < 1$ is a constant called the **"eccentricity"** of the ellipse. The fixed line, l, is called a **"directrix"** of the ellipse and the fixed point, S, is called a **"focus"** of the ellipse.



In fact, there are two foci and two directrices because the curve turns out to be symmetrical about a line parallel to l and the perpendicular line from S onto l. The diagram below illustrates two foci, S and S$'$, together with two directrices, l and l'. The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a\cos\theta, \quad y = b\sin\theta.$$

The curve clearly intersects the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm b)$. Whichever is the larger of a and b defines the length of the **"semi-major axis"** and whichever is the smaller defines the length of the **"semi-minor axis"**.

For the sake of completeness, it may further be shown that the eccentricity, ϵ, is obtainable from the formula

$$b^2 = a^2\left(1 - \epsilon^2\right)$$

and, having done so, the foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$. However, in these units, students will not normally be expected to determine the eccentricity, foci or directrices of an ellipse.

5.7.2 A MORE GENERAL FORM FOR THE EQUATION OF AN ELLIPSE

The equation of an ellipse, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is easily obtainable from the standard form of equation by a temporary change of origin to the point (h, k). We obtain

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with associated parametric equations

$$x = h + a\cos\theta, \quad y = k + b\sin\theta.$$

Ellipses will usually be encountered in the expanded form of the above cartesian equation and it will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the ellipse. The expanded form will be similar in appearance to that of a circle but the coefficients of x^2 and y^2, though both of the same sign, will not be equal to each other.

EXAMPLE

Determine the co-ordinates of the centre and the lengths of the semi-axes of the ellipse whose equation is

$$3x^2 + y^2 + 12x - 2y + 1 = 0.$$

Solution

Completing the square in the x terms gives

$$3x^2 + 12x \equiv 3\left[x^2 + 4x\right] \equiv 3\left[(x+2)^2 - 4\right] \equiv 3(x+2)^2 - 12.$$

Completing the square in the y terms gives

$$y^2 - 2y \equiv (y-1)^2 - 1.$$

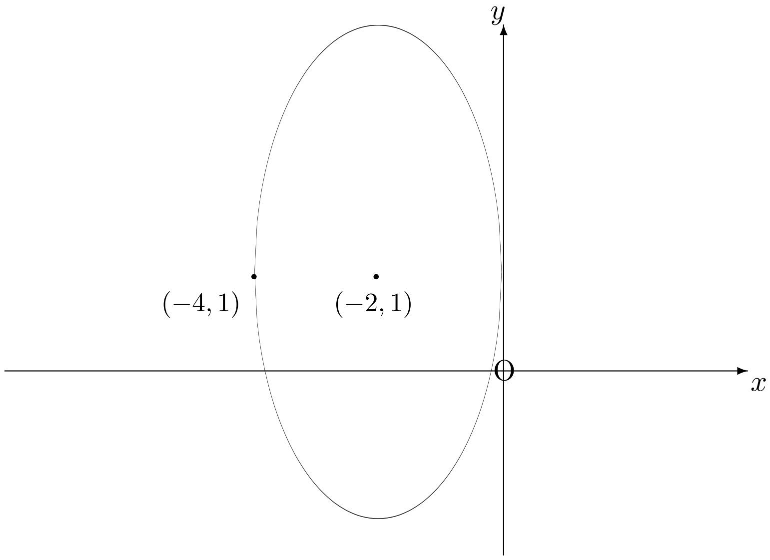
Hence, the equation of the ellipse becomes

$$3(x+2)^2 + (y-1)^2 = 12.$$

That is,

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{12} = 1.$$

The centre is thus located at the point $(-2, 1)$ and the semi-axes have lengths $a = 2$ and $b = \sqrt{12}$.



5.7.3 EXERCISES

1. For each of the following ellipses, determine the co-ordinates of the centre and give a sketch of the curve:

   (a)

   $$x^2 + 4y^2 - 4x - 8y + 4 = 0;$$

   (b)

   $$x^2 + 4y^2 + 16y + 12 = 0;$$

   (c)

   $$x^2 + 4y^2 + 6x - 8y + 9 = 0.$$

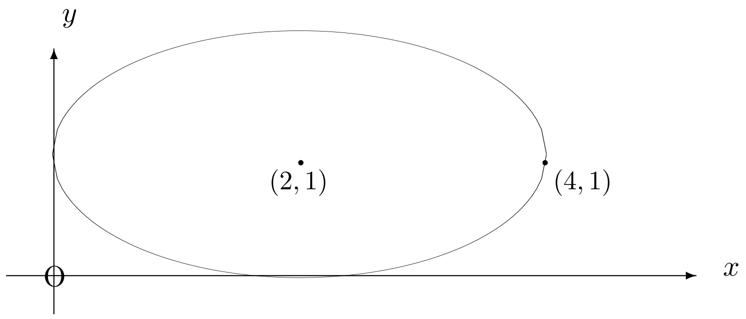
2. Determine the lengths of the semi-axes of the ellipse whose equation is

   $$9x^2 + 25y^2 = 225$$

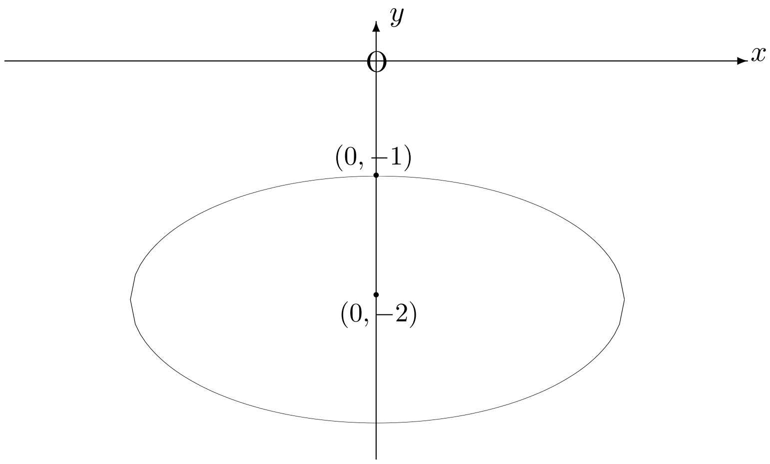
   and write down also a pair of parametric equations for this ellipse.

5.7.4 ANSWERS TO EXERCISES

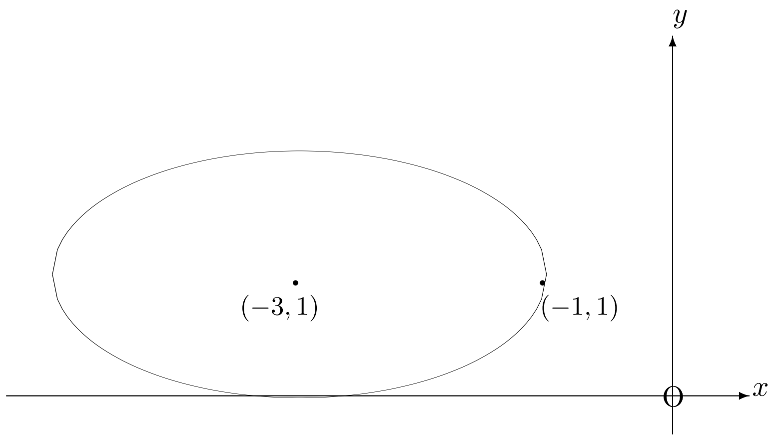
1. (a) Centre $(2, 1)$ with $a = 2$ and $b = 1$.



(b) Centre $(0, -2)$ with $a = 2$ and $b = 1$.



(c) Centre $(-3, 1)$ with $a = 2$ and $b = 1$.



2. $a = 5$ and $b = 3$, giving the parametric equations $x = 5\cos\theta$, $\quad y = 3\sin\theta$.

"JUST THE MATHS"

UNIT NUMBER

5.8

GEOMETRY 8
(Conic sections - the hyperbola)

by

A.J.Hobson

UNIT 5.8 - GEOMETRY 8

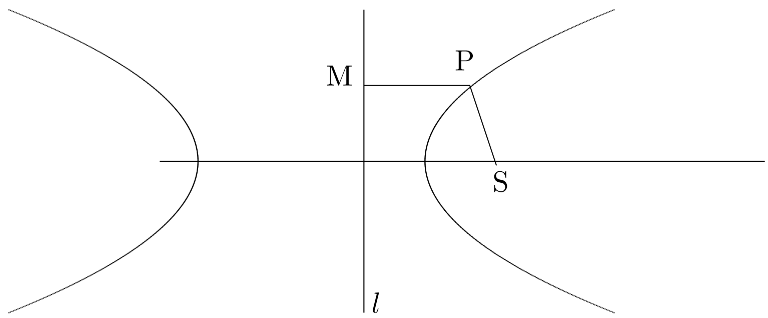
CONIC SECTIONS - THE HYPERBOLA

5.8.1 INTRODUCTION

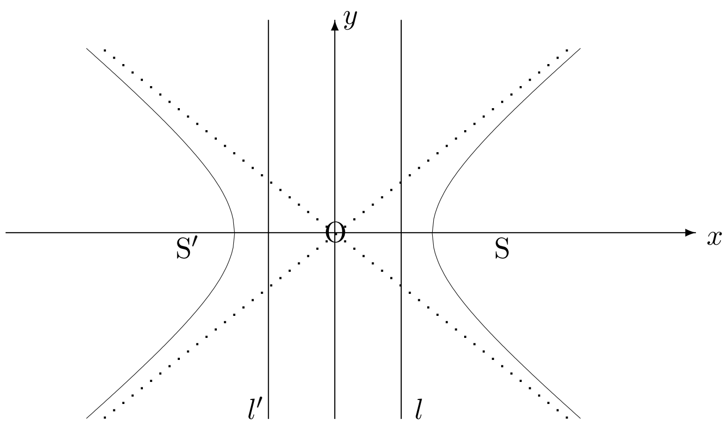
DEFINITION
The hyperbola is the path traced out by (or **"locus"** of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

where $\epsilon > 1$ is a constant called the **"eccentricity"** of the hyperbola. The fixed line, l, is called a **"directrix"** of the hyperbola and the fixed point, S, is called a **"focus"** of the hyperbola.



In fact, there are two foci and two directrices because the curve turns out to be symmetrical about a line parallel to l and the perpendicular line from S onto l. The diagram below illustrates two foci S and S$'$ together with two directrices l and l'. The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the hyperbola has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a\sec\theta, \quad y = b\tan\theta$$

although, for students who meet "hyperbolic functions", a better set of parametric equations would be

$$x = a\cosh t, \quad y = b\sinh t.$$

The curve clearly intersects the x-axis at $(\pm a, 0)$ but does not intersect the y-axis at all.

For the sake of completeness, it may further be shown that the eccentricity, ϵ, is obtainable from the formula

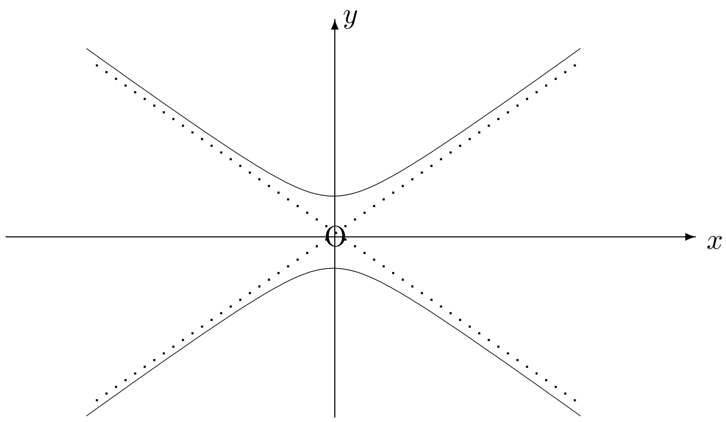
$$b^2 = a^2\left(\epsilon^2 - 1\right)$$

and, having done so, the foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$. However, in these units, students will not normally be expected to determine the eccentricity, foci or directrices of a hyperbola

Note:
A similar hyperbola to the one above, but intersecting the y-axis rather than the x-axis, has equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

The roles of x and y are simply reversed.



5.8.2 ASYMPTOTES

A special property of the hyperbola is that, at infinity, it approaches two straight lines through the centre of the hyperbola called **"asymptotes"**.

It can be shown that both of the hyperbolae

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

have asymptotes whose equations are:

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0.$$

These are easily obtained by factorising the **left hand side** of the equation of the hyperbola, then equating each factor to zero.

5.8.3 MORE GENERAL FORMS FOR THE EQUATION OF A HYPERBOLA

The equation of a hyperbola, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is easily obtainable from one of the standard forms of equation by a temporary change of origin to the point (h, k). We obtain either

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

with associated parametric equations

$$x = h + a\text{sec}\theta, \quad y = k + b\tan\theta$$

or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1,$$

with associated parametric equations

$$x = h + a\tan\theta, \quad y = k + b\text{sec}\theta.$$

Hyperbolae will usually be encountered in the expanded form of the above cartesian equations and it will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the hyperbola. The expanded form will be similar in appearance to that of a circle but the coefficients of x^2 and y^2 will be different numerically and opposite in sign.

EXAMPLE

Determine the co-ordinates of the centre and the equations of the asymptotes of the hyperbola whose equation is

$$4x^2 - y^2 + 16x + 6y + 6 = 0.$$

Solution

Completing the square in the x terms gives

$$4x^2 + 16x \equiv 4\left[x^2 + 4x\right] \equiv 4\left[(x+2)^2 - 4\right] \equiv 4(x+2)^2 - 16.$$

Completing the square in the y terms gives

$$-y^2 + 6y \equiv -\left[y^2 - 6y\right] \equiv -\left[(y-3)^2 - 9\right] \equiv -(y-3)^2 + 9.$$

Hence the equation of the hyperbola becomes

$$4(x+2)^2 - (y-3)^2 = 1$$

or

$$\frac{(x+2)^2}{\left(\frac{1}{2}\right)^2} - \frac{(y-3)^2}{1^2} = 1.$$

The centre is thus located at the point $(-2, 3)$.

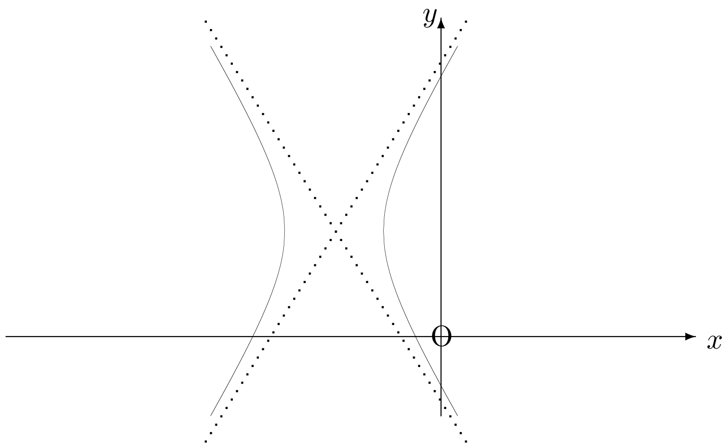
The asymptotes are best obtained by factorising the left hand side of the penultimate version of the equation of the hyperbola, then equating each factor to zero. We obtain

$$2(x+2) - (y-3) = 0 \quad \text{and} \quad 2(x+2) + (y-3) = 0.$$

In other words,

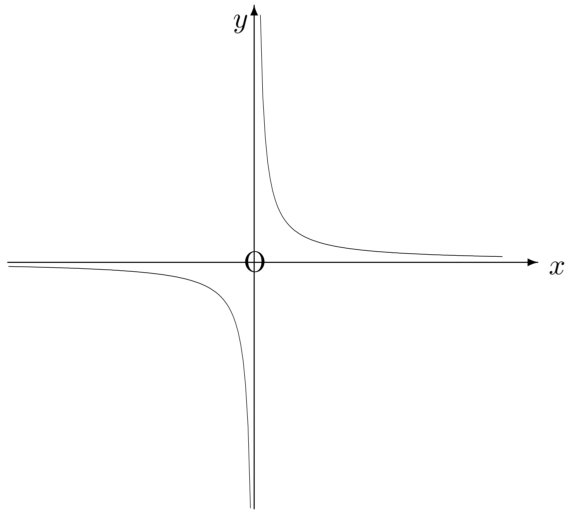
$$2x - y + 7 = 0 \quad \text{and} \quad 2x + y + 1 = 0.$$

To sketch the graph of a hyperbola, it is not always enough to have the position of the centre and the equations of the asymptotes. It may also be necessary to investigate some of the intersections of the curve with the co-ordinate axes. In the current example, by substituting first $y = 0$ and then $x = 0$ into the equation of the hyperbola, it is possible to determine intersections at $(-0.84, 0)$, $(-7.16, 0)$, $(0, -0.87)$ and $(0, 6.87)$.

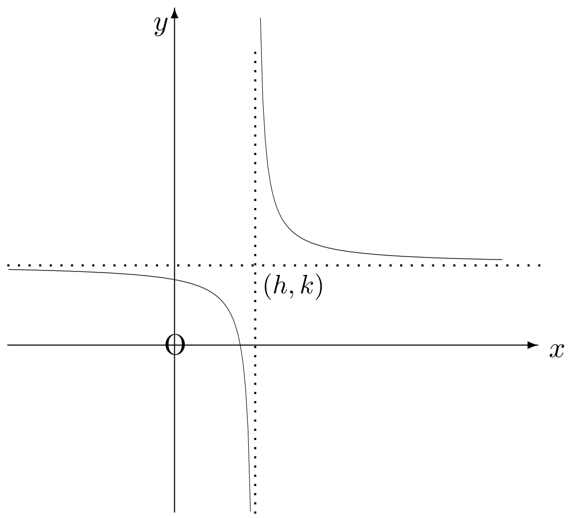


5.8.4 THE RECTANGULAR HYPERBOLA

For some hyperbolae, it will turn out that the asymptotes are at right-angles to each other; in which case, the **asymptotes themselves** could be used as the x-axis and y-axis. When this choice of reference system is made for a hyperbola with centre at the origin, it can be shown that hyperbola has the simpler equation $xy = C$, where C is a constant.



Similarly, for a rectangular hyperbola with centre at the point (h, k) and asymptotes used as the axes of reference, the equation will be $(x-h)(y-k) = C$.



Note:
A suitable pair of parametric equations for the rectangular hyperbola $(x-h)(y-k) = C$ are

$$x = t + h, \quad y = k + \frac{C}{t}.$$

EXAMPLES

1. Determine the centre of the rectangular hyperbola whose equation is

$$7x - 3y + xy - 31 = 0.$$

Solution

The equation factorises into the form

$$(x-3)(y+7) = 10.$$

Hence, the centre is located at the point $(3, -7)$.

2. A certain rectangular hyperbola has parametric equations

$$x = 1 + t, \quad y = 3 - \frac{1}{t}.$$

Determine its points of intersection with the straight line $x + y = 4$.

Solution

Substituting for x and y into the equation of the straight line, we obtain

$$1 + t + 3 - \frac{1}{t} = 4 \text{ or } t^2 - 1 = 0.$$

Hence, $t = \pm 1$ giving points of intersection at $(2, 2)$ and $(0, 4)$.

5.8.5 EXERCISES

1. For each of the following hyperbolae, determine the co-ordinates of the centre and the equations of the asympotes. Give a sketch of the curve, indicating where appropriate, the co-ordinates of its points of intersection with the x-axis and y-axis:

(a)

$$x^2 - y^2 - 2y = 0;$$

(b)

$$y^2 - x^2 - 6x = 10;$$

(c)

$$x^2 - y^2 - 2x - 2y = 4;$$

(d)

$$y^2 - x^2 - 6x + 4y = 14;$$

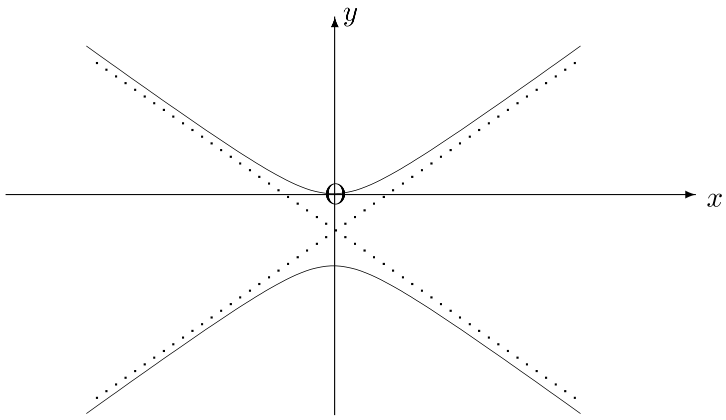
(e)

$$9x^2 - 4y^2 + 18x - 16y = 43.$$

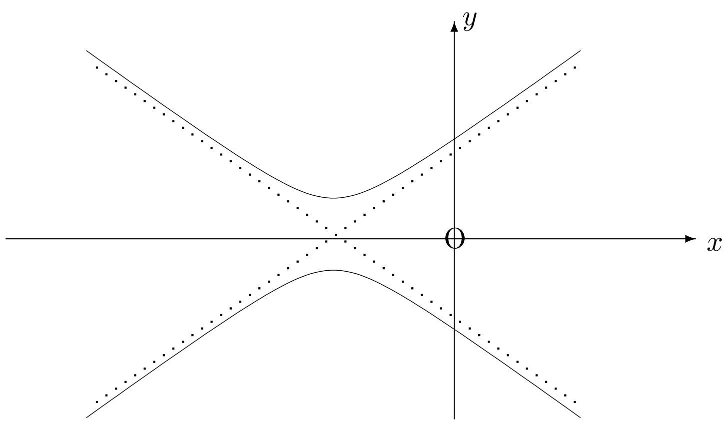
2. Determine a pair of parametric equations for the rectangular hyperbola whose equation is $xy - x + 2y - 6 = 0$ and hence obtain its points of intersection with the straight line $y = x + 3$. Sketch the hyperbola and the straight line on the same diagram.

5.8.6 ANSWERS TO EXERCISES

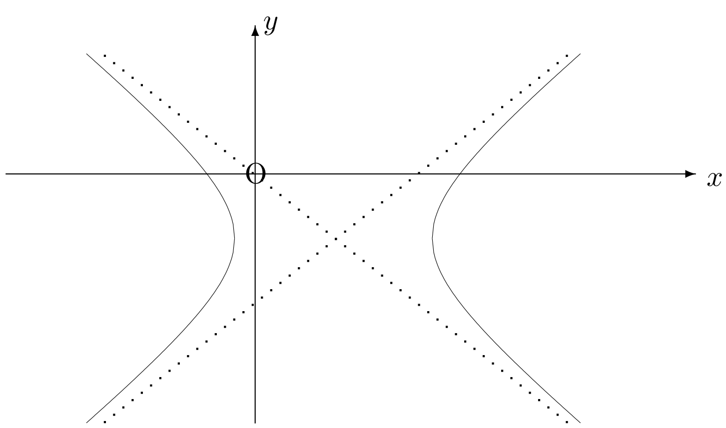
1. (a) Centre $(0, -1)$ with asymptotes $y = x - 1$ and $y = -x - 1$;



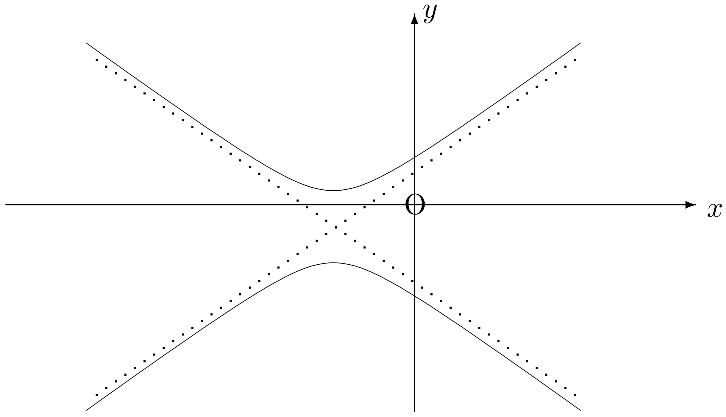
(b) Centre $(-3, 0)$ with asymptotes $y = x + 3$ and $y = -x - 3$;



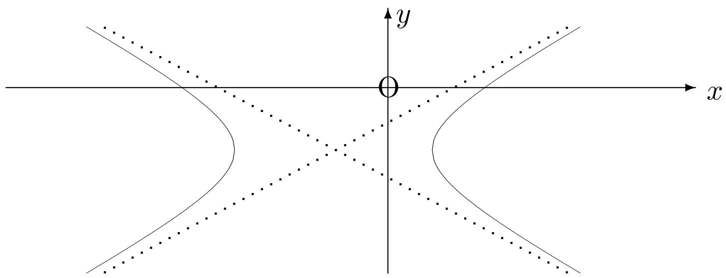
(c) Centre $(1, -1)$ with asymptotes $y = -x$ and $y = x - 2$;



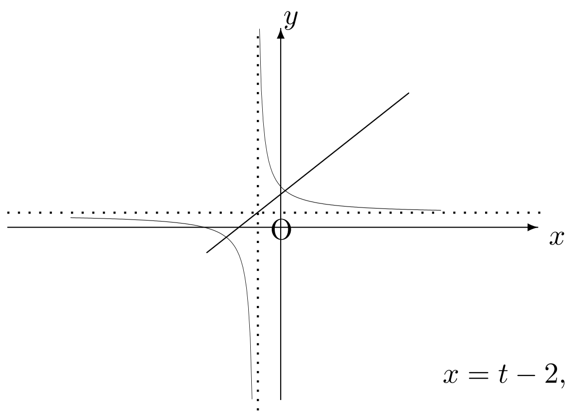
(d) Centre $(-3,-2)$ with asymptotes $y=x+1$ and $y=-x-5$;



(e) Centre $(-1,-2)$ with asymptotes $3x-2y=1$ and $3x+2y=-7$.



2. Centre $(-2,1)$ with asymptotes $x=-2$ and $y=1$. Intersections $(0,3)$ and $(-4,-1)$.



$x=t-2, \quad y=\frac{4}{t}+1$

"JUST THE MATHS"

UNIT NUMBER

5.9

GEOMETRY 9
(Curve sketching in general)

by

A.J.Hobson

UNIT 5.9 - GEOMETRY 9

CURVE SKETCHING IN GENERAL

Introduction

The content of the present section is concerned with those situations where it is desirable to find out the approximate shape of a curve whose equation is known, but not necessarily to determine an accurate "plot" of the curve.

In becoming accustomed to the points discussed below, the student should not feel that **every one** has to be used for a particular curve; merely enough of them to give a satisfactory impression of what the curve looks like.

5.9.1 SYMMETRY

A curve is symmetrical about the x-axis if its equation contains only even powers of y. It is symmetrical about the y-axis if its equation contains only even powers of x.

We say also that a curve is symmetrical with respect to the origin if its equation is unaltered when both x and y are changed in sign. In other words, if a point (x, y) lies on the curve, so does the point $(-x, -y)$.

ILLUSTRATIONS

1. The curve whose equation is
$$x^2\left(y^2 - 2\right) = x^4 + 4$$
is symmetrical about to both the x-axis and the y axis. This means that, once the shape of the curve is known in the first quadrant, the rest of the curve is obtained from this part by reflecting it in both axes.
The curve is also symmetrical with respect to the origin.

2. The curve whose equation is
$$xy = 5$$
is symmetrical with respect to the origin but not about either of the co-ordinate axes.

5.9.2 INTERSECTIONS WITH THE CO-ORDINATE AXES

Any curve intersects the x-axis where $y = 0$ and the y-axis where $x = 0$; but sometimes the curve has no intersection with one or more of the co-ordinate axes. This will be borne out by an inability to solve for x when $y = 0$ or for y when $x = 0$ (or both).

ILLUSTRATION

The circle

$$x^2 + y^2 - 4x - 2y + 4 = 0$$

meets the x-axis where

$$x^2 - 4x + 4 = 0.$$

That is,

$$(x - 2)^2 = 0,$$

giving a double intersection at the point $(2, 0)$. This means that the circle **touches** the x-axis at $(2, 0)$.

The circle meets the y-axis where

$$y^2 - 2y + 4 = 0.$$

That is,

$$(y - 1)^2 = -3,$$

which is impossible, since the left hand side is bound to be positive when y is a real number.

Thus, there are no intersections with the y-axis.

5.9.3 RESTRICTIONS ON THE RANGE OF EITHER VARIABLE

It is sometimes possible to detect a range of x values or a range of y values outside of which the equation of a curve would be meaningless in terms of real geometrical points of the cartesian diagram. Usually, this involves ensuring that neither x nor y would have to assume complex number values; but other kinds of restriction can also occur.

ILLUSTRATIONS

1. The curve whose equation is
$$y^2 = 4x$$
requires that x shall not be negative; that is, $x \geq 0$.
2. The curve whose equation is
$$y^2 = x\left(x^2 - 1\right)$$
requires that the right hand side shall not be negative; and from the methods of Unit 1.10, this will be so when either $x \geq 1$ or $-1 \leq x \leq 0$.

5.9.4 THE FORM OF THE CURVE NEAR THE ORIGIN

For small values of x (or y), the higher powers of the variable can often be usefully neglected to give a rough idea of the shape of the curve near to the origin.

This method is normally applied to curves which pass **through** the origin, although the behaviour near to other points can be considered by using a temporary change of origin.

ILLUSTRATION

The curve whose equation is

$$y = 3x^3 - 2x$$

approximates to the straight line

$$y = -2x$$

for very small values of x.

5.9.5 ASYMPTOTES

DEFINITION
An **"asymptote"** is a straight line which is approached by a curve at a very great distance from the origin.

(i) Asymptotes Parallel to the Co-ordinate Axes

Consider, by way of illustration, the curve whose equation is

$$y^2 = \frac{x^3(3-2y)}{x-1}.$$

(a) By inspection, we see that the straight line $x = 1$ "meets" this curve at an infinite value of y, making it an asymptote parallel to the y-axis.

(b) Now suppose we re-write the equation as

$$x^3 = \frac{y^2(x-1)}{3-2y}.$$

Inspection, this time, suggests that the straight line $y = \frac{3}{2}$ "meets" the curve at an infinite value of x, making it an asymptote parallel to the x axis.

(c) Another way of arriving at the conclusions in (a) and (b) is to write the equation of the curve in a form without fractions, namely

$$y^2(x-1) - x^3(3-2y) = 0,$$

then equate to zero the coefficients of the highest powers of x and y. That is,

The coefficient of y^2 gives $x - 1 = 0$.

The coefficient of x^3 gives $3 - 2y = 0$.

It can be shown that this method may be used with any curve to find asymptotes parallel to the co-ordinate axes. Of course, there may not be any, in which case the method will not work.

(ii) Asymptotes in General for a Polynomial Curve

This paragraph requires a fairly advanced piece of algebraical argument, but an outline proof will be included, for the sake of completeness.

Suppose a given curve has an equation of the form

$$P(x, y) = 0$$

where $P(x, y)$ is a polynomial in x and y.

Then, to find the intersections with this curve of a straight line

$$y = mx + c,$$

we substitute $mx + c$ in place of y into the equation of the curve, obtaining a polynomial equation in x, say

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n = 0.$$

In order for the line $y = mx + c$ to be an asymptote, this polynomial equation must have **coincident solutions at infinity**.

But now let us replace x by $\frac{1}{u}$ giving, after multiplying throughout by u^n, the new polynomial equation

$$a_0u^n + a_1u^{n-1} + a_2u^{n-2} + ... + a_{n-1}u + a_n = 0$$

This equation must have coincident solutions at $u = 0$ which will be the case provided

$$a_n = 0 \quad \text{and} \quad a_{n-1} = 0.$$

Conclusion

To find the asymptotes (if any) to a polynomial curve, we first substitute $y = mx + c$ into the equation of the curve. Then, in the polynomial equation obtained, we **equate to zero the two leading coefficents**; (that is, the coeffiecients of the highest two powers of x) and solve for m and c.

EXAMPLE

Determine the equations of the asymptotes to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Solution

Substituting $y = mx + c$ gives

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1.$$

That is,

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - \frac{2mcx}{b^2} - \frac{c^2}{b^2} - 1 = 0.$$

Equating to zero the two leading coefficients; that is, the coefficients of x^2 and x, we obtain

$$\frac{1}{a^2} - \frac{m^2}{b^2} = 0 \text{ and } \frac{2mc}{b^2} = 0.$$

No solution is obtainable if we let $m = 0$ in the second of these statements since it would imply $\frac{1}{a^2} = 0$ in the first statement, which is impossible. Therefore we must let $c = 0$ in the second statement, and $m = \pm\frac{b}{a}$ in the first statement.

The asymptotes are therefore

$$y = \pm\frac{b}{a}x.$$

In other words,

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ or } \frac{x}{a} + \frac{y}{b} = 0,$$

as used earlier in the section on the hyperbola.

5.9.6 EXERCISES

1. Sketch the graphs of the following equations:

   (a)
$$y = x + \frac{1}{x};$$

   (b)
$$y = \frac{1}{x^2+1};$$

   (c)
$$y^2 = \frac{x}{x-2};$$

   (d)
$$y = \frac{(x-1)(x+4)}{(x-2)(x-3)};$$

   (e)
$$y(x+2) = (x+3)(x-4);$$

   (f)
$$x^2\left(y^2-25\right) = y;$$

   (g)
$$y = 6 - e^{-2x}.$$

2. For each of the following curves, determine the equations of the asymptotes which are parallel to either the x-axis or the y-axis:

   (a)
$$xy^2 + x^2 - 1 = 0;$$

   (b)
$$x^2y^2 = 4\left(x^2+y^2\right);$$

(c)

$$y = \frac{x^2 - 3x + 5}{x - 3}.$$

3. Determine all the asymptotes of the following curves:

(a)

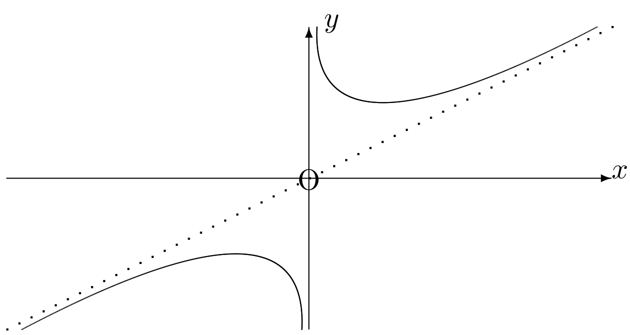
$$x^3 - xy^2 + 4x - 16 = 0;$$

(b)

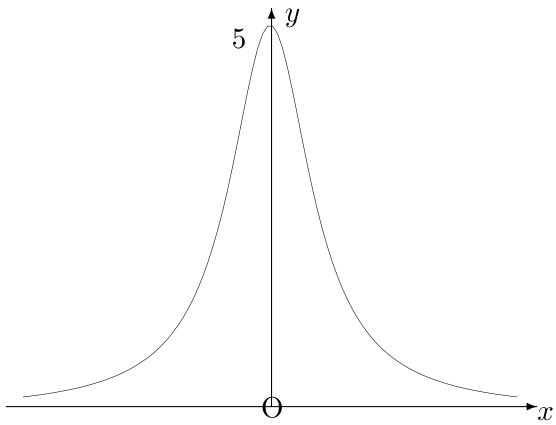
$$y^3 + 2y^2 - x^2y + y - x + 4 = 0.$$

5.9.7 ANSWERS TO EXERCISES

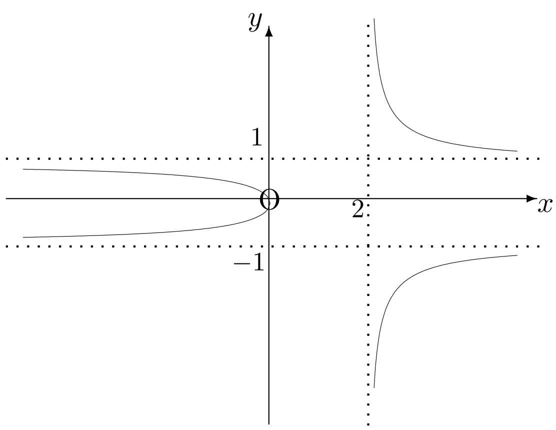
1.(a)



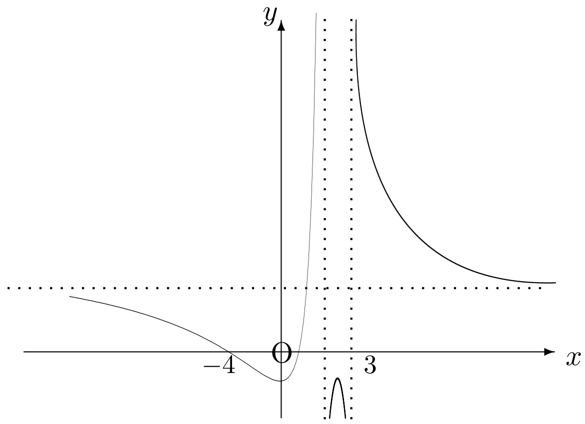
1.(b)



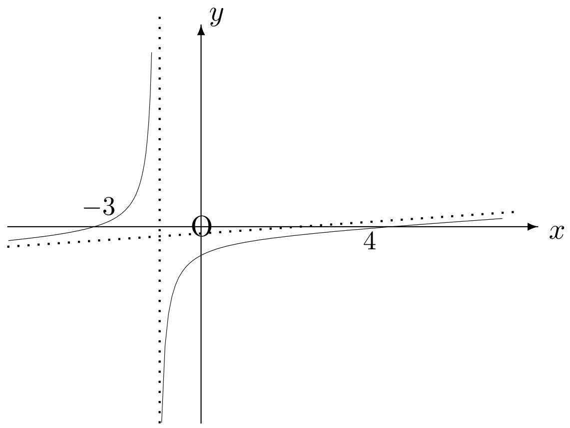
1.(c)



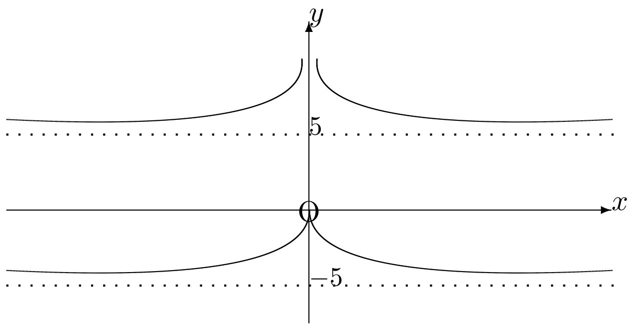
1.(d)



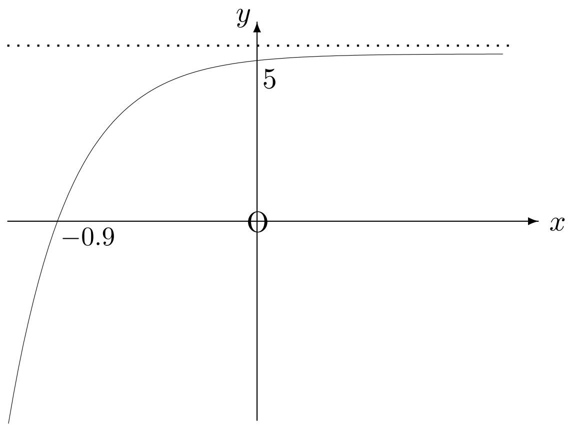
1.(e)



1.(f)



1.(g)



2.(a) $x = 0$, (b) $x = \pm 2$ and $y = \pm 2$, (c) $x = 3$

3.(a) $y = x$, $y = -x$ and $x = 0$; (b) $y = 0$, $y = x - 1$ and $y = -x - 1$.

"JUST THE MATHS"

UNIT NUMBER

5.10

GEOMETRY 10
(Graphical solutions)

by

A.J.Hobson

UNIT 5.10 - GEOMETRY 10

GRAPHICAL SOLUTIONS

5.10.1 INTRODUCTION

An algebraic equation in a variable quantity, x, may be written in the general form

$$f(x) = 0,$$

where $f(x)$ is an algebraic expression involving x; we call it a **"function of x"** (see Unit 10.1).

In the work which follows, $f(x)$ will usually be either a **linear** function of the form $ax + b$, where a and b are constants, or a **quadratic** function of the form $ax^2 + bx + c$ where a, b and c are constants.

The solutions of the equation $f(x) = 0$ consist of those values of x which, when substituted into the function $f(x)$, cause it to take the value zero.

The solutions may also be interpreted as the values of x for which the graph of the equation

$$y = f(x)$$

meets the x-axis since, at any point of this axis, y is equal to zero.

5.10.2 THE GRAPHICAL SOLUTION OF LINEAR EQUATIONS

To solve the equation

$$ax + b = 0,$$

we may plot the graph of the equation $y = ax + b$ to find the point at which it meets the x-axis.

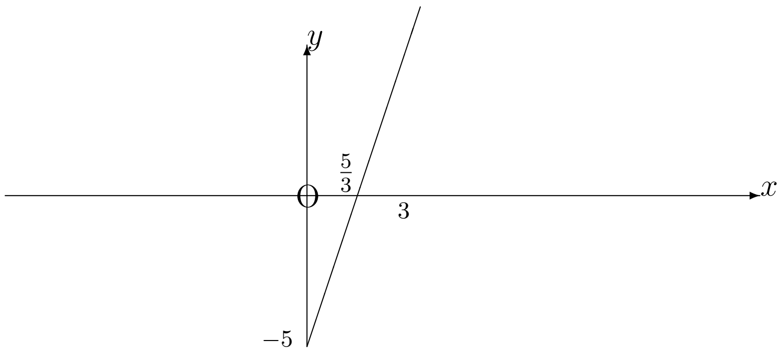
EXAMPLES

1. By plotting the graph of $y = 3x - 5$ from $x = 0$ to $x = 3$, solve the linear equation

$$3x - 5 = 0.$$

Solution

x	0	1	2	3
y	-5	-2	1	4



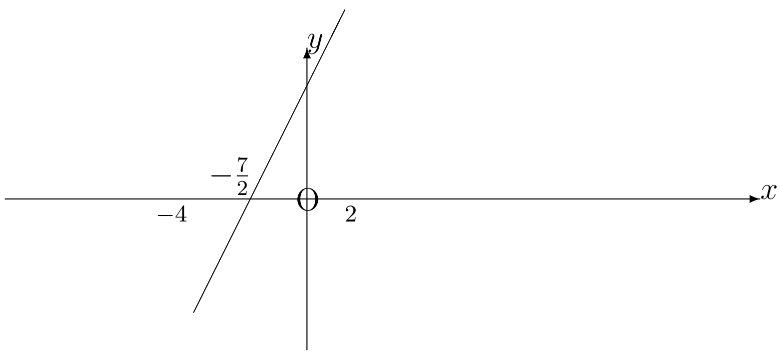
Hence $x \simeq 1.7$

2. By plotting the graph of $y = 2x + 7$ from $x = -4$ to $x = 2$, solve the linear equation

$$2x + 7 = 0$$

Solution

x	-4	-3	-2	-1	0	1	2
y	-1	1	3	5	7	10	11



Hence $x = -3.5$

5.10.3 THE GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS

To solve the quadratic equation

$$ax^2 + bx + c = 0$$

by means of a graph, we may plot the graph of the equation $y = ax^2 + bx + c$ and determine the points at which it crosses the x-axis.

An alternative method is to plot the graphs of the two equations $y = ax^2 + bx$ and $y = -c$ in order to determine their points of intersection. This method is convenient since the first graph has the advantage of passing through the origin.

EXAMPLE

By plotting the graph of $y = x^2 - 4x$ from $x = -2$ to $x = 6$, solve the quadratic equations

(a)

$$x^2 - 4x = 0;$$

(b)

$$x^2 - 4x + 2 = 0;$$

(c)

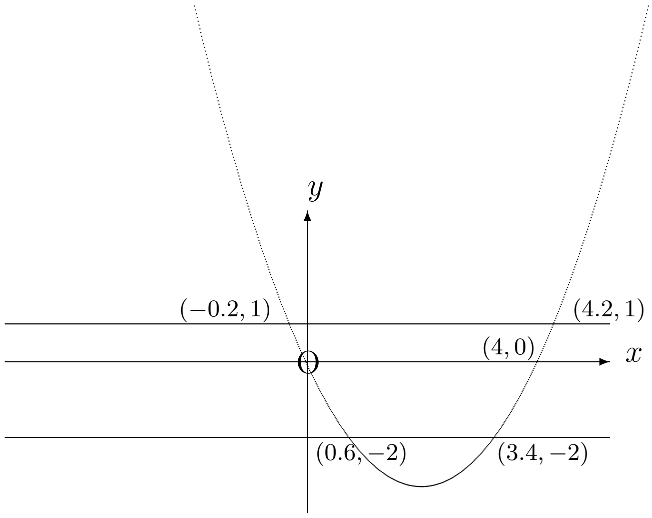
$$x^2 - 4x - 1 = 0.$$

Solution

A table of values for the graph of $y = x^2 - 4x$ is

x	-2	-1	0	1	2	3	4	5	6
y	12	5	0	-3	-2	-3	0	5	12

For parts (b) and (c), we shall also need the graphs of $y = -2$ and $y = 1$.



Hence, the three sets of solutions are:

(a)
$$x = 0 \quad \text{and} \quad x = 4;$$

(b)
$$x \simeq 3.4 \quad \text{and} \quad x \simeq 0.6;$$

(c)
$$x \simeq 4.2 \quad \text{and} \quad x \simeq -0.2$$

5.10.4 THE GRAPHICAL SOLUTION OF SIMULTANEOUS EQUATIONS

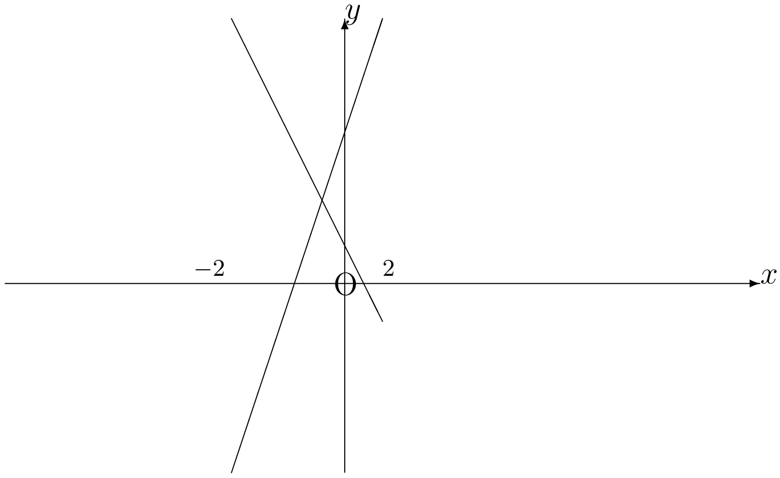
A simple extension of the ideas covered in the previous paragraphs is to solve either a pair of simultaneous linear equations or a pair of simultaneous equations consisting of one linear and one quadratic equation. More complicated cases can also be dealt with by a graphical method but we shall limit the discussion to the simpler ones.

EXAMPLES

1. By plotting the graphs of $5x + y = 2$ and $-3x + y = 6$ from $x = -2$ to $x = 2$, determine the common solution of the two equations.

   Solution

x	-2	-1	0	1	2
$y_1 = 2 - 5x$	12	7	2	-3	-8
$y_2 = 6 + 3x$	0	3	6	9	12



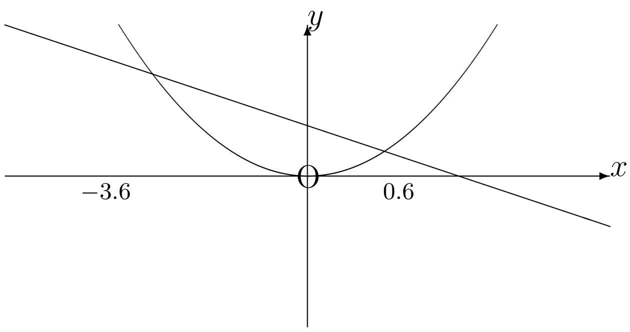
Hence, $x = -0.5$ and $y = 4.5$.

2. By plotting the graphs of the equations $y = x^2$ and $y = 2 - 3x$ from $x = -4$ to $x = 2$ determine their common solutions and hence solve the quadratic equation

$$x^2 + 3x - 2 = 0.$$

Solution

x	-4	-3	-2	-1	0	1	2
$y_1 = x^2$	16	9	4	1	0	1	4
$y_2 = 2 - 3x$	14	11	8	5	2	-1	-4



Hence $x \simeq 0.6$ and $x \simeq -3.6$.

5.10.5 EXERCISES

In these exercises, state your answers correct to one place of decimals.

1. Use a graphical method to solve the following linear equations:
   (a)
   $$8x - 3 = 0;$$
   (b)
   $$8x = 7.$$
2. Use a graphical method to solve the following quadratic equations:
   (a)
   $$2x^2 - x = 0;$$
   (b)
   $$2x^2 - x + 3 = 10;$$
   (c)
   $$2x^2 - x = 11.$$
3. Use a graphical method to solve the follwing pairs of simultaneous equations:
   (a)
   $$3x - y = 6 \quad \text{and} \quad x + y = 0;$$
   (b)
   $$x + 2y = 13 \quad \text{and} \quad 2x - 3y = 14;$$
   (c)
   $$y = 3x^2 \quad \text{and} \quad y = -5x + 1.$$

5.10.6 ANSWERS TO EXERCISES

1. (a)

$$x \simeq 0.4;$$

(b)

$$x \simeq 0.9$$

2. (a)

$$x = 0 \quad \text{and} \quad x = 2;$$

(b)

$$x \simeq 2.1 \quad \text{and} \quad x \simeq -1.6;$$

(c)

$$x \simeq 2.6 \quad \text{and} \quad x \simeq -2.1$$

3. (a)

$$x = 1.2 \quad \text{and} \quad y = -1.2;$$

(b)

$$x \simeq 9.6 \quad \text{and} \quad y \simeq 1.7;$$

(c)

$$x \simeq 0.18 \quad \text{and} \quad y \simeq 0.1 \quad \text{or} \quad x \simeq -1.8 \quad \text{and} \quad y \simeq 10.2$$

"JUST THE MATHS"

UNIT NUMBER

5.11

GEOMETRY 11
(Polar curves)

by

A.J.Hobson

UNIT 5.11 - GEOMETRY 11 - POLAR CURVES

5.11.1 INTRODUCTION

The concept of polar co-ordinates was introduced in Unit 5.1 as an alternative method, to cartesian co-ordinates, of specifying the position of a point in a plane. It was also seen that a relationship between cartesian co-ordinates, x and y, may be converted into an equivalent relationship between polar co-ordinates, r and θ by means of the formulae,

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta,$$

while the reverse process may be carried out using the formulae

$$r^2 = x^2 + y^2 \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

Sometimes the reverse process may be simplified by using a mixture of both sets of formulae.

In this Unit, we shall consider the graphs of certain relationships between r and θ without necessarily refering to the equivalent of those relationships in cartesian co-ordinates. The graphs obtained will be called **"polar curves"**.

Note:

In Unit 5.1, no consideration was given to the possibility of **negative** values of r; in fact, when polar co-ordinates are used in the subject of complex numbers (see Units 6.1 - 6.6) r is **not** allowed to take negative values.

However, for the present context it will be necessary to assign a meaning to a point (r, θ), in polar co-ordinates, when r is negative.

We simply plot the point at a distance of $|r|$ along the $\theta - 180^\circ$ line; and, of course, this implies that, when r is negative, the point (r, θ) is the same as the point $(|r|, \theta - 180^\circ)$

5.11 2 THE USE OF POLAR GRAPH PAPER

For equations in which r is expressed in terms of θ, it is convenient to plot values of r against values of θ using a special kind of graph paper divided into small cells by concentric circles and radial lines.

The radial lines are usually spaced at intervals of 15° and the concentric circles allow a scale to be chosen by which to measure the distances, r, from the pole.

We illustrate with examples:

EXAMPLES

1. Sketch the graph of the equation

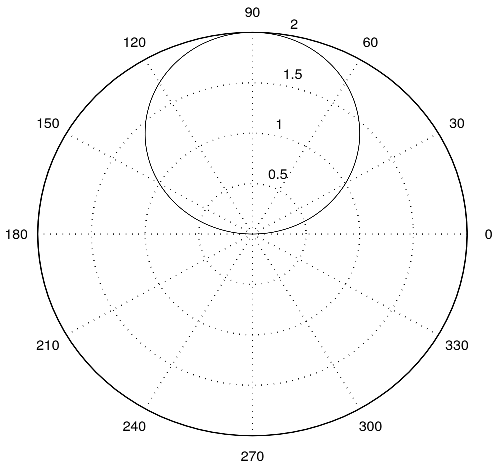
$$r = 2\sin\theta.$$

Solution

First we construct a table of values of r and θ, in steps of $15°$, from $0°$ to $360°$.

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$	$195°$
r	0	0.52	1	1.41	1.73	1.93	2	1.93	1.73	1.41	1	0.52	0	-0.52

θ	$210°$	$225°$	$240°$	$255°$	$270°$	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	-1	-1.41	-1.73	-1.93	-2	-1.93	-1.73	-1.41	-1	-0.52	0



Notes:

(i) The curve, in this case, is a circle whose cartesian equation turns out to be

$$x^2 + y^2 - 2y = 0.$$

(ii) The fact that half of the values of r are negative means, here, that the circle is described twice over. For example, the point $(-0.52, 195°)$ is the same as the point $(0.52, 15°)$.

2. Sketch the graph of the following equations:

(a)

$$r = 2(1 + \cos\theta);$$

(b)

$$r = 1 + 2\cos\theta;$$

(c)

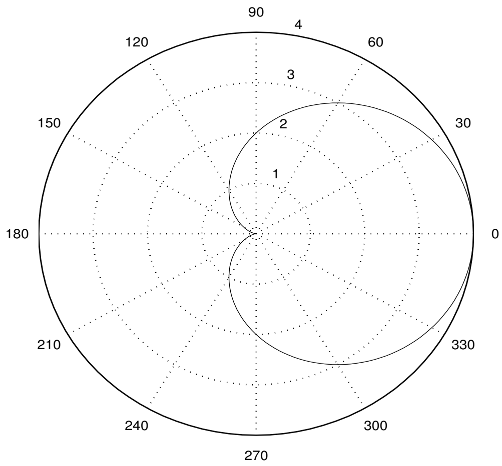
$$r = 5 + 3\cos\theta.$$

Solution

(a) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	4	3.93	3.73	3.42	3	2.52	2	1.48	1	0.59	0.27	0.07	0

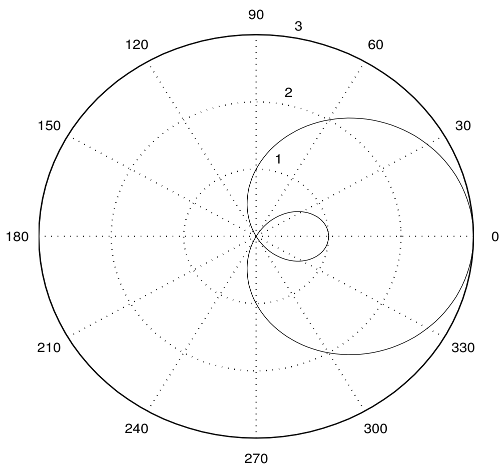
θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	0.07	0.27	0.59	1	1.48	2	2.52	3	3.42	3.73	3.93	4



(b) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	3	2.93	2.73	2.41	2	1.52	1	0.48	0	-0.41	-0.73	-0.93	-1

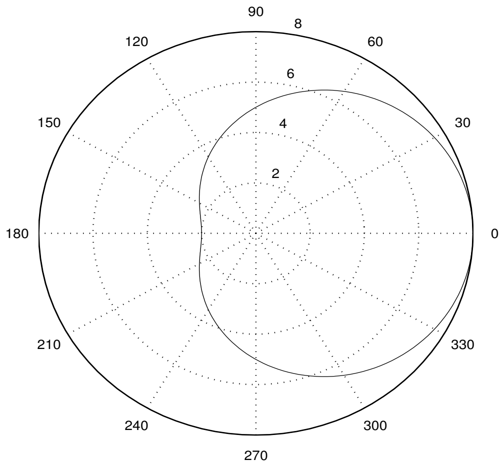
θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	-0.93	-0.73	-0.41	0	0.48	1	1.52	2	2.41	2.73	2.93	3



(c) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	8	7.90	7.60	7.12	6.5	5.78	5	4.22	3.5	2.88	2.40	2.10	2

θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	2.10	2.40	2.88	3.5	4.22	5	5.78	6.5	7.12	7.60	7.90	8



Note:

Each of the three curves in the above example is known as a **"limacon"** and they illustrate special cases of the more general curve, $r = a + b\cos\theta$, as follows:

(i) If $a = b$, the limacon may also be called a **"cardioid"**; that is, a heart-shape. At the pole, the curve possesses a **"cusp"**.

(ii) If $a < b$, the limacon contains a **"re-entrant loop"**.

(iii) If $a > b$, the limacon contains neither a cusp nor a re-entrant loop.

Other well-known polar curves, together with any special titles associated with them, may be found in the answers to the exercises at the end of this unit.

5.11.3 EXERCISES

Plot the graphs of the following polar equations:

1.
$$r = 3\cos\theta.$$

2.
$$r = \sin 3\theta.$$

3.
$$r = \sin 2\theta.$$

4.
$$r = 4\cos 3\theta.$$

5.
$$r = 5\cos 2\theta.$$

6.
$$r = 2\sin^2\theta.$$

7.
$$r = 2\cos^2\theta.$$

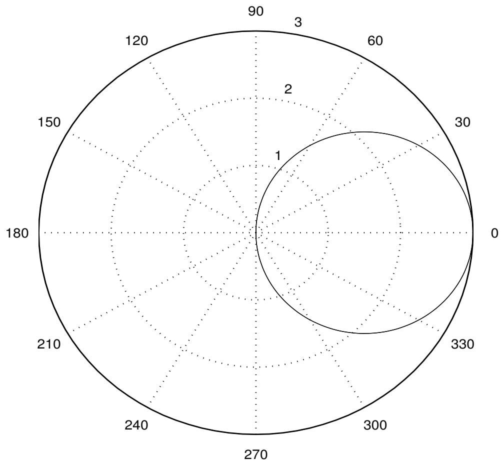
8.
$$r^2 = 25\cos 2\theta.$$

9.
$$r^2 = 16\sin 2\theta.$$

10.
$$r = 2\theta.$$

5.11.4 ANSWERS TO EXERCISES

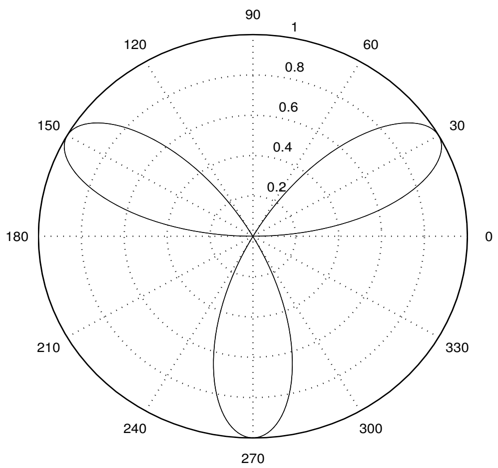
1. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\cos\theta$, which is a circle.

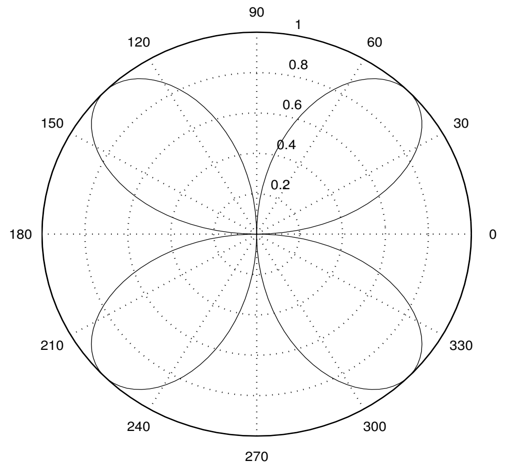
2. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\sin n\theta$, where n is **odd**. It is an **"n-leaved rose"**.

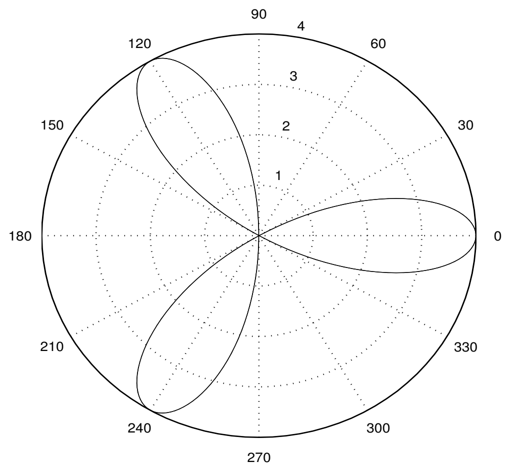
3. The graph is as follows:



Note:

This is an example of the more general curve, $r = a \sin n\theta$, where n is **even**. It is a **"$2n$-leaved rose"**.

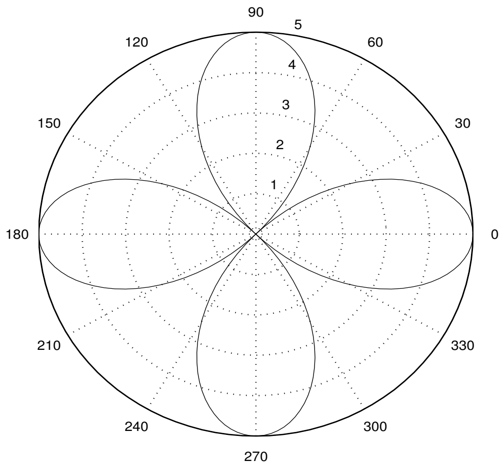
4. The graph is as follows:



Note:

This is an example od the more general curve, $r = a \cos n\theta$, where n is **odd**. It is an **"n-leaved rose"**.

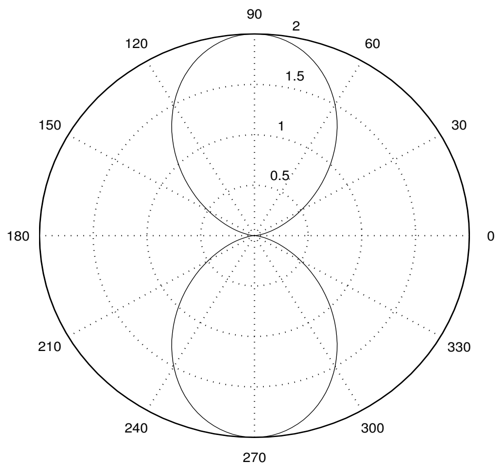
5. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\cos n\theta$, where n is **even**. It is a **"$2n$-leaved rose"**.

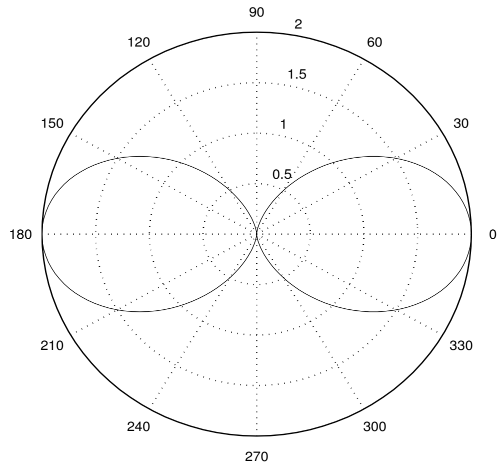
6. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\sin^2\theta$, which is called a **"lemniscate"**.

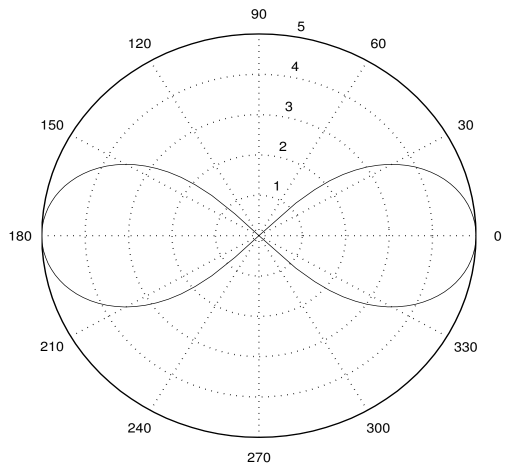
7. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\cos^2\theta$, which is also called a **"lemniscate"**.

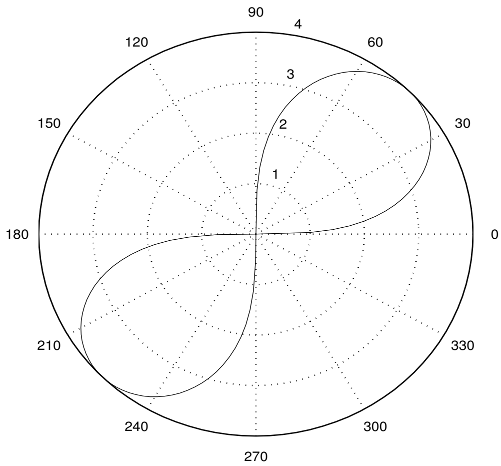
8. The graph is as follows:



Note:

This is an example of the more general curve, $r^2 = a^2 \cos 2\theta$. It is another example of a **"lemniscate"**; but, since r^2 cannot be negative, there are no points on the curve in the intervals $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ and $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$.

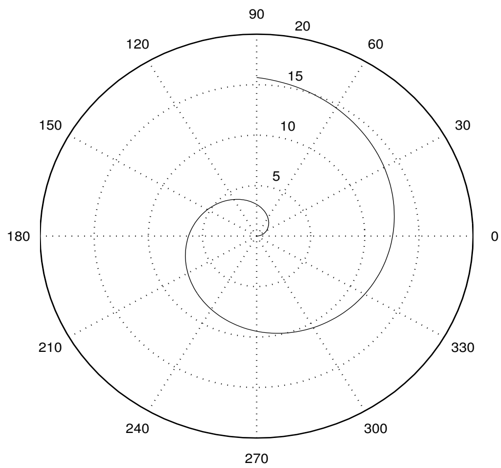
9. The graph is as follows:



Note:

This is an example of the more general curve, $r^2 = a^2 \sin 2\theta$. It is another example of a **"lemniscate"**; but, since r^2 cannot be negative, there are no points on the curve in the intervals $\frac{\pi}{2} < \theta < \pi$ and $\frac{3\pi}{2} < \theta < 2\pi$.

10. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\theta$, $a > 0$, which is called an **"Archimedean spiral"**.

"JUST THE MATHS"

UNIT NUMBER

6.1

COMPLEX NUMBERS 1 (Definitions and algebra)

by

A.J.Hobson

UNIT 6.1 - COMPLEX NUMBERS 1 - DEFINITIONS AND ALGEBRA

6.1.1 THE DEFINITION OF A COMPLEX NUMBER

Students who are already familiar with the Differential Calculus may appreciate that equations of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

which are called "Differential Equations", have wide-reaching applications in science and engineering. They are particularly applicable to problems involving either electrical circuits or mechanical vibrations.

It is possible to show that, in order to determine a formula (without derivatives) giving the variable y in terms of the variable x, one method is to solve, first, the quadratic equation whose coefficents are a, b and c and whose solutions are therefore

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note:
Students who are **not** already familiar with the Differential Calculus should consider only the quadratic equation whose coefficients are a, b and c, ignoring references to differential equations.

ILLUSTRATION

One method of solving the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13 = 2\sin x$$

would be to solve, first, the quadratic equation whose coefficients are 1, -6 and 13.

Its solutions are

$$\frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$

which clearly do not exist since we cannot find the square root of a negative number in elementary arithmetic.

However, if we assume that the differential equation represents a genuine scientific problem with a genuine scientific solution, we cannot simply dismiss the result obtained from the quadratic formula.

The difficulty seems to be, not so much with the -16 but with the minus sign in front of the 16. We shall therefore write the solutions in the form

$$\frac{6 \pm 4\sqrt{-1}}{2} = 3 \pm 2\sqrt{-1}.$$

Notes:
(i) The symbol $\sqrt{-1}$ will be regarded as an **"imaginary"** number.

(ii) In mathematical work, $\sqrt{-1}$ is normally denoted by i but, in scientific work it is denoted by j in order to avoid confusion with other quantities (eg. electric current) which could be denoted by the same symbol.

(iii) Whenever the imaginary quantity $j = \sqrt{-1}$ occurs in the solutions of a quadratic equation, those solutions will always be of the form $a + bj$ (or $a + jb$), where a and b are ordinary numbers of elementary arithmetic.

DEFINITIONS

1. The term **"complex number"** is used to denote any expression of the form $a + bj$ or $a + jb$ where a and b are ordinary numbers of elementary arithmetic (including zero) and j denotes the imaginary number $\sqrt{-1}$; i.e. $j^2 = -1$.
2. If the value a happens to be zero, then the complex number $a + bj$ or $a + jb$ is called **"purely imaginary"** and is written bj or jb.
3. If the value b happens to be zero, then the complex number $a + bj$ or $a + jb$ is defined to be the same as the number a and is called **"real"**. That is $a + j0 = a + 0j = a$.
4. For the complex number $a + bj$ or $a + jb$, the value a is called the **"real part"** and the value b is called the **"imaginary part"**. Notice that the imaginary part is b and **not** jb.
5. The complex numbers $a \pm bj$ are said to form a pair of **"complex conjugates"** and similarly $a \pm jb$ form a pair of complex conjugates. Alternatively, we may say, for instance, that $a - jb$ is the complex conjugate of $a + jb$ and $a + jb$ is the complex conjugate of $a - jb$.

Note:
In some work on complex numbers, especially where many complex numbers may be under

discussion at the same time, it is convenient to denote real and imaginary parts by the symbols x and y respectively, rather than a and b. It is also convenient, on some occasions, to denote the whole complex number $x + jy$ by the symbol z in which case the conjugate, $x - jy$, will be denoted by $\bar{z}$.

6.1.2 THE ALGEBRA OF COMPLEX NUMBERS

INTRODUCTION

An **"Algebra"** (coming from the Arabic word AL-JABR) refers to any mathematical system which uses the concepts of equality, addition, subtraction, multiplication and division. For example, the algebra of real numbers is what we normally call **"arithmetic"**; but algebraical concepts can be applied to other mathematical systems of which the system of complex numbers is one.

In meeting a new mathematical system for the first time, the concepts of equality, addition, subtraction, multiplication and division need to be properly defined, and that is the purpose of the present section. In some cases, the definitions are fairly obvious, but need to be made without contradicting ideas already established in the system of real numbers which complex numbers include.

(a) EQUALITY

Unlike a real number, a complex number does not have a "value"; and so the word "equality" must take on a meaning, here, which is different from that used in elementary arithmetic. In fact two complex numbers are defined to be equal if they have the same real part <u>and</u> the same imaginary part.

That is

$$a + jb = c + jd \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d$$

EXAMPLE

Determine x and y such that

$$(2x - 3y) + j(x + 5y) = 11 - j14.$$

Solution

From the definition of equality, we may

EQUATE REAL AND IMAGINARY PARTS.

Thus,

$$\begin{aligned} 2x - 3y &= 11, \\ x + 5y &= -14 \end{aligned}$$

These simultaneous linear equations are satisfied by $x = 1$ and $y = -3$.

(b) ADDITION AND SUBTRACTION

These two concepts are very easily defined. We simply add (or subtract) the real parts and the imaginary parts of the two complex numbers whose sum (or difference) is required.

That is,

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

and

$$(a + jb) - (c + jd) = (a - c) + j(b - d).$$

EXAMPLE

$$(-7 + j2) + (10 - j5) = 3 - j3 = 3(1 - j)$$

and

$$(-7 + j2) - (10 - j5) = -17 + j7.$$

(c) MULTIPLICATION

The definition of multiplication essentially treats j in the same way as any other algebraic symbol, but uses the fact that $j^2 = -1$.

Thus,

$$(a + jb)(c + jd) = (ac - bd) + j(bc + ad);$$

but this is not so much a formula to be learned off-by-heart as a technique to be applied in future examples.

EXAMPLES

1.

$$(5 + j9)(2 + j6) = (10 - 54) + j(18 + 30) = -44 + j48.$$

2.

$$(3 - j8)(1 + j4) = (3 + 32) + j(-8 + 12) = 35 + j4.$$

3.

$$(a + jb)(a - jb) = a^2 + b^2.$$

Note:
The third example above will be useful in the next section. It shows that **the product of a complex number and its complex conjugate is always a real number consisting of the sum of the squares of the real and imaginary parts**.

(d) DIVISION

The objective here is to make a definition which provides the real and imaginary parts of the complex expression

$$\frac{a + jb}{c + jd}.$$

Once again, we make this definition in accordance with what would be obtained algebraically by treating j in the same way as any other algebraic symbol, but using the fact that $j^2 = -1$.

The method is to multiply both the numerator and the denominator of the complex ratio by the conjugate of the denominator giving

$$\frac{a+jb}{c+jd}.\frac{c-jd}{c-jd} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2}.$$

The required definition is thus

$$\frac{a+jb}{c+jd} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2},$$

which, again, is not so much a formula to be learned off-by-heart as a technique to be applied in future examples.

EXAMPLES

1.

$$\frac{5+j3}{2+j7} = \frac{5+j3}{2+j7}.\frac{2-j7}{2-j7}$$

$$= \frac{(10+21)+j(6-35)}{2^2+7^2} = \frac{31-j29}{53}.$$

Hence, the real part is $\frac{31}{53}$ and the imaginary part is $-\frac{29}{53}$.

2.

$$\frac{6+j}{j2-4} = \frac{6+j}{j2-4}.\frac{-j2-4}{-j2-4}$$

$$= \frac{(-24+2)+j(-4-12)}{(-2)^2+(-4)^2} = \frac{-22-j16}{20}.$$

Hence, the real part is $-\frac{22}{20} = -\frac{11}{10}$ and the imaginary part is $-\frac{16}{20} = -\frac{4}{5}$.

6.1.3 EXERCISES

1. Simplify the following:
   (a) j^3; (b) j^4; (c) j^5; (d) j^{15}; (e) j^{22}.

2. If $z_1 = 2 - j5$, $z_2 = 1 + j7$ and $z_3 = -3 - j4$, determine the following in the form $a + jb$:
   (a)
   $$z_1 - z_2 + z_3;$$
   (b)
   $$2z_1 + z_2 - z_3;$$
   (c)
   $$z_1 - (4z_2 - z_3);$$
   (d)
   $$\frac{z_1}{z_2};$$
   (e)
   $$\frac{z_2}{z_3};$$
   (f)
   $$\frac{z_3}{z_1}.$$

3. Determine the values of x and y such that
   $$(3x - 5y) + j(x + 3y) = 20 + j2.$$

4. Determine the real and imaginary parts of the expression
   $$(1 - j3)^2 + j(2 + j5) - \frac{3(4 - j)}{1 - j}.$$

5. If $z \equiv x + jy$ and $\overline{z} \equiv x - jy$ are conjugate complex numbers, determine the values of x and y such that
   $$4z\overline{z} - 3(z - \overline{z}) = 2 + j.$$

6.1.4 ANSWERS TO EXERCISES

1. (a) $-j$; (b) 1; (c) j; (d) $-j$; (e) -1.
2. (a)

$$4 - j16;$$

(b)

$$8 + j;$$

(c)

$$-5 - j37;$$

(d)

$$-0.66 - j0.38;$$

(e)

$$-1.24 - j0.68;$$

(f)

$$0.48 - j0.79$$

3.

$$x = 5 \quad \text{and} \quad y = -1.$$

4. The real part = -20.5; the imaginary part = -8.5
5.

$$x = \pm\frac{1}{2} \quad \text{and} \quad y = -\frac{1}{2}.$$

"JUST THE MATHS"

UNIT NUMBER

6.2

COMPLEX NUMBERS 2
(The Argand Diagram)

by

A.J.Hobson

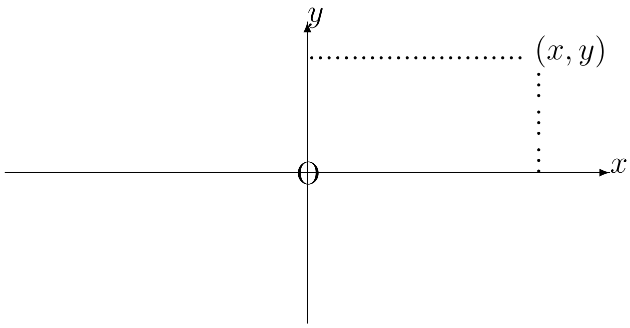
UNIT 6.2 - COMPLEX NUMBERS 2

THE ARGAND DIAGRAM

6.2.1 INTRODUCTION

It may be observed that a complex number $x + jy$ is completely specified if we know the values of x and y in the correct order. But the same is true for the cartesian co-ordinates, (x, y), of a point in two dimesions. There is therefore a **"one-to-one correspondence"** between the complex number $x + jy$ and the point with co-ordinates (x, y).

Hence it is possible to represent the complex number $x+jy$ by the point (x, y) in a geometrical diagram called the Argand Diagram:



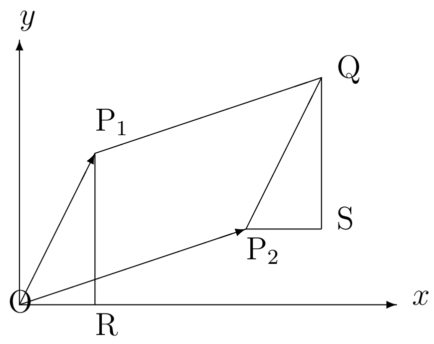
DEFINITIONS:

1. The x-axis is called the **"real axis"** since the points on it represent real numbers.
2. The y-axis is called the **"imaginary axis"** since the points on it represent purely imaginary numbers.

6.2.2 GRAPHICAL ADDITON AND SUBTRACTION

If two complex numbers, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, are represented in the Argand Diagram by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively, then the sum, $z_1 + z_2$, of the complex numbers will be represented by the point $Q(x_1 + x_2, y_1 + y_2)$.

If O is the origin, it is possible to show that Q is the fourth vertex of the parallelogram having OP_1 and OP_2 as adjacent sides.



In the diagram, the triangle ORP_1 has exactly the same shape as the triangle P_2SQ. Hence, the co-ordinates of Q must be $(x_1 + x_2, y_1 + y_2)$.

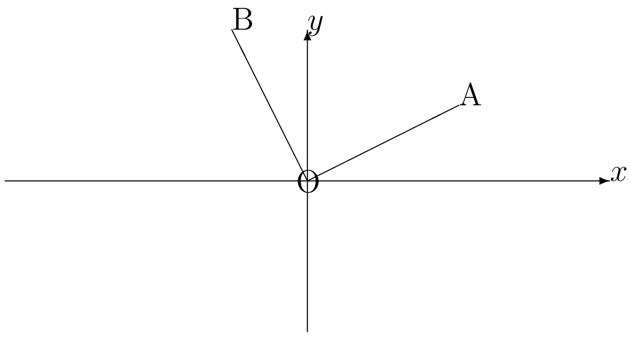
Note:
The difference $z_1 - z_2$ of the two complex numbers may similarly be found by completing the parallelogram of which two adjacent sides are the straight line segments joining the origin to the points with co-ordinates (x_1, y_1) and $(-x_2, -y_2)$.

6.2.3 MULTIPLICATION BY j OF A COMPLEX NUMBER

Given any complex number $z = x + jy$, we observe that

$$jz = j(x + jy) = -y + jx.$$

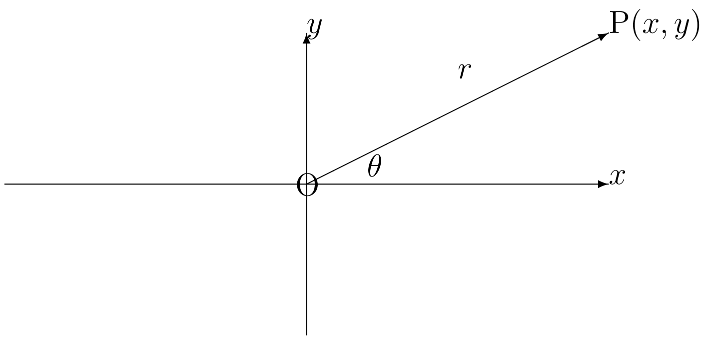
Thus, if z is represented in the Argand Diagram by the point with co-ordinates $A(x, y)$, then jz is represented by the point with co-ordinates $B(-y, x)$.



But OB is in the position which would be occupied by OA if it were rotated through 90° in a counter-clockwise direction.

We conclude that, in the Argand Diagram, multiplication by j of a complex number rotates, through 90° in a counter-clockwise direction, the straight line segment joining the origin to the point representing the complex number.

6.2.4 MODULUS AND ARGUMENT



(a) Modulus

If a complex number, $z = x + jy$ is represented in the Argand Diagram by the point, P, with cartesian co-ordinates (x, y) then the distance, r, of P from the origin is called the **"modulus"** of z and is denoted by either $|z|$ or $|x + jy|$.

Using the theorem of Pythagoras in the diagram, we conclude that

$$r = |z| = |x + jy| = \sqrt{x^2 + y^2}.$$

Note:
This definition of modulus is consistent with the definition of modulus for real numbers (which are included in the system of complex numbers). For any real number x, we may say that

$$|x| = |x + j0| = \sqrt{x^2 + 0^2} = \sqrt{x^2},$$

giving the usual numerical value of x.

ILLUSTRATIONS

1.
$$|3 - j4| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

2.
$$|1 + j| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

3.
$$|j7| = |0 + j7| = \sqrt{0^2 + 7^2} = \sqrt{49} = 7.$$

Note:
The result of the last example above is obvious from the Argand Diagram since the point on the y-axis representing $j7$ is a distance of exactly 7 units from the origin. In the same way, a real number is represented by a point on the x-axis whose distance from the origin is the numerical value of the real number.

(b) Argument

The **"argument"** (or **"amplitude"**) of a complex number, z, is defined to be the angle, θ, which the straight line segment OP makes with the positive real axis (measuring θ positively from this axis in a counter-clockwise sense).

In the diagram,

$$\tan\theta = \frac{y}{x}; \quad \text{that is,} \quad \theta = \tan^{-1}\frac{y}{x}.$$

Note:
For a given complex number, there will be infinitely many possible values of the argument, any two of which will differ by a whole multiple of 360°. The complete set of possible values is denoted by Arg$z$, using an upper-case A.

The particular value of the argument which lies in the interval $-180° < \theta \leq 180°$ is called the **"principal value"** of the argument and is denoted by $\arg z$ using a lower-case a. The particular value, 180°, in preference to −180°, represents the principal value of the argument of a negative real number.

ILLUSTRATIONS

1.

$$\text{Arg}(\sqrt{3} + j) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30° + k360°,$$

where k may be any integer. But we note that

$$\arg(\sqrt{3} + j) = 30° \quad \text{only.}$$

2.

$$\text{Arg}(-1 + j) = \tan^{-1}(-1) = 135° + k360°$$

but **not** $-45° + k360°$, since the complex number $-1 + j$ is represented by a point in the second quadrant of the Argand Diagram.
We note also that

$$\arg(-1 + j) = 135° \quad \text{only.}$$

3.

$$\text{Arg}(-1 - j) = \tan^{-1}(1) = 225° + k360° \quad \text{or} \quad -135° + k360°$$

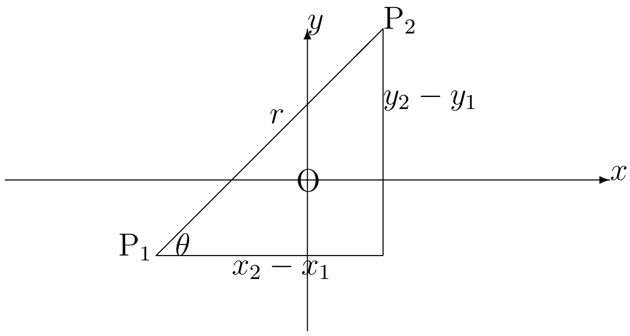
but **not** $45° + k360°$ since the complex number $-1 - j$ is represented by a point in the third quadrant of the Argand Diagram.
We note also that

$$\arg(-1 - j) = -135° \quad \text{only.}$$

Note:
It is worth mentioning here that, in the Argand Diagram, the directed straight line segment described from the point P_1 (representing the complex number $z_1 = x_1 + jy_1$) to the point P_2 (representing the complex number $z_2 = x_2 + jy_2$) has length, r, equal to $|z_2 - z_1|$, and is inclined to the positive direction of the real axis at an angle, θ, equal to $\arg(z_2 - z_1)$. This follows from the relationship

$$z_2 - z_1 = (x_2 - x_1) + j(y_2 - y_1)$$

in which $x_2 - x_1$ and $y_2 - y_1$ are the distances separating the two points, parallel to the real axis and the imaginary axis respectively.



6.2.5 EXERCISES

1. Determine the modulus (in decimals, where appropriate, correct to three significant figures) and the principal value of the argument (in degrees, correct to the nearest degree) of the following complex numbers:

   (a)
   $$1 - j;$$

   (b)
   $$-3 + j4;$$

   (c)
   $$-\sqrt{2} - j\sqrt{2};$$

(d)

$$\frac{1}{2} - j\frac{\sqrt{3}}{2};$$

(e)

$$-7 - j9.$$

2. If $z = 4 - j5$, verify that jz has the same modulus as z but that the principal value of the argument of jz is greater, by $90°$ than the principal value of the argument of z.

3. Illustrate the following statements in the Argand Diagram:

(a)

$$(6 - j11) + (5 + j3) = 11 - j8;$$

(b)

$$(6 - j11) - (5 + j3) = -1 - j14.$$

6.2.6 ANSWERS TO EXERCISES

1. (a) 1.41 and $-45°$;
   (b) 5 and $127°$;
   (c) 2 and $-135°$;
   (d) 1 and $-60°$;
   (e) 11.4 and $-128°$.

2. $4 - j5$ has modulus $\sqrt{41}$ and argument $-51°$;
   $j(4 - j5) = 5 + j4$ has modulus $\sqrt{41}$ and argument $39° = -51° + 90°$.

3. Construct the graphical sum and difference of the two complex numbers.

"JUST THE MATHS"

UNIT NUMBER

6.3

COMPLEX NUMBERS 3
(The polar & exponential forms)

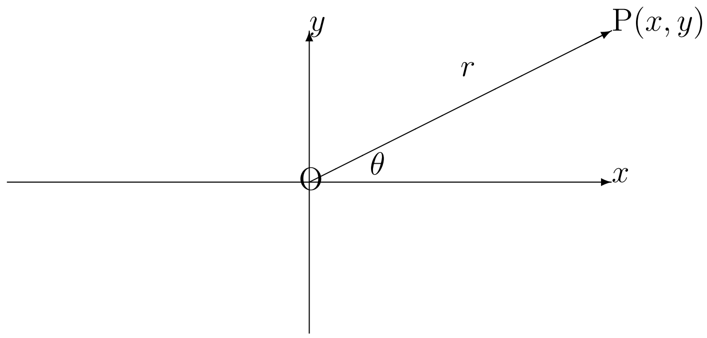
by

A.J.Hobson

UNIT 6.3 - COMPLEX NUMBERS 3

THE POLAR AND EXPONENTIAL FORMS

6.3.1 THE POLAR FORM



From the above diagram, we may observe that

$$\frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{y}{r} = \sin\theta.$$

Hence, the relationship between x, y, r and θ may also be stated in the form

$$x = r\cos\theta, \quad y = r\sin\theta,$$

which means that the complex number $x + jy$ may be written as $r\cos\theta + jr\sin\theta$. In other words,

$$x + jy = r(\cos\theta + j\sin\theta).$$

The left-hand-side of this relationship is called the **"rectangular form"** or **"cartesian form"** of the complex number while the right-hand-side is called the **"polar form"**.

Note:
For convenience, the polar form may be abbreviated to $r\angle\theta$, where θ may be positive, negative or zero and may be expressed in either degrees or radians.

EXAMPLES

1. Express the complex number $z = \sqrt{3} + j$ in polar form.
   Solution

$$|z| = r = \sqrt{3+1} = 2$$

and

$$\text{Arg}z = \theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ + k360^\circ,$$

where k may be any integer.
Alternatively, using radians,

$$\text{Arg}z = \frac{\pi}{6} + k2\pi,$$

where k may be any integer.
Hence, in polar form,

$$z = 2(\cos[30^\circ + k360^\circ] + j\sin[30^\circ + k360^\circ]) = 2\angle[30^\circ + k360^\circ]$$

or

$$z = 2\left(\cos\left[\frac{\pi}{6} + k2\pi\right] + j\sin\left[\frac{\pi}{6} + k2\pi\right]\right) = 2\angle\left[\frac{\pi}{6} + k2\pi\right].$$

2. Express the complex number $z = -1 - j$ in polar form.
   Solution

$$|z| = r = \sqrt{1+1} = \sqrt{2}$$

and

$$\text{Arg}z = \theta = \tan^{-1}(1) = -135^\circ + k360^\circ,$$

where k may be any integer.

Alternatively,

$$\text{Arg}z = -\frac{3\pi}{4} + k2\pi,$$

where k may be any integer.

Hence, in polar form,

$$z = \sqrt{2}(\cos[-135^\circ + k360^\circ] + j\sin[-135^\circ + k360^\circ]) = \sqrt{2}\angle[-135^\circ + k360^\circ]$$

or

$$z = \sqrt{2}\left(\cos\left[-\frac{3\pi}{4} + k2\pi\right] + j\sin\left[-\frac{3\pi}{4} + k2\pi\right]\right) = \sqrt{2}\angle\left[-\frac{3\pi}{4} + k2\pi\right].$$

Note:
If it is required that the polar form should contain only the **principal** value of the argument, θ, then, provided $-180^\circ < \theta \leq 180^\circ$ or $-\pi < \theta \leq \pi$, the component $k360^\circ$ or $k2\pi$ of the result is simply omitted.

6.3.2 THE EXPONENTIAL FORM

Using some theory from the differential calculus of complex variables (not included here) it is possible to show that, for any complex number, z,

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$$

These are, in fact, taken as the **definitions** of the functions $e^z, \sin z$ and $\cos z$.

Students who are already familiar with the differential calculus of a real variable, x, may recognise similarities between the above formulae and the "MacLaurin Series" for the functions $e^x, \sin x$ and $\cos x$. In the case of the series for $\sin x$ and $\cos x$, the value, x, must be expressed in **radians and not degrees**.

A useful deduction can be made from the three formulae if we make the substitution $z = j\theta$ into the first one, obtaining:

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + ...$$

and, since $j^2 = -1$, this gives

$$e^{j\theta} = 1 + j\frac{\theta}{1!} - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ...$$

On regrouping this into real and imaginary parts, then using the sine and cosine series, we obtain

$$e^{j\theta} = \cos\theta + j\sin\theta,$$

provided θ is expressed in radians and not degrees.

The complex number $x+jy$, having modulus r and argument $\theta+k2\pi$, may thus be expressed not only in polar form but also in

the exponential form, $re^{j\theta}$.

ILLUSTRATIONS

Using the examples of the previous section

1. $$\sqrt{3} + j = 2e^{j\left(\frac{\pi}{6}+k2\pi\right)}.$$

2. $$-1 + j = \sqrt{2}e^{j\left(\frac{3\pi}{4}+k2\pi\right)}.$$

3. $$-1 - j = \sqrt{2}e^{-j\left(\frac{3\pi}{4}+k2\pi\right)}.$$

Note:
If it is required that the exponential form should contain only the **principal** value of the argument, θ, then, provided $-\pi < \theta \leq \pi$, the component $k2\pi$ of the result is simply omitted.

6.3.3 PRODUCTS AND QUOTIENTS IN POLAR FORM

Let us suppose that two complex numbers z_1 and z_2 have already been expressed in polar form, so that

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1) = r_1\angle\theta_1$$

and

$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2) = r_2\angle\theta_2.$$

It is then possible to establish very simple rules for determining both the product and the quotient of the two complex numbers. The explanation is as follows:

(a) The Product

$$z_1.z_2 = r_1.r_2(\cos\theta_1 + j\sin\theta_1).(\cos\theta_2 + j\sin\theta_2).$$

That is,

$$z_1.z_2 = r_1.r_2([\cos\theta_1.\cos\theta_2 - \sin\theta_1.\sin\theta_2] + j[\sin\theta_1.\cos\theta_2 + \cos\theta_1.\sin\theta_2]).$$

Using trigonometric identities, this reduces to

$$z_1.z_2 = r_1.r_2(\cos[\theta_1 + \theta_2] + j\sin[\theta_1 + \theta_2]) = r_1.r_2\angle[\theta_1 + \theta_2].$$

We have shown that, to determine the product of two complex numbers in polar form, we construct the product of their modulus values and the sum of their argument values.

(b) The Quotient

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + j\sin\theta_1)}{(\cos\theta_2 + j\sin\theta_2)}.$$

On multiplying the numerator and denominator by $\cos\theta_2 - j\sin\theta_2$, we obtain

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}([\cos\theta_1.\cos\theta_2 + \sin\theta_1.\sin\theta_2] + j[\sin\theta_1.\cos\theta_2 - \cos\theta_1.\sin\theta_2]).$$

Using trigonometric identities, this reduces to

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos[\theta_1 - \theta_2] + j\sin[\theta_1 - \theta_2]) = \frac{r_1}{r_2}\angle[\theta_1 - \theta_2].$$

We have shown that, to determine the quotient of two complex numbers in polar form, we construct the quotient of their modulus values and the difference of their argument values.

ILLUSTRATIONS

Using results from earlier examples:

1.

$$(\sqrt{3} + j).(-1 - j) = 2\angle 30^\circ.\sqrt{2}\angle(-135^\circ) = 2\sqrt{2}\angle(-105^\circ).$$

   We notice that, for all of the complex numbers in this example, including the result, the argument appears as the principal value.

2.

$$\frac{\sqrt{3} + j}{-1 - j} = \frac{2\angle 30^\circ}{\sqrt{2}\angle(-135^\circ)} = \sqrt{2}\angle 165^\circ.$$

   Again, for all of the complex numbers in this example, including the result, the argument appears as the principal value.

   Note:
   It will not always turn out that the argument of a product or quotient of two complex numbers appears as the principal value. For instance,

3.

$$(-1 - j).(-\sqrt{3} - j) = \sqrt{2}\angle(-135^\circ).2\angle(-150^\circ) = 2\sqrt{2}\angle(-285^\circ),$$

   which must be converted to $2\sqrt{2}\angle(75^\circ)$ if the principal value of the argument is required.

6.3.4 EXERCISES

In the following cases, express the complex numbers z_1 and z_2 in

(a) the polar form, $r\angle\theta$

and

(b) the exponential form, $re^{j\theta}$

using only the principal value of θ.

(c) For each case, determine also the product, $z_1.z_2$, and the quotient, $\frac{z_1}{z_2}$, in polar form using only the principal value of the argument.

1.
$$z_1 = 1 + j, \quad z_2 = \sqrt{3} - j.$$

2.
$$z_1 = -\sqrt{2} + j\sqrt{2}, \quad z_2 = -3 - j4.$$

3.
$$z_1 = -4 - j5, \quad z_2 = 7 - j9.$$

6.3.5 ANSWERS TO EXERCISES

1. (a)
$$z_1 = \sqrt{2}\angle 45^\circ \quad z_2 = 2\angle(-30^\circ);$$

   (b)
$$z_1 = \sqrt{2}e^{j\frac{\pi}{4}} \quad z_2 = 2e^{-j\frac{\pi}{6}};$$

   (c)
$$z_1.z_2 = 2\sqrt{2}\angle 15^\circ \quad \frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\angle 75^\circ.$$

2. (a)
$$z_1 = 2\angle(135^\circ) \quad z_2 = 5\angle(-127^\circ);$$

(b)

$$z_1 = 2e^{j\frac{3\pi}{4}} \quad z_2 = 5e^{-j2.22};$$

(c)

$$z_1.z_2 = 10\angle 8^\circ \quad \frac{z_1}{z_2} = \frac{2}{5}\angle(-98^\circ).$$

3. (a)

$$z_1 = 6.40\angle(-128.66^\circ) \quad z_2 = 11.40\angle(-55.13^\circ);$$

(b)

$$z_1 = 6.40e^{-j2.25} \quad z_2 = 11.40e^{-j0.96};$$

(c)

$$z_1.z_2 = 72.96\angle 176.21^\circ \quad \frac{z_1}{z_2} = 0.56\angle(-73.53^\circ).$$

"JUST THE MATHS"

UNIT NUMBER

6.4

COMPLEX NUMBERS 4
(Powers of complex numbers)

by

A.J.Hobson

UNIT 6.4 - COMPLEX NUMBERS 4

POWERS OF COMPLEX NUMBERS

6.4.1 POSITIVE WHOLE NUMBER POWERS

As an application of the rule for multiplying together complex numbers in polar form, it is a simple matter to multiply a complex number by itself any desired number of times.

Suppose that
$$z = r\angle\theta.$$

Then,

$$z^2 = r.r\angle(\theta + \theta) = r^2\angle 2\theta;$$

$$z^3 = z.z^2 = r.r^2\angle(\theta + 2\theta) = r^3\angle 3\theta;$$

and, by continuing this process,

$$z^n = r^n\angle n\theta.$$

This result is due to De Moivre, but other aspects of it will need to be discussed before we may formalise what is called **"De Moivre's Theorem"**.

EXAMPLE

$$\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)^{19} = (1\angle\left[\frac{\pi}{4}\right])^{19} = 1\angle\left[\frac{19\pi}{4}\right] = 1\angle\left[\frac{3\pi}{4}\right] = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}.$$

6.4.2 NEGATIVE WHOLE NUMBER POWERS

If n is a <u>negative</u> whole number, we shall suppose that
$$n = -m,$$
where m is a <u>positive</u> whole number.

Thus, if $z = r\angle\theta$,

$$z^n = z^{-m} = \frac{1}{z^m} = \frac{1}{r^m \angle m\theta}.$$

In more detail,

$$z^n = \frac{1}{r^m(\cos m\theta + j\sin m\theta)},$$

giving

$$z^n = \frac{1}{r^m}.\frac{(\cos m\theta - j\sin m\theta)}{\cos^2 m\theta + \sin^2 m\theta} = r^{-m}(\cos[-m\theta] + j\sin[-m\theta]).$$

But $-m = n$, and so

$$z^n = r^n(\cos n\theta + j\sin n\theta) = r^n \angle n\theta,$$

showing that the result of the previous section remains true for negative whole number powers.

EXAMPLE

$$(\sqrt{3} + j)^{-3} = (2\angle 30^\circ)^{-3} = \frac{1}{8}\angle(-90^\circ) = -\frac{j}{8}.$$

6.4.3 FRACTIONAL POWERS AND DE MOIVRE'S THEOREM

To begin with, here, we consider the complex number

$$z^{\frac{1}{n}},$$

where n is a positive whole number and $z = r\angle\theta$.

We define $z^{\frac{1}{n}}$ to be any complex number which gives z itself when raised to the power n. Such a complex number is called **"an n-th root of z"**.

Certainly one such possibility is

$$r^{\frac{1}{n}}\angle\frac{\theta}{n},$$

by virtue of the paragraph dealing with positive whole number powers.

But the general expression for z is given by

$$z = r\angle(\theta + k360°),$$

where k may be any integer; and this suggests other possibilities for $z^{\frac{1}{n}}$, namely

$$r^{\frac{1}{n}}\angle\frac{\theta + k360°}{n}.$$

However, this set of n-th roots is not an infinite set because the roots which are given by $k = 0, 1, 2, 3............n - 1$ are also given by $k = n, n + 1, n + 2, n + 3, ..., 2n - 1, 2n,$ $2n + 1, 2n + 2, 2n + 3, ...$ and so on, respectively.

We conclude that there are precisely n n-th roots given by $k = 0, 1, 2, 3........., n - 1$.

EXAMPLE

Determine the cube roots (i.e. 3rd roots) of the complex number $j8$.

Solution

We first write

$$j8 = 8\angle(90° + k360°).$$

Hence,

$$(j8)^{\frac{1}{3}} = 8^{\frac{1}{3}}\angle\frac{(90° + k360°)}{3},$$

where $k = 0, 1, 2$

The three distinct cube roots are therefore

$$2\angle 30°, 2\angle 150° \text{ and } 2\angle 270° = 2\angle(-90°).$$

They all have the same modulus of 2 but their arguments are spaced around the Argand Diagram at regular intervals of $\frac{360°}{3} = 120°$.

Notes:

(i) In general, the n-th roots of a complex number will all have the same modulus, but their arguments will be spaced at regular intervals of $\frac{360°}{n}$.

(ii) Assuming that $-180° < \theta \leq 180°$; that is, assuming that the polar form of z uses the principal value of the argument, then the particular n-th root of z which is given by $k = 0$ is called the **"principal n-th root"**.

(iii) If $\frac{m}{n}$ is a fraction in its lowest terms, we define

$$z^{\frac{m}{n}}$$

to be either $\left(z^{\frac{1}{n}}\right)^m$ or $(z^m)^{\frac{1}{n}}$ both of which turn out to give the same set of n distinct results.

The discussion, so far, on powers of complex numbers leads us to the following statement:

DE MOIVRE'S THEOREM

If $z = r\angle\theta$, then, for any rational number n, **one value** of z^n is $r^n\angle n\theta$.

6.4.4 EXERCISES

1. Determine the following in the form $a + jb$, expressing a and b in decimals correct to four significant figures:

   (a)
   $$(1 + j\sqrt{3})^{10};$$

   (b)
   $$(2 - j5)^{-4}.$$

2. Determine the fourth roots of $j81$ in exponential form $re^{j\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$.
3. Determine the fifth roots of the complex number $-4 + j4$ in the form $a + jb$ expressing a and b in decimals, where appropriate, correct to two places. State also which root is the principal root.