Underdamped or Oscillatory Motion In this case $b^2 < \omega^2$ so $\sqrt{b^2-\omega^2}$ is imaginary. Let $\beta = \sqrt{\omega^2 - b^2}$; then $\sqrt{b^2-\omega^2} = i\beta$ and the roots (5.29) of the auxiliary equation are $-b \pm i\beta$. The general solution in the form (5.17) is then

$$y = e^{-bt}(A \sin \beta t + B \cos \beta t) \tag{5.32}$$

This result is more in accord with what we know actually happens to the mass m; because of the factor e^{-bt}, the oscillations in this case decrease in amplitude as time goes on. Also note that the frequency of the damped vibrations, namely $\beta = \sqrt{\omega^2 - b^2}$, is less than the frequency ω of the undamped vibrations.

Although we have stated a rather special physical problem, the mathematics we have just discussed applies to a great variety of problems. First, there are many kinds of mechanical vibrations besides a mass attached to a spring. Think of a tuning fork, a pendulum, the needle on the scale of a measuring device, and as more involved examples, the vibrations of complicated structures such as bridges or airplanes, and the vibrations of atoms in a crystal lattice. In such problems, we need to solve differential equations similar to the ones we have discussed. Differential equations of the same form arise in electricity. Consider equations (1.2) and (1.3) when $V = 0$. Remembering that $I = dq/dt$, we can write (1.2) as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \tag{5.33}$$

and (1.3) as

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0. \tag{5.34}$$

Both these equations are of the form (5.27) which we have solved. Thus there is an analogy between a series circuit and the motion of a mass m described by (5.26); L corresponds to m, R to the "friction" constant l, and $1/C$ to the spring constant k.

▶ PROBLEMS, SECTION 5

Solve the following differential equations by the methods discussed above and compare computer solutions.

1. $y'' + y' - 2y = 0$

2. $y'' - 4y' + 4y = 0$

3. $y'' + 9y = 0$

4. $y'' + 2y' + 2y = 0$

5. $(D^2 - 2D + 1)y = 0$

6. $(D^2 + 16)y = 0$

7. $(D^2 - 5D + 6)y = 0$

8. $D(D + 5)y = 0$

9. $(D^2 - 4D + 13)y = 0$

10. $y'' - 2y' = 0$

11. $4y'' + 12y' + 9 = 0$

12. $(2D^2 + D - 1)y = 0$

Recall from Chapter 3, equation (8.5), that a set of functions is linearly independent if their Wronskian is not identically zero. Calculate the Wronskian of each of the following sets to show that in each case they are linearly independent. For each set, write the differential equation of which they are solutions. Also note that each set of functions is a set of basis functions for a linear vector space (see Chapter 3, Section 14, Example 2) and that the general solution of the differential equation gives all vectors of the vector space.

13. e^{-x}, e^{-4x}

14. $e^{ax}, e^{bx}, a \neq b$ (a, b, real or complex)

15. e^{ax}, xe^{ax}

16. $\sin \beta x, \cos \beta x$

17. $1, x, x^2$

18. $e^{ax}, xe^{ax}, x^2 e^{ax}$

19. Solve the algebraic equation

$$D^2 + (1+2i)D + i - 1 = 0$$

(note the complex coefficients) and observe that the roots are complex but not complex conjugates. Show that the method of solution of (5.6) (case of unequal roots) is correct here, and so find the general solution of

$$y'' + (1+2i)y' + (i-1)y = 0.$$

20. As in Problem 19, solve $y'' + (1-i)y' - iy = 0$. *Hint:* See Chapter 2, Section 10, for a method of finding the square root of a complex number.

21. By the method used in solving (5.4) to get (5.9), show that the solution of the third-order equation

$$(D-a)(D-b)(D-c)y = 0$$

is

$$y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx}$$

if a, b, c are all different, and find the solutions if two or three of the roots of the auxiliary equation are equal. Generalize the result to higher-order equations. State your results in vector space language [see comment following equation (5.9)].

Use the results of Problem 21 to find the general solutions of the following equations and compare computer solutions.

22. $(D-1)(D+3)(D+5)y = 0$

23. $(D^2+1)(D^2-1)y = 0$ *Hint:* $D^2 + 1 = (D+i)(D-i)$.

24. $y''' + y = 0$

25. $(D^3 + D^2 - 6D)y = 0$

26. $y''' - 3y'' - 9y' - 5y = 0$

27. $D^2(D-1)^2(D+2)^3 y = 0$

28. $(D^4 + 4)y = 0$ *Hint:* Find the four 4th roots of -4 (see Chapter 2, Section 10).

29. $(D+1)^2(D^4 - 16)y = 0$

30. $(D^4 - 1)^2 y = 0$

31. Let D stand for d/dx, that is, $Dy = dy/dx$; then

$$D^2 y = D(Dy) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}, \quad D^3 y = \frac{d^3y}{dx^3}, \text{ etc.}$$

D (or an expression involving D) is called a differential operator. Two operators are equal if they give the same results when they operate on y. For example,

$$D(D+x)y = \frac{d}{dx}\left(\frac{dy}{dx} + xy\right) = \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = (D^2 + xD + 1)y$$

so we say that

$$D(D+x) = D^2 + xD + 1.$$

In a similar way show that:

(a) $(D-a)(D-b) = (D-b)(D-a) = D^2 - (b+a)D + ab$ for constant a and b.

(b) $D^3 + 1 = (D+1)(D^2 - D + 1)$.

(c) $Dx = xD + 1$. (Note that D and x do not commute, that is, $Dx \neq xD$.)

(d) $(D-x)(D+x) = D^2 - x^2 + 1$, but $(D+x)(D-x) = D^2 - x^2 - 1$.

Comment: The operator equations in (c) and (d) are useful in quantum mechanics; see Chapter 12, Section 22.

32. In Example 3, we used the second solution in (5.24), and obtained (5.25) as the particular solution satisfying the given initial conditions. Show that the first and third solutions in (5.24) also give the particular solution (5.25) satisfying the given initial conditions.

33. A particle moves along the x axis subject to a force toward the origin proportional to x (say $-kx$). Show that the particle executes simple harmonic motion (Example 3). Find the kinetic energy $\frac{1}{2}mv^2$ and the potential energy $\frac{1}{2}kx^2$ as functions of t and show that the total energy is constant. Find the time averages of the potential energy and the kinetic energy and show that these averages are each equal to one-half the total energy (see average values, Chapter 7, Section 4).

34. Find the equation of motion of a simple pendulum (see Chapter 7, Problem 2.13), that is, the differential equation for θ as a function of t. Show that, for small θ, this is approximately a simple harmonic motion equation, and find θ if $\theta = \theta_0$, $d\theta/dt = 0$ when $t = 0$.

35. The gravitational force on a particle of mass m inside the earth at a distance r from the center ($r <$ the radius of the earth R) is $F = -mgr/R$ (Chapter 6, Section 8, Problem 21). Show that a particle placed in an evacuated tube through the center of the earth would execute simple harmonic motion. Find the period of this motion.

36. Find (in terms of L and C) the frequency of electrical oscillations in a series circuit (Figure 1.1) if $R = 0$ and $V = 0$, but $I \neq 0$. (When you tune a radio, you are adjusting C and/or L to make this frequency equal to that of the radio station.)

37. A block of wood is floating in water; it is depressed slightly and then released to oscillate up and down. Assume that the top and bottom of the block are parallel planes which remain horizontal during the oscillations and that the sides of the block are vertical. Show that the period of the motion (neglecting friction) is $2\pi\sqrt{h/g}$, where h is the vertical height of the part of the block under water when it is floating at rest. *Hint:* Recall that the buoyant force is equal to the weight of displaced water.

38. Solve the RLC circuit equation [(5.33) or (5.34)] with $V = 0$ as we did (5.27), and write the conditions and solutions for overdamped, critically damped, and underdamped electrical oscillations in terms of the quantities R, L, and C.

39. (a) Find numerical values of the constants and computer plot together on the same axes graphs of (5.30), (5.31) and (5.32) in order to compare overdamped, critically damped, and oscillatory motion. *Suggested numbers:* Let $\omega = 1$, and $b = 13/5$, 1, 5/13 for the three kinds of motion. Let $y(0) = 1$ and $y'(0) = 0$.

(b) Repeat the problem with the same set of ω and b values and with $y(0) = 1$, but with $y'(0) = 1$.

(c) Again repeat, with $y'(0) = -1$.

40. The natural period of an undamped system is 3 sec, but with a damping force proportional to the velocity, the period becomes 5 sec. Find the differential equation of motion of the system and its solution.

6. SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO

So far we have considered second-order linear equations with constant coefficients and zero right-hand side (5.1). Such equations describe *free vibrations* or oscillations of mechanical or electrical systems. But often such systems are not free but are subject to an applied force or emf. The vibrations are then called *forced vibrations* and the differential equation describing the system is of the form

$$\begin{aligned} &a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = f(x), \quad \text{or} \\ &\frac{d^2y}{dx^2} + \frac{a_1}{a_2}\frac{dy}{dx} + \frac{a_0}{a_2}y = F(x). \end{aligned} \tag{6.1}$$

The function $f(x)$ is often called the forcing function; it represents the applied force or emf. We want to find the general solution of equations of the form (6.1).

Example 1. Consider the equation

$$(D^2 + 5D + 4)y = \cos 2x. \tag{6.2}$$

We already know (from Section 5, Example 1) the general solution of the corresponding equation (5.2) with the right-hand side equal to zero. This solution (5.9) is called the *complementary function*; it is not a solution of (6.2) but is related to it as we shall see. We shall denote the complementary function by y_c. Thus for equation (6.2) the complementary function is

$$y_c = Ae^{-x} + Be^{-4x}. \tag{6.3}$$

Now suppose we know just any solution of (6.2); we call this solution a *particular solution* and denote it by y_p. You can easily verify that

$$y_p = \tfrac{1}{10}\sin 2x \tag{6.4}$$

is a particular solution of (6.2), and we shall soon consider ways of finding such solutions. Then we have

$$(D^2 + 5D + 4)y_p = \cos 2x \tag{6.5}$$

and from Section 5, Example 1,

$$(D^2 + 5D + 4)y_c = 0. \tag{6.6}$$

Adding (6.5) and (6.6), we find

$$(D^2 + 5D + 4)(y_p + y_c) = \cos 2x + 0 = \cos 2x.$$

Thus

$$y = y_c + y_p = Ae^{-x} + Be^{-4x} + \tfrac{1}{10}\sin 2x \tag{6.7}$$

is a solution of (6.2). In fact, it is the general solution of (6.2) since it contains two independent arbitrary constants (Problem 27).

Thus we see how to solve 6.1):

> The general solution of an equation of the form (6.1) is
>
> (6.8) $$y = y_c + y_p$$
>
> where the complementary function y_c is the general solution of the homogeneous equation (as in Section 5) and y_p is a particular solution of (6.1).

We shall now discuss some ways of finding particular solutions. It is worthwhile to know about this even if you are using a computer to find the solution. When you know what to expect, you are better able to judge whether a computer solution is in the best form for your purposes, and if not, to find a better form. (See problems.)

Inspection If there *is* a very simple particular solution, we may be able to guess and verify it.

▶ **Example 2.** Consider the equation $y'' - 2y' + 3y = 5$.
It is easy to see that $y_p = \frac{5}{3}$ is a particular solution of this equation since if y is constant, y'' and y' are zero.

▶ **Example 3.** As a less trivial problem, consider

(6.9) $$y'' - 6y' + 9y = 8e^x.$$

We might suspect that a multiple of e^x is a solution of this equation, and it is easy to verify that $y = 2e^x$ is a solution. But trying the same method for the equation

(6.10) $$y'' + y' - 2y = e^x,$$

we fail to find a particular solution since e^x satisfies

$$y'' + y' - 2y = 0.$$

The method of inspection is very good in simple cases where it gives us an answer quickly, but usually we need other methods.

Successive Integration of Two First-Order Equations This is a straightforward method which can always be used to solve equations of the form (6.1). In practice, however, it often involves more work than various special methods; we shall find it particularly useful in deriving the special methods.

▶ **Example 4.** Let's solve (6.10) again. We can write this differential equation as

(6.11) $$(D - 1)(D + 2)y = e^x.$$

Let

(6.12) $$u = (D + 2)y.$$

Then the differential equation (6.11) becomes

$$(D-1)u = e^x \qquad \text{or} \quad u' - u = e^x. \tag{6.13}$$

This is a first-order linear differential equation which we solve as in Section 3.

$$\begin{aligned} I &= \int -dx = -x, \\ ue^{-x} &= \int e^{-x}e^x\,dx = x + c_1, \\ u &= xe^x + c_1e^x. \end{aligned} \tag{6.14}$$

Then the differential equation for y becomes

$$(D+2)y = xe^x + c_1e^x \qquad \text{or} \quad y' + 2y = xe^x + c_1e^x.$$

This is again a linear first-order equation which we solve as follows:

$$\begin{aligned} I &= \int 2\,dx = 2x, \\ ye^{2x} &= \int e^{2x}(xe^x + c_1e^x)\,dx = \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x} + \tfrac{1}{3}c_1e^{3x} + c_2 \\ &= \frac{1}{3}xe^{3x} + c_1'e^{3x} + c_2, \\ y &= \tfrac{1}{3}xe^x + c_1'e^x + c_2e^{-2x}. \end{aligned} \tag{6.15}$$

Notice that here we have obtained the general solution all in one process rather than finding the complementary function plus a particular solution in two separate processes. However, we could have obtained just the particular solution $xe^x/3$ by omitting the arbitrary constant at each integration (these led to the complementary function) and also dropping terms which are already in the complementary function ($-e^x/9$ in this example). Since it is easy to write the complementary function (by Section 5), it saves time to omit those terms when we are finding a particular solution. You may find that your computer gives a more complicated particular solution by including terms of the complementary function in the particular solution. Now that you know to watch for this, you can simplify a computer solution by removing those terms.

Exponential Right-Hand Side Let us consider how to find a particular solution when the right-hand side of (6.1) is $F(x) = ke^{cx}$ where k and c are given constants. Observe that c may be complex; we shall be especially interested in this case later. Let a and b be the roots of the auxiliary equation of (6.1); then (6.1) becomes

$$(D-a)(D-b)y = F(x) = ke^{cx}. \tag{6.16}$$

Let us first suppose that c is not equal to either a or b. Solving (6.16) by successive integration of two first-order equations as in the last paragraph is straightforward (Problem 28) and gives the result that the particular solution in this case is simply a multiple of e^{cx}. It is not necessary to remember the formula for the constant factor or to go through this process each time. Now that we know the form of the particular solution, we simply assume a solution of this form and solve for the constant.

▶ **Example 5.** Solve the equation

$$(D-1)(D+5)y = 7e^{2x}. \tag{6.17}$$

We observe that $c = 2$ is not equal to either of the roots of the auxiliary equation. To find a particular solution we substitute $y_p = Ce^{2x}$ into (6.17) and get

$$y_p'' + 4y_p' - 5y_p = C(4e^{2x} + 8e^{2x} - 5e^{2x}) = 7e^{2x}.$$

Thus we must have $C = 1$, and the general solution of (6.17) is

$$y = Ae^x + Be^{-5x} + e^{2x}.$$

We have already seen in solving (6.11) that if c is equal to either a or b ($a \neq b$), the particular solution is of the form Cxe^{cx}. By the same method used for (6.11), you can easily discover that if $a = b = c$, the particular solution is of the form Cx^2e^{cx} (Problem 28c). In practice, then, we find a particular solution of (6.16) by assuming a solution of the form:

$$\begin{cases} Ce^{cx} & \text{if } c \text{ is not equal to either } a \text{ or } b; \\ Cxe^{cx} & \text{if } c \text{ equals } a \text{ or } b,\ a \neq b; \\ Cx^2e^{cx} & \text{if } c = a = b. \end{cases} \tag{6.18}$$

Now that we know this, we would solve (6.10) as follows. Substitute

$$y_p = Cxe^x, \quad y_p' = C(xe^x + e^x), \quad y_p'' = C(xe^x + 2e^x)$$

into (6.10) and get

$$y_p'' + y_p' - 2y_p = C(xe^x + 2e^x + xe^x + e^x - 2xe^x) = e^x.$$

Thus we find $C = \frac{1}{3}$ as in (6.15) (but with much less work).

Use of Complex Exponentials In applied problems, the function $F(x)$ on the right-hand side of (6.1) is very often a sine or a cosine representing alternating emf or a periodic force. We *could* find y_p for such a problem either by the method of integrating two successive first-order equations or by replacing the sine or cosine by its complex exponential form and using the method of the last paragraph. There is a still more efficient variation of the latter method which we shall now show.

▶ **Example 6.** Solve

$$y'' + y' - 2y = 4\sin 2x. \tag{6.19}$$

Instead of tackling this problem directly, we are first going to solve the equation

$$Y'' + Y' - 2Y = 4e^{2ix}. \tag{6.20}$$

Since $e^{2ix} = \cos 2x + i\sin 2x$ is complex, the solution Y may be complex also. Then if $Y = Y_R + iY_I$, (6.20) is equivalent to two equations

$$\begin{aligned} Y_R'' + Y_R' - 2Y_R &= \operatorname{Re} 4e^{2ix} = 4\cos 2x, \\ Y_I'' + Y_I' - 2Y_I &= \operatorname{Im} 4e^{2ix} = 4\sin 2x. \end{aligned} \tag{6.21}$$

Since the second equation in (6.21) is the same as (6.19), we see that the solution of (6.19) is the imaginary part of Y. Thus to find y_p for (6.19), we find Y_p for (6.20) and take its imaginary part. We observe that $2i$ is not equal to either of the roots of the auxiliary equation in (6.20). Following the method of the last paragraph, we assume a solution of the form

$$Y_p = Ce^{2ix}$$

and substitute it into (6.20) to get

$$\begin{aligned}(-4 + 2i - 2)Ce^{2ix} &= 4e^{2ix},\\ C = \frac{4}{2i-6} = \frac{4(-2i-6)}{40} &= -\tfrac{1}{5}(i+3),\\ Y_p &= -\tfrac{1}{5}(i+3)e^{2ix}.\end{aligned}$$

Taking the imaginary part of Y_p, we find y_p for (6.19):

$$y_p = -\tfrac{1}{5}\cos 2x - \tfrac{3}{5}\sin 2x. \tag{6.22}$$

We summarize the method of complex exponentials:

(6.23) To find a particular solution of

$$(D-a)(D-b)y = \begin{cases} k\sin\alpha x,\\ k\cos\alpha x,\end{cases}$$

first solve

$$(D-a)(D-b)y = ke^{i\alpha x}$$

and then take the real or imaginary part.

Method of Undetermined Coefficients The method we have just discussed of assuming an exponential solution and determining the constant factor C is an example (and in practice the most important case) of the *method of undetermined coefficients*. In (6.18) we outlined the form of y_p to assume for equation (6.16), that is, when the right-hand side of (6.1) is an exponential. It is straightforward but tedious (Problems 29 and 32) to find the corresponding result (6.24) when the right-hand side is an exponential times a polynomial.

A particular solution y_p of $(D-a)(D-b)y = e^{cx}P_n(x)$ where $P_n(x)$ is a polynomial of degree n is

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if } c \text{ is not equal to either } a \text{ or } b,\\ xe^{cx}Q_n(x) & \text{if } c \text{ equals } a \text{ or } b,\ a \neq b,\\ x^2e^{cx}Q_n(x) & \text{if } c = a = b,\end{cases} \tag{6.24}$$

where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$ with *undetermined coefficients* to be found to satisfy the given differential equation. Note that sines and cosines are included in e^{cx} by use of complex exponentials as in (6.19) to (6.23). (Also see Problem 29.)

▶ **Example 7.** To illustrate using (6.24), let's find a particular solution of

$$(D-1)(D+2)y = y'' + y' - 2y = 18xe^x. \tag{6.25}$$

In the notation of (6.24) we have $a = 1$, $b = -2$, $c = 1$; also $P_n(x) = 18x = P_1(x)$ is a polynomial of degree 1. Then Q_1 is a polynomial of degree 1, namely $Ax + B$. Since $c = a \neq b$, we see by (6.24) that the form to assume for a particular solution of (6.25) is

$$y_p = xe^x(Ax + B) = e^x(Ax^2 + Bx).$$

We substitute this into (6.25) and find A and B so that we have an identity.

$$\begin{aligned} {y_p}' &= e^x(Ax^2 + Bx + 2Ax + B), \\ {y_p}'' &= e^x(Ax^2 + Bx + 4Ax + 2B + 2A) \\ {y_p}'' + {y_p}' - 2y_p &= e^x(6Ax + 3B + 2A) \equiv 18xe^x \end{aligned}$$

To make this an identity, we must have

$$6A = 18, \; 3B + 2A = 0, \quad \text{or} \quad A = 3, \; B = -2, \quad \text{so}$$

$$y_p = (3x^2 - 2x)e^x. \tag{6.26}$$

A computer solution may add to this a constant times e^x, but this is an unnecessary complication since e^x is a term in the complementary function.

If the right-hand side of a differential equation is a polynomial, then $c = 0$ in (6.24), and we assume for y_p a polynomial as indicated in (6.24).

▶ **Example 8.** To solve

$$(D-1)(D+2)y = y'' + y' - 2y = x^2 - x \tag{6.27}$$

we assume $y_p = Ax^2 + Bx + C$, and find the particular solution

$$y_p = -\tfrac{1}{2}(x^2 + 1). \tag{6.28}$$

A computer solution gives the same result.

▶ PROBLEMS, SECTION 6

Find the general solution of the following differential equations (complementary function + particular solution). Find the particular solution by inspection or by (6.18), (6.23), or (6.24). Also find a computer solution and reconcile differences if necessary, noticing especially whether the particular solution is in simplest form [see (6.26) and the discussion after (6.15)].

1. $y'' - 4y = 10$

2. $(D-2)^2y = 16$

3. $y'' + y' - 2y = e^{2x}$

4. $(D+1)(D-3)y = 24e^{-3x}$

5. $(D^2 + 1)y = 2e^x$

6. $y'' + 6y' + 9y = 12e^{-x}$

7. $y'' - y' - 2y = 3e^{2x}$

8. $y'' - 16y = 40e^{4x}$

9. $(D^2 + 2D + 1)y = 2e^{-x}$ **10.** $(D - 3)^2 y = 6e^{3x}$

11. $y'' + 2y' + 10y = 100 \cos 4x$ *Hint:* First solve $y'' + 2y' + 10y = 100e^{4ix}$.

12. $(D^2 + 4D + 12)y = 80 \sin 2x$ **13.** $(D^2 - 2D + 1)y = 2 \cos x$

14. $y'' + 8y' + 25y = 120 \sin 5x$ **15.** $5y'' + 12y' + 20y = 120 \sin 2x$

16. $(D^2 + 9)y = 30 \sin 3x$ **17.** $y'' + 16y = 16 \cos 4x$

18. $(D^2+2D+17)y = 60e^{-4x} \sin 5x$ *Hint:* First solve $(D^2+2D+17)y = 60e^{(-4+5i)x}$.

19. $(4D^2 + 4D + 5)y = 40e^{-3x/2} \sin 2x$ **20.** $y'' + 4y' + 8y = 30e^{-x/2} \cos 5x/2$

21. $5y'' + 6y' + 2y = x^2 + 6x$ **22.** $2y'' + y' = 2x$

23. $y'' + y = 2xe^x$ **24.** $y'' - 6y' + 9y = 12xe^{3x}$

25. $(D - 3)(D + 1)y = 16x^2 e^{-x}$ **26.** $(D^2 + 1)y = 8x \sin x$

27. Verify that (6.4) is a particular solution of (6.2). Verify that another particular solution of (6.2) is

$$y_p = \tfrac{1}{10} \sin 2x - e^{-x}.$$

Observe that we obtain the same general solution (6.7) whichever particular solution we use [since $(A - 1)$ is just as good an arbitrary constant as A]. Show in general that the difference between two particular solutions of $(a_2 D^2 + a_1 D + a_0)y = f(x)$ is always a solution of the homogeneous equation $(a_2 D^2 + a_1 D + a_0)y = 0$, and thus show that the general solution is the same for all choices of a particular solution.

28. Solve (6.16) by the method used in solving (6.11), for the following three cases, to obtain the result (6.18).

(a) c is not equal to either a or b;

(b) $a \neq b$, $c = a$;

(c) $a = b = c$.

29. Consider the differential equation $(D - a)(D - b)y = P_n(x)$, where $P_n(x)$ is a polynomial of degree n. Show that a particular solution of this equation is given by (6.24) with $c = 0$; that is, y_p is

$$\begin{cases} \text{a polynomial } Q_n(x) \text{ of degree } n \text{ if } a \text{ and } b \text{ are both different from zero;} \\ xQ_n(x) \quad \text{if } a \neq 0, \text{ but } b = 0; \\ x^2 Q_n(x) \quad \text{if } a = b = 0. \end{cases}$$

Hint: To show that $Q_n(x) = \sum a_n x^n$ is a solution of the differential equation for a given $P_n = \sum b_n x^n$, you have only to show that the coefficients a_n can be found so that $(D - a)(D - b)Q_n(x) \equiv P_n(x)$. Equate coefficients of x^n, x^{n-1}, $\cdots$, to see that this is always possible if $a \neq b$. For $b = 0$, the differential equation becomes $(D - a)Dy = P_n$; what is Dy if $y = xQ_n$? Similarly, consider $D^2 y$ if $y = x^2 Q_n$.

30. (a) Show that

$$\begin{aligned} (D - a)e^{cx} &= (c - a)e^{cx}; \\ (D^2 + 5D - 3)e^{cx} &= (c^2 + 5c - 3)e^{cx}; \\ L(D)e^{cx} &= L(c)e^{cx}, \text{ where } L(D) \text{ is any polynomial in } D; \\ (D - c)xe^{cx} &= e^{cx}; \\ (D - c)^2 x^2 e^{cx} &= 2e^{cx}. \end{aligned}$$

(b) Define the expression $y = [1/L(D)]u(x)$ to mean a solution of the differential equation $L(D)y = u$. Using part (a), show that

$$\frac{1}{D-a}e^{cx} = \frac{e^{cx}}{c-a}, \quad c \neq a;$$
$$\frac{1}{D^2+5D-3}e^{cx} = \frac{e^{cx}}{c^2+5c-3};$$
$$\frac{1}{L(D)}e^{cx} = \frac{e^{cx}}{L(c)}, \quad L(c) \neq 0;$$
$$\frac{1}{D-c}e^{cx} = xe^{cx};$$
$$\frac{1}{(D-c)^2}e^{cx} = \tfrac{1}{2}x^2e^{cx}.$$

(c) The expressions $1/L(D)$ in (b) are called inverse operators. They can be used to find particular solutions of differential equations. As an example consider Problem 3. We write

$$(D^2 + D - 2)y = e^{2x},$$
$$y = \frac{1}{D^2+D-2}e^{2x} = \frac{e^{2x}}{2^2+2-2} = \frac{e^{2x}}{4}.$$

Using inverse operators, find particular solutions of Problems 4 to 20. Be careful to use parts 4 or 5 of (b) if c is a root of the auxiliary equation. For example,

$$\frac{1}{(D-a)(D-c)}e^{cx} = \frac{1}{D-c}\frac{1}{D-a}e^{cx} = \frac{1}{D-c}\frac{e^{cx}}{c-a} = \frac{xe^{cx}}{c-a}.$$

31. (a) Show that

$$D(e^{ax}y) = e^{ax}(D+a)y,$$
$$D^2(e^{ax}y) = e^{ax}(D+a)^2y,$$

and so on; that is, for any positive integral n,

$$D^n(e^{ax}y) = e^{ax}(D+a)^ny.$$

Thus show that if $L(D)$ is any polynomial in the operator D, then

$$L(D)(e^{ax}y) = e^{ax}L(D+a)y.$$

This is called the *exponential shift*.

(b) Use (a) to show that

$$(D-1)^3(e^xy) = e^xD^3y,$$
$$(D^2+D-6)(e^{-3x}y) = e^{-3x}(D^2-5D)y.$$

(c) Replace D by $D-a$, to obtain

$$e^{ax}P(D)y = P(D-a)e^{ax}y.$$

This is called the *inverse exponential shift*.

(d) Using (c), we can change a differential equation whose right-hand side is an exponential times a polynomial, to one whose right-hand side is just a polynomial. For example, consider $(D^2 - D - 6)y = 10xe^{3x}$; multiplying both sides by e^{-3x} and using (c), we get

$$\begin{aligned} e^{-3x}(D^2 - D - 6)y &= [(D+3)^2 - (D+3) - 6]ye^{-3x} \\ &= (D^2 + 5D)ye^{-3x} = 10x. \end{aligned}$$

Show that a solution of $(D^2+5D)u = 10x$ is $u = x^2 - \frac{2}{5}x$; then $ye^{-3x} = x^2 - \frac{2}{5}x$ or $y = e^{3x}(x^2 - \frac{2}{5}x)$. Use this method to solve Problems 23 to 26.

32. Using Problems 29 and 31b, show that equation (6.24) is correct.

Several Terms on the Right-Hand Side: Principle of Superposition So far we have brushed over a question which may have occurred to you: What do we do if there are several terms on the right-hand side of the equation involving different exponentials?

Example 9. As an artificial problem to illustrate the ideas, consider the equation

$$y'' + y' - 2y = (D-1)(D+2)y = [e^x] + [4\sin 2x] + [x^2 - x]. \tag{6.29}$$

We have already solved differential equations with the same left-hand sides as (6.29) and with right-hand sides equal in turn to each of the three expressions in brackets in (6.29) [see (6.11) to (6.15), (6.19) to (6.22), (6.27), and (6.28)]. Thus we know that

$(D-1)(D+2)y = e^x$ has the particular solution $y_{p1} = \frac{1}{3}xe^x$;
$(D-1)(D+2)y = 4\sin 2x$ has the particular solution $y_{p2} = -\frac{1}{5}\cos 2x - \frac{3}{5}\sin 2x$;
$(D-1)(D+2)y = x^2 - x$ has the particular solution $y_{p3} = -\frac{1}{2}(x^2 + 1)$.

Adding these three solutions, we see that

$$y_p = y_{p1} + y_{p2} + y_{p3} = \tfrac{1}{3}xe^x - \tfrac{1}{5}\cos 2x - \tfrac{3}{5}\sin 2x - \tfrac{1}{2}(x^2+1) \tag{6.30}$$

is a particular solution of (6.29).

This is the easiest way of handling a complicated right-hand side: Solve a separate equation for *each different exponential* and add the solutions. The fact that this is correct for a linear equation is often called the *principle of superposition*. As we can see from (6.29) and (6.30), this amounts to a fancy name for the fact that the derivative (of any order) of a sum of terms is equal to the sum of the derivatives of the individual terms. Notice that the principle holds only for *linear* equations; for example, if the equation contained ${y'}^2$, the principle would not hold since $(y_1' + y_2')^2$ is not equal to ${y_1'}^2 + {y_2'}^2$. In fact, an operator (such as the D operators we have been using) which satisfies the principle of superposition is called a *linear operator*. [See Chapter 3, equation (7.4) and Problem 7.12.] Linear operators are of particular importance because they obey the principle of superposition; for example, $D^2(y_1 + y_2) = y_1'' + y_2'' = D^2y_1 + D^2y_2$, so D^2 is a linear operator. We shall make use of this principle shortly in our discussion of the use of Fourier series in finding particular solutions.

Forced Vibrations Let's return now to the physical problem we considered at the end of Section 5. There we set up and solved the differential equation which describes the free (zero right-hand side, no forcing function) vibrations of a damped oscillator. We commented that the same mathematics applies to a variety of mechanics problems and also to a simple RLC series electric circuit. As we know from experiment and as we can see from (5.30), (5.31), and (5.32), the free vibrations we considered in Section 5 die out as time passes. Such oscillations are referred to as *transients*. We next want to consider the vibrations obtained when a periodic force (or emf in the electric case) is applied. This means mathematically that we want to solve (5.27) with a function of t on the right-hand side. The solution will contain the appropriate one of (5.30), (5.31), (5.32); this is the complementary function and it is also the transient since it tends to zero as t tends to infinity. The solution will also contain a particular solution which does not tend to zero as t tends to infinity; this is the *steady-state solution* which we want to find.

▶ **Example 10.** Let us solve

(6.31) $$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = F\sin\omega' t \qquad (F = \text{const.}).$$

By the method of complex exponentials, we solve first

(6.32) $$\frac{d^2Y}{dt^2} + 2b\frac{dY}{dt} + \omega^2 Y = Fe^{i\omega' t}.$$

Substitute

(6.33) $$Y_p = Ce^{i\omega' t}$$

into (6.32) to get

(6.34) $$(-\omega'^2 + 2bi\omega' + \omega^2)Ce^{i\omega' t} = Fe^{i\omega' t},$$
$$C = \frac{F}{(\omega^2 - \omega'^2) + 2bi\omega'} = \frac{[(\omega^2 - \omega'^2) - 2bi\omega']F}{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}.$$

It is convenient to write the complex number C in the $re^{i\theta}$ form. We have

(6.35) $$|C| = \frac{F}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}},$$
angle of $C = -\phi$, where ϕ is given by Figure 6.1.

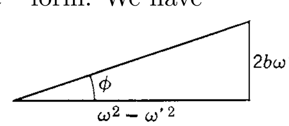


Figure 6.1

Thus

(6.36) $$C = \frac{F}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}e^{-i\phi}$$

and from (6.33)

(6.37) $$Y_p = \frac{F}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}e^{i(\omega' t - \phi)}$$

To find y_p we take the imaginary part of Y_p:

$$y_p = \frac{F}{\sqrt{\left(\omega^2 - \omega'^2\right)^2 + 4b^2\omega'^2}} \sin(\omega' t - \phi). \tag{6.38}$$

This is the steady-state solution, so-called because as t increases, the rest of the solution [given by (5.30), (5.31), or (5.32)] becomes negligible. For example, when you turn on an electric light, the current is given by (5.32) plus (6.38). The transient (5.32) tends to zero rapidly and the steady-state solution (6.38) becomes essentially the whole solution.

Resonance We note, by comparing (6.38) and the forcing function in (6.31), that the applied force (or emf) and the solution y (which represents displacement, current, etc.) are *out of phase*; that is, their maximum values do not occur at the same time because of the phase angle ϕ. We also see from (6.38) that for a given forcing frequency ω', the largest amplitude of y (also of dy/dt, Problem 40) occurs if the natural (undamped) frequency ω is equal to ω'. This situation is often called *resonance*. In the RLC series circuit problem, y represents the charge q on the capacitor if the forcing function is the emf, and y represents the current $I = dq/dt$ if the forcing function is the time derivative of the emf. For such a circuit, given the frequency ω' of the applied emf, the current (or charge) will have the largest amplitude when the natural (undamped) frequency ω is equal to ω'. This is almost always called the resonance condition for the electrical case. However, there is another question we could ask here which is of particular interest in mechanics. Given the natural (undamped) frequency ω of the system, what frequency of the forcing function will produce the largest amplitude of y? In (6.38), we want to maximize the coefficient of the sine; we can instead minimize the square of the denominator of the coefficient; that is, we want to find the value of ω' which minimizes $(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2$ for given ω. Setting the derivative of this function (with respect to ω') equal to zero and solving for ω', we get

$$\begin{aligned} 2(\omega^2 - \omega'^2)(-2\omega') + 8b^2\omega' &= 0, \\ \omega'^2 &= \omega^2 - 2b^2. \end{aligned} \tag{6.39}$$

Note that this value of ω' is not equal to either the natural undamped frequency ω or the natural damped frequency β where $\beta^2 = \omega^2 - b^2$ [see (5.32)]. However, if we define *resonance* as the situation in which we get the maximum amplitude for y for a given value of ω, then the resonance condition is (6.39). (The maximum amplitude for the velocity—or current in the electrical case—is still obtained for $\omega' = \omega$; Problem 40.) The resonance condition (6.39) is of particular importance in mechanics where we are apt to be interested in the displacement y of a given system under the action of various forces. For example, consider a bridge; we would want to avoid periodic forces with an ω' given by (6.39) since such forces would produce large vibrations. In this case resonance is undesirable. It may in other cases be desirable; for example, when you tune your radio to the frequency of a given station, you are given ω' and you adjust the circuit in your radio to make its natural frequency ω equal to the given ω'.

Use of Fourier Series in Finding Particular Solutions In simple problems, the forcing function in either the electrical or mechanical case is just a sine or cosine and the problem can be solved as we have just done. In more complicated (and realistic) cases, however, the forcing function may very well be some more complicated function; it is often a periodic function, however, and we shall assume this. Suppose, for example, that the periodic emf applied to a circuit is given by one of the graphs in Figure 3.2 of Chapter 7. We learned in Chapter 7 how to expand such a function in a Fourier series. Let us suppose that this has been done, using for definiteness the complex exponential form of the Fourier series. Then we can write (6.1) as

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{6.40}$$

We know how to solve the equation

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = c_n e^{inx} \tag{6.41}$$

with the right-hand side equal to any one term of the series. If we now add the solutions of all the equations (6.41) for all n, we have a solution of (6.40) (see *principle of superposition* above).

Example 11. Solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = f(t), \tag{6.42}$$

where $f(t)$ is a function of period 2π and

$$f(t) = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < 2\pi. \end{cases}$$

The auxiliary equation is

$$D^2 + 2D + 10 = 0;$$

its roots are

$$D = -1 \pm 3i,$$

so the complementary function is

$$y_c = e^{-t}(A\cos 3t + B\sin 3t).$$

To find a particular solution, we first expand $f(t)$ in a Fourier series; from Chapter 7, equation (7.8), we have

$$f(t) = \frac{1}{2} + \frac{1}{i\pi}[e^{it} - e^{-it} + \tfrac{1}{3}(e^{3it} - e^{-3it}) + \cdots]. \tag{6.43}$$

We next write and solve a whole set of differential equations like (6.42) but each having just one term of the series (6.43) on the right-hand side. For the first term (namely $\frac{1}{2}$) we see by inspection that a particular solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = \frac{1}{2}$$

is $y = \frac{1}{20}$. All the other terms of (6.43) are of the form $(1/ik\pi)e^{ikt}$, where k is a positive or negative odd integer. To solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = \frac{1}{ik\pi}e^{ikt}, \tag{6.44}$$

we substitute

$$y = Ce^{ikt} \tag{6.45}$$

into (6.44) and get

$$(-k^2 + 2ik + 10)Ce^{ikt} = \frac{1}{ik\pi}e^{ikt}.$$

Then we have

$$C = \frac{1}{ik\pi}\frac{1}{(10 - k^2) + 2ik} = \frac{1}{ik\pi}\frac{(10 - k^2) - 2ik}{(10 - k^2)^2 + 4k^2}. \tag{6.46}$$

By letting $k = \pm 1, \pm 3, \cdots$, and substituting the values of C thus obtained into (6.45), we obtain the solutions of (6.44) for the various k values corresponding to the terms of the series (6.43). The sum of all the solutions corresponding to all the terms is the desired particular solution of (6.42). Thus

$$\begin{aligned} y_p &= \frac{1}{20} + \frac{1}{i\pi}\frac{9 - 2i}{85}e^{it} - \frac{1}{i\pi}\frac{9 + 2i}{85}e^{-it} \\ &\quad + \frac{1}{3i\pi}\frac{1 - 6i}{37}e^{3it} - \frac{1}{3i\pi}\frac{1 + 6i}{37}e^{-3it} + \cdots \\ &= \frac{1}{20} + \frac{2}{\pi}\frac{9}{85}\left(\frac{e^{it} - e^{-it}}{2i}\right) - \frac{2}{\pi}\frac{2}{85}\left(\frac{e^{it} + e^{-it}}{2}\right) \\ &\quad + \frac{2}{3\pi}\frac{1}{37}\left(\frac{e^{3it} - e^{-3it}}{2i}\right) - \frac{2}{3\pi}\frac{6}{37}\left(\frac{e^{3it} + e^{-3it}}{2}\right) + \cdots \\ &= \frac{1}{20} + \frac{2}{85\pi}(9\sin t - 2\cos t) + \frac{2}{111\pi}(\sin 3t - 6\cos 3t) + \cdots \end{aligned} \tag{6.47}$$

is a particular solution of (6.42).

PROBLEMS, SECTION 6

In Problem 33 to 38, solve the given differential equations by using the principle of superposition [see the solution of equation (6.29)]. For example, in Problem 33, solve three differential equations with right-hand sides equal to the three different brackets. Note that terms with the *same exponential factor* are kept together; thus a polynomial of any degree is kept together in one bracket.

33. $y'' + y = [x^3 - 1] + [2\cos x] + [(2 - 4x)e^x]$

34. $y'' - 5y' + 6y = 2e^x + 6x - 5$

35. $(D^2 - 1)y = \sinh x$

36. $(D^2 + 1)y = 2\sin x + 4x\cos x$

37. $(D - 1)^2 y = 4e^x + (1 - x)(e^{2x} - 1)$

38. $y'' - 2y' = 9xe^{-x} - 6x^2 + 4e^{2x}$

39. Find the solutions of (1.2) (put $I = dq/dt$) and (1.3), if $V = V_0 \sin \omega' t$ (ω' =const.).

40. In (6.38), show that for a given forcing frequency ω', the displacement y and the velocity dy/dt have their largest amplitude when $\omega = \omega'$.

For a given ω, we have shown in Section 6 that the maximum amplitude of y does not correspond to $\omega' = \omega$. Show, however, that the maximum amplitude of dy/dt for a given ω does correspond to $\omega' = \omega$.

State the corresponding results for an electric circuit in terms of L, R, C.

Solve Problems 41 and 42 by use of Fourier series. Assume in each case that the right-hand side is a periodic function whose values are stated for one period.

41. $y'' + 2y' + 2y = |x|, \qquad -\pi < x < \pi.$

42. $y'' + 9y = \begin{cases} x, & 0 < x < 1, \\ 0, & -1 < x < 0. \end{cases}$

43. Consider an equation for damped forced vibrations (mechanical or electrical) in which the right-hand side is a sum of several forces or emfs of different frequencies. For example, in (6.32) let the right-hand side be

$$F_1 e^{i\omega_1' t} + F_2 e^{i\omega_2' t} + F_3 e^{i\omega_3' t}.$$

Write the solution by the principle of superposition. Suppose, for given ω_1', ω_2', ω_3', that we adjust the system so that $\omega = \omega_1'$; show that the principal term in the solution is then the first one. Thus the system acts as a “filter” to select vibrations of one frequency from a given set (for example, a radio tuned to one station selects principally the vibrations of the frequency of that station).

▶ 7. OTHER SECOND-ORDER EQUATIONS

Although second-order linear equations with constant coefficients are the ones used most frequently in applications, there are a few other kinds of second-order equations and methods of solving them which are also important. We shall discuss several of these here, namely (a) equations with y missing; (b) equations with x missing; (c) equations of the form $y'' + f(y) = 0$; (d) Euler-Cauchy equations; (e) reduction of order. For still more methods, see Section 9 (Laplace transforms), Section 12 (Green functions), Problem 12.14b (variation of parameters), and Chapter 12 (special functions, series solutions, ladder operators). You can also find computer solutions but, as we have said, they may not always be in the simplest form or the form you need. Comparing hand solutions can show you what to expect and help you make more efficient use of computer solutions.

To solve either (a) or (b), we make the substitution

$$y' = p. \tag{7.1}$$

Case (a): Dependent variable y missing.

$$y' = p, \quad y'' = p'. \tag{7.2}$$

After these substitutions, an equation of the type (a) is of the first order with p

as the dependent variable and x as the independent variable. First, we solve it for p as a function of x; then we put back $p = y'$ and solve the resulting first-order equation for y.

Case (b): Independent variable x missing.

$$y' = p, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}. \tag{7.3}$$

What we are doing here is to change the independent variable from x to y. Observe that there is *one* independent variable in an ordinary differential equation. We were originally thinking of x as the independent variable with y and $p\,dy/dx$ as functions of x. Now we think of y as the independent variable with p a function of y; (7.3) is just the chain rule (Chapter 4, Section 5) for differentiating a function $p(y)$ with respect to x if y is a function of x. With the substitutions (7.3), a differential equation with x missing becomes a first-order equation with p as the dependent variable and y as the independent variable.

Example 1. In Section 5, we discussed the motion of a mass m subject to a restoring force $-ky$ and a damping force $l(dy/dt)$. Let us now consider a similar problem but with the damping force proportional to the square of the velocity. The differential equation of motion is then [compare (5.26)]

$$m\frac{d^2y}{dt^2} \pm l\left(\frac{dy}{dt}\right)^2 + ky = 0 \qquad (l > 0), \tag{7.4}$$

where the plus or minus sign must be chosen correctly at each stage of the motion so that the retarding force opposes the motion. Let us solve the following special case of this problem. Discuss the motion of a particle which is released from rest at the point $y = 1$ when $t = 0$, and obeys the equation of motion

$$4\frac{d^2y}{dt^2} \pm 2\left(\frac{dy}{dt}\right)^2 + y = 0. \tag{7.5}$$

This is an example of case (b) (for "x missing" read "t missing," that is, the independent variable missing). Using (7.3) (with x replaced by t), we have

$$\begin{aligned} \frac{dy}{dt} &= p, \\ \frac{d^2y}{dt^2} &= p\frac{dp}{dy}, \end{aligned} \tag{7.6}$$

so (7.5) becomes

$$4p\frac{dp}{dy} \pm 2p^2 + y = 0 \qquad \text{or} \qquad \frac{dp}{dy} \pm \tfrac{1}{2}p = -\tfrac{1}{4}yp^{-1}. \tag{7.7}$$

This is a Bernoulli equation [compare (4.1) with y replaced by p, and P and Q functions of y]. We have $n = -1$, and the substitution (4.2) is

$$z = p^2. \tag{7.8}$$

Then

$$\frac{dz}{dy} = 2p\frac{dp}{dy}$$

and (7.7) becomes

$$\frac{dz}{dy} \pm z = -\tfrac{1}{2}y. \tag{7.9}$$

This is a first-order linear equation; solving it (see Section 3), we get

$$\begin{aligned} ze^{\pm y} &= -\tfrac{1}{2}\int ye^{\pm y}\,dy = -\tfrac{1}{2}e^{\pm y}(\pm y - 1) + c, \\ z &= -\tfrac{1}{2}(\pm y - 1) + ce^{\mp y}. \end{aligned} \tag{7.10}$$

Since initially $dy/dt = 0$ and $y > 0$, we see from (7.5) that the initial acceleration is in the negative direction; since the particle starts from rest, its velocity for small t is also in the negative direction. Then the damping force must be in the positive direction so we must use the lower sign in (7.10) for the first part of the motion. Thus we have

$$z = \tfrac{1}{2}(y + 1) + ce^{y} \qquad \text{(for small } t\text{)}. \tag{7.11}$$

We determine c from the initial conditions $dy/dt = 0$, $y = 1$, at $t = 0$; we have $z = p^2 = (dy/dt)^2 = 0$ when $y = 1$; therefore from (7.11) we get

$$0 = 1 + ce, \quad c = -e^{-1}.$$

Then we have

$$z = \left(\frac{dy}{dt}\right)^2 = \tfrac{1}{2}(y + 1) - e^{y-1} \qquad \text{(for small } t\text{)}. \tag{7.12}$$

This is a valid solution as long as $dy/dt < 0$ (this is what small t means). Thus the particle initially moves in the negative direction for a while. To continue the problem we would need to find whether it stops and if so where. This means solving a transcendental equation which has to be done by some approximation method. It turns out that when it stops, y is negative; at this point the force $-y$ is in the positive direction and the particle, after stopping, moves in the positive direction. The solution for $(dy/dt)^2$ is then given by (7.10) with the upper sign. After another interval of time, the particle again reverses its motion and we again use the solution (7.11) (with a different c), and so on, the total motion appearing something like a damped vibration. We shall not continue the details further here since we have already accomplished our purpose of illustrating case (b) and the solution of a Bernoulli equation.

Case (c) appears to be very special and is obviously included by (b); however, it is very important to know the easy way to solve it because it so frequently arises in applications. The trick is simply to multiply the equation by y'; we can then integrate each term.

Case (c): To solve $y'' + f(y) = 0$, multiply by y'.

$$y'y'' + f(y)y' = 0, \qquad \text{or} \quad y'\,dy' + f(y)\,dy = 0.$$

Then integrate to get

$$\tfrac{1}{2}y'^2 + \int f(y)\,dy = \text{const.} \tag{7.13}$$

This equation is separable and so can be solved (except for possible difficulty in evaluating the integrals). We say that the problem is reduced to *quadratures* (indicated integrations); this means that we can write the answer in terms of integrals which may or may not be easy to evaluate!

Example 2. Consider a particle of mass m moving along the x axis under the action of a force $F(x)$. Then the equation of motion is

$$m\frac{d^2x}{dt^2} = F(x). \tag{7.14}$$

If we multiply this equation by $v = dx/dt$ and integrate with respect to t, we get

$$mv\frac{dv}{dt} = F(x)\frac{dx}{dt} \qquad \text{or} \quad mv\,dv = F(x)\,dx,$$

$$\tfrac{1}{2}mv^2 = \int F(x)\,dx + \text{const.} \tag{7.15}$$

Recall (Chapter 6, Section 8) that the potential energy of a particle is the negative of the work done by the force. Thus

$$\tfrac{1}{2}mv^2 - \int F(x)\,dx \tag{7.16}$$

is the kinetic energy plus the potential energy; equation (7.15) expresses the law of conservation of energy for this problem. This energy equation is often of more interest than the equation of motion (x as a function of t) and so it is useful to be able to find it directly, as we have done, without solving the differential equation for x. Equation (7.15) is known as a *first integral* of the differential equation since we have integrated a second-order equation *once* to get it.

Case (d): An equation of the form

$$a_2x^2\frac{d^2y}{dx^2}+a_1x\frac{dy}{dx}+a_0y=f(x) \tag{7.17}$$

(called an Euler or Cauchy equation) can be reduced to a linear equation with constant coefficients by changing the independent variable from x to z where

$$x=e^z. \tag{7.18}$$

For then we have (see Problem 14 and also Chapter 4, Section 11)

$$x\frac{dy}{dx}=\frac{dy}{dz} \qquad \text{and} \quad x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}. \tag{7.19}$$

Substituting (7.18) and (7.19) into (7.17) gives

$$a_2\frac{d^2y}{dz^2}+(a_1-a_2)\frac{dy}{dz}+a_0y=f(e^z). \tag{7.20}$$

This is a linear equation with constant coefficients which can be solved by the methods of Sections 5 and 6.

It is worth noting that the solutions of (7.17) when $f(x)=0$ are often powers of x, so a way to solve this case is to assume $y=x^k$ and solve the resulting quadratic equation for k. However, if the values of k turn out to be complex, or equal, or if $f(x)\neq 0$, you may find it easier to use (7.18) which reduces the problem to a familiar one. (See Problems 15 to 23.)

Case (e): Reduction of order. To find a second solution of

$$y''+f(x)y'+g(x)y=0 \tag{7.21}$$

given one solution $u(x)$, substitute

$$y=u(x)v(x) \tag{7.22}$$

into (7.21) and solve for $v(x)$.

You can verify that when you substitute (7.22) into (7.21), the coefficient of $v(x)$ is $u''+f(x)u'+g(x)u$. This expression is equal to zero because we assumed that $u(x)$ is a solution of (7.21). Then the equation for $v'(x)$ is a separable first-order equation (Problem 24).

▶ **Example 3.** Solve $x^3y''+xy'-y=0$, given that $u=x$ is a solution.

We let $y=uv=xv$. Then $y'=xv'+v$, $y''=xv''+2v'$, and the differential equation becomes

$$x^3(xv''+2v')+x(xv'+v)-xv=0 \qquad \text{or}$$
$$x^4v''+(2x^3+x^2)v'=0.$$

Separating variables and integrating, we find

$$\frac{dv'}{v'} = -\left(\frac{2}{x} + \frac{1}{x^2}\right)dx, \quad \ln v' = -2\ln x + \frac{1}{x} + \ln K.$$

Solving for v', integrating again, and writing $y = uv$ gives

$$v' = \frac{K}{x^2}e^{1/x}, \quad v = -Ke^{1/x}, \quad y = -Kxe^{1/x}.$$

Thus the general solution of the given equation is $y = Ax + Bxe^{1/x}$.

PROBLEMS, SECTION 7

Solve the following differential equations by method (a) or (b) above.

1. $y''+yy' = 0$. Find a solution satisfying each of the following sets of initial conditions. If your computer says there is no such solution, don't believe it—do it by hand.

(a) $y(0) = 5, \quad y'(0) = 0$ (b) $y(0) = 2, \quad y'(0) = -2$

(c) $y(0) = 1, \quad y'(0) = -1$ (d) $y(0) = 0, \quad y'(0) = 2$

2. $y'' + 2xy' = 0$ *Hint:* The solution is $y = c_1 \operatorname{erf} x + c_2$; see Chapter 11, Section 9 for the definition of $\operatorname{erf} x$.

3. $2yy'' = y'^2$ **4.** $xy'' = y' + y'^3$

5. The differential equation of a hanging chain supported at its ends is

$$y''^2 = k^2\left(1 + y'^2\right).$$

Solve the equation to find the shape of the chain.

6. The curvature of a curve in the (x, y) plane is

$$K = y''\left(1 + y'^2\right)^{-3/2}.$$

With $K =$ const., solve this differential equation to show that curves of constant curvature are circles (or straight lines).

7. Solve $y'' + \omega^2 y = 0$ by method (c) above and compare with the solution as a linear equation with constant coefficients.

8. The force of gravitational attraction on a mass m at distance r from the center of the earth ($r >$ radius R of the earth) is mgR^2/r^2. Then the differential equation of motion of a mass m projected radially outward from the surface of the earth, with initial velocity v_0, is

$$md^2r/dt^2 = -mgR^2/r^2.$$

Use method (c) above to find v as a function of r if $v = v_0$ initially (that is, when $r = R$). Find the maximum value of r for a given v_0, that is, the value of r when $v = 0$. Find the *escape velocity*, that is, the smallest value of v_0 for which r can tend to infinity.

9. Show that (7.15) is a separable equation. [You may find it helpful to write $\int F(x)\,dx = f(x)$.] Thus solve (7.14) in terms of quadratures (that is, indicated integrations) as in Problem 2.

In Problems 10 and 11, solve (7.14) to find $v(x)$ and then $x(t)$ for the given $F(x)$ and initial conditions.

10. $F(x) = m/x^3$, $v = 0$, $x = 1$, at $t = 0$.

11. $F(x) = -2m/x^5$, $v = -1$, $x = 1$, at $t = 0$.

12. In Problem 11, find $v(x)$ if $v = 0$, $x = 1$, at $t = 0$. Then write an integral for $t(x)$.

13. The exact equation of motion of a simple pendulum is $d^2\theta/dt^2 = -\omega^2 \sin\theta$ where $\omega^2 = g/l$. By method (c) above, integrate this equation once to find $d\theta/dt$ if $d\theta/dt = 0$ when $\theta = 90°$. Write a formula for $t(\theta)$ as an integral. See Problem 5.34.

14. Verify (7.19) and (7.20). *Hint:* $dy/dz = (dy/dx)(dx/dz)$; write the first equation of (7.19) as $xD_x = D_z$, and find D_z^2.

15. If you solve (7.17) when $f(x) = 0$ by assuming a solution $y = x^k$, show that the quadratic equation for k is the same as the auxiliary equation for the z equation (7.20). Thus show (see Section 5) that if the two values of k are equal, the second solution is not a power of x but is $x^k \ln x$. Also show that if k is complex, say $k = a \pm bi$, the solutions are $x^a \cos(b \ln x)$ and $x^a \sin(b \ln x)$ or other equivalent forms [see (5.16) to (5.18)].

16. Solve the following equations either by method (d) above or by assuming $y = x^k$ (or try both methods to compare them). See Problem 15.

(a) $x^2y'' + 3xy' - 3y = 0$ (b) $x^2y'' + xy' - 4y = 0$

(c) $x^2y'' + 7xy' + 9y = 0$ (d) $x^2y'' - xy' + 6y = 0$

Solve the following equations using method (d) above.

17. $x^2y'' + xy' - 16y = 8x^4$ **18.** $x^2y'' + xy' - y = x - x^{-1}$

19. $x^2y'' - 5xy' + 9y = 2x^3$ **20.** $x^2y'' - 3xy' + 4y = 6x^2 \ln x$

21. $x^2y'' + y = 3x^2$ **22.** $x^2y'' + xy' + y = 2x$

23. Solve the two differential equations in Problem 5.11 of Chapter 13.

24. Substitute (7.22) into (7.21) to obtain the equation for $v'(x)$. Show that this equation is separable.

For the following problems, verify the given solution and then, by method (e) above, find a second solution of the given equation.

25. $x^2(2 - x)y'' + 2xy' - 2y = 0, \quad u = x$

26. $(x^2 + 1)y'' - 2xy' + 2y = 0, \quad u = x$

27. $xy'' - 2(x + 1)y' + (x + 2)y = 0, \quad u = e^x$

28. $3xy'' - 2(3x - 1)y' + (3x - 2)y = 0, \quad u = e^x$

29. $x^2y'' + (x + 1)y' - y = 0, \quad u = x + 1$

30. $x(x + 1)y'' - (x - 1)y' + y = 0, \quad u = x - 1$

8. THE LAPLACE TRANSFORM

As you will see in Section 9, Laplace transforms are useful in solving differential equations (for other uses see end of Section 9, page 442). Here we want to define the Laplace transform and obtain some needed formulas. We define $L(f)$, the Laplace transform of $f(t)$ [also written $F(p)$ since it is a function of p], by the equation

$$L(f) = \int_0^\infty f(t)e^{-pt}\,dt = F(p). \tag{8.1}$$

This is an example of an *integral transform* (also see Fourier transforms, Chapter 7, Section 12, and Hilbert transforms, Chapter 14, page 698). If we start with a function $f(t)$, multiply by a function of t and p, and find a definite integral with respect to t, we have a function $F(p)$ which is called an integral transform of $f(t)$. There are many named integral transforms which you may discover in tables and computer. Observe the notation for Laplace transforms in (8.1): we shall consistently use a small letter for the function of t, and the corresponding capital letter for the transform which is a function of p, for example $f(t)$ and $F(p)$, or $g(t)$ and $G(p)$, etc. Also note from (8.1) that since we integrate from 0 to ∞, $F(p)$ is the same no matter how $f(t)$ is defined for negative t. However, it is desirable to define $f(t) = 0$ for $t < 0$ (see footnote, page 447; also see Bromwich integral, page 696).

It is very convenient to have a table of corresponding $f(t)$ and $F(p)$ when we are using Laplace transforms to solve problems. Let us calculate some of the entries in the table of Laplace transforms at the end of the chapter (pages 469 to 471). Note that numbers preceded by L ($L1$, $L2$, $\cdots$, $L35$) refer to entries in the Laplace transform table.

Example 1. To obtain $L1$ in the table, we substitute $f(t) = 1$ into (8.1) and find

$$F(p) = \int_0^\infty 1 \cdot e^{-pt}\,dt = -\frac{1}{p}\,e^{-pt}\Big|_0^\infty = \frac{1}{p}, \qquad p > 0. \tag{8.2}$$

We have assumed $p > 0$ to make e^{-pt} zero at the upper limit; if p is complex, as it may be, then the real part of p must be positive ($\operatorname{Re} p > 0$), and this is the restriction we have stated in the table for $L1$.

Example 2. For $L2$, we have

$$\begin{aligned} &f(t) = e^{-at},\\ &F(p) = \int_0^\infty e^{-(a+p)t}\,dt = \frac{1}{p+a}, \qquad \operatorname{Re}(p+a) > 0. \end{aligned} \tag{8.3}$$

We could continue in this way to obtain the function $F(p)$ corresponding to each $f(t)$ by using (8.1) and evaluating the integral. However, there are some easier methods which we now illustrate. First observe that the Laplace transform of a sum of two functions is the sum of their Laplace transforms; also the transform of

$cf(t)$ is $cL(f)$ when c is a constant:

$$(8.4)\qquad \begin{aligned} L[f(t)+g(t)] &= \int_0^\infty [f(t)+g(t)]e^{-pt}\,dt \\ &= \int_0^\infty f(t)e^{-pt}\,dt + \int_0^\infty g(t)e^{-pt}\,dt = L(f)+L(g), \\ L[cf(t)] &= \int_0^\infty cf(t)e^{-pt}\,dt = c\int_0^\infty f(t)e^{-pt}\,dt = cL(f). \end{aligned}$$

In mathematical language, we say that the Laplace transform is *linear* (or is a *linear operator*—see Chapter 3, Section 7).

▶ **Example 3.** Now let us verify $L3$. In (8.3), replace the a by $-ia$; then we have

$$(8.5)\qquad \begin{aligned} f(t) &= e^{iat} = \cos at + i\sin at, \\ F(p) &= \frac{1}{p-ia} = \frac{p+ia}{p^2+a^2}, \qquad \operatorname{Re}(p-ia)>0. \end{aligned}$$

Remembering (8.4), we can write (8.5) as

$$(8.6)\qquad L(\cos at + i\sin at) = L(\cos at) + iL(\sin at) = \frac{p}{p^2+a^2} + i\frac{a}{p^2+a^2}.$$

Similarly, replacing a by ia in (8.3), we get

$$(8.7)\qquad L(\cos at - i\sin at) = \frac{p}{p^2+a^2} - i\frac{a}{p^2+a^2}, \qquad \operatorname{Re}(p+ia)>0.$$

Adding (8.6) and (8.7), we get $L4$; by subtracting, we get $L3$.

▶ **Example 4.** To verify $L11$, start with $L4$, namely

$$(8.8)\qquad L(\cos at) = \int_0^\infty e^{-pt}\cos at\,dt = \frac{p}{p^2+a^2}.$$

Differentiate (8.8) with respect to the parameter a to get

$$\int_0^\infty e^{-pt}(-t\sin at)\,dt = \frac{p(-2a)}{(p^2+a^2)^2}$$

or

$$\int_0^\infty e^{-pt} t\sin at\,dt = \frac{2pa}{(p^2+a^2)^2}$$

which is $L11$. Ways of finding other entries in the table are outlined in the problems.

▶ PROBLEMS, SECTION 8

1. For integral k, verify $L5$ and $L6$ in the Laplace transform table. *Hint:* From $L2$, you can write: $\int_0^\infty e^{-pt}e^{-at}\,dt = 1/(p+a)$. Differentiate this equation repeatedly with respect to p. (See Chapter 4, Section 12, Example 4, page 235.) Also note $L32$. For the Γ function results in $L5$ and $L6$, see Chapter 11, Problem 5.7.

2. By using $L2$, verify $L7$ and $L8$ in the Laplace transform table.

3. Using either $L2$, or $L3$ and $L4$, verify $L9$ and $L10$.

4. By differentiating the appropriate formula with respect to a, verify $L12$.

5. By integrating the appropriate formula with respect to a, verify $L19$.

6. By replacing a in $L2$ by $a+ib$ and then by $a-ib$, and adding and subtracting the results [as in (8.6) and (8.7)], verify $L13$ and $L14$.

7. Verify $L15$ to $L18$, by combining appropriate preceding formulas using (8.4).

Find the inverse transforms of the functions $F(p)$ in Problems 8 to 13.

8. $\dfrac{1+p}{(p+2)^2}$ *Hint:* Use $L6$ and $L18$.

9. $\dfrac{5-2p}{p^2+p-2}$ *Hint:* Use $L7$ and $L8$.

10. $\dfrac{2p-1}{p^2-2p+10}$ *Hint:* You *can* use $L7$ and $L8$ with complex a and b, but $L13$ and $L14$ are more direct.

11. $\dfrac{3p+2}{3p^2+5p-2}$ **12.** $\dfrac{3p+10}{p^2-25}$ **13.** $\dfrac{6-p}{p^2+4p+20}$

14. Show that a combination of entries $L3$ to $L10$, $L13$, $L14$ and $L18$ in the table, will give the inverse transform of any function of the form

$$\frac{Ap+B}{Cp^2+Ep+F}, \quad \text{where } A, B, C, E, \text{and } F \text{ are constants.}$$

15. Prove $L32$ for $n=1$. *Hint:* Differentiate equation (8.1) with respect to p.

16. Use $L32$ and $L3$ to obtain $L11$.

17. Use $L32$ and $L11$ to obtain $L(t^2 \sin at)$.

18. Use $L31$ to derive $L21$.

Table entries $L28$ and $L29$ are known as *translation* or *shifting* theorems. Do Problems 19 to 27 about them.

19. Prove the general formula $L29$ using (8.1).

20. Use $L29$ to verify $L6$, $L13$, $L14$, and $L18$.

21. Use $L29$ and $L11$ to obtain $L(te^{-at}\sin bt)$ which is not in the table.

22. Obtain $L(te^{-at}\cos bt)$ as in Problem 21.

23. Use the results which you have obtained in Problems 21 and 22 to find the inverse transform of $(p^2+2p-1)/(p^2+4p+5)^2$.

24. Sketch on the same axes graphs of $\sin t$, $\sin(t-\pi/2)$, and $\sin(t+\pi/2)$, and observe which way the graph shifts. *Hint:* You can, of course, have your calculator or computer plot these for you, but it's simpler and much more useful to do it in your head. *Hint:* What values of t make the sines equal to zero? For an even simpler example, sketch on the same axes $y=t$, $y=t-\pi/2$, $y=t+\pi/2$.

25. Use $L28$ to find the Laplace transform of

$$f(t)=\begin{cases}\sin(t-\pi/2), & t>\pi/2,\\ 0, & t<\pi/2.\end{cases}$$

26. Use $L28$ and $L4$ to find the inverse transform of $pe^{-p\pi}/(p^2+1)$.

27. Find the transform of

$$f(t)=\begin{cases}\sin(x-vt), & t>x/v,\\ 0, & t<x/v,\end{cases}$$

where x and v are constants.

▶ 9. SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

We are going to discuss the solution of linear differential equations with constant coefficients (see Sections 5 and 6). Laplace transforms can reduce such an equation to an algebraic equation and so simplify solving it. Also, since Laplace transforms automatically use given values of initial conditions, we find immediately a desired particular solution without the extra step of determining constants to satisfy the initial conditions. Discontinuous forcing functions are messy to deal with by Section 6 methods; the Laplace transform method handles them easily.

We are going to take Laplace transforms of the terms in differential equations; to do this we need to know the transforms of derivatives $y' = dy/dt, y'' = d^2y/dt^2$, etc. To find $L(y')$, we use the definition (8.1) and integrate by parts, as follows

$$\begin{aligned} L(y') &= \int_0^\infty y'(t)e^{-pt}dt = e^{-pt}y(t)\Big|_0^\infty - (-p)\int_0^\infty y(t)e^{-pt}dt \\ &= -y(0) + pL(y) = pY - y_0 \end{aligned} \tag{9.1}$$

where for simplicity we have written $L(y) = Y$ and $y(0) = y_0$. To find $L(y'')$, we think of y'' as $(y')'$, and substitute y' for y in (9.1) to get

$$L(y'') = pL(y') - y'(0).$$

Using (9.1) again to eliminate $L(y')$, we finally have

$$L(y'') = p^2L(y) - py(0) - y'(0) = p^2Y - py_0 - y_0'. \tag{9.2}$$

Continuing this process, we obtain the transforms of the higher-order derivatives (Problem 1 and $L35$).

We are now ready to solve differential equations. We illustrate the method by some examples.

▶ **Example 1.** Solve $y'' + 4y' + 4y = t^2e^{-2t}$ with initial conditions $y_0 = 0$, $y_0' = 0$.

We take the Laplace transform of each term in the equation, using $L35$ and $L6$ in the table of Laplace transforms. We get

$$p^2Y - py_0 - y_0' + 4pY - 4y_0 + 4Y = L(t^2e^{-2t}) = \frac{2}{(p+2)^3}.$$

But the initial conditions are $y_0 = y_0' = 0$. Thus we have

$$(p^2 + 4p + 4)Y = \frac{2}{(p+2)^3} \quad \text{or} \quad Y = \frac{2}{(p+2)^5}.$$

Now we want y, which is the inverse Laplace transform of Y. We look in the table for the inverse transform of $2/(p+2)^5$. By $L6$, we get

$$y = \frac{2t^4e^{-2t}}{4!} = \frac{t^4e^{-2t}}{12}.$$

This is much simpler than the general solution; we have obtained just the solution satisfying the given initial conditions.

Example 2. Solve $y'' + 4y = \sin 2t$, subject to the initial conditions $y_0 = 10$, $y'_0 = 0$.
Using the table, take the Laplace transform of each term of the equation to get

$$p^2Y - py_0 - y'_0 + 4Y = L(\sin 2t) = \frac{2}{p^2+4}.$$

Then we substitute the initial conditions and solve for Y as follows:

$$(p^2+4)Y - 10p = \frac{2}{p^2+4},$$
$$Y = \frac{10p}{p^2+4} + \frac{2}{(p^2+4)^2}.$$

Finally, taking the inverse transform using $L4$ and $L17$, we have the desired solution:

$$y = 10\cos 2t + \tfrac{1}{8}(\sin 2t - 2t\cos 2t) = 10\cos 2t + \tfrac{1}{8}\sin 2t - \tfrac{1}{4}t\cos 2t.$$

Example 3. Solve $y'' + 4y' + 13y = 20e^{-t}, \quad y_0 = 1, y'_0 = 3$.
We take the transform of each term and solve for Y as follows:

$$p^2Y - p - 3 + 4pY - 4 + 13Y = \frac{20}{p+1},$$
$$Y = \frac{1}{p^2+4p+13}\left(\frac{20}{p+1} + p + 7\right) = \frac{p^2+8p+27}{(p+1)(p^2+4p+13)}.$$

Since this Y is not in our table, we can either use a larger table, or use partial fractions to split Y into fractions which are in our table (which you can do by computer) or find the inverse transform by computer. We find:

$$Y = \frac{2}{p+1} + \frac{-p+1}{p^2+4p+13} = \frac{2}{p+1} + \frac{3}{(p+2)^2+9} - \frac{p+2}{(p+2)^2+9}$$

and by $L2$, $L13$, and $L14$,

$$y = 2e^{-t} + e^{-2t}\sin 3t - e^{-2t}\cos 3t.$$

Sets of simultaneous differential equations can also be solved by using Laplace transforms. Here is an example.

Example 4. Solve the set of equations

$$y' - 2y + z = 0,$$
$$z' - y - 2z = 0,$$

subject to the initial conditions $y_0 = 1, z_0 = 0$.

We shall call $L(z) = Z$ and $L(y) = Y$ as before. We take the Laplace transform of each of the equations to get

$$pY - y_0 - 2Y + Z = 0,$$
$$pZ - z_0 - Y - 2Z = 0.$$

After substituting the initial conditions and collecting terms, we have

$$\begin{aligned}(p-2)Y + Z &= 1,\\ Y - (p-2)Z &= 0.\end{aligned}$$

We solve this set of algebraic equations simultaneously for Y and Z (by any of the methods usually used for a pair of simultaneous equations—elimination, determinants, etc.). For example, we may multiply the first equation by $(p-2)$ and add the second to get

$$[(p-2)^2+1]Y = p-2 \qquad \text{or} \qquad Y = \frac{p-2}{(p-2)^2+1}.$$

We find y by looking up the inverse transform of Y using $L14$. We get

$$y = e^{2t}\cos t.$$

Similarly, solving for Z and looking up the inverse transform, we find

$$Z = \frac{1}{(p-2)^2+1}, \qquad z = e^{2t}\sin t$$

. Alternatively, we could find z from the first differential equation by substituting the y solution:

$$z = 2y - y' = 2e^{2t}\cos t + e^{2t}\sin t - 2e^{2t}\cos t = e^{2t}\sin t.$$

Solving linear differential equations with constant coefficients is not the only use of Laplace transforms. As you will see in Chapter 13, Section 10, we may solve some kinds of partial differential equations by Laplace transforms. Also a table of Laplace transforms can be used to evaluate definite integrals of the type $\int_0^\infty e^{-pt}f(t)dt$.

▶ **Example 5.** By $L15$ with $a=3$ and $p=2$, we have

$$\int_0^\infty e^{-2t}(1-\cos 3t)\,dt = \frac{3^2}{2(2^2+3^2)} = \frac{9}{26}.$$

Actually, there is more to the subject than this. Although we are discussing in this chapter the use of Laplace transforms as a tool, they also can play a more theoretical role in applied problems. It is often possible to find desired information about a problem directly from the Laplace transform of the solution without ever finding the solution. Thus the use of Laplace transforms may lead to a better understanding of a problem or a simpler method of solution. (Compare the use of matrices, for example, or the use of Fourier transforms.)

▶ PROBLEMS, SECTION 9

1. Continuing the method used in deriving (9.1) and (9.2), verify the Laplace transforms of higher-order derivatives of y given in the table ($L35$).

By using Laplace transforms, solve the following differential equations subject to the given initial conditions.

2. $y' - y = 2e^t$, $\qquad y_0 = 3$

3. $y'' + 4y' + 4y = e^{-2t}$, $\quad y_0 = 0,\ y_0' = 4$

4. $y'' + y = \sin t$, $\quad y_0 = 1,\ y_0' = 0$

5. $y'' + y = \sin t$, $\quad y_0 = 0,\ y_0' = -\frac{1}{2}$

6. $y'' - 6y' + 9y = te^{3t}$, $\quad y_0 = 0,\ y_0' = 5$

7. $y'' - 4y' + 4y = 4$, $\quad y_0 = 0,\ y_0' = -2$

8. $y'' + 16y = 8\cos 4t$, $\quad y_0 = y_0' = 0$

9. $y'' + 16y = 8\cos 4t$, $\quad y_0 = 0,\ y_0' = 8$

10. $y'' - 4y' + 4y = 6e^{2t}$, $\quad y_0 = y_0' = 0$

11. $y'' - 4y = 4e^{2t}$, $\quad y_0 = 0,\ y_0' = 1$

12. $y'' - y = e^{-t} - 2te^{-t}$, $\quad y_0 = 1,\ y_0' = 2$

13. $y'' + y = 5\sinh 2t$, $\quad y_0 = 0,\ y_0' = 2$

14. $y'' - 4y' = -4te^{2t}$, $\quad y_0 = 0,\ y_0' = 1$

15. $y'' + 9y = \cos 3t$, $\quad y_0 = 0,\ y_0' = 6$

16. $y'' + 9y = \cos 3t$, $\quad y_0 = 2,\ y_0' = 0$

17. $y'' + 5y' + 6y = 12$, $\quad y_0 = 2,\ y_0' = 0$

18. $y'' - 4y = 3e^{-t}$, $\quad y_0 = 1,\ y_0' = -3$

19. $y'' + y' - 5y = e^{2t}$, $\quad y_0 = 1,\ y_0' = 2$

20. $y'' - 8y' + 16y = 32t$, $\quad y_0 = 1,\ y_0' = 2$

21. $y'' + 4y' + 5y = 26e^{3t}$, $\quad y_0 = 1,\ y_0' = 5$

22. $y''+2y'+5y = 10\cos t$, $\quad y_0 = 2,\ y_0' = 1$

23. $y''+2y'+5y = 10\cos t$, $\quad y_0 = 0,\ y_0' = 3$

24. $y'' - 2y' + y = 2\cos t$, $\quad y_0 = 5,\ y_0' = -2$

25. $y'' + 4y' + 5y = 2e^{-2t}\cos t$, $\quad y_0 = 0,\ y_0' = 3$

26. $y'' + 2y' + 10y = -6e^{-t}\sin 3t$, $\quad y_0 = 0,\ y_0' = 1$

Solve the following sets of equations by the Laplace transform method.

27. $y' + z' - 3z = 0$, $\quad y_0 = y_0' = 0$
$y'' + z' = 0$, $\quad z_0 = \frac{4}{3}$

28. $y' + z = 2\cos t$, $\quad y_0 = -1$
$z' - y = 1$, $\quad z_0 = 1$

29. $y' + z' - 2y = 1$, $\quad y_0 = z_0 = 1$
$z - y' = t$

30. $y' + 2z = 1$, $\quad y_0 = 0$
$2y - z' = 2t$, $\quad z_0 = 1$

31. $y'' + z'' - z' = 0$, $\quad y_0 = 0,\quad y_0' = 1$
$y' + z' - 2z = 1 - e^t$, $\quad z_0 = 1,\quad z_0' = 1$

32. $z' + 2y = 0$, $\quad y_0 = z_0 = 0$
$y' - 2z = 2$

33. $y' - z' - y = \cos t \qquad y_0 = -1$
$y' + y - 2z = 0 \qquad z_0 = 0$

Evaluate each of the following definite integrals by using the Laplace transform table.

34. $\int_0^\infty e^{-2t} \sin 3t \, dt = \frac{3}{13}$. *Hint:* In (8.1), let $p = 2$, $f(t) = \sin 3t$; use $L3$ with $a = 3$.

35. $\int_0^\infty te^{-t} \sin 5t \, dt$

36. $\int_0^\infty \frac{e^{-3t} \sin 2t}{t} \, dt$

37. $\int_0^\infty t^5 e^{-2t} \, dt$

38. $\int_0^\infty e^{-t}(1 - \cos 2t) \, dt$

39. $\int_0^\infty \frac{e^{-t} - e^{-2t}}{t} \, dt$

40. $\int_0^\infty \frac{e^{-2t} - e^{-2et}}{t} \, dt$

41. $\int_0^\infty \frac{1}{t} e^{-2t} \sin(t\sqrt{2}) \, dt$

42. $\int_0^\infty \frac{1}{t} e^{-t\sqrt{3}} \sin 2t \cos t \, dt$

▶ 10. CONVOLUTION

In solving differential equations by Laplace transforms in Section 9, we found Y and then found the inverse transform y either in a table or by computer. We had no way of writing a formula for y. We now want to consider another way of finding inverse transforms. (Also see Bromwich integral, Chapter 14, page 696.)

Let us first see why the method we are going to discuss in this section is useful. Consider differential equations of the kind discussed in Sections 5 and 6, namely linear second-order equations with constant coefficients. Recall that such equations describe the vibrations or oscillations of either a mechanical or an electrical system. If the right-hand side of the equation is a function of t, called the *forcing function*, then the differential equation describes forced vibrations.

▶ **Example 1.** Let us solve the following representative equation by Laplace transforms, assuming that the system is initially at rest and that the force $f(t)$ starts being applied at $t = 0$.

$$Ay'' + By' + Cy = f(t), \qquad y_0 = y_0' = 0. \tag{10.1}$$

We take the Laplace transform of each term, substitute the initial conditions, and solve for Y as follows:

$$Ap^2Y + BpY + CY = L(f) = F(p), \qquad Y = \frac{1}{Ap^2 + Bp + C} F(p). \tag{10.2}$$

Note that Y is a product of two functions of p. We know the inverse transform of $F(p)$, namely $f(t)$. The factor $T(p) = (Ap^2 + Bp + C)^{-1}$ (called the *transfer function*) can always be written as

$$T(p) = \frac{1}{A(p + a)(p + b)}$$

by factoring the quadratic expression in the denominator. Hence by $L7$ (or $L6$ if $a = b$) we can find the inverse transform of $T(p)$ for any problem. Then y

[the inverse transform of Y in (10.2)] is the inverse transform of a product of two functions whose inverse transforms we know. We are going to show how to write y as an integral (that is, we are going to verify $L34$ in the table).

Let $G(p)$ and $H(p)$ be the transforms of $g(t)$ and $h(t)$. We want the inverse transform of the product $G(p)H(p)$. By the definition (8.1)

$$G(p)H(p) = \int_0^\infty e^{-pt} g(t)\,dt \cdot \int_0^\infty e^{-pt} h(t)\,dt. \tag{10.3}$$

Let us rewrite (10.3) replacing t by different dummy variables of integration so that we can write the product of the two integrals as a double integral. We then have

$$\begin{aligned} G(p)H(p) &= \int_0^\infty e^{-p\sigma} g(\sigma)\,d\sigma \cdot \int_0^\infty e^{-p\tau} h(\tau)\,d\tau \\ &= \int_0^\infty \int_0^\infty e^{-p(\sigma+\tau)} g(\sigma)h(\tau)\,d\sigma\,d\tau. \end{aligned} \tag{10.4}$$

Now we make a change of variables; in the σ integral (that is, with τ fixed), let $\sigma+\tau = t$. Then $\sigma = t-\tau$, $d\sigma = dt$, and the range of integration with respect to t is from $t=\tau$ (corresponding to $\sigma = 0$) to $t=\infty$ (corresponding to $\sigma=\infty$). Making these substitutions into (10.4), we get

$$G(p)H(p) = \int_{\tau=0}^\infty \int_{t=\tau}^\infty e^{-pt} g(t-\tau)h(\tau)\,dt\,d\tau. \tag{10.5}$$

Next we want to change the order of integration. From Figure 10.1, we see that the double integral in (10.5) is over the triangle in the first quadrant below the line $t=\tau$. The t integral ranges from the line $t=\tau$ to $t=\infty$ (indicated by a horizontal strip of width $d\tau$ from $t=\tau$ to ∞) and then the τ integral sums over the horizontal strips from $\tau=0$ to $\tau=\infty$ covering the whole infinite triangle. Let us integrate with respect to τ first; τ then ranges from 0 to the line $\tau = t$ [indicated by a vertical strip in Figure 10.1] and then the t integral sums over the vertical strips from $t=0$ to ∞. Making this change in (10.5), we get

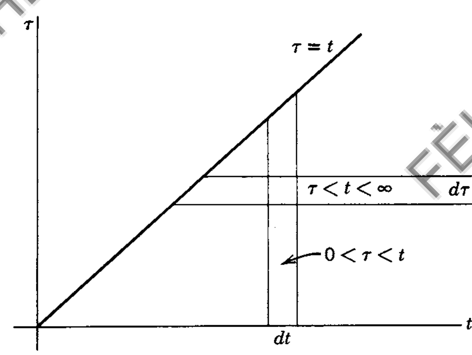


Figure 10.1

$$\begin{aligned} G(p)H(p) &= \int_{t=0}^\infty \int_{\tau=0}^t e^{-pt} g(t-\tau)h(\tau)\,d\tau\,dt \\ &= \int_0^\infty e^{-pt}\left[\int_0^t g(t-\tau)h(\tau)\,d\tau\right]dt \\ &= L\left[\int_0^t g(t-\tau)h(\tau)\,d\tau\right]. \qquad \text{(See } L34.\text{)} \end{aligned} \tag{10.6}$$

The last step follows from the definition (8.1) of a Laplace transform.

Definition of Convolution The integral

$$\int_0^t g(t-\tau)h(\tau)\,d\tau = g * h \tag{10.7}$$

is called the *convolution* of g and h (or the *resultant* or the *Faltung*). Note the abbreviation $g * h$ for the convolution integral, and do not confuse the symbol $*$, written *on* the line, with a star used as a superscript meaning complex conjugate. It is easy to show (Problem 1) that $g * h = h * g$; this result and (10.6) and (10.7) give $L34$ in the table.

Now let's see how to use (10.6) or $L34$ to solve the kind of problem indicated in (10.1) and (10.2).

▶ **Example 2.** Solve $y'' + 3y' + 2y = e^{-t}$, $\quad y_0 = y_0' = 0$.

Taking the Laplace transform of each term, substituting the initial conditions, and solving for Y, we get

$$p^2Y + 3pY + 2Y = L(e^{-t}),$$
$$Y = \frac{1}{p^2+3p+2}L(e^{-t}).$$

Since we are intending to use the convolution integral, we do not bother to look up the transform of e^{-t}. We do want, however, the inverse transform of $1/(p^2+3p+2)$; by $L7$, this is $e^{-t} - e^{-2t}$, so we have

$$Y = L(e^{-t} - e^{-2t})L(e^{-t}) = G(p)H(p),$$

with $g(t) = e^{-t} - e^{-2t}$ and $h(t) = e^{-t}$. We now use $L34$ to find y. Observe from $L34$ that we may use either $g(t-\tau)h(\tau)$ or $g(\tau)h(t-\tau)$ in the integral. It is well to choose whichever form is easier to integrate; usually it is best to put $(t-\tau)$ in the simpler function [here $h(t)$]. Then we have

$$\begin{aligned} y &= \int_0^t g(\tau)h(t-\tau)\,d\tau = \int_0^t (e^{-\tau} - e^{-2\tau})(e^{-(t-\tau)})\,d\tau \\ &= e^{-t}\int_0^t (1-e^{-\tau})\,d\tau = e^{-t}(\tau + e^{-\tau})\Big|_0^t \\ &= e^{-t}(t + e^{-t} - 1) = te^{-t} + e^{-2t} - e^{-t}. \end{aligned}$$

It is not always as easy to evaluate the convolution integral as it was in this example. However, let us observe that, at the very worst, we can always write the solution to a forced vibrations problem [equation (10.1)] as an integral (which can, if necessary, be evaluated numerically). This is true because, as we showed just after (10.2), we can always find the inverse transform of the transfer function $T(p)$, and so have Y as a product of two functions whose inverse transforms we know. Then y is given by the convolution (10.7) of the forcing function $f(t)$ and the inverse transform of the the transfer function. Also note (Problem 16) that a combination of $L6$, $L7$, $L8$ and $L18$ will handle any terms arising in a problem with nonzero initial conditions.

Fourier Transform of a Convolution We have shown that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. There is a similar theorem for Fourier transforms; let us see what it says. Let $g_1(\alpha)$ and $g_2(\alpha)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$. By analogy with equations (10.3), (10.4), (10.5), and (10.6), we might expect the product $g_1(\alpha) \cdot g_2(\alpha)$ to be the Fourier transform of something; let's investigate this idea. Assuming that $\int_{-\infty}^{\infty} |f_1(x)f_2(x)|dx$ is finite, then by the definition of a Fourier transform [Chapter 7, equation (12.2)], we have

$$\begin{aligned}(10.8) \qquad g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(v)e^{-i\alpha v}\,dv \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(u)e^{-i\alpha u}\,du \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\alpha(v+u)} f_1(v)f_2(u)\,dv\,du.\end{aligned}$$

[We have used different dummy integration variables as in (10.4).] Next we make the change of variables $x = v + u$, $dx = dv$, in the v integral, to get

$$\begin{aligned}(10.9) \qquad g_1(\alpha)g_2(\alpha) &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\alpha x} f_1(x-u)f_2(u)\,dx\,du \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{-i\alpha x}\left[\int_{-\infty}^{\infty} f_1(x-u)f_2(u)\,du\right] dx.\end{aligned}$$

If we define the convolution of $f_1(x)$ and $f_2(x)$ by

$$(10.10) \qquad f_1 * f_2 = \int_{-\infty}^{\infty} f_1(x-u)f_2(u)\,du,^{\dagger}$$

then (10.9) becomes

$$(10.11) \quad g_1 \cdot g_2 = \frac{1}{2\pi}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f_1 * f_2\, e^{-i\alpha x}\,dx\right] = \frac{1}{2\pi}\cdot \text{Fourier transform of } f_1 * f_2.$$

In other words,

$$(10.12) \qquad g_1 \cdot g_2 \text{ and } \frac{1}{2\pi} f_1 * f_2 \quad \text{are a pair of Fourier transforms.}$$

Because of the symmetry of the $f(x)$ and $g(\alpha)$ integrals, there is a similar result relating $f_1 \cdot f_2$ and the convolution of g_1 and g_2. We find that (Problem 19)

$$(10.13) \qquad g_1 * g_2 \quad \text{and} \quad f_1 \cdot f_2 \quad \text{are a pair of Fourier transforms.}$$

As discussed in Chapter 7, after (12.2) and after (12.10), various references differ

†Note that (10.10) is really the same as (10.7) if we agree that, for Laplace transforms, $f(t) = 0$ when $t < 0$ (see the first paragraph of Section 8, page 437). For then in (10.7), $h(\tau) = 0$ for $\tau < 0$ and $g(t-\tau) = 0$ for $\tau > t$, so the integral would not really be different if written with infinite limits (in fact, it is sometimes written that way).

in the position of the factor $1/(2\pi)$. Some authors include factors of $1/(2\pi)$ or $1/\sqrt{2\pi}$ in the convolution definition (10.10); this definition as well as Chapter 7, equation (12.2), affects (10.12) and (10.13). Check the notation in any reference you are using.

PROBLEMS, SECTION 10

1. Show that $g * h = h * g$ as claimed in *L*34. *Hint:* Let $u = t - \tau$ in (10.7).

2. Use *L*34 and *L*2 to find the inverse transform of $G(p)H(p)$ when $G(p) = 1/(p+a)$ and $H(p) = 1/(p+b)$; your result should be *L*7.

Use the convolution integral to find the inverse transforms of:

3. $\dfrac{p}{(p^2-1)^2} = \dfrac{p}{p^2-1} \cdot \dfrac{1}{p^2-1}$

4. $\dfrac{1}{(p+a)(p+b)^2}$

5. $\dfrac{p}{(p+a)(p+b)^2}$

6. $\dfrac{1}{(p+a)(p^2-b^2)}$

7. $\dfrac{p}{(p+a)(p^2-b^2)}$

8. $\dfrac{1}{(p+a)(p+b)(p+c)}$

9. $\dfrac{2}{p^3(p+2)}$

10. $\dfrac{1}{p(p^2+a^2)^2}$

11. $\dfrac{p}{(p^2+a^2)(p^2+b^2)}$

12. $\dfrac{1}{p(p^2+a^2)(p^2+b^2)}$

Hint: In Problems 11 and 12 use $2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi)$.

13. Use the Laplace transform table to find $f(t) = \int_0^t e^{-\tau}\sin(t-\tau)\,d\tau$. *Hint:* In *L*34, let $g(t) = e^{-t}$ and $h(t) = \sin t$, and find $G(p)H(p)$ which is the Laplace transform of the integral you want. Break the result into partial fractions and look up the inverse transforms.

Use the convolution integral (see Example 2) to solve the following differential equations.

14. $y'' + 5y' + 6y = e^{-2t}, \quad y_0 = y_0' = 0.$

15. $y'' + 3y' - 4y = e^{3t}, \quad y_0 = y_0' = 0.$

16. Consider solving an equation like (10.1) but with nonzero initial conditions.

(a) Write the corrected form of (10.2), writing the transfer function in factored form as indicated just after (10.2). Consider the extra terms in Y which arise from the initial conditions; show that the inverse transforms of such terms can always be found from *L*6, *L*7, *L*8, and *L*18.

(b) Find the explicit form of the inverse transform of the transfer function for $a \neq b$ (use *L*7), and so write the general solution of (10.2) with nonzero initial conditions as a convolution integral plus the terms which you found in (a).

17. Solve the differential equation $y'' - a^2 y = f(t)$, where

$$f(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0, \end{cases} \qquad \text{and} \quad y_0 = y_0' = 0.$$

Hint: Use the convolution integral as in the example.

18. A mechanical or electrical system is described by the differential equation $y'' + \omega^2 y = f(t)$. Find y if

$$f(t) = \begin{cases} 1, & 0 < t < a, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \quad y_0 = y_0' = 0.$$

Hint: Use the convolution integral carefully. Consider $t < a$ and $t > a$ separately, remembering that $f(t) = 0$ for $t > a$. Show that

$$y = \begin{cases} \dfrac{1}{\omega^2}(1 - \cos \omega t), & t < a, \\ \dfrac{1}{\omega^2}[\cos \omega(t-a) - \cos \omega t], & t > a. \end{cases}$$

Sketch the motion if $a = \frac{1}{3}T$ where T is the period for free vibrations of the system; if $a = \frac{3}{2}T$; if $a = \frac{1}{10}T$.

19. Following the method of equations (10.8) to (10.12), show that $f_1 f_2$ and $g_1 * g_2$ are a pair of Fourier transforms.

11. THE DIRAC DELTA FUNCTION

In mechanics we consider the idea of an impulsive force such as a hammer blow which lasts for a very short time. We usually do not know the exact shape of the force function $f(t)$, and so we proceed as follows. Let the impulsive force $f(t)$ lasting from $t = t_0$ till $t = t_1$ be applied to a mass m; then by Newton's second law we have

$$\int_{t_0}^{t_1} f(t)\,dt = \int_{t_0}^{t_1} m\frac{dv}{dt}\,dt = \int_{v_0}^{v_1} m\,dv = m(v_1 - v_0). \tag{11.1}$$

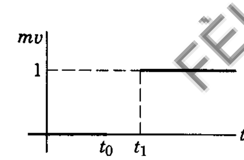


Figure 11.1

This says that the integral of $f(t)$ [called the impulse of $f(t)$] is equal to the change in the momentum of m, and we note that the result is independent of the shape of $f(t)$ but depends only on the area under the $f(t)$ curve. If this area is 1, we call the impulse a *unit impulse*. If $t_1 - t_0$ is very small, we may simply ignore the motion of m during this small time, and say only that the momentum jumped from mv_0 to mv_1 during the time $t_1 - t_0$. If $v_0 = 0$, the graph of the momentum as a function of time would be as in Figure 11.1, where we have simply omitted the (unknown) part of the graph between t_0 and t_1. We note that if $t_1 - t_0$ is very small, the graph in Figure 11.1 is almost the unit step function ($L24$). Let us imagine making $t_1 - t_0$ smaller and smaller while keeping the jump in mv always 1.

In Figures 11.2, 11.3 and 11.4 we have sketched some possible sequences of functions $f_n(t)$ which would do this. We could draw many other similar sets of graphs; the essential requirement is that $f(t)$ should become taller and narrower (that is, that the force should become more intense but act over a shorter time) in such a way that the impulse [area under the $f(t)$ curve] remains 1. We might then consider the limiting case in which Figure 11.1 has a jump of 1 at t_0; the force $f(t)$ required to produce this result would have to be infinite and act instantaneously. Also from equation (11.1), we see that the function $f(t)$ is the slope of the mv graph; thus we are asking for $f(t)$ to be the derivative of a step function at the jump. We see immediately that no ordinary function has these properties. However, we also note

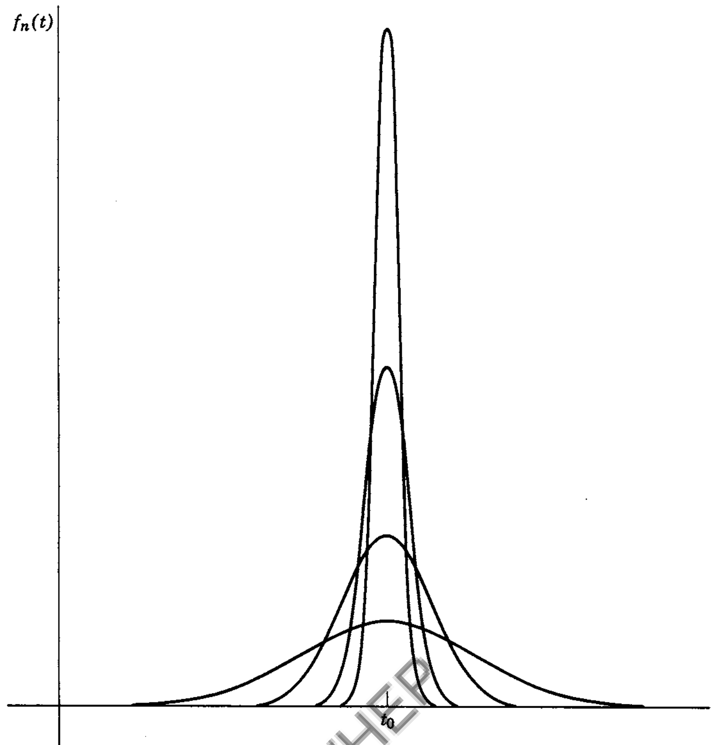


Figure 11.2

that we are not so much interested in $f(t)$ as in the results it produces. Figure 11.1 with a jump at t_0 makes perfectly good sense; for any $t > t_0$ we could choose a sufficiently tall and narrow $f_n(t)$ so that mv would already have its final value. We shall see that it is convenient to introduce a symbol $\delta(t - t_0)$ to represent the force which produces a jump of 1 in mv at t_0; $\delta(t - t_0)$ is called the *Dirac delta function* although it is not an ordinary function as we have seen. (It may properly be called a *generalized function* or a *distribution*, and is one of a whole class of such functions.) Introducing and using this symbol is much like introducing and using the symbol ∞. It is convenient to write equations like $1/\infty = 0$, but we must not write $\infty/\infty = 1$; that is, such symbolic equations must be abbreviations for correct limiting processes. Let us investigate, then, how we can use the δ function correctly.

▶ **Example 1.** Consider the differential equation

$$y'' + \omega^2 y = f(t), \qquad y_0 = y_0' = 0. \tag{11.2}$$

This equation might describe the oscillations of a mass suspended by a spring, or a simple series electric circuit with negligible resistance. Let us assume that the system is initially at rest ($y_0 = y_0' = 0$); then suppose that, at $t = t_0$ the mass is struck a sharp blow, or a sudden short surge of current is sent through the electric circuit. The function $f(t)$ may be one of those shown in Figures 11.2 to 11.4 or another similar function. Let us solve (11.2) with $f(t)$ equal to one of the functions in Figure 11.3, that is, $f(t) = ne^{-n(t-t_0)}, t > t_0$. Using Laplace transforms $L28$

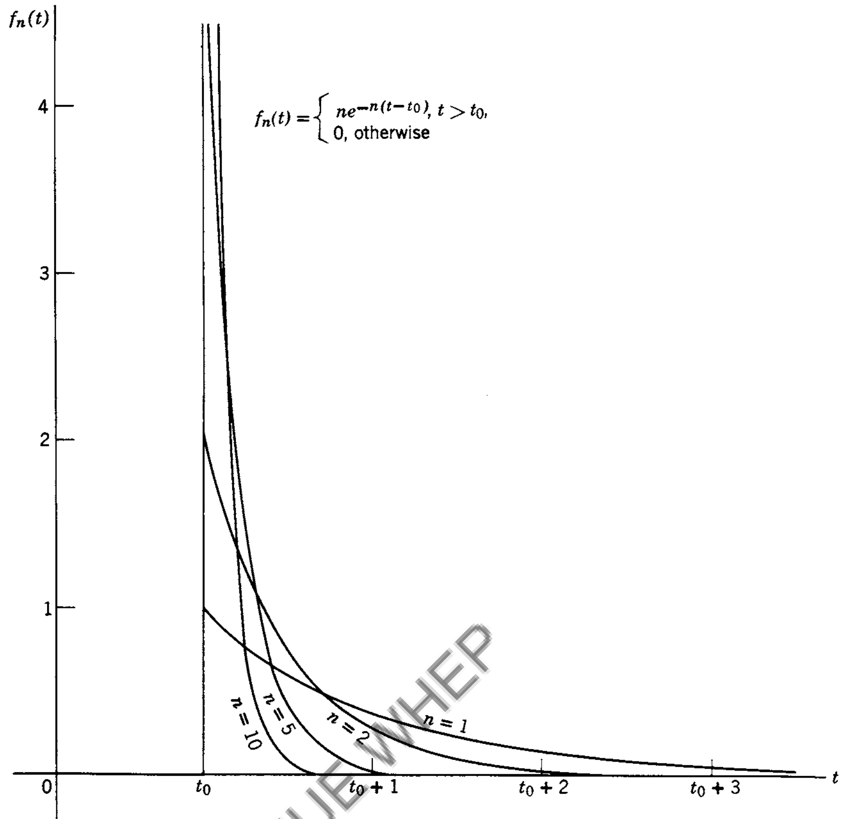


Figure 11.3

and $L2$ we find

$$(11.3)\qquad \begin{aligned} (p^2+\omega^2)Y &= L(ne^{-n(t-t_0)}) = n\cdot\frac{e^{-pt_0}}{p+n}, \\ Y = n\cdot\frac{e^{-pt_0}}{(p+n)(p^2+\omega^2)} &= \frac{ne^{-pt_0}}{(n^2+\omega^2)}\left[\frac{1}{p+n}+\frac{n}{p^2+\omega^2}-\frac{p}{p^2+\omega^2}\right]. \end{aligned}$$

(You can easily verify the partial fractions expansion in the last step.) Then by $L28$ with $a = t_0$, $L2$ with $a = n$, and $L3$, $L4$, with $a = \omega$, we find:

$$(11.4)\qquad y = n\left(\frac{e^{-n(t-t_0)}}{n^2+\omega^2}+\frac{n\sin\omega(t-t_0)}{(n^2+\omega^2)\omega}-\frac{\cos\omega(t-t_0)}{n^2+\omega^2}\right), \qquad t > t_0.$$

(Of course, $y = 0$ for $t < t_0$.) By making $f(t)$ sufficiently narrow and peaked (that is, by making n large enough), we can make the first and third terms in y negligible, and the coefficient of $\sin\omega(t-t_0)$ approximately equal to $1/\omega$. Thus the solution is approximately

$$(11.5)\qquad y = \frac{1}{\omega}\sin\omega(t-t_0), \qquad t > t_0,$$

for a unit impulse of very short duration at $t = t_0$. (We have shown this only for the functions of Figure 11.3; however, the same result would be found for other sets of functions, such as those in Figure 11.4, for example—see Problem 5.)

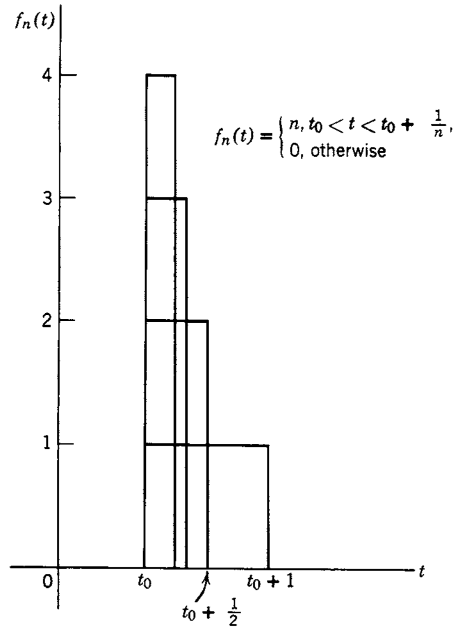


Figure 11.4

Now we would like to be able to find (11.5) without finding (11.4), in fact, without choosing a specific set of functions $f_n(t)$. Our discussion above suggests that we try using the symbol $\delta(t-t_0)$ for $f(t)$ on the right-hand side of (11.2). In solving the equation, we would then like to take the Laplace transform of $\delta(t-t_0)$.

Laplace Transform of a δ Function Let us investigate whether we can make sense out of the Laplace transform of $\delta(t-t_0)$. More generally, let us try to attach meaning to the integral $\int \phi(t)\delta(t-t_0)\,dt$, where $\phi(t)$ is any continuous function and $\delta(t-t_0)$ is the symbol indicating an impulse at t_0. We consider the integrals $\int \phi(t)f_n(t-t_0)\,dt$, where the functions $f_n(t-t_0)$ are more and more strongly peaked at t_0 as n increases (Figure 11.5), but the area under each graph is 1. When $f_n(t-t_0)$ is so narrow that $\phi(t)$ is essentially constant [equal to $\phi(t_0)$] over the width of $f_n(t-t_0)$, the integral becomes nearly $\phi(t_0)\int f_n(t-t_0)\,dt = \phi(t_0)\cdot 1 = \phi(t_0)$; that is, the sequence of integrals $\int \phi(t)f_n(t-t_0)\,dt$ tends to $\phi(t_0)$ as n tends to infinity. It then seems reasonable to say that

$$\text{(11.6)} \qquad \int_a^b \phi(t)\delta(t-t_0)\,dt = \begin{cases} \phi(t_0), & a < t_0 < b, \\ 0, & \text{otherwise.} \end{cases}$$

Equation (11.6) is the defining equation for the δ function; when we operate with δ functions, we use them in integrals and (11.6) tells us the value of the integral. The integral in (11.6) is not a Riemann integral; it is just a very useful symbol indicating that we have found the limit of $\int \phi(t)f_n(t-t_0)\,dt$ as $n \to \infty$. You may then ask how we can carry out familiar operations like integration by parts. When you treat an integral containing a δ function as an ordinary integral, you can, if you like, think

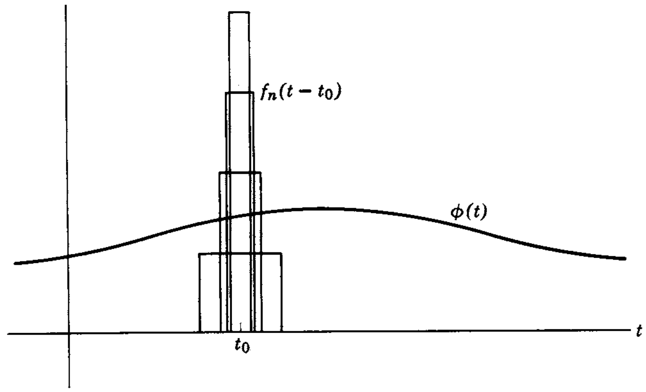


Figure 11.5

that you are really working with the functions $f_n(t-t_0)$ and then taking the limit at the end. Of course, all this needs mathematical justification which exists but is beyond our scope. (For two different mathematical developments of generalized functions, see Lighthill, and Chapter 9 of Folland.) Our purpose is just to make understandable the δ function formulas which are so useful in applications.

Example 2. We can now easily find the Laplace transform of $\delta(t-t_0)$. In the notation used in $L27$ (which we are about to derive), we have, using (8.1),

$$L[\delta(t-a)] = \int_0^\infty \delta(t-a)e^{-pt}\,dt = e^{-pa}, \qquad a>0, \tag{11.7}$$

since, by 11.6, the integral of the product of $\delta(t-a)$ and a function "picks out" the value of the function at $t=a$. Now let us use our results to obtain (11.5) more easily.

Example 3. Solve

$$y'' + \omega^2 y = \delta(t-t_0), \qquad y_0 = y_0' = 0. \tag{11.8}$$

Taking Laplace transforms and using (11.7), we get

$$(p^2+\omega^2)Y = L[\delta(t-t_0)] = e^{-pt_0}. \tag{11.9}$$

Then

$$Y = \frac{e^{-pt_0}}{p^2+\omega^2} \tag{11.10}$$

and, by $L3$ and $L28$,

$$y = \frac{1}{\omega}\sin\omega(t-t_0), \qquad t>t_0, \tag{11.11}$$

as in (11.5).

Fourier Transform of a δ Function Using (11.6) and the definition of a Fourier transform [Chapter 7, equation (12.2)], we may write

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-a)e^{-i\alpha x}\,dx = \frac{1}{2\pi}e^{-i\alpha a}. \tag{11.12}$$

Formally, then (12.2) of Chapter 7 would give for the inverse transform

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x-a)}\,d\alpha. \tag{11.13}$$

We say "formally" because the integral in (11.13) does not converge. However, if we replace the limits $-\infty, \infty$ by $-n, n$, we obtain a set of functions (Problem 12) which, like the functions $f_n(t)$ in Figures 11.2 to 11.4, are increasingly peaked around $x = a$ as n increases, but all have area 1. In this sense, then, (11.13) is a representation of the δ function. Equations (11.12) and (11.13) are useful in quantum mechanics.

Another Physical Application of δ functions What is the density (mass per unit length) of a point mass on the x axis? Compare the concept of a point mass with our discussion of δ functions. We could think of a point mass as corresponding to the limiting case of a density function like those in Figures 11.2 to 11.4. A point mass at $x = a$ requires that the density be zero everywhere except at $x = a$ but the integral of the density function across $x = a$ should be the mass m. Thus we can write the density function for a point mass m at $x = a$ as $m\delta(x-a)$. Similarly, we can represent the charge density for point electrical charges using δ functions.

▶ **Example 4.** The charge density for a charge of 2 at $x = 3$, a charge of -5 at $x = 7$ and a charge of 3 at $x = -4$ would be $2\delta(x-3) - 5\delta(x-7) + 3\delta(x+4)$.

Derivatives of the δ Function To see that we can attach a meaning to the derivative of $\delta(x-a)$, we write $\int_{-\infty}^{\infty}\phi(x)\delta'(x-a)\,dx$ and integrate by parts to get

$$\int_{-\infty}^{\infty}\phi(x)\delta'(x-a)\,dx = \phi(x)\delta(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\phi'(x)\delta(x-a)\,dx = -\phi'(a). \tag{11.14}$$

The integrated term is zero at $\pm\infty$, and we evaluated the integral using equation (11.6). Thus, just as $\delta(x-a)$ "picks out" the value of $\phi(x)$ at $x = a$ [see equation (11.6)], so $\delta'(x-a)$ picks out the negative of $\phi'(x)$ at $x = a$. Integrating by parts twice (Problem 14), we find

$$\int_{-\infty}^{\infty}\phi(x)\delta''(x-a)\,dx = \phi''(a). \tag{11.15}$$

Repeated integrations by parts gives the formula for the derivative of any order of the δ function (Problem 14):

$$\int_{-\infty}^{\infty}\phi(x)\delta^{(n)}(x-a)\,dx = (-1)^n\phi^{(n)}(a). \tag{11.16}$$

We have written the integrals in (11.14) to (11.16) with limits $-\infty$ to ∞, but all that is necessary is that the range of integration include $x = a$; otherwise, as for (11.6), the integrals are zero.

Some Formulas Involving δ Functions Our discussion at the beginning of this section (see Figure 11.1 and $L24$ in the Laplace transform table) implied that the derivative at $x = a$ of the unit step function ought to be $\delta(x-a)$.

$$\text{(11.17)} \qquad (a) \quad u(x-a) = \begin{cases} 1, & x > a \\ 0, & x < a \end{cases}$$
$$(b) \quad u'(x-a) = \delta(x-a).$$

What does the $u' = \delta$ equation mean? By definition, two generalized functions (distributions) are equal, say $G_1(x) = G_2(x)$, if $\int \phi(x)G_1\,dx = \int \phi(x)G_2\,dx$ for any test function $\phi(x)$. Test functions are assumed to be very well behaved functions; let's assume that they are continuous with continuous derivatives of all orders and that they are identically zero outside some finite interval so that the integrated term in an integration by parts is always zero. You can think of generalized functions as being operators; given a test function $\phi(x)$, they "operate" on it to produce a value such as $\phi(0)$. Compare the differential operators in Problem 5.31. We wrote $Dx = xD + 1$ where $D = d/dx$. This would be nonsense as an elementary calculus formula, but as an operator equation to be applied to $y(x)$, it means $D(xy) = (xD+1)y = xy' + y$ which is correct. In a similar way, two generalized functions are equal if they give the same results when they operate on any test function. Let's try this for (11.17b). We multiply u' by $\phi(x)$, integrate by parts (noting that the integrated term is zero because we require test functions to be zero for large $|x|$), substitute the values of $u(x-a)$, and integrate again to get

$$\int_{-\infty}^{\infty} \phi(x)u'(x-a)\,dx = -\int_{-\infty}^{\infty} \phi'(x)u(x-a)\,dx = -\int_{a}^{\infty} \phi'(x)\,dx = \phi(a).$$

This is indeed the value of the integral of $\phi(x)\delta(x-a)$, so $u'(x-a) = \delta(x-a)$ is a valid operator equation (generalized function equation).

Since we think of $\delta(x)$ and its derivatives as being zero except at the origin, and x is zero at the origin, it might seem plausible that $x\delta, x^2\delta, x\delta'$, etc. would be identically zero. It turns out that some of these are zero and some are not; to find out, we multiply by an arbitrary test function $\phi(x)$ and integrate. We state a few results; also see Problems 17 and 18.

$$\text{(11.18)} \qquad \begin{aligned} &(a) \quad x\delta(x) = 0 \\ &(b) \quad x\delta'(x) = -\delta(x) \\ &(c) \quad x^2\delta''(x) = 2\delta(x) \end{aligned}$$

To check (b), we multiply by $\phi(x)$ and integrate using (11.14) with $\phi(x)$ replaced by $x\phi(x)$.

$$\int_{-\infty}^{\infty} x\delta'(x)\phi(x)\,dx = -(x\phi)'\Big|_{x=0} = -(x\phi' + \phi)\Big|_{x=0} = -\phi(0) = -\int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx.$$

Here is another way to produce valid generalized function identities like those in (11.18). Suppose $G_1(x) = G_2(x)$; then we can show (Problem 19a) that $\frac{d}{dx}G_1(x) = \frac{d}{dx}G_2(x)$ and $x^nG_1(x) = x^nG_2(x)$. For example, if we differentiate (11.18a) we get $x\delta'(x) + \delta(x) = 0$, or $x\delta'(x) = -\delta(x)$ which is (11.18b).

We list a few more operator equations (see Problems 20 and 21).

$$
\text{(11.19)}\qquad
\begin{aligned}
&(a)\quad \delta(-x) = \delta(x) \text{ and } \delta(x-a) = \delta(a-x);\\
&(b)\quad \delta'(-x) = -\delta'(x) \text{ and } \delta'(x-a) = -\delta'(a-x);\\
&(c)\quad \delta(ax) = \frac{1}{|a|}\,\delta(x),\ \ a \neq 0;\\
&(d)\quad \delta[(x-a)(x-b)] = \frac{1}{|a-b|}[\delta(x-a)+\delta(x-b)],\ \ a \neq b;\\
&(e)\quad \delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \quad \text{if } f(x_i) = 0 \text{ and } f'(x_i) \neq 0.
\end{aligned}
$$

We first prove (c) when a is negative, say $a = -b$, $b > 0$. Let $u = -bx$, then $du = -b\,dx$, and the limits $x = -\infty, \infty$, become $u = \infty, -\infty$.

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)\delta(-bx)\,dx &= \int_{\infty}^{-\infty} \phi\left(\frac{u}{-b}\right)\delta(u)\left(\frac{du}{-b}\right) = \frac{1}{b}\int_{-\infty}^{\infty} \phi\left(\frac{u}{-b}\right)\delta(u)\,du\\
&= \frac{1}{b}\phi(0) = \frac{1}{|a|}\phi(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(x)\delta(x)\,dx.
\end{aligned}
$$

From the second integral to the third, we have reversed the order of integration (one minus sign) and also changed $\frac{du}{-b}$ to $\frac{du}{b}$ (another minus sign, which cancels the first). Now, if we repeat the calculation using $a > 0$ instead of $-b$, neither of these sign reversals occurs, and so we get the result $\frac{1}{a}\phi(0)$ instead of $\frac{1}{b}\phi(0)$. But when $a > 0$, a and $|a|$ are the same. Thus we get the result stated in (11.19c).

δ functions in 2 or 3 dimensions It is now straightforward to write the defining equations in rectangular coordinates for δ functions in 2 or 3 dimensions. We have

$$
\text{(11.20)}\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\delta(x-x_0)\delta(y-y_0)\,dx\,dy = \phi(x_0,y_0).
$$

$$
\text{(11.21)}\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y,z)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\,dx\,dy\,dz = \phi(x_0,y_0,z_0).
$$

As in one dimension, the delta function "picks out" the value of the test function ϕ at the "peak" of the δ function. The integrals need not be over all space, just over a region containing the point $\mathbf{r}_0$; otherwise the integral is zero. The abbreviations $\delta(\mathbf{r})$ or $\delta^3(\mathbf{r})$ are often used for $\delta(x)\delta(y)\delta(z)$, but note carefully that they do not mean functions of the vector $\mathbf{r}$, but rather functions of the components x, y, z of $\mathbf{r}$. Similarly you may see $\delta(\mathbf{r}-\mathbf{r}_0)$ or $\delta^3(\mathbf{r}-\mathbf{r}_0)$ meaning the δ function in (11.21).

In spherical coordinates, let's use f instead of ϕ to mean a test function (since ϕ is a spherical coordinate angle). By the definition of δ functions, we want $\iiint f(r,\theta,\phi)\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)\,dr\,d\theta\,d\phi = f(r_0,\theta_0,\phi_0)$. But since we would like to use the volume element $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi = r^2\,dr\,d\Omega$, we need to write

(also see Problem 22)

$$\delta(\mathbf{r}-\mathbf{r}_0)=\frac{\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{r^2\sin\theta};\qquad \text{then}$$
$$\iiint f(r,\theta,\phi)\delta(\mathbf{r}-\mathbf{r}_0)\,d\tau=f(r_0,\theta_0,\phi_0). \tag{11.22}$$

Similarly in cylindrical coordinates, with $d\tau = r\,dr\,d\theta\,dz$,

$$\delta(\mathbf{r}-\mathbf{r}_0)=\frac{\delta(r-r_0)\delta(\theta-\theta_0)\delta(z-z_0)}{r};\qquad \text{then}$$
$$\iiint f(r,\theta,z)\delta(\mathbf{r}-\mathbf{r}_0)\,d\tau=f(r_0,\theta_0,z_0). \tag{11.23}$$

Note that we can use these formulas to write mass density or charge density functions in the various coordinate systems.

Example 5. Suppose there is a unit charge or unit mass at the point $(x,y,z)=(-1,\sqrt{3},-2)$; then in rectangular coordinates, the density is

$$\rho=\delta(x+1)\delta(y-\sqrt{3})\delta(z+2).$$

In cylindrical coordinates the point is $(r,\theta,z)=(2,2\pi/3,-2)$ so in cylindrical coordinates the density is

$$\rho=\delta(r-2)\delta(\theta-2\pi/3)\delta(z+2)/r.$$

In spherical coordinates, the point is $(r,\theta,\phi)=(2\sqrt{2},3\pi/4,2\pi/3)$, so in spherical coordinates the density is

$$\rho=\delta(r-2\sqrt{2})\delta(\theta-3\pi/4)\delta(\phi-2\pi/3)/(r\sin\theta).$$

Finally, let's verify two useful operator equations for δ functions in 3 dimensions.

$$\nabla\cdot\frac{\mathbf{e}_r}{r^2}=4\pi\delta(\mathbf{r}); \tag{11.24}$$

$$\nabla^2\frac{1}{r}=-4\pi\delta(\mathbf{r}). \tag{11.25}$$

You can easily show (Problem 24a) that $\nabla\cdot(\mathbf{e}_r/r^2)$ is zero for any $r\neq 0$ (and undefined for $r=0$). Also, by the divergence theorem [Chapter 6, equation (10.17)] in spherical coordinates, we find

$$\iiint\limits_{\text{volume } \tau}\nabla\cdot\frac{\mathbf{e}_r}{r^2}\,d\tau=\iint\limits_{\substack{\text{surface}\\ \text{inclosing } \tau}}\frac{\mathbf{e}_r}{r^2}\cdot\mathbf{e}_r\,d\sigma=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{r^2}r^2\sin\theta\,d\theta\,d\phi=4\pi.$$

Thus $\nabla\cdot(\mathbf{e}_r/r^2)$ has the properties that it is zero for all $r>0$ but its integral over any volume including the origin $=4\pi$; this suggests that it is equal to $4\pi\delta(\mathbf{r})$. Let's verify that this is correct. (Compare Problem 25.) Since $\nabla\cdot(\mathbf{e}_r/r^2)$ depends only on r (Problem 24a), we use a test function $f(r)$. We want to show that

$\iiint f(r)\nabla\cdot(\mathbf{e}_r/r^2)\,d\tau$, over any volume containing the origin, is equal to $4\pi f(0)$. For convenience we integrate over the volume inside the sphere $r = a$. (Since the integrand is zero for $r > 0$, the answer is the same for any volume containing the origin.) By Problem 11.17(e) of Chapter 6 with $\phi = f$, $\mathbf{V} = \mathbf{e}_r/r^2$, and $\mathbf{n} = \mathbf{e}_r$, we find

$$\iiint_{\text{volume } r<a} f(r)\nabla\cdot\frac{\mathbf{e}_r}{r^2}\,d\tau = \iint_{\text{surface } r=a} f(r)\frac{\mathbf{e}_r}{r^2}\cdot\mathbf{e}_r\,d\sigma - \iiint_{\text{volume } r<a} \nabla f(r)\cdot\frac{\mathbf{e}_r}{r^2}\,d\tau.$$

On the surface $r = a$, the integrand of the surface integral is $f(a)\frac{1}{a^2}a^2\,d\Omega$, so the surface integral is $4\pi f(a)$. In the volume integral, $\nabla f(r)\cdot\mathbf{e}_r$ is the r component of $\nabla f(r)$; in spherical coordinates this is just $\partial f/\partial r$ (Chapter 6, equation 6.8). Thus the volume integral on the right-hand side is

$$\iiint_{\text{volume } r<a} \frac{\partial f}{\partial r}\frac{1}{r^2}r^2\,dr\,d\Omega = 4\pi f(r)\Big|_0^a = 4\pi[f(a) - f(0)]$$

and we have

$$\iiint_{\text{volume } r<a} f(r)\nabla\cdot\frac{\mathbf{e}_r}{r^2}\,d\tau = 4\pi f(a) - 4\pi[f(a) - f(0)] = 4\pi f(0)$$

as we expected. Thus (11.24) is a valid operator equation. You can show (Problem 24b) that $\nabla(1/r) = -\mathbf{e}_r/r^2$). Since $\nabla\cdot\nabla = \nabla^2$ (that is, div grad = Laplacian), we have $\nabla^2(1/r) = \nabla\cdot\nabla(1/r) = -\nabla\cdot(\mathbf{e}_r/r^2)$. Thus (11.25) is also valid.

We can write (11.24) and (11.25) with the peak of the δ function shifted from the origin to $\mathbf{r}_0$. The unit vector from $\mathbf{r}_0$ to $\mathbf{r}$ can be written as $(\mathbf{r}-\mathbf{r}_0)/|\mathbf{r}-\mathbf{r}_0|$. Then we have

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = -\nabla\cdot\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = -4\pi\delta(\mathbf{r}-\mathbf{r}_0). \tag{11.26}$$

It is now interesting to note that we have seen this before without recognizing that we were dealing with a δ function. Look back to Chapter 6, Equations (6.10.19) to (6.10.25). With $\mathbf{D}$ given by (6.10.19), we have from (8.11.24) above, $\nabla\cdot\mathbf{D} = \frac{q}{4\pi}4\pi\delta(\mathbf{r}) = q\delta(\mathbf{r})$ which we recognize as the charge density for a charge q at the origin. Thus, although we wrote (6.10.25) with ρ as the density of a charge distribution, we could now write it with ρ = charge density of a point charge = $q\delta(\mathbf{r})$, and then (6.10.25) becomes (6.10.22).

▶ PROBLEMS, SECTION 11

1. Find the inverse Laplace transform of e^{-2p}/p^2 in the following ways:
   (a) using $L5$ and $L27$ and the convolution integral of Section 10;
   (b) using $L28$.
2. Verify $L24$ in the table by using $L1$, $L27$, and the convolution integral.
3. Verify $L28$ in the table by using $L27$ and the convolution integral.
4. Show that $\int_{-\infty}^{\infty} f_n(t)\,dt = 1$ for the functions $f_n(t)$ in Figures 11.3 and 11.4.

5. Solve the differential equation $y'' + \omega^2 y = f(t)$, $y_0 = y_0' = 0$, with $f(t)$ given by the functions in Figure 11.4, by the following methods.

(a) Use the convolution integral, being careful to consider separately the three intervals 0 to t_0, t_0 to $t_0 + 1/n$, and $t_0 + 1/n$ to ∞.

(b) Write $f_n(t)$ as a difference of unit step functions as in $L25$, and use $L25$ to find $L(f_n)$. Expand $\dfrac{1}{p(p^2+\omega^2)}$ by partial fractions and use $L28$ to find $y_n(t)$. Your result should agree with (a).

(c) Let $n \to \infty$ and show that your solution in (a) and (b) tends to the same solution (11.5) obtained using the functions of Figure 11.3; that is, either set of functions gives, in the limit, the same solution (11.11) obtained by using the δ function. Note that, when you let $n \to \infty$, you do not need to consider the interval t_0 to $t_0 + 1/n$ since, if $t > t_0$, then for sufficiently large n, $t > t_0 + 1/n$.

6. (a) Let a mechanical or electrical system be described by the differential equation $Ay'' + By' + Cy = f(t)$, $y_0 = y_0' = 0$. As in Problem 10.16b, write the solution as a convolution (assume $a \neq b$). Let $f(t)$ be one of the functions in Figure 11.4 and Problem 5. Find y and then let $n \to \infty$.

(b) Solve (a) with $f(t) = \delta(t - t_0)$; your result should be the same as in (a).

(c) The solution y as found in (a) and (b) is called the *response* of the system to a unit impulse. Show that the response of a system to a unit impulse at $t_0 = 0$ is the inverse Laplace transform of the transfer function.

Using the δ function method, find the response (see Problem 6c) of each of the following systems to a unit impulse.

7. $y'' + 2y' + y = \delta(t - t_0)$

8. $y'' + 4y' + 5y = \delta(t - t_0)$

9. $y'' + 2y' + 10y = \delta(t - t_0)$

10. $y'' - 9y = \delta(t - t_0)$

11. $\dfrac{d^4 y}{dt^4} - y = \delta(t - t_0)$

12. Evaluate the functions $f_n(x-a)$ defined by the integral in (11.13) with limits $-n$, n. Show that $\int_{-\infty}^{\infty} f_n(x-a)\,dx = 1$ for all n. Sketch or computer plot graphs of several f_n's to show that as n increases the functions $f_n(x)$ are increasingly peaked around $x = a$, and that as $|x - a|$ increases, they oscillate with decreasing amplitude.

13. Using δ functions, write the following mass or charge density functions.

(a) Mass 5 at $x = 2$, and mass 3 at $x = -7$.

(b) Charge 3 at $x = -5$ and charge -4 at $x = 10$.

14. Integrate by parts as we did for (11.14) to obtain (11.15) and (11.16).

15. Use (11.6) and (11.14) to (11.16) to evaluate the following integrals. *Warning hint:* See comments just after (11.6) and (11.16) about the range of integration.

(a) $\displaystyle\int_0^{\pi} \sin x \;\, \delta(x - \frac{\pi}{2})\,dx$ (b) $\displaystyle\int_0^{\pi} \sin x \;\, \delta(x + \frac{\pi}{2})\,dx$

(c) $\displaystyle\int_{-1}^{1} e^{3x}\delta'(x)\,dx$ (d) $\displaystyle\int_0^{\pi} \cosh x \;\, \delta''(x-1)\,dx$

16. Verify the operator equation $\frac{d}{dx}\text{sgn } x = 2\delta(x)$ where the function signum x, meaning "sign of x," and abbreviated sgn x, is defined by

$$\text{sgn } x = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

17. Verify (11.18a) and (11.18c) by multiplying by a test function and integrating.

18. Use equation (11.16) to generalize the operator equations (11.18) as follows:

(a) Show that $x^m \delta^{(n)}(x) = 0$ if $m > n$; compare equation (11.18a).

(b) Show that $x^n \delta^{(n)}(x) = (-1)^n n!\, \delta(x)$; compare (11.18b) and (11.18c).

(c) Show that $x^m \delta^{(n)}(x) = (-1)^m \frac{n!}{(n-m)!}\, \delta^{(n-m)}(x)$, $m \le n$.

(d) Use the results in (a) and (b) to show that

$$(x^2 + y^2 + z^2)\nabla^2[\delta(x)\delta(y)\delta(z)] = 6\delta(x)\delta(y)\delta(z).$$

19. (a) Show that you can differentiate a generalized function equation or multiply it by a power of x. This means to show that if $\int \phi(x)G_1(x)\,dx = \int \phi(x)G_2(x)\,dx$ for all test functions ϕ, then

$$\int \phi(x)G_1'(x)\,dx = \int \phi(x)G_2'(x)\,dx \qquad \text{and}$$

$$\int \phi(x)x^n G_1(x)\,dx = \int \phi(x)x^n G_2(x)\,dx.$$

Hints: For the differentiation proof, integrate by parts. For the multiplication by x^n proof, consider whether $x^n\phi(x)$ is a test function if ϕ is. See comment just after equation (11.17).

(b) Multiply (11.18b) by x and use (11.18a). Differentiate the result and simplify to get (11.18c).

(c) Multiply (11.18c) by x, use (11.18a), differentiate and simplify to find $x^3\delta'''(x)$ in terms of $\delta(x)$. Check your result by Problem (18b).

(d) Try a few more examples as in (b) and (c) and check your results by Problem 18.

20. Verify the operator equations in (11.19) not done in text.
Hints for (a) and (b): Follow the text method of proof of (c), making the change of variable $u = -x$ or $u = a - x$. *Hints* for (c) and (d): Split the integral into a sum of integrals each including just one x_i. In (d), what is the value of $(x - a)$ when x is in the vicinity of b? Use part (c).

21. Make use of the operator equations (11.19) and previous equations to evaluate the following integrals.

(a) $\displaystyle\int_0^3 (5x-2)\delta(2-x)\,dx$ (b) $\displaystyle\int_0^\infty \phi(x)\delta(x^2-a^2)\,dx$

(c) $\displaystyle\int_{-1}^1 \cos x\,\delta(-2x)\,dx$ (d) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\,\delta(\sin x)\,dx$

22. You may find the spherical coordinate δ function written as

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0)/r^2.$$

Show that this equation is equivalent to (11.22). *Hints:* You want to show that $\delta(\cos\theta - \cos\theta_0) = \delta(\theta - \theta_0)/\sin\theta$. See (11.19e). Also note that it doesn't really matter whether we write $r^2\sin\theta$ or $r_0^2\sin\theta_0$ in the denominator of (11.22) since the δ functions are zero unless $r = r_0$ and $\theta = \theta_0$.

23. Write a formula in rectangular coordinates, in cylindrical coordinates, and in spherical coordinates for the density of a unit point charge or mass at the point with the given rectangular coordinates:

(a) $(-5, 5, 0)$ (b) $(0, -1, -1)$

(c) $(-2, 0, 2\sqrt{3})$ (d) $(3, -3, -\sqrt{6})$

24. (a) Show that $\nabla \cdot (\mathbf{e}_r/r^2) = 0$ for $r > 0$. *Hint:* You can do this in rectangular coordinates, but it is easier in spherical coordinates. See Chapter 6, equation (7.9). Show that $\nabla \cdot [\mathbf{e}_r F(r)]$, for any $F(r)$, is a function of r only.

(b) Show that $\nabla(1/r) = -\mathbf{e}_r/r^2$. See Chapter 6, equation (6.8).

25. Let

$$F(x) = \begin{cases} x - 2, & x > 0, \\ 0, & x < 0. \end{cases}$$

Show that $F''(x) = 0$ for all $x \neq 0$, and $\int_{-\infty}^{\infty} F''(x)\,dx = 1$, which leads you to think that $F''(x)$ might $= \delta(x)$. Show in two ways, as outlined in (a) and (b), that this is not true.

(a) Show that $\int_{-\infty}^{\infty} \phi(x)F''(x)\,dx = \phi(0) + 2\phi'(0)$, where ϕ is any test function. Then by (11.6) and (11.14), what is $F''(x)$?

(b) Show that $F(x) = (x - 2)u(x)$ where $u(x)$ is the unit step function in (11.17). Differentiate this equation twice and simplify using (11.17) and (11.18). Compare your result in (a).

(c) As in (a) and (b), find $G''(x)$ in terms of δ and δ' if

$$G(x) = \begin{cases} 3x + 1, & x > 0, \\ 2x - 4, & x < 0. \end{cases}$$

12. A BRIEF INTRODUCTION TO GREEN FUNCTIONS

Let's do some examples to see what a Green function is and how we can use it to solve ordinary differential equations. Also see Chapter 13, Section 8, for an application to partial differential equations. (You might find it interesting to read "The Green of Green Functions", Physics Today, December 2003, 41–46.)

Example 1. We reconsider the differential equation (11.2), namely

$$y'' + \omega^2 y = f(t), \quad y_0 = y_0' = 0 \tag{12.1}$$

where $f(t)$ is some given forcing function. Using (11.6), we can write

$$f(t) = \int_0^{\infty} f(t')\delta(t' - t)\,dt', \tag{12.2}$$

that is, we can think of the force $f(t)$ as (a limiting case of) a whole sequence of impulses. (You might reflect that, on the molecular level, air pressure is the force per unit area due to a tremendous number of impacts of individual molecules.) Now suppose that we have solved (12.1) with $f(t)$ replaced by $\delta(t' - t)$, that is, we find the response of the system to a unit impulse at t'. Let us call this response $G(t, t')$, that is, $G(t, t')$ is the solution of

$$\frac{d^2}{dt^2}G(t, t') + \omega^2 G(t, t') = \delta(t' - t). \tag{12.3}$$

Then, given some forcing function $f(t)$, we try to find a solution of (12.1) by "adding up" the responses of many such impulses. We shall show that this solution is

$$y(t) = \int_0^\infty G(t,t')f(t')\,dt'. \tag{12.4}$$

Substituting (12.4) into (12.1) and using (12.3) and (12.2), we find

$$\begin{aligned} y'' + \omega^2 y &= \left(\frac{d^2}{dt^2} + \omega^2\right) y = \left(\frac{d^2}{dt^2} + \omega^2\right) \int_0^\infty G(t,t')f(t')\,dt' \\ &= \int_0^\infty \left(\frac{d^2}{dt^2} + \omega^2\right) G(t,t')f(t')\,dt' = \int_0^\infty \delta(t'-t)f(t')\,dt' = f(t). \end{aligned}$$

Thus (12.4) is a solution of (12.1).

The function $G(t,t')$ is called a *Green function* (or Green's function). The Green function is the response of the system to a unit impulse at $t = t'$. Solving (12.3) with initial conditions $G = 0$ and $dG/dt = 0$ at $t = 0$, we find (Problem 1)

$$G(t,t') = \begin{cases} 0, & 0 < t < t', \\ \frac{1}{\omega}\sin\omega(t-t'), & 0 < t' < t. \end{cases} \tag{12.5}$$

Then (12.4) gives the solution of (12.1) with $y_0 = y_0' = 0$, namely

$$y(t) = \int_0^t \frac{1}{\omega}\sin\omega(t-t')f(t')\,dt'. \tag{12.6}$$

(The upper limit is $t' = t$ since $G = 0$ for $t' > t$.) Thus, given a forcing function $f(t)$, we can find the response $y(t)$ of the system (12.1) by integrating (12.6) (see Problems 2 to 5). Similarly for other differential equations we can find the solution in terms of an appropriate Green function (see Problems 6 to 8).

▶ **Example 2.** As we will see later (Chapter 13, Section 8), in using Green functions in three-dimensional problems, we usually want a solution which is zero on the boundary of some region. In order to have a similar problem here, let us ask for a solution of

$$y'' + y = f(x) \tag{12.7}$$

such that $y = 0$ at $x = 0$ and at $x = \pi/2$. A physical interpretation of this problem may be useful. If a string is stretched along the x axis from $x = 0$ to $x = \pi/2$, and then caused to vibrate by a force proportional to $-f(x)\sin\omega t$, then $|y(x)|$ in (12.7) gives the amplitude of small vibrations.

We first find a solution of [compare (12.3)]

$$\frac{d^2}{dx^2}G(x,x') + G(x,x') = \delta(x'-x) \tag{12.8}$$

satisfying $G(0,x') = G(\pi/2,x') = 0$; this solution is the Green function for our problem. Then [compare (12.4)]

$$y(x) = \int_0^{\pi/2} G(x,x')f(x')\,dx' \tag{12.9}$$

gives a solution of (12.7) satisfying the conditions $y(0) = y(\pi/2) = 0$ (Problem 9).

To construct the desired Green function, we first note that for any $x \neq x'$, the equation (12.8) becomes

$$\frac{d^2}{dx^2}G(x,x') + G(x,x') = 0, \qquad x \neq x'. \tag{12.10}$$

The solutions of (12.10) are $\sin x$ and $\cos x$; we observe that $\sin x = 0$ at $x = 0$ and $\cos x = 0$ at $x = \pi/2$. Thus we try to find a Green function of the form

$$G(x,x') = \begin{cases} A(x')\sin x, & 0 < x < x' < \pi/2, \\ B(x')\cos x, & 0 < x' < x < \pi/2. \end{cases} \tag{12.11}$$

The next step may be clarified by thinking about the string problem. If the string is oscillated by a concentrated force at x' [see (12.8)], then the amplitude of the vibration given by (12.11) is shown in Figure 12.1. At $x = x'$, $G(x,x')$ is continuous, that is, from (12.11)

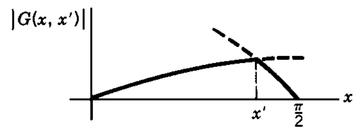


Figure 12.1

$$A(x')\sin x' = B(x')\cos x'. \tag{12.12}$$

However (see Figure 12.1), the slope changes abruptly at x'. From (12.11), we find

$$\frac{d}{dx}\,G(x,x') = \begin{cases} A(x')\cos x, & x < x', \\ -B(x')\sin x, & x > x'. \end{cases}$$

$$\text{Change in } \frac{dG}{dx} \text{ at } x' \text{ is } -B(x')\sin x' - A(x')\cos x'. \tag{12.13}$$

We can evaluate this change in dG/dx by integrating (12.8) from $x = x' - \epsilon$ to $x = x' + \epsilon$ and letting $\epsilon \to 0$. Since $\int d^2G/dx^2 = dG/dx$, we find

$$\left.\frac{dG}{dx}\right|_{x'-\epsilon}^{x'+\epsilon} + \int_{x'-\epsilon}^{x'+\epsilon} G(x,x')\,dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x'-x)\,dx = 1,$$

or, letting $\epsilon \to 0$:

$$\left(\text{Change in slope } \frac{dG}{dx} \text{ at } x'\right) \text{ is } 1.$$

Then from (12.13)

$$-B(x')\sin x' - A(x')\cos x' = 1. \tag{12.14}$$

We solve (12.12) and (12.14) for $A(x')$ and $B(x')$ (Problem 10) and get

$$A(x') = -\cos x', \qquad B(x') = -\sin x'. \tag{12.15}$$

Thus we have

$$G(x,x') = \begin{cases} -\cos x' \sin x, & 0 < x < x' < \pi/2, \\ -\sin x' \cos x, & 0 < x' < x < \pi/2. \end{cases} \tag{12.16}$$

Then from (12.9), the solution of (12.7) with $y(0) = y(\pi/2) = 0$ is

$$y(x) = -\cos x \int_0^x (\sin x')f(x')\,dx' - \sin x \int_x^{\pi/2} (\cos x')f(x')\,dx'. \tag{12.17}$$

▶ **Example 3.** If $f(x) = \csc x$, we find from (12.17):

$$y(x) = -\cos x \int_0^x \sin x' \csc x' \, dx' - \sin x \int_x^{\pi/2} \cos x' \csc x' \, dx'$$
$$= (-\cos x)(x) - (\sin x)(\ln \sin x')\Big|_x^{\pi/2} = -x\cos x + (\sin x)(\ln \sin x).$$

It is interesting to note that we can use the Green function method to obtain a particular solution of a nonhomogeneous differential equation (nonzero right-hand side) when we know the solutions of the corresponding homogeneous equation (zero right-hand side). (See Problems 14 to 18.) In (12.17) each integral gives a function of x minus a constant (from the constant limits); these constants times $\sin x$ and $\cos x$ give a solution of the homogeneous equation. Thus the remaining terms give a particular solution of the nonhomogeneous equation. We can write this particular solution in a simple form by changing $\int_x^{\pi/2}$ to $-\int_{\pi/2}^x$, dropping the constant limits and writing indefinite integrals. Then a particular solution $y_p(x)$ of (12.7) is given by

$$y_p(x) = -\cos x \int (\sin x) f(x)\, dx + \sin x \int (\cos x) f(x)\, dx. \tag{12.18}$$

▶ **Example 4.** By the same methods used above, you can verify (Problem 14) that a solution of the differential equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{12.19}$$

with $y(a) = y(b) = 0$ is given by

$$y(x) = y_2(x) \int_a^x \frac{y_1(x')f(x')}{W(x')}\, dx' + y_1(x) \int_x^b \frac{y_2(x')f(x')}{W(x')}\, dx', \tag{12.20}$$

where $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation with $y_1(a) = 0$, $y_2(b) = 0$, and W is the Wronskian of $y_1(x)$ and $y_2(x)$ [See Chapter 3, equation (8.5)]. Just as in (12.18), we find that a particular solution y_p of (12.19) is

$$y_p(x) = y_2(x) \int \frac{y_1(x)f(x)}{W(x)}\, dx - y_1(x) \int \frac{y_2(x)f(x)}{W(x)}\, dx. \tag{12.21}$$

The particular solution (12.18) and (12.21) are exactly the same as those obtained by the method of *variation of parameters* (see Problem 14b) but the Green function method may seem less arbitrary.

▶ PROBLEMS, SECTION 12

1. Solve (12.3) if $G = 0$ and $dG/dt = 0$ at $t = 0$ to obtain (12.5). *Hint:* Use $L28$ and $L3$ to find the inverse transform.

In Problems 2 and 3, use (12.6) to solve (12.1) when $f(t)$ is as given.

2. $f(t) = \sin \omega t$ **3.** $f(t) = e^{-t}$

4. Use equation (12.6) to solve Problem 10.18.

5. Obtain (12.6) by using the convolution integral to solve (12.1).

6. For Problem 10.17, show (as in Problem 1) that the Green function is

$$G(t,t') = \begin{cases} 0, & 0 < t < t', \\ (1/a)\sinh a(t-t'), & 0 < t' < t. \end{cases}$$

Thus write the solution of Problem 10.17 as an integral [similar to (12.6)] and evaluate it.

7. Use the Green function of Problem 6 to solve

$$y'' - a^2 y = e^{-t}, \qquad y_0 = y_0' = 0.$$

8. Solve the differential equation $y'' + 2y' + y = f(t)$, $y_0 = y_0' = 0$, where

$$f(t) = \begin{cases} 1, & 0 < t < a, \\ 0, & t > a. \end{cases}$$

As in Problems 6 and 7, find the Green function for the problem and use it in equation (12.4). Consider the cases $t < a$ and $t > a$ separately.

9. Following the proof of (12.4), show that (12.9) gives a solution of (12.7).

10. Solve (12.12) and (12.14) to get (12.15). *Hint:* Use Cramer's rule (Chapter 3, Section 3); note that the denominator determinant is the Wronskian [Chapter 3, equation (8.5)] of the functions $\sin x$ and $\cos x$.

In Problems 11 to 13, use (12.17) to find the solution of (12.7) with $y(0) = y(\pi/2) = 0$ when the forcing function is given $f(x)$.

11. $f(x) = \sin 2x$

12. $f(x) = \sec x$

13. $f(x) = \begin{cases} x, & 0 < x < \pi/4 \\ \pi/2 - x, & \pi/4 < x < \pi/2. \end{cases}$

Hint: Write separate formulas for $y(x)$ for $x < \pi/4$ and $x > \pi/4$.

14. (a) Given that $y_1(x)$ and $y_2(x)$ are solutions of (12.19) with $f(x) = 0$, and that $y_1(a) = 0$, $y_2(b) = 0$, find the Green function [as in (12.11) to (12.16)] and so obtain the solution (12.20). Then find the particular solution (12.21) as discussed for (12.18) and (12.21).

(b) The method of *variation of parameters* is an elementary way of finding a particular solution of (12.19) when you know the solutions of the homogeneous equation. Show as follows that this method leads to the same result (12.21) as the Green function method. Start with the known solution of the homogeneous equation, say $y = c_1 y_1 + c_2 y_2$ and allow the "constants" to be functions of x to be determined so that y satisfies (12.19). (The c's are the "parameters" which are to be "varied" in the expression "variation of parameters".) You want to find y' and y'' to substitute into (12.19). First find y' and set the sum of the terms involving derivatives of the c's equal to zero. Differentiate the rest of y' again to get y''. Now substitute y, y' and y'' into (12.19) and use the fact that y_1 and y_2 both satisfy the homogeneous equation [that is, (12.19) with $f(x) = 0$]. You should have the two equations:

$$c_1' y_1 + c_2' y_2 = 0,$$
$$c_1' y_1' + c_2' y_2' = f(x).$$

Solve this pair of equations for c_1' and c_2' [say by determinants, and note that the denominator determinant is the Wronskian as in (12.20) and (12.21)]. Write the indefinite integrals for c_1 and c_2, and write $y = c_1 y_1 + c_2 y_2$ to get (12.21).

In Problems 15 to 18, use the given solutions of the homogeneous equation to find a particular solution of the given equation. You can do this either by the Green function formulas in the text or by the method of variation of parameters in Problem 14b.

15. $y'' - y = \operatorname{sech} x; \quad \sinh x,\ \cosh x$

16. $x^2y'' - 2xy' + 2y = x \ln x; \quad x,\ x^2$

17. $y'' - 2(\csc^2 x)y = \sin^2 x; \quad \cot x,\ 1 - x\cot x$

18. $(x^2+1)y'' - 2xy' + 2y = (x^2+1)^2; \quad x,\ 1 - x^2$

▶ 13. MISCELLANEOUS PROBLEMS

Identify each of the differential equations in Problems 1 to 24 as to type (for example, separable, linear first order, linear second order, etc.), and then solve it.

1. $x^2y' - xy = 1/x$

2. $x(\ln y)y' - y\ln x = 0$

3. $y''' + 2y'' + 2y' = 0$

4. $\dfrac{d^2r}{dt^2} - 6\dfrac{dr}{dt} + 9r = 0$

5. $(2x - y\sin 2x)\,dx = (\sin^2 x - 2y)\,dy$

6. $y'' + 2y' + 2y = 10e^x + 6e^{-x}\cos x$

7. $3x^3y^2y' - x^2y^3 = 1$

8. $x^2y'' - xy' + y = x$

9. $dy - (2y + y^2e^{3x})\,dx = 0$

10. $u(1-v)\,dv + v^2(1-u)\,du = 0$

11. $(y + 2x)\,dx - x\,dy = 0$

12. $xy'' + y' = 4x$

13. $y'' + 4y' + 5y = 26e^{3x}$

14. $y'' + 4y' + 5y = 2e^{-2x}\cos x$

15. $y'' - 4y' + 4y = 6e^{2x}$

16. $y'' - 5y' + 6y = e^{2x}$

17. $(2x + y)\,dy - (x - 2y)\,dx = 0$

18. $(x\cos y - e^{-\sin y})\,dy + dx = 0$

19. $\sin^2 x\,dy + [\sin^2 x + (x+y)\sin 2x]\,dx = 0$

20. $y'' - 2y' + 5y = 5x + 4e^x(1 + \sin 2x)$

21. $y' + xy = x/y$

22. $(D-2)^2(D^2+9)y = 0$

23. $\sin\theta\cos\theta\,dr - \sin^2\theta\,d\theta = r\cos^2\theta\,d\theta$

24. $x(yy'' + y'^2) = yy'$ *Hint:* Let $u = yy'$.

In Problems 25 to 28, find a particular solution satisfying the given conditions.

25. $3x^2y\,dx + x^3\,dy = 0, \quad y = 2$ when $x = 1$.

26. $xy' - y = x^2, \quad y = 6$ when $x = 2$

27. $y'' + y' - 6y = 6, \quad y = 1,\ y' = 4$ when $x = 0$

28. $yy'' + y'^2 + 4 = 0 \quad y = 3,\ y' = 0$ when $x = 1$

29. If 10 kg of rock salt is placed in water, it dissolves at a rate proportional to the amount of salt still undissolved. If 2kg dissolve during the first 10 minutes, how long will it be until only 2kg remain undissolved?

30. A mass m falls under gravity (force mg) through a liquid whose viscosity is decreasing so that the retarding force is $-2mv/(1+t)$, where v is the speed of m. If the mass starts from rest, find its speed, its acceleration, and how far it has fallen (in terms of g) when $t = 1$.

31. The acceleration of an electron in the electric field of a positively charged sphere is inversely proportional to the square of the distance between the electron and the center of the sphere. Let an electron fall from rest at infinity to the sphere. What is the electron's velocity when it reaches the surface of the sphere?

32. Suppose that the rate at which you work on a hot day is inversely proportional to the excess temperature above 75°. One day the temperature is rising steadily, and you start studying at 2 p.m. You cover 20 pages the first hour and 10 pages the second hour. At what time was the temperature 75°?

33. Compare the temperatures of your cup of coffee at time t

(a) if you add cream and let the mixture cool;

(b) if you let the coffee and cream sit on the table and mix them at time t.

Hints: Assume Newton's law of cooling (Problem 2.27) for both coffee and cream (where it is a law of heating). Combine n' units of cream initially at temperature T_0' with n units of coffee initially at temperature T_0, and find the temperature at time t in (a) and in (b) assuming that the air temperature remains a constant T_a, and that the proportionality constant in the law of cooling is the same for both coffee and cream.

34. A flexible chain of length l is hung over a peg with one end of the chain slightly longer than the other. Assuming that the chain slides off with no friction, write and solve the differential equation of motion to show that $y = y_0 \cosh t\sqrt{2g/l}$, $0 < y < l/2$, where $2y$ is the difference in length of the two ends, and $y = y_0$ when $t = 0$.

35. A raindrop falls through a cloud, increasing in size as it picks up moisture. Assume that its shape always remains spherical. Also assume that the rate of increase of its volume with respect to distance fallen is proportional to the cross-sectional area of the drop at any time (that is, the mass increase $dm = \rho dV$ is proportional to the volume $\pi r^2\, dy$ swept out by the drop as it falls a distance dy). Show that the radius r of the drop is proportional to the distance y the drop has fallen if $r = 0$ when $y = 0$. Recall that when m is not constant, Newton's second law is properly stated as $(d/dt)(mv) = F$. Use this equation to find the distance y which the drop falls in time t under the force of gravity, if $y = \dot{y} = 0$ at $t = 0$. Show that the acceleration of the drop is $g/7$ where g is the acceleration of gravity.

36. (a) A rocket of (variable) mass m is propelled by steadily ejecting part of its mass at velocity u (constant with respect to the rocket). Neglecting gravity, the differential equation of the rocket is $m(dv/dm) = -u$ as long as $v \ll c$, $c =$ speed of light. Find v as a function of m if $m = m_0$ when $v = 0$.

(b) In the relativistic region (v/c not negligible), the rocket equation is

$$m\frac{dv}{dm} = -u\left(1 - \frac{v^2}{c^2}\right).$$

Solve this differential equation to find v as a function of m. Show that $v/c = (1-x)/(1+x)$, where $x = (m/m_0)^{2u/c}$.

37. The differential equation for the path of a planet around the sun (or any object in an inverse square force field) is, in polar coordinates,

$$\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)-\frac{1}{r^3}=-\frac{k}{r^2}.$$

Make the substitution $u = 1/r$ and solve the equation to show that the path is a conic section.

38. Use $L15$ and $L31$ to find the Laplace transform of $(1-\cos at)/t$.

39. Use $L32$ and $L9$ to find the Laplace transform of $t\sinh at$. Verify your result by finding its inverse transform using the convolution integral.

Use the Laplace transform table to evaluate:

40. $\displaystyle\int_0^\infty t^3 e^{-4t}\sinh 2t\,dt$

41. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\int_n^{n+1} te^{-2t}\,dt$

Find the inverse Laplace transform of:

42. $\dfrac{p}{(p+a)^3}$

43. $\dfrac{p^2}{(p^2+a^2)^2}$

44. $\dfrac{1}{(p^2+a^2)^3}$

45. Prove the following shifting or translation theorems for Fourier transforms. If $g(\alpha)$ is the Fourier transform of $f(x)$, then

(a) the Fourier transform of $f(x-a)$ is $e^{-i\alpha a}g(\alpha)$;

(b) the Fourier transform of $e^{i\beta x}f(x)$ is $g(\alpha-\beta)$.

Compare Problems 8.19 to 8.27.

46. Use the table of Laplace transforms to find the sine and cosine Fourier transforms of e^{-x}; of xe^{-x}.

Solve Problems 47 and 48 either by Laplace transforms and the convolution integral or by Green functions.

47. $y'' + y = \sec^2 t$

48. $y'' + y = t\sin t$

Table of Laplace Transforms

	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt$	
$L1$	1	$\frac{1}{p}$	$\text{Re } p > 0$
$L2$	e^{-at}	$\frac{1}{p+a}$	$\text{Re } (p+a) > 0$
$L3$	$\sin at$	$\frac{a}{p^2+a^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L4$	$\cos at$	$\frac{p}{p^2+a^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L5$	$t^k,\ k > -1$	$\frac{k!}{p^{k+1}}$ or $\frac{\Gamma(k+1)}{p^{k+1}}$	$\text{Re } p > 0$
$L6$	$t^k e^{-at},\ k > -1$	$\frac{k!}{(p+a)^{k+1}}$ or $\frac{\Gamma(k+1)}{(p+a)^{k+1}}$	$\text{Re } (p+a) > 0$
$L7$	$\frac{e^{-at} - e^{-bt}}{b-a}$	$\frac{1}{(p+a)(p+b)}$	$\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
$L8$	$\frac{ae^{-at} - be^{-bt}}{a-b}$	$\frac{p}{(p+a)(p+b)}$	$\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
$L9$	$\sinh at$	$\frac{a}{p^2-a^2}$	$\text{Re } p > \lvert \text{Re } a \rvert$
$L10$	$\cosh at$	$\frac{p}{p^2-a^2}$	$\text{Re } p > \lvert \text{Re } a \rvert$
$L11$	$t \sin at$	$\frac{2ap}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L12$	$t \cos at$	$\frac{p^2-a^2}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L13$	$e^{-at} \sin bt$	$\frac{b}{(p+a)^2+b^2}$	$\text{Re } (p+a) > \lvert \text{Im } b \rvert$
$L14$	$e^{-at} \cos bt$	$\frac{p+a}{(p+a)^2+b^2}$	$\text{Re } (p+a) > \lvert \text{Im } b \rvert$
$L15$	$1 - \cos at$	$\frac{a^2}{p(p^2+a^2)}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L16$	$at - \sin at$	$\frac{a^3}{p^2(p^2+a^2)}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L17$	$\sin at - at \cos at$	$\frac{2a^3}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$

Table of Laplace Transforms (continued)

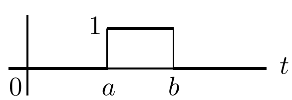
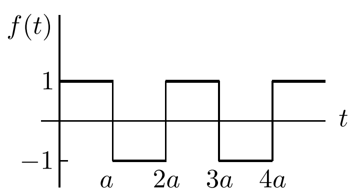
	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt$	
$L18$	$e^{-at}(1 - at)$	$\dfrac{p}{(p+a)^2}$	Re $(p+a) > 0$
$L19$	$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{p}$	Re $p > \lvert \text{Im } a \rvert$
$L20$	$\dfrac{1}{t} \sin at \cos bt,$ $a > 0,\ b > 0$	$\dfrac{1}{2}\left(\arctan \dfrac{a+b}{p} + \arctan \dfrac{a-b}{p}\right)$	Re $p > 0$
$L21$	$\dfrac{e^{-at} - e^{-bt}}{t}$	$\ln \dfrac{p+b}{p+a}$	Re $(p+a) > 0$ Re $(p+b) > 0$
$L22$	$1 - \text{erf}\left(\dfrac{a}{2\sqrt{t}}\right), \quad a > 0$ (See Chapter 11, Section 9)	$\dfrac{1}{p}\, e^{-a\sqrt{p}}$	Re $p > 0$
$L23$	$J_0(at)$ (See Chapter 12, Section 12)	$(p^2 + a^2)^{-1/2}$	Re $p > \lvert \text{Im } a \rvert$; or Re $p \geq 0$ for real $a \neq 0$
$L24$	$u(t-a) = \begin{cases} 1, & t > a > 0 \\ 0, & t < a \end{cases}$ (unit step, or Heaviside function)	$\dfrac{1}{p}\, e^{-pa}$	Re $p > 0$
$L25$	$f(t) = u(t-a) - u(t-b)$ 	$\dfrac{e^{-ap} - e^{-bp}}{p}$	All p
$L26$		$\dfrac{1}{p} \tanh\left(\frac{1}{2}ap\right)$	Re $p > 0$
$L27$	$\delta(t-a),\ a \geq 0$ (See Section 11)	e^{-pa}	
$L28$	$f(t) = \begin{cases} g(t-a), & t > a > 0 \\ 0, & t < a \end{cases}$ $= g(t-a)u(t-a)$	$e^{-pa}G(p)$ $[G(p)$ means $L(g)$.]	
$L29$	$e^{-at}g(t)$	$G(p+a)$	

Table of Laplace Transforms (continued)

	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\, dt$
$L30$	$g(at),\ a > 0$	$\frac{1}{a} G\left(\frac{p}{a}\right)$
$L31$	$\frac{g(t)}{t}$ (if integrable)	$\int_p^\infty G(u)\, du$
$L32$	$t^n g(t)$	$(-1)^n \frac{d^n G(p)}{dp^n}$
$L33$	$\int_0^t g(\tau)\, d\tau$	$\frac{1}{p} G(p)$
$L34$	$\int_0^t g(t-\tau)h(\tau)\, d\tau = \int_0^t g(\tau)h(t-\tau)\, d\tau$ (convolution of g and h, often written as $g * h$; see Section 10)	$G(p)H(p)$
$L35$	Transforms of derivatives of y (see Section 9):	

$$
\begin{aligned}
L(y') &= pY - y_0 \\
L(y'') &= p^2Y - py_0 - y_0' \\
L(y''') &= p^3Y - p^2y_0 - py_0' - y_0'', \text{ etc.} \\
L(y^{(n)}) &= p^nY - p^{n-1}y_0 - p^{n-2}y_0' - \cdots - y_0^{(n-1)}
\end{aligned}
$$

CHAPTER **9**

Calculus of Variations

1. INTRODUCTION

What is the shortest distance between two points? You probably laugh at such a simple question because you know the answer so well. Can you prove it? We shall see how to prove it shortly. Meanwhile we ask the same question about a sphere, for example, the earth. What is the shortest distance between two points on the surface of the earth, measured along the surface? Again you probably know that the answer is the distance measured along a great circle. But suppose you were asked the same question about some other surface, say an ellipsoid or a cylinder or a cone. The curve along a surface which marks the shortest distance between two neighboring points is called a *geodesic* of the surface. Finding geodesics is one of the problems which we can solve using the calculus of variations.

There are many others. To understand what the basic problem is, think about finding maximum and minimum values of $f(x)$ in ordinary calculus. You find $f'(x)$ and set it equal to zero. The values of x you find may correspond to maximum points , minimum points , or points of inflection with a horizontal tangent . Suppose that in solving a given physical problem you want the minimum values of a function $f(x)$. The equation $f'(x) = 0$ is a necessary (but not a sufficient) condition for an interior minimum point. To find the desired minimum, you would find all the values of x such that $f'(x) = 0$, and then rely on the physics or on further mathematical tests to sort out the minimum points. We use the general term *stationary point* to mean simply that $f'(x) = 0$ there; that is, stationary points include maximum points, minimum points, and points of inflection with horizontal tangent. In the calculus of variations, we often state problems by saying that a certain quantity is to be minimized. However, what we actually always do is something similar to putting $f'(x) = 0$, above; that is, we make the quantity stationary. The question of whether we have a maximum, a minimum, or neither, is, in general, a difficult mathematical problem (see calculus of variations texts) so we shall rely on the physics or geometry. Fortunately, in many applications, "stationary" is all that is required (Fermat's principle, Problems 1 to 3; Lagrange's equations, Section 5).

Now what is the quantity which we want to make stationary? It is an integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx \qquad \left(\text{where } y' = \frac{dy}{dx}\right), \tag{1.1}$$

and our problem is this: Given the points (x_1, y_1) and (x_2, y_2) and the form of the function F of x, y, and y', find the curve $y = y(x)$ (passing through the given points) which makes the integral I have the smallest possible value (or stationary value). Before we try to do this, let us look at several examples.

Example 1. Geodesics: Find the equation $y = y(x)$ of a curve joining two points (x_1, y_1) and (x_2, y_2) in the plane so that the distance between the points measured along the curve (arc length) is a minimum. Thus we want to minimize

$$I = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx; \tag{1.2}$$

this is equation (1.1) with $F(x, y, y') = \sqrt{1 + y'^2}$. See Section 2 and Example 3.4.

Example 2. The famous brachistochrone problem (from the Greek: *brachistos* = shortest, *chronos* = time, as in chronometer): Find the shape of a wire joining two given points so that a bead will slide down under gravity from one point to the other (without friction) in the shortest time. Here we must minimize $\int dt$. If ds is an element of arc length, then the velocity of the particle is $v = ds/dt$. Then we have

$$dt = \frac{1}{v}\, ds = \frac{1}{v}\sqrt{1 + y'^2}\, dx.$$

We shall see later that (using the law of conservation of energy) we can find v as a function of x and y. Then the integral which we want to minimize, namely

$$\int dt = \int \frac{1}{v}\sqrt{1 + y'^2}\, dx,$$

is of the form (1.1). See Section 4.

Example 3. Soap film problem: Suppose a soap film is suspended between two circular wire hoops as shown in Figure 1.1; what is the shape of the surface? It is clear from symmetry that it is a surface of revolution (neglecting gravity), and it is known that the soap film will adjust itself so that the surface area is a minimum. The surface area can be written as an integral and again our problem is to minimize an integral. See Section 4.

Figure 1.1

There are many other examples from physics. A chain suspended between two points hangs so that its center of gravity is as low as possible; the z coordinate of the center of gravity is given by an integral. Fermat's principle in optics says that light traveling between two given points follows the path requiring the least time. (This is a simple, but inaccurate, statement; we *should* say that $t = \int dt$ is stationary—there are examples where it is a maximum! See Problem 3). Various other basic principles in physics are stated in the form that certain integrals have stationary values.

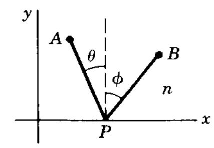
▶ PROBLEMS, SECTION 1

The speed of light in a medium of index of refraction n is $v = ds/dt = c/n$. Then the time of transit from A to B is $t = \int_A^B dt = c^{-1}\int_A^B n\,ds$. By Fermat's principle above, t is stationary. If the path consists of two straight line segments with n constant over each segment, then

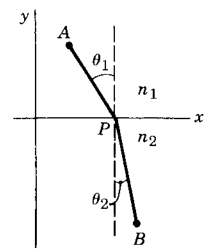
$$\int_A^B n\,ds = n_1 d_1 + n_2 d_2,$$

and the problem can be done by ordinary calculus. Thus solve the following problems:

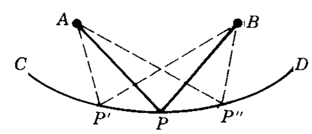
1. Derive the optical law of reflection. *Hint:* Let light go from the point $A = (x_1, y_1)$ to $B = (x_2, y_2)$ via an arbitrary point $P = (x, 0)$ on a mirror along the x axis. Set $dt/dx = (n/c)dD/dx = 0$, where $D =$ distance APB, and show that then $\theta = \phi$.



2. Derive Snell's law of refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$ (see figure).



3. Show that the actual path is not necessarily one of minimum time. *Hint:* In the diagram, A is a source of light; CD is a cross section of a reflecting surface, and B is a point to which a light ray is to be reflected. APB is to be the actual path and $AP'B$, $AP''B$ represent varied paths. Then show that the varied paths:



(a) Are the same length as the actual path if CD is an ellipse with A and B as foci.

(b) Are longer than the actual path if CD is a line tangent at P to the ellipse in (a).

(c) Are shorter than the actual path if CD is an arc of a curve tangent to the ellipse at P and lying inside it. Note that in this case the time is a *maximum*!

(d) Are longer on one side and shorter on the other if CD crosses the ellipse at P but is tangent to it (that is, CD has a point of inflection at P).

▶ 2. THE EULER EQUATION

Before we do the general problem, let us first do the problem of a geodesic on a plane; we shall show that a straight line gives the shortest distance between two points. (The reason for doing this is to clarify the theory; you will not do problems this way.) Our problem is to find $y = y(x)$ which will make

$$I = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx$$

as small as possible. The $y(x)$ which does this is called an *extremal*. Now we want some way to represent algebraically all the curves passing through the given endpoints, but differing from the (as yet unknown) extremal by small amounts. (We

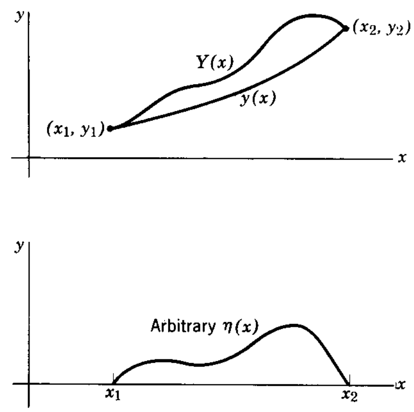


Figure 2.1

assume that all the curves have continuous second derivatives so that we can carry out needed differentiations later.) These curves are called varied curves; there are infinitely many of them as close as we like to the extremal. We construct a function representing these varied curves in the following way (Figure 2.1). Let $\eta(x)$ represent a function of x which is zero at x_1 and x_2, and has a continuous second derivative in the interval x_1 to x_2, but is otherwise completely arbitrary. We define the function $Y(x)$ by the equation

$$Y(x) = y(x) + \epsilon\eta(x), \tag{2.1}$$

where $y(x)$ is the desired extremal and ϵ is a parameter. Because of the arbitrariness of $\eta(x)$, $Y(x)$ represents any (single-valued) curve (with continuous second derivative) you want to draw through (x_1, y_1) and (x_2, y_2). Out of all these curves $Y(x)$ we want to pick the one curve that makes

$$I = \int_{x_1}^{x_2} \sqrt{1 + Y'^2}\, dx \tag{2.2}$$

a minimum. Now I is a function of the parameter ϵ; when $\epsilon = 0$, $Y = y(x)$, the desired extremal. Our problem then is to make $I(\epsilon)$ take its minimum value when $\epsilon = 0$. In other words, we want

$$\frac{dI}{d\epsilon} = 0 \quad \text{when } \epsilon = 0. \tag{2.3}$$

Differentiating (2.2) under the integral sign with respect to the parameter ϵ, we get

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \frac{1}{2} \frac{1}{\sqrt{1 + Y'^2}} 2Y' \left(\frac{dY'}{d\epsilon}\right) dx. \tag{2.4}$$

Differentiating (2.1) with respect to x, we get

$$Y'(x) = y'(x) + \epsilon\eta'(x). \tag{2.5}$$

Then from (2.5) we have

$$\frac{dY'}{d\epsilon} = \eta'(x). \tag{2.6}$$

We see from (2.1) that putting $\epsilon = 0$ means putting $Y(x) = y(x)$. Then substituting (2.6) into (2.4) and putting $dI/d\epsilon$ equal to zero when $\epsilon = 0$, we get

$$\left(\frac{dI}{d\epsilon}\right)_{\epsilon=0} = \int_{x_1}^{x_2} \frac{y'(x)\eta'(x)}{\sqrt{1 + y'^2}}\, dx = 0. \tag{2.7}$$

We can integrate this by parts (since we assumed that η and y have continuous second derivatives). Let

$$u = y'/\sqrt{1 + y'^2}, \quad dv = \eta'(x)dx.$$

Then

$$du = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)dx, \quad v = \eta(x),$$

and

$$\left(\frac{dI}{d\epsilon}\right)_{\epsilon=0} = \frac{y'}{\sqrt{1+y'^2}}\eta(x)\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)dx.$$

The first term is zero because $\eta(x) = 0$ at the endpoints. In the second term, recall that $\eta(x)$ is an arbitrary function. This means that

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0, \tag{2.8}$$

for otherwise we could select some function $\eta(x)$ so that the integral would not be zero. Notice carefully here that we are *not* saying that when an integral is zero, the integrand is also zero; this is not true (as, for example $\int_0^{2\pi} \sin x\,dx = 0$ shows). What we *are* saying is that the only way $\int_{x_1}^{x_2} f(x)\eta(x)\,dx$ can *always* be zero for *every* $\eta(x)$ is for $f(x)$ to be zero. You can prove this by contradiction in the following way. If $f(x)$ is not zero, then, since $\eta(x)$ is arbitrary, choose η to be positive where f is positive and negative where f is negative. Then $f\eta$ is positive, so its integral is not zero, in contradiction to the statement that $\int f\eta\,dx = 0$ for every η.

Integrating (2.8) with respect to x, we get

$$\frac{y'}{\sqrt{1+y'^2}} = \text{const.}$$

or $y' = \text{const.}$ Thus the slope of $y(x)$ is constant, so $y(x)$ is a straight line as we expected.

Now we *could* go through this process with every calculus of variations problem. It is much simpler to do the general problem once for all and find a differential equation which we can use to solve later problems. The problem is to find the y which will make stationary the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx, \tag{2.9}$$

where F is a given function. The $y(x)$ which makes I stationary is called an extremal whether I is a maximum or minimum or neither. The method is the one we have just used with the straight line. We consider a set of varied curves

$$Y(x) = y(x) + \epsilon\eta(x)$$

just as before. Then we have

$$I(\epsilon) = \int_{x_1}^{x_2} F(x,\ Y,\ Y')\,dx, \tag{2.10}$$

and we want $(d/d\epsilon)I(\epsilon) = 0$ when $\epsilon = 0$. Remembering that Y and Y' are functions of ϵ, and differentiating under the integral sign with respect to ϵ, we get

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\epsilon}\right)dx. \tag{2.11}$$

Substituting (2.1) and (2.5) into (2.11), we have

$$\text{(2.12)} \qquad \frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y}\,\eta(x) + \frac{\partial F}{\partial Y'}\,\eta'(x) \right] dx.$$

We want $dI/d\epsilon = 0$ at $\epsilon = 0$; recall that $\epsilon = 0$ means $Y = y$. Then (2.12) gives

$$\text{(2.13)} \qquad \left(\frac{dI}{d\epsilon}\right)_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\,\eta(x) + \frac{\partial F}{\partial y'}\,\eta'(x) \right] dx = 0.$$

Assuming that y'' is continuous, we can integrate the second term by parts just as in the straight-line problem:

$$\text{(2.14)} \qquad \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\,\eta'(x)\,dx = \frac{\partial F}{\partial y'}\,\eta(x)\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \eta(x)\,dx.$$

The integrated term is zero as before because $\eta(x)$ is zero at x_1 and x_2. Then we have

$$\text{(2.15)} \qquad \left(\frac{dI}{d\epsilon}\right)_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right] \eta(x)\,dx = 0.$$

As before, since $\eta(x)$ is arbitrary, we must have

$$\text{(2.16)} \qquad \frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0. \qquad \text{Euler equation}$$

This is the Euler (or Euler-Lagrange) equation.

Any problem in the calculus of variations, then, is solved by setting up the integral which is to be stationary, writing what the function F is, substituting it into the Euler equation, and solving the resulting differential equation.

Example. Let's find the geodesics in a plane again, this time using the Euler equation as you will do in problems.

We are to minimize

$$\int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx,$$

so we have $F = \sqrt{1 + y'^2}$. Then

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}, \qquad \frac{\partial F}{\partial y} = 0,$$

and the Euler equation gives

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0,$$

as we had in (2.8).

▶ PROBLEMS, SECTION 2

Write and solve the Euler equations to make the following integrals stationary. In solving the Euler equations, the integrals in Chapter 5, Section 1, may be useful.

1. $\displaystyle\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}\,dx$ **2.** $\displaystyle\int_{x_1}^{x_2} \frac{ds}{x}$ **3.** $\displaystyle\int_{x_1}^{x_2} x\sqrt{1-y'^2}\,dx$

4. $\displaystyle\int_{x_1}^{x_2} x\,ds$ **5.** $\displaystyle\int_{x_1}^{x_2} (y'^2+y^2)\,dx$ **6.** $\displaystyle\int_{x_1}^{x_2} (y'^2+\sqrt{y}\,)\,dx$

7. $\int_{x_1}^{x_2} e^x\sqrt{1+y'^2}\,dx$ *Hint:* In the last integration, let $u = e^x$ and see Chapter 5, Problem 1.6.

8. $\displaystyle\int_{x_1}^{x_2} x\sqrt{y'^2+x^2}\,dx$ **9.** $\displaystyle\int_{x_1}^{x_2} (1+yy')^2\,dx$ **10.** $\displaystyle\int_{x_1}^{x_2} \frac{x^2\,dx}{xy'+1}$

▶ 3. USING THE EULER EQUATION

Other Variables We have used x and y as our variables. But the mathematics is just the same if we use some other letters, for example polar coordinates r and θ. To minimize (make stationary) the integral

$$\int F(r,\theta,\theta')\,dr \qquad \text{where} \qquad \theta' = d\theta/dr,$$

we solve the Euler equation

$$\frac{d}{dr}\left(\frac{\partial F}{\partial \theta'}\right) - \frac{\partial F}{\partial \theta} = 0. \tag{3.1}$$

To minimize $\int F(t,x,\dot{x})\,dt$ where $\dot{x} = dx/dt$, we solve

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} - \frac{\partial F}{\partial x} = 0. \tag{3.2}$$

Notice that the first derivative in the Euler equation [d/dx in (2.16), d/dr in (3.1), d/dt in (3.2)] is with respect to the integration variable in the integral. The partial derivatives are with respect to the other variable and its derivative [y and y' in (2.16), θ and θ' in (3.1), x and $\dot{x}$ in (3.2)].

▶ **Example 1.** Find the path followed by a light ray if the index of refraction (in polar coordinates) is proportional to r^{-2}. We want to make stationary

$$\int n\,ds \quad \text{or} \quad \int r^{-2}\,ds = \int r^{-2}\sqrt{dr^2+r^2\,d\theta^2} = \int r^{-2}\sqrt{1+r^2\theta'^2}\,dr.$$

The Euler equation is then (3.1) with $F = r^{-2}\sqrt{1+r^2\theta'^2}$. Since $\partial F/\partial\theta = 0$, we have

$$\frac{d}{dr}\left(\frac{r^{-2}r^2\theta'}{\sqrt{1+r^2\theta'^2}}\right) = 0 \qquad \text{or} \qquad \frac{\theta'}{\sqrt{1+r^2\theta'^2}} = \text{const.} = K.$$

Solve for θ' and integrate (see Chapter 5, Problem 1.5):

$$\begin{aligned}
\theta'^2 &= K^2(1+r^2\theta'^2) \quad \text{so} \quad \theta'^2(1-K^2r^2) = K^2,\\
\theta' &= \frac{d\theta}{dr} = \frac{K}{\sqrt{1-K^2r^2}},\\
\theta &= \arcsin Kr + \text{const.}
\end{aligned}$$

First Integrals of the Euler Equation In some problems the integrand F in I [see equation (1.1)] does not contain y (that is, F does not contain the dependent variable). Then $\partial F/\partial y = 0$ and the Euler equation becomes

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = 0, \qquad \frac{\partial F}{\partial y'} = \text{const.}$$

This happened in the example and most of your problems in Section 2. Because $\partial F/\partial y$ was zero, we were able to integrate the Euler equation *once*; the equation $\partial F/\partial y' = \text{const.}$ is for this reason called a *first integral* of the Euler equation.

There is another less obvious case in which we can easily find a first integral of the Euler equation. Let us show this by an example (the soap film problem mentioned in Section 1).

Example 2. Our problem is this: Given two points P_1 and P_2 (not too far apart), we are going to draw a curve joining P_1 and P_2 and revolve it about the x axis to form a surface of revolution. We want the equation of the curve so that the surface area will be a minimum. That is, we want to minimize $I = \int 2\pi y\, ds$. We usually write $ds = \sqrt{1+y'^2}\,dx$. Instead, let us write $ds = \sqrt{1+x'^2}\,dy$, where $x' = dx/dy$. Then $I = \int 2\pi y\sqrt{1+x'^2}\,dy$. Recall from (3.1) and (3.2) and the discussion following them how to write the Euler equation in various sets of variables. Here y is the variable of integration, $F = y\sqrt{1+x'^2}$, and the Euler equation is

$$\frac{d}{dy}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0. \tag{3.3}$$

Since $\partial F/\partial x = 0$, (3.3) becomes

$$\frac{d}{dy}\left(\frac{yx'}{\sqrt{1+x'^2}}\right) = 0.$$

This is the simplified equation we wanted. We integrate once, solve for x' and integrate again (see Chapter 5, Problem 1.3):

$$\frac{yx'}{\sqrt{1+x'^2}} = c_1,$$
$$x' = \frac{dx}{dy} = \frac{c_1}{\sqrt{y^2 - c_1{}^2}},$$
$$x = c_1 \cosh^{-1}\frac{y}{c_1} + c_2,$$
$$y = c_1 \cosh\frac{x - c_2}{c_1}.$$

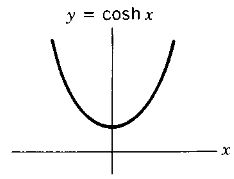


Figure 3.1

The graph of this equation is called a *catenary*; it is shown in Figure 3.1 for the special case $c_1 = 1$, $c_2 = 0$, $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$. The catenary does not always give the solution to the soap film problem. If the given points (or the hoops in Figure 1.1) are too far apart, the soap film may break into two parts (circular films on the hoops). For further discussion see Courant and Robbins, Chapter 7, Section 11, and Arfken and Weber, Chapter 17. For another problem involving a catenary, see Problem 6.4.

Observe that the method used in this example will simplify any problem in which $I = \int F(y, y')\,dx$ does not have the independent variable x in the integrand. We change to y as the integration variable making the substitutions

$$(3.4) \qquad x' = \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}, \quad y' = \frac{1}{x'}, \quad dx = \frac{dx}{dy}\,dy = x'\,dy$$

in I. Then the integrand is a function of y and x', so the Euler equation [now (3.3)] simplifies since $\partial F/\partial x = 0$. (See also Problem 8.1.)

▶ **Example 3.** Find a first integral of the Euler equation to make stationary the integral

$$(3.5) \qquad I = \int \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx.$$

Since x is missing in the integrand, we change to y as the integration variable; then by (3.4)

$$\sqrt{1+y'^2}\,dx = \sqrt{1+y'^2}\,x'\,dy = \sqrt{x'^2+1}\,dy,$$
$$I = \int \frac{\sqrt{x'^2+1}}{\sqrt{y}}\,dy = \int F(y, x')\,dy.$$

We see that $\partial F/\partial x = 0$; from (3.3) the Euler equation is

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = \frac{d}{dy}\left(\frac{x'}{\sqrt{y}\sqrt{x'^2+1}}\right) = 0.$$

The first integral of the Euler equation is, then,

$$\frac{x'}{\sqrt{y}\sqrt{x'^2+1}} = \text{const.} \tag{3.6}$$

Example 4. Find the geodesics on the cone $z^2 = 8(x^2+y^2)$. Using cylindrical coordinates, we have $z^2 = 8r^2$, $z = r\sqrt{8}$, $dz = dr\sqrt{8}$, so

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 = dr^2 + r^2 d\theta^2 + 8\,dr^2 = 9\,dr^2 + r^2 d\theta^2.$$

We want to minimize

$$I = \int ds = \int \sqrt{9\,dr^2 + r^2 d\theta^2} = \int \sqrt{9 + r^2\theta'^2}\,dr.$$

Note that we use r as the integration variable since the integrand contains r but not θ. Then $\partial F/\partial\theta = 0$, and we can immediately write a first integral of the Euler equation:

$$\frac{d}{dr}\left(\frac{\partial F}{\partial \theta'}\right) = 0, \qquad \frac{\partial F}{\partial \theta'} = \frac{r^2\theta'}{\sqrt{9 + r^2\theta'^2}} = \text{const.} = K.$$

We solve for θ' and integrate again.

$$\begin{aligned} r^4\theta'^2 &= K^2(9 + r^2\theta'^2), \\ \theta'^2(r^4 - K^2r^2) &= 9K^2, \\ \int d\theta &= \int \frac{3K\,dr}{r\sqrt{r^2 - K^2}}. \end{aligned}$$

From computer or tables (or see Chapter 5, Problem 1.6):

$$\theta + \alpha = 3\arccos\frac{K}{r} \qquad (\alpha = \text{const. of integration})$$

$$\cos\left(\frac{\theta+\alpha}{3}\right) = \frac{K}{r} \quad \text{or} \quad r\cos\left(\frac{\theta+\alpha}{3}\right) = K.$$

PROBLEMS, SECTION 3

Change the independent variable to simplify the Euler equation, and then find a first integral of it.

1. $\displaystyle\int_{x_1}^{x_2} y^{3/2}\,ds$

2. $\displaystyle\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{y^2}\,dx$

3. $\displaystyle\int_{y_1}^{y_2} \frac{x'^2}{\sqrt{x'^2+x^2}}\,dy$

4. $\displaystyle\int_{x_1}^{x_2} y\sqrt{y'^2+y^2}\,dx$

Write and solve the Euler equations to make the following integrals stationary. Change the independent variable, if needed, to make the Euler equation simpler.

5. $\displaystyle\int_{x_1}^{x_2} \sqrt{1+y^2y'^2}\,dx$

6. $\displaystyle\int_{x_1}^{x_2} \frac{yy'^2}{1+yy'}\,dx$

7. $\displaystyle\int_{x_1}^{x_2} (y'^2 + y^2)\,dx$

8. $\displaystyle\int_{\theta_1}^{\theta_2} \sqrt{r'^2 + r^2}\,d\theta, \quad r' = dr/d\theta$

9. $\displaystyle\int_{\phi_1}^{\phi_2} \sqrt{\theta'^2 + \sin^2\theta}\,d\phi, \quad \theta' = d\theta/d\phi$

10. $\displaystyle\int_{t_1}^{t_2} s^{-1}\sqrt{s^2 + s'^2}\,dt, \quad s' = ds/dt$

Use Fermat's principle to find the path followed by a light ray if the index of refraction is proportional to the given function.

11. $x + 1$ **12.** y^{-1} **13.** $\sqrt{y}$ **14.** r^{-1}

15. Find the geodesics on a plane using polar coordinates.

16. Show that the geodesics on a circular cylinder (with elements parallel to the z axis) are helices $az + b\theta = c$, where a, b, c are constants depending on the given endpoints. (*Hint:* Use cylindrical coordinates.) Note that the equation $az + b\theta = c$ includes the circles $z =$ const. (for $b = 0$), straight lines $\theta =$ const. (for $a = 0$), and the special helices $az + b\theta = 0$.

17. Find the geodesics on the cone $x^2 + y^2 = z^2$. *Hint:* Use cylindrical coordinates.

18. Find the geodesics on a sphere. *Hints:* Use spherical coordinates with constant $r = a$. Choose your integration variable so that you can write a first integral of the Euler equation. For the second integration, make the change of variable $w = \cot\theta$. To recognize your result as a great circle, find, in terms of spherical coordinates θ and ϕ, the equation of intersection of the sphere with a plane through the origin.

4. THE BRACHISTOCHRONE PROBLEM; CYCLOIDS

We have already mentioned this problem in Section 1. We are given the points (x_1, y_1) and (x_2, y_2); we choose axes through the point 1 with the y axis positive downward as shown in Figure 4.1. Our problem is to find the curve joining the two points, down which a bead will slide (from rest) in the least time; that is, we want to minimize $\int dt$. Let $v = 0$ initially, and let $y = 0$ be our reference level for potential energy. Then at the point (x, y) we have

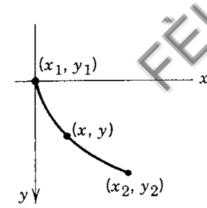


Figure 4.1

$$\text{kinetic energy } = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2,$$
$$\text{potential energy} = -mgy.$$

The sum of the two energies is zero initially and therefore zero at any time since the total energy is constant when there is no friction. Hence we have

$$\frac{1}{2}mv^2 - mgy = 0 \quad \text{or} \quad v = \sqrt{2gy}.$$

Then the integral which we want to minimize is

$$\int dt = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}}\int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\,dx.$$

This is the integral (3.5) in Example 3, Section 3. Then the first integral of the Euler equation is given by (3.6):

$$\frac{x'}{\sqrt{y}\sqrt{x'^2+1}} = \sqrt{c}.$$

Solving for x', we get

$$x' = \frac{dx}{dy} = \sqrt{\frac{cy}{1-cy}}. \tag{4.1}$$

This simplifies if we let $cy = \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$. We find (Problem 1)

$$\begin{aligned} dx &= \frac{1}{c}\sin^2\frac{\theta}{2}\,d\theta = \frac{1}{2c}(1-\cos\theta)\,d\theta,\\ x &= \frac{1}{2c}(\theta - \sin\theta) + c'. \end{aligned} \tag{4.2}$$

The equations for x and y as functions of θ are parametric equations of the curve along which the particle slides in minimum time. Since we have chosen axes to make the curve pass through the origin, $x = y = 0$ must satisfy the equations of the curve, so $c' = 0$, and we have

$$\begin{aligned} x &= \frac{1}{2c}(\theta - \sin\theta),\\ y &= \frac{1}{2c}(1-\cos\theta). \end{aligned} \tag{4.3}$$

We shall now show that these are the parametric equations of a cycloid. Imagine a circle of radius a (say a wheel) in the (x, y) plane rolling along the x axis. Let it start tangent to the x axis at the origin O in Figure 4.2. Place a mark on the circle at O. As the circle rolls, the mark traces out a cycloid as shown in Figure 4.3. Let point P in Figure 4.2 be the position of the mark when the circle is tangent to the x axis at A; let (x, y) be the coordinates of P. Since the circle rolled, $OA = PA = a\theta$ with θ in radians. Then from Figure 4.2 we have

$$\begin{aligned} x &= OA - PB = a\theta - a\sin\theta = a(\theta - \sin\theta),\\ y &= AB = AC - BC = a - a\cos\theta = a(1-\cos\theta). \end{aligned} \tag{4.4}$$

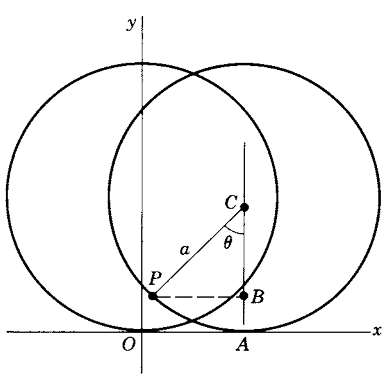


Figure 4.2

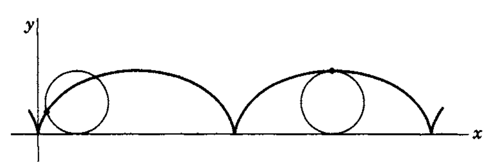


Figure 4.3

Equations (4.4) are the parametric equations of a cycloid. Comparing (4.3), we see that the brachistochrone is a cycloid as we claimed. Note that, since we have taken the y axis positive down (Figures 4.1 and 4.4), the circle which generates the brachistochrone rolls along the under side of the x axis.

From either (4.3) or (4.4), we see that all cycloids are similar; that is they differ from each other only in size (determined by a or c) and not in shape. Figure 4.4 is a sketch of a cycloid for arbitrary a. If the given endpoints for the wire along which the bead slides are O and P_3, we see that the particle slides down to P_2 and back up to P_3 in minimum time! At point P_2 the circle has rolled halfway around so $OA = \frac{1}{2}\cdot 2\pi a = \pi a$. For any point P_1 on arc OP_2, P_1 is below the line OP_2, and the coordinates (x_1, y_1) of P_1 have

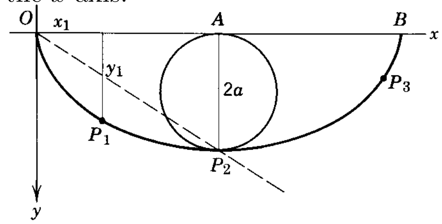


Figure 4.4

$$\frac{y_1}{x_1} > \frac{P_2A}{AO} = \frac{2a}{\pi a} = \frac{2}{\pi}$$

or $x_1/y_1 < \pi/2$. For points like P_3 on P_2B, $x_3/y_3 > \pi/2$, whereas *at* P_2, we have $x_2/y_2 = \pi/2$ (Problem 2). Then if the right-hand endpoint is (x, y) and the origin is the left-hand endpoint, we can say that the bead just slides down, or slides down and back up, depending on whether x/y is less than or greater than $\pi/2$ (Problem 2).

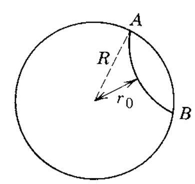
▶ PROBLEMS, SECTION 4

1. Verify equations (4.2).

2. Show, in Figure 4.4, that for a point like P_3, $x_3/y_3 > \pi/2$ and for P_2, $x_2/y_2 = \pi/2$.

3. In the brachistochrone problem, show that if the particle is given an initial velocity $v_0 \neq 0$, the path of minimum time is still a cycloid.

4. Consider a rapid transit system consisting of frictionless tunnels bored through the earth between points A and B on the earth's surface (see figure). The unpowered passenger trains would move under gravity. Using polar coordinates, set up $\int dt$ to be minimized to find the path through the earth requiring the least time. See Chapter 6, Problem 8.21, for the potential inside the earth. Find a first integral of the Euler equation. Evaluate the constant of integration using $dr/d\theta = 0$ when $r = r_0$ (where r_0 is the deepest point of the tunnel—see figure). Now solve for $\theta' = d\theta/dr$ as a function of r. Substitute this into the integral for t and evaluate the integral to show that the transit time is



$$T = \pi\sqrt{\frac{R^2 - {r_0}^2}{gR}}. \quad \text{Hint: Find } 2\int_{r=r_0}^{R} dt.$$

Evaluate T for $r_0 = 0$ (path through the center of the earth—see Chapter 8, Problem 5.35); for $r_0 = 0.99R$. [For more detail, see Am. J. Phys. **34** 701–704 (1966).]

In Problems 5 to 7, use Fermat's principle to find the path followed by a light ray if the index of refraction is proportional to the given function.

5. $x^{-1/2}$ **6.** $(y-1)^{-1/2}$ **7.** $(2x+5)^{-1/2}$

5. SEVERAL DEPENDENT VARIABLES; LAGRANGE'S EQUATIONS

It is not necessary to restrict ourselves to problems with one dependent variable y. Recall that in ordinary calculus problems the necessary condition for a minimum point on $z = z(x)$ is $dz/dx = 0$; for a function of two variables $z = z(x, y)$, we have the two conditions $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$. We have a somewhat analogous situation in the calculus of variations. Suppose that we are given an F which is a function of y, z, dy/dx, dz/dx, and x, and we want to find *two* curves $y = y(x)$ and $z = z(x)$ which make $I = \int F\,dx$ stationary. Then the value of the integral I depends on *both* $y(x)$ and $z(x)$ and you might very well guess that in this case we would have two Euler equations, one for y and one for z, namely

$$\begin{aligned} &\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0,\\ &\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = 0. \end{aligned} \tag{5.1}$$

By carrying through calculations similar to those we used in deriving the single Euler equation for the one dependent variable case you can show (Problem 1(a)) that this guess is correct. If there are still more dependent variables (but one independent variable), then we write an Euler equation for each dependent variable. It is also possible to consider a problem with more than one independent variable (see Problem 1(b)) or with F depending on y'' as well as x, y, y' (see Problem 1(c)).

There is a very important application of equations like (5.1) to mechanics. In elementary physics, Newton's second law $\mathbf{F} = m\mathbf{a}$ is a fundamental equation. In more advanced mechanics, it is often useful to start from a different assumption (which can be proved equivalent to Newton's law; see mechanics text books.) This assumption is called *Hamilton's principle.* It says that any particle or system of particles always moves in such a way that $I = \int_{t_1}^{t_2} L\,dt$ is stationary, where $L = T - V$ is called the *Lagrangian*; T is the kinetic energy, and V is the potential energy of the particle or system.

Example 1. Use Hamilton's principle to find the equations of motion of a single particle of mass m moving (near the earth) under gravity.

We first write the formulas for the kinetic energy T and the potential energy V of the particle. (It is convenient to use a dot to mean a derivative with respect to t just as we use a prime to indicate a derivative with respect to x; thus $dx/dt = \dot{x}$, $dy/dt = \dot{y}$, $dy/dx = y'$, $d^2x/dt^2 = \ddot{x}$, etc.) The equations for T, V, and $L = T - V$, are:

$$\begin{aligned} &T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),\\ &V = mgz,\\ &L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz. \end{aligned} \tag{5.2}$$

Here t is the independent variable; x, y, and z are the dependent variables, and L corresponds to what we have called F previously. Then to make $I = \int_{t_1}^{t_2} L\,dt$ stationary, we write the corresponding Euler equations. There are three Euler equations, one for x, one for y, and one for z. The Euler equations are called *Lagrange's equations* in mechanics [see (5.3) next page].

$$(5.3) \qquad \begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0,\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= 0,\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} &= 0. \end{aligned} \qquad \text{Lagrange's equations}$$

Substituting L in (5.2) into Lagrange's equations (5.3), we get

$$(5.4) \qquad \begin{cases} \dfrac{d}{dt}(m\dot{x}) = 0 \\ \dfrac{d}{dt}(m\dot{y}) = 0 \\ \dfrac{d}{dt}(m\dot{z}) + mg = 0 \end{cases} \quad \text{or} \quad \begin{cases} \dot{x} = \text{const.}, \\ \dot{y} = \text{const.}, \\ \ddot{z} = -g. \end{cases}$$

These are just the familiar equations obtained from Newton's law; they say that in the gravitational field near the surface of the earth, the horizontal velocity is constant and the vertical acceleration is $-g$. In this problem you may say that it would have been simpler just to write the equations from Newton's law in the first place! This is true in simple cases, but in more complicated problems it may be much simpler to find one scalar function (that is, L) than to find six functions (that is, the components of the two vectors, force and acceleration). For example, the acceleration components in spherical coordinates are quite complicated to derive by elementary methods (see mechanics text books), but you should have no trouble deriving the equations of motion in polar, cylindrical or spherical coordinates using the Lagrangian. Let's do some examples.

▶ **Example 2.** Use Lagrange's equations to find the equations of motion of a particle in terms of the polar coordinate variables r and θ.

The element of arc length in polar coordinates is ds where

$$(5.5) \qquad ds^2 = dr^2 + r^2\,d\theta^2.$$

The velocity of a moving particle is ds/dt; from (5.5) we get

$$(5.6) \qquad v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \dot{r}^2 + r^2\dot{\theta}^2.$$

The kinetic energy is $\frac{1}{2}mv^2$, so we have

$$(5.7) \qquad \begin{aligned} T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),\\ L = T - V &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta), \end{aligned}$$

where $V(r, \theta)$ is the potential energy of the particle. Lagrange's equations in the variables r, θ are:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} &= 0, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0. \end{aligned} \tag{5.8}$$

Substituting L from (5.7) into (5.8), we get

$$\begin{aligned} \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} &= 0, \\ \frac{d}{dt}(mr^2\dot{\theta}) + \frac{\partial V}{\partial \theta} &= 0. \end{aligned} \tag{5.9}$$

The r equation of motion is, then,

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r}. \tag{5.10}$$

The θ equation is

$$m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = -\frac{\partial V}{\partial \theta},$$

or, dividing by r,

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -\frac{1}{r}\frac{\partial V}{\partial \theta}. \tag{5.11}$$

Now the quantities $-\partial V/\partial r$ and $-(1/r)(\partial V/\partial \theta)$ are the components of the force ($\mathbf{F} = -\nabla V$) on the particle in the r and θ directions. (See Chapter 6.) Then equations (5.10) and (5.11) are just the components of $m\mathbf{a} = \mathbf{F}$; the acceleration components are then

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2, \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta}. \end{aligned}$$

The second term in a_r is a familiar one; it is just the centripetal acceleration v^2/r when $v = r\dot{\theta}$ (the minus sign indicates that it is toward the origin). The second term in a_θ is called the Coriolis acceleration.

We show by an example another important point about Lagrange's equations.

Example 3. A mass m_1 moves without friction on the surface of the cone shown (Figure 5.1). Mass m_2 is joined to m_1 by a string of constant length; m_2 can move only vertically up and down. Find the Lagrange equations of motion of the system.

Let's use spherical coordinates ρ, θ, ϕ for m_1, and coordinate z for m_2. Then for m_1, $v^2 = (ds/dt)^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \rho^2 \sin^2\theta\,\dot{\phi}^2$ [Chapter 5, equation (4.20)], and for m_2, $v^2 = \dot{z}^2$. The potential energy mgh of m_1 is $m_1 g\rho\cos\theta$ and of m_2 is $m_2 gz$. Note that we have used

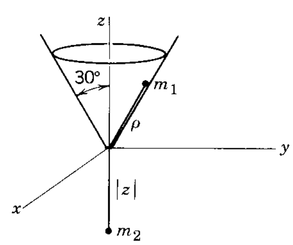


Figure 5.1

four variables: ρ, θ, ϕ, z; however, there are *not* four Lagrange equations We must use the equation of the cone ($\theta = 30°$) and the equation $\rho + |z| = l$ (string of constant length) to eliminate θ and either ρ or z. The Lagrangian L must always be written using the smallest possible number of variables (we say that we eliminate the constraint equations). Then, with $\theta = 30°$, $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{1}{2}\sqrt{3}$, $\dot{\theta} = 0$, and $z = -|z| = -(l - \rho)$, we find L in terms of ρ and ϕ:

$$L = \frac{1}{2}m_1(\dot{\rho}^2 + \rho^2\dot{\phi}^2/4) + \frac{1}{2}m_2\dot{\rho}^2 - \frac{1}{2}m_1 g\rho\sqrt{3} + m_2 g(l - \rho).$$

Thus the Lagrange equations are

$$\frac{d}{dt}(m_1\dot{\rho} + m_2\dot{\rho}) - m_1\rho\dot{\phi}^2/4 + \frac{1}{2}m_1 g\sqrt{3} + m_2 g = 0,$$
$$\frac{d}{dt}(m_1\rho^2\dot{\phi}/4) = 0 \qquad \text{or} \qquad \rho^2\dot{\phi} = \text{const.}$$

▶ PROBLEMS, SECTION 5

1. (a) Consider the case of two dependent variables. Show that if $F = F(x, y, z, y', z')$ and we want to find $y(x)$ and $z(x)$ to make $I = \int_{x_1}^{x_2} F\,dx$ stationary, then y and z should each satisfy an Euler equation as in (5.1). *Hint:* Construct a formula for a varied path Y for y as in Section 2 [$Y = y + \epsilon\eta(x)$ with $\eta(x)$ arbitrary] and construct a similar formula for z [let $Z = z + \epsilon\zeta(x)$, where $\zeta(x)$ is *another* arbitrary function]. Carry through the details of differentiating with respect to ϵ, putting $\epsilon = 0$, and integrating by parts as in Section 2; then use the fact that *both* $\eta(x)$ and $\zeta(x)$ are arbitrary to get (5.1).

(b) Consider the case of two independent variables. You want to find the function $u(x, y)$ which makes stationary the double integral

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} F(u, x, y, u_x, u_y)\,dx\,dy.$$

Hint: Let the varied $U(x, y) = u(x, y) + \epsilon\eta(x, y)$ where $\eta(x, y) = 0$ at $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$, but is otherwise arbitrary. As in Section 2, differentiate with respect to ϵ, set $\epsilon = 0$, integrate by parts, and use the fact that η is arbitrary. Show that the Euler equation is then

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial F}{\partial u_y} - \frac{\partial F}{\partial u} = 0.$$

(c) Consider the case in which F depends on x, y, y', and y''. Assuming zero values of the variation $\eta(x)$ and its derivative at the endpoints x_1 and x_2, show that then the Euler equation becomes

$$\frac{d^2}{dx^2}\frac{\partial F}{\partial y''} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial y} = 0.$$

2. Set up Lagrange's equations in cylindrical coordinates for a particle of mass m in a potential field $V(r, \theta, z)$. *Hint:* $v = ds/dt$; write ds in cylindrical coordinates.

3. Do Problem 2 in spherical coordinates.

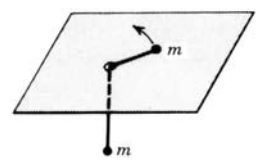
4. Use Lagrange's equations to find the equation of motion of a simple pendulum. (See Chapter 7, Problem 2.13.)

5. Find the equation of motion of a particle moving along the x axis if the potential energy is $V = \frac{1}{2}kx^2$. (This is a simple harmonic oscillator.)

6. A particle moves on the surface of a sphere of radius a under the action of the earth's gravitational field. Find the θ, ϕ equations of motion. (*Comment:* This is called a spherical pendulum. It is like a simple pendulum suspended from the center of the sphere, except that the motion is not restricted to a plane.)

7. Prove that a particle constrained to stay on a surface $f(x, y, z) = 0$, but subject to no other forces, moves along a geodesic of the surface. *Hint:* The potential energy V is constant, since constraint forces are normal to the surface and so do no work on the particle. Use Hamilton's principle and show that the problem of finding a geodesic and the problem of finding the path of the particle are identical mathematics problems.

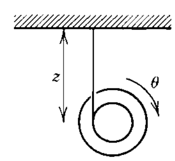
8. Two particles each of mass m are connected by an (inextensible) string of length l. One particle moves on a horizontal table (assume no friction), The string passes through a hole in the table and the particle at the lower end moves up and down along a vertical line. Find the Lagrange equations of motion of the particles. *Hint:* Let the coordinates of the particle on the table be r and θ, and let the coordinate of the other particle be z. Eliminate one variable from L (using $r + |z| = l$) and write two Lagrange equations.



9. A mass m moves without friction on the surface of the cone $r = z$ under gravity acting in the negative z direction. Here r is the cylindrical coordinate $r = \sqrt{x^2 + y^2}$. Find the Lagrangian and Lagrange's equations in terms of r and θ (that is, eliminate z).

10. Do Example 3 above, using cylindrical coordinates for m_1. *Hint:* Use z_1 and z_2 for the z coordinates of m_1 and m_2. What is the equation of the cone in terms of r and z_1? Note that $r \neq \rho$, and θ in cylindrical coordinates is not the same as in spherical coordinates (see Chapter 5, Figures 4.4 and 4.5).

11. A yo-yo (as shown) falls under gravity. Assume that it falls straight down, unwinding as it goes. Find the Lagrange equation of motion. *Hints:* The kinetic energy is the sum of the translational energy $\frac{1}{2}m\dot{z}^2$ and the rotational energy $\frac{1}{2}I\dot{\theta}^2$ where I is the moment of inertia. What is the relation between $\dot{z}$ and $\dot{\theta}$? Assume the yo-yo is a solid cylinder with inner radius a and outer radius b.



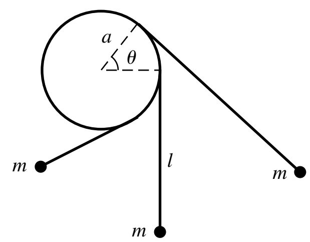
12. Find the Lagrangian and Lagrange's equations for a simple pendulum (Problem 4) if the cord is replaced by a spring with spring constant k. *Hint:* If the unstretched spring length is r_0, and the polar coordinates of the mass m are (r, θ), the potential energy of the spring is $\frac{1}{2}k(r - r_0)^2$.

13. A particle moves without friction under gravity on the surface of the paraboloid $z = x^2 + y^2$. Find the Lagrangian and the Lagrange equations of motion. Show that motion in a horizontal circle is possible and find the angular velocity of this motion. Use cylindrical coordinates.

14. A hoop of mass M and radius a rolls without slipping down an inclined plane of angle α. Find the Lagrangian and the Lagrange equation of motion. *Hint:* The kinetic energy of a body which is both translating and rotating is a sum of two terms: the translational kinetic energy $\frac{1}{2}Mv^2$ where v is the velocity of the center of mass, and the rotational kinetic energy $\frac{1}{2}I\omega^2$ where ω is the angular velocity and I is the moment of inertia around the rotation axis through the center of mass.

15. Generalize Problem 14 to any mass M of circular cross section and moment of inertia I. Consider a hoop, a disk, a spherical shell, a solid spherical ball; order them as to which would first reach the bottom of the inclined plane. (For moments of inertia, see Chapter 5, Section 4.)

16. Find the Lagrangian and the Lagrange equation for the pendulum shown. The vertical circle is fixed. The string winds up or unwinds as the mass m swings back and forth. Assume that the unwound part of the string at any time is in a straight line tangent to the circle. Let l be the length of unwound string when the pendulum hangs straight down.

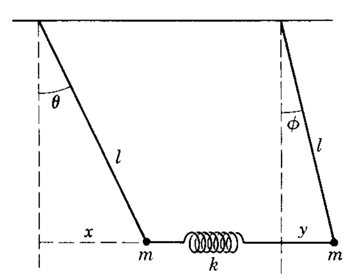


17. A simple pendulum (Problem 4) is suspended from a mass M which is free to move without friction along the x axis. The pendulum swings in the xz plane and gravity acts in the negative z direction. Find the Lagrangian and Lagrange's equations for the system.

18. A hoop of mass m in a vertical plane rests on a frictionless table. A thread is wound many times around the circumference of the hoop. The free end of the thread extends from the bottom of the hoop along the table, passes over a pulley (assumed weightless), and then hangs straight down with a mass m (equal to the mass of the hoop) attached to the end of the thread. Let x be the length of thread between the bottom of the hoop and the pulley, let y be the length of thread between the pulley and the hanging mass m, and let θ be the angle of rotation of the hoop about its center if the thread unwinds. What is the relation between x, y, and θ? Find the Lagrangian and Lagrange's equations for the system. If the system starts from rest, how does the hoop move?

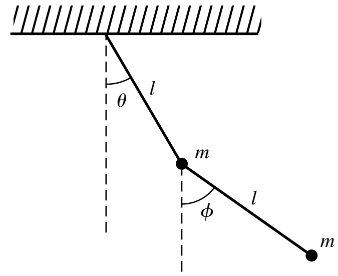
For the following problems, use the Lagrangian to find the equations of motion and then refer to Chapter 3, Section 12.

19. For small vibrations, find the characteristic frequencies and the characteristic modes of vibration of the coupled pendulums shown. All motion takes place in a single vertical plane. Assume the spring unstretched when both pendulums hang vertically, and take the spring constant as $k = mg/l$ to simplify the algebra. *Hints*: Write the kinetic and potential energies in terms of the rectangular coordinates of the masses relative to their positions hanging at rest. Don't forget the gravitational potential energies. Then write the rectangular coordinates x and y in terms of θ and ϕ, and for small vibrations approximate $\sin\theta = \theta$, $\cos\theta = 1 - \theta^2/2$, and similar equations for ϕ.



20. Do Problem 19 if the spring constant is $k = 3mg/l$.

21. Find the Lagrangian and Lagrange's equations for the double pendulum shown. All motion takes place in a single vertical plane. *Hint*: See the hint in Problem 19.



22. Do Problem 21 if the two masses are different. Let m be the lower mass and let M be the sum of the two masses.

23. For small oscillations of the double pendulum in Problem 22, let $M = 4m$ and find the characteristic frequencies and characteristic modes of vibration.

24. Do Problem 23 if $M/m = 9/4$

25. Do Problem 23 in general, that is, in terms of the ratio M/m. *Hint*: You may find it helpful to use a single letter to represent $\sqrt{m/M}$, say $\alpha^2 = m/M$.

6. ISOPERIMETRIC PROBLEMS

Recall that in ordinary calculus we sometimes want to maximize a quantity subject to a condition (for example, find the volume of the largest box you can make with given surface area). Also recall that the method of Lagrange multipliers was useful in such problems (see Chapter 4, Section 9). There are similar problems in the calculus of variations. The original question which gave this class of problems its name was this: Of all the closed plane curves of given perimeter (isoperimetric = same perimeter), which one incloses the largest area? To solve this problem, we must maximize the area, $\int y\,dx$, subject to the condition that the arc, $\int ds$, is the given length l. In other words, we want to maximize an integral subject to the condition that another integral has a given (constant) value; any such problem is called an isoperimetric problem. Let

$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

be the integral we want to make stationary; at the same time,

$$J = \int_{x_1}^{x_2} G(x, y, y')\,dx,$$

with the same integration variable and the same limits, is to have a given constant value. (This means that the allowed varied paths must be paths for which J has the given value.) By using the Lagrange multiplier method, it can be shown that the desired condition is that

$$\int_{x_1}^{x_2} (F + \lambda G)\,dx$$

should be stationary, that is, that $F + \lambda G$ should satisfy the Euler equation. The Lagrange multiplier λ is a constant. It will appear in the solution $y(x)$ of the Euler equation; having found $y(x)$, we can substitute it into $\int_{x_1}^{x_2} G(x, y, y')\,dx =$ const. and so find λ if we like. However, for many purposes we do not need to find λ.

▶ **Example 1.** Given two points x_1 and x_2 on the x axis, and an arc length $l > x_2 - x_1$, find the shape of the curve of length l joining the given points which, with the x axis, incloses the largest area.

We want to maximize $I = \int_{x_1}^{x_2} y\,dx$ subject to the condition $J = \int_{x_1}^{x_2} ds = l$. Here $F = y$ and $G = \sqrt{1+y'^2}$ so

$$F + \lambda G = y + \lambda\sqrt{1+y'^2}. \tag{6.1}$$

We want the Euler equation for $F + \lambda G$. Since

$$\frac{\partial}{\partial y'}(F+\lambda G) = \frac{\lambda y'}{\sqrt{1+y'^2}} \quad \text{and} \quad \frac{\partial}{\partial y}(F+\lambda G) = 1,$$

the Euler equation is

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) - 1 = 0. \tag{6.2}$$

The solution of (6.2) is (Problem 7):

$$(x+c)^2 + (y+c')^2 = \lambda^2 \tag{6.3}$$

We see that the answer to our problem is an arc of a circle passing through the two given points, and the Lagrange multiplier λ is the radius of the circle. The center and radius of the circle are determined by the given points x_1 and x_2, and the given arc length l (Problem 7).

▶ PROBLEMS, SECTION 6

In Problems 1 and 2, given the length l of a curve joining two given points, find the equation of the curve so that:

1. The surface of revolution formed by rotating the curve about the x axis has minimum area.

2. The plane area between the curve and a straight line joining the points is a maximum.

3. Given 10 cc of lead, find how to form it into a solid of revolution of height 1 cm and minimum moment of inertia about its axis.

4. A uniform flexible chain of given length is suspended at given points (x_1, y_1) and (x_2, y_2). Find the curve in which it hangs. *Hint:* It will hang so that its center of gravity is as low as possible.

5. A curve $y = y(x)$, joining two points x_1 and x_2 on the x axis, is revolved around the x axis to produce a surface and a volume of revolution. Given the surface area, find the shape of the curve $y = y(x)$ to maximize the volume. *Hint*: You should find a first integral of the Euler equation of the form $yf(y, x', \lambda) = C$. Since $y = 0$ at the endpoints, $C = 0$. Then either $y = 0$ for all x, or $f = 0$. But $y \equiv 0$ gives zero volume of the solid of revolution, so for maximum volume you want to solve $f = 0$.

6. In Problem 5, given the volume, find the shape of the curve $y = y(x)$ to minimize the surface area. *Hint:* See the hint in Problem 5.

7. Integrate (6.2), simplify the result and integrate again to get (6.3) where c and c' are constants of integration. If $x_1 = -\sqrt{3}$, $x_2 = \sqrt{3}$, and $l = 4\pi/3$, show that the center and radius of the circle are $(0, -1)$ and $\lambda =$ radius $= 2$.

7. VARIATIONAL NOTATION

The symbol δ was used in the early days of the development of the calculus of variations to indicate what we have called differentiation with respect to the parameter ϵ. It is just like the symbol d in a differential except that it warns you that ϵ and not x is the differentiation variable. The δ notation is not used much any more in mathematics, but you will find it in applications and so should understand its meaning. The quantity δI is just the differential

$$\delta I = \frac{dI}{d\epsilon}d\epsilon,$$

where $dI/d\epsilon$ is evaluated for $\epsilon = 0$. The symbol δ (read "the variation of") is also treated as a differential operator acting on F, y, and y'; we shall define δy, $\delta y'$, and δF in terms of our previous notation. We had in Section 2:

$$\begin{aligned} Y(x,\epsilon) &= y(x) = \epsilon\eta(x), \\ Y'(x,\epsilon) &= y'(x) + \epsilon\eta'(x). \end{aligned} \tag{7.1}$$

Then the meaning of δy is

$$\delta y = \left(\frac{\partial Y}{\partial \epsilon}\right)_{\epsilon=0} d\epsilon = \eta(x)d\epsilon; \tag{7.2}$$

this is just like a differential dY if ϵ is the variable. The meaning of $\delta y'$ is

$$\delta y' = \left(\frac{\partial Y'}{\partial \epsilon}\right)_{\epsilon=0} d\epsilon = \eta'(x)d\epsilon. \tag{7.3}$$

This is identical with

$$\frac{d}{dx}(\delta y) = \frac{d}{dx}[\eta(x)d\epsilon] = \eta'(x)d\epsilon \tag{7.4}$$

since x and ϵ are independent variables; in other words, d and δ commute. The meaning of δF is

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'; \tag{7.5}$$

this is just a total differential $dF = (\partial F/\partial \epsilon)_{\epsilon=0} d\epsilon$ of the function $F[x, Y(x,\epsilon), Y'(x,\epsilon)]$ at $\epsilon = 0$ with ϵ considered the only variable. Then the variation in I is

$$\begin{aligned} \delta I &= \delta \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \delta F\,dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right) dx \\ &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x)d\epsilon + \frac{\partial F}{\partial y'}\eta'(x)d\epsilon\right] dx. \end{aligned} \tag{7.6}$$

If you compare (7.6) with (2.13), you find that the following two statements about $I = \int F(x,y,y')dx$ mean the same thing:

(a) I is stationary; that is, $dI/d\epsilon = 0$ at $\epsilon = 0$ as in (2.13).

(b) The variation of I is zero; that is, $\delta I = 0$ as in (7.6).

▶ 8. MISCELLANEOUS PROBLEMS

1. (a) In Section 3, we showed how to obtain a first integral of the Euler equation when $F = F(y, y')$. There is an alternative method of handling this case. You can show that if $F = F(y, y')$, then $F - y'\partial F/\partial y' = \text{const.}$ To prove this, differentiate the left-hand side with respect to x, and show that the result is zero if F satisfies the Euler equation. Note that what you have is a first integral of the Euler equation.

(b) Use the method of (a) to do the problems at the end of Section 3.

(c) Consider the motion of a particle along the x axis; then $L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$. Note that L does not contain the independent variable t; this corresponds to the case $F = F(y, y')$ in (a). Show that the first integral found in (a) is just the equation of conservation of energy for the mechanics problem.

Find a first integral of the Euler equation to make stationary the integrals in Problems 2 to 4.

2. $\displaystyle\int_a^b \frac{x^2\,dy}{\sqrt{1+x'^2}}$ **3.** $\displaystyle\int_a^b \frac{yy'^2\,dx}{\sqrt{1+y'^2}}$ **4.** $\displaystyle\int_\alpha^\beta \sqrt{r^2r'^2+r^4}\,d\theta$

Write and solve the Euler equations to make stationary the integrals in Problems 5 to 7.

5. $\displaystyle\int_a^b \sqrt{\frac{y'^2}{y^2}+1}\,dx$ **6.** $\displaystyle\int_a^b \frac{\sqrt{1+y'^2}}{1+y}\,dx$ **7.** $\displaystyle\int_a^b \sqrt{1+x}\sqrt{1+x'^2}\,dy$

8. Find the geodesics on the cylinder $r = 1 + \cos\theta$.

9. Find the geodesics on the cone $z = r\cot\alpha$, where $r^2 = x^2 + y^2$.

10. Find the geodesics on the parabolic cylinder $y = x^2$.

In Problems 11 to 18, use Fermat's principle to find the path of a light ray through a medium of index of refraction proportional to the given function.

11. $r^{-1/2}$ **12.** e^y **13.** $(2x+3)^{-1}$ **14.** $(y+2)^{1/2}$

15. $x^{1/3}$ *Hint:* In the last integration, let $x = u^3$.

16. r *Hint:* In the last integration, let $u = r^2$.

17. $r^{-1}\ln r$ *Hint:* In the last integration, let $u = \ln r$.

18. $(x+y)^{1/2}$ *Hint:* Make the change of variables (45° rotation)

$$X = \frac{1}{\sqrt{2}}(x+y), \quad Y = \frac{1}{\sqrt{2}}(x-y); \quad \text{what is} \quad dX^2 + dY^2?$$

19. Find Lagrange's equations in polar coordinates for a particle moving in a plane if the potential energy is $V = \frac{1}{2}kr^2$.

20. Repeat Problem 19 if $V = -K/r$.

21. Write Lagrange's equations in cylindrical coordinates for a particle moving in the gravitational field $V = mgz$.

22. In spherical coordinates, find the θ Lagrange equation for a particle moving in the potential field $V = V(r, \theta, \phi)$. What is the θ component of the acceleration? *Hint:* The θ Lagrange equation is the θ component of $m\mathbf{a} = \mathbf{F} = -\nabla V$; for components of ∇V, see Chapter 6, end of Section 6, or Chapter 10, Section 9.

23. A particle slides without friction around a vertical circle under the force of gravity. Set up the Lagrange equation of motion.

24. Write and simplify the Euler equation to make stationary the integral

$$\int_a^b [P(x,y) + Q(x,y)y']\, dx.$$

Show that if the Euler equation is satisfied, the integral has the same value for *all* paths joining a and b. (See Problem 1.3. Also see Chapter 6, Section 9, Example 2.)

25. Find the shape of a curve of minimum length which will inclose a given area A lying in the (x, y) plane. Find the length in terms of A.

26. A wire carrying a uniform distribution of positive charge lies in the (x, y) plane and joins two given points. Find its shape to minimize the electrostatic potential at the origin.

27. Find a first integral of the Euler equation for Problem 26 if the length of the wire is given.

28. Write the θ Lagrange equation for a particle moving in a plane if $V = V(r)$ (that is, a central force). Use the θ equation to show that:

(a) The angular momentum $\mathbf{r} \times m\mathbf{v}$ is constant.

(b) The vector $\mathbf{r}$ sweeps out equal areas in equal times (Kepler's second law).

CHAPTER 10

Tensor Analysis

1. INTRODUCTION

You already know something about tensors although you may not have used the term tensor. Tensors of *rank* (or *order*) zero are just scalars and tensors of rank one are just vectors; you are already familiar with these. In 3-dimensional space a scalar has $3^0 = 1$ components and a vector has $3^1 = 3$ components; a second-rank tensor has $3^2 = 9$ components; and in general a tensor of rank n has 3^n components. After scalars and vectors, second-rank tensors are the ones you are most likely to find in applications, so let's consider an example of such a tensor.

Example 1. Think of a beam carrying a load; there are stresses and strains in the material of the beam. If we imagine cutting the beam in two by a plane perpendicular to the x direction, we realize that there is a force per unit area exerted *by* the material on one side of our imaginary cut *on* the other side. This is a vector, so it has three components P_{xx}, P_{xy}, P_{xz}, where the first subscript x is to emphasize that this is a force across a plane perpendicular to the x direction. Similarly, if we consider a plane perpendicular to the y direction, there is a force per unit area across this plane with components P_{yx}, P_{yy}, P_{yz}; and finally across a plane perpendicular to the z direction there is a force per unit area with components P_{zx}, P_{zy}, P_{zz}. At a point in the material, then, we have a set of nine quantities which could be displayed as a matrix:

$$\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix} \tag{1.1}$$

This is a second-rank tensor known as the stress tensor. The forces (per unit area) P_{xx}, P_{yy}, P_{zz} are pressures or tensions; the others are shear forces (per unit area). For example P_{zy} is a force per unit area in the y direction acting across a plane perpendicular to the z direction; this force tends to shear the beam.

So far, we have simply indicated the number of components that tensors of the various ranks have. This is not the whole story. To see what else is required, let us talk about first-rank tensors, that is vectors, which are already familiar to you. In elementary work a vector is usually defined either as a magnitude and a direction, or as a set of three components. To see that we need to give a more careful definition, consider this example.

Example 2. We can draw an arrow to represent a given rotation of a rigid body in the following way. Draw the arrow along the axis of rotation, make its length equal to the rotation angle in radians, and let its sense be given by the right-hand rule. Then, apparently, a rotation is a vector according to the magnitude and direction definition. But this is not so! Take a book and rotate it 90° about the x axis, then 90° about the y axis. (See Chapter 3, Problem 7.31.) Repeat, rotating this time first about the y axis and then about the x axis. The final positions of the book are different. But the sum of the two vectors does not depend on the order in which they are added (in mathematical language, vector addition is commutative). The arrows associated with rotations are not vectors.

Example 3. Now let us consider the idea of a vector as a set of three components. In order to talk about components, we must have a coordinate system. There are infinitely many coordinate systems—even for rectangular axes (x, y, z) there are infinitely many sets of rotated axes. Thus we must say that a vector consists of a set of three components *in each coordinate system.* If the components of a vector relative to one set of axes are given, we know from elementary vector analysis that the component of the vector in any direction, or its components relative to any rotated set of axes, can be found by taking projections. Then the new components are definite combinations of the old components. This fact allows us to decide whether a physical quantity is really a vector or not. There is a similar requirement for tensors, for example the second-rank stress tensor we have described. We could imagine cutting the beam by a plane oriented in any given direction and ask for the force per unit area acting across this plane. It can be shown (see Section 7) that each component of this force is a certain combination of the nine components of the stress tensor (1.1). Thus the components of the stress tensor in any other coordinate system are definite combinations of the nine components of the tensor relative to the (x, y, z) axes. In other words, tensors of all ranks, like vectors, have a physical meaning which is independent of the reference coordinate system and there are definite mathematical laws which relate their components in two systems.

You may wonder why we cannot make just any set of components (3 for a vector, 9 for a second-rank tensor, etc.), given in *one* coordinate system, a tensor by *defining* its components in other systems by the correct transformation laws. Mathematically, we could! But for a physical entity, we are not free to define its components in various coordinate systems; they are determined by physical fact. We merely give a mathematical description of the entity and identify it as a scalar, a vector, a second-rank tensor, etc. (or perhaps none of these). We can see again now why an arrow associated with a rotation is not a vector. If we treat the arrow as a vector and take components of it, these component vectors do not represent rotations which can be combined to give the original rotation. Thus a vector which looks superficially like the arrow we have defined is not a correct mathematical representation of the physical entity (a rotation) we are trying to describe.

What is the relationship between the vectors we are going to define here and the vectors of a linear vector space (Chapter 3, Sections 10 and 14)? The ideas of an abstract vector space grew out of the geometry of three-dimensional displacement vectors. A change of coordinate system (for example, rotation of axes) corresponds to a change of basis in a vector space. Because the definitions of a vector space are set up to parallel the geometry, displacement vectors are vectors in the vector space sense. Whether other physical entities (force, temperature, stress, etc.) are properly modeled as vectors then depends on whether they transform under a change of coordinate system (that is, change of basis) in the same way that displacement vectors do. Here we want the word "vector" to refer to all physical quantities which transform properly. Thus we shall find the transformation law for a displacement vector, and then define a vector as any entity which obeys the same law.

A tensor which transforms properly under a rotation of rectangular (x, y, z) axes is called a Cartesian tensor; we will study these in some detail. Things become a little more complicated when we consider transformations to other coordinate systems such as spherical coordinates; we will consider this at the end of the chapter. But for Sections 1 to 7, the term tensor will mean Cartesian tensor.

2. CARTESIAN TENSORS

In Chapter 3, Section 7, we considered the effect of rotations on vectors, and emphasized active rotations (vector rotated, axes fixed). Now we want to consider passive rotations (vector fixed, axes rotated), in order to find how the components of a displacement vector in one coordinate system are related to its components in a rotated system. Let (x, y, z) be a set of rectangular axes and (x', y', z') another set obtained by rotating the axes in any manner keeping the origin fixed (Figure 2.1). In the table (2.1), we list the cosines of the nine angles between the (x, y, z) axes and the (x', y', z') axes.

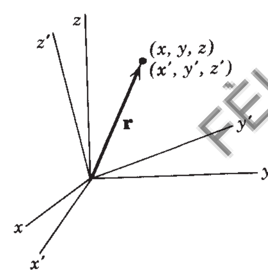


Figure 2.1

(2.1)

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

In the table, l_2 means the cosine of the angle between the x axis and the y' axis, etc. A vector $\mathbf{r}$ (Figure 2.1) has components x, y, z or x', y', z' relative to the two coordinate systems; we want to find the relations between the two sets of components.

Example 1. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit basis vectors along the (x, y, z) axes and $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ be unit basis vectors along the (x', y', z') axes. Then the vector $\mathbf{r}$ can be written in terms of either set of components and basis vectors as follows:

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z'. \tag{2.2}$$

Taking the dot product of this equation with $\mathbf{i}'$, we get

$$\mathbf{r} \cdot \mathbf{i}' = \mathbf{i} \cdot \mathbf{i}'x + \mathbf{j} \cdot \mathbf{i}'y + \mathbf{k} \cdot \mathbf{i}'z = x' \tag{2.3}$$

(since $\mathbf{i}'\cdot\mathbf{i}' = 1$, and $\mathbf{i}'\cdot\mathbf{j}' = \mathbf{i}'\cdot\mathbf{k}' = 0$). Now $\mathbf{i}\cdot\mathbf{i}'$ is the cosine of the angle between $\mathbf{i}$ and $\mathbf{i}'$, that is, between the x and x' axes, since $\mathbf{i}$ and $\mathbf{i}'$ are unit vectors; thus $\mathbf{i}\cdot\mathbf{i}' = l_1$ from the table (2.1). Similarly, $\mathbf{j}\cdot\mathbf{i}' = m_1$ and $\mathbf{k}\cdot\mathbf{i}' = n_1$ and (2.3) becomes

$$x' = l_1x + m_1y + n_1z. \tag{2.4}$$

Similarly, dotting $\mathbf{r}$ into $\mathbf{j}'$ nd $\mathbf{k}'$, and using (2.1) we get

$$\begin{aligned} y' &= l_2x + m_2y + n_2z,\\ z' &= l_3x + m_3y + n_3z. \end{aligned} \tag{2.5}$$

The equations (2.4) and (2.5) are called the transformation equations from the coordinate system (x, y, z) to (x', y', z').

▶ **Example 2.** In the same way, dotting $\mathbf{r}$ with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in turn, we get equations for x, y, z in terms of x', y', z':

$$\begin{aligned} x &= l_1x' + l_2y' + l_3z',\\ y &= m_1x' + m_2y' + m_3z',\\ z &= n_1x' + n_2y' + n_3z'. \end{aligned} \tag{2.6}$$

These transformation equations may be written more concisely in matrix notation. Equations (2.4) and (2.5) become the matrix equation:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1z \\ l_2 & m_2 & n_2z \\ l_3 & m_3 & n_3z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{or} \quad \mathrm{r}' = \mathrm{Ar}, \tag{2.7}$$

where r', A, and r stand for the matrices in (2.7). [Compare Chapter 3, equation (7.13) for the two-dimensional case.] Similarly, (2.6) becomes

$$\mathrm{r} = \mathrm{A}^{\mathrm{T}}\mathrm{r}' \tag{2.8}$$

where A^{T} is the transpose of A. Recall from Chapter 3, Sections 7 and 9, that a rotation matrix is an orthogonal matrix, and for an orthogonal matrix, $\mathrm{A}^{\mathrm{T}} = \mathrm{A}^{-1}$. Also see Problems 3 and 4.

Equations (2.7) or (2.8) tell us how displacement vectors in a rectangular coordinate system transform under a rotation of axes. We now use this result to define *Cartesian* vectors, that is, vectors which transform in the same way that displacement vectors do under rotations of rectangular (Cartesian) axes. We will then generalize this to define Cartesian tensors of other ranks.

Definition of Cartesian Vectors A Cartesian vector $\mathbf{V}$ consists of a set of three numbers (components) in *every* rectangular coordinate system; if V_x, V_y, V_z are the components in one system and V_x', V_y', V_z' are the components in a rotated system, these two sets of components are related by an equation similar to (2.7), namely,

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \mathrm{A}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \qquad \text{or} \quad \mathrm{V}' = \mathrm{AV}, \tag{2.9}$$

where A is the rotation matrix in (2.7). Alternatively, we could use (2.8) and require that $\mathrm{V} = \mathrm{A}^{\mathrm{T}}\mathrm{V}'$.

We can simplify our notation by making the following changes.

$$\begin{array}{lll}
 & \text{Replace } x, y, z & \text{by} \quad x_1, x_2, x_3 \\
 & \text{Replace } x', y', z' & \text{by} \quad x'_1, x'_2, x'_3 \\
 & \text{Replace } V_x, V_y, V_z & \text{by} \quad V_1, V_2, V_3 \\
(2.10) & \text{Replace } V'_x, V'_y, V'_z & \text{by} \quad V'_1, V'_2, V'_3 \\
 & \text{Replace A in (2.7)} & \text{by} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\end{array}$$

In this notation (2.7) and (2.9) become (2.11) and (2.12):

$$x'_i = \sum_{j=1}^{3} a_{ij} x_j, \qquad i = 1, 2, 3, \tag{2.11}$$

$$V'_i = \sum_{j=1}^{3} a_{ij} V_j, \qquad i = 1, 2, 3. \tag{2.12}$$

Alternatively, we could solve (2.11) for the x coordinates in terms of the x' coordinates as in (2.8), to get, in the summation form: $x_i = \sum_{j=1}^{3} a_{ji} x'_j$, and a similar companion formula to (2.12), namely

$$V_i = \sum_{j=1}^{3} a_{ji} V'_j. \tag{2.13}$$

Since we will occasionally want the transformation formula for a Cartesian vector solved for the unprimed components as in (2.13), you should be sure you understand (2.13). Compare carefully the indices in (2.12) and (2.13). In matrix form (2.12) is $\mathrm{V}' = \mathrm{AV}$ and (2.13) is $\mathrm{V} = \mathrm{A}^{\mathrm{T}}\mathrm{V}'$ [see equations (2.7) and (2.8)]. Now element i, j of A^{T} is the same as element j, i of A, so the coefficients in (2.13) are a_{ji} instead of a_{ij} as they were in (2.12). It is now straightforward to define tensors.

Definition of Cartesian Tensors A tensor of rank zero has one component which is unchanged by a rotation of axes; it is called an invariant or a scalar. Simple examples are the length of a vector, or the dot product of two vectors. A first rank tensor is just a vector. A tensor of second rank has nine components (in three dimensions) in every rectangular coordinate system. If we call the components in one system T_{ij}, the components T'_{kl} in a rotated coordinate system are given by (2.14), where the a's are the direction cosines in the rotation matrix A.

$$T'_{kl} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ki} a_{lj} T_{ij}, \qquad k, l = 1, 2, 3. \tag{2.14}$$

Direct Product We can give a very simple example of a second-rank tensor.

Example 3. Let $\mathbf{U}$ and $\mathbf{V}$ be vectors; we form the following array (in each coordinate system) from the components U_1, U_2, U_3 and V_1, V_2, V_3 of $\mathbf{U}$ and $\mathbf{V}$ (in that coordinate system):

$$\begin{matrix} U_1V_1 & U_1V_2 & U_1V_3 \\ U_2V_1 & U_2V_2 & U_2V_3 \\ U_3V_1 & U_3V_2 & U_3V_3 \end{matrix} \tag{2.15}$$

We can show that these nine quantities are the components of a second-rank tensor which we shall denote by $\mathbf{UV}$. Note that this is not a dot product or a cross product; it is called the *direct product* of $\mathbf{U}$ and $\mathbf{V}$ (or *outer product* or *tensor product*). Since $\mathbf{U}$ and $\mathbf{V}$ are vectors, their components in a rotated coordinate system are, by (2.12):

$$U'_k = \sum_{i=1}^{3} a_{ki}U_i, \qquad V'_l = \sum_{j=1}^{3} a_{lj}V_j. \tag{2.16}$$

Hence the components of the second-rank tensor $\mathbf{UV}$ are

$$U'_kV'_l = \sum_{i=1}^{3} a_{ki}U_i \sum_{j=1}^{3} a_{lj}V_j = \sum_{i,j=1}^{3} a_{ki}a_{lj}U_iV_j, \tag{2.17}$$

which is just (2.14) with $T_{ij} = U_iV_j$ and $T'_{kl} = U'_kV'_l$.

Equation (2.14) generalizes immediately. For example, a 4^{th}-rank Cartesian tensor is defined as a set of 3^4 or 81 components T_{ijkl}, in every rectangular coordinate system, which transform to a rotated coordinate system by the equations

$$T'_{\alpha\beta\gamma\delta} = \sum_{i,j,k,l} a_{\alpha i}a_{\beta j}a_{\gamma k}a_{\delta l}T_{ijkl}, \tag{2.18}$$

where i, j, k, l take the values 1, 2, 3. Note that a 4^{th}-rank tensor has 4 indices and requires four a's in its definition. Similarly, an n^{th}-rank tensor has n indices and requires n a's in its definition. Also we can generalize (2.17) to show, for example, that the direct product of a vector and a 3^{rd}-rank tensor produces a 4^{th}-rank tensor, and, in general, the direct product of tensors of ranks n and m is a tensor of rank $m + n$ (see Problem 7).

PROBLEMS, SECTION 2

1. Verify equations (2.6).

2. Show that the sum of the squares of the direction cosines of a line through the origin is equal to 1 *Hint:* Let (a, b, c) be a point on the line at distance 1 from the origin. Write the direction cosines in terms of (a, b, c).

3. Consider the matrix A in (2.7) or (2.10). Think of the elements in each row (or column) as the components of a vector. Show that the row vectors form an orthonormal triad (that is each is of unit length and they are all mutually orthogonal), and the column vectors form an orthonormal triad.

4. Any rotation of axes in three dimensions can be described by giving the nine direction cosines of the angle between the (x, y, z) axes and the (x', y', z') axes. Show that the matrix A of these direction cosines in (2.7) or (2.10) is an orthogonal matrix. *Hint:* See Chapter 3, Section 9. Find AA^{T} and use Problem 3.

5. Write equations (2.12) out in detail and solve the three simultaneous equations (say by determinants) for x_1, x_2, x_3 in terms of x_1', x_2', x_3' to verify equations (2.13). Use your results in Problem 4.

6. Write the transformation equation for a 3^{rd}-rank tensor; for a 5^{th}-rank tensor.

7. Following what we did in equations (2.14) to (2.17), show that the direct product of a vector and a 3^{rd}-rank tensor is a 4^{th}-rank tensor. Also show that the direct product of two 2^{nd}-rank tensors is a 4^{th}-rank tensor. Generalize this to show that the direct product of two tensors of ranks m and n is a tensor of rank $m + n$.

8. Write the equations in (2.16) and so in (2.17) solved for the unprimed components in terms of the primed components.

3. TENSOR NOTATION AND OPERATIONS

Summation Convention As you may have noticed in the last section, tensor equations use a lot of summation signs—it would be a simplification if we could get along without them. Using the *summation convention* (or Einstein summation convention), we omit the summation signs in equations like (2.11) to (2.14), and (2.16) to (2.18), and simply understand a summation over any index which appears exactly twice in one term. Here are some examples using summation convention (in three dimensions).

Examples.

$$a_{ii} \text{ or } a_{jj} \text{ or } a_{\beta\beta}, \text{ etc. means } a_{11} + a_{22} + a_{33};$$

$$x_i x_i \text{ or } x_\alpha x_\alpha, \text{ etc. means } x_1^2 + x_2^2 + x_3^2;$$

$$a_{ij} b_{jk} \text{ means } a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k};$$

$$\frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial x_i'} \text{ means } \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial x_i'} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial x_i'} + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial x_i'}; \tag{3.1}$$

$$T_{ijkl} S_{ij} V_k U_l \text{ means } \sum_i \sum_j \sum_k \sum_l T_{ijkl} S_{ij} V_k U_l;$$

and so on. The repeated index (which is summed over) is called a *dummy* index; like an integration variable in a definite integral, it does not matter what letter is used for it. An index which is not repeated is called a *free* index.

When summation convention is being used, we are not warned by a summation sign what letters to sum over; we just have to inspect the indices and see which ones appear twice. In writing terms using the summation convention, we must be careful not to re-use an index. For example, if we already have two i subscripts indicating a sum over i, and we want another sum in the same term, we must use a different dummy index, say j or m or α, etc. In the following discussion we will use summation convention; watch carefully for the repeated dummy indices.

Contraction The transformation equations for a 4^{th}-rank tensor are [see (2.18)]

$$T'_{\alpha\beta\gamma\delta} = a_{\alpha i} a_{\beta j} a_{\gamma k} a_{\delta l} T_{ijkl}. \tag{3.2}$$

(Note the sums over i, j, k, and l).

Example 1. Now suppose we put $\delta = \beta$ which, by summation convention, means a further sum over β. Then we have

$$T'_{\alpha\beta\gamma\beta} = a_{\alpha i}a_{\beta j}a_{\gamma k}a_{\beta l}T_{ijkl}. \tag{3.3}$$

Now $a_{\beta j}a_{\beta l}$ (summed over β) is the dot product of columns j and l of the rotation matrix A [see Problem (2.3)]. This dot product is 1 if $j = l$, and 0 otherwise. In other words $a_{\beta j}a_{\beta l} = \delta_{jl}$ [see Chapter 3, equation (9.4)]. Then $\delta_{jl}T_{ijkl}$ becomes T_{ijkj} since δ_{jl} is zero unless j and l are equal. (The repeated dummy index could be either j or l or anything else except the dummy indices i and k which are already used, and the free indices α and γ). Thus we have

$$T'_{\alpha\beta\gamma\beta} = a_{\alpha i}a_{\gamma k}\delta_{jl}T_{ijkl} = a_{\alpha i}a_{\gamma k}T_{ijkj} \tag{3.4}$$

Now (3.4) says that T_{ijkj} are the components of a 2^{nd}-rank tensor since there are two free indices and two a factors are required [compare equation (2.14)]. This process of setting two indices of a tensor equal to each other and then summing is called *contraction*. Contraction reduces the rank of a tensor by 2. Note that in (3.2) we started with a 4^{th}-rank tensor and after contracting we have a tensor of rank 2 in (3.4).

It is interesting to observe that the dot (or scalar or inner) product of two vectors in elementary vector analysis is an example of contraction. In Section 2 we showed that the direct product of two vectors [see (2.17)] is a 2^{nd}-rank tensor. If we contract U_iV_j to get U_iV_i we have the dot product of vectors $\mathbf{U}$ and $\mathbf{V}$, which is a scalar. Again note that contraction has reduced the rank of a tensor by 2 (a scalar is a tensor of rank zero).

Tensors and Matrices The components of first or second rank tensors can be displayed as matrices and this is often useful. We have frequently (see Chapter 3) written the components of a vector (1^{st}-rank tensor) as a column or row matrix. The components T_{ij} of a 2^{nd}-rank tensor can be written as the elements of a square matrix (see inertia matrix, Section 4). Then note that in the tensor equation, $U_i = T_{ij}V_j$, the contraction (sum on j) corresponds exactly to row times column multiplication for matrices.

Symmetric and Antisymmetric Tensors A 2^{nd}-rank tensor T_{ij} is called *symmetric* if $T_{ij} = T_{ji}$, and *antisymmetric* (or *skew symmetric*) if $T_{ij} = -T_{ji}$. Note that these agree with the corresponding definitions for matrices [Chapter 3, (9.2)]. Any 2^{nd}-rank tensor can be written as a sum of a symmetric tensor and an antisymmetric tensor as in (3.5) (Problem 13).

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}). \tag{3.5}$$

For tensors of higher rank, similar terminology is used. If an exchange of two indices leaves the tensor component unchanged, we say that the tensor is symmetric with respect to those two indices. If an exchange of two indices changes the tensor component to its negative, we say that the tensor is antisymmetric with respect to those two indices.

Combining tensors The sum or difference (in fact linear combination) of two tensors of rank n is a tensor of rank n (Problems 6 and 7). For example, $T_{ij}+R_{ijk}V_k$ is a tensor of rank 2. Note the summation convention and the contraction which makes $R_{ijk}V_k$ also a tensor of rank 2 so that we can add it to T_{ij}. (Addition is not defined for tensors of different ranks.)

Quotient Rule Let us suppose we know that, for every vector V_j, the quantities $U_i = T_{ij}V_j$ are the components of a non-zero vector and that this holds true in all rotated coordinate systems. Then we can prove that the quantities T_{ij} are the components of a 2^{nd}-rank tensor. This is an example of the *quotient rule*.

▶ **Example 2.** To prove this, we need the following equations:

$$\begin{aligned} T'_{\alpha\beta}V'_\beta &= U'_\alpha, && \text{given equation in rotated system;} \\ U'_\alpha &= a_{\alpha i}U_i, && \mathbf{U} \text{ is a vector;} \\ U_i &= T_{ij}V_j, && \text{given equation;} \\ V_j &= a_{\beta j}V'_\beta, && \mathbf{V} \text{ is a vector; see equation (2.13).} \end{aligned} \tag{3.6}$$

Now, putting this all together we have

$$T'_{\alpha\beta}V'_\beta = U'_\alpha = a_{\alpha i}U_i = a_{\alpha i}T_{ij}V_j = a_{\alpha i}T_{ij}a_{\beta j}V'_\beta. \tag{3.7}$$

Factoring out V'_β from the first and last steps, we have

$$(T'_{\alpha\beta} - a_{\alpha i}a_{\beta j}T_{ij})V'_\beta = 0 \quad \text{for } \textit{all} \text{ vectors } \mathbf{V}'. \tag{3.8}$$

Since $\mathbf{V}'$ is arbitrary, the parenthesis in (3.8) is equal to zero (Problem 8). Thus we have

$$T'_{\alpha\beta} = a_{\alpha i}a_{\beta j}T_{ij}. \tag{3.9}$$

Now (3.9) is the transformation equation for a 2^{nd}-rank tensor [compare (2.14)], so, as claimed, the quantities T_{ij} are the components of a 2^{nd}-rank tensor.

The quotient rule is useful in determining whether some given quantities are the components of a tensor. [As an example of this, see (4.1).] Suppose **X** is a set of 3^n components (the right number for a tensor of rank n in 3 dimensions). The quotient rule says that if the product of **X** and an arbitrary tensor is a non-zero tensor, then **X** is a tensor. The product may be either a direct product or a direct product combined with one or more contractions. We have proved the quotient rule for one case but the proof of any case follows this same pattern. Given $\mathbf{XA} = \mathbf{B}$, where **A** is an arbitrary tensor and **B** is a non-zero tensor, we use the transformation equations for **A** and **B**, and the fact that **A** is arbitrary, to find the transformation equations for **X** (see Problems 9 to 12).

PROBLEMS, SECTION 3

1. Write equations (2.11), (2.12), (2.13), (2.14), (2.16), (2.17), and (2.18) using summation convention.

2. Show that the fourth expression in (3.1) is equal to $\partial u/\partial x'_i$. By equations (2.6) and (2.10), show that $\partial x_j/\partial x'_i = a_{ij}$, so

$$\frac{\partial u}{\partial x'_i} = \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial x'_i} = a_{ij}\frac{\partial u}{\partial x_j}.$$

Compare this with equation (2.12) to show that $\boldsymbol{\nabla} u$ is a Cartesian vector. *Hint:* Watch the summation indices carefully and if it helps, put back the summation signs or write sums out in detail as in (3.1) until you get used to summation convention.

3. As we did in (3.3), show that the contracted tensor T_{iij} is a first-rank tensor, that is, a vector.

4. Show that the contracted tensor $T_{ijk}V_k$ is a 2^{nd}-rank tensor.

5. Show that $T_{ijklm}S_{lm}$ is a tensor and find its rank (assuming that **T** and **S** are tensors of the rank indicated by the indices).

6. Show that the sum of two 3^{rd}-rank tensors is a 3^{rd}-rank tensor. *Hint:* Write the transformation law for each tensor and then add your two equations. Divide out the a factors to leave the result $T'_{\alpha\beta\gamma} + S'_{\alpha\beta\gamma} = a_{\alpha i}a_{\beta j}a_{\gamma k}(T_{ijk} + S_{ijk})$ using summation convention.

7. As in problem 6, show that the sum of two 2^{nd}-rank tensors is a 2^{nd}-rank tensor; that the sum of two 4^{th}-rank tensors is a 4^{th}-rank tensor.

8. Show that (3.9) follows from (3.8). *Hint:* Give a proof by contradiction. Let $S_{\alpha\beta}$ be the parenthesis in (3.8); you may find it useful to think of the components written as a matrix. You want to prove that all 9 components of $S_{\alpha\beta}$ are zero. Suppose it is claimed that S_{12} is not zero. Since V'_β is an arbitrary vector, take it to be the vector $(0, 1, 0)$, and observe that $S_{\alpha\beta}V'_\beta$ is then not zero in contradiction to (3.8). Similarly show that all components of $S_{\alpha\beta}$ are zero as (3.9) claims.

Prove the quotient rule in each of the following problems, that is, given **XA** = **B** where **A** is any arbitrary tensor and **B** is a non-zero tensor, show that **X** is a tensor. *Hints:* Follow the general method in (3.6) to (3.9). See the last sentence of the section.

9. $X_iA_{ij} = B_j$

10. $X_iA_j = B_{ij}$

11. $X_{ij}A_k = B_{ijk}$

12. $X_{ijkl}A_{kl} = B_{ij}$

13. Show that the first parenthesis in (3.5) is a symmetric tensor and the second parenthesis is antisymmetric.

4. INERTIA TENSOR

Inertia tensor If a rigid body is rotating about a fixed axis, then from elementary mechanics we know that $\boldsymbol{\tau} = d\mathbf{L}/dt$ where $\boldsymbol{\tau}$ is the torque and $\mathbf{L}$ is the angular momentum about the rotation axis. The angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{L}$ are related by the equation $\mathbf{L} = I\boldsymbol{\omega}$ where I is the moment of inertia of the body about the rotation axis. For rotation about a fixed axis, $\mathbf{L}$ and $\boldsymbol{\omega}$ are parallel vectors, and I is a scalar. But if the rotation axis is not fixed, the angular velocity and the angular momentum may not be parallel.

▶ **Example 1.** Try the following experiment. Take a small book bound by a rubber band, hold it by one corner and toss it upward giving it a spin. As it falls observe that it tumbles, that is, the angular velocity $\boldsymbol{\omega}$ about the center of mass is not fixed in direction. However, by definition of the center of mass, the gravitational torque $\boldsymbol{\tau}$ about the center of mass is zero so $\boldsymbol{\tau} = d\mathbf{L}/dt = 0$. (We are neglecting air resistance.) Thus $\mathbf{L}$ is a constant vector, and a constant $\mathbf{L}$ and a changing $\boldsymbol{\omega}$ are not parallel. Then if the equation $\mathbf{L} = I\boldsymbol{\omega}$ is to be true, I cannot be a scalar.

We have seen this situation before; look at the discussion of the quotient rule in Section 3 and the proof of the case we have here in (3.5) to (3.8). Since $\mathbf{L}$ and $\boldsymbol{\omega}$ are vectors, we see by the quotient rule that (when $\mathbf{L}$ and $\boldsymbol{\omega}$ are not parallel) the scalar I must be replaced by a 2^{nd}-rank tensor with components I_{jk}. Then in component form we have

$$L_j = I_{jk}\omega_k \tag{4.1}$$

▶ **Example 2.** Next we want to find the components of the inertia tensor. For simplicity, first consider a point mass m at the tip of a vector $\mathbf{r}$ with tail at the origin O. From Chapter 6, end of Section 3, the angular momentum of m about the origin is $\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$ where $\boldsymbol{\omega}$ is the angular velocity of the mass m about O. (See Chapter 6, Figures 2.6 and 3.8.) We can expand the triple vector product [see Chapter 6, equation (3.8)] to get

$$\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = m[r^2\boldsymbol{\omega} - (\mathbf{r}\cdot\boldsymbol{\omega})\mathbf{r}] = m[r^2\boldsymbol{\omega} - (x\omega_x + y\omega_y + z\omega_z)\mathbf{r}]. \tag{4.2}$$

Next we write the components of $\mathbf{L}$ in terms of the components of $\boldsymbol{\omega}$. For example, taking the x component of (4.2), we find

$$L_x = m[r^2\omega_x - (x\omega_x + y\omega_y + z\omega_z)x] = m[(r^2 - x^2)\omega_x - xy\omega_y - xz\omega_z]. \tag{4.3}$$

Thus three components of the inertia tensor are

$$I_{xx} = m(r^2 - x^2) = m(y^2 + z^2),\ I_{xy} = -mxy,\ I_{xz} = -mxz. \tag{4.4}$$

The other 6 components can be found similarly by taking the y and z components of (4.2) (Problem 1).

▶ **Example 3.** If, instead of a single mass, we have a set of masses or an extended body, then the expressions for the components of the inertia tensor become sums or integrals.

$$\begin{aligned} I_{xx} &= \sum_i m_i(y_i^2 + z_i^2) \quad \text{or} \quad \int (y^2 + z^2)\,dm, \\ I_{xy} &= -\sum_i m_i x_i y_i \quad \text{or} \quad -\int xy\,dm, \text{ etc. (Problem 1.)} \end{aligned} \tag{4.5}$$

It is useful to write (4.1) as a matrix equation (see discussion in Section 3 about contraction). Then the inertia tensor components form a square matrix. This matrix is symmetric and so we know from Chapter 3, Section 11, that it can be diagonalized by an orthogonal similarity transformation. The new axes are called the *principal axes of inertia* and the three eigenvalues are called the *principal moments of inertia*. We see that the equations of motion are simpler relative to the principal axes.

Example 4. Find the inertia tensor about the origin for the mass distribution consisting of a mass 1 at $(0,1,1)$ and a mass 2 at $(1,-1,0)$. Find the principal moments of inertia and the principal axes.

Substituting $(x_1, y_1, z_1) = (0,1,1)$, $m_1 = 1$, and $(x_2, y_2, z_2) = (1,-1,0)$, $m_2 = 2$ into (4.5), we find $I_{xx} = (1^2 + 1^2) + 2(-1)^2 = 4, I_{xy} = I_{yx} = -0 - 2(-1) = 2$. Continuing in the same way, we can find the rest of the components (Problem 2) and write them as an inertia matrix

$$\mathrm{I} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 5 \end{pmatrix}$$

Either by hand or by computer we find that the eigenvalues of the matrix I are 6 and $3 \pm \sqrt{3}$; these are the principal moments of inertia. The corresponding eigenvectors are $(1,1,-1)$, $(-1-\sqrt{3}, 2+\sqrt{3}, 1)$, $(-1+\sqrt{3}, 2-\sqrt{3}, 1)$; these are vectors along the principal axes of inertia.

Example 5. Find the inertia tensor about the origin for a mass of uniform density $= 1$, inside the part of the unit sphere in the first octant, that is, $x > 0$, $y > 0$, $z > 0$.

We will write the integrals for the components of the inertia tensor first in rectangular coordinates and then switch to spherical coordinates [see Chapter 5, equation (4.5)] to evaluate them since the limits are then simpler. Satisfy yourself that in order to cover the required volume, the limits are: r from 0 to 1, θ from 0 to $\pi/2$, and ϕ from 0 to $\pi/2$. Then

$$I_{xx} = \iiint (r^2 - x^2)\, dV = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} (r^2 - r^2 \sin^2\theta \cos^2\phi) r^2 \sin\theta \, dr\, d\theta\, d\phi = \frac{\pi}{15}.$$

$$I_{xy} = \iiint (-xy)\, dV = -\int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} (r^2 \sin^2\theta \cos\phi \sin\phi) r^2 \sin\theta \, dr\, d\theta\, d\phi = -\frac{1}{15}.$$

Similarly, the other integrals can be written and evaluated (Problem 3). Alternatively, it may be clear that by symmetry the three diagonal components are all the same, and all the off-diagonal components are the same. Then the inertia matrix is

$$\mathrm{I} = \frac{1}{15} \begin{pmatrix} \pi & -1 & -1 \\ -1 & \pi & -1 \\ -1 & -1 & \pi \end{pmatrix}.$$

As in Example 4, we find (Problem 3):

Principal moments of inertia: $\dfrac{(\pi - 2, \pi + 1, \pi + 1)}{15}$

Principal axes of inertia: $(1,1,1)$, and any two orthogonal vectors in the plane $x + y + z = 0$, for example, $(1,-1,0)$ and $(1,1,-2)$.

▶ PROBLEMS, SECTION 4

1. As in (4.3) and (4.4), find the y and z components of (4.2) and the other 6 components of the inertia tensor. Write the corresponding components of the inertia tensor for a set of masses or an extended body as in (4.5).

2. Complete Example 4 to verify the rest of the components of the inertia tensor and the principal moments of inertia and principal axes. Verify that the three principal axes form an orthogonal triad.

3. As in Problem 2, complete Example 5.

4. Find the inertia tensor about the origin for a mass of uniform density = 1, inside the part of the unit sphere where $x > 0$, $y > 0$, and find the principal moments of inertia and the principal axes. Note that this is similar to Example 5 but the mass is both above and below the (x, y) plane. *Warning hint:* This time don't make the assumptions about symmetry that we did in Example 5.

For the mass distributions in Problems 5 to 7, find the inertia tensor about the origin, and find the principal moments of inertia and the principal axes.

5. Point masses 1 at $(1, 1, 1)$ and at $(-1, 1, 1)$.

6. Point masses 1 at $(1, 1, -2)$ and 2 at $(1, 1, 1)$.

7. Mass of uniform density = 1, bounded by the coordinate planes and the plane $x + y + z = 1$.

8. For the point mass m we considered in (4.2) to (4.4), the velocity is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ so the kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})$. Show that T can be written in matrix notation as $T = \frac{1}{2}\omega^{\mathrm{T}}\mathrm{I}\omega$ where I is the inertia matrix, ω is a column matrix, and ω^{T} is a row matrix with elements equal to the components of $\boldsymbol{\omega}$.

▶ 5. KRONECKER DELTA AND LEVI-CIVITA SYMBOL

The Kronecker δ is defined in Chapter 3, equation (9.3) but let's repeat it here for convenience.

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

The definition of the Levi-Civita symbol (or permutation symbol) is

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k = 1, 2, 3 \text{ or } 2, 3, 1 \text{ or } 3, 1, 2; \\ -1 & \text{if } i, j, k = 3, 2, 1 \text{ or } 2, 1, 3 \text{ or } 1, 3, 2; \\ 0 & \text{if any indices are repeated.} \end{cases} \tag{5.2}$$

Note in (5.2) that if you read the indices i, j, k, cyclically (as if they were written around a circle so you can start anywhere), then if the indices read in the direction $1, 2, 3, 1, 2, 3, 1, \cdots$, the result is $+1$; if the indices read in the opposite direction the result is -1.

We can say (5.2) in another way which is sometimes useful. Start with the fact that $\epsilon_{123} = +1$. Now if we exchange any two indices, we change the sign; for example $\epsilon_{321} = -1$ (we exchanged 1 and 3). If we now continue this process and exchange 1 and 2 in $\epsilon_{321} = -1$, we have $\epsilon_{312} = +1$. [Try a few more and compare with (5.2).] The result of an even number of exchanges in 123 is called an even permutation of 123 and the result of an odd number of exchanges is called an odd permutation of 123. Thus we could replace (5.2) by the definition

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k = \text{an even permutation of } 1,2,3; \\ -1 & \text{if } i,j,k = \text{an odd permutation of } 1,2,3; \\ 0 & \text{if any indices are repeated.} \end{cases} \tag{5.3}$$

We say that ϵ_{ijk} is *totally antisymmetric* (see Section 3), that is, it is antisymmetric with respect to every pair of indices, since each exchange of indices produces a change in sign.

Isotropic Tensors A Cartesian *isotropic* tensor means a tensor which has the same components in all rotated coordinate systems. The definitions (5.1) to (5.3) are general, independent of any reference system. Thus to show that δ_{ij} and ϵ_{ijk} are isotropic *tensors*, we need to show that a tensor transformation simply reproduces the tensor we start with, that is, $\delta' = \delta$ and $\epsilon' = \epsilon$. In this section we shall show this and develop some useful formulas.

Kronecker delta To show that δ_{ij} is an isotropic 2^{nd}-rank tensor, we write the tensor transformation to a rotated system and show that it gives $\delta' = \delta$.

$$\delta'_{mn} = a_{mi}a_{nj}\delta_{ij} = a_{mj}a_{nj} = \delta_{mn}. \tag{5.4}$$

Remember summation convention and follow carefully the sums in (5.4). Note in the second step that a_{mi} becomes a_{mj} because δ_{ij} is zero unless $i = j$. (We could just as well change a_{nj} to a_{ni} and sum on i.) In the last step, $a_{mj}a_{nj}$ (or $a_{mi}a_{ni}$) is the dot product of rows m and n of the rotation matrix and this is δ_{mn} (see Problem 2.3). Thus the Kronecker δ is a 2^{nd}-rank isotropic Cartesian tensor.

Determinants We can write a useful formula for the value of a 3-by-3 determinant using the Levi-Civita symbol:

$$\det \mathrm{A} = a_{1i}a_{2j}a_{3k}\epsilon_{ijk}. \tag{5.5}$$

It is straightforward to show (Problem 1) that (5.5) is equivalent to a Laplace development. Another useful formula is

$$\epsilon_{\alpha\beta\gamma} \det \mathrm{A} = a_{\alpha i}a_{\beta j}a_{\gamma k}\epsilon_{ijk}. \tag{5.6}$$

Again it is straightforward (although lengthy) to show that this is equivalent to a Laplace development (Problem 2).

Levi-Civita Symbol To show that ϵ_{ijk} is an isotropic tensor, we write the transformation equation to a rotated system. We find

$$\epsilon'_{\alpha\beta\gamma} = a_{\alpha i}a_{\beta j}a_{\gamma k}\epsilon_{ijk} = \epsilon_{\alpha\beta\gamma}. \tag{5.7}$$

In the last step we used (5.6) with $\det \mathrm{A} = 1$ (recall from Chapter 3, Section 7, that if A is a rotation matrix, $\det \mathrm{A} = 1$). Thus $\epsilon_{\alpha\beta\gamma}$ is a 3^{rd}-rank isotropic Cartesian tensor (assuming only rotated coordinate systems; for reflections see Section 6).

Products of Isotropic Tensors We can find other isotropic tensors from direct products of the two we have, or from direct products followed by contraction. Recall from Sections 2 and 3 that the direct product of two tensors of ranks n and m is a tensor of rank $n + m$ and that each contraction produces another tensor of rank smaller by 2. If the tensors you multiply are isotropic, the products are also isotropic (Problems 3 and 4).

To simplify products of two Levi-Civita tensors, the following formula is useful.

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{5.8}$$

Both sides of (5.8) are 4^{th}-rank tensors (contracted 6^{th}-rank on the left) with free indices j, k, m, n. We want to see that (5.8) is true for any choice of these four indices. Most choices will just give $0 = 0$; let's consider what is required for the product of ϵ's to be different from zero.

▶ **Example 1.** Remember that an ϵ is zero unless its three indices are all different. Since the first index is the same in ϵ_{ijk} and ϵ_{imn}, the product is different from zero only if the other two indices (j, k and m, n) are the same pair in both ϵ's. (For example, if $i = 1$, then j, k and m, n must be 2, 3 or 3, 2.) This means that either (1) $j = m$ and $k = n$, or (2) $j = n$ and $k = m$. In case (1), the two ϵ's are the same (both $= +1$ or both $= -1$) so the product is $+1$; this is the same as $\delta_{jm}\delta_{kn}$ on the right side of (5.8). In case (2), the indices on the two ϵ's are ijk and ikj so one of them is an even permutation of 1, 2, 3 and the other is an odd permutation. Thus the product of the two ϵ's is -1, and this is the same as $-\delta_{jn}\delta_{kn}$ on the right side of (5.8). Note that, given j, k, m, n satisfying either $j, k = m, n$ or $j, k = n, m$, only one term in the sum over i is different from zero, that is, the term with i different from either j or k. Also see Problem 5.

Now that we have (5.8), it is easy to write a similar formula with the contraction (sum) over a different pair of indices. Suppose we want $\epsilon_{abc}\epsilon_{pqb}$. Recall that an ϵ is not changed by cyclic permutation of its indices [see discussion after (5.2)]. Thus $\epsilon_{abc} = \epsilon_{bca}$ and $\epsilon_{pqb} = \epsilon_{bpq}$ [we have cyclically permuted the indices so that the summation index b appears as the first index for each ϵ as it does in (5.8)]. Now, this is the same pattern as in (5.8), with the sum over the first index of each ϵ [i in (5.8), b in the second step in (5.9)], so we have

$$\epsilon_{abc}\epsilon_{pqb} = \epsilon_{bca}\epsilon_{bpq} = \delta_{cp}\delta_{aq} - \delta_{cq}\delta_{ap} \tag{5.9}$$

It may be helpful in writing this to repeat what we said in getting (5.8). In (5.9), the product $\epsilon_{bca}\epsilon_{bpq}$ is zero unless either $c, a = p, q$, or $c, a = q, p$, as indicated by the right side of (5.9). For practice, do Problem 7.

We can further contract (5.8) to get (Problem 8).

$$\begin{cases} \epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}, \\ \epsilon_{ijk}\epsilon_{ijk} = 6. \end{cases} \tag{5.10}$$

Vector Identities The familiar formulas in vector analysis can be written in tensor form using δ_{ij} and ϵ_{ijk}. [See Am. J. Phys. **34**, 503–507 (1966).] We have already commented (Section 3) that the dot product $\mathbf{A}\cdot\mathbf{B}$ is the contracted direct product, A_iB_i. Now let's show that the components of the cross product of two vectors can be written as

$$(\mathbf{B}\times\mathbf{C})_i = \epsilon_{ijk}B_jC_k. \tag{5.11}$$

To see that this is correct we look at one component at a time and compare the result with Chapter 3, equation (4.19), replacing x, y, z by 1, 2, 3. To find the first component of $\mathbf{B}\times\mathbf{C}$ in (5.11), we let $i = 1$. Then the only nonzero terms on the right side of (5.11) are the two with $j, k = 2, 3$ or $3, 2$, so we find that the first component of $\mathbf{B}\times\mathbf{C}$ is $(B_2C_3 - B_3C_2)$, in agreement with Chapter 3. Similarly the other components agree with the vector analysis definition of a cross product (Problem 9a).

Example 2. Now let's use (5.11) to write a triple vector product in tensor form, and then use (5.8) or (5.9) to simplify it (Problem 9b).

$$\begin{aligned} [\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_n &= \epsilon_{nip}A_i(\mathbf{B}\times\mathbf{C})_p = \epsilon_{nip}A_i[\epsilon_{pjk}B_jC_k] \\ &= \epsilon_{nip}\epsilon_{pjk}A_iB_jC_k = \epsilon_{pni}\epsilon_{pjk}A_iB_jC_k \\ &= (\delta_{nj}\delta_{ik} - \delta_{nk}\delta_{ij})A_iB_jC_k = B_n(A_iC_i) - C_n(A_iB_i) \\ &= \text{components of } \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \end{aligned} \tag{5.12}$$

We recognize the final step as the formula [Chapter 6, (3.8)] for the triple vector product; we have just derived it in tensor form. Similarly we can prove other vector formulas (see Problems 10 to 13).

Recall from Chapter 6 that we treated $\boldsymbol{\nabla}$ as if it were "almost" a vector. Here we can similarly treat it as a first rank tensor, always remembering that it is also a differential operator. The components of $\boldsymbol{\nabla}$ are $\partial/\partial x_i$, so as in (5.11) we write

$$(\boldsymbol{\nabla}\times\mathbf{V})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}V_k. \tag{5.13}$$

Then following the method of (5.12), we next find the components of curl curl $\mathbf{V}$ in tensor form [compare part (e) in the table at the end of Chapter 6].

$$[\nabla \times (\nabla \times \mathbf{V})]_n = \epsilon_{nip}\frac{\partial}{\partial x_i}(\nabla \times \mathbf{V})_p \tag{5.14}$$
$$= \epsilon_{nip}\frac{\partial}{\partial x_i}[\epsilon_{pjk}\frac{\partial}{\partial x_j}V_k]$$
$$= \epsilon_{pni}\epsilon_{pjk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}V_k$$
$$= (\delta_{nj}\delta_{ik} - \delta_{nk}\delta_{ij})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}V_k$$
$$= \frac{\partial}{\partial x_n}\left(\frac{\partial}{\partial x_i}V_i\right) - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}V_n$$
$$= \text{components of } \nabla(\nabla \cdot \mathbf{V}) - \nabla^2\mathbf{V}.$$

Dual tensors Let T_{ij} be an antisymmetric 2^{nd}-rank tensor, that is, $T_{ij} = -T_{ji}$. If we display the components T_{ij} as elements of a matrix, it looks like this (see Problem 14).

$$\mathrm{T} = \begin{pmatrix} 0 & T_{12} & -T_{31} \\ -T_{12} & 0 & T_{23} \\ T_{31} & -T_{23} & 0 \end{pmatrix} \tag{5.15}$$

Observe that there are just 3 independent nonzero components, just enough to be the components of a vector. (Note that this happens only in 3 dimensions—see Problem 15.) If we define

$$V_i = \frac{1}{2}\epsilon_{ijk}T_{jk} \tag{5.16}$$

then we find (Problem 16)

$$V_1 = T_{23}, \quad V_2 = T_{31}, \quad V_3 = T_{12}. \tag{5.17}$$

Since ϵ_{ijk} and T_{jk} are tensors and V_i is a contracted direct product of them, we are assured that V_i is a first rank tensor, that is, a vector (but see Section 6). Thus the three quantities in (5.17) can be considered as the three independent components of an antisymmetric 2^{nd}-rank tensor T_{ij}, or as the three components of a vector V_k called the *dual* of T_{ij}. We can also start with a vector V_k and define T_{ij} in terms of it (Problem 16).

$$T_{ij} = \epsilon_{ijk}V_k \tag{5.18}$$

Now suppose A_j and B_k are vectors. Then $T_{jk} = A_jB_k - A_kB_j$ is a 2^{nd}-rank antisymmetric tensor, and the three independent components of T_{jk} are just the components of $\mathbf{A} \times \mathbf{B}$ (Problem 17). Thus we see that the vector product can be considered as either a vector or a 2^{nd}-rank antisymmetric tensor.

▶ PROBLEMS, SECTION 5

1. Verify that (5.5) agrees with a Laplace development, say on the first row (Chapter 3, Section 3). *Hints:* You will find 6 terms corresponding to the 6 non-zero values of ϵ_{ijk}. First let $i = 1$; then j, k can be $2, 3$ or $3, 2$. These two terms give you a_{11} times its cofactor. Next let $i = 2$ with $j, k = 1, 3$ and $3, 1$, and show that you get a_{12} times its cofactor. Finally let $i = 3$. Watch all the signs carefully.

2. Verify for a few representative cases that (5.6) gives the same results as a Laplace development. First note that if $\alpha, \beta, \gamma = 1, 2, 3$, then (5.6) is just (5.5). Then try letting $\alpha, \beta, \gamma =$ an even permutation of $1, 2, 3$, and then try an odd permutation, to see that the signs work out correctly. Finally try a case when $\epsilon_{\alpha\beta\gamma} = 0$ (that is when two of the indices are equal) to see that the right hand side of (5.6) is zero because you are evaluating a determinant which has two identical rows.

3. Show that $\delta_{ij}\epsilon_{klm}$ is an isotropic tensor of rank 5. *Hint:* Combine equations (5.4) and (5.7).

4. Generalize Problem 3 to see that the direct product of any two isotropic tensors (or a direct product contracted) is an isotropic tensor. For example show that $\epsilon_{ijk}\epsilon_{lmn}$ is an isotropic tensor (what is its rank?) and $\epsilon_{ijk}\epsilon_{lmn}\delta_{jn}$ is an isotropic tensor (what is its rank?).

5. Let T_{jkmn} be the tensor in (5.8). This is a 4^{th}-rank tensor and so has $3^4 = 81$ components. Most of the components are zero. Find the nonzero components and their values. *Hint:* See discussion after (5.8).

6. Evaluate:

(a) $\delta_{ij}\delta_{jk}\delta_{km}\delta_{im}$ (b) $\epsilon_{ijk}\delta_{jk}$
(c) $\epsilon_{jk2}\epsilon_{k2j}$ (d) $\epsilon_{3jk}\epsilon_{kj3}$
(e) $\epsilon_{23i}\epsilon_{2i3}$ (f) $\epsilon_{k31}\epsilon_{3k1}$

7. Write in terms of δ's as in (5.8) and (5.9):

(a) $\epsilon_{ijk}\epsilon_{pjq}$ (b) $\epsilon_{abc}\epsilon_{pqc}$

8. Show that the equations (5.10) are correct. *Hints:* You can do these by further contracting (5.8). You can also do them by direct argument as follows: In the first equation, why must $k = n$? If $k = n$, then how many choices are there for i and j? In the second equation, in how many ways can you arrange the three numbers 1, 2, 3, and for each arrangement, what is the product of the ϵ's?

9. (a) Finish the work of showing that the cross product components are correctly given by (5.11). *Hints:* Follow the text discussion just after (5.11). For the second component, let $i = 2$; etc.

(b) Go through the sums in (5.12) carefully to verify each step. *Hints:* Use (5.11) twice being careful about repeated indices, and look at the discussion after equation (5.4).

(c) Similarly check (5.14).

10. (a) Write the triple scalar product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ in tensor form and show that it is equal to the determinant in Chapter 6, equation (3.2). *Hint:* See (5.5).

(b) Write equation (3.2) of Chapter 6 in tensor form to show the equivalence of the various expressions for the triple scalar product. *Hint:* Change the dummy indices as needed.

11. Using problem 10, write $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A})$ in tensor notation and show that it is $= 0$.

12. Write and prove in tensor notation:

(a) Chapter 6, Problem 3.13.

(b) Chapter 6, Problem 3.14.

(c) Lagrange's identity: $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$.

(d) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{ABD})\mathbf{C} - (\mathbf{ABC})\mathbf{D}$, where the symbol $(\mathbf{XYZ})$ means the triple scalar product of the three vectors.

13. Write in tensor notation and prove the following vector operator identities in the table at the end of Chapter 6: parts (b), (d), (f), (g), (h), (k).

14. Show that the diagonal elements of an antisymmetric tensor are zero and that (5.15) is a correct display of the components of an antisymmetric 2^{nd}-rank tensor in 3 dimensions.

15. Write a 4-by-4 antisymmetric matrix to show that there are 6 different components, not the 4 components of a vector in 4 dimensions.

16. Verify that (5.16) gives (5.17). Also verify that (5.18) gives (5.17).

17. Write out the components of $T_{jk} = A_j B_k - A_k B_j$ to show that T_{jk} is a 2^{nd}-rank antisymmetric tensor with elements which are the components of $\mathbf{A} \times \mathbf{B}$.

6. PSEUDOVECTORS AND PSEUDOTENSORS

So far we have considered only rotations of rectangular coordinate systems in our definitions of tensors. Recall that an orthogonal transformation includes both rotations and reflections (Chapter 3, Sections 7 and 11). Now we want to consider how the entities we have called tensors behave under reflections. Remember that the determinant of an orthogonal matrix is $+1$ for a rotation (sometimes called a "proper" rotation) and the determinant is -1 if a reflection is involved (sometimes called an "improper" rotation).

When $\det A = -1$, at least one eigenvalue of matrix A is -1 (see Chapter 3, Section 11). The -1 eigenvalue corresponds to the reversal of one principal axis, that is, a reflection through the plane perpendicular to the axis [for example a reflection through the (x, y) plane which reverses the z axis]. The other two eigenvalues correspond to a rotation [see Chapter 3, equation (7.19)]; this includes the case of a 180° rotation which is equivalent to reversal of the other two axes (see Problems 1 and 2). So in thinking about reflections, we can think of reversing all three axes (called an inversion) or reversing just one, since a rotation doesn't affect the sign of $\det A$. It is important to realize that reversing either one or all three axes changes the coordinate system from a right-handed to a left-handed coordinate system.

▶ **Example 1.** Let's look at a simple example of something we usually think of as a vector (namely a cross product) which doesn't obey the vector transformation laws under reflections. Let $\mathbf{U}$ and $\mathbf{V}$ be displacement vectors. Recall (Section 2) that, by definition, a vector transforms the way displacement vectors do. Also remember that we are considering passive transformations: vectors remain fixed in space while the axes are changed (rotated or reflected). Now if the z axis is reversed [reflected through the (x, y) plane], then the z components of the displacement vectors $\mathbf{U}$ and $\mathbf{V}$ change signs; this is then a requirement for all vectors. But the z component of $\mathbf{U} \times \mathbf{V}$ (which is $U_x V_y - U_y V_x$) does not change sign (Problems 3 and 4). Thus $\mathbf{U} \times \mathbf{V}$ is not a vector under reflections. We call $\mathbf{U} \times \mathbf{V}$ a *pseudovector*. We will discover other pseudovectors as we continue.

Levi-Civita symbols We want to use (5.6) when the matrix A is the matrix of an orthogonal transformation. Remember (Chapter 3, Section 7) that if A is orthogonal, $\det \mathrm{A} = \pm 1$ so $(\det \mathrm{A})^2 = 1$. Multiply (5.6) by det A to get the equation $\epsilon_{\alpha\beta\gamma} = (\det \mathrm{A}) a_{\alpha i} a_{\beta j} a_{\gamma k} \epsilon_{ijk}$. Then the transformation which gives $\epsilon' = \epsilon$ (see isotropic tensors in Section 5) is

$$\epsilon'_{\alpha\beta\gamma} = (\det \mathrm{A}) a_{\alpha i} a_{\beta j} a_{\gamma k} \epsilon_{ijk} = \epsilon_{\alpha\beta\gamma}. \tag{6.1}$$

Now this is not the right transformation equation for a 3rd-rank tensor—the factor det A would not be there for, say, the direct product of three displacement vectors. Of course, we got away with calling ϵ_{ijk} a 3rd-rank tensor in Section 5 because we were discussing just rotations and $\det \mathrm{A} = 1$ if A is a rotation matrix. But now we are dealing with general orthogonal transformations, and when $\det \mathrm{A} = -1$ (reflection) there is an extra factor -1 in the transformation equation. We call ϵ_{ijk} a 3rd-rank *pseudotensor*. A *pseudovector* or *pseudotensor* obeys the tensor transformation equations under rotations (that is, $\det \mathrm{A} = 1$), but if the transformation includes a reflection (that is, $\det \mathrm{A} = -1$), then the transformation equation contains an extra factor of -1. If we have a direct product of two pseudotensors (or such a product contracted), this will be a tensor because the product of the two det A factors is $(\det \mathrm{A})^2 = 1$. (Problem 5).

Polar and Axial Vectors If a vector (under rotations) also satisfies the vector transformation equations (that is, behaves like a displacement vector) under reflections, it is called a *polar vector* (or true vector or just a vector). If there is a change in sign when $\det \mathrm{A} = -1$, it is called an *axial vector* (or pseudovector). In Example 1, **U** and **V** were polar vectors and **U** × **V** was an axial vector.

In order to understand pseudotensors we need to discuss left-handed coordinate systems. These are relatively unfamiliar in elementary work and for good reason. When we define a cross product or specify a vector to represent a rotation, the right hand rule is a part of our definition. It would be confusing to deal with this in a left-handed system so you are always warned to use right-handed systems. But we are now considering the general case of orthogonal transformations which includes reflections and so produces left-handed reference systems which we must learn to cope with.

Let's consider the physics and geometry of this by comparing linear velocity and angular velocity, both vectors under rotations. Is there a difference when we consider reflections and so have a left-handed coordinate system? The linear velocity vector indicates a path along which something moves; it has a direct physical meaning, and under passive transformations, it stays fixed in space. In the case of angular velocity, the physical motion is taking place in the plane perpendicular to the angular velocity vector, say a wheel rotating, or a mass or charge moving in a circle. The angular velocity "vector" is something *we choose via the right hand rule* to represent the motion. We might guess (correctly) that linear velocity is a vector (polar vector) and angular velocity is a pseudovector (axial vector). Remember that in Example 1 we found that the cross product (defined using the right hand rule) is a pseudovector. As we continue, watch for this; when the right hand rule is used in the *definition* of a vector, you suspect that it is a pseudovector.

Cross Product In Example 1, we found that the cross product of two displacement vectors does not satisfy the vector transformation equations under reflections. Now we want to write a formula to show exactly how a cross product transforms under a general orthogonal transformation. By (5.11), we write

$$(\mathbf{U}\times\mathbf{V})_i = \epsilon_{ijk}U_jV_k. \tag{6.2}$$

Then using (6.1), (6.2), and the vector transformation equations for the displacement vectors **U** and **V**, we find

$$\begin{aligned}(\mathbf{U}'\times\mathbf{V}')_\alpha &= \epsilon'_{\alpha\beta\gamma}U'_\beta V'_\gamma \\ &= (\det \mathrm{A})a_{\alpha i}a_{\beta j}a_{\gamma k}\epsilon_{ijk}a_{\beta m}U_m a_{\gamma p}V_p \\ &= (\det \mathrm{A})a_{\alpha i}\delta_{jm}\delta_{kp}\epsilon_{ijk}U_mV_p \\ &= (\det \mathrm{A})a_{\alpha i}(\epsilon_{ijk}U_jV_k) = (\det \mathrm{A})a_{\alpha i}(\mathbf{U}\times\mathbf{V})_i.\end{aligned} \tag{6.3}$$

If $\det \mathrm{A} = 1$ (no reflection, just a rotation), then (6.3) is the transformation equation for a vector. If $\det \mathrm{A} = -1$ (reflection) then the transformation has an extra -1 factor. Thus the vector product of two polar vectors is a pseudovector, as we have seen before and as we guessed from the fact that the right hand rule is used in defining cross product.

▶ **Example 2.** Find the triple scalar product of 3 polar vectors.

Here we have one $\det \mathrm{A}$ factor (from the cross product), so the triple scalar product of 3 polar vectors is a pseudoscalar (Problem 7).

▶ **Example 3.** What is the tensor character of $\mathbf{W}\times\mathbf{S}$ if **W** is a polar vector and **S** is a pseudovector?

In the transformation equation for $\mathbf{W}\times\mathbf{S}$, there is one factor of $\det \mathrm{A}$ for **S**, and another $\det \mathrm{A}$ for the cross product as in (6.3). The two minus signs cancel, so $\mathbf{W}\times\mathbf{S}$ is a polar vector (Problem 8).

▶ **Example 4.** Show that acceleration **a** and force **F** are polar vectors.

By definition, the displacement **r** is a polar vector (we define vectors as quantities which transform the way displacements do). Then the velocity $\mathbf{v} = d\mathbf{r}/dt$ and the acceleration $\mathbf{a} = d^2\mathbf{r}/dt^2$ are vectors (since time t is a scalar) and $\mathbf{F} = m\mathbf{a}$ is a vector since m is a scalar.

▶ **Example 5.** Find the tensor character of each symbol in $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$.

By Example 4, **v** is a vector so $\boldsymbol{\omega}\times\mathbf{r}$ must be a vector (both sides of a tensor equation must have the same tensor character). Then $\boldsymbol{\omega}$ must be a pseudovector so that there are two $\det \mathrm{A}$ factors, one from the cross product and one from $\boldsymbol{\omega}$. Recall that we predicted this because the right hand rule is used in defining angular velocity.

PROBLEMS, SECTION 6

1. Show that in 2 dimensions (say the x, y plane), an inversion through the origin (that is, $x' = -x, y' = -y$) is equivalent to a $180°$ rotation of the (x, y) plane about the z axis. *Hint:* Compare Chapter 3, equation (7.13) with the negative unit matrix.

2. In Chapter 3, we said that any 3-by-3 orthogonal matrix with determinant $= -1$ can be written in the form (7.19). Use this and Problem 1 to show that in 3 dimensions, an inversion (that is a reflection through the origin so that all three axes are reversed) is equivalent to a reflection through a plane combined with a rotation about the line perpendicular to the plane [say a reflection through the (x, y) plane—that is, a reversal of the z axis—and a rotation of the (x, y) plane about the z axis]. *Hint:* Consider the matrix B in Chapter 3, (7.19).

3. For Example 1, write out the components of $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{U} \times \mathbf{V}$ in the original right-handed coordinate system S and in the left-handed coordinate system S' with the z axis reflected. Show that each component of $\mathbf{U} \times \mathbf{V}$ in S' has the "wrong" sign to obey the vector transformation laws.

4. Do Example 1 and Problem 3 if the transformation to a left-handed system is an inversion (see Problem 2).

5. Write the tensor transformation equations for $\epsilon_{ijk}\epsilon_{mnp}$ to show that this is a (rank 6) tensor (*not* a pseudotensor). *Hint:* Write (6.1) for each ϵ and multiply them, being careful not to re-use a pair of summation indices.

6. Write the transformation equations to show that $\boldsymbol{\nabla} \times \mathbf{V}$ is a pseudovector if $\mathbf{V}$ is a vector. *Hint:* See equations (5.13), (6.2) and (6.3).

7. Write the transformation equations for the triple scalar product $\mathbf{W} \cdot (\mathbf{U} \times \mathbf{V})$ remembering that now $\det \mathrm{A} = -1$ if the transformation involves a reflection. Thus show that the triple scalar product of three polar vectors is a pseudoscalar as claimed in Example 2. *Hint:* Use the result in (6.3).

8. Write the transformation equations for $\mathbf{W} \times \mathbf{S}$ to verify the results of Example 3.

In the physics formulas of Problems 9 to 14, identify each symbol as a vector (polar vector) or a pseudovector (axial vector). Use results from the text and the fact that both sides of an equation must have the same tensor character. The definition of the symbols used is: $\mathbf{r}$ = displacement, t = time, m = mass, q = electric charge, $\mathbf{v}$ = velocity, $\mathbf{F}$ = force, $\boldsymbol{\omega}$ = angular velocity, $\boldsymbol{\tau}$ = torque, $\mathbf{L}$ = angular momentum, T = kinetic energy, $\mathbf{E}$ = electric field, $\mathbf{B}$ = magnetic field. Assume that t, m, and q are scalars. Note that we are working in 3 dimensional physical space and assuming classical (that is nonrelativistic) physics.

9. $\mathbf{E} = \dfrac{\mathbf{F}}{q}$

10. $\mathbf{L} = m\mathbf{r} \times \mathbf{v} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$

11. $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

12. $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

13. $\dfrac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E}$

14. $T = \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})$

15. In equation (5.12), find whether $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is a vector or a pseudovector assuming

(a) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are all vectors;

(b) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are all pseudovectors;

(c) $\mathbf{A}$ is a vector and $\mathbf{B}$ and $\mathbf{C}$ are pseudovectors.

Hint: Count up the number of $\det \mathrm{A}$ factors from pseudovectors and cross products.

16. In equation (5.14), is $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{V})$ a vector or a pseudovector?

17. In equation (5.16), show that if T_{jk} is a tensor (that is, not a pseudotensor), then V_i is a pseudovector (axial vector). Also show that if T_{jk} is a pseudotensor, then V_i is a vector (true or polar vector). You know that if V_i is a cross product of polar vectors, then it is a pseudovector. Is its dual T_{jk} a tensor or a pseudotensor?

7. MORE ABOUT APPLICATIONS

Stress Tensor We started our discussion of tensors with a description of the stress tensor (you may want to review this in Section 1). Now let's show that the nine quantities P_{ij} displayed in the matrix (1.1) really are the components of a 2^{nd}-rank tensor. For simplicity in notation (and to use summation convention), we make the replacements indicated in (2.10); we also replace $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ by $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. Our problem is to write the components $P'_{\alpha\beta}$ relative to a rotated coordinate system in terms of the components P_{ij} to show that $P'_{\alpha\beta} = a_{\alpha i}a_{\beta j}P_{ij}$ as in (2.14) or (3.9).

Figure 7.1 shows the unprimed axes and one of the rotated axes. (With $\alpha = 1, 2, 3$, the x'_α axis represents any one of the rotated axes.) We draw a slanted plane, as shown, perpendicular to the x'_α axis, and consider the forces on the small volume element dV bounded by the unprimed coordinate planes and the slanted plane. Recall (Section 1) that pressure is force per unit area, so the force acting across a face is the pressure times the area of the face. Let the area of the slanted face (call it face α) be dS. Then the area of the face perpendicular to the x_i axis (call it face i) is $a_{\alpha i}\,dS$ where $a_{\alpha i}$ [see (2.10)] is the cosine of the angle between the x'_α and x_i axes (Problem 1).

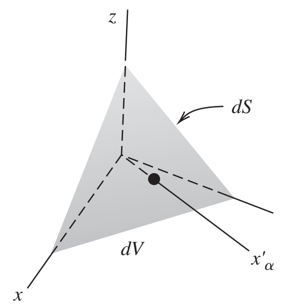


Figure 7.1

$$\text{Area of face } i \text{ is equal to } a_{\alpha i}\,dS. \tag{7.1}$$

The pressure across face i is $P_{ij}\mathbf{e}_j$ (note the sum on j and see Problem 2). Multiplying this by (7.1) (force = pressure times area of face) and summing on i, we find that the total force acting on the material in the volume element dV, across the three faces in the unprimed coordinate planes is

$$(P_{ij}\mathbf{e}_j)a_{\alpha i}\,dS. \tag{7.2}$$

For equilibrium, the sum of these three forces must be equal to the force acting across face α on the neighboring material. This force is

$$P'_{\alpha\beta}\mathbf{e}'_\beta\,dS \tag{7.3}$$

Setting (7.2) and (7.3) equal, taking the dot product of both sides with $\mathbf{e}'_\beta$, and canceling dS, we have (Problem 3)

$$P'_{\alpha\beta} = a_{\alpha i}a_{\beta j}P_{ij} \tag{7.4}$$

Thus we see that the stress P_{ij} is, as claimed, a 2^{nd}-rank tensor.

▶ **Example 1.** Suppose the following matrix is a display of the elements of a stress tensor.

$$\mathrm{P} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

We note that P is symmetric (this is true of stress tensors) so we can diagonalize P by an orthogonal transformation. In Chapter 3, Section 12, Example 2, we found that the eigenvalues of this matrix are 1, −4, 3. Thus a rotation of axes (matrix C in the Chapter 3 example) produces a stress tensor P′ with stress components only along the principal axes. The positive eigenvalues are tensions and the negative are compressions. Relative to the principal axes there are no shear forces.

Strain and Stress; Hooke's Law The strain tensor specifies the deformation of a solid body under stress. For a simple case such as a wire supporting a weight, strain (change in length per unit length) and stress (force per unit cross sectional area) are proportional (Hooke's Law). But for a 3 dimensional problem, stress is a 2^{nd}-rank tensor P_{ij} (as we have seen above), and strain is also a 2^{nd}-rank tensor S_{ij}. If the components of **P** are linear combinations of the components of **S**, then we can write

$$P_{ij} = C_{ijkm} S_{km} \tag{7.5}$$

By the quotient rule, C_{ijkm} is a 4^{th}-rank tensor (Problem 5). The components of C_{ijkm} depend on the kind of material under stress and are called the elastic constants of the material (see Problem 6).

Inertia Tensor Revisited In Section 4 we considered the inertia tensor using vector notation. Now let's look at it using the tensor form for vector identities that we discussed in Section 5.

Example 2. In (4.2) we had $\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$. Using (5.12) with $\mathbf{A} = \mathbf{C} = \mathbf{r}$ and $\mathbf{B} = \boldsymbol{\omega}$, we find

$$L_n = m[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_n = m(\delta_{nj}\delta_{ik} - \delta_{nk}\delta_{ij})x_i\omega_j x_k. \tag{7.6}$$

Now sum over i and k to get $\delta_{nj}\delta_{ik}x_i x_k = \delta_{nj}x_k x_k = \delta_{nj}r^2$ and $\delta_{nk}\delta_{ij}x_i x_k = x_j x_n$. Thus we have [compare (4.2)]

$$L_n = m(\delta_{nj}r^2 - x_n x_j)\omega_j. \tag{7.7}$$

The coefficient of ω_j is then the component I_{nj} of the inertia tensor.

$$\boxed{I_{nj} = m(\delta_{nj}\, r^2 - x_n x_j).} \tag{7.8}$$

We can easily verify that these components are the same as we found in Section 4. For example [compare (4.4)]:

$$I_{11} = m(r^2 - x_1^2), \quad I_{12} = -mx_1x_2, \quad I_{13} = -mx_1x_3, \tag{7.9}$$

and similarly for the other components (Problem 7).

Other Applications In your study of electric fields in matter, you will find the equation $\mathbf{P} = \chi\mathbf{E}$; this relates the electric field $\mathbf{E}$ applied to a dielectric and the resulting polarization $\mathbf{P}$ of the dielectric. For some materials it may be true that $\mathbf{P}$ and $\mathbf{E}$ are parallel vectors with $\chi =$ scalar, but for other materials $\mathbf{P}$ and $\mathbf{E}$ are not parallel. Now this should remind you of our work in Section 4 with the equation $\mathbf{L} = I\boldsymbol{\omega}$ when we realized that $\mathbf{L}$ and $\boldsymbol{\omega}$ are not always parallel. Just as we replaced the scalar I by a 2^{nd}-rank tensor, so we replace χ by a 2^{nd}-rank tensor. In the equation $P_i = \chi_{ij}E_j$, the quotient rule (see Section 3) tells us that χ_{ij} is a 2^{nd}-rank tensor. You will find other equations of this sort in various applications.

Tensor Fields Recall from Chapter 6 that a scalar field (temperature, for example) means a single number at each point, that is, a single function $f(x, y, z)$. A vector field (such as the electric field) means a set of three numbers at each point, that is, a set of three functions $V_i(x, y, z)$. Similarly, a 2^{nd}-rank tensor field means a set of 9 numbers at each point, that is, a set of 9 functions $T_{ij}(x, y, z)$. Think of our discussion of stress and strain. At every point in the material under stress, we can think of three vectors giving the force per unit area across the three perpendicular planes through the point, that is, a set of 9 functions. The 4^{th}-rank tensor C_{ijkm} in (7.5) is then a set of $3^4 = 81$ functions, and so on. (Of course, in order to be tensors these sets must transform properly under rotations as discussed in this chapter.)

▶ PROBLEMS, SECTION 7

1. Verify (7.1). *Hints:* In Figure 7.1, consider the projection of the slanted face of area dS onto the three unprimed coordinate planes. In each case, show that the projection angle is equal to an angle between the x'_α axis and one of the unprimed axes. Find the cosine of the angle from the matrix A in (2.10).

2. Write out the sums $P_{ij}\mathbf{e}_j$ for each value of i and compare the discussion of (1.1). *Hint:* For example, if $i = 2$ [or y in (1.1)], then the pressure across the face perpendicular to the x_2 axis is $P_{21}\mathbf{e}_1 + P_{22}\mathbf{e}_{22} + P_{23}\mathbf{e}_3$, or, in the notation of (1.1), $P_{yx}\mathbf{i} + P_{yy}\mathbf{j} + P_{yz}\mathbf{k}$.

3. Carry through the details of getting (7.4) from (7.2) and (7.3). *Hint:* You need the dot product of $\mathbf{e}'_\beta$ and $\mathbf{e}_j$. This is the cosine of an angle between two axes since each $\mathbf{e}$ is a unit vector. Identify the result from matrix A in (2.10).

4. Interpret the elements of the matrices in Chapter 3, Problems 11.18 to 11.21, as components of stress tensors. In each case diagonalize the matrix and so find the principal axes of the stress (along which the stress is pure tension or compression). Describe the stress relative to these axes. (See Example 1.)

5. Show by the quotient rule (Section 3) that C_{ijkm} in (7.5) is a 4^{th}-rank tensor.

6. If $\mathbf{P}$ and $\mathbf{S}$ are 2^{nd}-rank tensors, show that $9^2 = 81$ coefficients are needed to write each component of $\mathbf{P}$ as a linear combination of the components of $\mathbf{S}$. Show that $81 = 3^4$ is the number of components in a 4^{th}-rank tensor. If the components of the 4^{th}-rank tensor are C_{ijkm}, then equation (7.5) gives the components of $\mathbf{P}$ in terms of the components of $\mathbf{S}$. If $\mathbf{P}$ and $\mathbf{S}$ are both symmetric, show that we need only 36 different non-zero components in C_{ijkm}. *Hint:* Consider the number of different components in $\mathbf{P}$ and $\mathbf{S}$ when they are symmetric. *Comment:* The stress and strain tensors can both be shown to be symmetric. Further symmetry reduces the 36 components of $\mathbf{C}$ in (7.5) to 21 or less.

7. In (7.9) we have written the first row of elements in the inertia matrix. Write the formulas for the other 6 elements and compare with Section 4.

8. Do Problem 4.8 in tensor notation and compare the result with your solution of 4.8.

8. CURVILINEAR COORDINATES

Before we discuss non-Cartesian tensors we need to talk about some properties of curvilinear coordinate systems such as spherical or cylindrical coordinates. To make the discussion concrete, we shall illustrate the ideas involved by using two familiar coordinate systems—rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z). The elements of arc length in these two systems are given by

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \qquad \text{(rectangular coordinates)} \\ ds^2 &= dr^2 + r^2\,d\theta^2 + dz^2 \qquad \text{(cylindrical coordinates)} \end{aligned} \tag{8.1}$$

These expressions for ds are called *line elements*; they have much greater significance than just their use in computing arc lengths. First consider how we can find ds^2 for a given coordinate system. In the case of a well-known coordinate system, the answer may be obvious from the geometry. For example in polar coordinates in the plane we have (from Figure 8.1 and the Pythagorean theorem)

$$ds^2 = dr^2 + r^2\,d\theta^2. \tag{8.2}$$

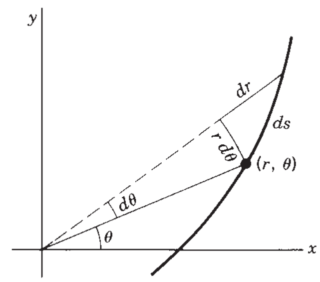


Figure 8.1

For an unfamiliar or complicated change of variables, however, we need a systematic method of finding ds; we illustrate the method by finding the value of ds^2 for cylindrical coordinates as given in (8.1). From the equations

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z, \end{aligned} \tag{8.3}$$

we get

$$\begin{aligned} dx &= \cos\theta\,dr - r\sin\theta\,d\theta, \\ dy &= \sin\theta\,dr + r\cos\theta\,d\theta, \\ dz &= dz. \end{aligned} \tag{8.4}$$

Squaring each equation in (8.4) and adding the results, we find

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\,d\theta^2 + dz^2. \tag{8.5}$$

Notice particularly here that all the cross products ($dr\,d\theta$, etc.) canceled out. This will not always happen, but it often does; when it does we call the coordinate system *orthogonal*. Such coordinate systems have some particularly simple and useful properties. Geometrically, an orthogonal system means that the *coordinate surfaces* are mutually perpendicular. For the cylindrical system (Figure 8.2), the coordinate surfaces are $r =$ const. (set of concentric cylinders), $\theta =$ const. (set of half-planes), and $z =$ const. (set of planes). The three coordinate surfaces through a given point intersect at right angles. The three curves of intersection of the coordinate surfaces in pairs intersect at right

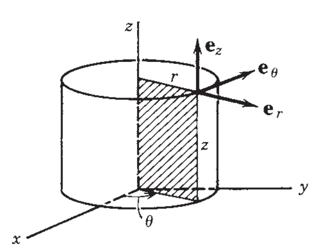


Figure 8.2

angles; these curves are called the *coordinate "lines"* or directions. We draw unit basis vectors tangent to the coordinate directions; for the cylindrical system (Figure 8.2) we might call them $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ ($\mathbf{e}_z$ is identical to $\mathbf{k}$). These unit vectors form an orthogonal triad like $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. We refer to such coordinate systems as *curvilinear coordinate systems* when the coordinate surfaces (or some of them) are not planes and the coordinate lines (or some of them) are curves rather than straight lines. We shall be principally interested in orthogonal curvilinear coordinate systems.

Scale Factors and Basis Vectors In the rectangular system, if x, y, z, are the coordinates of a particle, and x changes by dx with y and z constant, then the distance the particle moves is $ds = dx$. However, in the cylindrical system, if θ changes by $d\theta$ with r and z constant, the distance the particle moves is *not* $d\theta$, but $ds = r\,d\theta$. Factors like the r in $r\,d\theta$ which must multiply the differentials of the coordinates to get distances are known as *scale factors* and are very important as we shall see. A straightforward way to get them is to calculate ds^2 as we did in (8.5); if the transformation is orthogonal, then the scale factors can be read off from ds^2. (Note that the coefficients in ds^2 are the squares of the scale factors.) From (8.5), we see that the scale factors for cylindrical coordinates are 1, r, 1.

It is also useful to consider a vector $d\mathbf{s}$ which (in cylindrical coordinates) has components dr, $r\,d\theta$, dz in the coordinate directions $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$:

$$d\mathbf{s} = \mathbf{e}_r\,dr + \mathbf{e}_\theta\,r\,d\theta + \mathbf{e}_z\,dz. \tag{8.6}$$

Then $ds^2 = d\mathbf{s}\cdot d\mathbf{s}$ which gives (8.1), since the $\mathbf{e}$ vectors are orthonormal.

We can write the unit basis vectors of a curvilinear coordinate system ($\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ in cylindrical coordinates) in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. This is useful when we want to differentiate a vector which is expressed in terms of the curvilinear coordinate basis vectors. The unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are constant in magnitude *and direction*, but $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are not fixed in direction, so their derivatives are not zero. We illustrate an algebraic method of finding the relation between two sets of basis vectors by finding them for the cylindrical system. (Compare the geometrical method shown in Chapter 6, Section 4.)

▶ **Example 1.** Using (8.4) and collecting coefficients of dr, $d\theta$, and dz, we find

$$\begin{aligned} d\mathbf{s} &= \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz \\ &= \mathbf{i}(\cos\theta\,dr - r\sin\theta\,d\theta) + \mathbf{j}\,(\sin\theta\,dr + r\cos\theta\,d\theta) + \mathbf{k}\,dz \\ &= (\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)\,dr + (-\mathbf{i}r\sin\theta + \mathbf{j}\,r\cos\theta)\,d\theta + \mathbf{k}\,dz. \end{aligned} \tag{8.7}$$

Comparing (8.7) with (8.6), we have

$$\begin{aligned} \mathbf{e}_r &= \mathbf{i}\cos\theta + \mathbf{j}\sin\theta \\ r\mathbf{e}_\theta &= -\mathbf{i}r\sin\theta + \mathbf{j}\,r\cos\theta \\ \mathbf{e}_z &= \mathbf{k}. \end{aligned} \tag{8.8}$$

Notice that $\mathbf{e}_r$ is already a unit vector since $\sin^2\theta + \cos^2\theta = 1$, but $r\mathbf{e}_\theta$ must be divided by the scale factor r to get the unit vector $\mathbf{e}_\theta$. It is often convenient to use basis vectors which we shall call $\mathbf{a}_r$ and $\mathbf{a}_\theta$ (which are not necessarily of unit length), given by the coefficients of dr and $d\theta$ in (8.7). Then we just have to divide each $\mathbf{a}$ vector by its magnitude to get the corresponding $\mathbf{e}$ vector. Thus from (8.7)

$$\begin{aligned} &\mathbf{a}_r = \mathbf{e}_r \text{ is already a unit vector,} \\ (8.9)\qquad &\mathbf{a}_\theta = -\mathbf{i}r\sin\theta + \mathbf{j}\,r\cos\theta \text{ has magnitude } r, \text{ so} \\ &\mathbf{e}_\theta = \frac{1}{r}\mathbf{a}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta \end{aligned}$$

We can use these formulas to find the velocity and acceleration of a particle in cylindrical coordinates, and similar formulas for any coordinate system. The displacement of a particle from the origin at time t is, in cylindrical coordinates (Figure 8.3),

$$\mathbf{s} = r\mathbf{e}_r + z\mathbf{e}_z.$$

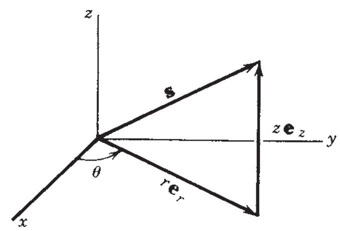


Figure 8.3

Then

$$\frac{d\mathbf{s}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d}{dt}(\mathbf{e}_r) + \frac{dz}{dt}\mathbf{e}_z.$$

By (8.8),

$$\frac{d}{dt}(\mathbf{e}_r) = -\mathbf{i}\sin\theta\frac{d\theta}{dt} + \mathbf{j}\cos\theta\frac{d\theta}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt},$$

so

$$(8.10)\qquad \frac{d\mathbf{s}}{dt} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z.$$

By differentiating again with respect to t and using (8.8) to find $(d/dt)(\mathbf{e}_\theta)$, we can find the acceleration $d^2\mathbf{s}/dt^2$ in cylindrical coordinates (Problem 2).

General Curvilinear Coordinates In general, let x_1, x_2, x_3 be the set of variables or coordinates we are considering (for example, in cylindrical coordinates, $x_1 = r$, $x_2 = \theta$, $x_3 = z$). Then the three sets of coordinate surfaces are $x_1 =$ const., $x_2 =$ const., $x_3 =$ const. The three coordinate surfaces through a given point intersect in three coordinate lines.

▶ **Example 2.** Given x, y, z as functions of x_1, x_2, x_3, we can find $d\mathbf{s}$ and the $\mathbf{a}$ vectors as we did for cylindrical coordinates in (8.7) and (8.9).

$$\begin{aligned} d\mathbf{s} &= \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz \\ (8.11)\qquad &= \mathbf{i}\frac{\partial x}{\partial x_n}\,dx_n + \mathbf{j}\frac{\partial y}{\partial x_n}\,dx_n + \mathbf{k}\frac{\partial z}{\partial x_n}\,dx_n \\ &= \mathbf{a}_1\,dx_1 + \mathbf{a}_2\,dx_2 + \mathbf{a}_3\,dx_3 = \mathbf{a}_n\,dx_n, \end{aligned}$$

where

$$(8.12)\qquad \mathbf{a}_n = \frac{\partial}{\partial x_n}\mathbf{s} = \mathbf{i}\frac{\partial x}{\partial x_n} + \mathbf{j}\frac{\partial y}{\partial x_n} + \mathbf{k}\frac{\partial z}{\partial x_n}.$$

Now defining $g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$, we can write $ds^2 = d\mathbf{s}\cdot d\mathbf{s}$ in matrix form as follows:

$$(8.13)\qquad ds^2 = \begin{pmatrix} dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix},$$

Note that g_{ij} is symmetric since the dot product of two vectors is the same in either order. In simpler form using summation convention (8.13) becomes

$$ds^2 = g_{ij}\, dx_i\, dx_j. \tag{8.14}$$

We will see later (Section 10) that the g_{ij} are the components of a tensor known as the *metric tensor*.

If the coordinate system is orthogonal, that is, if the basis vectors ($\mathbf{e}$ or $\mathbf{a}$) form an orthogonal triad, then $d\mathbf{s}$ and ds^2 can be written in terms of the scale factors as follows:

$$d\mathbf{s} = \mathbf{e}_1 h_1\, dx_1 + \mathbf{e}_2 h_2\, dx_2 + \mathbf{e}_3 h_3\, dx_3, \tag{8.15}$$

$$ds^2 = \begin{pmatrix} dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}. \tag{8.16}$$

Also note that the volume element in an orthogonal system is $h_1 h_2 h_3\, dx_1\, dx_2\, dx_3$ (volume of a small rectangular parallelepiped with edges $h_1\, dx_1$, $h_2\, dx_2$, $h_3\, dx_3$). For example, in cylindrical coordinates, the volume element is $dr \cdot r\, d\theta \cdot dz = r\, dr\, d\theta\, dz$.

▶ PROBLEMS, SECTION 8

1. Find ds^2 in spherical coordinates by the method used to obtain (8.5) for cylindrical coordinates. Use your result to find for spherical coordinates, the scale factors, the vector $d\mathbf{s}$, the volume element, the basis vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, $\mathbf{a}_\phi$ and the corresponding unit basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$. Write the g_{ij} matrix.

2. Observe that a simpler way to find the velocity $d\mathbf{s}/dt$ in (8.10) is to divide the vector $d\mathbf{s}$ in (8.6) by dt. Complete the problem to find the acceleration in cylindrical coordinates.

3. Use the results of Problem 1 to find the velocity and acceleration components in spherical coordinates. Find the velocity in two ways: starting with $d\mathbf{s}$ and starting with $\mathbf{s} = r\mathbf{e}_r$.

4. In the text and problems so far, we have found the $\mathbf{e}$ vectors for various coordinate systems in terms of $\mathbf{i}$ and $\mathbf{j}$ (or $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in three dimensions). We can solve these equations to find $\mathbf{i}$ and $\mathbf{j}$ in terms of the $\mathbf{e}$ vectors, and so express a vector given in rectangular form in terms of the basis vectors of another coordinate system. Carry out this process to express in cylindrical coordinates the vector $\mathbf{V} = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$. *Hint:* Use matrices (as in Chapter 3) to solve the set of equations for $\mathbf{i}$ and $\mathbf{j}$.

5. Using the results of Problem 1, express the vector in Problem 4 in spherical coordinates.

As in Problem 1, find ds^2, the scale factors, the vector $d\mathbf{s}$, the volume (or area) element, the $\mathbf{a}$ vectors, and the $\mathbf{e}$ vectors for each of the following coordinate systems.

6. Parabolic cylinder coordinates u, v, z:

$$x = \frac{1}{2}(u^2 - v^2),$$
$$y = uv,$$
$$z = z.$$

7. Elliptic cylinder coordinates u, v, z:

$$x = a \cosh u \cos v,$$
$$y = a \sinh u \sin v,$$
$$z = z.$$

8. Parabolic coordinates u, v, ϕ:

$$x = uv\cos\phi,\quad y = uv\sin\phi,\quad z = \frac{1}{2}(u^2 - v^2).$$

9. Bipolar coordinates u, v:

$$x = \frac{a\sinh u}{\cosh u + \cos v},\quad y = \frac{a\sin v}{\cosh u + \cos v}.$$

10. Sketch or computer plot the coordinate surfaces in Problems 6 to 9.

Using the expression you have found for $d\mathbf{s}$, and for the $\mathbf{e}$ vectors, find the velocity and acceleration components in the coordinate systems indicated.

11. Parabolic cylinder

12. Elliptic cylinder

13. Parabolic

14. Bipolar

15. Let $x = u + v$, $y = v$. Find $d\mathbf{s}$, the $\mathbf{a}$ vectors, and ds^2 for the u, v coordinate system and show that it is not an orthogonal system. *Hint:* Show that the $\mathbf{a}$ vectors are not orthogonal, and that ds^2 contains $du\,dv$ terms. Write the g_{ij} matrix and observe that it is symmetric but not diagonal. Sketch the lines $u =$ const. and $v =$ const. and observe that they are not perpendicular to each other.

9. VECTOR OPERATORS IN ORTHOGONAL CURVILINEAR COORDINATES

We have previously (Chapter 6, Sections 6 and 7) defined the gradient (∇u), the divergence ($\nabla \cdot \mathbf{V}$), the curl ($\nabla \times \mathbf{V}$), and the Laplacian ($\nabla^2 u$) in rectangular coordinates x, y, z. Since in many practical problems it is better to use some other coordinate system (cylindrical or spherical, for example), we need to see how to express the vector operators in terms of general orthogonal coordinates x_1, x_2, x_3. (We consider only orthogonal coordinate systems here; see Section 10 for the more general case.) We shall outline proofs of the formulas; some of the details of the proofs are left to the problems.

Gradient, ∇u. In Chapter 6, Section 6, we showed that the directional derivative du/ds in a given direction is the component of ∇u in that direction.

Example 1. In cylindrical coordinates, if we go in the r direction (θ and z constant), then by (8.5) $ds = dr$. Thus the r component of ∇u is du/ds when $ds = dr$, that is, $\partial u/\partial r$. Similarly, the θ component of ∇u is du/ds when $ds = r\,d\theta$, that is, $(1/r)(\partial u/\partial\theta)$. Thus ∇u in cylindrical coordinates is

$$\nabla u = \mathbf{e}_r\frac{\partial u}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial u}{\partial\theta} + \mathbf{e}_z\frac{\partial u}{\partial z}. \tag{9.1}$$

In general orthogonal coordinates x_1, x_2, x_3, the component of ∇u in the x_1 direction (x_2 and x_3 constant) is du/ds if $ds = h_1\,dx_1$ [from (8.11)]; that is, the component of ∇u in the direction $\mathbf{e}_1$ is $(1/h_1)(\partial u/\partial x_1)$. Similar formulas hold for the other components and we have

$$\begin{aligned}\nabla u &= \mathbf{e}_1\frac{1}{h_1}\frac{\partial u}{\partial x_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial u}{\partial x_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial u}{\partial x_3}\\ &= \sum_{i=1}^{3}\mathbf{e}_i\frac{1}{h_i}\frac{\partial u}{\partial x_i}.\end{aligned} \tag{9.2}$$

Divergence, $\nabla \cdot \mathbf{V}$ Let

$$\mathbf{V} = \mathbf{e}_1 V_1 + \mathbf{e}_2 V_2 + \mathbf{e}_3 V_3 \tag{9.3}$$

be a vector with components V_1, V_2, V_3 in an orthogonal system. We can prove (Problem 1) that

$$\nabla \cdot \left(\frac{\mathbf{e}_3}{h_1 h_2}\right) = 0, \qquad \nabla \cdot \left(\frac{\mathbf{e}_2}{h_1 h_3}\right) = 0, \qquad \nabla \cdot \left(\frac{\mathbf{e}_1}{h_2 h_3}\right) = 0. \tag{9.4}$$

Let us write (9.3) as

$$\mathbf{V} = \frac{\mathbf{e}_1}{h_2 h_3}(h_2 h_3 V_1) + \frac{\mathbf{e}_2}{h_1 h_3}(h_1 h_3 V_2) + \frac{\mathbf{e}_3}{h_1 h_2}(h_1 h_2 V_3). \tag{9.5}$$

We find $\nabla \cdot \mathbf{V}$ by taking the divergence of each term on the right side of (9.5). Using (7.6) of Chapter 6, namely

$$\nabla \cdot (\phi \mathbf{v}) = \mathbf{v} \cdot (\nabla \phi) + \phi \nabla \cdot \mathbf{v}, \tag{9.6}$$

with $\phi = h_2 h_3 V_1$ and $\mathbf{v} = \mathbf{e}_1 / h_2 h_3$, we find that the divergence of the first term on the right side of (9.5) is

$$\nabla \cdot \left(h_2 h_3 V_1 \frac{\mathbf{e}_1}{h_2 h_3}\right) = \frac{e_1}{h_2 h_3} \cdot \nabla (h_2 h_3 V_1) + h_2 h_3 V_1 \nabla \cdot \left(\frac{\mathbf{e}_1}{h_2 h_3}\right). \tag{9.7}$$

By (9.4), the last term in (9.7) is zero. In the first term on the right side of (9.7), the dot product of $\mathbf{e}_1$ with $\nabla (h_2 h_3 V_1)$ is the first component of $\nabla (h_2 h_3 V_1)$. By (9.2), this is

$$\frac{1}{h_1} \frac{\partial}{\partial x_1}(h_2 h_3 V_1).$$

Calculating the divergence of the other terms of (9.5) in a similar way, we get

$$\nabla \cdot \mathbf{V} = \frac{1}{h_2 h_3} \frac{1}{h_1} \frac{\partial}{\partial x_1}(h_2 h_3 V_1) + \frac{1}{h_1 h_3} \frac{1}{h_2} \frac{\partial}{\partial x_2}(h_1 h_3 V_2) + \frac{1}{h_1 h_2} \frac{1}{h_3} \frac{\partial}{\partial x_3}(h_1 h_2 V_3)$$

or

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1}(h_2 h_3 V_1) + \frac{\partial}{\partial x_2}(h_1 h_3 V_2) + \frac{\partial}{\partial x_3}(h_1 h_2 V_3)\right]. \tag{9.8}$$

▶ **Example 2.** In cylindrical coordinates, $h_1 = 1$, $h_2 = r$, $h_3 = 1$. By (9.8), the divergence in cylindrical coordinates is

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{1}{r} \left[\frac{\partial}{\partial r}(r V_r) + \frac{\partial}{\partial \theta}(V_\theta) + \frac{\partial}{\partial z}(r V_z)\right] \\ &= \frac{1}{r} \frac{\partial}{\partial r}(r V_r) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}. \end{aligned} \tag{9.9}$$

Laplacian, $\nabla^2 u$. Since $\nabla^2 u = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u$ we can find $\nabla^2 u$ by combining (9.2) and (9.8) with $\mathbf{V} = \boldsymbol{\nabla} u$. We get

(9.10)
$$\nabla^2 u = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right].$$

Example 3. In cylindrical coordinates, the Laplacian is then

$$\begin{aligned}\nabla^2 u &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}\right)\right]\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.\end{aligned}$$

Curl, $\boldsymbol{\nabla} \times \mathbf{V}$. By methods similar to those used in finding $\boldsymbol{\nabla} \cdot \mathbf{V}$ we can find $\boldsymbol{\nabla} \times \mathbf{V}$ (Problem 2). The result is

(9.11)
$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{V} &= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}\\ &= \frac{\mathbf{e}_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 V_3) - \frac{\partial}{\partial x_3}(h_2 V_2)\right]\\ &\quad + \frac{\mathbf{e}_2}{h_1 h_3}\left[\frac{\partial}{\partial x_3}(h_1 V_1) - \frac{\partial}{\partial x_1}(h_3 V_3)\right]\\ &\quad + \frac{\mathbf{e}_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 V_2) - \frac{\partial}{\partial x_2}(h_1 V_1)\right]\end{aligned}$$

Example 4. In cylindrical coordinates, we find

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{V} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}\\ &= \mathbf{e}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right) + \mathbf{e}_\theta\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \frac{1}{r}\mathbf{e}_z\left(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta}\right).\end{aligned}$$

PROBLEMS, SECTION 9

1. Prove (9.4) in the following way. Using (9.2) with $u = x_1$, show that $\boldsymbol{\nabla} x_1 = \mathbf{e}_1/h_1$. Similarly, show that $\boldsymbol{\nabla} x_2 = \mathbf{e}_2/h_2$ and $\boldsymbol{\nabla} x_3 = \mathbf{e}_3/h_3$. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in that order form a right-handed triad (so that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, etc.) and show that $\boldsymbol{\nabla} x_1 \times \boldsymbol{\nabla} x_2 = \mathbf{e}_3/(h_1 h_2)$. Take the divergence of this equation and, using the vector identities (h) and (b) in the table at the end of Chapter 6, show that $\boldsymbol{\nabla} \cdot (\mathbf{e}_3/h_1 h_2) = 0$. The other parts of (9.4) are proved similarly.

2. Derive the expression (9.11) for curl $\mathbf{V}$ in the following way. Show that $\boldsymbol{\nabla} x_1 = \mathbf{e}_1/h_1$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} x_1) = \boldsymbol{\nabla} \times (e_1/h_1) = 0$. Write $\mathbf{V}$ in the form

$$\mathbf{V} = \frac{\mathbf{e}_1}{h_1}(h_1V_1) + \frac{\mathbf{e}_2}{h_2}(h_2V_2) + \frac{\mathbf{e}_3}{h_3}(h_3V_3)$$

and use vector identities from Chapter 6 to complete the derivation.

3. Using cylindrical coordinates write the Lagrange equations for the motion of a particle acted on by a force $\mathbf{F} = -\boldsymbol{\nabla} V$, where V is the potential energy. Divide each Lagrange equation by the corresponding scale factor so that the components of $\mathbf{F}$ (that is, of $-\boldsymbol{\nabla} V$) appear in the equations. Thus write the equations as the component equations of $\mathbf{F} = m\mathbf{a}$, and so find the components of the acceleration $\mathbf{a}$. Compare the results with Problem 8.2.

4. Do Problem 3 in spherical coordinates; compare the results with Problem 8.3.

5. Write out $\boldsymbol{\nabla} U$, $\boldsymbol{\nabla} \cdot \mathbf{V}$, $\nabla^2 U$, and $\boldsymbol{\nabla} \times \mathbf{V}$ in spherical coordinates.

Do Problem 3 for the coordinate systems indicated in Problems 6 to 9. Compare the results with Problems 8.11 to 8.14.

6. Parabolic cylinder

7. Elliptic cylinder

8. Parabolic

9. Bipolar

Do Problem 5 for the coordinate systems indicated in Problems 10 to 13.

10. Parabolic cylinder

11. Elliptic cylinder

12. Parabolic

13. Bipolar

In each of the following coordinate systems, find the scale factors h_u and h_v; the basis vectors $\mathbf{e}_u$ and $\mathbf{e}_v$; the u and v Lagrange equations, and from them the acceleration components (see Problem 3).

14. $\begin{cases} x = u - v, \\ y = 2\sqrt{uv}. \end{cases}$

15. $\begin{cases} x = uv, \\ y = u\sqrt{1 - v^2}. \end{cases}$

Use equations (9.2), (9.8), and (9.11) to evaluate the following expressions

16. In cylindrical coordinates, $\boldsymbol{\nabla} \cdot \mathbf{e}_r$, $\boldsymbol{\nabla} \cdot \mathbf{e}_\theta$, $\boldsymbol{\nabla} \times \mathbf{e}_r$, $\boldsymbol{\nabla} \times \mathbf{e}_\theta$.

17. In spherical coordinates, $\boldsymbol{\nabla} \cdot \mathbf{e}_r$, $\boldsymbol{\nabla} \cdot \mathbf{e}_\theta$, $\boldsymbol{\nabla} \times \mathbf{e}_\theta$, $\boldsymbol{\nabla} \times \mathbf{e}_\phi$.

18. In cylindrical coordinates, $\boldsymbol{\nabla} \times \mathbf{k} \ln r$, $\boldsymbol{\nabla} \ln r$, $\boldsymbol{\nabla} \cdot (r\mathbf{e}_r + z\mathbf{e}_z)$.

19. In spherical coordinates, $\boldsymbol{\nabla} \times (r\mathbf{e}_\theta)$, $\boldsymbol{\nabla}(r \cos\theta)$, $\boldsymbol{\nabla} \cdot \mathbf{r}$.

20. In cylindrical coordinates, $\nabla^2 r$, $\nabla^2(1/r)$, $\nabla^2 \ln r$.

21. In spherical coordinates, $\nabla^2 r$, $\nabla^2(r^2)$, $\nabla^2(1/r^2)$, $\nabla^2 e^{ikr\cos\theta}$.

10. NON-CARTESIAN TENSORS

So far we have considered only the behavior of the rectangular components of tensors under orthogonal transformations. Now let's generalize this to include any change of variables.

Example 1. In spherical coordinates r, θ, ϕ,

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \tag{10.1}$$

This is not a linear transformation, and we cannot write equations like (2.4) to (2.9) for the relations between the *variables*. However, we *can* write such equations for the relations between the *differentials* of the variables. From (10.1), we find the differentials dx, dy, dz, in terms of dr, $d\theta$, $d\phi$:

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}. \tag{10.2}$$

Example 2. For general coordinates x_1, x_2, x_3, and x_1', x_2', x_3', if we are given the relations [like (10.1)] between the two sets of variables, we can write the relations between the two sets of differentials as follows:

$$\begin{pmatrix} dx_1' \\ dx_2' \\ dx_3' \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_1'}{\partial x_1} & \dfrac{\partial x_1'}{\partial x_2} & \dfrac{\partial x_1'}{\partial x_3} \\[2ex] \dfrac{\partial x_2'}{\partial x_1} & \dfrac{\partial x_2'}{\partial x_2} & \dfrac{\partial x_2'}{\partial x_3} \\[2ex] \dfrac{\partial x_3'}{\partial x_1} & \dfrac{\partial x_3'}{\partial x_2} & \dfrac{\partial x_3'}{\partial x_3} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}. \tag{10.3}$$

More simply, using index notation and summation convention, (10.3) becomes

$$dx_i' = \frac{\partial x_i'}{\partial x_j}\, dx_j. \tag{10.4}$$

Compare this with the transformation for the partial derivatives of a function u,

$$\frac{\partial u}{\partial x_i'} = \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'}\frac{\partial u}{\partial x_j}, \tag{10.5}$$

and compare both (10.4) and (10.5) with the transformation for a Cartesian vector

$$V_i' = a_{ij}V_j, \qquad \text{(Cartesian)}. \tag{10.6}$$

For Cartesian vectors you can easily verify that

$$\frac{\partial x_i'}{\partial x_j} = a_{ij} = \frac{\partial x_j}{\partial x_i'}, \qquad \text{(Cartesian)}, \tag{10.7}$$

since both the partial derivatives in (10.7) equal the cosine of the angle between the x_i' and the x_j axes (Problem 1). This is not true for general coordinate systems; for example, in (10.1), $\partial x/\partial\theta \neq \partial\theta/\partial x$ (see Problem 2). Thus in general we have two possible definitions of a vector, which become identical for Cartesian vectors.

Contravariant and Covariant Vectors By definition, $\mathbf{V}$ is a *contravariant vector* if its components transform like this:

$$V'_i = \frac{\partial x'_i}{\partial x_j} V_j, \qquad \text{(contravariant vector)}, \tag{10.8}$$

and $\mathbf{V}$ is a covariant vector if its components transform like this:

$$V'_i = \frac{\partial x_j}{\partial x'_i} V_j, \qquad \text{(covariant vector)}. \tag{10.9}$$

By comparing (10.4) and (10.8), we see that the differentials of the coordinates are the components of a contravariant vector. Similarly, by comparing (10.5) and (10.9), we see that the partial derivatives of a function are the components of a covariant vector.

Notation Before we define tensors in general, we need to discuss a few things about notation. It is customary to write the indices of contravariant vectors and tensors as superscripts rather than subscripts. Be careful not to confuse them with exponents! (You may find the mnemonic "low-co" useful; *lower* indices are *co*variant indices, so, of course, upper indices are contravariant indices.) In this notation, equation (10.8) for a contravariant vector becomes

$$V'^i = \frac{\partial x'_i}{\partial x_j} V^j, \qquad \text{(contravariant vector)}. \tag{10.10}$$

(In fact, to be strictly consistent, since the differentials are contravariant, we should write $\partial x'^i/\partial x^j$. For our purposes this seems unnecessary so we will leave the partial derivative notation as it is.) Also note that the summation convention now applies to a pair of indices, one upper and one lower. (An index in the denominator counts as a lower index and an index in the numerator counts as an upper index.) Note that this new rule about summation convention applies in (10.9) and (10.10) and watch for it in future formulas.

Components and basis vectors You may be wondering how the vectors you studied in vector analysis (Section 9 and Chapter 6) are related to covariant and contravariant vectors. Actually we should speak of covariant and contravariant components, but the former terminology is customary. Any vector has various sets of components relative to various sets of basis vectors. Let's discuss this for orthogonal coordinate systems where it is especially simple. Recall that in vector analysis, we use the unit basis vectors such as $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ or $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$; for example, the vectors $\mathbf{e}_i$ in Section 9 are all unit vectors. Then the components of a vector $\mathbf{V}$ in vector analysis are the projections $\mathbf{e}_i \cdot \mathbf{V}$ of the vector on the coordinate directions. To be able to refer to these components, let's call them the physical components (they have the right physical dimensions—see Problem 6). We would like to see the relation between the physical components and the covariant and contravariant components of a vector, and the relation between the unit basis vectors and the contravariant and covariant basis vectors.

Example 3. You have learned that, in polar coordinates, the (physical) components of $d\mathbf{s}$ are dr and $r\,d\theta$. Now (10.4) and (10.10) tell us that the contravariant components of $d\mathbf{s}$ are just dr and $d\theta$ (**not** $r\,d\theta$). Thus we may guess (correctly) that the contravariant components of a vector are the physical components divided by the scale factors. By considering the components of the gradient (Problem 4), you can show that the covariant components of a vector are the physical components multiplied by the scale factors.

Example 4. In polar coordinates we can write [see equation (8.9)]

$$d\mathbf{s} = \mathbf{e}_r\,dr + \mathbf{e}_\theta\,r\,d\theta = \mathbf{a}_r\,dr + \mathbf{a}_\theta\,d\theta. \tag{10.11}$$

We have written $d\mathbf{s}$ in terms of its physical components and the unit $\mathbf{e}_i$ vectors, and in terms of its contravariant components and the covariant $\mathbf{a}_i$ basis vectors. From (10.11) and from Section 8 we can see that the $\mathbf{a}_i$ basis vectors are the $\mathbf{e}_i$ unit vectors multiplied by the scale factors. Note that the components and the basis vectors used with them vary in opposite ways so that the scale factors cancel. Similarly we can write a vector in terms of its covariant components and the contravariant basis vectors $\mathbf{a}^i$ which are the unit vectors divided by the scale factors (Problem 5). Note carefully that what we have just said applies only to orthogonal coordinate systems. If a coordinate system is not orthogonal, then $\mathbf{a}_i$ and $\mathbf{a}^i$ are not in general parallel; see the discussion just after (10.19).

Definition of Tensors Tensors may be covariant of any rank, contravariant of any rank, or mixed. Here are some sample tensor definitions; you should be able to write the corresponding definitions for tensors of any rank or kind in a similar way (Problem 7).

$$\begin{aligned}
T'_{ij} &= \frac{\partial x_k}{\partial x'_i}\frac{\partial x_l}{\partial x'_j}T_{kl} && (2^{\text{nd}}\text{-rank covariant tensor}),\\
T'^{ijk} &= \frac{\partial x'_i}{\partial x_l}\frac{\partial x'_j}{\partial x_m}\frac{\partial x'_k}{\partial x_n}T^{lmn} && (3^{\text{rd}}\text{-rank contravariant tensor}),\\
T'^{ij}_{\ \ k} &= \frac{\partial x'_i}{\partial x_l}\frac{\partial x'_j}{\partial x_m}\frac{\partial x_n}{\partial x'_k}T^{lm}_{\ \ n} && (3^{\text{rd}}\text{-rank mixed tensor, one covariant and two contravariant indices}).
\end{aligned} \tag{10.12}$$

Kronecker delta We showed in Section 5 that δ_{ij} is a 2^{nd}-rank isotropic Cartesian tensor. In a general coordinate system, the 2^{nd}-rank tensor which is equal to 1 if $i = j$ and 0 otherwise in all coordinate systems, is a mixed tensor so we write it as δ^i_j. To show that this is correct we write the tensor transformation equation for δ^k_l to see that we get δ'^i_j.

$$\frac{\partial x'_i}{\partial x_k}\frac{\partial x_l}{\partial x'_j}\delta^k_l = \frac{\partial x'_i}{\partial x_k}\frac{\partial x_k}{\partial x'_j} = \frac{\partial x'_i}{\partial x'_j} = \delta'^i_j. \tag{10.13}$$

Thus we see that δ^i_j is an isotropic 2^{nd}-rank tensor in general coordinate systems.

Quotient Rule In Section 3, we discussed the quotient rule for Cartesian tensors. A similar rule applies in general. To give proofs, we must replace the a_{ij} by the appropriate partial derivatives, noting carefully that summation convention now applies to a sum over one lower and one upper index.

▶ **Example 5.** If we are given $T_{ij}V^j = U_i$ where $\mathbf{V}$ is an arbitrary contravariant vector and $\mathbf{U}$ is a non-zero covariant vector, we want to show that T_{ij} is a 2^{nd}-rank covariant tensor. We write [compare equations (3.6) to (3.9)]

$$T'_{\alpha\beta}V'^{\beta} = U'_{\alpha} = \frac{\partial x_i}{\partial x'_\alpha}U_i = \frac{\partial x_i}{\partial x'_\alpha}T_{ij}V^j = \frac{\partial x_i}{\partial x'_\alpha}T_{ij}\frac{\partial x_j}{\partial x'_\beta}V'^{\beta}. \tag{10.14}$$

Set the first and last steps equal; then since V'^{β} is arbitrary, its coefficient $= 0$ and we have

$$T'_{\alpha\beta} = \frac{\partial x_i}{\partial x'_\alpha}T_{ij}\frac{\partial x_j}{\partial x'_\beta}$$

which is the transformation equation for a 2^{nd}-rank covariant tensor.

Metric Tensor; Raising and Lowering Indices

▶ **Example 6.** From (8.14) we have (with the contravariant dx indices now written as superscripts)

$$ds^2 = g_{ij}\,dx^i\,dx^j. \tag{10.15}$$

Since ds^2 is a scalar, and each dx is a contravariant vector, it follows by the quotient rule (Problem 8) that g_{ij} is a 2^{nd}-rank covariant tensor. It is known as the *metric tensor*. If the elements of g_{ij} are written as a matrix [see (8.13)], then we define g^{ij} as the elements of the inverse matrix. We can interpret $g_{ij}g^{jk}$ as either the contracted direct product of two tensors, or as the row times column product of two matrices which are inverses of each other, that is, a unit matrix. Thus we can write

$$g_{ij}g^{jk} = \delta_i^k. \tag{10.16}$$

Then by (10.13) and the quotient rule, g^{ij} is a 2^{nd}-rank contravariant tensor.

▶ **Example 7.** If V^i is a contravariant vector then $V_i = g_{ij}V^j$ is a covariant vector (Problem 10). We can also show that $g^{ij}V_j$ gives back the V^i we started with:

$$g^{ij}V_j = g^{ij}g_{jk}V^k = \delta^i_k V^k = V^i. \tag{10.17}$$

This process of finding the contracted product of a vector (or tensor) with g^{ij} or g_{ij} is called *raising* or *lowering indices*. The vectors V^i and V_i are called the contravariant and covariant components of the vector $\mathbf{V}$.

In equation (8.12), we defined the covariant basis vectors $\mathbf{a}_i$ which we use with contravariant components to write a vector [see (8.11) for example, remembering that the differentials are the contravariant components of $d\mathbf{s}$]. The contravariant basis vectors to use with covariant components are given by $\mathbf{a}^i = g^{ij}\mathbf{a}_j$. We can then write a vector in two ways (Problem 11):

$$\mathbf{V} = \mathbf{a}_i V^i = \mathbf{a}^i V_i, \quad \text{where} \quad \begin{cases} V_i = g_{ij}V^j, & V^i = g^{ij}V_j, \\ \mathbf{a}^i = g^{ij}\mathbf{a}_j, & \mathbf{a}_i = g_{ij}\mathbf{a}^j. \end{cases} \tag{10.18}$$

It is interesting to consider the directions of the vectors $\mathbf{a}_i$ and $\mathbf{a}^i$. We have defined $\mathbf{a}^i = g^{ij}\mathbf{a}_j$ but you can show (Problem 12) that $\mathbf{a}^i = \boldsymbol{\nabla} x_i$. Thus we have

$$\begin{aligned} \mathbf{a}_i &= \frac{\partial}{\partial x_i}\mathbf{s} = \mathbf{i}\frac{\partial x}{\partial x_i} + \mathbf{j}\frac{\partial y}{\partial x_i} + \mathbf{k}\frac{\partial z}{\partial x_i}, \\ \mathbf{a}^i &= g^{ij}\mathbf{a}_j = \boldsymbol{\nabla} x_i = \mathbf{i}\frac{\partial x_i}{\partial x} + \mathbf{j}\frac{\partial x_i}{\partial y} + \mathbf{k}\frac{\partial x_i}{\partial z}. \end{aligned} \tag{10.19}$$

We see from the displacement vector $d\mathbf{s} = \mathbf{a}_i\, dx^i$ that the basis vectors $\mathbf{a}_i$ are tangent to the coordinate lines. The vectors $\mathbf{a}^i = \boldsymbol{\nabla} x_i$ are orthogonal to the coordinate surfaces $x_i = \text{const}$. (Recall that $\operatorname{grad} u$ is orthogonal to $u = \text{const}$.) For orthogonal coordinates, $\mathbf{a}_i$ and $\mathbf{a}^i$ are in the same direction. (For example, in spherical coordinates, $\mathbf{a}_r$ points in the radial direction, and $\mathbf{a}^r$ is orthogonal to the sphere $r = \text{const.}$; these are the same direction.) Thus for orthogonal coordinates, if we normalize each $\mathbf{a}^i$, we get the same set of unit basis vectors that we get if we normalize each $\mathbf{a}_i$. However, if the coordinate system is not orthogonal, then at each point we have two different sets of basis vectors $\mathbf{a}_i$ and $\mathbf{a}^i$ (see Problems 16 and 17).

Just as we did for vectors, any tensor, say T^i_{jk}, can be written in various different forms by raising and lowering indices to get T_{ijk}, T^{ijk}, T^{ij}_k. These tensors are called *associated tensors*. They really all represent the same tensor T, with components relative to various bases.

Orthogonal coordinate systems For orthogonal coordinate systems, formulas involving g_{ij} can be written in terms of the scale factors h_1, h_2, h_3 [compare (8.13) and (8.16)]. Remember that the g^{ij} matrix is the inverse of the g_{ij} matrix [see equation (10.16)]. Also let g represent the determinant of the g_{ij} matrix. Then you can show (Problem 13).

$$\begin{aligned} g_{ij} &= \begin{cases} 0, & i \neq j, \\ h_i^2, & i = j, \end{cases} \qquad g^{ij} = \begin{cases} 0, & i \neq j, \\ \dfrac{1}{h_i^2}, & i = j, \end{cases} \\ g &= h_1^2 h_2^2 h_3^2, \qquad \sqrt{g} = h_1 h_2 h_3 \end{aligned} \tag{10.20}$$

Vector Operators in Tensor Notation We state without proof the following tensor expressions for $\boldsymbol{\nabla} u$, $\boldsymbol{\nabla}\cdot\mathbf{V}$, and $\nabla^2 u$. They are correct for any coordinate system, orthogonal or not. Using (10.20), you can specialize them to orthogonal coordinate systems and so obtain the expressions given in Section 9. (Problems 14 and 15).

$$\text{The covariant components of } \boldsymbol{\nabla} u \text{ are } \frac{\partial u}{\partial x_i}. \tag{10.21}$$

$$\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}(\sqrt{g}\, V^i), \quad \text{where } V^i \text{ are contravariant components of } \mathbf{V}. \tag{10.22}$$

$$\nabla^2 u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij}\frac{\partial u}{\partial x_j}\right). \tag{10.23}$$

▶ PROBLEMS, SECTION 10

1. Verify equation (10.7). *Hint:* Use equations (2.4) to (2.6) and (2.10). For example, $\partial y'/\partial z = \partial z/\partial y' = n_2 = a_{23}$.

2. From (10.1) find $\partial\theta/\partial x = (1/r)\cos\theta\cos\phi$ and show that $\partial x/\partial\theta \neq \partial\theta/\partial x$. Note carefully that $\partial x/\partial\theta$ means that r and ϕ are constant, but $\partial\theta/\partial x$ means that y and z are constant. (See Chapter 4, Example 7.6 for further discussion.)

3. Divide equation (10.4) by dt to show that the velocity $\mathbf{v} = d\mathbf{s}/dt$ is a contravariant vector. Note that the contravariant components of the velocity in polar coordinates are $\dot{r}$ and $\dot{\theta}$ (*not* $\dot{r}$ and $r\dot{\theta}$ which are physical components). As we did in (10.11), write the velocity $\mathbf{v}$ in polar coordinates in terms of the unit $\mathbf{e}$ vectors and in terms of the covariant $\mathbf{a}$ vectors. Repeat the problem in spherical coordinates.

4. What are the physical components of the gradient in polar coordinates? [See (9.1)]. The partial derivatives in (10.5) are the covariant components of ∇u. What relation do you deduce between physical and covariant components? Answer the same questions for spherical coordinates, and for an orthogonal coordinate system with scale factors h_1, h_2, h_3.

5. Write ∇u in polar coordinates in terms of its physical components and the unit basis vectors $\mathbf{e}_i$, and in terms of its covariant components and the contravariant basis vectors $\mathbf{a}^i$. What is the relation between the contravariant basis vectors and the unit basis vectors? *Hint:* Compare equation (10.11) and our discussion of it.

6. Show that, in polar coordinates, the θ contravariant component of $d\mathbf{s}$ is $d\theta$ which is unitless, the θ physical component of $d\mathbf{s}$ is $r\,d\theta$ which has units of length, and the θ covariant component of $d\mathbf{s}$ is $r^2\,d\theta$ which has units (length)^2.

7. As in (10.12), write the transformation equations for the following tensors: 2^{nd}-rank contravariant, 3^{rd}-rank covariant, 4^{th}-rank mixed with 2 covariant and 2 contravariant indices.

8. Using (10.15) show that g_{ij} is a 2^{nd}-rank covariant tensor. *Hint:* Write the transformation equation for each dx, and set the scalar $ds'^2 = ds^2$ to find the transformation equation for g_{ij}.

9. If U^i is a contravariant vector and V_j is a covariant vector, show that U^iV_j is a 2^{nd}-rank mixed tensor. *Hint:* Write the transformation equations for $\mathbf{U}$ and $\mathbf{V}$ and multiply them.

10. Show that if V^i is a contravariant vector then $V_i = g_{ij}V^j$ is a covariant vector, and that if V_i is a covariant vector, then $V^i = g^{ij}V_j$ is a contravariant vector.

11. In (10.18), show by raising and lowering indices that $\mathbf{a}_iV^i = \mathbf{a}^iV_i$. Also write (10.18) for an orthogonal coordinate system with g_{ij} and g^{ij} written in terms of the scale factors.

12. Show that in a general coordinate system with variables x_1, x_2, x_3, the contravariant basis vectors are given by

$$\mathbf{a}^i = \nabla x_i = \mathbf{i}\frac{\partial x_i}{\partial x} + \mathbf{j}\frac{\partial x_i}{\partial y} + \mathbf{k}\frac{\partial x_i}{\partial z}.$$

Hint: Write the gradient in terms of its covariant components and the $\mathbf{a}^i$ basis vectors to get $\nabla u = \mathbf{a}^j\partial u/\partial x_j$ and let $u = x_i$.

13. Verify (10.20).

14. Using equations (10.20) to (10.23), write the gradient, divergence, and Laplacian in cylindrical coordinates and in spherical coordinates. Change covariant or contravariant components to physical components and compare with the formulas stated in Chapter 6, Sections 6 and 7.

15. Do Problem 14 for an orthogonal coordinate system with scale factors h_1, h_2, h_3, and compare with the Section 9 formulas.

16. Continue Problem 8.15 to find the g^{ij} matrix and the contravariant basis vectors. Check your result by solving the given equations for u and v in terms of x and y, and finding the contravariant basis vectors using Problem 12. On your Problem 8.15 sketches of the lines $u =$ const. and $v =$ const., also sketch the covariant and contravariant basis vectors. Observe that the covariant basis vectors lie along the lines $u =$ const. and $v =$ const. and the contravariant basis vectors lie along the normals to these lines.

17. Repeat Problems 8.15 and 10.16 above for the (u, v) coordinate system if $x = 2u - v$, $y = u - 2v$.

18. Using (10.19), show that $\mathbf{a}^i \cdot \mathbf{a}_i = \delta^i_j$.

11. MISCELLANEOUS PROBLEMS

1. Show that the transformation equation for a 2^{nd}-rank Cartesian tensor is equivalent to a similarity transformation. *Warning hint:* Note that the matrix C in Chapter 3, Section 11, is the inverse of the matrix A we are using in Chapter 10 (compare $\mathbf{r}' = \mathrm{A}\mathbf{r}$ and $\mathbf{r} = \mathrm{C}\mathbf{r}'$). Thus a similarity transformation of the matrix T with tensor components T_{ij} is $\mathrm{T}' = \mathrm{ATA}^{-1}$. Also see "Tensors and Matrices" in Section 3 and remember that A is orthogonal.

2. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be a set of orthogonal unit vectors forming a right-handed system if taken in cyclic order. Show that the triple scalar product $\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \epsilon_{ijk}$.

3. In Chapter 3, Problem 6.6, you are asked to prove some identities among the Pauli spin matrices (called A, B, C, in that problem). Call the Pauli spin matrices σ_1, σ_2, σ_3; then show that the identities can be written in the following summation forms:

$$\sigma_k \sigma_m = i\epsilon_{kmn}\sigma_n + \delta_{km};$$
$$\sigma_k \sigma_m \epsilon_{kmn} = 2i\sigma_n.$$

4. If $\mathbf{E} =$ electric field and $\mathbf{B} =$ magnetic field, is $\mathbf{E} \times \mathbf{B}$ a vector or a pseudovector? *Comment*: $\mathbf{E} \times \mathbf{B}/\mu_0$ is called the Poynting vector; it points in the direction of transfer of energy. Does that tell you from the physics whether it is a vector or a pseudovector?

Do Problems 5 to 8 for the (u, v) coordinate system if $x = u(1 - v)$, $y = u\sqrt{2v - v^2}$.

5. Find ds^2, the scale factors, the area element, the vector $d\mathbf{s}$, the unit basis vectors, and the covariant and contravariant basis vectors.

6. Use Lagrange's equations to find the u and v acceleration components.

7. Write $\boldsymbol{\nabla} U$, $\boldsymbol{\nabla} \cdot \mathbf{V}$, and $\nabla^2 U$.

8. Evaluate $\boldsymbol{\nabla} \cdot \mathbf{e}_u$, $\boldsymbol{\nabla} \times \mathbf{e}_v$, $\nabla^2 \ln u$.

9. If $\mathbf{u}$ is a vector specifying the displacement under stress of each point of a deformable medium, then $\boldsymbol{\nabla}\mathbf{u}$ is a 2^{nd}-rank Cartesian tensor (see Problem 11) which describes the strain at each point. Display the components of $\boldsymbol{\nabla}\mathbf{u}$ as a matrix. Write $\boldsymbol{\nabla}\mathbf{u}$ as the sum of a symmetric tensor and an antisymmetric tensor [see (3.5)]. *Comment*: The symmetric part of $\boldsymbol{\nabla}\mathbf{u}$ is called the *stress tensor* and the antisymmetric part the *rotation tensor*.

10. Show that elements R_{ij} of a rotation matrix are the elements of a Cartesian tensor. *Hints:* Could you use the quotient rule? Could you use Problem 1?

11. Show that the nine quantities $T_{ij} = \partial V_i/\partial x_j$ (which are the Cartesian components of $\nabla\mathbf{V}$ where $\mathbf{V}$ is a vector) satisfy the transformation equations (2.14) for a Cartesian 2^{nd}-rank tensor. Show that they do not satisfy the general tensor transformation equations as in (10.12). *Hint:* Differentiate (10.9) or (10.10) partially with respect to, say, x'_k. You should get the expected terms [as in (10.12)] plus some extra terms; these extraneous terms show that $\partial V_i/\partial x_j$ is not a tensor under general transformations. *Comment:* It is possible to express the components of $\nabla\mathbf{V}$ correctly in general coordinate systems by taking into account the variation of the basis vectors in length and direction.

12. The square matrix in equation (10.3) is called the Jacobian matrix J; the determinant of this matrix is the Jacobian $J = \det \mathrm{J}$ which we used in Chapter 5, Section 4 to find volume elements in multiple integrals. (Note that as in Chapter 3, J represents a matrix; J in italics is its determinant.) For the transformation to spherical coordinates in (10.1) and (10.2) show that $J = \det \mathrm{J} = r^2 \sin\theta$. Recall that the spherical coordinate volume element is $r^2 \sin\theta \, dr \, d\theta \, d\phi$. *Hint:* Find $\mathrm{J}^{\mathrm{T}}\mathrm{J}$ and note that $\det(\mathrm{J}^{\mathrm{T}}\mathrm{J}) = (\det \mathrm{J})^2$.

13. In equation (10.13), let the x' variables be rectangular coordinates x, y, z, and let x_1, x_2, x_3, be general curvilinear coordinates, orthogonal or not (see end of Section 8). Show that $\mathrm{J}^{\mathrm{T}}\mathrm{J}$ is the g_{ij} matrix in (8.13) [or in (8.16) for an orthogonal system]. Thus show that the volume element in a general coordinate system is $dV = \sqrt{g}\, dx_1 \, dx_2 \, dx_3$ where $g = \det(g_{ij})$, and that for an orthogonal system, this becomes [by (8.16) or (10.19)], $dV = h_1 h_2 h_3 \, dx_1 \, dx_2 \, dx_3$. *Hint:* To evaluate the products of partial derivatives in $\mathrm{J}^{\mathrm{T}}\mathrm{J}$, observe that the same expressions arise as in finding ds^2. In fact, from (8.11) and (8.12), you can show that row i times column j in $\mathrm{J}^{\mathrm{T}}\mathrm{J}$ is just $\mathbf{a}_i \cdot \mathbf{a}_j = g_{ij}$ in equations (8.11) to (8.14).

CHAPTER **11**

Special Functions

1. INTRODUCTION

The integrals and series and functions of this chapter arise in a variety of physical problems. Just as you learn about trigonometric functions, logarithms, etc., and use them in applied problems, so you should learn something about these special functions so that you can use them and understand their use as they come up in your more advanced work. An enormous amount of detail is known about these functions, and numerous formulas involving them exist and can be looked up in books or found in your computer program. Our purpose is not to study them intensively, but to give definitions and some of the simpler relations and show their use. This should develop your ability and confidence to cope with more complicated formulas and many other similar functions and relations that may crop up occasionally in texts or computer results.

Now you may be thinking that your computer will give you the answers for definite integrals and functions so you really don't need to bother with this chapter. If all you want is a numerical approximation, this may be true. However, in theoretical work, you often need an exact expression (say in terms of π or $\sqrt{3}$ or $\ln 2$) and your computer may not give you the form you need.

Example 1. Suppose you want $\int_0^{\pi/2} d\theta/\sqrt{\cos\theta}$. One computer program gives you the result $\sqrt{2}\,K(1/\sqrt{2})$ and another gives you $2\sqrt{\pi}\,\Gamma(5/4)/\Gamma(3/4)$. In books you find answers $\frac{1}{2}B(1/4, 1/2)$ and $[\Gamma(1/4)]^2/\sqrt{8\pi}$. What's going on here and which is right? They all are! And when you have studied the formulas in this chapter, you will be able to show this (Problem 12.21) just as you now recognize that $\sin^2\theta = 1 - \cos^2\theta$.

Also in some problems you may want an algebraic approximation for a complicated expression rather than a numerical answer.

Example 2. You will find in thermal physics the approximation $\ln N! \cong N\ln N - N$; we will discuss this approximation and its accuracy. (See Problem 11.3).

2. THE FACTORIAL FUNCTION

Let us calculate the values of some integrals. For $\alpha > 0$,

$$\int_0^\infty e^{-\alpha x} dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}. \tag{2.1}$$

Next differentiate both sides of this equation repeatedly with respect to α (see Chapter 4, Section 12):

$$\begin{aligned}
&\int_0^\infty -xe^{-\alpha x}\,dx = -\frac{1}{\alpha^2} \quad \text{or} \quad \int_0^\infty xe^{-\alpha x}\,dx = \frac{1}{\alpha^2},\\
&\int_0^\infty x^2e^{-\alpha x}\,dx = \frac{2}{\alpha^3},\\
&\int_0^\infty x^3e^{-\alpha x}\,dx = \frac{3!}{\alpha^4}.
\end{aligned}$$

or in general

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}. \tag{2.2}$$

Putting $\alpha = 1$, we get

$$\int_0^\infty x^n e^{-x}\,dx = n!, \quad n = 1, 2, 3, \cdots. \tag{2.3}$$

Thus we have a definite integral whose value is $n!$ for positive integral n. We can use (2.3) to give a meaning to $0!$. Putting $n = 0$ in (2.3), we get

$$0! = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1. \tag{2.4}$$

(This agrees with our previous definition of $0!$ in Chapter 1.)

PROBLEMS, SECTION 2

In Chapter 4, Section 12, do Problems 14 to 17.

3. DEFINITION OF THE GAMMA FUNCTION; RECURSION RELATION

So far n has been a nonnegative integer; it is natural to *define* the factorial function for nonintegral n by the definite integral (2.3). There is no real objection to the notation $n!$ for nonintegral n (and we shall occasionally use it), but it is customary to reserve the factorial notation for integral n and to call the corresponding function for nonintegral n the gamma (Γ) function. It is also rather common practice to replace n by the letter p when we do not necessarily mean an integer. Following these conventions, we define, for *any* $p > 0$

$$\boxed{\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx, \quad p > 0.} \tag{3.1}$$

For $0 < p < 1$, this is an improper integral because x^{p-1} becomes infinite at the lower limit. However, it is a convergent integral for $p > 0$ (Problem 1). For $p \le 0$, the integral diverges and so cannot be used to define $\Gamma(p)$; we shall see in Section 4 how to define $\Gamma(p)$ when $p \le 0$. Then from (3.1) and (2.3) we have

$$\begin{aligned} \Gamma(n) &= \int_0^\infty x^{n-1} e^{-x}\,dx = (n-1)!, \\ \Gamma(n+1) &= \int_0^\infty x^n e^{-x}\,dx = n!. \end{aligned} \tag{3.2}$$

Thus

$$\Gamma(1) = 0! = 1, \quad \Gamma(2) = 1! = 1, \quad \Gamma(3) = 2! = 2, \quad \Gamma(4) = 3! = 6, \quad \cdots,$$

with the usual meaning of factorial for positive integral n. The fact that $\Gamma(n) = (n-1)!$ and not $n!$ is unfortunate but that's the notation which is used, so watch out for it. Replacing p by $p+1$ in (3.1), we have

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx = p!, \quad p > -1. \tag{3.3}$$

Some authors use the factorial notation $p! = \Gamma(p+1)$ even though p is not an integer; this avoids the nuisance of the $p+1$.

Let us integrate (3.3) by parts, calling $x^p = u$, $e^{-x}dx = dv$; then we get

$$\begin{aligned} du &= p\,x^{p-1}\,dx, \quad v = -e^{-x}, \\ \Gamma(p+1) &= -x^p e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x}) p\,x^{p-1}\,dx \\ &= p\int_0^\infty x^{p-1} e^{-x}\,dx = p\Gamma(p). \end{aligned}$$

This equation

$$\Gamma(p+1) = p\Gamma(p) \tag{3.4}$$

is called the *recursion relation* for the Γ function. It can be used to simplify expressions involving Γ functions or to write them in a different form (much as you use trigonometric identities).

Example. By (3.4) we find $\Gamma(9/4) = (5/4)\Gamma(5/4) = (5/4)(1/4)\Gamma(1/4)$; then $\Gamma(1/4) \div \Gamma(9/4) = 16/5$.

PROBLEMS, SECTION 3

1. The integral in (3.1) is improper because of the infinite upper limit and it is also improper for $0 < p < 1$ because x^{p-1} becomes infinite at the lower limit. However, the integral is convergent for any $p > 0$. Prove this.

Use the recursion relation (3.4), and if needed, equation (3.2) to simplify:

2. $\Gamma(2/3)/\Gamma(5/3)$ **3.** $\Gamma(2/3)/\Gamma(8/3)$ **4.** $\Gamma(2/5)/\Gamma(12/5)$

5. $\Gamma(1/2)\Gamma(4)/\Gamma(9/2)$ **6.** $\Gamma(10)/\Gamma(8)$ **7.** $\Gamma(4)\Gamma(3/4)/\Gamma(7/4)$

Express each of the following integrals as a Γ function. By computer, evaluate numerically both the Γ function and the original integral.

8. $\displaystyle\int_0^\infty x^{2/3}e^{-x}\,dx$

9. $\displaystyle\int_0^\infty e^{-x^4}\,dx$ *Hint:* Put $x^4 = u$.

10. $\displaystyle\int_0^\infty x^{-2/5}e^{-x}\,dx$

11. $\displaystyle\int_0^\infty x^5e^{-x^2}dx$ *Hint:* Put $x^2 = u$.

12. $\displaystyle\int_0^\infty xe^{-x^3}dx$

13. $\displaystyle\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx$ *Hint:* Put $x = e^{-u}$.

14. $\displaystyle\int_0^1 \sqrt[3]{\ln x}\,dx$

15. $\displaystyle\int_0^\infty x^{-1/3}\,e^{-8x}\,dx$

16. A particle starting from rest at $x = 1$ moves along the x axis toward the origin. Its potential energy is $V = \frac{1}{2}m\ln x$. Write the Lagrange equation and integrate it to find the time required for the particle to reach the origin. *Caution:* $dx/dt < 0$. *Answer:* $\Gamma(\frac{1}{2})$.

17. Express as a Γ function

$$\int_0^1\left[\ln\left(\frac{1}{x}\right)\right]^{p-1}dx. \qquad \textit{Hint}\text{: See Problem 13.}$$

▶ 4. THE GAMMA FUNCTION OF NEGATIVE NUMBERS

For $p \le 0$, $\Gamma(p)$ has not so far been defined. We shall now define it by the recursion relation (3.4) solved for $\Gamma(p)$.

$$\Gamma(p) = \frac{1}{p}\,\Gamma(p+1) \tag{4.1}$$

defines $\Gamma(p)$ for $p < 0$.

▶ **Example.**

$$\Gamma(-0.3) = \frac{1}{-0.3}\Gamma(0.7), \quad \Gamma(-1.3) = \frac{1}{(-1.3)(-0.3)}\Gamma(0.7),$$

and so on. Since $\Gamma(1) = 1$, we see that

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \to \infty \quad \text{as } p \to 0.$$

From this and successive use of (4.1) it follows that $\Gamma(p)$ becomes infinite not only at zero but also at all the negative integers. In the intervals between the negative integers, it is alternately positive and negative, negative from 0 to -1, positive from -1 to -2, and so on, as you can see from computations like those for $\Gamma(-0.3)$ and $\Gamma(-1.3)$ above. See Problems 5.1 and 5.2.

5. SOME IMPORTANT FORMULAS INVOLVING GAMMA FUNCTIONS

First we evaluate $\Gamma\left(\frac{1}{2}\right)$. By definition

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{t}}\, e^{-t}\, dt. \tag{5.1}$$

(Note that it does not matter what letter we use for the dummy variable of integration in a definite integral.) Put $t = y^2$ in (5.1); then $dt = 2y\, dy$, and (5.1) becomes

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{y}\, e^{-y^2} 2y\, dy = 2\int_0^\infty e^{-y^2}\, dy$$

or, with x as the dummy integration variable,

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2}\, dx. \tag{5.2}$$

Multiply these two integrals for $\Gamma(\frac{1}{2})$ together and write the result as a double integral:

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

This is an integral over the first quadrant; it can be more easily evaluated in polar coordinates:

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} r\, dr\, d\theta = 4 \cdot \frac{\pi}{2} \cdot \left.\frac{e^{-r^2}}{-2}\right|_0^\infty = \pi.$$

Therefore

$$\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.} \tag{5.3}$$

We state here another important formula involving Γ functions (for proof, see Chapter 14, Section 7, Example 5):

$$\boxed{\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}.} \tag{5.4}$$

Notice that (5.4) also gives $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ if we put $p = \frac{1}{2}$.

▶ PROBLEMS, SECTION 5

1. Using (5.3) with (3.4) and (4.1), find $\Gamma(3/2)$, $\Gamma(-1/2)$, and $\Gamma(-3/2)$ in terms of $\sqrt{\pi}$.

2. Without computer or tables, but just using facts you know, sketch a quick rough graph of the Γ function from -2 to 3. *Hint:* This is easy; don't make a big job of it. From Section 3, you know the values of the Γ function at the positive integers in terms of factorials. From Problem 1, you can easily find and plot the Γ function at $\pm 1/2$, $\pm 3/2$. (Approximate $\sqrt{\pi}$ as a little less than 2.) From (4.1) and the discussion following it, you know that the Γ function tends to plus or minus infinity at 0 and the negative integers, and you know the intervals where it is positive or negative. After sketching your graph, make a computer plot of the Γ function from -5 to 5 and compare your sketch.

3. In Chapter 1, equations (13.5) and (13.6), we defined the binomial coefficients $\binom{p}{n}$ where n is a non-negative integer but p may be negative or fractional. Show that $\binom{p}{n}$ can be written in terms of Γ functions as

$$\binom{p}{n} = \frac{\Gamma(p+1)}{n!\,\Gamma(p-n+1)}.$$

4. Prove that, for positive integral n:

$$\Gamma(n+\tfrac{1}{2}) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi}.$$

5. Use (5.4) to show that

(a) $\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}+n) = (-1)^n\pi$ if n = a positive integer;

(b) $(z!)(-z)! = \pi z/\sin \pi z$, where z is not necessarily an integer; see comment after equation (3.3).

6. Prove that

$$\frac{d}{dp}\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\ln x\,dx,$$
$$\frac{d^n}{dp^n}\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}(\ln x)^n\,dx.$$

7. In the Table of Laplace Transforms (end of Chapter 8, page 469), verify the Γ function results for $L5$ and $L6$. Also show that $L(1/\sqrt{t}\,) = \sqrt{\pi/p}$.

▶ 6. BETA FUNCTIONS

The *beta function* is also defined by a definite integral:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \quad p>0, \quad q>0. \tag{6.1}$$

There are a number of simple transformations of (6.1) which are useful to know [see (6.3), (6.4), (6.5)]. It is easy to show that (Problem 1)

$$B(p,q) = B(q,p). \tag{6.2}$$

The range of integration in (6.1) can be changed by putting $x = y/a$; then $x = 1$ corresponds to $y = a$, and (6.1) becomes

$$(6.3) \qquad B(p,q) = \int_0^a \left(\frac{y}{a}\right)^{p-1}\left(1-\frac{y}{a}\right)^{q-1}\frac{dy}{a} = \frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}\,dy.$$

To obtain the trigonometric form of the beta function, let $x = \sin^2\theta$; then

$$dx = 2\sin\theta\cos\theta\,d\theta, \quad (1-x) = 1-\sin^2\theta = \cos^2\theta,$$
$$x = 1 \text{ corresponds to } \theta = \pi/2.$$

With these substitutions, (6.1) becomes

$$B(p,q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1}(\cos^2\theta)^{q-1} 2\sin\theta\cos\theta\,d\theta \quad \text{or}$$

$$(6.4) \qquad B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta.$$

Finally, let $x = y/(1+y)$ in (6.1); then we get (Problem 2):

$$(6.5) \qquad B(p,q) = \int_0^\infty \frac{y^{p-1}dy}{(1+y)^{p+q}}.$$

PROBLEMS, SECTION 6

1. Prove that $B(p,q) = B(q,p)$. *Hint:* Put $x = 1-y$ in Equation (6.1).
2. Prove equation (6.5).
3. Show that for integral n, m,

$$1/B(n,m) = m\binom{n+m-1}{n-1} = n\binom{n+m-1}{m-1}.$$

Hint: See Chapter 1, Section 13C, Problem 13.3.

7. BETA FUNCTIONS IN TERMS OF GAMMA FUNCTIONS

Beta functions are easily expressed in terms of Γ functions. We shall show that

$$(7.1) \qquad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Thus we can evaluate a B function in terms of Γ functions (see example below).

To prove (7.1), we start with

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt$$

and put $t = y^2$. Then we have

$$\Gamma(p) = 2\int_0^\infty y^{2p-1} e^{-y^2}\, dy. \tag{7.2}$$

Similarly (remember that the dummy integration variable can be any letter),

$$\Gamma(q) = 2\int_0^\infty x^{2q-1} e^{-x^2}\, dx.$$

Next we multiply these two equations together and change to polar coordinates:

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\int_0^\infty \int_0^\infty x^{2q-1} y^{2p-1} e^{-(x^2+y^2)}\, dx\, dy\\ &= 4\int_0^\infty \int_0^{\pi/2} (r\cos\theta)^{2q-1}(r\sin\theta)^{2p-1} e^{-r^2}\, r\, dr\, d\theta\\ &= 4\int_0^\infty r^{2p+2q-1} e^{-r^2}\, dr \int_0^{\pi/2} (\cos\theta)^{2q-1}(\sin\theta)^{2p-1}\, d\theta.\end{aligned} \tag{7.3}$$

The r integral in (7.3) is $\frac{1}{2}\Gamma(p+q)$ by (7.2). The θ integral in (7.3) is $\frac{1}{2}B(p,q)$ by (6.4). Then $\Gamma(p)\Gamma(q) = 4\cdot\frac{1}{2}\Gamma(p+q)\cdot\frac{1}{2}B(p,q)$ and (7.1) follows.

▶ **Example.** Find

$$I = \int_0^\infty \frac{x^3\, dx}{(1+x)^5}.$$

This is (6.5) with $(p+q) = 5$, $p-1 = 3$ or $p = 4$, $q = 1$. Then $I = B(4,1)$. By (7.1), this is

$$\frac{\Gamma(4)\Gamma(1)}{\Gamma(5)} = \frac{3!}{4!} = \frac{1}{4}.$$

▶ PROBLEMS, SECTION 7

Express the following integrals as B functions, and then, by (7.1), in terms of Γ functions. When possible, use Γ function formulas to write an exact answer in terms of π, $\sqrt{2}$, etc. Compare your answers with computer results and reconcile any discrepancies.

1. $\displaystyle\int_0^1 \frac{x^4\, dx}{\sqrt{1-x^2}}$

2. $\displaystyle\int_0^{\pi/2} \sqrt{\sin^3 x\, \cos x}\, dx$

3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

4. $\displaystyle\int_0^1 x^2(1-x^2)^{3/2}\, dx$

5. $\displaystyle\int_0^\infty \frac{y^2\, dy}{(1+y)^6}$

6. $\displaystyle\int_0^\infty \frac{y\, dy}{(1+y^3)^2}$

7. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$

8. $\displaystyle\int_0^2 \frac{x^2\, dx}{\sqrt{2-x}}$

9. Prove $B(n,n) = B(n, \frac{1}{2})/2^{2n-1}$. *Hint:* In (6.4), use the identity $2\sin\theta\cos\theta = \sin 2\theta$ and put $2\theta = \phi$. Use this result and (5.3) to derive the *duplication formula* for Γ functions:

$$\Gamma(2n) = \frac{1}{\sqrt{\pi}} 2^{2n-1}\Gamma(n)\Gamma(n+\tfrac{1}{2}).$$

Check this formula for the case $n = \frac{1}{4}$ by using (5.4).

Computer plot the graph of $x^3 + y^3 = 8$. Write the integrals for the following quantities (see Chapter 5 if needed) and evaluate them as B functions.

10. The first quadrant area bounded by the curve.

11. The centroid of this area.

12. The volume generated when the area is revolved about the y axis.

13. The moment of inertia of this volume about its axis.

8. THE SIMPLE PENDULUM

A simple pendulum means a mass m suspended by a string (or weightless rod) of length l so that it can swing in a plane, as shown in Figure 8.1. The kinetic energy of m is then

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2. \tag{8.1}$$

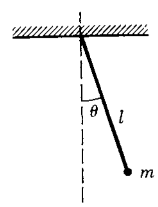


Figure 8.1

If the potential energy is zero when the string is horizontal, then at angle θ it is

$$V = -mgl\cos\theta.$$

Then the Lagrangian is (see Chapter 9, Section 5)

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta,$$

and the Lagrange equation of motion is

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

or

$$\ddot{\theta} = -\frac{g}{l}\sin\theta. \tag{8.2}$$

Example 1. Suppose the pendulum executes such small vibrations that $\sin\theta$ can be approximated by θ. Then (8.2) becomes the usual equation for the simple harmonic motion of a pendulum executing small vibrations, namely

$$\ddot{\theta} = -\frac{g}{l}\,\theta. \tag{8.3}$$

The solutions of (8.3) are $\sin\omega t$ and $\cos\omega t$ where $\omega = 2\pi\nu = \sqrt{g/l}$; the period of the motion is then (see Chapter 7, Problem 2.13, and Chapter 8, Problem 5.34)

$$T = \frac{1}{\nu} = 2\pi\sqrt{l/g}. \tag{8.4}$$

We now want to replace this approximate solution by one which is exact even for large θ.

▶ **Example 2.** Going back to the differential equation of motion (8.2), we multiply both sides of it by $\dot{\theta}$ and integrate, thus obtaining

$$\dot{\theta}\ddot{\theta} = -\frac{g}{l}\sin\theta\,\dot{\theta} \qquad \text{or} \qquad \dot{\theta}\,d\dot{\theta} = -\frac{g}{l}\sin\theta\,d\theta;$$

$$\frac{1}{2}\dot{\theta}^2 = \frac{g}{l}\cos\theta + \text{const.} \tag{8.5}$$

We shall come back to the general solution of this equation when we discuss elliptic integrals; for now let us find the period for 180° swings (back and forth from $-90°$ to $+90°$). For this case, $\dot{\theta} = 0$ when $\theta = 90°$, so the constant in (8.5) is zero, and we have (compare Chapter 8, Problem 7.13)

$$\frac{1}{2}\dot{\theta}^2 = \frac{g}{l}\cos\theta, \qquad \frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\cos\theta}, \qquad \frac{d\theta}{\sqrt{\cos\theta}} = \sqrt{\frac{2g}{l}}\,dt.$$

From $\theta = 0$ to $\theta = 90°$ is one-quarter of a period; hence the period for 180° swings is given by T in the equation

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}} = \sqrt{\frac{2g}{l}}\int_0^{T/4}dt = \sqrt{\frac{2g}{l}}\cdot\frac{T}{4}.$$

Then the period is

$$T = 4\sqrt{\frac{l}{2g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}}. \tag{8.6}$$

We can see by comparing (8.6) with (6.4) that this is a B function. By computer or tables we find that $T \cong 7.42\sqrt{l/g}$ (see Problem 1 and Problem 12.21). We can find the period for only this one special case (180° swings) by B functions; the general case gives an elliptic integral (Section 12).

▶ PROBLEMS, SECTION 8

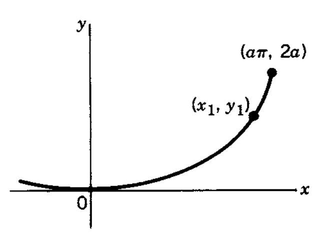
1. Complete the pendulum problem to find the period for 180° swings as a multiple of $\sqrt{l/g}$ [that is, evaluate the integral in (8.6)].

2. Suppose that a car with a door open at right angles ($\theta = 90°$) starts up and accelerates at a constant rate $a = 1$ mph/sec. The differential equation for $\theta(t)$ is $\ddot{\theta} = -A\sin\theta$ where $A = 3a/2w$ for a uniform door of width w. If $w = 3.5$ ft, find how long it takes for the door to close.

3. The figure is part of a cycloid with parametric equations

$$x = a(\theta + \sin\theta), \qquad y = a(1 - \cos\theta).$$

(The graph shown is like Figure 4.4 of Chapter 9 with the origin shifted to P_2.) Show that the time



for a particle to slide without friction along the curve from (x_1, y_1) to the origin is given by

$$t = \sqrt{\frac{a}{g}} \int_0^{y_1} \frac{dy}{\sqrt{y(y_1 - y)}}.$$

Hint: Show that the arc length element is $ds = \sqrt{2a/y}\, dy$. Evaluate the integral to show that the time is independent of the starting height y_1.

9. THE ERROR FUNCTION

You will meet this function in probability theory (Chapter 15, Section 8), and consequently in statistical mechanics and other applications of probability theory. You have probably heard of "grading on a curve." The "curve" means the bell-shaped graph of $y = e^{-x^2}$ (see Problem 1); the error function is the area under part of this curve. We define the error function as

$$\text{(9.1)} \qquad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Although this is the usual definition of $\operatorname{erf}(x)$, there are other closely related integrals which are used and sometimes referred to as the error function. Consequently, you need to look carefully at the definition in the reference you are using (text, tables, computer). Here are some integrals you may find and their relation to (9.1) (see Problem 2).

The standard normal or Gaussian cumulative distribution function $\Phi(x)$ [see Chapter 15, equation (8.5)]:

$$\text{(9.2a)} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}(x/\sqrt{2}\,),$$

$$\text{(9.2b)} \qquad \Phi(x) - \tfrac{1}{2} = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt = \tfrac{1}{2}\operatorname{erf}(x/\sqrt{2}\,).$$

The complementary error function:

$$\text{(9.3a)} \qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = 1 - \operatorname{erf}(x),$$

$$\text{(9.3b)} \qquad \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt.$$

We can also use (9.2) to write $\operatorname{erf}(x)$ in terms of the standard normal cumulative distribution function [Chapter 15, equation (8.5)].

$$\text{(9.4)} \qquad \operatorname{erf}(x) = 2\Phi(x\sqrt{2}\,) - 1.$$

We next consider several useful facts about the error function. You can easily prove that the error function is odd; that is, $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ (Problem 3). We show that $\operatorname{erf}(\infty) = 1$ as follows:

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\,\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{2}{\sqrt{\pi}}\,\tfrac{1}{2}\sqrt{\pi} = 1 \tag{9.5}$$

by (5.2) and (5.3). For very small values of x, $\operatorname{erf}(x)$ can be approximated by expanding e^{-t^2} in a power series and integrating term by term. We get

$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} \int_0^{x} \left(1 - t^2 + \frac{t^4}{2!} - \cdots\right) dt \\ &= \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \cdots\right). \end{aligned} \tag{9.6}$$

[Use this when $|x| \ll 1$. Compare (10.4).]

For large x, say $x > 3$, $\operatorname{erf}(x)$ differs from $\operatorname{erf}(\infty) = 1$ [see (9.5)] by less than 10^{-4} (and of course even less for larger x). We are then usually interested in $1 - \operatorname{erf}(x) = \operatorname{erfc}(x)$. This is best approximated by an asymptotic series; we shall discuss such expansions in Section 10.

The function $\operatorname{erfi}(x)$, called the imaginary error function, is similar to the error function but with a positive exponential. We define

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{t^2}\,dt. \tag{9.7}$$

You can show (Problem 5) that $\operatorname{erf}(ix) = i\operatorname{erfi}(x)$.

The Fresnel integrals (Chapter 1, Section 15) are related to the error function (Problem 6). Also see Section 10, Problem 3 for other relations involving error functions.

▶ PROBLEMS, SECTION 9

1. Sketch or computer plot a graph of the function $y = e^{-x^2}$.

2. Verify equations (9.2), (9.3), and (9.4). *Hint:* In (9.2a), you want to write $\Phi(x)$ in terms of an error function. Make the change of variable $t = u\sqrt{2}$ in the $\Phi(x)$ integral. *Warning:* Don't forget to adjust the limits; when $t = x$, $u = x/\sqrt{2}$.

3. Prove that $\operatorname{erf}(x)$ is an odd function of x. *Hint:* Put $t = -s$ in (9.1).

4. Show that $\displaystyle\int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}$

(a) by using (9.5) and (9.2a);

(b) by reducing it to a Γ function and using (5.3).

5. Replace x by ix in (9.1) and let $t = iu$ to show that $\operatorname{erf}(ix) = i\operatorname{erfi}(x)$, where $\operatorname{erfi}(x)$ is defined in (9.7).

6. Assuming that x is real, show the following relation between the error function and the Fresnel integrals.

$$\operatorname{erf}\left(\frac{1-i}{\sqrt{2}}x\right) = (1-i)\sqrt{\frac{2}{\pi}}\int_0^x (\cos u^2 + i\sin u^2)\,du.$$

Hint: In (9.1), make the change of variables $t = \dfrac{1-i}{\sqrt{2}}u$.

10. ASYMPTOTIC SERIES

Since you have spent some time learning to test series for convergence, it may surprise you to learn that there are divergent series which can be of practical use. We can show this best by an example.

Example 1. From (9.3a)

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{10.1}$$

We are going to expand the integral in (10.1) in a series of inverse powers of x. To do this we write

$$e^{-t^2} = \frac{1}{t}te^{-t^2} = \frac{1}{t}\frac{d}{dt}\left(-\frac{1}{2}e^{-t^2}\right) \tag{10.2}$$

and integrate by parts as follows:

$$\begin{aligned}\int_x^\infty e^{-t^2}\,dt &= \int_x^\infty \frac{1}{t}\frac{d}{dt}\left(-\frac{1}{2}e^{-t^2}\right)dt\\ &= \frac{1}{t}\left(-\frac{1}{2}e^{-t^2}\right)\Big|_x^\infty - \int_x^\infty\left(-\frac{1}{2}e^{-t^2}\right)\left(-\frac{1}{t^2}\right)dt\\ &= \frac{1}{2x}e^{-x^2} - \frac{1}{2}\int_x^\infty \frac{1}{t^2}e^{-t^2}\,dt.\end{aligned} \tag{10.3}$$

Now in the last integral in (10.3), write $(1/t^2)e^{-t^2} = (1/t^3)(d/dt)(-\frac{1}{2}e^{-t^2})$, and again integrate by parts:

$$\begin{aligned}\int_x^\infty \frac{1}{t^2}e^{-t^2}\,dt &= \int_x^\infty \frac{1}{t^3}\frac{d}{dt}\left(-\frac{1}{2}e^{-t^2}\right)dt\\ &= \frac{1}{t^3}\left(-\frac{1}{2}e^{-t^2}\right)\Big|_x^\infty - \int_x^\infty\left(-\frac{1}{2}e^{-t^2}\right)\left(-\frac{3}{t^4}\right)dt\\ &= \frac{1}{2x^3}e^{-x^2} - \frac{3}{2}\int_x^\infty \frac{1}{t^4}e^{-t^2}\,dt.\end{aligned}$$

Continue this process, and substitute (10.3) and the following steps into (10.1) to get (Problem 1)

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \cdots\right). \tag{10.4}$$

[Use this when $|x| \gg 1$. Compare (9.6).]

(We shall explain the exact meaning of the symbol $\sim$ shortly.) This series diverges for every x because of the factors in the numerator. However, suppose we stop after a few terms and keep the integral at the end so that we have an exact equation. If we stop after the second term, we have

$$\text{(10.5)} \qquad \operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2}\right) + \frac{3}{2\sqrt{\pi}}\int_x^\infty t^{-4}e^{-t^2}\,dt.$$

There is no approximation here. This is not an infinite series so there is no question of convergence. However, we shall show that the integral at the end is negligible for large enough x; this will then make it possible for us to use the rest of (10.5) [that is, the first two terms of (10.4)] as a good approximation for $\operatorname{erfc}(x)$ for large x. This is the meaning of an asymptotic series. As an infinite series it may diverge, but we do not use the infinite series. Instead, using an exact equation [like (10.5) for this example], we show that the first few terms which we *do* use give a good approximation if x is large.

▶ **Example 2.** Now let's look at the integral in (10.5); we want to estimate its size for large x. The t in the integrand takes values from x to ∞; therefore $t \geq x$ or $1/x \geq 1/t$ for all values of t from x to ∞. Let us write the integral as

$$\int_x^\infty t^{-4}e^{-t^2}\,dt = \int_x^\infty \frac{1}{t^5}\left(te^{-t^2}\right)dt.$$

We *increase* the value of this integral if we replace $1/t^5$ by $1/x^5$ since $1/x \geq 1/t$. Thus

$$\begin{aligned}\int_x^\infty t^{-4}e^{-t^2}\,dt &< \int_x^\infty \frac{1}{x^5}\left(te^{-t^2}\right)dt = \frac{1}{x^5}\int_x^\infty te^{-t^2}\,dt\\ &= \frac{1}{x^5}\left(-\frac{1}{2}e^{-t^2}\right)\Bigg|_x^\infty = \frac{e^{-x^2}}{2x^5}.\end{aligned}$$

When we stop in (10.5) with the term in e^{-x^2}/x^3, the error is of the order of e^{-x^2}/x^5, which becomes much smaller than e^{-x^2}/x^3 as x increases. Thus we have shown that two terms of (10.4) give a good approximation for $\operatorname{erfc}(x)$ when $x \gg 1$. A similar result can be shown for an approximation using any number of terms of the asymptotic series (10.4) with the error depending on the "left-over" integral and the value of x.

We can make the above discussion more precise. For (10.4), we have seen that if we stop after the term in $x^{-3}e^{-x^2}$, the error is of the order of $x^{-5}e^{-x^2}$. Then the ratio of the error to the last term kept (namely $x^{-5}e^{-x^2} \div x^{-3}e^{-x^2} = x^{-2}$) tends to zero as x tends to infinity, that is, the approximation becomes increasingly good for larger x as we have said. The "error" in an asymptotic expansion means in general the difference between the function being expanded and a partial sum (first N terms) of the series. A series is called an asymptotic expansion (about ∞) of a function $f(x)$ if, for each fixed N, the ratio of the error to the last (nonzero) term kept, tends to zero as $x \to \infty$. In symbols

$$f(x) \sim \sum_{n=0}^{\infty} \phi_n(x)$$

(10.6) $$\left(\text{read } \sum_{n=0}^{\infty} \phi_n(x) \text{ is an asymptotic expansion of } f(x)\right)$$

if for each fixed N

$$\left| f(x) - \sum_{n=0}^{N} \phi_n(x) \right| \div \phi_N(x) \to 0 \quad \text{as } x \to \infty.$$

Frequently, the terms of an asymptotic series (about ∞) are inverse powers of x. [We could write (10.4) this way by multiplying through by e^{x^2}.] Then (10.6) becomes

(10.7) $$f(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^n}$$

if for each fixed N

$$\left| f(x) - \sum_{n=0}^{N} \frac{a_n}{x^n} \right| \cdot x^N \to 0 \quad \text{as } x \to \infty.$$

We can also have asymptotic series about the origin (or any point—compare Taylor series). We say that

(10.8) $$f(x) \sim \sum_{n=0}^{\infty} a_n x^n$$

if for each fixed N

$$\left| f(x) - \sum_{n=0}^{N} a_n x^n \right| \div x^N \to 0 \quad \text{as } x \to 0.$$

Although we have discussed the particularly interesting case of *divergent* asymptotic series, it is not necessary for such series to diverge. Note that to test a series for convergence, we fix x and let n tend to infinity; to see if a series is asymptotic, we fix n and let x tend to a limit. A given series may meet both tests, or only one or the other (or neither).

PROBLEMS, SECTION 10

1. Carry through the algebra to get equation (10.4).

2. The integral $\int_x^\infty t^{p-1}e^{-t}\,dt = \Gamma(p,x)$ is called an *incomplete* Γ function. [Note that if $x = 0$, this integral is $\Gamma(p)$.] By repeated integration by parts, find several terms of the asymptotic series for $\Gamma(p,x)$.

3. Express the complementary error function $\mathrm{erfc}(x)$ as an incomplete Γ function (see Problem 2) and use your result in Problem 2 to obtain (again) the asymptotic expansion of $\mathrm{erfc}(x)$ as in (10.4).

4. $E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}\,dt$, $n = 0, 1, 2, \cdots$, and $\mathrm{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt$, and other similar integrals are called *exponential integrals*. By making appropriate changes of variable, show that

(a) $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$ (b) $\mathrm{Ei}(x) = -\int_{-x}^\infty \frac{e^{-t}}{t}\,dt$

(c) $E_1(x) = -\mathrm{Ei}(-x)$ (d) $\int_0^x \frac{e^{1/t}}{t}\,dt = E_1(-1/x)$

(*Caution:* Various notations are used; check carefully the notation in references you are using.)

5. (a) Express $E_1(x)$ as an incomplete Γ function.

(b) Find the asymptotic series for $E_1(x)$.

6. The *logarithmic integral* is $\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}$. Express as exponential integrals

(a) $\mathrm{li}(x)$ (b) $\mathrm{li}(e^x)$ (c) $\int_0^x \frac{dt}{\ln(1/t)}$

7. Computer plot graphs of

(a) $E_n(x)$ for $n = 0$ to 10 and $x = 0$ to 2;

(b) $E_1(x)$ and $\mathrm{Ei}(x)$ for $x = 0$ to 2;

(c) the *sine integral* $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$ and the *cosine integral* $\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt$ for $x = 0$ to 4π.

11. STIRLING'S FORMULA

Formulas involving $n!$ or $\Gamma(p)$ are not convenient to simplify algebraically or to differentiate. Here is an approximate formula for the factorial or Γ function known as Stirling's formula which can be used to simplify formulas involving factorials:

(11.1) $n! \sim n^n e^{-n}\sqrt{2\pi n}$ or $\Gamma(p+1) \sim p^p e^{-p}\sqrt{2\pi p}$. Stirling's formula

The sign $\sim$ (read "is asymptotic to") means that the ratio of the two sides

$$\frac{n!}{n^n e^{-n}\sqrt{2\pi n}}$$

tends to 1 as $n \to \infty$. Thus we get better approximations to $n!$ as n becomes large. Actually the absolute error (difference between the Stirling approximation and the correct value) *increases*, but the relative error (ratio of the error to the value of $n!$) tends to zero as n increases. To get some idea of how this formula arises, we outline what could, with a little more detail, be a derivation of it. (For more detail, consult advanced calculus books.) Start with

$$\Gamma(p+1) = p\,! = \int_0^\infty x^p e^{-x}\,dx = \int_0^\infty e^{p\ln x - x}\,dx. \tag{11.2}$$

Substitute a new variable y such that

$$x = p + y\sqrt{p}.$$

Then

$$dx = \sqrt{p}\,dy,$$
$$x = 0 \text{ corresponds to } y = -\sqrt{p},$$

and (11.2) becomes

$$p\,! = \int_{-\sqrt{p}}^{\infty} e^{p\ln(p+y\sqrt{p}\,)-p-y\sqrt{p}}\sqrt{p}\,dy. \tag{11.3}$$

For large p, the logarithm can be expanded in the following power series:

$$\ln(p+y\sqrt{p}\,) = \ln p + \ln\left(1+\frac{y}{\sqrt{p}}\right) = \ln p + \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \cdots. \tag{11.4}$$

Substituting (11.4) into (11.3), we get

$$\begin{aligned} p\,! &\sim \int_{-\sqrt{p}}^{\infty} e^{p\ln p + y\sqrt{p}-(y^2/2)-p-y\sqrt{p}}\sqrt{p}\,dy \\ &= e^{p\ln p - p}\sqrt{p}\int_{-\sqrt{p}}^{\infty} e^{-y^2/2}\,dy \\ &= p^p e^{-p}\sqrt{p}\left[\int_{-\infty}^{\infty} e^{-y^2/2}\,dy - \int_{-\infty}^{-\sqrt{p}} e^{-y^2/2}\,dy\right]. \end{aligned}$$

The first integral is easily shown to be $\sqrt{2\pi}$ (Problem 9.4). The second integral tends to zero as $p \to \infty$, and we have

$$p! \sim p^p e^{-p}\sqrt{2\pi p}$$

which is (11.1). With more work, it is possible to find an asymptotic expansion for $\Gamma(p+1)$:

$$\Gamma(p+1) = p! = p^p e^{-p}\sqrt{2\pi p}\left(1+\frac{1}{12p}+\frac{1}{288p^2}+\cdots\right). \tag{11.5}$$

This is another example of an asymptotic series which is divergent as an infinite series; however, the first term alone (Stirling's formula) is a good approximation when p is large, and the second term can be used to estimate the relative error (Problem 1).

PROBLEMS, SECTION 11

1. Use the term $1/(12p)$ in (11.5) to show that the error in Stirling's formula (11.1) is $< 10\%$ for $p > 1$; $< 1\%$ for $p > 10$; $< 0.1\%$ for $p > 100$; $< 0.01\%$ for $p > 1000$.

2. (a) To see the results in Problem 1 graphically, computer plot the percentage error in Stirling's formula as a function of p for values of p from 1 to 1000. Make separate plots, say for $p = 1$ to 10, 10 to 100, 100 to 1000, to make it easier to read values from your plots.

(b) Repeat part (a) for the percentage error in (11.5) using two terms of the asymptotic series, that is, Stirling's formula times $[1 + 1/(12p)]$.

3. In statistical mechanics, we frequently use the approximation $\ln N! = N\ln N - N$, where N is of the order of Avogadro's number. Write out $\ln N!$ using Stirling's formula, compute the approximate value of each term for $N = 10^{23}$, and so justify this commonly used approximation.

4. Use Stirling's formula to evaluate $\lim_{n\to\infty} \dfrac{(2n)!\,\sqrt{n}}{2^{2n}(n!)^2}$.

5. Use Stirling's formula to evaluate $\lim_{n\to\infty} \dfrac{\Gamma(n+\frac{3}{2})}{\sqrt{n}\,\Gamma(n+1)}$.

6. Use equations (3.4) and (11.5) to show that $\Gamma(p) \sim p^p e^{-p}\sqrt{2\pi/p}\left(1 + \frac{1}{12p} + \cdots\right)$.

7. The function $\psi(p) = \frac{d}{dp}\ln\Gamma(p)$ is called the digamma function, and the polygamma functions are defined by $\psi^n(p) = \frac{d^n}{dp^n}\psi(p)$. [*Warning:* Some authors define $\psi(p)$ as $\frac{d}{dp}\ln p! = \frac{d}{dp}\ln\Gamma(p+1)$.]

(a) Show that $\psi(p+1) = \psi(p) + \frac{1}{p}$. *Hint:* See (3.4).

(b) Use Problem 6 to obtain $\psi(p) \sim \ln p - \frac{1}{2p} - \frac{1}{12p^2}\cdots$.

8. Sketch or computer plot a graph of $y = \ln x$ for $x > 0$. Show that $\ln n!$ is between the values of the integrals $\int_2^{n+1} \ln x\, dx$ and $\int_1^n \ln x\, dx$. (*Hint:* $\ln n! = \ln 1 + \ln 2 + \ln 3 + \cdots$ is the sum of the areas of rectangles of width 1 and height up to the $\ln x$ curve at $x = 1, 2, 3, \cdots$.) By considering the values of the two integrals for very large n as in Problem 3, show that $\ln n! = n\ln n - n$ approximately for large n.

9. The following expression occurs in statistical mechanics:

$$P = \frac{n!}{(np+u)!\,(nq-u)!}p^{np+u}q^{nq-u}.$$

Use Stirling's formula to show that

$$\frac{1}{P} \sim x^{npx}y^{nqy}\sqrt{2\pi npqxy},$$

where $x = 1 + \dfrac{u}{np}$, $y = 1 - \dfrac{u}{nq}$, and $p + q = 1$. *Hint:* Show that

$$(np)^{np+u}(nq)^{nq-u} = n^n p^{np+u}q^{nq-u}$$

and divide numerator and denominator of P by this expression.

10. Use Stirling's formula to find $\lim_{n\to\infty}(n!)^{1/n}/n$.

▶ 12. ELLIPTIC INTEGRALS AND FUNCTIONS

This is another collection of integrals and related functions which may arise in applied problems and as computer answers (see problems). We shall merely summarize the basic definitions and properties—there are whole books on the subject—and you may find useful formulas and information in your computer program and in reference books and tables.

Legendre Forms The *Legendre* forms of the elliptic integrals of the first and second kinds are:

$$(12.1)\qquad \begin{aligned} F(\phi,k) &= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad 0\le k\le 1,\\ E(\phi,k) &= \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta, \quad 0\le k\le 1. \end{aligned}$$

There is also an elliptic integral of the third kind which occurs less frequently. In (12.1), ϕ is called the *amplitude* and k is called the *modulus* of the elliptic integral.

Jacobi Forms If we put $t=\sin\theta$, $x=\sin\phi$ in the *Legendre* forms (12.1), we obtain the *Jacobi* forms of the elliptic integrals of the first and second kind:

$$t = \sin\theta,$$

$$dt = \cos\theta\,d\theta \quad \text{or} \quad d\theta = \frac{dt}{\cos\theta} = \frac{dt}{\sqrt{1-t^2}}.$$

The limits $\theta = 0$ to ϕ become $t = 0$ to x.

Then

$$(12.2)\qquad \begin{aligned} F(\phi,k) &= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^x \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}\\ E(\phi,k) &= \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt. \end{aligned}$$

Complete Elliptic Integrals The *complete* elliptic integrals of the first and second kind are the values of F and E when $\phi = \pi/2$ or $x = \sin\phi = 1$:

$$(12.3)\qquad \begin{aligned} K \text{ or } K(k) = F\left(\frac{\pi}{2},k\right) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}},\\ E \text{ or } E(k) = E\left(\frac{\pi}{2},k\right) &= \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt. \end{aligned}$$

Warning: The notation used for elliptic integrals is not uniform. Most references use F and E, but you may find ϕ replaced by $x=\sin\phi$, and instead of k you may find $m=k^2$, or $\sin^{-1}k$. Also (ϕ,k) may be written as (k,ϕ), and other variations exist. So check carefully the notation of any book or computer program you are using and reconcile the results with the notation used here.

Example 1. $\displaystyle\int_0^{\pi/3} \sqrt{1-(1/2)\sin^2\theta}\,d\theta = E(\phi,k) = E(\pi/3, 1/\sqrt{2}\,)$ in our notation. Other books or computer programs might give: $E(\phi,m) = E(\pi/3,1/2)$, or $E(x,k) = E(\sqrt{3}/2, 1/\sqrt{2}\,)$ or $E(\phi,\sin^{-1}k) = E(\pi/3,\pi/4)$, etc. Of course, all of them will give the same numerical approximation 0.964951.

Many integrals can be written in the form of one of the integrals in (12.2).

▶ **Example 2.** $\int_0^{\pi/3} \sqrt{16 - 8\sin^2\theta}\, d\theta$ becomes 4 times the integral in Example 1 if we divide out a factor of 4 to get $4\int_0^{\pi/3} \sqrt{1 - (1/2)\sin^2\theta}\, d\theta$.

▶ **Example 3.** $\int_0^{2/5} \frac{dt}{\sqrt{1-t^2}\sqrt{1-4t^2}} = F(\phi, k) = F(\sin^{-1}\frac{2}{5}, 2)$ in the notation of (12.2), except that we have previously required $k < 1$, and here $k = 2$. However, we can put this integral in the standard form with $k < 1$ by making the change of variable $4t^2 = r^2$, or $r = 2t$. Substituting this into the given integral gives

$$\int_0^{4/5} \frac{dr/2}{\sqrt{1 - r^2/4}\sqrt{1 - r^2}}$$

which, by (12.2), is $\frac{1}{2}F(\phi, k) = \frac{1}{2}F(\sin^{-1}\frac{4}{5}, \frac{1}{2})$. (See Problem 24.)

It is sometimes useful to note that the integrands in elliptic integrals are all functions of $\sin^2\theta$ and so are even functions of θ. Thus an elliptic integral from $-\phi_1$ to ϕ_2 (ϕ_1 and ϕ_2 both positive) is equal to the integral from 0 to ϕ_1 plus the integral from 0 to ϕ_2 and we have

$$\int_{-\phi_1}^{\phi_2} \sqrt{1 - k^2\sin^2\theta}\, d\theta = E(\phi_1, k) + E(\phi_2, k)$$

and a similar formula for $F(\phi, k)$. Also we may note that a function of $\sin^2\theta$ has period π and is symmetric about $\theta = n\pi + \pi/2$ (look at a graph of $\sin^2\theta$). Thus, using the complete elliptic integrals in (12.3), we can write (Problem 2)

$$\begin{aligned} F(n\pi \pm \phi, k) &= 2nK \pm F(\phi, k), \\ E(n\pi \pm \phi, k) &= 2nE \pm E(\phi, k). \end{aligned} \tag{12.4}$$

Since $k^2\sin^2\theta < 1$ (for $k^2 < 1$), we get convergent infinite series for elliptic integrals by expanding their integrands using the binomial theorem, and then integrating term by term (Problem 1). For small k these series converge rapidly and provide a good method for approximating elliptic integrals when $k \ll 1$.

Here are some examples where elliptic integrals occur.

▶ **Example 4.** Find the arc length of an ellipse. This is the problem that gave elliptic integrals their name. We write the equation of the ellipse in the parametric form

$$\begin{aligned} x &= a\sin\theta, \\ y &= b\cos\theta, \end{aligned}$$

for the case $a > b$. (If $b > a$, use the form $x = a\cos\theta$, $y = b\sin\theta$; see Problem 15.) Then for $a > b$, we have

$$ds^2 = dx^2 + dy^2 = (a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta^2.$$

Since $a^2 - b^2 > 0$, we can write

$$\int ds = \int \sqrt{a^2 - (a^2 - b^2)\sin^2\theta}\, d\theta = a\int \sqrt{1 - \frac{a^2 - b^2}{a^2}\sin^2\theta}\, d\theta.$$

This is an elliptic integral of the second kind where $k^2 = (a^2 - b^2)/a^2 = e^2$ (e is the eccentricity of the ellipse in analytic geometry). If we want the complete circumference, θ goes from 0 to 2π, and the answer is $4aE(\pi/2, k) = 4aE(k)$. For a smaller arc, we use the appropriate limits ϕ_1 and ϕ_2 and obtain $E(\phi_2, k) - E(\phi_1, k)$. For any given ellipse (that is, given a and b), we can find the numerical value of the desired arc length from computer or tables.

Example 5. Let a pendulum swing through large angles. We had in Section 8

$$\dot{\theta}^2 = \frac{2g}{l}\cos\theta + \text{ const.,} \tag{12.5}$$

and we considered 180° swings, that is of amplitude 90°. Now we want to consider swings of any amplitude, say α; then $\dot{\theta} = 0$ when $\theta = \alpha$, and (12.5) becomes

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\alpha). \tag{12.6}$$

Integrating (12.6), we get

$$\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{l}}\frac{T_\alpha}{4}, \tag{12.7}$$

where T_α is the period for swings from $-\alpha$ to $+\alpha$ and back. This integral can be written as an elliptic integral; its value (Problem 17) is

$$\sqrt{2}\,K\left(\sin\frac{\alpha}{2}\right). \tag{12.8}$$

Then (12.7) gives for the period

$$T_\alpha = 4\sqrt{\frac{l}{2g}}\sqrt{2}\,K\left(\sin\frac{\alpha}{2}\right) = 4\sqrt{\frac{l}{g}}\,K\left(\sin\frac{\alpha}{2}\right).$$

For α not too large (say $\alpha < 90°$, $\frac{1}{2}\alpha < 45°$, so that $\sin^2(\alpha/2) < \frac{1}{2}$), we can get a good approximation to T_α by series (Problem 1):

$$T_\alpha = 4\sqrt{\frac{l}{g}}\frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2\sin^2\frac{\alpha}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\sin^4\frac{\alpha}{2} + \cdots\right). \tag{12.9}$$

For α small enough so that $\sin\alpha/2$ can be approximated by $\alpha/2$, we can write

$$T_\alpha = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\alpha^2}{16} + \cdots\right). \tag{12.10}$$

For very small α, we get the familiar formula for simple harmonic motion, $T = 2\pi\sqrt{l/g}$ independent of α. For somewhat larger α, say $\alpha = \frac{1}{2}$ radian (about 30°), we get

$$T_{\alpha=1/2} = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{64} + \cdots\right). \tag{12.11}$$

This would mean that a pendulum started at 30° would get exactly out of phase with one of very small amplitude in about 32 periods.

For another physics problem giving rise to an elliptic integral, see Am. J. Phys. **55**, 763 (1987).

Elliptic Functions Recall that

$$u = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} x$$

defines u as a function of x, or x as a function of u; in fact $x = \sin u$. In a similar way, $u = F(\phi, k)$ in (12.2) defines u as a function of ϕ (or of $x = \sin\phi$) or it defines x or ϕ as functions of u (we are assuming k fixed). We write

$$u = \int_0^x \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = \operatorname{sn}^{-1} x. \tag{12.12}$$

or $x = \operatorname{sn} u$. The function $\operatorname{sn} u$ (read ess-en of u) is an elliptic function. Since $x = \sin\phi$, we have

$$x = \operatorname{sn} u = \sin\phi. \tag{12.13}$$

There are other elliptic functions, related to $\operatorname{sn} u$; you will notice [in (12.14)] that they have some resemblance to the trigonometric functions. We define

$$\begin{aligned} \operatorname{cn} u &= \cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-\operatorname{sn}^2 u} = \sqrt{1-x^2}, \\ \operatorname{dn} u &= \frac{d\phi}{du} = \frac{1}{du/d\phi} = \sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2\operatorname{sn}^2 u} = \sqrt{1-k^2x^2}. \end{aligned} \tag{12.14}$$

[The value of $du/d\phi$ is found from $u = F(\phi, k)$ in (12.2).] There are many formulas relating these functions—for example, addition formulas, integrals, derivatives, etc. These can be looked up or, in some cases, easily worked out. For example, since $\operatorname{sn} u = \sin\phi$, we have

$$\frac{d}{du}(\operatorname{sn} u) = \frac{d}{du}(\sin\phi) = \cos\phi\frac{d\phi}{du} = \operatorname{cn} u \operatorname{dn} u.$$

For a physical problem using elliptic functions, see Am. J. Phys. **68**, 888–895 (2000).

▶ PROBLEMS, SECTION 12

1. Expand the integrands of K and E [see (12.3)] in power series in $k^2\sin^2\theta$ (assuming small k), and integrate term by term to find power series approximations for the complete elliptic integrals K and E.

2. Use a graph of $\sin^2\theta$ and the text discussion just before (12.4) to verify the equations (12.4). Note that the area under the $\sin^2\theta$ graph from 0 to $\pi/2$ and the area from $\pi/2$ to π are mirror images of each other, and this will be true also for any function of $\sin^2\theta$.

3. Computer plot graphs of $K(k)$ and $E(k)$ in (12.3) for k from 0 to 1. Also plot 3D graphs of $F(\phi, k)$ and $E(\phi, k)$ in (12.1) for k from 0 to 1 and ϕ from 0 to $\pi/2$ and also from 0 to 2π. *Warning:* Be sure you understand the notation used by your computer program; see text discussion just after (12.3) and Example 1.

In Problems 4 to 13, identify each of the integrals as an elliptic integral (see Examples 1 and 2). Learn the notation of your computer program (see Problem 3) and then evaluate the integral by computer.

4. $\displaystyle\int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-t^2/4}}$

5. $\displaystyle\int_0^{\pi/2} \sqrt{1-\frac{1}{9}\sin^2\theta}\,d\theta$

6. $\displaystyle\int_0^{\pi/3} \frac{d\theta}{\sqrt{9-\sin^2\theta}}$

7. $\displaystyle\int_0^{5\pi/4} \sqrt{25-\sin^2\theta}\,d\theta$

8. $\displaystyle\int_0^{\sqrt{3}/2} \frac{\sqrt{49-4t^2}}{\sqrt{1-t^2}}\,dt$

9. $\displaystyle\int_{-1/2}^{1/2} \frac{dt}{\sqrt{1-t^2}\sqrt{4-3t^2}}$

10. $\displaystyle\int_0^{\pi/4} \frac{d\theta}{\sqrt{4-\sin^2\theta}}$

11. $\displaystyle\int_{-\pi/2}^{3\pi/8} \frac{d\theta}{\sqrt{1-\frac{9}{10}\sin^2\theta}}$

12. $\displaystyle\int_0^{1/2} \frac{\sqrt{100-t^2}}{\sqrt{1-t^2}}\,dt$

13. $\displaystyle\int_{-1/2}^{3/4} \frac{\sqrt{9-4t^2}}{\sqrt{1-t^2}}\,dt$

14. Find the circumference of the ellipse $4x^2+9y^2=36$.

15. Find the length of arc of the ellipse $x^2+(y^2/4)=1$ between $(0,2)$ and $(\frac{1}{2},\sqrt{3})$. (Note that here $b>a$; see Example 4.)

16. Find the arc length of one arch of $y=\sin x$.

17. Write the integral in equation (12.7) as an elliptic integral and show that (12.8) gives its value. *Hints:* Write $\cos\theta = 1-2\sin^2(\theta/2)$ and a similar equation for $\cos\alpha$. Then make the change of variable $x=\sin(\theta/2)/\sin(\alpha/2)$.

18. Computer plot graphs of $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$, for several values of k, say, for example, $k=1/4,\ 1/2,\ 3/4,\ 0.9,\ 0.99$. Also plot 3D graphs of sn, cn, and dn as functions of u and k.

19. If $u=\ln(\sec\phi+\tan\phi)$, then ϕ is a function of u called the *Gudermannian* of u, $\phi=\operatorname{gd} u$. Prove that:

$$u=\ln\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right),\quad \tan\operatorname{gd} u=\sinh u,\quad \sin\operatorname{gd} u=\tanh u,\quad \frac{d}{du}\operatorname{gd} u=\operatorname{sech} u.$$

20. Show that for $k=0$:

$$u=F(\phi,0)=\phi,\quad \operatorname{sn} u=\sin u,\quad \operatorname{cn} u=\cos u,\quad \operatorname{dn} u=1;$$

and for $k=1$:

$$u=F(\phi,1)=\ln(\sec\phi+\tan\phi)\quad\text{or}\quad \phi=\operatorname{gd} u\quad\text{(Problem 19)},$$
$$\operatorname{sn} u=\tanh u,\quad \operatorname{cn} u=\operatorname{dn} u=\operatorname{sech} u.$$

21. Show that the four answers given in Section 1 for $\int_0^{\pi/2} d\theta/\sqrt{\cos\theta}$ are all correct. *Hints:* For the beta function result, use (6.4). Then get the gamma function results by using (7.1) and the various Γ function formulas. For the elliptic integral, use the hint of Problem 17 with $\alpha=\pi/2$.

22. In the pendulum problem, $\theta = \alpha \sin \sqrt{g/l}\, t$ is an approximate solution when the amplitude α is small enough for the motion to be considered simple harmonic. Show that the corresponding exact solution when α is not small is

$$\sin \frac{\theta}{2} = \sin \frac{\alpha}{2} \operatorname{sn} \sqrt{\frac{g}{l}}\, t$$

where $k = \sin(\alpha/2)$ is the modulus of the elliptic function. Show that this reduces to the simple harmonic motion solution for small amplitude α.

23. A uniform solid sphere of density $\frac{1}{2}$ is floating in water. (Compare Chapter 8, Problem 5.37.) It is pushed down just under water and released. Write the differential equation of motion (neglecting friction) and solve it to obtain the period in terms of $K(5^{-1/2})$. Show that this period is approximately 1.16 times the period for small oscillations.

24. Sometimes you may find the notation $F(\phi, k)$ in (12.2) used when $k > 1$. Allowing this notation, show that $\frac{1}{3}F(\sin^{-1}\frac{3}{5}, \frac{4}{3}) = \frac{1}{4}F(\sin^{-1}\frac{4}{5}, \frac{3}{4})$. *Hints:* Using the Jacobi form of F in (12.2), write the integral which is equal to $\frac{1}{3}F(\sin^{-1}\frac{3}{5}, \frac{4}{3})$. Follow Example 3 to make a change of variable, write the corresponding integral, and verify that it is equal to $\frac{1}{4}F(\sin^{-1}\frac{4}{5}, \frac{3}{4})$.

25. As in Problem 24, show that $\frac{1}{2}F(\sin^{-1}\frac{4}{15}, \frac{5}{2}) = \frac{1}{5}F(\sin^{-1}\frac{2}{3}, \frac{2}{5})$.

13. MISCELLANEOUS PROBLEMS

1. Show that

$$\int_0^\infty \frac{y^m\,dy}{(1+y)^{n+1}} = \frac{1}{(n-m)C(n,m)}$$

for positive integral m and n, $n > m$, where $C(n,m) = \binom{n}{m}$.

2. Show that $B(m,n)B(m+n,k) = B(n,k)B(n+k,m)$.

3. Use Stirling's formula to show that

$$\lim_{n\to\infty} n^x B(x,n) = \Gamma(x).$$

4. Verify the asymptotic series

$$\int_0^\infty \frac{e^{-t}\,dt}{(1+xt)} \sim \sum (-1)^n n!\, x^n$$

[see equation (10.8)]. *Hint:* Integrate by parts repeatedly, integrating $e^{-t}\,dt$ and differentiating the powers of $(1+xt)^{-1}$.

5. Use gamma and beta function formulas to show that $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi$.

6. Generalize Problem 5 to show that $\displaystyle\int_0^\infty \frac{dx}{(1+x)x^p} = \frac{\pi}{\sin \pi p}$, $\quad 0 < p < 1$.

Identify each of the following integrals or expressions as one of the functions of this chapter. Check your work by evaluating both your answer and the original problem by computer. Be sure you understand your computer program's notation.

7. $\displaystyle\int_0^\infty x^3 e^{-x}\,dx$ **8.** $\displaystyle\int_0^1 e^{-x^2}\,dx$ **9.** $\displaystyle\int_0^1 \sqrt{\frac{4-3x^2}{1-x^2}}\,dx$

10. $\displaystyle\int_{-\pi/4}^{3\pi/4} \frac{d\phi}{\sqrt{1+\cos^2\phi}}$

11. $\displaystyle\int_0^{3/5} \frac{dt}{\sqrt{1-t^2}\,\sqrt{16-25t^2}}$

12. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{2-\sin^2 x}}$

13. $\displaystyle\frac{d}{du}(\operatorname{cn} u)$

14. $\displaystyle\int_1^{\infty} e^{-x^2/2}\,dx$

15. $\displaystyle\int_0^{\infty} x^{5/2}e^{-x}\,dx$

16. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$

17. $\displaystyle\int_0^{\pi/2} \sqrt{\sin^3\theta\cos^5\theta}\,d\theta$

18. $\displaystyle\int_0^{\infty} \frac{e^{-x}\,dx}{x^{1/4}}$

19. $\displaystyle\int_5^{\infty} e^{-x^2}\,dx$

20. $\displaystyle\int_0^{\pi/2} (\cos x)^{5/2}\,dx$

21. $\displaystyle\int_0^{5} x^{-1/3}(5-x)^{10/3}\,dx$

22. $\displaystyle\int_0^{7\pi/8} \sqrt{4-\sin^2 x}\,dx$

23. Find an expression for the exact value of $\Gamma(55.5)$ in terms of double factorials (!!), powers of 2 and $\sqrt{\pi}$. For !!, see Chapter 1, Section 13C, Example 2.

24. Using your result in Problem 23 and equation (5.4), find an expression for the exact value of $\Gamma(-54.5)$.

25. As in problems 23 and 24, find expressions for the exact values of $\Gamma(28.5)$ and $\Gamma(-27.5)$.

CHAPTER 12

Series Solutions of Differential Equations; Legendre, Bessel, Hermite, and Laguerre Functions

1. INTRODUCTION

By now you are well aware that physical problems in many fields lead to differential equations to be solved. In Chapter 13, we will discuss a variety of physical problems which lead to partial differential equations. To solve them, we will need the solutions of some ordinary differential equations which cannot be solved in terms of elementary functions. So in this chapter we will learn about these equations and their solutions. However, if you would prefer to see some of the physics before you study the math, and if you've studied Chapters 7 and 8, you could first do Sections 1 to 4 of Chapter 13, and then come back to Chapter 12 to learn the material needed for the rest of Chapter 13. (See the Preface.)

Now you may be thinking that your computer will give you the solutions of these differential equations so you don't need to study this. What your computer may give you is the *name* of a function. What you need to know is something about the function: graphs; formulas for derivatives and integrals; formulas that correspond to trigonometric identities for sine and cosine functions; and other useful information so that you can work with these named functions which occur often in applications. This is what we will discuss in this chapter.

The differential equations we are going to solve are linear, like the equations of Chapter 8, Section 5, but with coefficients which are functions of x instead of constants, that is, of the form $y'' + f(x)y' + g(x)y = 0$. A method of solving such equations which we will find useful is to assume an infinite series solution.

Example 1. We illustrate the method of series solution by solving the following simple equation (which you can easily solve by elementary methods also!):

$$y' = 2xy. \tag{1.1}$$

We assume a solution of this differential equation in the form of a power series, namely

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \tag{1.2}$$
$$= \sum_{n=0}^{\infty} a_nx^n,$$

where the a's are to be found. Differentiating (1.2) term by term, we get

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots \tag{1.3}$$
$$= \sum_{n=1}^{\infty} na_nx^{n-1}.$$

We substitute (1.2) and (1.3) into the differential equation (1.1); we then have two power series equal to each other. Now the original differential equation is to be satisfied for all values of x, that is, y' and $2xy$ are to be the same function of x. Since a given function has only one series expansion in powers of x (see Chapter 1, Section 11), the two series must be identical, that is, the coefficients of corresponding powers of x must be equal. We get the following set of equations for the a's:

$$a_1 = 0, \qquad a_2 = a_0, \qquad a_3 = \tfrac{2}{3}a_1 = 0, \qquad a_4 = \tfrac{1}{2}a_0, \tag{1.4}$$

or in general:

$$na_n = 2a_{n-2}, \qquad a_n = \begin{cases} 0, & \text{odd } n, \\ \frac{2}{n}a_{n-2}, & \text{even } n. \end{cases} \tag{1.5}$$

Putting $n = 2m$ (since only even terms appear in this series), we get

$$a_{2m} = \frac{2}{2m}a_{2m-2} = \frac{1}{m}a_{2m-2} = \frac{1}{m}\frac{1}{m-1}a_{2m-4} = \cdots = \frac{1}{m!}a_0. \tag{1.6}$$

Substituting these values of the coefficients into the assumed solution (1.2) gives the solution

$$y = a_0 + a_0x^2 + \frac{1}{2!}a_0x^4 + \cdots + \frac{1}{m!}a_0x^{2m} + \cdots \quad = a_0\sum_{m=0}^{\infty}\frac{x^{2m}}{m!}. \tag{1.7}$$

Example 2. Compare this with the solution by an elementary method (in this case, separation of variables):

$$\frac{dy}{y} = 2x\,dx, \quad \ln y = x^2 + \ln c, \quad y = ce^{x^2}.$$

Expanding this in a series of powers of x^2, we get:

$$y = c\left(1 + x^2 + \frac{x^4}{2!} + \cdots\right) = c\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$

which, with $c = a_0$, is the same as the series solution (1.7).

You cannot always expect to find the closed form of a power series solution (that is, an elementary function for which your series solution is the power series expansion), but in simple cases you may recognize it. Of course, in that case, the problem could have been done without series; the real need for series is in problems for which there is no closed form in terms of elementary functions. Also you should realize that not all solutions have series expansions in powers of x, for example, $\ln x$ or $1/x^2$. All we can say is that if there is a solution which can be represented by a convergent power series this method will find it. We shall discuss later (Section 21) some theorems which tell us when we can expect to find such a solution.

In the following sections we consider some differential equations which occur frequently in applied problems and which are usually solved by series methods.

PROBLEMS, SECTION 1

Solve the following differential equations by series and also by an elementary method and verify that your solutions agree. Note that the goal of these problems is not to get the answer (that's easy by computer or by hand) but to become familiar with the method of series solutions which we will be using later. Check your results by computer.

1. $xy' = xy + y$

2. $y' = 3x^2y$

3. $xy' = y$

4. $y'' = -4y$

5. $y'' = y$

6. $y'' - 2y' + y = 0$

7. $x^2y'' - 3xy' + 3y = 0$

8. $(x^2 + 2x)y'' - 2(x + 1)y' + 2y = 0$

9. $(x^2 + 1)y'' - 2xy' + 2y = 0$

10. $y'' - 4xy' + (4x^2 - 2)y = 0$

2. LEGENDRE'S EQUATION

The Legendre differential equation is

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0, \tag{2.1}$$

where l is a constant. This equation arises in the solution of partial differential equations in spherical coordinates (see Problem 10.2 and Chapter 13, Section 7) and so in problems in mechanics, quantum mechanics, electromagnetic theory, heat, etc., with spherical symmetry. Also see an application in Section 5.

Although the most useful solutions of this equation are polynomials (called the *Legendre polynomials*), one way to find them is to assume a series solution of the differential equation, and show that the series terminates after a finite number of terms. [There are other ways of finding the Legendre polynomials; see Sections 4 and 5, and Chapter 3, Section 14, Example 6.] We assume the series solution (1.2) for y and differentiate it term by term twice to get y' and y'':

$$\begin{cases} y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n + \cdots , \\ y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1} + \cdots , \\ y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots + n(n-1)a_nx^{n-2} + \cdots . \end{cases} \tag{2.2}$$

We substitute (2.2) into (2.1) and collect the coefficients of the various powers of x; it is convenient to tabulate them as follows:

	const.	x	x^2	x^3	$\cdots\ x^n\ \cdots$
y''	$2a_2$	$6a_3$	$12a_4$	$20a_5$	$(n+2)(n+1)a_{n+2}$
$-x^2y''$			$-2a_2$	$-6a_3$	$-n(n-1)a_n$
$-2xy'$		$-2a_1$	$-4a_2$	$-6a_3$	$-2na_n$
$l(l+1)y$	$l(l+1)a_0$	$l(l+1)a_1$	$l(l+1)a_2$	$l(l+1)a_3$	$l(l+1)a_n$

Next we set the total coefficient of each power of x equal to zero [because, as discussed in Section 1, y must satisfy (2.1) identically]. For the first few powers of x we get

$$\begin{aligned} &2a_2 + l(l+1)a_0 = 0 && \text{or} \quad a_2 = -\frac{l(l+1)}{2}a_0; \\ &6a_3 + (l^2+l-2)a_1 = 0 && \text{or} \quad a_3 = -\frac{(l-1)(l+2)}{6}a_1; \\ &12a_4 + (l^2+l-6)a_2 = 0 && \text{or} \quad a_4 = -\frac{(l-2)(l+3)}{12}a_2 \\ & && \qquad\quad = \frac{l(l+1)(l-2)(l+3)}{4!}a_0; \end{aligned} \tag{2.3}$$

and from the x^n coefficient we get

$$(n+2)(n+1)a_{n+2} + (l^2+l-n^2-n)a_n = 0. \tag{2.4}$$

The coefficient of a_n in (2.4) can be factored to give

$$l^2 - n^2 + l - n = (l+n)(l-n) + (l-n) = (l-n)(l+n+1). \tag{2.5}$$

Then we can write a general formula for a_{n+2} in terms of a_n. This formula (2.6) includes the formulas (2.3) for a_2, a_3, and a_4, and makes it possible for us to find any even coefficient as a multiple of a_0, and any odd coefficient as a multiple of a_1. Solving (2.4) for a_{n+2} and using (2.5), we have

$$a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)}a_n. \tag{2.6}$$

The general solution of (2.1) is then a sum of two series containing (as the solution of a second-order differential equation should) two constants a_0 and a_1 to be determined by the given initial conditions:

$$\begin{aligned} y = a_0 &\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \cdots\right] \\ &+ a_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5 - \cdots\right]. \end{aligned} \tag{2.7}$$

From equation (2.6) you can see by the ratio test that these series converge for $x^2 < 1$. It can be shown that, in general, they do not converge for $x^2 = 1$.

▶ **Example.** Consider the a_1 series for $l = 0$. If $x^2 = 1$, this series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots$, which is divergent by the integral test (Chapter 1, Section 6B). Now in many applications x is the cosine of an angle θ, and l is a (nonnegative) integer. We want a solution which converges for all θ, that is, a solution which converges at $x = \pm 1$ as well as for $|x| < 1$. We can always find one (but not two) such solutions when l is an integer; let us see how.

Legendre Polynomials We have seen that for $l = 0$ the a_1 series in (2.7) diverges. But look at the a_0 series; it gives just $y = a_0$ for $l = 0$ since all the rest of the terms contain the factor l. If $l = 1$, the a_0 series is divergent at $x^2 = 1$, but the a_1 series stops with $y = a_1 x$ [since all the rest of the terms in the a_1 series contain the factor $(l - 1)$]. For any integral l, one series terminates giving a polynomial solution; the other series is divergent at $x^2 = 1$. (Negative integral values of l would simply give solutions already obtained for positive l's; for example, $l = -2$ gives the polynomial solution $y = a_1 x$ which is the same as the $l = 1$ solution. Consequently, it is customary to restrict l to nonnegative values.) Thus we obtain a set of polynomial solutions of the Legendre equation, one for each nonnegative integral l. Each solution contains an arbitrary constant factor (a_0 or a_1); for $l = 0$, $y = a_0$; for $l = 1$, $y = a_1 x$, and so on. If the value of a_0 or a_1 in each polynomial is selected so that $y = 1$ when $x = 1$, the resulting polynomials are called *Legendre Polynomials*, written $P_l(x)$. From (2.6) and (2.7) and the requirement $P_l(1) = 1$, we find the following expressions for the first few Legendre polynomials:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1). \tag{2.8}$$

Finding a few more Legendre polynomials by this method and other methods will be left to the problems. Although $P_l(x)$ for any integral l may be found by this method, simpler ways of obtaining the Legendre polynomials for larger l will be outlined in Sections 4 and 5. Of course, if you just want the formula for a particular P_l, you can find it by computer or in reference books.

Eigenvalue Problems In finding the Legendre polynomials as solutions of Legendre's equation (2.1), we have solved an *eigenvalue problem*. (See Chapter 3, Sections 11 and 12.) Recall that in an eigenvalue problem we are given an equation or a set of equations containing a parameter, and we want solutions that satisfy some special requirement; in order to obtain such solutions we must choose particular values (called eigenvalues) for the parameter in the problem. In finding the Legendre polynomials, we asked for series solutions of Legendre's equation (2.1) which converged at $x = \pm 1$. We saw that we could obtain such solutions if the parameter took on any integral value. The values of l, namely $0, 1, 2, \cdots$, are called *eigenvalues* (or *characteristic values*); the corresponding solutions $P_l(x)$ are called *eigenfunctions* (or *characteristic functions*).

Note the parallel between the eigenvalue-eigenvector problems of Chapter 3 and the eigenvalue-eigenfunction problems of this chapter. Recall that in Chapter 3, we wrote an eigenvalue equation as $\mathrm{M}\mathrm{r} = \lambda \mathrm{r}$ where M was a matrix operator which operated on the eigenvector r to produce a multiple of r. The Legendre equation is of the form $f(D)y(x) = l(l+1)y(x)$ where $f(D)$ is a differential operator which operates on the eigenfunction $y(x)$ to produce a multiple of $y(x)$. See Section 22 and Chapter 13 for further examples of differential equations whose solutions are eigenfunctions.

The Legendre polynomials are also called Legendre functions of the first kind. The second solution for each l, which is an infinite series (convergent for $x^2 < 1$), is called a Legendre function of the second kind and is denoted by $Q_l(x)$ (See Problem 4.) The functions $Q_l(x)$ are not used as frequently as the polynomials $P_l(x)$. For fractional l both solutions are infinite series; these again occur less frequently in applications.

PROBLEMS, SECTION 2

1. Using (2.6) and (2.7) and the requirement that $P_l(l) = 1$, find $P_2(x)$, $P_3(x)$, and $P_4(x)$. Check your results by computer.

2. Show that $P_l(-1) = (-1)^l$. *Hint:* When is $P_l(x)$ an even function and when is it an odd function?

3. Computer plot graphs of $P_l(x)$ for $l = 0, 1, 2, 3, 4$, and x from -1 to 1.

4. Use the method of reduction of order [Chapter 8, Section 7(e)] and the known solution $P_l(x)$ of Legendre's equation to find the second solution $Q_l(x)$ (in terms of an integral). Evaluate the integral for the cases $l = 0$ and $l = 1$ to find Q_0 and Q_1. Note the divergence of the logarithms at $x = \pm 1$. Expand the logarithms in Q_0 to get the divergent series mentioned above [a_1 series in (2.7) with $l = 0$, $x^2 = 1$].

3. LEIBNIZ' RULE FOR DIFFERENTIATING PRODUCTS

Let us digress for a moment to discuss a very useful formula called *Leibniz rule* for finding a high order derivative of a product. We shall first illustrate this by a numerical example. We could, of course, do a numerical problem by computer, but our purpose is to understand the general formula which we will need in derivations. Also, when you know Leibniz rule, you may find in simple numerical cases that you can write down the answer for a high order derivative of a product faster than you can type the problem into the computer (see Problems 2 to 5).

Example. Find $(d^9/dx^9)(x \sin x)$.

Leibniz rule says that the answer is

$$x\frac{d^9}{dx^9}(\sin x) + 9\frac{d}{dx}(x)\frac{d^8}{dx^8}(\sin x) + \frac{9\cdot 8}{2!}\frac{d^2}{dx^2}(x)\frac{d^7}{dx^7}(\sin x) + \cdots. \tag{3.1}$$

This should remind you of a binomial expansion

$$(a+b)^9 = a^0b^9 + 9ab^8 + \frac{9\cdot 8}{2!}a^2b^7 + \cdots.$$

The coefficients in (3.1) are, in fact, binomial coefficients, and the sum of the orders of the two derivatives in each term is 9. (You may find the second hint in Problem 6 useful in understanding and remembering this.) Now if it happens, as here, that the derivatives of one factor become zero after the first few, the rule saves much work. In (3.1), $(d^2/dx^2)(x) = 0$ and all higher derivatives of x are zero so we get

$$\frac{d^9}{dx^9}(x\sin x) = x\frac{d^9}{dx^9}(\sin x) + 9\frac{d^8}{dx^8}(\sin x) = x\cos x + 9\sin x.$$

▶ PROBLEMS, SECTION 3

1. By Leibniz' rule, write the formula for $(d^n/dx^n)(uv)$.

Use Problem 1 to find the following derivatives.

2. $(d^{10}/dx^{10})(xe^x)$

3. $(d^6/dx^6)(x^2 \sin x)$

4. $d^{25}/dx^{25})(x \cos x)$

5. $d^{100}/dx^{100})(x^2 e^{-x})$

6. Verify Problem 1. *Hints:* One method is to use mathematical induction. Another method is to write

$$\frac{d}{dx}(uv) = D(uv) = (D_u + D_v)(uv),$$

where D_u acts only on u and D_v acts only on v, that is, $D_u(uv)$ means $v(du/dx)$, etc. Then

$$\frac{d^n}{dx^n}(uv) = (D_u + D_v)^n(uv).$$

Expand $(D_u + D_v)^n$ by the binomial theorem and interpret the terms to get Leibniz' rule.

▶ 4. RODRIGUES' FORMULA

We have obtained the Legendre polynomials as solutions of Legendre's equation when l is an integer; there are other ways of obtaining them. We shall prove that Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2 - 1)^l \tag{4.1}$$

gives correctly the Legendre polynomials $P_l(x)$. There are two parts to the proof. First we show that if

$$v = (x^2 - 1)^l, \tag{4.2}$$

then $d^l v/dx^l$ is a solution of Legendre's equation; then we show that $P_l(1) = 1$ in (4.1). To prove the first part, find dv/dx in (4.2) and multiply it by $x^2 - 1$:

$$(x^2 - 1)\frac{dv}{dx} = (x^2 - 1)l(x^2 - 1)^{l-1} \cdot 2x = 2lxv. \tag{4.3}$$

Differentiate (4.3) $l + 1$ times by Leibniz' rule:

$$\begin{aligned}(x^2 - 1)\frac{d^{l+2}v}{dx^{l+2}} + (l+1)(2x)\frac{d^{l+1}v}{dx^{l+1}} + \frac{(l+1)l}{2!} \cdot 2 \cdot \frac{d^l v}{dx^l}& \\ = 2lx\frac{d^{l+1}v}{dx^{l+1}} + 2l(l+1)\frac{d^l v}{dx^l}&.\end{aligned} \tag{4.4}$$

Simplifying (4.4), we get (Problem 1)

$$(1 - x^2)\left(\frac{d^l v}{dx^l}\right)'' - 2x\left(\frac{d^l v}{dx^l}\right)' + l(l+1)\frac{d^l v}{dx^l} = 0. \tag{4.5}$$

This is just Legendre's equation (2.1) with $y = d^l v/dx^l$; thus we see that $d^l v/dx^l = (d^l/dx^l)(x^2-1)^l$ is a solution of Legendre's equation as we claimed. It *is* a polynomial of degree l, and since we have previously called the polynomial solution of degree l the Legendre polynomial $P_l(x)$, this must be it with the possible exception of the numerical factor which must give $P_l(1) = 1$. A simple method of showing that $P_l(1) = 1$ for the functions $P_l(x)$ in (4.1) is outlined in Problem 2.

PROBLEMS, SECTION 4

1. Verify equations (4.4) and (4.5).

2. Show that $P_l(1) = 1$, with $P_l(x)$ given by (4.1), in the following way. Factor $(x^2-1)^l$ into $(x+1)^l(x-1)^1$ and differentiate the product l times by Leibniz' rule. Without writing out very many terms you should see that every term but one contains the factor $x-1$ and so becomes zero when $x = 1$. Use this to evaluate $P_l(x)$ in (4.1) when $x = 1$ to get $P_l(1) = 1$.

3. Find $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$ from Rodrigues' formula (4.1). Check your results by computer.

4. Show that $\int_{-1}^{1} x^m P_l(x)\,dx = 0$ if $m < l$. *Hint:* Use Rodrigues' formula (4.1) and integrate repeatedly by parts, differentiating the power of x and integrating the derivative each time.

5. GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

The expression

$$\Phi(x,h) = (1-2xh+h^2)^{-1/2}, \qquad |h| < 1, \tag{5.1}$$

is called the generating function for Legendre polynomials. We shall show that

$$\Phi(x,h) = P_0(x) + hP_1(x) + h^2P_2(x) + \cdots = \sum_{l=0}^{\infty} h^l P_l(x), \tag{5.2}$$

where the functions $P_l(x)$ are the Legendre polynomials. (For discussion of convergence of the series, see Chapter 14, Problem 2.43.) Let us first verify a few terms of (5.2). For simplicity put $2xh - h^2 = y$ into (5.1), expand $(1-y)^{-1/2}$ in powers of y, then substitute back $y = 2xh - h^2$ and collect powers of h to get

$$\begin{aligned}
\Phi &= (1-y)^{-1/2} = 1 + \tfrac{1}{2}y + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}y^2 + \cdots \\
&= 1 + \tfrac{1}{2}(2xh-h^2) + \tfrac{3}{8}(2xh-h^2)^2 + \cdots \\
&= 1 + xh - \tfrac{1}{2}h^2 + \tfrac{3}{8}(4x^2h^2 - 4xh^3 + h^4) + \cdots \\
&= 1 + xh + h^2(\tfrac{3}{2}x^2 - \tfrac{1}{2}) + \cdots \\
&= P_0(x) + hP_1(x) + h^2P_2(x) + \cdots .
\end{aligned} \tag{5.3}$$

This is not a proof that the functions called $P_l(x)$ in (5.2) are really Legendre polynomials, but merely a verification of the first few terms. To prove in general that the polynomials called $P_l(x)$ in (5.2) are Legendre polynomials we must show that they satisfy Legendre's equation and that they have the property $P_l(1) = 1$. The latter is easy to prove; putting $x = 1$ in (5.1) and (5.2), we get

$$\begin{aligned}\Phi(1,h) &= (1-2h+h^2)^{-1/2} = \frac{1}{1-h} = 1 + h + h^2 + \cdots \\ &\equiv P_0(1) + P_1(1)h + P_2(1)h^2 + \cdots .\end{aligned} \tag{5.4}$$

Since this is an identity in h, the functions $P_l(x)$ in (5.2) have the property $P_l(1) = 1$. To show that they satisfy Legendre's equation, we shall use the following identity which can be verified from (5.1) by straightforward differentiation and some algebra (Problem 2):

$$(1-x^2)\frac{\partial^2 \Phi}{\partial x^2} - 2x\frac{\partial \Phi}{\partial x} + h\frac{\partial^2}{\partial h^2}(h\Phi) = 0. \tag{5.5}$$

Substituting the series (5.2) for Φ into (5.5), we get

$$(1-x^2)\sum_{l=0}^{\infty} h^l P_l''(x) - 2x\sum_{l=0}^{\infty} h^l P_l'(x) + \sum_{l=0}^{\infty} l(l+1)h^l P_l(x) = 0. \tag{5.6}$$

This is an identity in h, so the coefficient of each power of h must be zero. Setting the coefficient of h^l equal to zero, we get

$$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0. \tag{5.7}$$

This is Legendre's equation, so we have proved that the functions $P_l(x)$ in (5.2) satisfy it as claimed.

Recursion Relations The generating function is useful in deriving the *recursion relations* (also called *recurrence relations*) for Legendre polynomials. These recursion relations are identities in x and are used (as trigonometric identities are) to simplify work and to help in proofs and derivations. Some examples of recursion relations are:

$$\begin{aligned}
&\text{(a)} \quad lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x),\\
&\text{(b)} \quad xP_l'(x) - P_{l-1}'(x) = lP_l(x),\\
&\text{(c)} \quad P_l'(x) - xP_{l-1}'(x) = lP_{l-1}(x),\\
&\text{(d)} \quad (1-x^2)P_l'(x) = lP_{l-1}(x) - lxP_l(x),\\
&\text{(e)} \quad (2l+1)P_l(x) = P_{l+1}'(x) - P_{l-1}'(x),\\
&\text{(f)} \quad (1-x^2)P_{l-1}'(x) = lxP_{l-1}(x) - lP_l(x).
\end{aligned} \tag{5.8}$$

We shall now derive (5.8a); the problems outline derivations of the other equations.

From (5.1) we get

$$\begin{aligned}\frac{\partial \Phi}{\partial h} &= -\tfrac{1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h);\\ (1-2xh+h^2)\frac{\partial \Phi}{\partial h} &= (x-h)\Phi.\end{aligned} \tag{5.9}$$

Substituting the series (5.2) and its derivative with respect to h into (5.9), we get

$$(1-2xh+h^2)\sum_{l=1}^{\infty} lh^{l-1}P_l(x) = (x-h)\sum_{l=0}^{\infty} h^l P_l(x).$$

This is an identity in h so we equate coefficients of h^{l-1}. Carefully adjusting indices so that we select the term in h^{l-1} each time, we find

$$lP_l(x) - 2x(l-1)P_{l-1}(x) + (l-2)P_{l-2}(x) = xP_{l-1}(x) - P_{l-2}(x) \tag{5.10}$$

which simplifies to (5.8a). The recursion relation (5.8a) gives the simplest way of finding any Legendre polynomial when we know the Legendre polynomials for smaller l (Problem 3).

Expansion of a Potential The generating function is useful in problems involving the potential associated with any inverse square force. Recall that the gravitational force between two point masses separated by a distance d is proportional to $1/d^2$ and the associated potential energy is proportional to $1/d$. Similarly, the electrostatic force between two electric charges a distance d apart is proportional to $1/d^2$ and the associated electrostatic potential energy is proportional to $1/d$.

Example 1. In either case we can write the potential as

$$V = \frac{K}{d}, \tag{5.11}$$

where K is an appropriate constant. In Figure 5.1, let the two masses (or charges) be at the heads of vectors $\mathbf{r}$ and $\mathbf{R}$. Then, by the law of cosines, the distance between them is

$$\begin{aligned} d &= |\mathbf{R}-\mathbf{r}| \\ &= \sqrt{R^2 - 2Rr\cos\theta + r^2} \\ &= R\sqrt{1 - 2\frac{r}{R}\cos\theta + \left(\frac{r}{R}\right)^2} \end{aligned} \tag{5.12}$$

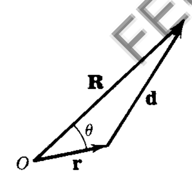


Figure 5.1

and the gravitational or electric potential is

$$V = \frac{K}{R}\left[1 - \frac{2r}{R}\cos\theta + \left(\frac{r}{R}\right)^2\right]^{-1/2}. \tag{5.13}$$

For $|\mathbf{r}| < |\mathbf{R}|$, we make the change of variables

$$\begin{aligned} h &= \frac{r}{R}, \\ x &= \cos\theta. \end{aligned} \tag{5.14}$$

(*Note:* x is *not* a coordinate but just a new variable standing for $\cos\theta$.) Then in terms of the generating function Φ in (5.1) we have

$$\begin{aligned} d &= R\sqrt{1-2hx+h^2} \\ V &= \frac{K}{R}(1-2hx+h^2)^{-1/2} = \frac{K}{R}\Phi. \end{aligned} \tag{5.15}$$

Using (5.2), we can write the potential V as an infinite series

$$V = \frac{K}{R}\sum_{l=0}^{\infty} h^l P_l(x) \tag{5.16}$$

or in terms of r and θ [using (5.14)]

$$V = \frac{K}{R}\sum_{l=0}^{\infty} \frac{r^l P_l(\cos\theta)}{R^l} = K\sum_{l=0}^{\infty}\frac{r^l P_l(\cos\theta)}{R^{l+1}}. \tag{5.17}$$

In many applications the distance $|\mathbf{R}|$ is much larger than $|\mathbf{r}|$. Then the terms of the series (5.17) decrease rapidly in magnitude because of the factor $(r/R)^l$, and the potential can be approximated by using only a few terms in the series.

We can make (5.17) more general and useful by considering the following problem. (We shall discuss the electrical case for definiteness—the gravitational case could be discussed in parallel fashion.)

▶ **Example 2.** Suppose there are a large number of charges q_i at points $\mathbf{r}_i$. The electrostatic potential V_i at the point $\mathbf{R}$ due to the charge q_i at $\mathbf{r}_i$ means the electrostatic potential energy of a pair of charges, namely, a *unit* charge at $\mathbf{R}$ and the charge q_i at $\mathbf{r}_i$; this is given by (5.11) and (5.12), or by (5.17), with $r = r_i$, $\theta = \theta_i$, and $K = q_i \cdot 1 \cdot K'$, where K' is a numerical constant depending on the choice of units:

$$V_i = K' q_i \sum_{l=0}^{\infty} \frac{r_i^l P_l(\cos\theta_i)}{R^{l+1}}. \tag{5.18}$$

The total potential V at $\mathbf{R}$ due to all the charges q_i is then a sum over i of all the series (5.18), namely

$$V = \sum_i V_i = K'\sum_i q_i \sum_{l=0}^{\infty}\frac{r_i^l P_l(\cos\theta_i)}{R^{l+1}} = K'\sum_{l=0}^{\infty}\frac{\sum_i q_i r_i^l P_l(\cos\theta_i)}{R^{l+1}}. \tag{5.19}$$

▶ **Example 3.** If, instead of a set of discrete charges, we have a continuous charge distribution, then the sum over i becomes an integral, namely

$$\int r^l P_l(\cos\theta)\,dq \qquad \text{or} \qquad \iiint r^l P_l(\cos\theta)\rho\,d\tau, \tag{5.20}$$

where ρ is the charge density, and the integral is over the space occupied by the charge distribution. Then (5.19) becomes

$$V = K'\sum_l \frac{1}{R^{l+1}}\iiint r^l P_l(\cos\theta)\rho\,d\tau. \tag{5.21}$$

The terms of the series (5.21) can be interpreted physically. The $l = 0$ term is

$$\frac{1}{R}\iiint \rho\,d\tau = \frac{1}{R}\cdot(\text{total charge}). \tag{5.22}$$

Thus if R is large enough compared to all the r_i or all the values of r at points of the charge distribution, we can approximate the potential of the distribution as that of a single charge at the origin of magnitude equal to the total charge of the distribution. The $l = 1$ term of the series (5.21) is

$$\frac{1}{R^2}\iiint r\cos\theta\,\rho\,d\tau. \tag{5.23}$$

To interpret this recall that the *electric dipole moment* of a pair of charges $+q$ and $-q$ a distance d apart (as in Figure 5.2) is defined as the vector $q\mathbf{d}$, where $\mathbf{d}$ is the vector from $-q$ to $+q$. Since the vector $q\mathbf{d}$ is equal to $q(\mathbf{r}_1 - \mathbf{r}_2) = q\mathbf{r}_1 - q\mathbf{r}_2$, we often call $q\mathbf{r}_1$ and $-q\mathbf{r}_2$ the dipole moments of $+q$ and $-q$ about O; then the total dipole moment due to the two charges is just the sum of the two moments. Suppose we calculate the dipole moment about O of all the charges q_i; this is the vector sum $\sum_i q_i r_i \cos\theta_i$, since θ_i is the angle between $\mathbf{R}$ and $\mathbf{r}_i$. In the case of a continuous charge distribution this sum becomes

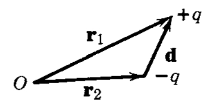


Figure 5.2

$$\iiint r\cos\theta\,\rho\,d\tau. \tag{5.24}$$

Thus we see from (5.23) and (5.24) that the second term of the series (5.21) is $1/R^2$ times the component in the $\mathbf{R}$ direction of the dipole moment of the charge distribution. If you consider the fact that the first term of (5.21) involves the total charge (a scalar, that is, a tensor of rank zero) and the second term involves the dipole moment (a vector, that is, a tensor of rank one), it may not surprise you to learn that the third term involves a 2^{nd}-rank tensor known as the quadrupole moment of the charge distribution, the fourth term involves a 3^{rd}-rank tensor known as the octopole moment, etc. (See Problem 15 for more detail.)

▶ **Example 4.** Given a charge or mass distribution, the moments of various ranks and the terms in (5.21) can be computed. The opposite process is often of great interest in applied problems. Consider a satellite circling the earth; it is moving in the gravitational field of the earth's mass. If the mass distribution of the earth were spherically symmetric, then only the first term would appear in the series for the gravitational potential [this series would be (5.21) with ρ a mass density instead of a charge density]. But since the earth is not a perfect sphere (equatorial bulge, etc.), other terms are present in (5.21) and the corresponding forces affect the motion of satellites. From accurate measurements of the satellite orbits, it is now possible to calculate many terms of the series (5.21). Similarly, in the electrical case, experimental measurements give us information about the distribution of electric charge inside atoms and nuclei; our discussion here and equation (5.21) provide the basis for the interpretation of such measurements, and the terminology used in discussing them.

▶ PROBLEMS, SECTION 5

1. Find $P_3(x)$ by getting one more term in the generating function expansion (5.3).
2. Verify (5.5) using (5.1).
3. Use the recursion relation (5.8a) and the values of $P_0(x)$ and $P_1(x)$ to find $P_2(x)$, $P_3(x)$, $P_4(x)$, $P_5(x)$, and $P_6(x)$. [After you have found $P_3(x)$, use it to find $P_4(x)$, and so on for the higher order polynomials.]

4. Show from (5.1) that

$$(x-h)\frac{\partial \Phi}{\partial x} = h\frac{\partial \Phi}{\partial h}.$$

Substitute the series (5.2) for Φ, and so prove the recursion relation (5.8b).

5. Differentiate the recursion relation (5.8a) and use the recursion relation (5.8b) with l replaced by $l-1$ to prove the recursion relation (5.8c).

6. From (5.8b) and (5.8c), obtain (5.8d) and (5.8f). Then differentiate (5.8d) with respect to x and eliminate $P'_{l-1}(x)$ using (5.8b). Your result should be the Legendre equation. The derivation of Problems 4 to 6 constitutes an alternative proof [to that of equations (5.5) to (5.7)] that the functions $P_l(x)$ in (5.2) are Legendre polynomials.

7. Write (5.8c) with l replaced by $l+1$ and use it to eliminate the $xP'_l(x)$ term in (5.8b). You should get (5.8e).

Express each of the following polynomials as linear combinations of Legendre polynomials. *Hint:* Start with the highest power of x and work down in finding the correct combination.

8. $5-2x$ **9.** $3x^2+x-1$ **10.** x^4

11. $x-x^3$ **12.** $7x^4-3x+1$ **13.** x^5

14. Show that any polynomial of degree n can be written as a linear combination of Legendre polynomials with $l \le n$.

15. Expand the potential $V = K/d$ in (5.11) in the following way in order to see how the terms depend on the tensors mentioned above. In Figure 5.1 let $\mathbf{R}$ have the coordinates X, Y, Z and $\mathbf{r}$ have coordinates x, y, z. [*Note:* The coordinate x here is *not* the x in (5.14).] Then

$$V = \frac{K}{d} = K[(X-x)^2 + (Y-y)^2 + (Z-z)^2]^{-1/2}.$$

Consider X, Y, Z as constants and expand $V(x, y, z)$ in a three variable power series about the origin. (See Chapter 4, Section 2, for discussion of two-variable power series and generalize the method.) You should find

$$\begin{aligned} V = &\frac{K}{R} + \frac{K}{R^2}\left(\frac{X}{R}x + \cdots\right) \\ &+ \frac{K}{R^3}\left[\left(\frac{3}{2}\frac{X^2}{R^2} - \frac{1}{2}\right)x^2 + \cdots + \frac{3}{2}\frac{X}{R}\frac{Y}{R}2xy + \cdots\right] + \cdots \end{aligned}$$

and similar terms in y, z, y^2, xz, and so on. Now letting $\mathbf{r} = \mathbf{r}_i$ and $K = K'q_i$ for a charge distribution as in (5.18), and summing (or integrating) over the charge distribution, show that: the first term is just $(K'/R)\cdot$ total charge; the next group of terms (in x, y, z) involve the three components of the electric dipole moment; the sum of these terms is $(K'/R^2)\cdot$component of the dipole moment in the $\mathbf{R}$ direction; the next group (quadratic terms) involve six quantities of the form

$$\iiint x^2 \rho \, d\tau \qquad \text{and similar } y,\ z \text{ integrals,}$$

$$\iiint 2xy\rho \, d\tau \qquad \text{and similar } xz,\ yz \text{ integrals.}$$

If we split the $2xy$ term into xy and yx (and similarly for the $2xz$ and $2yz$ terms), we have the nine components of a 2^{nd}-rank tensor called the quadrupole moment. Use the "direct product" method of Chapter 10, Section 2 to show that it *is* a 2^{nd}-rank tensor. (Remember from Chapter 10 that, by definition, x, y, z are the components of a vector, that is, a 1^{st}-rank tensor.) Just as two charges $+q$ and $-q$ form an electric dipole, so four charges like this $\begin{smallmatrix}+\bullet\bullet-\\-\bullet\bullet+\end{smallmatrix}$ form an electric quadrupole and the quadratic terms in the V series give the potential of such a charge configuration. Again using Chapter 10, Section 2, show that the third-order terms in x, y, z form a 3^{rd}-rank tensor; this is known as the octopole moment. It can be represented physically by two quadrupoles side by side just as the quadrupole above was formed by two dipoles side by side.

6. COMPLETE SETS OF ORTHOGONAL FUNCTIONS

Orthogonal functions Two vectors **A** and **B** are orthogonal (perpendicular) if their scalar product is zero, that is, if

$$\sum_i A_i B_i = 0. \tag{6.1}$$

[See Chapter 3, equations (4.12) and (10.3).] Recall from Chapter 3, Section 14, that we can think of functions as elements of a vector space. Then by analogy with (6.1) we say that two functions $A(x)$ and $B(x)$ are orthogonal on (a, b) if

$$\int_a^b A(x)B(x)\,dx = 0. \tag{6.2}$$

If the functions $A(x)$ and $B(x)$ are complex, the definition of orthogonality is [see Chapter 3, equation (14.3)]

> $A(x)$ and $B(x)$ are orthogonal on (a, b) if
>
> $$\int_a^b A^*(x)B(x)\,dx = 0, \tag{6.3}$$
>
> where $A^*(x)$ is the complex conjugate of $A(x)$ (see Problem 1).

Since (6.3) is identical with (6.2) if $A(x)$ and $B(x)$ are real, we can take (6.3) as the general definition of orthogonality of $A(x)$ and $B(x)$ on (a, b).

If we have a whole set of functions $A_n(x)$ where $n = 1, 2, 3, \cdots$, and

$$\int_a^b A_n^*(x)A_m(x)\,dx = \begin{cases} 0 & \text{if } m \neq n, \\ \text{const.} \neq 0 & \text{if } m = n, \end{cases} \tag{6.4}$$

we call the functions $A_n(x)$ a *set of orthogonal functions*. We have already used such sets of functions in Fourier series. Recall that [Chapter 7, equation (5.2)]

$$\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n \neq 0. \end{cases} \tag{6.5}$$

Thus $\sin nx$ is a set of orthogonal functions on $(-\pi, \pi)$, or in fact on any other interval of length 2π. Similarly, the functions $\cos nx$ are orthogonal on $(-\pi, \pi)$.

Also the whole set consisting of $\sin nx$ *and* $\cos nx$ is a set of orthogonal functions on $(-\pi, \pi)$ since

$$\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0 \qquad \text{for } any\ n \text{ and } m.$$

We have used complex functions also, namely the set e^{inx}. For this set the orthogonality property is given by (6.4), namely

$$\text{(6.6)} \qquad \int_{-\pi}^{\pi} (e^{inx})^* e^{imx}\, dx = \int_{-\pi}^{\pi} e^{-inx} e^{imx}\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2\pi & \text{if } m = n. \end{cases}$$

Recall that $\sin nx$ and $\cos nx$ (or e^{inx}) were the functions used in a Fourier series expansion on $(-\pi, \pi)$. You should now realize that it was the orthogonality property that we used in getting the coefficients. When we multiplied the equation $f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imx}$ by e^{-inx} and integrated, the integrals of all the terms in the series except the c_n term were zero by the orthogonality property (6.6). There are many other sets of orthogonal functions besides the trigonometric or exponential ones. Just as we used the sine-cosine or exponential set to expand a function in a Fourier series, so we can expand a function in a series using other sets of orthogonal functions. We shall show this for the functions $P_l(x)$ after we prove that they are orthogonal.

Complete sets There is another important point to consider when we want to expand a function in terms of a set of orthogonal functions. Again let us consider the vector analogy. We write vectors in terms of their components and the basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. In two dimensions we need only two basis vectors, say $\mathbf{i}$ and $\mathbf{j}$. But if we tried to write three-dimensional vectors in terms of just $\mathbf{i}$ and $\mathbf{j}$, there would be some vectors we could not represent; we say that (in three dimensions) $\mathbf{i}$ and $\mathbf{j}$ are not a *complete* set of basis vectors. A simple way of expressing this (which generalizes to n dimensions) is to say that there is another vector (namely $\mathbf{k}$) which is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$. Thus we define a set of orthogonal basis vectors as complete if there is no other vector orthogonal to all of them (in the space of the number of dimensions we are considering). By analogy, we define a set of orthogonal functions as *complete* on a given interval if there is no other function orthogonal to all of them on that interval. Now it is easy to see that there are some vectors in three dimensions which cannot be represented using only $\mathbf{i}$ and $\mathbf{j}$. Similarly, there are functions which cannot be represented by a series using an incomplete set of orthogonal functions. We have discussed one example of this in Fourier series (Chapter 7, Section 11). If we are trying to represent a sound wave by a Fourier series, we must not leave out any of the harmonics; that is, the set of functions $\sin nx$, $\cos nx$ on $(-\pi, \pi)$ would not be complete if we left out some of the values of n. As another example, the set of functions $\sin nx$ is an orthogonal set on $(-\pi, \pi)$. However, it is not complete; to have a complete set we must include also the functions $\cos nx$, and you should recall that this is what we did in Fourier series. On the other hand, $\sin nx$ is a complete set on $(0, \pi)$; we used this fact when we started with a function given on $(0, \pi)$, defined it on $(-\pi, 0)$ to make it odd, and then expanded it in a sine series. Similarly, $\cos nx$ is a complete set on $(0, \pi)$. In this chapter, we are particularly interested in the fact (which we state without proof) that the Legendre polynomials are a complete set on $(-1, 1)$.

PROBLEMS, SECTION 6

1. Show that if $\int_a^b A^*(x)B(x)\,dx = 0$ [see (6.3)], then $\int_a^b A(x)B^*(x)\,dx = 0$, and vice versa.
2. Show that the functions $e^{in\pi x/l}$, $n = 0, \pm 1, \pm 2, \cdots$, are a set of orthogonal functions on $(-l, l)$.
3. Show that the functions x^2 and $\sin x$ are orthogonal on $(-1, 1)$. *Hint:* See Chapter 7, Section 9.
4. Show that the functions $f(x)$ and $g(x)$ are orthogonal on $(-a, a)$ if $f(x)$ is even and $g(x)$ is odd. (See Problem 3.)
5. Evaluate $\int_{-1}^{1} P_0(x)P_2(x)\,dx$ to show that these functions are orthogonal on $(-1, 1)$.
6. Show in two ways that $P_l(x)$ and $P_l'(x)$ are orthogonal on $(-1, 1)$. *Hint:* See Problem 4 and Problem 4.4.
7. Show that the set of functions $\sin nx$ is not a complete set on $(-\pi, \pi)$ by trying to expand the function $f(x) = 1$ on $(-\pi, \pi)$ in terms of them.
8. Show that the functions $\cos{(n + \frac{1}{2})x}$, $n = 0, 1, 2, \cdots$, are orthogonal on $(0, \pi)$. Expand the function $f(x) = 1$ on $(0, \pi)$ in terms of them. (Is it a complete set? See Chapter 7, end of Section 11.)
9. Show in two ways that $\int_{-1}^{1} P_{2n+1}(x)\,dx = 0$.

7. ORTHOGONALITY OF THE LEGENDRE POLYNOMIALS

We are going to show that the Legendre polynomials are a set of orthogonal functions on $(-1, 1)$, that is, that

$$\int_{-1}^{1} P_l(x)P_m(x)\,dx = 0 \qquad \text{unless } l = m. \tag{7.1}$$

To prove this we rewrite the Legendre differential equation (2.1) in the form

$$\frac{d}{dx}[(1 - x^2)P_l'(x)] + l(l + 1)P_l(x) = 0. \tag{7.2}$$

Write (7.2) for $P_l(x)$ and for $P_m(x)$; multiply the $P_l(x)$ equation by $P_m(x)$, and the $P_m(x)$ equation by $P_l(x)$ and subtract to get

$$\begin{aligned} P_m(x)\frac{d}{dx}[(1 - x^2)P_l'(x)] - P_l(x)\frac{d}{dx}[(1 - x^2)P_m'(x)]& \\ + [l(l + 1) - m(m + 1)]P_m(x)P_l(x) &= 0. \end{aligned} \tag{7.3}$$

The first two terms of (7.3) can be written as

$$\frac{d}{dx}[(1 - x^2)(P_mP_l' - P_lP_m')] \tag{7.4}$$

where, for simplicity, we have used $P_l = P_l(x)$, and so forth. Integrating (7.3) between -1 and 1 and using (7.4), we get

$$(1 - x^2)(P_mP_l' - P_lP_m')\Big|_{-1}^{1} + [l(l + 1) - m(m + 1)]\int_{-1}^{1} P_m(x)P_l(x)\,dx = 0. \tag{7.5}$$

The integrated term is zero because $(1-x^2)=0$ at $x=\pm 1$, and $P_m(x)$ and $P_l(x)$ are finite. The bracket in front of the integral is not zero unless $m=l$. Therefore the integral must be zero for $l \neq m$ and we have (7.1).

The method we have used here is a standard one which can be used for many other sets of orthogonal functions to prove the orthogonality property by using the differential equation satisfied by the functions. (See Problems 1 and 2; also see Section 19 and Problems 10.3, 22.7, 22.16, 22.24, 23.24b, and 23.25.)

Recall (Section 5, Problems 8 to 14) that we can write any polynomial of degree n as a linear combination of Legendre polynomials of degree $\leq n$. Thus, by (7.1), any polynomial of degree $< l$ is orthogonal to $P_l(x)$:

$$\int_{-1}^{1} P_l(x) \cdot (\text{any polynomial of degree } < l)\, dx = 0. \tag{7.6}$$

PROBLEMS, SECTION 7

1. By a method similar to that we used to show that the P_l's are an orthogonal set of functions on $(-1,1)$, show that the solutions of $y_n'' = -n^2 y_n$ are an orthogonal set on $(-\pi,\pi)$. *Hint:* You should know what functions the solutions y_n are; do not use the functions themselves, but you may use their values and the values of their derivatives at $-\pi$ and π to evaluate the integrated part of your equation.

2. Following the method in (7.2) to (7.5), show that the solutions of the differential equation
$$(1-x^2)y'' - 2xy' + [l(l+1) - (1-x^2)^{-1}]y = 0$$
are a set of orthogonal functions on $(-1,1)$.

3. Use Problem 4.4 to show that $\int_{-1}^{1} P_m(x)P_l(x)\,dx = 0$ if $m < l$. *Comment:* This amounts to a different proof of orthogonality—via Rodrigues' formula instead of the differential equation.

4. Use equation (7.6) to show that $\int_{-1}^{1} P_l(x)P_{l-1}'(x)\,dx = 0$. *Hint:* What is the degree of $P_{l-1}'(x)$? Also show that $\int_{-1}^{1} P_l'(x)P_{l+1}(x)\,dx = 0$.

5. Show that $\int_{-1}^{1} P_l(x)\,dx = 0,\ l > 0$. *Hint:* Consider $\int_{-1}^{1} P_l(x)P_0(x)\,dx$.

6. Show that $P_1(x)$ is orthogonal to $[P_l(x)]^2$ on $(-1,1)$. *Hint:* See Problem 6.4.

8. NORMALIZATION OF THE LEGENDRE POLYNOMIALS

If we take the scalar product of a vector with itself, $\mathbf{A}\cdot\mathbf{A} = A^2$, we get the square of the length (or norm) of the vector. If we divide $\mathbf{A}$ by its length, we get a unit vector. In Chapter 3, Section 14 we showed that we can think of functions as the vectors of a vector space and we defined the *norm* N of a function $A(x)$ on (a,b) by [see Chapter 3, equation (14.2)]

$$\int_a^b A^*(x)A(x)\,dx = \int_a^b |A(x)|^2\,dx = N^2.$$

We also say that the function $N^{-1}A(x)$ is *normalized*; like a unit vector, a normalized function has norm $= 1$. The factor N^{-1} is called the *normalization factor*. For example, $\int_0^\pi \sin^2 nx\,dx = \pi/2$. Then the norm of $\sin nx$ on $(0,\pi)$ is $\sqrt{\pi/2}$, and the functions $\sqrt{2/\pi}\sin nx$ have norm 1 on $(0,\pi)$, that is, they are normalized. A set of normalized orthogonal functions is called *orthonormal*. For example, $\sqrt{2/\pi}\sin nx$ is an orthonormal set on $(0,\pi)$.

Such a set of orthonormal functions may remind us of $\mathbf{i},\mathbf{j},\mathbf{k}$; like these unit vectors, the functions are orthogonal and have norm $= 1$. If the elements of a vector space are functions, we can then use a (complete) orthonormal subset of the functions as the basis vectors of the space. We think of expanding other functions in terms of them (by analogy with writing a three-dimensional vector in terms of $\mathbf{i},\mathbf{j},\mathbf{k}$). For example, suppose we have expanded a given function $f(x)$ on $(0,\pi)$ in a Fourier sine series:

$$f(x) = \sum B_n \sqrt{\frac{2}{\pi}}\sin nx.$$

We call $f(x)$ a vector with components B_n in terms of the basis vectors $\sqrt{2/\pi}\sin nx$. Thus, in quantum mechanics, we often refer to a function which describes the state of a physical system as either a state function or a state vector. Just as we can write a three-dimensional vector in terms of $\mathbf{i},\mathbf{j},\mathbf{k}$, or in terms of another basis, say $\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_\phi$, so we can expand a given $f(x)$ in terms of another orthonormal set of functions and find its components relative to this new basis. In Section 9, we shall see how to expand functions in Legendre series.

Just as we needed the norm of $\sin nx$ in Fourier series, so we shall need the norm of $P_l(x)$ in expanding functions in Legendre series. We shall prove that

$$\int_{-1}^{1}[P_l(x)]^2\,dx = \frac{2}{2l+1}. \tag{8.1}$$

Then the functions $\sqrt{(2l+1)/2}\,P_l(x)$ are an orthonormal set of functions on $(-1,1)$.

To prove (8.1), we use the recursion relation (5.8b), namely,

$$lP_l(x) = xP_l'(x) - P_{l-1}'(x). \tag{8.2}$$

Multiply (8.2) by $P_l(x)$ and integrate to get

$$l\int_{-1}^{1}[P_l(x)]^2\,dx = \int_{-1}^{1} xP_l(x)P_l'(x)\,dx - \int_{-1}^{1} P_l(x)P_{l-1}'(x)\,dx. \tag{8.3}$$

The last integral is zero by Problem 7.4. To evaluate the middle integral in (8.3), we integrate by parts:

$$\begin{aligned}\int_{-1}^{1} xP_l(x)P_l'(x)\,dx &= \frac{x}{2}[P_l(x)]^2\Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}[P_l(x)]^2\,dx\\ &= 1 - \frac{1}{2}\int_{-1}^{1}[P_l(x)]^2\,dx\end{aligned}$$

(see Problem 2.2). Then (8.3) gives

$$l\int_{-1}^{1}[P_l(x)]^2\,dx = 1 - \frac{1}{2}\int_{-1}^{1}[P_l(x)]^2\,dx$$

which simplifies to (8.1). We can combine (7.1) and (8.1) to write

$$\int_{-1}^{1} P_l(x)P_m(x)\,dx = \frac{2}{2l+1}\,\delta_{lm}. \tag{8.4}$$

▶ PROBLEMS, SECTION 8

Find the norm of each of the following functions on the given interval and state the normalized function.

1. $\cos nx$ on $(0, \pi)$

2. $P_2(x)$ on $(-1, 1)$

3. $xe^{-x/2}$ on $(0, \infty)$

4. $e^{-x^2/2}$ on $(-\infty, \infty)$

5. $xe^{-x^2/2}$ on $(0, \infty)$ *Hint:* See Chapter 4, Section 12.

6. Give another proof of (8.1) as follows. Multiply (5.8e) by $P_l(x)$ and integrate from -1 to 1. To evaluate the middle term, integrate by parts. Then use Problem 7.4.

7. Using (8.1), write the first four normalized Legendre polynomials and compare with the answers we found by a different method in Chapter 3, Section 14, Example 6.

▶ 9. LEGENDRE SERIES

Since the Legendre polynomials form a complete orthogonal set on $(-1, 1)$, we can expand functions in Legendre series just as we expanded functions in Fourier series.

▶ **Example 1.** Expand in a Legendre series the function $f(x)$ given by

$$f(x) = \begin{cases} 0, & -1 < x < 0, \\ 1, & 0 < x < 1, \end{cases} \tag{9.1}$$

(see Figure 9.1). We put

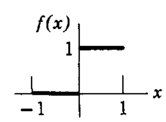


Figure 9.1

$$f(x) = \sum_{l=0}^{\infty} c_l P_l(x). \tag{9.2}$$

Our problem is to find the coefficients c_l. We do this by a method parallel to the one we used in finding the formulas for the coefficients in a Fourier series. We multiply both sides of (9.2) by $P_m(x)$ and integrate from -1 to 1. Because the Legendre polynomials are orthogonal, all the integrals on the right are zero except the one containing c_m, and we can evaluate it by (8.1). Thus we get

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \sum_{l=0}^{\infty} c_l \int_{-1}^{1} P_l(x)P_m(x)\,dx = c_m \cdot \frac{2}{2m+1}. \tag{9.3}$$

Using this result in our example (9.1), we find

$$\int_{-1}^{1} f(x)P_0(x)\,dx = c_0 \int_{-1}^{1} [P_0(x)]^2\,dx \quad \text{or} \quad \int_0^1 dx = c_0 \cdot 2, \quad c_0 = \tfrac{1}{2};$$

$$\int_{-1}^{1} f(x)P_1(x)\,dx = c_1 \int_{-1}^{1} [P_1(x)]^2\,dx \quad \text{or} \quad \int_0^1 x\,dx = c_1 \cdot \tfrac{2}{3}, \quad c_1 = \tfrac{3}{4};$$

$$\int_{-1}^{1} f(x)P_2(x)\,dx = c_2 \int_{-1}^{1} [P_2(x)]^2\,dx \quad \text{or} \quad \int_0^1 (\tfrac{3}{2}x^2 - \tfrac{1}{2})\,dx = c_2 \cdot \tfrac{2}{5}, \quad c_2 = 0.$$

Continuing in this way we find for the function given in (9.1)

$$f(x) = \tfrac{1}{2}P_0(x) + \tfrac{3}{4}P_1(x) - \tfrac{7}{16}P_3(x) + \tfrac{11}{32}P_5(x) + \cdots. \tag{9.4}$$

It is unnecessary for $f(x)$ to be continuous as it must be for expansion in a Maclaurin series. Just as for Fourier series, the Dirichlet conditions (see Chapter 7, Section 6) are a convenient set of sufficient conditions for a function $f(x)$ to be expandable in a Legendre series. If $f(x)$ satisfies the Dirichlet conditions on $(-1, 1)$, then at points inside $(-1, 1)$ (not necessarily at the endpoints), the Legendre series converges to $f(x)$ anywhere $f(x)$ is continuous and converges to the midpoint of the jump at discontinuities.

Example 2. Here is an interesting fact about Legendre series. Sometimes we want to fit a given curve as closely as possible by a polynomial of a given degree, say a cubic. The criterion of "Least Squares" is often used to determine the best fit. This means that if, say, we want to fit a given $f(x)$ on $(-1, 1)$ by a cubic, we find the coefficients a, b, c, d so that

$$\int_{-1}^{1} [f(x) - (ax^3 + bx^2 + cx + d)]^2\,dx \tag{9.5}$$

is as small as possible. Then

$$f(x) \cong ax^3 + bx^2 + cx + d \tag{9.6}$$

is called the best approximation (by a cubic) in the least squares sense. It can be proved that an expansion (as far as the desired degree of the polynomial approximation) in Legendre polynomials gives this best least squares approximation (Problem 16).

PROBLEMS, SECTION 9

Expand the following functions in Legendre series.

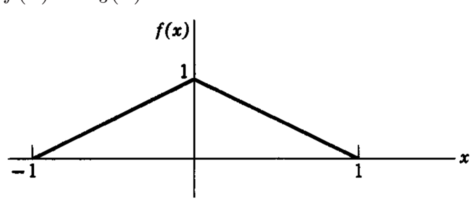
1. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$

2. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$

3. $f(x) = P_3'(x)$

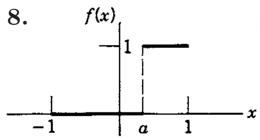
4. $f(x) = \arcsin x$

5.



6. $f(x) = \begin{cases} 0 & \text{on } (-1,0) \\ \left(\ln \frac{1}{x}\right)^2 & \text{on } (0,1) \end{cases}$ *Hint:* See Chapter 11, Section 3, Problem 13.

7. $f(x) = \begin{cases} 0 & \text{on } (-1,0) \\ \sqrt{1-x} & \text{on } (0,1) \end{cases}$ *Hint:* See Chapter 11, Sections 6 and 7.

8. 

Hint: Solve the recursion relation (5.8e) for $P_l(x)$ and show that

$$\int_a^1 P_l(x)\,dx = \frac{1}{2l+1}[P_{l-1}(a) - P_{l+1}(a)].$$

9. $f(x) = P_n'(x)$. *Hint:* For $l \geq n$, $\int_{-1}^{1} P_n'(x)P_l(x)\,dx = 0$ (Why?); for $l < n$, integrate by parts.

Expand each of the following polynomials in a Legendre series. You should get the same results that you got by a different method in the corresponding problems in Section 5.

10. $3x^2 + x - 1$ **11.** $7x^4 - 3x + 1$ **12.** $x - x^3$

Find the best (in the least squares sense) second-degree polynomial approximation to each of the given functions over the interval $-1 < x < 1$. (See Problem 16.)

13. x^4 **14.** $|x|$ **15.** $\cos \pi x$

16. Prove the least squares approximation property of Legendre polynomials [see (9.5) and (9.6)] as follows. Let $f(x)$ be the given function to be approximated. Let the functions $p_l(x)$ be the normalized Legendre polynomials, that is,

$$p_l(x) = \sqrt{\frac{2l+1}{2}}P_l(x) \qquad \text{so that} \qquad \int_{-1}^{1}[p_l(x)]^2\,dx = 1.$$

Show that the Legendre series for $f(x)$ as far as the $p_2(x)$ term is

$$f(x) = c_0p_0(x) + c_1p_1(x) + c_2p_2(x) \qquad \text{with} \quad c_l = \int_{-1}^{1} f(x)p_l(x)\,dx.$$

Write the quadratic polynomial satisfying the least squares condition as $b_0p_0(x) + b_1p_1(x) + b_2p_2(x)$ (by Problem 5.14 any quadratic polynomial can be written in this form). The problem is to find b_0, b_1, b_2 so that

$$I = \int_{-1}^{1}[f(x) - (b_0p_0(x) + b_1p_1(x) + b_2p_2(x))]^2\,dx$$

is a minimum. Square the bracket and write I as a sum of integrals of the individual terms. Show that some of the integrals are zero by orthogonality, some are 1 because the p_l's are normalized, and others are equal to the coefficients c_l. Add and subtract $c_0^2 + c_1^2 + c_2^2$ and show that

$$I = \int_{-1}^{1}[f^2(x) + (b_0 - c_0)^2 + (b_1 - c_1)^2 + (b_2 - c_2)^2 - c_0^2 - c_1^2 - c_2^2]\,dx.$$

Now determine the values of the b's to make I as small as possible. (*Hint:* The smallest value the square of a real number can have is zero.) Generalize the proof to polynomials of degree n.

10. THE ASSOCIATED LEGENDRE FUNCTIONS

A differential equation closely related to the Legendre equation is

$$(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0 \tag{10.1}$$

with $m^2 \le l^2$. We *could* solve this equation by series; however, it is more useful to know how the solutions are related to Legendre polynomials, so we shall simply verify the known solution. First we substitute

$$y = (1-x^2)^{m/2} u \tag{10.2}$$

into (10.1) and obtain (Problem 1)

$$(1-x^2)u'' - 2(m+1)xu' + [l(l+1) - m(m+1)]u = 0. \tag{10.3}$$

For $m = 0$, this is Legendre's equation with solutions $P_l(x)$. Differentiate (10.3), obtaining (Problem 1)

$$(1-x^2)(u')'' - 2[(m+1)+1]x(u')' + [l(l+1) - (m+1)(m+2)]u' = 0. \tag{10.4}$$

But this is just (10.3) with u' in place of u, and $(m+1)$ in place of m. In other words, if $P_l(x)$ is a solution of (10.3) with $m = 0$, $P_l'(x)$ is a solution of (10.3) with $m = 1$, $P_l''(x)$ is a solution with $m = 2$, and in general for integral m, $0 \le m \le l$, $(d^m/dx^m)P_l(x)$ is a solution of (10.3). Then

$$y = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \tag{10.5}$$

is a solution of (10.1). The functions in (10.5) are called *associated Legendre functions* are are denoted by

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \qquad \text{Associated Legendre functions} \tag{10.6}$$

[Some authors include a factor $(-1)^m$ in the definition of $P_l^m(x)$.]

A negative value for m in (10.1) does not change m^2, so a solution of (10.1) for positive m is also a solution for the corresponding negative m. Thus many references define $P_l^m(x)$ for $-l \le m \le l$ as equal to $P_l^{|m|}(x)$. Alternatively, we may use Rodrigues' formula (4.1) for $P_l(x)$ in (10.6) to get

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l. \tag{10.7}$$

It can be shown that (10.7) is a solution of (10.1) for either positive or negative m; however $P_l^{-m}(x)$ and $P_l^m(x)$ are then proportional rather than equal (see Problem 8).

For each m, the functions $P_l^m(x)$ are a set of orthogonal functions on $(-1,1)$ (Problem 3). The normalization constants can be evaluated; for the definition (10.7) we find (Problem 10)

$$\int_{-1}^{1} [P_l^m(x)]^2\,dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}. \tag{10.8}$$

The associated Legendre functions arise in many of the same problems in which Legendre polynomials appear (see the first paragraph of Section 2); in fact, the Legendre polynomials are just the special case of the functions $P_l^m(x)$ when $m=0$.

▶ PROBLEMS, SECTION 10

1. Verify equations (10.3) and (10.4).

2. The equation for the associated Legendre functions (and for Legendre functions when $m=0$) usually arises in the form (see, for example, Chapter 13, Section 7)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]y=0.$$

Make the change of variable $x=\cos\theta$, and obtain (10.1).

3. Show that the functions $P_l^m(x)$ for each m are a set of orthogonal functions on (-1,1), that is, show that

$$\int_{-1}^{1} P_l^m(x)P_n^m(x)\,dx = 0, \qquad l \neq n.$$

Hint: Use the differential equations (10.1) and follow the method of Section 7.

Substitute the $P_l(x)$ you found in Problems 4.3 or 5.3 into equation (10.6) to find $P_l^m(x)$; then let $x=\cos\theta$ to evaluate:

4. $P_1^1(\cos\theta)$ **5.** $P_4^1(\cos\theta)$ **6.** $P_3^2(\cos\theta)$

7. Show that

$$\frac{d^{\,l-m}}{dx^{l-m}}(x^2-1)^l = \frac{(l-m)!}{(l+m)!}(x^2-1)^m\frac{d^{\,l+m}}{dx^{l+m}}(x^2-1)^l.$$

Hint: Write $(x^2-1)^l=(x-1)^l(x+1)^l$ and find the derivatives by Leibniz' rule.

8. Write (10.7) with m replaced by $-m$; then use Problem 7 to show that

$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x).$$

Comment: This shows that (10.7) is a solution of (10.1) when m is negative.

9. Use Problem 7 to show that

$$P_l^m(x)=(-1)^m\frac{(l+m)!}{(l-m)!}\frac{(1-x^2)^{-m/2}}{2^l l!}\frac{d^{\,l-m}}{dx^{l-m}}(x^2-1)^l.$$

10. Derive (10.8) as follows: Multiply together the two formulas for $P_l^m(x)$ given in (10.7) and Problem 9. Then integrate by parts repeatedly lowering the $l+m$ derivative and raising the $l-m$ derivative until both are l derivatives. Then use (8.1).

11. GENERALIZED POWER SERIES OR THE METHOD OF FROBENIUS

It may happen that the solution of a differential equation is not a power series $\sum_{n=0}^{\infty} a_n x^n$ but may either

(a) contain some negative powers of x, for example,

$$y = \frac{\cos x}{x^2} = \frac{1}{x^2} - \frac{1}{2!} + \frac{x^2}{4!} - \cdots$$

or

(b) have a fractional power of x as a factor, for example,

$$y = \sqrt{x}\sin x = x^{1/2}\left(x - \frac{x^3}{3!} + \cdots\right).$$

Both these cases (and others—see Section 21) are covered by a series of the form

$$y = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s}, \tag{11.1}$$

where s is a number to be found to fit the problem; it may be either positive or negative and it may be a fraction. (In fact, it may even be complex, but we shall not consider this case.) Since $a_0 x^s$ is to be the first term of the series, we assume that a_0 is not zero. The series (11.1) is called a *generalized power series.* We shall consider some differential equations which can be solved by assuming a series of the form (11.1); this way of solving differential equations is called the *method of Frobenius.*

Example 1. As an illustration of this method we solve the equation

$$x^2y'' + 4xy' + (x^2 + 2)y = 0. \tag{11.2}$$

From (11.1) we have

$$\begin{aligned}
y &= a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + \cdots = \sum_{n=0}^{\infty} a_n x^{n+s},\\
y' &= s a_0 x^{s-1} + (s+1)a_1 x^s + (s+2)a_2 x^{s+1} + \cdots\\
&= \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}, \qquad (11.3)\\
y'' &= s(s-1)a_0 x^{s-2} + (s+1)s a_1 x^{s-1} + (s+2)(s+1)a_2 x^s + \cdots\\
&= \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}.
\end{aligned}$$

We substitute (11.3) into (11.2) and set up a table of powers of x as we did for the Legendre equation:

	x^s	x^{s+1}	x^{s+2}	$\cdots$	x^{n+s}
x^2y''	$s(s-1)a_0$	$(s+1)sa_1$	$(s+2)(s+1)a_2$		$(n+s)(n+s-1)a_n$
$4xy'$	$4sa_0$	$4(s+1)a_1$	$2(s+2)a_2$		$4(n+s)a_n$
x^2y			a_0		a_{n-2}
$2y$	$2a_0$	$2a_1$	$2a_2$		$2a_n$

The total coefficient of each power of x must be zero. From the coefficient of x^s we get $(s^2 + 3s + 2)a_0 = 0$, or since $a_0 \neq 0$ by hypothesis,

$$s^2 + 3s + 2 = 0. \tag{11.4}$$

This equation for s is called the *indicial* equation. We solve it and find

$$s = -2, \qquad s = -1.$$

From here on we solve two separate problems, one when $s = -2$, and another when $s = -1$; a linear combination of the two solutions so obtained is then the general solution just as $A \sin x + B \cos x$ is the general solution of $y'' + y = 0$.

▶ **Example 2.** For $s = -1$, the coefficient of x^{s+1} in the table gives $a_1 = 0$. From the x^{s+2} column on, we can use the general formula given by the last column. Notice, however, that the first two columns in the table do not contain the a_{n-2} term, so you must be careful about using the general term at first (Problems 13 and 14). From the general column with $s = -1$, we have

$$a_n[(n-1)(n+2) + 2] = -a_{n-2}$$

or

$$a_n = \frac{-a_{n-2}}{n(n+1)} \qquad \text{for } n \geq 2.$$

Since $a_1 = 0$, this gives all odd a's equal to zero. For even a's:

$$a_2 = -\frac{a_0}{3!}, \qquad a_4 = \frac{a_0}{5!}, \qquad a_6 = -\frac{a_0}{7!}, \qquad \cdots. \tag{11.5}$$

Then one solution of (11.2) is

$$\begin{aligned} y &= a_0 x^{-1} - \frac{a_0}{3!}x + \frac{a_0}{5!}x^3 - \cdots \\ &= a_0 x^{-2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = \frac{a_0 \sin x}{x^2}. \end{aligned} \tag{11.6}$$

The other solution, when $s = -2$, will be left to Problem 1.

▶ PROBLEMS, SECTION 11

1. Finish the solution of equation (11.2) when $s = -2$. Write your solution in closed form as in (11.6). To avoid confusion with the a_n values we found when $s = -1$, you may want to call the coefficients in your series a_n' or b_n; however, this is not essential as long as you realize that there are two separate problems, one when $s = -1$ and one when $s = -2$, and each series has its own coefficients.

Solve the following differential equations by the method of Frobenius (generalized power series). Remember that the point of doing these problems is to learn about the method (which we will use later), not just to find a solution. You may recognize some series [as we did in (11.6)] or you can check your series by expanding a computer answer.

2. $x^2y'' + xy' - 9y = 0$

3. $x^2y'' + 2xy' - 6y = 0$

4. $x^2y'' - 6y = 0$

5. $2xy'' + y' + 2y = 0$

6. $3xy'' + (3x + 1)y' + y = 0$

7. $x^2y'' - (x^2 + 2)y = 0$

8. $x^2y'' + 2x^2y' - 2y = 0$

9. $xy'' - y' + 9x^5y = 0$

10. $2xy'' - y' + 2y = 0$

11. $36x^2y'' + (5 - 9x^2)y = 0$

12. $3xy'' - 2(3x-1)y' + (3x-2)y = 0$

Consider each of the following problems as illustrations showing that, in a power series solution, we must be cautious about using the general recursion relation between the coefficients for the first few terms of the series.

13. Solve $y'' + y'/x^2 = 0$ by power series to find the relation

$$a_{n+1} = -\frac{n(n-1)}{n+1}a_n.$$

If, without thinking carefully, we test the series $\sum_{n=0}^{\infty} a_n x^n$ for convergence by the ratio test, we find

$$\lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \infty. \qquad \text{(Show this.)}$$

Thus we might conclude that the series diverges and that there is no power series solution of this equation. Show why this is wrong, and that the power series solution is $y = \text{const.}$

14. Solve $y'' = -y$ by the Frobenius method. You should find that the roots of the indicial equation are $s = 0$ and $s = 1$. The value $s = 0$ leads to the solutions $\cos x$ and $\sin x$ as you would expect. For $s = 1$, call the series $y = \sum_{n=0}^{\infty} b_n x^{n+1}$, and find the relation

$$b_{n+2} = -\frac{b_n}{(n+3)(n+2)}.$$

Show that the b_0 series obtained from this relation is just $\sin x$, but that the b_1 series is *not* a solution of the differential equation. What is wrong?

12. BESSEL'S EQUATION

Like Legendre's equation, Bessel's equation is another of the "named" equations which have been studied extensively. There are whole books on Bessel functions, and you will find numerous formulas, graphs, and numerical values available in your computer program and in reference tables. You can think of Bessel functions as being something like damped sines and cosines. In fact, if you had first learned about $\sin nx$ and $\cos nx$ as power series solutions of $y'' = -n^2 y$ instead of in elementary trigonometry, you would not feel that Bessel functions were appreciably more difficult or strange than trigonometric functions. Like sines and cosines, Bessel functions are solutions of a differential equation; they can be represented by power series, their graphs can be drawn, and many formulas involving them (compare trigonometric identities) are known. Of special interest to science students is the fact that they occur in many applications. The following list of some of the problems in which they arise will give you an idea of the great range of topics which may involve Bessel functions: problems in electricity, heat, hydrodynamics, elasticity, wave motion, quantum mechanics, etc., involving cylindrical symmetry (for this reason Bessel functions are sometimes called cylinder functions); the motion of a pendulum whose length increases steadily; the small oscillations of a flexible chain; railway transition curves; the stability of a vertical wire or beam; Fresnel integrals in optics; the current distribution in a conductor; Fourier series for the arc of a circle. We shall discuss some of these applications later (see Section 18, and Chapter 13, Sections 5 and 6).

Bessel's equation in the usual standard form is

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \tag{12.1}$$

where p is a constant (not necessarily an integer) called the *order* of the Bessel function y which is the solution of (12.1). You can easily verify that $x(xy')' = x^2y''+xy'$, so we can write (12.1) in the simpler form

$$x(xy')' + (x^2 - p^2)y = 0. \tag{12.2}$$

We find a generalized power series for (12.2) in the same way that we solved (11.2). [In fact, (11.2) is a form of Bessel's equation! See Problems 16.1 and 17.1.] Writing only the general terms in the series for y and the derivatives we need in (12.2), we have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+s} \\ y' &= \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1} \\ xy' &= \sum_{n=0}^{\infty} a_n (n+s) x^{n+s} \\ (xy')' &= \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1} \\ x(xy')' &= \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s} \end{aligned} \tag{12.3}$$

We substitute (12.3) into (12.2) and tabulate the coefficients of powers of x:

	x^s	x^{s+1}	x^{s+2}	$\cdots$	x^{s+n}
$x(xy')'$	s^2a_0	$(1+s)^2a_1$	$(2+s)^2a_2$		$(n+s)^2a_n$
x^2y			a_0		a_{n-2}
$-p^2y$	$-p^2a_0$	$-p^2a_1$	$-p^2a_2$		$-p^2a_n$

The coefficient of x^s gives the indicial equation and the values of s:

$$s^2 - p^2 = 0, \qquad s = \pm p.$$

The coefficient of x^{s+1} gives $a_1 = 0$. The coefficient of x^{s+2} gives a_2 in terms of a_0, etc., but we may as well write the general formula from the last column at this point. We get

$$[(n+s)^2 - p^2]a_n + a_{n-2} = 0$$

or

$$a_n = -\frac{a_{n-2}}{(n+s)^2 - p^2}. \tag{12.4}$$

First we shall find the coefficients for the case $s = p$. From (12.4) we have

$$a_n = -\frac{a_{n-2}}{(n+p)^2 - p^2} = -\frac{a_{n-2}}{n^2 + 2np} = -\frac{a_{n-2}}{n(n+2p)}. \tag{12.5}$$

Since $a_1 = 0$, all odd a's are zero. For even a's it is convenient to replace n by $2n$; then from (12.5) we have

$$a_{2n} = -\frac{a_{2n-2}}{2n(2n+2p)} = -\frac{a_{2n-2}}{2^2 n(n+p)}. \tag{12.6}$$

The formulas for the coefficients can be simplified by the use of the Γ function notation (Chapter 11, Sections 2 to 5) as you can see by examining (12.7) below. Recall that $\Gamma(p+1) = p\Gamma(p)$ for any p, so,

$$\begin{aligned}\Gamma(p+2) &= (p+1)\Gamma(p+1),\\ \Gamma(p+3) &= (p+2)\Gamma(p+2) = (p+2)(p+1)\Gamma(p+1),\end{aligned}$$

and so on. Then from (12.6) we find

$$\begin{aligned}a_2 &= -\frac{a_0}{2^2(1+p)} = -\frac{a_0\Gamma(1+p)}{2^2\Gamma(2+p)},\\ a_4 &= -\frac{a_2}{2^3(2+p)} = \frac{a_0}{2!2^4(1+p)(2+p)} = \frac{a_0\Gamma(1+p)}{2!2^4\Gamma(3+p)},\\ a_6 &= -\frac{a_4}{3!2(3+p)} = -\frac{a_0}{3!2^6(1+p)(2+p)(3+p)}\\ &= -\frac{a_0\Gamma(1+p)}{3!2^6\Gamma(4+p)},\end{aligned} \tag{12.7}$$

and so on. Then the series solution (for the $s = p$ case) is

$$\begin{aligned}y &= a_0 x^p \Gamma(1+p)\left[\frac{1}{\Gamma(1+p)} - \frac{1}{\Gamma(2+p)}\left(\frac{x}{2}\right)^2\right.\\ &\qquad \left. + \frac{1}{2!\Gamma(3+p)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\Gamma(4+p)}\left(\frac{x}{2}\right)^6 + \cdots\right]\\ &= a_0 2^p\left(\frac{x}{2}\right)^p \Gamma(1+p)\left[\frac{1}{\Gamma(1)\Gamma(1+p)} - \frac{1}{\Gamma(2)\Gamma(2+p)}\left(\frac{x}{2}\right)^2\right.\\ &\qquad \left. + \frac{1}{\Gamma(3)\Gamma(3+p)}\left(\frac{x}{2}\right)^4 - \frac{1}{\Gamma(4)\Gamma(4+p)}\left(\frac{x}{2}\right)^6 + \cdots\right].\end{aligned} \tag{12.8}$$

We have inserted $\Gamma(1)$ and $\Gamma(2)$ (which are both equal to 1) in the first two terms and written $x^p = 2^p(x/2)^p$ to make the series appear more systematic. If we take

$$a_0 = \frac{1}{2^p\Gamma(1+p)} \qquad \text{or } \frac{1}{2^p p!},$$

then y is called the Bessel function of the first kind of order p, and written $J_p(x)$.

$$J_p(x) = \frac{1}{\Gamma(1)\Gamma(1+p)}\left(\frac{x}{2}\right)^p - \frac{1}{\Gamma(2)\Gamma(2+p)}\left(\frac{x}{2}\right)^{2+p}$$

$$(12.9)\qquad + \frac{1}{\Gamma(3)\Gamma(3+p)}\left(\frac{x}{2}\right)^{4+p} - \frac{1}{\Gamma(4)\Gamma(4+p)}\left(\frac{x}{2}\right)^{6+p} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)}\left(\frac{x}{2}\right)^{2n+p}.$$

▶ PROBLEMS, SECTION 12

1. Show by the ratio test that the infinite series (12.9) for $J_p(x)$ converges for all x.

Use (12.9) to show that:

2. $J_2(x) = (2/x)J_1(x) - J_0(x)$

3. $J_1(x) + J_3(x) = (4/x)J_2(x)$

4. $(d/dx)J_0(x) = -J_1(x)$

5. $(d/dx)[xJ_1(x)] = xJ_0(x)$

6. $J_0(x) - J_2(x) = 2(d/dx)J_1(x)$

7. $\lim_{x\to 0} J_1(x)/x = \frac{1}{2}$

8. $\lim_{x\to 0} x^{-3/2}J_{3/2}(x) = 3^{-1}\sqrt{2/\pi}$ *Hint:* See Chapter 11, equations (3.4) and (5.3).

9. $\sqrt{\pi x/2}\, J_{1/2}(x) = \sin x$

▶ 13. THE SECOND SOLUTION OF BESSEL'S EQUATION

We have found just one of the two solutions of Bessel's equation, that is, the one when $s = p$; we must next find the solution when $s = -p$. It is unnecessary to go through the details again; we can just replace p by $-p$ in (12.9). In fact, the solution when $s = -p$ is usually written J_{-p}. From (12.9) we have

$$(13.1)\qquad J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n-p+1)}\left(\frac{x}{2}\right)^{2n-p}$$

If p is not an integer, $J_p(x)$ is a series starting with x^p and J_{-p} is a series starting with x^{-p}. Then $J_p(x)$ and $J_{-p}(x)$ are two independent solutions and a linear combination of them is a general solution. But if p is an integer, then the first few terms in J_{-p} are zero because $\Gamma(n-p+1)$ in the denominator is Γ of a negative integer, which is infinite. You can show (Problem 2) that $J_{-p}(x)$ starts with the term x^p (for integral p) just as $J_p(x)$ does, and that

$$(13.2)\qquad J_{-p}(x) = (-1)^p J_p(x) \qquad \text{for integral } p;$$

thus $J_{-p}(x)$ is not an independent solution when p is an integer. The second solution in this case is not a Frobenius series (11.1) but contains a logarithm. $J_p(x)$ is finite at the origin, but the second solution is infinite and so is useful only in applications involving regions not containing the origin.

Although $J_{-p}(x)$ is a satisfactory second solution when p is not an integer, it is customary to use a linear combination of $J_p(x)$ and $J_{-p}(x)$ as the second solution.

This is much as if $\sin x$ and $(2\sin x - 3\cos x)$ were used as the two solutions of $y'' + y = 0$ instead of $\sin x$ and $\cos x$. Remember that the general solution of this differential equation is a linear combination of $\sin x$ and $\cos x$ with arbitrary coefficients. But $A\sin x + B(2\sin x - 3\cos x)$ is just as good a linear combination as $c_1 \sin x + c_2 \cos x$. Similarly, any combination of $J_p(x)$ and $J_{-p}(x)$ is a satisfactory second solution of Bessel's equation. The combination which is used is called either the Neumann or the Weber function and is denoted by either N_p or Y_p:

$$N_p(x) = Y_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{13.3}$$

For integral p this expression is an indeterminate form $0/0$. However, for any $x \neq 0$ it has a limit (as p tends to an integral value) which gives a second solution. This is why the special form (13.3) is used; it is valid for any p. N_p or Y_p are called Bessel functions of the second kind. The general solution of Bessel's equation (12.1) or (12.2) may then be written as

$$y = AJ_p(x) + BN_p(x), \tag{13.4}$$

where A and B are arbitrary constants.

PROBLEMS, SECTION 13

1. Using equations (12.9) and (13.1), write out the first few terms of $J_0(x)$, $J_1(x)$, $J_{-1}(x)$, $J_2(x)$, $J_{-2}(x)$. Show that $J_{-1}(x) = -J_1(x)$ and $J_{-2}(x) = J_2(x)$.
2. Show that, in general for integral n, $J_{-n}(x) = (-1)^n J_n(x)$, and $J_n(-x) = (-1)^n J_n(x)$.

Use equations (12.9) and (13.1) to show that:

3. $\sqrt{\pi x/2}\, J_{-1/2}(x) = \cos x$
4. $J_{3/2}(x) = x^{-1} J_{1/2}(x) - J_{-1/2}(x)$.
5. Using equation (13.3), show that $N_{1/2}(x) = -J_{-1/2}(x)$; that $N_{3/2}(x) = J_{-3/2}(x)$.
6. Show from (13.3) that $N_{(2n+1)/2}(x) = (-1)^{n+1} J_{-(2n+1)/2}(x)$.

14. GRAPHS AND ZEROS OF BESSEL FUNCTIONS

You can find the values of Bessel functions both from your computer program and in reference books, and you can use your computer to plot graphs of Bessel functions (see problems). Except for $J_0(x)$, all the J_p's start at the origin behaving like x^p and then oscillate something like $\sin x$ but with decreasing amplitude. $J_0(x)$ is equal to 1 at $x = 0$ and so looks something like a damped cosine. All the N's are $\pm\infty$ at the origin, but away from it they also oscillate with decreasing amplitude.

The values of x for which $\sin x = 0$ (called the zeros for $\sin x$) do not need to be computed because they are just $x = n\pi$ for $n = 0, 1, 2, \cdots$. The zeros of the Bessel functions, however, do not occur at regular intervals; they have to be computed numerically. You can find their values by computer or in tables. It is worth noticing that the difference between two successive zeros becomes approximately π (as it is for $\sin x$ and $\cos x$) when x is large. You can see this from graphs of the functions or from tables of the zeros, or from the approximate formulas for the Bessel functions when x is large. (See Section 20).

PROBLEMS, SECTION 14

1. By computer, plot graphs of $J_p(x)$ for $p = 0, 1, 2, 3$, and x from 0 to 15.

2. From the graphs in Problem 1, read approximate values of the first three zeros of each of the functions. Then, by computer, find more accurate values of the zeros.

3. By computer, plot $N_0(x)$ for x from 0 to 15, and $N_p(x)$ for $p = 1, 2, 3$, and x from 1 to 15.

4. From the graphs in Problem 3, read approximate values of the first three zeros of each of the functions, and then find more accurate values by computer.

5. By computer, plot $\sqrt{x}\, J_{1/2}(x)$ for x from 0 to 4π. Do you recognize the curve? See Problem 12.9.

6. By computer, find 30 zeros of J_0 and note that the spacing between consecutive zeros is tending to π.

15. RECURSION RELATIONS

The following useful relations hold among Bessel functions and their derivatives. Although we state them and outline proofs for $J_p(x)$, they also hold for $N_p(x)$.

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x), \tag{15.1}$$

$$\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x), \tag{15.2}$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x), \tag{15.3}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x), \tag{15.4}$$

$$J_p'(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x). \tag{15.5}$$

To prove (15.1), first multiply (12.9) by x^p and differentiate to get

$$\begin{aligned}\frac{d}{dx}[x^p J_p(x)] &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \frac{x^{2n+2p}}{2^{2n+p}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p)}{\Gamma(n+1)\Gamma(n+1+p)} \frac{x^{2n+2p-1}}{2^{2n+p}}.\end{aligned}$$

Use the fact that $\Gamma(n+1+p) = (n+p)\Gamma(n+p)$, and cancel the factors 2 and $(n+p)$ to get

$$\frac{d}{dx}[x^p J_p(x)] = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)} \frac{x^{2x+2p-1}}{2^{2n+p-1}}.$$

Divide by x^p and compare with (12.9); this gives

$$\frac{1}{x^p}\frac{d}{dx}[x^p J_p(x)] = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)} \left(\frac{x}{2}\right)^{2n+p-1} = J_{p-1}(x),$$

since this series is just (12.9) with p replaced by $p-1$. Proofs of the other relations are outlined in Problems 1 to 3.

PROBLEMS, SECTION 15

1. Prove equation (15.2) by a method similar to the one used above to prove (15.1).

2. Solve equations (15.1) and (15.2) for $J_{p+1}(x)$ and $J_{p-1}(x)$. Add and subtract these two equations to get (15.3) and (15.4).

3. Carry out the differentiation in equations (15.1) and (15.2) to get (15.5).

4. Use equations (15.1) to (15.5) to do Problems 12.2 to 12.6.

5. Using equations (15.4) and (15.5), show that $J_0(x) = J_2(x)$ at every maximum or minimum of $J_1(x)$, and $J_0(x) = -J_2(x) = J_1'(x)$ at every positive zero of $J_1(x)$. Computer plot $J_0(x)$, $J_1(x)$, and $J_2(x)$ on the same axes, and verify that these results are true.

6. As in Problem 5, show that $J_{p-1}(x) = J_{p+1}(x)$ at every maximum or minimum of $J_p(x)$, and $J_{p-1}(x) = -J_{p+1}(x) = J_p'(x)$ at every positive zero of $J_p(x)$. Computer plot, say, J_2, J_3, and J_4 on the same axes, (or any other set of three consecutive J's or three consecutive N's) and check to see that the results are true.

7. (a) Using (15.2), show that

$$\int_0^\infty J_1(x)\,dx = -J_0(x)\Big|_0^\infty = 1.$$

(b) Use L23 of the Laplace Transform Table (page 469) to show that $\int_0^\infty J_0(t)\,dt = 1$. (Also see Problem 23.29.)

8. From equation (15.4), show that

$$\int_0^\infty J_1(x)\,dx = \int_0^\infty J_3(x)\,dx = \cdots = \int_0^\infty J_{2n+1}(x)\,dx,$$

and

$$\int_0^\infty J_0(x)\,dx = \int_0^\infty J_2(x)\,dx = \cdots = \int_0^\infty J_{2n}(x)\,dx.$$

Then, by Problem 7, show that

$$\int_0^\infty J_n(x)\,dx = 1 \qquad \text{for all integral } n.$$

9. Use L23 and L32 of the Laplace Transform Table (page 469) to evaluate $\int_0^\infty tJ_0(2t)e^{-t}\,dt$.

16. DIFFERENTIAL EQUATIONS WITH BESSEL FUNCTION SOLUTIONS

Many differential equations occur in practice that are not of the standard form (12.1) but whose solutions can be written in terms of Bessel functions. It can be shown (Problem 13) that the differential equation

$$y'' + \frac{1-2a}{x}y' + \left[(bcx^{c-1})^2 + \frac{a^2-p^2c^2}{x^2}\right]y = 0 \tag{16.1}$$

has the solution

$$y = x^a Z_p(bx^c), \tag{16.2}$$

where Z stands for J or N or any linear combination of them, and a, b, c, p are constants. To see how to use this, let us "solve" the differential equation

$$y'' + 9xy = 0. \tag{16.3}$$

If (16.3) is of the type (16.1), then we must have

$$1 - 2a = 0, \qquad (bc)^2 = 9, \qquad 2(c-1) = 1, \qquad a^2 - p^2c^2 = 0.$$

From these equations we find

$$a = \tfrac{1}{2}, \qquad c = \tfrac{3}{2}, \qquad b = 2, \qquad p = \frac{a}{c} = \frac{1}{3}.$$

Then the solution of (16.3) is

$$y = x^{1/2} Z_{1/3}(2x^{3/2}). \tag{16.4}$$

This means that the general solution of (16.3) is

$$y = x^{1/2}[AJ_{1/3}(2x^{3/2}) + BN_{1/3}(2x^{3/2})],$$

where A and B are arbitrary constants.

It is useful to write the differential equation whose solutions are $J_p(Kx)$ and $N_p(Kx)$, where K is a constant. We substitute Kx for x in (12.2). Then $x(dy/dx)$ becomes $Kx[dy/d(Kx)] = x(dy/dx)$ and similarly, $x(xy')'$ is unchanged. Thus the only change in (12.2) is to replace $x^2 - p^2$ by $K^2x^2 - p^2$ and we have:

(16.5) $\quad x(xy')' + (K^2x^2 - p^2)y = 0 \quad$ has solutions $J_p(Kx)$ and $N_p(Kx)$.

▶ PROBLEMS, SECTION 16

Find the solutions of the following differential equations in terms of Bessel functions by using equations (16.1) and (16.2).

1. Equation (11.2)

2. $y'' + 4x^2y = 0$

3. $xy'' + 2y' + 4y = 0$

4. $3xy'' + 2y' + 12y = 0$

5. $y'' - \dfrac{1}{x}y' + \left(4 + \dfrac{1}{x^2}\right)y = 0$

6. $4xy'' + y = 0$

7. $xy'' + 3y' + x^3y = 0$

8. $y'' + xy = 0$

9. $3xy'' + y' + 12y = 0$

10. $xy'' - y' + 9x^2y = 0$

11. $xy'' + 5y' + xy = 0$

12. $4xy'' + 2y' + y = 0$

13. Verify by direct substitution that the text solution of equation (16.3) and your solutions in the problems above are correct. Also prove in general that the solution (16.2) given for (16.1) is correct. *Hint:* These are exercises in partial differentiation. To verify the solution (16.4) of (16.3), we would change variables from x, y to say z, u where

$$y = x^{1/2}u, \qquad u = J_{1/3}(z), \qquad z = 2x^{3/2},$$

and show that if x, y satisfy (16.3), then u, z satisfy (12.1), that is,

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - \tfrac{1}{9})u = 0.$$

Use (16.5) to write the solutions of the following problems. Remember that $x(xy')' = x^2y'' + xy'$.

14. $x^2y'' + xy' + (4x^2 - 9)y = 0$

15. $x(xy')' + (25x^2 - 4)y = 0$

16. $x^2y'' + xy' + (16x^2 - 1)y = 0$

17. $xy'' + y' + 9xy = 0$

17. OTHER KINDS OF BESSEL FUNCTIONS

We have discussed $J_p(x)$ and $N_p(x)$ which are called *Bessel functions of the first and second kinds*, respectively. Since Bessel's equation is of second order, there are, of course, only two independent solutions. However, there are a number of related functions which are also called Bessel functions. Here again there is a close analogy to sines and cosines. We may think of $\cos x$ and $\sin x$ as the solutions of $y'' + y = 0$. But $\cos x \pm i \sin x$ are also solutions which we usually write as $e^{\pm ix}$. If we replace x by ix, we get the functions e^x, e^{-x}, $\cosh x$, $\sinh x$, which are solutions of $y'' - y = 0$. We list a number of Bessel functions which are frequently used and their trigonometric analogues:

Hankel Functions or Bessel Functions of the Third Kind

$$\begin{aligned} H_p^{(1)}(x) &= J_p(x) + iN_p(x), \\ H_p^{(2)}(x) &= J_p(x) - iN_p(x). \end{aligned} \tag{17.1}$$

(Compare $e^{\pm ix} = \cos x \pm i \sin x$.)

Modified or Hyperbolic Bessel Functions The solutions of

$$x^2y'' + xy' - (x^2 + p^2)y = 0 \tag{17.2}$$

are, by (16.1) $Z_p(ix)$. (Compare this with the standard Bessel equation and by analogy consider the relation between $y'' + y = 0$ and $y'' - y = 0$.) The two independent solutions of (17.2) which are ordinarily used are

$$\begin{aligned} I_p(x) &= i^{-p} J_p(ix), \\ K_p(x) &= \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix). \end{aligned} \tag{17.3}$$

These should be compared with $\sinh x = -i \sin(ix)$ and $\cosh x = \cos(ix)$; because of the analogy, I and K are called hyperbolic Bessel functions. The i factors are adjusted to make I and K real for real x.

Spherical Bessel Functions If $p = (2n+1)/2 = n + \frac{1}{2}$, n an integer, then $J_p(x)$ and $N_p(x)$ are called Bessel functions of half-odd integral order; they can be expressed in terms of $\sin x$, $\cos x$, and powers of x. The spherical Bessel functions are closely related to them as you can see from the formulas (17.4) below. Spherical Bessel functions arise in a variety of vibration problems especially when spherical coordinates are used. We define the spherical Bessel functions $j_n(x)$, $y_n(x)$, $h_n^{(1)}(x)$, $h_n^{(2)}(x)$, for $n = 0, 1, 2, \cdots$, and state their values in terms of elementary functions (see Problems 2 and 3). For the use of these functions, see Chapter 13, Problems 7.15, 7.16, 7.19, and 10.20.

$$\begin{aligned}
&j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{(2n+1)/2}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right),\\
(17.4)\qquad &y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{(2n+1)/2}(x) = -x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\cos x}{x}\right),\\
&h_n^{(1)} = j_n(x) + iy_n(x),\\
&h_n^{(2)} = j_n(x) - iy_n(x).
\end{aligned}$$

The Kelvin Functions A standard method of solving vibration problems is to assume a solution involving $e^{i\omega t}$; the resulting equation may contain imaginary terms. As an example, the following equation arises in the problem of the distribution of alternating current in wires (skin effect) (Relton, p. 177):

$$y'' + \frac{1}{x}y' - iy = 0. \tag{17.5}$$

The solution of this equation is (Problem 8a)

$$y = Z_0(i^{3/2}x). \tag{17.6}$$

This is complex, and it is customary to separate it into its real and imaginary parts, called (for $Z = J$) ber and bei; these stand for Bessel-real and Bessel-imaginary. We define the ber, bei, ker, kei functions by

$$\begin{aligned}
J_0(i^{3/2}x) &= \operatorname{ber} x + i \operatorname{bei} x,\\
K_0(i^{1/2}x) &= \operatorname{ker} x + i \operatorname{kei} x.
\end{aligned} \tag{17.7}$$

There are also similar functions for $n \neq 0$. These functions occur in problems in heat flow and in the theory of viscous fluids, as well in electrical engineering.

The Airy Functions The Airy differential equation is

$$y'' - xy = 0. \tag{17.8}$$

By Section 16, the solutions are (Problem 8b)

$$\sqrt{x}\, Z_{1/3}(\tfrac{2}{3}ix^{3/2}), \tag{17.9}$$

so by (17.3) they can be written in terms of $I_{1/3}$ and $K_{1/3}$. The Airy functions are defined as

$$\text{(17.10)}\qquad \begin{aligned} \mathrm{Ai}(x) &= \frac{1}{\pi}\sqrt{\frac{x}{3}}\,K_{1/3}\left(\tfrac{2}{3}x^{3/2}\right), \\ \mathrm{Bi}(x) &= \sqrt{\frac{x}{3}}\left[I_{-1/3}\left(\tfrac{2}{3}x^{3/2}\right) + I_{1/3}\left(\tfrac{2}{3}x^{3/2}\right)\right]. \end{aligned}$$

For negative x, Ai and Bi can be expressed in terms of $J_{1/3}$ and $N_{1/3}$, or the Hankel functions (17.1) of order 1/3. Airy functions are of use in electrodynamics and quantum mechanics.

PROBLEMS, SECTION 17

1. Write the solutions of Problem 16.1 as spherical Bessel functions using the definitions (17.4) of $j_n(x)$ and $y_n(x)$ in terms of $J_{(2n+1)/2}(x)$ and $Y_{(2n+1)/2}(x)$. Then, using (17.4), obtain the solutions in terms of $\sin x$ and $\cos x$. Compare with the answers in equation (11.6) and Problem 11.1.

2. From Problem 12.9, $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$. Use (15.2) to obtain $J_{3/2}(x)$ and $J_{5/2}(x)$. Substitute your results for the J's into (17.4) to verify the formulas stated for j_0, j_1, and j_2 in terms of $\sin x$ and $\cos x$.

3. From Problems 13.3 and 13.5, $Y_{1/2}(x) = -\sqrt{2/\pi x}\,\cos x$. As in Problem 2, obtain $Y_{3/2}$ and $Y_{5/2}$ and verify the formulas (17.4) for y_0, y_1, and y_2 in terms of $\sin x$ and $\cos x$.

4. Using (17.3) and the results stated in Problems 2 and 3 for $J_{1/2}$ and $Y_{1/2}$ $(= N_{1/2})$, show that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sinh x, \qquad \text{and} \qquad K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}.$$

5. Show from (17.4) that $h_n^{(1)}(x) = -ix^n\left(-\dfrac{1}{x}\dfrac{d}{dx}\right)^n\left(\dfrac{e^{ix}}{x}\right)$.

6. Using (16.1) and (17.4) show that the spherical Bessel functions satisfy the differential equation

$$x^2y'' + 2xy' + [x^2 - n(n+1)]y = 0.$$

7. (a) Solve the differential equation $xy'' = y$ using (16.1), and then express the answer in terms of a function I_p by (17.3).

(b) As in (a), find a solution of of $y'' - x^4y = 0$.

8. Using (16.1) and (16.2), verify that

(a) the solution of (17.5) is (17.6);

(b) the solution of (17.8) is (17.9).

9. Using (17.3) and (15.1) to (15.5), find the recursion relations for $I_p(x)$. In particular, show that $I_0' = I_1$.

10. Computer plot

(a) $I_0(x)$, $I_1(x)$, $I_2(x)$, from $x = 0$ to 2.

(b) $K_0(x)$, $K_1(x)$, $K_2(x)$, from $x = 0.1$ to 2.

(c) $\mathrm{Ai}(x)$ from $x = -10$ to 10.

(d) $\mathrm{Bi}(x)$ from $x = -10$ to 1.

11. From (17.4), show that $h_0^{(1)}(ix) = -e^{-x}/x$.

Use the Section 15 recursion relations and (17.4) to obtain the following recursion relations for spherical Bessel functions. We have written them for j_n, but they are valid for y_n and for the h_n's.

12. $j_{n-1}(x) + j_{n+1}(x) = (2n+1)j_n(x)/x$

13. $(d/dx)j_n(x) = nj_n(x)/x - j_{n+1}(x)$

14. $(d/dx)j_n(x) = j_{n-1}(x) - (n+1)j_n(x)/x$

15. $(d/dx)[x^{n+1}j_n(x)] = x^{n+1}j_{n-1}(x)$

16. $(d/dx)[x^{-n}j_n(x)] = -x^{-n}j_{n+1}(x)$

18. THE LENGTHENING PENDULUM

As an example of the use of Bessel functions we consider the following problem. Suppose that a simple pendulum (see Chapter 11, Section 8) has the length l of its string increased at a steady rate (for example, a weight swaying as it is lowered by a crane). (This problem was considered as early as 1707; see L. LeCornu, *Acta Mathematica* 19 (1895), 201–249. Also see Relton, and Problem 8.) Find the equation of motion and the solution for small oscillations.

From Chapter 11, Section 8, we have the equation of motion

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0. \tag{18.1}$$

Let the length of the string at time t be

$$l = l_0 + vt, \tag{18.2}$$

and change from t to l as the independent variable. For small oscillations, we may replace $\sin\theta$ by θ. Then (18.1) becomes (Problem 1):

$$l\frac{d^2\theta}{dl^2} + 2\frac{d\theta}{dl} + \frac{g}{v^2}\theta = 0. \tag{18.3}$$

(This equation could also describe the damped vibration of a variable mass, or an RLC circuit with variable L.)

We solve (18.3) by comparing it with the standard equation (16.1) to get (Problem 2)

$$\theta = l^{-1/2}Z_1(bl^{1/2}) \qquad \text{where } b = 2g^{1/2}/v. \tag{18.4}$$

To simplify the notation, let

$$u = b\,l^{1/2} = (2g^{1/2}/v)l^{1/2}. \tag{18.5}$$

The general solution of (18.3) is then

$$\theta = Au^{-1}J_1(u) + Bu^{-1}N_1(u). \tag{18.6}$$

We can find $d\theta/du$ from (18.6) using (15.2):

$$\frac{d\theta}{du} = -[Au^{-1}J_2(u) + Bu^{-1}N_2(u)]. \tag{18.7}$$

The constants A and B must be found from the starting conditions just as they are for the ordinary simple pendulum with constant l. For example, in the ordinary case, if $\theta = \theta_0$ and $\dot{\theta} = 0$ at $t = 0$, then the general solution $\theta = A\cos\omega t + B\sin\omega t$ becomes just $\theta = \theta_0 \cos\omega t$. For the lengthening pendulum, let's take the same simple initial conditions, namely $\theta = \theta_0$ and $\dot{\theta} = 0$ at $t = 0$. For these initial conditions, we find (after some calculations—see Problems 3 to 6)

$$A = -\frac{\pi u_0^2}{2}\,\theta_0 N_2(u_0), \qquad B = \frac{\pi u_0^2}{2}\,\theta_0 J_2(u_0). \tag{18.8}$$

The solution has a particularly simple form if we adjust the constants v and l_0 so that

$$u_0 = 2(gl_0)^{1/2}/v \qquad \text{is a zero of } J_2(u). \tag{18.9}$$

Then $B = 0$ and the second term of (18.6) is zero, so we have

$$\theta = Au^{-1}J_1(u) = Cl^{-1/2}J_1(b\,l^{1/2}), \tag{18.10}$$

where (Problem 7)

$$b = \frac{2g^{1/2}}{v} = \frac{u_0}{l_0^{1/2}}, \qquad C = \frac{\theta_0 l_0^{1/2}}{J_1(u_0)}. \tag{18.11}$$

For this simple case, $\dot{\theta}$ is a multiple of $J_2(u)$ (Problem 8); thus $\theta = 0$ corresponds to zeros of $J_1(u)$ and $\dot{\theta} = 0$ corresponds to zeros of $J_2(u)$. A "quarter" period corresponds to the time from $\theta = 0$ to $\dot{\theta} = 0$, or $\dot{\theta} = 0$ to $\theta = 0$. These quarter periods can be found from the zeros of $J_1(u)$ and $J_2(u)$ (Problem 8).

PROBLEMS, SECTION 18

1. Verify equation (18.3). *Hint:* From equation (18.2), $dl = v\,dt$, so

$$\frac{d}{dt} = v\frac{d}{dl}.$$

2. Solve equation (18.3) to get equation (18.4).

3. Prove

$$J_p(x)J'_{-p}(x) - J_{-p}(x)J'_p(x) = -\frac{2}{\pi x}\sin p\pi$$

as follows: Write Bessel's equation (12.1) with $y = J_p$ and with $y = J_{-p}$; multiply the J_p equation by J_{-p} and the J_{-p} equation by J_p and subtract to get

$$\frac{d}{dx}[x(J_pJ'_{-p} - J_{-p}J'_p)] = 0.$$

Then $J_pJ'_{-p} - J_{-p}J'_p = c/x$. To find c, use equation (12.9) for each of the four functions and pick out the $1/x$ terms in the products. Then use equation (5.4) of Chapter 11.

4. Using equation (13.3) and Problem 3, show that

$$J_p(x)N'_p(x) - J'_p(x)N_p(x) = \frac{J'_p(x)J_{-p}(x) - J_p(x)J'_{-p}(x)}{\sin p\pi} = \frac{2}{\pi x}.$$

5. Use the recursion relations of Section 15 (for N's as well as for J's) and Problem 4 to show that

$$J_n(x)N_{n+1}(x) - J_{n+1}(x)N_n(x) = -\frac{2}{\pi x}.$$

Hint: Do it first for $n = 0$; then use the result in proving the $n = 1$ case, and so on.

6. For the initial conditions $\theta = \theta_0$, $\dot{\theta} = 0$, show that the constants A and B in equations (18.6) and (18.7) are as given in (18.8). *Hints:* Show that $d\theta/du = 0$ if $\dot{\theta} = 0$. In equations (18.6) and (18.7), set $\theta = \theta_0$ and $d\theta/du = 0$ when $u = u_0$ and solve for A and B. Then use the formula in Problem 5 to simplify your results to get equation (18.8).

7. Verify the values of b and C given in equation (18.11). Note that C can be found in two ways: (1) in equation (18.10), $u = bl^{1/2}$, so $Au^{-1} = (A/b)l^{-1/2}$, $C = A/b$. Use Problem 5 to simplify this. (2) Set $\theta = \theta_0$, $u = u_0$, $l = l_0$ in equation (18.10) and solve for C.

8. Find

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{du}\frac{du}{dl}\frac{dl}{dt}$$

either from equations (18.10) and (15.2) or from equation (18.7) with $B = 0$. Thus show that $\theta = 0$ when $J_1(u) = 0$ and $\dot{\theta} = 0$ when $J_2(u) = 0$. Show that the successive (variable) quarter periods of the lengthening pendulum are $(v/4g)(r_2^2 - r_1^2)$ or $(v/4g)(r_1^2 - r_2^2)$, where r_1 and r_2 are successive zeros of J_1 and J_2. Use a computer or tables to find the needed zeros and calculate several quarter periods (as multiples of $v/(4g)$). Observe that an inward swing takes longer than either the preceding or the following outward swing. [This result is proved by Ll. G. Chambers, *Proceedings of the Edinburgh Mathematical Society* (2) 12, 17–18 (1960).]

9. Consider the "shortening pendulum" problem. Follow the method in the text but with $l = l_0 - vt$. Does the θ amplitude of the vibration increase or decrease as the pendulum shortens? Restate the result of Problem 8 about quarter periods for this case.

10. The differential equation for transverse vibrations of a string whose density increases linearly from one end to the other is $y'' + (Ax + B)y = 0$, where A and B are constants. Find the general solution of this equation in terms of Bessel functions. *Hint:* Make the change of variable $Ax + B = Au$.

11. A straight wire clamped vertically at its lower end stands vertically if it is short, but bends under its own weight if it is long. It can be shown that the greatest length for vertical equilibrium is l, where $kl^{3/2}$ is the first zero of $J_{-1/3}$ and

$$k = \frac{4}{3r^2}\sqrt{\frac{\rho g}{\pi Y}},$$

r = radius of the wire, ρ = linear density, g = acceleration of gravity, Y = Young's modulus. Find l for a steel wire of radius 1 mm; for a lead wire of the same radius.

19. ORTHOGONALITY OF BESSEL FUNCTIONS

You may expect here that we are going to prove that two J_p's for different p values are orthogonal. However, this is *not* what we are going to do—as a matter of fact it isn't true! To see what we *are* going to prove, look at the following comparison between Bessel functions and sines and cosines.

	Two functions: $\sin x$ and $\cos x$.	Two functions for each p: $J_p(x)$ and $N_p(x)$.
	Consider just $\sin x$.	Consider just $J_p(x)$ for one value of p.
(19.1)	At the zeros of $\sin x$, namely, $x = n\pi$, $\sin x = 0$.	At the zeros of $J_p(x)$, say $x = \alpha, \beta, \cdots$, $J_p(x) = 0$.
	At $x = 1$, $\sin n\pi x = 0$.	At $x = 1$, $J_p(\alpha x) = 0$, $J_p(\beta x) = 0, \cdots$.
	The differential equation satisfied by $y = \sin n\pi x$ is $y'' + (n\pi)^2 y = 0$.	The differential equation satisfied by $y = J_p(\alpha x)$ is [see (16.5)] $x(xy')' + (\alpha^2 x^2 - p^2)y = 0$.

(In comparing the differential equations remember that p is a fixed constant. The correspondence is between the zeros of $\sin x$, namely $n\pi$, and the zeros of $J_p(x)$, namely, α, β, etc.)

We have proved (Chapter 7):	We shall prove:
$\int_0^1 \sin n\pi x \sin m\pi x\, dx = 0$ for $n \neq m$.	$\int_0^1 x J_p(\alpha x) J_p(\beta x)\, dx = 0$ for $\alpha \neq \beta$.

By (16.5), the differential equation satisfied by $J_p(\alpha x)$ is

$$x(xy')' + (\alpha^2 x^2 - p^2)y = 0 \tag{19.2}$$

and the differential equation satisfied by $J_p(\beta x)$ is

$$x(xy')' + (\beta^2 x^2 - p^2)y = 0. \tag{19.3}$$

Let us for simplicity call $J_p(\alpha x) = u$ and $J_p(\beta x) = v$; then (19.2) and (19.3) become

$$\begin{aligned} x(xu')' + (\alpha^2 x^2 - p^2)u &= 0, \\ x(xv')' + (\beta^2 x^2 - p^2)v &= 0. \end{aligned} \tag{19.4}$$

We are going to use equations (19.4) to prove the last equation in (19.1) by a method parallel to that used in proving the orthogonality of Legendre polynomials (Section 7). Multiply the first equation of (19.4) by v, the second by u, subtract the two equations and cancel an x to get

$$v(xu')' - u(xv')' + (\alpha^2 - \beta^2)xuv = 0. \tag{19.5}$$

The first two terms of (19.5) are equal to

$$\frac{d}{dx}(vxu' - uxv'). \tag{19.6}$$

Using (19.6) and integrating (19.5), we get

$$(vxu' - uxv')\Big|_0^1 + (\alpha^2 - \beta^2)\int_0^1 xuv\,dx = 0. \tag{19.7}$$

At the lower limit the integrated term is zero because $x = 0$ and u, v, u', v' are finite. To evaluate the integrated term at the upper limit, recall that $u = J_p(\alpha x)$, $v = J_p(\beta x)$; then at $x = 1$, $u = J_p(\alpha) = 0$, $v = J_p(\beta) = 0$ since α and β are zeros of J_p. The integrated term is therefore zero at the upper limit also. Thus (19.7) becomes

$$(\alpha^2 - \beta^2)\int_0^1 xuv\,dx = 0 \tag{19.8}$$

or

$$(\alpha^2 - \beta^2)\int_0^1 xJ_p(\alpha x)J_p(\beta x)\,dx = 0. \tag{19.9}$$

If $\alpha \neq \beta$, that is, if α and β are different zeros of J_p, the integral must be zero. If $\alpha = \beta$, the integral is not zero; it can be evaluated, but we shall just state the answer (see Problem 1):

$$\int_0^1 xJ_p(\alpha x)J_p(\beta x)\,dx = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \frac{1}{2}J_{p+1}^2(\alpha) = \frac{1}{2}J_{p-1}^2(\alpha) = \frac{1}{2}J_p'^{\,2}(\alpha) & \text{if } \alpha = \beta, \end{cases} \tag{19.10}$$

where α and β are zeros of $J_p(x)$.

[You can see that the three answers for the case $\alpha = \beta$ are equal by equations (15.3) to (15.5), remembering that α is a zero of J_p.]

We can state (19.10) in words in two different ways; if α_n, $n = 1, 2, 3, \cdots$, are the zeros of $J_p(x)$, then we say either that

(a) the functions $\sqrt{x}J_p(\alpha_n x)$ are orthogonal on $(0, 1)$;

or that

(b) the functions $J_p(\alpha_n x)$ are orthogonal on $(0, 1)$ with respect to the *weight function* x.

You may meet other sets of functions which are orthogonal with respect to a weight function. (See, for example, Section 22.) In general, we say that $y_n(x)$ is a set of orthogonal functions on (x_1, x_2) with respect to the weight function $w(x)$ if

$$\int_{x_1}^{x_2} y_n(x)y_m(x)w(x)\,dx = 0 \qquad \text{for} \quad n \neq m.$$

The fact that the Bessel functions $J_p(\alpha_n x)$ obey (19.10) makes it possible to expand a given function in a series of Bessel functions much as we expand functions in Fourier series and Legendre series. We shall do this later (Chapter 13) when we need it in a physical example.

Just as we generalized Fourier series to an interval $(0, l)$, here we can generalize (19.10) to an interval $(0, a)$. In (19.10), let $x = r/a$. Then the limits are $x = r/a = 0$ to 1, that is, $r = 0$ to a. The integral in (19.10) becomes

$$\int_0^a (r/a)J_p(\alpha r/a)J_p(\beta r/a)\,d(r/a) = \frac{1}{a^2}\int_0^a rJ_p(\alpha r/a)J_p(\beta r/a)\,dr.$$

Thus we have

$$(19.11)\quad \int_0^a rJ_p(\alpha r/a)J_p(\beta r/a)\,dr = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \frac{a^2}{2}J_{p+1}^2(\alpha) = \frac{a^2}{2}J_{p-1}^2(\alpha) = \frac{a^2}{2}{J'_p}^2(\alpha) & \text{if } \alpha = \beta. \end{cases}$$

PROBLEMS, SECTION 19

1. Prove equation (19.10) in the following way. First note that (19.2) and (19.3) and therefore (19.7) hold whether α and β are zeros of $J_p(x)$ or not. Let α be a zero, but let β be just any number. From (19.7) show that then

$$\int_0^1 xuv\,dx = \frac{J_p(\beta)\alpha J'_p(\alpha)}{\beta^2 - \alpha^2}.$$

Now let $\beta \to \alpha$ and evaluate the indeterminate form by L'Hôpital's rule (that is, differentiate numerator and denominator with respect to β and let $\beta \to \alpha$). Hence find

$$\int_0^1 xuv\,dx = \tfrac{1}{2}{J'_p}^2(\alpha)$$

for $\alpha = \beta$, that is, for $u = v = J_p(\alpha x)$ as in (19.10). Use equations (15.3) to (15.5) to show that the other two expressions given in (19.10) are equivalent.

2. Given that

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right),$$

use (19.10) to evaluate

$$\int_0^1 \left(\frac{\sin \alpha x}{\alpha x} - \cos \alpha x\right)^2 dx$$

where α is a root of the equation $\tan x = x$.

3. Use (17.4) and (19.10) to write the orthogonality condition and the normalization integral for the spherical Bessel functions $j_n(x)$.

4. Define $J_p(z)$ for complex z by the power series (12.9) with x replaced by z. (By Problem 12.1, the series converges for all z.) Show by (19.10) that all the zeros of $J_p(z)$ are real. *Hint:* Suppose α and β in (19.10) were a complex conjugate pair; show that then the integrand would be positive so the integral could not be zero.

5. We obtained (19.10) for $J_p(x)$, $p \geq 0$. It is, however, valid for $p > -1$, that is for $N_p(x)$, $0 \leq p < 1$. The difficulty in the proof occurs just after (19.7); we said that u, v, u', v' are finite at $x = 0$ which is not true for $N_p(x)$. However, the negative powers of x cancel if $p < 1$. Show this for $p = \frac{1}{2}$ by using two terms of the power series (12.9) or (13.1) for the function $N_{1/2}(x) = -J_{-1/2}(x)$ [see (13.3)].

6. By Problem 5, $\int_0^1 xN_{1/2}(\alpha x)N_{1/2}(\beta x)\,dx = 0$ if α and β are different zeros of $N_{1/2}(x)$. Using (17.4), find $N_{1/2}(x)$ in terms of $\cos x$ and so find the zeros of $N_{1/2}(x)$. Show that the functions $\cos(n + \frac{1}{2})\pi x$ are an orthogonal set on $(0, 1)$. Use (19.10) to find the normalization constant. (Compare Problem 6.8.)

▶ 20. APPROXIMATE FORMULAS FOR BESSEL FUNCTIONS

There are often cases in which it is useful to have an approximate formula giving the behavior of a Bessel function when x is near zero or when x is very large. We list some of these formulas here for reference. The symbol $O(x^n)$ is read "terms of the order of x^n or less," and means that the error in the given approximation is less than a constant times x^n; thus $O(1)$ means bounded terms. Note that $p \geq 0$.

Function	*Small x*	*Large x (asymptotic formulas)*
$J_p(x)$	$\dfrac{1}{\Gamma(p+1)}\left(\dfrac{x}{2}\right)^p + O(x^{p+2})$	$\sqrt{\dfrac{2}{\pi x}}\cos\left(x - \dfrac{2p+1}{4}\pi\right) + O(x^{-3/2})$
$N_p(x)$	$\begin{cases} p = 0 & \dfrac{2}{\pi}\ln x + O(1) \\ p > 0 & -\dfrac{\Gamma(p)}{\pi}\left(\dfrac{2}{x}\right)^p + \begin{cases} O(x^p), & p < 1 \\ O(x \ln \frac{1}{x}), & p = 1 \\ O(x^{2-p}), & p > 1 \end{cases} \end{cases}$	$\sqrt{\dfrac{2}{\pi x}}\sin\left(x - \dfrac{2p+1}{4}\pi\right) + O(x^{-3/2})$
$H_p^{(1) \text{ or } (2)}(x)$	Like $\pm i N_p(x)$	$\sqrt{\dfrac{2}{\pi x}}\, e^{\pm i[x-(2p+1)\pi/4]} + O(x^{-3/2})$
$I_p(x)$	Like $J_p(x)$	$\dfrac{1}{\sqrt{2\pi x}}\, e^x + O\left(\dfrac{e^x}{x}\right)$
$K_p(x)$	Like $-\dfrac{\pi}{2}N_p(x)$	$\sqrt{\dfrac{\pi}{2x}}\, e^{-x} + O\left(\dfrac{e^{-x}}{x}\right)$
$j_n(x)$	$\dfrac{x^n}{(2n+1)!!} + O(x^{n+2})$	$\dfrac{1}{x}\sin\left(x - \dfrac{n\pi}{2}\right) + O(x^{-2})$
$y_n(x)$	$\dfrac{-(2n-1)!!}{x^{n+1}} + O(x^{1-n})$	$-\dfrac{1}{x}\cos\left(x - \dfrac{n\pi}{2}\right) + O(x^{-2})$

Note: $(2n+1)!!$ means $1\cdot 3\cdot 5\cdot 7 \cdots (2n+1) = \dfrac{(2n+1)!}{2^n n!}$. See Chapter 1, Section 13C.

▶ PROBLEMS, SECTION 20

Use the table above to evaluate the following limits:

1. $\lim_{x\to 0} J_4(x)/[J_2(x)]^2$ **2.** $\lim_{x\to\infty} I_3(x)/I_5(x)$ **3.** $\lim_{x\to 0} N_0(x^2)/\ln(x)$

4. $\lim_{x\to 0} J_p(x)/N_p(x)$ **5.** $\lim_{x\to\infty} xI_p(x)K_p(x)$ **6.** $\lim_{x\to 0} xj_n(x)y_n(x)$

Use the table above and the definitions in Section 17 to find approximate formulas for large x for:

7. $h_n^{(1)}(x)$ **8.** $h_n^{(2)}(x)$

9. $h_n^{(1)}(ix)$ **10.** $h_n^{(2)}(ix)$

To study the approximations in the table, computer plot on the same axes the given function together with its small x approximation and its asymptotic approximation. Use an interval large enough to show the asymptotic approximation agreeing with the function for large x. If the small x approximation is not clear, plot it alone with the function over a small interval.

11. $J_1(x)$ **12.** $J_2(x)$ **13.** $J_3(x)$ **14.** $N_2(x)$

15. $N_3(x)$ **16.** $j_1(x)$ **17.** $j_2(x)$ **18.** $y_2(x)$

19. Computer plot on the same axes several $I_p(x)$ functions together with their common asymptotic approximation. Then computer plot each function with its small x approximation.

20. As in Problem 19, study the $K_p(x)$ functions. It is interesting to note (see Problem 17.4) that $K_{1/2}(x)$ is equal to the asymptotic approximation.

21. SERIES SOLUTIONS; FUCHS'S THEOREM

We have discussed two examples of differential equations solvable by the Frobenius method (Legendre and Bessel equations). There are many other "named" equations and the corresponding "named" functions which are their solutions. (See a few more examples in Section 22.) All of them have much in common with our two examples and you should not hesitate to look them up and use them without a formal introduction when you run into them on your computer or in a text or reference book. You may discover any or all of the following things about such a new (to you) set of functions: that they are the set of solutions of a differential equation with one or more parameters (like the p in Bessel's equation); that the values of the functions, their derivatives, their zeros, and many formulas involving them are available in references (tables and computer); that they have orthogonality properties, perhaps with respect to a weight function, and consequently (suitably restricted) functions can be expanded in series of them; that there is a generating function for the set of functions; that there are physical problems whose solutions involve the functions, often in the solution of a partial differential equation; etc.

Now you may wonder whether all differential equations can be solved by the Frobenius method. A general theorem due to Fuchs tells when this method will work; we shall state it for second-order differential equations which are the most important ones in applications. Write the differential equation as

$$y'' + f(x)y' + g(x)y = 0. \tag{21.1}$$

If $xf(x)$ and $x^2g(x)$ are expandable in convergent power series $\sum_{n=0}^{\infty} a_n x^n$, we say that the differential equation (21.1) is regular (or has a nonessential singularity) at the origin. Let us call these the Fuchsian conditions. Fuchs's theorem says that these conditions are necessary and sufficient for the general solution of (21.1) to consist of either

(1) two Frobenius series, or

(2) one solution $S_1(x)$ which is a Frobenius series, and a second solution which is $S_1(x)\ln x + S_2(x)$ where $S_2(x)$ is another Frobenius series.

Case (2) occurs only when the roots of the indicial equation are equal or differ by an integer, and not always then. [See, for example, equation (11.2) and Problems 11.1 to 11.4, and 11.7 to 11.9.] Note the *necessary* condition: If the Fuchsian conditions are not met, we cannot find the general solution by the method of generalized power series (see Problems 11 to 13). However, the equations most commonly found in applications do meet these conditions.

If the first Frobenius series $S_1(x)$ happens to break off, or you can easily write its sum in closed form, then the method of "reduction of order" [Chapter 8, Section 7(e)] gives a way of finding the second solution without using infinite series (see Problems 1 to 4). However, note that our main interest in series solutions is not to solve differential equations this way in general, but to study sets of functions (like Legendre polynomials and Bessel functions) which are solutions of differential equations that occur in applications. So the purpose in using series to solve a few simple differential equations (for which there are easier methods) is to learn how and when the series method works—to watch Fuchs's theorem in action (see problems).

▶ PROBLEMS, SECTION 21

For Problems 1 to 4, find one (simple) solution of each differential equation by series, and then find the second solution by the "reduction of order" method, Chapter 8, Section 7(e).

1. $(x^2+1)y'' - xy' + y = 0$

2. $x^2y'' + (x+1)y' - y = 0$

3. $x^2y'' + x^2y' - 2y = 0$

4. $(x-1)y'' - xy' + y = 0$

Solve the differential equations in Problems 5 to 10 by the Frobenius method; observe that you get only one solution. (Note, also, that the two values of s are equal or differ by an integer, and in the latter case the larger s gives the one solution.) Show that the conditions of Fuchs's theorem are satisfied. Knowing that the second solution is $\ln x$ times the solution you have, plus another Frobenius series, find the second solution.

5. $x(x+1)y'' - (x-1)y' + y = 0$

6. $4x^2(x+1)y'' - 4x^2y' + (3x+1)y = 0$

7. $x(x-1)^2y'' - 2y = 0$

8. $xy'' + xy' - 2y = 0$

9. $x^2y'' + (x^2-3x)y' + (4-2x)y = 0$

10. $x^2(x-1)y'' - x(5x-4)y' + (9x-6)y = 0$

11. For the differential equation in Problem 2, verify that it does not satisfy the Fuchsian conditions, and that your second solution cannot be expanded in a Frobenius series.

12. Verify that the differential equation $x^4y'' + y = 0$ is not Fuchsian; that it has the two independent solutions $x\sin(1/x)$ and $x\cos(1/x)$; and that these solutions are not expandable in Frobenius series.

13. Verify that the the differential equation in Problem 11.13 is not Fuchsian. Solve it by separation of variables to find the obvious solution $y =$ const. and a second solution in the form of an integral. Show that the second solution is not expandable in a Frobenius series.

22. HERMITE FUNCTIONS; LAGUERRE FUNCTIONS; LADDER OPERATORS

In this section, we shall outline some of the important formulas for two more sets of named functions. Both Hermite and Laguerre functions are of interest in quantum mechanics where they arise as solutions of eigenvalue problems (see Problem 27, and Chapter 13, Problems 7.20 to 7.22). We shall also consider an operator method which is a useful alternative to series solution for some differential equations.

Hermite Functions The differential equation for Hermite functions is

$$y_n'' - x^2 y_n = -(2n+1)y_n, \qquad n = 0, 1, 2, \cdots . \tag{22.1}$$

This equation can be solved by power series (Problem 5), but here we shall consider an operator method which is particularly efficient for this equation. Let's use the operator D to mean d/dx; then (see Problem 5.31 of Chapter 8)

$$\begin{aligned} &(D-x)(D+x)y = \left(\frac{d}{dx} - x\right)(y' + xy) = y'' - x^2 y + y, \quad \text{and similarly} \\ &(D+x)(D-x)y = y'' - x^2 y - y. \end{aligned} \tag{22.2}$$

Using (22.2), we can write (22.1) in two ways:

$$(D-x)(D+x)y_n = -2ny_n \quad \text{or} \tag{22.3}$$

$$(D+x)(D-x)y_n = -2(n+1)y_n. \tag{22.4}$$

Now let us operate on (22.3) with $(D+x)$ and on (22.4) with $(D-x)$, and change n to m for later convenience:

$$(D+x)(D-x)[(D+x)y_m] = -2m[(D+x)y_m], \tag{22.5}$$

$$(D-x)(D+x)[(D-x)y_m] = -2(m+1)[(D-x)y_m]. \tag{22.6}$$

(The brackets have been inserted to clarify our next step.)

Now compare (22.3) and (22.6); if $y_n = [(D-x)y_m]$ and $n = m+1$, the equations are identical. We write

$$y_{m+1} = (D-x)y_m \tag{22.7}$$

and we see that, given a solution y_m of (22.1) for one value of n, namely $n = m$, we can find a solution when $n = m+1$ by applying the "raising operator" $(D-x)$ to y_m. Similarly, from (22.4) and (22.5), we find that (Problem 1)

$$y_{m-1} = (D+x)y_m. \tag{22.8}$$

We may call $(D+x)$ a "lowering operator"; these operators are called *creation* and *annihilation* operators in quantum theory. Operators of this kind (see Problems 29, 30, and 23.27 for other examples) are called *ladder operators* since, like the rungs of a ladder, they enable us to go up or down in a set of functions.

Now if $n = 0$, we find a solution of (22.3) [and therefore of (22.1)] by requiring

$$(D+x)y_0 = 0. \tag{22.9}$$

We solve this equation (Problem 2) to get

$$y_0 = e^{-x^2/2}. \tag{22.10}$$

Then, by (22.7), $y_n = (D - x)^n e^{-x^2/2}$. These are the Hermite functions; they can be written in the simpler form $y_n = e^{x^2/2}(d^n/dx^n)e^{-x^2}$ (Problem 3):

(22.11)
$$y_n = (D - x)^n e^{-x^2/2} \quad \text{or} \quad y_n = e^{x^2/2}(d^n/dx^n)e^{-x^2}.$$
Hermite functions

If we multiply (22.11) by $(-1)^n e^{x^2/2}$, we obtain the *Hermite polynomials*; the following equation may be called a Rodrigues formula for them:

(22.12)
$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$
Hermite polynomials

We find (Problems 4 and 5):

(22.13)
$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2.$$

The Hermite polynomials satisfy the differential equation (Problem 6):

(22.14)
$$y'' - 2xy' + 2ny = 0.$$
Hermite equation

Using the differential equation, we can prove (Problem 7) that the Hermite polynomials are orthogonal on $(-\infty, \infty)$ with respect to the weight function e^{-x^2}. The normalization integral can be evaluated (Problem 10). Thus we have:

(22.15)
$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0, & n \neq m, \\ \sqrt{\pi}\, 2^n n! & n = m. \end{cases}$$

The generating function for the Hermite polynomials is (Problem 8):

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!}. \tag{22.16}$$

The generating function can be used to derive recursion relations for the Hermite polynomials. Two useful relations are (Problem 9):

$$\begin{aligned} &\text{(a)}\quad H_n'(x) = 2nH_{n-1}(x),\\ &\text{(b)}\quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \end{aligned} \tag{22.17}$$

Laguerre functions The *Laguerre polynomials* may be defined by a Rodrigues formula:

$$L_n(x) = \frac{1}{n!}e^x\frac{d^n}{dx^n}(x^n e^{-x}). \tag{22.18}$$

Carrying out the differentiation (Problem 12), we find:

$$\begin{aligned} L_n(x) &= 1 - nx + \frac{n(n-1)}{2!}\frac{x^2}{2!} - \frac{n(n-1)(n-2)}{3!}\frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!}\\ &= \sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{x^m}{m!}. \qquad \text{Laguerre polynomials} \end{aligned} \tag{22.19}$$

The symbol $\binom{n}{m}$ is a binomial coefficient (see Chapter 1, Section 13C). Some authors omit the $1/n!$ in (22.18); then the series in (22.19) is multiplied by $n!$. It is convenient to note that the series in (22.19) is like the binomial expansion of $(1-x)^n$ except that each power of x, say x^m, is divided by an extra $m!$. We find (Problem 13):

$$L_0(x) = 1, \qquad L_1(x) = 1 - x, \qquad L_2(x) = 1 - 2x + x^2/2. \tag{22.20}$$

The Laguerre polynomials are solutions of the differential equation (Problems 14 and 15):

$$xy'' + (1-x)y' + ny = 0, \qquad y = L_n(x). \tag{22.21}$$

Using the differential equation, we can prove (Problem 16) that the Laguerre polynomials are orthogonal on $(0, \infty)$ with respect to the weight function e^{-x}. In fact, we find (Problem 19) that, with the definition (22.18), the functions $e^{-x/2}L_n(x)$ are an orthonormal set on $(0, \infty)$.

$$\int_0^\infty e^{-x} L_n(x) L_k(x)\, dx = \delta_{nk} = \begin{cases} 0, & n \neq k, \\ 1, & n = k. \end{cases} \tag{22.22}$$

The generating function for the Laguerre polynomials is (Problem 17):

$$\Phi(x, h) = \frac{e^{-xh/(1-h)}}{1-h} = \sum_{n=0}^{\infty} L_n(x) h^n. \tag{22.23}$$

Using it, we can derive recursion relations; some examples are (Problem 18):

$$\begin{aligned} &\text{(a)} \quad L'_{n+1}(x) - L'_n(x) + L_n(x) = 0, \\ &\text{(b)} \quad (n+1)L_{n+1}(x) - (2n+1-x)L_n(x) + nL_{n-1}(x) = 0, \\ &\text{(c)} \quad xL'_n(x) - nL_n(x) + nL_{n-1}(x) = 0. \end{aligned} \tag{22.24}$$

Warning: These formulas will be different if the factor $1/n!$ is omitted in the definition (22.18), so check the notation of any reference you are using (computer, text, tables).

Derivatives of the Laguerre polynomials are called associated Laguerre polynomials; they may be found by differentiating (22.18), (22.19), or (22.20) (Problem 20). We define:

$$L_n^k(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x). \qquad \text{Associated Laguerre polynomials} \tag{22.25}$$

Warning: The notation in various references may be confusing; some authors define $L_n^k(x)$ as $(d^k/dx^k)L_n(x)$ [compare our definition in (22.25)], so read carefully the definition in the reference you are using. For example, associated Laguerre polynomials are used in the theory of the hydrogen atom in quantum mechanics. In various references you will find them denoted by $L_{n-l-1}^{2l+1}(x)$ and by $L_{n+l}^{2l+1}(x)$; both these notations mean (except for sign) $(d^{2l+1}/dx^{2l+1})L_{n+l}(x)$. (See Problems 26 to 28.)

By differentiating the Laguerre equation (22.21), we find the differential equation satisfied by the polynomials $L_n^k(x)$ (Problem 21):

$$xy'' + (k+1-x)y' + ny = 0, \qquad y = L_n^k(x). \tag{22.26}$$

The polynomials $L_n^k(x)$ may also be found from the Rodrigues formula (Problem 22):

$$L_n^k(x) = \frac{x^{-k}e^x}{n!}\frac{d^n}{dx^n}(x^{n+k}e^{-x}). \tag{22.27}$$

Note that in this form k does not have to be an integer; in fact, (22.27) is used to define $L_n^k(x)$ for any $k > -1$.

Recursion relations for the polynomials $L_n^k(x)$ may be found by differentiating recursion relations for the Laguerre polynomials. Some examples are (Problem 23):

$$\text{(22.28)}\quad \begin{aligned} &\text{(a)}\quad (n+1)L_{n+1}^k(x) - (2n+k+1-x)L_n^k(x) + (n+k)L_{n-1}^k(x) = 0,\\ &\text{(b)}\quad x\frac{d}{dx}L_n^k(x) - nL_n^k(x) + (n+k)L_{n-1}^k(x) = 0. \end{aligned}$$

Using the differential equation (22.26), we can show (Problem 24) that the functions $L_n^k(x)$ are orthogonal on $(0, \infty)$ with respect to the weight function $x^k e^{-x}$. We find (Problem 25):

$$\int_0^\infty x^k e^{-x} L_n^k(x) L_m^k(x)\,dx = \begin{cases} 0, & m \neq n,\\ \frac{(n+k)!}{n!}, & m = n. \end{cases} \tag{22.29}$$

The normalization integral needed in the theory of the hydrogen atom is not (22.29), but instead has the factor x^{k+1}. We find (see Problems 25 to 27):

$$\int_0^\infty x^{k+1} e^{-x} [L_n^k(x)]^2\,dx = (2n+k+1)\frac{(n+k)!}{n!}. \tag{22.30}$$

Again warning: The formulas (22.28), (22.29), and (22.30) will be different in references which omit the $1/n!$ in (22.18) and/or use a different definition of $L_n^k(x)$ in (22.25).

PROBLEMS, SECTION 22

1. Verify equations (22.2), (22.3), (22.4), and (22.8).

2. Solve (22.9) to get (22.10). If needed, see Chapter 8, Section 2.

3. Show that $e^{x^2/2}D[e^{-x^2/2}f(x)] = (D-x)f(x)$. Now set

$$f(x) = (D-x)g(x) = e^{x^2/2}D[e^{-x^2/2}g(x)]$$

to get

$$(D-x)^2 g(x) = e^{x^2/2}D^2[e^{-x^2/2}g(x)].$$

Continue this process to show that

$$(D-x)^n F(x) = e^{x^2/2}D^n[e^{-x^2/2}F(x)]$$

for any $F(x)$. Then let $F(x) = e^{-x^2/2}$ to get (22.11).

4. Using (22.12) find the Hermite polynomials given in (22.13). Then use (22.17b) to find $H_3(x)$ and $H_4(x)$.

5. By power series, solve the Hermite differential equation

$$y'' - 2xy' + 2py = 0$$

You should find an a_0 series and an a_1 series as for the Legendre equation in Section 2. Show that the a_0 series terminates when p is an even integer, and the a_1 series terminates when p is an odd integer. Thus for each integer n, the differential equation (22.14) has one polynomial solution of degree n. These polynomials with a_0 or a_1 chosen so that the highest order term is $(2x)^n$ are the Hermite polynomials. Find $H_0(x)$, $H_1(x)$, and $H_2(x)$. Observe that you have solved an eigenvalue problem (see end of Section 2), namely to find values of p for which the given differential equation has polynomial solutions, and then to find the corresponding solutions (eigenfunctions).

6. Substitute $y_n = e^{-x^2/2}H_n(x)$ into (22.1) to show that the differential equation satisfied by $H_n(x)$ is (22.14).

7. Prove that the functions $H_n(x)$ are orthogonal on $(-\infty, \infty)$ with respect to the weight function e^{-x^2}. *Hint:* Write the differential equation (22.14) as

$$e^{x^2}\frac{d}{dx}(e^{-x^2}y') + 2ny = 0,$$

and see Sections 7 and 19.

8. In the generating function (22.16), expand the exponential in a power series and collect powers of h to obtain the first few Hermite polynomials. Verify the identity

$$\frac{\partial^2\Phi}{\partial x^2} - 2x\frac{\partial\Phi}{\partial x} + 2h\frac{\partial\Phi}{\partial h} = 0.$$

Substitute the series in (22.16) into this identity to prove that the functions $H_n(x)$ in (22.16) satisfy equation (22.14). Verify that the highest term in $H_n(x)$ in (22.16) is $(2x)^n$. [You have then proved that the functions called $H_n(x)$ are really the Hermite polynomials since, by Problem 5, (22.14) has just one polynomial solution of degree n.]

9. Use the generating function to prove the recursion relations in (22.17). *Hint for* (a): Differentiate (22.16) with respect to x and equate coefficients of h^n. *Hint for* (b): Differentiate (22.16) with respect to h and equate coefficients of h^n.

10. Evaluate the normalization integral in (22.15). *Hint:* Use (22.12) for one of the $H_n(x)$ factors, integrate by parts, and use (22.17a); then use your result repeatedly.

11. Show that we have solved the following eigenvalue problem (see Problem 5 and end of Section 2): Given the differential equation $y'' + (\epsilon - x^2)y = 0$ [compare equation (22.1)]. find the possible values of ϵ (eigenvalues) such that the solutions $y(x)$ of the given differential equation tend to zero as $x \to \pm\infty$; for these values of ϵ, find the eigenfunctions $y(x)$. What is ϵ, and what are the eigenfunctions?

12. Using Leibniz' rule (Section 3), carry out the differentiation in (22.18) to obtain (22.19).

13. Using (22.19), verify (22.20) and also find $L_3(x)$ and $L_4(x)$.

14. Show that $y = L_n(x)$ given in (22.18) satisfies (22.21). *Hint:* Follow a method similar to that used in Section 4. Let $v = x^n e^{-x}$ and show that $xv' = (n-x)v$. Differentiate this last equation $(n+1)$ times by Leibniz' rule, and use $d^n v/dx^n = n!\, e^{-x}L_n(x)$ from (22.18).

15. Solve the Laguerre differential equation

$$xy'' + (1-x)y' + py = 0$$

by power series. Show that the a_0 series terminates if p is an integer. Thus for each integer n, the differential equation (22.21) has one solution which is a polynomial of degree n. These polynomials with $a_0 = 1$ are the Laguerre polynomials $L_n(x)$. Find $L_0(x)$, $L_1(x)$, $L_2(x)$, and $L_3(x)$. (This is an eigenvalue problem—compare Problem 5 and Section 2.)

16. Prove that the functions $L_n(x)$ are orthogonal on $(0, \infty)$ with respect to the weight function e^{-x}. *Hint:* Write the differential equation (22.21) as

$$e^x \frac{d}{dx}(xe^{-x}y') + ny = 0,$$

and see Sections 7 and 19.

17. In (22.23), write the series for the exponential and collect powers of h to verify the first few terms of the series. Verify the identity

$$x\frac{\partial^2 \Phi}{\partial x^2} + (1-x)\frac{\partial \Phi}{\partial x} + h\frac{\partial \Phi}{\partial h} = 0.$$

Substitute the series in (22.23) into this identity to show that the functions $L_n(x)$ in (22.23) satisfy Laguerre's equation (22.21). Verify that the constant term is 1 by putting $x = 0$ in the generating function. [You have then proved that the functions called $L_n(x)$ in (22.23) are really Laguerre polynomials since, by Problem 15, (22.21) has just one polynomial solution of degree n.]

18. Verify the recursion relations (22.24) as follows:

(a) Differentiate (22.23) with respect to x to get $h\Phi = (h-1)(\partial\Phi/\partial x)$; equate coefficients of h^{n+1}.

(b) Differentiate (22.23) with respect to h to get $(1-h)^2(\partial\Phi/\partial h) = (1-h-x)\Phi$; equate coefficients of h^n.

(c) Combine (a) and (b) to get $x(\partial\Phi/\partial x) + h\Phi - h(1-h)\partial\Phi/\partial h = 0$. Substitute the series for Φ and equate coefficients of h^n.

19. Evaluate the normalization integral in (22.22). *Hint:* Use (22.18) for one of the $L_n(x)$ factors; integrate by parts n times. Use (22.19) to find $(d^n/dx^n)L_n(x)$ and Chapter 11, Section 3, to evaluate $\int_0^\infty x^n e^{-x}\,dx$.

20. Using (22.25), (22.20), and Problem 13, find $L_n^k(x)$ for $n = 0, 1, 2$, and $k = 1, 2$.

21. Verify that the polynomials $L_n^k(x)$ in (22.25) satisfy (22.26). *Hint:* Write (22.21) with n replaced by $n+k$ and differentiate k times by Leibniz' rule.

22. Verify that the polynomials given by (22.27) are the same as the $L_n^k(x)$ defined in (22.25). *Hints:* Show that the functions in (22.27) satisfy (22.26) as follows. Let $v = e^{-x}x^{n+k}$ and show that $xv' = (n+k-x)v$. (Compare Problem 14.) Differentiate this equation $n+1$ times by Leibniz' rule, and use $d^n v/dx^n = n!\,e^{-x}x^k L_n^k(x)$ from (22.27). Also show that the coefficient of x^n in both (22.25) and (22.27) is $(-1)^n/n!$ [Thus, assuming that (22.26) for one k has only one polynomial solution of degree n (which can be shown by series solution), (22.27) gives the same polynomials as (22.25) for integral k.]

23. Verify the recursion relation relations (22.28) as follows:

(a) In (22.24b), replace n by $n+k$ and differentiate k times by Leibniz' rule; in (22.24a), replace k by $n+k$ and differentiate $k-1$ times. Subtract k times the second result from the first.

(b) In (22.24c), replace n by $n+k$ and differentiate k times.

24. Show that the functions $L_n^k(x)$ are orthogonal on $(0,\infty)$ with respect to the weight function $x^k e^{-x}$. *Hint:* Write the differential equation (22.26) as

$$x^{-k}e^x \frac{d}{dx}(x^{k+1}e^{-x}y') + ny = 0$$

and see Sections 7 and 19.

25. Evaluate the normalization integrals (22.29) and (22.30). *Hints:* Use (22.27) for one of the $L_n^k(x)$ factors in (22.29); integrate by parts n times. Use (22.25) and then (22.19) to evaluate $d^n/dx^n L_n^k(x)$. Compare Problem 19. To evaluate (22.30), multiply (22.28a) by $x^k e^{-x}$ and integrate; use (22.29) both for $m \neq n$ and $m = n$.

26. Solve the following eigenvalue problem (see end of Section 2 and Problem 11): Given the differential equation

$$y'' + \left(\frac{\lambda}{x} - \frac{1}{4} - \frac{l(l+1)}{x^2}\right) y = 0$$

where l is an integer ≥ 0, find values of λ such that $y \to 0$ as $x \to \infty$, and find the corresponding eigenfunctions. *Hint:* let $y = x^{l+1}e^{-x/2}v(x)$, and show that $v(x)$ satisfies the differential equation

$$xv'' + (2l + 2 - x)v' + (\lambda - l - 1)v = 0.$$

Compare (22.26) to show that if λ is an integer $> l$, there is a polynomial solution $v(x) = L_{\lambda-l-1}^{2l+1}(x)$.

27. The functions which are of interest in the theory of the hydrogen atom are

$$f_n(x) = x^{l+1}e^{-x/2n}L_{n-l-1}^{2l+1}\left(\frac{x}{n}\right)$$

where n and l are integers with $0 \leq l \leq n-1$. (Note that here $k = 2l+1$, and we have replaced n by $n-l-1$; in this problem L_2^3 , say, means $l = 1, n = 4$.) For $l = 1$, show that

$$f_2(x) = x^2e^{-x/4}, \quad f_3(x) = x^2e^{-x/6}\left(4 - \frac{x}{3}\right), \quad f_4(x) = x^2e^{-x/8}\left(10 - \frac{5x}{4} + \frac{x^2}{32}\right).$$

Hint: Find the polynomials L_0^3, L_1^3, L_2^3 as in Problem 20 (with $k = 3$) and then replace x by x/n. The functions $f_n(x)$ are very different from those in (22.29) since x/n changes from one function to the next. However, it can be shown (Problem 23.25) that for one fixed l, the set of functions $f_n(x), n \geq l+1$, is an orthogonal set on $(0,\infty)$. Verify this for these three functions. *Hint:* The integrals are Γ functions—see Chapter 11, Section 3.

28. Repeat Problem 27 for $l = 0$, $n = 1, 2, 3$.

29. Show that $R_p = \frac{p}{x} - D$ and $L_p = \frac{p}{x} + D$ where $D = d/dx$, are raising and lowering operators for Bessel functions, that is, show that $R_pJ_p(x) = J_{p+1}(x)$ and $L_pJ_p(x) = J_{p-1}(x)$. *Hint:* Use equations (15.5). Note that these operators depend on p as well as x, so they are not as simple as the Hermite function raising and lowering operators (22.7) and (22.8). If you want to operate, say, on J_{p+1}, you must change p in R or L to $p+1$, etc. Making this adjustment, show that the equations $LRJ_p = J_p$ and $RLJ_p = J_p$ both give Bessel's equation.

30. Find raising and lowering operators (see Problem 29) for spherical Bessel functions. *Hint:* See problems 17.15 and 17.16.

23. MISCELLANEOUS PROBLEMS

1. Use the generating function (5.1) to find the normalizing factor for Legendre polynomials. *Hint:* Square equation (5.2) with Φ as in (5.1) and integrate from -1 to 1. Expand the integral of Φ^2 (after integrating) in powers of h and equate coefficients.

2. Use the generating function to show that

$$P_{2n+1}(0) = 0 \qquad \text{and} \quad P_{2n}(0) = \binom{-1/2}{n} = \frac{(-1)^n(2n-1)!!}{2^n n!};$$

Hints: Expand (5.1) for $x = 0$ in powers of h and equate coefficients of powers of h in (5.2). See Chapter 1, Section 13C.

3. Use (5.8e) to show that $\int_0^1 P_l(x)\,dx = [P_{l-1}(0) - P_{l+1}(0)]/(2l+1)$. Then use the result of Problem 2 and Chapter 1, Section 13C to show that

$$\int_0^1 P_{2n}(x)\,dx = 0, n > 0, \qquad \text{and} \qquad \int_0^1 P_{2n+1}(x)\,dx = \frac{(-1)^n(2n-1)!!}{2^{n+1}(n+1)!} = \binom{1/2}{n+1}.$$

4. Obtain the binomial coefficient result in Problem 3 directly by integrating the generating function from 0 to 1 and expanding the result in powers of h. Equate the coefficients of h^l in the identity obtained by integrating (5.2) from 0 to 1, and use Chapter 1, Section 13C.

5. Show that $\sum_0^n (2l+1)P_l(x) = P'_n(x) + P'_{n+1}(x)$. *Hint:* Use mathematical induction as follows:

(a) Verify the formula for $n = 0$.

(b) Assuming that the formula is true for $l = n - 1$, show [using (5.8e)] that it is true for $l = n$.

6. Using (10.6), (5.8), and Problem 2, evaluate $P^1_{2n+1}(0)$.

7. Show that, for $l > 0$, $\int_a^b P_l(x)\,dx = 0$ if a and b are any two maximum or minimum points of $P_l(x)$, or ± 1. *Hint:* Integrate (7.2).

8. Show that $(2l+1)(x^2-1)P'_l(x) = l(l+1)[P_{l+1}(x) - P_{l-1}(x)]$. *Hint:* Integrate (5.8e) and (7.2) and combine the results. Thus show that $P_{l+1}(x) = P_{l-1}(x)$ at maximum and minimum points of $P_l(x)$ and at ± 1.

9. Evaluate $\int_{-1}^1 xP_l(x)P_n(x)\,dx$, $n \le l$. *Hint:* Write (5.8a) with l replaced by $l+1$, multiply by $P_n(x)$ and integrate.

Use the recursion relations of Section 15 (and, as needed, Sections 12, 13, 17, and 20) to verify the formulas in Problems 10 to 14.

10. $\displaystyle\int_0^\infty x^{-p} J_{p+1}(x)\,dx = \frac{1}{2^p\Gamma(1+p)}.$

11. $\displaystyle\int_0^\infty x^{-n} j_{n+1}(x)\,dx = \frac{1}{(2n+1)!!}.$

12. $\displaystyle\frac{d}{dx}K_p(x) = -\tfrac{1}{2}[K_{p-1}(x) + K_{p+1}(x)].$

13. $\displaystyle\frac{d}{dx}j_n(x) = [nj_{n-1}(x) - (n+1)j_{n+1}(x)]/(2n+1).$

14. $\displaystyle\int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x).$

15. Use the result of Problem 18.4 and equations (17.4) to show that

$$j_n(x)y_n'(x) - y_n(x)j_n'(x) = \frac{1}{x^2}.$$

Then use Problem 17.14 (for y's as well as j's) to show that

$$j_n(x)y_{n-1}(x) - y_n(x)j_{n-1}(x) = \frac{1}{x^2}.$$

16. Use (15.2) repeatedly to show that

$$J_1(x) = x\left(-\frac{1}{x}\frac{d}{dx}\right)J_0(x), \qquad J_2(x) = x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^2 J_0(x),$$

and, in general,

$$J_n(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n J_0(x).$$

17. Let α be the first positive zero of $J_1(x)$ and let β_n be the zeros of $J_0(x)$. In terms of α and β_n, find the values of x at the maximum and minimum points of the function $y = xJ_1(\alpha x)$. By computer or tables, find the needed zeros and compute the coordinates of the maximum and minimum points on the graph of $y(x)$ for x between 0 and 5. Computer plot y from $x = 0$ to 5 and compare your computed maximum and minimum points with what the plot shows.

18. (a) Make the change of variables $z = e^x$ in the differential equation $y'' + e^{2x}y = 0$, and so find a solution of the differential equation in terms of Bessel functions.

(b) Make the change of variables $z = e^{x^2/2}$ in the differential equation $xy'' - y' + x^3(e^{x^2} - p^2)y = 0$, and solve the equation in terms of Bessel functions.

19. (a) The generating function for Bessel functions of integral order $p = n$ is

$$\Phi(x,h) = e^{(1/2)x(h-h^{-1})} = \sum_{n=-\infty}^{\infty} h^n J_n(x).$$

By expanding the exponential in powers of $x(h - h^{-1})$ show that the $n = 0$ term is $J_0(x)$ as claimed.

(b) Show that

$$x^2\frac{\partial^2\Phi}{\partial x^2} + x\frac{\partial\Phi}{\partial x} + x^2\Phi - \left(h\frac{\partial}{\partial h}\right)^2\Phi = 0.$$

Use this result and $\Phi(x,h) = \sum_{n=-\infty}^{\infty} h^n J_n(x)$ to show that the functions $J_n(x)$ satisfy Bessel's equation. By considering the terms in h^n in the expansion of $e^{(1/2)x(h-h^{-1})}$ in part (a), show that the coefficient of h^n is a series starting with the term $(1/n!)(x/2)^n$. (You have then proved that the functions called $J_n(x)$ in the expansion of $\Phi(x,h)$ are indeed the Bessel functions of integral order previously defined by (12.9) and (13.1) with $p = n$.)

20. In the generating function equation of Problem 19, put $h = e^{i\theta}$ and separate real and imaginary parts to derive the equations

$$\begin{aligned}\cos(x\sin\theta) &= J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots \\ &= J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos 2n\theta, \\ \sin(x\sin\theta) &= 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots] \\ &= 2\sum_{n=0}^{\infty} J_{2n+1}(x)\sin(2n+1)\theta.\end{aligned}$$

These are Fourier series with Bessel functions as coefficients. (In fact the J_n's for integral n are often called Bessel coefficients because they occur in many series like these.) Use the formulas for the coefficients in a Fourier series to find integrals representing J_n for even n and for odd n. Show that these results can be combined to give

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$$

for *all* integral n. These series and integrals are of interest in astronomy and in the theory of frequency modulated waves.

21. In the generating function equation, Problem 19, put $x = iy$ and $h = -ik$ and show that

$$e^{(1/2)y(k+k^{-1})} = \sum_{n=-\infty}^{\infty} k^n I_n(y).$$

22. In the $\cos(x\sin\theta)$ series of Problem 20, let $\theta = 0$, and then let $\theta = \pi/2$, and add the results to show that (recall Problem 13.2)

$$\sum_{n=-\infty}^{\infty} J_{4n}(x) = \tfrac{1}{2}(1 + \cos x).$$

23. Solve by power series $(1 - x^2)y'' - xy' + n^2 y = 0$. The polynomial solutions of this equation with coefficients determined to make $y(1) = 1$ are called Chebyshev polynomials $T_n(x)$. Find T_0, T_1, and T_2.

24. (a) The following differential equation is often called a Sturm-Liouville equation:

$$\frac{d}{dx}[A(x)y'] + [\lambda B(x) + C(x)]y = 0$$

(λ is a constant parameter). This equation includes many of the differential equations of mathematical physics as special cases. Show that the following equations can be written in the Sturm-Liouville form: the Legendre equation (7.2); Bessel's equation (19.2) for a *fixed* p, that is, with the parameter λ corresponding to α^2; the simple harmonic motion equation $y'' = -n^2 y$; the Hermite equation (22.14); the Laguerre equations (22.21) and (22.26).

(b) By following the methods of the orthogonality proofs in Sections 7 and 19, show that if y_1 and y_2 are two solutions of the Sturm-Liouville equation (corresponding to the two values λ_1 and λ_2 of the parameter λ), then y_1 and y_2 are orthogonal on (a, b) with respect to the weight function $B(x)$ if $A(x)(y_1' y_2 - y_2' y_1)|_a^b = 0$.

25. In Problem 22.26, replace x by x/n in the y differential equation and set $\lambda = n$ to show that the differential equation satisfied by the functions $f_n(x)$ in Problem 22.27 is

$$y'' + \left(\frac{1}{x} - \frac{1}{4n^2} - \frac{l(l+1)}{x^2}\right) y = 0.$$

Hence show by Problem 24 that the functions $f_n(x)$ are orthogonal on $(0, \infty)$.

26. Verify *Bauer's formula* $e^{ixw} = \sum_0^\infty (2l+1)i^l j_l(x)P_l(w)$ as follows. Write the integral for the coefficients c_l in the Legendre series for $e^{ixw} = \sum c_l P_l(w)$. You want to show that $c_l(x) = (2l+1)i^l j_l(x)$. First show that $y = c_l(x)$ satisfies the differential equation (Problem 17.6) for spherical Bessel functions. *Hints:* Differentiate with respect to x under the integral sign to find y' and y''; substitute into the left side of the differential equation. Now integrate by parts with respect to w to show that the integrand is zero because $P_l(w)$ satisfies Legendre's equation. Thus $c_l(x)$ must be a linear combination of $j_l(x)$ and $n_l(x)$. Now consider the $c_l(x)$ integral for small x; expand e^{iwx} in series and evaluate the lowest term (which is x^l since $\int_{-1}^1 w^n P_l(w)\,dw = 0$ for $n < l$). Compare with the approximate formulas for $j_1(x)$ and $n_l(x)$ in Section 20.

27. Show that $R = lx - (1-x^2)D$ and $L = lx + (1-x^2)D$, where $D = d/dx$, are raising and lowering operators for Legendre polynomials [compare Hermite functions, (22.1) to (22.11) and Bessel functions, Problems 22.29 and 22.30]. More precisely, show that $RP_{l-1}(x) = lP_l(x)$ and $LP_l(x) = lP_{l-1}(x)$. *Hint:* Use equations (5.8d) and (5.8f). Note that, unlike the raising and lowering operators for Hermite functions, here R and L depend on l as well as x, so you must be careful about indices. The L operator operates on P_l, but the R operator as given operates on P_{l-1} to produce lP_l. [If you prefer, you could replace l by $l+1$ to rewrite R as $(l+1)x - (1-x^2)D$; then it operates on P_l to produce $(l+1)P_{l+1}$.] Assuming that all $P_l(1) = 1$, solve $LP_0(x) = 0$ to find $P_0(x) = 1$, and then use raising operators to find $P_1(x)$ and $P_2(x)$.

28. Show that the functions $J_0(t)$ and $J_0(\pi - t)$ are orthogonal on $(0, \pi)$. *Hints:* See the Laplace transform table (page 469), L23 and L24 with $g = h = J_0$. What is the inverse transform of $(p^2 + a^2)^{-1}$?

29. Show that the Fourier cosine transform (Chapter 7, Section 12) of $J_0(x)$ is

$$\begin{cases} \sqrt{\dfrac{2}{\pi}}\dfrac{1}{\sqrt{1-\alpha^2}}, & 0 \le \alpha < 1, \\ 0, & \alpha > 1. \end{cases}$$

Hence show that $\int_0^\infty J_0(x)\,dx = 1$. *Hints:* Show that the integral in Problem 20 gives $J_0(x) = (2/\pi)\int_0^{\pi/2} \cos(x\sin\theta)\,d\theta$. (Replace θ by $\pi - \theta$ in the $\pi/2$ to π integral.) Let $\sin\theta = \alpha$ to find J_0 as a cosine transform; write the inverse transform. Now let $\alpha = 0$.

30. Use the results of Chapter 7, Problems 12.18 and 13.19 to evaluate $\int_0^\infty [j_1(\alpha)]^2\,d\alpha$.

CHAPTER 13

Partial Differential Equations

1. INTRODUCTION

Many of the problems of mathematical physics involve the solution of partial differential equations. The same partial differential equation may apply to a variety of physical problems; thus the mathematical methods which you will learn in this chapter apply to many more problems than those we shall discuss in the illustrative examples. Let us outline the partial differential equations we shall consider, and the kinds of physical problems which lead to each of them.

(1.1) **Laplace's equation** $\nabla^2 u = 0$

The function u may be the gravitational potential in a region containing no mass, the electrostatic potential in a charge-free region, the steady-state temperature (that is, temperature not changing with time) in a region containing no sources of heat, or the velocity potential for an incompressible fluid with no vortices and no sources or sinks.

(1.2) **Poisson's equation** $\nabla^2 u = f(x, y, z)$

The function u may represent the same physical quantities listed for Laplace's equation, but in a region containing mass, electric charge, or sources of heat or fluid, respectively, for the various cases. The function $f(x, y, z)$ is called the source density; for example, in electricity it is proportional to the density of the electric charge.

(1.3) **The diffusion or heat flow equation** $\nabla^2 u = \dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t}$

Here u may be the non-steady-state temperature (that is, temperature varying with time) in a region with no heat sources; or it may be the concentration of a diffusing substance (for example, a chemical, or particles such as neutrons). The quantity α^2 is a constant known as the diffusivity.

(1.4) **Wave equation** $\nabla^2 u = \dfrac{1}{v^2}\dfrac{\partial^2 u}{\partial t^2}$

Here u may represent the displacement from equilibrium of a vibrating string or membrane or (in acoustics) of the vibrating medium (gas, liquid, or solid); in electricity u may be the current or potential along a transmission line; or u may be a component of $\mathbf{E}$ or $\mathbf{B}$ in an electromagnetic wave (light, radio waves, etc.). The quantity v is the speed of propagation of the waves; for example, for light in a vacuum it is c, the speed of light, and for sound waves it is the speed at which sound travels in the medium under consideration. The operator $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called the d'Alembertian.

(1.5) **Helmholtz equation** $\quad \nabla^2 F + k^2 F = 0$

As you will see later, the function F here represents the space part (that is, the time-independent part) of the solution of either the diffusion or the wave equation.

(1.6) **Schrödinger equation** $\quad -\dfrac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\dfrac{\partial}{\partial t}\Psi$

This is the wave equation of quantum mechanics. In this equation, $\hbar$ is Planck's constant divided by 2π, m is the mass of a particle, $i = \sqrt{-1}$, and V is the potential energy of the particle. The wave function Ψ is complex, and its absolute square is proportional to the position probability of the particle.

We shall be principally concerned with the solution of these equations rather than their derivation. If you like, you could say that it is true experimentally that the physical quantities mentioned above satisfy the given equations. However, it is also true that the equations can be derived from somewhat simpler experimental assumptions. Let us indicate briefly an example of how this can be done. In Chapter 6, Sections 10 and 11, we considered the flow of a fluid. We showed (Chapter 6, Problem 10.15) that $\nabla \cdot \mathbf{v} = 0$ for an incompressible fluid in a region containing no sources or sinks. If it is also true that there are no vortices (that is, the flow is irrotational), then $\operatorname{curl} \mathbf{v} = 0$, and $\mathbf{v}$ can be written as the gradient of a scalar function: $\mathbf{v} = \nabla u$. Combining these two equations, we have $\nabla \cdot \nabla u = \nabla^2 u = 0$. The function u is called the velocity potential and we see that (under the given conditions) it satisfies Laplace's equation as we claimed. A few more examples of such derivations are outlined in the problems.

In the following sections, we shall consider a number of physical problems to illustrate the very useful method of solving partial differential equations known as *separation of variables* (no relation to the same term used in ordinary differential equations, Chapter 8). In Sections 2 to 4, we consider problems in rectangular coordinates leading to Fourier series solutions—problems similar to those solved by Fourier. In later sections, we consider use of other coordinate systems (cylindrical, spherical) leading to solutions using Legendre or Bessel series.

▶ PROBLEMS, SECTION 1

1. Assume from electrostatics the equations $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ and $\mathbf{E} = -\nabla\phi$ ($\mathbf{E}$ = electric field, ρ = charge density, ϵ_0 = constant, ϕ = electrostatic potential). Show that the electrostatic potential satisfies Laplace's equation (1.1) in a charge-free region and satisfies Poisson's equation (1.2) in a region of charge density ρ.

2. (a) Show that the expression $u = \sin(x - vt)$ describing a sinusoidal wave (see Chapter 7, Figure 2.3), satisfies the wave equation (1.4). Show that, in general,

$u = f(x - vt)$ and $u = f(x + vt)$ satisfy the wave equation, where f is any function with a second derivative. This is the d'Alembert solution of the wave equation. (See Chapter 4, Section 11, Example 1.) The function $f(x - vt)$ represents a wave moving in the positive x direction and $f(x + vt)$ represents a wave moving in the opposite direction.

(b) Show that $u(r, t) = (1/r)f(r - vt)$ and $u(r, t) = (1/r)f(r + vt)$ satisfy the wave equation in spherical coordinates. [Use the first term of (7.1) for $\nabla^2 u$ since here u is independent of θ and ϕ.] These functions represent spherical waves spreading out from the origin or converging on the origin.

3. Assume from electrodynamics the following equations which are valid in free space. (They are called Maxwell's equations.)

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, and c is the speed of light in a vacuum. From them show that any component of $\mathbf{E}$ or $\mathbf{B}$ satisfies the wave equation (1.4) with $v = c$.

4. Obtain the heat flow equation (1.3) as follows: The quantity of heat Q flowing across a surface is proportional to the normal component of the (negative) temperature gradient, $(-\boldsymbol{\nabla} T) \cdot \mathbf{n}$. Compare Chapter 6, equation (10.4), and apply the discussion of flow of water given there to the flow of heat. Thus show that the rate of gain of heat per unit volume per unit time is proportional to $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} T$. But $\partial T/\partial t$ is proportional to this gain in heat; thus show that T satisfies (1.3).

2. LAPLACE'S EQUATION; STEADY-STATE TEMPERATURE IN A RECTANGULAR PLATE

We want to solve the following problem: A long rectangular metal plate has its two long sides and the far end at 0° and the base at 100° (Figure 2.1). The width of the plate is 10 cm. Find the steady-state temperature distribution inside the plate. (This problem is mathematically identical to the problem of finding the electrostatic potential in the region $0 < x < 10, y > 0$, if the given temperatures are replaced by potentials—see, for example, Jackson, 3rd edition, p.73)

To simplify the problem, we shall assume at first that the plate is so long compared to its width that we may make the mathematical approximation that it extends to infinity in the y direction. It is then called a semi-infinite plate. This is a good approximation if we are interested in temperatures not too near the far end.

The temperature T satisfies Laplace's equation inside the plate where there are no sources of heat, that is,

$$\text{(2.1)} \qquad \nabla^2 T = 0 \qquad \text{or} \qquad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

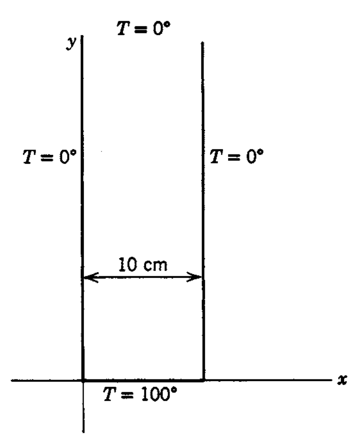


Figure 2.1

We have written ∇^2 in rectangular coordinates because the boundary of the plate is rectangular and we have omitted the z term because the plate is in two dimensions. To solve this equation, we are going to *try* a solution of the form

$$T(x, y) = X(x)Y(y), \tag{2.2}$$

where, as indicated, X is a function only of x, and Y is a function only of y. Immediately you may raise the question: But how do we know that the solution is of this form? The answer is that it is not! However, as you will see, once we have solutions of the form (2.2) we can combine them to get the solution we want. [Note that a sum of solutions of (2.1) is a solution of (2.1).] Substituting (2.2) into (2.1), we have

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0. \tag{2.3}$$

(Ordinary instead of partial derivatives are now correct since X depends only on x, and y depends only on y.) Divide (2.3) by XY to get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0. \tag{2.4}$$

The next step is really the key to the process of *separation of variables*. We are going to say that each of the terms in (2.4) is a constant because the first term is a function of x alone and the second term is a function of y alone. Why is this correct? Recall that when we say $u = \sin t$ is a *solution* of $\ddot{u} = -u$, we mean that if we substitute $u = \sin t$ into the differential equation, we get an identity ($\ddot{u} = -u$ becomes $-\sin t = -\sin t$), which is true for all values of t. Although we speak of an *equation*, when we substitute the solution into a differential equation, we have an *identity* in the independent variable. (We made use of this fact in series solutions of differential equations in Chapter 12, Sections 1 and 2.) In (2.1) to (2.4) we have two independent variables, x and y. Saying that (2.2) is a solution of (2.1) means that (2.4) is an identity in the two independent variables x and y [recall that (2.4) was obtained by substituting (2.2) into (2.1)]. In other words, if (2.2) is a solution of (2.1), then (2.4) must be true for any and all values of the two independent variables x and y. Since X is a function only of x and Y of y, the first term of (2.4) is a function only of x and the second term is a function only of y. Suppose we substitute a particular x into the first term; that term is then some numerical constant. To have (2.4) satisfied, the second term must be minus the same constant. While x remains fixed, let y vary (remember that x and y are independent). We have said that (2.4) is an identity; it is then true for our fixed x and *any* y. Thus the second term remains constant as y varies. Similarly, if we fix y and let x vary, we see that the first term of (2.4) is a constant. To say this more concisely, the equation $f(x) = g(y)$, with x and y independent variables, is an identity only if both functions are the same constant; this is the basis of the process of separation of variables. From (2.4) we then write

$$\begin{gathered}\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \text{const.} = -k^2, \qquad k \geq 0, \qquad \text{or} \\ X'' = -k^2X \qquad \text{and} \qquad Y'' = k^2Y.\end{gathered} \tag{2.5}$$

The constant k^2 is called the *separation constant*. The solutions of (2.5) are

$$X = \begin{cases}\sin kx, \\ \cos kx,\end{cases} \qquad Y = \begin{cases}e^{ky}, \\ e^{-ky},\end{cases} \tag{2.6}$$

and the solutions of (2.1) [of the form (2.2)] are

$$T = XY = \begin{Bmatrix} e^{ky} \\ e^{-ky} \end{Bmatrix} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}. \tag{2.7}$$

None of the four solutions in (2.7) satisfies the given boundary temperatures. What we must do now is to take a combination of the solutions (2.7), with the constant k properly selected, which *will* satisfy the given boundary conditions. [Any linear combination of solutions of (2.1) is a solution of (2.1) because the differential equation (2.1) is *linear*; see Chapter 3, Section 7, and Chapter 8, Sections 1 and 6.] We first discard the solutions containing e^{ky} since we are given $T \to 0$ as $y \to \infty$. (We are assuming $k > 0$; see Problem 5.) Next we discard solutions containing $\cos kx$ since $T = 0$ when $x = 0$. This leaves us just $e^{-ky} \sin kx$, but the value of k is still to be determined. When $x = 10$, we are to have $T = 0$; this will be true if $\sin(10k) = 0$, that is, if $k = n\pi/10$ for $n = 1, 2, \cdots$. Thus for any integral n, the solution

$$T = e^{-n\pi y/10} \sin \frac{n\pi x}{10} \tag{2.8}$$

satisfies the given boundary conditions on the three $T = 0$ sides.

Finally, we must have $T = 100$ when $y = 0$; this condition is not satisfied by (2.8) for any n. But a linear combination of solutions like (2.8) is a solution of (2.1); let us try to find such a combination which does satisfy $T = 100$ when $y = 0$. In order to allow all possible n's we write an infinite series for T, namely

$$T = \sum_{n=1}^{\infty} b_n e^{-n\pi y/10} \sin \frac{n\pi x}{10}. \tag{2.9}$$

For $y = 0$, we must have $T = 100$; from (2.9) with $y = 0$ we get

$$T_{y=0} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} = 100. \tag{2.10}$$

But this is just the Fourier sine series (Chapter 7, Section 9) for $f(x) = 100$ with $l = 10$. We can find the coefficients b_n, as we did in Chapter 7; we get

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{10} \int_0^{10} 100 \sin \frac{n\pi x}{10}\, dx = \begin{cases} \frac{400}{n\pi}, & \text{odd } n, \\ 0, & \text{even } n. \end{cases} \tag{2.11}$$

Then (2.9) becomes

$$T = \frac{400}{\pi} \left(e^{-\pi y/10} \sin \frac{\pi x}{10} + \frac{1}{3} e^{-3\pi y/10} \sin \frac{3\pi x}{10} + \cdots \right). \tag{2.12}$$

Equation (2.12) can be used for computation if $\pi y/10$ is not too small since then the series converges rapidly. (See also Problem 6.) For example, at $x = 5$ (central line of the plate) and $y = 5$, we find

$$T = \frac{400}{\pi} \left(e^{-\pi/2} \sin \frac{\pi}{2} + \frac{1}{3} e^{-3\pi/2} \sin \frac{3\pi}{2} + \cdots \right) \simeq 26.1^\circ. \tag{2.13}$$

To see how the temperature varies with x and y over a rectangle, you can computer plot a 3-dimensional graph of several terms of $T(x,y)$ in (2.12). Or you can make a 2-dimensional contour plot which shows the isothermals (curves of constant T). If the temperature on the bottom edge is any function $f(x)$ instead of 100° (with the other three sides at 0° as before), we can do the problem by the same method. We have only to expand the given $f(x)$ in a Fourier sine series and substitute the coefficients into (2.9).

Next, let us consider a finite plate of height 30 cm with the top edge at $T = 0°$, and other dimensions and temperatures as in Figure 2.1. We no longer have any reason to discard the e^{ky} solution since y does not become infinite. We now replace e^{-ky} by a linear combination $ae^{-ky} + be^{ky}$ which is zero when $y = 30$. The most convenient way to do this is to use the combination

$$\tfrac{1}{2}e^{k(30-y)} - \tfrac{1}{2}e^{-k(30-y)} \tag{2.14}$$

(that is, let $a = \frac{1}{2}e^{30k}$ and $b = -\frac{1}{2}e^{-30k}$). Then, when $y = 30$, (2.14) gives $e^0 - e^0 = 0$ as we wanted. Now (2.14) is just $\sinh k(30-y)$ (see Chapter 2, Section 12), so for the finite plate, we can write the solution as [compare (2.9)]

$$T = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{10}(30-y) \sin \frac{n\pi x}{10}. \tag{2.15}$$

Each term of this series is zero on the three $T = 0$ sides of the plate. When $y = 0$, we want $T = 100$:

$$T_{y=0} = 100 = \sum_{n=1}^{\infty} B_n \sinh(3n\pi) \sin \frac{n\pi x}{10} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \tag{2.16}$$

where $b_n = B_n \sinh 3n\pi$ or $B_n = b_n / \sinh 3n\pi$. We find b_n, solve for B_n and substitute into (2.15) to get the temperature distribution in the finite plate:

$$T = \sum_{\text{odd } n} \frac{400}{n\pi \sinh 3n\pi} \sinh \frac{n\pi}{10}(30-y) \sin \frac{n\pi x}{10}. \tag{2.17}$$

In (2.12) and (2.17) we have found functions $T(x,y)$ satisfying both (2.1) and all the given boundary conditions. For a bounded region with given boundary temperatures, it is an experimental fact (and it can also be shown mathematically—see Problem 16 and Chapter 14, Problem 11.38) that there is only one $T(x,y)$ satisfying Laplace's equation and the given boundary conditions. Thus (2.17) is the desired solution for the rectangular plate. It can also be shown that there is only one solution for the semi-infinite plate provided $T \to 0$ at ∞; thus (2.12) is the solution for that case.

It may have occurred to you to wonder why we took the constant in (2.5) to be $-k^2$ and what would happen if we took $+k^2$ instead. As far as getting solutions of the differential equation is concerned it would be perfectly correct to use $+k^2$; we would get instead of (2.7):

$$T = XY = \begin{Bmatrix} e^{kx} \\ e^{-kx} \end{Bmatrix} \begin{Bmatrix} \sin ky \\ \cos ky \end{Bmatrix}. \tag{2.18}$$

[We are assuming that k is real; an imaginary k in (2.18) would simply give combinations of the solutions (2.7) over again. Also see Problem 5.] The solutions (2.18) would not be of any use for the semi-infinite plate problem since none of them tends to zero as $y \to \infty$, and a linear combination of e^{kx} and e^{-kx} cannot be zero both at $x = 0$ and at $x = 10$. However, if we had considered a semi-infinite plate with its long sides parallel to the x axis instead of the y axis, and $T = 100°$ along the short end on the y axis, the solutions (2.18) would have been the ones needed. Or, for the finite plate, if the $100°$ side were along the y axis, then we would want (2.18).

Finally, let us see how to find the temperature distribution in a plate if two adjacent sides are held at $100°$ and the other two at $0°$ (or, in general, if any values are given for the four sides). We can find the solution to this problem by a combination of the results we have already obtained. Let us call the sides of the rectangular plate A, B, C, D (Figure 2.2). If sides A, B, and C are held at $0°$, and D at $100°$, we can find the temperature distribution by the same method we used in finding (2.17) if we take the x axis along D. Next suppose that for the same plate (Figure 2.2) sides A, B, and D are held at $0°$ and C at $100°$. This is the same kind of problem over again, but this time we want to use the solutions (2.18). [Or to shorten the work, we could write the solution like (2.17) with the x axis taken along C and then interchange x and y in the result to agree with Figure 2.2.] Having obtained the two solutions (one for C at $100°$ and one for D at $100°$), let us add these two answers. The result is a solution of the differential equation (2.1) (linearity: the sum of any two solutions is a solution). The temperatures on the boundary (as well as inside) are the sums of the temperatures in the two solutions we added, that is, $0°$ on A, $0°$ on B, $0° + 100°$ on C, and $100° + 0°$ on D. These are the given boundary conditions we wanted to satisfy. Thus the sum of the solutions of two simple problems gives the answer to the more complicated one (see Problems 11 to 13).

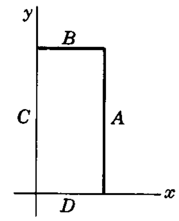


Figure 2.2

Before solving more problems, let us stop for a moment to summarize this process of separation of variables which is basically the same for all the partial differential equations we shall discuss. We first assume a solution which is a product of functions of the independent variables [like (2.2)], and separate the partial differential equation into several ordinary differential equations [like (2.5)]. We solve these ordinary differential equations; the solutions may be exponential functions, trigonometric functions, powers (positive or negative), Bessel functions, Legendre polynomials, etc. Any linear combination of these solutions, with any values of the separation constants, is a solution of the partial differential equation. The problem is to determine both the values of the separation constants and the correct linear combination to fit the given boundary or initial conditions.

The problem of finding the solution of a given differential equation subject to given boundary conditions is called a *boundary value problem*. Such problems often lead to *eigenvalue problems*. Recall (Chapter 3, Section 11, and Chapter 12, end of Section 2) that in an eigenvalue (or characteristic value) problem, there is a parameter whose values are to be selected so that the solutions of the problem meet some given requirements. The separation constants we have been using are just such parameters; their values are determined by demanding that the solutions satisfy some of the boundary conditions. [For example, we found $k = n\pi/10$ just before (2.8) by requiring that $T = 0$ when $x = 10$.] The resulting values of the separation constants are called *eigenvalues* and the solutions of the differential equation [for example (2.8)] corresponding to the eigenvalues are called *eigenfunctions*. It may also happen that in addition to the separation constants there is a parameter in the original partial differential equation [for example, E in the Schrödinger equation (1.6)]. Again the possible values of this parameter (for which the equation has solutions meeting specified requirements) are called eigenvalues, and the corresponding solutions are called eigenfunctions.

Having found the eigenfunctions, the next step is to expand the given function (boundary or initial conditions) in terms of them. [See, for example (2.10) and (2.16) and many examples in later sections.] As we have discussed (see Chapter 7, Section 8, and Chapter 12, Section 6), the eigenfunctions are a set of basis functions for this expansion. Thus we select the functions [for example $e^{-ky}\sin kx$ in (2.7)] and values of the separation constants (eigenvalues) to fit the given boundary (or initial) conditions; this determines the basis functions for a problem.

▶ PROBLEMS, SECTION 2

After you find the series solution of a problem, make computer plots of your results as discussed just after equation (2.13).

1. Find the steady-state temperature distribution for the semi-infinite plate problem if the temperature of the bottom edge is $T = f(x) = x$ (in degrees; that is, the temperature at x cm is x degrees), the temperature of the other sides is $0°$, and the width of the plate is 10 cm.

Answer: $T = \dfrac{20}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n\pi y/10}\sin(n\pi x/10).$

2. Solve the semi-infinite plate problem if the bottom edge of width 20 is held at

$$T = \begin{cases} 0°, & 0 < x < 10, \\ 100°, & 10 < x < 20, \end{cases}$$

and the other sides are at $0°$.

3. Solve the semi-infinite plate problem if the bottom edge of width π is held at $T = \cos x$ and the other sides are at $0°$.

Answer: $T = \dfrac{4}{\pi}\displaystyle\sum_{\text{even } n}\frac{n}{n^2-1}e^{-ny}\sin nx.$

4. Solve the semi-infinite plate problem if the bottom edge of width 30 is held at

$$T = \begin{cases} x, & 0 < x < 15, \\ 30 - x, & 15 < x < 30, \end{cases}$$

and the other sides are at $0°$.

5. Show that the solutions of (2.5) can also be written as

$$X = \begin{cases} e^{ikx}, \\ e^{-ikx}, \end{cases} \qquad Y = \begin{cases} \sinh ky, \\ \cosh ky. \end{cases}$$

Also show that these solutions are equivalent to (2.7) if k is real and equivalent to (2.18) if k is pure imaginary. (See Chapter 2, Section 12.) Also show that $X = \sin k(x - a), Y = \sinh k(y - b)$ are solutions of (2.5).

6. Show that the series in (2.12) can be summed to get

$$T = \frac{200}{\pi} \arctan\left(\frac{\sin(\pi x/10)}{\sinh(\pi y/10)}\right)$$

(with the arc tangent in radians). Use this formula to check the value $T = 26.1°$ at $x = y = 5$. *Hints for summing the series:* Use $\sin(n\pi x/10) = \operatorname{Im} e^{in\pi x/10}$ to write the series as $\operatorname{Im}\sum_{\text{odd } n} z^n/n$. (What is z?) Compare this with the series for $\ln[(1+z)/(1-z)]$ (see Chapter 1, Problem 13.17). Then use (13.5) of Chapter 2.

7. Solve Problem 3 if the plate is cut off at height 1 and the temperature at $y = 1$ is held at $0°$.

Answer: $T = \dfrac{4}{\pi} \displaystyle\sum_{\text{even } n} \frac{n}{(n^2-1)\sinh n} \sinh n(1-y) \sin nx.$

8. Find the steady-state temperature distribution in a rectangular plate 30 cm by 40 cm given that the temperature is $0°$ along the two long sides and along one short end; the other short end along the x axis has temperature

$$T = \begin{cases} 100°, & 0 < x < 10, \\ 0°, & 10 < x < 30. \end{cases}$$

9. Solve Problem 2 if the plate is cut off at height 10 and the temperature of the top edge is $0°$.

10. Find the steady-state temperature distribution in a metal plate 10 cm square if one side is held at $100°$ and the other three sides at $0°$. Find the temperature at the center of the plate.

Answer: $T = \displaystyle\sum_{\text{odd } n} \frac{400}{n\pi \sinh n\pi} \sinh \frac{n\pi}{10}(10-y) \sin \frac{n\pi x}{10},$

$T(5, 5) \simeq 25°.$

11. Find the steady-state temperature distribution in the plate of Problem 10 if two adjacent sides are at $100°$ and the other two at $0°$. *Hint:* Use your solution of Problem 10. You should not have to do any calculation—just write down the answer!

12. Find the temperature distribution in a rectangular plate 10 cm by 30 cm if two adjacent sides are held at $100°$ and the other two sides at $0°$.

13. Find the steady-state temperature distribution in a rectangular plate covering the area $0 < x < 10$, $0 < y < 20$, if the two adjacent sides along the axes are held at temperatures $T = x$ and $T = y$ and the other two sides at $0°$.

14. In the rectangular plate problem, we have so far had the temperature specified all around the boundary. We could, instead, have some edges insulated. The heat flow across an edge is proportional to $\partial T/\partial n$, where n is a variable in the direction normal to the edge (see normal derivatives, Chapter 6, Section 6). For example, the heat flow across an edge lying along the x axis is proportional to $\partial T/\partial y$. Since the heat flow across an insulated edge is zero, we must have not T, but a partial derivative of T, equal to zero on an insulated boundary. Use this fact to find the steady-state temperature distribution in a semi-infinite plate of width 10 cm if the two long sides are insulated, the far end (at ∞ as in Figure 2.1) is at $0°$, and the bottom edge is at $T = f(x) = x - 5$.

Note that you used $T \to 0$ as $y \to \infty$ only to discard the solutions e^{+ky}; it would be just as satisfactory to say that T does not become infinite as $y \to \infty$. Actually, the temperature (assumed finite) as $y \to \infty$ in this problem is determined by the given temperature at $y = 0$. Let $T = f(x) = x$ at $y = 0$, repeat your calculations above to find the temperature distribution, and find the value of T for large y. Don't forget the $k = 0$ term in the series!

15. Consider a finite plate, 10 cm by 30 cm, with two insulated sides, one end at 0° and the other at a given temperature $T = f(x)$. Try $f(x) = 100°$; $f(x) = x$. You should convince yourself that this problem cannot be done using just the solutions (2.7). To see what is wrong, go back to the differential equations (2.5) and solve them if $k = 0$. You should find solutions x, y, xy and constant [the constant is already contained in (2.7) for $k = 0$, but the other three solutions are not]. Now go back over each of the problems we have done so far and see why we could ignore these $k = 0$ solutions; then including the $k = 0$ solutions, finish the problem of the finite plate with insulated sides.

For the case $f(x) = x$, the answer is:

$$T = \frac{1}{6}(30 - y) - \frac{40}{\pi^2} \sum_{\text{odd } n} \frac{1}{n^2 \sinh 3n\pi} \sinh \frac{n\pi}{10}(30 - y) \cos \frac{n\pi x}{10}.$$

16. Show that there is only one function u which takes given values on the (closed) boundary of a region and satisfies Laplace's equation $\nabla^2 u = 0$ in the interior of the region. *Hints:* Suppose u_1 and u_2 are both solutions with the same boundary conditions so that $U = u_1 - u_2 = 0$ on the boundary. In Green's first identity (Chapter 6, Problem 10.16), let $\phi = \Psi = U$ to show that $\boldsymbol{\nabla} U \equiv 0$. Thus show $U \equiv 0$ everywhere inside the region.

3. THE DIFFUSION OR HEAT FLOW EQUATION; THE SCHRÖDINGER EQUATION

The heat flow equation is

$$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}, \tag{3.1}$$

where u is the temperature and α^2 is a constant characteristic of the material through which heat is flowing. It is worthwhile to do first a partial separation of (3.1) into a space equation and a time equation; the space equation in more than one dimension then must be further separated into ordinary differential equations in x and y, or x, y, and z, or r, θ, ϕ, etc. We assume a solution of (3.1) of the form

$$u = F(x, y, z)T(t). \tag{3.2}$$

(Note the change in meaning of T; we have previously used it for temperature; here u is temperature and T is the time-dependent factor in u.) Substitute (3.2) into (3.1); we get

$$T\nabla^2 F = \frac{1}{\alpha^2} F \frac{dT}{dt}. \tag{3.3}$$

Next divide (3.3) by FT to get

$$\frac{1}{F}\nabla^2 F = \frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt}. \tag{3.4}$$

The left side of this identity is a function only of the space variables x, y, z, and the right side is a function only of time. Therefore both sides are the same constant and we can write

$$\begin{aligned} &\frac{1}{F}\nabla^2 F = -k^2 \qquad \text{or} \qquad \nabla^2 F + k^2 F = 0 \qquad \text{and} \\ &\frac{1}{\alpha^2}\frac{1}{T}\frac{dT}{dt} = -k^2 \qquad \text{or} \qquad \frac{dT}{dt} = -k^2\alpha^2 T. \end{aligned} \tag{3.5}$$

The time equation can be integrated to give

$$T = e^{-k^2\alpha^2 t}. \tag{3.6}$$

We can see a physical reason here for choosing the separation constant $(-k^2)$ to be negative. As t increases, the temperature of a body might decrease to zero as in (3.6), but it could not increase to infinity as it would if we had used $+k^2$ in (3.5) and (3.6). The space equation in (3.5) is the Helmholtz equation (1.5) as promised. You will find (Problem 10) that the space part of the wave equation is also the Helmholtz equation.

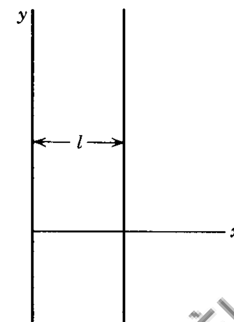


Figure 3.1

▶ **Example 1.** Let us now consider the flow of heat through a slab of thickness l (for example, the wall of a refrigerator). We shall assume that the faces of the slab are so large that we may neglect any end effects and assume that heat flows only in the x direction (Figure 3.1). This problem is then identical to the problem of heat flow in a bar of length l with insulated sides, because in both cases the heat flow is just in the x direction. Suppose the slab has initially a steady-state temperature distribution with the $x = 0$ wall at 0° and the $x = l$ wall at 100°. From $t = 0$ on, let the $x = l$ wall (as well as the $x = 0$ wall) be held at 0°. We want to find the temperature at any x (in the slab) at any later time.

First, we find the initial steady-state temperature distribution. You probably already know that this is linear, but it is interesting to see this from our equations. The initial steady-state temperature u_0 satisfies Laplace's equation, which in this one-dimensional case is $d^2u_0/dx^2 = 0$. The solution of this equation is $u_0 = ax + b$, where a and b are constants which must be found to fit the given conditions. Since $u_0 = 0$ at $x = 0$ and $u_0 = 100$ at $x = l$, we have

$$u_0 = \frac{100}{l}x. \tag{3.7}$$

From $t = 0$ on, u satisfies the heat flow equation (3.1). We have already separated this; the solutions are (3.2) where $T(t)$ is given by (3.6) and $F(x)$ satisfies the first of equations (3.5), namely

$$\nabla^2 F + k^2 F = 0 \qquad \text{or} \qquad \frac{d^2F}{dx^2} + k^2 F = 0. \tag{3.8}$$

(For this one-dimensional problem, F is a function only of x.) The solutions of (3.8) are

$$F(x) = \begin{cases} \sin kx, \\ \cos kx, \end{cases} \tag{3.9}$$

and the solutions (3.2) are

$$u = \begin{cases} e^{-k^2\alpha^2 t}\sin kx \\ e^{-k^2\alpha^2 t}\cos kx \end{cases} \tag{3.10}$$

We discard the $\cos kx$ solution for this problem because we are given $u = 0$ at $x = 0$. Also we want $u = 0$ at $x = l$; this will be true if $\sin kl = 0$, that is, $kl = n\pi$, or $k = n\pi/l$ (eigenvalues). Our basis functions (eigenfunctions) are then

$$u = e^{-(n\pi\alpha/l)^2 t}\sin\frac{n\pi x}{l} \tag{3.11}$$

and the solution of our problem will be the series

$$u = \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/l)^2 t}\sin\frac{n\pi x}{l}. \tag{3.12}$$

At $t = 0$, we want $u = u_0$ as in (3.7), that is,

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = u_0 = \frac{100}{l}x. \tag{3.13}$$

This means finding the Fourier sine series for $(100/l)x$ on $(0, l)$; the result (from Problem 1) for the coefficients is

$$b_n = \frac{100}{l}\frac{2l}{\pi}\frac{1}{n}(-1)^{n-1} = \frac{200}{\pi}\frac{(-1)^{n-1}}{n}. \tag{3.14}$$

Then we get the final solution by substituting (3.14) into (3.12); this gives

$$u = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{-(n\pi\alpha/l)^2 t}\sin\frac{n\pi x}{l}. \tag{3.15}$$

▶ **Example 2.** We can now do some variations of this problem. Suppose the final temperatures of the faces are given as two different constant values different from zero. Then, as for the initial steady state, the final steady state is a linear function of distance. The series (3.12) tends to a final steady state of zero; to obtain a solution tending to some other final steady state, we add to (3.12) the linear function u_f representing the correct final steady state. Thus we write instead of of (3.12)

$$u = \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/l)^2 t}\sin\frac{n\pi x}{l} + u_f. \tag{3.16}$$

Then for $t = 0$, the equation corresponding to (3.13) is

$$u_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} + u_f \tag{3.17}$$

or

$$u_0 - u_f = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}. \tag{3.18}$$

Thus when $u_f \neq 0$, it is $u_0 - u_f$ rather than u_0 which must be expanded in a Fourier series.

Insulated boundaries So far we have had the boundary temperatures given. We could, instead, have the faces insulated; then no heat flows in or out of the body. This will be true if the normal derivative $\partial u/\partial n$ (see Problem 2.14) of the temperature is zero at the boundary. (When the boundary values of u are given, the problem is called a *Dirichlet problem;* when the boundary values of the normal derivative $\partial u/\partial n$ are given, the problem is called a *Neumann problem.*) For the one-dimensional case we have considered, we replace the condition $u = 0$ at $x = 0$ and l by the condition $\partial u/\partial x = 0$ at $x = 0$ and l if the faces are insulated. This means that the useful solution in (3.10) is now the one containing $\cos kx$; note carefully that we must include the constant term (corresponding to $k = 0$). See Problem 7.

The Schrödinger Equation Compare equations (1.3) and (1.6). If $V = 0$ in (1.6), the two equations have the same form (a ∇^2 term and a first partial with respect to t). For future reference (see problems, Section 7), let's first separate variables in the general equation (1.6). We assume [compare (3.2)]

$$\Psi = \psi(x, y, z)T(t). \tag{3.19}$$

Substitute (3.19) into (1.6) and divide by Ψ to get

$$-\frac{\hbar^2}{2m}\frac{1}{\psi}\nabla^2\psi + V = i\hbar\frac{1}{T}\frac{dT}{dt} = E \tag{3.20}$$

where E is the separation constant [compare (3.5)]. (In quantum mechanics, E has the meaning of energy of the particle.) Then integrating the time equation gives (compare 3.6)

$$T = e^{-iEt/\hbar} \tag{3.21}$$

and the space equation (called the *time-independent Schrödinger equation*) is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi. \qquad \text{Time-independent Schrödinger equation} \tag{3.22}$$

For the one-dimensional problems that we consider in this section, and with $V = 0$, we have

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \text{or} \qquad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \tag{3.23}$$

which is (3.8) with $k^2 = \frac{2mE}{\hbar^2}$. Thus the solutions of (3.23) are the same as in (3.9) and the corresponding Ψ solutions are

$$\Psi = \psi(x)T(t) = \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} e^{-iEt/\hbar}. \tag{3.24}$$

▶ **Example 3.** The "particle in a box" problem in quantum mechanics requires the solution of the Schrödinger equation with $V = 0$ on $(0, l)$ and $\Psi = 0$ at the endpoints $x = 0$ and $x = l$ for all t. (The wave function Ψ then describes a particle trapped between 0 and l.) As in the heat flow problem, $\Psi = 0$ at $x = 0$ requires the sine solutions in (3.24) and $\Psi = 0$ at $x = l$ requires $k = n\pi/l$. Since $k^2 = 2mE/\hbar^2$, we find $E = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{l^2}$ which we will call E_n. (The meaning of this equation in quantum mechanics is that the energy of a particle trapped between 0 and l can have only a discrete set of values called eigenvalues. We say that the energy is quantized.) The basis functions for this problem are then the eigenfunctions

$$\Psi_n = \sin\frac{n\pi x}{l} e^{-iE_n t/\hbar}, \tag{3.25}$$

and we write $\Psi(x, t)$ as a linear combination of them.

$$\Psi(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-iE_n t/\hbar}, \tag{3.26}$$

(compare (3.12) for the heat flow problem). If the initial state $\Psi(x, 0)$ is the same function as in (3.7), the b_n coefficients are the same as in (3.14), so we have

$$\Psi(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{l} e^{-iE_n t/\hbar}. \tag{3.27}$$

See Problems 11 and 12; also see Problems 6.6 to 6.8, and 7.17 to 7.22.

▶ PROBLEMS, SECTION 3

As in Section 2, make computer plots of your results.

1. Verify the coefficients in equation (3.14).

2. A bar 10 cm long with insulated sides is initially at 100°. Starting at $t = 0$, the ends are held at 0°. Find the temperature distribution in the bar at time t.

Answer: $u = \dfrac{400}{\pi}\displaystyle\sum_{\text{odd } n}\frac{1}{n}e^{-(n\pi\alpha/10)^2 t}\sin\frac{n\pi x}{10}.$

3. In the initial steady state of an infinite slab of thickness l, the face $x = 0$ is at 0° and the face $x = l$ is at 100°. From $t = 0$ on, the $x = 0$ face is held at 100° and the $x = l$ face at 0°. Find the temperature distribution at time t.

Answer: $u = 100 - \dfrac{100x}{l} - \dfrac{400}{\pi}\displaystyle\sum_{\text{even } n}\frac{1}{n}e^{-(n\pi\alpha/l)^2 t}\sin\frac{n\pi x}{l}.$

4. At $t = 0$, two flat slabs each 5 cm thick, one at 0° and one at 20°, are stacked together, and then the surfaces are kept at 0°. Find the temperature as a function of x and t for $t > 0$.

5. Two slabs, each 1 inch thick, each have one surface at 0° and the other surface at 100°. At $t = 0$, they are stacked with their 100° faces together and then the outside surfaces are held at 100°. Find $u(x, t)$ for $t > 0$.

6. Show that the following problem is easily solved using (3.15): The ends of a bar are initially at 20° and 150°; at $t = 0$ the 150° end is changed to 50°. Find the time-dependent temperature distribution.

7. A bar of length l with insulated sides has its ends also insulated from time $t = 0$ on. Initially the temperature is $u = x$, where x is the distance from one end. Determine the temperature distribution inside the bar at time t. *Hints and comments:* See the discussion above and also Problem 2.14. Show that the $k = 0$ solutions are x and constant (time independent). Note that here (unlike Problem 2.15) you do not need the extra solution (namely x) for $k = 0$ since the final steady state is a constant and this is included in the solutions (3.10). Also note that we *did* need the $k = 0$ solutions in the discussion following (3.15) but were able to simplify the work by observing that these linear solutions simply give the final steady state.

Answer: $u = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{\text{odd } n} \frac{1}{n^2} \cos \frac{n\pi x}{l} e^{-(n\pi\alpha/l)^2 t}$.

8. A bar of length 2 is initially at 0°. From $t = 0$ on, the $x = 0$ end is held at 0° and the $x = 2$ end at 100°. Find the time-dependent temperature distribution.

9. Solve Problem 8 if, for $t > 0$, the $x = 0$ end of the bar is insulated and the $x = 2$ end is held at 100°. See Problem 7 above, and Chapter 7, end of Section 11.

10. Separate the wave equation (1.4) into a space equation and a time equation as we did the heat flow equation, and show that the space equation is the Helmholtz equation for this case also.

11. Solve the "particle in a box" problem to find $\Psi(x, t)$ if $\Psi(x, 0) = 1$ on $(0, \pi)$. What is E_n? The function of interest here which you should plot is $|\Psi(x, t)|^2$.

12. Do Problem 11 if $\Psi(x, 0) = \sin^2 \pi x$ on $(0, 1)$.

4. THE WAVE EQUATION; THE VIBRATING STRING

Let a string (for example, a piano or violin string) be stretched tightly and its ends fastened to supports at $x = 0$ and $x = l$. When the string is vibrating, its vertical displacement y from its equilibrium position along the x axis depends on x and t. We assume that the displacement y is always very small and that the slope $\partial y/\partial x$ of the string at any point at any time is small. In other words, we assume that the string never gets very far away from its stretched equilibrium position; in fact, we do not distinguish between the length of the string and the distance between the supports, although it is clear that the string must stretch a little as it vibrates out of its equilibrium position. Under these assumptions, the displacement $y(x, t)$ satisfies the (one-dimensional) wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}. \tag{4.1}$$

The constant v depends on the tension and the linear density of the string; it is called the wave velocity because it is the velocity with which a disturbance at one point of the string would travel along the string. To separate the variables, we substitute

$$y = X(x)T(t) \tag{4.2}$$

into (4.1) and get (Problem 3.10)

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{v^2} \frac{1}{T} \frac{d^2 T}{dt^2} = -k^2,$$

or

$$X'' + k^2 X = 0, \qquad\qquad (4.3)$$
$$\ddot{T} + k^2 v^2 T = 0.$$

We can see from the physical problem why we use a negative separation constant here; the solutions are to describe vibrations which are represented by sines and cosines, not by real exponentials. Of course, if we tried using $+k^2$ with k real, we would also discover mathematically that we could not satisfy the boundary conditions.

Recall the following notation used in discussing wave phenomena (see Chapter 7, Problem 2.17):

$\nu =$ frequency (sec^{-1})	$\omega = 2\pi\nu =$ angular frequency (radians)
$\lambda =$ wavelength	$k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi\nu}{v} = \dfrac{\omega}{v} =$ wave number
$v = \lambda\nu$	

The solutions of the two equations in (4.3) are

$$X = \begin{cases} \sin kx, \\ \cos kx, \end{cases} \qquad T = \begin{cases} \sin kvt = \sin \omega t, \\ \cos kvt = \cos \omega t, \end{cases} \qquad (4.4)$$

and so the solutions (4.2) for y are

$$y = \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} \left\{ \begin{array}{c} \sin \omega t \\ \cos \omega t \end{array} \right\} \qquad \text{where } \omega = kv. \qquad (4.5)$$

Since the string is fastened at $x = 0$ and $x = l$, we must have $y = 0$ for these values of x and all t. This means that we want only the $\sin kx$ factors in (4.5), and also we select k so that $\sin kl = 0$ or $k = n\pi/l$. The solutions then become

$$y = \begin{cases} \sin \dfrac{n\pi x}{l} \sin \dfrac{n\pi vt}{l}, \\[2ex] \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi vt}{l}. \end{cases} \qquad (4.6)$$

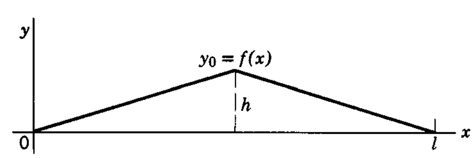


Figure 4.1

The particular combination of solutions (4.6) that we should take to solve a given problem depends on the initial conditions. For example, suppose the string is started vibrating by plucking (that is, pulling it aside a small distance h at the center and letting go). Then we are given the shape of the string at $t = 0$, namely $y_0 = f(x)$ as in Figure 4.1, and also the fact that the velocity $\partial y/\partial t$ of points on the string is zero at $t = 0$. (Do not confuse $\partial y/\partial t$ with the wave velocity v; there is no relation between them.) In (4.6) we must then discard the term containing $\sin(n\pi vt/l)$ since its time derivative is not zero when $t = 0$. Thus the basis functions for this problem are $\sin(n\pi x/l)\cos(n\pi vt/l)$ and we write the solution in the form

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}. \tag{4.7}$$

The coefficients b_n are to be determined so that at $t = 0$ we have $y_0 = f(x)$, that is,

$$y_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = f(x). \tag{4.8}$$

As in previous problems, we find the coefficients in the Fourier sine series for the given $f(x)$ and substitute them into (4.7). The result is (Problem 1)

$$y = \frac{8h}{\pi^2}\left(\sin\frac{\pi x}{l}\cos\frac{\pi vt}{l} - \frac{1}{9}\sin\frac{3\pi x}{l}\cos\frac{3\pi vt}{l} + \dots\right). \tag{4.9}$$

Another way to start the string vibrating is to hit it (a piano string, for example). In this case the initial conditions would be $y = 0$ at $t = 0$, with the velocity $\partial y/\partial t$ at $t = 0$ given as a function of x (that is, the velocity of each point of the string is given at $t = 0$). This time we discard in (4.6) the term containing $\cos(n\pi vt/l)$ because it is not zero at $t = 0$. Then, for this problem, the basis functions are $\sin(n\pi x/l)\sin(n\pi vt/l)$ and the solution is of the form

$$y = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}. \tag{4.10}$$

Here the coefficients must be determined so that

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n_1}^{\infty} B_n \frac{n\pi v}{l}\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = V(x), \tag{4.11}$$

that is, $V(x)$, the given initial velocity, must be expanded in a Fourier sine series (see Problems 5 to 8).

Suppose the string is vibrating in such a way that, instead of an infinite series for y, we have just one of the solutions (4.6), say

$$y = \sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l} \tag{4.12}$$

for some one value of n. The largest value of $\sin(n\pi vt/l)$, for any t, is 1, and the shape of the string then is

$$y = \sin\frac{n\pi x}{l}. \tag{4.13}$$

Graphs of (4.13) are sketched in Figure 4.2 for $n = 1, 2, 3, 4$. (The graphs are exaggerated! Remember that the displacements are actually very small.) Consider