which, from Trigonometric Identities (Unit 3.5), becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}.$$

Finally, using the standard limit (Unit 10.1),

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x.$$

Note:
The derivative of $\cos x$ may be obtained in the same way (see EXERCISES 10.2.5, question 2) but it will also be possible to obtain this later (Unit 10.3) by regarding $\cos x$ as $\sin\left(\frac{\pi}{2} - x\right)$.

3. Differentiate from first principles the function

$$\log_b x$$

where b is any base of logarithms.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_b x] = \lim_{\delta x \to 0} \frac{\log_b(x + \delta x) - \log_b x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\log_b\left(1 + \frac{\delta x}{x}\right)}{\delta x}.$$

But writing

$$\frac{\delta x}{x} = r \quad \text{that is} \quad \delta x = rx,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_b x] = \frac{1}{x}\lim_{r \to 0} \frac{\log_b(1 + r)}{r}$$

$$= \frac{1}{x} \lim_{r \to 0} \log_b (1 + r)^{\frac{1}{r}}.$$

For convenience, we may choose a base of logarithms which causes the limiting value above to equal 1; and this will occur when

$$b = \lim_{r \to 0} (1 + r)^{\frac{1}{r}}.$$

The appropriate value of b turns out to be approximately 2.71828 and this is the standard base of natural logarithms denoted by e.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_e x] = \frac{1}{x}.$$

Note:
In scientific work, the natural logarithm of x is usually denoted by $\ln x$ and this notation will be used in future.

10.2.5 EXERCISES

1. Differentiate from first principles the function $x^3 + 2$.
2. Differentiate from first principles the function $\cos x$.
3. Differentiate from first principles the function $\sqrt{x}$.
   Hint:
   $$(\sqrt{x + \delta x} - \sqrt{x})(\sqrt{x + \delta x} + \sqrt{x}) = \delta x.$$

10.2.6 ANSWERS TO EXERCISES

1. $3x^2$.
2. $-\sin x$.
3. $\frac{1}{2\sqrt{x}}$.

"JUST THE MATHS"

UNIT NUMBER

10.3

DIFFERENTIATION 3
(Elementary techniques of differentiation)

by

A.J.Hobson

UNIT 10.3 - DIFFERENTIATION 3

ELEMENTARY TECHNIQUES OF DIFFERENTIATION

10.3.1 STANDARD DERIVATIVES

In Unit 10.2, reference was made to the use of a table of standard derivatives and such a table can be found in the appendix at the end of this Unit.

However, for the time being, a very short list of standard derivatives is all that is necessary since other derivatives may be developed from them using techniques to be discussed later in this and subsequent Units.

$f(x)$	$f'(x)$
a const.	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$

Note:
In the work which now follows, standard derivatives may be used which have not, here, been obtained from first principles; but the student is expected to be able to quote results from a table of derivatives including those for which no proof has been given.

10.3.2 RULES OF DIFFERENTIATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[Af(x)+Bg(x)\right]=A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]+B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

Proof:

The left-hand side is equivalent to

$$\lim_{\delta x\to 0}\frac{[Af(x+\delta x)+Bg(x+\delta x)]-[Af(x)+Bg(x)]}{\delta x}$$

$$=A\left[\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}\right]+B\left[\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$= A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

The result, so far, deals with a **"linear combination"** of **two** functions of x but is easily extended to linear combinations of **three or more** functions of x.

EXAMPLES

1. Write down the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = 6x^2 + 2x^6 + 13x - 7.$$

Solution

Using the linearity property, the standard derivative of x^n, and the derivative of a constant, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\frac{\mathrm{d}}{\mathrm{d}x}[x^2] + 2\frac{\mathrm{d}}{\mathrm{d}x}[x^6] + 13\frac{\mathrm{d}}{\mathrm{d}x}[x^1] - \frac{\mathrm{d}}{\mathrm{d}x}[7]$$

$$= 12x + 12x^5 + 13.$$

2. Write down the derivative with respect to x of the function

$$\frac{5}{x^2} - 4\sin x + 2\ln x.$$

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{5}{x^2} - 4\sin x + 2\ln x\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[5x^{-2} - 4\sin x + 2\ln x\right]$$

$$= -10x^{-3} - 4\cos x + \frac{2}{x}$$

$$= \frac{-10}{x^3} - 4\cos x + \frac{2}{x}.$$

(b) Composite Functions (or Functions of a Function)

(i) Functions of a Linear Function

Expressions such as $(5x+2)^{16}$, $\sin(2x+3)$ and $\ln(7-4x)$ may be called **"functions of a linear function"** and have the general form

$$f(ax+b),$$

where a and b are constants. The function $f(x)$ would, of course, be the one obtained on replacing $ax+b$ by a single x; hence, in the above illustrations, $f(x)$ would be x^{16}, $\sin x$ and $\ln x$, respectively.

Functions of a linear function may be differentiated as easily as $f(x)$ itself on the strength of the following argument:

Suppose we write

$$y = f(u) \quad \text{where} \quad u = ax+b.$$

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δy in y and δu in u. Then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u}\frac{\delta u}{\delta x}.$$

Assuming that δy and δu tend to zero as δx tends to zero, we can say that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x}.$$

This rule is called the **"Function of a Function Rule"** or **"Composite Function Rule"** or **"Chain Rule"** and has applications to a much wider class of composite functions than has so far been discussed. But, for the moment we restrict the discussion to functions of a linear function.

EXAMPLES

1. Determine $\frac{dy}{dx}$ when $y = (5x+2)^{16}$.

   Solution

   First, we write $y = u^{16}$ where $u = 5x + 2$.

   Then, $\frac{dy}{du} = 16u^{15}$ and $\frac{du}{dx} = 5$.

   Hence, $\frac{dy}{dx} = 16u^{15}.5 = 80(5x+2)^{15}$.

2. Determine $\frac{dy}{dx}$ when $y = \sin(2x+3)$.

   Solution

   First, we write $y = \sin u$ where $u = 2x + 3$.

   Then, $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2$.

   Hence, $\frac{dy}{dx} = \cos u.2 = 2\cos(2x+3)$.

3. Determine $\frac{dy}{dx}$ when $y = \ln(7-4x)$.

   Solution

   First, we write $y = \ln u$ where $u = 7 - 4x$.

   Then, $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = -4$.

   Hence, $\frac{dy}{dx} = \frac{1}{u}.(-4) = \frac{-4}{7-4x}$.

Note:
It is hoped that the student will quickly appreciate how the fastest way to obtain the derivative of a function of a linear function is to treat the expression $ax + b$ initially as if it were a single x; then, multiply the final result by the constant value, a.

(ii) Functions of a Function in general

The formula

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

is in no way dependent on the fact that the examples so far used to illustrate it have involved functions of a linear function. Exactly the same formula may be used for the composite function

$$f[g(x)],$$

whatever the functions $f(x)$ and $g(x)$ happen to be. All we need to do is to write

$$y = f(u) \quad \text{where} \quad u = g(x),$$

then apply the formula.

EXAMPLES

1. Determine an expression for $\frac{dy}{dx}$ in the case when

$$y = (x^2 + 7x - 3)^4.$$

Solution
Letting $y = u^4$ where $u = x^2 + 7x - 3$, we have

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 4u^3.(2x + 7)$$
$$= 4(x^2 + 7x - 3)^3(2x + 7).$$

2. Determine an expression for $\frac{dy}{dx}$ in the case when

$$y = \ln(x^2 - 3x + 1).$$

Solution
Letting $y = \ln u$ where $u = x^2 - 3x + 1$, we have

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{u}.(2x - 3) = \frac{2x - 3}{x^2 - 3x + 1}.$$

3. Determine the value of $\frac{dy}{dx}$ at $x = 1$ in the case when

$$y = 2\sin(5x^2 - 1) + 19x.$$

Solution
Consider, first, the function $2\sin(5x^2 - 1)$ which we shall call z.
Its derivative is $\frac{dz}{dx}$, where $z = 2\sin(5x^2 - 1)$.

Let $z = 2\sin u$ where $u = 5x^2 - 1$; then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos u.10x = 20x\cos(5x^2 - 1).$$

Hence, the complete derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 20x\cos(5x^2 - 1) + 19.$$

Finally, when $x = 1$, this derivative has the value $20\cos 4 + 19$, which is approximately equal to 5.927, remembering that the calculator must be in **radian mode**.

Note:
Again, it is hoped that the student will appreciate how there is a faster way of differentiating composite functions in general. We simply treat $g(x)$ initially as if it were a single x, then multiply by $g'(x)$ afterwards.

For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^3 x\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[(\sin x)^3\right] = 3(\sin x)^2.\cos x = 3\sin^2 x.\cos x.$$

10.3.3 EXERCISES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases:

   (a)
   $$y = 3x^3 - 8x^2 + 11x + 9;$$

   (b)
   $$y = 10\cos x + 5\sin x - 14x^7;$$

   (c)
   $$y = (2x - 7)^5;$$

   (d)
   $$y = (2 - 5x)^{-\frac{5}{2}};$$

(e)

$$y = \sin\left(\frac{\pi}{2} - x\right);\ \text{that is, } \cos x;$$

(f)

$$y = \cos(4x + 1);$$

(g)

$$y = \ln(4 - 2x);$$

(h)

$$y = \ln\left[\frac{3x - 8}{6x + 2}\right].$$

2. Determine an expression for $\frac{dy}{dx}$ in the cases when

(a)

$$y = (4 - 7x^3)^8;$$

(b)

$$y = (x^2 + 1)^{\frac{3}{2}};$$

(c)

$$y = \cos^5 x;$$

(d)

$$y = \ln(\ln x).$$

3. If $y = \sin(\cos x)$, evaluate $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$.

4. If $y = \cos(7x^5 - 3)$, evaluate $\frac{dy}{dx}$ at $x = 1$.

10.3.4 ANSWERS TO EXERCISES

1. (a)

$$9x^2 - 16x + 11;$$

(b)

$$-10\sin x + 5\cos x - 98x^6;$$

(c)

$$10(2x-7)^4;$$

(d)

$$\frac{25}{2}(2-5x)^{-\frac{7}{2}};$$

(e)

$$-\cos\left(\frac{\pi}{2}-x\right);\ \text{that is,}\ \ -\sin x;$$

(f)

$$-4\sin(4x+1);$$

(g)

$$\frac{-2}{4-2x}\ \text{or}\ \frac{2}{2x-4};$$

(h)

$$\frac{3}{3x-8}-\frac{6}{6x+2}=\frac{54}{(3x-8)(6x+2)}.$$

2. (a)

$$-168x^2(4-7x^3)^7;$$

(b)

$$3x(x^2+1)^{\frac{1}{2}};$$

(c)

$$-5\cos^4 x.\sin x;$$

(d)

$$\frac{1}{x\ln x}.$$

3.

$$-1$$

4.

$$-35\sin 4 \cong 26.488$$

APPENDIX - A Table of Standard Derivatives

$f(x)$	$f'(x)$
a (const.)	0
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\text{sec}^2 ax$
$\cot ax$	$-a\text{cosec}^2\ ax$
$\text{sec}\ ax$	$a\text{sec}\ ax.\tan ax$
$\text{cosec}\ ax$	$-a\text{cosec}\ ax.\cot ax$
$\ln x$	$1/x$
e^{ax}	ae^{ax}
a^x	$a^x.\ln a$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\text{sech}^2\ ax$
$\text{sech}\ ax$	$-a\text{sech}\ ax.\tanh ax$
$\text{cosech}\ ax$	$-a\text{cosech}\ ax.\coth x$
$\ln(\sin x)$	$\cot x$
$\ln(\cos x)$	$-\tan x$
$\ln(\sinh x)$	$\coth x$
$\ln(\cosh x)$	$\tanh x$
$\sin^{-1}(x/a)$	$1/\sqrt{(a^2-x^2)}$
$\cos^{-1}(x/a)$	$-1/\sqrt{(a^2-x^2)}$
$\tan^{-1}(x/a)$	$a/(a^2+x^2)$
$\sinh^{-1}(x/a)$	$1/\sqrt{(x^2+a^2)}$
$\cosh^{-1}(x/a)$	$1/\sqrt{(x^2-a^2)}$
$\tanh^{-1}(x/a)$	$a/(a^2-x^2)$

"JUST THE MATHS"

UNIT NUMBER

10.4

DIFFERENTIATION 4
(Products and quotients)
&
(Logarithmic differentiation)

by

A.J.Hobson

UNIT 10.4 - DIFFERENTIATION 4

PRODUCTS, QUOTIENTS AND LOGARITHMIC DIFFERENTIATION

10.4.1 PRODUCTS

Suppose
$$y = u(x)v(x),$$
where $u(x)$ and $v(x)$ are two functions of x.

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δu in u, δv in v and δy in y.

Then,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(u + \delta u)(v + \delta v) - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{uv + u\delta v + v\delta u + \delta u \delta v - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} \right].$$

Hence,
$$\frac{\mathrm{d}}{\mathrm{d}x}[u.v] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Hint: Think of this as
(FIRST x DERIVATIVE OF SECOND) + (SECOND x DERIVATIVE OF FIRST)

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when
$$y = x^7 \cos 3x.$$

   Solution
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^7. - 3\sin 3x + \cos 3x.7x^6 = x^6[7\cos 3x - 3x\sin 3x].$$

2. Evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = -1$ in the case when

$$y = (x^2 - 8)\ln(2x + 3).$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2 - 8).\frac{1}{2x+3}.2 + \ln(2x+3).2x = 2\left[\frac{x^2-8}{2x+3} + x\ln(2x+3)\right].$$

When $x = -1$, this has value -14 since $\ln 1 = 0$.

10.4.2 QUOTIENTS

Suppose, this time, that

$$y = \frac{u(x)}{v(x)}.$$

Then, we may write

$$y = u(x).[v(x)]^{-1}$$

in order to use the rule already known for products.

We obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u.(-1)[v]^{-2}.\frac{\mathrm{d}v}{\mathrm{d}x} + v^{-1}.\frac{\mathrm{d}u}{\mathrm{d}x},$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

EXAMPLES

1. Using the formula for the derivative of a quotient, show that the derivative with respect to x of the function $\tan x$ is the function $\sec^2 x$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin x}{\cos x}\right] = \frac{(\cos x).(\cos x) - (\sin x).(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

2. Determine an expression for $\frac{dy}{dx}$ in the case when

$$y = \frac{2x+1}{(5x-3)^3}.$$

Solution

Using $u(x) \equiv 2x + 1$ and $v(x) \equiv (5x - 3)^3$, we have

$$\frac{dy}{dx} = \frac{(5x-3)^3.2 - (2x+1).3(5x-3)^2.5}{(5x-3)^6}.$$

The expression $(5x - 3)^2$ may be cancelled as a common factor of both numerator and denominator, leaving

$$\frac{dy}{dx} = \frac{(5x-3).2 - 15(2x+1)}{(5x-3)^4} = -\frac{20x+21}{(5x-3)^4}.$$

Note:

The step in the second example above, where a common factor could be cancelled, may be avoided if we use a modified version of the rule for quotients when the function can be considered in the form

$$\frac{u}{v^n}.$$

It can be shown that, if

$$y = \frac{u}{v^n},$$

then,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - nu\frac{dv}{dx}}{v^{n+1}}.$$

For instance, in Example 2 above, we could write

$$u \equiv 2x+1 \quad v \equiv 5x-3 \quad \text{and} \quad n = 3$$

Hence,

$$\frac{dy}{dx} = \frac{(5x-3).2 - 3(2x+1).5}{(5x-3)^4},$$

as before.

10.4.3 LOGARITHMIC DIFFERENTIATION

The algebraic properties of natural logarithms (see Unit 1.4), together with the standard derivative of $\ln x$ and the rules of differentiation, enable us to differentiate two specific kinds of function as described below:

(a) Functions containing a variable index

The most familiar function with which to introduce this technique is the **"exponential function"**, e^x.

Suppose we let

$$y = e^x;$$

then, by properties of natural logarithms, we can write

$$\ln y = x;$$

and, if we differentiate both sides **with respect to** x, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y = e^x.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x\right] = e^x.$$

Notes:

(i) After taking logarithms, we could have differentiated the statement $x = \ln y$ with respect to y, obtaining

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{y}.$$

But it can be shown that, for most functions,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

so that the same result is obtained as before.

(ii) The derivative of e^x may easily be used to establish the standard derivatives of the hyperbolic functions, $\sinh x$, $\cosh x$ and $\tanh x$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh x] = \cosh x, \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cosh x] = \sinh x, \quad \frac{\mathrm{d}}{\mathrm{d}x}[\tanh x] = \mathrm{sech}^2\ x.$$

The first two of these follow from the definitions

$$\sinh x \equiv \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x \equiv \frac{e^x + e^{-x}}{2},$$

while the third may be obtained using the definition

$$\tanh x \equiv \frac{\sinh x}{\cosh x},$$

together with the Quotient Rule.

FURTHER EXAMPLES

1. Write down the derivative with respect to x of the function

$$e^{\sin x}$$

Solution

All that is required in this example is the standard derivative of e^x together with the Function of a Function Rule. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\sin x}\right] = e^{\sin x}.\cos x.$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = (3x+2)^x.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x\ln(3x+2).$$

Differentiating both sides with respect to x and using the Product Rule gives

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x.\frac{3}{3x+2} + \ln(3x+2).1.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x+2)^x\left[\frac{3x}{3x+2} + \ln(3x+2)\right].$$

(b) Products or Quotients with more than two elements

We have already discussed the rules for differentiating products and quotients; but, in certain cases, it is easier to make use of logarithmic differentiation. Essentially, we use this alternative method when a product or a quotient involves more than the two functions $u(x)$ and $v(x)$ mentioned earlier.

We illustrate with examples:

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = e^{x^2}.\cos x.(x+1)^5.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x^2 + \ln(\cos x) + 5\ln(x+1).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{\sin x}{\cos x} + \frac{5}{x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2}.\cos x.(x+1)^5\left[2x - \tan x + \frac{5}{x+1}\right].$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \frac{e^x.\sin x}{(7x+1)^4}.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x + \ln(\sin x) - 4\ln(7x+1).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{\cos x}{\sin x} - 4.\frac{7}{7x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x.\sin x}{(7x+1)^4}\left[1 + \cot x - \frac{28}{7x+1}\right].$$

Note:
In all examples on logarithmic differentiation, the original function will appear as a factor at the beginning of its derivative.

10.4.4 EXERCISES

1. Differentiate the following functions with respect to x:

   (a)
   $$\sin x.\cos x;$$

   (b)
   $$(x^2+3).\sin 2x;$$

   (c)
   $$x.(x^2+1)^{\frac{1}{2}};$$

   (d)
   $$x^2\ln(1-2x).$$

2. Differentiate the following functions with respect to x:

   (a)
   $$\frac{\cos x}{\sin x} \quad \text{(that is,} \quad \cot x);$$

(b)

$$\frac{x^2-2}{(x+1)^2};$$

(c)

$$\frac{\cos x+\sin x}{\cos x-\sin x};$$

(d)

$$\frac{x}{(2x-x^2)^{\frac{1}{2}}}.$$

3. Differentiate the following functions with respect to x:

(a)

$$e^{x^2+1};$$

(b)

$$e^{1-x-x^2};$$

(c)

$$(2x+1)e^{4-x^3};$$

(d)

$$\frac{e^{1-7x}}{3x+2};$$

(e)

$$x.\sinh(x^2+1);$$

(f)

$$\text{sech } x.$$

4. Use logarithms to differentiate the following functions with respect to x:

(a)

$$a^x \quad (a \text{ constant});$$

(b)

$$(x^2+1)^{3x};$$

(c)

$$(\sin x)^x;$$

(d)

$$\frac{x(x-2)}{(x+1)(x+3)};$$

(e)

$$\frac{e^{2x}.\ln x}{(x-1)^3}.$$

10.4.5 ANSWERS TO EXERCISES

1. (a)

$$\cos^2 x - \sin^2 x \quad \text{(or} \quad \cos 2x);$$

(b)

$$2(x^2+3)\cos 2x + 2x\sin 2x;$$

(c)

$$\frac{2x^2+1}{(x^2+1)^{\frac{1}{2}}};$$

(d)

$$2x\ln(1-2x) - \frac{2x^2}{1-2x}.$$

2. (a)

$$-\mathrm{cosec}^2 x;$$

(b)

$$\frac{4+2x}{(x+1)^3};$$

(c)

$$\frac{2}{(\cos x - \sin x)^2};$$

(d)

$$\frac{x}{(2x - x^2)^{\frac{3}{2}}}.$$

3. (a)

$$2xe^{x^2+1};$$

(b)

$$-(1+2x)e^{1-x-x^2};$$

(c)

$$2.e^{4-x^3} - 3x^2(2x+1)e^{4-x^3};$$

(d)

$$-\frac{e^{1-7x}.(21x+17)}{(3x+2)^2};$$

(e)

$$\sinh(x^2+1) + 2x^2\cosh(x^2+1);$$

(f)

$$-\text{cosech}^2\ x.$$

4. (a)

$$a^x.\ln a;$$

(b)

$$(x^2+1)^{3x}\left[3\ln(x^2+1) + \frac{6x^2}{x^2+1}\right];$$

(c)

$$(\sin x)^x\left[\ln\sin x + x\cot x\right];$$

(d)

$$\frac{x(x-2)}{(x+1)(x+3)}\left[\frac{1}{x} + \frac{1}{x-2} - \frac{1}{x+1} - \frac{1}{x+3}\right];$$

(e)

$$\frac{e^{2x}.\ln x}{(x-1)^3}\left[2 + \frac{1}{x\ln x} - \frac{3}{x-1}\right].$$

“JUST THE MATHS”

UNIT NUMBER

10.5

DIFFERENTIATION 5
(Implicit and parametric functions)

by

A.J.Hobson

UNIT 10.5 - DIFFERENTIATION 5
IMPLICIT AND PARAMETRIC FUNCTIONS

10.5.1 IMPLICIT FUNCTIONS

Some relationships between two variables x and y do not give y explicitly in terms of x (or x explicitly in terms of y); but, nevertheless, it is **implied** that one of the two variables is a function of the other. In the work which follows, we shall normally assume that y is a function of x.

Consider, for instance, the relationship

$$x^2 + y^2 = 16,$$

which is not explicit for either x or y but could, if desired, be written in one of the two forms

$$y = \pm\sqrt{16 - x^2} \quad \text{or} \quad x = \pm\sqrt{16 - y^2}.$$

By contrast, consider the relationship

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

In this case, there is no apparent way of stating either variable explicitly in terms of the other; yet we may still wish to calculate $\frac{dy}{dx}$ or even $\frac{dx}{dy}$.

Such relationships between x and y are said to be **"implicit relationships"** and, in the technique of **"implicit differentiation"**, we simply differentiate each term in the relationship with respect to the same variable without attempting to rearrange the formula.

EXAMPLES

1. Determine an expression for $\frac{dy}{dx}$ in the case when

$$x^2 + y^2 = 16.$$

Solution

Treating y^2 as a function of a function, we have

$$2x + 2y\frac{dy}{dx} = 0.$$

Hence,

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

It is perfectly acceptable that the result is expressed in terms of both x and y; this will normally happen.

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2 + 2xy^3 + y^5 = 4.$$

Solution

Treating y^3 and y^5 as functions of a function and using the Product Rule in the second term on the left hand side,

$$2x + 2\left[x.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.1\right] + 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

On rearrangement,

$$\left[6xy^2 + 5y^4\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -(2x + 2y^3).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + 2y^3}{6xy^2 + 5y^4}.$$

3. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

Solution

Differentiating throughout with respect to x and using both the Product Rule and the Function of a Function Rule, we obtain

$$x^2.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.2x + 9\cos(5x - 7y).\left[5 - 7\frac{\mathrm{d}y}{\mathrm{d}x}\right] = 0.$$

On rearrangment,

$$\left[3x^2y^2 - 63\cos(5x - 7y)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[2xy^3 + 45\cos(5x - 7y)\right].$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^3 + 45\cos(5x - 7y)}{3x^2y^2 - 63\cos(5x - 7y)}.$$

10.5.2 PARAMETRIC FUNCTIONS

In the geometry of straight lines, circles etc, we encounter **"parametric equations"** in which the variables x and y, related to each other by a formula, may each be expressed individually in terms of a third variable, usually t or θ, called a **"parameter"**.

In general, we write

$$x = x(t) \quad \text{and} \quad y = y(t);$$

but, in theory, we can imagine that t could be expressed explicitly in terms of x; so, essentially, y is a function of t, where t is a function of x. Hence, from the Function of a Function Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x}.$$

However, we are **not** given t explicitly in terms of x and it may not be practical to obtain it in this form. Therefore, we write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t}.$$

This is the standard formula for differentiating y with respect to x from a pair of parametric equations.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t in the case when

$$x = 3t^2 \quad \text{and} \quad y = 6t.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6 \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 6t.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{6t} = \frac{1}{t}.$$

2. Determine an expression for $\frac{dy}{dx}$ in terms of θ in the case when

$$x = \sin^3\theta \quad \text{and} \quad y = \cos^3\theta.$$

Solution

$$\frac{dx}{d\theta} = 3\sin^2\theta.\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = -3\cos^2\theta.\sin\theta.$$

Hence,

$$\frac{dy}{dx} = \frac{-3\cos^2\theta.\sin\theta}{3\sin^2\theta.\cos\theta}.$$

That is,

$$\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$$

10.5.3 EXERCISES

1. Determine an expression for $\frac{dy}{dx}$ in the following cases:

   (a)
   $$x^2 + y^2 = 10x;$$
   (b)
   $$x^3 + y^3 - 3xy^2 = 8;$$
   (c)
   $$x^4 + 2x^2y^2 + y^4 = x;$$
   (d)
   $$xe^y = \cos y.$$

2. Determine an expression for $\frac{dy}{dx}$ in terms of the appropriate parameter in the following cases:

   (a)
   $$x = 3\sin\theta \quad \text{and} \quad y = 4\cos\theta;$$
   (b)
   $$x = 4t \quad \text{and} \quad y = \frac{4}{t};$$

(c)

$$x = (1-t)^{\frac{1}{2}} \text{ and } y = (1-t^2)^{\frac{1}{2}}.$$

10.5.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5-x}{y};$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-x^2}{y^2-2xy};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-4x^3-4xy^2}{4\left(x^2y+y^3\right)};$$

(d)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^y}{xe^y+\sin y}.$$

2. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{3}\tan\theta;$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t}{(1+t)^{\frac{1}{2}}}.$$

"JUST THE MATHS"

UNIT NUMBER

10.6

DIFFERENTIATION 6
(Inverse trigonometric functions)

by

A.J.Hobson

UNIT 10.6 - DIFFERENTIATION 6

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

10.6.1 SUMMARY OF RESULTS

The derivatives of inverse trigonometric functions should be considered as standard results. They will be stated here first, before their proofs are discussed.

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}},$$
where $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}},$$
where $0 \leq \cos^{-1}x \leq \pi$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2},$$
where $-\frac{\pi}{2} \leq \tan^{-1}x \leq \frac{\pi}{2}$.

10.6.2 THE DERIVATIVE OF AN INVERSE SINE

We shall consider the formula
$$y = \mathrm{Sin}^{-1}x$$
and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case S in the formula; the reason will be explained later.

The formula is equivalent to
$$x = \sin y,$$
so we may say that
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \equiv \pm\sqrt{1-\sin^2 y} \equiv \pm\sqrt{1-x^2}.$$

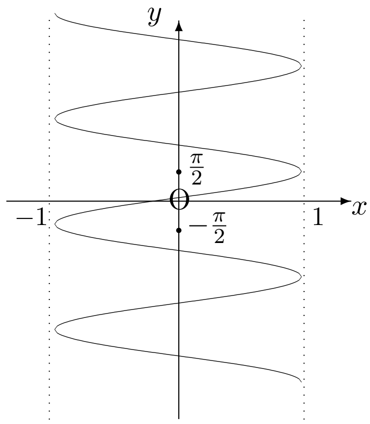
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Sin}^{-1}x,$$

which may be obtained from the graph of $y = \sin x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i)The variable x must lie in the interval $-1 \leq x \leq 1$;

(ii) For each value of x in the interval $-1 \leq x \leq 1$, the variable y has infinitely many values which are spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each value of x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is positive and the other negative.

(iv) By restricting the discussion to the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y and one (positive) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of x in the interval $-1 \leq x \leq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse sine function and is denoted by $\sin^{-1}x$ using a lower-case s.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}.$$

10.6.3 THE DERIVATIVE OF AN INVERSE COSINE

We shall consider the formula

$$y = \mathrm{Cos}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case C in the formula; the reason will be explained later.

The formula is equivalent to

$$x = \cos y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y \equiv \pm\sqrt{1-\cos^2 y} \equiv \pm\sqrt{1-x^2}.$$

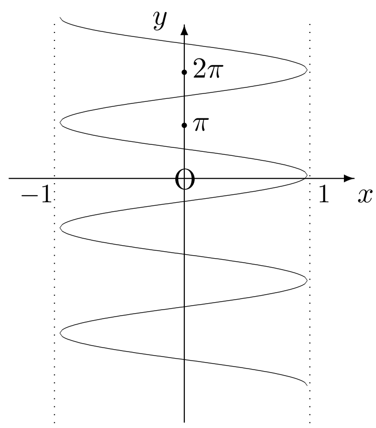
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Cos}^{-1}x$$

which may be obtained from the graph of $y = \cos x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x must lie in the interval $-1 \leq x \leq 1$;

(ii) For each value of x in the interval $-1 \leq x \leq 1$, the variable y has infinitely many values which are spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each value of x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is positive and the other negative.

(iv) By restricting the discussion to the part of the graph from $y = 0$ to $y = \pi$, we may distinguish the results from those of the inverse sine function; and there will be only one value of y with one (negative) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of x in the interval $-1 \leq x \leq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse cosine function and is denoted by $\cos^{-1}x$ using a lower-case c.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}}.$$

10.6.4 THE DERIVATIVE OF AN INVERSE TANGENT

We shall consider the formula

$$y = \mathrm{Tan}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case T in the formula; the reason will be explained later.

The formula is equivalent to

$$x = \tan y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sec^2 y \equiv 1 + \tan^2 y \equiv 1 + x^2.$$

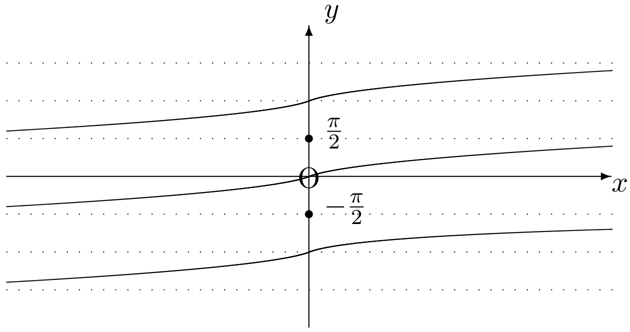
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tan}^{-1}x,$$

which may be obtained from the graph of $y = \tan x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x may lie anywhere in the interval $-\infty < x < \infty$;

(ii) For each value of x, the variable y has infinitely many values which are spaced at regular intervals of π.

(iii) For each value of x, there is only possible value of $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is positive.

(iv) By restricting the discussion to the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y for each value of x.

The restricted part of the graph defines what is called the **"principal value"** of the inverse tangent function and is denoted by $\tan^{-1}x$ using a lower-case t.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2}.$$

ILLUSTRATIONS

1.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}2x] = \frac{2}{\sqrt{1-4x^2}}.$$

2.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}(x+3)] = -\frac{1}{\sqrt{1-(x+3)^2}}.$$

3.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}(\sin x)] = \frac{\cos x}{1+\sin^2 x}.$$

4.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}\left(x^5\right)] = \frac{5x^4}{\sqrt{1-x^{10}}} \quad \text{(real only if} \quad -1 < x < 1).$$

10.6.5 EXERCISES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases, assuming any necessary restrictions on the values of x:

   (a)

   $$y = \cos^{-1} 7x;$$

   (b)

   $$y = \tan^{-1}(\cos x);$$

   (c)

   $$y = \sin^{-1}(3-2x).$$

2. Show that

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)\right] = 1.$$

3. If

   $$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}},$$

   show that

   (a)

   $$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 1;$$

   (b)

   $$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}y}{\mathrm{d}x} = y.$$

10.6.6 ANSWERS TO EXERCISES

1. (a)

$$-\frac{7}{\sqrt{1-49x^2}} \quad \left(\text{real only if} \quad -\frac{1}{7} < x < \frac{1}{7}\right);$$

(b)

$$-\frac{\sin x}{1+\cos^2 x};$$

(c)

$$-\frac{2}{\sqrt{1-(3-2x)^2}} \quad (\text{real only if} \quad 1 < x < 2).$$

"JUST THE MATHS"

UNIT NUMBER

10.7

DIFFERENTIATION 7
(Inverse hyperbolic functions)

by

A.J.Hobson

UNIT 10.7 - DIFFERENTIATION

DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

10.7.1 SUMMARY OF RESULTS

The derivatives of inverse trigonometric and inverse hyperbolic functions should be considered as standard results. They will be stated here, first, before their proofs are discussed.

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}},$$
where $-\infty < \sinh^{-1}x < \infty$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2-1}},$$
where $\cosh^{-1}x \geq 0$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh^{-1}x] = \frac{1}{1-x^2},$$
where $-\infty < \tanh^{-1}x < \infty$.

10.7.2 THE DERIVATIVE OF AN INVERSE HYPERBOLIC SINE

We shall consider the formula
$$y = \mathrm{Sinh}^{-1}x$$
and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
The use of the upper-case S in the formula is temporary; and the reason will be explained shortly.

The formula is equivalent to
$$x = \sinh y,$$
so we may say that
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cosh y \equiv \sqrt{1+\sin^2 y} \equiv \sqrt{1+x^2},$$

noting that $\cosh y$ is never negative.

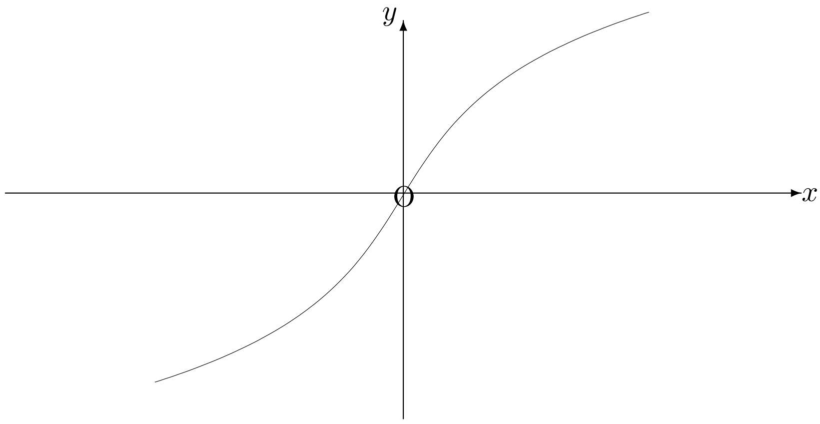
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1+x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Sinh}^{-1}x$$

which may be obtained from the graph of $y = \sinh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x may lie anywhere in the interval $-\infty < x < \infty$.

(ii) For each value of x, the variable y has only one value.

(iii) For each value of x, there is only one possible value of $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is positive.

(iv) In this case (unlike the case of an inverse sine, in Unit 10.6) there is no need to distinguish between a general value and a principal value of the inverse hyperbolic sine function. This is because there is only one value of both the function and its derivative.

However, it is customary to denote the inverse function by $\sinh^{-1}x$, using a lower-case s rather than an upper-case S.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}}.$$

10.7.3 THE DERIVATIVE OF AN INVERSE HYPERBOLIC COSINE

We shall consider the formula

$$y = \mathrm{Cosh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case C in the formula; the reason will be explained shortly.

The formula is equivalent to

$$x = \cosh y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sinh y \equiv \pm\sqrt{\cosh^2 y - 1} \equiv \pm\sqrt{x^2 - 1}.$$

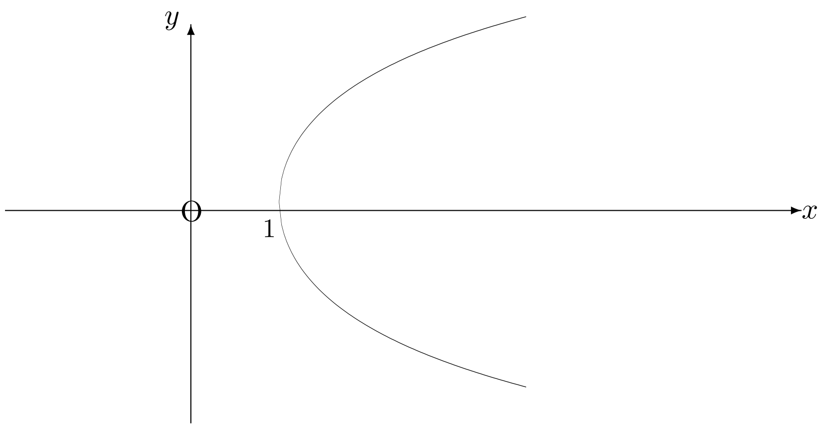
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - 1}}.$$

Consider now the graph of the formula

$$y = \mathrm{Cosh}^{-1}x,$$

which may be obtained from the graph of $y = \cosh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x must lie in the interval $x \geq 1$.

(ii) For each value of x in the interval $x > 1$, the variable y has two values one of which is positive and the other negative.

(iii) For each value of x in the interval $x > 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is postive and the other negative.

(iv) By restricting the discussion to the part of the graph for which $y \geq 0$, there will be only one value of y with one (positive) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of x in the interval $x \geq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse cosine function and is denoted by $\cosh^{-1}x$, using a lower-case c.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2 - 1}}.$$

10.7.4 THE DERIVATIVE OF AN INVERSE HYPERBOLIC TANGENT

We shall consider the formula

$$y = \mathrm{Tanh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
The use of the upper-case T in the formula is temporary; and the reason will be explained later.

The formula is equivalent to

$$x = \tanh y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{sech}^2 y \equiv 1 - \tanh^2 y \equiv 1 - x^2.$$

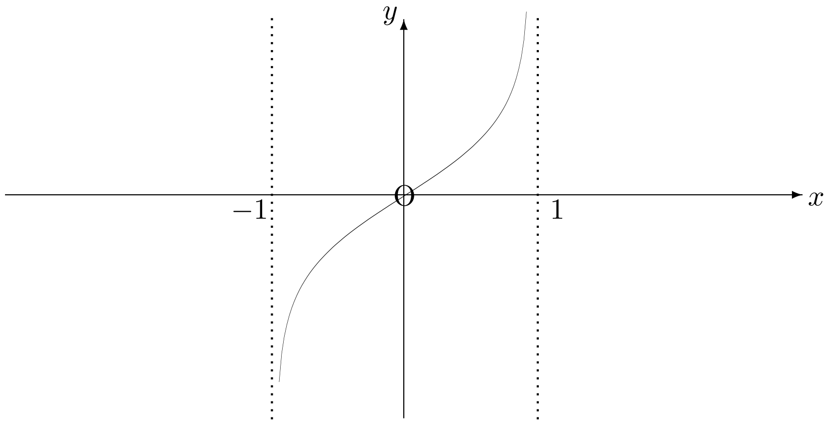
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 - x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tanh}^{-1}x,$$

which may be obtained from the graph of $y = \tanh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x must lie in the interval $-1 < x < 1$.

(ii) For each value of x in the interval $-1 < x < 1$, the variable y has just one value.

(iii) For each value of x in the interval $-1 < x < 1$, there is only possible value of $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is positive.

(iv) In this case (unlike the case of an inverse tangent in Unit 10.6) there is no need to distinguish between a general value and a principal value of the inverse hyperbolic tangent function. This is because there is only one value of both the function and its derivative.

However, it is customary to denote the inverse hyperbolic tangent by $\tan^{-1}x$ using a lower-case t rather than an upper-case T.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh^{-1}x] = \frac{1}{1-x^2}.$$

ILLUSTRATIONS

1.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}(\tanh x)] = \frac{\mathrm{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \mathrm{sech} x.$$

2.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}(5x-4)] = \frac{5}{\sqrt{(5x-4)^2-1}},$$

assuming that $5x - 4 \geq 1$; that is, $x \geq 1$.

10.7.5 EXERCISES

Obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases:

1.

$$y = \cosh^{-1}(4+3x),$$

assuming that $4 + 3x \geq 1$; that is, $x \geq -1$.

2.

$$y = \cos^{-1}(\sinh x),$$

assuming that $-1 \leq \sinh x \leq 1$.

3.

$$y = \tanh^{-1}(\cos x).$$

4.

$$y = \cosh^{-1}(x^3),$$

assuming that $x \geq 1$.

5.

$$y = \tanh^{-1}\frac{2x}{1+x^2}.$$

10.7.6 ANSWERS TO EXERCISES

1.

$$\frac{3}{\sqrt{(4+3x)^2-1}}.$$

2.

$$-\frac{\cosh x}{\sqrt{1-\sinh^2 x}}.$$

3.

$$-\text{cosec}x.$$

4.

$$\frac{3x^2}{\sqrt{x^6-1}}.$$

5.

$$\frac{2}{1-x^2}.$$

"JUST THE MATHS"

UNIT NUMBER

10.8

DIFFERENTIATION 8
(Higher derivatives)

by

A.J.Hobson

UNIT 10.8 - DIFFERENTIATION 8

HIGHER DERIVATIVES

10.8.1 THE THEORY

In most of the examples (seen in earlier Units) on differentiating a function of x with respect to x, the result obtained has been **another** function of x. In general, this **will** be the case and the possibilty arises of differentiating again with respect to x.

This would occur, for example, in the case when the formula

$$y = f(x)$$

represents the distance, y, travelled by a moving object at time, x.

The **speed** of the moving object is the rate of increase of distance with respect to time; that is, $\frac{\mathrm{d}y}{\mathrm{d}x}$. But a second quantity called **acceleration** is defined as the rate of increase of speed with respect to time. It is therefore represented by the symbol

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right];$$

but this is usually written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

and is pronounced "d two y by dx squared".

We could, if necessary, differentiate over and over again to obtain the derivatives of order three, four, etc., namely

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \text{ and } \frac{\mathrm{d}^4y}{\mathrm{d}x^4}, \quad \text{etc.}$$

EXAMPLES

1. If $y = \sin 2x$, show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 0.$$

Solution

Firstly,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x,$$

so that, on differentiating a second time, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\sin 2x = -4y.$$

Hence, the result follows.

2. If $y = x^4$, show that every derivative of y with respect to x after the fourth derivative is zero.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3;$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2;$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 24x;$$

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 24.$$

We now have a constant function, so that all future derivatives will be zero.

Note:

In general, every derivative of $y = x^n$ after the n-th derivative will be zero.

3. If $x = 3t^2$ and $y = 6t$, obtain an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{6t} = \frac{1}{t}.$$

In order to differentiate again with respect to x, we observe that, in the formula for the first derivative with respect to x, we need to replace y with $\frac{\mathrm{d}y}{\mathrm{d}x}$.

That is,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\mathrm{d}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\mathrm{d}x}.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

In the present example, therefore, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}\right]}{6t} = \frac{\frac{-1}{t^2}}{6t} = -\frac{1}{6t^3}.$$

Note:

For a function $f(x)$, an alternative notation for the derivatives of order two, three, four, etc. is

$$f''(x),\ f'''(x),\ f^{(iv)}(x),\ \text{etc.}$$

10.8.2 EXERCISES

1. Obtain expressions for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in the following cases:

   (a)

   $$y = 4x^3 - 7x^2 + 5x - 17;$$

   (b)

   $$y = (3x-2)^{10};$$

   (c)

   $$y = x^2e^{4x};$$

   (d)

   $$y = \frac{x-1}{x+1}.$$

2. If $y = \sin 3x$, evaluate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = \frac{\pi}{4}$.

3. If $x^2 + y^2 - 2x + 2y = 23$ determine the values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point where $x = -2$ and $y = 3$.

4. If $x = 3(1-\cos\theta)$ and $y = 3(\theta - \sin\theta)$, show that

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\frac{\theta}{2};$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{12\sin\frac{\theta}{2}\cos^3\frac{\theta}{2}}.$$

5. If $y = 3e^{2x}\cos(2x-3)$, verify that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 0.$$

10.8.3 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 14;$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 810(3x-2)^8;$$

(c)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = e^{4x}\left[16x^2 + 16x + 2\right];$$

(d)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{4}{(x+1^3}.$$

2.

$$-\frac{9}{\sqrt{2}}.$$

3.

$$\frac{3}{4} \text{ and } -\frac{25}{64}.$$

"JUST THE MATHS"

UNIT NUMBER

11.1

DIFFERENTIATION APPLICATIONS 1
(Tangents and normals)

by

A.J.Hobson

UNIT 11.1 - APPLICATIONS OF DIFFERENTIATION 1

TANGENTS AND NORMALS

11.1.1 TANGENTS

In the definition of a derivative (Unit 10.2), it is explained that the derivative of the function $f(x)$ can be interpreted as the gradient of the tangent to the curve $y = f(x)$ at the point (x, y).

We may now use this information, together with the geometry of the straight line, in order to determine the equation of the tangent to a given curve at a particular point on it.

We illustrate with examples which will then be used also in the subsequent paragraph dealing with normals.

EXAMPLES

1. Determine the equation of the tangent at the point $(-1, 2)$ to the curve whose equation is
$$y = 2x^3 + 5x^2 - 2x - 3.$$

   Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 10x - 2,$$

   which takes the value -6 when $x = -1$.

   Hence the tangent is the straight line passing through the point $(-1, 2)$ having gradient -6. Its equation is therefore
$$y - 2 = -6(x + 1).$$

   That is,
$$6x + y + 4 = 0.$$

2. Determine the equation of the tangent at the point $(2, -2)$ to the curve to the curve whose equation is
$$x^2 + y^2 + 3xy + 4 = 0.$$

   Solution

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3\left[x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right] = 0.$$

   That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x+3y}{3x+2y},$$

which takes the value -2 at the point $(2, -2)$.

Hence, the equation of the tangent is

$$y + 2 = -2(x - 2).$$

That is,

$$2x + y - 2 = 0.$$

3. Determine the equation of the tangent at the point where $t = 2$ to the curve given parametrically by

$$x = \frac{3t}{1+t} \quad \text{and} \quad y = \frac{t^2}{1+t}.$$

Solution

We note first that the point at which $t = 2$ has co-ordinates $\left(2, \frac{4}{3}\right)$.

Furthermore,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{(1+t)^2} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2t+t^2}{(1+t)^2},$$

by the quotient rule.

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t+t^2}{3},$$

which takes the value $\frac{8}{3}$ when $t = 2$.

Hence, the equation of the tangent is

$$y - \frac{4}{3} = \frac{8}{3}(x - 2).$$

That is,

$$3y + 12 = 8x.$$

11.1.2 NORMALS

The normal to a curve at a point on it is defined to be a straight line passing through this point and perpendicular to the tangent there.

Using previous work on perpendicular lines (Unit 5.2), if the gradient of the tangent is m, then the gradient of the normal will be $-\frac{1}{m}$.

EXAMPLES

In the examples of section 11.1.1, therefore, the normals to each curve at the point given will have equations as follows:

1.
$$y - 2 = \frac{1}{6}(x + 1).$$
That is,
$$6y = x + 13.$$

2.
$$y + 2 = \frac{1}{2}(x - 2).$$
That is,
$$2y = x - 6.$$

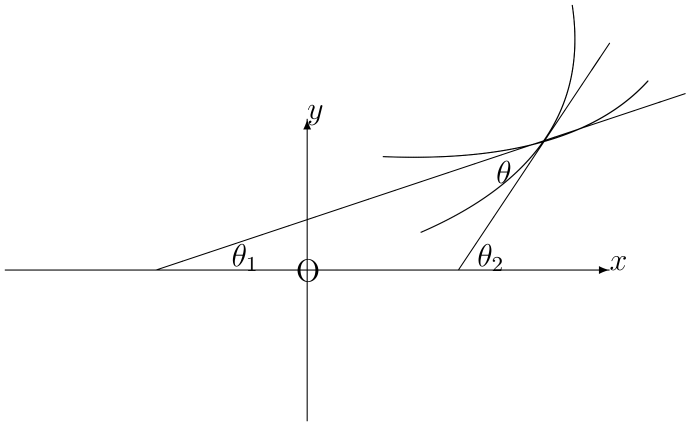
3.
$$y - \frac{4}{3} = -\frac{3}{8}(x - 2).$$
That is,
$$24y + 9x = 50.$$

Note:
It may occasionally be required to determine the angle, θ, between two curves at one of their points of intersection. This is defined to be the angle between the tangents at this point; and, if the gradients of the tangents are $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, then the angle $\theta \equiv \theta_2 - \theta_1$ and is given by

$$\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1}.$$

That is,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$



11.1.3 EXERCISES

1. Determine the equations of the tangent and normal to the following curves at the point given:

   (a)

$$8y = x^3 \quad \text{at} \quad (2, 1);$$

   (b)

$$y = \frac{e^{2x}\cos x}{(1+x)^3} \quad \text{at} \quad (0, 1).$$

2. The parametric equations of a curve are

$$x = 1 + \sin 2t, \quad y = 1 + \cos t + \cos 2t.$$

   Determine the equation of the tangent to the curve at the point for which $t = \frac{\pi}{2}$.

3. Determine the quation of the tangent at the point $(2, 3)$ to the curve whose equation is
$$3x^2 + 2y^2 = 30.$$

4. Determine the equation of the normal at the point $(-1, 2)$ to the curve whose equation is
$$2xy + 3xy^2 - x^2 + y^3 + 9 = 0.$$

11.1.4 ANSWERS TO EXERCISES

1. (a) The tangent is
$$2y = 3x - 4,$$
and the normal is
$$2x + 3y = 7;$$
(b) The tangent is
$$y = 1 - x,$$
and the normal is
$$y = x + 1.$$

2. The tangent is
$$2y = x - 1.$$

3. The tangent is
$$x + y = 5.$$

4. The normal is
$$x + 9y = 17.$$

"JUST THE MATHS"

UNIT NUMBER

11.2

DIFFERENTIATION APPLICATIONS 2
(Local maxima and local minima)
&
(Points of inflexion)

by

A.J.Hobson

UNIT 11.2 - APPLICATIONS OF DIFFERENTIATION 2

LOCAL MAXIMA, LOCAL MINIMA AND POINTS OF INFLEXION

11.2.1 INTRODUCTION

(a) Let us first suppose that the formula

$$s = f(t)$$

represents the distance s, travelled in time t, by a moving object from some previously chosen point on its journey.

The derivative , $\frac{ds}{dt}$, of s with respect to t gives the speed of the object at time t and can be represented by the slope of the tangent at the point (t, s) to the curve whose equation is $s = f(t)$.

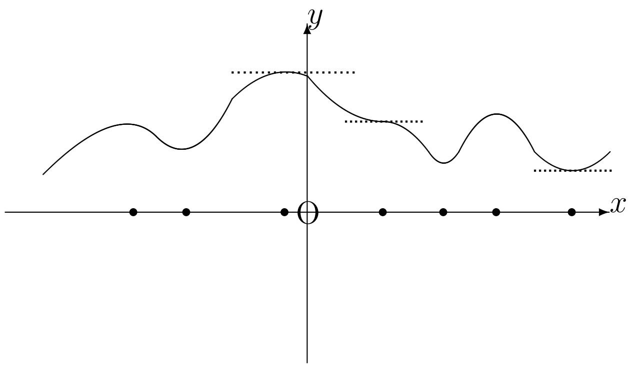
For any instant, t_0, of time, at which the object is stationary, the value of the derivative will be zero and hence, the slope of the tangent will be zero.

The corresponding point (t_0, s_0), on the graph may thus be called a **"stationary point"**.

(b) More generally, any relationship,

$$y = f(x),$$

between two variable quantities, x and y, can usually be represented by a graph of y against x and any point (x_0, y_0) on the graph at which $\frac{dy}{dx}$ takes the value zero is called a "stationary point". The tangent to the curve at the point (x_0, y_0) will be parallel to x-axis.



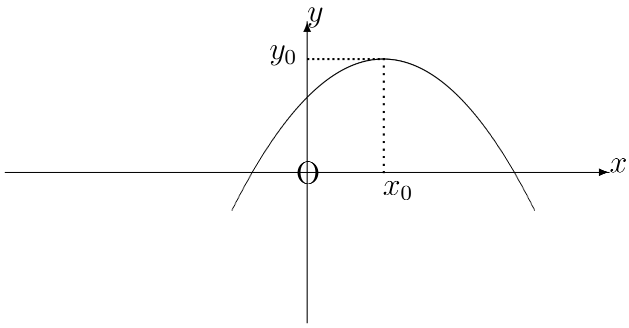
In the paragraphs which follow, we shall discuss the definitions and properties of particular kinds of stationary point.

11.2.2 LOCAL MAXIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local maximum"** if y_0 is greater than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



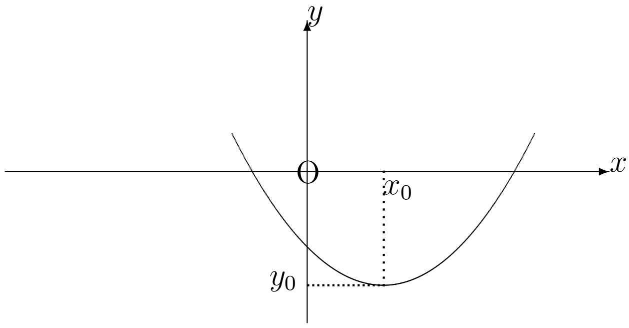
Note:
It may well happen that, for points on the curve which are some distance away from (x_0, y_0), their y co-ordinates are greater than y_0; hence, the definition of a local maximum point must refer to the behaviour of y in the immediate neighbourhood of the point.

11.2.3 LOCAL MINIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local minimum"** if y_0 is less than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



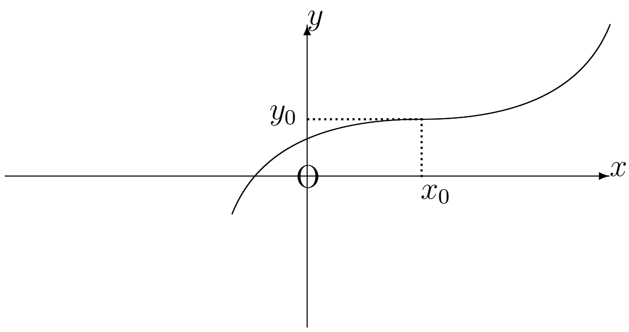
Note:
It may well happen that, for points on the curve which are some distance away from (x_0, y_0), their y co-ordinates are less than y_0; hence, the definition of a local minimum point must refer to the behaviour of y in the immediate neighbourhood of the point.

11.2.4 POINTS OF INFLEXION

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"point of inflexion"** if the curve exhibits a change in the direction bending there.



11.2.5 THE LOCATION OF STATIONARY POINTS AND THEIR NATURE

In order to determine the location of any stationary points on the curve whose equation is

$$y = f(x),$$

we simply obtain an expression for the derivate of y with respect to x, then equate it to zero. That is, we solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Having located a stationary point (x_0, y_0), we may then determine whether it is a local maximum, a local minimum, or a point of inflexion using two alternative methods. These methods will be illustrated by examples:

METHOD 1. - The "First Derivative" Method

Suppose ϵ denotes a number which is relatively small compared with x_0.

If we examine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$, first at $x = x_0 - \epsilon$ and then at $x = x_0 + \epsilon$, the following conclusions may be drawn:

(a) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from positive to negative, there is a local maximum at (x_0, y_0).

(b) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from negative to positive, there is a local minimum at (x_0, y_0).

(c) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not change, there is a point of inflexion at (x_0, y_0).

EXAMPLES

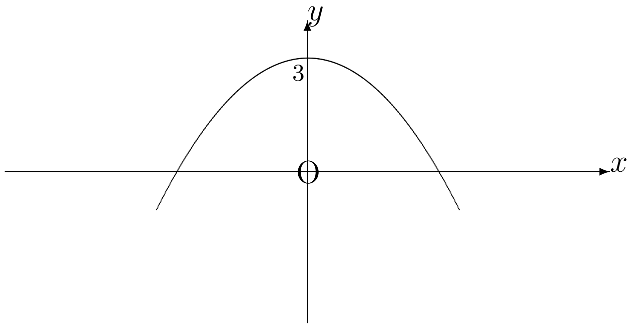
1. Determine the stationary point on the graph whose equation is

$$y = 3 - x^2.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x,$$

which is equal to zero at the point where $x = 0$ and hence, $y = 3$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and
If $x = 0 + \epsilon$, (for example $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.
Hence, there is a local maximum at the point $(0, 3)$.



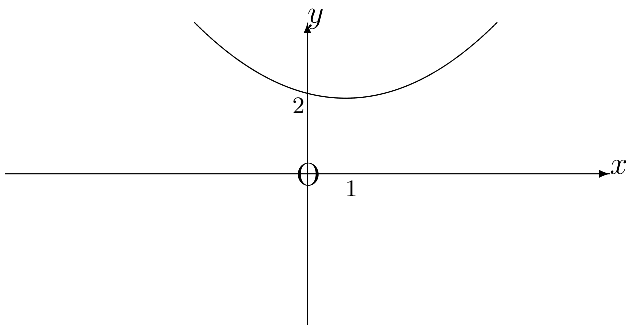
2. Determine the stationary point on the graph whose equation is

$$y = x^2 - 2x + 3.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2,$$

which is equal to zero at the point where $x = 1$ and hence, $y = 2$.
If $x = 1 - \epsilon$, (for example, $x = 1 - 0.01 = 0.99$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ and
If $x = 1 + \epsilon$, (for example, $x = 1 + 0.01) = 1.01$, then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a local minimum at the point $(1, 2)$.



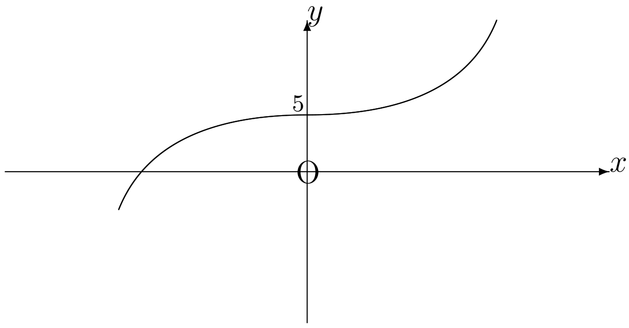
3. Determine the stationary point on the graph whose equation is

$$y = 5 + x^3.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

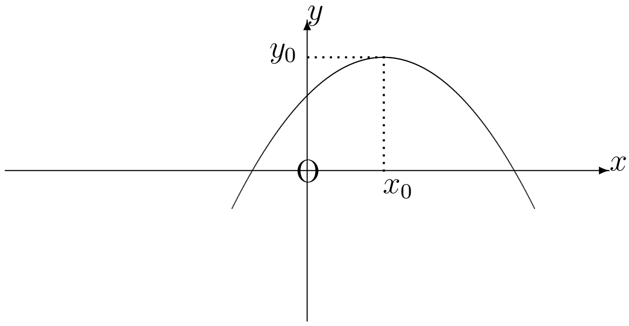
which is equal to zero at the point where $x = 0$ and hence, $y = 5$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and
If $x = 0 + \epsilon$, (for example, $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a point of inflexion at $(0, 5)$.



METHOD 2. - The "Second Derivative" Method

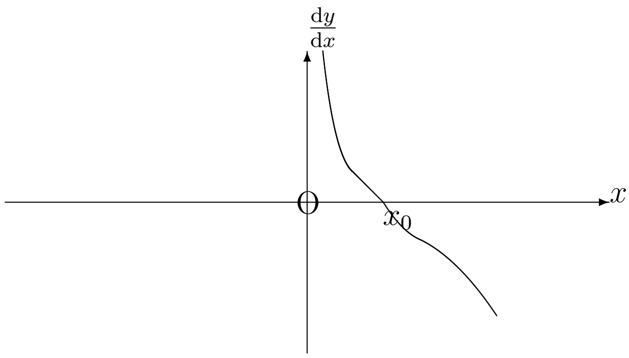
This method considers the general appearance of the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x, which is called the **"first derived curve"**. The properties of the first derived curve in the neighbourhood of a stationary point (x_0, y_0) may be used to predict the nature of this point.

(a) Local Maxima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ steadily decrease from large positive values to large negative values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going downwards"** tendency at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is **negative**. In other words,

$$\frac{d^2y}{dx^2} < 0 \quad \text{at} \quad x = x_0.$$

This is the second derivative test for a local maximum.

EXAMPLE

For the curve whose equation is

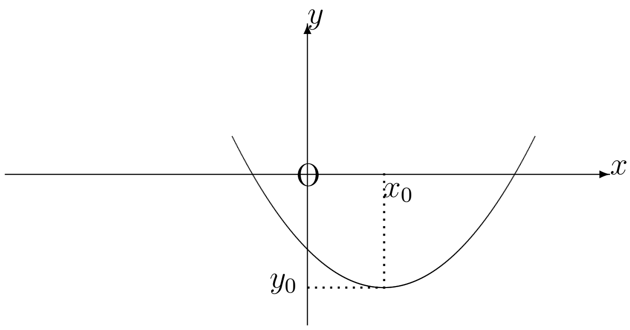
$$y = 3 - x^2,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2.$$

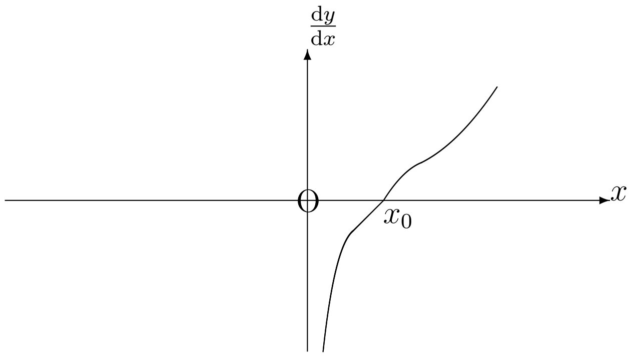
The second derivative is negative everywhere, so it is certainly negative at the stationary point $(0,3)$ obtained in the previous method. Hence, $(0,3)$ is a local maximum.

(b) Local Minima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ steadily increase from large negative values to large positive values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going upwards"** tendency at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is **positive**. In other words,

$$\frac{d^2y}{dx^2} > 0 \quad \text{at} \quad x = x_0.$$

This is the second derivative test for a local minimum.

EXAMPLE

For the curve whose equation is

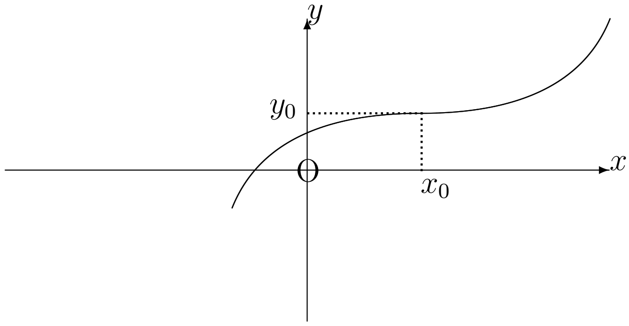
$$y = x^2 - 2x + 3,$$

we have

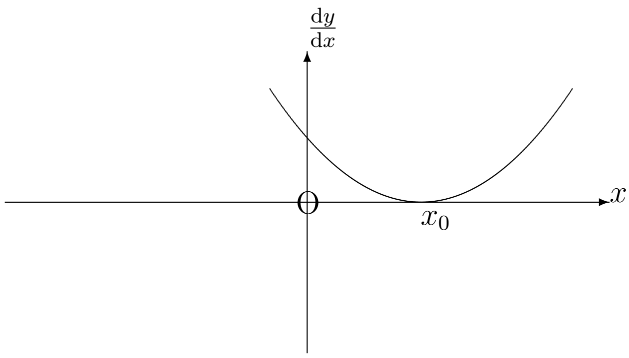
$$\frac{dy}{dx} = 2x - 2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 2.$$

The second derivative is positive everywhere, so it is certainly positive at the stationary point $(1, 2)$ obtained in the previous method. Hence, $(1, 2)$ is a local minimum.

(c) Points of inflexion



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ appear to reach either a minimum or a maximum value at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is zero and changes sign as x passes through the value x_0.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \text{at} \quad x = x_0 \quad \text{and} \quad \text{changes} \quad \text{sign.}$$

This is the second derivative test for a point of inflexion.

EXAMPLE

For the curve whose equation is

$$y = 5 + x^3,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

The second derivative is zero when $x = 0$ and changes sign as x passes through the value zero.

Hence, the stationary point $(0, 5)$ found previously is a point of inflexion.

Notes:
(i) For a stationary point of inflexion, it is not enough that

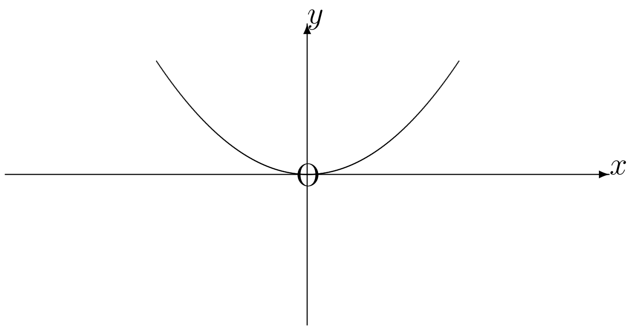
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

without also the change of sign.

For example, the curve whose equation is

$$y = x^4$$

is easily shown (by Method 1) to have a local minimum at the point $(0, 0)$; and yet, for this curve, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$.



(ii) Some curves contain what are called **"ordinary points of inflexion"**. They are not stationary points and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$; but the rest of the condition for a point of inflexion still holds. That is,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \text{and} \quad \text{changes} \quad \text{sign.}$$

EXAMPLE

For the curve whose equation is

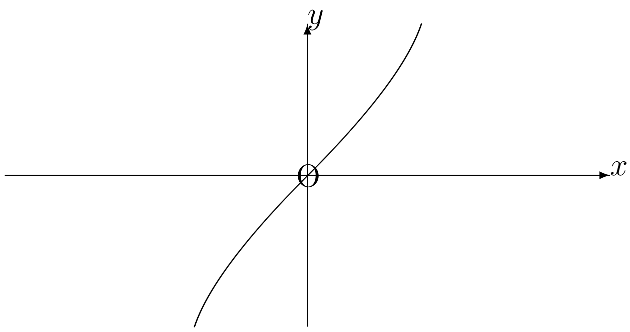
$$y = x^3 + x,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

Hence, there are no stationary points at all; but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$ and changes sign as x passes through $x = 0$.

That is, there is an ordinary point of inflexion at $(0, 0)$.



Notes:
(i) In any interval of the x-axis, the greatest value of a function of x will be either the greatest maximum or possibly the value at one end of the interval. Similarly, the least value of the function will be either the smallest minimum or possibly the value at one end of the interval.

(ii) In sketching a curve whose maxima, minima and points of inflexion are known, it may also be necessary to determine, from the equation of the curve, its points of intersection with the axes of reference.

11.2.6 EXERCISES

1. Determine the local maxima, local minima and points of inflexion (including ordinary points of inflexion) on the curves whose equations are given in the following:

   (a)
   $$y = x^3 - 6x^2 + 9x + 6;$$

   (b)
   $$y = x + \frac{1}{x}.$$

   In each case, give also a sketch of the curve.

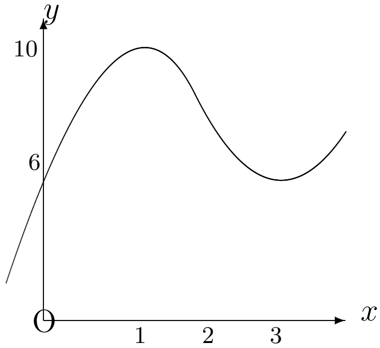
2. Show that the curve whose equation is
   $$y = \frac{1}{2x+1} + \ln(2x+1)$$
   has a local minimum at a point on the y-axis.

3. The horse-power, P, transmitted by a belt is given by
   $$P = k\left[Tv - \frac{wv^3}{g}\right],$$
   where k is a constant, v is the speed of the belt, T is the tension on the driving side and w is the weight per unit length of the belt. Determine the speed for which the horse-power is a maximum.

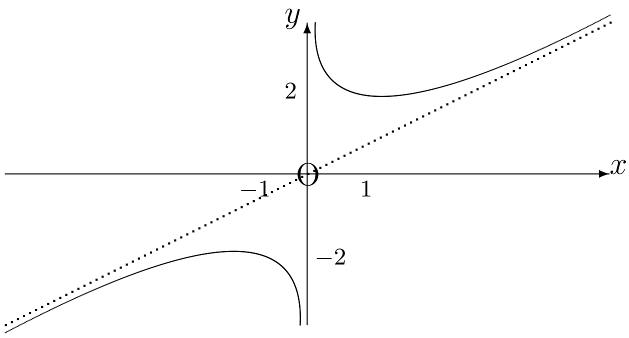
4. For x lying in the interval $-3 \leq x \leq 5$, determine the least and greatest values of the function
   $$x^3 - 12x + 20$$

11.2.7 ANSWERS TO EXERCISES

1. (a) Local maximum at $(1, 10)$, local minumum at $(3, 6)$, ordinary point of inflexion at $(2, 8)$;



(b) Local maximum at $(-1, -2)$, local minimum at $(1, 2)$.



2. Local minimum at the point $(0, 1)$.
3. The horse-power is maximum when

$$v = \sqrt{\frac{gT}{2w}}.$$

4. The greatest value is 85 at $(5, 85)$; the least value is 4 at $(2, 4)$.

"JUST THE MATHS"

UNIT NUMBER

11.3

DIFFERENTIATION APPLICATIONS 3 (Curvature)

by

A.J.Hobson

UNIT 11.3 - DIFFERENTIATION APPLICATIONS 3

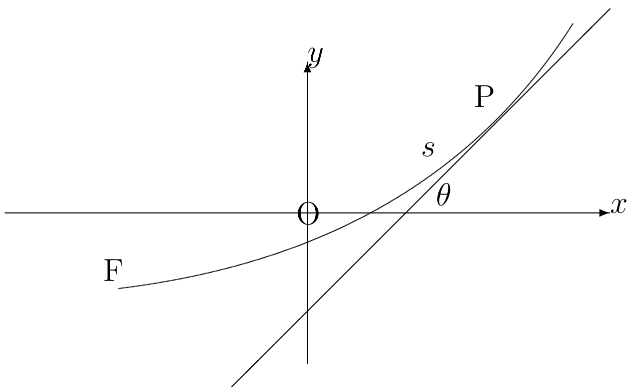
CURVATURE

11.3.1 INTRODUCTION

In the discussion which follows, consideration will be given to a method of measuring the **"tightness of bends"** on a curve. This measure will be called **"curvature"** and its definition will imply that very tight bends have large curvature.

We shall also need to distinguish between curves which are **"concave upwards"** ($\cup$), having positive curvature, and curves which are **"concave downwards"** ($\cap$), having negative curvature.

DEFINITION
Suppose we are given a curve whose equation is $y = f(x)$; and suppose that θ is the angle made with the positive x-axis by the tangent to the curve at a point, P(x, y), on it. If s is the distance to P, measured along the curve from some fixed point, F, on it then the curvature, κ, at P, is defined as the rate of increase of θ with respect to s.

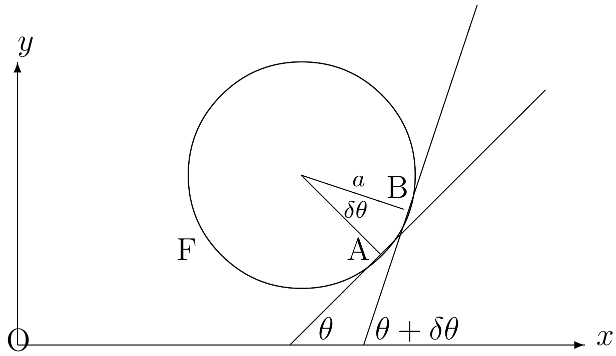


$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

EXAMPLE

Determine the curvature at any point of a circle with radius a.

Solution



We shall let A be a point on the circle at which the tangent is inclined to the positive x-axis at an angle, θ, and let B be a point (close to A) at which the tangent is inclined to the positive x-axis at an angle, $\theta + \delta\theta$. The length of the arc, AB, will be called δs, where we shall assume that distances, s, are measured along the circle in a counter-clockwise sense from the fixed point, F.

The diagram shows that $\delta\theta$ is both the angle between the two tangents **and** the angle subtended at the centre of the circle by the arc, AB.

Thus, $\delta s = a\delta\theta$ which can be written

$$\frac{\delta\theta}{\delta s} = \frac{1}{a}.$$

Allowing $\delta\theta$, and hence δs, to approach zero, we conclude that

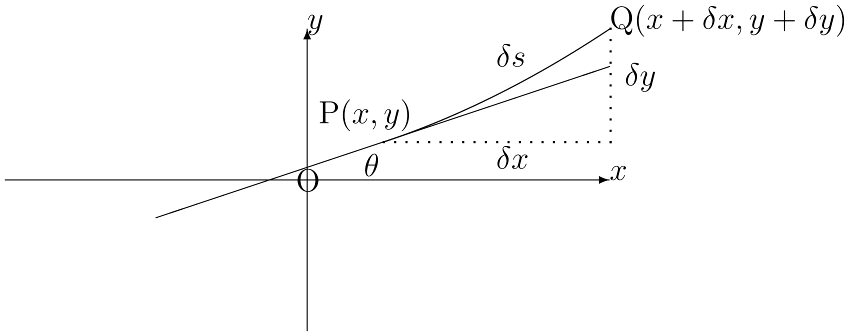
$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{a}.$$

We note, however, that, for the lower half of the circle, θ **increases** as s increases, while, in the upper half of the circle, θ **decreases** as s increases. The curvature will therefore be positive for the lower half (which is concave upwards) and negative for the upper half (which is concave downwards).

Summary
The curvature at any point of a circle is numerically equal to the reciprocal of the radius.

11.3.2 CURVATURE IN CARTESIAN CO-ORDINATES

Given a curve whose equation is $y = f(x)$, suppose $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ are two neighbouring points on it which are separated by a distance of δs along the curve.



In this diagram, we may observe that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \tan\theta$$

and also that

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lim_{\delta s \to 0} \frac{\delta x}{\delta s} = \cos\theta.$$

The curvature may therefore be evaluated as follows:

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\cos\theta.$$

But,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{1}{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Finally,

$$\cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}};$$

and so,

$$\kappa = \pm\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

Notes:

(i) For a curve which is concave upwards at a particular point, the gradient, $\frac{dy}{dx}$, will **increase** as x increases through the point. Hence, $\frac{d^2y}{dx^2}$ will be positive at the point.

(ii) For a curve which is concave downwards at a particular point, the gradient, $\frac{dy}{dx}$, will **decrease** as x increases through the point. Hence, $\frac{d^2y}{dx^2}$ will be negative at the point.

(ii) In future, therefore, we may allow the value of the curvature to take the same sign as $\frac{d^2y}{dx^2}$, giving the formula

$$\kappa = \frac{\frac{d^2y}{dx^2}}{\left[1+\frac{dy}{dx}^2\right]^{\frac{3}{2}}}.$$

EXAMPLE

Use the cartesian formula to determine the curvature at any point on the circle, centre $(0, 0)$ with radius a.

Solution

The equation of the circle is

$$x^2 + y^2 = a^2,$$

which means that, for the upper half,

$$y = \sqrt{a^2 - x^2}$$

and, for the lower half,

$$y = -\sqrt{a^2 - x^2}.$$

Considering, firstly, the upper half,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{a^2 - x^2}}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2-x^2}}}{a^2 - x^2} = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Therefore,

$$\kappa = \frac{-\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}}{\left(1 + \frac{x^2}{a^2-x^2}\right)^{\frac{3}{2}}} = -\frac{a^2}{a^3} = -\frac{1}{a}.$$

For the lower half of the circle,

$$\kappa = \frac{1}{a}.$$

11.3.3 EXERCISES

In the following questions, state your answer in decimals correct to three places of decimals:

1. Calculate the curvature at the point $(-1, 3)$ on the curve whose equation is
$$y = x + 3x^2 - x^3.$$
2. Calculate the curvature at the origin on the curve whose equation is
$$y = \frac{x - x^2}{1 + x^2}.$$
3. Calculate the curvature at the point $(1, 1)$ on the curve whose equation is
$$x^3 - 2xy + y^3 = 0.$$
4. Calculate the curvature at the point for which $\theta = 30^\circ$ on the curve whose parametric equations are
$$x = 1 + \sin\theta \quad \text{and} \quad y = \sin\theta - \frac{1}{2}\cos 2\theta.$$

11.3.4 ANSWERS TO EXERCISES

1. $\kappa = 0.023$
2. $\kappa = -0.707$
3. $\kappa = -5.650$
4. $\kappa = 0.179$

"JUST THE MATHS"

UNIT NUMBER

11.4

DIFFERENTIATION APPLICATIONS 4
(Circle, radius & centre of curvature)

by

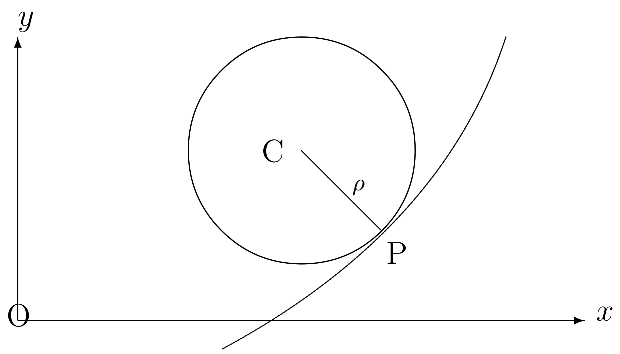
A.J.Hobson

UNIT 11.4 DIFFERENTIATION APPLICATIONS 4

CIRCLE, RADIUS AND CENTRE OF CURVATURE

11.4.1 INTRODUCTION

At a point, P, on a given curve, suppose we were to draw a circle which **just touches** the curve and has the same value of the curvature (including its sign). This circle is called the **"circle of curvature at P"**. Its radius, ρ, is called the **"radius of curvature at P"** and its centre is called the **"centre of curvature at P"**.



11.4.2 RADIUS OF CURVATURE

Using the earlier examples on the circle (Unit 11.3), we conclude that, if the curvature at P is κ, then $\rho = \frac{1}{\kappa}$ and, hence,

$$\rho = \frac{ds}{d\theta}.$$

Furthermore, in cartesian co-ordinates,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Note:
If we are interested in the radius of curvature simply as a length, then, for curves with

negative curvature, we would use only the **numerical** value obtained in the above formula. However, in a later discussion, it is necessary to use the appropriate sign for the radius of curvature.

EXAMPLE

Calculate the radius of curvature at the point $(0.5, -1)$ of the curve whose equation is

$$y^2 = 2x.$$

Solution

Differentiating implicitly,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}.$$

Also

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^2}.\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y^3}.$$

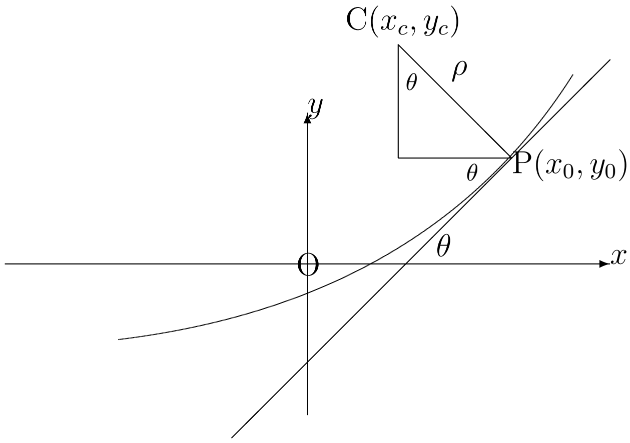
Hence, at the point $(0.5, -1)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$.

We conclude that

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{1} = 2\sqrt{2}.$$

11.4.3 CENTRE OF CURVATURE

We shall consider a point, (x_0, y_0), on an arc of a curve whose equation is $y = f(x)$ and for which the curvature is positive, the arc lying in the first quadrant. But it may be shown that the formulae obtained for the co-ordinates, (x_c, y_c), of the centre of curvature apply in any situation, provided that the curvature is associated with its appropriate sign.



From the diagram,

$$\begin{aligned} x_c &= x_0 - \rho \sin\theta, \\ y_c &= y_0 + \rho \cos\theta. \end{aligned}$$

Note:
Although the formulae apply in any situation, it is a good idea to sketch the curve in order estimate, roughly, where the centre of curvature is going to be. This is especially important where there is uncertainty about the precise value of the angle θ.

EXAMPLE

Determine the centre of curvature at the point $(0.5, -1)$ of the curve whose equation is

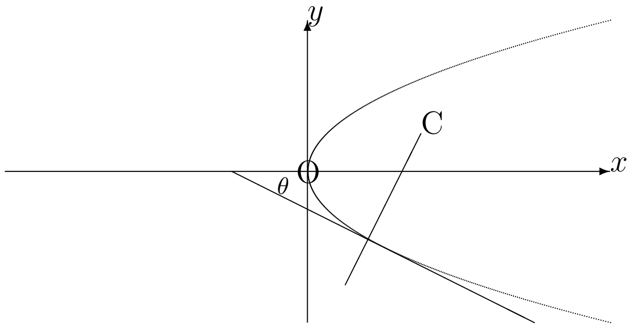
$$y^2 = 2x.$$

Solution

From the earlier example on calculating radius of curvature,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^3},$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$ and $\rho = 2\sqrt{2}$ at the point $(0.5, -1)$.



The diagram shows that the co-ordinates, (x_c, y_c), of the centre of curvature will be such that $x_c > 0.5$ and $y_c > -1$. This will be so provided that the angle, θ, is a negative acute angle; (that is, its cosine will be positive and its sine will be negative).

In fact,

$$\theta = \tan^{-1}(-1) = -45^\circ.$$

Hence,

$$\begin{aligned} x_c &= 0.5 - 2\sqrt{2}\sin(-45^\circ), \\ y_c &= -1 + 2\sqrt{2}\cos(-45^\circ). \end{aligned}$$

That is,

$$x_c = 2.5 \quad \text{and} \quad y_c = 1.$$

11.4.4 EXERCISES

In the following questions, state your results in decimals correct to three places of decimals:

1. Calculate the radius of curvature at the point $(-1, 3)$ on the curve whose equation is
$$y = x + 3x^2 - x^3$$
and hence obtain the co-ordinates of the centre of curvature.

2. Calculate the radius of curvature at the origin on the curve whose equation is
$$y = \frac{x - x^2}{1 + x^2}$$
and hence obtain the co-ordinates of the centre of curvature.

3. Calculate the radius of curvature at the point $(1, 1)$ on the curve whose equation is
$$x^3 - 2xy + y^3 = 0$$
and hence obtain the co-ordinates of the centre of curvature.

4. Calculate the radius of curvature at the point for which $\theta = 30°$ on the curve whose parametric equations are
$$x = 1 + \sin\theta \quad \text{and} \quad y = \sin\theta - \frac{1}{2}\cos 2\theta$$
and hence obtain the co-ordinates of the centre of curvature.

11.4.5 ANSWERS TO EXERCISES

1. $\rho = 43.6705, \quad (x_c, y_c) = (42.333, 8.417)$.
2. $\rho = -1.414 \quad (x_c, y_c) = (1, -1)$.
3. $\rho = -0.177 \quad (x_c, y_c) = (0.875, 0.875)$.
4. $\rho = 0.590 \quad (x_c, y_c = (-3.500, 2.750)$.

"JUST THE MATHS"

UNIT NUMBER

11.5

DIFFERENTIATION APPLICATIONS 5
(Maclaurin's and Taylor's series)

by

A.J.Hobson

UNIT 11.5 - DIFFERENTIATION APPLICATIONS 5

MACLAURIN'S AND TAYLOR'S SERIES

11.5.1 MACLAURIN'S SERIES

One of the simplest kinds of function to deal with, in either algebra or calculus, is a polynomial (see Unit 1.8). Polynomials are easy to substitute numerical values into and they are easy to differentiate.

One useful application of the present section is to approximate, to a polynomial, functions which are not already in polynomial form.

THE GENERAL THEORY

Suppose $f(x)$ is a given function of x which is not in the form of a polynomial, and let us assume that it may be expressed in the form of an infinite series of ascending powers of x; that is, a **"power series"**, (see Unit 2.4).

More specifically, we assume that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ \ldots$$

This assumption cannot be justified unless there is a way of determining the **"coefficients"**, a_0, a_1, a_2, a_3, a_4, etc.; but this is possible as an application of differentiation as we now show:

(a) Firstly, if we substitute $x = 0$ into the assumed formula for $f(x)$, we obtain $f(0) = a_0$; in other words,

$$a_0 = f(0).$$

(b) Secondly, if we differentiate the assumed formula for $f(x)$ once with respect to x, we obtain

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$$

which, on substituting $x = 0$, gives $f'(0) = a_1$; in other words,

$$a_1 = f'(0).$$

(c) Differentiating a second time leads to the result that

$$f''(x) = 2a_2 + (3 \times 2)a_3x + (4 \times 3)a_4x^2 + \ldots$$

which, on substituting $x = 0$ gives $f''(0) = 2a_2$; in other words,

$$a_2 = \frac{1}{2}f''(0).$$

(d) Differentiating yet again leads to the result that

$$f'''(x) = (3 \times 2)a_3 + (4 \times 3 \times 2)a_4x + \ldots$$

which, on substituting $x = 0$ gives $f'''(0) = (3 \times 2)a_3$; in other words,

$$a_3 = \frac{1}{3!}f'''(0).$$

(e) Continuing this process with further differentiation will lead to the general formula

$$a_n = \frac{1}{n!}f^{(n)}(0),$$

where $f^{(n)}(0)$ means the value, at $x = 0$ of the n-th derivative of $f(x)$.

Summary

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

This is called the **"Maclaurin's series for** $f(x)$**"**.

Notes:

(i) We must assume, ofcourse, that all of the derivatives of $f(x)$ exist at $x = 0$ in the first place; otherwise the above result is invalid.
It is also necessary to examine, for convergence or divergence, the Maclaurin's series obtained

for a particular function. The result may not be used when the series diverges; (see Units 2.3 and 2.4).

(b) If x is small and it is possible to neglect powers of x after the n-th power, then Maclaurin's series approximates $f(x)$ to a polynomial of degree n.

11.5.2 STANDARD SERIES

Here, we determine the Maclaurin's series for some of the functions which occur frequently in the applications of mathematics to science and engineering. The ranges of values of x for which the results are valid will be stated without proof.

1. **The Exponential Series**

   (i) $f(x) \equiv e^x$; hence, $f(0) = e^0 = 1$.

   (ii) $f'(x) = e^x$; hence, $f'(0) = e^0 = 1$.

   (iii) $f''(x) = e^x$; hence, $f''(0) = e^0 = 1$.

   (iv) $f'''(x) = e^x$; hence, $f'''(0) = e^0 = 1$.

   (v) $f^{(iv)}(x) = e^x$; hence, $f^{(iv)}(0) = e^0 = 1$.

   Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots$$

   and it may be shown that this series is valid for all values of x.

2. **The Sine Series**

   (i) $f(x) \equiv \sin x$; hence, $f(0) = \sin 0 = 0$.

   (ii) $f'(x) = \cos x$; hence, $f'(0) = \cos 0 = 1$.

   (iii) $f''(x) = -\sin x$; hence, $f''(0) = -\sin 0 = 0$.

   (iv) $f'''(x) = -\cos x$; hence, $f'''(0) = -\cos 0 = -1$.

   (v) $f^{(iv)}(x) = \sin x$; hence, $f^{(iv)}(0) = \sin 0 = 0$.

   (vi) $f^{(v)}(x) = \cos x$; hence, $f^{(v)}(0) = \cos 0 = 1$.

   Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots$$

   and it may be shown that this series is valid for all values of x.

3. The Cosine Series

(i) $f(x) \equiv \cos x$; hence, $f(0) = \cos 0 = 1$.

(ii) $f'(x) = -\sin x$; hence, $f'(0) = -\sin 0 = 0$.

(iii) $f''(x) = -\cos x$; hence, $f''(0) = -\cos 0 = -1$.

(iv) $f'''(x) = \sin x$; hence, $f'''(0) = \sin 0 = 0$.

(v) $f^{(iv)}(x) = \cos x$; hence, $f^{(iv)}(0) = \cos 0 = 1$.

Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ \ldots$$

and it may be shown that this series is valid for all values of x.

4. The Logarithmic Series

It is not possible to find a Maclaurin's series for the function $\ln x$, since neither the function nor its derivatives exist at $x = 0$.

As an alternative, we may consider the function $\ln(1 + x)$ instead.

(i) $f(x) \equiv \ln(1 + x)$; hence, $f(0) = \ln 1 = 0$.

(ii) $f'(x) = \frac{1}{1+x}$; hence, $f'(0) = 1$.

(iii) $f''(x) = -\frac{1}{(1+x)^2}$; hence, $f''(0) = 1$.

(iv) $f'''(x) = \frac{2}{(1+x)^3}$; hence, $f'''(0) = 2$.

(v) $f^{(iv)}(x) = -\frac{2\times 3}{(1+x)^4}$; hence, $f^{(iv)}(0) = -(2 \times 3)$.

Thus,

$$\ln(1 + x) = x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - (2 \times 3)\frac{x^4}{4!} + \ \ldots$$

which simplifies to

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots$$

and it may be shown that this series is valid for $-1 < x \leq 1$.

5. The Binomial Series

The statement of the Binomial Formula has already appeared in Unit 2.2; and it was seen there that

(a) When n is a positive integer, the expansion of $(1 + x)^n$ in ascending powers of x is a **finite** series;

(b) When n is a negative integer or a fraction, the expansion of $(1+x)^n$ in ascending powers of x is an **infinite** series.

Here, we examine the proof of the Binomial Formula.

(i) $f(x) \equiv (1+x)^n$; hence, $f(0) = 1$.

(ii) $f'(x) = n(1+x)^{n-1}$; hence, $f'(0) = n$.

(iii) $f''(x) = n(n-1)(1+x)^{n-2}$; hence, $f''(0) = n(n-1)$.

(iv) $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$; hence, $f'''(0) = n(n-1)(n-2)$.

(v) $f^{(iv)}(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$; hence, $f^{(iv)}(0) = n(n-1)(n-2)(n-3)$.

Thus,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ \ldots$$

If n is a positive integer, all of the derivatives of $(1+x)^n$ after the n-th derivative are identically equal to zero; so the series is a finite series ending with the term in x^n.

In all other cases, the series is an infinite series and it may be shown that it is valid whenever $-1 < x \leq 1$.

EXAMPLES

1. Use the Maclaurin's series for $\sin x$ to evaluate

$$\lim_{x\to 0} \frac{x + \sin x}{x(x+1)}.$$

Solution

Substituting the series for $\sin x$ gives

$$\lim_{x\to 0} \frac{x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots}{x^2 + x}$$

$$= \lim_{x\to 0} \frac{2x - \frac{x^3}{6} + \frac{x^5}{120} - \ \ldots}{x^2 + x}$$

$$= \lim_{x\to 0} \frac{2 - \frac{x^2}{6} + \frac{x^4}{120} - \ \ldots}{x + 1} = 2.$$

2. Use a Maclaurin's series to evaluate $\sqrt{1.01}$ correct to six places of decimals.

   Solution

   We shall consider the expansion of the function $(1+x)^{\frac{1}{2}}$ and then substitute $x = 0.01$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ \dots$$

   That is,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ \dots$$

   Substituting $x = 0.01$ gives

$$\sqrt{1.01} = 1 + \frac{1}{2} \times 0.01 - \frac{1}{8} \times 0.0001 + \frac{1}{16} \times 0.000001 - \ \dots$$

$$= 1 + 0.005 - 0.0000125 + 0.0000000625 - \ \dots$$

   The fourth term will not affect the sixth decimal place in the result given by the first three terms; and this is equal to 1.004988 correct to six places of decimals.

3. Assuming the Maclaurin's series for e^x and $\sin x$ and assuming that they may be multiplied together term-by-term, obtain the expansion of $e^x \sin x$ in ascending powers of x as far as the term in x^5.

   Solution

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{120} + \dots\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24} + \ \dots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \ \dots$$

11.5.3 TAYLOR'S SERIES

A useful consequence of Maclaurin's series is known as Taylor's series and one form of it may be stated as follows:

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ . \ . \ .$$

Proof:

To obtain this result from Maclaurin's series, we simply let $f(x+h) \equiv F(x)$. Then,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \frac{x^3}{3!}F'''(0) + \ . \ .$$

But, $F(0) = f(h)$, $F'(0) = f'(h)$, $F''(0) = f''(h)$, $F'''(0) = f'''(h)$,. . . which proves the result.

Note: An alternative form of Taylor's series, often used for approximations, may be obtained by interchanging the symbols x and h to give

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ . \ . \ .$$

EXAMPLE

Given that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, use Taylor's series to evaluate $\sin(x+h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{4}$ and $h = 0.01$.

Solution

Using the sequence of derivatives as in the Maclaurin's series for $\sin x$, we have

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + ...$$

Subsituting $x = \frac{\pi}{4}$ and $h = 0.01$, we obtain

$$\sin\left(\frac{\pi}{4} + 0.01\right) = \frac{1}{\sqrt{2}}\left(1 + 0.01 - \frac{(0.01)^2}{2!} - \frac{(0.01)^3}{3!} + \ . \ . \ .\right)$$

$$= \frac{1}{\sqrt{2}}(1 + 0.01 - 0.00005 - 0.000000017 + ...)$$

The fourth term does not affect the fifth decimal place in the sum of the first three terms; and so

$$\sin\left(\frac{\pi}{4}+0.01\right) \simeq \frac{1}{\sqrt{2}} \times 1.00995 \simeq 0.71414$$

11.5.4 EXERCISES

1. Determine the first three non-vanishing terms of the Maclaurin's series for the function $\sec x$.
2. Determine the Maclaurin's series for the function $\tan x$ as far as the term in x^5.
3. Determine the Maclaurin's series for the function $\ln(1+e^x)$ as far as the term in x^4.
4. Use the Maclaurin's series for the function e^x to deduce the expansion, in ascending powers of x of the function e^{-x} and then use these two series to obtain the expansion, in ascending powers of x, of the functions

   (a)
   $$\frac{e^x+e^{-x}}{2}(\equiv \cosh x);$$

   (b)
   $$\frac{e^x-e^{-x}}{2}(\equiv \sinh x).$$

5. Use the Maclaurin's series for the function $\cos x$ and the Binomial Series for the function $\frac{1}{1+x}$ to obtain the expansion of the function
   $$\frac{\cos x}{1+x}$$
   in ascending powers of x as far as the term in x^4.
6. From the Maclaurin's series for the function $\cos x$, deduce the expansions of the functions $\cos 2x$ and $\sin^2 x$ as far as the term in x^4.

7. Use appropriate Maclaurin's series to evaluate the following limits:

   (a)

$$\lim_{x\to 0}\left[\frac{e^x+e^{-x}-2}{2\cos 2x-2}\right];$$

   (b)

$$\lim_{x\to 0}\left[\frac{\sin^2 x-x^2\cos x}{x^4}\right].$$

8. Use a Maclaurin's series to evaluate $\sqrt[3]{1.05}$ correct to four places of decimals.

9. Expand $\cos(x+h)$ as a series of ascending powers of h.

   Given that $\sin\frac{\pi}{6}=\frac{1}{2}$ and $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, evaluate $\cos(x+h)$, correct to five places of decimals, in the case when $x=\frac{\pi}{6}$ and $h=-0.05$.

11.5.5 ANSWERS TO EXERCISES

1.

$$1+\frac{x^2}{2}+\frac{x^4}{8}+\ ..$$

2.

$$x+\frac{x^3}{3}+\frac{2x^5}{15}+\ \ldots$$

3.

$$\ln 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ \ldots$$

4. (a)

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ \ldots;$$

   (b)

$$\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ ..$$

5.

$$1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{13x^4}{24} - \ \ldots$$

6.

$$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} - \ \ldots$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \ \ldots$$

7. (a) $-\frac{1}{4}$, (b) $\frac{1}{6}$

8. 1.0164

9. 0.74156

"JUST THE MATHS"

UNIT NUMBER

11.6

DIFFERENTIATION APPLICATIONS 6
(Small increments and small errors)

by

A.J.Hobson

UNIT 11.6 - DIFFERENTIATION APPLICATIONS 6

SMALL INCREMENTS AND SMALL ERRORS

11.6.1 SMALL INCREMENTS

Given that a dependent variable, y, and an independent variable, x are related by means of the formula

$$y = f(x),$$

suppose that x is subject to a small **"increment"**, δx,

In the present context we use the term "increment" to mean that δx is positive when x is **increased**, but negative when x is **decreased**.

The exact value of the corresponding increment, δy, in y is given by

$$\delta y = f(x + \delta x) - f(x),$$

but this can often be a cumbersome expression to evaluate.

However, since δx is small, we may recall, from the definition of a derivative (Unit 10.2), that

$$\frac{f(x + \delta x) - f(x)}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x}.$$

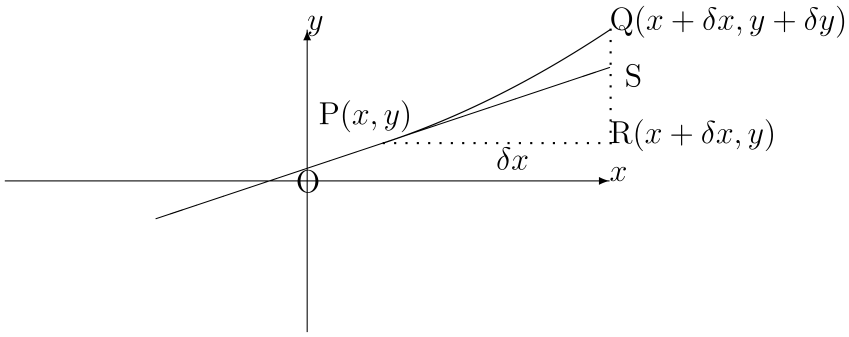
That is,

$$\frac{\delta y}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x};$$

and we may conclude that

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

For a diagramatic approach to this approximation for the increment in y, let us consider the graph of y against x in the neighbourhood of the two points $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ on the curve whose equation is $y = f(x)$.



In the diagram, PR $= \delta x$, QR $= \delta y$ and the gradient of the line PS is given by the value of $\frac{dy}{dx}$ at P.

Taking SR as an approximation to QR, we obtain

$$\frac{\text{SR}}{\text{PR}} = \left[\frac{dy}{dx}\right]_{\text{P}}.$$

In other words,

$$\frac{\text{SR}}{\delta x} = \left[\frac{dy}{dx}\right]_{\text{P}}.$$

Hence,

$$\delta y \simeq \left[\frac{dy}{dx}\right]_{\text{P}} \delta x,$$

which is the same result as before.

Notes:

(i) The quantity $\frac{dy}{dx}\delta x$ is known as the **"total differential of y"** (or simply the "differential of y"). It provides an approximation **(including the appropriate sign)** for the increment, δy, in y subject to an increment of δx in x.

(ii) It is important **not** to use the word "differential" when referring to a "derivative". Rather, the correct alternative to "derivative" is "differential coefficient".

(iii) A more rigorous approach to the calculation of δy is to use the result known as "Taylor's Theorem" (see Unit 11.5) which, in this context, would give the formula

$$f(x+\delta x) = f(x) + f'(x)\delta x + \frac{f''(x)}{2!}(\delta x)^2 + \frac{f'''(x)}{3!}(\delta x)^3 + \ldots .$$

Hence, if δx is small enough for powers of two and above to be neglected, then

$$f(x+\delta x) - f(x) \simeq f'(x)\delta x$$

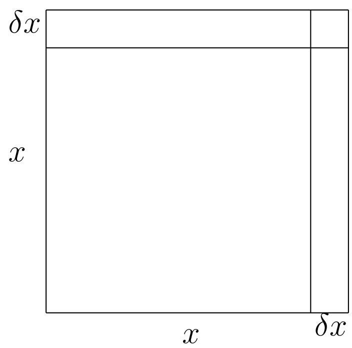
to the first order of approximation.

EXAMPLES

1. If a square has side $x$cms., determine both the exact and the approximate values of the increment in the area Acms^2. when x is increased by δx.

   Solution

   (a) Exact Method



The area is given by the formula

$$A = x^2.$$

If x increases by δx, then the increase, δA, in A may be obtained from the formula

$$A + \delta A = (x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2.$$

That is,

$$\delta A = 2x\delta x + (\delta x)^2.$$

(b) Approximate Method

Here, we use

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x$$

to give

$$\delta A \simeq 2x\delta x;$$

and we observe from the diagram that the two results differ only by the area of the small square, with side δx.

2. If

$$y = xe^{-x},$$

calculate, approximately, the change in y when x increases from 5 to 5.03.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}(1 - x),$$

so that

$$\delta y \simeq e^{-x}(1 - x)\delta x,$$

where $x = 5$ and $\delta x = 0.3$.

Hence,

$$\delta y \simeq e^{-5}.(1 - 5).(0.3) \simeq -0.00809,$$

showing a **decrease** of 0.00809 in y.

We may compare this with the exact value which is given by

$$\delta y = 5.3e^{-5.3} - 5e^{-5} \simeq -0.00723$$

3. If

$$y = xe^{-x},$$

determine, in terms of x, the percentage change in y when x is increased by 2%.

Solution

Once again, we have

$$\delta y = e^{-x}(1 - x)\delta x;$$

but, this time, $\delta x = 0.02x$, so that

$$\delta y = e^{-x}(1 - x) \times 0.02x.$$

The **percentage** change in y is given by

$$\frac{\delta y}{y} \times 100 = \frac{e^{-x}(1 - x) \times 0.02x}{xe^{-x}} \times 100 = 2(1 - x).$$

That is, y increases by $2(1-x)$%, which will be positive when $x < 1$ and negative when $x > 1$.

Note:

It is usually more meaningful to discuss increments in the form of a percentage, since this gives a better idea of how much a variable has changed in porportion to its original value.

11.6.2 SMALL ERRORS

In the functional relationship

$$y = f(x),$$

let us suppose that x is known to be subject to an error in measurement; then we consider what error will be likely in the calculated value of y.

In particular, suppose x is known to be **too large** by a small amount, δx, in which case the correct value of x could be obtained if we **decreased** it by δx; or, what amounts to the same thing, if we **increased** it by $-\delta x$.

Correspondingly, the value of y will **increase** by approximately $-\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; that is, y will **decrease** by approximately $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$.

Summary

We conclude that, if x is too large by an amount δx, then y is too large by approximately $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; though, ofcourse, if $\frac{\mathrm{d}y}{\mathrm{d}x}$ itself is negative, y will be too small when x is too large and vice versa.

EXAMPLES

1. If
$$y = x^2 \sin x,$$
calculate, approximately, the error in y when x is measured as 3, but this measurement is subsequently discovered to be too large by 0.06.

   Solution

   We have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \cos x + 2x \sin x$$
   and, hence,
$$\delta y \simeq (x^2 \cos x + 2x \sin x)\delta x,$$
   where $x = 3$ and $\delta x = 0.06$.

   The error in y is therefore given approximately by
$$\delta y \simeq (3^2 \cos 3 + 6 \sin 3) \times 0.06 \simeq -0.4838$$
   That is, y is too small by approximately 0.4838.

2. If
$$y = \frac{x}{1+x},$$
determine approximately, in terms of x, the percentage error in y when x is subject to an error of 5%.

   Solution

   We have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2},$$
   so that

$$\delta y \simeq \frac{1}{(1+x)^2}\delta x,$$

where $\delta x = 0.05x$.

The **percentage** error in y is thus given by

$$\frac{\delta y}{y} \times 100 \simeq \frac{1}{(1+x)^2} \times 0.05x \times \frac{x+1}{x} \times 100 = \frac{5}{1+x}.$$

Hence, y is too large by approximately $\frac{5}{1+x}$% which will be positive when $x > -1$ and negative when $x < -1$.

11.6.3 EXERCISES

1. If
$$y = \frac{e^{2x}}{x},$$
calculate, approximately, the change in y when x is increased from 1 to 1.0025.
State your answer correct to three significant figures.

2. If
$$y = (2x+1)^5,$$
determine approximately, in terms of x, the percentage change in y when x increases by 0.1%.

3. If
$$y = x^3 \ln x,$$
calculate approximately, correct to the nearest integer, the error in y when x is measured as 4, but this measurement is subsequently discovered to be too small by 0.12.

4. If
$$y = \cos(3x^2+2),$$
determine approximately, in terms of x, the percentage error in y if x is too large by 2%.
You may assume that $3x^2+2$ lies between π and $\frac{3\pi}{2}$.

11.6.4 ANSWERS TO EXERCISES

1. y increases by approximately 0.0185.
2. y increases by approximately $\frac{x}{(2x+1)}\%$
3. y is too small by approximately 10.
4. y is too small by approximately $-12x^2\tan(3x^2+2)$.

"JUST THE MATHS"

UNIT NUMBER

12.1

INTEGRATION 1
(Elementary indefinite integrals)

by

A.J.Hobson

UNIT 12.1 - INTEGRATION 1 - ELEMENTARY INDEFINITE INTEGRALS

12.1.1 THE DEFINITION OF AN INTEGRAL

In Differential Calculus, we are given functions of x and asked to obtain their derivatives; but, in Integral Calculus, we are given functions of x and asked what they are the derivatives of. The process of answering this question is called **"integration"**.

In other words **integration is the reverse of differentiation**.

DEFINITION
Given a function $f(x)$, another function, z, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x)$$

is called an integral of $f(x)$ with respect to x.

Notes:
(i) The above definition refers to **an** integral of $f(x)$ rather than **the** integral of $f(x)$. This is because, having found a possible function, z, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x),$$

$z + C$ is also an integral for any constant value, C.

(ii) We call $z + C$ the **"indefinite integral of $f(x)$ with respect to x"** and we write

$$\int f(x)\mathrm{d}x = z + C.$$

(iii) C is an **arbitrary constant** called the **"constant of integration"**.

(iv) The symbol $\mathrm{d}x$ does not denote a number; it is to be taken as a label indicating the variable with respect to which we are integrating. It may seem obvious that this will be x, but it could happen, for instance, that x is already dependent upon some other variable, t, in which case it would be vital to indicate the variable with respect to which we are integrating.

(v) In any integration problem, the function being integrated is called the **"integrand"**.

Result:
Two functions z_1 and z_2 are both integrals of the same function $f(x)$ if and only if they differ by a constant.

Proof:
(a) Suppose, firstly, that

$$z_1 - z_2 = C,$$

where C is a constant.

Then,

$$\frac{d}{dx}[z_1 - z_2] = 0.$$

That is,

$$\frac{dz_1}{dx} - \frac{dz_2}{dx} = 0,$$

or

$$\frac{dz_1}{dx} = \frac{dz_2}{dx}.$$

From our definition, this shows that both z_1 and z_2 are integrals of the same function.

(b) Secondly, suppose that z_1 and z_2 are integrals of the same function. Then

$$\frac{dz_1}{dx} = \frac{dz_2}{dx}.$$

That is,

$$\frac{dz_1}{dx} - \frac{dz_2}{dx} = 0,$$

or

$$\frac{d}{dx}[z_1 - z_2] = 0.$$

Hence,

$$z_1 - z_2 = C,$$

where C may be any constant.

ILLUSTRATIONS
Any result so far encountered in differentiation could be re-stated in reverse as a result on integration as shown in the following illustrations:

1.

$$\int 3x^2 dx = x^3 + C.$$

2.

$$\int x^2 dx = \frac{x^3}{3} + C.$$

3.

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad \text{Provided} \quad n \neq -1.$$

4.

$$\int \frac{1}{x}\mathrm{d}x \quad \text{i.e.} \quad \int x^{-1}\mathrm{d}x = \ln x + C.$$

5.

$$\int e^x \mathrm{d}x = e^x + C.$$

6.

$$\int \cos x \mathrm{d}x = \sin x + C.$$

7.

$$\int \sin x \mathrm{d}x = -\cos x + C.$$

Note:
Basic integrals of the above kinds may simply be quoted from a table of standard integrals in a suitable formula booklet. More advanced integrals are obtainable using the rules to be discussed below.

12.1.2 ELEMENTARY TECHNIQUES OF INTEGRATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants. Then,

$$\int [Af(x) + Bg(x)]\mathrm{d}x = A\int f(x)\mathrm{d}x + B\int g(x)\mathrm{d}x.$$

The proof follows from the fact that differentiation is already linear and hence the derivative of the right hand side is the integrand of the left hand side. The result itself is easily extended to linear combinations of three or more functions.

ILLUSTRATIONS

1.

$$\int (x^2 + 3x - 7)\mathrm{d}x = \frac{x^3}{3} + 3\frac{x^2}{2} - 7x + C.$$

2.

$$\int (3\cos x + 4\sec^2 x)\mathrm{d}x = 3\sin x + 4\tan x + C.$$

(b) Functions of a Linear Function

(i) Inspection Method
Provided the method of **differentiating** functions of a linear function has been fully understood, the fastest method of **integrating** such functions is to examine, by inspection, what needs to be differentiated in order to arrive at them.

EXAMPLES

1. Determine the indefinite integral

$$\int (2x+3)^{12}\mathrm{d}x.$$

Solution

In order to arrive at the function $(2x+3)^{12}$ by a differentiation process, we would have to begin with a function related to $(2x+3)^{13}$.

In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+3)^{13}\right] = 13(2x+3)^{12}.2 = 26(2x+3)^{12}.$$

Since this is 26 times the function we are trying to integrate, we may say that

$$\int (2x+3)^{12} = \frac{(2x+3)^{13}}{26} + C.$$

2. Determine the indefinite integral

$$\int \cos(3-5x)\mathrm{d}x.$$

Solution

In order to arrive at the function $\cos(3-5x)$ by a differentiation process, we would have to begin with a function related to $\sin(3-5x)$.

In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(3-5x)] = \cos(3-5x).-5 = -5\cos(3-5x).$$

Since this is -5 times the function we are trying to integrate, we may say that

$$\int \cos(3-5x) = -\frac{\sin(3-5x)}{5} + C.$$

3. Determine the indefinite integral
$$\int e^{4x+1}\mathrm{d}x.$$

Solution

In order to arrive at the function e^{4x+1} by a differentiation process, we would have to begin with a function related to e^{4x+1} itself because the derivative of a power of e always contains the **same** power of e.

In fact
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{4x+1}\right] = e^{4x+1}.4.$$

Since this is 4 times the function we are trying to integrate, we may say that
$$\int e^{4x+1}\mathrm{d}x = \frac{e^{4x+1}}{4} + C.$$

4. Determine the indefinite integral
$$\int \frac{1}{7x+3}\mathrm{d}x.$$

Solution

In order to arrive at the function $\frac{1}{7x+3}$ by a differentiation process, we would have to begin with a function related to $\ln(7x+3)$.

In fact,
$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(7x+3)] = \frac{1}{7x+3}.7 = \frac{7}{7x+3}$$

Since this is 7 times the function we are trying to integrate, we may say that
$$\int \frac{1}{7x+3}\mathrm{d}x = \frac{\ln(7x+3)}{7} + C.$$

Note:
In each of these examples, we are essentially treating the linear function as if it were a single x, then dividing the result by the coefficient of x in that linear function.

(ii) Substitution Method
The method to be discussed here will eventually be applied to functions other than functions of a linear function; but the latter serve as a useful way of introducing the technique of **"Integration by Substitution"**.

In the integral of the form $\int f(ax+b)\mathrm{d}x$, we may substitute $u = ax+b$ proceeding as follows:

Suppose

$$z = \int f(ax+b)\mathrm{d}x.$$

Then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(ax+b).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(u).$$

But,

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{\mathrm{d}z}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}u} = f(u).\frac{\mathrm{d}x}{\mathrm{d}u}.$$

Hence,

$$z = \int f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u.$$

Note:
The secret of this integration by substitution formula is that, apart from putting $u = ax+b$ into $f(ax+b)$, we replace the symbol $\mathrm{d}x$ with $\frac{\mathrm{d}x}{\mathrm{d}u}.\mathrm{d}u$; almost as if we had divided $\mathrm{d}x$ by $\mathrm{d}u$ then immediately multiplied by it again, though, strictly, this would not be allowed since $\mathrm{d}x$ and $\mathrm{d}u$ are not numbers.

EXAMPLES

1. Determine the indefinite integral

$$z = \int (2x+3)^{12}\mathrm{d}x.$$

Solution
Putting $u = 2x+3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}$.
Thus,

$$z = \int u^{12}.\frac{1}{2}\mathrm{d}u = \frac{u^{13}}{13} \times \frac{1}{2} + C.$$

That is,

$$z = \frac{(2x+3)^{13}}{26} + C,$$

as before.

2. Determine the indefinite integral

$$z = \int \cos(3 - 5x)\mathrm{d}x.$$

Solution

Putting $u = 3 - 5x$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = -5$ and hence $\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{5}$.

Thus,

$$z = \int \cos u. -\frac{1}{5}\mathrm{d}u = -\frac{1}{5}\sin u + C.$$

That is,

$$z = -\frac{1}{5}\sin(3 - 5x) + C,$$

as before.

3. Determine the indefinite integral

$$z = \int e^{4x+1}\mathrm{d}x.$$

Solution

Putting $u = 4x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 4$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{4}$.

Thus,

$$z = \int e^u.\frac{1}{4}\mathrm{d}u = \frac{e^u}{4} + C.$$

That is,

$$z = \frac{e^{4x+1}}{4} + C,$$

as before.

4. Determine the indefinite integral

$$z = \int \frac{1}{7x + 3}\mathrm{d}x.$$

Solution

Putting $u = 7x + 3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 7$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{7}$

Thus,

$$z = \int \frac{1}{u}.\frac{1}{7}\mathrm{d}u = \frac{1}{7}\ln u + C.$$

That is,

$$z = \frac{1}{7}\ln(7x + 3) + C,$$

as before.

12.1.3 EXERCISES

1. Integrate the following functions with respect to x:

   (a)
$$x^5;$$
   (b)
$$x^{\frac{3}{2}};$$
   (c)
$$\frac{1}{x^6};$$
   (d)
$$2x^2 - x + 5;$$
   (e)
$$x^3 - 7x^2 + x + 1.$$

2. Use a substitution of the form $u = ax + b$ in order to determine the following integrals:

   (a)
$$\int \sin(5x - 6)\mathrm{d}x;$$
   (b)
$$\int e^{2x+11}\mathrm{d}x;$$
   (c)
$$\int (3x + 2)^6\mathrm{d}x.$$

3. Write down, by inspection, the indefinite integrals with respect to x of the following functions:

   (a)
$$(1 + 2x)^{10};$$

(b)

$$e^{12x+4};$$

(c)

$$\frac{1}{3x-1};$$

(d)

$$\sin(3-5x);$$

(e)

$$\frac{9}{(4-x)^5};$$

(f)

$$\text{cosec}^2(7x+1).$$

12.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{x^6}{6}+C;$$

(b)

$$\frac{2}{5}x^{\frac{5}{2}}+C;$$

(c)

$$-\frac{1}{5x^5}+C;$$

(d)

$$\frac{2}{3}x^3-\frac{1}{2}x^2+5x+C;$$

(e)

$$\frac{1}{4}x^4-\frac{7}{3}x^3+\frac{1}{2}x^2+x+C.$$

2. (a)

$$-\frac{1}{5}\cos(5x-6)+C;$$

(b)

$$\frac{1}{2}e^{2x+11}+C;$$

(c)

$$\frac{1}{21}(3x+2)^7+C.$$

3. (a)

$$\frac{1}{22}(1+2x)^{11}+C;$$

(b)

$$\frac{1}{12}e^{12x+4}+C;$$

(c)

$$\frac{1}{3}\ln(3x-1)+C;$$

(d)

$$\frac{1}{5}\cos(3-5x)+C;$$

(e)

$$\frac{9}{4(4-x)^4}+C;$$

(f)

$$-\frac{1}{7}\cot(7x+1)+C.$$

APPENDIX - A Table of Standard Integrals

$f(x)$	$\int\ f(x)\ \mathrm{d}x$
a (const.)	ax
x^n	$x^{n+1}/(n+1)\ n \neq -1$
$1/x$	$\ln x$
$\sin ax$	$-(1/a)\cos ax$
$\cos ax$	$(1/a)\sin ax$
$\sec^2 ax$	$(1/a)\tan ax$
$\mathrm{cosec}^2\ ax$	$-(1/a)\cot ax$
$\sec\ ax.\tan ax$	$(1/a)\mathrm{sec}ax$
$\mathrm{cosec}\ ax.\cot ax$	$-(1/a)\mathrm{cosec}ax$
e^{ax}	$(1/a)e^{ax}$
a^x	$a^x/\ln a$
$\sinh ax$	$(1/a)\cosh ax$
$\cosh ax$	$(1/a)\sinh ax$
$\mathrm{sech}^2\ ax$	$(1/a)\tanh ax$
$\mathrm{sech}\ ax.\tanh ax$	$-(1/a)\mathrm{sech}ax$
$\mathrm{cosech}\ ax.\coth ax$	$-(1/a)\mathrm{cosech}ax$
$\cot ax$	$(1/a)\ln(\sin ax)$
$\tan ax$	$-(1/a)\ln(\cos ax)$
$\tanh ax$	$(1/a)\ln(\cosh ax)$
$\coth ax$	$(1/a)\ln(\sinh ax)$
$1/\sqrt{(a^2-x^2)}$	$\sin^{-1}(x/a)$
$1/(a^2+x^2)$	$(1/a)\tan^{-1}(x/a)$
$1/\sqrt{(x^2+a^2)}$	$\sinh^{-1}(x/a)$ or $\ln(x+\sqrt{x^2+a^2})$
$1/\sqrt{(x^2-a^2)}$	$\cosh^{-1}(x/a)$ or $\ln(x+\sqrt{x^2-a^2})$
$1/(a^2-x^2)$	$(1/a)\tanh^{-1}(x/a)$ or $\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ when $\|x\| < a$, $\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right)$ when $\|x\| > a$.

"JUST THE MATHS"

UNIT NUMBER

12.2

INTEGRATION 2
(Introduction to definite integrals)

by

A.J.Hobson

UNIT 12.2 - INTEGRATION 2
INTRODUCTION TO DEFINITE INTEGRALS

12.2.1 DEFINITION AND EXAMPLES

So far, all the integrals considered have been **"indefinite integrals"** since each result has contained an arbitrary constant which cannot be assigned a value without further information.

In practical applications of integration, however, a different kind of integral, called a **"definite integral"**, is encountered and is represented by a numerical value rather than a function plus an arbitrary constant.

Suppose that

$$\int f(x)\mathrm{d}x = g(x) + C.$$

Then the symbol

$$\int_a^b f(x)\mathrm{d}x$$

is used to mean

(Value of $g(x) + C$ at $x = b$) minus (Value of $g(x) + C$ at $x = a$).

In other words, since C will cancel out,

$$\int_a^b f(x)\mathrm{d}x = g(b) - g(a).$$

The right hand side of this statement can also be written

$$[g(x)]_a^b,$$

a notation which is used as the middle stage of a definite integral calculation.

The values a and b are known as the **"lower limit"** and **"upper limit"**, respectively, of the definite integral (even when a is larger than b).

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos x\mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x = [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

2. Evaluate the definite integral

$$\int_1^3 (2x+1)^2 \mathrm{d}x.$$

Solution

Using the method of integrating a function of a linear function, we obtain

$$\int_1^3 (2x+1)^2 \mathrm{d}x = \left[\frac{(2x+1)^3}{6}\right]_1^3 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

Notes:
(i) If we had decided to multiply out the integrand $(2x+1)^2$ before integrating, giving

$$4x^2 + 4x + 1,$$

the integration process would have yielded the expression

$$4\frac{x^3}{3} + 2x^2 + x,$$

which differs only from the previous result by the constant value $\frac{1}{6}$; students may like to check this. Hence the numerical result for the definite integral will be the same.

(ii) An alternative method of evaluating the definite integral would be to make the substitution

$$u = 2x + 1.$$

But, whenever substitution is used for definite integrals, it is not necessary to return to the original variable at the end as long as the limits of integration are changed to the appropriate values for u.

Replacing $\mathrm{d}x$ by $\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ (that is, $\frac{1}{2}\mathrm{d}u$) and the limits $x = 1$ and $x = 3$ by $u = 2 \times 1 + 1 = 3$ and $u = 2 \times 3 + 1 = 7$, respectively, we obtain

$$\int_3^7 u^2 \frac{1}{2} \mathrm{d}u = \left[\frac{u^3}{6}\right]_3^7 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

12.2.2 EXERCISES

Evaluate the following definite integrals:

1.
$$\int_0^{\frac{\pi}{3}} \sin 4x \mathrm{d}x.$$

2.
$$\int_{-1}^{1} (x+1)^7 \mathrm{d}x.$$

3.
$$\int_0^{\frac{\pi}{4}} \sec^2(x+\pi)\ \mathrm{d}x.$$

4.
$$\int_1^2 \frac{1}{4+3x}\ \mathrm{d}x.$$

5.
$$\int_0^1 e^{1-7x}\ \mathrm{d}x.$$

12.2.3 ANSWERS TO EXERCISES

1.
$$0.375$$

2.
$$32.$$

3.
$$1.$$

4.
$$\frac{1}{3}\ln\frac{10}{7} \quad \text{or} \quad 0.119 \quad \text{approximately}$$

5.
$$0.388 \text{ approximately}$$

"JUST THE MATHS"

UNIT NUMBER

12.3

INTEGRATION 3
(The method of completing the square)

by

A.J.Hobson

UNIT 12.3 - INTEGRATION 3
THE METHOD OF COMPLETING THE SQUARE

12.3.1 INTRODUCTION AND EXAMPLES

A substitution such as $u = \alpha x + \beta$ may also be used with integrals of the form

$$\int \frac{1}{px^2 + qx + r}\mathrm{d}x \quad \text{and} \quad \int \frac{1}{\sqrt{px^2 + qx + r}}\mathrm{d}x,$$

where, in the first of these, we assume that the quadratic will not factorise into simple linear factors; otherwise the method of partial fractions would be used to integrate it (see Unit 12.6).

Note:
The two types of integral here are often written, for convenience, as

$$\int \frac{\mathrm{d}x}{px^2 + qx + r} \quad \text{and} \quad \int \frac{\mathrm{d}x}{\sqrt{px^2 + qx + r}}.$$

In order to deal with such functions, we shall need to quote standard results which may be deduced from previous ones developed in the differentiation of inverse trigonometric and hyperbolic functions.

They are as follows:

1.
$$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

2.
$$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} + C.$$

3.
$$\int \frac{1}{\sqrt{a^2 + x^2}}\,\mathrm{d}x = \sinh^{-1}\frac{x}{a} + C \quad \text{or} \quad \ln(x + \sqrt{x^2 + a^2}) + C.$$

4.
$$\int \frac{1}{\sqrt{x^2 - a^2}}\,\mathrm{d}x = \cosh^{-1}\frac{x}{a} + C \quad \text{or} \quad \ln(x + \sqrt{x^2 - a^2}) + C.$$

5.

$$\int \frac{1}{a^2 - x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C;$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

EXAMPLES

1. Determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{\sqrt{x^2 + 2x - 3}}.$$

Solution

Completing the square in the quadratic expression gives

$$x^2 + 2x - 3 \equiv (x+1)^2 - 4 \equiv (x+1)^2 - 2^2.$$

Hence,

$$z = \int \frac{\mathrm{d}x}{\sqrt{(x+1)^2 - 2^2}}.$$

Putting $u = x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.
Thus,

$$z = \int \frac{\mathrm{d}u}{\sqrt{u^2 - 2^2}},$$

giving

$$z = \ln\left[u + \sqrt{u^2 - 2^2}\right] + C.$$

Returning to the variable, x, we have

$$z = \ln\left[x + 1 + \sqrt{x^2 + 2x - 3}\right] + C.$$

2. Evaluate the definite integral

$$z = \int_3^7 \frac{\mathrm{d}x}{x^2 - 6x + 25}.$$

Solution

Completing the square in the quadratic expression gives

$$x^2 - 6x + 25 \equiv (x-3)^2 + 16.$$

Hence,

$$z = \int_3^7 \frac{\mathrm{d}x}{(x-3)^2 + 16}.$$

Putting $u = x - 3$, we obtain $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.

Thus,

$$z = \int_0^4 \frac{\mathrm{d}u}{u^2 + 16},$$

giving

$$z = \left[\frac{1}{4}\tan^{-1}\frac{u}{4}\right]_0^4 = \frac{\pi}{16}.$$

Alternatively, without changing the original limits of integration,

$$z = \left[\frac{1}{4}\tan^{-1}\frac{x-3}{4}\right]_3^7.$$

Note:

In cases like the two examples discussed above, when $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and therefore $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$, it seems pointless to go through the laborious process of actually **making** the substitution in detail. All we need to do is to treat the linear expression within the completed square as if it were a single x, then write the result straight down !

12.3.2 EXERCISES

1. Use a table of standard integrals to write down the indefinite integrals of the following functions:

   (a)

   $$\frac{1}{\sqrt{4 - x^2}};$$

   (b)

   $$\frac{1}{9 + x^2};$$

(c)

$$\frac{1}{\sqrt{x^2-7}}.$$

2. By completing the square, evaluate the following definite integrals:

(a)

$$\int_{-1}^{\sqrt{3}-1} \frac{\mathrm{d}x}{x^2+2x+2};$$

(b)

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}}.$$

12.3.3 ANSWERS TO EXERCISES

1. (a)

$$\sin^{-1}\frac{x}{2}+C;$$

(b)

$$\frac{1}{3}\tan^{-1}\frac{x}{3}+C;$$

(c)

$$\ln(x+\sqrt{x^2-7})+C.$$

2. (a)

$$\left[\tan^{-1}(x+1)\right]_{-1}^{\sqrt{3}-1}=\frac{\pi}{3};$$

(b)

$$\left[\sin^{-1}\frac{x+1}{2}\right]_0^1=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}.$$

"JUST THE MATHS"

UNIT NUMBER

12.4

INTEGRATION 4
(Integration by substitution in general)

by

A.J.Hobson

UNIT 12.4 - INTEGRATION 4
INTEGRATION BY SUBSTITUTION IN GENERAL

12.4.1 EXAMPLES USING THE STANDARD FORMULA

With any integral

$$\int f(x)\mathrm{d}x,$$

it may be convenient to make some kind of substitution relating the variable, x, to a new variable, u. In such cases, we may use the formula discussed in Unit 12.1, namely

$$\int f(x)\mathrm{d}x = \int f(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u,$$

where it is assumed that, on the right hand side, the integrand has been expressed wholly in terms of u.

For this Unit, substitutions other than linear ones will be given in the problems to be solved.

EXAMPLES

1. Use the substitution $x = a\sin u$ to show that

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$$

Solution

To be precise, we shall assume for simplicity that u is the **acute** angle for which $x = a\sin u$. In effect, we shall be making the substitution $u = \sin^{-1}\frac{x}{a}$ using the principal value of the inverse function; we can certainly do this because the expression $\sqrt{a^2 - x^2}$ requires that $-a < x < a$.

If $x = a\sin u$, then $\frac{\mathrm{d}x}{\mathrm{d}u} = a\cos u$, so that the integral becomes

$$\int \frac{a\cos u}{\sqrt{a^2 - a^2\sin^2 u}}\mathrm{d}u.$$

But, from trigonometric identities,

$$\sqrt{a^2 - a^2\sin^2 u} \equiv a\cos u,$$

both sides being positive when u is an acute angle.

We are thus left with

$$\int 1\mathrm{d}u = u + C = \sin^{-1}\frac{x}{a} + C.$$

2. Use the substitution $u = \frac{1}{x}$ to determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{x\sqrt{1+x^2}}.$$

Solution

Converting the substitution to the form

$$x = \frac{1}{u},$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2}.$$

Hence,

$$z = \int \frac{1}{\frac{1}{u}\sqrt{1+\frac{1}{u^2}}}. - \frac{1}{u^2}\mathrm{d}u$$

That is,

$$z = \int -\frac{1}{\sqrt{u^2+1}} = -\ln(u + \sqrt{u^2+1}) + C.$$

Returning to the original variable, x, we have

$$z = -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) + C.$$

Note:

This example is somewhat harder than would be expected under examination conditions.

12.4.2 INTEGRALS INVOLVING A FUNCTION AND ITS DERIVATIVE

The method of integration by substitution provides two useful results applicable to a wide range of problems. They are as follows:

(a)

$$\int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C$$

provided $n \neq -1$.

(b)

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x) + C.$$

These two results are readily established by means of the substitution

$$u = f(x).$$

In both cases $\frac{du}{dx} = f'(x)$ and hence $\frac{dx}{du} = \frac{1}{f'(x)}$. This converts the integrals, respectively, into

(a)

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

and (b)

$$\int \frac{1}{u} du = \ln u + C.$$

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \sin^3 x. \cos x \; dx.$$

Solution

In this example we can consider $\sin x$ to be $f(x)$ and $\cos x$ to be $f'(x)$.

Thus, by quoting result (a), we obtain

$$\int_0^{\frac{\pi}{3}} \sin^3 x. \cos x dx = \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{3}} = \frac{9}{64},$$

using $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

2. Integrate the function

$$\frac{2x+1}{x^2+x-11}$$

with respect to x.

Solution

Here, we can identify $x^2 + x - 11$ with $f(x)$ and $2x+1$ with $f'(x)$.

Thus, by quoting result (b), we obtain

$$\int \frac{2x+1}{x^2+x-11} dx = \ln(x^2 + x - 11) + C.$$

12.4.3 EXERCISES

1. Use the substitution $u = x + 3$ in order to determine the indefinite integral

$$\int x\sqrt{3+x}\ \mathrm{d}x.$$

2. Use the substitution $u = x^2 - 1$ in order to evaluate the definite integral

$$\int_1^5 x\sqrt{x^2-1}\ \mathrm{d}x.$$

3. Integrate the following functions with respect to x:

(a)

$$\sin^7 x.\cos x;$$

(b)

$$\cos^5 x.\sin x;$$

(c)

$$\frac{4x-3}{2x^2-3x+13};$$

(d)

$$\cot x.$$

12.4.4 ANSWERS TO EXERCISES

1.

$$\frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.$$

2.

$$\left[\frac{1}{3}(x^2-1)^{\frac{3}{2}}\right]_1^5 = \frac{1}{3}24^{\frac{3}{2}} \simeq 39.192$$

3. (a)

$$\frac{\sin^8 x}{8} + C;$$

(b)

$$-\frac{\cos^6 x}{6} + C;$$

(c)

$$\ln(2x^2 - 3x + 13) + C;$$

(d)

$$\ln \sin x + C.$$

"JUST THE MATHS"

UNIT NUMBER

12.5

INTEGRATION 5
(Integration by parts)

by

A.J.Hobson

UNIT 12.5 - INTEGRATION 5

INTEGRATION BY PARTS

12.5.1 THE STANDARD FORMULA

The technique to be discussed here provides a convenient method for integrating the product of two functions. However, it is possible to develop a suitable formula by considering, instead, the **derivative** of the product of two functions.

We consider, first, the following comparison:

$\frac{\mathrm{d}}{\mathrm{d}x}[x\sin x] = x\cos x + \sin x$	$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
$x\cos x = \frac{\mathrm{d}}{\mathrm{d}x}[x\sin x] - \sin x$	$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[uv] - v\frac{\mathrm{d}u}{\mathrm{d}x}$
$\int x\cos x\ \mathrm{d}x = x\sin x - \int \sin x\ \mathrm{d}x$	$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$
$= x\sin x + \cos x + C$	

We see that, by labelling the product of two given functions as $u\frac{\mathrm{d}v}{\mathrm{d}x}$, we may express the integral of this product in terms of another integral which, it is anticipated, will be simpler than the original.

To summarise, the formula for **"integration by parts"** is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x.$$

EXAMPLES

1. Determine

$$I = \int x^2e^{3x}\ \mathrm{d}x.$$

Solution

In theory, it does not matter which element of the product x^2e^{3x} is labelled as u and which is labelled as $\frac{\mathrm{d}v}{\mathrm{d}x}$; but the integral obtained on the right-hand-side of the integration by parts formula must be simpler than the original.

In this case we shall take

$$u = x^2 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Hence,

$$I = x^2\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.2x \text{ d}x.$$

That is,

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x} \text{ d}x.$$

The integral on the right-hand-side still contains the product of two functions and so we must use integration by parts a second time, setting

$$u = x \text{ and } \frac{\text{d}v}{\text{d}x} = e^{3x}.$$

Thus,

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.1 \text{ d}x\right].$$

The integration may now be completed to obtain

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C,$$

or

$$I = \frac{e^{3x}}{27}\left[9x^2 - 6x + 2\right] + C.$$

2. Determine

$$I = \int x\ln x \text{ d}x.$$

Solution

In this case, we cannot effectively choose $\frac{\text{d}v}{\text{d}x} = \ln x$ since we would need to know the integral of $\ln x$ in order to find v. Hence, we choose

$$u = \ln x \text{ and } \frac{\text{d}v}{\text{d}x} = x,$$

obtaining

$$I = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}.\frac{1}{x}\ \mathrm{d}x.$$

That is,

$$I = \frac{1}{2}x^2 \ln x - \int \frac{x}{2}\ \mathrm{d}x,$$

giving

$$I = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C.$$

3. Determine

$$I = \int \ln x\ \mathrm{d}x.$$

Solution
It is possible to regard this as an integration by parts problem if we set

$$u = \ln x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = x \ln x - \int x.\frac{1}{x}\ \mathrm{d}x,$$

giving

$$I = x \ln x - x + C.$$

4. Evaluate

$$I = \int_0^1 \sin^{-1}x\ \mathrm{d}x.$$

Solution
In a similar way to the previous example, it is possible to regard this as an integration by parts problem if we set

$$u = \sin^{-1}x \ \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = [x\sin^{-1}x]_0^1 - \int_0^1 x.\frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x.$$

That is,

$$I = [x\sin^{-1}x + \sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1.$$

5. Determine

$$I = \int e^{2x}\cos x\ \mathrm{d}x.$$

Solution

In this example, it makes little difference whether we choose e^{2x} or $\cos x$ to be u; but we shall set

$$u = e^{2x} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x.$$

Hence,

$$I = e^{2x}\sin x - \int (\sin x).2e^{2x}\ \mathrm{d}x.$$

That is,

$$I = e^{2x}\sin x - 2\int e^{2x}\sin x\ \mathrm{d}x.$$

Now we need to integrate by parts again, setting

$$u = e^{2x} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x.$$

Therefore,

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x - \int (-\cos x).2e^{2x}\ \mathrm{d}x\right].$$

In other words, the original integral has appeared again on the right hand side to give

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x + 2I\right].$$

On simplification,

$$5I = e^{2x}\sin x + 2e^{2x}\cos x,$$

so that

$$I = \frac{1}{5}e^{2x}[\sin x + 2\cos x] + C.$$

Note:

The above examples suggest a priority order for choosing u in a typical integration by parts problem. For example, if the product to be integrated contains a logarithm or an inverse function, then we must choose the logarithm or the inverse function as u; but if there are powers of x without logarithms or inverse functions, then we choose the power of x to be u.

The order of priorities is as follows:

1. LOGARITHMS or INVERSE FUNCTIONS;
2. POWERS OF x;
3. POWERS OF e.

12.5.2 EXERCISES

1. Use integration by parts to evaluate the definite integral

$$\int_0^1 x^3e^{2x}\ \mathrm{d}x.$$

2. Use integration by parts to integrate the following functions with respect to x:

(a)

$$x^2\cos 2x;$$

(b)

$$x^5\ln x;$$

(c)

$$\tan^{-1}x;$$

(d)

$$x\tan^{-1}x.$$

3. Use integration by parts to evaluate the definite integral

$$\int_0^\pi e^{-2x}\sin 3x \ \mathrm{d}x.$$

12.5.3 ANSWERS TO EXERCISES

1.

$$\left[\frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3)\right]_0^1 = \frac{1}{8}\left(e^2 + 3\right) \simeq 1.299$$

2. (a)

$$\frac{1}{4}[2x^2\sin 2x + 2x\cos 2x - \sin 2x] + C;$$

(b)

$$\frac{x^6}{36}[6\ln x - 1] + C;$$

(c)

$$x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C;$$

(d)

$$\frac{1}{2}[x^2\tan^{-1}x - x + \tan^{-1}x] + C.$$

3.

$$\left[\frac{e^{-2x}}{13}(3\cos 3x - 2\sin 3x)\right]_0^\pi = -\frac{3}{13}(e^{-2\pi} + 1) \simeq -0.231$$

"JUST THE MATHS"

UNIT NUMBER

12.6

INTEGRATION 6
(Integration by partial fractions)

by

A.J.Hobson

UNIT 12.6 - INTEGRATION 6
INTEGRATION BY PARTIAL FRACTIONS

12.6.1 INTRODUCTION AND ILLUSTRATIONS

If the ratio of two polynomials, whose denominator has been factorised, is expressed as a sum of partial fractions, each partial fraction will be of a type whose integral can be determined by the methods of preceeding sections of this chapter.

The following summary of results will cover most elementary problems involving partial fractions:

RESULTS

1.
$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln(ax+b) + C.$$

2.
$$\int \frac{1}{(ax+b)^n}\,\mathrm{d}x = \frac{1}{a}.\frac{(ax+b)^{-n+1}}{-n+1} + C \quad \text{provided} \quad n \neq 1.$$

3.
$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

4.
$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C,$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

5.
$$\int \frac{2ax+b}{ax^2+bx+c}\,\mathrm{d}x = \ln(ax^2+bx+c) + C.$$

ILLUSTRATIONS

We use some of the results of examples on partial fractions in Unit 1.8

1.

$$\int \frac{7x+8}{(2x+3)(x-1)}\,\mathrm{d}x = \int \left[\frac{1}{2x+3} + \frac{3}{x-1}\right]\,\mathrm{d}x$$

$$= \frac{1}{2}\ln(2x+3) + 3\ln(x-1) + C.$$

2.

$$\int_6^8 \frac{3x^2+9}{(x-5)(x^2+2x+7)}\,\mathrm{d}x = \int_6^8 \left[\frac{2}{x-5} + \frac{x+1}{x^2+2x+7}\right]\,\mathrm{d}x$$

$$= \left[2\ln(x-5) + \frac{1}{2}\ln(x^2+2x+7)\right]_6^8 \simeq 2.427$$

3.

$$\int \frac{9}{(x+1)^2(x-2)} = \int \left[\frac{-1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}\right]\,\mathrm{d}x$$

$$= -\ln(x+1) + \frac{3}{x+1} + \ln(x-2) + C.$$

4.

$$\int \frac{4x^2+x+6}{(x-4)(x^2+4x+5)}\,\mathrm{d}x = \int \left[\frac{2}{x-4} + \frac{2x+1}{x^2+4x+5}\right]\mathrm{d}x$$

$$= 2\ln(x-4) + \ln(x^2+4x+5) - 3\tan^{-1}(x+2) + C.$$

Note:
In the last example above, the second partial fraction has a numerator of $2x+1$ which is not the derivative of x^2+4x+5. But we simply rearrange the numerator as $(2x+4)-3$ to give a third integral which requires the technique of completing the square (discussed in Unit 12.3).

12.6.2 EXERCISES

Integrate the following functions with respect to x:

1. (a)

$$\frac{3x+5}{(x+1)(x+2)};$$

(b)

$$\frac{17x+11}{(x+1)(x-2)(x+3)};$$

(c)

$$\frac{3x^2-8}{(x-1)(x^2+x-7)}.$$

(d)

$$\frac{2x+1}{(x+2)^2(x-3)};$$

(e)

$$\frac{9+11x-x^2}{(x+1)^2(x+2)};$$

(f)

$$\frac{x^5}{(x+2)(x-4)}.$$

2. Evaluate the following definite integrals

(a)

$$\int_2^5 \frac{7x^2+11x+47}{(x-1)(x^2+2x+10)}\,\mathrm{d}x;$$

(b)

$$\int_1^3 \frac{4x^2+1}{x(2x-1)^2}\,\mathrm{d}x.$$

12.6.3 ANSWERS TO EXERCISES

1. (a)

$$2\ln(x+1)+\ln(x+2)+C;$$

(b)

$$\ln(x+1)+3\ln(x-2)-4\ln(x+3)+C;$$

(c)

$$\ln(x-1)+\ln(x^2+x-7)+C;$$

(d)

$$-\frac{3}{5(x+2)}-\frac{7}{25}\ln(x+2)+\frac{7}{25}\ln(x-3)+C;$$

(e)

$$-\frac{3}{(x+1)^2}+\frac{16}{x+1}-\frac{17}{x+2}$$

$$\frac{3}{x+1}+16\ln(x+1)-17\ln(x+2)+C;$$

(f)

$$\frac{x^4}{4}+\frac{2x^3}{3}+6x^2+40x+\frac{16}{3}\ln(x+2)+\frac{512}{3}\ln(x-4)+C.$$

2. (a)

$$\left[5\ln(x-1)+\ln(x^2+2x+10)+\frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_2^5\simeq -2.726;$$

(b)

$$\left[\ln x-\frac{2}{2x-1}\right]_1^3\simeq 2.699$$

"JUST THE MATHS"

UNIT NUMBER

12.7

INTEGRATION 7
(Further trigonometric functions)

by

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UNIT 12.7 - INTEGRATION 7 - FURTHER TRIGONOMETRIC FUNCTIONS

12.7.1 PRODUCTS OF SINES AND COSINES

In order to integrate the product of a sine and a cosine, or two cosines, or two sines, we may use one of the following trigonometric identities:

$$\text{sinAcosB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B}) + \sin(\text{A}-\text{B})\right];$$

$$\text{cosAsinB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B}) - \sin(\text{A}-\text{B})\right];$$

$$\text{cosAcosB} \equiv \frac{1}{2}\left[\cos(\text{A}+\text{B}) + \cos(\text{A}-\text{B})\right];$$

$$\text{sinAsinB} \equiv \frac{1}{2}\left[\cos(\text{A}-\text{B}) - \cos(\text{A}+\text{B})\right].$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin 2x \cos 5x \ \mathrm{d}x.$$

Solution

$$\int \sin 2x \cos 5x \ \mathrm{d}x = \frac{1}{2}\int \left[\sin 7x - \sin 3x\right] \mathrm{d}x$$

$$= -\frac{\cos 7x}{14} + \frac{\cos 3x}{6} + C.$$

2. Determine the indefinite integral

$$\int \sin 3x \sin x \ \mathrm{d}x.$$

Solution

$$\int \sin 3x \sin x \ \mathrm{d}x = \frac{1}{2}\int \left[\cos 2x - \cos 4x\right] \mathrm{d}x$$

$$= \frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C.$$

12.7.2 POWERS OF SINES AND COSINES

In this section, we consider the two integrals,

$$\int \sin^n x \, dx \quad \text{and} \quad \int \cos^n \, dx,$$

where n is a positive integer.

(a) The Complex Number Method

A single method which will cover both of the above integrals requires us to use the methods of Unit 6.5 in order to express $\cos^n x$ and $\sin^n x$ as a sum of whole multiples of sines or cosines of whole multiples of x.

EXAMPLE

Determine the indefinite integral

$$\int \sin^4 x \, dx.$$

Solution

By the complex number method,

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

The Working:

$$j^4 2^4 \sin^4 x \equiv \left(z - \frac{1}{z}\right)^4,$$

where $z \equiv \cos x + j \sin x$.

That is,

$$16\sin^4 x \equiv z^4 - 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 - 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4;$$

or, after cancelling common factors,

$$16\sin^4 x \equiv z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \equiv z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6,$$

which gives

$$16\sin^4 x \equiv 2\cos 4x - 8\cos 2x + 6,$$

or

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

Hence,

$$\int \sin^4 x \, \mathrm{d}x = \frac{1}{8}\left[\frac{\sin 4x}{4} - 4\frac{\sin 2x}{2} + 3x\right] + C$$

$$= \frac{1}{32}[\sin 4x - 8\sin 2x + 12x] + C.$$

(b) Odd Powers of Sines and Cosines

The following method uses the facts that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = \sin x.$$

We illustrate with examples in which use is made of the trigonometric identity

$$\cos^2 A + \sin^2 A \equiv 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^3 x \, \mathrm{d}x.$$

Solution

$$\int \sin^3 x \ \mathrm{d}x = \int \sin^2 x. \sin x \ \mathrm{d}x.$$

That is,

$$\int \sin^3 x \mathrm{d}x = \int \left(1 - \cos^2 x\right) \sin x \ \mathrm{d}x$$

$$= \int \left(\sin x - \cos^2 x. \sin x\right) \ \mathrm{d}x$$

$$= -\cos x + \frac{\cos^3 x}{3} + C.$$

2. Determine the indefinite integral

$$\int \cos^7 x \ \mathrm{d}x.$$

Solution

$$\int \cos^7 x \ \mathrm{d}x = \int \cos^6 x. \cos x \ \mathrm{d}x.$$

That is,

$$\int \cos^7 x \ \mathrm{d}x = \int \left(1 - \sin^2 x\right)^3 . \cos x \ \mathrm{d}x$$

$$= \int \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right) . \cos x \ \mathrm{d}x$$

$$= \sin x - \sin^3 x + 3.\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

(c) Even Powers of Sines and Cosines

The method illustrated here becomes tedious if the even power is higher than 4. In such cases, it is best to use the complex number method in paragraph (a) above.

In the examples which follow, we shall need the trigonometric identity

$$\cos 2A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^2 x \, dx.$$

Solution

$$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \, dx$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

3. Determine the indefinite integral

$$\int \cos^4 x \, dx.$$

Solution

$$\int \cos^4 x \, dx = \int \left[\cos^2 x\right]^2 dx = \int \left[\frac{1}{2}(1 + \cos 2x)\right]^2 dx.$$

That is,

$$\int \cos^4 x \, dx = \int \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) \, dx$$

$$= \int \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2}[1 + \cos 4x]\right) \, dx$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

12.7.3 EXERCISES

1. Determine the indefinite integral

$$\int \cos x \cos 3x \ \mathrm{d}x.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \cos 4x \sin 2x \ \mathrm{d}x.$$

3. Determine the following indefinite integrals:

   (a)

$$\int \sin^5 x \ \mathrm{d}x;$$

   (b)

$$\int \cos^3 x \ \mathrm{d}x.$$

4. Evaluate the following definite integrals:

   (a)

$$\int_0^{\frac{\pi}{8}} \sin^4 x \ \mathrm{d}x;$$

   (b)

$$\int_0^{\frac{\pi}{2}} \cos^6 x \ \mathrm{d}x.$$

12.7.4 ANSWERS TO EXERCISES

1.

$$\frac{\sin 4x}{8} + \frac{\sin 2x}{4} + C.$$

2.

$$-\frac{\sqrt{3}}{4} \simeq -0.433$$

3. (a)

$$-\frac{\cos^5 x}{5} + 2\frac{\cos^3 x}{3} - \cos x + C,$$

or

$$\frac{1}{16}\left[-\frac{\cos 5x}{5} + \frac{5\cos 3x}{3} - 10\cos x\right] + C \quad \text{by complex numbers;}$$

(b)

$$\sin x - \frac{\sin^3 x}{3} + C,$$

or

$$\frac{1}{4}\left[\frac{\sin 3x}{3} - 3\sin x\right] + C \quad \text{by complex numbers.}$$

4. (a)

$$1.735 \times 10^{-3} \text{ approx;}$$

(b)

$$-1.$$

"JUST THE MATHS"

UNIT NUMBER

12.8

INTEGRATION 8
(The tangent substitutions)

by

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UNIT 12.8 - INTEGRATION 8

THE TANGENT SUBSTITUTIONS

There are two types of integral, involving sines and cosines, which require a special substitution using a tangent function. They are described as follows:

12.8.1 THE SUBSTITUTION $t = \tan x$

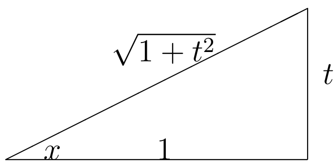
This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin^2 x + c\cos^2 x}\,\mathrm{d}x,$$

where a, b and c are constants; though, in most exercises, at least one of these three constants will be zero.

A simple right-angled triangle will show that, if $t = \tan x$, then

$$\sin x \equiv \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos x \equiv \frac{1}{\sqrt{1+t^2}}.$$



Furthermore,

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \sec^2 x \equiv 1 + t^2 \quad \text{so that} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{1}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{4 - 3\sin^2 x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4-3\sin^2 x}\,\mathrm{d}x$$

$$= \int \frac{1}{4-\frac{3t^2}{1+t^2}}.\frac{1}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{4+t^2}\,\mathrm{d}t$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2}+C = \frac{1}{2}\tan^{-1}\left[\frac{\tan x}{2}\right]+C.$$

2. Determine the indefinite integral

$$\int \frac{1}{\sin^2 x+9\cos^2 x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{\sin^2 x+9\cos^2 x}\,\mathrm{d}x$$

$$= \int \frac{1}{\frac{t^2}{1+t^2}+\frac{9}{1+t^2}}.\frac{1}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{t^2+9}\,\mathrm{d}t$$

$$= \frac{1}{3}\tan^{-1}\frac{t}{3}+C = \frac{1}{3}\tan^{-1}\left[\frac{\tan x}{3}\right]+C.$$

12.8.2 THE SUBSTITUTION $t = \tan(x/2)$

This substitution is used for integrals of the form

$$\int \frac{1}{a+b\sin x+c\cos x}\,\mathrm{d}x,$$

where a, b and c are constants; though, in most exercises, one or more of these constants will be zero.

In order to make the substitution, we make the following observations:

(i)

$$\sin x \equiv 2\sin(x/2).\cos(x/2) \equiv 2\tan(x/2).\cos^2(x/2) \equiv \frac{2\tan(x/2)}{\sec^2(x/2)} \equiv \frac{2\tan(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\sin x \equiv \frac{2t}{1+t^2}.$$

(ii)

$$\cos x \equiv \cos^2(x/2) - \sin^2(x/2) \equiv \cos^2(x/2)\left[1-\tan^2(x/2)\right] \equiv \frac{1-\tan^2(x/2)}{\sec^2(x/2)} \equiv \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\cos x \equiv \frac{1-t^2}{1+t^2}.$$

(iii)

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \frac{1}{2}\sec^2(x/2) \equiv \frac{1}{2}\left[1+\tan^2(x/2)\right] \equiv \frac{1}{2}[1+t^2].$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{2}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

Solution

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{1+t^2+2t}\,\mathrm{d}t$$

$$= \int \frac{2}{(1+t)^2}\,\mathrm{d}t$$

$$= -\frac{2}{1+t} + C = -\frac{2}{1+\tan(x/2)} + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{4\frac{1-t^2}{1+t^2} - \frac{6t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{4-4t^2-6t}\,\mathrm{d}t = \int -\frac{1}{2t^2+3t-2}\,\mathrm{d}t$$

$$= \int -\frac{1}{(2t-1)(t+2)}\,\mathrm{d}t$$

$$= \int \frac{1}{5}\left[\frac{1}{t+2} - \frac{2}{2t-1}\right]\,\mathrm{d}t$$

$$= \frac{1}{5}[\ln(t+2) - \ln(2t-1)] + C = \frac{1}{5}\ln\left[\frac{\tan(x/2)+2}{2\tan(x/2)-1}\right] + C.$$

12.8.3 EXERCISES

1. Determine the indefinite integral

$$\int \frac{1}{4+12\cos^2 x}\,\mathrm{d}x.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{4}} \frac{1}{5\cos^2 x+3\sin^2 x}\,\mathrm{d}x.$$

3. Determine the indefinite integral

$$\int \frac{1}{5+3\cos x}\,\mathrm{d}x.$$

4. Evaluate the definite integral

$$\int_3^{3.1} \frac{1}{12\sin x+5\cos x}\,\mathrm{d}x.$$

12.8.4 ANSWERS TO EXERCISES

1.

$$\frac{1}{4}\tan^{-1}\left[\frac{\tan x}{2}\right]+C.$$

2.

$$\left[\frac{1}{\sqrt{15}}\tan^{-1}\left(\sqrt{\frac{3}{5}}\tan x\right)\right]_0^{\frac{\pi}{4}} \simeq 0.1702$$

3.

$$\frac{1}{2}\tan^{-1}\left[\frac{\tan(x/2)}{2}\right]+C.$$

4.

$$\left[\frac{1}{13}[5\ln(5\tan(x/2)+1)-\ln(\tan(x/2)-5)\right]_3^{3.1} \simeq 0.348$$

"JUST THE MATHS"

UNIT NUMBER

12.9

INTEGRATION 9
(Reduction formulae)

by

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UNIT 12.9 - INTEGRATION 9

REDUCTION FORMULAE

INTRODUCTION

For certain integrals, both definite and indefinite, the function being integrated (that is, the "integrand") consists of a product of two functions, one of which involves an unspecified integer, say n. Using the method of integration by parts, it is sometimes possible to express such an integral in terms of a similar integral where n has been replaced by $(n-1)$, or sometimes $(n-2)$. The relationship between the two integrals is called a **"reduction formula"** and, by repeated application of this formula, the original integral may be determined in terms of n.

12.9.1 INDEFINITE INTEGRALS

The method will be illustrated by examples.

EXAMPLES

1. Obtain a reduction formula for the indefinite integral

$$I_n = \int x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^x$, we obtain

$$I_n = x^n e^x - \int e^x . n x^{n-1} \, \mathrm{d}x.$$

That is,

$$I_n = x^n e^x - n I_{n-1}.$$

Substituting $n = 3$,

$$I_3 = x^3 e^x - 3I_2,$$

where

$$I_2 = x^2 e^x - 2I_1$$

and

$$I_1 = xe^x - I_0.$$

But

$$I_0 = \int e^x \, \mathrm{d}x = e^x + \text{constant},$$

which leads us to the conclusion that

$$I_3 = x^3 e^x - 3\left[x^2 e^x - 2\left(xe^x - e^x\right)\right] + \text{constant}.$$

In other words,

$$I_3 = e^x\left[x^3 - 3x^2 + 6x - 6\right] + C,$$

where C is an arbitrary constant.

2. Obtain a reduction formula for the indefinite integral

$$I_n = \int x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = x^n \sin x - \int \sin x . n x^{n-1} \, \mathrm{d}x = x^n \sin x - n \int x^{n-1} \sin x \, \mathrm{d}x.$$

Using integration by parts in this last integral, with $u = x^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = x^n \sin x - n\left\{-x^{n-1}\cos x + \int \cos x . (n-1) x^{n-2} \, \mathrm{d}x\right\}.$$

That is,

$$I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = x^2 \sin x + 2x \cos x - 2I_0,$$

where

$$I_0 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x + C,$$

where C is an arbitrary constant.
Also, substituting $n = 3$,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 3.2.I_1,$$

where

$$I_1 = \int x \cos x \, \mathrm{d}x = x \sin x + \cos x + \text{constant}.$$

Therefore,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 6x \sin x - 6 \cos x + D,$$

where D is an arbitrary constant.

12.9.2 DEFINITE INTEGRALS

Integrals of the type encountered in the previous section may also include upper and lower limits of integration. The process of finding a reduction formula is virtually the same, except that the limits of integration are inserted where appropriate. Again, the method is illustrated by examples.

EXAMPLES

1. Obtain a reduction formula for the definite integral

$$I_n = \int_0^1 x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

From the first example in section 12.9.1,

$$I_n = [x^n e^x]_0^1 - nI_{n-1} = e - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = e - 3I_2,$$

where

$$I_2 = e - 2I_1$$

and

$$I_1 = e - I_0.$$

But

$$I_0 = \int_0^1 e^x \, \mathrm{d}x = e - 1,$$

which leads us to the conclusion that

$$I_3 = e - 3e + 6e - 6e + 6 = 6 - 2e.$$

2. Obtain a reduction formula for the definite integral

$$I_n = \int_0^\pi x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

From the second example in section 12.9.1,

$$I_n = \left[x^n \sin x + nx^{n-1} \cos x\right]_0^\pi - n(n-1)I_{n-2} = -n\pi^{n-1} - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = -2\pi - 2I_0,$$

where

$$I_0 = \int_0^\pi \cos x \, \mathrm{d}x = [\sin x]_0^\pi = 0.$$

Hence,

$$I_2 = -2\pi.$$

Also, substituting $n = 3$,

$$I_3 = -3\pi^2 - 3.2.I_1,$$

where

$$I_1 = \int_0^{\pi} x\cos x \ \mathrm{d}x = [x\sin x + \cos x]_0^{\pi} = -2.$$

Therefore,

$$I_3 = -3\pi^2 + 12.$$

12.9.3 EXERCISES

1. Obtain a reduction formula for

$$I_n = \int \ x^n e^{2x} \ \mathrm{d}x$$

when $n \geq 1$ and, hence, determine I_3.

2. Obtain a reduction formula for

$$I_n = \int_0^1 \ x^n e^{2x} \mathrm{d}x$$

when $n \geq 1$ and, hence, evaluate I_4.

3. Obtain a reduction formula for

$$I_n = \int \ x^n \sin x \ \mathrm{d}x$$

when $n \geq 1$ and, hence, determine I_4.

4. Obtain a reduction formula for

$$I_n = \int_0^{\pi} \ x^n \sin x \ \mathrm{d}x$$

when $n \geq 1$ and, hence, evaluate I_3.

5. If

$$I_n = \int (\ln x)^n \, \mathrm{d}x,$$

where $n \geq 1$, show that

$$I_n = x(\ln x)^n - nI_{n-1}$$

and, hence, determine I_3.

6. If

$$I_n = \int \left(x^2 + a^2\right)^n \, \mathrm{d}x,$$

show that

$$I_n = \frac{1}{2n+1}\left[x\left(x^2 + a^2\right)^n + 2na^2 I_{n-1}\right].$$

Hint: Write $(x^2 + a^2)^n$ as $1.(x^2 + a^2)^n$.

12.9.4 ANSWERS TO EXERCISES

1.

$$I_n = \frac{1}{2}\left[x^n e^{2x} - nI_{n-1}\right],$$

giving

$$I_3 = \frac{e^{2x}}{8}\left[4x^3 - 6x^2 + 6x - 3\right] + C.$$

2.

$$I_n = \frac{1}{2}\left[e^2 - nI_{n-1}\right],$$

giving

$$I_4 = \frac{1}{4}\left[e^2 - 3\right].$$

3.

$$I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1) I_{n-2},$$

giving

$$I_4 = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.$$

4.

$$I_n = \pi^n - n(n-1) I_{n-2},$$

giving

$$I_3 = \pi^3 - 6\pi.$$

5.

$$I_3 = x \Big[\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6 \Big] + C.$$

"JUST THE MATHS"

UNIT NUMBER

12.10

INTEGRATION 10
(Further reduction formulae)

by

A.J.Hobson

UNIT 12.10 - INTEGRATION 10

FURTHER REDUCTION FORMULAE

INTRODUCTION

As an extension to the idea of reduction formulae, there are two particular definite integrals which are worthy of special consideration. They are

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x.$$

But, first, we shall establish the reduction formulae for the equivalent indefinite integrals.

12.10.1 INTEGER POWERS OF A SINE

Suppose that

$$I_n = \int \sin^n x \, \mathrm{d}x;$$

then, by writing the integrand as the product of two functions, we have

$$I_n = \int \sin^{n-1} x \sin x \, \mathrm{d}x.$$

Using integration by parts, with $u = \sin^{n-1} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = \sin^{n-1} x(-\cos x) + \int (n-1)\sin^{n-2} x \cos^2 x \, \mathrm{d}x.$$

But, since $\cos^2 x \equiv 1 - \sin^2 x$, this becomes

$$I_n = -\sin^{n-1} x \cos x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n}\left[-\sin^{n-1} x \cos x + (n-1)I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \sin^6 x \, \mathrm{d}x.$$

Solution

$$I_6 = \frac{1}{6}\left[-\sin^5 x \cos x + 5I_4\right],$$

where

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x + 3I_2\right], \quad I_2 = \frac{1}{2}\left[-\sin x \cos x + I_0\right]$$

and

$$I_0 = \int \mathrm{d}x = x + \text{constant}.$$

Hence,

$$I_2 = \frac{1}{2}\left[-\sin x \cos x + x + \text{constant}\right];$$

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x - \frac{3}{2}\sin x \cos x + \frac{3}{2}x + \text{constant}\right];$$

$$I_6 = \frac{1}{6}\left[-\sin^5 x \cos x - \frac{5}{4}\sin^3 x \cos x - \frac{15}{8}\sin x \cos x + \frac{15}{8}x + \text{constant}\right].$$

$$\text{Thus,} \quad \int \sin^6 x \, \mathrm{d}x = -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5x}{16} + C,$$

where C is an arbitrary constant.

12.10.2 INTEGER POWERS OF A COSINE

Suppose that

$$I_n = \int \cos^n x \ \mathrm{d}x;$$

then, by writing the integrand as the product of two functions, we have

$$I_n = \int \cos^{n-1} x \cos x \ \mathrm{d}x.$$

Using integration by parts, with $u = \cos^{n-1} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x \ \mathrm{d}x.$$

But, since $\sin^2 x \equiv 1 - \cos^2 x$, this becomes

$$I_n = \cos^{n-1} x \sin x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n}\left[\cos^{n-1} x \sin x + (n-1)I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \cos^5 x \ \mathrm{d}x.$$

Solution

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + 4I_3\right],$$

where

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2I_1\right]$$

and

$$I_1 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2\sin x + \text{constant}\right];$$

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + \frac{4}{3}\cos^2 x \sin x + \frac{8}{3}\sin x + \text{constant}\right];$$

We conclude that

$$\int \cos^5 x \, \mathrm{d}x = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + C,$$

where C is an arbitrary constant.

12.10.3 WALLIS'S FORMULAE

Here, we consider the definite integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x.$$

Denoting either of these integrals by I_n, the reduction formula reduces to

$$I_n = \frac{n-1}{n} I_{n-2}$$

in both cases, from the previous two sections.

Convenient results may be obtained from this formula according as n is an odd number or an even number, as follows:

(a) n is an odd number

Repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}. \ldots \frac{6}{7}.\frac{4}{5}.\frac{2}{3}.I_1.$$

But

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \; \mathrm{d}x \quad \text{or} \quad I_1 = \int_0^{\frac{\pi}{2}} \cos x \; \mathrm{d}x,$$

both of which have a value of 1.

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5). \ldots 6.4.2}{n(n-2)(n-4). \ldots 7.5.3},$$

which is **the first of "Wallis's formulae"**.

(b) n is an even number

This time, repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}. \ldots \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.I_0.$$

But

$$I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}.$$

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5). \ldots 5.3.1}{n(n-2)(n-4). \ldots 6.4.2}\frac{\pi}{2},$$

which is **the second of "Wallis's formulae"**.

EXAMPLES

1. Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x.$$
Solution
$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x = \frac{4.2}{5.3} = \frac{8}{15}.$$
2. Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x.$$
Solution
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x = \frac{3.1}{4.2}\frac{\pi}{2} = \frac{3\pi}{16}.$$

12.10.4 COMBINATIONS OF SINES AND COSINES

Another type of problem to which Wallis's formulae may be applied is of the form

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, \mathrm{d}x,$$

where either m or n (or both) is an even number. We simply use $\sin^2 x \equiv 1 - \cos^2 x$ or $\cos^2 x \equiv 1 - \sin^2 x$ in order to convert the problem to several integrals of the types already discussed.

EXAMPLE

Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^5 x \left(1 - \cos^2 x\right) \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\cos^5 x - \cos^7 x\right) \, \mathrm{d}x,$$

which may be interpreted as

$$I_5 - I_7 = \frac{4.2}{5.3} - \frac{5.4.3}{6.4.2} = \frac{8}{15} - \frac{16}{35} = \frac{8}{105}.$$

12.10.5 EXERCISES

1. Determine the indefinite integral

$$\int \cos^4 x \, \mathrm{d}x.$$

2. Determine the indefinite integral

$$\int \sin^7 x \, \mathrm{d}x.$$

3. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^6 x \, \mathrm{d}x.$$

4. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^9 x \, \mathrm{d}x.$$

5. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \sin^6 x \, \mathrm{d}x.$$

6. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x \, \mathrm{d}x.$$

12.10.6 ANSWERS TO EXERCISES

1.

$$\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3x}{8} + C.$$

2.

$$-\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x - \frac{24}{105}\sin^2 x \cos x - \frac{16}{35}\cos x + C.$$

3.

$$\frac{5\pi}{32}.$$

4.

$$\frac{128}{315}.$$

5.

$$\frac{5\pi}{32}.$$

6.

$$-\frac{4}{105}.$$

"JUST THE MATHS"

UNIT NUMBER

13.1

INTEGRATION APPLICATIONS 1
(The area under a curve)

by

A.J.Hobson

UNIT 13.1 - INTEGRATION APPLICATIONS 1

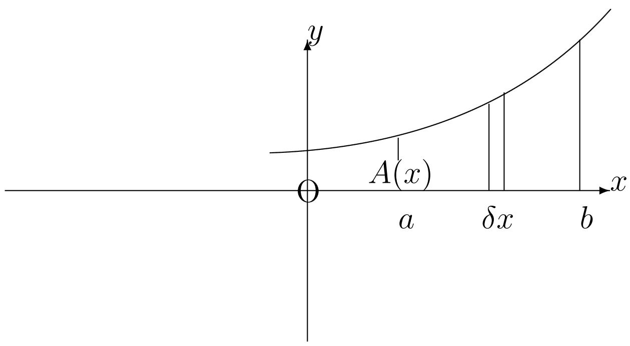
THE AREA UNDER A CURVE

13.1.1 THE ELEMENTARY FORMULA

We shall consider, here, a method of calculating the area contained between the x-axis of a cartesian co-ordinate system and the arc, from $x = a$ to $x = b$, of the curve whose equation is

$$y = f(x).$$

Suppose that $A(x)$ represents the area contained between the curve, the x-axis, the y-axis and the ordinate at some arbitrary value of x.



A small increase of δx in x will lead to a corresponding increase of δA in A approximating in area to that of a narrow rectangle whose width is δx and whose height is $f(x)$.

Thus,

$$\delta A \simeq f(x)\delta x,$$

which may be written

$$\frac{\delta A}{\delta x} \simeq f(x).$$

By allowing δx to tend to zero, the approximation disappears to give

$$\frac{\mathrm{d}A}{\mathrm{d}x} = f(x).$$

Hence, on integrating both sides with respect to x,

$$A(x) = \int f(x)\ \mathrm{d}x.$$

The constant of integration would need to be such that $A = 0$ when $x = 0$; but, in fact, we do not need to know the value of this constant because the required area, from $x = a$ to $x = b$, is given by

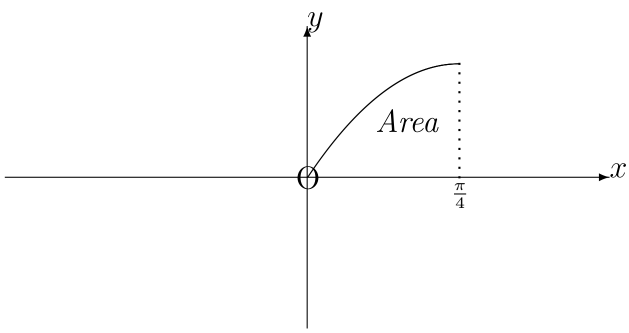
$$A(b) - A(a) = \int_a^b f(x)\ \mathrm{d}x.$$

EXAMPLES

1. Determine the area contained between the x-axis and the curve whose equation is $y = \sin 2x$, from $x = 0$ to $x = \frac{\pi}{4}$.

   Solution

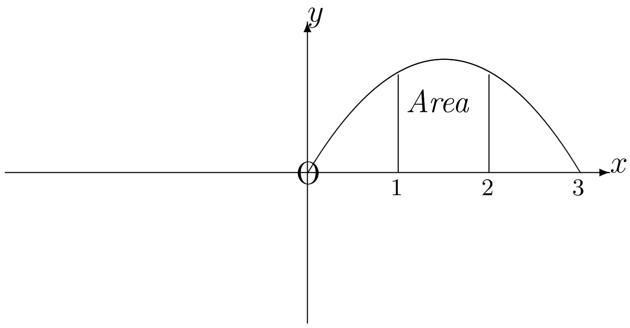
$$\int_0^{\frac{\pi}{4}} \sin 2x\ \mathrm{d}x = \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$



2. Determine the area contained between the x-axis and the curve whose equation is $y = 3x - x^2$, from $x = 1$ to $x = 2$.

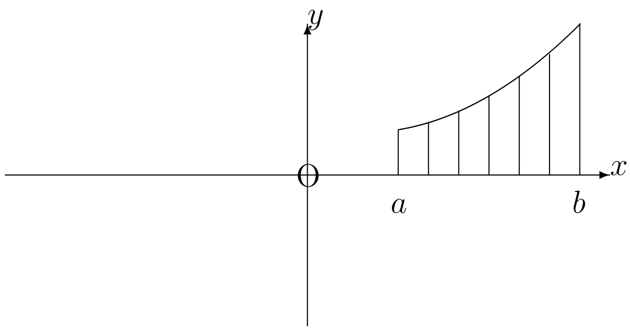
Solution

$$\int_1^2 (3x - x^2)\,\mathrm{d}x = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^2 = \left(6 - \frac{8}{3}\right) - \left(\frac{3}{2} - \frac{1}{3}\right) = \frac{13}{6}.$$



13.1.2 DEFINITE INTEGRATION AS A SUMMATION

Consider, now, the same area as in the previous section, but regarded (approximately) as the sum of a large number of narrow rectangles with typical width δx and typical height $f(x)$. The narrower the strips, the better will be the approximation.



Hence, we may state an alternative expression for the area from $x = a$ to $x = b$ in the form

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x.$$

Since this new expression represents the same area as before, we may conclude that

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x = \int_a^b f(x)\delta x.$$

Notes:

(i) The above result shows that an area which lies wholly **below** the x-axis will be **negative** and so care must be taken with curves which cross the x-axis between $x = a$ and $x = b$.

(ii) If c is any value of x between $x = a$ and $x = b$, the above result shows that

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x.$$

(iii) To calculate the TOTAL area contained between the x-axis and a curve which crosses the x-axis between $x = a$ and $x = b$, account must be taken of any parts of the area which are negative.

(iv) It is usually a good idea to sketch the area under consideration before evaluating the appropriate definite integrals.

(v) It will be seen shortly that the formula obtained for definite integration as a summation has a wider field of application than simply the calculation of areas.

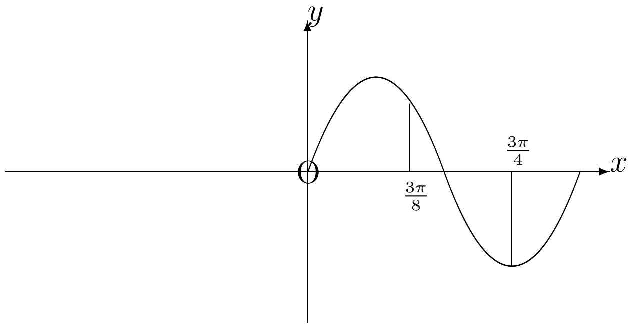
EXAMPLES

1. Determine the total area between the x-axis and the curve whose equation is $y = \sin 2x$, from $x = \frac{3\pi}{8}$ and $x = \frac{3\pi}{4}$.
   Solution

$$\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sin 2x\ \mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2x\ \mathrm{d}x.$$

   That is,

$$\left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}} - \left[-\frac{\cos 2x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right) - \left(0 - \frac{1}{2}\right) = 1 - \frac{1}{2\sqrt{2}}.$$



2. Evaluate the definite integral,

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \ \mathrm{d}x.$$

Solution

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \ \mathrm{d}x = \left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} = -\frac{1}{2\sqrt{2}}.$$

13.1.3 EXERCISES

1. Determine the areas bounded by the following curves and the x-axis between the ordinates $x = 1$ and $x = 3$:

   (a)
   $$y = 2x^2 + x + 1;$$
   (b)
   $$y = (1 - x)^2;$$
   (c)
   $$y = 2\sqrt{x}.$$

2. Sketch the curve whose equation is

   $$y = (1 - x)(2 + x)$$

   and determine the area contained between the x-axis and the portion of the curve above the x-axis.

3. To the nearest whole number, determine the area bounded between $x = 1$ and $x = 2$ by the curves whose equations are

$$y = 3e^{2x} \text{ and } y = 3e^{-x}.$$

4. Determine the area bounded between $x = 0$ and $x = \frac{\pi}{3}$ by the curves whose equations are

$$y = \sin x \text{ and } y = \sin 2x.$$

5. Determine the total area, from $x = 0$ to $x = \frac{3\pi}{10}$, contained between the x-axis and the curve whose equation is

$$y = \cos 5x.$$

13.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{70}{3};$$

(b)

$$\frac{8}{3};$$

(c)

$$4\sqrt{3} - \frac{4}{3}.$$

2.

$$\frac{9}{2}.$$

3.

$$70.$$

4.

$$0.25$$

5.

$$\frac{2\sqrt{2}-1}{5\sqrt{2}} - \simeq 0.259$$

"JUST THE MATHS"

UNIT NUMBER

13.2

INTEGRATION APPLICATIONS 2
(Mean values)
&
(Root mean square values)

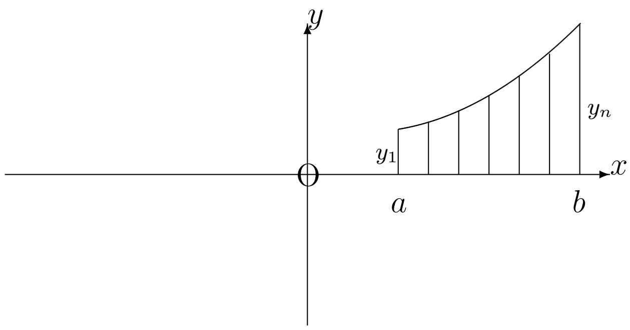
by

A.J.Hobson

UNIT 13.2 - INTEGRATION APPLICATIONS 2

MEAN AND ROOT MEAN SQUARE VALUES

13.2.1 MEAN VALUES



On the curve whose equation is

$$y = f(x),$$

suppose that $y_1, y_2, y_3, ..., y_n$ are the y-coordinates which correspond to n different x-coordinates, $a = x_1, x_2, x_3, ..., x_n = b$.

The average (that is, the arithmetic mean) of these n y-coordinates is

$$\frac{y_1 + y_2 + y_3 + ... + y_n}{n}.$$

But now suppose that we wished to determine the average (arithmetic mean) of **all** the y-coordinates, from $x = a$ to $x = b$ on the curve whose equation is $y = f(x)$.

We could make a reasonable approximation by taking a very **large** number, n, of y-coordinates separated in the x-direction by very **small** distances. If these distances are typically represented by δx, then the required mean value could be written

$$\frac{y_1\delta x + y_2\delta x + y_3\delta x + ... + y_n\delta x}{n\delta x},$$

in which the denominator is equivalent to $(b - a + \delta x)$, since there are only $n - 1$ spaces between the n y-coordinates.

Allowing the number of y-coordinates to increase indefinitely, δx will to tend to zero and we obtain the formula for the **"Mean Value"** in the form

$$\text{M.V.} = \frac{1}{b-a} \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x.$$

That is,

$$\text{M.V.} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x.$$

Note:
In cases where the definite integral in this formula represents the area between the curve and the x-axis, the Mean Value provides the height of a rectangle, with base $b - a$, having the same area as that represented by the definite integral.

EXAMPLE

Determine the Mean Value of the function

$$f(x) \equiv x^2 - 5x$$

from $x = 1$ to $x = 4$.

Solution

The Mean Value is given by

$$\text{M.V.} = \frac{1}{4-1} \int_1^4 (x^2 - 5x) \, \mathrm{d}x = \frac{1}{3} \left[\frac{x^3}{3} - \frac{5x^2}{2} \right]_1^4 =$$

$$\frac{1}{3} \left[\left(\frac{64}{3} - 40 \right) - \left(\frac{1}{3} - \frac{5}{2} \right) \right] = -\frac{33}{2}.$$

13.2.2 ROOT MEAN SQUARE VALUES

It is sometimes convenient to use an alternative kind of average for the values of a function, $f(x)$, between $x = a$ and $x = b$.

The **"Root Mean Square Value"** provides a measure of "central tendency" for the **numerical** values of $f(x)$ and is defined to be the square root of the Mean Value of $f(x)$ from $x = a$ to $x = b$.

Hence,

$$\text{R.M.S.V.} = \sqrt{\frac{1}{b-a}\int_a^b \ [f(x)]^2 \ \mathrm{d}x}.$$

EXAMPLE

Determine the Root Mean Square Value of the function, $f(x) \equiv x^2 - 5$, from $x = 1$ to $x = 3$.

Solution

The Root Mean Square Value is given by

$$\text{R.M.S.V.} = \sqrt{\frac{1}{3-1}\int_1^3 \ (x^2-5)^2 \ \mathrm{d}x}.$$

Temporarily ignoring the square root, we obtain the **"Mean Square Value"**,

$$\text{M.S.V.} = \frac{1}{2}\int_1^3 \ (x^4 - 10x^2 + 25) \ \mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{10x^3}{3} + 25x\right]_1^3 = \frac{1}{2}\left[\left(\frac{243}{5} - \frac{270}{3} + 75\right) - \left(\frac{1}{5} - \frac{10}{3} + 25\right)\right] = \frac{176}{30}.$$

Thus,

$$\text{R.M.S.V.} = \sqrt{\frac{176}{30}} \simeq 2.422$$

13.2.3 EXERCISES

1. (a) Determine the Mean Value of the function, $(x-1)(x-2)$, from $x=1$ to $x=2$;
   (b) Determine, correct to three significant figures, the Mean Value of the function, $\frac{1}{2x+5}$, from $x=3$ to $x=5$;
   (c) Determine the Mean Value of the function, $\sin 2t$, from $t=0$ to $t=\frac{\pi}{2}$;
   (d) Determine, correct to three places of decimals, the Mean Value of the function, e^{-x}, from $x=1$ to $x=5$;
   (e) Determine, correct to three significant figures, the mean value of the function, xe^{-2x}, from $x=0$ to $x=2$.
2. (a) Determine the Root Mean Square Value of the function, $3x+1$, from $x=-2$ to $x=2$;
   (b) Determine the Root Mean Square Value, of the function, e^{x}, from $x=0$ to $x=1$, correct to three decimal places;
   (c) Determine the Root Mean Square Value of the function, $\cos x$, from $x=\frac{\pi}{2}$ to $x=\pi$;
   (d) Determine the Root Mean Square Value of the function, $(4x-5)^{\frac{3}{2}}$, from $x=1.25$ to $x=1.5$.

13.2.4 ANSWERS TO EXERCISES

1. (a) $-\frac{1}{6}$;
   (b) 0.0775;
   (c) $\frac{2}{\pi}$;
   (d) -0.076;
   (e) 0.114
2. (a) $\sqrt{13} \simeq= 3.606$;
   (b) 1.787;
   (c) $\frac{1}{\sqrt{2}}$;
   (d) $\frac{1}{2}$.

"JUST THE MATHS"

UNIT NUMBER

13.3

INTEGRATION APPLICATIONS 3
(Volumes of revolution)

by

A.J.Hobson

UNIT 13.3 - INTEGRATION APLICATIONS 3

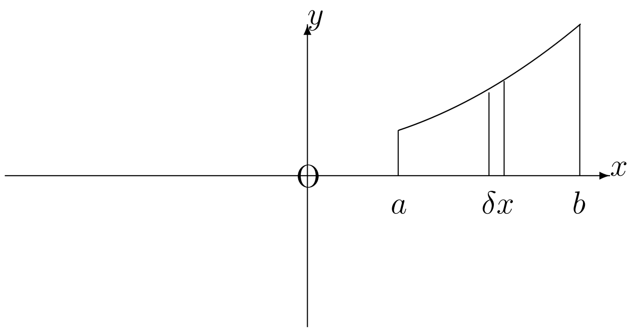
VOLUMES OF REVOLUTION

13.3.1 VOLUMES OF REVOLUTION ABOUT THE X-AXIS

Suppose that the area between a curve whose equation is

$$y = f(x)$$

and the x-axis, from $x = a$ to $x = b$, lies wholly above the x-axis; suppose, also, that this area is rotated through 2π radians about the x-axis. Then a solid figure is obtained whose volume may be determined as an application of definite integration.



When a narrow strip of width, δx, and height, y, is rotated through 2π radians about the x-axis, we obtain a disc whose volume, δV, is given approximately by

$$\delta V \simeq \pi y^2 \delta x.$$

Thus, the total volume, V, obtained is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x.$$

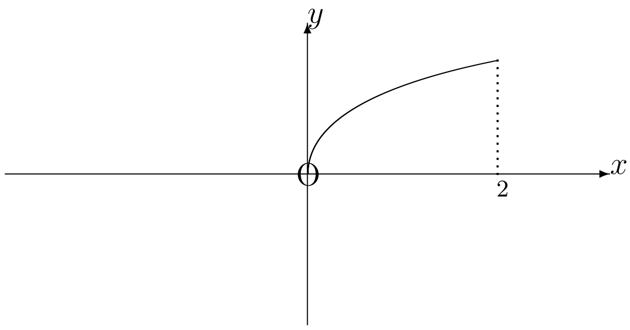
That is,

$$V = \int_a^b \pi y^2 \; \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line, $x = 2$, and the parabola, $y^2 = 8x$, is rotated through 2π radians about the x-axis.

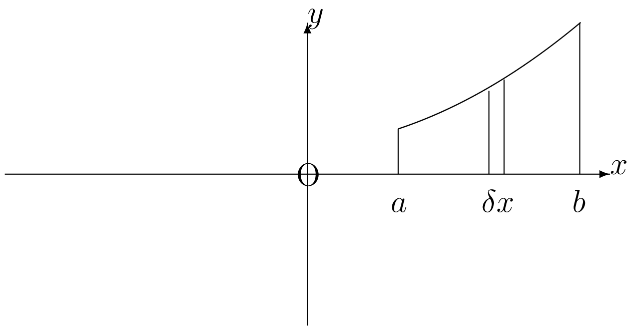
Solution



$$V = \int_0^2 \pi \times 8x \ \mathrm{d}x = [4\pi x^2]_0^2 = 16\pi.$$

13.3.2 VOLUMES OF REVOLUTION ABOUT THE Y-AXIS

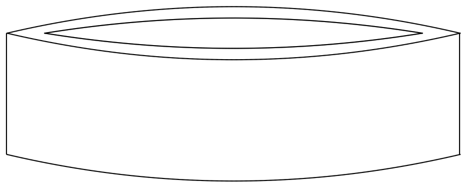
First we consider the same diagram as in the previous section:



This time, if the narrow strip of width, δx, is rotated through 2π radians about the y-axis, we obtain, approximately, a cylindrical shell of internal radius, x, external radius, $x + \delta x$ and height, y.

The volume, δV, of the shell is thus given by

$$\delta V \simeq 2\pi x y \delta x.$$



The total volume is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x y \delta x.$$

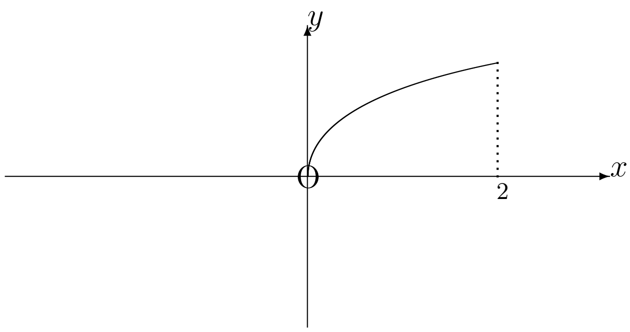
That is,

$$V = \int_a^b 2\pi x y \ \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line $x = 2$ and the parabola $y^2 = 8x$ is rotated through 2π radians about the y-axis.

Solution



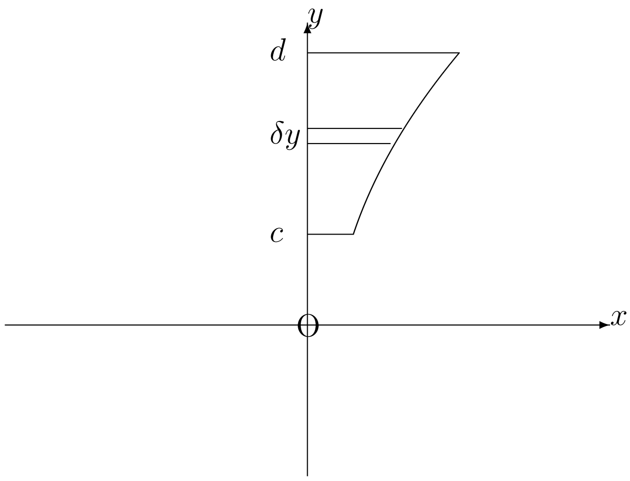
$$V = \int_0^2 2\pi x \times \sqrt{8x}\ \mathrm{d}x.$$

In other words,

$$V = \pi 4\sqrt{2}\int_0^2 x^{\frac{3}{2}}\mathrm{d}x = \pi 4\sqrt{2}\left[\frac{2x^{\frac{5}{2}}}{5}\right]_0^2 = \frac{64\pi}{5}.$$

Note:

It may be required to find the volume of revolution about the y-axis of an area which is contained between a curve and the y-axis from $y = c$ to $y = d$.



But here we simply interchange the roles of x and y in the original formula for rotation about the x-axis; that is

$$V = \int_c^d \pi x^2 \, \mathrm{d}y.$$

Similarly, the volume of rotation of the above area about the x-axis is given by

$$V = \int_c^d 2\pi y x \, \mathrm{d}y.$$

13.3.3 EXERCISES

1. By using a straight line through the origin, obtain a formula for the volume, V, of a solid right-circular cone with height, h, and base radius, r.
2. Determine the volume obtained when the segment straight line

   $$y = 5 - 4x,$$

   lying between $x = 0$ and $x = 1$, is rotated through 2π radians about (a) the x-axis and (b) the y-axis.
3. Determine the volume obtained when the part of the curve

   $$y = \cos 3x,$$

   lying between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$, is rotated through 2π radians about the x-axis.

4. Determine the volume obtained when the part of the curve
$$y = \frac{1}{x\sqrt{2+x}},$$
lying between $x = 2$ and $x = 7$, is rotated through 2π radians about the x-axis.

5. Determine the volume obtained when the part of the curve
$$y = \frac{1}{(x-1)(x-5)},$$
lying between $x = 6$ and $x = 8$, is rotated through 2π radians about the y-axis.

6. Determine the volume obtained when the part of the curve
$$x = ye^{-y},$$
lying between $y = 0$ and $y = 1$, is rotated through 2π radians about the y-axis.

7. Determine the volume obtained when the part of the curve
$$y = \sin 2x,$$
lying between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$, is rotated through 2π radians about the y-axis.

8. Determine the volume obtained when the part of the curve
$$y = x(1-x^3)^{\frac{1}{4}},$$
lying between $x = 0$ and $x = 1$, is rotated through 2π radians about the x-axis.

9. Determine the volume obtained when the part of the curve
$$x = (4-y^2)^2,$$
lying between $y = 1$ and $y = 2$, is rotated through 2π radians about the x-axis.

10. Determine the volume obtained when the part of the curve
$$y = x\sec(x^3),$$
lying between $x = 0$ and $x = 0.5$, is rotated through 2π radians about the x-axis.

11. Determine the volume obtained when the part of the curve
$$y = \frac{1}{x^2-1},$$
lying between $x = 2$ and $x = 3$ is rotated through 2π radians about the y-axis.

13.3.4 ANSWERS TO EXERCISES

1.

$$V = \frac{1}{3}\pi r^2 h.$$

2.

$$\text{(a)} \;\; \frac{\pi}{3} \simeq 1.047 \quad \text{(b)} \;\; \frac{7\pi}{3} \simeq 7.330$$

3.

$$\frac{\pi^2}{12} \simeq 0.822$$

4.

0.214 approximately.

5.

8.010 approximately.

6.

0.254 approximately.

7.

3.364 approximately.

8.

$$\frac{2\pi}{9} \simeq 0.698$$

9.

$$9\pi \simeq 28.274$$

10.

0.132 approximately.

11.

3.081 approximately.

"JUST THE MATHS"

UNIT NUMBER

13.4

INTEGRATION APPLICATIONS 4
(Lengths of curves)

by

A.J.Hobson

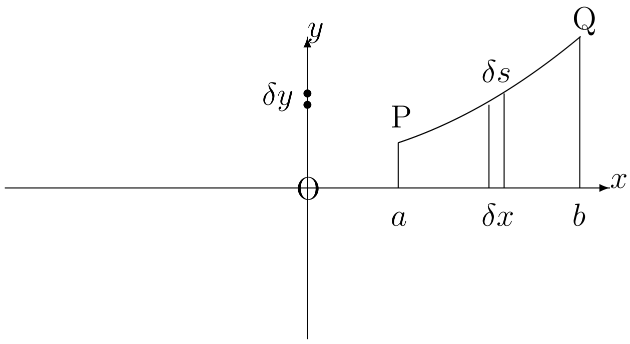
UNIT 13.4 - INTEGRATION APPLICATIONS 4 - LENGTHS OF CURVES

13.4.1 THE STANDARD FORMULAE

The problem, in this unit, is to calculate the length of the arc of the curve with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

The total length, s, of arc is thus given by

$$s = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

That is,

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Notes:
(i) If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$s = \pm \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{dx}{dt}$ is positive or negative.

(ii) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y, so that the length of the arc is given by

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy.$$

EXAMPLES

1. A curve has equation

$$9y^2 = 16x^3.$$

Determine the length of the arc of the curve between the point $\left(1, \frac{4}{3}\right)$ and the point $\left(4, \frac{32}{3}\right)$.

Solution

We may write the equation of the curve in the form

$$y = \frac{4x^{\frac{3}{2}}}{3};$$

and so,

$$\frac{dy}{dx} = 2x^{\frac{1}{2}}.$$

Hence,

$$s = \int_1^4 \sqrt{1 + 4x}\ dx = \left[\frac{(1+4x)^{\frac{3}{2}}}{6}\right]_1^4 = \frac{17^{\frac{3}{2}}}{6} - \frac{5^{\frac{3}{2}}}{6} \simeq 13.55$$

2. A curve is given parametrically by

$$x = t^2 - 1, \quad y = t^3 + 1.$$

Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 1$.

Solution

Since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2,$$

we have

$$s = \int_0^1 \sqrt{4t^2 + 6t^4}\, \mathrm{d}t = \int_0^1 t\sqrt{4 + 6t^2}\, \mathrm{d}t = \left[\frac{1}{18}\left(4 + 6t^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{1}{18}\left(10^{\frac{3}{2}} - 8\right) \simeq 1.31$$

13.4.2 EXERCISES

1. A straight line has equation

$$y = 3x + 2.$$

Use (a) elementary trigonometry and (b) definite integration to determine the length of the line segment joining the point where $x = 3$ and the point where $x = 7$.

2. A curve has equation

$$y = \frac{1}{2}x^2 - \frac{1}{4}\ln x.$$

Determine the length of the arc of the curve between $x = 1$ and $x = e$.

3. A curve has equation

$$x = 2(y + 3)^{\frac{3}{2}}.$$

Determine the length of the arc of the curve between $y = -2$ and $y = 1$, stating your answer in decimals correct to four significant figures.

4. A curve is given parametrically by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

   Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 2\pi$.

5. A curve is given parametrically by

$$x = 4(\cos\theta + \theta\sin\theta), \quad y = 4(\sin\theta - \theta\cos\theta).$$

   Determine the length of the arc of the curve between the point where $\theta = 0$ and the point where $\theta = \frac{\pi}{4}$.

6. A curve is given parametrically by

$$x = e^u \sin u, \quad y = e^u \cos u.$$

   Determine the length of the arc of the curve between the point where $u = 0$ and the point where $u = 1$.

13.4.3 ANSWERS TO EXERCISES

1.
$$4\sqrt{10} \simeq 12.65$$

2.
$$\frac{2e^2 - 1}{4} \simeq 3.44$$

3.
$$14.33$$

4.
$$8.$$

5.
$$\frac{\pi^2}{8}.$$

6.
$$\sqrt{2}(e - 1) \simeq 2.43$$

"JUST THE MATHS"

UNIT NUMBER

13.5

INTEGRATION APPLICATIONS 5
(Surfaces of revolution)

by

A.J.Hobson

UNIT 13.5 - INTEGRATION APPLICATIONS 5

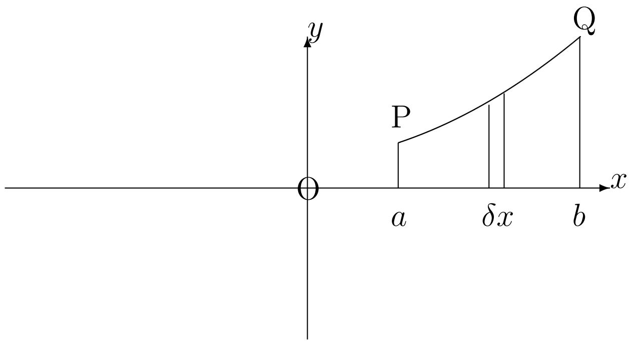
SURFACES OF REVOLUTION

13.5.1 SURFACES OF REVOLUTION ABOUT THE X-AXIS

The problem, in this unit, is to calculate the surface area obtained when the arc of the curve, with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$, is rotated through 2π radians about the x-axis or the y-axis.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis, respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

When the arc, of length δs, is rotated through 2π radians about the x-axis, it generates a thin band whose area is, approximately,

$$2\pi y \delta s = 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

The total surface area, S, is thus given by

$$S = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

That is,

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$S = \pm\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

EXAMPLES

1. A curve has equation

$$y^2 = 2x.$$

Determine the surface area obtained when the arc of the curve between the point $(2, 2))$ and the point $(8, 4)$ is rotated through 2π radians about the x-axis.

Solution

We may write the equation of the arc of the curve in the form

$$y = \sqrt{2x} = \sqrt{2}x^{\frac{1}{2}};$$

and so,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{2x}}.$$

Hence,

$$S = \int_2^8 2\pi\sqrt{2x}\sqrt{1+\frac{1}{2x}}\ \mathrm{d}x = \int_2^8 \sqrt{2x+1}\ \mathrm{d}x = \left[\frac{(2x+1)^{\frac{3}{2}}}{3}\right]_2^8.$$

Thus,

$$S = \frac{17^{\frac{3}{2}}}{3} - \frac{5^{\frac{3}{2}}}{3} \simeq 19.64$$

2. A curve is given parametrically by

$$x = \sqrt{2}\cos\theta, \quad y = \sqrt{2}\sin\theta.$$

Determine the surface area obtained when the arc of the curve between the point $(0, \sqrt{2})$ and the point $(1, 1)$ is rotated through 2π radians about the x-axis.

Solution

The parameters of the two points are $\frac{\pi}{2}$ and $\frac{\pi}{4}$, respectively; and, since

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sqrt{2}\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \sqrt{2}\cos\theta,$$

we have

$$S = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\sqrt{2}\pi\sin\theta\sqrt{2\sin^2\theta + 2\cos^2\theta}\ \mathrm{d}\theta = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\pi\sin\theta\ \mathrm{d}\theta.$$

Thus,

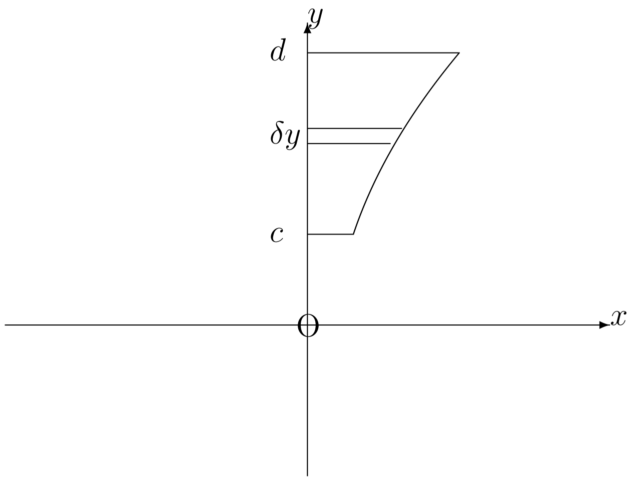
$$S = -\left[-4\pi\cos\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{4\pi}{\sqrt{2}} \simeq 8.89$$

13.5.2 SURFACES OF REVOLUTION ABOUT THE Y-AXIS

For a curve whose equation is of the form $x = g(y)$, the surface of revolution about the y-axis of an arc joining the two points at which $y = c$ and $y = d$ is given by

$$S = \int_a^b 2\pi x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$

We simply reverse the roles of x and y in the previous section.



Alternatively, if the curve is given parametrically,

$$S = \pm \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

EXAMPLE

If the arc of the parabola, with equation

$$x^2 = 2y,$$

joining the two points $(2, 2)$ and $(4, 8)$, is rotated through 2π radians about the y-axis, determine the surface area obtained.

Solution

Using the result from the previous section, the surface area obtained is given by

$$S = \int_2^8 2\pi\sqrt{2y}\sqrt{1 + \frac{1}{2y}}\ \mathrm{d}y \simeq 19.64$$

13.5.3 EXERCISES

1. Use a straight line through the origin to determine the surface area of a right-circular cone with height, h, and base radius, r.
2. Determine the surface area obtained when the arc of the curve $x = y^3$, between $y = 0$ and $y = 1$, is rotated through 2π radians about the y-axis.
3. A curve is given parametrically by
$$x = t - \sin t, \quad y = 1 - \cos t.$$
Determine the surface area obtained when the arc of the curve between the point where $t = 0$ and the point where $t = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.
State your answer correct to three places of decimals.
4. A curve is given parametrically by
$$x = 4(\cos\theta + \theta\sin\theta), \quad y = 4(\sin\theta - \theta\cos\theta).$$
Determine the surface area obtained when the arc of the curve between the point where $\theta = 0$ and the point where $\theta = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.
5. A curve is given parametrically by
$$x = e^u\cos u, \quad y = e^u\sin u.$$
Determine the surface area obtained when the arc of the curve between the point where $u = 0$ and the point where $u = \frac{\pi}{4}$ is rotated through 2π radians about the y-axis.
State your answer correct to three places of decimals.

13.5.4 ANSWERS TO EXERCISES

1.

$$\pi r\sqrt{r^2+h^2}.$$

2.

$$\frac{\pi(10\sqrt{10}-1)}{27} \simeq 3.56$$

3.

$$3.891$$

4.

$$32\pi\left(3-\left(\frac{\pi}{2}\right)^2\right) \simeq 53.54$$

5.

$$1.037$$

"JUST THE MATHS"

UNIT NUMBER

13.6

INTEGRATION APPLICATIONS 6
(First moments of an arc)

by

A.J.Hobson

UNIT 13.6 - INTEGRATION APPLICATIONS 6

FIRST MOMENTS OF AN ARC

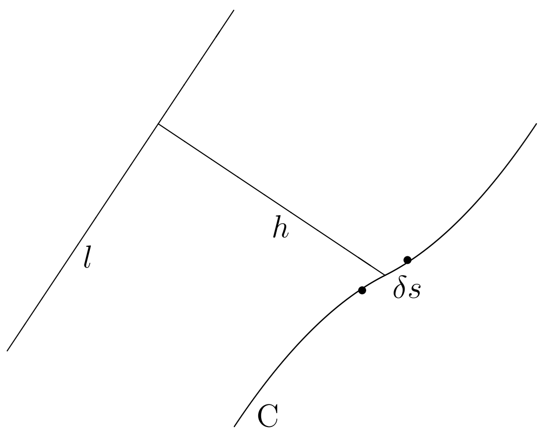
13.6.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates; and suppose that δs is the length of a small element of this arc.

Then the **"first moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} h\delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

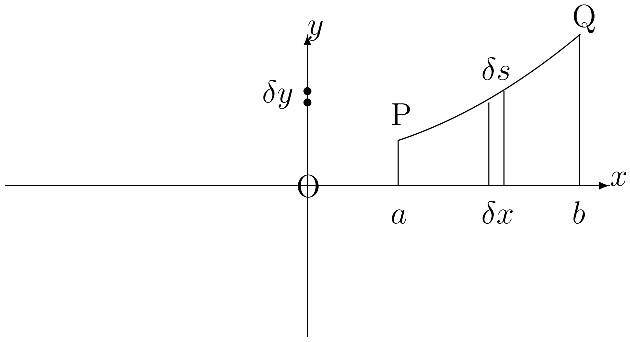


13.6.2 FIRST MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve, whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The first moment of each element about the y-axis is x times the length of the element; that is $x\delta s$, implying that the total first moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} x\delta s.$$

But, from Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

so that the first moment of the arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the y-axis is given by

$$\pm \int_{t_1}^{t_2} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.6.3 FIRST MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the first moment about the x-axis will be

$$\int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the first moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

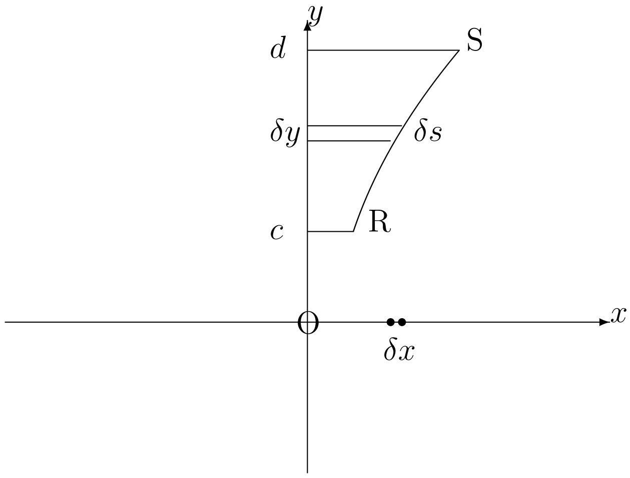
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.6.2 so that the first moment of the arc about the x-axis is given by

$$\int_c^d y\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the x-axis is given by

$$\pm\int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

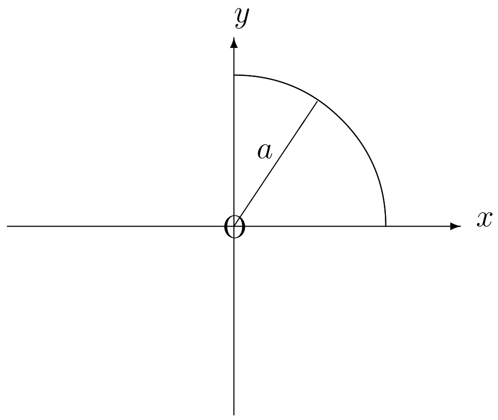
EXAMPLES

1. Determine the first moments about the x-axis and the y-axis of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The first moment of the arc about the y-axis is therefore given by

$$\int_0^a x\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a \frac{x}{y}\sqrt{x^2 + y^2}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$ and $y = \sqrt{a^2 - x^2}$.

Hence,

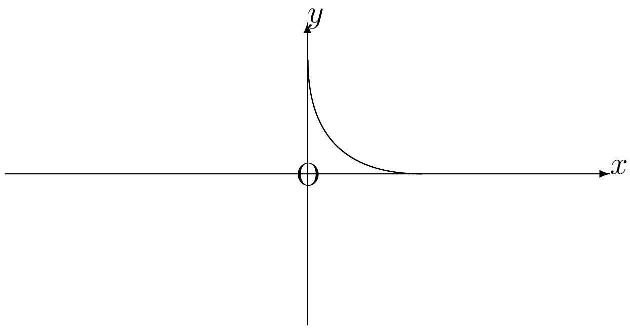
$$\text{first moment} = \int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\ \mathrm{d}x = \left[-a\sqrt{(a^2 - x^2)}\right]_0^a = a^2.$$

By symmetry, the first moment of the arc about the x-axis will also be a^2.

2. Determine the first moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} y\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\ \mathrm{d}\theta$$

$$= 3a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5}.$$

Similarly, the first moment of the arc about the y-axis is given by

$$\int_0^{\frac{\pi}{2}} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} a\cos^3\theta.(3a\cos\theta\sin\theta)\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\ \mathrm{d}\theta = 3a^2\left[-\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5},$$

though, again, this second result could be deduced, by symmetry, from the first.

13.6.4 THE CENTROID OF AN ARC

Having calculated the first moments of an arc about both the x-axis and the y-axis it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $s\overline{x}$, where s is the total length of the arc;

and

(b) The first moment about the x-axis is given by $s\overline{y}$, where s is the total length of the arc.

The point is called the **"centroid"** or the **"geometric centre"** of the arc and, for an arc of the curve with equation $y = f(x)$, between $x = a$ and $x = b$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}.$$

Notes:

(i) The first moment of an arc about an axis through its centroid will, by definition, be zero. In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δs, to the y-axis, the first moment about the given axis will be

$$\sum_{\mathrm{C}} (x-\overline{x})\delta s = \sum_{\mathrm{C}} x\delta s - \overline{x}\sum_{\mathrm{C}} \delta s = s\overline{x} - s\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole arc at a single point for the purposes of considering first moments. In practice, it corresponds, for example, to the position of the centre of mass of a thin wire with uniform density.

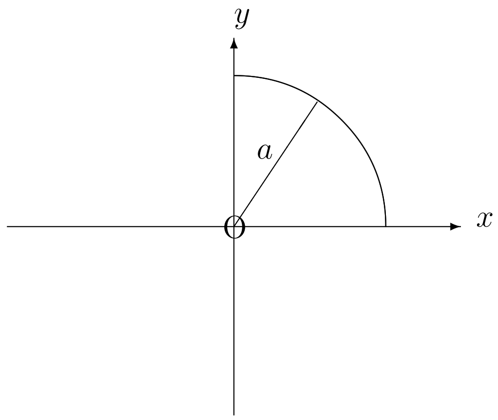
EXAMPLES

1. Determine the cartesian co-ordinates of the centroid of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From an earlier example in this unit, we know that the first moments of the arc about the x-axis and the y-axis are both equal to a^2.

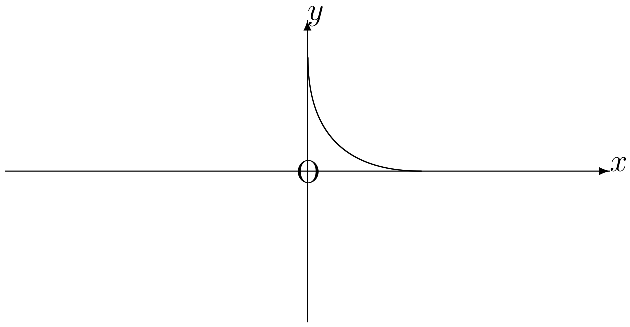
Also, the length of the arc is $\frac{\pi a}{2}$, which implies that

$$\overline{x} = \frac{2a}{\pi} \text{ and } \overline{y} = \frac{2a}{\pi}.$$

2. Determine the cartesian co-ordinates of the centroid of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From an earlier example in this unit, we know that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

and that the first moments of the arc about the x-axis and the y-axis are both equal to $\frac{3a^2}{5}$.
Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus,

$$\overline{x} = \frac{2a}{5} \quad \text{and} \quad \overline{y} = \frac{2a}{5}.$$

13.6.5 EXERCISES

1. Determine the first moment about the y-axis of the arc of the curve with equation
$$y = x^2,$$
lying between $x = 0$ and $x = 1$.

2. Determine the first moment about the x-axis of the arc of the curve with equation
$$x = 5y^2,$$
lying between $y = 0.1$ and $y = 0.5$.

3. Determine the first moment about the x-axis of the arc of the curve with equation
$$y = 2\sqrt{x},$$
lying between $x = 3$ and $x = 24$.

4. Verify, using integration, that the centroid of the straight line segment, defined by the equation
$$y = 3x + 2,$$
from $x = 0$ to $x = 1$, lies at its centre point.

5. Determine the cartesian co-ordinates of the centroid of the arc of the circle given parametrically by
$$x = 5\cos\theta, \quad y = 5\sin\theta,$$
from $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$.

6. For the curve whose equation is
$$9y^2 = x(3 - x)^2,$$
show that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$
Hence show that the centroid of the first quadrant arch of this curve lies at the point $\left(\frac{7}{5}, \frac{\sqrt{3}}{4}\right)$.

13.6.6 ANSWERS TO EXERCISES

1.

$$\frac{5\sqrt{5}-1}{12} \simeq 0.85$$

2.

$$\frac{13\sqrt{26}-\sqrt{2}}{150} \simeq 0.43$$

3.

$$156.$$

4.

$$\overline{x} = \frac{1}{2} \text{ and } \overline{y} = \frac{7}{2}.$$

5.

$$\overline{x} = \frac{15}{\pi} \simeq 4.77, \quad \overline{y} = 0.$$

"JUST THE MATHS"

UNIT NUMBER

13.7

INTEGRATION APPLICATIONS 7
(First moments of an area)

by

A.J.Hobson

UNIT 13.7 - INTEGRATION APPLICATIONS 7

FIRST MOMENTS OF AN AREA

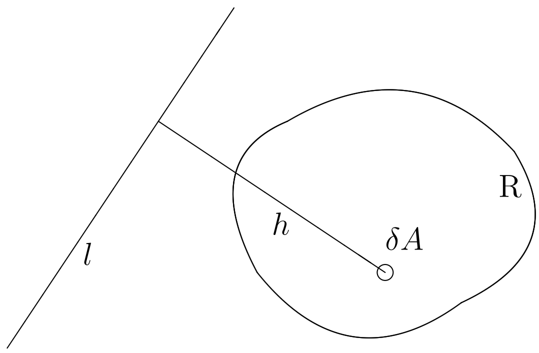
13.7.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy-plane of cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the **"first moment"** of R about a fixed line, l, in the plane of R is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h\delta A,$$

where h is the perpendicular distance, from l, of the element with area, δA.



13.7.2 FIRST MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$