"JUST THE MATHS"

UNIT NUMBER

14.3

PARTIAL DIFFERENTIATION 3
(Small increments and small errors)

by

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UNIT 14.3 - PARTIAL DIFFERENTIATION 3

SMALL INCREMENTS AND SMALL ERRORS

14.3.1 FUNCTIONS OF ONE INDEPENDENT VARIABLE - A RECAP

For functions of **one** independent variable, a discussion of small increments and small errors has already taken place in Unit 11.6.

It was established that, if a dependent variable, y, is related to an independent variable, x, by means of the formula

$$y = f(x),$$

then

(a) The **increment**, δy, in y, due to an increment of δx, in x is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x;$$

and, in much the same way,

(b) The **error**, δy, in y, due to an error of δx in x, is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

14.3.2 FUNCTIONS OF MORE THAN ONE INDEPENDENT VARIABLE

Let us consider, first, a function, z, of two independent variables, x and y, given by the formula

$$z = f(x, y).$$

If x is subject to a small increment (or a small error) of δx, while y remains constant, then the corrresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x.$$

Similarly, if y is subject to a small increment (or a small error) of δy, while x remains constant, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial y}\delta y.$$

It seems reasonable to assume, therefore, that, when x is subject to a small increment (or a small error) of δx **and** y is subject to a small increment (or a small error) of δy, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

It may be shown that, to the first order of approximation, this is indeed true.

Notes:

(i) To prove more rigorously that the above result is true, use would have to be made of the result known as **"Taylor's Theorem"** for a function of two independent variables.

In the present case, where $z = f(x, y)$, it would give

$$f(x+\delta x, y+\delta y) = f(x, y) + \left(\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right) + \left(\frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\delta x\delta y + \frac{\partial^2 z}{\partial y^2}(\delta y)^2\right) + \ldots,$$

which shows that

$$\delta z = f(x+\delta x, y+\delta y) - f(x, y) \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

to the first order of approximation.

(ii) The formula for a function of two independent variables may be extended to functions of a greater number of independent variables by simply adding further appropriate terms to the right hand side.

For example, if

$$w = F(x, y, z),$$

then

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

EXAMPLES

1. A rectangle has sides of length $x$cms. and $y$cms.

   Determine, approximately, in terms of x and y, the increment in the area, A, of the rectangle when x and y are subject to increments of δx and δy, respectively.

   Solution

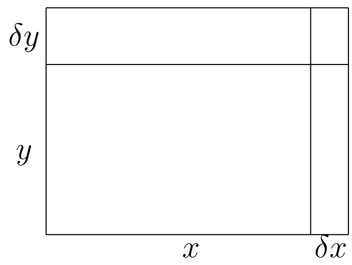
   The area, A, is given by

   $$A = xy,$$

   so that

   $$\delta A \simeq \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y = y\delta x + x\delta y.$$

   Note:
   The exact value of δA may be seen in the following diagram:



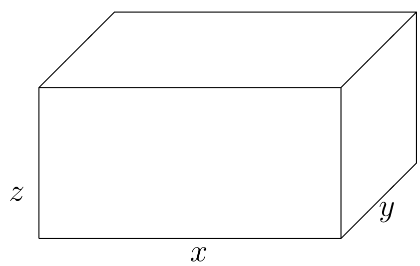
The difference between the approximate value and the exact value is represented by the area of the small rectangle having sides δxcms. and δycms.

2. In measuring a rectangular block of wood, the dimensions were found to be 10cms., 12cms and 20cms. with a possible error of $\pm 0.05cms.$ in each.

   Calculate, approximately, the greatest possible error in the surface area, S, of the block and the percentage error so caused.

   Solution

   First, we may denote the lengths of the edges of the block by x, y and z.



The surface area, S, is given by

$$S = 2(xy + yz + zx),$$

which has the value 1120cms^2 when $x = 10$cms., $y = 12$cms. and $z = 20$cms.

Also,

$$\delta S \simeq \frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y + \frac{\partial S}{\partial z}\delta z,$$

which gives

$$\delta S \simeq 2(y+z)\delta x + 2(x+z)\delta y + 2(y+x)\delta z;$$

and, on substituting $x = 10$, $y = 12$, $z = 20$, $\delta x = \pm 0.05$, $\delta y = \pm 0.05$ and $\delta z = \pm 0.05$, we obtain

$$\delta S \simeq \pm 2(12+20)(0.05) \pm 2(10+20)(0.05) \pm 2(12+10)(0.05).$$

The greatest error will occur when all the terms of the above expression have the same sign. Hence, the greatest error is given by

$$\delta S_{\max} \simeq \pm 8.4 \text{cms.}^2;$$

and, since the originally calculated value was 1120, this represents a percentage error of approximately

$$\pm\frac{8.4}{1120} \times 100 = \pm 0.75$$

3. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

We have

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

That is,

$$\delta w \simeq \frac{3x^2 z}{y^4}\delta x - \frac{4x^3 z}{y^5}\delta y + \frac{x^3}{y^4}\delta z,$$

where

$$\delta x = -\frac{3x}{100}, \quad \delta y = \frac{y}{100} \quad \text{and} \quad \delta z = \frac{2z}{100}.$$

Thus,

$$\delta w \simeq \frac{x^3 z}{y^4}\left[-\frac{9}{100} - \frac{4}{100} + \frac{2}{100}\right] = -\frac{11w}{100}.$$

The percentage error in w is given approximately by

$$\frac{\delta w}{w} \times 100 = -11.$$

That is, w is too small by approximately 11%.

14.3.3 THE LOGARITHMIC METHOD

In this section we consider again examples where it is required to calculate either a percentage increment or a percentage error.

We may conveniently use logarithms if the right hand side of the formula for the dependent variable involves a product, a quotient, or a combination of these two in which the independent variables are separated. This would be so, for instance, in the final example of the previous section.

The method is to take the natural logarithms of both sides of the equation before considering any partial derivatives; and we illustrate this, firstly, for a function of **two** independent variables.

Suppose that

$$z = f(x, y)$$

where $f(x, y)$ is the type of function described above.

Then,

$$\ln z = \ln f(x, y);$$

and, if we temporarily replace $\ln z$ by w, we have a new formula

$$w = \ln f(x, y).$$

The increment (or the error) in w, when x and y are subject to increments (or errors) of δx and δy respectively, is given by

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y.$$

That is,

$$\delta w \simeq \frac{1}{f(x,y)}\frac{\partial f}{\partial x}\delta x + \frac{1}{f(x,y)}\frac{\partial f}{\partial y}\delta y = \frac{1}{f(x,y)}\left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right].$$

In other words,

$$\delta w \simeq \frac{1}{z}\left[\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right].$$

We conclude that

$$\delta w \simeq \frac{\delta z}{z},$$

which means that the fractional increment (or error) in z approximates to the actual increment (or error) in $\ln z$. Multiplication by 100 will, of course, convert the fractional increment (or error) into a percentage.

Note:
The logarthmic method will apply equally well to a function of more than two independent variables where it takes the form of a product, a quotient, or a combination of these two.

EXAMPLES

1. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

Taking the natural logarithm of both sides of the given formula,

$$\ln w = 3\ln x + \ln z - 4\ln y,$$

giving

$$\frac{\delta w}{w} \simeq 3\frac{\delta x}{x} + \frac{\delta z}{z} - 4\frac{\delta y}{y},$$

where

$$\frac{\delta x}{x} = -\frac{3}{100}, \quad \frac{\delta y}{y} = \frac{1}{100} \quad \text{and} \quad \frac{\delta z}{z} = \frac{2}{100}.$$

Hence,

$$\frac{\delta w}{w} \times 100 = -9 + 2 - 4 = -13.$$

Thus, w is too small by approximately 11%, as before.

2. In the formula,

$$w = \sqrt{\frac{x^3}{y}},$$

x is subjected to an increase of 2%. Calculate, approximately, the percentage change needed in y to ensure that w remains unchanged.

Solution

Taking the natural logarithm of both sides of the formula,

$$\ln w = \frac{1}{2}[3\ln x - \ln y].$$

Hence,

$$\frac{\delta w}{w} \simeq \frac{1}{2}\left[3\frac{\delta x}{x} - \frac{\delta y}{y}\right],$$

where $\frac{\delta x}{x} = 0.02$, and we require that $\delta w = 0$.
Thus,

$$0 = \frac{1}{2}\left[0.06 - \frac{\delta y}{y}\right],$$

giving

$$\frac{\delta y}{y} = 0.06,$$

which means that y must be approximately 6% too large.

14.3.4 EXERCISES

1. A triangle is such that two of its sides (of length 6cms. and 8cms.) are at right-angles to each other.

   Calculate, approximately, the change in the length of the hypotenuse of the triangle when the shorter side is lengthened by 0.25cms. and the longer side is shortened by 0.125cms.

2. Two sides of a triangle are measured as $x = 150$cms. and $y = 200$cms. while the angle included between them is measured as $\theta = 60°$. Calculate the area of the triangle.

   If there are possible errors of ± 0.2cms. in the measurement of the sides and $\pm 1°$ in the angle, determine, approximately, the maximum possible error in the calculated area of the triangle.

   State your answers correct to the nearest whole number.

   (**Hint** use the formula, Area $= \frac{1}{2}xy\sin\theta$).

3. Given that the volume of a segment of a sphere is $\frac{1}{6}x(x^2 + 3y^2)$ where x is the height and y is the radius of the base, obtain, in terms of x and y, the percentage error in the volume when x is too large by 1% and y is too small by 0.5%.

4. If

   $$z = kx^{0.01}y^{0.08},$$

   where k is a constant, calculate, approximately, the percentage change in z when x is increased by 2% and y is decreased by 1%.

5. If

$$w = \frac{5xy^4}{z^3},$$

calculate, approximately, the maximum percentage error in w if x, y and z are subject to errors of $\pm 3\%$, $\pm 2.5\%$ and $\pm 4\%$, respectively.

6. If

$$w = 2xyz^{-\frac{1}{2}},$$

where x and z are subject to errors of 0.2%, calculate, approximately, the percentage error in y which results in w being without error.

14.3.5 ANSWERS TO EXERCISES

1. 0.05cms.
2. 12990cms.^{2} and 161cms.^{2}
3. $\frac{3x^2}{x^2+3y^2}$.
4. z decreases by 0.06%.
5. 25%.
6. -0.1%.

"JUST THE MATHS"

UNIT NUMBER

14.4

PARTIAL DIFFERENTIATION 4
(Exact differentials)

by

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UNIT 14.4 - PARTIAL DIFFERENTIATION 4

EXACT DIFFERENTIALS

14.4.1 TOTAL DIFFERENTIALS

In Unit 14.3, use was made of expressions of the form,

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

as an approximation for the increment (or error), δf, in the function, $f(x, y, ...)$, when x, y etc. are subject to increments (or errors) of δx, δy etc., respectively.

The expression may be called the **"total differential"** of $f(x, y, ...)$ and may be denoted by $\mathrm{d}f$, giving

$$\mathrm{d}f \simeq \delta f.$$

OBSERVATIONS

Consider the formula,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ \ldots$$

(a) In the special case when $f(x, y, ...) \equiv x$, we may conclude that $\mathrm{d}f = \delta x$ or, in other words,

$$\mathrm{d}x = \delta x.$$

(b) In the special case when $f(x, y, ...) \equiv y$, we may conclude that $\mathrm{d}f = \delta y$ or, in other words,

$$\mathrm{d}y = \delta y.$$

(c) Observations (a) and (b) imply that the total differential of each **independent** variable is the same as the small increment (or error) in that variable; but the total differential of the **dependent** variable is only approximately equal to the increment (or error) in that variable.

(d) All of the previous observations may be summarised by means of the formula

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \ldots$$

14.4.2 TESTING FOR EXACT DIFFERENTIALS

In general, an expression of the form

$$P(x, y, ...)dx + Q(x, y, ...)dy + \ldots$$

will not be the total differential of a function, $f(x, y, ...)$, unless the functions, $P(x, y, ...)$, $Q(x, y, ...)$ etc. can be identified with $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc., respectively.

If this **is** possible, then the expression is known as an **"exact differential"**.

RESULTS

(i) The expression

$$P(x, y)dx + Q(x, y)dy$$

is an exact differential if and only if

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

Proof:

(a) If the expression,

$$P(x, y)dx + Q(x, y)dy,$$

is an exact differential, df, then

$$\frac{\partial f}{\partial x} \equiv P(x, y) \text{ and } \frac{\partial f}{\partial y} \equiv Q(x, y).$$

Hence, it must be true that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \left(\equiv \frac{\partial^2 f}{\partial x \partial y} \right).$$

(b) Conversely, suppose that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

We can certainly say that

$$P(x, y) \equiv \frac{\partial u}{\partial x}$$

for <u>some</u> function $u(x, y)$, since $P(x, y)$ could be integrated partially with respect to x.

But then,

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y} \equiv \frac{\partial^2 u}{\partial y \partial x};$$

and, on integrating partially with respect to x, we obtain

$$Q(x, y) = \frac{\partial u}{\partial y} + A(y),$$

where $A(y)$ is an **arbitrary** function of y.

Thus,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = \frac{\partial u}{\partial x}\mathrm{d}x + \left(\frac{\partial u}{\partial y} + A(y) \right) \mathrm{d}y;$$

and the right-hand side is the exact differential of the function,

$$u(x, y) + \int A(y)\ \mathrm{d}y.$$

(ii) By similar reasoning, it may be shown that the expression

$$P(x, y, z)\mathrm{d}x + Q(x, y, z)\mathrm{d}y + R(x, y, z)\mathrm{d}z$$

is an exact differential, provided that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

ILLUSTRATIONS

1.

$$x\mathrm{d}x + y\mathrm{d}y = \mathrm{d}\left[\frac{1}{2}\left(x^2 + y^2\right)\right].$$

2.

$$y\mathrm{d}x + x\mathrm{d}y = \mathrm{d}[xy].$$

3.

$$y\mathrm{d}x - x\mathrm{d}y$$

is not an exact differential since

$$\frac{\partial y}{\partial y} = 1 \quad \text{and} \quad \frac{\partial(-x)}{\partial x} = -1.$$

4.

$$2\ln y\mathrm{d}x + (x + z)\mathrm{d}y + z^2\mathrm{d}z$$

is not an exact differential since

$$\frac{\partial(2\ln y)}{\partial y} = \frac{2}{y}, \quad \text{and} \quad \frac{\partial(x + z)}{\partial x} = 1.$$

14.4.3 INTEGRATION OF EXACT DIFFERENTIALS

In section 14.4.2, the second half of the proof of the condition for the expression,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y,$$

to be an exact differential suggests, also, a method of determining which function, $f(x, y)$, it is the total differential of. The method may be illustrated by the following examples:

EXAMPLES

1. Verify that the expression,

$$(x + y\cos x)\mathrm{d}x + (1 + \sin x)\mathrm{d}y,$$

is an exact differential, and obtain the function of which it is the total differential.
Solution
Firstly,

$$\frac{\partial}{\partial y}(x + y\cos x) \equiv \frac{\partial}{\partial x}(1 + \sin x) \equiv \cos x;$$

and, hence, the expression is an exact differential.
Secondly, suppose that the expression is the total differential of the function, $f(x, y)$.
Then,

$$\frac{\partial f}{\partial x} \equiv x + y\cos x \quad - - - - - - - - - - - (1)$$

and

$$\frac{\partial f}{\partial y} \equiv 1 + \sin x. \quad - - - - - - - - - - - -(2)$$

Integrating (1) partially with respect to x gives

$$f(x, y) \equiv \frac{x^2}{2} + y\sin x + A(y),$$

where $A(y)$ is an **arbitrary** function of y only.

Substituting this result into (2) gives

$$\sin x + \frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1 + \sin x.$$

That is,

$$\frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1;$$

and, hence,

$$A(y) \equiv y + \text{constant}.$$

We conclude that

$$f(x, y) \equiv \frac{x^2}{2} + y \sin x + y + \text{constant}.$$

2. Verify that the expression,

$$(yz + 2)\mathrm{d}x + (xz + 6y)\mathrm{d}y + (xy + 3z^2)\mathrm{d}z,$$

is an exact differential and obtain the function of which it is the total differential.
Solution
Firstly,

$$\frac{\partial}{\partial y}(yz + 2) \equiv \frac{\partial}{\partial x}(xz + 6y) \equiv z,$$

$$\frac{\partial}{\partial z}(xz + 6y) \equiv \frac{\partial}{\partial y}(xy + 3z^2) \equiv x,$$

and

$$\frac{\partial}{\partial x}(xy + 3z^2) \equiv \frac{\partial}{\partial z}(yz + 2) \equiv y,$$

so that the given expression is an exact differential.

Suppose it is the total differential of the function, $F(x, y, z)$.
Then,

$$\begin{aligned}\frac{\partial F}{\partial x} &\equiv yz + 2, \quad ---------(1)\\ \frac{\partial F}{\partial y} &\equiv xz + 6y, \quad --------(2)\\ \frac{\partial F}{\partial z} &\equiv xy + 3z^2. \quad --------(3)\end{aligned}$$

Integrating (1) partially with respect to x gives

$$F(x, y, z) \equiv xyz + 2x + A(y, z),$$

where $A(y, z)$ is an arbitrary function of y and z only.
Substituting this result into both (2) and (3) gives

$$\begin{aligned}xz + \frac{\partial A}{\partial y} &\equiv xz + 6y,\\ xy + \frac{\partial A}{\partial z} &\equiv xy + 3z^2.\end{aligned}$$

That is,

$$\begin{aligned}\frac{\partial A}{\partial y} &\equiv 6y, \quad ------(4)\\ \frac{\partial A}{\partial z} &\equiv 3z^2. \quad ------(5)\end{aligned}$$

Integrating (4) partially with respect to y gives

$$A(y, z) \equiv 3y^2 + B(z),$$

where $B(z)$ is an arbitrary function of z only.
Substituting this result into (5) gives

$$\frac{\mathrm{d}B}{\mathrm{d}z} \equiv 3z^2,$$

which implies that

$$B(z) \equiv z^3 + \text{constant}.$$

We conclude that

$$F(x, y, z) \equiv xyz + 2x + 3y^2 + z^3 + \text{constant}.$$

14.4.4 EXERCISES

1. Verify which of the following are exact differentials and integrate those which are:

   (a)

   $$5x + 12y - 9)\text{d}x + (2x + 5y - 4)\text{d}y;$$

   (b)

   $$12x + 5y - 9)\text{d}x + (5x + 2y - 4)\text{d}y;$$

   (c)

   $$(3x^2 + 2y + 1)\text{d}x + (2x + 6y^2 + 2)\text{d}y;$$

   (d)

   $$(y - e^x)\text{d}x + x\text{d}y;$$

   (e)

   $$\frac{1}{x}\text{d}x - \left(\frac{y}{x^2} + 2x\right)\text{d}y;$$

   (f)

   $$\cos(x + y)\text{d}x + \cos(y - x)\text{d}y;$$

   (g)

   $$(1 - \cos 2x)\text{d}y + 2y \sin 2x\text{d}x.$$

2. Verify that the expression,

   $$3x^2\text{d}x + 2yz\text{d}y + y^2\text{d}z,$$

   is an exact differential and obtain the function of which it is the total differential.

3. Verify that the expression,

$$e^{xy}[y \sin z\mathrm{d}x + x \sin z\mathrm{d}y + \cos z\mathrm{d}z],$$

is an exact differential and obtain the function of which it is the total differential.

14.4.5 ANSWERS TO EXERCISES

1. (a) Not exact;

   (b)

$$6x^2 + 5xy - 9x + y^2 - 4y + \text{constant};$$

   (c)

$$x^3 + 2xy + x + 2y^3 + 2y + \text{constant};$$

   (d)

$$xy - e^x + \text{constant};$$

   (e) Not exact;

   (f) Not exact;

   (g)

$$y(1 - \cos 2x) + \text{constant}.$$

2.

$$x^3 + y^2 z;$$

3.

$$e^{xy} \sin z.$$

"JUST THE MATHS"

UNIT NUMBER

14.5

PARTIAL DIFFERENTIATION 5
(Partial derivatives of composite functions)

by

A.J.Hobson

UNIT 14.5 - PARTIAL DIFFERENTIATION 5

PARTIAL DERIVATIVES OF COMPOSITE FUNCTIONS

14.5.1 SINGLE INDEPENDENT VARIABLES

In this Unit, we shall be concerned with functions, $f(x, y...)$, of two or more variables in which those variables are not independent, but are themselves dependent on some other variable, t.

The problem is to calculate the rate of increase (positive or negative) of such functions with respect to t.

Let us suppose that the variable, t, is subject to a small increment of δt, so that the variables $x, y...$ are subject to small increments of $\delta x,\ \delta y,\ \ldots$, respectively. Then the corresponding increment, δf, in $f(x, y...)$ is given by

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

where we note that no label other than f is being used, here, for the function of several variables. That is, it is not essential to use a specific **formula**, such as $w = f(x, y..)$.

Dividing throughout by δt gives

$$\frac{\delta f}{\delta t} \simeq \frac{\partial f}{\partial x}.\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}.\frac{\delta y}{\delta t} + \ldots$$

Allowing δt to tend to zero, we obtain the standard result for the **"total derivative"** of $f(x, y..)$ with respect to t, namely

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \ldots$$

This rule may be referred to as the **"chain rule"**, but more advanced versions of it will appear later.

EXAMPLES

1. A point, P, is moving along the curve of intersection of the surface whose cartesian equation is

$$\frac{x^2}{16} - \frac{y^2}{9} = z \quad \text{(a Paraboloid)}$$

and the surface whose cartesian equation is

$$x^2 + y^2 = 5 \quad \text{(a Cylinder).}$$

If x is increasing at 0.2 cms/sec, how fast is z changing when $x = 2$?

Solution

We may use the formula

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t},$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2 \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

But, from the equation of the paraboloid,

$$\frac{\partial z}{\partial x} = \frac{x}{8} \ \text{ and } \ \frac{\partial z}{\partial y} = -\frac{2y}{9};$$

and, from the equation of the cylinder,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Substituting $x = 2$ gives $y = \pm 1$ on the curve of intersection, so that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{2}{8}\right)(0.2) + \left(-\frac{2}{9}\right)(\pm 1)(0.2)\left(\frac{-2}{\pm 1}\right) = 0.2\left(\frac{1}{4} + \frac{4}{9}\right) = \frac{5}{36} \text{ cms/sec.}$$

2. Determine the total derivative of u with respect to t in the case when

$$u = xy + yz + zx, \quad x = e^t, \quad y = e^{-t} \quad \text{and} \quad z = x + y.$$

Solution

We may use the formula

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}.\frac{\mathrm{d}z}{\mathrm{d}t},$$

where

$$\frac{\partial u}{\partial x} = y + z, \quad \frac{\partial u}{\partial y} = z + x, \quad \frac{\partial u}{\partial z} = x + y$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^t = x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -e^{-t} = -y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = e^t - e^{-t} = x - y.$$

Hence,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (y + z)x - (z + x)y + (x + y)(x - y)$$

$$= -zy + zx + x^2 - y^2$$

$$= z(x - y) + (x - y)(x + y).$$

That is,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (x - y)(x + y + z).$$

14.5.2 SEVERAL INDEPENDENT VARIABLES

We may now extend the work of the previous section to functions, $f(x, y..)$, of two or more variables in which $x, y..$ are each dependent on two or more variables, $s, t..$

Since the function, $f(x, y..)$, is dependent on $s, t..$, we may wish to determine its **partial** derivatives with respect to any one of these (independent) variables.

The result previously established for a **single** independent variable may easily be adapted as follows:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial s} + \ldots$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial t} + \ldots$$

Again, this is referred to as the **"chain rule"**.

EXAMPLES

1. Determine the first-order partial derivatives of z with respect to r and θ in the case when

$$z = x^2 + y^2, \quad \text{where} \quad x = r\cos\theta \quad \text{and} \quad y = r\sin 2\theta.$$

Solution
We may use the formulae

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \theta}.$$

These give

$$\text{(i)} \quad \frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin 2\theta$$

$$= 2r\left(\cos^2\theta + \sin^2 2\theta\right)$$

and

$$\text{(ii)}\ \frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) + 2y(2r\cos 2\theta)$$

$$= 2r^2\left(2\cos 2\theta\sin 2\theta - \cos\theta\sin\theta\right).$$

2. Determine the first-order partial derivatives of w with respect to u, θ and ϕ in the case when

$$w = x^2 + 2y^2 + 2z^2,$$

where

$$x = u\sin\phi\cos\theta,\quad y = u\sin\phi\sin\theta \quad \text{and} \quad z = u\cos\phi.$$

Solution

We may use the formulae

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \theta}$$

and

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \phi}.$$

These give

$$\text{(i)}\ \frac{\partial w}{\partial u} = 2x\sin\phi\cos\theta + 4y\sin\phi\sin\theta + 4z\cos\phi$$

$$= 2u\sin^2\phi\cos^2\theta + 4u\sin^2\phi\sin^2\theta + 4u\cos^2\phi;$$

$$\text{(ii)} \quad \frac{\partial w}{\partial \theta} = -2xu\sin\phi\sin\theta + 4yu\sin\phi\cos\theta$$

$$= -2u^2\sin^2\phi\sin\theta\cos\theta + 4u^2\sin^2\phi\sin\theta\cos\theta$$

$$= 2u^2\sin^2\phi\sin\theta\cos\theta;$$

$$\text{(iii)} \quad \frac{\partial w}{\partial \phi} = 2xu\cos\phi\cos\theta + 4yu\cos\phi\sin\theta - 4zu\sin\phi$$

$$= 2u^2\sin\phi\cos\phi\cos^2\theta + 4u^2\sin\phi\cos\phi\sin^2\theta - 4u^2\sin\phi\cos\phi$$

$$= 2u^2\sin\phi\cos\phi\left(\cos^2\theta + 2\sin^2\theta - 2\right).$$

14.5.3 EXERCISES

1. Determine the total derivative of z with respect to t in the cases when

   (a)
   $$z = x^2 + 3xy + 5y^2, \quad \text{where} \quad x = \sin t \quad \text{and} \quad y = \cos t;$$

   (b)
   $$z = \ln(x^2 + y^2), \quad \text{where} \quad x = e^{-t} \quad \text{and} \quad y = e^t;$$

   (c)
   $$z = x^2y^2 \quad \text{where} \quad x = 2t^3 \quad \text{and} \quad y = 3t^2.$$

2. If $z = f(x, y)$, show that, when y is a function of x,
   $$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x}.$$

   Hence, determine $\frac{\mathrm{d}z}{\mathrm{d}x}$ in the case when $z = xy + x^2y$ and $y = \ln x$.

3. The base radius, r, of a cone is decreasing at a rate of 0.1cms/sec while the perpendicular height, h, is increasing at a rate of 0.2cms/sec. Determine the rate at which the volume, V, is changing when $r = 2$cm and $h = 3$cm. (**Hint:** $V = (\pi r^2 h)/3$).

4. A rectangular solid has sides of lengths 3cms, 4cms and 5cms. Determine the rate of increase of the length of the diagonal of the solid if the sides are increasing at rates of $\frac{1}{3}$cms./sec, $\frac{1}{4}$cms./sec and $\frac{1}{5}$cms/sec, respectively.

5. If
$$z = (2x + 3y)^2 \quad \text{where} \quad x = r^2 - s^2 \quad \text{and} \quad y = 2rs,$$
determine, in terms of r and s the first-order partial derivatives of z with respect to r and s.

6. If
$$z = f(x, y) \quad \text{where} \quad x = e^u \cos v \quad \text{and} \quad y = e^u \sin v,$$
show that
$$\frac{\partial z}{\partial u} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial z}{\partial v} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

7. If
$$w = 5x - 3y^2 + 7z^3 \quad \text{where} \quad x = 2s + 3t, \quad y = s - t \quad \text{and} \quad z = 4s + t,$$
determine, in terms of s and t, the first order partial derivatives of w with respect to s and t.

14.5.4 ANSWERS TO EXERCISES

1. (a)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 3\cos 2t - 4\sin 2t;$$
(b)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 2\left[\frac{e^{4t} - 1}{e^{4t} + 1}\right];$$
(c)
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 360t^9.$$

2.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^2 + 2xy + 1 + x.$$

3. The volume is decreasing at a rate of approximately 0.42 cubic centimetres per second.
4. The diagonal is increasing at a rate of approximately 0.42 centimetres per second.
5.

$$\frac{\partial z}{\partial r} = 8(r^2 - s^2 + 3rs)(2r + 3s) \quad \text{and} \quad \frac{\partial z}{\partial s} = 8(r^2 - s^2 + 3rs)(3r - 2s).$$

6. Results follow immediately from the formulae

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}.$$

7.

$$\frac{\partial w}{\partial s} = 1344s^2 + 672st + 84t^2 - 6s + 6t + 10$$

and

$$\frac{\partial w}{\partial t} = 336s^2 + 168st + 21t^2 + 6s - 6t + 15.$$

"JUST THE MATHS"

UNIT NUMBER

14.6

PARTIAL DIFFERENTIATION 6
(Implicit functions)

by

A.J.Hobson

UNIT 14.6 - PARTIAL DIFFERENTIATION 6

IMPLICIT FUNCTIONS

14.6.1 FUNCTIONS OF TWO VARIABLES

The chain rule, encountered earlier, has a convenient application to implicit relationships of the form,

$$f(x, y) = \text{constant},$$

between two independent variables, x and y.

It provides a means of determining the total derivative of y with respect to x.

Explanation

Taking x as the single independent variable, we may interpret $f(x, y)$ as a function of x and y in which both x and y are functions of x.

Differentiating both sides of the relationship, $f(x, y) = \text{constant}$, with respect to x gives

$$\frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

In other words,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

EXAMPLES

1. If
$$f(x,y) \equiv x^3 + 4x^2y - 3xy + y^2 = 0,$$
determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Solution
$$\frac{\partial f}{\partial x} = 3x^2 + 8xy - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x^2 - 3x + 2y.$$
Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 + 8xy - 3y}{4x^2 - 3x + 2y}.$$

2. If
$$f(x,y) \equiv x\sin(2x - 3y) + y\cos(2x - 3y),$$
determine and expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Solution
$$\frac{\partial f}{\partial x} = \sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)$$
and
$$\frac{\partial f}{\partial y} = -3x\cos(2x - 3y) + \cos(2x - 3y) + 3y\sin(2x - 3y).$$
Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)}{3x\cos(2x - 3y) - \cos(2x - 3y) - 3y\sin(2x - 3y)}.$$

14.6.2 FUNCTIONS OF THREE VARIABLES

For relationships of the form,

$$f(x,y,z) = \text{constant},$$

let us suppose that x and y are independent of each other.

Then, regarding $f(x,y,z)$ as a function of x, y and z, where x, y and z are **all** functions of x and y, the chain rule gives

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0.$$

But,

$$\frac{\partial x}{\partial x}=1 \text{ and } \frac{\partial y}{\partial x}=0.$$

Hence,

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0,$$

giving

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}};$$

and, similarly,

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

EXAMPLES

1. If
$$f(x,y,z)\equiv z^2xy+zy^2x+x^2+y^2=5,$$
determine expressions for $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$.
Solution
$$\frac{\partial f}{\partial x}=z^2y+zy^2+2x,$$

$$\frac{\partial f}{\partial y} = z^2x + 2zyx + 2y$$

and

$$\frac{\partial f}{\partial z} = 2zxy + y^2x.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{z^2y + zy^2 + 2x}{2zxy + y^2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{z^2x + 2zyx + 2y}{2zxy + y^2x}.$$

2. If

$$f(x, y, z) \equiv xe^{y^2+2z},$$

determine expressions for $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$.

Solution

$$\frac{\partial f}{\partial x} = e^{y^2+2z},$$

$$\frac{\partial f}{\partial y} = 2yxe^{y^2+2z},$$

and

$$\frac{\partial f}{\partial z} = 2xe^{y^2+2z}.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{e^{y^2+2z}}{2xe^{y^2+2z}} = -\frac{1}{2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{2yxe^{y^2+2z}}{2xe^{y^2+2z}} = -y.$$

14.6.3 EXERCISES

1. Use partial differentiation to determine expressions for $\frac{dy}{dx}$ in the following cases:

   (a)
$$x^3 + y^3 - 2x^2y = 0;$$

   (b)
$$e^x \cos y = e^y \sin x;$$

   (c)
$$\sin^2 x - 5\sin x \cos y + \tan y = 0.$$

2. If
$$x^2y + y^2z + z^2x = 10,$$
   where x and y are independent, determine expressions for
$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

3. If
$$xyz - 2\sin(x^2 + y + z) + \cos(xy + z^2) = 0,$$
   where x and y are independent, determine expressions for
$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

4. If
$$r^2 \sin\theta = (r\cos\theta - 1)z,$$
   where r and θ are independent, determine expressions for
$$\frac{\partial z}{\partial r} \text{ and } \frac{\partial z}{\partial \theta}.$$

14.6.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4xy - 3x^2}{3y^2 - 2x^2};$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x \cos y - e^y \cos x}{x^x \sin y + e^y \sin x};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5\cos x \cos y - 2\sin x \cos x}{5\sin x siny + \sec^2 y}.$$

2.

$$\frac{\partial z}{\partial x} = -\frac{2xy + z^2}{y^2 + 2zx}$$

and

$$\frac{\partial z}{\partial y} = -\frac{x^2 + 2yz}{y^2 + 2zx}.$$

3.

$$\frac{\partial z}{\partial x} = -\frac{yz - 4x\cos(x^2 + y + z) - y\sin(xy + z^2)}{xy - 2\cos(x^2 + y + z) - 2z\sin(xy + z^2)}$$

and

$$\frac{\partial z}{\partial y} = -\frac{xz - 2\cos(x^2 + y + z) - x\sin(xy + z^2)}{xy - 2\cos(x^2 + y + z) - 2z\sin(xy + z^2)}.$$

4.

$$\frac{\partial z}{\partial r} = \frac{2r\sin\theta - z\cos\theta}{r\cos\theta - 1}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{r^2\cos\theta + rz\sin\theta}{r\cos\theta - 1}.$$

"JUST THE MATHS"

UNIT NUMBER

14.7

PARTIAL DIFFERENTIATION 7
(Change of independent variable)

by

A.J.Hobson

UNIT 14.7 - PARTIAL DIFFERENTIATON 7

CHANGE OF INDEPENDENT VARIABLE

14.7.1 ILLUSTRATIONS OF THE METHOD

In the theory of **"partial differential equations"** (that is, equations which involve partial derivatives) it is sometimes required to express a given equation in terms of a new set of independent variables. This would be necessary, for example, in changing a discussion from one geometrical reference system to another. The method is an application of the chain rule for partial derivatives and we illustrate it with examples.

EXAMPLES

1. Express, in plane polar co-ordinates, r and θ, the following partial differential equations:

   (a)

$$\frac{\partial V}{\partial x} + 5\frac{\partial V}{\partial y} = 1;$$

   (b)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

   Solution

   Both differential equations involve a function, $V(x, y)$, where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

   Hence,

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial r},$$

   or

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cos\theta + \frac{\partial V}{\partial y}\sin\theta$$

   and

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial \theta},$$

or

$$\frac{\partial V}{\partial \theta} = -\frac{\partial V}{\partial x} r \sin\theta + \frac{\partial V}{\partial y} r \cos\theta.$$

Now, we may eliminate, first $\frac{\partial V}{\partial y}$, and then $\frac{\partial V}{\partial x}$ to obtain

$$\frac{\partial V}{\partial x} = \cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}$$

and

$$\frac{\partial V}{\partial y} = \sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}.$$

Hence, differential equation, (a), becomes

$$(\cos\theta + 5\sin\theta)\frac{\partial V}{\partial r} + \left(\frac{5\cos\theta}{r} - \sin\theta\right)\frac{\partial V}{\partial \theta} = 1.$$

In order to find the <u>second</u>-order derivatives of V with respect to x and y, it is necessary to write the formulae for the first-order derivatives in the form

$$\frac{\partial}{\partial x}[V] = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)[V]$$

and

$$\frac{\partial}{\partial y}[V] = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)[V].$$

From these, we obtain

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right),$$

which gives

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta \frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Adding these together gives the differential equation, (b), in the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0.$$

2. Express the differential equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

(a) in cylindrical polar co-ordinates,

and

(b) in spherical polar co-ordinates.

Solution

(a) Using

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z,$$

we may use the results of the previous example to give

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

(b) Using

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta, \quad \text{and} \quad z = u\cos\phi,$$

we could write out three formulae for $\frac{\partial V}{\partial u}$, $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi}$ and then solve for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$; but this is complicated.

However, the result in part (a) provides a shorter method as follows:

Cylindrical polar co-ordinates are expressible in terms of spherical polar co-ordinates by the formulae

$$z = u\cos\phi, \quad r = u\sin\phi, \quad \theta = \theta.$$

Hence, by using the previous example with z, r, θ in place of x, y, z respectively and u, ϕ in place of r, θ, respectively, we obtain

$$\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial r^2}=\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}.$$

Therefore, to complete the conversion we need only to consider $\frac{\partial V}{\partial r}$; and, by using r, u, ϕ in place of y, r, θ, respectively, the previous formula for $\frac{\partial V}{\partial y}$ gives

$$\frac{\partial V}{\partial r}=\sin\phi\frac{\partial V}{\partial u}+\frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}.$$

The given differential equation thus becomes

$$\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{1}{u^2\sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{u\sin\phi}\left[\sin\phi\frac{\partial V}{\partial u}+\frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}\right]=0.$$

That is,

$$\frac{\partial^2 V}{\partial u^2}+\frac{2}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\cot\phi}{u^2}\frac{\partial V}{\partial \phi}+\frac{1}{u^2\sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}=0.$$

14.7.2 EXERCISES

1. Express the partial differential equation,

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial x}=0,$$

in plane polar co-ordinates, r and θ, where

$$x=r\cos\theta \quad \text{and} \quad y=r\sin\theta.$$

2. Express the differential equation,

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=\frac{\partial V}{\partial z},$$

in spherical polar co-ordinates u, θ and ϕ, where

$$x=u\sin\phi\cos\theta, \quad y=u\sin\phi\sin\theta \quad \text{and} \quad z=u\cos\phi.$$

3. A function $\phi(x,t)$ satisfies the partial differential equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{k^2}\frac{\partial^2\phi}{\partial t^2},$$

where k is a constant.

Express this equation in terms of new independent variables, u and v, where

$$x = \frac{1}{2}(u+v) \quad \text{and} \quad t = \frac{1}{2k}(u-v).$$

4. A function $\theta(x,y)$ satisfies the partial differential equation,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Express this equation in terms of new independent variables, s and t, where

$$x = \ln u \quad \text{and} \quad y = \ln v.$$

Determine, also, an expression for $\frac{\partial^2\theta}{\partial x\partial y}$ in terms of θ, u and v.

14.7.3 ANSWERS TO EXERCISES

1.

$$\frac{\partial V}{\partial r} = 0.$$

2.

$$(\sin\phi - \cos\phi)\frac{\partial V}{\partial u} - \frac{\cos\phi + \sin\phi}{u}\frac{\partial V}{\partial \phi} = 0.$$

3.

$$\frac{\partial^2\phi}{\partial u\partial v} = 0.$$

4.

$$u^2\frac{\partial^2\theta}{\partial x^2} + v^2\frac{\partial^2\theta}{\partial v^2} + u\frac{\partial\theta}{\partial u} + v\frac{\partial\theta}{\partial v} = 0,$$

and

$$\frac{\partial^2\theta}{\partial x\partial y} = uv\frac{\partial^2\theta}{\partial u\partial v}.$$

"JUST THE MATHS"

UNIT NUMBER

14.8

PARTIAL DIFFERENTIATION 8
(Dependent and independent functions)

by

A.J.Hobson

UNIT 14.8 - PARTIAL DIFFERENTIATION 8

DEPENDENT AND INDEPENDENT FUNCTIONS

14.8.1 THE JACOBIAN

Suppose that

$$u \equiv u(x,y) \quad \text{and} \quad v \equiv v(x,y)$$

are two functions of two independent variables, x and y; then, in general, it is not possible to express u solely in terms of v, nor v solely in terms of u.

However, on occasions, it may be possible, as the following illustrations demonstrate:

ILLUSTRATIONS

1. If

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y},$$

then

$$u \equiv 1 + \frac{x}{y} \quad \text{and} \quad v \equiv \frac{x}{y} - 1,$$

which gives

$$(u-1)(v+1) \equiv \frac{x}{y}.\frac{y}{x} \equiv 1.$$

Hence,

$$u \equiv 1 + \frac{1}{v+1} \quad \text{and} \quad v \equiv \frac{1}{u-1} - 1.$$

2. If

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2,$$

then

$$v \equiv u^2 \quad \text{and} \quad u \equiv \pm\sqrt{v}.$$

If u and v are **not** connected by an identical relationship, they are said to be **"independent functions"**.

THEOREM

Two functions, $u(x, y)$ and $v(x, y)$, are independent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \not\equiv 0.$$

Proof:

We prove an equivalent statement, namely that $u(x, y)$ and $v(x, y)$ are dependent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0.$$

(a) Suppose that v is dependent on u by virtue of the relationship

$$v \equiv v(u).$$

By expressing $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we shall establish that the determinant, J, is identically equal to zero.

We have

$$\frac{\partial v}{\partial x} \equiv \frac{\mathrm{d}v}{\mathrm{d}u} \cdot \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} \equiv \frac{\mathrm{d}v}{\mathrm{d}u} \cdot \frac{\partial u}{\partial y}.$$

Thus,

$$\frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}} \equiv \frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}$$

or

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial y} \equiv 0,$$

which means that the determinant, J, is identically equal to zero.

(b) Secondly, let us suppose that

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0,$$

and attempt to prove that $u(x, y)$ and $v(x, y)$ are dependent.

In theory, we could express v in terms of u and x by eliminating y between $u(x, y)$ and $v(x, y)$.

We shall assume that

$$v \equiv A(u, x).$$

By expressing $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we may show that $A(u, x)$ does not contain x.

We have

$$\left(\frac{\partial v}{\partial x}\right)_y \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u$$

and

$$\left(\frac{\partial v}{\partial y}\right)_x \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x.$$

Hence, if the determinant, J, is identically equal to zero, we may say that

$$\begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u & \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x \end{vmatrix} \equiv 0;$$

and, on expansion, this gives

$$\left(\frac{\partial u}{\partial y}\right)_x \cdot \left(\frac{\partial A}{\partial x}\right)_u \equiv 0.$$

If the first of these two is equal to zero, then u contains only x and, hence, x could be expressed in terms of u, giving v as a function of u only. If the second is equal to zero, then A contains no x's and, again, v is a function of u only.

Notes:

(i) The determinant

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

may also be denoted by

$$\frac{\partial(u, v)}{\partial(x, y)}$$

and is called the **"Jacobian determinant"** or simply the **"Jacobian"** of u and v with respect to x and y.

(ii) Similar Jacobian determinants may be used to test for the dependence or independence of three functions of three variables, four functions of four variables, and so on.

For example, the three functions

$$u \equiv u(x, y, z), \quad v \equiv v(x, y, z) \quad \text{and} \quad w \equiv w(x, y, z)$$

are independent if and only if

$$J \equiv \frac{\partial(u, v, w)}{\partial(x, y, z)} \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \not\equiv 0.$$

ILLUSTRATIONS

1.

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y}$$

are **not** independent, since

$$J \equiv \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} \equiv \frac{1}{xy} - \frac{1}{xy} \equiv 0$$

2.

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 1 & 1 \\ 2x+2y & 2x+2y \end{vmatrix} \equiv 0.$$

3.

$$u \equiv x^2 + 2y \quad \text{and} \quad v \equiv xy$$

are independent, since

$$J \equiv \begin{vmatrix} 2x & 2 \\ y & x \end{vmatrix} \equiv 2x^2 - 2y \not\equiv 0.$$

4.

$$u \equiv x^2 - 2y + z, \quad v \equiv x + 3y^2 - 2z, \quad \text{and} \quad w \equiv 5x + y + z^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 2x & -2 & 1 \\ 1 & 6y & -2 \\ 5 & 1 & 2z \end{vmatrix} \equiv 24xyz + 4x - 30y + 4z + 25 \not\equiv 0.$$

14.8.2 EXERCISES

1. Determine which of the following pairs of functions are independent:

   (a)
$$u \equiv x\cos y \quad \text{and} \quad v \equiv x\sin y;$$

   (b)
$$u \equiv x + y \quad \text{and} \quad v \equiv \frac{y}{x+y};$$

   (c)
$$u \equiv x - 2y \quad \text{and} \quad v \equiv x^2 + 4y^2 - 4xy + 3x - 6y;$$

   (d)
$$u \equiv x + 2y \quad \text{and} \quad v \equiv x^2 - y^2 + 2xy - x.$$

2. Show that
$$u \equiv x + y + z, \quad v \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$
and
$$w \equiv x^3 + y^3 + z^3 - 3xyz$$
are dependent.
Show also that w may be expressed as a linear combination of u^3 and uv.

3. Given that
$$x + y + z \equiv u, \quad y + z \equiv uv \quad \text{and} \quad z \equiv uvw,$$
express x and y in terms of u, v and w.
Hence, show that
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv u^2 v.$$

14.8.3 ANSWERS TO EXERCISES

1. (a) Independent, since $J \equiv x$;
   (b) Independent, since $J \equiv 1/(x+y)$;
   (c) Dependent, since $J \equiv 0$;
   (d) Independent, since $J \equiv 2 - 2x - 6y$.
2. $$w \equiv \frac{1}{4}\left[u^3 + 3uv\right].$$
3. $$x \equiv u - uv \quad \text{and} \quad y \equiv uv - uvw.$$

"JUST THE MATHS"

UNIT NUMBER

14.9

PARTIAL DIFFERENTIATION 9
(Taylor's series)
for
(Functions of several variables)

by

A.J.Hobson

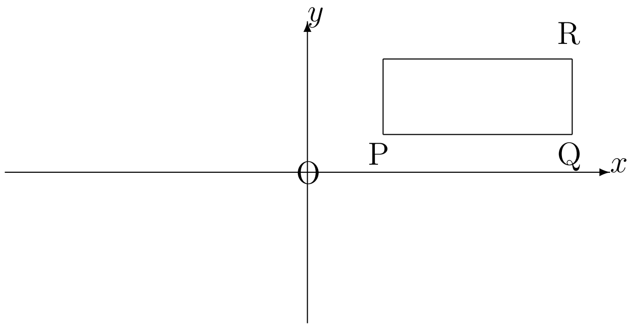
UNIT 14.9 - PARTIAL DIFFERENTIATION 9

TAYLOR'S SERIES FOR FUNCTIONS OF SEVERAL VARIABLES

14.9.1 THE THEORY AND FORMULA

Initially, we shall consider a function, $f(x, y)$, of **two** independent variables, x, y, and obtain a formula for $f(x+h, y+k)$ in terms of $f(x, y)$ and its partial derivatives.

Suppose that P,Q and R denote the points with cartesian co-ordinates, (x, y), $(x+h, y)$ and $(x+h, y+k)$, respectively.



(a) As we move in a straight line from P to Q, y remains constant so that $f(x, y)$ behaves as a function of x only.

Hence, by Taylor's theorem for one independent variable,

$$f(x+h, y) = f(x, y) + f_x(x, y) + \frac{h^2}{2!} f_{xx}(x, y) + \ldots,$$

where $f_x(x, y)$ and $f_{xx}(x, y)$ mean $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ respectively, with similar notations encountered in what follows.

In abbreviated notation,

$$f(\mathrm{Q}) = f(\mathrm{P}) + h f_x(\mathrm{P}) + \frac{h^2}{2!} f_{xx}(\mathrm{P}) + \ . \ .$$

(b) As we move in a straight line from Q to R, x remains constant so that $f(x, y)$ behaves as a function of y only.

Hence,

$$f(x+h, y+k) = f(x+h, y) + k f_x(x+h, y) + \frac{k^2}{2!} f_{xx}(x+h, y) + \ldots;$$

or, in abbreviated notation,

$$f(\mathrm{R}) = f(\mathrm{Q}) + k f_y(\mathrm{Q}) + \frac{k^2}{2!} f_{yy}(\mathrm{Q}) + \ldots$$

(c) From the result in (a)

$$f_y(\mathrm{Q}) = f_y(\mathrm{P}) + h f_{yx}(\mathrm{P}) + \frac{h^2}{2!} f_{yxx}(\mathrm{P}) + \ldots$$

and

$$f_{yy}(\mathrm{Q}) = f_{yy}(\mathrm{P}) + h f_{yyx}(\mathrm{P}) + \frac{h^2}{2!} f_{yyxx}(\mathrm{Q}) + \ldots$$

(d) Substituting the results into (b) gives

$$f(\mathrm{R}) = f(\mathrm{P}) + h f_x(\mathrm{P}) + k f_y(\mathrm{P}) + \frac{1}{2!}\left[h^2 f_{xx}(\mathrm{P}) + 2hk f_{yx}(\mathrm{P}) + k^2 f_{yy}(\mathrm{P})\right] + \ldots$$

It may be shown that the complete result can be written as

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x, y) + \ \ldots$$

Notes:

(i) The equivalent of this result for a function of three variables would be

$$f(x+h,y+k,z+l)=f(x,y,z)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)f(x,y,z)+$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)^2 f(x,y,z)+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)^3 f(x,y,z)+\ \ldots$$

(ii) Alternative versions of Taylor's theorem may be obtained by interchanging $x, y, z...$ with $h, k, l....$

For example,

$$f(x+h,y+k)=f(h,k)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(h,k)+$$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(h,k)+\frac{1}{3!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^3 f(h,k)+\ \ldots$$

(iii) Replacing x with $x-h$ and y with $y-k$ in (ii) gives the formula,

$$f(x,y)=f(h,k)+\left((x-h)\frac{\partial}{\partial x}+(y-k)\frac{\partial}{\partial y}\right)f(h,k)+$$

$$\frac{1}{2!}\left((x-h)\frac{\partial}{\partial x}+(y-k)\frac{\partial}{\partial y}\right)^2 f(h,k)+\frac{1}{3!}\left((x-h)\frac{\partial}{\partial x}+(y-k)\frac{\partial}{\partial y}\right)^3 f(h,k)+\ \ldots$$

This is called the **"Taylor expansion of $f(x,y)$ about the point (a,b)"**

(iv) A special case of Taylor's series (for two independent variables) is obtained by putting $h = 0$ and $k = 0$ in (ii) to give

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) + \ldots$$

This is called a **"MacLaurin's series"** but is also the Taylor expansion of $f(x, y)$ about the point $(0, 0)$.

EXAMPLE

Determine the Taylor series expansion of the function $f\left(x + 1, y + \frac{\pi}{3}\right)$ in ascending powers of x and y when

$$f(x, y) \equiv \sin xy,$$

neglecting terms of degree higher than two.

Solution

We use the result that

$$f\left(x + 1, y + \frac{\pi}{3}\right) = f\left(1, \frac{\pi}{3}\right) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f\left(1, \frac{\pi}{3}\right) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f\left(1, \frac{\pi}{3}\right) + \ldots,$$

in which the first term on the right has value $\sqrt{3}/2$.

The partial derivatives required are as follows:

$$\frac{\partial f}{\partial x} \equiv y \cos xy \;\; \text{giving} \;\; -\frac{\pi}{6} \;\; \text{at} \;\; x = 1, \;\; y = \frac{\pi}{3};$$

$$\frac{\partial f}{\partial y} \equiv x \cos xy \;\; \text{giving} \;\; \frac{1}{2} \;\; \text{at} \;\; x = 1, \;\; y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x^2} \equiv -y^2 \sin xy \;\; \text{giving} \;\; -\frac{\pi^2\sqrt{3}}{18} \;\; \text{at} \;\; x = 1, \;\; y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \cos xy - xy \sin xy \;\; \text{giving} \;\; \frac{1}{2} - \frac{\pi\sqrt{3}}{6} \;\; \text{at} \;\; x = 1, \;\; y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial y^2} \equiv -x^2 \sin xy \;\; \text{giving} \;\; -\frac{\sqrt{3}}{2} \;\; \text{at} \;\; x = 1, \;\; y = \frac{\pi}{3}.$$

Neglecting terms of degree higher than two, we have

$$\sin xy = \frac{\sqrt{3}}{2} + \frac{\pi}{6}x + \frac{1}{2}y - \frac{\sqrt{3}\pi^2}{36}x^2 + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)xy - \frac{\sqrt{3}}{4}y^2 + \ldots$$

14.9.2 EXERCISES

1. If $f(x, y) \equiv x^3 - 3xy^2$, show that

$$f(2 + h, 1 + k) = 2 + 9h - 12k + 6\left(h^2 - hk - k^2\right) + h^3 - 3hk^2.$$

2. If $f(x, y) \equiv \sin x \cosh y$, evaluate all the partial derivatives of $f(x, y)$ up to order five at the point, $(x, y) = (0, 0)$, and, hence, show that

$$\sin x \cosh y = x - \frac{1}{6}\left(x^3 - 3xy^2\right) + \frac{1}{120}\left(x^5 - 10x^3y^2 + 5xy^4\right) + \ldots$$

3. If z is a function of two independent variables, x and y, where $y \equiv z - x \sin z$, evaluate all the partial derivatives of $z(x, y)$ up to order three at the point, $(x, y) = (0, 0)$, and, hence, show that

$$z(x, y) = y + xy + x^2y + \ldots$$

"JUST THE MATHS"

UNIT NUMBER

14.10

PARTIAL DIFFERENTIATION 10
(Stationary values)
for
(Functions of two variables)

by

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1727

UNIT 14.10 - PARTIAL DIFFERENTIATION 10

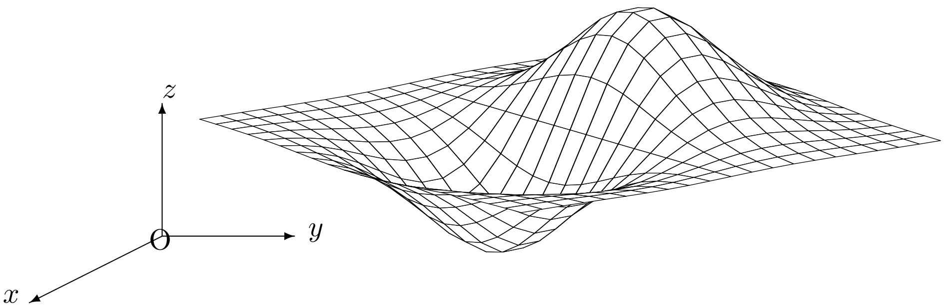
STATIONARY VALUES FOR FUNCTIONS OF TWO VARIABLES

14.10.1 INTRODUCTION

If $f(x, y)$ is a function of the two independent variables, x and y, then the equation,

$$z = f(x, y),$$

will normally represent some surface in space, referred to cartesian axes, Ox, Oy and Oz.



DEFINITION 1

The **"stationary points"**, on a surface whose equation is $z = f(x, y)$, are defined to be the points for which

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0.$$

DEFINITION 2

The function, $z = f(x, y)$, is said to have a **"local maximum"** at a point, P(x_0, y_0, z_0), if z_0 is larger than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

DEFINITION 3

The function $z = f(x, y)$ is said to have a **"local minimum"** at a point, P(x_0, y_0, z_0), if z_0 is smaller than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

Note:
At a stationary point, $\mathrm{P}(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, each of the planes, $x = x_0$ and $y = y_0$, intersect the surface in a curve which has a stationary point at P.

14.10.2 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA

A complete explanation of the conditions for a function, $z = f(x, y)$, to have a local maximum or a local minimum at a particular point require the use of Taylor's theorem for two variables.

At this stage, we state the standard set of sufficient conditions without proof.

(a) Sufficient conditions for a local maximum

A point, $\mathrm{P}(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local maximum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} < 0 \ \text{ or } \ \frac{\partial^2 z}{\partial y^2} < 0$$

and

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

(b) Sufficient conditions for a local minimum

A point, $\mathrm{P}(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local minimum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} > 0 \ \text{ or } \ \frac{\partial^2 z}{\partial y^2} > 0$$

and

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

Notes:

(i) If $\frac{\partial^2 z}{\partial x^2}$ is positive (or negative) and also $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, then $\frac{\partial^2 z}{\partial y^2}$ is automatically positive (or negative).

(ii) If it turns out that $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ is **negative** at P, we have what is called a **"saddle-point"**, irrespective of what $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ themselves are.

(iii) The values of z at the local maxima and local minima of the function, $z = f(x, y)$, may also be called the **"extreme values"** of the function, $f(x, y)$.

EXAMPLES

1. Determine the extreme values and the co-ordinates of any saddle-points of the function,

$$z = x^3 + x^2 - xy + y^2 + 4.$$

Solution

(i) First, we determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = 3x^2 + 2x - y \quad \text{and} \quad \frac{\partial z}{\partial y} = -x + 2y.$$

(ii) Secondly, we solve the equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ for x and y.

$$\begin{aligned} 3x^2 + 2x - y &= 0, - - - - - - - - - - -(1) \\ -x + 2y &= 0. - - - - - - - - - - - (2) \end{aligned}$$

Substituting equation (2) into equation (1) gives

$$3x^2 + 2x - \frac{1}{2}x = 0.$$

That is,

$$6x^2 + 3x = 0 \quad \text{or} \quad 3x(2x + 1) = 0.$$

Hence, $x = 0$ or $x = -\frac{1}{2}$, with corresponding values, $y = 0$, $z = 4$ and $y = -\frac{1}{4}$, $z = -\frac{65}{16}$, respectively.

The stationary points are thus $(0, 0, 4)$ and $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$.

(iii) Thirdly, we evaluate $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at each stationary point.

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

(a) At the point $(0, 0, 4)$,

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} > 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 3 > 0$$

and, therefore, the point, $(0, 0, 4)$, is a local minimum, with z having a corresponding extreme value of 4.

(b) At the point $\left(-\frac{1}{2}, -\frac{1}{4}, \frac{65}{16}\right)$,

$$\frac{\partial^2 z}{\partial x^2} = -1, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} < 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -3 < 0$$

and, therefore, the point, $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$, is a saddle-point.

2. Determine the stationary points of the function,

$$z = 2x^3 + 6xy^2 - 3y^3 - 150x,$$

and determine their nature.

Solution

Following the same steps as in the previous example, we have

$$\frac{\partial z}{\partial x} = 6x^2 + 6y^2 - 150 \quad \text{and} \quad \frac{\partial z}{\partial y} = 12xy - 9y^2.$$

Hence, the stationary points occur where x and y are the solutions of the simultaneous equations,

$$\begin{aligned} x^2 + y^2 &= 25, - - - - - - - - - - (1) \\ y(4x - 3y) &= 0. - - - - - - - - - - - (2) \end{aligned}$$

From the second equation, $y = 0$ or $4x = 3y$.

Putting $y = 0$ in the first equation gives $x = \pm 5$ and, with these values of x and y, we obtain stationary points at $(5, 0, -500)$ and $(-5, 0, 500)$.

Putting $x = \frac{3}{4}y$ into the first equation gives $y = \pm 4$, $x = \pm 3$ and, with these values of x and y, we obtain stationary points at $(3, 4, -300)$ and $(-3, -4, 300)$.

To classify the stationary points we require

$$\frac{\partial^2 z}{\partial x^2} = 12x, \quad \frac{\partial^2 z}{\partial y^2} = 12x - 18y, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = 12y,$$

and the conclusions are given in the following table:

Point	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial y^2}$	$\frac{\partial^2 z}{\partial x \partial y}$	$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$	Nature
$(5, 0, -500)$	60	60	0	positive	minimum
$(-5, 0, 500)$	-60	-60	0	positive	maximum
$(3, 4, -300)$	36	-36	48	negative	saddle-point
$(-3, -4, 300)$	-36	36	-48	negative	saddle-point

Note:

The conditions used in the examples above are only **sufficient** conditions; that is, if the conditions are satisfied, we may make a conclusion. But it may be shown that there are stationary points which do **not** satisfy the conditions.

Outline proof of the sufficient conditions

From Taylor's theorem for two variables,

$$f(a+h,b+k)-f(a,b) = hf_x(a,b)+kf_y(a,b)+\frac{1}{2}\left(h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right)+\ldots,$$

where h and k are small compared with a and b, f_x means $\frac{\partial f}{\partial x}$, f_y means $\frac{\partial f}{\partial y}$, f_{xx} means $\frac{\partial^2 f}{\partial x^2}$, f_{yy} means $\frac{\partial^2 f}{\partial y^2}$ and f_{xy} means $\frac{\partial^2 f}{\partial x \partial y}$.

If $f_x(a,b) = 0$ and $f_y(a,b) = 0$, the conditions for a local minimum at the point $(a, b, f(a,b))$ will be satisfied when the second term on the right-hand side is positive; and the conditions for a local maximum at this point are satisfied when the second term on the right is negative.

We assume, here, that later terms of the Taylor series expansion are negligible.

Also, it may be shown that a quadratic expression of the form

$$Lh^2 + 2Mhk + Nk^2$$

is positive when $L > 0$ or $N > 0$ and $LN - M^2 > 0$; but negative when $L < 0$ or $N < 0$ and $LN - M^2 > 0$.

If it happens that $LN - M^2 < 0$, then it may be shown that the quadratic expression may take both positive and negative values.

Finally, replacing L, M and N by $f_{xx}(a,b)$, $f_{yy}(a,b)$ and $f_{xy}(a,b)$ respectively, the sufficient conditions for local maxima, local minima and saddle-points follow.

14.10.3 EXERCISES

1. Show that the function,

$$z = 3x^3 - y^3 - 4x + 3y,$$

has a local minimum value when $x = \frac{2}{3}$, $y = -1$ and calculate this minimum value. What other stationary points are there, and what is their nature ?

2. Determine the smallest value of the function,

$$z = 2x^2 + y^2 - 4x + 8y.$$

3. Show that the function,

$$z = 2x^2y^2 + x^2 + 4y^2 - 12xy,$$

has three stationary points and determine their nature.

4. Investigate the local extreme values of the function,

$$z = x^3 + y^3 + 9(x^2 + y^2) + 12xy.$$

5. Discuss the stationary points of the following functions and, where possible, determine their nature:

   (a)

$$z = x^2 - 2xy + y^2;$$

   (b)

$$z = xy.$$

Note:
It will not be possible to use all of the standard conditions; and a geometrical argument will be necessary.

14.10.4 ANSWERS TO EXERCISES

1. $\left(\frac{2}{3}, -1, -\frac{34}{9}\right)$ is a local minimum;
   $\left(-\frac{2}{3}, 1, \frac{34}{9}\right)$ is a local maximum;
   $\left(\frac{2}{3}, 1, \frac{2}{9}\right)$ is a saddle-point;
   $\left(-\frac{2}{3}, -1, -\frac{2}{9}\right)$ is a saddle-point.

2. The smallest value is -18, since there is a single local minimum at the point $(1, -4, -18)$.

3. $(0, 0, 0)$ is a saddle-point;
   $(2, 1, -8)$ is a local minimum; (**Hint:** try $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$)
   $(-2, -1, -8)$ is a local minimum.

4. $(0, 0, 0)$ is a local minimum;
   $(-10, -10, 1000)$ is a local maximum; (**Hint**: try $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$)
   $(-4, 2, 28)$ is a saddle-point;
   $(2, -4, 28)$ is a saddle-point.

5. (a) Points $(\alpha, \alpha, 0)$ are such that $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$; but $\frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$. In fact the surface is a "parabolic cylinder" which contains the straight line $x = y,\ z = 0$ and is symmetrical about the plane $x = y$.
   (b) $(0, 0, 0)$ is a saddle-point since z may have both positive and negative values in the neighbourhood of this point.

1735

"JUST THE MATHS"

UNIT NUMBER

14.11

PARTIAL DIFFERENTIATION 11
(Constrained maxima and minima)

by

A.J.Hobson

UNIT 14.11 - PARTIAL DIFFERENTIATION 11

CONSTRAINED MAXIMA AND MINIMA

Having discussed the determination of local maxima and local minima for a function, $f(x, y, ...)$, of several independent variables, we shall now consider that an additional constraint is imposed in the form of a relationship, $g(x, y, ...) = 0$.

This would occur, for example, if we wished to construct a container with the largest possible volume for a fixed value of the surface area.

14.11.1 THE SUBSTITUTION METHOD

The following examples illustrate a technique which may be used in elementary cases:

EXAMPLES

1. Determine any local maxima or local minima of the function,

$$f(x, y) \equiv 3x^2 + 2y^2,$$

subject to the constraint that $x + 2y - 1 = 0$.

Solution

In this kind of example, it is possible to eliminate either x or y by using the constraint. If we eliminate x, for instance, we may write $f(x, y)$ as a function, $F(y)$, of y only.

In fact,

$$f(x, y) \equiv F(y) \equiv 3(1 - 2y)^2 + 2y^2 \equiv 3 - 12y + 14y^2.$$

Using the principles of maxima and minima for functions of a single independent variable, we have

$$F'(y) \equiv 28y - 12 \quad \text{and} \quad F'' \equiv 28$$

and, hence, a local minimum occurs when $y = 3/7$ and hence, $x = 1/7$.

The corresponding local minimum value of $f(x, y)$ is

$$3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{21}{49} = \frac{3}{7}.$$

2. Determine any local maxima or local minima of the function,

$$f(x, y, z) \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Eliminating x, we may write $f(x, y, z)$ as a function, $F(y, z)$, of y and z only.
In fact,

$$f(x, y, z) \equiv F(y, z) \equiv (1 - 2y - 3z)^2 + y^2 + z^2.$$

That is,

$$F(y, z) \equiv 1 - 4y - 6z + 12yz + 5y^2 + 10z^2.$$

Using the principles of maxima and minima for functions of two independent variables we have,

$$\frac{\partial F}{\partial y} \equiv -4 + 12z + 10y \ \text{ and } \ \frac{\partial F}{\partial z} \equiv -6 + 12y + 20z,$$

and a stationary value will occur when these are both equal to zero.
Thus,

$$\begin{aligned} 5y + 6z &= 2, \\ 6y + 10z &= 3, \end{aligned}$$

which give $y = 1/7$ and $z = 3/14$, on solving simultaneously.
The corresponding value of x is $1/14$, which gives a stationary value, for $f(x, y, z)$, of $14/(14)^2 = \frac{1}{14}$.
Also, we have

$$\frac{\partial^2 F}{\partial y^2} \equiv 10 > 0, \quad \frac{\partial^2 F}{\partial z^2} \equiv 20 > 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial y \partial z} \equiv 12,$$

which means that

$$\frac{\partial^2 F}{\partial y^2} . \frac{\partial^2 F}{\partial z^2} - \left(\frac{\partial^2 F}{\partial y \partial z}\right)^2 = 200 - 144 > 0.$$

Hence there is a local minimum value, $\frac{1}{14}$, of $x^2+y^2+z^2$, subject to the constraint that $x+2y+3z=1$, at the point where

$$x=\frac{1}{14}, \quad y=\frac{1}{7}, \quad \text{and} \quad z=\frac{3}{14}.$$

Note:
Geometrically, this example is calculating the square of the shortest distance from the origin onto the plane whose equation is $x+2y+3z=1$.

14.11.2 THE METHOD OF LAGRANGE MULTIPLIERS

In determining the local maxima and local minima of a function, $f(x,y,...)$, subject to the constraint that $g(x,y,...)=0$, it may be inconvenient (or even impossible) to eliminate one of the variables, $x,y,....$

An alternative method may be illustrated by means of the following steps for a function of two independent variables:

(a) Suppose that the function, $z\equiv f(x,y)$, is subject to the constraint that $g(x,y)=0$.

Then, since z is effectively a function of x only, its stationary values will be determined by the equation

$$\frac{\mathrm{d}z}{\mathrm{d}x}=0.$$

(b) From Unit 14.5 (Exercise 2), the total derivative of $z\equiv f(x,y)$ with respect to x, when x and y are not independent of each other, is given by the formula,

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(c) From the constraint that $g(x,y)=0$, the process used in **(b)** gives

$$\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

and, hence, for all points on the surface with equation, $g(x,y)=0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}.$$

Thus, throughout the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} - \left(\frac{\partial f}{\partial y}\right)\frac{\left(\frac{\partial g}{\partial x}\right)}{\left(\frac{\partial g}{\partial y}\right)}.$$

(d) Stationary values of z, subject to the constraint that $g(x, y) = 0$, will, therefore, occur when

$$\frac{\partial f}{\partial x}.\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}.\frac{\partial g}{\partial x} = 0.$$

But this may be interpreted as the condition that the two equations,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 \ \text{ and } \ \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0,$$

should have a common solution for λ.

(e) Suppose that

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y).$$

Then $\phi(x, y, \lambda)$ would have stationary values whenever its first order partial derivatives with respect to x, y and λ were equal to zero.

In other words,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0, \quad \text{and} \quad g(x, y) = 0.$$

Conclusion

The stationary values of the function, $z \equiv f(x, y)$, subject to the constraint that $g(x, y) = 0$, occur at the points for which the function

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y)$$

has stationary values.

The number, λ, is called a **"Lagrange multiplier"**.

Notes:

(i) In order to determine the nature of the stationary values of z, it will usually be necessary to examine the geometrical conditions in the neighbourhood of the stationary points.

(ii) The Lagrange multiplier method may also be applied to functions of three or more independent variables.

EXAMPLES

1. Determine any local maxima or local minima of the function,

$$z \equiv 3x^2 + 2y^2,$$

subject to the constraint that $x + 2y - 1 = 0$.

Solution

Firstly, we write

$$\phi(x, y, \lambda) \equiv 3x^2 + 2y^2 + \lambda(x + 2y - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 6x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 4y + 2\lambda \ \text{ and } \ \frac{\partial \phi}{\partial \lambda} \equiv x + 2y - 1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 6x + \lambda &= 0, \\ 2y + \lambda &= 0. \end{aligned}$$

Eliminating λ shows that $6x - 2y = 0$, or $y = 3x$; and, if we substitute this into the constraint, we obtain $7x - 1 = 0$.

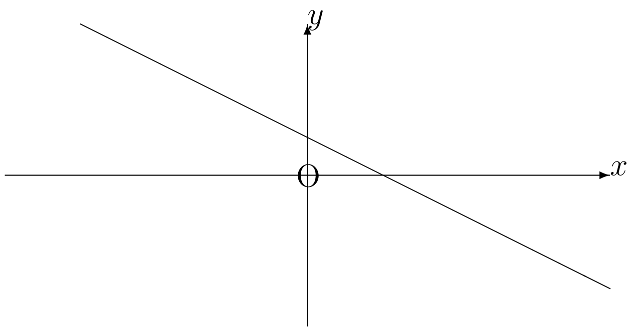
Hence,

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \quad \text{and} \quad \lambda = -\frac{6}{7}.$$

A single stationary point therefore occurs at the point where

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \quad \text{and} \quad z = 3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{3}{7}.$$

Finally, the geometrical conditions imply that the stationary value of z occurs at a point on the straight line whose equation is $x + 2y - 1 = 0$.



The stationary point is, in fact, a **minimum** value of z, since the function, $3x^2 + 2y^2$, has values larger than $3/7 \simeq 0.429$ at any point either side of the point, $(1/7, 3/7) = (0.14, 0.43)$, on the line whose equation is $x + 2y - 1 = 0$.

For example, at the points, $0.12, 0.44)$ and $(0.16, 0.42)$, on the line, the values of z are 0.4304 and 0.4296, respectively.

2. Determine the maximum and minimum values of the function, $z \equiv 3x + 4y$, subject to the constraint that $x^2 + y^2 = 1$.

   Solution

   Firstly, we write

$$\phi(x, y, \lambda) \equiv 3x + 4y + \lambda(x^2 + y^2 - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 3 + 2\lambda x, \quad \frac{\partial \phi}{\partial y} \equiv 4 + 2\lambda y \quad \text{and} \quad \frac{\partial \phi}{\partial \lambda} \equiv x^2 + y^2 - 1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 3 + 2\lambda x &= 0, \\ 2 + \lambda y &= 0. \end{aligned}$$

Thus,

$$x = -\frac{3}{2\lambda} \quad \text{and} \quad y = \frac{2}{\lambda},$$

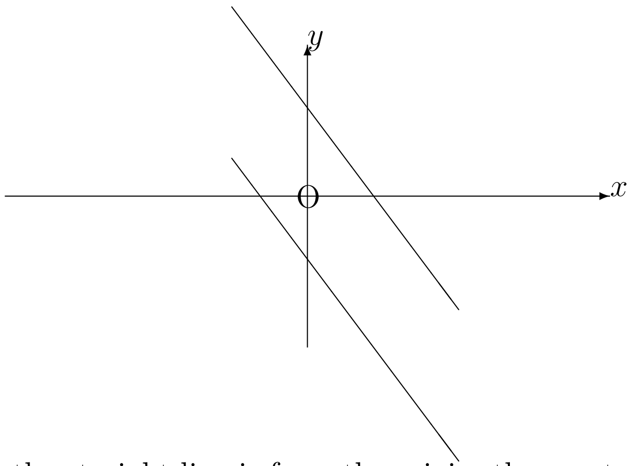
which we may substitute into the constraint to give

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1.$$

That is,

$$9 + 16 = 4\lambda^2 \quad \text{and} \quad \text{hence} \quad \lambda = \pm\frac{5}{2}.$$

We may deduce that $x = \pm\frac{3}{5}$ and $y = \pm\frac{4}{5}$, giving stationary values, ± 5, of z.
Finally, the geometrical conditions suggest that we consider a straight line with equation $3x + 4y = c$ (a constant) moving across the circle with equation $x^2 + y^2 = 1$.



The further the straight line is from the origin, the greater is the value of the constant, c.

The maximum and minimum values of $3x+4y$, subject to the constraint that $x^2+y^2=1$ will occur where the straight line touches the circle; and we have shown that these are the points, $(3/5, 4/5)$ and $(-3/5, -4/5)$.

3. Determine any local maxima or local minima of the function,

$$w \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Firstly, we write

$$\phi(x, y, z, \lambda) \equiv x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 2x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 2y + 2\lambda, \quad \frac{\partial \phi}{\partial z} \equiv 2z + 3\lambda \ \text{ and } \ \frac{\partial \phi}{\partial \lambda} \equiv x + 2y + 3z - 1.$$

The fourth of these is already equal to zero; but we equate the first three to zero, giving

$$\begin{aligned} 2x + \lambda &= 0, \\ y + \lambda &= 0, \\ 2z + 3\lambda &= 0. \end{aligned}$$

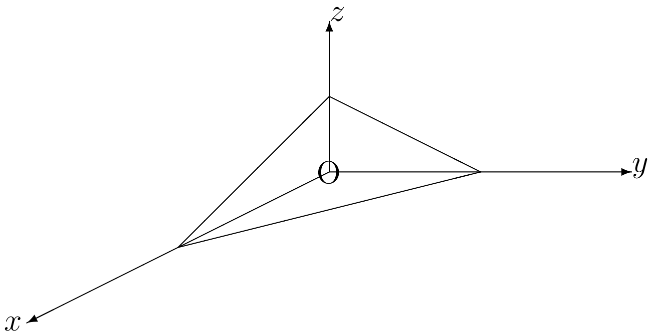
Eliminating λ shows that $2x - y = 0$, or $y = 2x$, and $6x - 2z = 0$, or $z = 3x$.
Substituting these into the constraint gives $14x = 1$.
Hence,

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \quad \text{and} \quad \lambda = -\frac{1}{7}.$$

A single stationary point therefore occurs at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \quad \text{and} \quad w = \left(\frac{1}{14}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2 = \frac{1}{14}.$$

Finally, the geometrical conditions imply that the stationary value of w occurs at a point on the plane whose equation is $x + 2y + 3z = 1$.



The stationary point must give a **minimum** value of w since the function, $x^2 + y^2 + z^2$, represents the square of the distance of a point, (x, y, z), from the origin; and, if the point is constrained to lie on a plane, this distance is bound to have a minimum value.

14.11.3 EXERCISES

1. In the following exercises, use both the substitution method and the Lagrange multiplier method:

   (a) Determine the minimum value of the function,

   $$z \equiv x^2 + y^2,$$

   subject to the constraint that $x + y = 1$.

   (b) Determine the maximum value of the function,

   $$z \equiv xy,$$

   subject to the constraint that $x + y = 15$.

(c) Determine the maximum value of the function,

$$z \equiv x^2 + 3xy - 5y^2,$$

subject to the constraint that $2x + 3y = 6$.

2. In the following exercises, use the Lagrange multiplier method:

(a) Determine the maximum and minimum values of the function,

$$w \equiv x - 2y + 5z,$$

subject to the constraint that $x^2 + y^2 + z^2 = 30$.

(b) If $x > 0$, $y > 0$ and $z > 0$, determine the maximum value of the function,

$$w \equiv xyz,$$

subject to the constraint that $x + y + z^2 = 16$.

(c) Determine the maximum value of the function,

$$w \equiv 8x^2 + 4yz - 16z + 600,$$

subject to the constraint that $4x^2 + y^2 + 4z^2 = 16$.

14.11.4 ANSWERS TO EXERCISES

1. (a) The minimum value is $z = 1/2$, and occurs when $x = y = 1/2$;
   (b) The maximum value is $z \simeq 56.25$, and occurs when $x = y = 15/2$;
   (c) The maximum value is $z = 9$, and occurs when $x = 3$ and $y = 0$.

2. (a) The maximum value is 30, and occurs when $x = 1$, $y = -2$ and $z = 5$;
   The minimum value is -30, and occurs when $x = -1$, $y = 2$ and $z = -5$;
   (b) The maximum value is

$$\frac{4096}{25\sqrt{5}} \simeq 73.27,$$

   and occurs when $x = 32/\sqrt{5}$, $y = 32/\sqrt{5}$ and $z = 4/\sqrt{5}$;
   (c) The maximum value is approximately 613.86, and occurs when $x = 0$, $y = -2$ and $z = \sqrt{3}$.

"JUST THE MATHS"

UNIT NUMBER

14.12

PARTIAL DIFFERENTIATION 12
(The principle of least squares)

by

A.J.Hobson

UNIT 14.12 - PARTIAL DIFFERENTIATION 12

THE PRINCIPLE OF LEAST SQUARES

14.12.1 THE NORMAL EQUATIONS

Suppose two variables, x and y, are known to obey a **"straight line law"**, of the form $y = a + bx$, where a and b are constants to be found.

Suppose also that, in an experiment to test this law, we obtain n pairs of values, (x_i, y_i), where i = 1,2,3,...,n.

If the values x_i are **assigned** values, they are likely to be free from error, whereas the **observed** values, y_i, will be subject to experimental error.

The principle underlying the straight line of **"best fit"** is that, in its most likely position, the sum of the squares of the y-deviations, from the line, of all observed points is a minimum.

The Calculation

The y-deviation, ϵ_i, of the point, (x_i, y_i), is given by

$$\epsilon_i = y_i - (a + bx_i).$$

Hence,

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} [y_i - (a + bx_i]^2 = P \ \text{ say.}$$

Regarding P as a function of a and b, it will be a minimum when

$$\frac{\partial P}{\partial a} = 0, \ \ \frac{\partial P}{\partial b} = 0, \ \ \frac{\partial^2 P}{\partial a^2} > 0 \ \text{ or } \ \frac{\partial^2 P}{\partial b^2} > 0, \ \text{ and } \ \frac{\partial^2 P}{\partial a^2}.\frac{\partial^2 P}{\partial b^2} - \left(\frac{\partial^2 P}{\partial a \partial b}\right)^2 > 0.$$

For these conditions, we have

$$\frac{\partial P}{\partial a} = -2\sum_{i=1}^{n} [y_i - (a + bx_i] \ \text{ and } \ \frac{\partial P}{\partial b} = -2\sum_{i=1}^{n} x_i[y_i + bx_i],$$

and these will be zero when

$$\sum_{i=1}^{n} [y_i - (a + bx_i] = 0 \quad - - - (1)$$

and

$$\sum_{i=1}^{n} x_i[y_i + bx_i] = 0 \quad - - - (2).$$

From (1),

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - \sum_{i=1}^{n} bx_i = 0.$$

That is,

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \quad - - - (3).$$

From (2),

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \quad - - - (4).$$

The statements (3) and (4) are two simultaneous equations which may be solved for a and b.

They are called the **"normal equations"**

A simpler notation for the normal equations is

$$\Sigma\ y = na + b\Sigma\ x;$$

$$\Sigma xy = a\Sigma x + b\Sigma\ x^2.$$

By eliminating a and b in turn, we obtain the solutions

$$a = \frac{\Sigma\ x^2.\Sigma\ y - \Sigma x.\Sigma\ xy}{n\Sigma\ x^2 - (\Sigma\ x)^2} \quad \text{and} \quad b = \frac{n\Sigma\ xy - \Sigma\ x.\Sigma\ y}{n\Sigma\ x^2 - (\Sigma\ x)^2}.$$

With these values of a and b, the straight line with equation, $y = a + bx$, is called the **"regression line of y on x"**.

Note:
To verify that the y-deviations from the regression line have indeed been minimised, we also need the results that

$$\frac{\partial^2 P}{\partial a^2} = \sum_{i=1}^{n} 2 = 2n, \quad \frac{\partial^2 P}{\partial b^2} = \sum_{i=1}^{n} 2x_i^2, \quad \text{and} \quad \frac{\partial^2 P}{\partial a \partial b} = \sum_{i=1}^{n} 2x_i.$$

The first two of these are clearly positive; and it may be shown that

$$\frac{\partial^2 P}{\partial a^2}.\frac{\partial^2 P}{\partial b^2} - \left(\frac{\partial^2 P}{\partial a \partial b}\right)^2 > 0.$$

EXAMPLE

Determine the equation of the regression line of y on x for the following data, which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

x	y	xy	x^2
45	6.53	293.85	2025
42	6.30	264.60	1764
56	9.52	533.12	3136
48	7.50	360.00	2304
42	6.99	293.58	1764
35	5.90	206.50	1225
58	9.49	550.42	3364
40	6.20	248.00	1600
39	6.55	255.45	1521
50	8.72	436.00	2500
455	73.70	3441.52	21203

The regression line of y on x thus has equation $y = a + bx$ where

$$a = \frac{(21203)(73.70) - (455)(3441.52)}{(10)(21203) - (455)^2} \simeq -0.645$$

and

$$b = \frac{(10)(3441.52) - (455)(73.70)}{(10)21203) - (455)^2} \simeq 0.176$$

Thus,

$$y = 0.176x - 0.645$$

14.12.2 SIMPLIFIED CALCULATION OF REGRESSION LINES

A simpler method of determining the regression line of y on x for a given set of data, is to consider a temporary change of origin to the point $(\overline{x}, \overline{y})$, where $\overline{x}$ is the arithmetic mean of the values x_i and $\overline{y}$ is the arithmetic mean of the values y_i.

RESULT

The regression line of y on x contains the point $(\overline{x}, \overline{y})$.

Proof:

From the first of the normal equations,

$$\frac{\Sigma\ y}{n} = a + b\frac{\Sigma\ x}{n}.$$

That is,

$$\overline{y} = a + b\overline{x}.$$

A change of origin to the point $(\overline{x}, \overline{y})$, with new variables X and Y, is associated with the formulae

$$X = x - \overline{x} \ \text{ and } \ Y = y - \overline{y}$$

and, in this system of reference, the regression line will pass through the origin.

Its equation is therefore

$$Y = BX,$$

where

$$B = \frac{n\Sigma\ XY - \Sigma\ X.\Sigma\ Y}{n\Sigma\ X^2 - (\Sigma\ X)^2}.$$

However,

$$\Sigma\, X = \Sigma\, (x - \overline{x}) = \Sigma\, x - \Sigma\, \overline{x} = n\overline{x} - n\overline{x} = 0$$

and

$$\Sigma\, Y = \Sigma\, (y - \overline{y}) = \Sigma\, y - \Sigma\, \overline{y} = n\overline{y} - n\overline{y} = 0.$$

Thus,

$$B = \frac{\Sigma\, XY}{\Sigma\, X^2}.$$

Note:
In a given problem, we make a table of values of x_i, y_i, X_i, Y_i, X_iY_i and X_i^2.

The regression line is then

$$y - \overline{y} = B(x - \overline{x}) \quad \text{or} \quad y = BX + (\overline{y} - B\overline{x});$$

though, there may be slight differences in the result obtained compared with that from the earlier method.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

The arithmetic mean of the x values is $\overline{x} = 45.5$

The arithmetic mean of the y values is $\overline{y} = 7.37$

This gives the following table:

x	y	$X = x - \overline{x}$	$Y = y - \overline{y}$	XY	X^2
45	6.53	-0.5	-0.84	0.42	0.25
42	6.30	-3.5	-1.07	3.745	12.25
56	9.52	10.5	2.15	22.575	110.25
48	7.50	2.5	0.13	0.325	6.25
42	6.99	-3.5	-0.38	1.33	12.25
35	5.90	-10.5	-1.47	15.435	110.25
58	9.49	12.5	2.12	26.5	156.25
40	6.20	-5.5	-1.17	6.435	30.25
39	6.55	-6.5	-0.82	5.33	42.25
50	8.72	4.5	1.35	6.075	20.25
455	73.70			88.17	500.5

Hence,

$$B = \frac{88.17}{500.5} \simeq 0.176$$

and so the regression line has equation

$$y = 0.176x + (7.37 - 0.176 \times 45.5).$$

That is,

$$y = 0.176x - 0.638$$

14.12.3 EXERCISES

1. For the following tables, determine the regression line of y on x, assuming that $y = a + bx$.

(a)

x	0	2	3	5	6
y	6	-1	-3	-10	-16

(b)

x	0	20	40	60	80
y	54	65	75	85	96

(c)

x	1	3	5	10	12
y	58	55	40	37	22

2. To determine the relation between the normal stress and the shear resistance of soil, a shear-box experiment was performed, giving the following results:

Normal Stress, x p.s.i.	11	13	15	17	19	21
Shear Stress, y p.s.i.	15.2	17.7	19.3	21.5	23.9	25.4

If $y = a + bx$, determine the regression line of y on x.

3. Fuel consumption, y miles per gallon, at speeds of x miles per hour, is given by the following table:

x	20	30	40	50	60	70	80	90
y	18.3	18.8	19.1	19.3	19.5	19.7	19.8	20.0

Assuming that

$$y = a + \frac{b}{x},$$

determine the most probable values of a and b.

14.12.4 ANSWERS TO EXERCISES

1. (a)

$$y = 6.46 - 3.52x;$$

(b)

$$y = 54.20 + 0.52x;$$

(c)

$$y = 60.78 - 2.97x.$$

2.

$$y = 4.09 + 1.03x.$$

3.

$$a \simeq -42 \quad \text{and} \quad b \simeq 20.$$

"JUST THE MATHS"

UNIT NUMBER

15.1

ORDINARY DIFFERENTIAL EQUATIONS 1 (First order equations (A))

by

A.J.Hobson

UNIT 15.1 - ORDINARY DIFFERENTIAL EQUATIONS 1

FIRST ORDER EQUATIONS (A)

15.1.1 INTRODUCTION AND DEFINITIONS

1. An **ordinary differential equation** is a relationship between an independent variable (such as x), a dependent variable (such as y) and one or more ordinary derivatives of y with respect to x.

   There is no discussion, in Units 15, of **partial** differential equations, which involve partial derivatives (see Units 14). Hence, in what follows, we shall refer simply to "differential equations".

   For example,

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-2x}, \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = y, \quad x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0 \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$$

   are differential equations.

2. The **"order"** of a differential equation is the order of the highest derivative which appears in it.

3. The **"general solution"** of a differential equation is the most general algebraic relationship between the dependent and independent variables which satisfies the differential equation.

   Such a solution will not contain any derivatives; but we shall see that it will contain one or more arbitrary constants (the number of these constants being equal to the order of the equation). The solution need not be an explicit formula for one of the variables in terms of the other.

4. A **"boundary condition"** is a numerical condition which must be obeyed by the solution. It usually amounts to the substitution of particular values of the dependent and independent variables into the general solution.

5. An **"initial condition"** is a boundary condition in which the independent variable takes the value zero.

6. A **"particular solution"** (or **"particular integral"**) is a solution which contains no arbitrary constants.

   Particular solutions are usually the result of appplying a boundary condition to a general solution.

15.1.2 EXACT EQUATIONS

The simplest kind of differential equation of the first order is one which has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

It is an elementary example of an **"exact differential equation"** because, to find its solution, all that it is necessary to do is integrate both sides with respect to x.

In other cases of exact differential equations, the terms which are not just functions of the independent variable only, need to be recognised as the exact derivative with respect to x of some known function (possibly involving both of the variables).

The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 5,$$

subject to the boundary condition that $y = 2$ when $x = 1$.

Solution

By direct integration, the general solution is

$$y = x^3 - 3x^2 + 5x + C,$$

where C is an arbitrary constant.

From the boundary condition,

$$2 = 1 - 3 + 5 + C, \text{ so that } C = -1.$$

Thus the particular solution obeying the given boundary condition is

$$y = x^3 - 3x^2 + 5x - 1.$$

2. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3,$$

subject to the boundary condition that $y = 4$ when $x = 2$.

Solution

The left hand side of the differential equation may be recognised as the exact derivative with respect to x of the function xy.

Hence, we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3;$$

and, by direct integration, this gives

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

That is,

$$y = \frac{x^3}{4} + \frac{C}{x}.$$

Applying the boundary condition,

$$4 = 2 + \frac{C}{2},$$

which implies that $C = 4$ and the particular solution is

$$y = \frac{x^3}{4} + \frac{4}{x}.$$

3. Determine the general solution to the differential equation

$$\sin x + \sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

The second and third terms on the right hand side may be recognised as the exact derivative of the function $x\sin y$; and, hence, we may write

$$\sin x + \frac{\mathrm{d}}{\mathrm{d}x}(x \sin y) = 0.$$

By direct integration, we obtain

$$-\cos x + x \sin y = C,$$

where C is an arbitrary constant.

This result counts as the general solution without further modification; but an explicit formula for y in terms of x may, in this case, be written in the form

$$y = \mathrm{Sin}^{-1}\left[\frac{C + \cos x}{x}\right].$$

15.1.3 THE METHOD OF SEPARATION OF THE VARIABLES

The method of this section relates to differential equations of the first order which may be written in the form

$$P(y)\frac{\mathrm{d}y}{\mathrm{d}x} = Q(x).$$

Integrating both sides with respect to x gives

$$\int P(y)\frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = \int Q(x)\ \mathrm{d}x.$$

But, from the formula for integration by substitution in Units 12.3 and 12.4, this simplifies to

$$\int P(y)\ \mathrm{d}y = \int Q(x)\ \mathrm{d}x.$$

Note:
The way to remember this result is to treat dx and dy, in the given differential equation, as if they were separate numbers; then rearrange the equation so that one side contains only y while the other side contains only x; that is, we **separate the variables**. The process is completed by putting an integral sign in front of each side.

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

subject to the boundary condition that $y = 6$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x};$$

and, hence,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{x}\ \mathrm{d}x,$$

giving

$$\ln y = \ln x + C.$$

Applying the boundary condition,

$$\ln 6 = \ln 2 + C,$$

so that

$$C = \ln 6 - \ln 2 = \ln\left(\frac{6}{2}\right) = \ln 3.$$

The particular solution is therefore

$$\ln y = \ln x + \ln 3 \quad \text{or} \quad y = 3x.$$

Note:

In a general solution where most of the terms are logarithms, the calculation can be made simpler by regarding the arbitrary constant itself as a logarithm, calling it $\ln A$, for instance, rather than C. In the above example, we would then write

$$\ln y = \ln x + \ln A \quad \text{simplifying to} \quad y = Ax.$$

On applying the boundary condition, $6 = 2A$, so that $A = 3$ and the particular solution is the same as before.

2. Solve the differential equation

$$x(4-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to the boundary condition that $y = 7$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x(4-x)}.$$

Hence,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x(4-x)}\,\mathrm{d}x;$$

or, using the theory of partial fractions,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \left[\frac{\frac{1}{4}}{x} + \frac{\frac{1}{4}}{4-x}\right]\,\mathrm{d}x.$$

The general solution is therefore

$$\ln y = \frac{1}{4}\ln x - \frac{1}{4}\ln(4-x) + \ln A$$

or

$$y = A\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

Applying the boundary condition, $7 = A$, so that the particular solution is

$$y = 7\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

15.1.4 EXERCISES

1. Determine the general solution of the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^5 + 3e^{-2x}.$$

2. Given that differential equation
$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \sin x$$
is exact, determine its general solution.

3. Given that the differential equation
$$\tan x\frac{\mathrm{d}y}{\mathrm{d}x} + y\mathrm{sec}^2 x = \cos 2x$$
is exact, determine the particular solution for which $y = 1$ when $x = \frac{\pi}{4}$.

4. Use the method of separation of the variables to determine the general solution of each of the following differential equations:

   (a)
$$\frac{\mathrm{d}x}{\mathrm{d}y} = (x-1)(x+2);$$

   (b)
$$x(y-3)\frac{\mathrm{d}y}{\mathrm{d}x} = 4y.$$

5. Use the method of separation of the variables to solve the following differential equations subject to the given boundary condition:

   (a)
$$(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y,$$
   where $y = 2$ when $x = 1$;

   (b)
$$x^3 + (y+1)^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$
   where $y = 0$ when $x = 0$.

15.1.5 ANSWERS TO EXERCISES

1.

$$y = \frac{x^6}{6} - \frac{3e^{-2x}}{2} + C.$$

2.

$$y = \frac{C - \cos x}{x^2}.$$

3.

$$y = \frac{3}{2}\cot x - \cos^2 x.$$

4. (a)

$$y = \ln\left[A\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}}\right];$$

(b)

$$y = \ln[Ax^4y^3].$$

5. (a)

$$y^3 = 4(1 + x^3);$$

(b)

$$4[1 - (y+1)^3] = 3x^4.$$

"JUST THE MATHS"

UNIT NUMBER

15.2

ORDINARY DIFFERENTIAL EQUATIONS 2 (First order equations (B))

by

A.J.Hobson

UNIT 15.2 - ORDINARY DIFFERENTIAL EQUATIONS 2

FIRST ORDER EQUATIONS (B)

15.2.1 HOMOGENEOUS EQUATIONS

A differential equation of the first order is said to be **"homogeneous"** if, on replacing x by λx and y by λy in all the parts of the equation except $\frac{dy}{dx}$, λ may be removed from the equation by cancelling a common factor of λ^n, for some integer n.

Note:
Some examples of homogeneous equations would be

$$(x+y)\frac{dy}{dx}+(4x-y)=0, \quad \text{and} \quad 2xy\frac{dy}{dx}+(x^2+y^2)=0,$$

where, from the first of these, a factor of λ could be cancelled and, from the second, a factor of λ^2 could be cancelled.

15.2.2 THE STANDARD METHOD

It turns out that the substitution

$$\boxed{y=vx} \quad \left(\text{giving } \frac{dy}{dx}=v+x\frac{dv}{dx}\right),$$

always converts a homogeneous differential equation into one in which the variables can be separated. The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation
$$x\frac{dy}{dx}=x+2y,$$
subject to the condition that $y=6$ when $x=6$.

   Solution

   If $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$, so that the differential equation becomes
$$x\left(v+x\frac{dy}{dx}\right)=x+2vx.$$

That is,

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 2v$$

or

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v.$$

On separating the variables,

$$\int \frac{1}{1+v}\ \mathrm{d}v = \int \frac{1}{x}\ \mathrm{d}x,$$

giving

$$\ln(1+v) = \ln x + \ln A,$$

where A is an arbitrary constant.
An alternative form of this solution, without logarithms, is

$$Ax = 1 + v$$

and, substituting back $v = \frac{y}{x}$, the solution becomes

$$Ax = 1 + \frac{y}{x}$$

or

$$y = Ax^2 - x.$$

Finally, if $y = 6$ when $x = 1$, we have $6 = A - 1$ and, hence, $A = 7$.
The required particular solution is thus

$$y = 7x^2 - x.$$

2. Determine the general solution of the differential equation

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x - y) = 0.$$

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$(x + vx)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (4x - vx) = 0.$$

That is,

$$(1 + v)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (4 - v) = 0$$

or

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{v + 1}.$$

On further rearrangement, we obtain

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{v + 1} - v = \frac{-4 - v^2}{v + 1};$$

and, on separating the variables,

$$\int \frac{v + 1}{4 + v^2}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$$

or

$$\frac{1}{2}\int \left[\frac{2v}{4 + v^2} + \frac{2}{4 + v^2}\right]\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x.$$

Hence,

$$\frac{1}{2}\left[\ln(4 + v^2) + \tan^{-1}\frac{v}{2}\right] = -\ln x + C,$$

where C is an arbitrary constant.

Substituting back $v = \frac{y}{x}$, gives the general solution

$$\frac{1}{2}\left[\ln\left(4 + \frac{y^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{2x}\right)\right] = -\ln x + C.$$

3. Determine the general solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2) = 0.$$

Solution
If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$2vx^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (x^2 + v^2x^2) = 0.$$

That is,

$$2v\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (1 + v^2) = 0$$

or

$$2vx\frac{\mathrm{d}v}{\mathrm{d}x} = -(1 + 3v^2).$$

On separating the variables, we obtain

$$\int \frac{2v}{1 + 3v^2}\ \mathrm{d}x = -\int \frac{1}{x}\ \mathrm{d}x,$$

which gives

$$\frac{1}{3}\ln(1 + 3v^2) = -\ln x + \ln A,$$

where A is an arbitrary constant.
Hence,

$$(1 + 3v^2)^{\frac{1}{3}} = \frac{A}{x}$$

or, on substituting back $v = \frac{y}{x}$,

$$\left(\frac{x^2 + 3y^2}{x^2}\right)^{\frac{1}{3}} = Ax,$$

which can be written

$$x^2 + 3y^2 = Bx^5,$$

where $B = A^3$.

15.2.3 EXERCISES

Use the substitution $y = vx$ to solve the following differential equations subject to the given boundary condition:

1.
$$(2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + y,$$
where $y = 3$ when $x = -2$.

2.
$$(x^2 - y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$
where $y = 5$ when $x = 0$.

3.
$$x^3 + y^3 = 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x},$$
where $y = 1$ when $x = 2$.

4.
$$x(x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^3,$$
where $y = 2$ when $x = 1$.

5.
$$x\frac{\mathrm{d}y}{\mathrm{d}x} - (y + \sqrt{x^2 - y^2}) = 0,$$
where $y = 0$ when $x = 1$.

15.2.4 ANSWERS TO EXERCISES

1.

$$y^2 - xy - x^2 = 11.$$

2.

$$y = 5e^{-\frac{x^2}{2y^2}}.$$

3.

$$x^3 - 2y^3 = 3x.$$

4.

$$3x^2y = 2(y^2 - x^2).$$

5.

$$e^{\sin^{-1}\frac{y}{x}} = x.$$

"JUST THE MATHS"

UNIT NUMBER

15.3

ORDINARY DIFFERENTIAL EQUATIONS 3 (First order equations (C))

by

A.J.Hobson

UNIT 15.3 - ORDINARY DIFFERENTIAL EQUATIONS 3

FIRST ORDER EQUATIONS (C)

15.3.1 LINEAR EQUATIONS

For certain kinds of first order differential equation, it is possible to multiply the equation throughout by a suitable factor which converts it into an exact differential equation.

For instance, the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2$$

may be multiplied throughout by x to give

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3.$$

It may now be written

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3$$

and, hence, it has general solution

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

Notes:

(i) The factor, x which has multiplied both sides of the differential equation serves as an **"integrating factor"**, but such factors cannot always be found by inspection.

(ii) In the discussion which follows, we shall develop a formula for determining integrating factors, in general, for what are known as **"linear differential equations"**.

DEFINITION

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

is said to be **"linear"**.

RESULT

Given the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

the function

$$e^{\int P(x)\ \mathrm{d}x}$$

is always an integrating factor; and, on multiplying the differential equation throughout by this factor, its left hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{\int P(x)\ \mathrm{d}x}\right].$$

Proof

Suppose that the function, $R(x)$, is an integrating factor; then, in the equation

$$R(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)P(x)y = R(x)Q(x),$$

the left hand side must be the exact derivative of some function of x.

Using the formula for differentiating the product of two functions of x, we can **make** it the derivative of $R(x)y$ provided we can arrange that

$$R(x)P(x) = \frac{\mathrm{d}}{\mathrm{d}x}[R(x)].$$

But this requirement can be interpreted as a differential equation in which the variables $R(x)$ and x may be separated as follows:

$$\int \frac{1}{R(x)}\ \mathrm{d}R(x) = \int\ P(x)\ \mathrm{d}x.$$

Hence,

$$\ln R(x) = \int\ P(x)\ \mathrm{d}x.$$

That is,

$$R(x) = e^{\int\ P(x)\ \mathrm{d}x},$$

as required.

The solution is obtained by integrating the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times R(x)] = R(x)P(x).$$

Note:
There is no need to include an arbitrary constant, C, when $P(x)$ is integrated, since it would only serve to introduce a constant factor of e^C in the above result, which would then immediately cancel out on multiplying the differential equation by $R(x)$.

EXAMPLES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2.$$

Solution
An integrating factor is

$$e^{\int\ \frac{1}{x}\ \mathrm{d}x} = e^{\ln x} = x.$$

On multiplying throughout by the integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times x] = x^3;$$

and so,

$$yx = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2e^{-x^2}.$$

Solution
An integrating factor is

$$e^{\int\ 2x\ \mathrm{d}x} = e^{x^2}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{x^2}\right] = 2,$$

giving

$$ye^{x^2} = 2x + C,$$

where C is an arbitrary constant.

15.3.2 BERNOUILLI'S EQUATION

A similar type of differential equation to that in the previous section has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n.$$

It is called **"Bernouilli's Equation"** and may be converted to a linear differential equation by making the substitution

$$z = y^{1-n}.$$

Proof

The differential equation may be rewritten as

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x).$$

Also,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence the differential equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}x} + P(x)z = Q(x).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)P(x)z = (1-n)Q(x),$$

which is a linear differential equation.

Note:
It is better not to regard this as a standard formula, but to apply the method of obtaining it in the case of particular examples.

EXAMPLES

1. Determine the general solution of the differential equation

$$xy - \frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{-x^2}.$$

Solution

The differential equation may be rewritten

$$-y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + x.y^{-2} = e^{-x^2}.$$

Substituting $z = y^{-2}$, we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$ and, hence,

$$\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}x} + xz = e^{-x^2}$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 2e^{-x^2}.$$

An integrating factor for this equation is

$$e^{\int\ 2x\ \mathrm{d}x} = e^{x^2}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{x^2}\right) = 2,$$

giving

$$ze^{x^2} = 2x + C,$$

where C is an arbitrary constant.
Finally, replacing z by y^{-2},

$$y^2 = \frac{e^{x^2}}{2x + C}.$$

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = xy^2.$$

Solution

The differential equation may be rewritten

$$y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}.y^{-1} = x.$$

On substituting $z = y^{-1}$ we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$ so that

$$-\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{1}{x}.z = x$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{x}.z = -x.$$

An integrating factor for this equation is

$$e^{\int \left(-\frac{1}{x}\right)\, \mathrm{d}x} = e^{-\ln x} = \frac{1}{x}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(z \times \frac{1}{x}\right) = -1,$$

giving

$$\frac{z}{x} = -x + C,$$

where C is an arbitrary constant.

The general solution of the given differential equation is therefore

$$\frac{1}{xy} = -x + C \quad \text{or} \quad y = \frac{1}{Cx - x^2}.$$

15.3.3 EXERCISES

Use an integrating factor to solve the following differential equations subject to the given boundary condition:

1.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$

where $y = 10$ when $x = 0$.

2.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 10,$$

where $y = 4$ when $x = 0$.

3.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 3x,$$

where $y = 2$ when $x = -1$.

4.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{1-x} = 1 - x^2,$$

where $y = 0$ when $x = -1$.

5.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \cos x,$$

where $y = \frac{5}{2}$ when $x = \frac{\pi}{2}$.

6.

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = x,$$

where $y = 0$ when $x = 1$.

7.

$$3y - 2\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{4x},$$

where $y = 1$ when $x = 0$.

8.

$$2y - x\frac{\mathrm{d}y}{\mathrm{d}x} = x(x-1)y^4,$$

where $y^3 = 14$ when $x = 1$.

15.3.4 ANSWERS TO EXERCISES

1.
$$y = 10e^{-\frac{2}{3}x}.$$

2.
$$y = 6e^{\frac{5}{3}x} - 2.$$

3.
$$yx = x^3 - 1.$$

4.
$$y = \frac{1}{2}(1-x)(1+x)^2.$$

5.
$$y = \frac{\sin x}{2} + \frac{2}{\sin x}.$$

6.
$$y = 1 + x^2 - \sqrt{2(1+x^2)}.$$

7.
$$y^2 = \frac{7e^{3x}}{e^{7x}+6}.$$

8.
$$y^3 = \frac{56x^6}{21x^6 - 24x^7 + 7}.$$

"JUST THE MATHS"

UNIT NUMBER

15.4

ORDINARY DIFFERENTIAL EQUATIONS 4 (Second order equations (A))

by

A.J.Hobson

UNIT 15.4 - ORDINARY DIFFERENTIAL EQUATIONS 4

SECOND ORDER EQUATIONS (A)

15.4.1 INTRODUCTION

In the discussion which follows, we shall consider a particular kind of second order ordinary differential equation which is called **"linear, with constant coefficients"**; it has the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

where a, b and c are the constant coefficients.

The various cases of solution which arise depend on the values of the coefficients, together with the type of function, $f(x)$, on the right hand side. These cases will now be dealt with in turn.

15.4.2 SECOND ORDER HOMOGENEOUS EQUATIONS

The term **"homogeneous"**, in the context of second order differential equations, is used to mean that the function, $f(x)$, on the right hand side is zero. It should not be confused with the previous use of this term in the context of first order differential equations.

We therefore consider equations of the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

Note:
A very simple case of this equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0,$$

which, on integration twice, gives the general solution

$$y = Ax + B,$$

where A and B are arbitrary constants. We should therefore expect two arbitrary constants in the solution of any second order linear differential equation with constant coefficients.

The Standard General Solution

The equivalent of

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

in the discussion of first order differential equations would have been

$$b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0; \quad \text{that is,} \quad \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{b}y = 0$$

and this could have been solved using an integrating factor of

$$e^{\int \frac{c}{b} \, \mathrm{d}x} = e^{\frac{c}{b}x},$$

giving the general solution

$$y = Ae^{-\frac{c}{b}x},$$

where A is an arbitrary constant.

It seems reasonable, therefore, to make a trial solution of the form $y = Ae^{mx}$, where $A \neq 0$, in the second order case.

We shall need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ame^{mx} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Am^2e^{mx}.$$

Hence, on substituting the trial solution, we require that

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0;$$

and, by cancelling Ae^{mx}, this condition reduces to

$$am^2 + bm + c = 0,$$

a quadratic equation, called the **"auxiliary equation"**, having the same (constant) coefficients as the orginal differential equation.

In general, it will have two solutions, say $m = m_1$ and $m = m_2$, giving corresponding solutions $y = Ae^{m_1 x}$ and $y = Be^{m_2 x}$ of the differential equation.

However, the linearity of the differential equation implies that the sum of any two solutions is also a solution, so that

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

is another solution; and, since this contains two arbitrary constants, we shall take it to be the general solution.

Notes:

(i) It may be shown that there are no solutions other than those of the above form though special cases are considered later.

(ii) It will be possible to determine particular values of A and B if an appropriate number of boundary conditions for the differential equation are specified. These will usually be a set of given values for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a certain value of x.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

and also the particular solution for which $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -5$ when $x = 0$.

Solution

The auxiliary equation is $m^2 + 5m + 6 = 0$,

which can be factorised as

$$(m+2)(m+3) = 0.$$

Its solutions are therefore $m = -2$ and $m = -3$.

Hence, the differential equation has general solution

$$y = Ae^{-2x} + Be^{-3x},$$

where A and B are arbitrary constants.

Applying the boundary conditions, we shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2Ae^{-2x} - 3Be^{-3x}.$$

Hence,

$$\begin{aligned} 2 &= A + B, \\ -5 &= -2A - 3B \end{aligned}$$

giving $A = 1$, $B = 1$ and a particular solution

$$y = e^{-2x} + e^{-3x}.$$

15.4.3 SPECIAL CASES OF THE AUXILIARY EQUATION

(a) The auxilary equation has coincident solutions

Suppose that both solutions of the auxiliary equation are the same number, m_1.

In other words, the quadratic expression $am^2 + bm + c$ is a **"perfect square"**, which means that it is actually $a(m - m_1)^2$.

Apparently, the general solution of the differential equation is

$$y = Ae^{m_1 x} + Be^{m_1 x},$$

which does not genuinely contain two arbitrary constants since it can be rewritten as

$$y = Ce^{m_1 x} \quad \text{where} \quad C = A + B.$$

It will not, therefore, count as the general solution, though the fault seems to lie with the constants A and B rather than with m_1.

Consequently, let us now examine a new trial solution of the form

$$y = ze^{m_1 x},$$

where z denotes a function of x rather than a constant.

We shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = zm_1 e^{m_1 x} + e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = zm_1^2 e^{m_1 x} + 2m_1 e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x} + e^{m_1 x}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}.$$

On substituting these into the differential equation, we obtain the condition that

$$e^{m_1 x}\left[a\left(zm_1^2 + 2m_1\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right) + b\left(zm_1 + \frac{\mathrm{d}z}{\mathrm{d}x}\right) + cz\right] = 0$$

or

$$z(am_1^2 + bm_1 + c) + \frac{\mathrm{d}z}{\mathrm{d}x}(2am_1 + b) + a\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0.$$

The first term on the left hand side of this condition is zero since m_1 is already a solution of the auxiliary equation; and the second term is also zero since the auxiliary equation, $am^2 + bm + c = 0$, is equivalent to $a(m - m_1)^2 = 0$; that is, $am^2 - 2am_1 m + am_1^2 = 0$. Thus $b = -2am_1$.

We conclude that $\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0$ with the result that $z = Ax + B$, where A and B are arbitrary constants.

The general solution of the differential equation in the case of coincident solutions to the auxiliary equation is therefore

$$y = (Ax + B)e^{m_1 x}.$$

EXAMPLE

Determine the general solution of the differential equation

$$4\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Solution

The auxilary equation is

$$4m^2 + 4m + 1 = 0 \quad \text{or} \quad (2m + 1)^2 = 0$$

and it has coincident solutions at $m = -\frac{1}{2}$.

The general solution is therefore

$$y = (Ax + B)e^{-\frac{1}{2}x}.$$

(b) The auxiliary equation has complex solutions

If the auxiliary equation has complex solutions, they will automatically appear as a pair of **"complex conjugates"**, say $m = \alpha \pm j\beta$.

Using these two solutions instead of the previous m_1 and m_2, the general solution of the differential equation will be

$$y = Pe^{(\alpha + j\beta)x} + Qe^{(\alpha - j\beta)x},$$

where P and Q are arbitrary constants.

But, by properties of complex numbers, a neater form of this result is obtainable as follows:

$$y = e^{\alpha x}[P(\cos\beta x + j\sin\beta x) + Q(\cos\beta x - j\sin\beta x)]$$

or

$$y = e^{\alpha x}[(P+Q)\cos\beta x + j(P-Q)\sin\beta x].$$

Replacing $P+Q$ and $j(P-Q)$ (which are just arbitrary quantities) by A and B, we obtain the standard general solution for the case in which the auxiliary equation has complex solutions. It is

$$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x].$$

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = 0.$$

Solution

The auxiliary equation is

$$m^2 - 6m + 13 = 0,$$

which has solutions given by

$$m = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\times 13\times 1}}{2\times 1} = \frac{6\pm j4}{2} = 3 \pm j2.$$

The general solution is therefore

$$y = e^{3x}[A\cos 2x + B\sin 2x],$$

where A and B are arbitrary constants.

15.4.4 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 12y = 0;$$

   (b)
   $$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} + 6\frac{\mathrm{d}r}{\mathrm{d}\theta} + 9r = 0;$$

   (c)
   $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 4\frac{\mathrm{d}\theta}{\mathrm{d}t} + 5\theta = 0.$$

2. Solve the following differential equations, subject to the given boundary conditions:

   (a)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$
   where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$;

   (b)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0,$$
   where $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ when $t = 0$;

   (c)
   $$4\frac{\mathrm{d}^2z}{\mathrm{d}s^2} - 12\frac{\mathrm{d}z}{\mathrm{d}s} + 9z = 0,$$
   where $z = 1$ and $\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{5}{2}$ when $s = 0$;

(d)

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} - 2\frac{\mathrm{d}r}{\mathrm{d}\theta} + 2r = 0,$$

where $r = 5$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 7$ when $\theta = 0$.

15.4.5 ANSWERS TO EXERCISES

1. (a)

$$y = Ae^{-3x} + Be^{-4x};$$

(b)

$$r = (A\theta + B)e^{-3\theta};$$

(c)

$$\theta = e^{-2t}[A\cos 2t + B\sin 2t].$$

2. (a)

$$y = 3e^x - e^{2x};$$

(b)

$$x = 2e^t + e^{3t};$$

(c)

$$z = (s+1)e^{\frac{3}{2}s};$$

(d)

$$r = e^{\theta}[5\cos\theta + 2\sin\theta].$$

"JUST THE MATHS"

UNIT NUMBER

15.5

ORDINARY DIFFERENTIAL EQUATIONS 5 (Second order equations (B))

by

A.J.Hobson

UNIT 15.5 - ORDINARY DIFFERENTIAL EQUATIONS 5

SECOND ORDER EQUATIONS (B)

15.5.1 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

The following discussion will examine the solution of the second order linear differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=f(x),$$

in which a, b and c are constants, but $f(x)$ is not identically equal to zero.

The Particular Integral and Complementary Function

(i) Suppose that $y=u(x)$ is any particular solution of the differential equation; that is, it contains no arbitrary constants. In the present context, we shall refer to such particular solutions as **"particular integrals"** and systematic methods of finding them will be discussed later.

It follows that

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+b\frac{\mathrm{d}u}{\mathrm{d}x}+cu=f(x).$$

(ii) Suppose also that we make the substitution $y=u(x)+v(x)$ in the original differential equation to give

$$a\frac{\mathrm{d}^2(u+v)}{\mathrm{d}x^2}+b\frac{\mathrm{d}(u+v)}{\mathrm{d}x}+c(u+v)=f(x).$$

That is,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+b\frac{\mathrm{d}u}{\mathrm{d}x}+cu+a\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+b\frac{\mathrm{d}v}{\mathrm{d}x}+cv=f(x);$$

and, hence,

$$a\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+b\frac{\mathrm{d}v}{\mathrm{d}x}+cv=0.$$

This means that the function $v(x)$ is the general solution of the homogeneous differential equation whose auxiliary equation is

$$am^2 + bm + c = 0.$$

In future, $v(x)$ will be called the **"complementary function"** in the general solution of the original (non-homogeneous) differential equation. It complements the particular integral to provide the general solution.

Summary

$$\boxed{\textbf{General solution = particular integral + complementary function.}}$$

15.5.2 DETERMINATION OF SIMPLE PARTICULAR INTEGRALS

(a) Particular integrals, when $f(x)$ is a constant, k.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = k,$$

it is easy to see that a particular integral will be $y = \frac{k}{c}$, since its first and second derivatives are both zero, while $cy = k$.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 20.$$

Solution

(i) By inspection, we may observe that a particular integral is $y = 2$.

(ii) The auxiliary equation is

$$m^2 + 7m + 10 = 0 \quad \text{or} \quad (m+2)(m+5) = 0,$$

having solutions $m = -2$ and $m = -5$.

(iii) The complementary function is

$$Ae^{-2x} + Be^{-5x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 2 + Ae^{-2x} + Be^{-5x}.$$

(b) Particular integrals, when $f(x)$ is of the form $px + q$.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = px + q,$$

it is possible to determine a particular integral by assuming one which has the same form as the right hand side; that is, in this case, another expression consisting of a multiple of x and constant term. The method is, again, illustrated by an example.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 11\frac{\mathrm{d}y}{\mathrm{d}x} + 28y = 84x - 5.$$

Solution

(i) First, we assume a particular integral of the form

$$y = \alpha x + \beta,$$

which implies that $\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

Substituting into the differential equation, we require that

$$-11\alpha + 28(\alpha x + \beta) \equiv 84x - 5.$$

Hence, $28\alpha = 84$ and $-11\alpha + 28\beta = -5$, giving $\alpha = 3$ and $\beta = 1$.

Thus, the particular integral is

$$y = 3x + 1.$$

(ii) The auxiliary equation is

$$m^2 - 11m + 28 = 0 \quad \text{or} \quad (m-4)(m-7) = 0,$$

having solutions $m = 4$ and $m = 7$.

(iii) The complementary function is

$$Ae^{4x} + Be^{7x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 3x + 1 + Ae^{4x} + Be^{7x}.$$

Note:
In examples of the above types, the complementary function <u>must not</u> be prefixed by "$y =$", since the given differential equation, as a whole, is not normally satisfied by the complementary function alone.

15.5.3 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6;$$

   (b)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 16y = 7;$$

   (c)
$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - y = x + 1;$$

   (d)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 18x + 28.$$

2. Solve, completely, the following differential equations, subject to the given boundary condidtions:

   (a)
$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 100,$$

   where $y = -26$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 5$ when $x = 0$;

   (b)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 12x + 16,$$

   where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$;

   (c)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 10x + 14,$$

   where $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.5.4 ANSWERS TO EXERCISES

1. (a)

$$y = -3 + Ae^{x} + Be^{2x};$$

(b)

$$y = \frac{7}{16} + A\cos 4x + B\sin 4x;$$

(c)

$$y = 1 - x + Ae^{x} + Be^{-\frac{1}{3}x};$$

(d)

$$y = 2x + 5 + (Ax + B)e^{3x}.$$

2. (a)

$$y = -25 + e^{4x} - 2e^{\frac{1}{2}x};$$

(b)

$$y = 3x + 1 - (x + 1)e^{-2x};$$

(c)

$$y = x + 2 + e^{3x}(\cos x - 2\sin x).$$

"JUST THE MATHS"

UNIT NUMBER

15.6

ORDINARY DIFFERENTIAL EQUATIONS 6 (Second order equations (C))

by

A.J.Hobson

UNIT 15.6 - ORDINARY DIFFERENTIAL EQUATIONS 6

SECOND ORDER EQUATIONS (C)

15.6.1 RECAP

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

it was seen, in Unit 15.5, that

(a) when $f(x) \equiv k$, a given **constant**, a particular integral is $y = \frac{k}{c}$;

(b) when $f(x) \equiv px+q$, a **linear** function in which p and q are given constants, it is possible to obtain a particular integral by assuming that y also has the form of a linear function; that is, we make a **"trial solution"**, $y = \alpha x + \beta$.

15.6.2 FURTHER TYPES OF PARTICULAR INTEGRAL

We now examine particular integrals for other cases of $f(x)$, the method being illustrated by examples. Also, for reasons relating to certain problematic cases discussed in Unit 15.7, we shall determine the complementary function **before** determining the particular integral.

1. $f(x) \equiv px^2 + qx + r$, a **quadratic** function in which p, q and r are given constants; $p \neq 0$.

$$\text{Trial solution :} \quad y = \alpha x^2 + \beta x + \gamma.$$

Note:
This is the trial solution even if q or r (or both) are zero.

EXAMPLE

Determine the general solution of the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 4x^2 + 10x - 23.$$

Solution

The auxilary equation is

$$2m^2 - 7m - 4 = 0 \ \text{ or } \ (2m+1)(m-4) = 0,$$

having solutions $m = 4$ and $m = -\frac{1}{2}$.
Thus, the complementary function is

$$Ae^{4x} + Be^{-\frac{1}{2}x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form $y = \alpha x^2 + \beta x + \gamma$, giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x + \beta$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha$.
We thus require that

$$4\alpha - 14\alpha x - 7\beta - 4\alpha x^2 - 4\beta x - 4\gamma \equiv 4x^2 + 10x - 23.$$

That is,

$$-4\alpha x^2 - (14\alpha + 4\beta)x + 4\alpha - 7\beta - 4\gamma \equiv 4x^2 + 10x - 23.$$

Comparing corresponding coefficents on both sides, this means that

$$-4\alpha = 4, \quad -(14\alpha + 4\beta) = 10 \;\text{ and }\; 4\alpha - 7\beta - 4\gamma = -23,$$

which give $\alpha = -1$, $\beta = 1$ and $\gamma = 3$.
Hence, the particular integral is

$$y = 3 + x - x^2.$$

Finally, the general solution is

$$y = 3 + x - x^2 + Ae^{4x} + Be^{-\frac{1}{2}x}.$$

2. $f(x) \equiv p \sin kx + q \cos kx$, a **trigonometric** function in which p, q and k are given constants.

$$\text{Trial solution :} \quad y = \alpha \sin kx + \beta \cos kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 8\cos 3x - 19\sin 3x.$$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0,$$

which has complex number solutions given by

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm j.$$

Hence, the complementary function is

$$e^x(A\cos x + B\sin x),$$

where A and B are arbitrary constants.

To determine a particular integral, we may make a trial solution of the form

$$y = \alpha\sin 3x + \beta\cos 3x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha\cos 3x - 3\beta\sin 3x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\alpha\sin 3x - 9\beta\cos 3x$.

We thus require that

$$-9\alpha\sin 3x - 9\beta\cos 3x - 6\alpha\cos 3x + 6\beta\sin 3x + 2\alpha\sin 3x + 2\beta\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

That is,

$$(-9\alpha + 6\beta + 2\alpha)\sin 3x + (-9\beta - 6\alpha + 2\beta)\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

Comparing corresponding coefficients on both sides, we have

$$\begin{aligned} -7\alpha + 6\beta &= -19, \\ -6\alpha - 7\beta &= 8. \end{aligned}$$

These equations are satisfied by $\alpha = 1$ and $\beta = -2$, so that the particular integral is

$$y = \sin 3x - 2\cos 3x.$$

Finally, the general solution is

$$y = \sin 3x - 2\cos 3x + e^x(A\cos x + B\sin x).$$

3. $f(x) \equiv pe^{kx}$, an **exponential** function in which p and k are given constants.

$$\text{Trial solution :} \quad y = \alpha e^{kx}.$$

EXAMPLE

Determine the general solution of the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y = 50e^{3x}.$$

Solution

The auxiliary equation is

$$9m^2 + 6m + 1 = 0 \ \text{ or } \ (3m+1)^2 = 0,$$

which has coincident solutions at $m = -\frac{1}{3}$.
The complementary function is therefore

$$(Ax + B)e^{-\frac{1}{3}x}.$$

To find a particular integral, we may make a trial solution of the form

$$y = \alpha e^{3x},$$

which gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha e^{3x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 9\alpha e^{3x}$.
Hence, on substituting into the differential equation, it is necessary that

$$81\alpha e^{3x} + 18\alpha e^{3x} + \alpha e^{3x} = 50e^{3x}.$$

That is, $100\alpha = 50$, from which we deduce that $\alpha = \frac{1}{2}$ and a particular integral is

$$y = \frac{1}{2}e^{3x}.$$

Finally, the general solution is

$$y = \frac{1}{2}e^{3x} + (Ax + B)e^{-\frac{1}{3}x}.$$

4. $f(x) \equiv p\sinh kx + q\cosh kx$, a **hyperbolic** function in which p, q and k are given constants.

$$\text{Trial solution :} \quad y = \alpha\sinh kx + \beta\cosh kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 93\cosh 5x - 75\sinh 5x.$$

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \ \text{ or } \ (m-2)(m-3) = 0,$$

which has solutions $m = 2$ and $m = 3$ so that the complementary function is

$$Ae^{2x} + Be^{3x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form

$$y = \alpha\sinh 5x + \beta\cosh 5x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 5\alpha\cosh 5x + 5\beta\sinh 5x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 25\alpha\sinh 5x + 25\beta\cosh 5x$.
Substituting into the differential equation, the left-hand-side becomes

$$25\alpha\sinh 5x + 25\beta\cosh 5x - 25\alpha\cosh 5x - 25\beta\sinh 5x + 6\alpha\sinh 5x + 6\beta\cosh 5x.$$

This simplifies to

$$(31\alpha - 25\beta)\sinh 5x + (31\beta - 25\alpha)\cosh 5x,$$

so that we require

$$\begin{aligned} 31\alpha - 25\beta &= -75, \\ -25\alpha + 31\beta &= 93, \end{aligned}$$

and these are satisfied by $\alpha = 0$ and $\beta = 3$.
The particular integral is thus

$$y = 3\cosh 5x$$

and, hence, the general solution is

$$y = 3\cosh 5x + Ae^{2x} + Be^{3x}.$$

5. Combinations of Different Types of Function

In cases where $f(x)$ is the sum of two or more of the various types of function discussed previously, then the particular integrals for each type (determined separately) may be added together to give an overall particular integral.

15.6.3 EXERCISES

1. Determine the general solution for each of the following differential equations:

(a)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 4x^2 + 2x - 4;$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 8\cos 2x - \sin 2x;$$

(c)

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 12\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 27e^{-x};$$

(d)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = \cosh 3x - \sinh 3x.$$

2. Solve completely the following differential equations subject to the given boundary conditions:

   (a)
$$\frac{d^2y}{dx^2} - y = 10 - 5x^2 - x + 16e^{-3x},$$
   where $y = 13$ and $\frac{dy}{dx} = -2$ when $x = 0$;

   (b)
$$4\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 3y = 9x + 6\cos x - 19\sin x,$$
   where $y = -2$ and $\frac{dy}{dx} = 0$ when $x = 0$.

15.6.4 ANSWERS TO EXERCISES

1. (a)
$$y = x^2 - 2x + 1 + Ae^{-x} + Be^{-4x};$$
   (b)
$$y = \sin 2x + e^{-2x}(A\cos x + B\sin x);$$
   (c)
$$y = 3e^{-x} + (Ax + B)e^{-\frac{3}{2}x};$$
   (d)
$$y = \frac{1}{8}(\cosh 3x - \sinh 3x) + Ae{-}2x + Be^{5x}.$$
2. (a)
$$y = 5x^2 + x + 2e^{-3x} + 3e^x - 2e^{-x};$$
   (b)
$$y = 3x - 8 + 2\cos x + \sin x + 2e^{-\frac{1}{2}x} + 2e^{-\frac{3}{2}x}.$$

"JUST THE MATHS"

UNIT NUMBER

15.7

ORDINARY DIFFERENTIAL EQUATIONS 7 (Second order equations (D))

by

A.J.Hobson

UNIT 15.7 - ORDINARY DIFFERENTIAL EQUATIONS 7

SECOND ORDER EQUATIONS (D)

15.7.1 PROBLEMATIC CASES OF PARTICULAR INTEGRALS

Difficulties can arise if all or part of any trial solution would already be included in the complementary function. We illustrate with some examples:

EXAMPLES

1. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{2x}.$$

Solution

The auxiliary equation is $m^2 - 3m + 2 = 0$, with solutions $m = 1$ and $m = 2$ and hence the complementary function is $Ae^x + Be^{2x}$, where A and B are arbitrary constants.

A trial solution of $y = \alpha e^{2x}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha e^{2x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\alpha e^{2x}$$

and, on substituting these into the differential equation, it is necessary that

$$4\alpha e^{2x} - 6\alpha e^{2x} + 2\alpha e^{2x} \equiv e^{2x}.$$

That is, $0 \equiv e^{2x}$ which is impossible.

However, if $y = \alpha e^{2x}$ has proved to be unsatisfactory, let us investigate, as an alternative, $y = F(x)e^{2x}$ (where $F(x)$ is a function of x instead of a constant).

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2F(x)e^{2x} + F'(x)e^{2x}$$

and, hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4F(x)e^{2x} + 2F'(x)e^{2x} + F''(x)e^{2x} + 2F'(x)e^{2x}.$$

On substituting these into the differential equation, it is necessary that

$$(4F(x) + 2F'(x) + F''(x) + 2F'(x) - 6F(x) - 3F'(x) + 2F(x))\, e^{2x} \equiv e^{2x}.$$

That is,

$$F''(x) + F'(x) = 1,$$

which is satisfied by the function $F(x) \equiv x$ and thus a suitable particular integral is

$$y = xe^{2x}.$$

Note:
It may be shown, in other cases too that, if the standard trial solution is already contained in the complementary function, then it is is necessary to multiply it by x in order to obtain a suitable particular integral.

2. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin x.$$

Solution
The auxilary equation is $m^2 + 1 = 0$, with solutions $m = \pm j$ and, hence, the complementary function is $A\sin x + B\cos x$, where A and B are arbitrary constants.
A trial solution of $y = \alpha\sin x + \beta\cos x$ gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\alpha\sin x - \beta\cos x;$$

and, on substituting into the differential equation, it is necessary that $0 \equiv \sin x$, which is impossible.
Here, we may try $y = x(\alpha\sin x + \beta\cos x)$, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha\sin x + \beta\cos x + x(\alpha\cos x - \beta\sin x) = (\alpha - \beta x)\sin x + (\beta + \alpha x)\cos x$$

and, therefore,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\alpha-\beta x)\cos x - \beta\sin x - (\beta+\alpha x)\sin x + \alpha\cos x = (2\alpha-\beta x)\cos x - (2\beta+\alpha x)\sin x.$$

Substituting into the differential equation, we thus require that

$$(2\alpha - \beta x)\cos x - (2\beta + \alpha x)\sin x + x(\alpha \sin x + \beta \cos x) \equiv \sin x,$$

which simplifies to

$$2\alpha \cos x - 2\beta \sin x \equiv \sin x.$$

Thus $2\alpha = 0$ and $-2\beta = 1$.

An appropriate particular integral is now

$$y = -\frac{1}{2} x \cos x.$$

3. Determine the complementary function and a particular integral for the differential equation

$$9\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y == 50e^{-\frac{1}{3}x}.$$

Solution

The auxiliary equation is $9m^2 + 6m + 1 = 0$, or $(3m+1)^2 = 0$, which has coincident solutions $m = -\frac{1}{3}$ and so the complementary function is

$$(Ax + B)e^{-\frac{1}{3}x}.$$

In this example, both $e^{-\frac{1}{3}x}$ **and** $xe^{-\frac{1}{3}x}$ are contained in the complementary function.

Thus, in the trial solution, it is necessary to multiply by a **further** x, giving

$$y = \alpha x^2 e^{-\frac{1}{3}x}.$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x e^{-\frac{1}{3}x} - \frac{1}{3}x^2 e^{\frac{1}{3}x}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\alpha e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} + \frac{1}{9}\alpha x^2 e^{-\frac{1}{3}x}.$$

Substituting these into the differential equation, it is necessary that

$$\left(18\alpha - 12\alpha x + \alpha x^2 + 12\alpha x - 2\alpha x^2 + \alpha x^2\right) e^{-\frac{1}{3}x} = 50e^{-\frac{1}{3}x}$$

and, hence, $18\alpha = 50$ or $\alpha = \frac{25}{9}$.
An appropriate particular integral is

$$y = \frac{25}{9}x^2 e^{-\frac{1}{3}x}.$$

4. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \sinh 2x.$$

Solution
The auxiliary equation is $m^2 - 5m + 6 = 0$ or $(m-2)(m-3) = 0$ which has solutions $m = 2$ and $m = 3$ and, hence, the complementary function is

$$Ae^{2x} + Be^{3x}.$$

However, since $\sinh 2x \equiv \frac{1}{2}(e^{2x} - e^{-2x})$, **part** of it is contained in the complementary function and we must find a particular integral for each part separately.
(a) For $\frac{1}{2}e^{2x}$, we may try

$$y = x\alpha e^{2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha e^{2x} + 2x\alpha e^{2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{2x} + 2\alpha e^{2x} + 4x\alpha e^{2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\alpha + 4x\alpha - 5\alpha - 10x\alpha + 6x\alpha)\, e^{2x} \equiv \frac{1}{2}e^{2x},$$

which gives $\alpha = -\frac{1}{2}$.

(b) For $-\frac{1}{2}e^{-2x}$, we may try

$$y = \beta e^{-2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\beta e^{-2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\beta e^{-2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\beta + 10\beta + 6\beta)e^{-2x} \equiv -\frac{1}{2}e^{-2x},$$

which gives $\beta = -\frac{1}{40}$.
The overall particular integral is thus

$$y = -\frac{1}{2}xe^{2x} - \frac{1}{40}e^{-2x}.$$

15.7.2 EXERCISES

Solve completely the following differential equations subject to the given boundary conditions:

1.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-x},$$

where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{2}$ when $x = 0$.

2.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y = 2\sin 3x,$$

where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8}{3}$ when $x = 0$.

3.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 10\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 8e^{3x} + 25x^2 - 20x + 27,$$

where $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 13$ when $x = 0$.

4.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cosh x,$$

where $y = \frac{7}{12}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$.

5.

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 24e^{-\frac{1}{2}x}$$

where $y = 6$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.7.3 ANSWERS TO EXERCISES

1.

$$y = \frac{1}{2}xe^{-x} + Ae^{-x} + Be^{-3x}.$$

2.

$$y = -\frac{1}{3}x\cos 3x + 2\cos 3x + \sin 3x.$$

3.

$$y = 2e^{3x} + x^2 + 1 + (2 - 3x)e^{5x}.$$

4.

$$y = \frac{1}{12}\left(e^{-x} - 6xe^{x} - e^{x} + 7e^{2x}\right).$$

5.

$$y = 3x^2e^{-\frac{1}{2}x} + (5x + 6)e^{-\frac{1}{2}x}.$$

"JUST THE MATHS"

UNIT NUMBER

15.8

ORDINARY DIFFERENTIAL EQUATIONS 8 (Simultaneous equations (A))

by

A.J.Hobson

UNIT 15.8 - ORDINARY DIFFERENTIAL EQUATIONS 8

SIMULTANEOUS EQUATIONS (A)

15.8.1 THE SUBSTITUTION METHOD

The methods discussed in previous Units for the solution of second order ordinary linear differential equations with constant coefficients may now be used for cases of two first order differential equations which must be satisfied simultaneously. The technique will be illustrated by the following examples:

EXAMPLES

1. Determine the general solutions for y and z in the case when

$$\begin{aligned} 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + 4y - z &= e^{-x}, \text{------------} (1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 3z &= 5e^{-x}. \text{-----------} (2) \end{aligned}$$

Solution

First, we eliminate one of the dependent variables from the two equations; in this case, we eliminate z.

From equation (2),

$$z = \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right)$$

and, on substituting this into equation (1), we obtain

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{3}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5e^{-x}\right) + 4y - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right) = e^{-x}.$$

That is, $$-\frac{2}{3}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{2}{3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{3}y = \frac{8}{3}e^{-x}$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = -4e^{-x}.$$

The auxiliary equation is

$$m^2 + m - 2 = 0 \quad \text{or} \quad (m-1)(m+2) = 0,$$

giving a complementary function of $Ae^{x} + Be^{-2x}$, where A and B are arbitrary constants. A particular integral will be of the form ke^{-x}, where $k - k - 2k = -4$ and hence $k = 2$. Thus,

$$y = 2e^{-x} + Ae^{x} + Be^{-2x}.$$

Finally, from the formula for z in terms of y,

$$z = \frac{1}{3}\left(-2e^{-x} + Ae^{x} - 2Be^{-2x} + 16e^{-x} + 8Ae^{x} + 8Be^{-2x} - 5e^{-x}\right).$$

That is,

$$z = 3e^{-x} + 3Ae^{-x} + 2Be^{-2x}.$$

Note:
The above example would have been a little more difficult if the second differential equation had contained a term in $\frac{dz}{dx}$. But, if this were the case, we could eliminate $\frac{dz}{dx}$ between the two equations in order to obtain a statement with the same form as Equation (2).

2. Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{dz}{dx} + 2y &= e^{x}, - - - - - - - - - (1) \\ \frac{dy}{dx} - 2z &= 1 + x, - - - - - - - -(2) \end{aligned}$$

given that $y = 1$ and $z = 2$ when $x = 0$.

Solution:

From equation (2), we have

$$z = \frac{1}{2}\left[\frac{dy}{dx} - 1 - x\right].$$

Substituting into the first differential equation gives

$$\frac{1}{2}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 1\right] + 2y = e^x$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 2e^x + 1.$$

The auxiliary equation is therefore $m^2+4=0$, having solutions $m = \pm j2$, which means that the complementary function is

$$A\cos 2x + B\sin 2x,$$

where A and B are arbitrary constants.
The particular integral will be of the form $y = pe^x + q$,
where

$$pe^x + 4pe^x + 4q = 2e^x + 1.$$

We require, then, that $5p = 2$ and $4q = 1$; and so the general solution for y is

$$y = A\cos 2x + B\sin 2x + \frac{2}{5}e^x + \frac{1}{4}.$$

Using the earlier formula for z, we obtain

$$z = \frac{1}{2}\left[-2A\sin 2x + 2B\cos 2x + \frac{2}{5}e^x - 1 - x\right] = B\cos 2x - A\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

Applying the boundary conditions,

$$1 = A + \frac{2}{5} + \frac{1}{4} \quad \text{giving} \quad A = \frac{7}{20}$$

and

$$2 = B + \frac{1}{5} - \frac{1}{2} \quad \text{giving} \quad B = \frac{23}{10}.$$

The required solutions are therefore

$$y = \frac{7}{20}\cos 2x + \frac{23}{10}\sin 2x + \frac{2}{5}e^x + \frac{1}{4}$$

and

$$z = \frac{23}{10}\cos 2x - \frac{7}{20}\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

15.8.2 EXERCISES

Solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + 2z &= e^{-x}, \\ \frac{\mathrm{d}z}{\mathrm{d}x} + 3z &= y, \end{aligned}$$

given that $y = 1$ and $z = 0$ when $x = 0$.

2.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} - z &= \sin x, \\ \frac{\mathrm{d}z}{\mathrm{d}x} + y &= \cos x, \end{aligned}$$

given that $y = 3$ and $z = 4$ when $x = 0$.

3.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + 2y - 3z &= 1, \\ \frac{\mathrm{d}z}{\mathrm{d}x} - y &= e^{-2x}, \end{aligned}$$

given that $y = 0$ and $z = 0$ when $x = 0$.

4.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 2z, \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= 8y, \end{aligned}$$

given that $y = 1$ and $z = 0$ when $x = 0$.

5.

$$\begin{aligned} \frac{dy}{dx} + 4\frac{dz}{dx} + 6z &= 0, \\ 5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y &= 0, \end{aligned}$$

given that $y = 3$ and $z = 0$ when $x = 0$.

Hint: First eliminate the $\frac{dz}{dx}$ terms to obtain a formula for z in terms of y and $\frac{dy}{dx}$.

6.

$$\begin{aligned} 10\frac{dy}{dx} - 3\frac{dz}{dx} + 6y + 5z &= 0, \\ 2\frac{dy}{dx} - \frac{dz}{dx} + 2y + z &= 2e^{-x}, \end{aligned}$$

given that $y = 2$ and $z = -1$ when $x = 0$.

Hint: First, eliminate the $\frac{dy}{dx}$ and z terms in one step, to obtain a formula for y in terms of $\frac{dz}{dx}$ and x.

15.8.3 ANSWERS TO EXERCISES

1.

$$y = (2x+1)e^{-x} \quad \text{and} \quad z = xe^{-x}.$$

2.

$$y = (x+4)\sin x + 3\cos x \quad \text{and} \quad z = (x+4)\cos x - 3\sin x.$$

3.

$$y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-3x} - e^{-2x} \quad \text{and} \quad z = \frac{1}{2}e^{x} - \frac{1}{6}e^{-3x} - \frac{1}{3}.$$

4.

$$y = \frac{1}{2}e^{4x} - \frac{1}{2}e^{-4x} \equiv \sinh 4x \quad \text{and} \quad z = e^{4x} + e^{-4x} \equiv 2\cosh 4x.$$

5.

$$y = 2e^{-x} + e^{-2x} \quad \text{and} \quad z = e^{-x} - e^{-2x}.$$

6.

$$y = \sin x + 2e^{-x} \quad \text{and} \quad z = e^{-x} - 2\cos x.$$

"JUST THE MATHS"

UNIT NUMBER

15.9

ORDINARY DIFFERENTIAL EQUATIONS 9 (Simultaneous equations (B))

by

A.J.Hobson

UNIT 15.9 - ORDINARY DIFFERENTIAL EQUATIONS 9

SIMULTANEOUS EQUATIONS (B)

15.9.1 INTRODUCTION

For students who have studied the principles of eigenvalues and eigenvectors (see Unit 9.6), a second method of solving two simultaneous linear differential equations is to interpret them as a single equation using matrix notation. The discussion will be limited to the simpler kinds of example, and we shall find it convenient to use t, x_1 and x_2 rather than x, y and z.

15.9.2 MATRIX METHODS FOR HOMOGENEOUS SYSTEMS

To introduce the technique, we begin by considering two simultaneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2.\end{aligned}$$

which are of the "homogeneous" type, since no functions of t, other than x_1 and x_2, appear on the right hand sides.

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathrm{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(ii) Secondly, in a similar way to the method appropriate to a single differential equation, we make a trial solution of the form

$$\mathrm{X} = \mathrm{K}e^{\lambda t},$$

where

$$\mathrm{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

is a constant matrix of order 2×1.

This requires that

$$\lambda \mathrm{K} e^{\lambda t} = \mathrm{MK} e^{\lambda t} \quad \text{or} \quad \lambda \mathrm{K} = \mathrm{MK},$$

which we may recognise as the condition that λ is an eigenvalue of the matrix M, and K is an eigenvector of M.

The solutions for λ are obtained from the "characteristic equation"

$$|\mathrm{M} - \lambda \mathrm{I}| = 0.$$

In other words,

$$\begin{vmatrix} a - \lambda & b \\ c & b - \lambda \end{vmatrix} = 0,$$

leading to a quadratic equation having real and distinct solutions ($\lambda = \lambda_1$ and $\lambda = \lambda_2$), real and coincident solutions (λ only) or conjugate complex solutions ($\lambda = l \pm jm$).

(iii) The possibilities for the matrix K are obtained by solving the homogeneous linear equations

$$\begin{aligned} (a - \lambda_1 k_1 + b k_2 &= 0, \\ c k_1 + (d - \lambda_1) k_2 &= 0, \end{aligned}$$

giving $k_1 : k_2 = 1 : \alpha$ (say)

and

$$
\begin{aligned}
(a-\lambda_2)k_1 + bk_2 &= 0, \\
ck_1 + (d-\lambda_2)k_2 &= 0,
\end{aligned}
$$

giving $k_1 : k_2 = 1 : \beta$ (say).

Finally, it may be shown that, according to the types of solution to the auxiliary equation, the solution of the differential equation will take one of the following three forms, in which A and B are arbitrary constants:

(a)

$$A\begin{bmatrix}1\\ \alpha\end{bmatrix}e^{\lambda_1 t} + B\begin{bmatrix}1\\ \beta\end{bmatrix}e^{\lambda_2 t};$$

(b)

$$\left\{(At+B)\begin{bmatrix}1\\ \alpha\end{bmatrix} + \frac{A}{b}\begin{bmatrix}0\\ 1\end{bmatrix}\right\}e^{\lambda t};$$

or

(c)

$$e^{lt}\left\{\begin{bmatrix}A\\ pA+qB\end{bmatrix}\cos mt + \begin{bmatrix}B\\ pB-qA\end{bmatrix}\sin mt\right\},$$

where, in (c), $1 : \alpha = 1 : p + jq$ and $1 : \beta = 1 : p - jq$.

EXAMPLES

1. Determine the general solution of the simultaneous differential equations

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -4x_1 + 5x_2, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= -x_1 + 2x_2.
\end{aligned}
$$

Solution

The characteristic equation is

$$\begin{vmatrix} -4-\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 + 2\lambda - 3 = 0 \quad \text{or} \quad (\lambda - 1)(\lambda + 3) = 0.$$

When $\lambda = 1$, we need to solve the homogeneous equations

$$\begin{aligned} -5k_1 + 5k_2 &= 0, \\ -k_1 + k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : 1$.

When $\lambda = -3$, we need to solve the homogeneous equations

$$\begin{aligned} -k_1 + 5k_2 &= 0, \\ -k_1 + 5k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{1}{5}$.

The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} e^{-3t}$$

or, alternatively,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{-3t},$$

where A and B are arbitrary constants.

2. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 - x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 + 3x_2.\end{aligned}$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 4 = 0 \quad \text{or} \quad (\lambda - 2)^2 = 0.$$

When $\lambda = 2$, we need to solve the homogeneous equations

$$\begin{aligned}-k_1 - k_2 &= 0,\\ k_1 + k_2 &= 0,\end{aligned}$$

both of which give $k_1 : k_2 = 1 : -1$.

The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ (At + B)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - A\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} e^{2t},$$

where A and B are arbitrary constants.

3. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 - 5x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 2x_1 + 3x_2.\end{aligned}$$

Solution
The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 13 = 0,$$

which gives $\lambda = 2 \pm j3$.
When $\lambda = 2 + j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1-j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1-j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1-j3}{5}$.
When $\lambda = 2 - j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1+j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1+j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1+j3}{5}$.
The general solution is therefore

$$\frac{e^{2t}}{5}\left\{\begin{bmatrix} A \\ -A+3B \end{bmatrix}\cos 3t + \begin{bmatrix} B \\ -B-3A \end{bmatrix}\sin 3t\right\},$$

where A and B are arbitrary constants.
Note:
From any set of simultaneous differential equations of the form

$$\begin{aligned} a\frac{\mathrm{d}x_1}{\mathrm{d}t} + b\frac{\mathrm{d}x_2}{\mathrm{d}t} + cx_1 + dx_2 &= 0, \\ a'\frac{\mathrm{d}x_1}{\mathrm{d}t} + b'\frac{\mathrm{d}x_2}{\mathrm{d}t} + b'x_1 + c'x_2 &= 0, \end{aligned}$$

it is possible to eliminate $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ in turn, in order to obtain two equivalent equations of the form discussed in the above examples.

15.9.3 EXERCISES

1. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 + 3x_2,\\ \frac{dx_2}{dt} &= 3x_1 + x_2.\end{aligned}$$

2. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 + 2x_2,\\ \frac{dx_2}{dt} &= 4x_1 + x_2,\end{aligned}$$

given that $x_1 = 3$ and $x_2 = -3$ when $t = 0$.

3. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + 9x_2,\\ \frac{dx_2}{dt} &= 11x_1 + x_2,\end{aligned}$$

given that $x_1 = 20$ and $x_2 = 20$ when $t = 0$.

4. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} - \frac{dx_2}{dt} + 2x_1 - 2x_2 &= 0,\\ \frac{dx_1}{dt} + 2\frac{dx_2}{dt} - 7x_1 - 5x_2 &= 0,\end{aligned}$$

given that $x_1 = 2$ and $x_2 = 0$ when $t = 0$.

5. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2.\end{aligned}$$

6. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2.\end{aligned}$$

15.9.4 ANSWERS TO EXERCISES

1.

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

2.

$$\begin{bmatrix}1\\1\end{bmatrix}e^{5t} + 2\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}.$$

3.

$$2\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t} + \begin{bmatrix}18\\22\end{bmatrix}e^{10t}.$$

4.

$$\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + \begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

5.

$$\left\{(At+B)\begin{bmatrix}1\\-1\end{bmatrix} + \frac{A}{2}\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}.$$

6.

$$e^{7t}\left\{\begin{bmatrix}A\\-A+2B\end{bmatrix}\cos 2t + \begin{bmatrix}B\\-B-2A\end{bmatrix}\sin 2t\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

15.10

ORDINARY DIFFERENTIAL EQUATIONS 10 (Simultaneous equations (C))

by

A.J.Hobson

UNIT 15.10 - ORDINARY DIFFERENTIAL EQUATIONS 10

SIMULTANEOUS EQUATIONS (C)

15.10.1 MATRIX METHODS FOR NON-HOMOGENEOUS SYSTEMS

In Units 15.5, 15.6 and 15.7 , it was seen that, for a single linear differential equation with constant coefficients, the general solution is made up of a particular integral and a complementary function (the latter being the general solution of the corresponding homogeneous differential equation).

In the work which follows, a similar principle is applied to a pair of simultaneous non-homogeneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2 + f(t),\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2 + g(t).\end{aligned}$$

The method will be illustrated by the following example, in which $f(t) \equiv 0$:

EXAMPLE

Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, - - - - - - - - - - - - - (1)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -4x_1 - 5x_2 + g(t), - - - - - (2)\end{aligned}$$

where $g(t)$ is (a) t, (b) e^{2t} (c) $\sin t$, (d) e^{-t}.

Solutions

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t),$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t) \ \text{ where } \ \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \ \text{ and } \ \mathrm{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Secondly, we consider the corresponding "homogeneous" system

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX},$$

for which the characteristic equation is

$$\begin{vmatrix} 0-\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0,$$

and gives

$$\lambda(5+\lambda)+4 = 0 \ \text{ or } \ \lambda^2+5\lambda+4 = 0 \ \text{ or } \ (\lambda+1)(\lambda+4) = 0.$$

(iii) The eigenvectors of M are obtained from the homogeneous equations

$$\begin{aligned} -\lambda k_1 + k_2 &= 0, \\ -4k_1 - (5+\lambda)k_2 &= 0. \end{aligned}$$

Hence, in the case when $\lambda = -1$, we solve

$$\begin{aligned} k_1 + k_2 &= 0, \\ -4k_1 - 4k_2 &= 0, \end{aligned}$$

and these are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -1$.

Also, when $\lambda = -4$, we solve

$$\begin{aligned} 4k_1 + k_2 &= 0, \\ -4k_1 - k_2 &= 0 \end{aligned}$$

which are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -4$.

The complementary function may now be written in the form

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t},$$

where A and B are arbitrary constants.

(iv) In order to obtain a particular integral for the equation

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t),$$

we note the second term on the right hand side and investigate a trial solution of a similar form. The three cases in this example are as follows:

(a) $g(t) \equiv t$

$$\text{Trial solution } \mathrm{X} = \mathrm{P} + \mathrm{Q}t,$$

where P and Q are constant matrices of order 2×1.

We require that

$$\mathrm{Q} = \mathrm{M}(\mathrm{P} + \mathrm{Q}t) + \mathrm{N}t,$$

whereupon, equating the matrix coefficients of t and the constant matrices,

$$\mathrm{MQ} + \mathrm{N} = \mathbf{0} \text{ and } \mathrm{Q} = \mathrm{MP},$$

giving

$$\mathrm{Q} = -\mathrm{M}^{-1}\mathrm{N} \quad \text{and} \quad \mathrm{P} = \mathrm{M}^{-1}\mathrm{Q}.$$

Thus, using

$$\mathrm{M}^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix},$$

we obtain

$$\mathrm{Q} = -\frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

and

$$\mathrm{P} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} t.$$

(b) $g(t) \equiv e^{2t}$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P}e^{2t}$$

We require that

$$2\mathrm{P}e^{2t} = \mathrm{MP}e^{2t} + \mathrm{N}e^{2t}.$$

That is,

$$2\mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The matrix, P, may now be determined from the formula

$$(2\mathrm{I} - \mathrm{M})\mathrm{P} = \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix} .\mathrm{P} = \mathrm{N}.$$

Hence,

$$\mathrm{P} = \frac{1}{18}\begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix} e^{2t}.$$

(c) $g(t) \equiv \sin t$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P}\sin t + \mathrm{Q}\cos t.$$

We require that

$$\mathrm{P}\cos t - \mathrm{Q}\sin t = \mathrm{M}(\mathrm{P}\sin t + \mathrm{Q}\cos t) + \mathrm{N}\sin t.$$

Equating the matrix coefficients of $\cos t$ and $\sin t$,

$$\mathrm{P} = \mathrm{MQ} \quad \text{and} \quad -\mathrm{Q} = \mathrm{MP} + \mathrm{N},$$

which means that

$$-\mathrm{Q} = \mathrm{M}^2\mathrm{Q} + \mathrm{N} \quad \text{or} \quad (\mathrm{M}^2 + \mathrm{I})\mathrm{Q} = -\mathrm{N}.$$

Thus,

$$Q = -(M^2 + I)^{-1}N,$$

where

$$M^2 + I = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 20 & 22 \end{bmatrix}$$

and, hence,

$$Q = -\frac{1}{34} \begin{bmatrix} 22 & 5 \\ -20 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

Also,

$$P = MQ = \frac{1}{34} \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin t + \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix} \cos t.$$

(d) $g(t) \equiv e^{-t}$

In this case, the function, $g(t)$, is already included in the complementary function and it becomes necessary to assume a particular integral of the form

$$X = (P + Qt)e^{-t},$$

where P and Q are constant matrices of order 2×1.

We require that

$$Qe^{-t} - (P + Qt)e^{-t} = M(P + Qt)e^{-t} + Ne^{-t},$$

whereupon, equating the matrix coefficients of te^{-t} and e^{-t}, we obtain

$$-\mathrm{Q} = \mathrm{MQ} \quad \text{and} \quad \mathrm{Q} - \mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The first of these conditions shows that Q is an eigenvector of the matrix M corresponding the eigenvalue -1 and so, from earlier work,

$$\mathrm{Q} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any constant k.

Also,

$$(\mathrm{M} + \mathrm{I})\mathrm{P} = \mathrm{Q} - \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} p_1 + p_2 &= k, \\ -4p_1 - 4p_2 &= -k - 1. \end{aligned}$$

Using $p_1 + p_2 = k$ and $p_1 + p_2 = \frac{k+1}{4}$, we deduce that $k = \frac{1}{3}$ and that the matrix P is given by

$$\mathrm{P} = \begin{bmatrix} l \\ \frac{1}{3} - l \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + l\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any number, l.

Taking $l = 0$ for simplicity, a particular integral is therefore

$$X = \frac{1}{3}\left\{\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}t\right\}e^{-t}.$$

and the general solution is

$$X = A\begin{bmatrix}1\\-1\end{bmatrix}e^{-t} + B\begin{bmatrix}1\\-4\end{bmatrix}e^{-4t} + \frac{1}{3}\left\{\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}t\right\}e^{-t}.$$

Note:
In examples for which neither $f(t)$ nor $g(t)$ is identically equal to zero, the particular integral may be found by adding together the separate forms of particular integral for $f(t)$ and $g(t)$ and writing the system of differential equations in the form

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}_1 f(t) + \mathrm{N}_2 g(t),$$

where

$$\mathrm{N}_1 = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } \mathrm{N}_2 = \begin{bmatrix}0\\1\end{bmatrix}.$$

For instance, if $f(t) \equiv t$ and $g(t) \equiv e^{2t}$, the particular integral would take the form

$$\mathrm{X} = \mathrm{P} + \mathrm{Q}t + \mathrm{R}e^{2t},$$

where P, Q and R are matrices of order 2×1.

15.10.2 EXERCISES

1. Determine the general solutions of the following systems of simultaneous differential equations:

   (a)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + 3x_2 + 5t,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 3x_1 + x_2 + e^{3t}.\end{aligned}$$

(b)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 3x_1 + 2x_2 + t^2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 4x_1 + x_2 + e^{-2t}.\end{aligned}$$

2. Determine the complete solutions of the following systems of differential equations, subject to the conditions given:

(a)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 + 9x_2 + 3,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 11x_1 + x_2 + e^{10t},\end{aligned}$$

given that $x_1 = \frac{1}{225}$ and $x_2 = -\frac{1}{100}$ when $t = 0$.

(b)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2 + 2t^2 + t,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2,\end{aligned}$$

given that $x_1 = \frac{32}{27}$ and $x_2 = -\frac{12}{27}$ when $t = 0$.

(c)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2 + \sin t,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2 + \cos t,\end{aligned}$$

given that $x_1 = 0$ and $x_2 = 0$ when $t = 0$.

15.10.3 ANSWERS TO EXERCISES

1. (a)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t}+B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}+\frac{1}{32}\begin{bmatrix}-25\\15\end{bmatrix}+\frac{1}{8}\begin{bmatrix}5\\-15\end{bmatrix}t-\frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}e^{3t};$$

(b)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{5t}+B\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}+\frac{2}{125}\begin{bmatrix}41\\-84\end{bmatrix}+\frac{2}{25}\begin{bmatrix}-9\\16\end{bmatrix}t+\frac{1}{5}\begin{bmatrix}1\\-4\end{bmatrix}t^2+\frac{1}{7}\begin{bmatrix}2\\-5\end{bmatrix}e^{-2t}.$$

2. (a)

$$-\frac{7}{45}\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t}+\frac{13}{900}\begin{bmatrix}9\\11\end{bmatrix}e^{10t}+\frac{3}{100}\begin{bmatrix}1\\-11\end{bmatrix}+\frac{1}{180}\begin{bmatrix}0\\1\end{bmatrix}e^{10t}+\frac{1}{20}\begin{bmatrix}9\\11\end{bmatrix}te^{10t};$$

(b)

$$\left\{(2t+1)\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}+\frac{1}{27}\begin{bmatrix}5\\-12\end{bmatrix}+\frac{1}{27}\begin{bmatrix}1\\-22\end{bmatrix}t-\frac{2}{9}\begin{bmatrix}1\\2\end{bmatrix}t^2;$$

(c)

$$\frac{1}{145}\left\{e^{7t}\left(\begin{bmatrix}-1\\25\end{bmatrix}\cos 2t+\begin{bmatrix}-12\\-10\end{bmatrix}\sin 2t\right)+\begin{bmatrix}-17\\-10\end{bmatrix}\sin t+\begin{bmatrix}1\\-25\end{bmatrix}\cos t\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

16.1

LAPLACE TRANSFORMS 1
(Definitions and rules)

by

A.J.Hobson

UNIT 16.1 - LAPLACE TRANSFORMS 1 - DEFINITIONS AND RULES

16.1.1 INTRODUCTION

The theory of **"Laplace Transforms"** to be discussed in the following notes will be for the purpose of solving certain kinds of **"differential equation"**; that is, an equation which involves a derivative or derivatives.

The particular differential equation problems to be encountered will be limited to the two types listed below:

(a) Given the **"first order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

together with the value of x when $t = 0$ (that is, $x(0)$), determine a formula for x in terms of t, which does not include any derivatives.

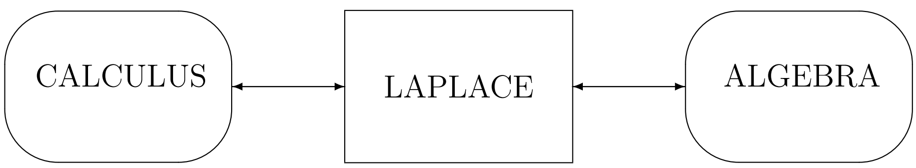
(b) Given the **"second order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

together with the values of x and $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $t = 0$ (that is, $x(0)$ and $x'(0)$), determine a formula for x in terms of t which does not include any derivatives.

Roughly speaking, the method of Laplace Transforms is used to convert a calculus problem (the differential equation) into an algebra problem (frequently an exercise on partial fractions and/or completing the square).

The solution of the algebra problem is then fed backwards through what is called an **"Inverse Laplace Transform"** and the solution of the differential equation is obtained.



The background to the development of Laplace Transforms would be best explained using certain other techniques of solving differential equations which may not have been part of earlier work. This background will therefore be omitted here.

DEFINITION
The Laplace Transform of a given function $f(t)$, defined for $t > 0$, is defined by the definite integral

$$\int_0^\infty e^{-st} f(t)\ \mathrm{d}t,$$

where s is an **arbitrary positive number**.

Notes
(i) The Laplace Transform is usually denoted by $L[f(t)]$ or $F(s)$, since the result of the definite integral in the definition will be an expression involving s.

(ii) Although s is an arbitrary positive number, it is occasionally necessary to assume that it is large enough to avoid difficulties in the calculations; (see the note to the second standard result below).

16.1.2 LAPLACE TRANSFORMS OF SIMPLE FUNCTIONS

The following is a list of standard results on which other Laplace Transforms will be based:

1. $f(t) \equiv t^n$.

$$F(s) = \int_0^\infty e^{-st} t^n\ \mathrm{d}t = I_n \ \text{ say}.$$

Hence,

$$I_n = \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\ \mathrm{d}t = \frac{n}{s}.I_{n-1},$$

using the fact that e^{-st} tends to zero much faster than any other function of t can tend to infinity. That is, a decaying exponential will always have the dominating effect.

We conclude that

$$I_n = \frac{n}{s}.\frac{(n-1)}{s}.\frac{(n-2)}{s}....\frac{2}{s}.\frac{1}{s}.I_0 = \frac{n!}{s^n}.I_0.$$

But,

$$I_0 = \int_0^\infty e^{-st}\ \mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}.$$

Thus,

$$L[t^n] = \frac{n!}{s^{n+1}}.$$

Note:
This result also shows that

$$L[1] = \frac{1}{s},$$

since $1 = t^0$.

2. $f(t) \equiv e^{-at}$.

$$F(s) = \int_0^\infty e^{-st}e^{-at}\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}\ \mathrm{d}t$$
$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}.$$

Hence,

$$L[e^{-at}] = \frac{1}{s+a}.$$

Note:
A slightly different form of this result, less commonly used in applications to science and engineering, is

$$L[e^{bt}] = \frac{1}{s-b};$$

but, to obtain this result by integration, we would need to assume that $s > b$ to ensure that $e^{-(s-b)t}$ is genuinely a **decaying** exponential.

3. $f(t) \equiv \cos at$.

$$F(s) = \int_0^\infty e^{-st}\cos at\ \mathrm{d}t = \left[\frac{e^{-st}\sin at}{a}\right]_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at\ \mathrm{d}t$$

using Integration by Parts, once.

Using Integration by Parts a second time,

$$F(s) = 0 + \frac{s}{a}\left\{\left[-\frac{e^{-st}\cos at}{a}\right]_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos at\ \mathrm{d}t\right\},$$

which gives

$$F(s) = \frac{s}{a^2} - \frac{s^2}{a^2}.F(s).$$

That is,

$$F(s) = \frac{s}{s^2+a^2}.$$

In other words,

$$L[\cos at] = \frac{s}{s^2+a^2}.$$

4. $f(t) \equiv \sin at$.
   The method is similar to that for $\cos at$, and we obtain
$$L[\sin at] = \frac{a}{s^2 + a^2}.$$

16.1.3 ELEMENTARY LAPLACE TRANSFORM RULES

The following list of results is of use in finding the Laplace Transform of a function which is made up of **basic** functions, such as those encountered in the previous section.

1. LINEARITY

If A and B are constants, then
$$L[Af(t) + Bg(t)] = AL[f(t)] + BL[g(t)].$$

Proof:

This follows easily from the linearity of an integral.

EXAMPLE

Determine the Laplace Transform of the function,
$$2t^5 + 7\cos 4t - 1.$$

Solution
$$L[2t^5 + 7\cos 4t - 1] = 2.\frac{5!}{s^6} + 7.\frac{s}{s^2 + 4^2} - \frac{1}{s} = \frac{240}{s^6} + \frac{7s}{s^2 + 16} - \frac{1}{s}.$$

2. THE TRANSFORM OF A DERIVATIVE

The two results which follow are of special use when solving first and second order differential equations. We shall begin by discussing them in relation to an arbitrary function, $f(t)$; then we shall restate them in the form which will be needed for solving differential equations.

(a)
$$L[f'(t)] = sL[f(t)] - f(0).$$

Proof:
$$L[f'(t)] = \int_0^\infty e^{-st}f'(t)\ \mathrm{d}t = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)\ \mathrm{d}t$$

using integration by parts.

Thus,
$$L[f'(t)] = -f(0) + sL[f(t)],$$

as required.

(b)

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0).$$

Proof:
Treating $f''(t)$ as the first derivative of $f'(t)$, we have

$$L[f''(t)] = sL[f'(t)] - f'(0),$$

which gives the required result on substituting from (a) the expression for $L[f'(t)]$.
Alternative Forms (Using $L[x(t)] = X(s)$)**:**
(i)

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

(ii)

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \text{ or } s[sX(s) - x(0)] - x'(0).$$

3. THE (First) SHIFTING THEOREM

$$L[e^{-at}f(t)] = F(s+a).$$

Proof:

$$L[e^{-at}f(t)] = \int_0^\infty e^{-st}e^{-at}f(t)\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}f(t)\ \mathrm{d}t,$$

which can be regarded as the effect of replacing s by $s+a$ in $L[f(t)]$. In other words, $F(s+a)$.
Notes:
(i) Sometimes, this result is stated in the form

$$L[e^{bt}f(t)] = F(s-b)$$

but, in science and engineering, the exponential is more likely to be a **decaying** exponential.
(ii) There is, in fact, a Second Shifting Theorem, encountered in more advanced courses; but we do not include it in this Unit (see Unit 16.5).
EXAMPLE
Determine the Laplace Transform of the function, $e^{-2t}\sin 3t$.
Solution
First of all, we note that

$$L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.$$

Replacing s by $(s+2)$ in this result, the First Shifting Theorem gives

$$L[e^{-2t}\sin 3t] = \frac{3}{(s+2)^2+9}.$$

4. MULTIPLICATION BY t

$$L[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}[F(s)].$$

Proof:

It may be shown that

$$\frac{\mathrm{d}}{\mathrm{d}s}[F(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\mathrm{d}t = \int_0^\infty -te^{-st}f(t)\ \mathrm{d}t = -\ L[tf(t)].$$

EXAMPLE

Determine the Laplace Transform of the function,

$$t\cos 7t.$$

Solution

$$L[t\cos 7t] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2+7^2}\right] = -\ \frac{(s^2+7^2).1 - s.2s}{(s^2+7^2)^2} = \frac{s^2-49}{(s^2+49)^2}.$$

THE USE OF A TABLE OF LAPLACE TRANSFORMS AND RULES

For the purposes of these Units, the following **brief** table may be used to determine the Laplace Transforms of functions of t without having to use integration:

$f(t)$	$L[f(t)] = F(s)$
K (a constant)	$\frac{K}{s}$
e^{-at}	$\frac{1}{s+a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\text{sinh}at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\text{cosh}at$	$\frac{s}{s^2-a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
$\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$

16.1.4 FURTHER LAPLACE TRANSFORM RULES

1.
$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

2.
$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \quad \text{or} \quad s[sX(s) - x(0)] - x'(0).$$

3. **The Initial Value Theorem**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s),$$

provided that the indicated limits exist.

4. **The Final Value Theorem**

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s),$$

provided that the indicated limits exist.

5. **The Convolution Theorem**

$$L\left[\int_0^t f(T)g(t-T)\,\mathrm{d}T\right] = F(s)G(s).$$

16.1.5 EXERCISES

1. Use a table the table of Laplace Transforms to find $L[f(t)]$ in the following cases:

(a)

$$3t^2 + 4t - 1;$$

(b)

$$t^3 + 3t^2 + 3t + 1 \quad (\equiv (t+1)^3);$$

(c)

$$2e^{5t} - 3e^t + e^{-7t};$$

(d)

$$2\sin 3t - 3\cos 2t;$$

(e)

$$t\sin 6t;$$

(f)

$$t(e^t + e^{-2t});$$

(g)

$$\frac{1}{2}(1 - \cos 2t) \quad (\equiv \sin^2 t).$$

2. Using the First Shifting Theorem, obtain the Laplace Transforms of the following functions of t:

(a)

$$e^{-3t}\cos 5t;$$

(b)

$$t^2 e^{2t};$$

(c)

$$e^{-2t}\left(2t^3+3t-2\right);$$

(d)

$$\cosh 2t.\sin t;$$

(e)

$$e^{-at}f'(t),$$

where $L[f(t)] = F(s)$.

3. (a) If

$$x = t^3 e^{-t},$$

determine the Laplace Transform of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ without differentiating x more than once with respect to t.

(b) If

$$\frac{\mathrm{d}x}{\mathrm{d}t} + x = e^t,$$

where $x(0) = 0$, show that

$$X(s) = \frac{1}{s^2-1}.$$

4. Verify the Initial and Final Value Theorems for the function

$$f(t) = te^{-3t}.$$

16.1.6 ANSWERS TO EXERCISES

1. (a)

$$\frac{6}{s^3} + \frac{4}{s^2} - \frac{1}{s};$$

(b)

$$\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s};$$

(c)

$$\frac{2}{s-5} - \frac{3}{s-1} + \frac{1}{s+7};$$

(d)

$$\frac{6}{s^2+9} - \frac{3s}{s^2+4};$$

(e)

$$\frac{12s}{(s^2+36)^2};$$

(f)

$$\frac{1}{(s-1)^2}+\frac{1}{(s+2)^2};$$

(g)

$$\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right].$$

2. (a)

$$\frac{s+3}{(s+3)^2+25};$$

(b)

$$\frac{2}{(s-2)^3};$$

(c)

$$\frac{12}{(s+2)^4}+\frac{3}{(s+2)^2}-\frac{2}{s+2};$$

(d)

$$\frac{1}{2}\left[\frac{1}{(s-2)^2+1}+\frac{1}{(s+2)^2+1}\right];$$

(e)

$$(s+a)F(s+a)-f(0).$$

3. (a)

$$\frac{6s^2}{(s+1)^4};$$

(b) On the left hand side, use the formula for $L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]$.

4.

$$\lim_{t\to 0} f(t)=0 \quad \text{and} \quad \lim_{t\to\infty} f(t)=0.$$

"JUST THE MATHS"

UNIT NUMBER

16.2

LAPLACE TRANSFORMS 2
(Inverse Laplace Transforms)

by

A.J.Hobson

UNIT 16.2 - LAPLACE TRANSFORMS 2
INVERSE LAPLACE TRANSFORMS

In order to solve differential equations, we now examine how to determine a function of the variable, t, whose Laplace Transform is already known.

16.2.1 THE DEFINITION OF AN INVERSE LAPLACE TRANSFORMS

A function of t, whose Laplace Transform is the given expression, $F(s)$, is called the **"Inverse Laplace Transform"** of $f(t)$ and may be denoted by the symbol

$$L^{-1}[F(s)].$$

Notes:
(i) Since two functions which coincide for $t > 0$ will have the same Laplace Transform, we can determine the Inverse Laplace Transform of $F(s)$ only for **positive** values of t.

(ii) Inverse Laplace Transforms are **linear** since

$$L^{-1}\left[AF(s) + BG(s)\right]$$

is a function of t whose Laplace Transform is

$$AF(s) + BG(s);$$

and, by the linearity of Laplace Transforms, discussed in Unit 16.1, such a function is

$$AL^{-1}[F(s)] + BL^{-1}[G(s)].$$

16.2.2 METHODS OF DETERMINING AN INVERSE LAPLACE TRANSFORM

The type of differential equation to be encountered in simple practical problems usually lead to Laplace Transforms which are **"rational functions of s"**. We shall restrict the discussion to such cases, as illustrated in the following examples, where the table of standard Laplace Transforms is used whenever possible. The partial fractions are discussed in detail, but other, shorter, methods may be used if known (for example, the "Cover-up Rule" and "Keily's Method"; see Unit 1.9)

EXAMPLES

1. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{s^3} + \frac{4}{s-2}.$$

Solution

$$f(t) = \frac{3}{2}t^2 + 4e^{2t} \quad t > 0$$

2. Determine the Inverse Laplace Transform of

$$F(s) = \frac{2s+3}{s^2+3s} = \frac{2s+3}{s(s+3)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{2s+3}{s(s+3)} \equiv \frac{A}{s} + \frac{B}{s+3},$$

giving

$$2s+3 \equiv A(s+3) + Bs$$

Note:

Although the s of a Laplace Transform is an arbitrary **positive** number, we may temporarily ignore that in order to complete the partial fractions. Otherwise, entire partial fractions exercises would have to be carried out by equating coefficients of appropriate powers of s on both sides.

Putting $s = 0$ and $s = -3$ gives

$$3 = 3A \text{ and } -3 = -3B;$$

so that

$$A = 1 \text{ and } B = 1.$$

Hence,

$$F(s) = \frac{1}{s} + \frac{1}{s+3}$$

Finally,

$$f(t) = 1 + e^{-3t} \quad t > 0.$$

3. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{s^2+9}.$$

Solution

$$f(t) = \frac{1}{3}\sin 3t \quad t > 0.$$

4. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+2}{s^2+5}.$$

Solution

$$f(t) = \cos t\sqrt{5} + \frac{2}{\sqrt{5}}\sin t\sqrt{5} \quad t > 0.$$

5. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3s^2 + 2s + 4}{(s+1)(s^2+4)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{3s^2 + 2s + 4}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs + C}{s^2+4}.$$

That is,

$$3s^2 + 2s + 4 \equiv A(s^2+4) + (Bs+C)(s+1).$$

Substituting $s = -1$, we obtain

$$5 = 5A \text{ which implies that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$3 = A + B \text{ so that } B = 2.$$

Equating constant terms on both sides,

$$4 = 4A + C \text{ so that } C = 0.$$

We conclude that

$$F(s) = \frac{1}{s+1} + \frac{2s}{s^2+4}.$$

Hence,

$$f(t) = e^{-t} + 2\cos 2t \quad t > 0.$$

6. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s+2)^5}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{n!}{s^{n+1}}$, we obtain

$$f(t) = \frac{1}{24}t^4 e^{-2t} \quad t > 0.$$

7. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{(s-7)^2 + 9}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{a}{s^2+a^2}$, we obtain

$$f(t) = e^{7t} \sin 3t \quad t > 0.$$

8. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s}{s^2 + 4s + 13}.$$

Solution

The denominator will not factorise conveniently, so we **complete the square**, giving

$$F(s) = \frac{s}{(s+2)^2 + 9}.$$

In order to use the First Shifting Theorem, we must try to include $s+2$ in the numerator; so we write

$$F(s) = \frac{(s+2) - 2}{(s+2)^2 + 9} = \frac{s+2}{(s+2)^2 + 3^2} - \frac{2}{3}.\frac{3}{(s+2)^2 + 3^2}.$$

Hence,

$$f(t) = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t = \frac{1}{3}e^{-2t}\left[3\cos 3t - 2\sin 3t\right] \quad t > 0.$$

9. Determine the Inverse Laplace Transform of

$$F(s) = \frac{8(s+1)}{s(s^2 + 4s + 8)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{8(s+1)}{s(s^2 + 4s + 8)} \equiv \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}.$$

Mutiplying up, we obtain

$$8(s+1) \equiv A(s^2 + 4s + 8) + (Bs + C)s.$$

Substituting $s = 0$ gives

$$8 = 8A \text{ so that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B \text{ which gives } B = -1.$$

Equating coefficients of s on both sides,

$$8 = 4A + C \text{ which gives } C = 4.$$

Thus,

$$F(s) = \frac{1}{s} + \frac{-s + 4}{s^2 + 4s + 8}.$$

The quadratic denominator will not factorise conveniently, so we complete the square to give

$$F(s) = \frac{1}{s} + \frac{-s+4}{(s+2)^2+4},$$

which, on rearrangement, becomes

$$F(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+2^2} + \frac{6}{(s+2)^2+2^2}.$$

Thus, from the First Shifting Theorem,

$$f(t) = 1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t \quad t > 0.$$

10. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+10}{s^2-4s-12}.$$

Solution

This time, the denominator **will** factorise, into $(s+2)(s-6)$, and partial fractions give

$$\frac{s+10}{(s+2)(s-6)} \equiv \frac{A}{s+2} + \frac{B}{s-6}.$$

Hence,

$$s + 10 \equiv \mathrm{A}(s-6) + \mathrm{B}(s+2).$$

Putting $s = -2$,

$$8 = -8A \text{ giving } A = -1.$$

Putting $s = 6$,

$$16 = 8B \text{ giving } B = 2.$$

We conclude that

$$F(s) = \frac{-1}{s+2} + \frac{2}{s-6}.$$

Finally,

$$f(t) = -e^{-2t} + 2e^{6t} \quad t > 0.$$

However, if we did not factorise the denominator, a different form of solution could be obtained as follows:

$$F(s) = \frac{(s-2)+12}{(s-2)^2-4^2} = \frac{s-2}{(s-2)^2-4^2} + 3.\frac{4}{(s-2)^2+4^2}.$$

Hence,

$$f(t) = e^{2t}[\mathrm{cosh}4t + 3\mathrm{sinh}4t] \quad t > 0.$$

11. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s-1)(s+2)}.$$

Solution

The Inverse Laplace Transform of this function could certainly be obtained by using partial fractions, but we note here how it could be obtained from the Convolution Theorem.

Writing

$$F(s) = \frac{1}{(s-1)}.\frac{1}{(s+2)},$$

we obtain

$$f(t) = \int_0^t e^T.e^{-2(t-T)}\ \mathrm{d}T = \int_0^t e^{(3T-2t)}\ \mathrm{d}T = \left[\frac{e^{3T-2t}}{3}\right]_0^t.$$

That is,

$$f(t) = \frac{e^t}{3} - \frac{e^{-2t}}{3} \quad t > 0.$$

16.2.3 EXERCISES

Determine the Inverse Laplace Transforms of the following rational functions of s:

1. (a)

$$\frac{1}{(s-1)^2};$$

(b)

$$\frac{1}{(s+1)^2+4};$$

(c)

$$\frac{s+2}{(s+2)^2+9};$$

(d)

$$\frac{s-2}{(s-3)^3};$$

(e)

$$\frac{1}{(s^2+4)^2};$$

(f)

$$\frac{s+1}{s^2+2s+5};$$

(g)

$$\frac{s-3}{s^2-4s+5};$$

(h)

$$\frac{s-3}{(s-1)^2(s-2)};$$

(i)

$$\frac{5}{(s+1)(s^2-2s+2)};$$

(j)

$$\frac{2s-9}{(s-3)(s+2)};$$

(k)

$$\frac{3}{s(s^2+9)};$$

(l)

$$\frac{2s-1}{(s-1)(s^2+2s+2)}.$$

2. Use the Convolution Theorem to obtain the Inverse Laplace Transform of

$$\frac{s}{(s^2+1)^2}.$$

16.2.4 ANSWERS TO EXERCISES

1. (a)

$$te^t \quad t > 0;$$

(b)

$$\frac{1}{2}e^{-t}\sin 2t \quad t > 0;$$

(c)

$$e^{-2t}\cos 3t \quad t > 0;$$

(d)

$$e^{3t}\left[t + \frac{1}{2}t^2\right] \quad t > 0;$$

(e)

$$\frac{1}{16}[\sin 2t - 2t\cos 2t] \quad t > 0;$$

(f)

$$e^{-t}\cos 2t \quad t > 0;$$

(g)

$$e^{2t}[\cos t - \sin t] \quad t > 0;$$

(h)

$$2te^{t} + e^{t} - e^{2t} \quad t > 0;$$

(i)

$$e^{-t} + e^{t}[2\sin t - \cos t] \quad t > 0;$$

(j)

$$\frac{1}{5}[13e^{-2t} - 3e^{3t}] \quad t > 0;$$

(k)

$$\frac{1}{3}[1 - \cos 3t] \quad t > 0;$$

(l)

$$\frac{1}{5}[e^{t} - e^{-t}\cos t + 8e^{-t}\sin t] \quad t > 0.$$

2.

$$\frac{1}{2}t\sin t \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.3

LAPLACE TRANSFORMS 3
(Differential equations)

by

A.J.Hobson

UNIT 16.3 - LAPLACE TRANSFORMS 3 - DIFFERENTIAL EQUATIONS

16.3.1 EXAMPLES OF SOLVING DIFFERENTIAL EQUATIONS

In the work which follows, the problems considered will usually take the form of a linear differential equation of the second order with constant coefficients.

That is,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

However, the method will apply equally well to the corresponding first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t).$$

The technique will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 0,$$

given that $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 3] + 4[sX(s) - 3] + 13X(s) = 0.$$

Hence,

$$(s^2 + 4s + 13)X(s) = 3s + 12,$$

giving

$$X(s) \equiv \frac{3s + 12}{s^2 + 4s + 13}.$$

The denominator does not factorise, therefore we complete the square to obtain

$$X(s) \equiv \frac{3s + 12}{(s + 2)^2 + 9} \equiv \frac{3(s + 2) + 6}{(s + 2)^2 + 9} \equiv 3.\frac{s + 2}{(s + 2)^2 + 9} + 2.\frac{3}{(s + 2)^2 + 9}.$$

Thus,

$$x(t) = 3e^{-2t}\cos 3t + 2e^{-2t}\sin 3t \quad t > 0$$

or

$$x(t) = e^{-2t}[3\cos 3t + 2\sin 3t] \quad t > 0.$$

2. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 50\sin t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] - 4 + 6[sX(s) - 1] + 9X(s) = \frac{50}{s^2+1},$$

giving

$$(s^2 + 6s + 9)X(s) = \frac{50}{s^2+1} + s + 10.$$

Hint: Do <u>not</u> combine the terms on the right into a single fraction - it won't help !

Thus,

$$X(s) \equiv \frac{50}{(s^2+6s+9)(s^2+1)} + \frac{s+10}{s^2+6s+9}$$

or

$$X(s) \equiv \frac{50}{(s+3)^2(s^2+1)} + \frac{s+10}{(s+3)^2}.$$

Using the principles of partial fractions in the first term on the right,

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}.$$

Hence,

$$50 \equiv A(s^2+1) + B(s+3)(s^2+1) + (Cs+D)(s+3)^2.$$

Substituting $s = -3$,

$$50 = 10A \text{ giving } A = 5.$$

Equating coefficients of s^3 on both sides,

$$0 = B + C. \quad (1)$$

Equating the coefficients of s on both sides (we shall not need the s^2 coefficients in this example),

$$0 = B + 9C + 6D. \qquad (2)$$

Equating the constant terms on both sides,

$$50 = A + 3B + 9D = 5 + 3B + 9D. \qquad (3)$$

Putting $C = -B$ into (2), we obtain

$$-8B + 6D = 0, \qquad (4)$$

and we already have

$$3B + 9D = 45. \qquad (3)$$

These last two solve easily to give $B = 3$ and $D = 4$ so that $C = -3$.

We conclude that

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{5}{(s+3)^2} + \frac{3}{s+3} + \frac{-3s+4}{s^2+1}.$$

In addition to this, we also have

$$\frac{s+10}{(s+3)^2} \equiv \frac{s+3}{(s+3)^2} + \frac{7}{(s+3)^2} \equiv \frac{1}{s+3} + \frac{7}{(s+3)^2}.$$

The total for $X(s)$ is therefore given by

$$X(s) \equiv \frac{12}{(s+3)^2} + \frac{4}{s+3} - 3.\frac{s}{s^2+1} + 4.\frac{1}{s^2+1}.$$

Finally,

$$x(t) = 12te^{-3t} + 4e^{-3t} - 3\cos t + 4\sin t \quad t > 0.$$

3. Solve the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 3x = 4e^t,$$

given that $x = 1$ and $\frac{dx}{dt} = -2$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] + 2 + 4[sX(s) - 1] - 3X(s) = \frac{4}{s-1}.$$

This gives

$$(s^2+4s-3)X(s) = \frac{4}{s-1} + s + 2.$$

Therefore,

$$X(s) \equiv \frac{4}{(s-1)(s^2+4s-3)} + \frac{s+2}{s^2+4s-3}.$$

Applying the principles of partial fractions,

$$\frac{4}{(s-1)(s^2+4s-3)} \equiv \frac{A}{s-1} + \frac{Bs+C}{s^2+4s-3}.$$

Hence,

$$4 \equiv A(s^2+4s-3) + (Bs+C)(s-1).$$

Substituting $s=1$, we obtain

$$4 = 2A; \quad \text{that is,} \quad A = 2.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B, \quad \text{so that} \quad B = -2.$$

Equating constant terms on both sides,

$$4 = -3A - C, \quad \text{so that} \quad C = -10.$$

Thus, in total,

$$X(s) \equiv \frac{2}{s-1} + \frac{-s-8}{s^2+4s-3} \equiv \frac{2}{s-1} + \frac{-s-8}{(s+2)^2-7}$$

or

$$X(s) \equiv \frac{2}{s-1} - \frac{s+2}{(s+2)^2-7} - \frac{6}{(s+2)^2-7}.$$

Finally,

$$x(t) = 2e^t - e^{-2t}\text{cosh}t\sqrt{7} - \frac{6}{\sqrt{7}}e^{-2t}\text{sinh}t\sqrt{7} \quad t > 0.$$

16.3.2 THE GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

On some occasions, we may either be given no boundary conditions at all; or else the boundary conditions given do not tell us the values of $x(0)$ and $x'(0)$.

In such cases, we simply let $x(0) = A$ and $x'(0) = B$ to obtain a solution in terms of A and B called the **"general solution"**.

If any non-standard boundary conditions are provided, we then substitute them into the general solution to obtain particular values of A and B.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 0$$

and, hence, determine the particular solution in the case when $x(\frac{\pi}{2}) = -3$ and $x'(\frac{\pi}{2}) = 10$.

Solution

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - A) - B + 4X(s) = 0.$$

That is,

$$(s^2 + 4)X(s) = As + B.$$

Hence,

$$X(s) \equiv \frac{As + B}{s^2 + 4} \equiv A.\frac{s}{s^2 + 4} + B.\frac{1}{s^2 + 4}.$$

This gives

$$x(t) = A\cos 2t + \frac{B}{2}\sin 2t \quad t > 0;$$

but, since A and B are **arbitrary** constants, this may be written in the simpler form

$$x(t) = A\cos 2t + B\sin 2t \quad t > 0,$$

in which $\frac{B}{2}$ has been rewritten as B.

To apply the boundary conditions, we require also the formula for $x'(t)$, namely

$$x'(t) = -\ 2A\sin 2t + 2B\cos 2t.$$

Hence, $-\ 3 = -\ A$ and $10 = -\ 2B$ giving $A = 3$ and $B =$ - 5.

Therefore, the particular solution is

$$x(t) = 3\cos 2t - 5\sin 2t \quad t > 0.$$

16.3.3 EXERCISES

1. Solve the following differential equations subject to the conditions given:

   (a)
   $$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 5x = 0,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (b)
   $$4\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 0,$$
   given that $x(0) = 4$ and $x'(0) = 1$;

   (c)
   $$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 8x = 2t,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (d)
   $$\frac{d^2x}{dt^2} - 4x = 2e^{2t},$$
   given that $x(0) = 1$ and $x'(0) = 10.5$;

   (e)
   $$\frac{d^2x}{dt^2} + 4x = 3\cos^2 t,$$
   given that $x(0) = 1$ and $x'(0) = 2$.
   Hint: $\cos 2t \equiv 2\cos^2 t - 1$.

2. Determine the particular solution of the differential equation
   $$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = e^t(t - 3)$$
   in the case when $x(0) = 2$ and $x(3) = -1$.
   Hint:
   Since $x(0)$ is given, just let $x'(0) = B$ to obtain a solution in terms of B; then substitute the second boundary condition at the end.

16.3.4 ANSWERS TO EXERCISES

1. (a)

$$X(s) = \frac{3s-5}{s^2-2s+5},$$

giving

$$x(t) = e^t(3\cos 2t - \sin 2t) \quad t > 0;$$

(b)

$$X(s) = \frac{4}{s+\frac{1}{2}} + \frac{3}{(s+\frac{1}{2})^2},$$

giving

$$x(t) = 4e^{-\frac{1}{2}t} + 3te^{-\frac{1}{2}t} = e^{-\frac{1}{2}t}[4+3t] \quad t > 0;$$

(c)

$$X(s) = \frac{27}{12}.\frac{1}{s-2} + \frac{39}{48}.\frac{1}{s+4} - \frac{1}{4}.\frac{1}{s^2} - \frac{1}{16}.\frac{1}{s},$$

giving

$$x(t) = \frac{27}{12}e^{2t} + \frac{39}{48}e^{-4t} - \frac{1}{4}t - \frac{1}{16} \quad t > 0;$$

(d)

$$X(s) = \frac{\frac{1}{2}}{(s-2)^2} + \frac{3}{s-2} - \frac{2}{s+2},$$

giving

$$x(t) = \frac{1}{2}te^{2t} + 3e^{2t} - 2e^{-2t} \quad t > 0;$$

(e)

$$X(s) = \frac{3}{2}.\frac{s}{(s^2+4)^2} + \frac{3}{8}.\frac{1}{s} + \frac{5}{8}.\frac{s}{s^2+4} + \frac{2}{s^2+4},$$

giving

$$x(t) = \frac{3}{8}t\sin 2t + \frac{3}{8} + \frac{5}{8}\cos 2t + \sin 2t \quad t > 0.$$

2.

$$x(t) = 3e^t - te^t - 1 \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.4

LAPLACE TRANSFORMS 4
(Simultaneous differential equations)

by

A.J.Hobson

UNIT 16.4 - LAPLACE TRANSFORMS 4
SIMULTANEOUS DIFFERENTIAL EQUATIONS

16.4.1 AN EXAMPLE OF SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In this Unit, we shall consider a pair of differential equations involving an independent variable, t, such as a time variable, and two dependent variables, x and y, such as electric currents or linear displacements.

The general format is as follows:

$$\begin{aligned} a_1\frac{\mathrm{d}x}{\mathrm{d}t} + b_1\frac{\mathrm{d}y}{\mathrm{d}t} + c_1x + d_1y &= f_1(t), \\ a_2\frac{\mathrm{d}x}{\mathrm{d}t} + b_2\frac{\mathrm{d}y}{\mathrm{d}t} + c_2x + d_2y &= f_2(t). \end{aligned}$$

To solve these equations simultaneously, we take the Laplace Transform of each equation obtaining two simultaneous algebraic equations from which we may determine $X(s)$ and $Y(s)$, the Laplace Transforms of $x(t)$ and $y(t)$ respectively.

EXAMPLE

Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= e^t, \\ \frac{\mathrm{d}x}{\mathrm{d}t} - 2y &= 1 + t, \end{aligned}$$

given that $x(0) = 1$ and $y(0) = 2$.

Solution

Taking the Laplace Transforms of the differential equations,

$$sY(s) - 2 + 2X(s) = \frac{1}{s-1},$$

$$sX(s) - 1 - 2Y(s) \quad = \quad \frac{1}{s} + \frac{1}{s^2}.$$

That is,

$$\begin{aligned} 2X(s) + sY(s) &= \frac{1}{s-1} + 2, && (1) \\ sX(s) - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2} + 1. && (2) \end{aligned}$$

Using $(1) \times 2 + (2) \times s$, we obtain

$$(4 + s^2)X(s) = \frac{2}{s-1} + 4 + 1 + \frac{1}{s} + s.$$

Hence,

$$X(s) = \frac{2}{(s-1)(s^2+4)} + \frac{5}{s^2+4} + \frac{1}{s(s^2+4)} + \frac{s}{s^2+4}.$$

Applying the methods of partial fractions, this gives

$$X(s) = \frac{2}{5}.\frac{1}{s-1} + \frac{7}{20}.\frac{s}{s^2+4} + \frac{23}{5}.\frac{1}{s^2+4} + \frac{1}{4}.\frac{1}{s}.$$

Thus,

$$x(t) = \frac{2}{5}e^t + \frac{1}{4} + \frac{7}{20}\cos 2t + \frac{23}{10}\sin 2t \quad t > 0.$$

We could now start again by eliminating x from equations (1) and (2) in order to calculate y, and this is often necessary; but, since

$$2y = \frac{\mathrm{d}x}{\mathrm{d}t} - 1 - t$$

in the current example,

$$y(t) = \frac{1}{5}e^t - \frac{1}{2} - \frac{7}{20}\sin 2t + \frac{23}{10}\cos 2t - \frac{t}{2} \quad t > 0.$$

16.4.2 EXERCISES

Use Laplace Transforms to solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + 2y &= e^{-t}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} + 3y &= x, \end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

2.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} - y &= \sin t, \\ \frac{\mathrm{d}y}{\mathrm{d}t} + x &= \cos t, \end{aligned}$$

given that $x = 3$ and $y = 4$ when $t = 0$.

3.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 3y &= 1, \\ \frac{\mathrm{d}y}{\mathrm{d}t} - x + 2y &= e^{-2t}, \end{aligned}$$

given that $x = 0$ and $y = 0$ when $t = 0$.

4.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 2y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 8x, \end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

5.

$$\begin{aligned} 10\frac{\mathrm{d}x}{\mathrm{d}t} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 6x + 5y &= 0, \\ 2\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} + 2x + y &= 2e^{-t}, \end{aligned}$$

given that $x = 2$ and $y = -1$ when $t = 0$.

6.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 6y &= 0, \\ 5\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 6x &= 0, \end{aligned}$$

given that $x = 3$ and $y = 0$ when $t = 0$.

7.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 2y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2z, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 2x, \end{aligned}$$

given that $x = 1$, $y = 0$ and $z = -1$ when $t = 0$.

16.4.3 ANSWERS TO EXERCISES

1.

$$x = (2t+1)e^{-t} \quad \text{and} \quad y = te^{-t}.$$

2.

$$x = (t+4)\sin t + 3\cos t \quad \text{and} \quad y = (t+4)\cos t - 3\sin t.$$

3.

$$x = 2 - e^{-2t}[1 + \sqrt{3}\sinh t\sqrt{3} + \cosh t\sqrt{3}]$$

and

$$y = 1 - e^{-2t}\left[\cosh t\sqrt{3} + \frac{1}{\sqrt{3}}\sinh t\sqrt{3}\right].$$

4.

$$x = \sinh 4t \quad \text{and} \quad y = 2\cosh 4t.$$

5.

$$x = 4\cos t - 2e^{-t} \quad \text{and} \quad y = e^{-t} - 2\cos t.$$

6.

$$x = 2e^{-t} + e^{-2t} \quad \text{amd} \quad y = e^{-t} - e^{-2t}.$$

7.

$$x = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} + \cos t\sqrt{3}\right],$$

$$y = \frac{-2}{\sqrt{3}}e^{-t}\sin t\sqrt{3}$$

and

$$z = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} - \cos t\sqrt{3}\right].$$

"JUST THE MATHS"

UNIT NUMBER

16.5

LAPLACE TRANSFORMS 5
(The Heaviside step function)

by

A.J.Hobson

UNIT 16.5 - LAPLACE TRANSFORMS 5

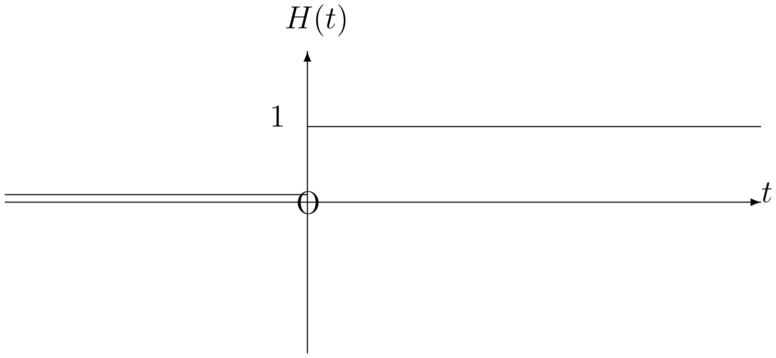
THE HEAVISIDE STEP FUNCTION

16.5.1 THE DEFINITION OF THE HEAVISIDE STEP FUNCTION

The Heaviside Step Function, $H(t)$, is defined by the statements

$$H(t) = \begin{cases} 0 & \text{for } t < 0; \\ 1 & \text{for } t > 0. \end{cases}$$

Note:
$H(t)$ is undefined when $t = 0$.

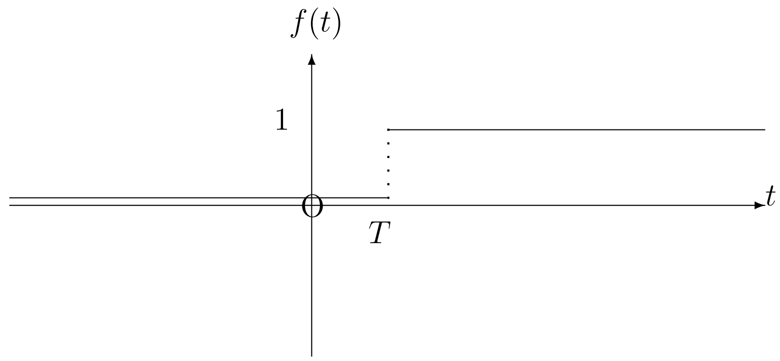


EXAMPLE

Express, in terms of $H(t)$, the function, $f(t)$, given by the statements

$$f(t) = \begin{cases} 0 & \text{for } t < T; \\ 1 & \text{for } t > T. \end{cases}$$

Solution



Clearly, $f(t)$ is the same type of function as $H(t)$, but we have effectively moved the origin to the point $(T, 0)$. Hence,

$$f(t) \equiv H(t - T).$$

Note:
The function $H(t-T)$ is of importance in constructing what are known as **"pulse functions"** (see later).

16.5.2 THE LAPLACE TRANSFORM OF H(t - T)

From the definition of a Laplace Transform,

$$L[H(t - T)] = \int_0^\infty e^{-st}H(t - T)\ \mathrm{d}t$$

$$= \int_0^T e^{-st}.0\ \mathrm{d}t + \int_T^\infty e^{-st}.1\ \mathrm{d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_T^\infty = \frac{e^{-sT}}{s}.$$

Note:
In the special case when $T = 0$, we have

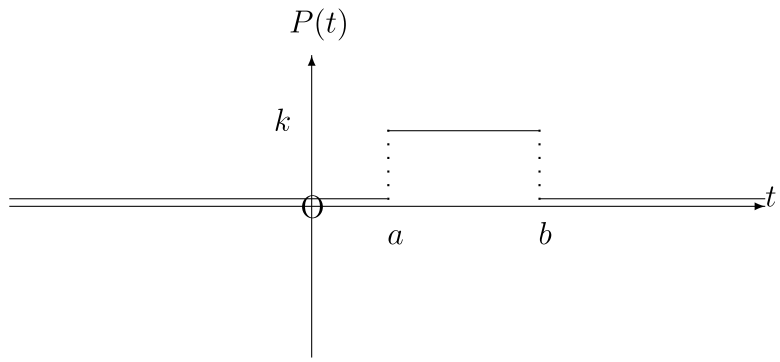
$$L[H(t)] = \frac{1}{s},$$

which can be expected since $H(t)$ and 1 are identical over the range of integration.

16.5.3 PULSE FUNCTIONS

If $a < b$, a **"rectangular pulse"**, $P(t)$, of duration, $b - a$, and magnitude, k, is defined by the statements,

$$P(t) = \begin{cases} k & \text{for } a < t < b; \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$



We can show that, in terms of Heaviside functions, the above pulse may be represented by

$$P(t) \equiv k[H(t-a) - H(t-b)].$$

Proof:

(i) If $t < a$, then $H(t-a) = 0$ and $H(t-b) = 0$. Hence, the above right-hand side $= 0$.

(ii) If $t > b$, then $H(t-a) = 1$ and $H(t-b) = 1$. Hence, the above right-hand side $= 0$.

(iii) If $a < t < b$, then $H(t-a) = 1$ and $H(t-b) = 0$. Hence, the above right-hand side $= k$.

EXAMPLE

Determine the Laplace Transform of a pulse, $P(t)$, of duration, $b - a$, having magnitude, k.

Solution

$$L[P(t)] = k\left[\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right] = k.\frac{e^{-sa} - e^{-sb}}{s}.$$

Notes:

(i) The **"strength"** of the pulse, described above, is defined as the area of the rectangle with base, $b - a$, and height, k. That is,

$$\text{strength} = k(b - a).$$

(ii) In general, the expression,

$$[H(t - a) - H(t - b)]f(t),$$

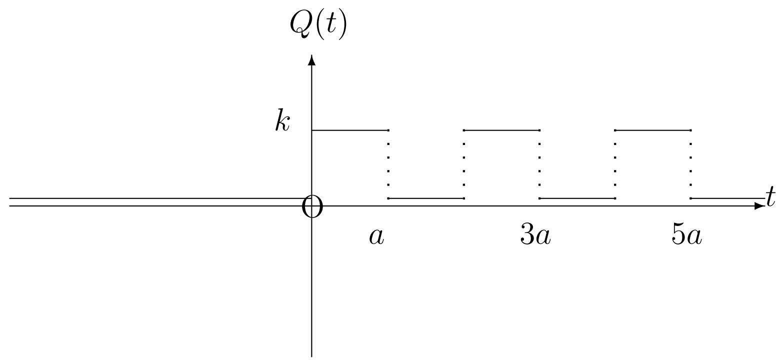
may be considered to **"switch on"** the function, $f(t)$, between $t = a$ and $t = b$ but **"switch off"** the function, $f(t)$, when $t < a$ or $t > b$.

(iii) Similarly, the expression,

$$H(t - a)f(t),$$

may be considered to **"switch on"** the function, $f(t)$, when $t > a$ but **"switch off"** the function, $f(t)$, when $t < a$.

For example, the train of rectangular pulses, $Q(t)$, in the following diagram:



may be represented by the function

$$Q(t) \equiv k\left\{[H(t) - H(t - a)] + [H(t - 2a) - H(t - 3a)] + [H(t - 4a) - H(t - 5a)] + ...\right\}.$$

16.5.4 THE SECOND SHIFTING THEOREM

THEOREM

$$L[H(t-T)f(t-T)] = e^{-sT}L[f(t)].$$

Proof:

Left-hand side =

$$\int_0^\infty e^{-st}H(t-T)f(t-T)\ \mathrm{d}t$$
$$= \int_0^T 0\ \mathrm{d}t + \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t$$
$$= \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t.$$

Making the substitution $u = t - T$, we obtain

$$\int_0^\infty e^{-s(u+T)}f(u)\ \mathrm{d}u$$
$$= e^{-sT}\int_0^\infty e^{-su}f(u)\ \mathrm{d}u = e^{-sT}L[f(t)].$$

EXAMPLES

1. Express, in terms of Heaviside functions, the function

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t > 1; \\ 0 & \text{for } 0 < t < 1. \end{cases}$$

and, hence, determine its Laplace Transform.

Solution

For values of $t > 0$, we may write

$$f(t) = (t-1)^2H(t-1).$$

Therefore, using $T = 1$ in the second shifting theorem,

$$L[f(t)] = e^{-s}L[t^2] = e^{-s}.\frac{2}{s^3}.$$

2. Determine the inverse Laplace Transform of the expression

$$\frac{e^{-7s}}{s^2+4s+5}.$$

Solution

First, we find the inverse Laplace Transform of the expression,

$$\frac{1}{s^2+4s+5} \equiv \frac{1}{(s+2)^2+1}.$$

From the first shifting theorem, this will be the function

$$e^{-2t}\sin t, \quad t > 0.$$

From the second shifting theorem, the required function will be

$$H(t-7)e^{-2(t-7)}\sin(t-7), \quad t > 0.$$

16.5.5 EXERCISES

1. (a) For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} e^{-t} & \text{for } 0 < t < 3; \\ 0 & \text{for } t > 3. \end{cases}$$

   (b) Determine the Laplace Transform of the function, $f(t)$, in part (a).

2. For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} f_1(t) & \text{for } 0 < t < a; \\ f_2(t) & \text{for } t > a. \end{cases}$$

3. For values of $t > 0$, express the following functions in terms of Heaviside functions:

   (a)

$$f(t) = \begin{cases} t^2 & \text{for } 0 < t < 2; \\ 4t & \text{for } t > 2. \end{cases}$$

   (b)

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi; \\ \sin 2t & \text{for } \pi < t < 2\pi; \\ \sin 3t & \text{for } t > 2\pi. \end{cases}$$

4. Use the second shifting theorem to determine the Laplace Transform of the function,

$$f(t) \equiv t^3 H(t-1).$$

Hint:
Write $t^3 \equiv [(t-1)+1]^3$.

5. Determine the inverse Laplace Transforms of the following:

(a)

$$\frac{e^{-2s}}{s^2};$$

(b)

$$\frac{8e^{-3s}}{s^2+4};$$

(c)

$$\frac{se^{-2s}}{s^2+3s+2};$$

(d)

$$\frac{e^{-3s}}{s^2-2s+5}.$$

6. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = H(t-2),$$

given that $x=0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t=0$.

16.5.6 ANSWERS TO EXERCISES

1. (a)

$$e^{-t}[H(t) - H(t-3)];$$

(b)

$$L[f(t)] = \frac{1-e^{-3(s+1)}}{s+1}.$$

2.

$$f(t) \equiv f_1(t)[H(t) - H(t-a)] + f_2(t)H(t-a)].$$

3. (a)

$$f(t) \equiv t^2[H(t) - H(t-2)] + 4tH(t-2);$$

(b)

$$f(t) \equiv \sin t[H(t) - H(t-\pi)] + \sin 2t[H(t-\pi) - H(t-2\pi)] + \sin 3t[H(t-2\pi).$$

4.

$$L[f(t)] = \left[\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right] e^{-s}.$$

5. (a)

$$H(t-2)(t-2);$$

(b)

$$4H(t-3)\sin 2(t-3);$$

(c)

$$H(t-2)[2e^{-2(t-2)} - e^{-(t-2)};$$

(d)

$$\frac{1}{2}H(t-3)e^{(t-3)}\sin 2(t-3).$$

6.

$$x = \frac{1}{2}\sin 2t + \frac{1}{4}H(t-2)[1 - \cos 2(t-2)].$$

1882

"JUST THE MATHS"

UNIT NUMBER

16.6

LAPLACE TRANSFORMS 6
(The Dirac unit impulse function)

by

A.J.Hobson

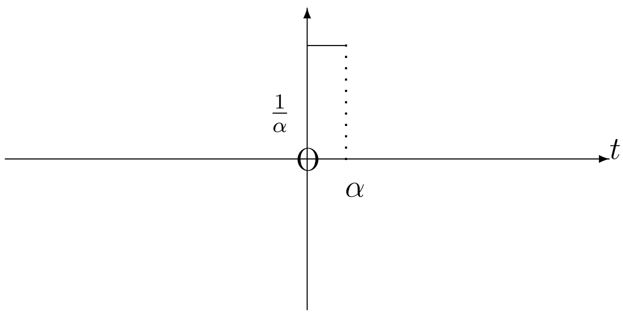
UNIT 16.6 - LAPLACE TRANSFORMS 6
THE DIRAC UNIT IMPULSE FUNCTION

16.6.1 THE DEFINITION OF THE DIRAC UNIT IMPULSE FUNCTION

A pulse of large magnitude, short duration and finite strength is called an **"impulse"**. In particular, a **"unit impulse"** is an impulse of strength 1.

ILLUSTRATION

Consider a pulse, of duration α, between $t = 0$ and $t = \alpha$, having magnitude, $\frac{1}{\alpha}$. The strength of the pulse is then 1.



From Unit 16.5, this pulse is given by

$$\frac{H(t) - H(t-\alpha)}{\alpha}.$$

If we now allow α to tend to zero, we obtain a unit impulse located at $t = 0$. This leads to the following definition:

DEFINITION 2

The **"Dirac unit impulse function"** , $\delta(t)$ is defined to be an impulse of unit strength located at $t = 0$. It is given by

$$\delta(t) = \lim_{\alpha \to 0} \frac{H(t) - H(t-\alpha)}{\alpha}.$$

Notes:

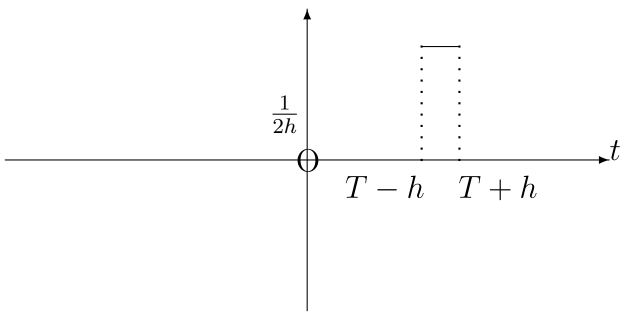
(i) An impulse of unit strength located at $t = T$ is represented by $\delta(t - T)$.

(ii) An alternative definition of the function $\delta(t - T)$ is as follows:

$$\delta(t - T) = \begin{cases} 0 & \text{for } t \neq T; \\ \infty & \text{for } t = T. \end{cases}$$

and

$$\lim_{h \to 0} \int_{T-h}^{T+h} \delta(t - T) \text{ d}t = 1.$$



THEOREM

$$\int_a^b f(t)\delta(t - T) \text{ d}t = f(T) \text{ if } a < T < b.$$

Proof:

Since $\delta(t - T)$ is equal to zero everywhere except at $t = T$, the left-hand side of the above formula reduces to

$$\lim_{h \to 0} \int_{T-h}^{T+h} f(t)\delta(t - T) \text{ d}t.$$

But, in the small interval from $T - h$ to $T + h$, $f(t)$ is approximately constant and equal to $f(T)$. Hence, the left-hand side may be written

$$f(T) \left[\lim_{h \to 0} \int_{T-h}^{T+h} \delta(T - T) \text{ d}t \right],$$

which reduces to $f(T)$, using note (ii) in the definition of the Dirac unit impulse function.

16.6.2 THE LAPLACE TRANSFORM OF THE DIRAC UNIT IMPULSE FUNCTION

RESULT

$$L[\delta(t-T)] = e^{-sT};$$

and, in particular,

$$L[\delta(t)] = 1.$$

Proof:

From the definition of a Laplace Transform,

$$L[\delta(t-T) = \int_0^\infty e^{-st}\delta(t-T)\ \mathrm{d}t.$$

But, from the Theorem discussed above, with $f(t) = e^{-st}$, we have

$$L[\delta(t-T)] = e^{-sT}.$$

EXAMPLES

1. Solve the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = \delta(t),$$

given that $x = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$3sX(s) + 4X(s) = 1.$$

That is,

$$X(s) = \frac{1}{3s+4} = \frac{1}{3}.\frac{1}{s+\frac{4}{3}}.$$

Hence,

$$x(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

2. Show that, for any function, $f(t)$,

$$\int_0^\infty f(t)\delta'(t-a)\ \mathrm{d}t = -f'(a).$$

Solution
Using Integration by Parts, the left-hand side of the formula may be written

$$[f(t)\delta(t-a)]_0^\infty - \int_0^\infty f'(t)\delta(t-a)\ \mathrm{d}t.$$

The first term of this reduces to zero, since $\delta(t-a)$ is equal to zero except when $t = a$. The required result follows from the Theorem discussed earlier, with $T = a$.

16.6.3 TRANSFER FUNCTIONS

In scientific applications, the solution of an ordinary differential equation having the form

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

is sometimes called the **"response of a system to the function** $f(t)$**"**.

The term **"system"** may, for example, refer to an oscillatory electrical circuit or a mechanical vibration.

It is also customary to refer to $f(t)$ as the **"input"** and $x(t)$ as the **"output"** of a system.

In the work which follows, we shall consider the special case in which $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$; that is, we shall assume zero initial conditions.

Impulse response function and transfer function

Consider, for the moment, the differential equation having the form,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + cu = \delta(t).$$

Here, we refer to the function, $u(t)$, as the **"impulse response function"** of the original system.

The Laplace Transform of its differential equation is given by

$$(as^2 + bs + c)U(s) = 1.$$

Hence,

$$U(s) = \frac{1}{as^2 + bs + c},$$

which is called the **"transfer function"** of the original system.

EXAMPLE

Determine the transfer function and impulse response function for the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = f(t),$$

assuming zero initial conditions.

Solution

To find $U(s)$ and $u(t)$, we have

$$3\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = \delta t,$$

so that

$$(3s + 4)U(s) = 1$$

and, hence, the transfer function is

$$U(s) = \frac{1}{3s + 4} = \frac{1}{3}.\frac{1}{s + \frac{4}{3}}.$$

Taking the inverse Laplace Transform of $U(s)$ gives the impulse response function,

$$u(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

System response for any input

Assuming zero initial conditions, the Laplace Transform of the differential equation

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + bx + cx = f(t)$$

is given by

$$(as^2 + bs + c)X(s) = F(s),$$

which means that

$$X(s) = \frac{F(s)}{as^2 + bs + c} = F(s).U(s).$$

In order to find the response of the system to the function $f(t)$, we need the inverse Laplace Transform of $F(s).U(s)$ which may possibly be found using partial fractions but may, if necessary, be found by using the Convolution Theorem referred to in Unit 16.1

The Convolution Theorem shows, in this case, that

$$L\left[\int_0^t f(T).u(t-T)\ \mathrm{d}T\right] = F(s).U(s);$$

in other words,

$$L^{-1}[F(s).U(s)] = \int_0^t f(T).u(t-T)\ \mathrm{d}T.$$

EXAMPLE

The impulse response of a system is known to be $u(t) = \frac{10e^{-t}}{3}$.

Determine the response, $x(t)$, of the system to an input of $f(t) \equiv \sin 3t$.

Solution

First, we note that

$$U(s) = \frac{10}{3(s+1)} \quad \text{and} \quad F(s) = \frac{3}{s^2+9}.$$

Hence,

$$X(s) = \frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} + \frac{-s+1}{s^2+9},$$

using partial fractions.

Thus

$$x(t) = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0.$$

Alternatively, using the Convolution Theorem,

$$x(t) = \int_0^t \sin 3T.\frac{10e^{-(t-T)}}{3}\ \mathrm{d}T;$$

but the integration here can be made simpler if we replace $\sin 3T$ by e^{j3T} and use the imaginary part, only, of the result.

Hence,

$$x(t) = I_m\left(\int_0^t \frac{10}{3}e^{-t}.e^{(1+j3)T}\ \mathrm{d}T\right)$$

$$= I_m\left(\frac{10}{3}\left[e^{-t}\frac{e^{(1+j3)T}}{1+j3}\right]_0^t\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{e^{-t}.e^{(1+j3)t} - e^{-t}}{1+j3}\right]\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{[(\cos 3t - e^{-t}) + j\sin 3t](1-j3)}{10}\right]\right)$$

$$= \frac{10}{3}\left[\frac{\sin 3t - 3\cos 3t + 3e^{-t}}{10}\right]$$

$$= e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0,$$

as before.

Note:
Clearly, in this example, the method using partial fractions is simpler.

16.6.4 STEADY-STATE RESPONSE TO A SINGLE FREQUENCY INPUT

In the differential equation,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

suppose that the quadratic denominator of the transfer function, $U(s)$, has negative real roots; that is, it gives rise to an impulse response, $u(t)$, involving negative powers of e and, hence, tending to zero as t tends to infinity.

Suppose also that $f(t)$ takes one of the forms, $\cos\omega t$ or $\sin\omega t$, which may be regarded, respectively, as the real and imaginary parts of the function, $e^{j\omega t}$.

It turns out that the response, $x(t)$, will consist of a **"transient"** part which tends to zero as t tends to infinity, together with a non-transient part forming the **"steady-state response"**.

We illustrate with an example:

EXAMPLE

Consider that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = e^{j7t},$$

where $x = 2$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - 2) - 1 + 3(sX(s) - 2) + 2X(s) = \frac{1}{s - j7}.$$

That is,

$$(s^2 + 3s + 2)X(s) = 2s + 7 + \frac{1}{s - j7},$$

giving

$$X(s) = \frac{2s + 7}{s^2 + 3s + 2} + \frac{1}{(s - j7)(s^2 + 3s + 2)} = \frac{2s + 7}{(s + 2)(s + 1)} + \frac{1}{(s - j7)(s + 2)(s + 1)}.$$

Using the "cover-up" rule for partial fractions, we obtain

$$X(s) = \frac{5}{s + 1} - \frac{3}{s + 2} + \frac{1}{(-1 - j7)(s + 1)} + \frac{1}{(2 + j7)(s + 2)} + \frac{U(j7)}{(s - j7)},$$

where

$$U(s) \equiv \frac{1}{s^2 + 3s + 2}$$

is the transfer function.

Taking inverse Laplace Transforms,

$$x(t) = 5e^{-t} - 3e^{-2t} + \frac{1}{-1-j7}e^{-t} + \frac{1}{2+j7}e^{-2t} + U(j7)e^{j7t}.$$

The first four terms on the right-hand side tend to zero as t tends to infinity, so that the final term represents the steady state response; we need its real part if $f(t) \equiv \cos 7t$ and its imaginary part if $f(t) \equiv \sin 7t$.

Summary

The above example illustrates the result that the steady-state response, $s(t)$, of a system to an input of $e^{j\omega t}$ is given by

$$s(t) = U(j\omega)e^{j\omega t}.$$

16.6.5 EXERCISES

1. Evaluate
$$\int_0^\infty e^{-4t}\delta'(t-2)\ \mathrm{d}t.$$

2. In the following cases, solve the differential equation
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = f(t),$$
where $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$:

   (a)
$$f(t) \equiv \delta(t);$$

   (b)
$$f(t) \equiv \delta(t-2).$$

3. Determine the transfer function and impulse response function for the differential equation,
$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t),$$
assuming zero initial conditions.

4. The impulse response function of a system is known to be $u(t) = e^{3t}$. Determine the response, $x(t)$, of the system to an input of $f(t) \equiv 6\cos 3t$.

5. Determine the steady-state response to the system

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t)$$

in the cases when

(a)

$$f(t) \equiv e^{j2t};$$

(b)

$$f(t) \equiv 3\cos 2t.$$

16.6.6 ANSWERS TO EXERCISES

1.

$$4e^{-8}.$$

2. (a)

$$x = \sin 2t \quad t > 0;$$

(b)

$$x = \sin t + H(t-2)\sin(t-2) \quad t \neq 2.$$

3.

$$U(s) = \frac{1}{2s^2 - 3s + 1} \quad \text{and} \quad u(t) = [e^t - e^{\frac{1}{2}t}].$$

4.

$$\frac{1}{13}\left[18e^{3t} - 18\cos 2t + 12\sin 2t\right] \quad t > 0.$$

5. (a)

$$\frac{(1-j6)e^{j2t}}{37} \quad t > 0;$$

(b)

$$\frac{1}{37}(\cos 2t + 6\sin 2t) \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.7

LAPLACE TRANSFORMS 7
(An appendix)

by

A.J.Hobson

One view of how Laplace Transforms might have arisen

UNIT 16.7 - LAPLACE TRANSFORMS 7 (AN APPENDIX)

ONE VIEW OF HOW LAPLACE TRANSFORMS MIGHT HAVE ARISEN.

(i) Let us consider that our main problem is to solve a second order linear differential equation with constant coefficients, the general form of which is

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+b\frac{\mathrm{d}x}{\mathrm{d}t}+cx=f(t).$$

(ii) Assuming that the solution of an equivalent first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t}+bx=f(t),$$

has already been included in previous knowledge, we examine a typical worked example as follows:

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t}+3x=e^{2t},$$

given that $x=0$ when $t=0$.

Solution

A method called the **"integrating factor method"** uses the coefficient of x to find a function of t which multiplies both sides of the given differential equation to convert it to an **"exact"** differential equation.

The integrating factor in the current example is e^{3t} since the coefficient of x is 3.

We obtain, therefore,

$$e^{3t}\left[\frac{\mathrm{d}x}{\mathrm{d}t}+3x\right]=e^{5t}.$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right]=e^{5t}.$$

On integrating both sides with respect to t,

$$xe^{3t}=\frac{e^{5t}}{5}+C$$

or

$$x = \frac{e^{2t}}{5} + Ce^{-3t}.$$

Putting $x = 0$ and $t = 0$, we have

$$0 = \frac{1}{5} + C.$$

Hence, $C = -\frac{1}{5}$ and the complete solution becomes

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5}.$$

(iii) As a lead up to what follows, we shall now examine a different way of setting out the above working in which we do not leave the substitution of the boundary condition until the very end.

We multiply both sides of the differential equation by e^{3t} as before, but we then integrate both sides of the new "exact" equation from 0 to t.

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right]\ \mathrm{d}t = \int_0^t e^{5t}\ \mathrm{d}t.$$

That is,

$$\left[xe^{3t}\right]_0^t = \left[\frac{e^{5t}}{5}\right]_0^t,$$

giving

$$xe^{3t} - 0 = \frac{e^{5t}}{5} - \frac{1}{5}.$$

since $x = 0$ when $t = 0$.

In other words,

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5},$$

as before.

(iv) Now let us consider whether an example of a second order linear differential equation could be solved by a similar method.

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x = e^{9t},$$

given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Supposing that there might be an integrating factor for this equation, we shall take it to be e^{st} where s, at present, is unknown, but assumed to be positive.

Multiplying throughout by e^{st} and integrating from 0 to t, as in the previous example,

$$\int_0^t e^{st}\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x\right]\,\mathrm{d}t = \int_0^t e^{(s+9)t}\,\mathrm{d}t = \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

Now, using integration by parts, with the boundary condition,

$$\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = e^{st}x - s\int_0^t e^{st}x\,\mathrm{d}t$$

and

$$\int_0^t e^{st}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - s\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - se^{st}x + s^2\int_0^t e^{st}x\,\mathrm{d}t.$$

On substituting these results into the differential equation, we may collect together (on the left hand side) terms which involve $\int_0^t e^{st}x\,\mathrm{d}t$ and e^{st} as follows:

$$(s^2 + 10s + 21)\int_0^t e^{st}x\,\mathrm{d}t + e^{st}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - (s+10)x\right] = \frac{e^{(s+9)t}}{s+9} - \frac{1}{s+9}.$$

(v) OBSERVATIONS

(a) If we had used e^{-st} instead of e^{st}, the quadratic expression in s, above, would have had the same coefficients as the original differential equation; that is, $(s^2 - 10s + 21)$.

(b) Using e^{-st} with $s > 0$, if we had integrated from 0 to ∞ instead of 0 to t, the second term on the left hand side above would have been absent, since $e^{-\infty} = 0$.

(vi) Having made our observations, we start again, multiplying both sides of the differential equation by e^{-st} and integrating from 0 to ∞ to obtain

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x \ \mathrm{d}t = \left[\frac{e^{(-s+9)t}}{-s+9}\right]_0^\infty = \frac{-1}{-s+9} = \frac{1}{s-9}.$$

Of course, this works only if $s > 9$, but we can easily assume that it is so. Hence,

$$\int_0^\infty e^{-st}x \ \mathrm{d}t = \frac{1}{(s-9)(s^2-10s+21)} = \frac{1}{(s-9)(s-3)(s-7)}.$$

Applying the principles of partial fractions, we obtain

$$\int_0^\infty e^{-st}x \ \mathrm{d}t = \frac{1}{12}.\frac{1}{s-9} + \frac{1}{24}.\frac{1}{s-3} - \frac{1}{8}.\frac{1}{s-7}.$$

(vii) But, finally, it can be shown by an independent method of solution that

$$x = \frac{e^{9t}}{12} + \frac{e^{3t}}{24} - \frac{e^{7t}}{8}.$$

and we may conclude that the solution of the differential equation is closely linked to the integral

$$\int_0^\infty e^{-st}x \ \mathrm{d}t,$$

which is called the **"Laplace Transform"** of $x(t)$.

"JUST THE MATHS"

UNIT NUMBER

16.8

Z-TRANSFORMS 1
(Definition and rules)

by

A.J.Hobson

UNIT 16.8 - Z TRANSFORMS 1 - DEFINITION AND RULES

16.8.1 INTRODUCTION - Linear Difference Equations

Closely linked with the concept of a linear differential equation with constant coefficients is that of a **"linear difference equation with constant coefficients"**.

Two particular types of difference equation to be discussed in the present section may be defined as follows:

DEFINITION 1

A first-order linear difference equation with constant coefficients has the general form,

$$a_1 u_{n+1} + a_0 u_n = f(n),$$

where a_0, a_1 are constants, n is a positive integer, $f(n)$ is a given function of n (possibly zero) and u_n is the general term of an infinite sequence of numbers, $\{u_n\} \equiv u_0,\ u_1,\ u_2,\ u_3,\ \ldots.$

DEFINITION 2

A second-order linear difference equation with constant coefficients has the general form,

$$a_2 u_{n+2} + a_1 u_{n+1} + a_0 u_n = f(n),$$

where a_0, a_1, a_2 are constants, n is an integer, $f(n)$ is a given function of n (possibly zero) and u_n is the general term of an infinite sequence of numbers, $\{u_n\} \equiv u_0,\ u_1,\ u_2\ , u_3,\ \ldots.$

Notes:

(i) We shall assume that the sequences under discussion are such that $u_n = 0$ whenever $n < 0$.

(ii) Difference equations are usually associated with given "boundary conditions", such as the value of u_0 for a first-order equation or the values of u_0 and u_1 for a second-order equation.

ILLUSTRATION

Certain **simple** difference equations may be solved by very elementary methods.

For example, suppose that we wish to solve the difference equation,

$$u_{n+1} - (n+1)u_n = 0,$$

subject to the boundary condition that $u_0 = 1$.

We may rewrite the difference equation as

$$u_{n+1} = (n+1)u_n$$

and, by using this formula repeatedly, we obtain

$$u_1 = u_0 = 1, \quad u_2 = 2u_1 = 2, \quad u_3 = 3u_2 = 3 \times 2, \quad u_4 = 4u_3 = 4 \times 3 \times 2, \quad . \ . \ . \ .$$

In general, for this illustration, $u_n = n!$.

However, not all difference equations can be solved as easily as this and we shall now discuss the Z-Transform method of solving more advanced types.

16.8.2 STANDARD DEFINITION AND RESULTS

THE DEFINITION OF A Z-TRANSFORM (WITH EXAMPLES)

The Z-Transform of the sequence of numbers, $\{u_n\} \equiv u_0, \ u_1, \ u_2, \ u_3, \ . \ . \ . \ .$, is defined by the formula,

$$Z\{u_n\} = \sum_{r=0}^{\infty} u_r z^{-r},$$

provided that the series converges (allowing for z to be a complex number if necessary).

EXAMPLES

1. Determine the Z-Transform of the sequence,

$$\{u_n\} \equiv \{a^n\},$$

where a is a non-zero constant.

Solution

$$Z\{a^n\} = \sum_{r=0}^{\infty} a^r z^{-r}.$$

That is,

$$Z\{a^n\} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ .\ .\ .\ . \ = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a},$$

by properties of infinite geometric series.

Thus,

$$Z\{a^n\} = \frac{z}{z-a}.$$

2. Determine the Z-Transform of the sequence,

$$\{u_n\} = \{n\}.$$

Solution

$$Z\{n\} = \sum_{r=0}^{\infty} r z^{-r}.$$

That is,

$$Z\{n\} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ .\ .\ .\ .\ ,$$

which may be rearranged as

$$Z\{n\} = \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ .\ .\ .\ .\right) + \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ .\ .\ .\ .\right) + \left(\frac{1}{z^3} + \frac{1}{z^4} + \ .\ .\ .\ .\right),$$

giving

$$Z\{n\} = \frac{\frac{1}{z}}{1 - \frac{1}{z}} + \frac{\frac{1}{z^2}}{1 - \frac{1}{z}} + \frac{\frac{1}{z^3}}{1 - \frac{1}{z}} + \ .\ .\ .\ . = \frac{1}{1 - \frac{1}{z}}\left[\frac{\frac{1}{z}}{1 - \frac{1}{z}}\right],$$

by properties of infinite geometric series.

Thus,

$$Z\{n\} = \frac{z}{(1-z)^2} = \frac{z}{(z-1)^2}.$$

Note:
Other Z-Transforms may be obtained, in the same way as in the above examples, from the definition.

We list, here, for reference, a short table of standard Z-Transforms, including those already proven:

A SHORT TABLE OF Z-TRANSFORMS

$\{u_n\}$	$Z\{u_n\}$	Region of Existence
$\{1\}$	$\frac{z}{z-1}$	$\|z\| > 1$
$\{a^n\}$ (a constant)	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$\{n\}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$\left\{e^{-nT}\right\}$ (T constant)	$\frac{z}{z-e^{-T}}$	$\|z\| > e^{-T}$
$\sin nT$ (T constant)	$\frac{z\sin T}{z^2-2z\cos T+1}$	$\|z\| > 1$
$\cos nT$ (T constant)	$\frac{z(z-\cos T)}{z^2-2z\cos T+1}$	$\|z\| > 1$
1 for $n = 0$ 0 for $n > 0$ (Unit pulse sequence)	1	All z
0 for $n = 0$ $\{a^{n-1}\}$ for $n > 0$	$\frac{1}{z-a}$	$\|z\| > \|a\|$

16.8.3 PROPERTIES OF Z-TRANSFORMS

(a) Linearity

If $\{u_n\}$ and $\{v_n\}$ are sequences of numbers, while A and B are constants, then

$$Z\{Au_n + Bv_n\} \equiv A.Z\{u_n\} + B.Z\{v_n\}.$$

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} (Au_r + Bv_r)z^{-r} \equiv A\sum_{r=0}^{\infty} u_r z^{-r} + B\sum_{r=0}^{\infty} v_r z^{-r},$$

which, in turn, is equivalent to the right-hand side.

EXAMPLE

$$Z\{5.2^n - 3n\} = \frac{5z}{z-2} - \frac{3z}{(z-1)^2}.$$

(b) The First Shifting Theorem

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.Z\{u_n\},$$

where $\{u_{n-1}\}$ denotes the sequence whose first term, corresponding to $n = 0$, is taken as zero and whose subsequent terms, corresponding to $n = 1, \ 2, \ 3, \ 4, \ \ldots.$, are the terms $u_0, \ u_1, \ u_3, \ u_4, \ \ \ldots$ of the original sequence.

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} u_{r-1}z^{-r} \equiv \frac{u_0}{z} + \frac{u_1}{z^2} + \frac{u_2}{z^3} + \frac{u_3}{z^4} + \ \ldots.,$$

since it is assumed that $u_n = 0$ whenever $n < 0$.

Thus,

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \ \ldots .\right],$$

which is equivalent to the right-hand side.

Note:
A more general form of the first shifting theorem states that

$$Z\{u_{n-k}\} \equiv \frac{1}{z^k}.Z\{u_n\},$$

where $\{u_{n-k}\}$ denotes the sequence whose first k terms, corresponding to $n = 0,\ 1,\ 2,\ \ldots.,k-1$, are taken as zero and whose subsequent terms, corresponding to $n = k,\ k+1,\ k+2,\ \ldots.$ are the terms $u_0,\ u_1,\ u_2,\ \ldots.$ of the original sequence.

ILLUSTRATION

Given that $\{u_n\} \equiv \{4^n\}$, we may say that

$$Z\{u_{n-2}\} \equiv \frac{1}{z^2}.Z\{u_n\} \equiv \frac{1}{z^2}.\frac{z}{z-4} \equiv \frac{1}{z(z-4)}.$$

Note:
In this illustration, the sequence, $\{u_{n-2}\}$ has terms $0,\ 0,\ 1,\ 4,\ 4^2,\ 4^3,\ \ldots$ and, by applying the definition of a Z-Transform directly, we would obtain

$$Z\{u_{n-2}\} = \frac{1}{z^2} + \frac{4}{z^3} + \frac{4^2}{z^4} + \frac{4^3}{z^5}\ \ldots.,$$

which gives

$$Z\{u_n\} \equiv \frac{1}{z^2}.\frac{1}{1-\frac{4}{z}} \equiv \frac{1}{z(z-4)},$$

by properties of infinite geometric series.

(c) The Second Shifting Theorem

$$Z\{u_{n+1}\} \equiv z.Z\{u_n\} - z.u_0$$

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} u_{r+1}z^{-r} \equiv u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \frac{u_4}{z^4} + \ \ldots .$$

This may be rearranged as

$$z.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \ \ldots .\right] - z.u_0$$

which, in turn, is equivalent to the right-hand side.

Note:
This **"recursive relationship"** may be applied repeatedly. For example, we may deduce that

$$Z\{u_{n+2}\} \equiv z.Z\{u_{n+1}\} - z.u_1 \equiv z^2.Z\{u_n\} - z^2.u_0 - z.u_1$$

16.8.4 EXERCISES

1. Determine, from first principles, the Z-Transforms of the following sequences, $\{u_n\}$:

   (a)

   $$\{u_n\} \equiv \{e^{-n}\};$$

   (b)

   $$\{u_n\} \equiv \{\cos \pi n\}.$$

2. Determine the Z-Transform of the following sequences:

   (a)

$$\{u_n\} \equiv \{7.(3)^n - 4.(-1)^n\};$$

   (b)

$$\{u_n\} \equiv \left\{6n + 2e^{-5n}\right\};$$

   (c)

$$\{u_n\} \equiv \{13 + \sin 2n - \cos 2n\}.$$

3. Determine the Z-Transform of $\{u_{n-1}\}$ and $\{u_{n-2}\}$ for the sequences in question 1.

4. Determine the Z-Transform of $\{u_{n+1}\}$ and $\{u_{n+2}\}$ for the sequences in question 1.

16.8.5 ANSWERS TO EXERCISES

1. (a)

$$\frac{ez}{ez-1};$$

   (b)

$$\frac{z}{z+1}.$$

2. (a)

$$\frac{7z}{z-3} - \frac{4z}{z+1};$$

   (b)

$$\frac{6z}{(z-1)^2} + \frac{2z}{z-e^{-5}};$$

   (c)

$$\frac{13z}{z-1} + \frac{z(\sin 2 + \cos 2 - z)}{z^2 - 2z\cos 2 + 1}.$$

3. (a)

$$Z\{u_{n-1}\} \equiv \frac{e}{ez-1}\ (n>0), \quad Z\{u_{n-2}\} \equiv \frac{e}{z(ez-1)}\ (n>1);$$

(b)

$$Z\{u_{n-1}\} \equiv \frac{1}{z+1}\ (n>0), \quad Z\{u_{n-2}\} \equiv \frac{1}{z(z+1)}\ (n>1).$$

Note:
$u_{-2} = 0$ and $u_{-1} = 0$.

4. (a)

$$Z\{u_{n+1}\} \equiv \frac{z}{ez-1}, \quad Z\{u_{n+2}\} \equiv \frac{z}{e(ez-1)};$$

(b)

$$Z\{u_{n+1}\} \equiv -\frac{z}{z+1}, \quad Z\{u_{n+2}\} \equiv \frac{z}{z+1}.$$

"JUST THE MATHS"

UNIT NUMBER

16.9

Z-TRANSFORMS 2
(Inverse Z-Transforms)

by

A.J.Hobson

UNIT 16.9 - Z TRANSFORMS 2

INVERSE Z - TRANSFORMS

16.9.1 THE USE OF PARTIAL FRACTIONS

When solving linear difference equations by means of Z-Transforms, it is necessary to be able to determine a sequence, $\{u_n\}$, of numbers, whose Z-Transform is a known function, $F(z)$, of z. Such a sequence is called the **"inverse Z-Transform of $F(z)$"** and may be denoted by $\mathrm{Z}^{-1}[F(z)]$.

For simple difference equations, the function $F(z)$ turns out to be a rational function of z, and the method of partial fractions may be used to determine the corresponding inverse Z-Transform.

EXAMPLES

1. Determine the inverse Z-Transform of the function

$$F(z) \equiv \frac{10z(z+5)}{(z-1)(z-2)(z+3)}.$$

Solution
Bearing in mind that

$$Z\{a^n\} = \frac{z}{z-a},$$

for any non-zero constant, a, we shall write

$$F(z) \equiv z.\left[\frac{10(z+5)}{(z-1)(z-2)(z+3)}\right],$$

which gives

$$F(z) \equiv z.\left[\frac{-15}{z-1} + \frac{14}{z-2} + \frac{1}{z+3}\right]$$

or

$$F(z) \equiv \frac{z}{z+3} + 14\frac{z}{z-2} - 15\frac{z}{z-1}.$$

Hence,

$$\mathrm{Z}^{-1}[F(z)] = \{(-3)^n + 14(2)^n - 15\}.$$

2. Determine the Inverse Z-Transform of the function

$$F(z) \equiv \frac{1}{z-a}.$$

Solution

In this example, there is no factor, z, in the function $F(z)$ and we shall see that it is necessary to make use of the first shifting theorem.

First, we may write

$$F(z) \equiv \frac{1}{z}\left[\frac{z}{z-a}\right]$$

and, since the inverse Z-Transform of the expression inside the brackets is a^n, the first shifting theorem tells us that

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ a^{n-1} & \text{when } n > 0. \end{cases}$$

Note:

This may now be taken as a standard result.

3. Determine the inverse Z-Transform of the function

$$F(z) \equiv \frac{4(2z+1)}{(z+1)(z-3)}.$$

Solution

Expressing $F(z)$ in partial fractions, we obtain

$$F(z) \equiv \frac{1}{z+1} + \frac{7}{(z-3)}.$$

Hence,

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ (-1)^{n-1} + 7.(3)^{n-1} & \text{when } n > 0. \end{cases}$$

16.9.2 EXERCISES

1. Determine the inverse Z-Transforms of each of the following functions, $F(z)$:

   (a)
$$F(z) \equiv \frac{z}{z-1};$$

   (b)
$$F(z) \equiv \frac{z}{z+1};$$

   (c)
$$F(z) \equiv \frac{2z}{2z-1};$$

   (d)
$$F(z) \equiv \frac{z}{3z+1};$$

   (e)
$$F(z) \equiv \frac{z}{(z-1)(z+2)};$$

   (f)
$$F(z) \equiv \frac{z}{(2z+1)(z-3)};$$

   (g)
$$F(z) \equiv \frac{z^2}{(2z+1)(z-1)}.$$

2. Determine the inverse Z-Transform of each of the following functions, $F(z)$, and list the first five terms of the sequence obtained:

   (a)
$$F(z) \equiv \frac{1}{z-1};$$

   (b)
$$F(z) \equiv \frac{z+2}{z+1};$$

(c)

$$F(z) \equiv \frac{z-3}{(z-1)(z-2)};$$

(d)

$$F(z) \equiv \frac{2z^2-7z+7}{(z-1)^2(z-2)}.$$

16.9.3 ANSWERS TO EXERCISES

1. (a)

$$\mathrm{Z}^{-1}[F(z)] = \{1\}$$

(b)

$$\mathrm{Z}^{-1}[F(z)] = \{(-1)^n\}$$

(c)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\left(\frac{1}{2}\right)^n\right\};$$

(d)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}\left(-\frac{1}{3}\right)^n\right\};$$

(e)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}[1-(-2)^n]\right\};$$

(f)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{7}\left[(3)^n - \left(-\frac{1}{2}\right)^n\right]\right\};$$

(g)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}+\frac{1}{6}\left(-\frac{1}{2}\right)^n\right\}.$$

2. (a)

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ 1 & \text{when } n > 0; \end{cases}$$

The first five terms are 0,1,1,1,1

(b)

$$Z^{-1}[F(z)] = \begin{cases} 1 & \text{when } n = 0; \\ (-1)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 1,1,−1,1,−1

(c)

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ 2-(2)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 0,1,0,−2,−6

(d)

$$Z^{-1}[F(z] = \begin{cases} 0 & \text{when } n = 0; \\ 3-2n+(2)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 0,2,1,1,3

"JUST THE MATHS"

UNIT NUMBER

16.10

Z-TRANSFORMS 3
(Solution of linear difference equations)

by

A.J.Hobson

UNIT 16.10 - Z TRANSFORMS 3

THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS

Linear difference equations may be solved by constructing the Z-Transform of both sides of the equation. The method will be illustrated with linear difference equations of the first and second orders (with constant coefficients).

16.10.1 FIRST ORDER LINEAR DIFFERENCE QUATIONS

EXAMPLES

1. Solve the linear difference equation,

$$u_{n+1} - 2u_n = (3)^{-n},$$

given that $u_0 = 2/5$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.\frac{2}{5}.$$

Taking the Z-Transform of the difference equation, we obtain

$$z.Z\{u_n\} - \frac{2}{5}.z - 2Z\{u_n\} = \frac{z}{z - \frac{1}{3}},$$

so that, on rearrangement,

$$Z\{u_n\} = \frac{2}{5}.\frac{z}{z-2} + \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}$$

$$\equiv \frac{2}{5}.\frac{z}{z-2} + z.\left[\frac{-\frac{3}{5}}{z - \frac{1}{3}} + \frac{\frac{3}{5}}{z-2}\right]$$

$$\equiv \frac{z}{z-2} - \frac{3}{5}.\frac{z}{z - \frac{1}{3}}.$$

Taking the inverse Z-Transform of this function of z gives the solution

$$\{u_n\} \equiv \left\{(2)^n - \frac{3}{5}(3)^{-n}\right\}.$$

2. Solve the linear difference equation,

$$u_{n+1} + u_n = f(n),$$

given that

$$f(n) \equiv \begin{cases} 1 & \text{when } n = 0; \\ 0 & \text{when } n > 0. \end{cases}$$

and $u_0 = 5$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.5$$

Taking the Z-Transform of the difference equation, we obtain

$$z.Z\{u_n\} - 5z + Z\{u_n\} = 1,$$

which, on rearrangement, gives

$$Z\{u_n\} = \frac{1}{z+1} + \frac{5z}{z+1}.$$

Hence,

$$\{u_n\} = \begin{cases} 5 & \text{when } n = 0; \\ (-1)^{n-1} + 5(-1)^n \equiv 4(-1)^n & \text{when } n > 0. \end{cases}$$

16.10.2 SECOND ORDER LINEAR DIFFERENCE EQUATIONS

EXAMPLES

1. Solve the linear difference equation

$$u_{n+2} = u_{n+1} + u_n,$$

given that $u_0 = 0$ and $u_1 = 1$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n - z.0 \equiv z.Z\{u_n\}$$

and

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z.1 \equiv z^2 Z\{u_n\} - z.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2.Z\{u_n\} - z = z.Z\{u_n\} + Z\{u_n\},$$

so that, on rearrangement,

$$Z\{u_n\} = \frac{z}{z^2 - z - 1},$$

which may be written

$$Z\{u_n\} = \frac{z}{(z-\alpha)(z-\beta)},$$

where, from the quadratic formula,

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

Using partial fractions,

$$Z\{u_n\} = \frac{1}{\alpha - \beta}\left[\frac{z}{z-\alpha} - \frac{z}{z-\beta}\right].$$

Taking the inverse Z-Transform of this function of z gives the solution

$$\{u_n\} \equiv \left\{\frac{1}{\alpha - \beta}\left[(\alpha)^n - (\beta)^n\right]\right\}.$$

2. Solve the linear difference equation

$$u_{n+2} - 7u_{n+1} + 10u_n = 16n,$$

given that $u_0 = 6$ and $u_1 = 2$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - 6z$$

and

$$Z\{u_{n+2}\} = z^2.Z\{u_n\} - 6z^2 - 2z.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2.Z\{u_n\} - 6z^2 - 2z - 7\left[z.Z\{u_n\} - 6z\right] + 10Z\{u_n\} = \frac{16z}{(z-1)^2},$$

which, on rearrangement, gives

$$Z\{u_n\}[z^2 - 7z + 10] - 6z^2 + 40z = \frac{16z}{(z-1)^2};$$

and, hence,

$$Z\{u_n\} = \frac{16z}{(z-1)^2(z-5)(z-2)} + \frac{6z^2 - 40z}{(z-5)(z-2)}.$$

Using partial fractions, we obtain

$$Z\{u_n\} = z.\left[\frac{4}{z-2} - \frac{3}{z-5} + \frac{4}{(z-1)^2} + \frac{5}{z-1}\right].$$

The solution of the difference equation is therefore

$$\{u_n\} \equiv \{4(2)^n - 3(5)^n + 4n + 5\}.$$

3. Solve the linear difference equation

$$u_{n+2} + 2u_n = 0$$

given that $u_0 = 1$ and $u_1 = \sqrt{2}$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z^2 - z\sqrt{2}.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2 Z\{u_n\} - z^2 - z\sqrt{2} + 2Z\{u_n\} = 0,$$

which, on rearrangement, gives

$$Z\{u_n\} = \frac{z^2 + z\sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{(z + j\sqrt{2})(z - j\sqrt{2})}.$$

Using partial fractions,

$$Z\{u_n\} = z\left[\frac{\sqrt{2}(1+j)}{j2\sqrt{2}(z - j\sqrt{2})} + \frac{\sqrt{2}(1-j)}{-j2\sqrt{2}(z + j\sqrt{2})}\right] \equiv z.\left[\frac{(1-j)}{2(z - j\sqrt{2})} + \frac{(1+j)}{2(z + j\sqrt{2})}\right],$$

so that

$$\{u_n\} \equiv \left\{\frac{1}{2}(1-j)(j\sqrt{2})^n + \frac{1}{2}(1+j)(-j\sqrt{2})^n\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[(1-j)(j)^n + (1+j)(-j)^n\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[\sqrt{2}e^{-j\frac{\pi}{4}}.e^{j\frac{n\pi}{2}} + \sqrt{2}e^{j\frac{\pi}{4}}.e^{-j\frac{n\pi}{2}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}\left[e^{j\frac{(2n-1)\pi}{4}} + e^{-j\frac{(2n-1)\pi}{4}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}.2\cos\frac{(2n-1)\pi}{4}\right\}$$

$$\equiv \left\{(\sqrt{2})^{n+1}\cos\frac{(2n-1)\pi}{4}\right\}.$$

16.10.3 EXERCISES

1. Solve the following first-order linear difference equations:

   (a)

   $$3u_{n+1} + 2u_n = (-1)^n,$$

   given that $u_0 = 0$;

   (b)

   $$u_{n+1} - 5u_n = 3(2)^n,$$

   given that $u_0 = 1$;

   (c)

   $$u_{n+1} + u_n = n,$$

   given that $u_0 = 1$;

   (d)

   $$u_{n+1} + 2u_n = f(n),$$

   where

   $$f(n) \equiv \begin{cases} 3 & \text{when } n = 0; \\ 0 & \text{when } n > 0; \end{cases}$$

   and $u_0 = 2$;

   (e)

   $$u_{n+1} - 3u_n = \sin\frac{n\pi}{2} + \frac{1}{2}\cos\frac{n\pi}{2},$$

   given that $u_0 = 0$.

2. Solve the following second-order linear difference equations:

   (a)

   $$u_{n+2} - 2u_{n+1} + u_n = 0,$$

   given that $u_0 = 0$ and $u_1 = 1$;

   (b)

   $$u_{n+2} - 4u_n = n,$$

   given that $u_0 = 0$ and $u_1 = 1$;

(c)

$$u_{n+2} - 8u_{n+1} - 9u_n = 24,$$

given that $u_0 = 2$ and $u_1 = 0$;

(d)

$$6u_{n+2} + 5u_{n+1} - u_n = 20,$$

given that $u_0 = 3$ and $u_1 = 8$;

(e)

$$u_{n+2} + 2u_{n+1} - 15u_n = 32\cos n\pi,$$

given that $u_0 = 0$ and $u_1 = 0$;

(f)

$$u_{n+2} - 3u_{n+1} + 3u_n = 5,$$

given that $u_0 = 5$ and $u_1 = 8$.

16.10.4 ANSWERS TO EXERCISES

1. (a)

$$\{u_n\} \equiv \left\{\left(-\frac{2}{3}\right)^n - (-1)^n\right\};$$

(b)

$$\{u_n\} \equiv \{2(5)^n - (2)^n\};$$

(c)

$$\{u_n\} \equiv \left\{\frac{1}{2}n - \frac{1}{4} + \frac{5}{4}(-1)^n\right\};$$

(d)

$$\{u_n\} \equiv 2(-2)^n + 3(-2)^{n-1} \quad \text{when} \quad n > 0;$$

(e)

$$\{u_n\} \equiv \Bigg\{\frac{1}{4}\Bigg[(3^n - \sqrt{2}\cos\frac{(2n-1)\pi}{4}\Bigg]\Bigg]\Bigg\}.$$

2. (a)

$$\{u_n\} \equiv \{n\};$$

(b)

$$\{u_n\} \equiv \left\{\frac{1}{2}(2)^n - \frac{1}{3}n - \frac{5}{18}(-2)^n - \frac{2}{9}\right\};$$

(c)

$$\{u_n\} \equiv \left\{\frac{1}{2}(9)^n + 3(-1)^n - \frac{3}{2}\right\};$$

(d)

$$\{u_n\} \equiv \left\{2 + (6)^{1-n} - 5(-1)^n\right\};$$

(e)

$$\{u_n\} \equiv \left\{2(-1)^{n+1} + (3)^n + (-5)^n\right\};$$

(f)

$$\{u_n\} \equiv \left\{5 + (2\sqrt{3})^{n+1} \cos\frac{(n-3)\pi}{6}\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

17.1

NUMERICAL MATHEMATICS 1
(Approximate solution of equations)

by

A.J.Hobson

UNIT 17.1 - NUMERICAL MATHEMATICS 1

THE APPROXIMATE SOLUTION OF ALGEBRAIC EQUATIONS

17.1.1 INTRODUCTION

In the work which follows, we shall consider the solution of the equation

$$f(x) = 0,$$

where $f(x)$ is a given function of x.

It is assumed that examples of such equations will have been encountered earlier at an elementary level; as, for instance, with quadratic equations where there is simple formula for obtaining solutions.

However, the equation

$$f(x) = 0$$

cannot, in general, be solved algebraically to give **exact** solutions and we have to be satisfied, at most, with **approximate** solutions. Nevertheless, it is often possible to find approximate solutions which are correct to any specified degree of accuracy; and this is satisfactory for the applications of mathematics to science and engineering.

It is certainly possible to consider **graphical** methods of solving the equation

$$f(x) = 0,$$

where we try to plot a graph of the equation

$$y = f(x),$$

then determine where the graph crosses the x-axis. But this method can be laborious and inaccurate and will not be discussed, here, as a viable method.

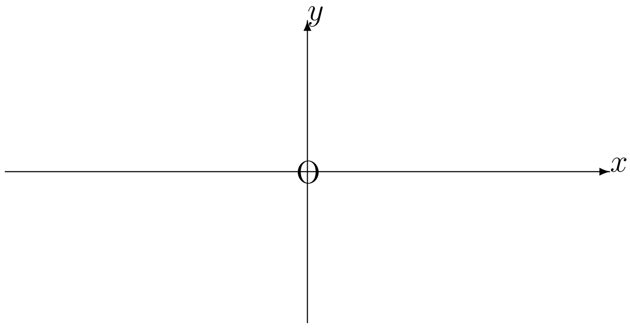
Three so called **"iterative"** methods will be included, below, where repeated use of the method is able to improve the accuracy of an approximate solution, already obtained.

17.1.2 THE BISECTION METHOD

Suppose a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. We may obtain these by trial and error or by sketching, roughly, the graph of the equation

$$y = f(x),$$

in order to estimate convenient values a and b between which the graph crosses the x-axis; whole numbers will usually suffice.



If we let $c = (a+b)/2$, there are three possibilities;

(i) $f(c) = 0$, in which case we have solved the equation;

(ii) $f(c) < 0$, in which case there is a solution between c and b enabling us repeat the procedure with these two numbers;

(iii) $f(c) > 0$, in which case there is a solution between c and a enabling us to repeat the procedure with these two numbers.

Each time we apply the method, we bisect the interval between the two numbers being used so that, eventually, the two numbers used will be very close together. The method stops when two consecutive values of the mid-point agree with each other to the required number of decimal places or significant figures.

Convenient labels for the numbers used at each stage (or iteration) are

$$a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, ..., a_n, b_n, c_n, ...$$

EXAMPLE

Determine, correct to three decimal places, the positive solution of the equation

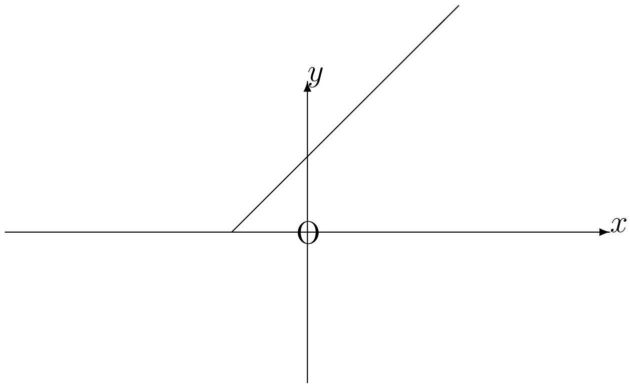
$$e^x = x + 2.$$

Solution

We could first observe, from a rough sketch of the graphs of

$$y = e^x \quad \text{and} \quad y = x + 2,$$

that the graphs intersect each other at a positive value of x. This confirms that there is indeed a positive solution to our equation.



But now let
$$f(x) = e^x - x - 2$$
and look for two numbers between which $f(x)$ changes sign from positive to negative. By trial and error, suitable numbers are 1 and 2, since
$$f(1) = e - 3 < 0 \quad \text{and} \quad f(2) = e^2 - 5 > 0.$$

The rest of the solution may be set out in the form of a table as follows:

n	a_n	b_n	c_n	$f(c_n)$
0	1.00000	2.00000	1.50000	0.98169
1	1.00000	1.50000	1.25000	0.24034
2	1.00000	1.25000	1.12500	- 0.04478
3	1.12500	1.25000	1.18750	0.09137
4	1.12500	1.18750	1.15625	0.02174
5	1.12500	1.15625	1.14062	- 0.01191
6	1.14063	1.15625	1.14844	0.00483
7	1.14063	1.14844	1.14454	- 0.00354
8	1.14454	1.14844	1.14649	0.00064
9	1.14454	1.14649	1.14552	- 0.00144

As a general rule, it is appropriate to work to two more places of decimals than that of the required accuracy; and so, in this case, we work to five.

We can stop at stage 9, since c_8 and c_9 are the same value when rounded to three places of decimals. The required solution is therefore $\boxed{x = 1.146}$

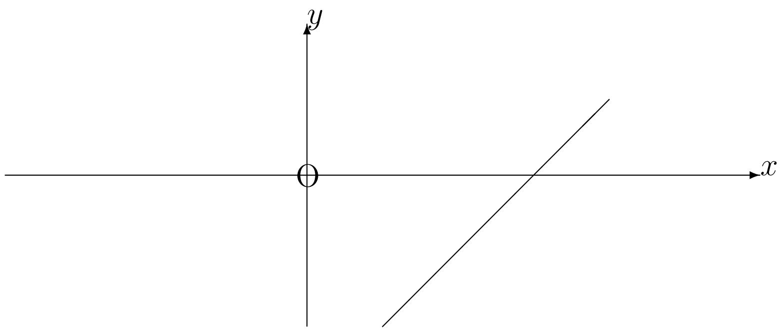
17.1.3 THE RULE OF FALSE POSITION

This method is commonly known by its Latin name, **"Regula Falsi"**, and tries to compensate a little for the shortcomings of the Bisection Method.

Instead of taking c as the average of a and b, we consider that the two points, $(a, f(a))$ and $(b, f(b))$, on the graph of the equation,

$$y = f(x),$$

are joined by a straight line; and the point at which this straight line crosses the x-axis is taken as c.



From elementary co-ordinate geometry, the equation of the straight line is given by

$$\frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}.$$

Hence, when $y = 0$, we obtain

$$x = a - \frac{(b - a)f(a)}{f(b) - f(a)}.$$

That is,

$$x = \frac{a[f(b) - f(a)] - (b - a)f(a)}{f(b) - f(a)}.$$

Hence,

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

In setting out the tabular form of a Regula Falsi solution, the c_n column uses the general formula
$$c_n = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}.$$

EXAMPLE

For the equation
$$f(x) \equiv x^3 + 2x - 1 = 0$$
use the Regula Falsi method with $a_0 = 0$ and $b_0 = 1$ to determine the first approximation, c_0, to the solution between $x = 0$ and $x = 1$.

Solution

We have $f(0) = -1$ and $f(1) = 2$, so that there is certainly a solution between $x = 0$ and $x = 1$.

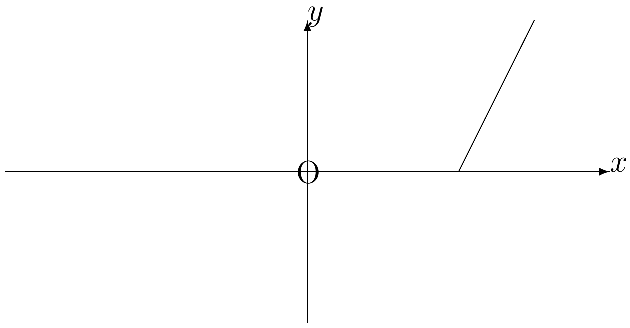
From the general formula,
$$c_0 = \frac{0 \times 2 - 1 \times (-1)}{2 - (-1)} = \frac{1}{3}$$
and, if we were to continue with the method, we would observe that $f(1/3) < 0$ so that $a_1 = 1/3$ and $b_1 = 1$.

Note:
The Bisection Method would have given $c_0 = \frac{1}{2}$.

17.1.4 THE NEWTON-RAPHSON METHOD

This method is based on the guessing of an approximate solution, $x = x_0$, to the equation $f(x) = 0$.

We then draw the tangent to the curve whose equation is
$$y = f(x)$$
at the point $x_0, f(x_0)$ to find out where this tangent crosses the x-axis. The point obtained is normally a better approximation x_1 to the solution.



In the diagram,

$$f'(x_0) = \frac{\text{AB}}{\text{AC}} = \frac{f(x_0)}{h}.$$

Hence,

$$h = \frac{f(x_0)}{f'(x_0)},$$

so that a better approximation to the exact solution at point D is given by

$$x_1 = x_0 - h.$$

Repeating the process, gives rise to the following iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Notes:

(i) To guess the starting approximation, x_0, it is normally sufficient to use a similar technique to that in the Bisection Method; that is, we find a pair of whole numbers, a and b, such that $f(a) < 0$ and $f(b) > 0$; then we take $x_0 = (a + b)/2$. In some exercises, however, an alternative starting approximation may be suggested in order to speed up the rate of convergence to the final solution.

(ii) There are situations where the Newton-Raphson Method fails to give a better approximation; as, for example, when the tangent to the curve has a very small gradient, and consequently meets the x-axis at a relatively great distance from the previous approximation. In this Unit, we shall consider only examples in which the successive approximations converge rapidly to the required solution.

EXAMPLE

Use the Newton-Raphson method to calculate $\sqrt{5}$, correct to three places of decimals.

Solution

We are required to solve the equation

$$f(x) \equiv x^2 - 5 = 0.$$

By trial and error, we find that a solution exists between $x = 2$ and $x = 3$ since $f(2) = -1 < 0$ and $f(3) = 4 > 0$. Hence, we use $x_0 = 2.5$

Furthermore,

$$f'(x) = 2x,$$

so that

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}.$$

Thus,

$$x_1 = 2.5 - \frac{1.25}{5} = 2.250,$$
$$x_2 = 2.250 - \frac{0.0625}{4.5} \simeq 2.236,$$
$$x_3 = 2.236 - \frac{-0.000304}{4.472} \simeq 2.236$$

At each stage, we round off the result to the required number of decimal places and use the rounded figure in the next iteration.

The last two iterations give the same result to three places of decimals and this is therefore the required result.

17.1.5 EXERCISES

1. Determine the smallest positive solution to the following equations (i) by the Bisection Method and (ii) by the Regula Falsi Method, giving your answers correct to four significant figures:

   (a)
   $$x - 2\sin^2 x = 0;$$
   (b)
   $$e^x - \cos(x^2) - 1 = 0.$$

2. Use the Newton-Raphson Method to determine the smallest positive solution to each of the following equations, correct to five decimal places:

   (a)
   $$x^4 = 5;$$
   (b)
   $$x^3 + x^2 - 4x + 1 = 0;$$
   (c)
   $$x - 2 = \ln x.$$

17.1.6 ANSWERS TO EXERCISES

1. (a)
   $$x \simeq 1.849;$$
   (b)
   $$x \simeq 0.6486$$

2. (a)
   $$x \simeq 1.49535;$$
   (b)
   $$x \simeq 0.27389;$$
   (c)
   $$x \simeq 3.14619$$

"JUST THE MATHS"

UNIT NUMBER

17.2

NUMERICAL MATHEMATICS 2
(Approximate integration (A))

by

A.J.Hobson

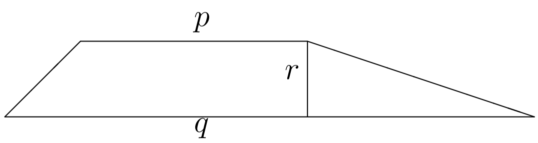
UNIT 17.2 - NUMERICAL MATHEMATICS 2

APPROXIMATE INTEGRATION (A)

17.2.1 THE TRAPEZOIDAL RULE

The rule which is explained below is based on the formula for the area of a trapezium. If the parallel sides of a trapezium are of length p and q while the perpendicular distance between them is r, then the area A is given by

$$A = \frac{r(p+q)}{2}.$$



Let us assume first that the curve whose equation is

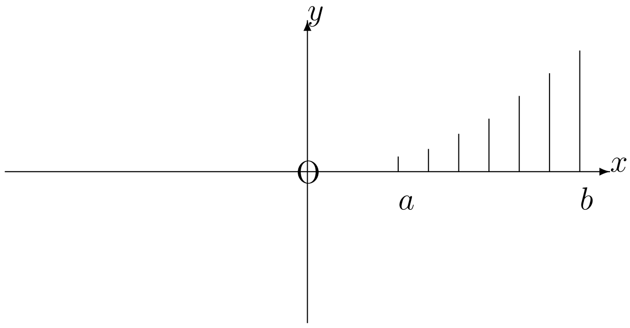
$$y = f(x)$$

lies wholly above the x-axis between $x = a$ and $x = b$. It has already been established, in Unit 13.1, that the definite integral

$$\int_a^b f(x)\ \mathrm{d}x$$

can be regarded as the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

However, suppose we divided this area into several narrow strips of equal width, h, by marking the values $x_1, x_2, x_3, ..., x_n$ along the x-axis (where $x_1 = a$ and $x_n = b$) and drawing in the corresponding lines of length $y_1, y_2, y_3, ..., y_n$ parallel to the y-axis.



Each narrow strip of width h may be considered approximately as a trapezium whose parallel sides are of lengths y_i and y_{i+1}, where $i = 1, 2, 3, ..., n-1$.

Thus, the area under the curve, and hence the value of the definite integral, approximates to

$$\frac{h}{2}[(y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + ... + (y_{n-1} + y_n)].$$

That is,

$$\int_a^b f(x) \ \mathrm{d}x \simeq \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + y_4 + ... + y_{n-1})];$$

or, what amounts to the same thing,

$$\int_a^b f(x) \ \mathrm{d}x = \frac{h}{2}[\text{First} + \text{Last} + 2 \times \ \text{The Rest}].$$

Note:
Care must be taken at the beginning to ascertain whether or not the curve $y = f(x)$ crosses the x-axis between $x = a$ and $x = b$. If it does, then allowance must be made for the fact that areas below the x-axis are negative and should be calculated separately from those above the x-axis.

EXAMPLE

Use the trapezoidal rule with five divisions of the x-axis in order to evaluate, approximately, the definite integral:

$$\int_0^1 e^{x^2} \ \mathrm{d}x.$$

Solution

First we make up a table of values as follows:

x	0	0.2	0.4	0.6	0.8	1.0
e^{x^2}	1	1.041	1.174	1.433	1.896	2.718

Then, using $h = 0.2$, we have

$$\int_0^1 e^{x^2}\ \mathrm{d}x \simeq \frac{0.2}{2}[1 + 2.718 + 2(1.041 + 1.174 + 1.433 + 1.896)] \simeq 1.481$$

17.2.2 EXERCISES

Use the trapezoidal rule with six divisions of the x-axis to determine an approximation for each of the following, working to three decimal places throughout:

1.
$$\int_1^7 x \ln x\ \mathrm{d}x.$$

2.
$$\int_{-2}^1 \frac{1}{5 - x^2}\ \mathrm{d}x.$$

3.
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\ \mathrm{d}x.$$

4.
$$\int_0^{\frac{\pi}{2}} \sin\sqrt{x^2 + 1}\ \mathrm{d}x.$$

5.
$$\int_0^{\frac{\pi}{2}} \ln(1 + \sin x)\ \mathrm{d}x.$$

17.2.3 ANSWERS TO EXERCISES

1. 35.836 2. 0.931 3. 0.348 4. 1.468 5. 0.737

"JUST THE MATHS"

UNIT NUMBER

17.3

NUMERICAL MATHEMATICS 3
(Approximate integration (B))

by

A.J.Hobson

UNIT 17.3 - NUMERICAL MATHEMATICS 3

APPROXIMATE INTEGRATION (B)

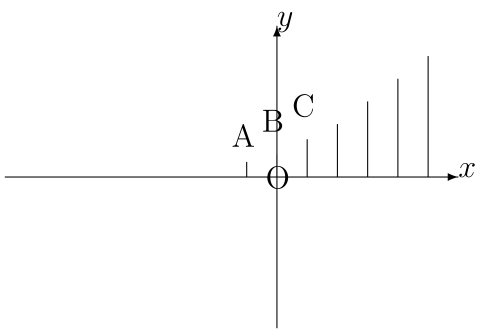
17.3.1 SIMPSON'S RULE

A better approximation to

$$\int_a^b f(x)\mathrm{d}x$$

than that provided by the Trapezoidal rule (Unit 17.2) may be obtained by usingan **even** number of narrow strips of width, h, and considering them in pairs.

To begin with, we examine a **special** case in which the first strip lies to the left of the y-axis as in the following diagram:



The arc of the curve passing through the points A$(-h, y_1)$, B$(0, y_2)$ and C(h, y_3) may be regarded as an arc of a parabola whose equation is

$$y = Lx^2 + Mx + N,$$

provided that the coefficients L, M and N satisfy the equations

$$\begin{aligned} y_1 &= Lh^2 - Mh + N, \\ y_2 &= N, \\ y_3 &= Lh^2 + Mh + N. \end{aligned}$$

Also, the area of the first pair of strips is given by

$$\text{Area} = \int_{-h}^{h} (Lx^2 + Mx + N)\ \mathrm{d}x$$

$$= \left[L\frac{x^3}{3} + M\frac{x^2}{2} + Nx\right]_{-h}^{h}$$

$$= \frac{2Lh^3}{3} + 2Nh$$

$$= \frac{h}{3}[2Lh^2 + 6N],$$

which, from the simultaneous equations earlier, gives

$$\text{Area} = \frac{h}{3}[y_1 + y_3 + 4y_2].$$

But the area of **every** pair of strips will be dependent only on the three corresponding y co-ordinates, together with the value of h.

Hence, the area of the next pair of strips will be

$$\frac{h}{3}[y_3 + y_5 + 4y_4],$$

and the area of the pair after that will be

$$\frac{h}{3}[y_5 + y_7 + 4y_6].$$

Thus, the total area is given by

$$\text{Area} = \frac{h}{3}[y_1 + y_n + 4(y_2 + y_4 + y_6 + \ldots) + 2(y_3 + y_5 + y_7 + \ldots),$$

usually interpreted as

$$\text{Area} = \frac{h}{3}[\text{First}+\text{Last}+4\times\text{The even numbered } y \text{ co} - \text{ords.}+2\times\text{The remaining } y \text{ co} - \text{ords.}]$$

or

$$\text{Area} = \frac{h}{3}[\text{F} + \text{L} + 4\text{E} + 2\text{R}]$$

This result is known as SIMPSON'S RULE.

Notes:

(i) Since the area of the pairs of strips depends only on the three corresponding y co-ordinates, together with the value of h, the Simpson's rule formula provides an approximate value of the definite integral

$$\int_a^b f(x)\ \mathrm{d}x$$

whatever the values of a and b are, as long as the curve does not cross the x-axis between $x = a$ and $x = b$.

(ii) If the curve **does** cross the x-axis between $x = a$ and $x = b$, it is necessary to consider separately the positive parts of the area above the x-axis and the negative parts below the x-axis.

(iii) The approximate evaluation, by Simpson's rule, of a definite integral should be set out in **tabular form**, as illustrated in the examples overleaf.

EXAMPLES

1. Working to a maximum of three places of decimals throughout, use Simpson's rule with ten divisions to evaluate, approximately, the definite integral

$$\int_0^1 e^{x^2} \, \mathrm{d}x.$$

Solution

x_i	$y_i = e^{x_i^2}$	F & L	E	R
0	1	1		
0.1	1.010		1.010	
0.2	1.041			1.041
0.3	1.094		1.094	
0.4	1.174			1.174
0.5	1.284		1.284	
0.6	1.433			1.433
0.7	1.632		1.632	
0.8	1.896			1.896
0.9	2.248		2.248	
1.0	2.718	2.718		
	F + L $\rightarrow$	3.718	7.268	5.544
	4E $\rightarrow$	29.072	$\times 4$	$\times 2$
	2R $\rightarrow$	11.088	29.072	11.088
	(F + L) + 4E + 2R $\rightarrow$	43.878	//////	//////

Hence,

$$\int_0^1 e^{x^2} \, \mathrm{d}x \simeq \frac{0.1}{3} \times 43.878 \simeq 1.463$$

2. Working to a maximum of three places of decimals throughout, use Simpson's rule with eight divisions between $x = -1$ and $x = 1$ and four divisions between $x = 1$ and $x = 2$ in order to evaluate, approximately, the area between the curve whose equation is

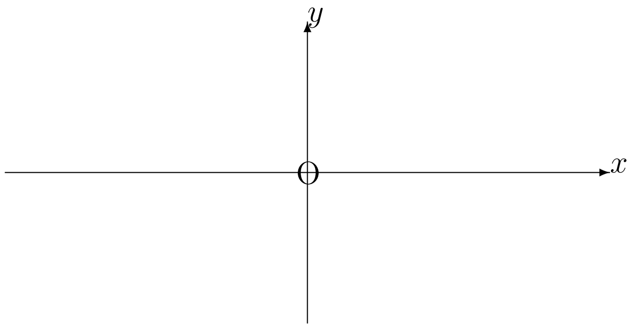
$$y = (x^2 - 1)e^{-x}$$

and the x-axis from $x = -1$ to $x = 2$.

Solution

We note that the curve crosses the x-axis when $x = -1$ and when $x = 1$, the y co-ordinates being negative in the interval between these two values of x and positive outside this interval.

Hence, we need to evaluate the negative area between $x = -1$ and $x = 1$ and the positive area between $x = 1$ and $x = 2$; then we add their numerical values together to find the total area.



(a) The Negative Area

x_i	$y_i = (x^2 - 1)e^{-x}$	F & L	E	R
-1	0	0		
-0.75	-0.926		-0.926	
-0.5	-1.237			-1.237
-0.25	-1.204		-1.204	
0	-1			-1
0.25	-0.730		-0.730	
0.50	-0.455			-0.455
0.75	-0.207		-0.207	
1	0	0		
	F + L $\rightarrow$	0	-2.860	-2.692
	4E $\rightarrow$	-11.440	$\times 4$	$\times 2$
	2R $\rightarrow$	-5.384	-11.440	-5.384
	(F + L) + 4E + 2R $\rightarrow$	-16.824	//////	//////

(b) The Positive Area

x_i	$y_i = (x^2 - 1)e^{-x}$	F & L	E	R
1	0	0		
1.25	0.161		0.161	
1.5	0.279			0.279
1.75	0.358		0.358	
2	0.406	0.406		
	F + L $\rightarrow$	0.406	0.519	0.279
	4E $\rightarrow$	2.076	$\times 4$	$\times 2$
	2R $\rightarrow$	0.558	2.076	0.558
	(F + L) + 4E + 2R $\rightarrow$	3.040	//////	//////

The total area is thus

$$\frac{0.25}{3} \times (16.824 + 3.040) \simeq 1.655$$

17.3.2 EXERCISES

Use Simpson's rule with six divisions of the x-axis to find an approximation for each of the following, working to a maximum of three decimal places throughout:

1.
$$\int_1^7 x \ln x \, \mathrm{d}x.$$

2.
$$\int_{-2}^1 \frac{1}{5 - x^2} \, \mathrm{d}x.$$

3.
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, \mathrm{d}x.$$

4.
$$\int_0^{\frac{\pi}{2}} \sin \sqrt{x^2 + 1} \, \mathrm{d}x.$$

5.
$$\int_0^{\frac{\pi}{2}} \ln(1 + \sin x) \, \mathrm{d}x.$$

17.3.3 ANSWERS TO EXERCISES

1. 35.678 2. 0.882 3. 0.347 4. 1.469 5. 0.743

"JUST THE MATHS"

UNIT NUMBER

17.4

NUMERICAL MATHEMATICS 4
(Further Gaussian elimination)

by

A.J.Hobson

UNIT 17.4 - NUMERICAL MATHEMATICS 4
FURTHER GAUSSIAN ELIMINATION

The **elementary** method of Gaussian Elimination, for simultaneous linear equations, was discussed in Unit 9.4. We introduce, here, a more **general** method, suitable for use with sets of equations having **decimal** coefficients.

17.4.1 GAUSSIAN ELIMINATION BY "PARTIAL PIVOTING" WITH A CHECK COLUMN

Let us first consider an example in which the coefficients are **integers**.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 2x + y + z &= 3, \\ x - 2y - z &= 2, \\ 3x - y + z &= 8. \end{aligned}$$

Solution

We may set out the solution, in the form of a **table** (rather than a **matrix**) indicating each of the **"pivot elements"** in a box as follows:

	x	y	z	constant	Σ
	$\boxed{2}$	1	1	3	7
$\frac{1}{2}$	1	-2	-1	2	0
$\frac{3}{2}$	3	-1	1	8	11
		$\boxed{\frac{-5}{2}}$	$\frac{-3}{2}$	$\frac{1}{2}$	$\frac{-7}{2}$
1		$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
			1	3	4

INSTRUCTIONS

(i) Divide the coefficients of x in lines 2 and 3 by the coefficient of x in line 1 and write the respective results at the side of lines 2 and 3; (that is, $\frac{1}{2}$ and $\frac{3}{2}$ in this case).

(ii) Eliminate x by subtracting $\frac{1}{2}$ times line 1 from line 2 and $\frac{3}{2}$ times line 1 from line 3.