### 2.3.2 The Casimir Effect

"I mentioned my results to Niels Bohr, during a walk. That is nice, he said, that is something new... and he mumbled something about zero-point energy."

*Hendrik Casimir*

Using the normal ordering prescription we can happily set $E_0 = 0$, while chanting the mantra that only energy differences can be measured. But we should be careful, for there is a situation where differences in the energy of vacuum fluctuations themselves can be measured.

To regulate the infra-red divergences, we'll make the $x^1$ direction periodic, with size $L$, and impose periodic boundary conditions such that

$$\varphi(\vec{x}) = \varphi(\vec{x} + L\vec{n}) \tag{2.31}$$

with $\vec{n} = (1, 0, 0)$. We'll leave $y$ and $z$ alone, but remember that we should compute all physical quantities per unit area $A$. We insert two reflecting plates, separated by a distance $d$ $L$ in the $x^1$ direction. The plates are such that they impose $\varphi(x) = 0$ at the position of the plates. The presence of these plates affects the Fourier decomposition of the field and, in particular, means that the momentum of the field inside the

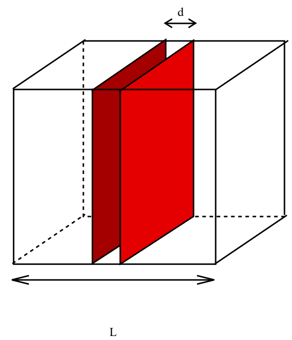


**Figure 3:**

plates is quantized as

$$\vec{p} = \frac{n\pi}{d}, p_y, p_z \qquad n \in \mathbf{Z}^+ \tag{2.32}$$

For a *massless* scalar field, the ground state energy between the plates is

$$\frac{E(d)}{} = \infty \int \frac{dp_y dp_z}{}$$

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$$A \quad {}_{n=1} \quad (2\pi)^2 \; \frac{1}{2} \quad \frac{n\pi}{d}^{\;2} + p_y^2 + p_z^2 \tag{2.33}$$

while the energy outside the plates is $E(L - d)$. The total energy is therefore

$$E = E(d) + E(L - d) \tag{2.34}$$

which – at least naively – depends on $d$. If this naive guess is true, it would mean that there is a force on the plates due to the fluctuations of the vacuum. This is the Casimir force, first predicted in 1948 and observed 10 years later. In the real world, the effect is due to the vacuum fluctuations of the electromagnetic field, with the boundary conditions imposed by conducting plates. Here we model this effect with a scalar.

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But there's a problem. $E$ is infinite! What to do? The problem comes from the arbitrarily high momentum modes. We could regulate this in a number of different ways. Physically one could argue that any real plate cannot reflect waves of arbitrarily high frequency: at some point, things begin to leak. Mathematically, we want to find a way to neglect modes of momentum $p \gg a^{-1}$ for some distance scale $a \ll d$, known as the ultra-violet (UV) cut-off. One way to do this is to change the integral (2.33) to,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y\, dp_z}{(2\pi)^2}\, \frac{1}{2} \sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}\; e^{-a\sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}} \tag{2.35}$$

which has the property that as $a \to 0$, we regain the full, infinite, expression (2.33). However (2.35) is finite, and gives us something we can easily work with. Of course, we made it finite in a rather ad-hoc manner and we better make sure that any physical quantity we calculate doesn't depend on the UV cut-off $a$, otherwise it's not something we can really trust.

The integral (2.35) is do-able, but a little complicated. It's a lot simpler if we look at the problem in $d = 1 + 1$ dimensions, rather than $d = 3 + 1$ dimensions. We'll find that all the same physics is at play. Now the energy is given by

$$E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n \tag{2.36}$$

We now regulate this sum by introducing the UV cutoff $a$ introduced above. This renders the expression finite, allowing us to start manipulating it thus,

$$\begin{aligned} E_{1+1}(d) &\to \frac{\pi}{2d} \sum_{n=1}^{\infty} n\, e^{-an\pi/d} \\ &= -\frac{1}{2}\frac{\partial}{\partial a} \sum_{n=1}^{\infty} e^{-an\pi/d} \\ &= -\frac{1}{2}\frac{\partial}{\partial a} \frac{1}{1 - e^{-a\pi/d}} \\ &= \frac{\pi}{2d} \frac{e^{a\pi/d}}{(e^{a\pi/d} - 1)^2} \\ &= \frac{d}{2\pi a^2} - \frac{\pi}{24d} + \mathcal{O}(a^2) \end{aligned} \tag{2.37}$$

where, in the last line, we've used the fact that $a \ll d$. We can now compute the full energy,

$$E_{1+1} = E_{1+1}(d) + E_{1+1}(L - d) = \frac{L}{2\pi a^2} - \frac{\pi}{24}\left(\frac{1}{d} + \frac{1}{L - d}\right) + \mathcal{O}(a^2) \tag{2.38}$$

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This is still infinite in the limit $a \to 0$, which is to be expected. However, the force is given by

$$\frac{\partial E_{1+1}}{\partial d} = \frac{\pi}{24d^2} + \dots \tag{2.39}$$

where the . . . include terms of size $d/L$ and $a/d$. The key point is that as we remove both the regulators, and take $a \to 0$ and $L \to \infty$, the force between the plates remains finite. This is the Casimir force[2].

If we ploughed through the analogous calculation in $d = 3 + 1$ dimensions, and performed the integral (2.35), we would find the result

$$\frac{1}{A}\frac{\partial E}{\partial d} = \frac{\pi^2}{480d^4} \tag{2.40}$$

The true Casimir force is twice as large as this, due to the two polarization states of the photon.

## 2.4 Particles

Having dealt with the vacuum, we can now turn to the excitations of the field. It's easy to verify that

$$[H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}}\, a^\dagger_{\vec{p}} \quad \text{and} \quad [H, a_{\vec{p}}] = -\omega_{\vec{p}}\, a_{\vec{p}} \tag{2.41}$$

which means that, just as for the harmonic oscillator, we can construct energy eigenstates by acting on the vacuum $|0\rangle$ with $a^\dagger_{\vec{p}}$. Let

$$|\vec{p}\rangle = a^\dagger_{\vec{p}}\, |0\rangle \tag{2.42}$$

This state has energy

$$H\,|\vec{p}\rangle = \omega_{\vec{p}}\,|\vec{p}\rangle \quad \text{with} \quad \omega^2_{\vec{p}} = \vec{p}^{\,2} + m^2 \tag{2.43}$$

But we recognize this as the relativistic dispersion relation for a particle of mass $m$ and 3-momentum $\vec{p}$,

$$E^2_{\vec{p}} = \vec{p}^{\,2} + m^2 \tag{2.44}$$

[2]The number 24 that appears in the denominator of the one-dimensional Casimir force plays a more famous role in string theory: the same calculation in that context is the reason the bosonic string lives in 26 = 24 + 2 spacetime dimensions. (The +2 comes from the fact the string itself is extended in one space and one time dimension). You will need to attend next term's "String Theory" course to see what on earth this has to do with the Casimir force.

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We interpret the state $|\vec p\,\rangle$ as the momentum eigenstate of a single particle of mass $m$. To stress this, from now on we'll write $E_{\vec p}$ everywhere instead of $\omega_{\vec p}$. Let's check this particle interpretation by studying the other quantum numbers of $|\vec p\,\rangle$. We may take the classical total momentum $\vec P$ given in (1.46) and turn it into an operator. After normal ordering, it becomes

$$\vec P = -\int d^3x\, \pi\, \vec\nabla\varphi = \int \frac{d^3p}{(2\pi)^3}\, \vec p\, a^\dagger_{\vec p}\, a_{\vec p} \tag{2.45}$$

Acting on our state $|\vec p\,\rangle$ with $\vec P$, we learn that it is indeed an eigenstate,

$$\vec P\, |\vec p\,\rangle = \vec p\, |\vec p\,\rangle \tag{2.46}$$

telling us that the state $|\vec p\,\rangle$ has momentum $\vec p$. Another property of $|\vec p\,\rangle$ that we can study is its angular momentum. Once again, we may take the classical expression for the total angular momentum of the field (1.55) and turn it into an operator,

$$J^i = \epsilon^{ijk} \int d^3x\, (\mathcal{J}^0)^{jk} \tag{2.47}$$

It's not hard to show that acting on the one-particle state with zero momentum, $J^i\, |\vec p = 0\rangle = 0$, which we interpret as telling us that the particle carries no internal angular momentum. In other words, quantizing a scalar field gives rise to a spin 0 particle.

**Multi-Particle States, Bosonic Statistics and Fock Space**

We can create multi-particle states by acting multiple times with $a^\dagger$'s. We interpret the state in which $n$ $a^\dagger$'s act on the vacuum as an $n$-particle state,

$$|\vec p_1, \ldots, \vec p_n\rangle = a^\dagger_{\vec p_1} \ldots a^\dagger_{\vec p_n} |0\rangle \tag{2.48}$$

Because all the $a^\dagger$'s commute among themselves, the state is symmetric under exchange of any two particles. For example,

$$|\vec p, \vec q\,\rangle = |\vec q, \vec p\,\rangle \tag{2.49}$$

This means that the particles are *bosons*.

The full Hilbert space of our theory is spanned by acting on the vacuum with all possible combinations of $a^\dagger$'s,

$$|0\rangle\,,\ a^\dagger_{\vec p}\,|0\rangle\,,\ a^\dagger_{\vec p}\, a^\dagger_{\vec q}\,|0\rangle\,,\ a^\dagger_{\vec p}\, a^\dagger_{\vec q}\, a^\dagger_{\vec r}\,|0\rangle \ldots \tag{2.50}$$

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This space is known as a *Fock space*. The Fock space is simply the sum of the $n$-particle Hilbert spaces, for all $n \geq 0$. There is a useful operator which counts the number of particles in a given state in the Fock space. It is called the *number operator N*

$$N = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}} \qquad (2.51)$$

and satisfies $N\,|\vec{p}_1, \ldots, \vec{p}_n\rangle = n\,|\vec{p}_1, \ldots, \vec{p}_n\rangle$. The number operator commutes with the Hamiltonian, $[N, H] = 0$, ensuring that particle number is conserved. This means that we can place ourselves in the $n$-particle sector, and stay there. This is a property of free theories, but will no longer be true when we consider interactions: interactions create and destroy particles, taking us between the different sectors in the Fock space.

### Operator Valued Distributions

Although we're referring to the states $|\vec{p}\rangle$ as "particles", they're not localized in space in any way — they are momentum eigenstates. Recall that in quantum mechanics the position and momentum eigenstates are not good elements of the Hilbert space since they are not normalizable (they normalize to delta-functions). Similarly, in quantum field theory neither the operators $\varphi(\vec{x})$, nor $a_{\vec{p}}$ are good operators acting on the Fock

space. This is because they don't produce normalizable states. For example,

$$\langle 0|\, a_{\vec{p}}\, a^\dagger_{\vec{p}}\, |0\rangle = \langle \vec{p}\,|\,\vec{p}\,\rangle = (2\pi)^3\, \delta(0) \text{ and } \quad \langle 0|\varphi(\vec{x})\, \varphi(\vec{x})\, |0\rangle = \langle \vec{x}\,|\,\vec{x}\,\rangle = \delta(0) \qquad (2.52)$$

They are operator valued distributions, rather than functions. This means that although $\varphi(\vec{x})$ has a well defined vacuum expectation value, $\langle 0|\,\varphi(\vec{x})\,|0\rangle = 0$, the fluc- tuations of the operator at a fixed point are infinite, $\langle 0|\,\varphi(\vec{x})\varphi(\vec{x})\,|0\rangle = \infty$. We can construct well defined operators by smearing these distributions over space. For exam- ple, we can create a wavepacket

$$|\phi\rangle = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{x}}\, \phi(\vec{p})\, |\vec{p}\,\rangle \qquad (2.53)$$

which is partially localized in both position and momentum space. (A typical state might be described by the Gaussian $\phi(\vec{p}) = \exp(-\vec{p}^{\,2}/2m^2)$).

### 2.4.1 Relativistic Normalization

We have defined the vacuum $|0\rangle$ which we normalize as $\langle 0|\,0\rangle = 1$. The one-particle states $|\vec{p}\,\rangle = a^\dagger_{\vec{p}}\,|0\rangle$ then satisfy

$$\langle \vec{p}\,|\,\vec{q}\,\rangle = (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{q}\,) \qquad (2.54)$$

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But is this Lorentz invariant? It's not obvious because we only have 3-vectors. What could go wrong? Suppose we have a Lorentz transformation

$$p^\mu \to (p')^\mu = \Lambda^\mu{}_\nu p^\nu \tag{2.55}$$

such that the 3-vector transforms as $\vec{p} \to \vec{p}\,'$. In the quantum theory, it would be preferable if the two states are related by a unitary transformation,

$$|\vec{p}\rangle \to |\vec{p}\,'\rangle = U(\Lambda)\,|\vec{p}\rangle \tag{2.56}$$

This would mean that the normalizations of $|\vec{p}\rangle$ and $|\vec{p}\,'\rangle$ are the same whenever $\vec{p}$ and $\vec{p}\,'$ are related by a Lorentz transformation. But we haven't been at all careful with normalizations. In general, we could get

$$|\vec{p}\rangle \to \lambda(\vec{p}, \vec{p}\,')\,|\vec{p}\,'\rangle \tag{2.57}$$

for some unknown function $\lambda(\vec{p}, \vec{p}\,')$. How do we figure this out? The trick is to look at an object which we know is Lorentz invariant. One such object is the identity operator on one-particle states (which is really the projection operator onto one-particle states). With the normalization (2.54) we know this is given by

$$1 = \int \frac{d^3p}{(2\pi)^3}\,|\vec{p}\rangle\langle\vec{p}| \tag{2.58}$$

This operator is Lorentz invariant, but it consists of two terms: the measure $\int d^3p$ and the projector $|\vec{p}\rangle\langle\vec{p}|$. Are these individually Lorentz invariant? In fact the answer is no.

**Claim** The Lorentz invariant measure is,

$$\int \frac{d^3p}{2E_{\vec{p}}} \tag{2.59}$$

**Proof:** $\int d^4p$ is obviously Lorentz invariant. And the relativistic dispersion relation for a massive particle,

$$p_\mu p^\mu = m^2 \quad \Rightarrow \quad p_0^2 = E^2 = \vec{p}^{\,2} + m^2 \tag{2.60}$$

is also Lorentz invariant. Solving for $p_0$, there are two branches of solutions: $p_0 = \pm E_{\vec{p}}$. But the choice of branch is another Lorentz invariant concept. So piecing everything together, the following combination must be Lorentz invariant,

$$\int d^4p\,\delta(p_0^2 - \vec{p}^{\,2} - m^2)\Big|_{p_0>0} = \int \frac{d^3p}{2p_0}\Big|_{p_0=E_{\vec{p}}} \tag{2.61}$$

which completes the proof.

From this result we can figure out everything else. For example, the Lorentz invariant $\delta$-function for 3-vectors is

$$2E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}) \tag{2.62}$$

which follows because

$$\int \frac{d^3p}{2E_{\vec{p}}}\, 2E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}) = 1 \tag{2.63}$$

So finally we learn that the relativistically normalized momentum states are given by

$$|p\rangle = \sqrt{2E_{\vec{p}}}\,|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\,a^\dagger_{\vec{p}}\,|0\rangle \tag{2.64}$$

Notice that our notation is rather subtle: the relativistically normalized momentum state $|p\rangle$ differs from $|\vec{p}\rangle$ by the factor $\sqrt{2E_{\vec{p}}}$. These states now satisfy

$$\langle p|\,q\rangle = (2\pi)^3 2E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}) \tag{2.65}$$

Finally, we can rewrite the identity on one-particle states as

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\,|p\rangle\,\langle p| \tag{2.66}$$

Some texts also define relativistically normalized creation operators by $a^\dagger(p) = \sqrt{2E_{\vec{p}}}\,a^\dagger_{\vec{p}}$. We won't make use of this notation here.

## 2.5 Complex Scalar Fields

Consider a complex scalar field $\psi(x)$ with Lagrangian

$$\mathcal{L} = \partial_\mu\psi^\star\partial^\mu\psi - M^2\psi^\star\psi \tag{2.67}$$

Notice that, in contrast to the Lagrangian (1.7) for a real scalar field, there is no factor of 1/2 in front of the Lagrangian for a complex scalar field. If we write $\psi$ in terms of real scalar fields by $\psi = (\phi_1 + i\phi_2)/\sqrt{2}$, we get the factor of 1/2 coming from the $1/\sqrt{2}$'s. The equations of motion are

$$\begin{aligned}\partial_\mu\partial^\mu\psi + M^2\psi &= 0\\ \partial_\mu\partial^\mu\psi^\star + M^2\psi^\star &= 0\end{aligned} \tag{2.68}$$

where the second equation is the complex conjugate of the first. We expand the complex field operator as a sum of plane waves as

$$\begin{aligned}\psi &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(b_{\vec{p}}\,e^{+i\vec{p}\cdot\vec{x}} + c^\dagger_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\right)\\ \psi^\dagger &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(b^\dagger_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}\,e^{+i\vec{p}\cdot\vec{x}}\right)\end{aligned} \tag{2.69}$$

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Since the classical field $\psi$ is not real, the corresponding quantum field $\psi$ is not hermitian. This is the reason that we have different operators $b$ and $c^\dagger$ appearing in the positive and negative frequency parts. The classical field momentum is $\pi = \partial \mathcal{L}/\partial\dot{\psi} = \dot{\psi}^\star$. We also turn this into a quantum operator field which we write as,

$$\pi = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \left( b_{\vec{p}}^\dagger \, e^{-i\vec{p}\cdot\vec{x}} - c_{\vec{p}} \, e^{+i\vec{p}\cdot\vec{x}} \right)$$

$$\pi^\dagger = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left( b_{\vec{p}} \, e^{+i\vec{p}\cdot\vec{x}} - c_{\vec{p}}^\dagger \, e^{-i\vec{p}\cdot\vec{x}} \right) \tag{2.70}$$

The commutation relations between fields and momenta are given by

$$[\psi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad \text{and} \quad [\psi(\vec{x}), \pi^\dagger(\vec{y})] = 0 \tag{2.71}$$

together with others related by complex conjugation, as well as the usual $[\psi(\vec{x}), \psi(\vec{y})] = [\psi(\vec{x}), \psi^\dagger(\vec{y})] = 0$, etc. One can easily check that these field commutation relations are equivalent to the commutation relations for the operators $b_{\vec{p}}$ and $c_{\vec{p}}$,

$$[b_{\vec{p}}, b_{\vec{q}}^\dagger] = (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{q})$$
$$[c_{\vec{p}}, c_{\vec{q}}^\dagger] = (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{q}) \tag{2.72}$$

and

$$[b_{\vec{p}}, b_{\vec{q}}] = [c_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c_{\vec{q}}^\dagger] = 0 \tag{2.73}$$

In summary, quantizing a complex scalar field gives rise to two creation operators, $b_{\vec{p}}^\dagger$ and $c_{\vec{p}}^\dagger$. These have the interpretation of creating two types of particle, both of mass $M$ and both spin zero. They are interpreted as particles and anti-particles. In contrast, for a real scalar field there is only a single type of particle: for a real scalar field, the particle is its own antiparticle.

Recall that the theory (2.67) has a classical conserved charge

$$Q = i \int d^3x \, (\dot{\psi}^\star \psi - \psi^\star \dot{\psi}) = i \int d^3x \, (\pi\psi - \psi^\star\pi^\star) \tag{2.74}$$

After normal ordering, this becomes the quantum operator

$$Q = \int \frac{d^3p}{(2\pi)^3} (c_{\vec{p}}^\dagger c_{\vec{p}} - b_{\vec{p}}^\dagger b_{\vec{p}}) = N_c - N_b \tag{2.75}$$

so $Q$ counts the number of anti-particles (created by $c^\dagger$) minus the number of particles (created by $b^\dagger$). We have $[H, Q] = 0$, ensuring that $Q$ is conserved quantity in the quantum theory. Of course, in our free field theory this isn't such a big deal because both $N_c$ and $N_b$ are separately conserved. However, we'll soon see that in interacting theories $Q$ survives as a conserved quantity, while $N_c$ and $N_b$ individually do not.

## 2.6 The Heisenberg Picture

Although we started with a Lorentz invariant Lagrangian, we slowly butchered it as we quantized, introducing a preferred time coordinate $t$. It's not at all obvious that the theory is still Lorentz invariant after quantization. For example, the operators $\varphi(\vec{x})$ depend on space, but not on time. Meanwhile, the one-particle states evolve in time by Schrödinger's equation,

$$i\frac{d|\vec{p}(t)\rangle}{dt} = H|\vec{p}(t)\rangle \;\Rightarrow\; |\vec{p}(t)\rangle = e^{-iE_{\vec{p}}t}|\vec{p}\rangle \tag{2.76}$$

Things start to look better in the Heisenberg picture where time dependence is assigned to the operators $O$,

$$O_H = e^{iHt}\,O_S\,e^{-iHt} \tag{2.77}$$

so that

$$\frac{dO_H}{dt} = i[H, O_H] \tag{2.78}$$

where the subscripts $S$ and $H$ tell us whether the operator is in the Schrödinger or Heisenberg picture. In field theory, we drop these subscripts and we will denote the picture by specifying whether the fields depend on space $\varphi(\vec{x})$ (the Schrödinger picture) or spacetime $\varphi(\vec{x}, t) = \varphi(x)$ (the Heisenberg picture).

The operators in the two pictures agree at a fixed time, say, $t = 0$. The commutation relations (2.2) become equal time commutation relations in the Heisenberg picture,

$$\begin{aligned}
[\varphi(\vec{x}, t), \varphi(\vec{y}, t)] &= [\pi(\vec{x}, t), \pi(\vec{y}, t)] = 0 \\
[\varphi(\vec{x}, t), \pi(\vec{y}, t)] &= i\delta^{(3)}(\vec{x} - \vec{y})
\end{aligned} \tag{2.79}$$

Now that the operator $\varphi(x) = \varphi(\vec{x}, t)$ depends on time, we can start to study how it evolves. For example, we have

$$\begin{aligned}
\dot{\varphi} = i[H, \varphi] &= \frac{i}{2}\left[\int d^3y\, \pi(y)^2 + \nabla\varphi(y)^2 + m^2\varphi(y)^2\,,\; \varphi(x)\right] \\
&= i\int d^3y\, \pi(y)\,(-i)\,\delta^{(3)}(\vec{y} - \vec{x}) = \pi(x)
\end{aligned} \tag{2.80}$$

Meanwhile, the equation of motion for $\pi$ reads,

$$\dot{\pi} = i[H, \pi] = \frac{i}{2}\left[\int d^3y\, \pi(y)^2 + \nabla\varphi(y)^2 + m^2\varphi(y)^2\,,\; \pi(x)\right]$$

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$$= -\frac{i}{2}\int d^3y\,\Big(\nabla_y[\varphi(y),\pi(x)]\,\nabla\varphi(y) + \nabla\varphi(y)\,\nabla_y[\varphi(y),\pi(x)] + 2im^2\varphi(y)\,\delta^{(3)}(\vec{x}-\vec{y})\Big)$$

$$= -\int d^3y\,\Big(\nabla_y\delta^{(3)}(\vec{x}-\vec{y})\,\nabla_y\varphi(y)\Big) - m^2\varphi(x)$$

$$= \nabla^2\varphi - m^2\varphi \tag{2.81}$$

where we've included the subscript $y$ on $\nabla_y$ when there may be some confusion about which argument the derivative is acting on. To reach the last line, we've simply integrated by parts. Putting (2.80) and (2.81) together we find that the field operator $\varphi$ satisfies the Klein-Gordon equation

$$\partial_\mu\partial^\mu\varphi + m^2\varphi = 0 \tag{2.82}$$

Things are beginning to look more relativistic. We can write the Fourier expansion of $\varphi(x)$ by using the definition (2.77) and noting,

$$e^{iHt}a_{\vec{p}}\,e^{-iHt} = e^{-iE_{\vec{p}}t}a_{\vec{p}} \quad \text{and} \quad e^{iHt}a^\dagger_{\vec{p}}\,e^{-iHt} = e^{+iE_{\vec{p}}t}a^\dagger_{\vec{p}} \tag{2.83}$$

which follows from the commutation relations $[H, a_{\vec{p}}] = -E_{\vec{p}}\,a_{\vec{p}}$ and $[H, a^\dagger_{\vec{p}}] = +E_{\vec{p}}\,a^\dagger_{\vec{p}}$. This then gives,

$$\varphi(\vec{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\Big(a_{\vec{p}}\,e^{-ip\cdot x} + a^\dagger_{\vec{p}}\,e^{+ip\cdot x}\Big) \tag{2.84}$$

which looks very similar to the previous expansion (2.18) except that the exponent is now written in terms of 4-vectors, $p\cdot x = E_{\vec{p}}\,t - \vec{p}\cdot\vec{x}$. (Note also that a sign has flipped in the exponent due to our Minkowski metric con- traction). It's simple to check that (2.84) indeed satisfies the Klein-Gordon equation (2.82).

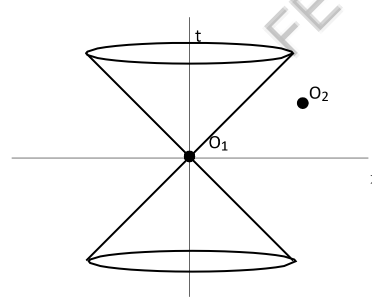


**Figure 4:**

### 2.6.1 Causality

We're approaching something Lorentz invariant in the Heisenberg picture, where $\varphi(x)$ now satisfies the Klein-Gordon equation. But there's still a hint of non-Lorentz invariance because $\varphi$ and $\pi$ satisfy *equal time* commutation relations,

$$[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{2.85}$$

FEUE WHEP

But what about arbitrary spacetime separations? In particular, for our theory to be *causal*, we must require that all spacelike separated operators commute,

$$[O_1(x), O_2(y)] = 0 \quad \forall \ (x - y)^2 < 0 \tag{2.86}$$

This ensures that a measurement at $x$ cannot affect a measurement at $y$ when $x$ and $y$ are not causally connected. Does our theory satisfy this crucial property? Let's define

$$\Delta(x - y) = [\varphi(x), \varphi(y)] \tag{2.87}$$

The objects on the right-hand side of this expression are operators. However, it's easy to check by direct substitution that the left-hand side is simply a c-number function with the integral expression

$$\Delta(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left( e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \tag{2.88}$$

What do we know about this function?

- It's Lorentz invariant, thanks to the appearance of the Lorentz invariant measure $\int d^3p/2E_{\vec{p}}$ that we introduced in (2.59).

- It doesn't vanish for timelike separation. For example, taking $x - y = (t, 0, 0, 0)$ gives $[\varphi(\vec{x}, 0), \varphi(\vec{x}, t)] \sim e^{-imt} - e^{+imt}$.

- It vanishes for space-like separations. This follows by noting that $\Delta(x - y) = 0$ at equal times for all $(x - y)^2 = -(\vec{x} - \vec{y})^2 < 0$, which we can see explicitly by writing

$$[\varphi(\vec{x}, t), \varphi(\vec{y}, t)] = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^{\,2} + m^2}} \left( e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right) \tag{2.89}$$

  and noticing that we can flip the sign of $\vec{p}$ in the last exponent as it is an integration variable. But since $\Delta(x - y)$ Lorentz invariant, it can only depend on $(x - y)^2$ and must therefore vanish for all $(x - y)^2 < 0$.

We therefore learn that our theory is indeed causal with commutators vanishing outside the lightcone. This property will continue to hold in interacting theories; indeed, it is usually given as one of the axioms of local quantum field theories. I should mention however that the fact that $[\varphi(x), \varphi(y)]$ is a c-number function, rather than an operator, is a property of free fields only.

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## 2.7 Propagators

We could ask a different question to probe the causal structure of the theory. Prepare a particle at spacetime point $y$. What is the amplitude to find it at point $x$? We can calculate this:

$$\begin{aligned}\langle 0|\,\varphi(x)\varphi(y)\,|0\rangle &= \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec p}E_{\vec p'}}}\langle 0|\, a_{\vec p}\, a^\dagger_{\vec p'}\,|0\rangle\, e^{-ip\cdot x+ip'\cdot y} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\, e^{-ip\cdot(x-y)} \equiv D(x-y) \end{aligned} \qquad (2.90)$$

The function $D(x-y)$ is called the *propagator*. For spacelike separations, $(x-y)^2 < 0$, one can show that $D(x-y)$ decays like

$$D(x-y) \sim e^{-m|\vec x-\vec y|} \qquad (2.91)$$

So it decays exponentially quickly outside the lightcone but, nonetheless, is non-vanishing! The quantum field appears to leak out of the lightcone. Yet we've just seen that space-like measurements commute and the theory is causal. How do we reconcile these two facts? We can rewrite the calculation (2.89) as

$$[\varphi(x), \varphi(y)] = D(x-y) - D(y-x) = 0 \;\text{ if }\; (x-y)^2 < 0 \qquad (2.92)$$

There are words you can drape around this calculation. When $(x-y)^2 < 0$, there is no Lorentz invariant way to order events. If a particle can travel in a spacelike direction from $x \to y$, it can just as easily travel from $y \to x$. In any measurement, the amplitudes for these two events cancel.

With a complex scalar field, it is more interesting. We can look at the equation $[\psi(x), \psi^\dagger(y)] = 0$ outside the lightcone. The interpretation now is that the amplitude for the particle to propagate from $x \to y$ cancels the amplitude for the *antiparticle* to travel from $y \to x$. In fact, this interpretation is also there for a real scalar field because the particle is its own antiparticle.

### 2.7.1 The Feynman Propagator

As we will see shortly, one of the most important quantities in interacting field theory is the *Feynman propagator*,

$$\Delta_F(x-y) = \langle 0|\, T\varphi(x)\varphi(y)\,|0\rangle = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & y^0 > x^0 \end{cases} \qquad (2.93)$$

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where $T$ stands for time ordering, placing all operators evaluated at later times to the left so,

$$T\varphi(x)\varphi(y) = \begin{cases} \varphi(x)\varphi(y) & x^0 > y^0 \\ \varphi(y)\varphi(x) & y^0 > x^0 \end{cases} \tag{2.94}$$

**Claim:** There is a useful way of writing the Feynman propagator in terms of a 4-momentum integral that shows that it is explicitly Lorentz invariant

$$\Delta_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)} \tag{2.95}$$

Notice that this is the first time in this course that we've integrated over 4-momentum. Until now, we integrated only over 3-momentum, with $p^0$ fixed by the mass-shell condition to be $p^0 = E_{\vec{p}}$. In the expression (2.95) for $\Delta_F$, we have no such condition on $p^0$. However, as it stands this integral is ill-defined because, for each value of $\vec{p}$, the denominator $p^2 - m^2 = (p^0)^2 - \vec{p}^{\,2} - m^2$ produces a pole when $p^0 = \pm E_{\vec{p}} = \pm\sqrt{\vec{p}^{\,2} + m^2}$.

We need a prescription for avoiding these singularities in the $p_0$ integral. To get the Feynman propagator, we must choose the contour to be

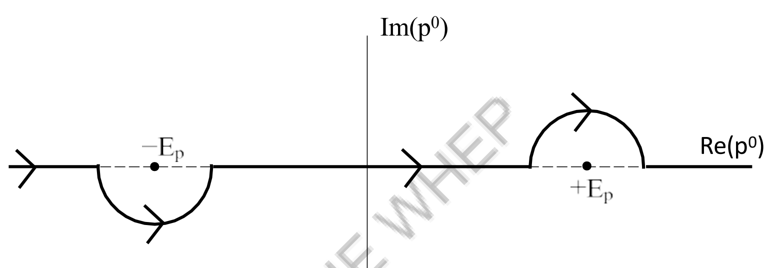



**Figure 5:** The contour for the Feynman propagator.

**Proof:**

$$\frac{1}{p^2 - m^2} = \frac{1}{(p^0)^2 - E_{\vec{p}}^2} = \frac{1}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})} \tag{2.96}$$

so the residue of the pole at $p^0 = \pm E_{\vec{p}}$ is $\pm 1/2E_{\vec{p}}$. When $x^0 > y^0$, we close the contour in the lower half plane, where $p^0 \to -i\infty$, ensuring that the integrand vanishes since $e^{-ip^0(x^0 - y^0)} \to 0$. The integral over $p^0$ then picks up the residue at $p^0 = +E_{\vec{p}}$ which is $-2\pi i/2E_{\vec{p}}$ where the minus sign arises because we took a clockwise contour. Hence when $x^0 > y^0$ we have

$$\Delta_F(x - y) = \int \frac{d^3p}{(2\pi)^4} \frac{-2\pi i}{2E_{\vec{p}}}\, i\, e^{-iE_{\vec{p}}(x^0 - y^0) + i\vec{p}\cdot(\vec{x} - \vec{y})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} = D(x-y) \qquad (2.97)$$

which is indeed the Feynman propagator for $x^0 > y^0$. In contrast, when $y^0 > x^0$, we close the contour in an anti-clockwise direction in the upper half plane to get,

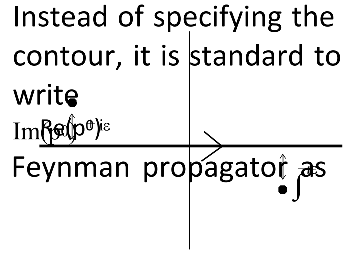
$$\Delta_F = \int \frac{d^3p}{(2\pi)^4} \frac{2\pi i}{(-2E_{\vec{p}})} \, i e^{+iE_{\vec{p}}(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-iE_{\vec{p}}(y^0-x^0)-i\vec{p}\cdot(\vec{y}-\vec{x})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(y-x)} = D(y-x) \qquad (2.98)$$

where to go to from the second line to the third, we have flipped the sign of $\vec{p}$ which is valid since we integrate over $d^3p$ and all other quantities depend only on $\vec{p}^{\,2}$. Once again we reproduce the Feynman propagator.

Instead of specifying the contour, it is standard to write the Feynman propagator as




$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+}$$

$i\epsilon$

(2.99)

with $\epsilon > 0$, and infinitesimal. This has the effect of shifting the poles slightly off the real axis, so the integral along the real $p^0$ axis is equivalent to the contour shown in Figure 5. This way of writing the propagator is, for obvious reasons, called the "$i\epsilon$ prescription".

**Figure 6:**

### 2.7.2 Green's Functions

There is another avatar of the propagator: it is a Green's function for the Klein-Gordon operator. If we stay away from the singularities, we have

$$(\partial_t^2 - \nabla^2 + m^2)\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}(-p^2+m^2)\,e^{-ip\cdot(x-y)} = -i\int\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot(x-y)} = -i\delta^{(4)}(x-y) \qquad (2.100)$$

Note that we didn't make use of the contour anywhere in this derivation. For some purposes it is also useful to pick other contours which also give rise to Green's functions.

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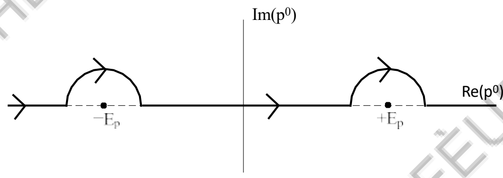


**Figure 7:** The retarded contour

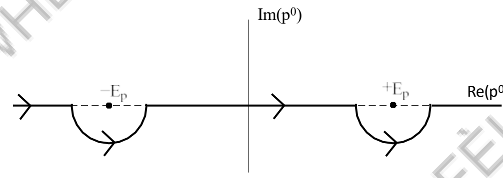


**Figure 8:** The advanced contour

For example, the *retarded* Green's function $\Delta_R(x-y)$ is defined by the contour shown in Figure 7 which has the property

$$\Delta_R(x-y) = \begin{cases} D(x-y) - D(y-x) & x^0 > y^0 \\ 0 & y^0 > x^0 \end{cases} \tag{2.101}$$

The retarded Green's function is useful in classical field theory if we know the initial value of some field configuration and want to figure out what it evolves into in the presence of a source, meaning that we want to know the solution to the inhomogeneous Klein-Gordon equation,

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = J(x) \tag{2.102}$$

for some fixed background function $J(x)$. Similarly, one can define the *advanced* Green's function $\Delta_A(x-y)$ which vanishes when $y^0 < x^0$, which is useful if we know the end point of a field configuration and want to figure out where it came from. Given that next term's course is called "Advanced Quantum Field Theory", there is an obvious name for the current course. But it got shot down in the staff meeting. In the quantum theory, we will see that the Feynman Green's function is most relevant.

## 2.8 Non-Relativistic Fields

Let's return to our classical complex scalar field obeying the Klein-Gordon equation. We'll decompose the field as

$$\psi(\vec{x}, t) = e^{-imt}\,\tilde{\psi}(\vec{x}, t) \tag{2.103}$$

Then the KG-equation reads

$$\partial_t^2 \psi - \nabla^2 \psi + m^2 \psi = e^{-imt}\left[\ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \nabla^2 \tilde{\psi}\right] = 0 \tag{2.104}$$

with the $m^2$ term cancelled by the time derivatives. The non-relativistic limit of a particle is $|\vec{p}| \ll m$. Let's look at what this does to our field. After a Fourier this is equivalent to saying that $|\ddot{\tilde\psi}| \quad m|\dot{\tilde\psi}|$. In this limit, we drop the term with two time derivatives and the KG equation becomes,

$$i\frac{\partial\tilde\psi}{\partial t} = -\frac{1}{2m}\nabla^2\tilde\psi \tag{2.105}$$

This looks very similar to the Schrödinger equation for a non-relativistic free particle of mass $m$. Except it doesn't have any probability interpretation — it's simply a classical field evolving through an equation that's first order in time derivatives.

We wrote down a Lagrangian in section 1.1.2 which gives rise to field equations which are first order in time derivatives. In fact, we can derive this from the relativistic Lagrangian for a scalar field by again taking the limit $\partial_t\psi \quad m\psi$. After losing the tilde, so $\tilde\psi \to \psi$, the non-relativistic Lagrangian becomes

$$\mathcal{L} = +i\psi^\star\dot\psi - \frac{1}{2m}\nabla\psi^\star\nabla\psi \tag{2.106}$$

where we've divided by $1/2m$. This Lagrangian has a conserved current arising from the internal symmetry $\psi \to e^{i\alpha}\psi$. The current has time and space components

$$j^\mu = \left(-\psi^\star\psi, \frac{i}{2m}(\psi^\star\nabla\psi - \psi\nabla\psi^\star)\right) \tag{2.107}$$

To move to the Hamiltonian formalism we compute the momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = i\psi^\star \tag{2.108}$$

This means that the momentum conjugate to $\psi$ is $i\psi^\star$. The momentum does not depend on time derivatives at all! This looks a little disconcerting but it's fully consistent for a theory which is first order in time derivatives. In order to determine the full trajectory of the field, we need only specify $\psi$ and $\psi^\star$ at time $t = 0$: no time derivatives on the initial slice are required.

Since the Lagrangian already contains a "$p\dot q$" term (instead of the more familiar $\frac{1}{2}p\dot q$ term), the time derivatives drop out when we compute the Hamiltonian. We get,

$$\mathcal{H} = \frac{1}{2m}\nabla\psi^\star\nabla\psi \tag{2.109}$$

To quantize we impose (in the Schrödinger picture) the canonical commutation relations

$$[\psi(\vec x), \psi(\vec y)] = [\psi^\dagger(\vec x), \psi^\dagger(\vec y)] = 0$$

$$[\psi(\vec x), \psi^\dagger(\vec y)] = \delta^{(3)}(\vec x - \tag{2.110}$$

We may expand $\psi(\vec{x})$ as a Fourier transform

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \tag{2.111}$$

where the commutation relations (2.110) require

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{q}) \tag{2.112}$$

The vacuum satisfies $a_{\vec{p}}|0\rangle = 0$, and the excitations are $a^\dagger_{\vec{p}_1} \cdots a^\dagger_{\vec{p}_n}|0\rangle$. The one-particle states have energy

$$H|\vec{p}\rangle = \frac{\vec{p}^{\,2}}{2m}|\vec{p}\rangle \tag{2.113}$$

which is the non-relativistic dispersion relation. We conclude that quantizing the first order Lagrangian (2.106) gives rise to non-relativistic particles of mass $m$. Some comments:

- We have a complex field but only a single type of particle. The anti-particle is not in the spectrum. The existence of anti-particles is a consequence of relativity.
- A related fact is that the conserved charge $Q = \int d^3x : \psi^\dagger\psi :$ is the particle number. This remains conserved even if we include interactions in the Lagrangian of the form

$$\Delta\mathcal{L} = V(\psi^\star\psi) \tag{2.114}$$

  So in non-relativistic theories, particle number is conserved. It is only with relativity, and the appearance of anti-particles, that particle number can change.
- There is no non-relativistic limit of a real scalar field. In the relativistic theory, the particles are their own anti-particles, and there can be no way to construct a multi-particle theory that conserves particle number.

### 2.8.1 Recovering Quantum Mechanics

In quantum mechanics, we talk about the position and momentum operators $\vec{X}$ and $\vec{P}$. In quantum field theory, position is relegated to a label. How do we get back to quantum mechanics? We already have the operator for the total momentum of the field

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}}\, a_{\vec{p}} \tag{2.115}$$

which, on one-particle states, gives $\vec{P}\,|\vec{p}\,\rangle = \vec{p}\,|\vec{p}\,\rangle$. It's also easy to construct the position operator. Let's work in the non-relativistic limit. Then the operator

$$\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \tag{2.116}$$

creates a particle with $\delta$-function localization at $\vec{x}$. We write $|\vec{x}\,\rangle = \psi^\dagger(\vec{x})\,|0\rangle$. A natural position operator is then

$$\vec{X} = \int d^3x\, \vec{x}\, \psi^\dagger(\vec{x})\, \psi(\vec{x}) \tag{2.117}$$

so that $\vec{X}\,|\vec{x}\,\rangle = \vec{x}\,|\vec{x}\,\rangle$.

Let's now construct a state $|\phi\rangle$ by taking superpositions of one-particle states $|\vec{x}\rangle$,

$$|\phi\rangle = \int d^3x\, \phi(\vec{x})\, |\vec{x}\,\rangle \tag{2.118}$$

The function $\phi(\vec{x})$ is what we would usually call the Schrödinger wavefunction (in the position representation). Let's make sure that it indeed satisfies all the right properties. Firstly, it's clear that acting with the position operator $\vec{X}$ has the right action of $\phi(\vec{x})$,

$$X^i\,|\phi\rangle = \int d^3x\, x^i\, \phi(\vec{x})\, |\vec{x}\,\rangle \tag{2.119}$$

but what about the momentum operator $\vec{P}$? We will now show that

$$P^i\,|\phi\rangle = \int d^3x \left(-i\frac{\partial\phi}{\partial x^i}\right) |\vec{x}\,\rangle \tag{2.120}$$

which tells us that $P^i$ acts as the familiar derivative on wavefunctions $|\phi\rangle$. To see that this is the case, we write

$$\begin{aligned} P^i\,|\phi\rangle &= \int \frac{d^3x\,d^3p}{(2\pi)^3}\, p^i\, a^\dagger_{\vec{p}} a_{\vec{p}}\, \phi(\vec{x})\, \psi^\dagger(\vec{x})\,|0\rangle \\ &= \int \frac{d^3x\,d^3p}{(2\pi)^3}\, p^i\, a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\, \phi(\vec{x})\,|0\rangle \end{aligned} \tag{2.121}$$

where we've used the relationship $[a_{\vec{p}}, \psi^\dagger(\vec{x})] = e^{-i\vec{p}\cdot\vec{x}}$ which can be easily checked. Proceeding with our calculation, we have

$$\begin{aligned} P^i\,|\phi\rangle &= \int \frac{d^3x\,d^3p}{(2\pi)^3}\, a^\dagger_{\vec{p}} \left(i\frac{\partial}{\partial x^i} e^{-i\vec{p}\cdot\vec{x}}\right) \phi(\vec{x})\,|0\rangle \\ &= \int \frac{d^3x\,d^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{x}} \left(-i\frac{\partial\phi}{\partial x^i}\right) a^\dagger_{\vec{p}}\,|0\rangle \\ &= \int d^3x \left(-i\frac{\partial\phi}{\partial x^i}\right) |\vec{x}\,\rangle \end{aligned} \tag{2.122}$$

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which confirms (2.120). So we learn that when acting on one-particle states, the operators $\vec{X}$ and $\vec{P}$ act as position and momentum operators in quantum mechanics, with $[X^i, P^j]\,|\phi\rangle = i\delta^{ij}\,|\phi\rangle$. But what about dynamics? How does the wavefunction $\phi(\vec{x}, t)$ change in time? The Hamiltonian (2.109) can be rewritten as

$$H = \int d^3x \, \frac{1}{2m} \nabla\psi^* \cdot \nabla\psi = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}^{\,2}}{2m} a^\dagger_{\vec{p}} a_{\vec{p}} \tag{2.123}$$

so we find that

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2m}\nabla^2\phi \tag{2.124}$$

But this is the same equation obeyed by the original field (2.105)! Except this time, it really is the Schrödinger equation, complete with the usual probabilistic interpretation for the wavefunction $\phi$. Note in particular that the conserved charge arising from the Noether current (2.107) is $Q = \int d^3x\, |\phi(\vec{x})|^2$ which is the total probability.

Historically, the fact that the equation for the classical field (2.105) and the one-particle wavefunction (2.124) coincide caused some confusion. It was thought that perhaps we are quantizing the wavefunction itself and the resulting name "second quantization" is still sometimes used today to mean quantum field theory. It's important to stress that, despite the name, we're not quantizing anything twice! We simply quantize a classical field once. Nonetheless, in practice it's useful to know that if we treat the one-particle Schrödinger equation as the equation for a quantum field then it will give the correct generalization to multi-particle states.

### Interactions

Often in quantum mechanics, we're interested in particles moving in some fixed background potential $V(\vec{x})$. This can be easily incorporated into field theory by working with a Lagrangian with explicit $\vec{x}$ dependence,

$$\mathcal{L} = i\psi^*\dot{\psi} - \frac{1}{2m}\nabla\psi^*\cdot\nabla\psi - V(\vec{x})\,\psi^*\psi \tag{2.125}$$

Note that this Lagrangian doesn't respect translational symmetry and we won't have the associated energy-momentum tensor. While such Lagrangians are useful in condensed matter physics, we rarely (or never) come across them in high-energy physics, where all equations obey translational (and Lorentz) invariance.

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One can also consider interactions *between* particles. Obviously these are only important for $n$ particle states with $n \geq 2$. We therefore expect them to arise from additions to the Lagrangian of the form

$$\Delta L = \psi^{\wedge}(\vec{x})\,\psi^{\wedge}(\vec{x})\,\psi(\vec{x})\,\psi(\vec{x}) \tag{2.126}$$

which, in the quantum theory, is an operator which destroys two particles before creat- ing two new ones. Such terms in the Lagrangian will indeed lead to inter-particle forces, both in the non-relativistic and relativistic setting. In the next section we explore these types of interaction in detail for relativistic theories.

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# 3. Interacting Fields

The free field theories that we've discussed so far are very special: we can determine their spectrum, but nothing interesting then happens. They have particle excitations, but these particles don't interact with each other.

Here we'll start to examine more complicated theories that include interaction terms. These will take the form of higher order terms in the Lagrangian. We'll start by asking what kind of *small* perturbations we can add to the theory. For example, consider the Lagrangian for a real scalar field,

$$\mathrm{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \sum_{n\geq 3}\frac{\lambda_n}{n!}\varphi^n \tag{3.1}$$

The coefficients $\lambda_n$ are called *coupling constants*. What restrictions do we have on $\lambda_n$ to ensure that the additional terms are small perturbations? You might think that we need simply make "$\lambda_n$ 1". But this isn't quite right. To see why this is the case, let's do some dimensional analysis. Firstly, note that the action has dimensions of angular momentum or, equivalently, the same dimensions as *k*. Since we've set *k* = 1, using the convention described in the introduction, we have $[S] = 0$. With $S = \int d^4x\,\mathrm{L}$, and $[d^4x] = -4$, the Lagrangian density must therefore have

$$[\mathrm{L}] = 4 \tag{3.2}$$

What does this mean for the Lagrangian (3.1)? Since $[\partial_\mu] = 1$, we can read off the mass dimensions of all the factors to find,

$$[\varphi] = 1 \quad , \quad [m] = 1 \quad , \quad [\lambda_n] = 4 - n \tag{3.3}$$

So now we see why we can't simply say we need $\lambda_n$ 1, because this statement only makes sense for dimensionless quantities. The various terms, parameterized by $\lambda_n$, fall into three different categories

- $[\lambda_3] = 1$: For this term, the dimensionless parameter is $\lambda_3/E$, where $E$ has dimensions of mass. Typically in quantum field theory, $E$ is the energy scale of the process of interest. This means that $\lambda_3\,\varphi^3/3!$ is a small perturbation at high energies $E$ $\lambda_3$, but a large perturbation at low energies $E$ $\lambda_3$. Terms that we add to the Lagrangian with this behavior are called *relevant* because they're most relevant at low energies (which, after all, is where most of the physics we see lies). In a relativistic theory, $E > m$, so we can always make this perturbation small by taking $\lambda_3$ $m$.

- $[\lambda_4] = 0$: this term is small if $\lambda_4 \ll 1$. Such perturbations are called *marginal*.
- $[\lambda_n] < 0$ for $n \geq 5$: The dimensionless parameter is $(\lambda_n E^{n-4})$, which is small at low-energies and large at high energies. Such perturbations are called *irrelevant*.

As you'll see later, it is typically impossible to avoid high energy processes in quantum field theory. (We've already seen a glimpse of this in computing the vacuum energy). This means that we might expect problems with irrelevant operators. Indeed, these lead to "non-renormalizable" field theories in which one cannot make sense of the infinities at arbitrarily high energies. This doesn't necessarily mean that the theory is useless; just that it is incomplete at some energy scale.

Let me note however that the naive assignment of relevant, marginal and irrelevant is not always fixed in stone: quantum corrections can sometimes change the character of an operator.

### An Important Aside: Why QFT is Simple

Typically in a quantum field theory, only the relevant and marginal couplings are important. This is basically because, as we've seen above, the irrelevant couplings become small at low-energies. This is a huge help: of the infinite number of interaction terms that we could write down, only a handful are actually needed (just two in the case of the real scalar field described above).

Let's look at this a little more. Suppose that we some day discover the true superduper "theory of everything unimportant" that describes the world at very high energy scales, say the GUT scale, or the Planck scale. Whatever this scale is, let's call it $\Lambda$. It is an energy scale, so $[\Lambda] = 1$. Now we want to understand the laws of physics down at our puny energy scale $E \ll \Lambda$. Let's further suppose that down at the energy scale $E$, the laws of physics are described by a real scalar field. (They're not of course: they're described by non-Abelian gauge fields and fermions, but the same argument applies in that case so bear with me). This scalar field will have some complicated interaction terms (3.1), where the precise form is dictated by all the stuff that's going on in the high energy superduper theory. What are these interactions? Well, we could write our dimensionful coupling constants $\lambda_n$ in terms of dimensionless couplings $g_n$, multiplied by a suitable power of the relevant scale $\Lambda$,

$$\lambda_n = \frac{g_n}{\Lambda^{n-4}} \tag{3.4}$$

The exact values of dimensionless couplings $g_n$ depend on the details of the high-energy superduper theory, but typically one expects them to be of order 1: $g_n \sim \mathcal{O}(1)$. This means that for experiments at small energies $E \ll \Lambda$, the interaction terms of the form $\varphi^n$ with $n > 4$ will be suppressed by powers of $(E/\Lambda)^{n-4}$. This is usually a suppression by many orders of magnitude. (e.g for the energies $E$ explored at the LHC, $E/M_{pl} \sim 10^{-16}$). It is this simple argument, based on dimensional analysis, that ensures that we need only focus on the first few terms in the interaction: those which are relevant and marginal. It also means that if we only have access to low-energy experiments (which we do!), it's going to be very difficult to figure out the high energy theory (which it is!), because its effects are highly diluted except for the relevant and marginal interactions. The discussion given above is a poor man's version of the ideas of *effective field theory* and *Wilson's renormalization group*, about which you can learn more in the "Statistical Field Theory" course.

### Examples of Weakly Coupled Theories

In this course we'll study only weakly coupled field theories i.e. ones that can truly be considered as small perturbations of the free field theory at all energies. In this section, we'll look at two types of interactions

**1) $\varphi^4$ theory:**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 \tag{3.5}$$

with $\lambda \ll 1$. We can get a hint for what the effects of this extra term will be. Expanding out $\varphi^4$ in terms of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$, we see a sum of interactions that look like

$$a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a^\dagger_{\vec{p}} \quad \text{and} \quad a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a_{\vec{p}} \quad \text{etc.} \tag{3.6}$$

These will create and destroy particles. This suggests that the $\varphi^4$ Lagrangian describes a theory in which particle number is not conserved. Indeed, we could check that the number operator $N$ now satisfies $[H, N] \neq 0$.

**2) Scalar Yukawa Theory**

$$\mathcal{L} = \partial_\mu\psi^\star\partial^\mu\psi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - M^2\psi^\star\psi - \frac{1}{2}m^2\varphi^2 - g\psi^\star\psi\varphi \tag{3.7}$$

with $g \ll M, m$. This theory couples a complex scalar $\psi$ to a real scalar $\varphi$. While the individual particle numbers of $\psi$ and $\varphi$ are no longer conserved, we do still have a symmetry rotating the phase of $\psi$, ensuring the existence of the charge $Q$ defined in (2.75) such that $[Q, H] = 0$. This means that the number of $\psi$ particles minus the number of $\psi$ anti-particles is conserved. It is common practice to denote the anti-particle as $\bar{\psi}$.

The scalar Yukawa theory has a slightly worrying aspect: the potential has a stable local minimum at $\varphi = \psi = 0$, but is unbounded below for large enough $-g\varphi$. This means we shouldn't try to push this theory too far.

### A Comment on Strongly Coupled Field Theories

In this course we restrict attention to weakly coupled field theories where we can use perturbative techniques. The study of strongly coupled field theories is much more difficult, and one of the major research areas in theoretical physics. For example, some of the amazing things that can happen include

- **Charge Fractionalization:** Although electrons have electric charge 1, under the right conditions the elementary excitations in a solid have fractional charge $1/N$ (where $N \in 2\mathbf{Z} + 1$). For example, this occurs in the fractional quantum Hall effect.
- **Confinement:** The elementary excitations of quantum chromodynamics (QCD) are quarks. But they *never* appear on their own, only in groups of three (in a baryon) or with an anti-quark (in a meson). They are confined.
- **Emergent Space:** There are field theories in four dimensions which at strong coupling become quantum gravity theories in ten dimensions! The strong coupling effects cause the excitations to act as if they're gravitons moving in higher dimensions. This is quite extraordinary and still poorly understood. It's called the AdS/CFT correspondence.

## 3.1 The Interaction Picture

There's a useful viewpoint in quantum mechanics to describe situations where we have small perturbations to a well-understood Hamiltonian. Let's return to the familiar ground of quantum mechanics with a finite number of degrees of freedom for a moment. In the Schrödinger picture, the states evolve as

$$i\frac{d|\psi\rangle_S}{dt} = H\,|\psi\rangle_S \tag{3.8}$$

while the operators $O_S$ are independent of time.

In contrast, in the Heisenberg picture the states are fixed and the operators change in time

$$\begin{aligned} O_H(t) &= e^{iHt}\, O_S\, e^{-iHt} \\ |\psi\rangle_H &= e^{iHt}\,|\psi\rangle_S \end{aligned} \tag{3.9}$$

The *interaction picture* is a hybrid of the two. We split the Hamiltonian up as

$$H = H_0 + H_{\text{int}} \tag{3.10}$$

The time dependence of operators is governed by $H_0$, while the time dependence of states is governed by $H_{\text{int}}$. Although the split into $H_0$ and $H_{\text{int}}$ is arbitrary, it's useful when $H_0$ is soluble (for example, when $H_0$ is the Hamiltonian for a free field theory). The states and operators in the interaction picture will be denoted by a subscript $I$ and are given by,

$$\begin{aligned} |\psi(t)\rangle_I &= e^{iH_0t}\,|\psi(t)\rangle_S \\ O_I(t) &= e^{iH_0t}\,O_S\,e^{-iH_0t} \end{aligned} \tag{3.11}$$

This last equation also applies to $H_{\text{int}}$, which is time dependent. The interaction Hamiltonian in the interaction picture is,

$$H_I \equiv (H_{\text{int}})_I = e^{iH_0t}(H_{\text{int}})_S\,e^{-iH_0t} \tag{3.12}$$

The Schrödinger equation for states in the interaction picture can be derived starting from the Schrödinger picture

$$\begin{aligned} i\frac{d|\psi\rangle_S}{dt} = H_S\,|\psi\rangle_S \quad &\Rightarrow \quad i\frac{d}{dt}\left(e^{-iH_0t}\,|\psi\rangle_I\right) = (H_0 + H_{\text{int}})_S\,e^{-iH_0t}\,|\psi\rangle_I \\ &\Rightarrow \quad i\frac{d|\psi\rangle_I}{dt} = e^{iH_0t}(H_{\text{int}})_S\,e^{-iH_0t}\,|\psi\rangle_I \end{aligned} \tag{3.13}$$

So we learn that

$$i\frac{d|\psi\rangle_I}{dt} = H_I(t)\,|\psi\rangle_I \tag{3.14}$$

### 3.1.1 Dyson's Formula

> "Well, Birmingham has much the best theoretical physicist to work with, Peierls; Bristol has much the best experimental physicist, Powell; Cambridge has some excellent architecture. You can make your choice."
>
> *Oppenheimer's advice to Dyson on which university position to accept.*

We want to solve (3.14). Let's write the solution as

$$|\psi(t)\rangle_I = U(t, t_0)\,|\psi(t_0)\rangle_I \tag{3.15}$$

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where $U(t, t_0)$ is a unitary time evolution operator such that $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$ and $U(t, t) = 1$. Then the interaction picture Schrödinger equation (3.14) requires that

$$i\frac{dU}{dt} = H_I(t)\, U \tag{3.16}$$

If $H_I$ were a function, then we could simply solve this by

$$U(t, t_0) \stackrel{?}{=} \exp\left(-i\int_{t_0}^{t} H_I(t')\, dt'\right) \tag{3.17}$$

But there's a problem. Our Hamiltonian $H_I$ is an operator, and we have ordering issues. Let's see why this causes trouble. The exponential of an operator is defined in terms of the expansion,

$$\exp\left(-i\int_{t_0}^{t} H_I(t')\, dt'\right) = 1 - i\int_{t_0}^{t} H_I(t')\, dt' + \frac{(-i)^2}{2}\left(\int_{t_0}^{t} H_I(t')\, dt'\right)^2 + \ldots \tag{3.18}$$

But when we try to differentiate this with respect to $t$, we find that the quadratic term gives us

$$-\frac{1}{2}\left(\int_{t_0}^{t} H_I(t')\, dt'\right) H_I(t) - \frac{1}{2} H_I(t)\left(\int_{t_0}^{t} H_I(t')\, dt'\right) \tag{3.19}$$

Now the second term here looks good, since it will give part of the $H_I(t)U$ that we need on the right-hand side of (3.16). But the first term is no good since the $H_I(t)$ sits on the wrong side of the integral term, and we can't commute it through because $[H_I(t'), H_I(t)] \neq 0$ when $t' \neq t$. So what's the way around this?

**Claim:** The solution to (3.16) is given by *Dyson's Formula*. (Essentially first figured out by Dirac, although the compact notation is due to Dyson).

$$U(t, t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t')\, dt'\right) \tag{3.20}$$

where $T$ stands for *time ordering* where operators evaluated at later times are placed to the left

$$T\left(O_1(t_1)\, O_2(t_2)\right) = \begin{cases} O_1(t_1)\, O_2(t_2) & t_1 > t_2 \\ O_2(t_2)\, O_1(t_1) & t_2 > t_1 \end{cases} \tag{3.21}$$

Expanding out the expression (3.20), we now have

$$U(t, t_0) = 1 - i\int_{t_0}^{t} dt'\, H_I(t') + \frac{(-i)^2}{2}\left(\int_{t_0}^{t} dt' \int_{t'}^{t} dt''\, H_I(t'')H_I(t') + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')H_I(t'')\right) + \ldots$$

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Actually these last two terms double up since

$$\int_{t_0}^{t} dt' \int_{t'}^{t} dt''\, H_I(t'')H_I(t') = \int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt'\, H_I(t'')H_I(t') = \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')H_I(t'') \tag{3.22}$$

where the range of integration in the first expression is over $t'' \geq t'$, while in the second expression it is $t' \leq t''$ which is, of course, the same thing. The final expression is the same as the second expression by a simple relabelling. This means that we can write

$$U(t, t_0) = 1 - i\int_{t_0}^{t} dt'\, H_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t')H_I(t'') + \ldots \tag{3.23}$$

**Proof:** The proof of Dyson's formula is simpler than explaining what all the notation means! Firstly observe that under the $T$ sign, all operators commute (since their order is already fixed by the $T$ sign). Thus

$$\begin{aligned} i\frac{\partial}{\partial t}\, T\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right) &= T\left[H_I(t)\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right)\right] \\ &= H_I(t)\, T\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right) \end{aligned} \tag{3.24}$$

since $t$, being the upper limit of the integral, is the latest time so $H_I(t)$ can be pulled out to the left.

Before moving on, I should confess that Dyson's formula is rather formal. It is typically very hard to compute time ordered exponentials in practice. The power of the formula comes from the expansion which is valid when $H_I$ is small and is very easily computed.

### 3.2 A First Look at Scattering

Let us now apply the interaction picture to field theory, starting with the interaction Hamiltonian for our scalar Yukawa theory,

$$H_{\text{int}} = g\int d^3x\; \psi^\dagger\psi\phi \tag{3.25}$$

Unlike the free theories discussed in Section 2, this interaction doesn't conserve particle number, allowing particles of one type to morph into others. To see why this is, we use

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the interaction picture and follow the evolution of the state: $|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$, where $U(t, t_0)$ is given by Dyson's formula (3.20) which is an expansion in powers of $H_{\text{int}}$. But $H_{\text{int}}$ contains creation and annihilation operators for each type of particle. In particular,

- $\varphi \sim a + a^\dagger$: This operator can create or destroy $\varphi$ particles. Let's call them *mesons*.
- $\psi \sim b + c^\dagger$ : This operator can destroy $\psi$ particles through $b$, and create anti-particles through $c^\dagger$. Let's call these particles *nucleons*. Of course, in reality nucleons are spin 1/2 particles, and don't arise from the quantization of a scalar field. But we'll treat our scalar Yukawa theory as a toy model for nucleons interacting with mesons.
- $\psi^\dagger \sim b^\dagger + c$: This operator can create nucleons through $b^\dagger$, and destroy anti-nucleons through $c$.

Importantly, $Q = N_c - N_b$ remains conserved in the presence of $H_{\text{int}}$. At first order in perturbation theory, we find terms in $H_{\text{int}}$ like $c^\dagger b^\dagger a$. This kills a meson, producing a nucleon-anti-nucleon pair. It will contribute to meson decay $\varphi \to \psi\bar{\psi}$.

At second order in perturbation theory, we'll have more complicated terms in $(H_{\text{int}})^2$, for example $(c^\dagger b^\dagger a)(cba^\dagger)$. This term will give contributions to scattering processes $\psi\bar{\psi} \to \varphi \to \psi\bar{\psi}$. The rest of this section is devoted to computing the quantum ampli- tudes for these processes to occur.

To calculate amplitudes we make an important, and slightly dodgy, assumption:

Initial and final states are eigenstates of the free theory

This means that we take the initial state $|i\rangle$ at $t \to -\infty$, and the final state $|f\rangle$ at $t \to +\infty$, to be eigenstates of the free Hamiltonian $H_0$. At some level, this sounds plausible: at $t \to -\infty$, the particles in a scattering process are far separated and don't feel the effects of each other. Furthermore, we intuitively expect these states to be eigenstates of the individual number operators $N$ , which commute with $H_0$, but not $H_{\text{int}}$. As the particles approach each other, they interact briefly, before departing again, each going on its own merry way. The amplitude to go from $|i\rangle$ to $|f\rangle$ is

$$\lim_{t_\pm \to \pm\infty} \langle f\,|\,U(t_+, t_-)\,|i\rangle \equiv \langle f\,|\,S\,|i\rangle \tag{3.26}$$

where the unitary operator $S$ is known as the S-matrix. (S is for scattering). There are a number of reasons why the assumption of non-interacting initial and final states is shaky:

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- Obviously we can't cope with bound states. For example, this formalism can't describe the scattering of an electron and proton which collide, bind, and leave as a Hydrogen atom. It's possible to circumvent this objection since it turns out that bound states show up as poles in the S-matrix.

- More importantly, a single particle, a long way from its neighbors, is never alone in field theory. This is true even in classical electrodynamics, where the electron sources the electromagnetic field from which it can never escape. In quantum electrodynamics (QED), a related fact is that there is a cloud of *virtual* photons surrounding the electron. This line of thought gets us into the issues of renormal- ization — more on this next term in the "AQFT" course. Nevertheless, motivated by this problem, after developing scattering theory using the assumption of non- interacting asymptotic states, we'll mention a better way.

### 3.2.1 An Example: Meson Decay

Consider the relativistically normalized initial and final states,

$$\begin{aligned} |i\rangle &= \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}}\, |0\rangle \\ |f\rangle &= \sqrt{4E_{\vec{q}_1} E_{\vec{q}_2}}\, b^\dagger_{\vec{q}_1} c^\dagger_{\vec{q}_2}\, |0\rangle \end{aligned} \tag{3.27}$$

The initial state contains a single meson of momentum $p$; the final state contains a nucleon-anti-nucleon pair of momentum $q_1$ and $q_2$. We may compute the amplitude for the decay of a meson to a nucleon-anti-nucleon pair. To leading order in $g$, it is

$$\langle f|\, S\, |i\rangle = -ig\, \langle f|\int d^4x\, \psi^\dagger(x)\psi(x)\phi(x)\, |i\rangle \tag{3.28}$$

Let's go slowly. We first expand out $\phi \sim a + a^\dagger$ using (2.84). (Remember that the $\phi$ in this formula is in the interaction picture, which is the same as the Heisenberg picture of the free theory). The $a$ piece will turn $|i\rangle$ into something proportional to $|0\rangle$, while the $a^\dagger$ piece will turn $|i\rangle$ into a two meson state. But the two meson state will have zero overlap with $\langle f|$, and there's nothing in the $\psi$ and $\psi^\dagger$ operators that lie between them to change this fact. So we have

$$\begin{aligned} \langle f|\, S\, |i\rangle &= -ig\, \langle f|\int d^4x\, \psi^\dagger(x)\psi(x) \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{2E_{\vec{p}}}}{\sqrt{2E_{\vec{k}}}}\, a_{\vec{k}}\, a^\dagger_{\vec{p}}\, e^{-ik\cdot x}\, |0\rangle \\ &= -ig\, \langle f|\int d^4x\, \psi^\dagger(x)\psi(x) e^{-ip\cdot x}\, |0\rangle \end{aligned} \tag{3.29}$$

where, in the second line, we've commuted $a_{\vec{k}}$ past $a^\dagger_{\vec{p}}$, picking up a $\delta^{(3)}(\vec{p}-\vec{k})$ delta-function which kills the $d^3k$ integral. We now similarly expand out $\psi \sim b + c^\dagger$ and

$\psi^\dagger \sim b^\dagger + c$. To get non-zero overlap with $\langle f|$, only the $b^\dagger$ and $c^\dagger$ contribute, for they create the nucleon and anti-nucleon from $|0\rangle$. We then have

$$\begin{aligned}\langle f|\,S\,|i\rangle &= -ig\,\langle 0| \int \frac{d^4x\, d^3k_1\, d^3k_2}{(2\pi)^6} \frac{\sqrt{E_{\vec{q}_1} E_{\vec{q}_2}}}{\sqrt{E_{\vec{k}_1} E_{\vec{k}_2}}}\, c_{\vec{q}_2} b_{\vec{q}_1} c^\dagger_{\vec{k}_1} b^\dagger_{\vec{k}_2} \,|0\rangle\, e^{i(k_1+k_2-p)\cdot x} \\ &= -ig\,(2\pi)^4\,\delta^{(4)}(q_1+q_2-p) \end{aligned} \tag{3.30}$$

and so we get our first quantum field theory amplitude.

Notice that the $\delta$-function puts constraints on the possible decays. In particular, the decay only happens at all if $m \geq 2M$. To see this, we may always boost ourselves to a reference frame where the meson is stationary, so $p = (m, 0, 0, 0)$. Then the delta function imposes momentum conservation, telling us that $\vec{q}_1 = -\vec{q}_2$ and $m = 2\sqrt{M^2 + |\vec{q}|^2}$.

Later you will learn how to turn this quantum amplitude into something more physical, namely the lifetime of the meson. The reason this is a little tricky is that we must square the amplitude to get the probability for decay, which means we get the square of a $\delta$-function. We'll explain how to deal with this in Section 3.6 below, and again in next term's "Standard Model" course.

### 3.3 Wick's Theorem

From Dyson's formula, we want to compute quantities like $\langle f|\,T\,\{H_I(x_1)\ldots H_I(x_n)\}\,|i\rangle$, where $|i\rangle$ and $|f\rangle$ are eigenstates of the free theory. The ordering of the operators is fixed by $T$, time ordering. However, since the $H_I$'s contain certain creation and annihilation operators, our life will be much simpler if we can start to move all annihilation operators to the right where they can start killing things in $|i\rangle$. Recall that this is the definition of normal ordering. Wick's theorem tells us how to go from time ordered products to normal ordered products.

#### 3.3.1 An Example: Recovering the Propagator

Let's start simple. Consider a real scalar field which we decompose in the Heisenberg picture as

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{3.31}$$

where

$$\begin{aligned}\phi^+(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}}\, a_{\vec{p}}\, e^{-ip\cdot x} \\ \phi^-(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}}\, a^\dagger_{\vec{p}}\, e^{+ip\cdot x}\end{aligned} \tag{3.32}$$

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where the ± signs on $\varphi^{\pm}$ make little sense, but apparently you have Pauli and Heisenberg to blame. (They come about because $\varphi^{+} \sim e^{-iEt}$, which is sometimes called the positive frequency piece, while $\varphi^{-} \sim e^{+iEt}$ is the negative frequency piece). Then choosing $x^0 > y^0$, we have

$$
\begin{aligned}
T\,\varphi(x)\varphi(y) &= \varphi(x)\varphi(y) \\
&= (\varphi^{+}(x) + \varphi^{-}(x))(\varphi^{+}(y) + \varphi^{-}(y)) \qquad (3.33)\\
&= \varphi^{+}(x)\varphi^{+}(y) + \varphi^{-}(x)\varphi^{+}(y) + \varphi^{-}(y)\varphi^{+}(x) + [\varphi^{+}(x), \varphi^{-}(y)] + \varphi^{-}(x)\varphi^{-}(y)
\end{aligned}
$$

where the last line is normal ordered, and for our troubles we have picked up the extra term $D(x-y) = [\varphi^{+}(x), \varphi^{-}(y)]$ which is the propagator we met in (2.90). So for $x^0 > y^0$ we have

$$T\,\varphi(x)\varphi(y) =\, :\varphi(x)\varphi(y): + D(x-y) \qquad (3.34)$$

Meanwhile, for $y^0 > x^0$, we may repeat the calculation to find

$$T\,\varphi(x)\varphi(y) =\, :\varphi(x)\varphi(y): + D(y-x) \qquad (3.35)$$

So putting this together, we have the final expression

$$T\,\varphi(x)\varphi(y) =\, :\varphi(x)\varphi(y): + \Delta_F(x-y) \qquad (3.36)$$

where $\Delta_F(x-y)$ is the Feynman propagator defined in (2.93), for which we have the integral representation

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \qquad (3.37)$$

Let me reiterate a comment from Section 2: although $T\,\varphi(x)\varphi(y)$ and $:\varphi(x)\varphi(y):$ are both operators, the difference between them is a c-number function, $\Delta_F(x-y)$.

**Definition:** We define the *contraction* of a pair of fields in a string of operators $\ldots\varphi(x_1)\ldots\varphi(x_2)\ldots$ to mean replacing those operators with the Feynman propagator, leaving all other operators untouched. We use the notation,

$$\ldots\overbrace{\varphi(x_1)\ldots\varphi(x_2)}\ldots \qquad (3.38)$$

to denote contraction. So, for example,

$$\overbrace{\varphi(x)\varphi(y)} = \Delta_F(x-y) \qquad (3.39)$$

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A similar discussion holds for complex scalar fields. We have

$$T\psi(x)\psi^\dagger(y) = :\psi(x)\psi^\dagger(y): + \Delta_F(x-y) \tag{3.40}$$

prompting us to define the contraction

$$\overline{\psi(x)\psi^\dagger(y)} = \Delta_F(x-y) \quad \text{and} \quad \overline{\psi(x)\psi(y)} = \overline{\psi^\dagger(x)\psi^\dagger(y)} = 0 \tag{3.41}$$

### 3.3.2 Wick's Theorem

For any collection of fields $\varphi_1 = \varphi(x_1)$, $\varphi_2 = \varphi(x_2)$, etc, we have

$$T(\varphi_1 \dots \varphi_n) = :\varphi_1 \dots \varphi_n: + :\text{all possible contractions}: \tag{3.42}$$

To see what the last part of this equation means, let's look at an example. For $n = 4$, the equation reads

$$\begin{aligned} T(\varphi_1\varphi_2\varphi_3\varphi_4) = {} & :\varphi_1\varphi_2\varphi_3\varphi_4: + \overline{\varphi_1\varphi_2}:\varphi_3\varphi_4: + \overline{\varphi_1\varphi_3}:\varphi_2\varphi_4: + \text{four similar terms} \\ & + \overline{\varphi_1\varphi_2}\,\overline{\varphi_3\varphi_4} + \overline{\varphi_1\varphi_3}\,\overline{\varphi_2\varphi_4} + \overline{\varphi_1\varphi_4}\,\overline{\varphi_2\varphi_3} \end{aligned} \tag{3.43}$$

**Proof:** The proof of Wick's theorem proceeds by induction and a little thought. It's true for $n = 2$. Suppose it's true for $\varphi_2 \dots \varphi_n$ and now add $\varphi_1$. We'll take $x_1^0 > x_k^0$ for all $k = 2, \dots, n$. Then we can pull $\varphi_1$ out to the left of the time ordered product, writing

$$T(\varphi_1\varphi_2 \dots \varphi_n) = (\varphi_1^+ + \varphi_1^-)\,(:\varphi_2 \dots \varphi_n: + :\text{contractions}:) \tag{3.44}$$

The $\varphi_1^-$ term stays where it is since it is already normal ordered. But in order to write the right-hand side as a normal ordered product, the $\varphi_1^+$ term has to make its way past the crowd of $\varphi_k^-$ operators. Each time it moves past $\varphi_k^-$, we pick up a factor of $\overline{\varphi_1\varphi_k} = \Delta_F(x_1 - x_k)$ from the commutator. (Try it!)

### 3.3.3 An Example: Nucleon Scattering

Let's look at $\psi\psi \to \psi\psi$ scattering. We have the initial and final states

$$\begin{aligned} |i\rangle &= \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}\, b^\dagger_{\vec{p}_1} b^\dagger_{\vec{p}_2}|0\rangle \equiv |p_1, p_2\rangle \\ |f\rangle &= \sqrt{2E_{\vec{p}\,'_1}}\sqrt{2E_{\vec{p}\,'_2}}\, b^\dagger_{\vec{p}\,'_1} b^\dagger_{\vec{p}\,'_2}|0\rangle \equiv |p'_1, p'_2\rangle \end{aligned} \tag{3.45}$$

We can then look at the expansion of $\langle f|\,S\,|i\rangle$. In fact, we really want to calculate $\langle f|\,S - 1\,|i\rangle$ since we're not interested in situations where no scattering occurs. At order $g^2$ we have the term

$$\frac{(-ig)^2}{2}\int d^4x_1 d^4x_2\, T\left[\psi^\dagger(x_1)\psi(x_1)\varphi(x_1)\psi^\dagger(x_2)\psi(x_2)\varphi(x_2)\right] \tag{3.46}$$

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Now, using Wick's theorem we see there is a piece in the string of operators which looks like

$$: \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : \; \overbrace{\phi(x_1)\,\phi(x_2)} \tag{3.47}$$

which will contribute to the scattering because the two $\psi$ fields annihilate the $\psi$ particles, while the two $\psi^\dagger$ fields create $\psi$ particles. Any other way of ordering the $\psi$ and $\psi^\dagger$ fields will give zero contribution. This means that we have

$$\begin{aligned}
&\langle p_1', p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1, p_2\rangle \\
&= \langle p_1', p_2'|\, \psi^\dagger(x_1)\psi^\dagger(x_2)\, |0\rangle \langle 0|\, \psi(x_1)\psi(x_2)\, |p_1, p_2\rangle \\
&= \left(e^{ip_1'\cdot x_1 + ip_2'\cdot x_2} + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1}\right)\left(e^{-ip_1\cdot x_1 - ip_2\cdot x_2} + e^{-ip_1\cdot x_2 - ip_2\cdot x_1}\right) \\
&= e^{ix_1\cdot(p_1'-p_1)+ix_2\cdot(p_2'-p_2)} + e^{ix_1\cdot(p_2'-p_1)+ix_2\cdot(p_1'-p_2)} + (x_1 \leftrightarrow x_2)
\end{aligned} \tag{3.48}$$

where, in going to the third line, we've used the fact that for relativistically normalized states,

$$\langle 0|\, \psi(x)\, |p\rangle = e^{-ip\cdot x} \tag{3.49}$$

Now let's insert this into (3.46), to get the expression for $\langle f|\, S\, |i\rangle$ at order $g^2$,

$$\frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 \left[e^{i\ldots} + e^{i\ldots} + (x_1 \leftrightarrow x_2)\right] \int \frac{d^4k}{(2\pi)^4}\, \frac{i e^{ik\cdot(x_1-x_2)}}{k^2 - m^2 + i\epsilon} \tag{3.50}$$

where the expression in square brackets is (3.48), while the final integral is the $\phi$ propagator which comes from the contraction in (3.47). Now the $(x_1 \leftrightarrow x_2)$ terms double up with the others to cancel the factor of 1/2 out front. Meanwhile, the $x_1$ and $x_2$ integrals give delta-functions. We're left with the expression

$$(-ig)^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{i(2\pi)^8}{k^2 - m^2 + i\epsilon} \Big[\delta^{(4)}(p_1' - p_1 + k)\,\delta^{(4)}(p_2' - p_2 - k) + \delta^{(4)}(p_2' - p_1 + k)\,\delta^{(4)}(p_1' - p_2 - k)\Big] \tag{3.51}$$

Finally, we can trivially do the $d^4k$ integral using the delta-functions to get

$$i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2 + i\epsilon} + \frac{1}{(p_1 - p_2')^2 - m^2 + i\epsilon}\right] (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

In fact, for this process we may drop the $+i\epsilon$ terms since the denominator is never zero. To see this, we can go to the center of mass frame, where $\vec{p}_1 = -\vec{p}_2$ and, by

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momentum conservation, $|\vec{p}_1| = |\vec{p}_1'|$. This ensures that the 4-momentum of the meson is $k = (0, \vec{p} - \vec{p}')$, so $k^2 < 0$. We therefore have the end result,

$$i(-ig)^2 \left[ \frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2} \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \tag{3.52}$$

We will see another, much simpler way to reproduce this result shortly using Feynman diagrams. This will also shed light on the physical interpretation.

This calculation is also relevant for other scattering processes, such as $\bar{\psi}\bar{\psi} \to \bar{\psi}\bar{\psi}$, $\psi\bar{\psi} \to \psi\bar{\psi}$. Each of these comes from the term (3.48) in Wick's theorem. However, we will never find a term that contributes to scattering $\psi\psi \to \bar{\psi}\bar{\psi}$, for this would violate
the conservation of $Q$ charge.

### Another Example: Meson-Nucleon Scattering

If we want to compute $\psi\varphi \to \psi\varphi$ scattering at order $g^2$, we would need to pick out the term

$$:\psi^\dagger(x_1)\varphi(x_1)\psi(x_2)\varphi(x_2):\ \overline{\psi(x_1)\psi^\dagger(x_2)} \tag{3.53}$$

and a similar term with $\psi$ and $\psi^\dagger$ exchanged. Once more, this term also contributes to similar scattering processes, including $\bar{\psi}\varphi \to \bar{\psi}\varphi$ and $\varphi\varphi \to \psi\bar{\psi}$.

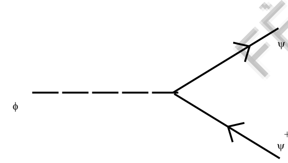
## 3.4 Feynman Diagrams

> "Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses."
>
> *Julian Schwinger*

As the above example demonstrates, to actually compute scattering amplitudes using Wick's theorem is rather tedious. There's a much better way. It requires drawing pretty pictures. These pictures represent the expansion of $\langle f | S | i\rangle$ and we will learn how to associate numbers (or at least integrals) to them. These pictures are called *Feynman diagrams*.

The object that we really want to compute is $\langle f | S - 1 | i\rangle$, since we're not interested in processes where no scattering occurs. The various terms in the perturbative expansion can be represented pictorially as follows

- Draw an external line for each particle in the initial state $|i\rangle$ and each particle in the final state $|f\rangle$. We'll choose dotted lines for mesons, and solid lines for nucleons. Assign a directed momentum $p$ to each line. Further, add an arrow to solid lines to denote its charge; we'll choose an incoming (outgoing) arrow in the initial state for $\psi$ ($\bar{\psi}$). We choose the reverse convention for the final state, where an outgoing arrow denotes $\psi$.

- Join the external lines together with trivalent vertices 

Each such diagram you can draw is in 1-1 correspondence with the terms in the expansion of $\langle f\,|\,S - 1\,|i\rangle$.

### 3.4.1 Feynman Rules

To each diagram we associate a number, using the *Feynman rules*

- Add a momentum $k$ to each internal line
- To each vertex, write down a factor of

$$(-ig)\,(2\pi)^4\,\delta^{(4)}\Big(\sum_i k_i\Big) \tag{3.54}$$

  where $\sum k_i$ is the sum of all momenta flowing *into* the vertex.

- For each internal dotted line, corresponding to a $\varphi$ particle with momentum $k$, we write down a factor of

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2 - m^2 + i\epsilon} \tag{3.55}$$

  We include the same factor for solid internal $\psi$ lines, with $m$ replaced by the nucleon mass $M$.

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## 3.5 Examples of Scattering Amplitudes

Let's apply the Feynman rules to compute the amplitudes for various processes. We start with something familiar:

### Nucleon Scattering Revisited

Let's look at how this works for the $\psi\psi \to \psi\psi$ scattering at order $g^2$. We can write down the two simplest diagrams contributing to this process. They are shown in Figure 9.

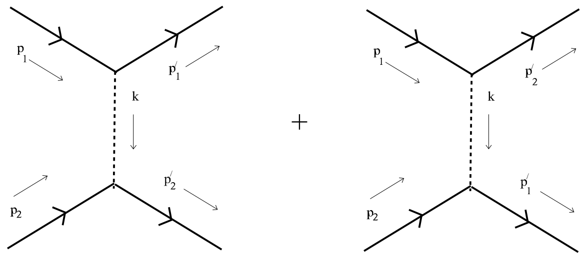


**Figure 9:** The two lowest order Feynman diagrams for nucleon scattering.

Applying the Feynman rules to these diagrams, we get

$$i(-ig)^2\left[\frac{1}{(p_1-p_1')^2-m^2}+\frac{1}{(p_1-p_2')^2-m^2}\right](2\pi)^4\,\delta^{(4)}(p_1+p_2-p_1'-p_2') \qquad (3.56)$$

which agrees with the calculation (3.51) that we performed earlier. There is a nice physical interpretation of these diagrams. We talk, rather loosely, of the nucleons exchanging a meson which, in the first diagram, has momentum $k = (p_1 - p_1') = (p_2' - p_2)$.

This meson doesn't satisfy the usual energy dispersion relation, because $k^2 \neq m^2$: the meson is called a *virtual particle* and is said to be *off-shell* (or, sometimes, off mass-shell). Heuristically, it can't live long enough for its energy to be measured to great accuracy. In contrast, the momentum on the external, nucleon legs all satisfy $p^2 = M^2$, the mass of the nucleon. They are *on-shell*. One final note: the addition of the two diagrams above ensures that the particles satisfy Bose statistics.

There are also more complicated diagrams which will contribute to the scattering process at higher orders. For example, we have the two diagrams shown in Figures 10 and 11, and similar diagrams with $p_1'$ and $p_2'$ exchanged. Using the Feynman rules, each of these diagrams translates into an integral that we will not attempt to calculate here. And so we go on, with increasingly complicated diagrams, all appearing at higher order in the coupling constant $g$.

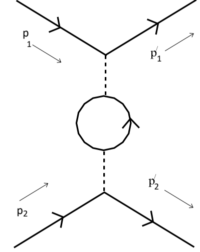

**Figure 10:** A contribution at $O(g^4)$.

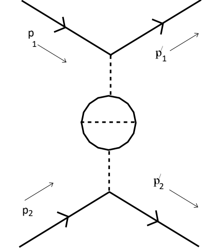

**Figure 11:** A contribution at $O(g^6)$

**Amplitudes**

Our final result for the nucleon scattering amplitude $\langle f \,|\, S - 1 \,|i\rangle$ at order $g^2$ was

$$i(-ig)^2 \left[ \frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2} \right] (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

The $\delta$-function follows from the conservation of 4-momentum which, in turn, follows from spacetime translational invariance. It is common to all S-matrix elements. We will define the amplitude $\mathcal{A}_{fi}$ by stripping off this momentum-conserving delta-function,

$$\langle f \,|\, S - 1 \,|i\rangle = i\, \mathcal{A}_{fi}\, (2\pi)^4 \delta^{(4)}(p_F - p_I) \tag{3.57}$$

where $p_I$ ($p_F$ ) is the sum of the initial (final) 4-momenta, and the factor of $i$ out front is a convention which is there to match non-relativistic quantum mechanics. We can now refine our Feynman rules to compute the amplitude $i\mathcal{A}_{fi}$ itself:

- Draw all possible diagrams with appropriate external legs and impose 4-momentum conservation at each vertex.
- Write down a factor of $(-ig)$ at each vertex.
- For each internal line, write down the propagator
- Integrate over momentum $k$ flowing through each loop $\int d^4k/(2\pi)^4$.

This last step deserves a short explanation. The diagrams we've computed so far have no loops. They are *tree level* diagrams. It's not hard to convince yourself that in tree diagrams, momentum conservation at each vertex is sufficient to determine the momentum flowing through each internal line. For diagrams with loops, such as those shown in Figures 10 and 11, this is no longer the case.

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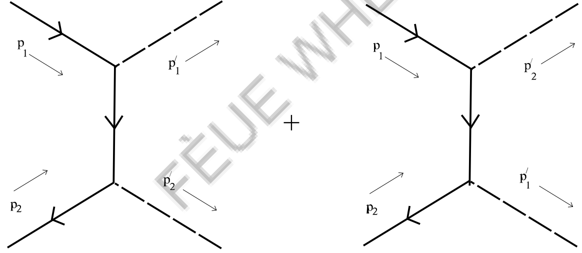



**Figure 12:** The two lowest order Feynman diagrams for nucleon to meson scattering.

### Nucleon to Meson Scattering

Let's now look at the amplitude for a nucleon-anti-nucleon pair to annihilate into a pair of mesons: $\psi\bar{\psi} \to \varphi\varphi$. The simplest Feynman diagrams for this process are shown in Figure 12 where the virtual particle in these diagrams is now the nucleon $\psi$ rather than the meson $\varphi$. This fact is reflected in the denominator of the amplitudes which are given by

$$iA = (-ig)^2 \left[ \frac{i}{(p_1 - p_1')^2 - M^2} + \frac{i}{(p_1 - p_2')^2 - M^2} \right] \tag{3.58}$$

As in (3.52), we've dropped the $i\epsilon$ from the propagators as the denominator never vanishes.

### Nucleon-Anti-Nucleon Scattering

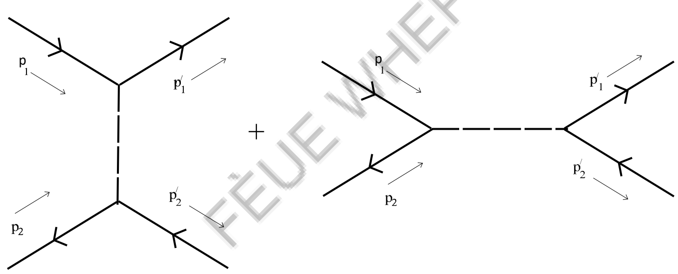



**Figure 13:** The two lowest order Feynman diagrams for nucleon-anti-nucleon scattering.

For the scattering of a nucleon and an anti-nucleon, $\psi\bar{\psi} \to \psi\bar{\psi}$, the Feynman diagrams are a little different. At lowest order, they are given by the diagrams of Figure 13. It is a simple matter to write down the amplitude using the Feynman rules,

$$iA = (-ig)^2 \left[ \frac{i}{(p_1 - p_1')^2 - m^2} + \frac{i}{(p_1 + p_2)^2 - m^2 + i\epsilon} \right] \tag{3.59}$$

Notice that the momentum dependence in the second term is different from that of nucleon-nucleon scattering ([3.56](#)), reflecting the different Feynman diagram that contributes to the process. In the center of mass frame, $\vec{p}_1 = -\vec{p}_2$, the denominator of the second term is $4(M^2 + \vec{p}_1^{\,2}) - m^2$. If $m < 2M$, then this term never vanishes and we may drop the $i\epsilon$. In contrast, if $m > 2M$, then the amplitude corresponding to the second diagram diverges at some value of $\vec{p}$. In this case it turns out that we may also neglect the $i\epsilon$ term, although for a different reason: the meson is unstable when $m > 2M$, a result we derived in ([3.30](#)). When correctly treated, this instability adds a finite imaginary piece to the denominator which

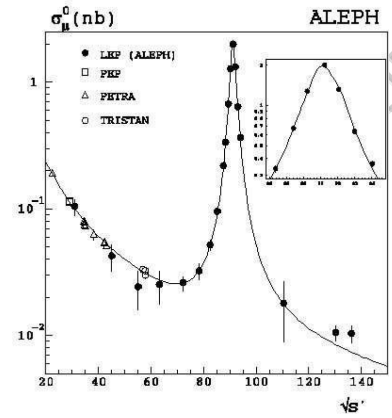
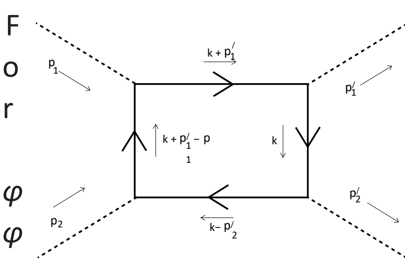



**Figure 14:**

overwhelms the $i\epsilon$. Nonetheless, the increase in the scattering amplitude which we see in the second diagram when $4(M^2 + \vec{p}^{\,2}) = m^2$ is what allows us to discover new particles: they appear as a resonance in the cross section. For example, the Figure 14 shows the cross-section (roughly the amplitude squared) plotted vertically for $e^+e^- \to \mu^+\mu^-$ scattering from the ALEPH experiment in CERN. The horizontal axis shows the center of mass energy. The curve rises sharply around 91 GeV, the mass of the Z-boson.

### Meson Scattering




For $\varphi\varphi \to \varphi\varphi$, th e simplest diagram we can write down has a single loop, and momentum conservation at each vertex is no longer sufficient to determine every momentum passing through the diagram. We choose to assign the single undetermined momentum $k$ to the right-hand propagator. All other momenta are then determined. The amplitude corresponding to the diagram shown in the figure is

**Figure 15:**

$$(-ig)^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2 + i\epsilon)((k + p'_1)^2 - M_1^2 + i\epsilon)} \times \frac{1}{((k + p'_1 - p_1)^2 - M^2 + i\epsilon)((k - p'_2)^2 - M^2 + i\epsilon)}$$

These integrals can be tricky. For large $k$, this integral goes as $\int d^4k/k^8$, which is at least convergent as $k \to \infty$. But this won't always be the case!

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### 3.5.1 Mandelstam Variables

We see that in many of the amplitudes above — in particular those that include the exchange of just a single particle — the same combinations of momenta are appearing frequently in the denominators. There are standard names for various sums and differences of momenta: they are known as *Mandelstam variables*. They are

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_1' + p_2')^2 \\
t &= (p_1 - p_1')^2 = (p_2 - p_2')^2 \\
u &= (p_1 - p_2')^2 = (p_2 - p_1')^2
\end{aligned}
\tag{3.60}
$$

where, as in the examples above, $p_1$ and $p_2$ are the momenta of the two initial particles, and $p_1'$ and $p_2'$ are the momenta of the final two particles. We can define these variables whether the particles involved in the scattering are the same or different. To get a feel for what these variables mean, let's assume all four particles are the same. We sit in the center of mass frame, so that the initial two particles have four-momenta

$$p_1 = (E, 0, 0, p) \quad \text{and} \quad p_2 = (E, 0, 0, -p) \tag{3.61}$$

The particles then scatter at some angle $\vartheta$ and leave with momenta

$$p_1' = (E, 0, p\sin\vartheta, p\cos\vartheta) \quad \text{and} \quad p_2' = (E, 0, -p\sin\vartheta, -p\cos\vartheta) \tag{3.62}$$

Then from the above definitions, we have that

$$s = 4E^2 \quad \text{and} \quad t = -2p^2(1 - \cos\vartheta) \quad \text{and} \quad u = -2p^2(1 + \cos\vartheta) \tag{3.63}$$

The variable $s$ measures the total center of mass energy of the collision, while the variables $t$ and $u$ are measures of the momentum exchanged between particles. (They are basically equivalent, just with the outgoing particles swapped around). Now the amplitudes that involve exchange of a single particle can be written simply in terms of the Mandelstam variables. For example, for nucleon-nucleon scattering, the amplitude (3.56) is schematically $\mathcal{A} \sim (t - m^2)^{-1} + (u - m^2)^{-1}$. For the nucleon-anti-nucleon scattering, the amplitude (3.59) is $\mathcal{A} \sim (t - m^2)^{-1} + (s - m^2)^{-1}$. We say that the first case involves "t-channel" and "u-channel" diagrams. Meanwhile the nucleon-anti- nucleon scattering is said to involve "t-channel" and "s-channel" diagrams. (The first diagram indeed includes a vertex that looks like the letter "T").

Note that there is a relationship between the Mandelstam variables. When all the masses are the same we have $s + t + u = 4M^2$. When the masses of all 4 particles differ,

this becomes $s + t + u = \sum_i M_i^2$.

### 3.5.2 The Yukawa Potential

So far we've computed the quantum amplitudes for various scattering processes. But these quantities are a little abstract. In Section 3.6 below (and again in next term's "Standard Model" course) we'll see how to turn amplitudes into measurable quantities such as cross-sections, or the lifetimes of unstable particles. Here we'll instead show how to translate the amplitude (3.52) for nucleon scattering into something familiar from Newtonian mechanics: a potential, or force, between the particles.

Let's start by asking a simple question in classical field theory that will turn out to be relevant. Suppose that we have a fixed $\delta$-function source for a real scalar field $\varphi$, that persists for all time. What is the profile of $\varphi(\vec{x})$? To answer this, we must solve the static Klein-Gordon equation,

$$-\nabla^2\varphi + m^2\varphi = \delta^{(3)}(\vec{x}) \tag{3.64}$$

We can solve this using the Fourier transform,

$$\varphi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\varphi}(\vec{k}) \tag{3.65}$$

Plugging this into (3.64) tells us that $(\vec{k}^{\,2} + m^2)\,\tilde{\varphi}(\vec{k}) = 1$, giving us the solution

$$\varphi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^{\,2}+m^2} \tag{3.66}$$

Let's now do this integral. Changing to polar coordinates, and writing $\vec{k}\cdot\vec{x} = kr\cos\vartheta$, we have

$$\begin{aligned}\varphi(\vec{x}) &= \frac{1}{(2\pi)^2}\int_0^\infty dk\, \frac{k^2}{k^2+m^2}\, \frac{2\sin kr}{kr} \\ &= \frac{1}{(2\pi)^2 r}\int_{-\infty}^{+\infty} dk\, \frac{k\sin kr}{k^2+m^2} \\ &= \frac{1}{2\pi r}\,\mathrm{Re}\int_{-\infty}^{+\infty}\frac{dk}{2\pi i}\,\frac{k e^{ikr}}{k^2+m^2}\end{aligned} \tag{3.67}$$

We compute this last integral by closing the contour in the upper half plane $k \to +i\infty$, picking up the pole at $k = +im$. This gives

$$\varphi(\vec{x}) = \frac{1}{4\pi r}e^{-mr} \tag{3.68}$$

The field dies off exponentially quickly at distances $1/m$, the Compton wavelength of the meson.

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Now we understand the profile of the $\varphi$ field, what does this have to do with the force between $\psi$ particles? We do very similar calculations to that above in electrostatics where a charged particle acts as a $\delta$-function source for the gauge potential: $-\nabla^2 A_0 = \delta^{(3)}(\vec{x})$, which is solved by $A_0 = 1/4\pi r$. The profile for $A_0$ then acts as the potential energy for another charged (test) particle moving in this background. Can we give the same interpretation to our scalar field? In other words, is there a classical limit of the scalar Yukawa theory where the $\psi$ particles act as $\delta$-function sources for $\varphi$, creating the profile (3.68)? And, if so, is this profile then felt as a static potential? The answer is essentially yes, at least in the limit $M \gg m$. But the correct way to describe the potential felt by the $\psi$ particles is not to talk about classical fields at all, but instead work directly with the quantum amplitudes.

Our strategy is to compare the nucleon scattering amplitude (3.52) to the corresponding amplitude in non-relativistic quantum mechanics for two particles interacting through a potential. To make this comparison, we should first take the non-relativistic limit of (3.52). Let's work in the center of mass frame, with $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$ and $\vec{p}\,' \equiv \vec{p}_1\,' = -\vec{p}_2\,'$. The non-relativistic limit means $|\vec{p}| \ll M$ which, by momentum conservation, ensures that $|\vec{p}\,'| \ll M$. In fact one can check that, for this particular example, this limit doesn't change the scattering amplitude (3.52): it's given by

$$i\mathcal{A} = +ig^2 \left[ \frac{1}{(\vec{p} - \vec{p}\,')^2 + m^2} + \frac{1}{(\vec{p} + \vec{p}\,')^2 + m^2} \right] \tag{3.69}$$

How do we compare this to scattering in quantum mechanics? Consider two particles, separated by a distance $\vec{r}$, interacting through a potential $U(\vec{r})$. In non-relativistic quantum mechanics, the amplitude for the particles to scatter from momentum states $\pm\vec{p}$ into momentum states $\pm\vec{p}\,'$ can be computed in perturbation theory, using the techniques described in Section 3.1. To leading order, known in this context as the Born approximation, the amplitude is given by

$$\langle \vec{p}\,' | U(\vec{r}) | \vec{p} \rangle = -i \int d^3r \, U(\vec{r}) e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{r}} \tag{3.70}$$

There's a relative factor of $(2M)^2$ that arises in comparing the quantum field theory amplitude $\mathcal{A}$ to $\langle \vec{p}\,' | U(\vec{r}) | \vec{p} \rangle$, that can be traced to the relativistic normalization of the states $|p_1, p_2\rangle$. (It is also necessary to get the dimensions of the potential to work out correctly). Including this factor, and equating the expressions for the two amplitudes, we get

$$\int d^3r \, U(\vec{r}) \, e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{r}} = \frac{-\lambda^2}{(\vec{p} - \vec{p}\,')^2 + m^2} \tag{3.71}$$

where we've introduced the dimensionless parameter $\lambda = g/2M$ . We can trivially invert this to find,

$$U(\vec{r}) = -\lambda^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{r}}}{\vec{p}^{\,2} + m^2} \tag{3.72}$$

But this is exactly the integral (3.66) we just did in the classical theory. We have

$$U(\vec{r}) = \frac{-\lambda^2}{4\pi r} e^{-mr} \tag{3.73}$$

This is the *Yukawa potential*. The force has a range $1/m$, the Compton wavelength of the exchanged particle. The minus sign tells us that the potential is attractive.

Notice that quantum field theory has given us an entirely new perspective on the nature of forces between particles. Rather than being a fundamental concept, the force arises from the virtual exchange of other particles, in this case the meson. In Section 6 of these lectures, we will see how the Coulomb force arises from quantum field theory due to the exchange of virtual photons.

We could repeat the calculation for nucleon-anti-nucleon scattering. The amplitude from field theory is given in (3.59). The first term in this expression gives the same result as for nucleon-nucleon scattering *with the same sign*. The second term vanishes in the non-relativisitic limit (it is an example of an interaction that doesn't have a simple Newtonian interpretation). There is no longer a factor of $1/2$ in (3.70), because the incoming/outgoing particles are not identical, so we learn that the potential between a nucleon and anti-nucleon is again given by (3.73). This reveals a key feature of forces arising due to the exchange of scalars: they are universally attractive. Notice that this is different from forces due to the exchange of a spin 1 particle — such as electromagnetism — where the sign flips when we change the charge. However, for forces due to the exchange of a spin 2 particle — i.e. gravity — the force is again universally attractive.

### 3.5.3 $\varphi^4$ Theory

Let's briefly look at the Feynman rules and scattering amplitudes for the interaction Hamiltonian

$$H_{\text{int}} = \frac{\lambda}{4!} \varphi^4 \tag{3.74}$$

The theory now has a single interaction vertex, which comes with a factor of $(-i\lambda)$, while the other Feynman rules remain the same. Note that we assign $(-i\lambda)$ to the vertex rather than $(-i\lambda/4!)$. To see why this is, we can look at $\varphi\varphi \to \varphi\varphi$ scattering, which has its lowest contribution at order $\lambda$, with the term

$$\frac{-i\lambda}{4!}\langle p_1', p_2'| :\varphi(x)\varphi(x)\varphi(x)\varphi(x): |p_1, p_2\rangle \tag{3.75}$$

Any one of the fields can do the job of annihilation or creation. This gives 4! different contractions, which cancels the 1/4! sitting out front.

Feynman diagrams in the $\varphi^4$ theory sometimes come with extra combinatoric factors (typically 2 or 4) which are known as symmetry factors that one must take into account. For more details, see the book by Peskin and Schroeder.

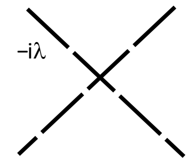


**Figure 16:**

Using the Feynman rules, the scattering amplitude for $\varphi\varphi \to \varphi\varphi$ is simply $i\mathcal{A} = -i\lambda$. Note that it doesn't depend on the angle at which the outgoing particles emerge: in $\varphi^4$ theory the leading order two-particle scattering occurs with equal probability in all directions. Translating this into a potential between two mesons, we have

$$U(\vec{r}) = \frac{\lambda}{(2m)^2}\int \frac{d^3p}{(2\pi)^3}\, e^{+i\vec{p}\cdot\vec{r}} = \frac{\lambda}{(2m)^2}\,\delta^{(3)}(\vec{r}) \tag{3.76}$$

So scattering in $\varphi^4$ theory is due to a $\delta$-function potential. The particles don't know what hit them until it's over.

### 3.5.4 Connected Diagrams and Amputated Diagrams

We've seen how one can compute scattering amplitudes by writing down all Feynman diagrams and assigning integrals to them using the Feynman rules. In fact, there are a couple of caveats about what Feynman diagrams you should write down. Both of these caveats are related to the assumption we made earlier that "initial and final states are eigenstates of the free theory" which, as we mentioned at the time, is not strictly accurate. The two caveats which go some way towards ameliorating the problem are the following

- We consider only connected Feynman diagrams, where every part of the diagram is connected to at least one external line. As we shall see shortly, this will be related to the fact that the vacuum $|0\rangle$ of the free theory is not the true vacuum
$|\Omega\rangle$ of the interacting theory. An example of a diagram that is not connected is shown in Figure 17.

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- We do not consider diagrams with loops on external lines, for example the diagram shown in the Figure 18. We will not explain how to take these into account in this course, but you will discuss them next term. They are related to the fact that the one-particle states of the free theory are not the same as the one-particle states of the interacting theory. In particular, correctly dealing with these diagrams will account for the fact that particles in interacting quantum field theories are never alone, but surrounded by a cloud of virtual particles. We will refer to diagrams in which all loops on external legs have been cut-off as "amputated".

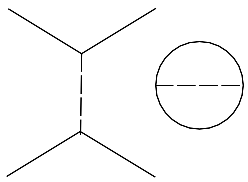

**Figure 17:** A disconnected diagram.

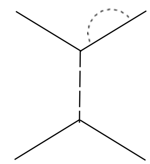

**Figure 18:** An un-amputated diagram

### 3.6 What We Measure: Cross Sections and Decay Rates

So far we've learnt to compute the quantum amplitudes for particles decaying or scattering. As usual in quantum theory, the probabilities for things to happen are the (modulus) square of the quantum amplitudes. In this section we will compute these probabilities, known as decay rates and cross sections. One small subtlety here is that the S-matrix elements $\langle f|\, S - 1\, |i\rangle$ all come with a factor of $(2\pi)^4\delta^{(4)}(p_F - p_I)$, so we end up with the square of a delta-function. As we will now see, this comes from the fact that we're working in an infinite space.

#### 3.6.1 Fermi's Golden Rule

Let's start with something familiar and recall how to derive Fermi's golden rule from Dyson's formula. For two energy eigenstates $|m\rangle$ and $|n\rangle$, with $E_m \neq E_n$, we have to leading order in the interaction,

$$\begin{aligned}
\langle m|\, U(t)\, |n\rangle &= -i \langle m| \int_0^t dt\, H_I(t)\, |n\rangle \\
&= -i \langle m|\, H_{int}\, |n\rangle \int_0^t dt'\, e^{i\omega t'} \\
&= -\langle m|\, H_{int}\, |n\rangle\, \frac{e^{i\omega t} - 1}{\omega}
\end{aligned} \tag{3.77}$$

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where $\omega = E_m - E_n$. This gives us the probability for the transition from $|n\rangle$ to $|m\rangle$ in time $t$, as

$$P_{n\to m}(t) = |\langle m| U(t) |n\rangle|^2 = 2|\langle m| H_{int} |n\rangle|^2 \frac{1-\cos\omega t}{\omega^2} \tag{3.78}$$

The function in brackets is plotted in Figure 19 for fixed $t$. We see that in time $t$, most transitions happen in a region between energy eigenstates separated by $\Delta E = 2\pi/t$. As $t \to \infty$, the function in the figure starts to approach a delta-function. To find the normalization, we can calculate

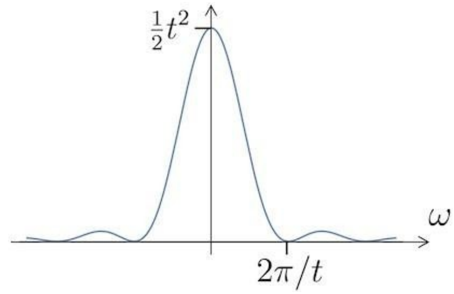


**Figure 19:**

$$\int_{-\infty}^{+\infty} d\omega \frac{1-\cos\omega t}{\omega^2} = \pi t$$

$$\Rightarrow \quad \frac{1-\cos\omega t}{\omega^2} \to \pi t\delta(\omega) \quad \text{as} \quad t \to \infty$$

Consider now a transition to a cluster of states with density $\rho(E)$. In the limit $t \to \infty$, we get the transition probability

$$\begin{aligned} P_{n\to m} &= \int dE_m\, \rho(E_m)\, 2|\langle m| H_{int} |n\rangle|^2 \frac{1-\cos\omega t}{\omega^2} \\ &\to 2\pi\, |\langle m| H_{int} |n\rangle|^2 \rho(E_n)t \end{aligned} \tag{3.79}$$

which gives a constant probability for the transition per unit time for states around the same energy $E_n \sim E_m = E$.

$$\dot{P}_{n\to m} = 2\pi|\langle m| H_{int} |n\rangle|^2 \rho(E) \tag{3.80}$$

This is Fermi's Golden Rule.

In the above derivation, we were fairly careful with taking the limit as $t \to \infty$. Suppose we were a little sloppier, and first chose to compute the amplitude for the state $|n\rangle$ at $t \to -\infty$ to transition to the state $|m\rangle$ at $t \to +\infty$. Then we get

$$-i\langle m| \int_{t=-\infty}^{t=+\infty} H_I(t)\, |n\rangle = -i\langle m| H_{int} |n\rangle\, 2\pi\delta(\omega) \tag{3.81}$$

Now when squaring the amplitude to get the probability, we run into the problem of the square of the delta-function: $P_{n\to m} = |\langle m| H_{int} |n\rangle|^2(2\pi)^2\delta(\omega)^2$. Tracking through the previous computations, we realize that the extra infinity is coming because $P_{m\to n}$

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is the probability for the transition to happen in infinite time $t \to \infty$. We can write the delta-functions as

$$(2\pi)^2\delta(\omega)^2 = (2\pi)\delta(\omega)\ T \tag{3.82}$$

where $T$ is shorthand for $t \to \infty$ (we used a very similar trick when looking at the vacuum energy in (2.25)). We now divide out by this power of $T$ to get the transition probability per unit time,

$$\dot{P}_{n\to m} = 2\pi |\langle m| H_{int} |n\rangle|^2\ \delta(\omega) \tag{3.83}$$

which, after integrating over the density of final states, gives us back Fermi's Golden rule. The reason that we've stressed this point is because, in our field theory calculations, we've computed the amplitudes in the same way as (3.81), and the square of the $\delta^{(4)}$-functions will just be re-interpreted as spacetime volume factors.

### 3.6.2 Decay Rates

Let's now look at the probability for a single particle $|i\rangle$ of momentum $p_I$ (I=initial) to decay into some number of particles $|f\rangle$ with momentum $p_i$ and total momentum $p_F = \sum_i p_i$. This is given by

$$P = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} \tag{3.84}$$

Our states obey the relativistic normalization formula (2.65),

$$\langle i|i\rangle = (2\pi)^3 2E_{\vec{p}_I}\delta^{(3)}(0) = 2E_{\vec{p}_I} V \tag{3.85}$$

where we have replaced $\delta^{(3)}(0)$ by the volume of 3-space. Similarly,

$$\langle f|f\rangle = \prod_{\text{final states}} 2E_{\vec{p}_i} V \tag{3.86}$$

If we place our initial particle at rest, so $\vec{p}_I = 0$ and $E_{\vec{p}_I} = m$, we get the probability for decay

$$P = \frac{|\mathcal{A}_{fi}|^2}{2mV}(2\pi)^4\delta^{(4)}(p_I - p_F)\, VT \prod_{\text{final states}} \frac{1}{2E_{\vec{p}_i}V} \tag{3.87}$$

where, as in the second derivation of Fermi's Golden Rule, we've exchanged one of the delta-functions for the volume of spacetime: $(2\pi)^4\delta^{(4)}(0) = VT$. The amplitudes $\mathcal{A}_{fi}$ are, of course, exactly what we've been computing. (For example, in (3.30), we saw that $\mathcal{A} = -g$ for a single meson decaying into two nucleons). We can now divide out by $T$ to get the transition function per unit time. But we still have to worry about summing over all final states. There are two steps: the first is to integrate over all possible momenta of the final particles: $V\int d^3p_i/(2\pi)^3$. The factors of spatial volume $V$ in this measure cancel those in (3.87), while the factors of $1/2E_{\vec{p}_i}$ in (3.87) conspire to produce the Lorentz invariant measure for 3-momentum integrals. The result is an expression for the density of final states given by the Lorentz invariant measure

$$d\Pi = (2\pi)^4\delta^{(4)}(p_F - p_I)\prod_{\text{final states}}\frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_{\vec{p}_i}} \tag{3.88}$$

The second step is to sum over all final states with different numbers (and possibly types) of particles. This gives us our final expression for the decay probability per unit time, $\Gamma = \dot{P}$.

$$\Gamma = \frac{1}{2m}\sum_{\text{final states}}\int |\mathcal{A}_{fi}|^2\, d\Pi \tag{3.89}$$

$\Gamma$ is called the width of the particle. It is equal to the reciprocal of the half-life $\tau = 1/\Gamma$.

### 3.6.3 Cross Sections

Collide two beams of particles. Sometimes the particles will hit and bounce off each other; sometimes they will pass right through. The fraction of the time that they collide is called the *cross section* and is denoted by $\sigma$. If the incoming flux $F$ is defined to be the number of incoming particles per area per unit time, then the total number of scattering events $N$ per unit time is given by,

$$N = F\sigma \tag{3.90}$$

We would like to calculate $\sigma$ from quantum field theory. In fact, we can calculate a more sensitive quantity $d\sigma$ known as the *differential cross section* which is the probability for a given scattering process to occur in the solid angle $(\vartheta, \varphi)$. More precisely

$$d\sigma = \frac{\text{Differential Probability}}{\text{Unit Time} \times \text{Unit Flux}} = \frac{1}{4E_1E_2V}\frac{1}{F}|\mathcal{A}_{fi}|^2\, d\Pi \tag{3.91}$$

where we've used the expression for probability per unit time that we computed in the previous subsection. $E_1$ and $E_2$ are the energies of the incoming particles. We now need an expression for the unit flux. For simplicity, let's sit in the center of mass frame of the collision. We've been considering just a single particle per spatial volume $V$,

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meaning that the flux is given in terms of the 3-velocities $\vec{v}_i$ as $F = |\vec{v}_1 - \vec{v}_2|/V$. This then gives,

$$d\sigma = \frac{1}{4E_1E_2} \frac{1}{|\vec{v}_1 - \vec{v}_2|} |\mathcal{A}_{fi}|^2 \, d\Pi \tag{3.92}$$

If you want to write this in terms of momentum, then recall from your course on special relativity that the 3-velocities $\vec{v}_i$ are related to the momenta by $\vec{v} = \vec{p}/m\sqrt{1 - v^2} = \vec{p}/p^0$.

Equation (3.92) is our final expression relating the S-matrix to the differential cross section. You may now take your favorite scattering amplitude, and compute the proba- bility for particles to fly out at your favorite angles. This will involve doing the integral over the phase space of final states, with measure $d\Pi$. Notice that different scattering amplitudes have different momentum dependence and will result in different angular dependence in scattering amplitudes. For example, in $\varphi^4$ theory the amplitude for tree level scattering was simply $\mathcal{A} = -\lambda$. This results in isotropic scattering. In contrast, for nucleon-nucleon scattering we have schematically $\mathcal{A} \sim (t - m^2)^{-1} + (u - m^2)^{-1}$. This gives rise to angular dependence in the differential cross-section, which follows from the fact that, for example, $t = -2|\vec{p}|^2 (1 - \cos\vartheta)$, where $\vartheta$ is the angle between the incoming and outgoing particles.

## 3.7 Green's Functions

So far we've learnt to compute scattering amplitudes. These are nice and physical (well – they're directly related to cross-sections and decay rates which are physical) but there are many questions we want to ask in quantum field theory that aren't directly related to scattering experiments. For example, we might want to compute the viscosity of the quark gluon plasma, or the optical conductivity in a tentative model of strange metals, or figure out the non-Gaussianity of density perturbations arising in the CMB from novel models of inflation. All of these questions are answered in the framework of quantum field theory by computing elementary objects known as *correlation functions*. In this section we will briefly define correlation functions, explain how to compute them using Feynman diagrams, and then relate them back to scattering amplitudes. We'll leave the relationship to other physical phenomena to other courses.

We'll denote the true vacuum of the interacting theory as $|\Omega\rangle$. We'll normalize $H$ such that

$$H\,|\Omega\rangle = 0 \tag{3.93}$$

and $\langle\Omega|\,\Omega\rangle = 1$. Note that this is different from the state we've called $|0\rangle$ which is the vacuum of the free theory and satisfies $H_0\,|0\rangle = 0$. Define

$$G^{(n)}(x_1, \ldots, x_n) = \langle\Omega|\, T\, \varphi_H(x_1) \ldots \varphi_H(x_n)\, |\Omega\rangle \tag{3.94}$$

where $\varphi_H$ is $\varphi$ in the Heisenberg picture of the full theory, rather than the interaction picture that we've been dealing with so far. The $G^{(n)}$ are called correlation functions, or *Green's functions*. There are a number of different ways of looking at these objects which tie together nicely. Let's start by asking how to compute $G^{(n)}$ using Feynman diagrams. We prove the following result

**Claim:** We use the notation $\varphi_1 = \varphi(x_1)$, and write $\varphi_{1H}$ to denote the field in the Heisenberg picture, and $\varphi_{1I}$ to denote the field in the interaction picture. Then

$$G^{(n)}(x_1, \ldots, x_n) = \langle\Omega|\, T\, \varphi_{1H} \ldots \varphi_{nH}\, |\Omega\rangle = \frac{\langle 0|\, T\varphi_{1I} \ldots \varphi_{nI}\, S\, |0\rangle}{\langle 0|\, S\, |0\rangle} \tag{3.95}$$

where the operators on the right-hand side are evaluated on $|0\rangle$, the vacuum of the free theory.

**Proof:** Take $t_1 > t_2 > \ldots > t_n$. Then we can drop the $T$ and write the numerator of the right-hand side as

$$\langle 0|\, U_I(+\infty, t_1)\varphi_{1I}\, U(t_1, t_2)\, \varphi_{2I} \ldots \varphi_{nI}\, U_I(t_n, -\infty)\, |0\rangle$$

We'll use the factors of $U_I(t_k, t_{k+1}) = T\exp(-i\int_{t_k}^{t_{k+1}} H_I)$ to convert each of the $\varphi_I$ into $\varphi_H$ and we choose operators in the two pictures to be equal at some arbitrary time $t_0$. Then we can write

$$\langle 0|\, U_I(+\infty, t_1)\varphi_{1I}\, U(t_1, t_2)\, \varphi_{2I} \ldots \varphi_{nI}\, U_I(t_n, -\infty)\, |0\rangle$$
$$= \langle 0|\, U_I(+\infty, t_0)\varphi_{1H} \ldots \varphi_{nH}\, U_I(t_0, -\infty)\, |0\rangle$$

Now let's deal with the two remaining $U(t_0, \pm\infty)$ at either end of the string of operators. Consider an arbitrary state $|\Psi\rangle$ and look at

$$\langle\Psi|\, U_I(t, -\infty)\, |0\rangle = \langle\Psi|\, U(t, -\infty)\, |0\rangle \tag{3.96}$$

where $U(t, -\infty)$ is the Schrödinger evolution operator, and the equality above follows because $H_0\,|0\rangle = 0$. Now insert a complete set of states, which we take to be energy eigenstates of $H = H_0 + H_{int}$,

$$\langle\Psi|\, U(t, -\infty)\, |0\rangle = \langle\Psi|\, U(t, -\infty) \left[ |\Omega\rangle\langle\Omega| + \sum_{n\neq 0} |n\rangle\langle n| \right] |0\rangle$$
$$= \langle\Psi|\,\Omega\rangle\langle\Omega|\,0\rangle + \lim_{t'\to-\infty} \sum_{n\neq 0} e^{iE_n(t'-t)}\, \langle\Psi|\, n\rangle\langle n|\, 0\rangle \tag{3.97}$$

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But the last term vanishes. This follows from the Riemann-Lebesgue lemma which says that for any well-behaved function

$$\lim_{\mu\to\infty} \int_a^b dx\, f(x) e^{i\mu x} = 0 \tag{3.98}$$

Why is this relevant? The point is that the $\sum_n$ in (3.97) is really an integral $\int dn$, because all states are part of a continuum due to the momentum. (There is a caveat here: we want the vacuum $|\Omega\rangle$ to be special, so that it sits on its own, away from the continuum of the integral. This means that we must be working in a theory with a mass gap – i.e. with no massless particles). So the Riemann-Lebesgue lemma gives us

$$\lim_{t'\to-\infty} \langle\Psi|\, U(t,t')\, |0\rangle = \langle\Psi|\,\Omega\rangle\,\langle\Omega|\,0\rangle \tag{3.99}$$

(Notice that to derive this result, Peskin and Schroeder instead send $t \to -\infty$ in a slightly imaginary direction, which also does the job). We now apply the formula (3.99), to the top and bottom of the right-hand side of (3.95) to find

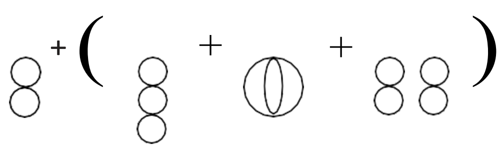
$$\frac{\langle 0|\,\Omega\rangle\,\langle\Omega|\, T\varphi_{1H} \ldots \varphi_{nH}\, |\Omega\rangle\,\langle\Omega|\,0\rangle}{\langle 0|\,\Omega\rangle\,\langle\Omega|\,\Omega\rangle\,\langle\Omega|\,0\rangle} \tag{3.100}$$

which, using the normalization $\langle\Omega|\,\Omega\rangle = 1$, gives us the left-hand side, completing the proof.

### 3.7.1 Connected Diagrams and Vacuum Bubbles

We're getting closer to our goal of computing the Green's functions $G^{(n)}$ since we can compute both $\langle 0|\, T\varphi_I(x_1) \ldots \varphi_I(x_n)\, S\, |0\rangle$ and $\langle 0|\, S\, |0\rangle$ using the same methods we developed for S-matrix elements; namely Dyson's formula and Wick's theorem or, alternatively, Feynman diagrams. But what about dividing one by the other? What's that all about? In fact, it has a simple interpretation. For the following discussion, we will work in $\varphi^4$ theory. Since there is no ambiguity in the different types of lines in Feynman diagrams, we will represent the $\varphi$ particles as solid lines, rather than the dashed lines that we used previously. Then we have the diagramatic expansion for $\langle 0|\, S\, |0\rangle$.

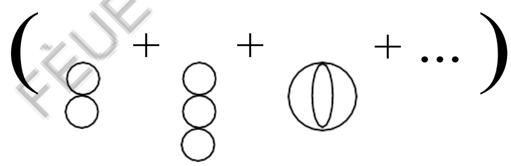
$$\langle 0|\, S\, |0\rangle = 1 +$$



$$+ \ldots \tag{3.101}$$

These diagrams are called vacuum bubbles. The combinatoric factors (as well as the symmetry factors) associated with each diagram are such that the whole series sums to an exponential,

$$\langle 0|\, S\, |0\rangle = \exp\Big( \quad + \quad + \quad + \ldots \Big) \qquad (3.102)$$



So the amplitude for the vacuum of the free theory to evolve into itself is $\langle 0|\, S\, |0\rangle =$ exp(all distinct vacuum bubbles). A similar combinatoric simplification occurs for generic correlation functions. Remarkably, the vacuum diagrams all add up to give the same exponential. With a little thought one can show that

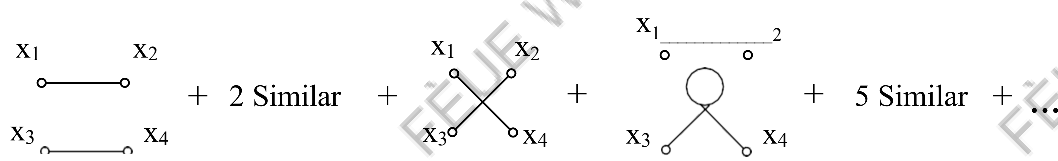
$$\langle 0|\, T\varphi_1 \ldots \varphi_n S\, |0\rangle = \sum \text{connected diagrams} \quad \langle 0|\, S\, |0\rangle \qquad (3.103)$$

where "connected" means that every part of the diagram is connected to at least one of the external legs. The upshot of all this is that dividing by $\langle 0|\, S\, |0\rangle$ has a very nice interpretation in terms of Feynman diagrams: we need only consider the connected Feynman diagrams, and don't have to worry about the vacuum bubbles. Combining this with (3.95), we learn that the Green's functions $G^{(n)}(x_1 \ldots , x_n)$ can be calculated by summing over all connected Feynman diagrams,

$$\langle \Omega|\, T\, \varphi_H(x_1) \ldots \varphi_H(x_n)\, |\Omega\rangle = \sum \text{Connected Feynman Graphs} \qquad (3.104)$$

**An Example: The Four-Point Correlator:** $\langle \Omega|\, T\varphi_H(x_1) \ldots \varphi_H(x_4)\, |\Omega\rangle$

As a simple example, let's look at the four-point correlation function in $\varphi^4$ theory. The sum of connected Feynman diagrams is given by,




All of these are connected diagrams, even though they don't look that connected! The point is that a connected diagram is defined by the requirement that every line is joined to an external leg. An example of a diagram that is not connected is shown in the figure. As we have seen, such diagrams are taken care of in shifting the vacuum from $|0\rangle$ to $|\Omega\rangle$.

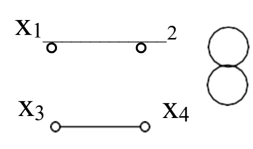



**Figure 20:**

### Feynman Rules

The Feynman diagrams that we need to calculate for the Green's functions depend on $x_1, \ldots , x_n$. This is rather different than the Feynman diagrams that we calculated for

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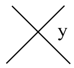
the S-matrix elements, where we were working primarily with momentum eigenstates, and ended up integrating over all of space. However, it's rather simple to adapt the Feynman rules that we had earlier in momentum space to compute $G^{(n)}(x_1 \dots , x_n)$. For $\varphi^4$ theory, we have

- Draw $n$ external points $x_1, \dots , x_n$, connected by the usual propagators and vertices. Assign a spacetime position $y$ to the end of each line.
- For each line $x$ ——— $y$ from $x$ to $y$ write down a factor of the Feynman propagator $\Delta_F(x - y)$.
- For each vertex  $y$ at position $y$, write down a factor of $-i\lambda \int d^4y$.

### 3.7.2 From Green's Functions to S-Matrices

Having described how to compute correlation functions using Feynman diagrams, let's now relate them back to the S-matrix elements that we already calculated. The first step is to perform the Fourier transform,

$$\tilde{G}^{(n)}(p_1, \dots, p_n) = \int \left[\prod_{i=1}^{n} d^4x_i\, e^{-ip_i \cdot x_i}\right] G^{(n)}(x_1, \dots , x_n) \tag{3.105}$$

These are very closely related to the S-matrix elements that we've computed above. The difference is that the Feynman rules for $G^{(n)}(x_1, \dots , x_n)$, effectively include propagators $\Delta_F$ for the external legs, as well as the internal legs. A related fact is that the 4-momenta assigned to the external legs is arbitrary: they are not on-shell. Both of these problems are easily remedied to allow us to return to the S-matrix elements: we need to simply cancel off the propagators on the external legs, and place their momentum back on shell. We have

$$\begin{aligned}\langle p'_1, \dots , p'_{n'} \,|\, S - 1 \,|\, p_1 \dots , p_n \rangle = (-i)^{n+n'} &\prod_{i=1}^{n'} (p'^2_i - m^2) \prod_{j=1}^{n} (p_j^2 - m^2) \\ &\times \tilde{G}^{(n+n')}(-p'_1, \dots , -p'_{n'}, p_1, \dots , p_n)\end{aligned} \tag{3.106}$$

Each of the factors $(p^2 - m^2)$ vanishes once the momenta are placed on-shell. This means that we only get a non-zero answer for diagrams contributing to $G^{(n)}(x_1, \dots , x_n)$ which have propagators for each external leg.

So what's the point of all of this? We've understood that ignoring the unconnected diagrams is related to shifting to the true vacuum $|\Omega\rangle$. But other than that, introducing the Green's functions seems like a lot of bother for little reward. The important point

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is that this provides a framework in which to deal with the true particle states in the interacting theory through renormalization. Indeed, the formula (3.106), suitably interpreted, remains true even in the interacting theory, taking into account the swarm of virtual particles surrounding asymptotic states. This is the correct way to consider scattering. In this context, (3.106) is known as the LSZ reduction formula. You will derive it properly next term.

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# 4. The Dirac Equation

> "A great deal more was hidden in the Dirac equation than the author had expected when he wrote it down in 1928. Dirac himself remarked in one of his talks that his equation was more intelligent than its author. It should be added, however, that it was Dirac who found most of the additional insights."
>
> *Weisskopf on Dirac*

So far we've only discussed scalar fields such that under a Lorentz transformation $x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu x^\nu$, the field transforms as

$$\varphi(x) \to \varphi'(x) = \varphi(\Lambda^{-1}x) \tag{4.1}$$

We have seen that quantization of such fields gives rise to spin 0 particles. But most particles in Nature have an intrinsic angular momentum, or spin. These arise naturally in field theory by considering fields which themselves transform non-trivially under the Lorentz group. In this section we will describe the Dirac equation, whose quantization gives rise to fermionic spin 1/2 particles. To motivate the Dirac equation, we will start by studying the appropriate representation of the Lorentz group.

A familiar example of a field which transforms non-trivially under the Lorentz group is the vector field $A_\mu(x)$ of electromagnetism,

$$A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \tag{4.2}$$

We'll deal with this in Section 6. (It comes with its own problems!). In general, a field can transform as

$$\varphi^a(x) \to D[\Lambda]^a{}_b \varphi^b(\Lambda^{-1}x) \tag{4.3}$$

where the matrices $D[\Lambda]$ form a *representation* of the Lorentz group, meaning that

$$D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2] \tag{4.4}$$

and $D[\Lambda^{-1}] = D[\Lambda]^{-1}$ and $D[1] = 1$. How do we find the different representations? Typically, we look at infinitesimal transformations of the Lorentz group and study the resulting Lie algebra. If we write,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{4.5}$$

for infinitesimal $\omega$, then the condition for a Lorentz transformation $\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho \eta^{\sigma\rho} = \eta^{\mu\nu}$ becomes the requirement that $\omega$ is anti-symmetric:

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0 \tag{4.6}$$

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Note that an antisymmetric 4 × 4 matrix has 4 × 3/2 = 6 independent components, which agrees with the 6 transformations of the Lorentz group: 3 rotations and 3 boosts. It's going to be useful to introduce a basis of these six 4 × 4 anti-symmetric matrices. We could call them $(M^A)^{\mu\nu}$, with $A = 1, \ldots, 6$. But in fact it's better for us (although initially a little confusing) to replace the single index $A$ with a pair of antisymmetric indices $[\rho\sigma]$, where $\rho, \sigma = 0, \ldots, 3$, so we call our matrices $(M^{\rho\sigma})^{\mu}{}_{\nu}$. The antisymmetry on the $\rho$ and $\sigma$ indices means that, for example, $M^{01} = -M^{10}$, etc, so that $\rho$ and $\sigma$ again label six different matrices. Of course, the matrices are also antisymmetric on the $\mu\nu$ indices because they are, after all, antisymmetric matrices. With this notation in place, we can write a basis of six 4 × 4 antisymmetric matrices as

$$(M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu} \tag{4.7}$$

where the indices $\mu$ and $\nu$ are those of the 4 × 4 matrix, while $\rho$ and $\sigma$ denote which basis element we're dealing with. If we use these matrices for anything practical (for example, if we want to multiply them together, or act on some field) we will typically need to lower one index, so we have

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = \eta^{\rho\mu}\delta^{\sigma}{}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}{}_{\nu} \tag{4.8}$$

Since we lowered the index with the Minkowski metric, we pick up various minus signs which means that when written in this form, the matrices are no longer necessarily antisymmetric. Two examples of these basis matrices are,

$$(M^{01})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (M^{12})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.9}$$

The first, $M^{01}$, generates boosts in the $x^1$ direction. It is real and symmetric. The second, $M^{12}$, generates rotations in the $(x^1, x^2)$-plane. It is real and antisymmetric. We can now write any $\omega^{\mu}{}_{\nu}$ as a linear combination of the $M^{\rho\sigma}$,

$$\omega^{\mu}{}_{\nu} = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^{\mu}{}_{\nu} \tag{4.10}$$

where $\Omega_{\rho\sigma}$ are just six numbers (again antisymmetric in the indices) that tell us what Lorentz transformation we're doing. The six basis matrices $M^{\rho\sigma}$ are called the *generators* of the Lorentz transformations. The generators obey the Lorentz Lie algebra relations,

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau} \tag{4.11}$$

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where we have suppressed the matrix indices. A finite Lorentz transformation can then be expressed as the exponential

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right) \tag{4.12}$$

Let me stress again what each of these objects are: the $M^{\rho\sigma}$ are six $4 \times 4$ basis elements of the Lorentz Lie algebra; the $\Omega_{\rho\sigma}$ are six numbers telling us what kind of Lorentz transformation we're doing (for example, they say things like rotate by $\vartheta = \pi/7$ about the $x^3$-direction and run at speed $v = 0.2$ in the $x^1$ direction).

## 4.1 The Spinor Representation

We're interested in finding other matrices which satisfy the Lorentz algebra commutation relations (4.11). We will construct the spinor representation. To do this, we start by defining something which, at first sight, has nothing to do with the Lorentz group. It is the *Clifford algebra*,

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}1 \tag{4.13}$$

where $\gamma^\mu$, with $\mu = 0, 1, 2, 3$, are a set of four matrices and the 1 on the right-hand side denotes the unit matrix. This means that we must find four matrices such that

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \text{when } \mu \neq \nu \tag{4.14}$$

and

$$(\gamma^0)^2 = 1 \quad , \quad (\gamma^i)^2 = -1 \qquad i = 1, 2, 3 \tag{4.15}$$

It's not hard to convince yourself that there are no representations of the Clifford algebra using $2 \times 2$ or $3 \times 3$ matrices. The simplest representation of the Clifford algebra is in terms of $4 \times 4$ matrices. There are many such examples of $4 \times 4$ matrices which obey (4.13). For example, we may take

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4.16}$$

where each element is itself a $2 \times 2$ matrix, with the $\sigma^i$ the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.17}$$

which themselves satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.

One can construct many other representations of the Clifford algebra by taking $V\gamma^\mu V^{-1}$ for any invertible matrix $V$. However, up to this equivalence, it turns out that there is a unique irreducible representation of the Clifford algebra. The matrices (4.16) provide one example, known as the *Weyl* or *chiral representation* (for reasons that will soon become clear). We will soon restrict ourselves further, and consider only representations of the Clifford algebra that are related to the chiral representation by a unitary transformation $V$.

So what does the Clifford algebra have to do with the Lorentz group? Consider the commutator of two $\gamma^\mu$,

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0 & \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{cases} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma} \tag{4.18}$$

Let's see what properties these matrices have:

**Claim 4.1:** $[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$

**Proof:** When $\mu \neq \nu$ we have

$$\begin{aligned}[S^{\mu\nu}, \gamma^\rho] &= \tfrac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] \\ &= \tfrac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \tfrac{1}{2}\gamma^\rho\gamma^\mu\gamma^\nu \\ &= \tfrac{1}{2}\gamma^\mu\{\gamma^\nu, \gamma^\rho\} - \tfrac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu - \tfrac{1}{2}\{\gamma^\rho, \gamma^\mu\}\gamma^\nu + \tfrac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu \\ &= \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}\end{aligned}$$

**Claim 4.2:** The matrices $S^{\mu\nu}$ form a representation of the Lorentz algebra (4.11), meaning

$$[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho} \tag{4.19}$$

**Proof:** Taking $\rho \neq \sigma$, and using Claim 4.1 above, we have

$$\begin{aligned}[S^{\mu\nu}, S^{\rho\sigma}] &= \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma] \\ &= \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \tfrac{1}{2}\gamma^\rho[S^{\mu\nu}, \gamma^\sigma] \\ &= \tfrac{1}{2}\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \tfrac{1}{2}\gamma^\nu\gamma^\sigma\eta^{\rho\mu} + \tfrac{1}{2}\gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \tfrac{1}{2}\gamma^\rho\gamma^\nu\eta^{\sigma\mu}\end{aligned} \tag{4.20}$$

Now using the expression (4.18) to write $\gamma^\mu\gamma^\sigma = 2S^{\mu\sigma} + \eta^{\mu\sigma}$, we have

$$[S^{\mu\nu}, S^{\rho\sigma}] = S^{\mu\sigma}\eta^{\nu\rho} - S^{\nu\sigma}\eta^{\rho\mu} + S^{\rho\mu}\eta^{\nu\sigma} - S^{\rho\nu}\eta^{\sigma\mu} \tag{4.21}$$

which is our desired expression.

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### 4.1.1 Spinors

The $S^{\mu\nu}$ are $4 \times 4$ matrices, because the $\gamma^\mu$ are $4 \times 4$ matrices. So far we haven't given an index name to the rows and columns of these matrices: we're going to call them $\alpha, \beta = 1, 2, 3, 4$.

We need a field for the matrices $(S^{\mu\nu})^\alpha{}_\beta$ to act upon. We introduce the Dirac *spinor* field $\psi^\alpha(x)$, an object with four complex components labelled by $\alpha = 1, 2, 3, 4$. Under Lorentz transformations, we have

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x) \tag{4.22}$$

where

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right) \tag{4.23}$$

$$S[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) \tag{4.24}$$

Although the basis of generators $M^{\rho\sigma}$ and $S^{\rho\sigma}$ are different, we use the same six numbers $\Omega_{\rho\sigma}$ in both $\Lambda$ and $S[\Lambda]$: this ensures that we're doing the same Lorentz transformation on $x$ and $\psi$. Note that we denote both the generator $S^{\rho\sigma}$ and the full Lorentz transformation $S[\Lambda]$ as "$S$". To avoid confusion, the latter will always come with the square brackets $[\Lambda]$.

Both $\Lambda$ and $S[\Lambda]$ are $4 \times 4$ matrices. So how can we be sure that the spinor representation is something new, and isn't equivalent to the familiar representation $\Lambda^\mu{}_\nu$? To see that the two representations are truly different, let's look at some specific transformations.

**Rotations**

$$S^{ij} = \frac{1}{2}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\right] = \frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad (\text{for } i \neq j) \tag{4.25}$$

If we write the rotation parameters as $\Omega_{ij} = -\epsilon_{ijk}\phi^k$ (meaning $\Omega_{12} = -\phi^3$, etc) then the rotation matrix becomes

$$S[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{+i\vec{\phi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\phi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{4.26}$$

where we need to remember that $\Omega_{12} = -\Omega_{21} = -\phi^3$ when following factors of 2.

Consider now a rotation by $2\pi$ about, say, the $x^3$-axis. This is achieved by $\vec{\phi} = (0, 0,$

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and the spinor rotation matrix becomes,

$$S[\Lambda] = \begin{pmatrix} e^{+i\pi\sigma^3} & 0 \\ 0 & e^{+i\pi\sigma^3} \end{pmatrix} = -1 \qquad (4.27)$$

Therefore under a $2\pi$ rotation

$$\psi^\alpha(x) \to -\psi^\alpha(x) \qquad (4.28)$$

which is definitely not what happens to a vector! To check that we haven't been cheating with factors of 2, let's see how a vector would transform under a rotation by $\vec{\phi} = (0, 0, \phi^3)$. We have

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \phi^3 & 0 \\ 0 & -\phi^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (4.29)$$

So when we rotate a vector by $\phi^3 = 2\pi$, we learn that $\Lambda = 1$ as you would expect. So $S[\Lambda]$ is definitely a different representation from the familiar vector representation $\Lambda^\mu{}_\nu$.

**Boosts**

$$S^{0i} = \frac{1}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \qquad (4.30)$$



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2 0

$\sigma^i$

Writing the boost parameter as $\Omega_{i0} = -\Omega_{0i} = \chi_i$, we have

$$S[\Lambda] = \begin{pmatrix} e^{+\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \qquad (4.31)$$

**Representations of the Lorentz Group are not Unitary**

Note that for rotations given in (4.26), $S[\Lambda]$ is unitary, satisfying $S[\Lambda]^\dagger S[\Lambda] = 1$. But for boosts given in (4.31), $S[\Lambda]$ is not unitary. In fact, there are *no* finite dimensional unitary representations of the Lorentz group. We have demonstrated this explicitly for the spinor representation using the chiral representation (4.16) of the Clifford algebra. We can get a feel for why it is true for a spinor representation constructed from any representation of the Clifford algebra. Recall that

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) \qquad (4.32)$$

so the representation is unitary if $S^{\mu\nu}$ are anti-hermitian, i.e. $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$. But we have

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger] \qquad (4.33)$$

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which can be anti-hermitian if all $\gamma^\mu$ are hermitian or all are anti-hermitian. However, we can never arrange for this to happen since

$$(\gamma^0)^2 = 1 \Rightarrow \text{Real Eigenvalues}$$
$$(\gamma^i)^2 = -1 \Rightarrow \text{Imaginary Eigenvalues} \quad (4.34)$$

So we could pick $\gamma^0$ to be hermitian, but we can only pick $\gamma^i$ to be anti-hermitian. Indeed, in the chiral representation (4.16), the matrices have this property: $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. In general there is no way to pick $\gamma^\mu$ such that $S^{\mu\nu}$ are anti-hermitian.

## 4.2 Constructing an Action

We now have a new field to work with, the Dirac spinor $\psi$. We would like to construct a Lorentz invariant equation of motion. We do this by constructing a Lorentz invariant action.

We will start in a naive way which won't work, but will give us a clue how to proceed. Define

$$\psi^\dagger(x) = (\psi^\wedge)^T(x) \quad (4.35)$$

which is the usual adjoint of a multi-component object. We could then try to form a Lorentz scalar by taking the product $\psi^\dagger\psi$, with the spinor indices summed over. Let's see how this transforms under Lorentz transformations,

$$\psi(x) \to S[\Lambda]\,\psi(\Lambda^{-1}x)$$
$$\psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)\,S[\Lambda]^\dagger \quad (4.36)$$

So $\psi^\dagger(x)\psi(x) \to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger S[\Lambda]\psi(\Lambda^{-1}x)$. But, as we have seen, for some Lorentz transformation $S[\Lambda]^\dagger S[\Lambda]$ /= 1 since the representation is not unitary. This means that $\psi^\dagger\psi$ isn't going to do it for us: it doesn't have any nice transformation under the Lorentz group, and certainly isn't a scalar. But now we see why it fails, we can also see how to proceed. Let's pick a representation of the Clifford algebra which, like the chiral representation (4.16), satisfies $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. Then for all $\mu = 0$, 1, 2, 3 we have

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \quad (4.37)$$

which, in turn, means that

$$(S^{\mu\nu})^\dagger = \frac{1}{}\,[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger] = -\gamma^0 S^{\mu\nu}\gamma^0 \quad (4.38)$$



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so that

$$S[\Lambda]^\dagger = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right) = \gamma^0 S[\Lambda]^{-1}\gamma^0 \tag{4.39}$$

With this in mind, we now define the *Dirac adjoint*

$$\bar{\psi}(x) = \psi^\dagger(x)\,\gamma^0 \tag{4.40}$$

Let's now see what Lorentz covariant objects we can form out of a Dirac spinor $\psi$ and its adjoint $\bar{\psi}$.

**Claim 4.3:** $\bar{\psi}\psi$ is a Lorentz scalar.

**Proof:** Under a Lorentz transformation,

$$\begin{aligned}
\bar{\psi}(x)\,\psi(x) = \psi^\dagger(x)\,\gamma^0\,\psi(x) &\to \psi^\dagger(\Lambda^{-1}x)\,S[\Lambda]^\dagger\,\gamma^0\,S[\Lambda]\psi(\Lambda^{-1}x) \\
&= \psi^\dagger(\Lambda^{-1}x)\,\gamma^0\,\psi(\Lambda^{-1}x) \\
&= \bar{\psi}(\Lambda^{-1}x)\,\psi(\Lambda^{-1}x)
\end{aligned} \tag{4.41}$$

which is indeed the transformation law for a Lorentz scalar.

**Claim 4.4:** $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector, which means that

$$\bar{\psi}(x)\,\gamma^\mu\,\psi(x) \to \Lambda^\mu{}_\nu\bar{\psi}(\Lambda^{-1}x)\,\gamma^\nu\,\psi(\Lambda^{-1}x) \tag{4.42}$$

This equation means that we can treat the $\mu = 0, 1, 2, 3$ index on the $\gamma^\mu$ matrices as a true vector index. In particular we can form Lorentz scalars by contracting it with other Lorentz indices.

**Proof:** Suppressing the $x$ argument, under a Lorentz transformation we have,

$$\bar{\psi}\,\gamma^\mu\,\psi \to \bar{\psi}\,S[\Lambda]^{-1}\,\gamma^\mu\,S[\Lambda]\psi \tag{4.43}$$

If $\bar{\psi}\,\gamma^\mu\,\psi$ is to transform as a vector, we must have

$$S[\Lambda]^{-1}\,\gamma^\mu\,S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu \tag{4.44}$$

We'll now show this. We work infinitesimally, so that

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \approx 1 + \tfrac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma} + \ldots \tag{4.45}$$

$$S[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) \approx 1 + \tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \ldots \tag{4.46}$$

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so the requirement (4.44) becomes

$$-[S^{\rho\sigma}, \gamma^\mu] = (\mathcal{M}^{\rho\sigma})^\mu_{\ \nu} \gamma^\nu \tag{4.47}$$

where we've suppressed the $\alpha$, $\beta$ indices on $\gamma^\mu$ and $S^{\mu\nu}$, but otherwise left all other indices explicit. In fact equation (4.47) follows from Claim 4.1 where we showed that $[S^{\rho\sigma}, \gamma^\mu] = \gamma^\rho\eta^{\sigma\mu} - \gamma^\sigma\eta^{\mu\rho}$. To see this, we write the right-hand side of (4.47) by expanding out $\mathcal{M}$,

$$\begin{aligned}(\mathcal{M}^{\rho\sigma})^\mu_{\ \nu} \gamma^\nu &= (\eta^{\rho\mu}\delta^\sigma_{\ \nu} - \eta^{\sigma\mu}\delta^\rho_{\ \nu})\gamma^\nu \\ &= \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho\end{aligned} \tag{4.48}$$

which means that the proof follows if we can show

$$-[S^{\rho\sigma}, \gamma^\mu] = \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho \tag{4.49}$$

which is exactly what we proved in Claim 4.1.

**Claim 4.5:** $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz tensor. More precisely, the symmet- ric part is a Lorentz scalar, proportional to $\eta^{\mu\nu}\bar{\psi}\psi$, while the antisymmetric part is a Lorentz tensor, proportional to $\bar{\psi}S^{\mu\nu}\psi$.

**Proof:** As above.

We are now armed with three bilinears of the Dirac field, $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma^\nu\psi$, each of which transforms covariantly under the Lorentz group. We can try to build a Lorentz invariant action from these. In fact, we need only the first two. We choose

$$S = \int d^4x\, \bar{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x) \tag{4.50}$$

This is the Dirac action. The factor of "i" is there to make the action real; upon complex conjugation, it cancels a minus sign that comes from integration by parts. (Said another way, it's there for the same reason that the Hermitian momentum operator $-i\nabla$ in quantum mechanics has a factor $i$). As we will see in the next section, after quantization this theory describes particles and anti-particles of mass $|m|$ and spin 1/2. Notice that the Lagrangian is first order, rather than the second order Lagrangians we were working with for scalar fields. Also, the mass appears in the Lagrangian as $m$, which can be positive or negative.

The equation of motion follows from the action (4.50) by varying with respect to $\psi$ and $\bar{\psi}$ independently. Varying with respect to $\bar{\psi}$, we have

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{4.51}$$

This is the *Dirac equation*. It's completely gorgeous. Varying with respect to $\psi$ gives the conjugate equation

$$i\partial_\mu \bar{\psi}\,\gamma^\mu + m\bar{\psi} = 0 \tag{4.52}$$

The Dirac equation is first order in derivatives, yet miraculously Lorentz invariant. If we tried to write down a first order equation of motion for a scalar field, it would look like $v^\mu \partial_\mu \varphi = \ldots$, which necessarily includes a privileged vector in spacetime $v^\mu$ and is not Lorentz invariant. However, for spinor fields, the magic of the $\gamma^\mu$ matrices means that the Dirac Lagrangian is Lorentz invariant.

The Dirac equation mixes up different components of $\psi$ through the matrices $\gamma^\mu$. However, each individual component itself solves the Klein-Gordon equation. To see this, write

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\,\psi = -\,\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2\,\psi = 0 \tag{4.53}$$

But $\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = \partial_\mu \partial^\mu$, so we get

$$-(\partial_\mu \partial^\mu + m^2)\psi = 0 \tag{4.54}$$

where this last equation has no $\gamma^\mu$ matrices, and so applies to each component $\psi^\alpha$, with $\alpha = 1, 2, 3, 4$.

### The Slash

Let's introduce some useful notation. We will often come across 4-vectors contracted with $\gamma^\mu$ matrices. We write

$$A_\mu \gamma^\mu \equiv A\!\!\!/ \tag{4.55}$$

so the Dirac equation reads

$$(i\,\partial\!\!\!/ - m)\psi = 0 \tag{4.56}$$

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## 4.4 Chiral Spinors

When we've needed an explicit form of the $\gamma^\mu$ matrices, we've used the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4.57}$$

In this representation, the spinor rotation transformation $S[\Lambda_{\rm rot}]$ and boost transformation $S[\Lambda_{\rm boost}]$ were computed in (4.26) and (4.31). Both are block diagonal,

$$S[\Lambda_{\rm rot}] = \begin{pmatrix} e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix} \quad \text{and} \quad S[\Lambda_{\rm boost}] = \begin{pmatrix} e^{+\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{4.58}$$

This means that the Dirac spinor representation of the Lorentz group is *reducible*. It decomposes into two irreducible representations, acting only on two-component spinors $u_\pm$ which, in the chiral representation, are defined by

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{4.59}$$

The two-component objects $u_\pm$ are called *Weyl spinors* or *chiral spinors*. They transform in the same way under rotations,

$$u_\pm \rightarrow e^{i\vec{\varphi}\cdot\vec{\sigma}/2}\, u_\pm \tag{4.60}$$

but oppositely under boosts,

$$u_\pm \rightarrow e^{\pm\vec{\chi}\cdot\vec{\sigma}/2}\, u_\pm \tag{4.61}$$

In group theory language, $u_+$ is in the $(\frac{1}{2}, 0)$ representation of the Lorentz group, while $u_-$ is in the $(0, \frac{1}{2})$ representation. The Dirac spinor $\psi$ lies in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation. (Strictly speaking, the spinor is a representation of the double cover of the Lorentz group $SL(2, \mathbf{C})$).

### 4.4.1 The Weyl Equation

Let's see what becomes of the Dirac Lagrangian under the decomposition (4.59) into Weyl spinors. We have

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = iu_-^\dagger \sigma^\mu \partial_\mu u_- + iu_+^\dagger \bar{\sigma}^\mu \partial_\mu u_+ - m(u_+^\dagger u_- + u_-^\dagger u_+) = 0 \tag{4.62}$$

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where we have introduced some new notation for the Pauli matrices with a $\mu = 0, 1, 2, 3$ index,

$$\sigma^\mu = (1, \sigma^i) \quad \text{and} \quad \bar{\sigma}^\mu = (1, -\sigma^i) \tag{4.63}$$

From (4.62), we see that a massive fermion requires both $u_+$ and $u_-$, since they couple through the mass term. However, a massless fermion can be described by $u_+$ (or $u_-$) alone, with the equation of motion

$$i\bar{\sigma}^\mu \partial_\mu u_+ = 0$$
$$\text{or} \quad i\sigma^\mu \partial_\mu u_- = 0 \tag{4.64}$$

These are the *Weyl equations*.

### Degrees of Freedom

Let me comment here on the degrees of freedom in a spinor. The Dirac fermion has 4 complex components = 8 real components. How do we count degrees of freedom? In classical mechanics, the number of degrees of freedom of a system is equal to the dimension of the configuration space or, equivalently, half the dimension of the phase space. In field theory we have an infinite number of degrees of freedom, but it makes sense to count the number of degrees of freedom per spatial point: this should at least be finite. For example, in this sense a real scalar field $\varphi$ has a single degree of freedom. At the quantum level, this translates to the fact that it gives rise to a single type of particle. A classical complex scalar field has two degrees of freedom, corresponding to the particle and the anti-particle in the quantum theory.

But what about a Dirac spinor? One might think that there are 8 degrees of freedom. But this isn't right. Crucially, and in contrast to the scalar field, the equation of motion is first order rather than second order. In particular, for the Dirac Lagrangian, the momentum conjugate to the spinor $\psi$ is given by

$$\pi_\psi = \partial \mathcal{L}/\partial \dot{\psi} = i\psi^\dagger \tag{4.65}$$

It is not proportional to the time derivative of $\psi$. This means that the phase space for a spinor is therefore parameterized by $\psi$ and $\psi^\dagger$, while for a scalar it is parameterized by $\varphi$ and $\pi = \dot{\varphi}$. So the *phase space* of the Dirac spinor $\psi$ has 8 real dimensions and correspondingly the number of real degrees of freedom is 4. We will see in the next section that, in the quantum theory, this counting manifests itself as two degrees of freedom (spin up and down) for the particle, and a further two for the anti-particle.

A similar counting for the Weyl fermion tells us that it has two degrees of freedom.

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### 4.4.2 $\gamma^5$

The Lorentz group matrices $S[\Lambda]$ came out to be block diagonal in (4.58) because we chose the specific representation (4.57). In fact, this is why the representation (4.57) is called the chiral representation: it's because the decomposition of the Dirac spinor $\psi$ is simply given by (4.59). But what happens if we choose a different representation $\gamma^\mu$ of the Clifford algebra, so that

$$\gamma^\mu \to U\gamma^\mu U^{-1} \quad \text{and} \quad \psi \to U\psi \quad ? \tag{4.66}$$

Now $S[\Lambda]$ will not be block diagonal. Is there an invariant way to define chiral spinors? We can do this by introducing the "fifth" gamma-matrix

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4.67}$$

You can check that this matrix satisfies

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \text{and} \quad (\gamma^5)^2 = +1 \tag{4.68}$$

The reason that this is called $\gamma^5$ is because the set of matrices $\tilde{\gamma}^A = (\gamma^\mu, i\gamma^5)$, with $A = 0, 1, 2, 3, 4$ satisfy the $d = 4 + 1$ Clifford algebra $\{\tilde{\gamma}^A, \tilde{\gamma}^B\} = 2\eta^{AB}$. (You might think that $\gamma^4$ would be a better name! But $\gamma^5$ is the one everyone chooses - it's a more sensible name in Euclidean space, where $A = 1, 2, 3, 4, 5$). You can also check that $[S_{\mu\nu}, \gamma^5] = 0$, which means that $\gamma^5$ is a scalar under rotations and boosts. Since $(\gamma^5)^2 = 1$, this means we may form the Lorentz invariant projection operators

$$P_\pm = \frac{1}{2}\left(1 \pm \gamma^5\right) \tag{4.69}$$

such that $P_+^2 = P_+$ and $P_-^2 = P_-$ and $P_+P_- = 0$. One can check that for the chiral representation (4.57),

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.70}$$

from which we see that the operators $P_\pm$ project onto the Weyl spinors $u_\pm$. However, for an arbitrary representation of the Clifford algebra, we may use $\gamma^5$ to define the chiral spinors,

$$\psi_\pm = P_\pm\psi \tag{4.71}$$

which form the irreducible representations of the Lorentz group. $\psi_+$ is often called a "left-handed" spinor, while $\psi_-$ is "right-handed". The name comes from the way the spin precesses as a massless fermion moves: we'll see this in Section 4.7.2.

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### 4.4.3 Parity

The spinors $\psi_\pm$ are related to each other by *parity*. Let's pause to define this concept. The Lorentz group is defined by $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ such that

$$\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \eta^{\nu\sigma} = \eta^{\mu\rho} \tag{4.72}$$

So far we have only considered transformations $\Lambda$ which are continuously connected to the identity; these are the ones which have an infinitesimal form. However there are also two discrete symmetries which are part of the Lorentz group. They are

$$\begin{aligned} \text{Time Reversal} \quad T &: x^0 \to -x^0 \; ; \; x^i \to x^i \\ \text{Parity} \quad P &: x^0 \to x^0 \; ; \; x^i \to -x^i \end{aligned} \tag{4.73}$$

We won't discuss time reversal too much in this course. (It turns out to be represented by an anti-unitary transformation on states. See, for example the book by Peskin and Schroeder). But parity has an important role to play in the standard model and, in particular, the theory of the weak interaction.

Under parity, the left and right-handed spinors are exchanged. This follows from the transformation of the spinors under the Lorentz group. In the chiral representation, we saw that the rotation (4.60) and boost (4.61) transformations for the Weyl spinors $u_\pm$ are

$$u_\pm \xrightarrow{\text{rot}} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} u_\pm \qquad \text{and} \qquad u_\pm \xrightarrow{\text{boost}} e^{\pm\vec{\chi}\cdot\vec{\sigma}/2} u_\pm \tag{4.74}$$

Under parity, rotations don't change sign. But boosts do flip sign. This confirms that parity exchanges right-handed and left-handed spinors, $P : u_\pm \to u_\mp$, or in the notation $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi$, we have

$$P : \psi_\pm(\vec{x}, t) \to \psi_\mp(-\vec{x}, t) \tag{4.75}$$

Using this knowledge of how chiral spinors transform, and the fact that $P^2 = 1$, we see that the action of parity on the Dirac spinor itself can be written as

$$P : \psi(\vec{x}, t) \to \gamma^0 \psi(-\vec{x}, t) \tag{4.76}$$

Notice that if $\psi(\vec{x}, t)$ satisfies the Dirac equation, then the parity transformed spinor $\gamma^0\psi(-\vec{x}, t)$ also satisfies the Dirac equation, meaning

$$(i\gamma^0\partial_t + i\gamma^i\partial_i - m)\gamma^0\psi(-\vec{x}, t) = \gamma^0(i\gamma^0\partial_t - i\gamma^i\partial_i - m)\psi(-\vec{x}, t) = 0 \tag{4.77}$$

where the extra minus sign from passing $\gamma^0$ through $\gamma^i$ is compensated by the derivative acting on $-\vec{x}$ instead of $+\vec{x}$.

### 4.4.4 Chiral Interactions

Let's now look at how our interaction terms change under parity. We can look at each of our spinor bilinears from which we built the action,

$$P: \bar{\psi}\psi(\vec{x}, t) \rightarrow \bar{\psi}\psi(-\vec{x}, t) \tag{4.78}$$

which is the transformation of a scalar. For the vector $\bar{\psi}\gamma^\mu\psi$, we can look at the temporal and spatial components separately,

$$\begin{aligned} P: \bar{\psi}\gamma^0\psi(\vec{x}, t) &\rightarrow \bar{\psi}\gamma^0\psi(-\vec{x}, t) \\ P: \bar{\psi}\gamma^i\psi(\vec{x}, t) &\rightarrow \bar{\psi}\gamma^0\gamma^i\gamma^0\psi(-\vec{x}, t) = -\bar{\psi}\gamma^i\psi(-\vec{x}, t) \end{aligned} \tag{4.79}$$

which tells us that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector, with the spatial part changing sign. You can also check that $\bar{\psi}S^{\mu\nu}\psi$ transforms as a suitable tensor.

However, now we've discovered the existence of $\gamma^5$, we can form another Lorentz scalar and another Lorentz vector,

$$\bar{\psi}\gamma^5\psi \quad \text{and} \quad \bar{\psi}\gamma^5\gamma^\mu\psi \tag{4.80}$$

How do these transform under parity? We can check:

$$P: \bar{\psi}\gamma^5\psi(\vec{x}, t) \rightarrow \bar{\psi}\gamma^0\gamma^5\gamma^0\psi(-\vec{x}, t) = -\bar{\psi}\gamma^5\psi(-\vec{x}, t) \tag{4.81}$$

$$P: \bar{\psi}\gamma^5\gamma^\mu\psi(\vec{x}, t) \rightarrow \bar{\psi}\gamma^0\gamma^5\gamma^\mu\gamma^0\psi(-\vec{x}, t) = \begin{cases} -\bar{\psi}\gamma^5\gamma^0\psi(-\vec{x}, t) & \mu = 0 \\ +\bar{\psi}\gamma^5\gamma^i\psi(-\vec{x}, t) & \mu = i \end{cases}$$

which means that $\bar{\psi}\gamma^5\psi$ transforms as a *pseudoscalar*, while $\bar{\psi}\gamma^5\gamma^\mu\psi$ transforms as an *axial vector*. To summarize, we have the following spinor bilinears,

$$\begin{aligned} \bar{\psi}\psi &: \quad \text{scalar} \\ \bar{\psi}\gamma^\mu\psi &: \quad \text{vector} \\ \bar{\psi}S^{\mu\nu}\psi &: \quad \text{tensor} \\ \bar{\psi}\gamma^5\psi &: \quad \text{pseudoscalar} \\ \bar{\psi}\gamma^5\gamma^\mu\psi &: \quad \text{axial vector} \end{aligned} \tag{4.82}$$

The total number of bilinears is $1 + 4 + (4 \times 3/2) + 4 + 1 = 16$ which is all we could hope for from a 4-component object.

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We're now armed with new terms involving $\gamma^5$ that we can start to add to our Lagrangian to construct new theories. Typically such terms will break parity invariance of the theory, although this is not always true. (For example, the term $\varphi\bar{\psi}\gamma^5\psi$ doesn't break parity if $\varphi$ is itself a pseudoscalar). Nature makes use of these parity violating interactions by using $\gamma^5$ in the weak force. A theory which treats $\psi_\pm$ on an equal footing is called a *vector-like theory*. A theory in which $\psi_+$ and $\psi_-$ appear differently is called a *chiral theory*.

### 4.5 Majorana Fermions

Our spinor $\psi^\alpha$ is a complex object. It has to be because the representation $S[\Lambda]$ is typically also complex. This means that if we were to try to make $\psi$ real, for example by imposing $\psi = \psi^*$, then it wouldn't stay that way once we make a Lorentz transformation. However, there is a way to impose a reality condition on the Dirac spinor $\psi$. To motivate this possibility, it's simplest to look at a novel basis for the Clifford algebra, known as the *Majorana basis*.

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

These matrices satisfy the Clifford algebra. What is special about them is that they are all pure imaginary $(\gamma^\mu)^* = -\gamma^\mu$. This means that the generators of the Lorentz group $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, and hence the matrices $S[\Lambda]$ are real. So with this basis of the Clifford algebra, we can work with a real spinor simply by imposing the condition,

$$\psi = \psi^* \tag{4.83}$$

which is preserved under Lorentz transformation. Such spinors are called *Majorana spinors*.

So what's the story if we use a general basis for the Clifford algebra? We'll ask only that the basis satisfies $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. We then define the *charge conjugate* of a Dirac spinor $\psi$ as

$$\psi^{(c)} = C\psi^* \tag{4.84}$$

Here $C$ is a $4 \times 4$ matrix satisfying

$$C^\dagger C = 1 \quad \text{and} \quad C^\dagger \gamma^\mu C = -(\gamma^\mu)^* \tag{4.85}$$

Let's firstly check that (4.84) is a good definition, meaning that $\psi^{(c)}$ transforms nicely under a Lorentz transformation. We have

$$\psi^{(c)} \rightarrow CS[\Lambda]^*\psi^* = S[\Lambda]C\psi^* = S[\Lambda]\psi^{(c)} \tag{4.86}$$

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where we've made use of the properties (4.85) in taking the matrix $C$ through $S[\Lambda]^\star$. In fact, not only does $\psi^{(c)}$ transform nicely under the Lorentz group, but if $\psi$ satisfies the Dirac equation, then $\psi^{(c)}$ does too. This follows from,

$$(i\partial\!\!\!/ - m)\psi = 0 \;\Rightarrow\; (-i\partial\!\!\!/^\star - m)\psi^\star = 0$$
$$\Rightarrow\; C(-i\partial\!\!\!/^\star - m)\psi^\star = (+i\partial\!\!\!/ - m)\psi^{(c)} = 0$$

Finally, we can now impose the Lorentz invariant reality condition on the Dirac spinor, to yield a Majorana spinor,

$$\psi^{(c)} = \psi \tag{4.87}$$

After quantization, the Majorana spinor gives rise to a fermion that is its own anti-particle. This is exactly the same as in the case of scalar fields, where we've seen that a real scalar field gives rise to a spin 0 boson that is its own anti-particle. (Be aware: In many texts an extra factor of $\gamma^0$ is absorbed into the definition of $C$).

So what is this matrix $C$? Well, for a given representation of the Clifford algebra, it is something that we can find fairly easily. In the Majorana basis, where the gamma matrices are pure imaginary, we have simply $C_{\rm Maj} = 1$ and the Majorana condition $\psi = \psi^{(c)}$ becomes $\psi = \psi^\star$. In the chiral basis (4.16), only $\gamma^2$ is imaginary, and we may take $C_{\rm chiral} = i\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$. (The matrix $i\sigma^2$ that appears here is simply the anti-symmetric matrix $\epsilon^{\alpha\beta}$). It is interesting to see how the Majorana condition (4.87) looks in terms of the decomposition into left and right handed Weyl spinors (4.59). Plugging in the various definitions, we find that $u_+ = i\sigma^2 u_-^\star$ and $u_- = -i\sigma^2 u_+^\star$. In other words, a Majorana spinor can be written in terms of Weyl spinors as

$$\psi = \begin{pmatrix} u_+ \\ -i\sigma^2 u_+^\star \end{pmatrix} \tag{4.88}$$

Notice that it's not possible to impose the Majorana condition $\psi = \psi^{(c)}$ at the same time as the Weyl condition ($u_- = 0$ or $u_+ = 0$). Instead the Majorana condition relates $u_-$ and $u_+$.

**An Aside: Spinors in Different Dimensions:** The ability to impose Majorana or Weyl conditions on Dirac spinors depends on both the dimension and the signature of spacetime. One can always impose the Weyl condition on a spinor in even dimensional Minkowski space, basically because you can always build a suitable "$\gamma^5$" projection matrix by multiplying together all the other $\gamma$-matrices. The pattern for when the Majorana condition can be imposed is a little more sporadic. Interestingly, although the Majorana condition and Weyl condition cannot be imposed simultaneously in four dimensions, you can do this in Minowski spacetimes of dimension $2, 10, 18, \ldots$.

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The Dirac Lagrangian enjoys a number of symmetries. Here we list them and compute the associated conserved currents.

### Spacetime Translations

Under spacetime translations the spinor transforms as

$$\delta\psi = \epsilon^\mu \partial_\mu \psi \tag{4.89}$$

The Lagrangian depends on $\partial_\mu\psi$, but not $\partial_\mu\bar{\psi}$, so the standard formula (1.41) gives us the energy-momentum tensor

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L} \tag{4.90}$$

Since a current is conserved only when the equations of motion are obeyed, we don't lose anything by imposing the equations of motion already on $T^{\mu\nu}$. In the case of a scalar field this didn't really buy us anything because the equations of motion are second order in derivatives, while the energy-momentum is typically first order. However, for a spinor field the equations of motion are first order: $(i\partial\!\!\!/ - m)\psi = 0$. This means we can set $\mathcal{L} = 0$ in $T^{\mu\nu}$, leaving

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi \tag{4.91}$$

In particular, we have the total energy

$$E = \int d^3x\, T^{00} = \int d^3x\, i\bar{\psi}\gamma^0\dot{\psi} = \int d^3x\, \psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi \tag{4.92}$$

where, in the last equality, we have again used the equations of motion.

### Lorentz Transformations

Under an infinitesimal Lorentz transformation, the Dirac spinor transforms as (4.22) which, in infinitesimal form, reads

$$\delta\psi^\alpha = -\omega^\mu{}_\nu\, x^\nu\partial_\mu\psi^\alpha + \tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\psi^\beta \tag{4.93}$$

where, following (4.10), we have $\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$, and $M^{\rho\sigma}$ are the generators of the Lorentz algebra given by (4.8)

$$(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu \tag{4.94}$$

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which, after direct substitution, tells us that $\omega^{\mu\nu} = \Omega^{\mu\nu}$. So we get

$$\delta\psi^\alpha = -\omega^{\mu\nu}\left[x_\nu\partial_\mu\psi^\alpha - \tfrac{1}{2}(S_{\mu\nu})^\alpha{}_\beta\psi^\beta\right] \tag{4.95}$$

The conserved current arising from Lorentz transformations now follows from the same calculation we saw for the scalar field (1.54) with two differences: firstly, as we saw above, the spinor equations of motion set $\mathcal{L} = 0$; secondly, we pick up an extra piece in the current from the second term in (4.95). We have

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} + i\bar{\psi}\gamma^\mu S^{\rho\sigma}\psi \tag{4.96}$$

After quantization, when $(\mathcal{J}^\mu)^{\rho\sigma}$ is turned into an operator, this extra term will be responsible for providing the single particle states with internal angular momentum, telling us that the quantization of a Dirac spinor gives rise to a particle carrying spin 1/2.

### Internal Vector Symmetry

The Dirac Lagrangian is invariant under rotating the phase of the spinor, $\psi \to e^{-i\alpha}\psi$. This gives rise to the current

$$j^\mu_V = \bar{\psi}\gamma^\mu\psi \tag{4.97}$$

where "$V$" stands for *vector*, reflecting the fact that the left and right-handed components $\psi_\pm$ transform in the same way under this symmetry. We can easily check that $j^\mu_V$ is conserved under the equations of motion,

$$\partial_\mu j^\mu_V = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi) = im\bar{\psi}\psi - im\bar{\psi}\psi = 0 \tag{4.98}$$

where, in the last equality, we have used the equations of motion $i\partial\!\!\!/\,\psi = m\psi$ and $i\partial_\mu\bar{\psi}\gamma^\mu = -m\bar{\psi}$. The conserved quantity arising from this symmetry is

$$Q = \int d^3x\, \bar{\psi}\gamma^0\psi = \int d^3x\, \psi^\dagger\psi \tag{4.99}$$

We will see shortly that this has the interpretation of electric charge, or particle number, for fermions.

### Axial Symmetry

When $m = 0$, the Dirac Lagrangian admits an extra internal symmetry which rotates left and right-handed fermions in opposite directions,

$$\psi \to e^{i\alpha\gamma^5}\psi \quad \text{and} \quad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma^5} \tag{4.100}$$

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Here the second transformation follows from the first after noting that $e^{-i\alpha\gamma^5}\gamma^0 = \gamma^0 e^{+i\alpha\gamma^5}$. This gives the conserved current,

$$j^\mu_A = \bar\psi\gamma^\mu\gamma^5\psi \tag{4.101}$$

where $A$ is for "axial" since $j^\mu_A$ is an axial vector. This is conserved only when $m = 0$. Indeed, with the full Dirac Lagrangian we may compute

$$\partial_\mu j^\mu_A = (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi = 2im\bar\psi\gamma^5\psi \tag{4.102}$$

which vanishes only for $m = 0$. However, in the quantum theory things become more interesting for the axial current. When the theory is coupled to gauge fields (in a manner we will discuss in Section 6), the axial transformation remains a symmetry of the classical Lagrangian. But it doesn't survive the quantization process. It is the archetypal example of an *anomaly*: a symmetry of the classical theory that is not preserved in the quantum theory.

## 4.7 Plane Wave Solutions

Let's now study the solutions to the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{4.103}$$

We start by making a simple ansatz:

$$\psi = u(\vec p)\,e^{-ip\cdot x} \tag{4.104}$$

where $u(\vec p)$ is a four-component spinor, independent of spacetime $x$ which, as the notation suggests, can depend on the 3-momentum $\vec p$. The Dirac equation then becomes

$$(\gamma^\mu p_\mu - m)u(\vec p) = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar\sigma^\mu & -m \end{pmatrix} u(\vec p) = 0 \tag{4.105}$$

where we're again using the definition,

$$\sigma^\mu = (1, \sigma^i) \quad \text{and} \quad \bar\sigma^\mu = (1, -\sigma^i) \tag{4.106}$$

**Claim:** The solution to (4.105) is

$$u(\vec p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \tag{4.107}$$

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for any 2-component spinor $\xi$ which we will normalize to $\xi^\dagger \xi = 1$.

**Proof:** Let's write $u(\vec{p})^T = (u_1, u_2)$. Then equation (4.105) reads

$$(p \cdot \sigma)\, u_2 = m u_1 \quad \text{and} \quad (p \cdot \bar{\sigma})\, u_1 = m u_2 \tag{4.108}$$

Either one of these equations implies the other, a fact which follows from the identity $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_0^2 - p_i p_j \sigma^i \sigma^j = p_0^2 - p_i p_j \delta^{ij} = p_\mu p^\mu = m^2$. To start with, let's try the ansatz $u_1 = (p \cdot \sigma)\xi'$ for some spinor $\xi'$. Then the second equation in (4.108) immediately tells us that $u_2 = m\xi'$. So we learn that any spinor of the form

$$u(\vec{p}) = A \begin{pmatrix} (p \cdot \sigma)\, \xi' \\ m\xi' \end{pmatrix} \tag{4.109}$$

with constant $A$ is a solution to (4.105). To make this more symmetric, we choose $A = 1/m$ and $\xi' = \sqrt{p \cdot \bar{\sigma}}\, \xi$ with constant $\xi$. Then $u_1 = (p \cdot \sigma)\sqrt{p \cdot \bar{\sigma}}\, \xi = m\sqrt{p \cdot \sigma}\, \xi$. So we get the promised result (4.107)

### Negative Frequency Solutions

We get further solutions to the Dirac equation from the ansatz

$$\psi = v(\vec{p})\, e^{+ip \cdot x} \tag{4.110}$$

Solutions of the form (4.104), which oscillate in time as $\psi \sim e^{-iEt}$, are called positive frequency solutions. If we compute the energy of these solutions using (4.92), we find that it is positive. Those of the form (4.110), which oscillate as $\psi \sim e^{+iEt}$, are negative frequency solutions. Now if we compute the energy using (4.92), it is negative.

The Dirac equation requires that the 4-component spinor $v(\vec{p})$ satisfies

$$(\gamma^\mu p_\mu + m)\, v(\vec{p}) = \begin{pmatrix} m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & m \end{pmatrix} v(\vec{p}) = 0 \tag{4.111}$$

which is solved by

$$v(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \eta \\ -\sqrt{p \cdot \bar{\sigma}}\, \eta \end{pmatrix} \tag{4.112}$$

for some 2-component spinor $\eta$ which we take to be constant and normalized to $\eta^\dagger \eta = 1$.

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### 4.7.1 Some Examples

Consider the positive frequency solution with mass $m$ and 3-momentum $\vec{p} = 0$,

$$u(\vec{p}) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \tag{4.113}$$

where $\xi$ is any 2-component spinor. Spatial rotations of the field act on $\xi$ by (4.26),

$$\xi \to e^{+i\vec{\varphi}\cdot\vec{\sigma}/2}\,\xi \tag{4.114}$$

The 2-component spinor $\xi$ defines the *spin* of the field. This should be familiar from quantum mechanics. A field with spin up (down) along a given direction is described by the eigenvector of the corresponding Pauli matrix with eigenvalue +1 (-1 respectively). For example, $\xi^T = (1, 0)$ describes a field with spin up along the z-axis. After quantization, this will become the spin of the associated particle. In the rest of this section, we'll indulge in an abuse of terminology and refer to the classical solutions to the Dirac equations as "particles", even though they have no such interpretation before quantization.

Consider now boosting the particle with spin $\xi^T = (1, 0)$ along the $x^3$ direction, with $p^\mu = (E, 0, 0, p^3)$. The solution to the Dirac equation becomes

$$u(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\begin{pmatrix}1\\0\end{pmatrix}\\ \sqrt{p\cdot\bar{\sigma}}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix}\\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \tag{4.115}$$

In fact, this expression also makes sense for a massless field, for which $E = p^3$. (We picked the normalization (4.107) for the solutions so that this would be the case). For a massless particle we have

$$u(\vec{p}) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{4.116}$$

Similarly, for a boosted solution of the spin down $\xi^T = (0, 1)$ field, we have

$$u(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\begin{pmatrix}0\\1\end{pmatrix}\\ \sqrt{p\cdot\bar{\sigma}}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix}\\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \xrightarrow{m\to 0} \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{4.117}$$

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### 4.7.2 Helicity

The helicity operator is the projection of the angular momentum along the direction of momentum,

$$h = \frac{i}{2}\epsilon_{ijk}\,\hat{p}^i S^{jk} = \frac{1}{2}\hat{p}^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{4.118}$$

where $S^{ij}$ is the rotation generator given in (4.25). The massless field with spin $\xi^T = (1, 0)$ in (4.116) has helicity $h = 1/2$: we say that it is *right-handed*. Meanwhile, the field (4.117) has helicity $h = -1/2$: it is *left-handed*.

### 4.7.3 Some Useful Formulae: Inner and Outer Products

There are a number of identities that will be very useful in the following section, regarding the inner (and outer) products of the spinors $u(\vec{p})$ and $v(\vec{p})$. It's firstly convenient to introduce a basis $\xi^s$ and $\eta^s$, $s = 1, 2$ for the two-component spinors such that

$$\xi^{r\,\dagger}\xi^s = \delta^{rs} \quad \text{and} \quad \eta^{r\,\dagger}\eta^s = \delta^{rs} \tag{4.119}$$

for example,

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.120}$$

and similarly for $\eta^s$. Let's deal first with the positive frequency plane waves. The two independent solutions are now written as

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \tag{4.121}$$

We can take the inner product of four-component spinors in two different ways: either as $u^\dagger\cdot u$, or as $\bar{u}\cdot u$. Of course, only the latter will be Lorentz invariant, but it turns out that the former is needed when we come to quantize the theory. Here we state both:

$$\begin{aligned} u^{r\,\dagger}(\vec{p})\cdot u^s(\vec{p}) &= \left(\xi^{r\,\dagger}\sqrt{p\cdot\sigma},\ \xi^{r\,\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \\ &= \xi^{r\,\dagger}p\cdot\sigma\,\xi^s + \xi^{r\,\dagger}p\cdot\bar{\sigma}\,\xi^s = 2\xi^{r\,\dagger}p_0\xi^s = 2p_0\delta^{rs} \end{aligned} \tag{4.122}$$

while the Lorentz invariant inner product is

$$\bar{u}^r(\vec{p})\cdot u^s(\vec{p}) = \left(\xi^{r\,\dagger}\sqrt{p\cdot\sigma},\ \xi^{r\,\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} = 2m\delta^{rs} \tag{4.123}$$

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We have analogous results for the negative frequency solutions, which we may write as

$$v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix} \quad \text{with} \quad v^{r\dagger}(\vec{p})\cdot v^s(\vec{p}) = 2p_0\delta^{rs} \text{ and } \bar{v}^r(\vec{p})\cdot v^s(\vec{p}) = -2m\delta^{rs} \tag{4.124}$$

We can also compute the inner product between $u$ and $v$. We have

$$\begin{aligned}\bar{u}^r(\vec{p})\cdot v^s(\vec{p}) &= \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\ \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\gamma^0\begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix} \\ &= \xi^{r\dagger}\sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)}\,\eta^s - \xi^{r\dagger}\sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)}\,\eta^s = 0\end{aligned} \tag{4.125}$$

and similarly, $\bar{v}^r(\vec{p})\cdot u^s(\vec{p}) = 0$. However, when we come to $u^\dagger\cdot v$, it is a slightly different combination that has nice properties (and this same combination appears when we quantize the theory). We look at $u^{r\dagger}(\vec{p})\cdot v^s(-\vec{p})$, with the 3-momentum in the spinor $v$ taking the opposite sign. Defining the 4-momentum $(p')^\mu = (p^0, -\vec{p})$, we have

$$\begin{aligned}u^{r\dagger}(\vec{p})\cdot v^s(-\vec{p}) &= \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\ \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\begin{pmatrix} \sqrt{p'\cdot\sigma}\,\eta^s \\ -\sqrt{p'\cdot\bar{\sigma}}\,\eta^s \end{pmatrix} \\ &= \xi^{r\dagger}\sqrt{(p\cdot\sigma)(p'\cdot\sigma)}\,\eta^s - \xi^{r\dagger}\sqrt{(p\cdot\bar{\sigma})(p'\cdot\bar{\sigma})}\,\eta^s\end{aligned} \tag{4.126}$$

Now the terms under the square-root are given by $(p\cdot\sigma)(p'\cdot\sigma) = (p_0 + p_i\sigma^i)(p_0 - p_i\sigma^i) = p_0^2 - \vec{p}^{\,2} = m^2$. The same expression holds for $(p\cdot\bar{\sigma})(p'\cdot\bar{\sigma})$, and the two terms cancel. We learn

$$u^{r\dagger}(\vec{p})\cdot v^s(-\vec{p}) = v^{r\dagger}(\vec{p})\cdot u^s(-\vec{p}) = 0 \tag{4.127}$$

**Outer Products**

There's one last spinor identity that we need before we turn to the quantum theory. It is:

**Claim:**

$$\sum_{s=1}^{2} u^s(\vec{p})\,\bar{u}^s(\vec{p}) = \not{p} + m \tag{4.128}$$

where the two spinors are not now contracted, but instead placed back to back to give a $4\times 4$ matrix. Also,

$$\sum_{s=1}^{2} v^s(\vec{p})\,\bar{v}^s(\vec{p}) = \not{p} - m \tag{4.129}$$

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**Proof:**

$$\sum_{s=1}^{2} u^s(\vec{p})\,\bar{u}^s(\vec{p}) = \sum_{s=1}^{2} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \left( \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}},\ \xi^{s\dagger}\sqrt{p\cdot\sigma} \right) \tag{4.130}$$

But $\sum_s \xi^s \xi^{s\dagger} = \mathbf{1}$, the $2\times 2$ unit matrix, which then gives us

$$\sum_{s=1}^{2} u^s(\vec{p})\,\bar{u}^s(\vec{p}) = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} \tag{4.131}$$

which is the desired result. A similar proof works for $\sum_s v^s(\vec{p})\,\bar{v}^s(\vec{p})$.

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# 5. Quantizing the Dirac Field

We would now like to quantize the Dirac Lagrangian,

$$\mathcal{L} = \bar{\psi}(x) \left( i \partial\!\!\!/ - m \right) \psi(x) \tag{5.1}$$

We will proceed naively and treat $\psi$ as we did the scalar field. But we'll see that things go wrong and we will have to reconsider how to quantize this theory.

## 5.1 A Glimpse at the Spin-Statistics Theorem

We start in the usual way and define the momentum,

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = i\psi^\dagger \tag{5.2}$$

For the Dirac Lagrangian, the momentum conjugate to $\psi$ is $i\psi^\dagger$. It does not involve the time derivative of $\psi$. This is as it should be for an equation of motion that is first order in time, rather than second order. This is because we need only specify $\psi$ and $\psi^\dagger$ on an initial time slice to determine the full evolution.

To quantize the theory, we promote the field $\psi$ and its momentum $\psi^\dagger$ to operators, satisfying the canonical commutation relations, which read

$$[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = [\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = 0$$
$$[\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x} - \vec{y}) \tag{5.3}$$

It's this step that we'll soon have to reconsider.

Since we're dealing with a free theory, where any classical solution is a sum of plane waves, we may write the quantum operators as

$$\psi(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[ b^s_{\vec{p}}\, u^s(\vec{p})\, e^{+i\vec{p}\cdot\vec{x}} + c^{s\,\dagger}_{\vec{p}}\, v^s(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \right]$$
$$\psi^\dagger(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[ b^{s\,\dagger}_{\vec{p}}\, u^s(\vec{p})^\dagger\, e^{-i\vec{p}\cdot\vec{x}} + c^{s}_{\vec{p}}\, v^s(\vec{p})^\dagger\, e^{+i\vec{p}\cdot\vec{x}} \right] \tag{5.4}$$

where the operators $b^{s\,\dagger}_{\vec{p}}$ create particles associated to the spinors $u^s(\vec{p})$, while $c^{s\,\dagger}_{\vec{p}}$ create particles associated to $v^s(\vec{p})$. As with the scalars, the commutation relations of the fields imply commutation relations for the annihilation and creation operators.

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**Claim:** The field commutation relations (5.3) are equivalent to

$$[b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}] = (2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q})$$
$$[c^r_{\vec{p}}, c^{s\,\dagger}_{\vec{q}}] = -(2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}) \qquad (5.5)$$

with all other commutators vanishing.

Note the strange minus sign in the $[c, c^\dagger]$ term. This means that we can't define the ground state $|0\rangle$ as something annihilated by $c^r_{\vec{p}}|0\rangle = 0$, because then the excited states $c^{s\,\dagger}_{\vec{p}}|0\rangle$ would have negative norm. To avoid this, we will have to flip the interpretation of $c$ and $c^\dagger$, with the vacuum defined by $c^{s\,\dagger}_{\vec{p}}|0\rangle = 0$ and the excited states by $c^r_{\vec{p}}|0\rangle$. This, as we will see, will be our undoing.

**Proof:** Let's show that the $[b, b^\dagger]$ and $[c, c^\dagger]$ commutators reproduce the field commutators (5.3),

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \sum_{r,s}\int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\Big([b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}]\,u^r(\vec{p})u^s(\vec{q})^\dagger e^{i(\vec{x}\cdot\vec{p}-\vec{y}\cdot\vec{q})}$$
$$+\,[c^{r\,\dagger}_{\vec{p}}, c^s_{\vec{q}}]\,v^r(\vec{p})v^s(\vec{q})^\dagger e^{-i(\vec{x}\cdot\vec{p}-\vec{y}\cdot\vec{q})}\Big)$$
$$= \sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\Big(u^s(\vec{p})\bar{u}^s(\vec{p})\gamma^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v^s(\vec{p})\bar{v}^s(\vec{p})\gamma^0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big) \qquad (5.6)$$

At this stage we use the outer product formulae (4.128) and (4.129) which tell us $\sum_s u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m$ and $\sum_s v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m$, so that

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\Big((\not{p}+m)\gamma^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + (\not{p}-m)\gamma^0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big)$$
$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\Big((p_0\gamma^0 + p_i\gamma^i + m)\gamma^0 + (p_0\gamma^0 - p_i\gamma^i - m)\gamma^0\Big)e^{+i\vec{p}\cdot(\vec{x}-\vec{y})}$$

where, in the second term, we've changed $\vec{p} \to -\vec{p}$ under the integration sign. Now, using $p_0 = E_{\vec{p}}$ we have

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \int\frac{d^3p}{(2\pi)^3}\, e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^{(3)}(\vec{x}-\vec{y}) \qquad (5.7)$$

as promised. Notice that it's a little tricky in the middle there, making sure that the $p_i\gamma^i$ terms cancel. This was the reason we needed the minus sign in the $[c, c^\dagger]$ commutator terms in (5.5).

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### 5.1.1 The Hamiltonian

To proceed, let's construct the Hamiltonian for the theory. Using the momentum $\pi = i\psi^\dagger$, we have

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}\,(-i\gamma^i\partial_i + m)\psi \tag{5.8}$$

which means that $H = \int d^3x\, \mathcal{H}$ agrees with the conserved energy computed using Noether's theorem (4.92). We now wish to turn the Hamiltonian into an operator. Let's firstly look at

$$(-i\gamma^i\partial_i + m)\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}(-\gamma^i p_i + m)u^s(\vec{p})\,e^{+i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}(\gamma^i p_i + m)v^s(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right]$$

where, for once we've left the sum over $s = 1, 2$ implicit. There's a small subtlety with the minus signs in deriving this equation that arises from the use of the Minkowski metric in contracting indices, so that $\vec{p}\cdot\vec{x} \equiv \sum_i x^i p^i = -x^i p_i$. Now we use the defining equations for the spinors $u^s(\vec{p})$ and $v^s(\vec{p})$ given in (4.105) and (4.111), to replace

$$(-\gamma^i p_i + m)u^s(\vec{p}) = \gamma^0 p_0\, u^s(\vec{p}) \quad\text{and}\quad (\gamma^i p_i + m)v^s(\vec{p}) = -\gamma^0 p_0\, v^s(\vec{p}) \tag{5.9}$$

so we can write

$$(-i\gamma^i\partial_i + m)\psi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\,\gamma^0\left[b^s_{\vec{p}}\,u^s(\vec{p})\,e^{+i\vec{p}\cdot\vec{x}} - c^{s\dagger}_{\vec{p}}\,v^s(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right] \tag{5.10}$$

We now use this to write the operator Hamiltonian

$$\begin{aligned}
H &= \int d^3x\,\psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi \\
&= \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}\sqrt{\frac{E_{\vec{p}}}{4E_{\vec{q}}}}\left[b^{r\dagger}_{\vec{q}}\,u^r(\vec{q})^\dagger e^{-i\vec{q}\cdot\vec{x}} + c^r_{\vec{q}}\,v^r(\vec{q})^\dagger e^{+i\vec{q}\cdot\vec{x}}\right]\cdot\left[b^s_{\vec{p}}\,u^s(\vec{p})\,e^{+i\vec{p}\cdot\vec{x}} - c^{s\dagger}_{\vec{p}}\,v^s(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\Big[b^{r\dagger}_{\vec{p}}b^s_{\vec{p}}[u^r(\vec{p})^\dagger\cdot u^s(\vec{p})] - c^r_{\vec{p}}c^{s\dagger}_{\vec{p}}[v^r(\vec{p})^\dagger\cdot v^s(\vec{p})] \\
&\qquad - b^{r\dagger}_{\vec{p}}c^{s\dagger}_{-\vec{p}}[u^r(\vec{p})^\dagger\cdot v^s(-\vec{p})] + c^r_{\vec{p}}b^s_{-\vec{p}}[v^r(\vec{p})^\dagger\cdot u^s(-\vec{p})]\Big]
\end{aligned}$$

where, in the last two terms we have relabelled $\vec{p} \to -\vec{p}$. We now use our inner product formulae (4.122), (4.124) and (4.127) which read

$$u^r(\vec{p})^\dagger\cdot u^s(\vec{p}) = v^r(\vec{p})^\dagger\cdot v^s(\vec{p}) = 2p_0\delta^{rs} \qquad\text{and}\qquad u^r(\vec{p})^\dagger\cdot v^s(-\vec{p}) = v^r(\vec{p})^\dagger\cdot u^s(-\vec{p}) = 0$$

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giving us

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left( b^{s\,\dagger}_{\vec p} b^s_{\vec p} - c^s_{\vec p} c^s_{\vec p} \right) \tag{5.11}$$

$$= \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left( b^{s\,\dagger}_{\vec p} b^s_{\vec p} - c^{s}_{\vec p} c^s_{\vec p} + (2\pi)^3 \delta^{(3)}(0) \right) \tag{5.12}$$

The $\delta^{(3)}$ term is familiar and easily dealt with by normal ordering. The $b^\dagger b$ term is familiar and we can check that $b^\dagger$ create positive energy states as expected,

$[H, b$ $_{\vec{p}}$ $_{\vec{p}}$

The minus sign in front of the $c^\dagger c$ term should make us nervous. If we think of $c^\dagger$ as creation operators then there's no problem since, using the commutation relation ([5.5](#)), we still find that $c^\dagger$ creates positive energy states,

$_{\vec{p}}$ $_{\vec{p}}$ [

However, as we noted after ([5.5](#)), these states have negative norm. To have a sensible Hilbert space, we need to interpret $c$ as the creation operator. But then the Hamiltonian is not bounded below because

$$[H, c^s_{\vec{p}}] = -E_{\vec{p}}\, c^s_{\vec{p}}$$

This is a disaster. Taken seriously it would tell us that we could tumble to states of lower and lower energy by continually producing $c$ particles. As the English would say, it's all gone a bit Pete Tong. (No relation).

Since the above calculation was a little tricky, you might think that it's possible to rescue the theory to get the minus signs to work out right. You can play around with different things, but you'll always find this minus sign cropping up somewhere. And, in fact, it's telling us something important that we missed.

## 5.2 Fermionic Quantization

The key piece of physics that we missed is that spin 1/2 particles are fermions, meaning that they obey Fermi-Dirac statistics with the quantum state picking up a minus sign upon the

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interchange of any two particles. This fact is embedded into the structure of relativistic quantum field theory: the spin-statistics theorem says that integer spin fields must be quantized as bosons, while half-integer spin fields must be quantized as fermions. Any attempt to do otherwise will lead to an inconsistency, such as the unbounded Hamiltonian we saw in (5.12).

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So how do we go about quantizing a field as a fermion? Recall that when we quantized the scalar field, the resulting particles obeyed bosonic statistics because the creation and annihilation operators satisfied the commutation relations,

$$[a^\dagger_{\vec p}, a^\dagger_{\vec q}] = 0 \quad \Rightarrow \quad a^\dagger_{\vec p} a^\dagger_{\vec q}\, |0\rangle \equiv |\vec p, \vec q\,\rangle = |\vec q, \vec p\,\rangle \tag{5.13}$$

To have states obeying fermionic statistics, we need anti-commutation relations, $\{A, B\} \equiv AB + BA$. Rather than (5.3), we will ask that the spinor fields satisfy

$$\begin{aligned} \{\psi_\alpha(\vec x), \psi_\beta(\vec y)\} &= \{\psi^\dagger_\alpha(\vec x), \psi^\dagger_\beta(\vec y)\} = 0 \\ \{\psi_\alpha(\vec x), \psi^\dagger_\beta(\vec y)\} &= \delta_{\alpha\beta}\, \delta^{(3)}(\vec x - \vec y) \end{aligned} \tag{5.14}$$

We still have the expansion (5.4) of $\psi$ and $\psi^\dagger$ in terms of $b$, $b^\dagger$, $c$ and $c^\dagger$. But now the same proof that led us to (5.5) tells us that

$$\begin{aligned} \{b^r_{\vec p}, b^{s\dagger}_{\vec q}\} &= (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec p - \vec q) \\ \{c^r_{\vec p}, c^{s\dagger}_{\vec q}\} &= (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec p - \vec q) \end{aligned} \tag{5.15}$$

with all other *anti-commutators* vanishing,

$$\{b^r_{\vec p}, b^s_{\vec q}\} = \{c^r_{\vec p}, c^s_{\vec q}\} = \{b^r_{\vec p}, c^{s\dagger}_{\vec q}\} = \{b^r_{\vec p}, c^s_{\vec q}\} = \ldots = 0 \tag{5.16}$$

The calculation of the Hamiltonian proceeds as before, all the way through to the penultimate line (5.11). At that stage, we get

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left[ b^{s\dagger}_{\vec p} b^s_{\vec p} - c^s_{\vec p} c^{s\dagger}_{\vec p} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left[ b^{s\dagger}_{\vec p} b^s_{\vec p} + c^{s\dagger}_{\vec p} c^s_{\vec p} - (2\pi)^3 \delta^{(3)}(0) \right] \end{aligned} \tag{5.17}$$

The anti-commutators have saved us from the indignity of a Hamiltonian unbounded below. Note that when normal ordering the Hamiltonian we now throw away a negative contribution $-(2\pi)^3\delta^{(3)}(0)$. In principle, this could partially cancel the positive contribution from bosonic fields. Cosmological constant problem anyone?!

### 5.2.1 Fermi-Dirac Statistics

Just as in the bosonic case, we define the vacuum $|0\rangle$ to satisfy,

$$b^s_{\vec p}\, |0\rangle = c^s_{\vec p}\, |0\rangle = 0 \tag{5.18}$$

Although $b$ and $c$ obey anti-commutation relations, the Hamiltonian (5.17) has nice commutation relations with them. You can check that

$$[H, b^r_{\vec{p}}] = -E_{\vec{p}}\, b^r_{\vec{p}} \quad \text{and} \quad [H, b^{r\,\dagger}_{\vec{p}}] = E_{\vec{p}}\, b^{r\,\dagger}_{\vec{p}}$$
$$[H, c^r_{\vec{p}}] = -E_{\vec{p}}\, c^r_{\vec{p}} \quad \text{and} \quad [H, c^{r\,\dagger}_{\vec{p}}] = E_{\vec{p}}\, c^{r\,\dagger}_{\vec{p}} \tag{5.19}$$

This means that we can again construct a tower of energy eigenstates by acting on the vacuum by $b^{r\,\dagger}_{\vec{p}}$ and $c^{r,\dagger}_{\vec{p}}$ to create particles and antiparticles, just as in the bosonic case.

For example, we have the one-particle states

$$|\vec{p}, r\rangle = b^{r\,\dagger}_{\vec{p}}|0\rangle \tag{5.20}$$

The two particle states now satisfy

$$|\vec{p}_1, r_1; \vec{p}_2, r_2\rangle \equiv b^{r_1\,\dagger}_{\vec{p}_1} b^{r_2\,\dagger}_{\vec{p}_2}|0\rangle = -|\vec{p}_2, r_2; \vec{p}_1, r_1\rangle \tag{5.21}$$

confirming that the particles do indeed obey Fermi-Dirac statistics. In particular, we have the Pauli-Exclusion principle $|\vec{p}, r; \vec{p}, r\rangle = 0$. Finally, if we wanted to be sure about the spin of the particle, we could act with the angular momentum operator (4.96) to confirm that a stationary particle $|\vec{p} = 0, r\rangle$ does indeed carry intrinsic angular momentum 1/2 as expected.

### 5.3 Dirac's Hole Interpretation

> "In this attempt, the success seems to have been on the side of Dirac rather than logic"
>
> *Pauli on Dirac*

Let's pause our discussion to make a small historical detour. Dirac originally viewed his equation as a relativistic version of the Schrödinger equation, with $\psi$ interpreted as the wavefunction for a single particle with spin. To reinforce this interpretation, he wrote $(i\partial\!\!\!/ - m)\psi = 0$ as

$$i\frac{\partial\psi}{\partial t} = -i\,\vec{\alpha}\cdot\vec{\nabla}\psi + m\beta\psi \equiv \hat{H}\psi \tag{5.22}$$

where $\vec{\alpha} = -\gamma^0\vec{\gamma}$ and $\beta = \gamma^0$. Here the operator $\hat{H}$ is interpreted as the one-particle Hamiltonian. This is a very different viewpoint from the one we now have, where $\psi$ is a classical field that should be quantized. In Dirac's view, the Hamiltonian of the system is $\hat{H}$ defined above, while for us the Hamiltonian is the field operator (5.17).

Let's see where Dirac's viewpoint leads.

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With the interpretation of $\psi$ as a single-particle wavefunction, the plane-wave solutions (4.104) and (4.110) to the Dirac equation are thought of as energy eigenstates, with

$$\begin{aligned} \psi = u(\vec{p})\, e^{-ip\cdot x} \quad &\Rightarrow \quad i\frac{\partial\psi}{\partial t} = E_{\vec{p}}\,\psi \\ \psi = v(\vec{p})\, e^{+ip\cdot x} \quad &\Rightarrow \quad i\frac{\partial\psi}{\partial t} = -E_{\vec{p}}\,\psi \end{aligned} \tag{5.23}$$

which look like positive and negative energy solutions. The spectrum is once again unbounded below; there are states $v(\vec{p})$ with arbitrarily low energy $-E_{\vec{p}}$. At first glance this is disastrous, just like the unbounded field theory Hamiltonian (5.12). Dirac postulated an ingenious solution to this problem: since the electrons are fermions (a fact which is put in by hand to Dirac's theory) they obey the Pauli-exclusion principle. So we could simply stipulate that in the true vacuum of the universe, all the negative energy states are filled. Only the positive energy states are accessible. These filled negative energy states are referred to as the *Dirac sea*. Although you might worry about the infinite negative charge of the vacuum, Dirac argued that only charge differences would be observable (a trick reminiscent of the normal ordering prescription we used for field operators).

Having avoided disaster by floating on an infinite sea comprised of occupied negative energy states, Dirac realized that his theory made a shocking prediction. Suppose that a negative energy state is excited to a positive energy state, leaving behind a hole. The hole would have all the properties of the electron, except it would carry positive charge. After flirting with the idea that it may be the proton, Dirac finally concluded that the hole is a new particle: the positron. Moreover, when a positron comes across an electron, the two can annihilate. Dirac had predicted anti-matter, one of the greatest achievements of theoretical physics. It took only a couple of years before the positron was discovered experimentally in 1932.

Although Dirac's physical insight led him to the right answer, we now understand that the interpretation of the Dirac spinor as a single-particle wavefunction is not really correct. For example, Dirac's argument for anti-matter relies crucially on the particles being fermions while, as we have seen already in this course, anti-particles exist for both fermions and bosons. What we really learn from Dirac's analysis is that there is no consistent way to interpret the Dirac equation as describing a single particle. It is instead to be thought of as a classical field which has only positive energy solutions because the Hamiltonian (4.92) is positive definite. Quantization of this field then gives rise to both particle and anti-particle excitations.

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This from Julian Schwinger:

> *"Until now, everyone thought that the Dirac equation referred directly to physical particles. Now, in field theory, we recognize that the equations refer to a sublevel. Experimentally we are concerned with particles, yet the old equations describe fields.... When you begin with field equations, you operate on a level where the particles are not there from the start. It is when you solve the field equations that you see the emergence of particles."*

### 5.4 Propagators

Let's now move to the Heisenberg picture. We define the spinors $\psi(\vec{x}, t)$ at every point in spacetime such that they satisfy the operator equation

$$\frac{\partial\psi}{\partial t} = i[H, \psi] \tag{5.24}$$

We solve this by the expansion

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[ b^s_{\vec{p}} u^s(\vec{p}) e^{-ip\cdot x} + c^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{+ip\cdot x} \right]$$

$$\psi^\dagger(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[ b^{s\dagger}_{\vec{p}} u^{s\dagger}(\vec{p}) e^{+ip\cdot x} + c^s_{\vec{p}} v^{s\dagger}(\vec{p}) e^{-ip\cdot x} \right] \tag{5.25}$$

Let's now look at the anti-commutators of these fields. We define the fermionic propagator to be

$$iS_{\alpha\beta} = \{\psi_\alpha(x), \bar{\psi}_\beta(y)\} \tag{5.26}$$

In what follows we will often drop the indices and simply write $iS(x-y) = \{\psi(x), \bar{\psi}(y)\}$, but you should remember that $S(x-y)$ is a 4×4 matrix. Inserting the expansion (5.25), we have

$$\begin{aligned} iS(x-y) &= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}} \Big[ \{b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\} u^s(\vec{p})\bar{u}^r(\vec{q}) e^{-i(p\cdot x - q\cdot y)} \\ &\qquad + \{c^{s\dagger}_{\vec{p}}, c^r_{\vec{q}}\} v^s(\vec{p})\bar{v}^r(\vec{q}) e^{+i(p\cdot x - q\cdot y)} \Big] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[ u^s(\vec{p})\bar{u}^s(\vec{p}) e^{-ip\cdot(x-y)} + v^s(\vec{p})\bar{v}^s(\vec{p}) e^{+ip\cdot(x-y)} \right] \\ &\quad \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \end{aligned}$$

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$$\not\mp \quad (p\!\!/ + m)e^{-ip\cdot(x-} \quad \not\mp (p\!\!/ - m)e^{+ip\cdot(x-} \tag{5.27}$$

$(2\pi)^3\ 2E$

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where to reach the final line we have used the outer product formulae (4.128) and (4.129). We can then write

$$iS(x-y) = (i\partial\!\!\!/_x + m)(D(x-y) - D(y-x)) \tag{5.28}$$

in terms of the propagator for a real scalar field $D(x-y)$ which, recall, can be written as (2.90)

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \tag{5.29}$$

Some comments:

- For spacelike separated points $(x-y)^2 < 0$, we have already seen that $D(x-y) - D(y-x) = 0$. In the bosonic theory, we made a big deal of this since it ensured that

$$[\varphi(x), \varphi(y)] = 0 \qquad (x-y)^2 < 0 \tag{5.30}$$

  outside the lightcone, which we trumpeted as proof that our theory was causal. However, for fermions we now have

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0 \qquad (x-y)^2 < 0 \tag{5.31}$$

  outside the lightcone. What happened to our precious causality? The best that we can say is that all our observables are bilinear in fermions, for example the Hamiltonian (5.17). These still commute outside the lightcone. The theory remains causal as long as fermionic operators are not observable. If you think this is a little weak, remember that no one has ever seen a physical measuring apparatus come back to minus itself when you rotate by 360 degrees!

- At least away from singularities, the propagator satisfies

$$(i\partial\!\!\!/_x - m)S(x-y) = 0 \tag{5.32}$$

  which follows from the fact that $(\partial\!\!\!/_x^{\,2} + m^2)D(x-y) = 0$ using the mass shell condition $p^2 = m^2$.

### 5.5 The Feynman Propagator

By a similar calculation to that above, we can determine the vacuum expectation value,

$$\begin{aligned}\langle 0|\, \psi_\alpha(x)\bar{\psi}_\beta(y)\, |0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (p\!\!\!/ + m)_{\alpha\beta}\, e^{-ip\cdot(x-y)} \\ \langle 0|\, \bar{\psi}_\beta(y)\psi_\alpha(x)\, |0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (p\!\!\!/ - m)_{\alpha\beta}\, e^{+ip\cdot(x-y)}\end{aligned} \tag{5.33}$$

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We now define the Feynman propagator $S_F(x-y)$, which is again a 4 × 4 matrix, as the time ordered product,

$$S_F(x-y) = \langle 0|\, T\psi(x)\bar{\psi}(y)\,|0\rangle \equiv \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle & x^0 > y^0 \\ \langle 0| - \bar{\psi}(y)\psi(x)\,|0\rangle & y^0 > x^0 \end{cases} \tag{5.34}$$

Notice the minus sign! It is necessary for Lorentz invariance. When $(x-y)^2 < 0$, there is no invariant way to determine whether $x^0 > y^0$ or $y^0 > x^0$. In this case the minus sign is necessary to make the two definitions agree since $\{\psi(x), \bar{\psi}(y)\} = 0$ outside the lightcone.

We have the 4-momentum integral representation for the Feynman propagator,

$$S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{\gamma\cdot p + m}{p^2 - m^2 + i\epsilon} \tag{5.35}$$

which satisfies $(i\partial\!\!\!/_x - m)S_F(x-y) = i\delta^{(4)}(x-y)$, so that $S_F$ is a Green's function for the Dirac operator.

The minus sign that we see in (5.34) also occurs for any string of operators inside a time ordered product $T(\ldots)$. While bosonic operators commute inside $T$, fermionic operators anti-commute. We have this same behaviour for normal ordered products as well, with fermionic operators obeying $:\psi_1\psi_2: = -:\psi_2\psi_1:$. With the understanding that all fermionic operators anti-commute inside $T$ and $::$, Wick's theorem proceeds just as in the bosonic case. We define the contraction

$$\overbrace{\psi(x)\bar{\psi}(y)} = T(\psi(x)\bar{\psi}(y)) - :\psi(x)\bar{\psi}(y): = S_F(x-y) \tag{5.36}$$

## 5.6 Yukawa Theory

The interaction between a Dirac fermion of mass $m$ and a real scalar field of mass $\mu$ is governed by the Yukawa theory,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{2}\mu^2\varphi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\varphi\bar{\psi}\psi \tag{5.37}$$

which is the proper version of the baby scalar Yukawa theory we looked at in Section 3. Couplings of this type appear in the standard model, between fermions and the Higgs boson. In that context, the fermions can be leptons (such as the electron) or quarks.

Yukawa originally proposed an interaction of this type as an effective theory of nuclear forces. With an eye to this, we will again refer to the $\varphi$ particles as mesons, and the $\psi$ particles as nucleons. Except, this time, the nucleons have spin. (This is still not a particularly realistic theory of nucleon interactions, not least because we're omitting isospin. Moreover, in Nature the relevant mesons are pions which are pseudoscalars, so a coupling of the form $\varphi\bar{\psi}\gamma^5\psi$ would be more appropriate. We'll turn to this briefly in Section 5.7.3).

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Note the dimensions of the various fields. We still have $[\phi] = 1$, but the kinetic terms require that $[\psi] = 3/2$. Thus, unlike in the case with only scalars, the coupling is dimensionless: $[\lambda] = 0$.

We'll proceed as we did in Section 3, firstly computing the amplitude of a particular scattering process then, with that calculation as a guide, writing down the Feynman rules for the theory. We start with:

### 5.6.1 An Example: Putting Spin on Nucleon Scattering

Let's study $\psi\psi \to \psi\psi$ scattering. This is the same calculation we performed in Section (3.3.3) except now the fermions have spin. Our initial and final states are

$$
\begin{aligned}
|i\rangle &= \sqrt{4E_{\vec p}E_{\vec q}}\, b^{s\dagger}_{\vec p}\, b^{r\dagger}_{\vec q}|0\rangle \equiv |\vec p, s; \vec q, r\rangle \\
|f\rangle &= \sqrt{4E_{\vec p\,'}E_{\vec q\,'}}\, b^{s'\dagger}_{\vec p\,'}\, b^{r'\dagger}_{\vec q\,'}|0\rangle \equiv |\vec p\,', s'; \vec q\,', r'\rangle
\end{aligned}
\tag{5.38}
$$

We need to be a little cautious about minus signs, because the $b^\dagger$'s now anti-commute. In particular, we should be careful when we take the adjoint. We have

$$
\langle f| = \sqrt{4E_{\vec p\,'}E_{\vec q\,'}}\,\langle 0|\, b^{r'}_{\vec q\,'}\, b^{s'}_{\vec p\,'}
\tag{5.39}
$$

We want to calculate the order $\lambda^2$ terms from the S-matrix element $\langle f|\, S - 1\, |i\rangle$.

$$
\frac{(-i\lambda)^2}{2}\int d^4x_1 d^4x_2\; T\Big(\bar\psi(x_1)\psi(x_1)\phi(x_1)\,\bar\psi(x_2)\psi(x_2)\phi(x_2)\Big)
\tag{5.40}
$$

where, as usual, all fields are in the interaction picture. Just as in the bosonic calculation, the contribution to nucleon scattering comes from the contraction

$$
:\bar\psi(x_1)\psi(x_1)\bar\psi(x_2)\psi(x_2):\;\overbrace{\phi(x_1)\,\phi(x_2)}
\tag{5.41}
$$

We just have to be careful about how the spinor indices are contracted. Let's start by looking at how the fermionic operators act on $|i\rangle$. We expand out the $\psi$ fields, leaving the $\bar\psi$ fields alone for now. We may ignore the $c^\dagger$ pieces in $\psi$ since they give no contribution at order $\lambda^2$. We have

$$
\begin{aligned}
:\bar\psi(x_1)\psi(x_1)\,\bar\psi(x_2)\psi(x_2):\; b^{s\dagger}_{\vec p}\, b^{r\dagger}_{\vec q}|0\rangle = \int \frac{d^3k_1\, d^3k_2}{(2\pi)^6}\, [\bar\psi(x_1)\cdot u^m(\vec k_1)]\,[\bar\psi(x_2)\cdot u^n(\vec k_2)] \\
\frac{e^{-ik_1\cdot x_1 - ik_2\cdot x_2}}{\sqrt{4E_{\vec k_1}E_{\vec k_2}}}\, b^m_{\vec k_1} b^n_{\vec k_2} b^{s\dagger}_{\vec p}\, b^{r\dagger}_{\vec q}|0\rangle
\end{aligned}
\tag{5.42}
$$

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where we've used square brackets $[\cdot]$ to show how the spinor indices are contracted. The minus sign that sits out front came from moving $\psi(x_1)$ past $\bar\psi(x_2)$. Now anti-commuting the $b$'s past the $b^\dagger$'s, we get

$$= \frac{1}{\sqrt{E_{\vec p}E_{\vec q}}}\Big([\bar\psi(x_1)\cdot u^r(\vec q)]\,[\bar\psi(x_2)\cdot u^s(\vec p)]\,e^{-ip\cdot x_2-iq\cdot x_1} - [\bar\psi(x_1)\cdot u^s(\vec p)]\,[\bar\psi(x_2)\cdot u^r(\vec q)]\,e^{-ip\cdot x_1-iq\cdot x_2}\Big)|0\rangle \quad (5.43)$$

Note, in particular, the relative minus sign that appears between these two terms. Now let's see what happens when we hit this with $\langle f|$. We look at

$$\langle 0|\, b^{r'}_{\vec q'} b^{s'}_{\vec p'}\, [\bar\psi(x_1)\cdot u^r(\vec q)]\,[\bar\psi(x_2)\cdot u^s(\vec p)]\,|0\rangle = \frac{e^{+ip'\cdot x_1+iq'\cdot x_2}}{\sqrt{E_{\vec p'}E_{\vec q'}}}[\bar u^{s'}(\vec p')\cdot u^r(\vec q)]\,[\bar u^{r'}(\vec q')\cdot u^s(\vec p)] - \frac{e^{+ip'\cdot x_2+iq'\cdot x_1}}{\sqrt{E_{\vec p'}E_{\vec q'}}}[\bar u^{r'}(\vec q')\cdot u^r(\vec q)]\,[\bar u^{s'}(\vec p')\cdot u^s(\vec p)]$$

The $[\bar\psi(x_1)\cdot u^s(\vec p)]\,[\bar\psi(x_2)\cdot u^r(\vec q)]$ term in (5.43) doubles up with this, cancelling the factor of 1/2 in front of (5.40). Meanwhile, the $1/\sqrt{E}$ terms cancel the relativistic state normalization. Putting everything together, we have the following expression for $\langle f|\,S-1\,|i\rangle$

$$(-i\lambda)^2\int\frac{d^4x_1\,d^4x_2\,d^4k}{(2\pi)^4}\,\frac{ie^{ik\cdot(x_1-x_2)}}{k^2-\mu^2+i\epsilon}\Big([\bar u^{s'}(\vec p')\cdot u^s(\vec p)]\,[\bar u^{r'}(\vec q')\cdot u^r(\vec q)]\,e^{+ix_1\cdot(q'-q)+ix_2\cdot(p'-p)} - [\bar u^{s'}(\vec p')\cdot u^r(\vec q)]\,[\bar u^{r'}(\vec q')\cdot u^s(\vec p)]\,e^{ix_1\cdot(p'-q)+ix_2\cdot(q'-p)}\Big)$$

where we've put the $\varphi$ propagator back in. Performing the integrals over $x_1$ and $x_2$, this becomes,

$$\int d^4k\,\frac{(2\pi)^4 i(-i\lambda)^2}{k^2-\mu^2+i\epsilon}\Big([\bar u^{s'}(\vec p')\cdot u^s(\vec p)]\,[\bar u^{r'}(\vec q')\cdot u^r(\vec q)]\,\delta^{(4)}(q'-q+k)\,\delta^{(4)}(p'-p-k) - [\bar u^{s'}(\vec p')\cdot u^r(\vec q)]\,[\bar u^{r'}(\vec q')\cdot u^s(\vec p)]\,\delta^{(4)}(p'-q+k)\,\delta^{(4)}(q'-p-k)\Big)$$

And we're almost there! Finally, writing the S-matrix element in terms of the amplitude in the usual way, $\langle f|\,S-1\,|i\rangle = i\mathcal{A}(2\pi)^4\delta^{(4)}(p+q-p'-q')$, we have

$$\mathcal{A} = (-i\lambda)^2\left(\frac{[\bar u^{s'}(\vec p')\cdot u^s(\vec p)]\,[\bar u^{r'}(\vec q')\cdot u^r(\vec q)]}{(p'-p)^2-\mu^2+i\epsilon} - \frac{[\bar u^{s'}(\vec p')\cdot u^r(\vec q)]\,[\bar u^{r'}(\vec q')\cdot u^s(\vec p)]}{(q'-p)^2-\mu^2+i\epsilon}\right)$$

which is our final answer for the amplitude.

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## 5.7 Feynman Rules for Fermions

It's important to bear in mind that the calculation we just did kind of blows. Thankfully the Feynman rules will once again encapsulate the combinatoric complexities and make life easier for us. The rules to compute amplitudes are the following

- To each incoming fermion with momentum $p$ and spin $r$, we associate a spinor $u^r(\vec{p})$. For outgoing fermions we associate $\bar{u}^r(\vec{p})$.

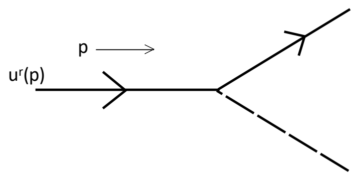



**Figure 21:** An incoming fermion

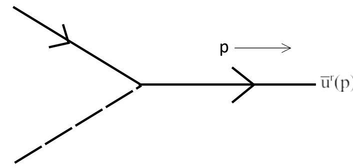



**Figure 22:** An outgoing fermion

- To each incoming anti-fermion with momentum $p$ and spin $r$, we associate a spinor $\bar{v}^r(\vec{p})$. For outgoing anti-fermions we associate $v^r(\vec{p})$.

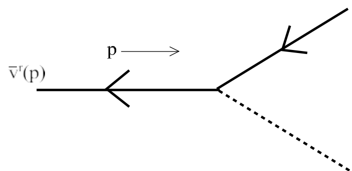



**Figure 23:** An incoming anti-fermion

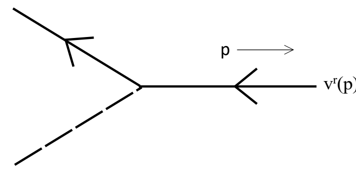
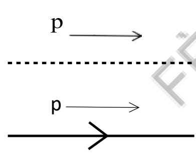



**Figure 24:** An outgoing anti-fermion

- Each vertex gets a factor of $-i\lambda$.
- Each internal line gets a factor of the relevant propagator.




$$\frac{i}{p^2-\mu^2+i\epsilon} \quad \text{for scalars}$$

$$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} \quad \text{for fermions} \tag{5.44}$$

The arrows on the fermion lines must flow consistently through the diagram (this ensures fermion number conservation). Note that the fermionic propagator is a 4×4 matrix. The matrix indices are contracted at each vertex, either with further propagators, or with external spinors $u$, $\bar{u}$, $v$ or $\bar{v}$.

- Impose momentum conservation at each vertex, and integrate over undetermined loop momenta.
- Add extra minus signs for statistics. Some examples will be given below.

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### 5.7.1 Examples

Let's run through the same examples we did for the scalar Yukawa theory. Firstly, we have

**Nucleon Scattering**

For the example we worked out previously, the two lowest order Feynman diagrams are shown in Figure 25. We've drawn the second Feynman diagram with the legs crossed

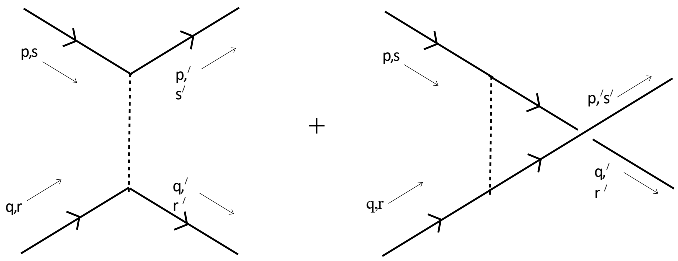


**Figure 25:** The two Feynman diagrams for nucleon scattering

to emphasize the fact that it picks up a minus sign due to statistics. (Note that the way the legs point in the Feynman diagram doesn't tell us the direction in which the particles leave the scattering event: the momentum label does that. The two diagrams above are different because the incoming legs are attached to different outgoing legs). Using the Feynman rules we can read off the amplitude.

$$\mathcal{A} = (-i\lambda)^2 \left( \frac{[\bar{u}^{s'}(\vec{p}\,') \cdot u^s(\vec{p})]\,[\bar{u}^{r'}(\vec{q}\,') \cdot u^r(\vec{q})]}{(p-p')^2-\mu^2} - \frac{[\bar{u}^{s'}(\vec{p}\,') \cdot u^r(\vec{q})]\,[\bar{u}^{r'}(\vec{q}\,') \cdot u^s(\vec{p})]}{(p-q')^2-\mu^2} \right) \tag{5.45}$$

The denominators in each term are due to the meson propagator, with the momentum determined by conservation at each vertex. This agrees with the amplitude we computed earlier using Wick's theorem.

**Nucleon to Meson Scattering**

Let's now look at $\psi\bar{\psi} \to \phi\phi$. The two lowest order Feynman diagrams are shown in Figure 26. Applying the Feynman rules, we have

$$\mathcal{A} = (-i\lambda)^2 \left( \frac{\bar{v}^r(\vec{q})[\gamma^\mu(p_\mu - p'_\mu) + m]u^s(\vec{p})}{(p-p')^2 - m^2} + \frac{\bar{v}^r(\vec{q})[\gamma^\mu(p_\mu - q'_\mu) + m]u^s(\vec{p})}{(p-q')^2 - m^2} \right)$$

Since the internal line is now a fermion, the propagator contains $\gamma_\mu(p^\mu - p'^\mu)+m$ factors. This is a $4 \times 4$ matrix which sits on the top, sandwiched between the two external spinors. Now the exchange statistics applies to the final meson states. These are bosons and, correspondingly, there is no relative minus sign between the two diagrams.

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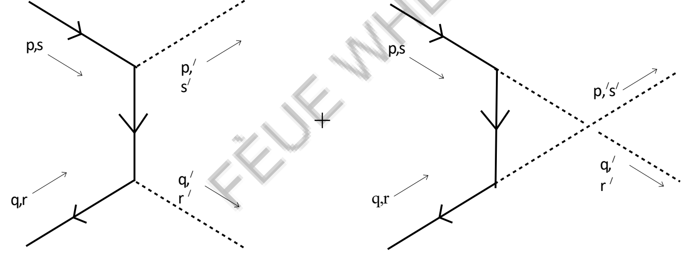



**Figure 26:** The two Feynman diagrams for nucleon to meson scattering

## Nucleon-Anti-Nucleon Scattering

For $\psi\bar{\psi} \to \psi\bar{\psi}$, the two lowest order Feynman diagrams are of two distinct types, just like in the bosonic case. They are shown in Figure 27.

The corresponding amplitude is given by,

$$\mathcal{A} = (-i\lambda)^2 \left( -\frac{[\bar{u}^{s'}(\vec{p}\,')\cdot u^{s}(\vec{p})]\,[\bar{v}^{r}(\vec{q})\cdot v^{r'}(\vec{q}\,')]}{(p-p')^2-\mu^2} + \frac{[\bar{v}^{r}(\vec{q})\cdot u^{s}(\vec{p})]\,[\bar{u}^{s'}(\vec{p}\,')\cdot v^{r'}(\vec{q}\,')]}{(p+q)^2-\mu^2+i\epsilon} \right) \tag{5.46}$$

As in the bosonic diagrams, there is again the difference in the momentum dependence in the denominator. But now the difference in the diagrams is also reflected in the spinor contractions in the numerator.

More subtle are the minus signs. The fermionic statistics mean that the first diagram has an extra minus sign relative to the $\psi\psi$ scattering of Figure 25. Since this minus sign will be important when we come to figure out whether the Yukawa force is attractive or repulsive, let's go back to basics and see where it comes from. The initial and final states for this scattering process are

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\; b^{s\,\dagger}_{\vec{p}}\, c^{r\,\dagger}_{\vec{q}}\,|0\rangle \equiv |\vec{p}, s; \vec{q}, r\rangle \\ |f\rangle &= \sqrt{4E_{\vec{p}\,'}E_{\vec{q}\,'}}\; b^{s'\,\dagger}_{\vec{p}\,'}\, c^{r'\,\dagger}_{\vec{q}\,'}\,|0\rangle \equiv |\vec{p}\,', s'; \vec{q}\,', r'\rangle \end{aligned} \tag{5.47}$$

The ordering of $b^\dagger$ and $c^\dagger$ in these states is crucial and reflects the scattering $\psi\bar{\psi} \to \psi\bar{\psi}$, as opposed to $\psi\bar{\psi} \to \bar{\psi}\psi$ which would differ by a minus sign. The first diagram in Figure 27 comes from the term in the perturbative expansion,

$$\langle f|:\bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2): b^{s\,\dagger}_{\vec{p}}\, c^{r\,\dagger}_{\vec{q}}\,|0\rangle \sim \langle f|\,[\bar{v}^m(\vec{k}_1)\cdot\psi(x_1)]\,[\bar{\psi}(x_2)\cdot u^n(\vec{k}_2)]\, c^m_{\vec{k}_1} b^n_{\vec{k}_2} b^{s\,\dagger}_{\vec{p}}\, c^{r\,\dagger}_{\vec{q}}\,|0\rangle$$

where we've neglected a bunch of objects in this equation like $\int d^4k_i$ and exponential factors because we only want to keep track of the minus signs. Moving the annihilation operators past the creation operators, we have

$$+\,\langle f|\,[\bar{v}^r(\vec{q})\cdot\psi(x_1)]\,[\bar{\psi}(x_2)\cdot u^s(\vec{p})]\,|0\rangle \tag{5.48}$$

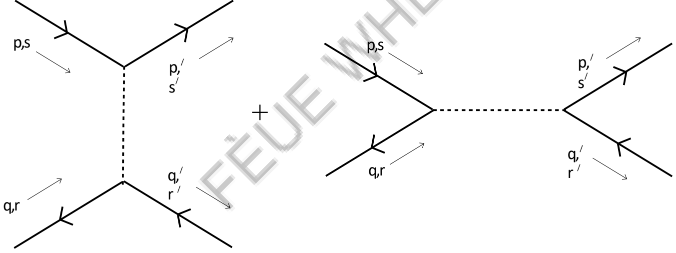



**Figure 27:** The two Feynman diagrams for nucleon-anti-nucleon scattering

Repeating the process by expanding out the $\psi(x_1)$ and $\bar\psi(x_2)$ fields and moving them to the left to annihilate $\langle f|$, we have

$$\langle 0|\, c_{\vec q\,'}^{r'} b_{\vec p\,'}^{s'} c_{\vec l_1}^{m\dagger} b_{\vec l_2}^{n\dagger}\, [\bar v^{m}(\vec l_1)\cdot v^r(\vec q)]\,[\bar u^{n}(\vec l_2)\cdot u^s(\vec p)]\,|0\rangle \sim -[\bar v^{r'}(\vec q\,')\cdot v^r(\vec q)]\,[\bar u^{s'}(\vec p\,')\cdot u^s(\vec p)]$$

where the minus sign has appeared from anti-commuting $c^{m\dagger}_{\vec l_1}$ past $b^{s'}_{\vec p\,'}$. This is the overall minus sign found in (5.46). One can also follow similar contractions to compute the second diagram in Figure 27.

### Meson Scattering

Finally, we can also compute the scattering of $\varphi\varphi \to \varphi\varphi$ which, as in the bosonic case, picks up its leading contribution at one-loop. The amplitude for the diagram shown in the figure is

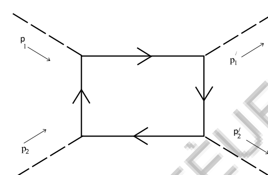
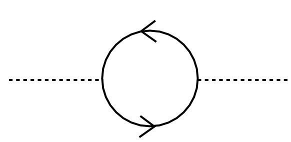



**Figure 28:**

$$i\mathcal{A} = -(-i\lambda)^4\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\,\frac{\not k+m}{(k^2-m^2+i\epsilon)}\,\frac{\not k+\not p_1'+m}{((k+p_1')^2-m^2+i\epsilon)}$$
$$\times\frac{\not k+\not p_1'-\not p_1+m}{((k+p_1'-p_1)^2-m^2+i\epsilon)}\,\frac{\not k-\not p_2'+m}{((k-p_2')^2-m^2+i\epsilon)}$$

Notice that the high momentum limit of the integral is $\int d^4k/k^4$, which is no longer finite, but diverges logarithmically. You will have to wait until next term to make sense of this integral.

There's an overall minus sign sitting in front of this amplitude. This is a generic feature of diagrams with fermions running in loops: each fermionic loop in a diagram gives rise to an extra minus sign. We can see this rather simply in the diagram



which involves the expression

$$\overline{\psi}_\alpha(x)\,\psi_\alpha(x)\overline{\psi}_\beta(y)\,\psi_\beta(y) = -\psi_\beta(y)\overline{\psi}_\alpha(x)\,\psi_\alpha(x)\overline{\psi}_\beta(y) = -\mathrm{Tr}\left(S_F(y-x)S_F(x-y)\right)$$

After passing the fermion fields through each other, a minus sign appears, sitting in front of the two propagators.

### 5.7.2 The Yukawa Potential Revisited

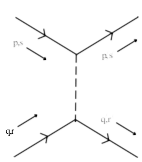
We saw in Section 3.5.2, that the exchange of a real scalar particle gives rise to a universally attractive Yukawa potential between two spin zero particles. Does the same hold for the spin 1/2 particles?

Recall that the strategy to compute the potential is to take the non-relativistic limit of the scattering amplitude, and compare with the analogous result from quantum mechanics. Our new amplitude now also includes the spinor degrees of freedom $u(\vec p)$ and $v(\vec p)$. In the non-relativistic limit, $p \to (m, \vec p)$, and

$$u(\vec p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \to \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$v(\vec p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \to \sqrt{m}\begin{pmatrix} \xi \\ -\xi \end{pmatrix} \tag{5.49}$$

In this limit, the spinor contractions in the amplitude for $\psi\psi \to \psi\psi$ scattering (5.45) become $\bar u^{s'} \cdot u^s = 2m\delta^{ss'}$ and the amplitude is



$$= -i(-i\lambda)^2\,(2m)\left(\frac{\delta^{s's}\delta^{r'r}}{(\vec p - \vec p\,')^2 + \mu^2} - \frac{\delta^{s'r}\delta^{r's}}{(\vec p - \vec q\,')^2 + \mu^2}\right) \tag{5.50}$$

The $\delta$ symbols tell us that spin is conserved in the non-relativistic limit, while the momentum dependence is the same as in the bosonic case, telling us that once again the particles feel an attractive Yukawa potential,

$$U(\vec r) = -\frac{\lambda^2 e^{-\mu r}}{4\pi r} \tag{5.51}$$

Repeating the calculation for $\psi\bar\psi \to \psi\bar\psi$, there are two minus signs which cancel each other. The first is the extra overall minus sign in the scattering amplitude (5.46), due to the fermionic nature of the particles. The second minus sign comes from the non-relativistic limit of the spinor contraction for anti-particles in (5.46), which is $\bar v^{s'} \cdot v^{s} = -2m\delta^{ss'}$. These two signs cancel, giving us once again an attractive Yukawa potential (5.51).

### 5.7.3 Pseudo-Scalar Coupling

Rather than the standard Yukawa coupling, we could instead consider

$$\mathcal{L}_{\text{Yuk}} = -\lambda\phi\bar\psi\gamma^5\psi \tag{5.52}$$

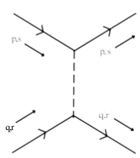
This still preserves parity if $\phi$ is a pseudoscalar, i.e.

$$P: \phi(\vec x, t) \to -\phi(-\vec x, t) \tag{5.53}$$

We can compute in this theory very simply: the Feynman rule for the interaction vertex is now changed to a factor of $-i\lambda\gamma^5$. For example, the Feynman diagrams for $\psi\psi \to \psi\psi$ scattering are again given by Figure 25, with the amplitude now

$$\mathcal{A} = (-i\lambda)^2 \left( \frac{[\bar u^{s'}(\vec p\,')\gamma^5 u^{s}(\vec p)]\,[\bar u^{r'}(\vec q\,')\gamma^5 u^{r}(\vec q)]}{(p-p')^2-\mu^2} - \frac{[\bar u^{s'}(\vec p\,')\gamma^5 u^{r}(\vec q)]\,[\bar u^{r'}(\vec q\,')\gamma^5 u^{s}(\vec p)]}{(p-q')^2-\mu^2} \right)$$

We could again try to take the non-relativistic limit for this amplitude. But this time, things work a little differently. Using the expressions for the spinors (5.49), we have $\bar u^{s'}\gamma^5 u^{s} \to 0$ in the non-relativistic limit. To find the non-relativistic amplitude, we must go to next to leading order. One can easily check that $\bar u^{s'}(\vec p\,')\gamma^5 u^{s}(\vec p) \to m\,\xi^{s'T}(\vec p - \vec p\,')\cdot\vec\sigma\,\xi^{s}$. So, in the non-relativistic limit, the leading order amplitude arising from pseudoscalar exchange is given by a spin-spin coupling,



$$\to +im(-i\lambda)^2\, \frac{[\xi^{s'T}(\vec p - \vec p\,')\cdot\vec\sigma\,\xi^{s}]\,[\xi^{r'T}(\vec p - \vec p\,')\cdot\vec\sigma\,\xi^{r}]}{(\vec p - \vec p\,')^2 + \mu^2} \tag{5.54}$$

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# 6. Quantum Electrodynamics

In this section we finally get to quantum electrodynamics (QED), the theory of light interacting with charged matter. Our path to quantization will be as before: we start with the free theory of the electromagnetic field and see how the quantum theory gives rise to a photon with two polarization states. We then describe how to couple the photon to fermions and to bosons.

## 6.1 Maxwell's Equations

The Lagrangian for Maxwell's equations in the absence of any sources is simply

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{6.1}$$

where the field strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.2}$$

The equations of motion which follow from this Lagrangian are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial_\mu F^{\mu\nu} = 0 \tag{6.3}$$

Meanwhile, from the definition of $F_{\mu\nu}$, the field strength also satisfies the Bianchi identity

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{6.4}$$

To make contact with the form of Maxwell's equations you learn about in high school, we need some 3-vector notation. If we define $A^\mu = (\varphi, \vec{A})$, then the electric field $\vec{E}$ and magnetic field $\vec{B}$ are defined by

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A} \tag{6.5}$$

which, in terms of $F_{\mu\nu}$, becomes

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{6.6}$$

The Bianchi identity (6.4) then gives two of Maxwell's equations,

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \tag{6.7}$$

These remain true even in the presence of electric sources. Meanwhile, the equations of motion give the remaining two Maxwell equations,

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B} \tag{6.8}$$

As we will see shortly, in the presence of charged matter these equations pick up extra terms on the right-hand side.

### 6.1.1 Gauge Symmetry

The massless vector field $A_\mu$ has 4 components, which would naively seem to tell us that the gauge field has 4 degrees of freedom. Yet we know that the photon has only two degrees of freedom which we call its polarization states. How are we going to resolve this discrepancy? There are two related comments which will ensure that quantizing the gauge field $A_\mu$ gives rise to 2 degrees of freedom, rather than 4.

- The field $A_0$ has no kinetic term $\dot{A}_0$ in the Lagrangian: it is not dynamical. This means that if we are given some initial data $A_i$ and $\dot{A}_i$ at a time $t_0$, then the field $A_0$ is fully determined by the equation of motion $\nabla \cdot \vec{E} = 0$ which, expanding out, reads

$$\nabla^2 A_0 + \nabla \cdot \frac{\partial \vec{A}}{\partial t} = 0 \tag{6.9}$$

This has the solution

$$A_0(\vec{x}) = \int d^3x' \, \frac{(\nabla \cdot \partial \vec{A}/\partial t)(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \tag{6.10}$$

So $A_0$ is not independent: we don't get to specify $A_0$ on the initial time slice. It looks like we have only 3 degrees of freedom in $A_\mu$ rather than 4. But this is still one too many.

- The Lagrangian (6.3) has a *very* large symmetry group, acting on the vector potential as

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x) \tag{6.11}$$

for any function $\lambda(x)$. We'll ask only that $\lambda(x)$ dies off suitably quickly at spatial $\vec{x} \to \infty$. We call this a *gauge symmetry*. The field strength is invariant under the gauge symmetry:

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu \lambda) - \partial_\nu(A_\mu + \partial_\mu \lambda) = F_{\mu\nu} \tag{6.12}$$

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So what are we to make of this? We have a theory with an infinite number of symmetries, one for each function $\lambda(x)$. Previously we only encountered symmetries which act the same at all points in spacetime, for example $\psi \to e^{i\alpha}\psi$ for a complex scalar field. Noether's theorem told us that these symmetries give rise to conservation laws. Do we now have an infinite number of conservation laws?

The answer is no! Gauge symmetries have a very different interpretation than the global symmetries that we make use of in Noether's theorem. While the latter take a physical state to another physical state with the same properties, the gauge symmetry is to be viewed as a redundancy in our description. That is, two states related by a gauge symmetry are to be identified: they are the same physical state. (There is a small caveat to this statement which is explained in Section 6.3.1). One way to see that this interpretation is necessary is to notice that Maxwell's equations are not sufficient to specify the evolution of $A_\mu$. The equations read,

$$[\eta_{\mu\nu}(\partial^\rho\partial_\rho) - \partial_\mu\partial_\nu]\, A^\nu = 0 \tag{6.13}$$

But the operator $[\eta_{\mu\nu}(\partial^\rho\partial_\rho) - \partial_\mu\partial_\nu]$ is not invertible: it annihilates any function of the form $\partial_\mu\lambda$. This means that given any initial data, we have no way to uniquely determine $A_\mu$ at a later time since we can't distinguish between $A_\mu$ and $A_\mu + \partial_\mu\lambda$. This would be problematic if we thought that $A_\mu$ is a physical object. However, if we're happy to identify $A_\mu$ and $A_\mu + \partial_\mu\lambda$ as corresponding to the same physical state, then our problems disappear.

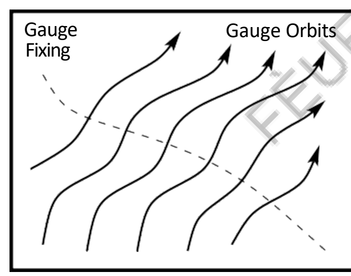


**Figure 29:**

Since gauge invariance is a redundancy of the system, we might try to formulate the theory purely in terms of the local, physical, gauge invariant objects $\vec{E}$ and $\vec{B}$. This is fine for the free classical theory: Maxwell's equations were, after all, first written in terms of $\vec{E}$ and $\vec{B}$. But it is not possible to describe certain quantum phenomena, such as the Aharonov-Bohm effect, without using the gauge potential $A_\mu$. We will see shortly that we also require the gauge potential to describe classically charged fields. To describe Nature, it appears that we have to introduce quantities $A_\mu$ that we can never measure.

The picture that emerges for the theory of electromagnetism is of an enlarged phase space, foliated by gauge orbits as shown in the figure. All states that lie along a given

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line can be reached by a gauge transformation and are identified. To make progress, we pick a representative from each gauge orbit. It doesn't matter which representative we pick — after all, they're all physically equivalent. But we should make sure that we pick a "good" gauge, in which we cut the orbits.

Different representative configurations of a physical state are called different *gauges*. There are many possibilities, some of which will be more useful in different situations. Picking a gauge is rather like picking coordinates that are adapted to a particular problem. Moreover, different gauges often reveal slightly different aspects of a problem. Here we'll look at two different gauges:

- **Lorentz Gauge:** $\partial_\mu A^\mu = 0$

  To see that we can always pick a representative configuration satisfying $\partial_\mu A^\mu = 0$, suppose that we're handed a gauge field ${}_\mu A^{r}$ satisfying $\partial_\mu (A^{r})^\mu = f(x)$. Then we choose $A_\mu = A^{r} + \partial_\mu \lambda$, where

  $$\partial_\mu \partial^\mu \lambda = -f \tag{6.14}$$

  This equation always has a solution. In fact this condition doesn't pick a unique representative from the gauge orbit. We're always free to make further gauge transformations with $\partial_\mu \partial^\mu \lambda = 0$, which also has non-trivial solutions. As the name suggests, the Lorentz gauge[3] has the advantage that it is Lorentz invariant.

- **Coulomb Gauge:** $\nabla \cdot \vec{A} = 0$

  We can make use of the residual gauge transformations in Lorentz gauge to pick $\nabla \cdot \vec{A} = 0$. (The argument is the same as before). Since $A_0$ is fixed by (6.10), we have as a consequence

  $$A_0 = 0 \tag{6.15}$$

  (This equation will no longer hold in Coulomb gauge in the presence of charged matter). Coulomb gauge breaks Lorentz invariance, so may not be ideal for some purposes. However, it is very useful to exhibit the physical degrees of freedom: the 3 components of $\vec{A}$ satisfy a single constraint: $\nabla \cdot \vec{A} = 0$, leaving behind just 2 degrees of freedom. These will be identified with the two polarization states of the photon. Coulomb gauge is sometimes called radiation gauge.

[3]Named after Lorenz who had the misfortune to be one letter away from greatness.

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## 6.2 The Quantization of the Electromagnetic Field

In the following we shall quantize free Maxwell theory twice: once in Coulomb gauge, and again in Lorentz gauge. We'll ultimately get the same answers and, along the way, see that each method comes with its own subtleties.

The first of these subtleties is common to both methods and comes when computing the momentum $\pi^\mu$ conjugate to $A_\mu$,

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$
$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} \equiv E^i \tag{6.16}$$

so the momentum $\pi^0$ conjugate to $A_0$ vanishes. This is the mathematical consequence of the statement we made above: $A_0$ is not a dynamical field. Meanwhile, the momentum conjugate to $A_i$ is our old friend, the electric field. We can compute the Hamiltonian,

$$H = \int d^3x\, \pi^i \dot{A}_i - \mathcal{L}$$
$$= \int d^3x\, \tfrac{1}{2}\vec{E}\cdot\vec{E} + \tfrac{1}{2}\vec{B}\cdot\vec{B} - A_0(\nabla\cdot\vec{E}) \tag{6.17}$$

So $A_0$ acts as a Lagrange multiplier which imposes Gauss' law

$$\nabla\cdot\vec{E} = 0 \tag{6.18}$$

which is now a constraint on the system in which $\vec{A}$ are the physical degrees of freedom. Let's now see how to treat this system using different gauge fixing conditions.

### 6.2.1 Coulomb Gauge

In Coulomb gauge, the equation of motion for $\vec{A}$ is

$$\partial_\mu \partial^\mu \vec{A} = 0 \tag{6.19}$$

which we can solve in the usual way,

$$\vec{A} = \int \frac{d^3p}{(2\pi)^3}\, \vec{\xi}(\vec{p})\, e^{ip\cdot x} \tag{6.20}$$

with $p_0^2 = |\vec{p}|^2$. The constraint $\nabla\cdot\vec{A} = 0$ tells us that $\vec{\xi}$ must satisfy

$$\vec{\xi}\cdot\vec{p} = 0 \tag{6.21}$$

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which means that $\vec\xi$ is perpendicular to the direction of motion $\vec p$. We can pick $\vec\xi(\vec p)$ to be a linear combination of two orthonormal vectors $\vec\epsilon_r$, $r = 1, 2$, each of which satisfies $\vec\epsilon_r(\vec p)\cdot\vec p = 0$ and

$$\vec\epsilon_r(\vec p)\cdot\vec\epsilon_s(\vec p) = \delta_{rs} \qquad r, s = 1, 2 \tag{6.22}$$

These two vectors correspond to the two polarization states of the photon. It's worth pointing out that you can't consistently pick a continuous basis of polarization vectors for every value of $\vec p$ because you can't comb the hair on a sphere. But this topological fact doesn't cause any complications in computing QED scattering processes.

To quantize we turn the Poisson brackets into commutators. Naively we would write

$$[A_i(\vec x), A_j(\vec y)] = [E^i(\vec x), E^j(\vec y)] = 0$$
$$[A_i(\vec x), E^j(\vec y)] = i\delta^j_i\,\delta^{(3)}(\vec x - \vec y) \tag{6.23}$$

But this can't quite be right, because it's not consistent with the constraints. We still want to have $\nabla\cdot\vec A = \nabla\cdot\vec E = 0$, now imposed on the operators. But from the commutator relations above, we see

$$[\nabla\cdot\vec A(\vec x), \nabla\cdot\vec E(\vec y)] = i\nabla^2\,\delta^{(3)}(\vec x - \vec y) \neq 0 \tag{6.24}$$

What's going on? In imposing the commutator relations (6.23) we haven't correctly taken into account the constraints. In fact, this is a problem already in the classical theory, where the Poisson bracket structure is already altered[4]. The correct Poisson bracket structure leads to an alteration of the last commutation relation,

$$[A_i(\vec x), E_j(\vec y)] = i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^{(3)}(\vec x - \vec y) \tag{6.25}$$

To see that this is now consistent with the constraints, we can rewrite the right-hand side of the commutator in momentum space,

$$[A_i(\vec x), E_j(\vec y)] = i\int\frac{d^3p}{(2\pi)^3}\left(\delta_{ij} - \frac{p_ip_j}{|\vec p|^2}\right)e^{i\vec p\cdot(\vec x - \vec y)} \tag{6.26}$$

which is now consistent with the constraints, for example

$$[\partial_iA_i(\vec x), E_j(\vec y)] = i\int\frac{d^3p}{(2\pi)^3}\left(\delta_{ij} - \frac{p_ip_j}{|\vec p|^2}\right)ip_i\,e^{i\vec p\cdot(\vec x - \vec y)} = 0 \tag{6.27}$$

[4]For a nice discussion of the classical and quantum dynamics of constrained systems, see the small book by Paul Dirac, "*Lectures on Quantum Mechanics*"

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We now write $\vec{A}$ in the usual mode expansion,

$$\vec{A}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_{r=1}^{2}\vec{\epsilon}_r(\vec{p})\left[a^r_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^{r\,\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$\vec{E}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{|\vec{p}|}{2}}\sum_{r=1}^{2}\vec{\epsilon}_r(\vec{p})\left[a^r_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^{r\,\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right] \tag{6.28}$$

where, as before, the polarization vectors satisfy

$$\vec{\epsilon}_r(\vec{p})\cdot\vec{p} = 0 \quad\quad \text{and} \quad\quad \vec{\epsilon}_r(\vec{p})\cdot\vec{\epsilon}_s(\vec{p}) = \delta_{rs} \tag{6.29}$$

It is not hard to show that the commutation relations (6.25) are equivalent to the usual commutation relations for the creation and annihilation operators,

$$[a^r_{\vec{p}}, a^s_{\vec{q}}] = [a^{r\,\dagger}_{\vec{p}}, a^{s\,\dagger}_{\vec{q}}] = 0$$

$$[a^r_{\vec{p}}, a^{s\,\dagger}_{\vec{q}}] = (2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}) \tag{6.30}$$

where, in deriving this, we need the completeness relation for the polarization vectors,

$$\sum_{r=1}^{2}\epsilon^i_r(\vec{p})\,\epsilon^j_r(\vec{p}) = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} \tag{6.31}$$

You can easily check that this equation is true by acting on both sides with a basis of vectors $(\vec{\epsilon}_1(\vec{p}), \vec{\epsilon}_2(\vec{p}), \vec{p})$.

We derive the Hamiltonian by substituting (6.28) into (6.17). The last term vanishes in Coulomb gauge. After normal ordering, and playing around with $\vec{\epsilon}_r$ polarization vectors, we get the simple expression

$$H = \int \frac{d^3p}{(2\pi)^3}\,|\vec{p}|\sum_{r=1}^{2} a^{r\,\dagger}_{\vec{p}}\, a^r_{\vec{p}} \tag{6.32}$$

The Coulomb gauge has the advantage that the physical degrees of freedom are manifest. However, we've lost all semblance of Lorentz invariance. One place where this manifests itself is in the propagator for the fields $A_i(x)$ (in the Heisenberg picture). In Coulomb gauge the propagator reads

$$D^{tr}_{ij}(x-y) \equiv \langle 0|\, T A_i(x) A_j(y)\, |0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}\left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}\right) e^{-ip\cdot(x-y)} \tag{6.33}$$

The *tr* superscript on the propagator refers to the "transverse" part of the photon. When we turn to the interacting theory, we will have to fight to massage this propagator into something a little nicer.

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### 6.2.2 Lorentz Gauge

We could try to work in a Lorentz invariant fashion by imposing the Lorentz gauge condition $\partial_\mu A^\mu = 0$. The equations of motion that follow from the action are then

$$\partial_\mu \partial^\mu A^\nu = 0 \tag{6.34}$$

Our approach to implementing Lorentz gauge will be a little different from the method we used in Coulomb gauge. We choose to change the theory so that (6.34) arises directly through the equations of motion. We can achieve this by taking the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 \tag{6.35}$$

The equations of motion coming from this action are

$$\partial_\mu F^{\mu\nu} + \partial^\nu (\partial_\mu A^\mu) = \partial_\mu \partial^\mu A^\nu = 0 \tag{6.36}$$

(In fact, we could be a little more general than this, and consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \tag{6.37}$$

with arbitrary $\alpha$ and reach similar conclusions. The quantization of the theory is independent of $\alpha$ and, rather confusingly, different choices of $\alpha$ are sometimes also referred to as different "gauges". We will use $\alpha = 1$, which is called "Feynman gauge". The other common choice, $\alpha = 0$, is called "Landau gauge".)

Our plan will be to quantize the theory (6.36), and only later impose the constraint $\partial_\mu A^\mu = 0$ in a suitable manner on the Hilbert space of the theory. As we'll see, we will also have to deal with the residual gauge symmetry of this theory which will prove a little tricky. At first, we can proceed very easily, because both $\pi^0$ and $\pi^i$ are dynamical:

$$\begin{aligned} \pi^0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu \\ \pi^i &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \partial^i A^0 - \dot{A}^i \end{aligned} \tag{6.38}$$

Turning these classical fields into operators, we can simply impose the usual commutation relations,

$$\begin{aligned} [A_\mu(\vec{x}), A_\nu(\vec{y})] &= [\pi^\mu(\vec{x}), \pi^\nu(\vec{y})] = 0 \\ [A_\mu(\vec{x}), \pi_\nu(\vec{y})] &= i\eta_{\mu\nu}\, \delta^{(3)}(\vec{x} - \vec{y}) \end{aligned} \tag{6.39}$$

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and we can make the usual expansion in terms of creation and annihilation operators and 4 polarization vectors $(\epsilon_\mu)^\lambda$, with $\lambda = 0, 1, 2, 3$.

$$A_\mu(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\vec{p}) \left[ a_{\vec{p}}^\lambda e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\lambda\,\dagger} e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\pi^\mu(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (+i) \sqrt{\frac{|\vec{p}|}{2}} \sum_{\lambda=0}^{3} (\epsilon^\mu)^\lambda \left[ a_{\vec{p}}^\lambda e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\lambda\,\dagger} e^{-i\vec{p}\cdot\vec{x}} \right] \tag{6.40}$$

Note that the momentum $\pi^\mu$ comes with a factor of $(+i)$, rather than the familiar $(-i)$ that we've seen so far. This can be traced to the fact that the momentum (6.38) for the classical fields takes the form $\pi^\mu = -\dot{A}^\mu + \ldots$ In the Heisenberg picture, it becomes clear that this descends to $(+i)$ in the definition of momentum.

There are now four polarization 4-vectors $\epsilon^\lambda(\vec{p})$, instead of the two polarization 3-vectors that we met in the Coulomb gauge. Of these four 4-vectors, we pick $\epsilon^0$ to be timelike, while $\epsilon^{1,2,3}$ are spacelike. We pick the normalization

$$\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'} \tag{6.41}$$

which also means that

$$(\epsilon_\mu)^\lambda (\epsilon_\nu)^{\lambda'} \eta_{\lambda\lambda'} = \eta_{\mu\nu} \tag{6.42}$$

The polarization vectors depend on the photon 4-momentum $p = (|\vec{p}|, \vec{p})$. Of the two spacelike polarizations, we will choose $\epsilon^1$ and $\epsilon^2$ to lie transverse to the momentum:

$$\epsilon^1 \cdot p = \epsilon^2 \cdot p = 0 \tag{6.43}$$

The third vector $\epsilon^3$ is the longitudinal polarization. For example, if the momentum lies along the $x^3$ direction, so $p \sim (1, 0, 0, 1)$, then

$$\epsilon^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.44}$$

For other 4-momenta, the polarization vectors are the appropriate Lorentz transformations of these vectors, since (6.43) are Lorentz invariant.

We do our usual trick, and translate the field commutation relations (6.39) into those for creation and annihilation operators. We find $[a_{\vec{p}}^\lambda, a_{\vec{q}}^{\lambda'}] = [a_{\vec{p}}^{\lambda\,\dagger}, a_{\vec{q}}^{\lambda'\,\dagger}] = 0$ and

$$[a_{\vec{p}}^\lambda, a_{\vec{q}}^{\lambda'\,\dagger}] = -\eta^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{6.45}$$

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The minus signs here are odd to say the least! For spacelike $\lambda = 1, 2, 3$, everything looks fine,

$$[a^{\lambda}_{\vec{p}}, a^{\lambda' \dagger}_{\vec{q}}] = \delta^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \quad \lambda, \lambda' = 1, 2, 3 \tag{6.46}$$

But for the timelike annihilation and creation operators, we have

$$[a^{0}_{\vec{p}}, a^{0 \dagger}_{\vec{q}}] = -(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{6.47}$$

This is very odd! To see just how strange this is, we take the Lorentz invariant vacuum $|0\rangle$ defined by

$$a^{\lambda}_{\vec{p}} |0\rangle = 0 \tag{6.48}$$

Then we can create one-particle states in the usual way,

$$|\vec{p}, \lambda\rangle = a^{\lambda \dagger}_{\vec{p}} |0\rangle \tag{6.49}$$

For spacelike polarization states, $\lambda = 1, 2, 3$, all seems well. But for the timelike polarization $\lambda = 0$, the state $|\vec{p}, 0\rangle$ has negative norm,

$$\langle \vec{p}, 0| \vec{q}, 0\rangle = \langle 0| a^{0}_{\vec{p}} a^{0\dagger}_{\vec{q}} |0\rangle = -(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{6.50}$$

Wtf? That's very very strange. A Hilbert space with negative norm means negative probabilities which makes no sense at all. We can trace this negative norm back to the wrong sign of the kinetic term for $A_0$ in our original Lagrangian: $\mathcal{L} = +\frac{1}{2}\dot{\vec{A}}^2 - \frac{1}{2}\dot{A}_0^2 + \ldots$

At this point we should remember our constraint equation, $\partial_\mu A^\mu = 0$, which, until now, we've not imposed on our theory. This is going to come to our rescue. We will see that it will remove the timelike, negative norm states, and cut the physical polarizations down to two. We work in the Heisenberg picture, so that

$$\partial_\mu A^\mu = 0 \tag{6.51}$$

makes sense as an operator equation. Then we could try implementing the constraint in the quantum theory in a number of different ways. Let's look at a number of increasingly weak ways to do this

- We could ask that $\partial_\mu A^\mu = 0$ is imposed as an equation on operators. But this can't possibly work because the commutation relations (6.39) won't be obeyed for $\pi^0 = -\partial_\mu A^\mu$. We need some weaker condition.

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- We could try to impose the condition on the Hilbert space instead of directly on the operators. After all, that's where the trouble lies! We could imagine that there's some way to split the Hilbert space up into good states $|\Psi\rangle$ and bad states that somehow decouple from the system. With luck, our bad states will include the weird negative norm states that we're so disgusted by. But how can we define the good states? One idea is to impose

$$\partial_\mu A^\mu |\Psi\rangle = 0 \tag{6.52}$$

on all good, physical states $|\Psi\rangle$. But this can't work either! Again, the condition is too strong. For example, suppose we decompose $A_\mu(x) = A^+_\mu(x) + A^-_\mu(x)$ with

$$A^+_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_{\lambda=0}^{3}\epsilon^\lambda_\mu a^\lambda_{\vec{p}}\, e^{-ip\cdot x}$$

$$A^-_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_{\lambda=0}^{3}\epsilon^\lambda_\mu a^{\lambda\dagger}_{\vec{p}}\, e^{+ip\cdot x} \tag{6.53}$$

Then, on the vacuum $A^+_\mu|0\rangle = 0$ automatically, but $\partial^\mu A^-_\mu|0\rangle \neq 1$. So not even the vacuum is a physical state if we use (6.52) as our constraint

- Our final attempt will be the correct one. In order to keep the vacuum as a good physical state, we can ask that physical states $|\Psi\rangle$ are defined by

$$\partial^\mu A^+_\mu |\Psi\rangle = 0 \tag{6.54}$$

This ensures that

$$\langle\Psi'|\,\partial_\mu A^\mu\,|\Psi\rangle = 0 \tag{6.55}$$

so that the operator $\partial_\mu A^\mu$ has vanishing matrix elements between physical states. Equation (6.54) is known as the *Gupta-Bleuler* condition. The linearity of the constraint means that the physical states $|\Psi\rangle$ span a physical Hilbert space $\mathcal{H}_{phys}$.

So what does the physical Hilbert space $\mathcal{H}_{phys}$ look like? And, in particular, have we rid ourselves of those nasty negative norm states so that $\mathcal{H}_{phys}$ has a positive definite inner product defined on it? The answer is actually no, but almost!

Let's consider a basis of states for the Fock space. We can decompose any element of this basis as $|\Psi\rangle = |\psi_T\rangle\,|\varphi\rangle$, where $|\psi_T\rangle$ contains only transverse photons, created by

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$a^{1,2\,\dagger}_{\vec{p}}$, while $|\varphi\rangle$ contains the timelike photons created by $a^{0\,\dagger}$ and longitudinal photons created by $a^{3\,\dagger}_{\vec{p}}$. The Gupta-Bleuler condition (6.54) requires

$$(a^3_{\vec{p}} - a^0_{\vec{p}})\,|\varphi\rangle = 0 \tag{6.56}$$

This means that the physical states must contain combinations of timelike and longitudinal photons. Whenever the state contains a timelike photon of momentum $\vec{p}$, it must also contain a longitudinal photon with the same momentum. In general $|\varphi\rangle$ will be a linear combination of states $|\varphi_n\rangle$ containing $n$ pairs of timelike and longitudinal photons, which we can write as

$$|\varphi\rangle = \sum_{n=0}^{\infty} C_n\,|\varphi_n\rangle \tag{6.57}$$

where $|\varphi_0\rangle = |0\rangle$ is simply the vacuum. It's not hard to show that although the condition (6.56) does indeed decouple the negative norm states, all the remaining states involving timelike and longitudinal photons have zero norm

$$\langle\varphi_m|\,\varphi_n\rangle = \delta_{n0}\delta_{m0} \tag{6.58}$$

This means that the inner product on $\mathcal{H}_{phys}$ is positive semi-definite. Which is an improvement. But we still need to deal with all these zero norm states.

The way we cope with the zero norm states is to treat them as gauge equivalent to the vacuum. Two states that differ only in their timelike and longitudinal photon content, $|\varphi_n\rangle$ with $n \geq 1$ are said to be physically equivalent. We can think of the gauge symmetry of the classical theory as descending to the Hilbert space of the quantum theory. Of course, we can't just stipulate that two states are physically identical unless they give the same expectation value for all physical observables. We can check that this is true for the Hamiltonian, which can be easily computed to be

$$H = \int \frac{d^3p}{(2\pi)^3}\,|\vec{p}|\left(\sum_{i=1}^{3} a^{i\,\dagger}_{\vec{p}}a^i_{\vec{p}} - a^{0\,\dagger}_{\vec{p}}a^0_{\vec{p}}\right) \tag{6.59}$$

But the condition (6.56) ensures that $\langle\Psi|\,a^{3\,\dagger}_{\vec{p}}a^3_{\vec{p}}\,|\Psi\rangle = \langle\Psi|\,a^{0\,\dagger}_{\vec{p}}a^0_{\vec{p}}\,|\Psi\rangle$ so that the contri- butions from the timelike and longitudinal photons cancel amongst themselves in the Hamiltonian. This also renders the Hamiltonian positive definite, leaving us just with the contribution from the transverse photons as we would expect.

In general, one can show that the expectation values of all gauge invariant operators evaluated on physical states are independent of the coefficients $C_n$ in (6.57).

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**Propagators**

Finally, it's a simple matter to compute the propagator in Lorentz gauge. It is given by

$$\langle 0|\, T A_\mu(x)A_\nu(y)\, |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{6.60}$$

This is a lot nicer than the propagator we found in Coulomb gauge: in particular, it's Lorentz invariant. We could also return to the Lagrangian (6.37). Had we pushed through the calculation with arbitrary coefficient $\alpha$, we would find the propagator,

$$\langle 0|\, T A_\mu(x)A_\nu(y)\, |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + i\epsilon} \left(\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}\right) e^{-ip\cdot(x-y)} \tag{6.61}$$

## 6.3 Coupling to Matter

Let's now build an interacting theory of light and matter. We want to write down a Lagrangian which couples $A_\mu$ to some matter fields, either scalars or spinors. For example, we could write something like

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{6.62}$$

where $j^\mu$ is some function of the matter fields. The equations of motion read

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{6.63}$$

so, for consistency, we require

$$\partial_\mu j^\mu = 0 \tag{6.64}$$

In other words, $j^\mu$ must be a conserved current. But we've got lots of those! Let's look at how we can couple two of them to electromagnetism.

### 6.3.1 Coupling to Fermions

The Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{6.65}$$

has an internal symmetry $\psi \to e^{-i\alpha}\psi$ and $\bar{\psi} \to e^{+i\alpha}\bar{\psi}$, with $\alpha \in \mathbf{R}$. This gives rise to the conserved current $j^\mu_V = \bar{\psi}\gamma^\mu\psi$. So we could look at the theory of electromagnetism coupled to fermions, with the Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu A_\mu\psi \tag{6.66}$$

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where we've introduced a coupling constant $e$. For the free Maxwell theory, we have seen that the existence of a gauge symmetry was crucial in order to cut down the physical degrees of freedom to the requisite 2. Does our interacting theory above still have a gauge symmetry? The answer is yes. To see this, let's rewrite the Lagrangian as

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i \not{D} - m)\psi \tag{6.67}$$

where $D_\mu \psi = \partial_\mu \psi + ieA_\mu \psi$ is called the *covariant derivative*. This Lagrangian is invariant under gauge transformations which act as

$$A_\mu \to A_\mu + \partial_\mu \lambda \quad \text{and} \quad \psi \to e^{-ie\lambda}\psi \tag{6.68}$$

for an arbitrary function $\lambda(x)$. The tricky term is the derivative acting on $\psi$, since this will also hit the $e^{-ie\lambda}$ piece after the transformation. To see that all is well, let's look at how the covariant derivative transforms. We have

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi + ieA_\mu \psi \\ &\to \partial_\mu (e^{-ie\lambda}\psi) + ie(A_\mu + \partial_\mu \lambda)(e^{-ie\lambda}\psi) \\ &= e^{-ie\lambda} D_\mu \psi \end{aligned} \tag{6.69}$$

so the covariant derivative has the nice property that it merely picks up a phase under the gauge transformation, with the derivative of $e^{-ie\lambda}$ cancelling the transformation of the gauge field. This ensures that the whole Lagrangian is invariant, since $\bar{\psi} \to e^{+ie\lambda(x)}\bar{\psi}$.

## Electric Charge

The coupling $e$ has the interpretation of the electric charge of the $\psi$ particle. This follows from the equations of motion of classical electromagnetism $\partial_\mu F^{\mu\nu} = j^\nu$: we know that the $j^0$ component is the charge density. We therefore have the total charge $Q$ given by

$$Q = e \int d^3x\, \bar{\psi}(\vec{x})\gamma^0 \psi(\vec{x}) \tag{6.70}$$

After treating this as a quantum equation, we have

$$Q = e \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} (b_{\vec{p}}^{s\,\dagger} b_{\vec{p}}^{s} - c_{\vec{p}}^{s\,\dagger} c_{\vec{p}}^{s}) \tag{6.71}$$

which is the number of particles, minus the number of antiparticles. Note that the particle and the anti-particle are required by the formalism to have opposite electric charge. For QED, the theory of light interacting with electrons, the electric charge is usually written in terms of the dimensionless ratio $\alpha$, known as the fine structure constant

$$\alpha = \frac{e^2}{4\pi k c} \approx \frac{1}{137} \tag{6.72}$$

Setting $k = c = 1$, we have $e = \sqrt{4\pi\alpha} \approx 0.3$.

There's a small subtlety here that's worth elaborating on. I stressed that there's a radical difference between the interpretation of a global symmetry and a gauge symmetry. The former takes you from one physical state to another with the same properties and results in a conserved current through Noether's theorem. The latter is a redundancy in our description of the system. Yet in electromagnetism, the gauge symmetry $\psi \to e^{+ie\lambda(x)}\psi$ seems to lead to a conservation law, namely the conservation of electric charge. This is because among the infinite number of gauge symmetries parameterized by a function $\lambda(x)$, there is also a single global symmetry: that with $\lambda(x)$ = constant. This is a true symmetry of the system, meaning that it takes us to another physical state. More generally, the subset of global symmetries from among the gauge symme- tries are those for which $\lambda(x) \to \alpha =$ constant as $x \to \infty$. These take us from one physical state to another.

Finally, let's check that the 4 × 4 matrix $C$ that we introduced in Section 4.5 really deserves the name "charge conjugation matrix". If we take the complex conjugation of the Dirac equation, we have

$$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0 \quad \Rightarrow \quad (-i(\gamma^\mu)^{\wedge}\partial_\mu - e(\gamma^\mu)^{\wedge}A_\mu - m)\psi^{\wedge} = 0$$

Now using the defining equation $C^\dagger\gamma^\mu C = -(\gamma^\mu)^{\wedge}$, and the definition $\psi^{(c)} = C\psi^{\wedge}$, we see that the charge conjugate spinor $\psi^{(c)}$ satisfies

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\psi^{(c)} = 0 \tag{6.73}$$

So we see that the charge conjugate spinor $\psi^{(c)}$ satisfies the Dirac equation, but with charge $-e$ instead of $+e$.

### 6.3.2 Coupling to Scalars

For a real scalar field, we have no suitable conserved current. This means that we can't couple a real scalar field to a gauge field.

Let's now consider a complex scalar field $\phi$. (For this section, I'll depart from our previous notation and call the scalar field $\phi$ to avoid confusing it with the spinor). We have a symmetry $\phi \to e^{-i\alpha}\phi$. We could try to couple the associated current to the gauge field,

$$\mathcal{L}_{int} = -i((\partial_\mu\phi^\dagger)\phi - \phi^\dagger\partial_\mu\phi)A^\mu \tag{6.74}$$

But this doesn't work because

- The theory is no longer gauge invariant
- The current $j^\mu$ that we coupled to $A_\mu$ depends on $\partial_\mu\phi$. This means that if we try to compute the current associated to the symmetry, it will now pick up a contribution from the $j^\mu A_\mu$ term. So the whole procedure wasn't consistent.

We solve both of these problems simultaneously by remembering the covariant derivative. In this scalar theory, the combination

$$D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi \tag{6.75}$$

again transforms as $D_\mu\phi \to e^{-ie\lambda}D_\mu\phi$ under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ and $\phi \to e^{-ie\lambda}\phi$. This means that we can construct a gauge invariant action for a charged scalar field coupled to a photon simply by promoting all derivatives to covariant derivatives

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^\dagger D^\mu\phi - m^2|\phi|^2 \tag{6.76}$$

In general, this trick works for any theory. If we have a $U(1)$ symmetry that we wish to couple to a gauge field, we may do so by replacing all derivatives by suitable covariant derivatives. This procedure is known as *minimal coupling*.

## 6.4 QED

Let's now work out the Feynman rules for the full theory of quantum electrodynamics (QED) – the theory of electrons interacting with light. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{6.77}$$

where $D_\mu = \partial_\mu + ieA_\mu$.

The route we take now depends on the gauge choice. If we worked in Lorentz gauge previously, then we can jump straight to Section 6.5 where the Feynman rules for QED are written down. If, however, we worked in Coulomb gauge, then we still have a bit of work in front of us in order to massage the photon propagator into something Lorentz invariant. We will now do that.

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In Coulomb gauge $\nabla\cdot\vec{A} = 0$, the equation of motion arising from varying $A_0$ is now

$$-\nabla^2 A_0 = e\psi^\dagger\psi \equiv ej^0 \tag{6.78}$$

which has the solution

$$A_0(\vec{x},t) = e\int d^3x'\,\frac{j^0(\vec{x}',t)}{4\pi|\vec{x}-\vec{x}'|} \tag{6.79}$$

In Coulomb gauge we can rewrite the Maxwell part of the Lagrangian as

$$\begin{aligned} L_{\text{Maxwell}} &= \int d^3x\; \tfrac{1}{2}\vec{E}^{\,2} - \tfrac{1}{2}\vec{B}^{\,2} \\ &= \int d^3x\; \tfrac{1}{2}(\dot{\vec{A}} + \nabla A_0)^2 - \tfrac{1}{2}\vec{B}^{\,2} \\ &= \int d^3x\; \tfrac{1}{2}\dot{\vec{A}}^{\,2} + \tfrac{1}{2}(\nabla A_0)^2 - \tfrac{1}{2}\vec{B}^{\,2} \end{aligned} \tag{6.80}$$

where the cross-term has vanished using $\nabla\cdot\vec{A} = 0$. After integrating the second term by parts and inserting the equation for $A_0$, we have

$$L_{\text{Maxwell}} = \int d^3x \left[\tfrac{1}{2}\dot{\vec{A}}^{\,2} - \tfrac{1}{2}\vec{B}^{\,2}\right] + \frac{e^2}{2}\int d^3x\, d^3x'\,\frac{j_0(\vec{x})\,j_0(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} \tag{6.81}$$

We find ourselves with a nonlocal term in the action. This is exactly the type of interaction that we boasted in Section 1.1.4 never arises in Nature! It appears here as an artifact of working in Coulomb gauge: it does not mean that the theory of QED is nonlocal. For example, it wouldn't appear if we worked in Lorentz gauge.

We now compute the Hamiltonian. Changing notation slightly from previous chapters, we have the conjugate momenta,

$$\begin{aligned} \vec{\Pi} &= \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = \dot{\vec{A}} \\ \pi_\psi &= \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \end{aligned} \tag{6.82}$$

which gives us the Hamiltonian

$$H = \int d^3x \left[\tfrac{1}{2}\dot{\vec{A}}^{\,2} + \tfrac{1}{2}\vec{B}^{\,2} + \bar{\psi}(-i\gamma^i\partial_i + m)\psi - e\vec{j}\cdot\vec{A}\right] + \frac{e^2}{2}\int d^3x'\,\frac{j^0(\vec{x})\,j^0(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$$

where $\vec{j} = \bar{\psi}\vec{\gamma}\psi$ and $j^0 =$

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### 6.4.1 Naive Feynman Rules

We want to determine the Feynman rules for this theory. For fermions, the rules are the same as those given in Section 5. The new pieces are:

- We denote the photon by a wavy line. Each end of the line comes with an $i, j = 1, 2, 3$ index telling us the component of $\vec{A}$. We calculated the transverse photon propagator in (6.33): it is  and contributes $D^{tr}_{ij} = \frac{i}{p^2 + i\epsilon}\left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}\right)$

- The vertex  contributes $-ie\gamma^i$. The index on $\gamma^i$ contracts with the index on the photon line.

- The non-local interaction which, in position space, is given by  contributes a factor of $\frac{i(e\gamma^0)^2\delta(x^0 - y^0)}{4\pi|\vec{x} - \vec{y}|}$

These Feynman rules are rather messy. This is the price we've paid for working in Coulomb gauge. We'll now show that we can massage these expressions into something much more simple and Lorentz invariant. Let's start with the offending instantaneous interaction. Since it comes from the $A_0$ component of the gauge field, we could try to redefine the propagator to include a $D_{00}$ piece which will capture this term. In fact, it fits quite nicely in this form: if we look in momentum space, we have

$$\frac{\delta(x^0 - y^0)}{4\pi|\vec{x} - \vec{y}|} = \int \frac{d^4p}{(2\pi)^4}\,\frac{e^{ip\cdot(x-y)}}{|\vec{p}|^2} \tag{6.83}$$

so we can combine the non-local interaction with the transverse photon propagator by defining a new photon propagator

$$D_{\mu\nu}(p) = \begin{cases} +\frac{i}{|\vec{p}|^2} & \mu, \nu = 0 \\ \frac{i}{p^2 + i\epsilon}\left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}\right) & \mu = i \neq 0,\ \nu = j \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.84}$$

With this propagator, the wavy photon line now carries a $\mu, \nu = 0, 1, 2, 3$ index, with the extra $\mu = 0$ component taking care of the instantaneous interaction. We now need to change our vertex slightly: the $-ie\gamma^i$ above gets replaced by $-ie\gamma^\mu$ which correctly accounts for the $(e\gamma^0)^2$ piece in the instantaneous interaction.

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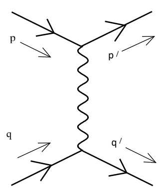
The $D_{00}$ piece of the propagator doesn't look a whole lot different from the transverse photon propagator. But wouldn't it be nice if they were both part of something more symmetric! In fact, they are. We have the following:

**Claim:** We can replace the propagator $D_{\mu\nu}(p)$ with the simpler, Lorentz invariant propagator

$$D_{\mu\nu}(p) = -i\frac{\eta_{\mu\nu}}{p^2} \tag{6.85}$$

**Proof:** There is a general proof using current conservation. Here we'll be more pedestrian and show that we can do this for certain Feynman diagrams. In particular, we focus on a particular tree-level diagram that contributes to $e^-e^- \to e^-e^-$ scattering,




$$\sim e^2 [\bar{u}(p')\gamma^\mu u(p)]\, D_{\mu\nu}(k)\, [\bar{u}(q')\gamma^\nu u(q)] \tag{6.86}$$

where $k = p - p' = q' - q$. Recall that $u(\vec{p})$ satisfies the equation

$$(\not{p} - m)u(\vec{p}) = 0 \tag{6.87}$$

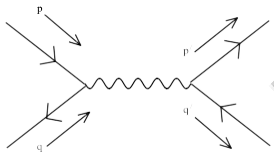
Let's define the spinor contractions $\alpha^\mu = \bar{u}(\vec{p}\,')\gamma^\mu u(\vec{p})$ and $\beta^\nu = \bar{u}(\vec{q}\,')\gamma^\nu u(\vec{q})$. Then since $k = p - p' = q' - q$, we have

$$k_\mu \alpha^\mu = \bar{u}(\vec{p}\,')(\not{p} - \not{p}\,')u(\vec{p}) = \bar{u}(\vec{p}\,')(m - m)u(\vec{p}) = 0 \tag{6.88}$$

and, similarly, $k_\nu \beta^\nu = 0$. Using this fact, the diagram can be written as

$$\begin{aligned}
\alpha^\mu D_{\mu\nu} \beta^\nu &= i\left(\frac{\vec{\alpha}\cdot\vec{\beta}}{k^2} - \frac{(\vec{\alpha}\cdot\vec{k})(\vec{\beta}\cdot\vec{k})}{k^2|\vec{k}|^2} + \frac{\alpha^0\beta^0}{|\vec{k}|^2}\right) \\
&= i\left(\frac{\vec{\alpha}\cdot\vec{\beta}}{k^2} - \frac{k_0^2\alpha_0\beta_0}{k^2|\vec{k}|^2} + \frac{\alpha_0\beta_0}{|\vec{k}|^2}\right) \\
&= i\left(\frac{\vec{\alpha}\cdot\vec{\beta}}{k^2} - \frac{1}{k^2|\vec{k}|^2}(k_0^2 - k^2)\,\alpha_0\beta_0\right) \\
&= -\frac{i}{k^2}\,\alpha\cdot\beta = \alpha^\mu\left(-\frac{i\eta_{\mu\nu}}{k^2}\right)\beta^\nu
\end{aligned} \tag{6.89}$$

which is the claimed result. You can similarly check that the same substitution is legal in the diagram



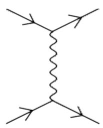
$$\sim e^2[\bar{v}(\vec{q})\gamma^\mu u(\vec{p})]D_{\mu\nu}(k)[\bar{u}(\vec{p}\,')\gamma^\nu v(\vec{q}\,')] \qquad (6.90)$$

In fact, although we won't show it here, it's a general fact that in every Feynman diagram we may use the very nice, Lorentz invariant propagator $D_{\mu\nu} = -i\eta_{\mu\nu}/p^2$.

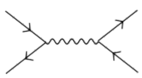
**Note:** This is the propagator we found when quantizing in Lorentz gauge (using the Feynman gauge parameter). In general, quantizing the Lagrangian (6.37) in Lorentz gauge, we have the propagator

$$D_{\mu\nu} = -\frac{i}{p^2}\left(\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}\right) \qquad (6.91)$$

Using similar arguments to those given above, you can show that the $p_\mu p_\nu/p^2$ term cancels in all diagrams. For example, in the following diagrams the $p_\mu p_\nu$ piece of the propagator contributes as



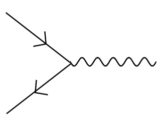
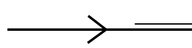
$$\sim \bar{u}(p')\gamma^\mu u(p)\, k_\mu = \bar{u}(p')(\not{p} - \not{p}')u(p) = 0$$



$$\sim \bar{v}(p)\gamma^\mu u(q)\, k_\mu = \bar{u}(p)(\not{p} + \not{q})u(q) = 0 \qquad (6.92)$$

## 6.5 Feynman Rules

Finally, we have the Feynman rules for QED. For vertices and internal lines, we write

- Vertex:  $-ie\gamma^\mu$
- Photon Propagator:  $-\dfrac{i\eta_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion Propagator:  $\dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

For external lines in the diagram, we attach

- Photons: We add a polarization vector $\epsilon^\mu_{in}/\epsilon^\mu_{out}$ for incoming/outgoing photons. In Coulomb gauge, $\epsilon^0 = 0$ and $\vec{\epsilon}\cdot\vec{p} = 0$.
- Fermions: We add a spinor $u^r(\vec{p})/\bar{u}^r(\vec{p})$ for incoming/outgoing fermions. We add a spinor $\bar{v}^r(\vec{p})/v^r(\vec{p})$ for incoming/outgoing anti-fermions.

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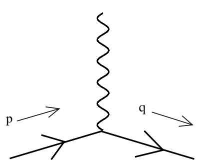
### 6.5.1 Charged Scalars

> "Pauli asked me to calculate the cross section for pair creation of scalar particles by photons. It was only a short time after Bethe and Heitler had solved the same problem for electrons and positrons. I met Bethe in Copenhagen at a conference and asked him to tell me how he did the calculations. I also inquired how long it would take to perform this task; he answered, "It would take me three days, but you will need about three weeks." He was right, as usual; furthermore, the published cross sections were wrong by a factor of four."
>
> *Viki Weisskopf*

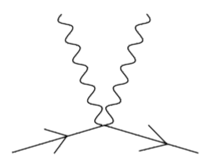
The interaction terms in the Lagrangian for charged scalars come from the covariant derivative terms,

$$\mathcal{L} = D_\mu\psi^\dagger D^\mu\psi = \partial_\mu\psi\,\partial^\dagger{}^\mu\psi - ieA_\mu(\psi^\dagger\partial^\mu\psi - \psi\partial^\mu\psi^\dagger) + e^2 A_\mu A^\mu\psi^\dagger\psi \tag{6.93}$$

This gives rise to two interaction vertices. But the cubic vertex is something we haven't seen before: it contains kinetic terms. How do these appear in the Feynman rules? After a Fourier transform, the derivative term means that the interaction is stronger for fermions with higher momentum, so we include a momentum factor in the Feynman rule. There is also a second, "seagull" graph. The two Feynman rules are




$-ie(p+q)_\mu$ and



$+2ie^2\eta_{\mu\nu}$

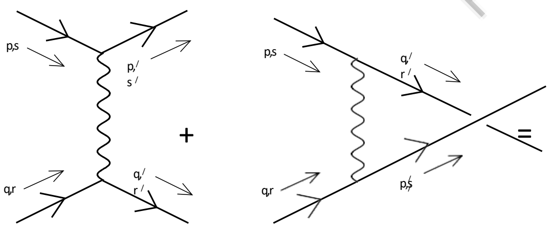
The factor of two in the seagull diagram arises because of the two identical particles appearing in the vertex. (It's the same reason that the 1/4! didn't appear in the Feynman rules for $\varphi^4$ theory).

## 6.6 Scattering in QED

Let's now calculate some amplitudes for various processes in quantum electrodynamics, with a photon coupled to a single fermion. We will consider the analogous set of processes that we saw in Section 3 and Section 5. We have

### Electron Scattering

Electron scattering $e^- e^- \to e^- e^-$ is described by the two leading order Feynman diagrams, given by

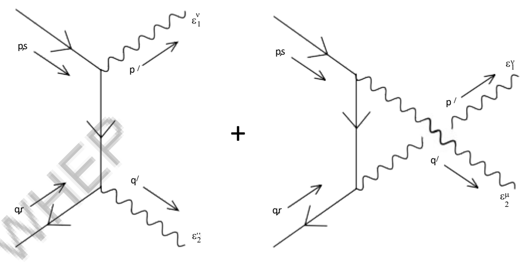


$$= -i(-ie)^2 \left( \frac{[\bar u^{s'}(\vec p\,')\gamma^\mu u^s(\vec p)]\,[\bar u^{r'}(\vec q\,')\gamma_\mu u^r(\vec q)]}{(p'-p)^2} - \frac{[\bar u^{s'}(\vec p\,')\gamma^\mu u^r(\vec q)]\,[\bar u^{r'}(\vec q\,')\gamma_\mu u^s(\vec p)]}{(p-q')^2} \right)$$

The overall $-i$ comes from the $-i\eta_{\mu\nu}$ in the propagator, which contract the indices on the $\gamma$-matrices (remember that it's really positive for $\mu, \nu = 1, 2, 3$).

### Electron Positron Annihilation

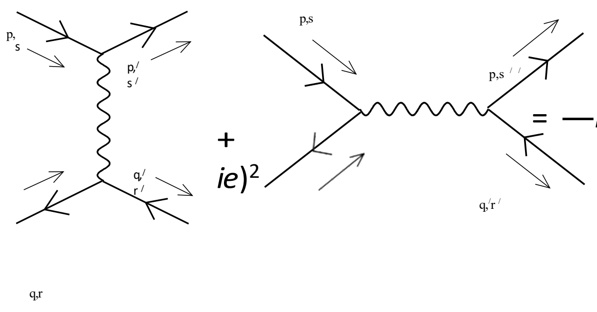
Let's now look at $e^- e^+ \to 2\gamma$, two gamma rays. The two lowest order Feynman diagrams are,



$$= i(-ie)^2 \bar v^r(\vec q) \left[ \frac{\gamma_\mu(\not p - \not p' + m)\gamma_\nu}{(p-p')^2 - m^2} + \frac{\gamma_\nu(\not p - \not q' + m)\gamma_\mu}{(p-q')^2 - m^2} \right] u^s(\vec p)\, \epsilon_1^\nu(\vec p\,')\, \epsilon_2^\mu(\vec q\,')$$

### Electron Positron Scattering

For $e^- e^+ \to e^- e^+$ scattering (sometimes known as Bhabha scattering) the two lowest order Feynman diagrams are



$$= -i(-ie)^2 \left( -\frac{[\bar u^{s'}(\vec p\,')\gamma^\mu u^s(\vec p)]\,[\bar v^r(\vec q)\gamma_\mu v^{r'}(\vec q\,')]}{(p-p')^2} + \frac{[\bar v^r(\vec q)\gamma^\mu u^s(\vec p)]\,[\bar u^{s'}(\vec p\,')\gamma_\mu v^{r'}(\vec q\,')]}{(p+q)^2} \right)$$

### Compton Scattering

The scattering of photons (in particular x-rays) off electrons $e^- \gamma \to e^- \gamma$ is known as Compton scattering. Historically, the change in wavelength of the photon in the

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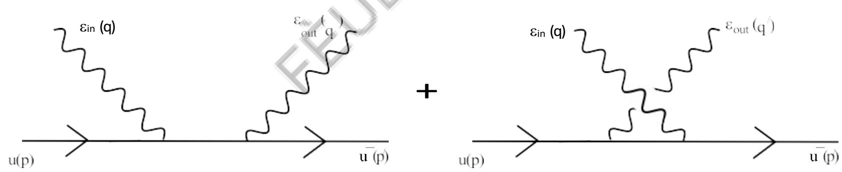
scattering process was one of the conclusive pieces of evidence that light could behave as a particle. The amplitude is given by,




$$= i(-ie)^2\, \bar{u}^{r'}(\vec{p}\,') \left[ \frac{\gamma^\mu(\not{p} + \not{q} + m)\gamma^\nu}{(p+q)^2 - m^2} + \frac{\gamma^\nu(\not{p} - \not{q}' + m)\gamma^\mu}{(p-q')^2 - m^2} \right] u^s(\vec{p})\, \epsilon^\nu_{in}\, \epsilon^\mu_{out}$$

This amplitude vanishes for longitudinal photons. For example, suppose $\epsilon_{in} \sim q$. Then, using momentum conservation $p + q = p' + q'$, we may write the amplitude as

$$i\mathcal{A} = i(-ie)^2\, \bar{u}^{r'}(\vec{p}\,') \left( \frac{\not{\epsilon}_{out}(\not{p} + \not{q} + m)\not{q}}{(p+q)^2 - m^2} + \frac{\not{q}(\not{p}' - \not{q} + m)\not{\epsilon}_{out}}{(p'-q)^2 - m^2} \right) u^s(\vec{p})$$

$$= i(-ie)^2\, \bar{u}^{r'}(\vec{p}\,')\, \not{\epsilon}_{out}\, u^s(\vec{p}) \left( \frac{2p\cdot q}{(p+q)^2 - m^2} + \frac{2p'\cdot q}{(p'-q)^2 - m^2} \right) \qquad (6.94)$$

where, in going to the second line, we've performed some $\gamma$-matrix manipulations, together with the spinor equations $(\not{p} - m)u(\vec{p})$ and $\bar{u}(\vec{p}\,')(\not{p}' - m) = 0$. We now recall the fact that $q$ is a null vector, while $p^2 = (p')^2 = m^2$ since the external legs are on mass- shell. This means that the two denominators in the amplitude read $(p+q)^2 - m^2 = 2p\cdot q$ and $(p' - q)^2 - m^2 = -2p' \cdot q$. This ensures that the combined amplitude vanishes for longitudinal photons as promised. A similar result holds when $\not{\epsilon}_{out} \sim q'$.

### Photon Scattering

In QED, photons no longer pass through each other unimpeded. At one-loop, there is a diagram which leads to photon scattering. Although naively logarithmically divergent, the diagram is actually rendered finite by gauge invariance.

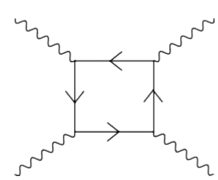


**Figure 30:**

### Adding Muons

Adding a second fermion into the mix, which we could identify as a muon, new processes become possible. For example, we can now have processes such as $e^-\mu^- \to e^-\mu^-$ scattering, and $e^+e^-$ annihilation into a muon anti-muon pair. Using our standard notation of $p$ and $q$ for incoming momenta, and $p'$ and $q'$ for outgoing

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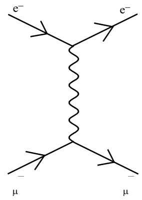
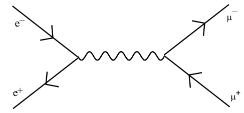
momenta, we have the amplitudes given by




$$\sim \frac{1}{(p - p')^2} \quad \text{and}$$

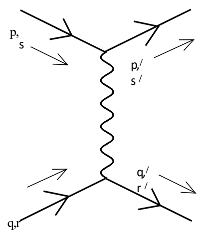



$$\sim \frac{1}{(p + q)^2} \tag{6.95}$$

### 6.6.1 The Coulomb Potential

We've come a long way. We've understood how to compute quantum amplitudes in a large array of field theories. To end this course, we use our newfound knowledge to rederive a result you learnt in kindergarten: Coulomb's law.

To do this, we repeat our calculation that led us to the Yukawa force in Sections 3.5.2 and 5.7.2. We start by looking at $e^- e^- \to e^- e^-$ scattering. We have



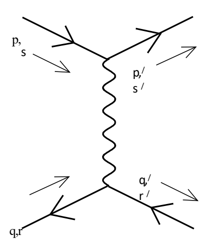

$$= -i(-ie)^2 \frac{[\bar{u}(\vec{p}')\gamma^\mu u(\vec{p})][\bar{u}(\vec{q}')\gamma_\mu u(\vec{q})]}{(p' - p)^2} \tag{6.96}$$

Following (5.49), the non-relativistic limit of the spinor is $u(p) \to \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$. This means that the $\gamma^0$ piece of the interaction gives a term $\bar{u}^s(\vec{p}')\gamma^0 u^r(\vec{q}') \to 2m\delta^{rs}$, while the spatial $\gamma^i$, $i = 1, 2, 3$ pieces vanish in the non-relativistic limit: $\bar{u}^s(\vec{p}')\gamma^i u^r(\vec{q}') \to 0$. Comparing the scattering amplitude in this limit to that of non-relativistic quantum mechanics, we have the effective potential between two electrons given by,

$$U(\vec{r}) = +e^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{r}}}{|\vec{p}|^2} = + \frac{e^2}{4\pi r} \tag{6.97}$$

We find the familiar repulsive Coulomb potential. We can trace the minus sign that gives a repulsive potential to the fact that only the $A_0$ component of the intermediate propagator $\sim -i\eta_{\mu\nu}$ contributes in the non-relativistic limit.

For $e^- e^+ \to e^- e^+$ scattering, the amplitude is




$$= +i(-ie)^2 \frac{[\bar{u}(\vec{p}')\gamma^\mu u(\vec{p})][\bar{v}(\vec{q})\gamma_\mu v(\vec{q}')]}{(p' - p)^2} \tag{6.98}$$

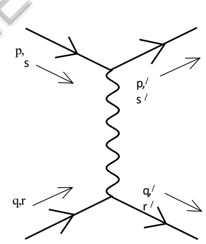
The overall + sign comes from treating the fermions correctly: we saw the same minus sign when studying scattering in Yukawa theory. The difference now comes from looking at the non-relativistic limit. We have $\bar v \gamma^0 v \to 2m$, giving us the potential between opposite charges,

$$U(\vec r) = -e^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec p\cdot\vec r}}{|\vec p|^2} = -\frac{e^2}{4\pi r} \tag{6.99}$$

Reassuringly, we find an attractive force between an electron and positron. The difference from the calculation of the Yukawa force comes again from the zeroth component of the gauge field, this time in the guise of the $\gamma^0$ sandwiched between $\bar v \gamma^0 v \to 2m$, rather than the $\bar v v \to -2m$ that we saw in the Yukawa case.

### The Coulomb Potential for Scalars

There are many minus signs in the above calculation which somewhat obscure the crucial one which gives rise to the repulsive force. A careful study reveals the offending sign to be that which sits in front of the $A_0$ piece of the photon propagator $-i\eta_{\mu\nu}/p^2$. Note that with our signature $(+---)$, the propagating $A_i$ have the correct sign, while $A_0$ comes with the wrong sign. This is simpler to see in the case of scalar QED, where we don't have to worry about the gamma matrices. From the Feynman rules of Section 6.5.1, we have the non-relativistic limit of scalar $e^-e^-$ scattering,




$$= -i\eta_{\mu\nu}(-ie)^2 \frac{(p+p')^\mu (q+q')^\nu}{(p'-p)^2} \to -i(-ie)^2 \frac{(2m)^2}{-(\vec p - \vec p\,')^2}$$

where the non-relativistic limit in the numerator involves $(p+p')\cdot(q+q') \approx (p+p')^0(q+q')_0 \approx (2m)^2$ and is responsible for selecting the $A_0$ part of the photon propagator rather than the $A_i$ piece. This shows that the Coulomb potential for spin 0 particles of the same charge is again repulsive, just as it is for fermions. For $e^-e^+$ scattering, the amplitude picks up an extra minus sign because the arrows on the legs of the Feynman rules in Section 6.5.1 are correlated with the momentum arrows. Flipping the arrows on one pair of legs in the amplitude introduces the relevant minus sign to ensure that the non-relativistic potential between $e^-e^+$ is attractive as expected.

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### 6.7 Afterword

In this course, we have laid the foundational framework for quantum field theory. Most of the developments that we've seen were already in place by the middle of the 1930s, pioneered by people such as Jordan, Dirac, Heisenberg, Pauli and Weisskopf [5].

Yet by the end of the 1930s, physicists were ready to give up on quantum field theory. The difficulty lies in the next terms in perturbation theory. These are the terms that correspond to Feynamn diagrams with loops in them, which we have scrupulously avoided computing in this course. The reason we've avoided them is because they typically give infinity! And, after ten years of trying, and failing, to make sense of this, the general feeling was that one should do something else. This from Dirac in 1937,

> *Because of its extreme complexity, most physicists will be glad to see the end of QED*

But the leading figures of the day gave up too soon. It took a new generation of postwar physicists — people like Schwinger, Feynman, Tomonaga and Dyson — to return to quantum field theory and tame the infinities. The story of how they did that will be told in next term's course.

[5]For more details on the history of quantum field theory, see the excellent book "QED and the Men who Made it" by Sam Schweber.

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# Chapter 2

# Basic Set Theory

**A set is a Many that allows itself to be thought of as a One.**
- *Georg Cantor*

This chapter introduces set theory, mathematical induction, and formalizes the notion of mathematical functions. The material is mostly elementary. For those of you new to abstract mathematics elementary does not mean *simple* (though much of the material is fairly simple). Rather, elementary means that the material requires very little previous education to understand it. Elementary material can be quite challenging and some of the material in this chapter, if not exactly rocket science, may require that you adjust you point of view to understand it. The single most powerful technique in mathematics is to adjust your point of view until the problem you are trying to solve becomes simple.

Another point at which this material may diverge from your previous experience is that it will require proof. In standard introductory classes in algebra, trigonometry, and calculus there is currently very little emphasis on the discipline of *proof.* Proof is, however, the central tool of mathematics. This text is for a course that is a students formal introduction to tools and methods of proof.

## 2.1 Set Theory

A *set* is a collection of distinct objects. This means that $\{1, 2, 3\}$ is a set but $\{1, 1, 3\}$ is not because 1 appears twice in the second collection. The second collection is called a *multiset.* Sets are often specified with curly brace notation. The set of even integers can be written:

$$\{2n : \text{n is an integer}\}$$

The opening and closing curly braces denote a set, $2n$ specifies the members of the set, the colon says "such that" or "where" and everything following the colon are conditions that explain or refine the membership. All correct mathematics can be spoken in English. The set definition above is spoken "The set of twice $n$ where $n$ is an integer".

The only problem with this definition is that we do not yet have a formal definition of the integers. The integers are the set of whole numbers, both positive and negative: $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$. We now introduce the operations used to manipulate sets, using the opportunity to practice curly brace notation.

**Definition 2.1** *The* **empty set** *is a set containing no objects. It is written as a pair of curly braces with nothing inside* $\{\}$ *or by using the symbol* $\emptyset$.

As we shall see, the empty set is a handy object. It is also quite strange. The set of all humans that weigh at least eight tons, for example, is the empty set. Sets whose definition contains a contradiction or impossibility are often empty.

**Definition 2.2** *The* **set membership symbol** $\in$ *is used to say that an object is a member of a set. It has a partner symbol* $\notin$ *which is used to say an object is not in a set.*

**Definition 2.3** *We say two sets are* **equal** *if they have exactly the same members.*



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**Example 2.1** *If*

$$S = \{1, 2, 3\}$$

*then* $3 \in S$ *and* $4 \notin S$. *The set membership symbol is often used in defining operations that manipulate sets. The set*

$$T = \{2, 3, 1\}$$

*is equal to* $S$ *because they have the same members: 1, 2, and 3. While we usually list the members of a set in a "standard" order (if one is available) there is no requirement to do so and sets are indifferent to the order in which their members are listed.*

**Definition 2.4** *The* **cardinality** *of a set is its* size. *For a finite set, the cardinality of a set is the number of members it contains. In symbolic notation the size of a set* $S$ *is written* $|S|$. *We will deal with the idea of the cardinality of an infinite set later.*

**Example 2.2 Set cardinality**
*For the set* $S = \{1, 2, 3\}$ *we show cardinality by writing* $|S| = 3$

We now move on to a number of *operations* on sets. You are already familiar with several operations on numbers such as addition, multiplication, and negation.

**Definition 2.5** *The* **intersection** *of two sets* $S$ *and* $T$ *is the collection of all objects that are in both sets. It is written* $S \cap T$. *Using curly brace notation*

$$S \cap T = \{x : (x \in S) \text{ and } (x \in T)\}$$

The symbol *and* in the above definition is an example of a Boolean or logical operation. It is only true when both the propositions it joins are also true. It has a symbolic equivalent $\wedge$. This lets us write the formal definition of intersection more compactly:

$$S \cap T = \{x : (x \in S) \wedge (x \in T)\}$$

**Example 2.3 Intersections of sets**
*Suppose* $S = \{1, 2, 3, 5\}$,
$T = \{1, 3, 4, 5\}$, *and* $U = \{2, 3, 4, 5\}$.
*Then:*

$S \cap T = \{1, 3, 5\}$,

$S \cap U = \{2, 3, 5\}$, *and*

$T \cap U = \{3, 4, 5\}$

**Definition 2.6** *If* $A$ *and* $B$ *are sets and* $A \cap B = \emptyset$ *then we say that* $A$ *and* $B$ *are* disjoint, *or* disjoint sets.

**Definition 2.7** *The* **union** *of two sets* $S$ *and* $T$ *is the collection of all objects that are in either set. It is written* $S \cup T$. *Using curly brace notion*

$$S \cup T = \{x : (x \in S) \text{ or } (x \in T)\}$$

The symbol *or* is another Boolean operation, one that is true if either of the propositions it joins are true. Its symbolic equivalent is $\vee$ which lets us re-write the definition of union as:

$$S \cup T = \{x : (x \in S) \vee (x \in T)\}$$

**Example 2.4 Unions of sets.**
*Suppose* $S = \{1, 2, 3\}$, $T = \{1, 3, 5\}$, *and* $U = \{2, 3, 4, 5\}$.
*Then:*

$S \cup T = \{1, 2, 3, 5\}$,

$S \cup U = \{1, 2, 3, 4, 5\}$, *and*

$T \cup U = \{1, 2, 3, 4, 5\}$

When performing set theoretic computations, you should declare the domain in which you are working. In set theory this is done by declaring a universal set.

**Definition 2.8** *The* **universal set**, *at least for a given collection of set theoretic computations, is the set of all possible objects.*

If we declare our universal set to be the integers then $\{\frac{1}{2}, \frac{2}{3}\}$ is not a well defined set because the objects used to define it are not members of the universal set. The symbols $\{\frac{1}{2}, \frac{2}{3}\}$ do define a set if a universal set that includes $\frac{1}{2}$ and $\frac{2}{3}$ is chosen. The problem arises from the fact that neither of these numbers are integers. The universal set is commonly written $\mathcal{U}$. Now that we have the idea of declaring a universal set we can define another operation on sets.

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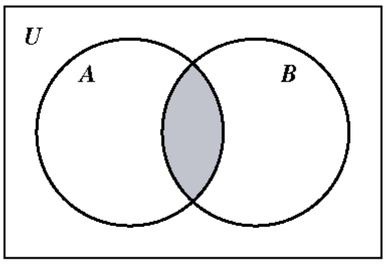
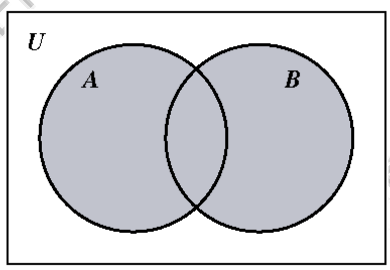
### 2.1.1 Venn Diagrams

A Venn diagram is a way of depicting the relationship between sets. Each set is shown as a circle and circles overlap if the sets intersect.

**Example 2.5** *The following are Venn diagrams for the intersection and union of two sets. The shaded parts of the diagrams are the intersections and unions respectively.*




$A \cap B$




$A \cup B$

*Notice that the rectangle containing the diagram is labeled with a $U$ representing the universal set.*

**Definition 2.9** *The* **compliment** *of a set $S$ is the collection of objects in the universal set that are not in $S$. The compliment is written $S^c$. In curly brace notation*

$$S^c = \{x : (x \in \mathcal{U}) \wedge (x \notin S)\}$$

*or more compactly as*

$$S^c = \{x : x \notin S\}$$

*however it should be apparent that the compliment of a set always depends on which universal set is chosen.*

There is also a Boolean symbol associated with the complementation operation: the *not* operation. The notation for not is $\neg$. There is not much savings in space as the definition of compliment becomes

$$S^c = \{x : \neg(x \in S)\}$$

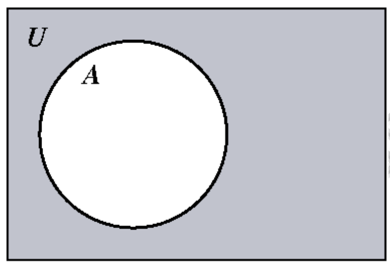
**Example 2.6 Set Compliments**

*(i)* *Let the universal set be the integers. Then the compliment of the even integers is the odd integers.*

*(ii)* *Let the universal set be $\{1, 2, 3, 4, 5\}$, then the compliment of $S = \{1, 2, 3\}$ is $S^c = \{4, 5\}$ while the compliment of $T = \{1, 3, 5\}$ is $T^c = \{2, 4\}$.*

*(iii)* *Let the universal set be the letters $\{a, e, i, o, u, y\}$. Then $\{y\}^c = \{a, e, i, o, u\}$.*

*The Venn diagram for $A^c$ is*




$A^c$

We now have enough set-theory operators to use them to define more operators quickly. We will continue to give English and symbolic definitions.

**Definition 2.10** *The* **difference** *of two sets $S$ and $T$ is the collection of objects in $S$ that are not in $T$. The difference is written $S - T$. In curly brace notation*

$$S - T = \{x : x \in (S \cap (T^c))\},$$

*or alternately*

$$S - T = \{x : (x \in S) \wedge (x \notin T)\}$$

Notice how intersection and complementation can be used together to create the difference operation and that the definition can be rephrased to use Boolean operations. There is a set of rules that reduces the number of parenthesis required. These are called **operator precedence rules**.

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(i) Other things being equal, operations are performed left-to-right.

(ii) Operations between parenthesis are done first, starting with the innermost of nested parenthesis.

(iii) All complimentations are computed next.

(iv) All intersections are done next.

(v) All unions are performed next.

(vi) Tests of set membership and computations, equality or inequality are performed last.

Special operations like the set difference or the symmetric difference, defined below, are not included in the precedence rules and thus always use parenthesis.

**Example 2.7 Operator precedence**
*Since complementation is done before intersection the symbolic definition of the difference of sets can be rewritten:*

$$S - T = \{x : x \in S \cap T^c\}$$

*If we were to take the set operations*

$$A \cup B \cap C^c$$

*and put in the parenthesis we would get*

$$(A \cup (B \cap (C^c)))$$

**Definition 2.11** *The* **symmetric difference** *of two sets $S$ and $T$ is the set of objects that are in one and only one of the sets. The symmetric difference is written $S\Delta T$. In curly brace notation:*

$$S\Delta T = \{(S - T) \cup (T - S)\}$$

**Example 2.8 Symmetric differences**
*Let $S$ be the set of non-negative multiples of two that are no more than twenty four. Let $T$ be the non-negative multiples of three that are no more than twenty four. Then*

$$S\Delta T = \{2, 3, 4, 8, 9, 10, 14, 15, 16, 20, 21, 22\}$$

*Another way to think about this is that we need numbers that are positive multiples of 2 or 3 (but not both) that are no more than 24.*

Another important tool for working with sets is the ability to compare them. We have already defined what it means for two sets to be equal, and so by implication what it means for them to be unequal. We now define another comparator for sets.

**Definition 2.12** *For two sets $S$ and $T$ we say that $S$* **is a subset of** *$T$ if each element of $S$ is also an element of $T$. In formal notation $S \subseteq T$ if for all $x \in S$ we have $x \in T$.*

If $S \subseteq T$ then we also say $T$ contains $S$ which can be written $T \supseteq S$. If $S \subseteq T$ and $S \neq T$ then we write $S \subset T$ and we say $S$ is a *proper* subset of $T$.

**Example 2.9 Subsets**
*If $A = \{a, b, c\}$ then $A$ has eight different subsets:*

$$\emptyset \qquad \{a\} \qquad \{b\} \qquad \{c\}$$

$$\{a, b\} \quad \{a, c\} \quad \{b, c\} \quad \{a, b, c\}$$

*Notice that $A \subseteq A$ and in fact each set is a subset of itself. The empty set $\emptyset$ is a subset of every set.*

We are now ready to prove our first proposition. Some new notation is required and we must introduce an important piece of mathematical culture. If we say "A if and only if B" then we mean that either A and B are both true or they are both false in any given circumstance. For example: "an integer x is even if and only if it is a multiple of 2". The phrase "if and only if" is used to establish *logical equivalence.* Mathematically, "A if and only if B" is a way of stating that A and B are simply different ways of saying the same thing. The phrase "if and only if" is abbreviated iff and is represented symbolically as the double arrow $\Leftrightarrow$. Proving an iff statement is done by independently demonstrating that each may be deduced from the other.

**Proposition 2.1** *Two sets are equal if and only if each is a subset of the other. In symbolic notation:*

$$(A = B) \;\Leftrightarrow\; (A \subseteq B) \wedge (B \subseteq A)$$

Proof:

Let the two sets in question be $A$ and $B$. Begin by assuming that $A = B$. We know that every set is

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a subset of itself so $A \subseteq A$. Since $A = B$ we may substitute into this expression on the left and obtain $B \subseteq A$. Similarly we may substitute on the right and obtain $A \subseteq B$. We have thus demonstrated that if $A = B$ then $A$ and $B$ are both subsets of each other, giving us the first half of the iff.

Assume now that $A \subseteq B$ and $B \subseteq A$. Then the definition of subset tells us that any element of $A$ is an element of $B$. Similarly any element of $B$ is an element of $A$. This means that $A$ and $B$ have the same elements which satisfies the definition of set equality. We deduce $A = B$ and we have the second half of the iff. □

A note on mathematical grammar: the symbol □ indicates the end of a proof. On a paper turned in by a student it is usually taken to mean "I think the proof ends here". Any proof should have a □ to indicate its end. The student should also note the lack of calculations in the above proof. If a proof cannot be read back in (sometimes overly formal) English then it is probably incorrect. Mathematical symbols should be used for the sake of brevity or clarity, not to obscure meaning.

**Proposition 2.2 De Morgan's Laws** *Suppose that $S$ and $T$ are sets. DeMorgan's Laws state that*

*(i)* $(S \cup T)^c = S^c \cap T^c$, *and*

*(ii)* $(S \cap T)^c = S^c \cup T^c$.

Proof:

Let $x \in (S \cup T)^c$; then $x$ is not a member of $S$ or $T$. Since $x$ is not a member of $S$ we see that $x \in S^c$. Similarly $x \in T^c$. Since $x$ is a member of both these sets we see that $x \in S^c \cap T^c$ and we see that $(S \cup T)^c \subseteq S^c \cap T^c$. Let $y \in S^c \cap T^c$. Then the definition of intersection tells us that $y \in S^c$ and $y \in T^c$. This in turn lets us deduce that $y$ is not a member of $S \cup T$, since it is not in either set, and so we see that $y \in (S \cup T)^c$. This demonstrates that $S^c \cap T^c \subseteq (S \cup T)^c$. Applying Proposition 2.1 we get that $(S \cup T)^c = S^c \cap T^c$ and we have proven part (i). The proof of part (ii) is left as an exercise. □

In order to prove a mathematical statement you must prove it is always true. In order to disprove a mathematical statement you need only find a single instance where it is false. It is thus possible for a false mathematical statement to be "true most of the time". In the next chapter we will develop the theory of prime numbers. For now we will assume the reader has a modest familiarity with the primes. The statement "Prime numbers are odd" is false once, because 2 is a prime number. All the other prime numbers are odd. The statement is a false one. This very strict definition of what makes a statement true is a convention in mathematics. We call 2 a *counter example*. It is thus necessary to find only one counter-example to demonstrate a statement is (mathematically) false.

**Example 2.10 Disproof by counter example**
*Prove that the statement $A \cup B = A \cap B$ is false.*

*Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Then $A \cap B = \emptyset$ while $A \cup B = \{1, 2, 3, 4\}$. The sets $A$ and $B$ form a counter-example to the statement.*

# Problems

**Problem 2.1** *Which of the following are sets? Assume that a proper universal set has been chosen and answer by listing the names of the collections of objects that are sets. Warning: at least one of these items has an answer that, while likely, is not 100% certain.*

(i) $A = \{2, 3, 5, 7, 11, 13, 19\}$

(ii) $B = \{A, E, I, O, U\}$

(iii) $C = \{\sqrt{x} : x < 0\}$

(iv) $D = \{1, 2, A, 5, B, Q, 1, V\}$

(v) $E$ is the list of first names of people in the 1972 phone book in Lawrence Kansas in the order they appear in the book. There were more than 35,000 people in Lawrence that year.

(vi) $F$ is a list of the weight, to the nearest kilogram, of all people that were in Canada at any time in 2007.

(vii) $G$ is a list of all weights, to the nearest kilogram, that at least one person in Canada had in 2007.

**Problem 2.2** *Suppose that we have the set* $U = \{n : 0 \leq n < 100\}$ *of whole numbers as our universal set. Let* $P$ *be the prime numbers in* $U$*, let* $E$ *be the even numbers in* $U$*, and let* $F = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89\}$*. Describe the following sets either by listing them or with a careful English sentence.*

(i) $E^c$,

(ii) $P \cap F$,

(iii) $P \cap E$,

(iv) $F \cap E \cup F \cap E^c$, and

(v) $F \cup F^c$.

**Problem 2.3** *Suppose that we take the universal set* $\mathcal{U}$ *to be the integers. Let* $S$ *be the even integers, let* $T$ *be the integers that can be obtained by tripling any one integer and adding one to it, and let* $V$ *be the set of numbers that are whole multiples of both two and three.*

(i) Write $S$, $T$, and $V$ using symbolic notation.

(ii) Compute $S \cap T$, $S \cap V$ and $T \cap V$ and give symbolic representations that do not use the symbols $S$, $T$, or $V$ on the right hand side of the equals sign.

**Problem 2.4** *Compute the cardinality of the following sets. You may use other texts or the internet.*

(i) Two digit positive odd integers.

(ii) Elements present in a sucrose molecule.

(iii) Isotopes of hydrogen that are not radioactive.

(iv) Planets orbiting the same star as the planet you are standing on that have moons. Assume that Pluto is a minor planet.

(v) Elements with seven electrons in their valence shell. Remember that Ununoctium was discovered in 2002 so be sure to use a relatively recent reference.

(vi) Subsets of $S = \{a, b, c, d\}$ with cardinality 2.

(vii) Prime numbers whose base-ten digits sum to ten. Be careful, some have three digits.

**Problem 2.5** *Find an example of an infinite set that has a finite complement, be sure to state the universal set.*

**Problem 2.6** *Find an example of an infinite set that has an infinite complement, be sure to state the universal set.*

**Problem 2.7** *Add parenthesis to each of the following expressions that enforce the operator precedence rules as in Example 2.7. Notice that the first three describe sets while the last returns a logical value (true of false).*

(i) $A \cup B \cup C \cup D$

(ii) $A \cup B \cap C \cup D$

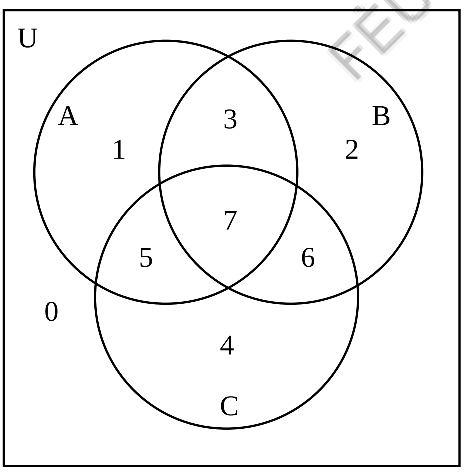
(iii) $A^c \cap B^c \cup C$

(iv) $A \cup B = A \cap C$

**Problem 2.8** *Give the Venn diagrams for the following sets.*

(i) $A - B$ (ii) $B - A$ (iii) $A^c \cap B$

(iv) $A \Delta B$ (v) $(A \Delta B)^c$ (vi) $A^c \cup B^c$




**Problem 2.9** *Examine the Venn diagram above. Notice that every combination of sets has a unique number in common. Construct a similar collection of four sets.*

**Problem 2.10** *Read Problem 2.9. Can a system of sets of this sort be constructed for any number of sets? Explain your reasoning.*

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**Problem 2.11** *Suppose we take the universal set to be the set of non-negative integers. Let $E$ be the set of even numbers, $O$ be the set of odd numbers and $F = \{0, 1, 2, 3, 5, 8, 13, 21, 34, 89, 144, ...\}$ be the set of Fibonacci numbers. The Fibonacci sequence is $0, 1, 1, 2, 3, 5, 8, \ldots$ in which the next term is obtained by adding the previous two.*

(i) Prove that the intersection of $F$ with $E$ and $O$ are both infinite.

(ii) Make a Venn diagram for the sets $E$, $F$, and $O$, and explain why this is a Mickey-Mouse problem.

**Problem 2.12** *A binary operation $\odot$ is* **commutative** *if $x \odot y = y \odot x$. An example of a commutative operation is multiplication. Subtraction is non-commutative. Determine, with proof, if union, intersection, set difference, and symmetric difference are commutative.*

**Problem 2.13** *An* **identity** *for an operation $\odot$ is an object $i$ so that, for all objects $x$, $i \odot x = x \odot i = x$. Find, with proof, identities for the operations* set union *and* set intersection.

**Problem 2.14** *Prove part (ii) of Proposition 2.2.*

**Problem 2.15** *Prove that*

$$A \cup (B \cup C) = (A \cup B) \cup C$$

**Problem 2.16** *Prove that*

$$A \cap (B \cap C) = (A \cap B) \cap C$$

**Problem 2.17** *Prove that*

$$A \Delta (B \Delta C) = (A \Delta B) \Delta C$$

**Problem 2.18** *Disprove that*

$$A \Delta (B \cup C) = (A \Delta B) \cup C$$

**Problem 2.19** *Consider the set $S = \{1, 2, 3, 4\}$. For each $k = 0, 1, \ldots, 4$ how many $k$ element subsets does $S$ have?*

**Problem 2.20** *Suppose we have a set $S$ with $n \geq 0$ elements. Find a formula for the number of different subsets of $S$ that have $k$ elements.*

**Problem 2.21** *For finite sets $S$ and $T$, prove*

$$|S \cup T| = |S| + |T| - |S \cap T|$$

## 2.2 Mathematical Induction

Mathematical induction is a technique used in proving mathematical assertions. The basic idea of induction is that we prove that a statement is true in one case and then also prove that if it is true in a given case it is true in the next case. This then permits the cases for which the statement is true to cascade from the initial true case. We will start with the mathematical foundations of induction.

We assume that the reader is familiar with the symbols $<$, $>$, $\leq$ and $\geq$. From this point on we will denote the set of integers by the symbol $\mathbb{Z}$. The non-negative integers are called the *natural numbers*. The symbol for the set of natural numbers is $\mathbb{N}$. Any mathematical system rests on a foundation of axioms. Axioms are things that we simply assume to be true. We will assume the truth of the following principle, adopting it as an axiom.

**The well-ordering principle:** Every non-empty set of natural numbers contains a smallest element.

The well ordering principle is an axiom that agrees with the common sense of most people familiar with the natural numbers. An empty set does not contain a smallest member because it contains no members at all. As soon as we have a set of natural numbers with some members then we can order those members in the usual fashion. Having ordered them, one will be smallest. This intuition agreeing with this latter claim depends strongly on the fact the integers are "whole numbers" spaced out in increments of one. To see why this is important consider the smallest positive distance. If such a distance existed, we could cut it in half to obtain a smaller distance - the quantity contradicts its own existence. The well-ordering principle can be used to prove the correctness of induction.

**Theorem 2.1 Mathematical Induction I** *Suppose that $P(n)$ is a proposition that it either true or false for any given natural numbers $n$. If*

*(i) $P(0)$ is true and,*

*(ii) when $P(n)$ is true so is $P(n+1)$*

*Then we may deduce that $P(n)$ is true for any natural number.*

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Proof:

Assume that (i) and (ii) are both true statements. Let $S$ be the set of all natural numbers for which $P(n)$ is false. If $S$ is empty then we are done, so assume that $S$ is not empty. Then, by the well ordering principle, $S$ has a least member $m$. By (i) above $m \neq 0$ and so $m-1$ is a natural number. Since $m$ is the smallest member of $S$ it follows that $P(m-1)$ is true. But this means, by (ii) above, that $P(m)$ is true. We have a contradiction and so our assumption that $S \neq \emptyset$ must be wrong. We deduce $S$ is empty and that as a consequence $P(n)$ is true for all $n \in \mathbb{N}$. □

The technique used in the above proof is called *proof by contradiction*. We start by assuming the logical opposite of what we want to prove, in this case that there is some $m$ for which $P(m)$ is false, and from that assumption we derive an impossibility. If an assumption can be used to demonstrate an impossibility then it is false and its logical opposite is true.

A nice problem on which to demonstrate mathematical induction is counting how many subsets a finite set has.

**Proposition 2.3 Subset counting**. *A set $S$ with $n$ elements has $2^n$ subsets.*

Proof:

First we check that the proposition is true when $n = 0$. The empty set has exactly one subset: itself. Since $2^0 = 1$ the proposition is true for $n = 0$. We now assume the proposition is true for some $n$. Suppose that $S$ is a set with $n+1$ members and that $x \in S$. Then $S-\{x\}$ (the set difference of $S$ and a set $\{x\}$ containing only $x$) is a set of $n$ elements and so, by the assumption, has $2^n$ subsets. Now every subset of $S$ either contains $x$ or it fails to. Every subset of $S$ that does not contain $x$ is a subset of $S-\{x\}$ and so there are $2^n$ such subsets of $S$. Every subset of $S$ that contains $x$ may be obtained in exactly one way from one that does not by taking the union with $\{x\}$. This means that the number of subsets of $S$ containing or failing to contain $x$ are equal. This means there are $2^n$ subsets of $S$ containing $x$. The total number of subsets of $S$ is thus $2^n + 2^n = 2^{n+1}$. So if we assume the proposition is true for $n$ we can demonstrate that it is also true for $n+1$. It follows by mathematical induction that the proposition is true for all natural numbers. □

The set of all subsets of a given set is itself an important object and so has a name.

**Definition 2.13** *The set of all subsets of a set $S$ is called the* **powerset** *of $S$. The notation for the powerset of $S$ is $\mathcal{P}(S)$.*

This definition permits us to rephrase Proposition 2.3 as follows: the power set of a set of $n$ elements has size $2^n$.

Theorem 2.1 lets us prove propositions that are true on the natural numbers, starting at zero. A small modification of induction can be used to prove statements that are true only for those $n \geq k$ for any integer $k$. All that is needed is to use induction on a proposition $Q(n-k)$ where $Q(n-k)$ is logically equivalent to $P(n)$. If $Q(n-k)$ is true for $n-k \geq 0$ then $P(n)$ is true for $n \geq k$ and we have the modified induction. The practical difference is that we start with $k$ instead of zero.

**Example 2.11** *Prove that $n^2 \geq 2n$ for all $n \geq 2$.*

*Notice that $2^2 = 4 = 2 \times 2$ so the proposition is true when $n = 2$. We next assume that $P(n)$ is true for some $n$ and we compute:*

$$\begin{array}{rcl} n^2 & \geq & 2n \\ n^2+2n+1 & \geq & 2n+2n+1 \\ (n+1)^2 & \geq & 2n+2n+1 \\ (n+1)^2 & \geq & 2n+1+1 \\ (n+1)^2 & \geq & 2n+2 \\ (n+1)^2 & \geq & 2(n+1) \end{array}$$

*To move from the third step to the fourth step we use the fact that $2n > 1$ when $n \geq 2$. The last step is $P(n+1)$, which means we have deduced $P(n+1)$ from $P(n)$. Using the modified form of induction we have proved that $n^2 \geq 2n$ for all $n \geq 2$.*

It is possible to formalize the procedure for using mathematical induction into a three-part process. Once we have a proposition $P(n)$,

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(i) First demonstrate a *base case* by directly demonstrating $P(k)$,

(ii) Next make the *induction hypothesis* that $P(n)$ is true for some $n$,

(iii) Finally, starting with the assumption that $P(n)$ is true, demonstrate $P(n+1)$.

These steps permit us to deduce that $P(n)$ is true for all $n \geq k$.

**Example 2.12** *Using induction, prove*

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

*In this case $P(n)$ is the statement*

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

**Base case:** $1 = \frac{1}{2}1(1+1)$, *so $P(1)$ is true.* **Induction hypothesis:** *for some $n$,*

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

*Compute:*

$$\begin{aligned}
1 + 2 + \cdots + (n+1) &= 1 + 2 + \cdots + n + (n+1) \\
&= \frac{1}{2}n(n+1) + (n+1) \\
&= \frac{1}{2}\left(n(n+1) + 2(n+1)\right) \\
&= \frac{1}{2}\left(n^2 + 3n + 2\right) \\
&= \frac{1}{2}(n+1)(n+2) \\
&= \frac{1}{2}(n+1)((n+1)+1)
\end{aligned}$$

*and so we have shown that if $P(n)$ is true then so is $P(n+1)$. We have thus proven that $P(n)$ is true for all $n \geq 1$ by mathematical induction.*

We now introduce *sigma notation* which makes problems like the one worked in Example 2.12 easier to state and manipulate. The symbol $\sum$ is used to add up lists of numbers. If we wished to sum some formula $f(i)$ over a range from $a$ to $b$, that is to say $a \leq i \leq b$, then we write :

$$\sum_{i=a}^{b} f(i)$$

On the other hand if $S$ is a set of numbers and we want to add up $f(s)$ for all $s \in S$ we write:

$$\sum_{s \in S} f(s)$$

The result proved in Example 2.12 may be stated in the following form using sigma notation.

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

**Proposition 2.4** *Suppose that $c$ is a constant and that $f(i)$ and $g(i)$ are formulas. Then*

*(i)* $\sum_{i=a}^{b}(f(i) + g(i)) = \sum_{i=a}^{b} f(i) + \sum_{i=a}^{b} g(i)$

*(ii)* $\sum_{i=a}^{b}(f(i) - g(i)) = \sum_{i=a}^{b} f(i) - \sum_{i=a}^{b} g(i)$

*(iii)* $\sum_{i=a}^{b} c \cdot f(i) = c \cdot \sum_{i=a}^{b} f(i)$.

Proof:

Part (i) and (ii) are both simply the associative law for addition: $a+(b+c) = (a+b)+c$ applied many times. Part (iii) is a similar multiple application of the distributive law $ca + cb = c(a+b)$. □

The sigma notation lets us work with indefinitely long (and even infinite) sums. There are other similar notations. If $A_1, A_2, \ldots, A_n$ are sets then the intersection or union of all these sets can be written:

$$\bigcap_{i=1}^{n} A_i$$

$$\bigcup_{i=1}^{n} A_i$$

Similarly if $f(i)$ is a formula on the integers then

$$\prod_{i=1}^{n} f(i)$$

is the notation for computing the product $f(1) \cdot f(2) \cdot \cdots \cdot f(n)$. This notation is called **Pi** notation.

**Definition 2.14** *When we solve an expression involving $\sum$ to obtain a formula that does not use $\sum$ or "..." as in Example 2.12 then we say we have found a* **closed form** *for the expression. Example 2.12 finds a closed form for $\sum_{i=1}^{n} i$.*

At this point we introduce a famous mathematical sequence in order to create an arena for practicing proofs by induction.

**Definition 2.15** *The* **Fibonacci numbers** *are defined as follows. $f_1 = f_2 = 1$ and, for $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$.*

**Example 2.13** *The Fibonacci numbers with four or fewer digits are: $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_6 = 8$, $f_7 = 13$, $f_8 = 21$, $f_9 = 34$, $f_{10} = 55$, $f_{11} = 89$, $f_{12} = 144$, $f_{13} = 233$, $f_{14} = 377$, $f_{15} = 610$, $f_{16} = 987$, $f_{17} = 1597$, $f_{18} = 2584$, $f_{19} = 4181$, and $f_{20} = 6765$.*

**Example 2.14** *Prove that the Fibonacci number $f_{3n}$ is even.*

**Solution:**

*Notice that $f_3 = 2$ and so the proposition is true when $n = 1$. Assume that the proposition is true for some $n \geq 1$. Then:*

$$\begin{aligned} f_{3(n+1)} &= f_{3n+3} && (2.1)\\ &= f_{3n+2} + f_{3n+1} && (2.2)\\ &= f_{3n+1} + f_{3n} + f_{3n+1} && (2.3)\\ &= 2 \cdot f_{3n+1} + f_{3n} && (2.4) \end{aligned}$$

*but this suffices because $f_{3n}$ is even by the induction hypothesis while $2 \cdot f_{3n+1}$ is also even. The sum is thus even and so $f_{3(n+1)}$ is even. If follows by induction that $f_{3n}$ is even for all $n$.* □

## Problems

**Problem 2.22** *Suppose that $S = \{a, b, c\}$. Compute and list explicitly the members of the powerset, $\mathcal{P}(S)$.*

**Problem 2.23** *Prove that for a finite set $X$ that*

$$|X| \leq |\mathcal{P}(X)|$$

**Problem 2.24** *Suppose that $X \subseteq Y$ with $|Y| = n$ and $|X| = m$. Compute the number of subsets of $Y$ that contain $X$.*

**Problem 2.25** *Compute the following sums.*

(i) $\sum_{i=1}^{20} i$,

(ii) $\sum_{i=10}^{30} i$, and

(iii) $\sum_{i=-20}^{21} \ i$.

**Problem 2.26** *Using mathematical induction, prove the following formulas.*

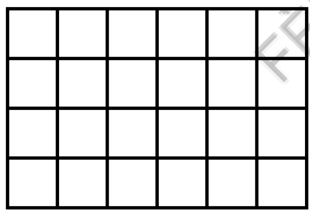
(i) $\sum_{i=1}^{n} 1 = n$,

(ii) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, and

(iii) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

**Problem 2.27** *If $f(i)$ and $g(i)$ are formulas and $c$ and $d$ are constants prove that*

$$\sum_{i=a}^{b}(c \cdot f(i) + d \cdot g(i)) = c \cdot \sum_{i=a}^{b} f(i) + d \cdot \sum_{i=a}^{b} g(i)$$



**Problem 2.28** *Suppose you want to break an $n \times m$ chocolate bar, like the $6 \times 4$ example shown above, into pieces corresponding to the small squares shown. What is the minimum number of breaks you can make? Prove your answer is correct.*

**Problem 2.29** *Prove by induction that the sum of the first $n$ odd numbers equals $n^2$.*

**Problem 2.30** *Compute the sum of the first $n$ positive even numbers.*

**Problem 2.31** *Find a closed form for*

$$\sum_{i=1}^{n} i^2 + 3i + 5$$

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**Problem 2.32** *Let $f(n,3)$ be the number of subsets of $\{1,2,\ldots,n\}$ of size 3. Using induction, prove that $f(n,3) = \frac{1}{6}n(n-1)(n-2)$.*

**Problem 2.33** *Suppose that we have sets $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ such that $X_i \subseteq Y_i$. Prove that the intersection of all the $X_i$ is a subset of the intersection of all the $Y_i$:*

$$\bigcap_{i=1}^{n} X_i \subseteq \bigcap_{i=1}^{n} Y_i$$

**Problem 2.34** *Suppose that $S_1, S_2, \ldots S_n$ are sets. Prove the following generalization of DeMorgan's laws:*

*(i)* $\left(\bigcap_{i=1}^{n} S_i\right)^c = \bigcup_{i=1}^{n} S_i^c$, *and*

*(ii)* $\left(\bigcup_{i=1}^{n} S_i\right)^c = \bigcap_{i=1}^{n} S_i^c$.

**Problem 2.35** *Prove by induction that the Fibonacci number $f_{4n}$ is a multiple of 3.*

**Problem 2.36** *Prove that if $r$ is a real number $r \neq 1$ and $r \neq 0$ then*

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

**Problem 2.37** *Prove by induction that the Fibonacci number $f_{5n}$ is a multiple of 5.*

**Problem 2.38** *Prove by induction that the Fibonacci number $f_n$ has the value*

$$f_n = \frac{\sqrt{5}}{5} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

**Problem 2.39** *Prove that for sufficiently large $n$ the Fibonacci number $f_n$ is the integer closest to*

$$\frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

*and compute the exact value of $f_{30}$. Show your work (i.e. don't look the result up on the net).*

**Problem 2.40** *Prove that $\frac{n(n-1)(n-2)(n-3)}{24}$ is a whole number for any whole number $n$.*

**Problem 2.41** *Consider the statement "All cars are the same color." and the following "proof".*

*Proof:*

*We will prove for $n \geq 1$ that for any set of $n$ cars all the cars in the set have the same color.*

- *Base Case: n=1 If there is only one car then clearly there is only one color the car can be.*
- *Inductive Hypothesis: Assume that for any set of n cars there is only one color.*
- *Inductive step: Look at any set of n + 1 cars. Number them: $1, 2, 3, \ldots, n, n+1$. Consider the sets $\{1, 2, 3, \ldots, n\}$ and $\{2, 3, 4, ..., n+1\}$. Each is a set of only n cars, therefore for each set there is only one color. But the $n^{th}$ car is in both sets so the color of the cars in the first set must be the same as the color of the cars in the second set. Therefore there must be only one color among all n + 1 cars.*
- *The proof follows by induction.* □

*What are the problems with this proof?*

## 2.3 Functions

In this section we will define functions and extend much of our ability to work with sets to infinite sets. There are a number of different types of functions and so this section contains a great deal of terminology.

Recall that two finite sets are the same size if they contain the same number of elements. It is possible to make this idea formal by using functions and, once the notion is formally defined, it can be applied to infinite sets.

**Definition 2.16** *An* **ordered pair** *is a collection of two elements with the added property that one element comes first and one element comes second. The set containing only $x$ and $y$ (for $x \neq y$) is written $\{x, y\}$. The ordered pair containing $x$ and $y$ with $x$ first is written $(x, y)$. Notice that while $\{x, x\}$ is not a well defined set, $(x, x)$ is a well defined ordered pair because the two copies of $x$ are different by virtue of coming first and second.*

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The reason for defining ordered pairs at this point is that it permits us to make an important formal definition that pervades the rest of mathematics.

**Definition 2.17** *A* **function** $f$ *with* **domain** $S$ *and* **range** $T$ *is a set of ordered pairs* $(s, t)$ *with first element from* $S$ *and second element from* $T$ *that has the property that every element of* $S$ *appears exactly once as the first element in some ordered pair. We write* $f : S \to T$ *for such a function.*

**Example 2.15** *Suppose that* $A = \{a, b, c\}$ *and* $B = \{0, 1\}$ *then*

$$f = \{(a, 0), (b, 1), (c, 0)\}$$

*is a function from* $A$ *to* $B$*. The function* $f : A \to B$ *can also be specified by saying* $f(a) = 0$*,* $f(b) = 1$ *and* $f(c) = 0$*.*

*The set of ordered pairs* $\{(a, 0), (b, 1)\}$ *is not a function from* $A$ *to* $B$ *because* $c$ *is not the first coordinate of any ordered pair. The set of ordered pairs* $\{(a, 0), (a, 1), (b, 0), (c, 0)\}$ *is not a function from* $A$ *to* $B$ *because* $a$ *appears as the first coordinate of two different ordered pairs.*

In calculus you may have learned the *vertical line rule* that states that the graph of a function may not intersect a vertical line at more than one point. This corresponds to requiring that each point in the domain of the function appear in only one ordered pair. In set theory, all functions are required to state their domain and range when they are defined. In calculus functions had a domain that was a subset of the real numbers and you were sometimes required to identify the subset.

**Example 2.16** *This example contrasts the way functions were treated in a typical calculus course with the way we treat them in set theory.*

**Calculus:** *find the domain of the function*

$$f(x) = \sqrt{x}$$

*Since we know that the square root function exists only for non-negative real numbers the domain is* $\{x : x \geq 0\}$*.*

**Set theory:** *the function* $f = \sqrt{x}$ *from the non-negative real numbers to the real numbers is the set of ordered pairs* $\{(r^2, r) : r \geq 0\}$*. This function is well defined because each non-negative real number is the square of some positive real number.*

The major contrasts between functions in calculus and functions in set theory are:

(i) The domain of functions in calculus are often specified only by implication (you have to know how all the functions used work) and are almost always a subset of the real numbers. The domain in set theory must be explicitly specified and may be any set at all.

(ii) Functions in calculus typically had graphs that you could draw and look at. Geometric intuition driven by the graphs plays a major role in our understanding of functions. Functions in set theory are seldom graphed and often don't have a graph.

A point of similarity between calculus and set theory is that the range of the function is not explicitly specified. When we have a function $f : S \to T$ then the range of $f$ is a subset of $T$.

**Definition 2.18** *If* $f$ *is a function then we denote the domain of* $f$ *by dom*$(f)$ *and the range of* $f$ *by rng*$(f)$

**Example 2.17** *Suppose that* $f(n) : \mathbb{N} \to \mathbb{N}$ *is defined by* $f(n) = 2n$*. Then the domain and range of* $f$ *are the integers: dom*$(f) =$ *rng*$(f) = \mathbb{N}$*. If we specify the ordered pairs of* $f$ *we get*

$$f = \{(n, 2n) : n \in \mathbb{N}\}$$

There are actually two definitions of range that are used in mathematics. The definition we are using, the set from which second coordinates of ordered pairs in a function are drawn, is also the definition typically using in computer science. The other definition is the set of second coordinates that actually appear in ordered pairs. This set, which we will define formally later, is the *image* of the function. To make matters even worse the set we are calling the range of a function is also called the *co-domain*. We include these confusing terminological notes for students that may try and look up supplemental material.

**Definition 2.19** *Let $X, Y$, and $Z$ be sets. The* **composition** *of two functions $f : X \to Y$ and $g : Y \to Z$ is a function $h : X \to Z$ for which $h(x) = g(f(x))$ for all $x \in X$. We write $g \circ f$ for the composition of $g$ with $f$.*

The definition of the composition of two functions requires a little checking to make sure it makes sense. Since *every* point must appear as a first coordinate of an ordered pair in a function, every result of applying $f$ to an element of $X$ is an element of $Y$ to which $g$ can be applied. This means that $h$ is a well-defined set of ordered pairs. Notice that the order of composition is important - if the sets $X$, $Y$, and $Z$ are distinct there is only one order in which composition even makes sense.

**Example 2.18** *Suppose that $f : \mathbb{N} \to \mathbb{N}$ is given by $f(n) = 2n$ while $g : \mathbb{N} \to \mathbb{N}$ is given by $g(n) = n + 4$. Then*

$$(g \circ f)(n) = 2n + 4$$

*while*

$$(f \circ g)(n) = 2(n + 4) = 2n + 8$$

We now start a series of definitions that divide functions into a number of classes. We will arrive at a point where we can determine if the mapping of a function is reversible, if there is a function that exactly reverses the action of a given function.

**Definition 2.20** *A function $f : S \to T$ is* **injective** *or* **one-to-one** *if no element of $T$ (no second coordinate) appears in more than one ordered pair. Such a function is called an* **injection**.

**Example 2.19** *The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = 2n$ is an injection. The ordered pairs of $f$ are $(n, 2n)$ and so any number that appears as a second coordinate does so once.*

*The function $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = n^2$ is not an injection. To see this notice that $g$ contains the ordered pairs $(1, 1)$ and $(-1, 1)$ so that 1 appears twice as the second coordinate of an ordered pair.*

**Definition 2.21** *A function $f : S \to T$ is* **surjective** *or* **onto** *if every element of $T$ appears in an ordered pair. Surjective functions are called* **surjections**.

We use the symbol $\mathbb{R}$ for the real numbers. We also assume familiarity with interval notation for contiguous subsets of the reals. For real numbers $a \leq b$

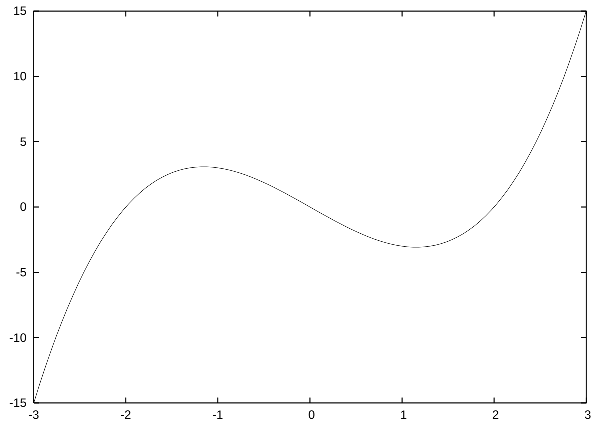
$$\begin{array}{lll} (a, b) & \text{is} & \{x : a < x < b\} \\ (a, b] & \text{is} & \{x : a < x \leq b\} \\ [a, b) & \text{is} & \{x : a \leq x < b\} \\ [a, b] & \text{is} & \{x : a \leq x \leq b\} \end{array}$$

**Example 2.20** *The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 5 - n$ is a surjection. If we set $m = 5 - n$ then $n = 5 - m$. This means that if we want to find some $n$ so that $f(n)$ is, for example, 8, then $5 - 8 = -3$ and we see that $f(-3) = 8$. This demonstrates that all $m$ have some $n$ so that $f(n) = m$, showing that all $m$ appear as the second coordinate of an ordered pair in $f$.*

*The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{x^2}{1+x^2}$ is not a surjection because $-1 < g(x) < 1$ for all $x \in \mathbb{R}$.*

**Definition 2.22** *A function that is both surjective and injective is said to be* **bijective**. *Bijective functions are called* **bijections**.

**Example 2.21** *The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n$ is a bijection. All of its ordered pairs have the same first and second coordinate. This function is called the* identity function.




*The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3 - 4x$ is not a bijection. It is not too hard to show that it is a surjection, but it fails to be an injection. The portion of the graph shown above demonstrates that $g(x)$ takes on the same value more than once. This means that*

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*some numbers appear twice as second coordinates of ordered pairs in $g$. We can use the graph because $g$ is a function from the real numbers to the real numbers.*

For a function $f : S \to T$ to be a bijection every element of $S$ appears in an ordered pair as the first member of an ordered pair and every element of $T$ appears in an ordered pair as the second member of an ordered pair. Another way to view a bijection is as a matching of the elements of $S$ and $T$ so that every element of $S$ is paired with an element of $T$. For finite sets this is clearly only possible if the sets are the same size and, in fact, this is the formal definition of "same size" for sets.

**Definition 2.23** *Two sets $S$ and $T$ are defined to be* **the same size** *or to have* **equal cardinality** *if there is a bijection $f : S \to T$.*

**Example 2.22** *The sets $A = \{a, b, c\}$ and $Z = \{1, 2, 3\}$ are the same size. This is obvious because they have the same number of elements, $|A| = |Z| = 3$ but we can construct an explicit bijection*

$$f = \{(a, 3), (b, 1), (c, 2)\}$$

*with each member of $A$ appearing once as a first coordinate and each member of $B$ appearing once as a second coordinate. This bijection is a* witness *that $A$ and $B$ are the same size.*

*Let $E$ be the set of even integers. Then the function*

$$g : \mathbb{Z} \to E$$

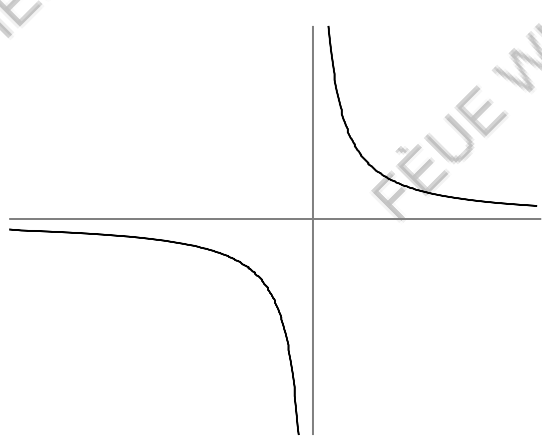
*in which $g(n) = 2n$ is a bijection. Notice that each integers can be put into $g$ and that each even integer has exactly one integer that can be doubled to make it. The existence of $g$ is a witness that the set of integers and the set of even integers are the same size. This may seem a bit bizarre because the set $\mathbb{Z} - E$ is the infinite set of odd integers. In fact one hallmark of an infinite set is that it can be the same size as a proper subset. This also means we now have an equality set for sizes of infinite sets. We will do a good deal more with this in Chapter 3.*

Bijections have another nice property: they can be unambiguously reversed.

**Definition 2.24** *The* **inverse** *of a function $f : S \to T$ is a function $g : T \to S$ so that for all $x \in S$, $g(f(x)) = x$ and for all $y \in T$, $f(g(y)) = y$.*

If a function $f$ has an inverse we use the notation $f^{-1}$ for that inverse. Since an exponent of $-1$ also means reciprocal in some circumstances this can be a bit confusing. The notational confusion is resolved by considering context. So long as we keep firmly in mind that functions are sets of ordered pairs it is easy to prove the proposition/definition that follows after the next example.

**Example 2.23** *If $E$ is the set of even integers then the bijection $f(n) = 2n$ from $\mathbb{Z}$ to $E$ has the inverse $f^{-1} : E \to \mathbb{Z}$ given by $g(2n) = n$. Notice that defining the rule for $g$ as depending on the argument $2n$ seamlessly incorporates the fact that the domain of $g$ is the even integers.*



*If $g(x) = \frac{x}{x-1}$, shown above with its asymptotes $x = 1$ and $y = 1$ then $f$ is a function from the set $H = \mathbb{R} - \{1\}$ to itself. The function was chosen to have asymptotes at equal $x$ and $y$ values; this is a bit unusual. The function $g$ is a bijection. Notice that the graph intersects any horizontal or vertical line in at most one point. Every value except $x = 1$ may be put into $g$ meaning that $g$ is a function on $H$. Since the vertical asymptote goes off to $\infty$ in both directions, all values in $H$ come out of $g$. This demonstrates $g$ is a bijection. This means that it has an inverse which we now compute using a standard* *technique from calculus classes.*

$$\begin{aligned} y &= \frac{x}{x-1} \\ y(x-1) &= x \\ xy - y &= x \\ xy - x &= y \\ x(y-1) &= y \\ x &= \frac{y}{y-1} \end{aligned}$$

*which tells us that* $g^{-1}(x) = \frac{x}{x-1}$ *so* $g = g^{-1}$*: the function is its own inverse.*

**Proposition 2.5** *A function has an inverse if and only if it is a bijection.*

Proof:

Suppose that $f : S \to T$ is a bijection. Then if $g : T \to S$ has ordered pairs that are the exact reverse of those given by $f$ it is obvious that for all $x \in S$, $g(f(x)) = x$, likewise that for all $y \in T$, $f(g(y)) = y$. We have that bijections posses inverses. It remains to show that non-bijections do not have inverses.

If $f : S \to T$ is not a bijection then either it is not a surjection or it is not an injection. If $f$ is not a surjection then there is some $t \in T$ that appears in no ordered pair of $f$. This means that no matter what $g(t)$ is, $f(g(t)) \neq t$ and we fail to have an inverse. If, on the other hand, $f : S \to T$ is a surjection but fails to be an injection then for some distinct $a, b \in S$ we have that $f(a) = t = f(b)$. For $g : T \to S$ to be an inverse of $f$ we would need $g(t) = a$ and $g(t) = b$, forcing $t$ to appear as the first coordinate of two ordered pairs in $g$ and so rendering $g$ a non-function. We thus have that non-bijections do not have inverses. $\square$

The type of inverse we are discussing above is a *two-sided inverse*. The functions $f$ and $f^{-1}$ are mutually inverses of one another. It is possible to find a function that is a one-way inverse of a function so that $f(g(x)) = x$ but $g(f(x))$ is not even defined. These are called *one-sided inverses*.

Note on mathematical grammar: Recall that when two notions, such as "bijection" and "has an inverse" are equivalent we use the phrase "if and only if" (abbreviated iff) to phrase a proposition declaring that the notions are equivalent. A proposition that $A$ iff $B$ is proven by first assuming $A$ and deducing $B$ and then separately assuming $B$ and deducing $A$. The formal symbol for $A$ iff $B$ is $A \Leftrightarrow B$. Likewise we have symbols for the ability to deduce $B$ given $A$, $A \Rightarrow B$ and vice-versa $B \Rightarrow A$. These symbols are spoken "A implies B" and "B implies A" respectively.

**Proposition 2.6** *Suppose that* $X$*,* $Y$*, and* $Z$ *are sets. If* $f : X \to Y$ *and* $g : Y \to Z$ *are bijections then so is* $g \circ f : X \to Z$*.*

Proof: *this proof is left as an exercise.*

**Definition 2.25** *Suppose that* $f : A \to B$ *is a function. The* **image of** $A$ **in** $B$ *is the subset of* $B$ *made of elements that appear as the second element of ordered pairs in* $f$*. Colloquially the image of* $f$ *is the set of elements of* $B$ *hit by* $f$*. We use the notation* $Im(f)$ *for images. In other words* $Im(f) = \{f(a) : a \in A\}$*.*

**Example 2.24** *If* $f : \mathbb{N} \to \mathbb{N}$ *is given by the rule* $f(n) = 3n$ *then the set* $T = \{0, 3, 6, \ldots\}$ *of natural numbers that are multiples of three is the image of* $f$*. Notation:* $Im(f) = T$*.*

*If* $g : \mathbb{R} \to \mathbb{R}$ *given by* $g(x) = x^2$ *then*

$$Im(g) = \{y : y \geq 0, y \in \mathbb{R}\}$$

There is a name for the set of all ordered pairs drawn from two sets.

**Definition 2.26** *If* $A$ *and* $B$ *are sets then the set of all ordered pairs with the first element from* $A$ *and the second from* $B$ *is called the* **Cartesian Product** *of* $A$ *and* $B$*.*

The notation for the Cartesian product of $A$ and $B$ is $A \times B$. using curly brace notation:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

**Example 2.25** *If* $A = \{1, 2\}$ *and* $B = \{x, y\}$ *then*

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

*The* **Cartesian plane** *is an example of a Cartesian product of the real numbers with themselves:* $\mathbb{R} \times \mathbb{R}$*.*

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### 2.3.1 Permutations

In this section we will look at a very useful sort of function, bijections of finite sets.

**Definition 2.27** *A* **permutation** *is a bijection of a finite set with itself. Likewise a bijection of a finite set $X$ with itself is called a* **permutation of X**.

**Example 2.26** *Let $A = \{a, b, c\}$ then the possible permutations of $A$ consist of the following six functions:*

| | |
|---|---|
| *{(a,a)(b,b)(c,c)}* | *{(a,a)(b,c)(c,b)}* |
| *{(a,b)(b,a)(c,c)}* | *{(a,b)(b,c)(c,a)}* |
| *{(a,c)(b,a)(c,b)}* | *{(a,c)(b,b)(c,a)}* |

Notice that the number of permutations of three objects does not depend on the identity of those objects. In fact there are always six permutations of any set of three objects. We now define a handy function that uses a rather odd notation. The method of showing permutations in Example 2.26, explicit listing of ordered pairs, is a bit cumbersome.

**Definition 2.28** *Assume that we have agreed on an order, e.g. a,b,c, for the members of a set $X = \{a, b, c\}$. Then* **one-line** *notation for a permutation $f$ consists of listing the first coordinate of the ordered pairs in the agreed on order. The table in Example 2.26 would become:*

| | |
|---|---|
| **abc** | **acb** |
| **bac** | **bca** |
| **cab** | **cba** |

*in one line notation. Notice the saving of space.*

**Definition 2.29** *The* **factorial** *of a natural number $n$ is the product*

$$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = \prod_{i=1}^{n} i$$

*with the convention that the factorial of 0 is 1. We denote the factorial of $n$ as $n!$, spoken "n factorial".*

**Example 2.27** *Here are the first few factorials:*

| *n* | *0* | *1* | *2* | *3* | *4* | *5* | *6* | *7* |
|---|---|---|---|---|---|---|---|---|
| *n!* | *1* | *1* | *2* | *6* | *24* | *120* | *720* | *5040* |

**Proposition 2.7** *The number of permutations of a finite set with $n$ elements is $n!$.*

Proof: *this proof is left as an exercise.*

Notice that one implication of Proposition 2.6 is that the composition of two permutations is a permutation. This means that the set of permutations of a set is *closed* under functional composition.

**Definition 2.30** *A* **fixed point** *of a function $f : S \to S$ is any $x \in S$ such that $f(x) = x$. We say that* **f fixes x**.

## Problems

**Problem 2.42** *Suppose for finite sets $A$ and $B$ that $f : A \to B$ is an injective function. Prove that*

$$|B| \geq |A|$$

**Problem 2.43** *Suppose that for finite sets $A$ and $B$ that $f : A \to B$ is a surjective function. Prove that $|A| \geq |B|$.*

**Problem 2.44** *Using functions from the integers to the integers give an example of*

(i) A function that is an injection but not a surjection.

(ii) A function that is a surjection but not an injection.

(iii) A function that is neither an injection nor a surjection.

(iv) A bijection that is not the identity function.

**Problem 2.45** *For each of the following functions from the real numbers to the real numbers say if the function is surjective or injective. It may be neither.*

(i) $f(x) = x^2$ (ii) $g(x) = x^3$

(iii) $h(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -\sqrt{-x} & x < 0 \end{cases}$

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## *Interlude*

**The Collatz Conjecture**

One of the most interesting features of mathematics is that it is possible to phrase problems in a few lines that turn out to be incredibly hard. The Collatz conjecture was first posed in 1937 by Lothar Collatz. Define the function $f$ from the natural numbers to the natural numbers with the rule

$$f(n) = \begin{cases} 3n+1 & \text{n odd} \\ \frac{n}{2} & \text{n even} \end{cases}$$

Collatz' conjecture is that if you apply $f$ repeatedly to a positive integer then the resulting sequence of numbers eventually arrives at one. If we start with 17, for example, the result of repeatedly applying $f$ is:

$$f(17) = 52, f(52) = 26, f(26) = 13, f(13) = 40, f(40) = 20, f(20) = 10,$$
$$f(10) = 5, f(5) = 16, f(16) = 8, f(8) = 4, f(4) = 2, f(2) = 1$$

The sequences of numbers generated by repeatedly applying $f$ to a natural number are called *hailstone sequences* with the collapse of the value when a large power of 2 appears being analogous to the impact of a hailstone. If we start with the number 27 then 111 steps are required to reach one and the largest intermediate number is 9232. This quite irregular behavior of the sequence is not at all apparent in the original phrasing of the problem.

The Collatz conjecture has been checked for numbers up to $5 \times 2^{61}$ (about $5.764 \times 10^{18}$) by using a variety of computational tricks. It has not, however, been proven or disproven. The very simple statement of the problem causes mathematicians to underestimate the difficulty of the problem. At one point a mathematician suggested that the problem might have been developed by the Russians as a way to slow American mathematical research. This was after several of his colleagues spent months working on the problem without obtaining results.

A simple (but incorrect) argument suggests that hailstone sequences ought to grow indefinitely. Half of all numbers are odd, half are even. The function $f$ slightly more than triples odd numbers and divides even numbers in half. Thus, on average, $f$ increases the value of numbers. The problem is this: half of all even numbers are multiples of four and so are divided in half twice. One-quarter of all even numbers are multiples of eight and so get divided in half three times, and so on. The net effect of factors that are powers of two is to defeat the simple argument that $f$ grows "on average".

**Problem 2.46** *True or false (and explain): The function* $f(x) = \frac{x-1}{x+1}$ *is a bijection from the real numbers to the real numbers.*

**Problem 2.47** *Find a function that is an injection of the integers into the even integers that does not appear in any of the examples in this chapter.*

**Problem 2.48** *Suppose that* $B \subset A$ *and that there exists a bijection* $f : A \to B$*. What may be reasonably deduced about the set* $A$*?*

**Problem 2.49** *Suppose that* $A$ *and* $B$ *are finite sets. Prove that* $|A \times B| = |A| \cdot |B|$*.*

**Problem 2.50** *Suppose that we define* $h : \mathbb{N} \to \mathbb{N}$ *as follows. If* $n$ *is even then* $h(n) = n/2$ *but if* $n$ *is odd then* $h(n) = 3n+1$*. Determine if* $h$ *is a (i) surjection or (ii) injection.*

**Problem 2.51** *Prove proposition 2.6.*

**Problem 2.52** *Prove or disprove: the composition of injections is an injection.*

**Problem 2.53** *Prove or disprove: the composition of surjections is a surjection.*

**Problem 2.54** *Prove proposition 2.7.*

**Problem 2.55** *List all permutations of*

$$X = \{1, 2, 3, 4\}$$

*using one-line notation.*

**Problem 2.56** *Suppose that* $X$ *is a set and that* $f$*,* $g$*, and* $h$ *are permutations of* $X$*. Prove that the equation* $f \circ g = h$ *has a solution* $g$ *for any given permutations* $f$ *and* $h$*.*

**Problem 2.57** *Examine the permutation* $f$ *of* $Q = \{a, b, c, d, e\}$ *which is* **bcaed** *in one line notation. If we create the series* $f, f \circ f, f \circ (f \circ f), \ldots$ *does the identity function,* **abcde***, ever appear in the series? If so, what is its first appearance? If not, why not?*

**Problem 2.58** *If* $f$ *is a permutation of a finite set, prove that the sequence* $f, f \circ f, f \circ (f \circ f), \ldots$ *must contain repeated elements.*

**Problem 2.59** *Suppose that* $X$ *and* $Y$ *are finite sets and that* $|X| = |Y| = n$*. Prove that there are* $n!$ *bijections of* $X$ *with* $Y$*.*

**Problem 2.60** *Suppose that* $X$ *and* $Y$ *are sets with* $|X| = n$*,* $|Y| = m$*. Count the number of functions from* $X$ *to* $Y$*.*

**Problem 2.61** *Suppose that* $X$ *and* $Y$ *are sets with* $|X| = n$*,* $|Y| = m$ *for* $m > n$*. Count the number of injections of* $X$ *into* $Y$*.*

**Problem 2.62** *For a finite set* $S$ *with a subset* $T$ *prove that the permutations of* $S$ *that have all members of* $T$ *as fixed points form a set that is closed under functional composition.*

**Problem 2.63** *Compute the number of permutations of a set* $S$ *with* $n$ *members that fix at least* $m < n$ *points.*

**Problem 2.64** *Using any technique at all, estimate the fraction of permutations of an* $n$*-element set that have no fixed points. This problem is intended as an exploration.*

**Problem 2.65** *Let* $X$ *be a finite set with* $|X| = n$*. Let* $C = X \times X$*. How many subsets of* $C$ *have the property that every element of* $X$ *appears once as a first coordinate of some ordered pair and once as a second coordinate of some ordered pair?*

**Problem 2.66** *An alternate version of Sigma* ($\sum$) *and Pi* ($\prod$) *notation works by using a set as an index. So if* $S = \{1, 3, 5, 7\}$ *then*

$$\sum_{s \in S} s = 16 \quad \textit{and} \quad \prod_{s \in S} s = 105$$

*Given all the material so far, give and defend reasonable values for the sum and product of an empty set.*

**Problem 2.67** *Suppose that* $f_\alpha : [0, 1] \to [0, 1]$ *for* $-1 < \alpha < \infty$ *is given by*

$$f_\alpha(x) = \frac{(\alpha + 1)x}{\alpha x + 1},$$

*prove that* $f_\alpha$ *is a bijection.*

**Problem 2.68** *Find, to five decimals accuracy:*

$$Ln(200!)$$

*Explain how you obtained the answer.*

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## 2.4 $\infty + 1$

We conclude the chapter with a brief section that demonstrates a strange thing that can be accomplished with set notation. We choose to represent the natural numbers $0, 1, 2, \ldots$ by sets that contain the number of elements counted by the corresponding natural number. We also choose to do so as simply as possible, using only curly braces and commas. Given this the numbers and their corresponding sets are:

0 : {}
1 : {{}} = {0}
2 : {{},{{}}}={0,1}
3 : {{},{{}},{{},{{}}}}={0,1,2}
4 : {{},{{}},{{},{{}}},{{},{{}},{{},{{}}}}}
={0,1,2,3}

The trick for the above representation is this. Zero is represented by the empty set. One is represented by the set of the only thing we have constructed - zero, represented as the empty set. Similarly the representation of two is the set of the representation of zero and one (the empty set and the set of the empty set). This representation is incredibly inefficient but it uses a very small number of symbols. This representation also has a useful property. As always, we will start with a definition.

**Definition 2.31** *The* **minimal set representation** *of the natural numbers is constructed as follows:*

*(i) Let 0 be represented by the empty set.*

*(ii) For $n > 0$ let $n$ be represented by the set $\{0, 1, \ldots, n-1\}$.*

The shorthand $\{0, 1\}$ for $\{\{\}, \{\{\}\}\}$ is called the *simplified notation* for the minimal set representation. We now give the useful property of the minimal set representation.

**Proposition 2.8** $n + 1 = n \cup \{n\}$

Proof:
This follows directly from Definition 2.31 by considering the set difference of the representations of $n$ and $n - 1$. □
The definition says that any set of the representations of consecutive natural numbers, starting at zero, is the representation of the next natural number. This permits us to conclude that the set of all natural numbers

$$\{0, 1, 2, \ldots\}$$

fits the definition of a natural number. Which natural number is it? It is easy to see, in the minimal set representation, that for natural numbers $m$ and $n$, $m < n$ implies that the representation of $m$ is a subset of the representation of $n$. Every finite natural number is a subset of the set of all natural numbers and so we conclude that $\{0, 1, 2, \ldots\}$ is an infinite natural number. The set notation thus permits us to construct an infinite number.

The set consisting of the representations of all finite natural numbers is an infinite natural number. The number has been given the name $\omega$, the lower-case omega. In addition to being a letter omega traditionally also means "the last". The number $\omega$ comes after all the finite natural numbers. If we now apply Proposition 2.8 we see that

$$\omega \cup \{\omega\} = \omega + 1$$

This means that we can add one to an infinite number. Is the resulting number $\omega + 1$ a different number from $\omega$? It turns out the answer is "'yes", because the representations of these numbers are different as sets. The representation of $\omega$ contains no infinite sets while the representation of $\omega + 1$ contains one.

# Problems

**Problem 2.69** *Find the representation for 5 using the curly-brace-and-comma notation.*

**Problem 2.70** *Give the minimal set representation of $\omega + 2$ using the simplified notation.*

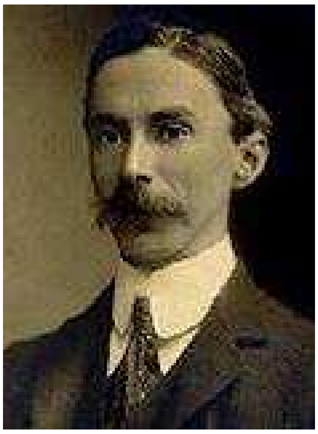
**Problem 2.71** *Suppose that $n > m$ are natural numbers and that $S$ is the minimal set representation of $n$ while $T$ is the minimal set representation of $m$. Is the representation of $n - m$ a member of the set difference $S - T$?*

**Problem 2.72** *Give a formula, as a function of $n$, for the number of times that the symbol* **{** *appears in the representation of $n$.*

**Problem 2.73** *Prove or disprove: there are an infinite number of distinct infinite numbers.*

# *Interlude*

## Russell's Paradox



**Bertrand Arthur William Russell, 3rd Earl Russell, OM, FRS (18 May 1872-2 February 1970), commonly known as simply Bertrand Russell, was a British philosopher, logician, mathematician, historian, religious skeptic, social reformer, socialist and pacifist. Although he spent the majority of his life in England, he was born in Wales, where he also died.**

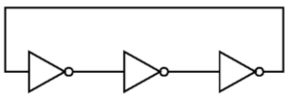
Let $Q$ be the set of all sets that do not contain themselves as a member. Consider the question: "Does $Q$ contain itself?" If the answer to this question is no then $Q$, by definition must contain itself. If, however, $Q$ contains itself then it is by definition unable to contain itself. This rather annoying contradiction, constructed by Russell, had a rather amusing side effect.

Friedrich Frege had just finished the second of a three volume set of works called the *Basic Laws of Arithmetic* that was supposed to remove all intuition from mathematics and place it on a purely logical basis. Russell wrote Frege, explaining his paradox. Frege added an appendix to his second volume that attempted to avoid Russell's paradox. The third volume was never published.

It is possible to resolve Russell's paradox by being much more careful about what objects may be defined to be sets; the *category* of all sets that do not contain themselves gives rise to no contradiction (it does give rise to an entire field of mathematics, category theory). The key to resolving the paradox from a set theoretic perspective is that one cannot assume that, for every property, there is a set of all things satisfying that property. This is a reason why it is important that a set is properly defined. Another consequence of Russell's paradox is a warning that self-referential statements are both potentially interesting and fairly dangerous, at least on the intellectual plane.

The original phrasing of Russell's paradox was in terms of normal and abnormal sets. A set is *normal* if it fails to contain itself and abnormal otherwise. Consider the set of all normal sets. If this set is abnormal, it contains itself but by definition the set contains only normal sets and hence it is itself normal. The normality of this set forces the set to contain itself, which makes it abnormal. This is simply a rephrasing of the original contradiction.

Puzzle: what does the circuit below have to do with Russell's paradox and what use is it?



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# String Theory

## University of Cambridge Part III Mathematical Tripos

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**Recommended Books and Resources**

- J. Polchinski, *String Theory*

This two volume work is the standard introduction to the subject. Our lectures will more or less follow the path laid down in volume one covering the bosonic string. The book contains explanations and descriptions of many details that have been deliberately (and, I suspect, at times inadvertently) swept under a very large rug in these lectures. Volume two covers the superstring.

- M. Green, J. Schwarz and E. Witten, *Superstring Theory*

Another two volume set. It is now over 20 years old and takes a slightly old-fashioned route through the subject, with no explicit mention of conformal field theory. However, it does contain much good material and the explanations are uniformly excellent. Volume one is most relevant for these lectures.

- B. Zwiebach, *A First Course in String Theory*

This book grew out of a course given to undergraduates who had no previous exposure to general relativity or quantum field theory. It has wonderful pedagogical discussions of the basics of lightcone quantization. More surprisingly, it also has some very clear descriptions of several advanced topics, even though it misses out all the bits in between.

- P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*

This big yellow book is affectionately known as the yellow pages. It's a great way to learn conformal field theory. At first glance, it comes across as slightly daunting because it's big. (And yellow). But you soon realise that it's big because it starts at the beginning and provides detailed explanations at every step. The material necessary for this course can be found in chapters 5 and 6.

Further References: "*String Theory and M-Theory*" by Becker, Becker and Schwarz and "*String Theory in a Nutshell*" (it's a big nutshell) by Kiritsis both deal with the bosonic string fairly quickly, but include more advanced topics that may be of interest. The book "*D-Branes*" by Johnson has lively and clear discussions about the many joys of D-branes. Links to several excellent online resources, including video lectures by Shiraz Minwalla, are listed on the course webpage.

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# Contents

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### Acknowledgements

These lectures are aimed at beginning graduate students. They assume a working knowledge of quantum field theory and general relativity. The lectures were given over one semester and are based broadly on Volume one of the book by Joe Polchinski. I inherited the course from Michael Green whose notes were extremely useful. I also benefited enormously from the insightful and entertaining video lectures by Shiraz Minwalla.

I'm grateful to Anirban Basu, Niklas Beisert, Joe Bhaseen, Diego Correa, Nick Dorey, Michael Green, Anshuman Maharana, Malcolm Perry and Martin Schnabl for discussions and help with various aspects of these notes. I'm also grateful to the students, especially Carlos Guedes, for their excellent questions and superhuman typo-spotting abilities. Finally, my thanks to Alex Considine for infinite patience and understanding over the weeks these notes were written. I am supported by the Royal Society.

## 0. Introduction

String theory is an ambitious project. It purports to be an all-encompassing theory of the universe, unifying the forces of nature, including gravity, in a single quantum mechanical framework.

The premise of string theory is that, at the fundamental level, matter does not consist of point-particles but rather of tiny loops of string. From this slightly absurd beginning, the laws of physics emerge. General relativity, electromagnetism and Yang-Mills gauge theories all appear in a surprising fashion. However, they come with baggage. String theory gives rise to a host of other ingredients, most strikingly extra spatial dimensions of the universe beyond the three that we have observed. The purpose of this course is to understand these statements in detail.

These lectures differ from most other courses that you will take in a physics degree. String theory is speculative science. There is no experimental evidence that string theory is the correct description of our world and scant hope that hard evidence will arise in the near future. Moreover, string theory is very much a work in progress and certain aspects of the theory are far from understood. Unresolved issues abound and it seems likely that the final formulation has yet to be written. For these reasons, I'll begin this introduction by suggesting some answers to the question: Why study string theory?

**Reason 1. String theory is a theory of quantum gravity**

String theory unifies Einstein's theory of general relativity with quantum mechanics. Moreover, it does so in a manner that retains the explicit connection with both quantum theory and the low-energy description of spacetime.

But quantum gravity contains many puzzles, both technical and conceptual. What does spacetime look like at the shortest distance scales? How can we understand physics if the causal structure fluctuates quantum mechanically? Is the big bang truely the beginning of time? Do singularities that arise in black holes really signify the end of time? What is the microscopic origin of black hole entropy and what is it telling us? What is the resolution to the information paradox? Some of these issues will be reviewed later in this introduction.

Whether or not string theory is the true description of reality, it offers a framework in which one can begin to explore these issues. For some questions, string theory has given very impressive and compelling answers. For others, string theory has been almost silent.



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**Reason 2. String theory may be *the* theory of quantum gravity**

With broad brush, string theory looks like an extremely good candidate to describe the real world. At low-energies it naturally gives rise to general relativity, gauge theories, scalar fields and chiral fermions. In other words, it contains all the ingredients that make up our universe. It also gives the only presently credible explanation for the value of the cosmological constant although, in fairness, I should add that the explanation is so distasteful to some that the community is rather amusingly split between whether this is a good thing or a bad thing. Moreover, string theory incorporates several ideas which do not yet have experimental evidence but which are considered to be likely candidates for physics beyond the standard model. Prime examples are supersymmetry and axions.

However, while the broad brush picture looks good, the finer details have yet to be painted. String theory does not provide unique predictions for low-energy physics but instead offers a bewildering array of possibilities, mostly dependent on what is hidden in those extra dimensions. Partly, this problem is inherent to any theory of quantum gravity: as we'll review shortly, it's a long way down from the Planck scale to the domestic energy scales explored at the LHC. Using quantum gravity to extract predictions for particle physics is akin to using QCD to extract predictions for how coffee makers work. But the mere fact that it's hard is little comfort if we're looking for convincing evidence that string theory describes the world in which we live.

While string theory cannot at present offer falsifiable predictions, it has nonetheless inspired new and imaginative proposals for solving outstanding problems in particle physics and cosmology. There are scenarios in which string theory might reveal itself in forthcoming experiments. Perhaps we'll find extra dimensions at the LHC, perhaps we'll see a network of fundamental strings stretched across the sky, or perhaps we'll detect some feature of non-Gaussianity in the CMB that is characteristic of D-branes at work during inflation. My personal feeling however is that each of these is a long shot and we may not know whether string theory is right or wrong within our lifetimes. Of course, the history of physics is littered with naysayers, wrongly suggesting that various theories will never be testable. With luck, I'll be one of them.

**Reason 3. String theory provides new perspectives on gauge theories**

String theory was born from attempts to understand the strong force. Almost forty years later, this remains one of the prime motivations for the subject. String theory provides tools with which to analyze down-to-earth aspects of quantum field theory that are far removed from high-falutin' ideas about gravity and black holes.



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Of immediate relevance to this course are the pedagogical reasons to invest time in string theory. At heart, it is the study of conformal field theory and gauge symmetry. The techniques that we'll learn are not isolated to string theory, but apply to countless systems which have direct application to real world physics.

On a deeper level, string theory provides new and very surprising methods to understand aspects of quantum gauge theories. Of these, the most startling is the *AdS/CFT correspondence*, first conjectured by Juan Maldacena, which gives a relationship between strongly coupled quantum field theories and gravity in higher dimensions. These ideas have been applied in areas ranging from nuclear physics to condensed matter physics and have provided qualitative (and arguably quantitative) insights into strongly coupled phenomena.

**Reason 4. String theory provides new results in mathematics**

For the past 250 years, the close relationship between mathematics and physics has been almost a one-way street: physicists borrowed many things from mathematicians but, with a few noticeable exceptions, gave little back. In recent times, that has changed. Ideas and techniques from string theory and quantum field theory have been employed to give new "proofs" and, perhaps more importantly, suggest new directions and insights in mathematics. The most well known of these is *mirror symmetry*, a relationship between topologically different Calabi-Yau manifolds.

The four reasons described above also crudely characterize the string theory community: there are "relativists" and "phenomenologists" and "field theorists" and "mathematicians". Of course, the lines between these different sub-disciplines are not fixed and one of the great attractions of string theory is its ability to bring together people working in different areas — from cosmology to condensed matter to pure mathematics — and provide a framework in which they can profitably communicate. In my opinion, it is this cross-fertilization between fields which is the greatest strength of string theory.

## 0.1 Quantum Gravity

This is a starter course in string theory. Our focus will be on the perturbative approach to the bosonic string and, in particular, why this gives a consistent theory of quantum gravity. Before we leap into this, it is probably best to say a few words about quantum gravity itself. Like why it's hard. And why it's important. (And why it's not).

The Einstein Hilbert action is given by

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\mathcal{R}$$

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Newton's constant $G_N$ can be written as

$$8\pi G_N = \frac{\hbar c}{M_{pl}^2}$$

Throughout these lectures we work in units with $\hbar = c = 1$. The Planck mass $M_{pl}$ defines an energy scale

$$M_{pl} \approx 2 \times 10^{18} \text{ GeV} .$$

(This is sometimes referred to as the reduced Planck mass, to distinguish it from the scale without the factor of $8\pi$, namely $\sqrt{1/G_N} \approx 1 \times 10^{19}$ GeV).

There are a couple of simple lessons that we can already take from this. The first is that the relevant coupling in the quantum theory is $1/M_{pl}$. To see that this is indeed the case from the perspective of the action, we consider small perturbations around flat Minkowski space,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{pl}} h_{\mu\nu}$$

The factor of $1/M_{pl}$ is there to ensure that when we expand out the Einstein-Hilbert action, the kinetic term for $h$ is canonically normalized, meaning that it comes with no powers of $M_{pl}$. This then gives the kind of theory that you met in your first course on quantum field theory, albeit with an infinite series of interaction terms,

$$S_{EH} = \int d^4x \; (\partial h)^2 + \frac{1}{M_{pl}} \, h \, (\partial h)^2 + \frac{1}{M_{pl}^2} \, h^2 \, (\partial h)^2 + \ldots$$

Each of these terms is schematic: if you were to do this explicitly, you would find a mess of indices contracted in different ways. We see that the interactions are suppressed by powers of $M_{pl}$. This means that quantum perturbation theory is an expansion in the dimensionless ratio $E^2/M_{pl}^2$, where $E$ is the energy associated to the process of interest. We learn that gravity is weak, and therefore under control, at low-energies. But gravitational interactions become strong as the energy involved approaches the Planck scale. In the language of the renormalization group, couplings of this type are known as *irrelevant*.

The second lesson to take away is that the Planck scale $M_{pl}$ is very very large. The LHC will probe the electroweak scale, $M_{EW} \sim 10^3$ GeV. The ratio is $M_{EW}/M_{pl} \sim 10^{-15}$. For this reason, quantum gravity will not affect your daily life, even if your daily life involves the study of the most extreme observable conditions in the universe.

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**Gravity is Non-Renormalizable**

Quantum field theories with irrelevant couplings are typically ill-behaved at high-energies, rendering the theory ill-defined. Gravity is no exception. Theories of this type are called *non-renormalizable*, which means that the divergences that appear in the Feynman diagram expansion cannot be absorbed by a finite number of counterterms. In pure Einstein gravity, the symmetries of the theory are enough to ensure that the one-loop S-matrix is finite. The first divergence occurs at two-loops and requires the introduction of a counterterm of the form,

$$\Gamma \sim \frac{1}{\epsilon}\frac{1}{M_{pl}^4}\int d^4x\sqrt{-g}\,\mathcal{R}^{\mu\nu}{}_{\rho\sigma}\mathcal{R}^{\rho\sigma}{}_{\lambda\kappa}\mathcal{R}^{\lambda\kappa}{}_{\mu\nu}$$

with $\epsilon = 4 - D$. All indications point towards the fact that this is the first in an infinite number of necessary counterterms.

Coupling gravity to matter requires an interaction term of the form,

$$S_{int} = \int d^4x\,\frac{1}{M_{pl}}\,h_{\mu\nu}T^{\mu\nu} + \mathcal{O}(h^2)$$

This makes the situation marginally worse, with the first divergence now appearing at one-loop. The Feynman diagram in the figure shows particle scattering through the exchange of two gravitons. When the momentum $k$ running in the loop is large, the diagram is badly divergent: it scales as

$$\frac{1}{M_{pl}^4}\int^{\infty} d^4k$$

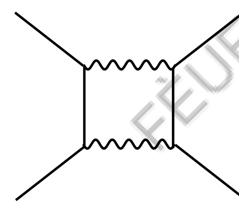

**Figure 1:**

Non-renormalizable theories are commonplace in the history of physics, the most commonly cited example being Fermi's theory of the weak interaction. The first thing to say about them is that they are far from useless! Non-renormalizable theories are typically viewed as *effective* field theories, valid only up to some energy scale $\Lambda$. One deals with the divergences by simply admitting ignorance beyond this scale and treating $\Lambda$ as a UV cut-off on any momentum integral. In this way, we get results which are valid to an accuracy of $E/\Lambda$ (perhaps raised to some power). In the case of the weak interaction, Fermi's theory accurately predicts physics up to an energy scale of $\sqrt{1/G_F} \sim 100$ GeV. In the case of quantum gravity, Einstein's theory works to an accuracy of $(E/M_{pl})^2$.

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However, non-renormalizable theories are typically unable to describe physics at their cut-off scale $\Lambda$ or beyond. This is because they are missing the true ultra-violet degrees of freedom which tame the high-energy behaviour. In the case of the weak force, these new degrees of freedom are the W and Z bosons. We would like to know what missing degrees of freedom are needed to complete gravity.

**Singularities**

Only a particle physicist would phrase all questions about the universe in terms of scattering amplitudes. In general relativity we typically think about the geometry as a whole, rather than bastardizing the Einstein-Hilbert action and discussing perturbations around flat space. In this language, the question of high-energy physics turns into one of short distance physics. Classical general relativity is not to be trusted in regions where the curvature of spacetime approaches the Planck scale and ultimately becomes singular. A quantum theory of gravity should resolve these singularities.

The question of spacetime singularities is morally equivalent to that of high-energy scattering. Both probe the ultra-violet nature of gravity. A spacetime geometry is made of a coherent collection of gravitons, just as the electric and magnetic fields in a laser are made from a collection of photons. The short distance structure of spacetime is governed – after Fourier transform – by high momentum gravitons. Understanding spacetime singularities and high-energy scattering are different sides of the same coin.

There are two situations in general relativity where singularity theorems tell us that the curvature of spacetime gets large: at the big bang and in the center of a black hole. These provide two of the biggest challenges to any putative theory of quantum gravity.

**Gravity is Subtle**

It is often said that general relativity contains the seeds of its own destruction. The theory is unable to predict physics at the Planck scale and freely admits to it. Problems such as non-renormalizability and singularities are, in a Rumsfeldian sense, known unknowns. However, the full story is more complicated and subtle. On the one hand, the issue of non-renormalizability may not quite be the crisis that it first appears. On the other hand, some aspects of quantum gravity suggest that general relativity isn't as honest about its own failings as is usually advertised. The theory hosts a number of unknown unknowns, things that we didn't even know that we didn't know. We won't have a whole lot to say about these issues in this course, but you should be aware of them. Here I mention only a few salient points.

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Firstly, there is a key difference between Fermi's theory of the weak interaction and gravity. Fermi's theory was unable to provide predictions for any scattering process at energies above $\sqrt{1/G_F}$. In contrast, if we scatter two objects at extremely high-energies in gravity — say, at energies $E \gg M_{pl}$ — then we know exactly what will happen: we form a big black hole. We don't need quantum gravity to tell us this. Classical general relativity is sufficient. If we restrict attention to scattering, the crisis of non-renormalizability is not problematic at ultra-high energies. It's troublesome only within a window of energies around the Planck scale.

Similar caveats hold for singularities. If you are foolish enough to jump into a black hole, then you're on your own: without a theory of quantum gravity, no one can tell you what fate lies in store at the singularity. Yet, if you are smart and stay outside of the black hole, you'll be hard pushed to see any effects of quantum gravity. This is because Nature has conspired to hide Planck scale curvatures from our inquisitive eyes. In the case of black holes this is achieved through cosmic censorship which is a conjecture in classical general relativity that says singularities are hidden behind horizons. In the case of the big bang, it is achieved through inflation, washing away any traces from the very early universe. Nature appears to shield us from the effects of quantum gravity, whether in high-energy scattering or in singularities. I think it's fair to say that no one knows if this conspiracy is pointing at something deep, or is merely inconvenient for scientists trying to probe the Planck scale.

While horizons may protect us from the worst excesses of singularities, they come with problems of their own. These are the unknown unknowns: difficulties that arise when curvatures are small and general relativity says "trust me". The entropy of black holes and the associated paradox of information loss strongly suggest that local quantum field theory breaks down at macroscopic distance scales. Attempts to formulate quantum gravity in de Sitter space, or in the presence of eternal inflation, hint at similar difficulties. Ideas of holography, black hole complimentarity and the AdS/CFT correspondence all point towards non-local effects and the emergence of spacetime. These are the deep puzzles of quantum gravity and their relationship to the ultra-violet properties of gravity is unclear.

As a final thought, let me mention the one observation that has an outside chance of being related to quantum gravity: the cosmological constant. With an energy scale of $\Lambda \sim 10^{-3}$ eV it appears to have little to do with ultra-violet physics. If it does have its origins in a theory of quantum gravity, it must either be due to some subtle "unknown unknown", or because it is explained away as an environmental quantity as in string theory.

### Is the Time Ripe?

Our current understanding of physics, embodied in the standard model, is valid up to energy scales of $10^3$ GeV. This is 15 orders of magnitude away from the Planck scale. Why do we think the time is now ripe to tackle quantum gravity? Surely we are like the ancient Greeks arguing about atomism. Why on earth do we believe that we've developed the right tools to even address the question?

The honest answer, I think, is hubris.

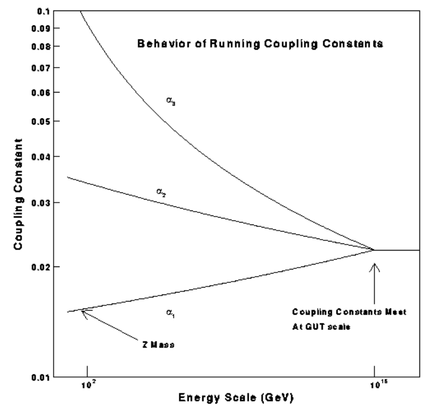


**Figure 2:**

However, there is mild circumstantial evidence that the framework of quantum field theory might hold all the way to the Planck scale without anything very dramatic happening in between. The main argument is unification. The three coupling constants of Nature run logarithmically, meeting miraculously at the GUT energy scale of $10^{15}$ GeV. Just slightly later, the fourth force of Nature, gravity, joins them. While not overwhelming, this does provide a hint that perhaps quantum field theory can be taken seriously at these ridiculous scales. Historically I suspect this was what convinced large parts of the community that it was ok to speak about processes at $10^{18}$ GeV.

Finally, perhaps the most compelling argument for studying physics at the Planck scale is that string theory *does* provide a consistent unified quantum theory of gravity and the other forces. Given that we have this theory sitting in our laps, it would be foolish not to explore its consequences. The purpose of these lecture notes is to begin this journey.



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# 1. The Relativistic String

All lecture courses on string theory start with a discussion of the point particle. Ours is no exception. We'll take a flying tour through the physics of the relativistic point particle and extract a couple of important lessons that we'll take with us as we move onto string theory.

## 1.1 The Relativistic Point Particle

We want to write down the Lagrangian describing a relativistic particle of mass $m$. In anticipation of string theory, we'll consider $D$-dimensional Minkowski space $\mathbf{R}^{1,D-1}$. Throughout these notes, we work with signature

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, \ldots, +1)$$

Note that this is the opposite signature to my quantum field theory notes.

If we fix a frame with coordinates $X^\mu = (t, \vec{x})$ the action is simple:

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}} \, . \tag{1.1}$$

To see that this is correct we can compute the momentum $\vec{p}$, conjugate to $\vec{x}$, and the energy $E$ which is equal to the Hamiltonian,

$$\vec{p} = \frac{m\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}} \quad , \quad E = \sqrt{m^2 + \vec{p}^{\,2}} \, ,$$

both of which should be familiar from courses on special relativity.

Although the Lagrangian (1.1) is correct, it's not fully satisfactory. The reason is that time $t$ and space $\vec{x}$ play very different roles in this Lagrangian. The position $\vec{x}$ is a dynamical degree of freedom. In contrast, time $t$ is merely a parameter providing a label for the position. Yet Lorentz transformations are supposed to mix up $t$ and $\vec{x}$ and such symmetries are not completely obvious in (1.1). Can we find a new Lagrangian in which time and space are on equal footing?

One possibility is to treat both time and space as labels. This leads us to the concept of field theory. However, in this course we will be more interested in the other possibility: we will promote time to a dynamical degree of freedom. At first glance, this may appear odd: the number of degrees of freedom is one of the crudest ways we have to characterize a system. We shouldn't be able to add more degrees of freedom

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at will without fundamentally changing the system that we're talking about. Another way of saying this is that the particle has the option to move in space, but it doesn't have the option to move in time. It *has* to move in time. So we somehow need a way to promote time to a degree of freedom without it really being a true dynamical degree of freedom! How do we do this? The answer, as we will now show, is gauge symmetry.

Consider the action,

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} \ , \tag{1.2}$$

where $\mu = 0, \ldots, D-1$ and $\dot{X}^\mu = dX^\mu/d\tau$. We've introduced a new parameter $\tau$ which labels the position along the worldline of the particle as shown by the dashed lines in the figure. This action has a simple interpretation: it is just the proper time $\int ds$ along the worldline.

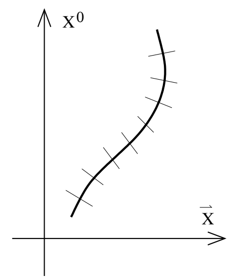


**Figure 3:**

Naively it looks as if we now have $D$ physical degrees of freedom rather than $D-1$ because, as promised, the time direction $X^0 \equiv t$ is among our dynamical variables: $X^0 = X^0(\tau)$. However, this is an illusion. To see why, we need to note that the action (1.2) has a very important property: reparameterization invariance. This means that we can pick a different parameter $\tilde{\tau}$ on the worldline, related to $\tau$ by any monotonic function

$$\tilde{\tau} = \tilde{\tau}(\tau) \ .$$

Let's check that the action is invariant under transformations of this type. The integration measure in the action changes as $d\tau = d\tilde{\tau}\,|d\tau/d\tilde{\tau}|$. Meanwhile, the velocities change as $dX^\mu/d\tau = (dX^\mu/d\tilde{\tau})\,(d\tilde{\tau}/d\tau)$. Putting this together, we see that the action can just as well be written in the $\tilde{\tau}$ reparameterization,

$$S = -m \int d\tilde{\tau} \sqrt{-\frac{dX^\mu}{d\tilde{\tau}} \frac{dX^\nu}{d\tilde{\tau}} \eta_{\mu\nu}} \ .$$

The upshot of this is that not all $D$ degrees of freedom $X^\mu$ are physical. For example, suppose you find a solution to this system, so that you know how $X^0$ changes with $\tau$ and how $X^1$ changes with $\tau$ and so on. Not all of that information is meaningful because $\tau$ itself is not meaningful. In particular, we could use our reparameterization invariance to simply set

$$\tau = X^0(\tau) \equiv t \tag{1.3}$$

If we plug this choice into the action (1.2) then we recover our initial action (1.1). The reparameterization invariance is a *gauge symmetry* of the system. Like all gauge symmetries, it's not really a symmetry at all. Rather, it is a redundancy in our description. In the present case, it means that although we seem to have $D$ degrees of freedom $X^\mu$, one of them is fake.

The fact that one of the degrees of freedom is a fake also shows up if we look at the momenta,

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m\dot{X}^\nu \eta_{\mu\nu}}{\sqrt{-\dot{X}^\lambda \dot{X}^\rho\, \eta_{\lambda\rho}}} \tag{1.4}$$

These momenta aren't all independent. They satisfy

$$p_\mu p^\mu + m^2 = 0 \tag{1.5}$$

This is a constraint on the system. It is, of course, the mass-shell constraint for a relativistic particle of mass $m$. From the worldline perspective, it tells us that the particle isn't allowed to sit still in Minkowski space: at the very least, it had better keep moving in a timelike direction with $(p^0)^2 \geq m^2$.

One advantage of the action (1.2) is that the Poincaré symmetry of the particle is now manifest, appearing as a global symmetry on the worldline

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{1.6}$$

where $\Lambda$ is a Lorentz transformation satisfying $\Lambda^\mu{}_\nu \eta^{\nu\rho} \Lambda^\sigma{}_\rho = \eta^{\mu\sigma}$, while $c^\mu$ corresponds to a constant translation. We have made all the symmetries manifest at the price of introducing a gauge symmetry into our system. A similar gauge symmetry will arise in the relativistic string and much of this course will be devoted to understanding its consequences.

### 1.1.1 Quantization

It's a trivial matter to quantize this action. We introduce a wavefunction $\Psi(X)$. This satisfies the usual Schrödinger equation,

$$i\,\frac{\partial \Psi}{\partial \tau} = H\Psi\ .$$

But, computing the Hamiltonian $H = \dot{X}^\mu p_\mu - L$, we find that it vanishes: $H = 0$. This shouldn't be surprising. It is simply telling us that the wavefunction doesn't depend on



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$\tau$. Since the wavefunction is something physical while, as we have seen, $\tau$ is not, this is to be expected. Note that this doesn't mean that time has dropped out of the problem. On the contrary, in this relativistic context, time $X^0$ is an operator, just like the spatial coordinates $\vec{x}$. This means that the wavefunction $\Psi$ is immediately a function of space and time. It is not like a static state in quantum mechanics, but more akin to the fully integrated solution to the non-relativistic Schrödinger equation.

The classical system has a constraint given by (1.5). In the quantum theory, we impose this constraint as an operator equation on the wavefunction, namely $(p^\mu p_\mu + m^2)\Psi = 0$. Using the usual representation of the momentum operator $p_\mu = -i\partial/\partial X^\mu$, we recognize this constraint as the Klein-Gordon equation

$$\left(-\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial X^\nu}\eta^{\mu\nu} + m^2\right)\Psi(X) = 0 \tag{1.7}$$

Although this equation is familiar from field theory, it's important to realize that the interpretation is somewhat different. In relativistic field theory, the Klein-Gordon equation is the equation of motion obeyed by a scalar field. In relativistic quantum mechanics, it is the equation obeyed by the wavefunction. In the early days of field theory, the fact that these two equations are the same led people to think one should view the wavefunction as a classical field and quantize it a second time. This isn't correct, but nonetheless the language has stuck and it is common to talk about the point particle perspective as "first quantization" and the field theory perspective as "second quantization".

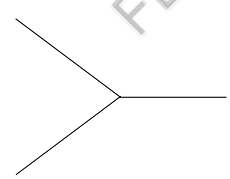

**Figure 4:**

So far we've considered only a free point particle. How can we introduce interactions into this framework? We would have to first decide which interactions are allowed: perhaps the particle can split into two; perhaps it can fuse with other particles? Obviously, there is a huge range of options for us to choose from. We would then assign amplitudes for these processes to happen. There would be certain restrictions coming from the requirement of unitarity which, among other things, would lead to the necessity of anti-particles. We could draw diagrams associated to the different interactions — an example is given in the figure — and in this manner we would slowly build up the Feynman diagram expansion that is familiar from field theory. In fact, this was pretty much the way Feynman himself approached the topic of QED. However, in practice we rarely construct particle interactions in this way because the field theory framework provides a much better way of looking at things. In contrast, this way of building up interactions is exactly what we will later do for strings.

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### 1.1.2 Ein Einbein

There is another action that describes the relativistic point particle. We introduce yet another field on the worldline, $e(\tau)$, and write

$$S = \frac{1}{2} \int d\tau \, \left( e^{-1} \dot{X}^2 - e m^2 \right) \, , \tag{1.8}$$

where we've used the notation $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}$. For the rest of these lectures, terms like $X^2$ will always mean an implicit contraction with the spacetime Minkowski metric.

This form of the action makes it look as if we have coupled the worldline theory to 1d gravity, with the field $e(\tau)$ acting as an einbein (in the sense of vierbeins that are introduced in general relativity). To see this, note that we could change notation and write this action in the more suggestive form

$$S = -\frac{1}{2} \int d\tau \sqrt{-g_{\tau\tau}} \left( g^{\tau\tau} \dot{X}^2 + m^2 \right) \, . \tag{1.9}$$

where $g_{\tau\tau} = (g^{\tau\tau})^{-1}$ is the metric on the worldline and $e = \sqrt{-g_{\tau\tau}}$

Although our action appears to have one more degree of freedom, $e$, it can be easily checked that it has the same equations of motion as (1.2). The reason for this is that $e$ is completely fixed by its equation of motion, $\dot{X}^2 + e^2 m^2 = 0$. Substituting this into the action (1.8) recovers (1.2)

The action (1.8) has a couple of advantages over (1.2). Firstly, it works for massless particles with $m = 0$. Secondly, the absence of the annoying square root means that it's easier to quantize in a path integral framework.

The action (1.8) retains invariance under reparameterizations which are now written in a form that looks more like general relativity. For transformations parameterized by an infinitesimal $\eta$, we have

$$\tau \to \tilde{\tau} = \tau - \eta(\tau) \quad , \quad \delta e = \frac{d}{d\tau}(\eta(\tau) e) \quad , \quad \delta X^\mu = \frac{dX^\mu}{d\tau} \eta(\tau) \tag{1.10}$$

The einbein $e$ transforms as a density on the worldline, while each of the coordinates $X^\mu$ transforms as a worldline scalar.

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### 1.2 The Nambu-Goto Action

A particle sweeps out a worldline in Minkowski space. A string sweeps out a *worldsheet*. We'll parameterize this worldsheet by one timelike coordinate $\tau$, and one spacelike coordinate $\sigma$. In this section we'll focus on closed strings and take $\sigma$ to be periodic, with range

$$\sigma \in [0, 2\pi) \ . \tag{1.11}$$

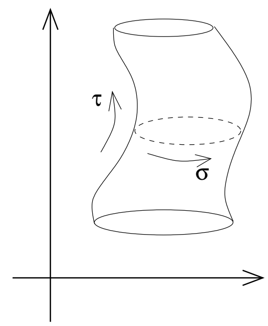


**Figure 5:**

We will sometimes package the two worldsheet coordinates together as $\sigma^\alpha = (\tau, \sigma)$, $\alpha = 0, 1$. Then the string sweeps out a surface in spacetime which defines a map from the worldsheet to Minkowski space, $X^\mu(\sigma, \tau)$ with $\mu = 0, \ldots, D-1$. For closed strings, we require

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau) \ .$$

In this context, spacetime is sometimes referred to as the *target space* to distinguish it from the worldsheet.

We need an action that describes the dynamics of this string. The key property that we will ask for is that nothing depends on the coordinates $\sigma^\alpha$ that we choose on the worldsheet. In other words, the string action should be reparameterization invariant. What kind of action does the trick? Well, for the point particle the action was proportional to the length of the worldline. The obvious generalization is that the action for the string should be proportional to the area, $A$, of the worldsheet. This is certainly a property that is characteristic of the worldsheet itself, rather than any choice of parameterization.

How do we find the area $A$ in terms of the coordinates $X^\mu(\sigma, \tau)$? The worldsheet is a curved surface embedded in spacetime. The induced metric, $\gamma_{\alpha\beta}$, on this surface is the pull-back of the flat metric on Minkowski space,

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \ . \tag{1.12}$$

Then the action which is proportional to the area of the worldsheet is given by,

$$S = -T \int d^2\sigma \ \sqrt{- \det \gamma} \ . \tag{1.13}$$

Here $T$ is a constant of proportionality. We will see shortly that it is the *tension* of the string, meaning the mass per unit length.

We can write this action a little more explicitly. The pull-back of the metric is given by,

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix} .$$

where $\dot{X}^\mu = \partial X^\mu/\partial\tau$ and $X^{\mu\prime} = \partial X^\mu/\partial\sigma$. The action then takes the form,

$$S = -T \int d^2\sigma \sqrt{-(\dot{X})^2\,(X')^2 + (\dot{X}\cdot X')^2}\ . \tag{1.14}$$

This is the *Nambu-Goto* action for a relativistic string.

**Action = Area: A Check**

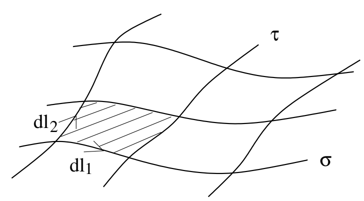


**Figure 6:**

If you're unfamiliar with differential geometry, the argument about the pull-back of the metric may be a bit slick. Thankfully, there's a more pedestrian way to see that the action (1.14) is equal to the area swept out by the worldsheet. It's slightly simpler to make this argument for a surface embedded in Euclidean space rather than Minkowski space. We choose some parameterization of the sheet in terms of $\tau$ and $\sigma$, as drawn in the figure, and we write the coordinates of Euclidean space as $\vec{X}(\sigma,\tau)$. We'll compute the area of the infinitesimal shaded region. The vectors tangent to the boundary are,

$$\vec{dl}_1 = \frac{\partial\vec{X}}{\partial\sigma} \quad , \quad \vec{dl}_2 = \frac{\partial\vec{X}}{\partial\tau} .$$

If the angle between these two vectors is $\theta$, then the area is then given by

$$ds^2 = |\vec{dl}_1||\vec{dl}_2|\sin\theta = \sqrt{dl_1^2\,dl_2^2(1-\cos^2\theta)} = \sqrt{dl_1^2\,dl_2^2 - (\vec{dl}_1\cdot\vec{dl}_2)^2} \tag{1.15}$$

which indeed takes the form of the integrand of (1.14).

**Tension and Dimension**

Let's now see that $T$ has the physical interpretation of tension. We write Minkowski coordinates as $X^\mu = (t,\vec{x})$. We work in a gauge with $X^0 \equiv t = R\tau$, where $R$ is a constant that is needed to balance up dimensions (see below) and will drop out at the end of the argument. Consider a snapshot of a string configuration at a time when



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$d\vec{x}/d\tau = 0$ so that the instantaneous kinetic energy vanishes. Evaluating the action for a time $dt$ gives

$$S = -T \int d\tau d\sigma R \sqrt{(d\vec{x}/d\sigma)^2} = -T \int dt \, (\text{spatial length of string}) \ . \qquad (1.16)$$

But, when the kinetic energy vanishes, the action is proportional to the time integral of the potential energy,

$$\text{potential energy} = T \times (\text{spatial length of string}) \ .$$

So $T$ is indeed the energy per unit length as claimed. We learn that the string acts rather like an elastic band and its energy increases linearly with length. (This is different from the elastic bands you're used to which obey Hooke's law where energy increased quadratically with length). To minimize its potential energy, the string will want to shrink to zero size. We'll see that when we include quantum effects this can't happen because of the usual zero point energies.

There is a slightly annoying way of writing the tension that has its origin in ancient history, but is commonly used today

$$T = \frac{1}{2\pi\alpha'} \qquad (1.17)$$

where $\alpha'$ is pronounced "alpha-prime". In the language of our ancestors, $\alpha'$ is referred to as the "universal Regge slope". We'll explain why later in this course.

At this point, it's worth pointing out some conventions that we have, until now, left implicit. The spacetime coordinates have dimension $[X] = -1$. In contrast, the worldsheet coordinates are taken to be dimensionless, $[\sigma] = 0$. (This can be seen in our identification $\sigma \equiv \sigma + 2\pi$). The tension is equal to the mass per unit length and has dimension $[T] = 2$. Obviously this means that $[\alpha'] = -2$. We can therefore associate a length scale, $l_s$, by

$$\alpha' = l_s^2 \qquad (1.18)$$

The *string scale* $l_s$ is the natural length that appears in string theory. In fact, in a certain sense (that we will make more precise later in the course) this length scale is the only parameter of the theory.



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### Actual Strings vs. Fundamental Strings

There are several situations in Nature where string-like objects arise. Prime examples include magnetic flux tubes in superconductors and chromo-electric flux tubes in QCD. Cosmic strings, a popular speculation in cosmology, are similar objects, stretched across the sky. In each of these situations, there are typically two length scales associated to the string: the tension, $T$ and the width of the string, $L$. For all these objects, the dynamics is governed by the Nambu-Goto action as long as the curvature of the string is much greater than $L$. (In the case of superconductors, one should work with a suitable non-relativistic version of the Nambu-Goto action).

However, in each of these other cases, the Nambu-Goto action is not the end of the story. There will typically be additional terms in the action that depend on the width of the string. The form of these terms is not universal, but often includes a *rigidity* piece of form $L \int K^2$, where $K$ is the extrinsic curvature of the worldsheet. Other terms could be added to describe fluctuations in the width of the string.

The string scale, $l_s$, or equivalently the tension, $T$, depends on the kind of string that we're considering. For example, if we're interested in QCD flux tubes then we would take

$$T \sim (1 \text{ GeV})^2 \tag{1.19}$$

In this course we will consider *fundamental strings* which have zero width. What this means in practice is that we take the Nambu-Goto action as the complete description for all configurations of the string. These strings will have relevance to quantum gravity and the tension of the string is taken to be much larger, typically an order of magnitude or so below the Planck scale.

$$T \lesssim M_{pl}^2 = (10^{18} \text{ GeV})^2 \tag{1.20}$$

However, I should point out that when we try to view string theory as a fundamental theory of quantum gravity, we don't really know what value $T$ should take. As we will see later in this course, it depends on many other aspects, most notably the string coupling and the volume of the extra dimensions.

### 1.2.1 Symmetries of the Nambu-Goto Action

The Nambu-Goto action has two types of symmetry, each of a different nature.

- Poincaré invariance of the spacetime (1.6). This is a global symmetry from the perspective of the worldsheet, meaning that the parameters $\Lambda^\mu{}_\nu$ and $c^\mu$ which label



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the symmetry transformation are constants and do not depend on worldsheet coordinates $\sigma^\alpha$.

- Reparameterization invariance, $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$. As for the point particle, this is a gauge symmetry. It reflects the fact that we have a redundancy in our description because the worldsheet coordinates $\sigma^\alpha$ have no physical meaning.

### 1.2.2 Equations of Motion

To derive the equations of motion for the Nambu-Goto string, we first introduce the momenta which we call $\Pi$ because there will be countless other quantities that we want to call $p$ later,

$$\Pi^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T\,\frac{(\dot{X}\cdot X')X'_\mu - (X'^2)\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2\, X'^2}} \ .$$

$$\Pi^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} = -T\,\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X}^2)X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2\, X'^2}} \ .$$

The equations of motion are then given by,

$$\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma} = 0$$

These look like nasty, non-linear equations. In fact, there's a slightly nicer way to write these equations, starting from the earlier action (1.13). Recall that the variation of a determinant is $\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma}\,\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}$. Using the definition of the pull-back metric $\gamma_{\alpha\beta}$, this gives rise to the equations of motion

$$\partial_\alpha(\sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\partial_\beta X^\mu) = 0 \ , \tag{1.21}$$

Although this notation makes the equations look a little nicer, we're kidding ourselves. Written in terms of $X^\mu$, they are still the same equations. Still nasty.

## 1.3 The Polyakov Action

The square-root in the Nambu-Goto action means that it's rather difficult to quantize using path integral techniques. However, there is another form of the string action which is classically equivalent to the Nambu-Goto action. It eliminates the square root at the expense of introducing another field,

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\,\partial_\alpha X^\mu\partial_\beta X^\nu\,\eta_{\mu\nu} \tag{1.22}$$

where $g \equiv \det g$. This is the *Polyakov* action. (Polyakov didn't discover the action, but he understood how to work with it in the path integral and for this reason it carries his name. The path integral treatment of this action will be the subject of Chapter 5).

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The new field is $g_{\alpha\beta}$. It is a dynamical metric on the worldsheet. From the perspective of the worldsheet, the Polyakov action is a bunch of scalar fields $X$ coupled to 2d gravity.

The equation of motion for $X^\mu$ is

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta X^\mu) = 0 \ , \tag{1.23}$$

which coincides with the equation of motion (1.21) from the Nambu-Goto action, except that $g_{\alpha\beta}$ is now an independent variable which is fixed by its own equation of motion. To determine this, we vary the action (remembering again that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} = +\frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}$),

$$\delta S = -\frac{T}{2}\int d^2\sigma \ \delta g^{\alpha\beta}\left(\sqrt{-g}\,\partial_\alpha X^\mu\partial_\beta X^\nu - \tfrac{1}{2}\sqrt{-g}\,g_{\alpha\beta}g^{\rho\sigma}\partial_\rho X^\mu\partial_\sigma X^\nu\right)\eta_{\mu\nu} = 0 \ . \tag{1.24}$$

The worldsheet metric is therefore given by,

$$g_{\alpha\beta} = 2f(\sigma)\,\partial_\alpha X\cdot\partial_\beta X \ , \tag{1.25}$$

where the function $f(\sigma)$ is given by,

$$f^{-1} = g^{\rho\sigma}\,\partial_\rho X\cdot\partial_\sigma X$$

A comment on the potentially ambiguous notation: here, and below, any function $f(\sigma)$ is always short-hand for $f(\sigma,\tau)$: it in no way implies that $f$ depends only on the spatial worldsheet coordinate.

We see that $g_{\alpha\beta}$ isn't quite the same as the pull-back metric $\gamma_{\alpha\beta}$ defined in equation (1.12); the two differ by the conformal factor $f$. However, this doesn't matter because, rather remarkably, $f$ drops out of the equation of motion (1.23). This is because the $\sqrt{-g}$ term scales as $f$, while the inverse metric $g^{\alpha\beta}$ scales as $f^{-1}$ and the two pieces cancel. We therefore see that Nambu-Goto and the Polyakov actions result in the same equation of motion for $X$.

In fact, we can see more directly that the Nambu-Goto and Polyakov actions coincide. We may replace $g_{\alpha\beta}$ in the Polyakov action (1.22) with its equation of motion $g_{\alpha\beta} = 2f\,\gamma_{\alpha\beta}$. The factor of $f$ also drops out of the action for the same reason that it dropped out of the equation of motion. In this manner, we recover the Nambu-Goto action (1.13).

### 1.3.1 Symmetries of the Polyakov Action

The fact that the presence of the factor $f(\sigma,\tau)$ in (1.25) didn't actually affect the equations of motion for $X^\mu$ reflects the existence of an extra symmetry which the Polyakov action enjoys. Let's look more closely at this. Firstly, the Polyakov action still has the two symmetries of the Nambu-Goto action,

- Poincaré invariance. This is a global symmetry on the worldsheet.

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \ .$$

- Reparameterization invariance, also known as diffeomorphisms. This is a gauge symmetry on the worldsheet. We may redefine the worldsheet coordinates as $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$. The fields $X^\mu$ transform as worldsheet scalars, while $g_{\alpha\beta}$ transforms in the manner appropriate for a 2d metric.

$$\begin{aligned} X^\mu(\sigma) &\to \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma) \\ g_{\alpha\beta}(\sigma) &\to \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}\, g_{\gamma\delta}(\sigma) \end{aligned}$$

  It will sometimes be useful to work infinitesimally. If we make the coordinate change $\sigma^\alpha \to \tilde{\sigma}^\alpha = \sigma^\alpha - \eta^\alpha(\sigma)$, for some small $\eta$. The transformations of the fields then become,

$$\begin{aligned} \delta X^\mu(\sigma) &= \eta^\alpha\partial_\alpha X^\mu \\ \delta g_{\alpha\beta}(\sigma) &= \nabla_\alpha\eta_\beta + \nabla_\beta\eta_\alpha \end{aligned}$$

  where the covariant derivative is defined by $\nabla_\alpha\eta_\beta = \partial_\alpha\eta_\beta - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$ with the Levi-Civita connection associated to the worldsheet metric given by the usual expression,

$$\Gamma^\sigma_{\alpha\beta} = \tfrac{1}{2}\, g^{\sigma\rho}(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta})$$

Together with these familiar symmetries, there is also a new symmetry which is novel to the Polyakov action. It is called *Weyl invariance*.

- Weyl Invariance. Under this symmetry, $X^\mu(\sigma) \to X^\mu(\sigma)$, while the metric changes as

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma)\, g_{\alpha\beta}(\sigma) \ . \qquad (1.26)$$

  Or, infinitesimally, we can write $\Omega^2(\sigma) = e^{2\phi(\sigma)}$ for small $\phi$ so that

$$\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma)\, g_{\alpha\beta}(\sigma) \ .$$



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It is simple to see that the Polyakov action is invariant under this transformation: the factor of $\Omega^2$ drops out just as the factor of $f$ did in equation (1.25), canceling between $\sqrt{-g}$ and the inverse metric $g^{\alpha\beta}$. This is a gauge symmetry of the string, as seen by the fact that the parameter $\Omega$ depends on the worldsheet coordinates $\sigma$. This means that two metrics which are related by a Weyl transformation (1.26) are to be considered as the same physical state.

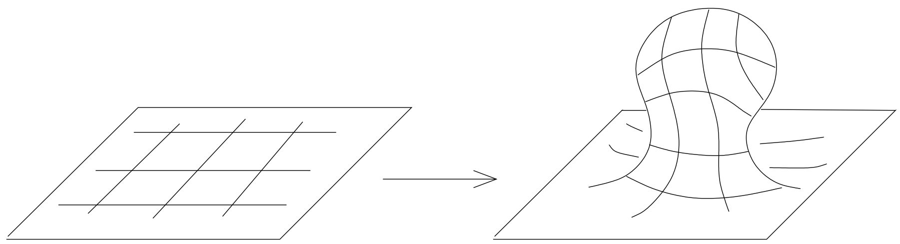

**Figure 7:** An example of a Weyl transformation

How should we think of Weyl invariance? It is not a coordinate change. Instead it is the invariance of the theory under a local change of scale which preserves the angles between all lines. For example the two worldsheet metrics shown in the figure are viewed by the Polyakov string as equivalent. This is rather surprising! And, as you might imagine, theories with this property are extremely rare. It should be clear from the discussion above that the property of Weyl invariance is special to two dimensions, for only there does the scaling factor coming from the determinant $\sqrt{-g}$ cancel that coming from the inverse metric. But even in two dimensions, if we wish to keep Weyl invariance then we are strictly limited in the kind of interactions that can be added to the action. For example, we would not be allowed a potential term for the worldsheet scalars of the form,

$$\int d^2\sigma \sqrt{-g}\, V(X) \ .$$

These break Weyl invariance. Nor can we add a worldsheet cosmological constant term,

$$\mu \int d^2\sigma \sqrt{-g} \ .$$

This too breaks Weyl invariance. We will see later in this course that the requirement of Weyl invariance becomes even more stringent in the quantum theory. We will also see what kind of interactions terms can be added to the worldsheet. Indeed, much of this course can be thought of as the study of theories with Weyl invariance.



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### 1.3.2 Fixing a Gauge

As we have seen, the equation of motion (1.23) looks pretty nasty. However, we can use the redundancy inherent in the gauge symmetry to choose coordinates in which they simplify. Let's think about what we can do with the gauge symmetry.

Firstly, we have two reparameterizations to play with. The worldsheet metric has three independent components. This means that we expect to be able to set any two of the metric components to a value of our choosing. We will choose to make the metric locally conformally flat, meaning

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta} \ , \tag{1.27}$$

where $\phi(\sigma, \tau)$ is some function on the worldsheet. You can check that this is possible by writing down the change of the metric under a coordinate transformation and seeing that the differential equations which result from the condition (1.27) have solutions, at least locally. Choosing a metric of the form (1.27) is known as *conformal gauge*.

We have only used reparameterization invariance to get to the metric (1.27). We still have Weyl transformations to play with. Clearly, we can use these to remove the last independent component of the metric and set $\phi = 0$ such that,

$$g_{\alpha\beta} = \eta_{\alpha\beta} \ . \tag{1.28}$$

We end up with the flat metric on the worldsheet in Minkowski coordinates.

#### A Diversion: How to make a metric flat

The fact that we can use Weyl invariance to make any two-dimensional metric flat is an important result. Let's take a quick diversion from our main discussion to see a different proof that isn't tied to the choice of Minkowski coordinates on the worldsheet. We'll work in 2d Euclidean space to avoid annoying minus signs. Consider two metrics related by a Weyl transformation, $g'_{\alpha\beta} = e^{2\phi}g_{\alpha\beta}$. One can check that the Ricci scalars of the two metrics are related by,

$$\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\phi) \ . \tag{1.29}$$

We can therefore pick a $\phi$ such that the new metric has vanishing Ricci scalar, $R' = 0$, simply by solving this differential equation for $\phi$. However, in two dimensions (but not in higher dimensions) a vanishing Ricci scalar implies a flat metric. The reason is simply that there aren't too many indices to play with. In particular, symmetry of the Riemann tensor in two dimensions means that it must take the form,

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \ .$$



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So $R' = 0$ is enough to ensure that $R'_{\alpha\beta\gamma\delta} = 0$, which means that the manifold is flat. In equation (1.28), we've further used reparameterization invariance to pick coordinates in which the flat metric is the Minkowski metric.

### The equations of motion and the stress-energy tensor

With the choice of the flat metric (1.28), the Polyakov action simplifies tremendously and becomes the theory of $D$ free scalar fields. (In fact, this simplification happens in any conformal gauge).

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\ \partial_\alpha X \cdot \partial^\alpha X\ , \tag{1.30}$$

and the equations of motion for $X^\mu$ reduce to the free wave equation,

$$\partial_\alpha \partial^\alpha X^\mu = 0\ . \tag{1.31}$$

Now that looks too good to be true! Are the horrible equations (1.23) really equivalent to a free wave equation? Well, not quite. There is something that we've forgotten: we picked a choice of gauge for the metric $g_{\alpha\beta}$. But we must still make sure that the equation of motion for $g_{\alpha\beta}$ is satisfied. In fact, the variation of the action with respect to the metric gives rise to a rather special quantity: it is the stress-energy tensor, $T_{\alpha\beta}$. With a particular choice of normalization convention, we define the stress-energy tensor to be

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}}\ .$$

We varied the Polyakov action with respect to $g_{\alpha\beta}$ in (1.24). When we set $g_{\alpha\beta} = \eta_{\alpha\beta}$ we get

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \tfrac{1}{2}\eta_{\alpha\beta}\,\eta^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X\ . \tag{1.32}$$

The equation of motion associated to the metric $g_{\alpha\beta}$ is simply $T_{\alpha\beta} = 0$. Or, more explicitly,

$$\begin{aligned} T_{01} &= \dot{X} \cdot X' = 0 \\ T_{00} = T_{11} &= \tfrac{1}{2}(\dot{X}^2 + X'^2) = 0\ . \end{aligned} \tag{1.33}$$

We therefore learn that the equations of motion of the string are the free wave equations (1.31) subject to the two constraints (1.33) arising from the equation of motion $T_{\alpha\beta} = 0$.

### Getting a feel for the constraints

Let's try to get some intuition for these constraints. There is a simple meaning of the first constraint in (1.33): we must choose our parameterization such that lines of constant $\sigma$ are perpendicular to the lines of constant $\tau$, as shown in the figure.

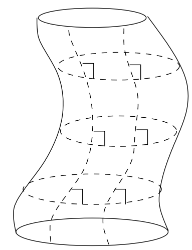

**Figure 8:**

But we can do better. To gain more physical insight, we need to make use of the fact that we haven't quite exhausted our gauge symmetry. We will discuss this more in Section 2.2, but for now one can check that there is enough remnant gauge symmetry to allow us to go to static gauge,

$$X^0 \equiv t = R\tau \ ,$$

so that $(X^0)' = 0$ and $\dot{X}^0 = R$, where $R$ is a constant that is needed on dimensional grounds. The interpretation of this constant will become clear shortly. Then, writing $X^\mu = (t, \vec{x})$, the equation of motion for spatial components is the free wave equation,

$$\ddot{\vec{x}} - \vec{x}'' = 0$$

while the constraints become

$$\begin{aligned} \dot{\vec{x}} \cdot \vec{x}' &= 0 \\ \dot{\vec{x}}^{\,2} + \vec{x}'^{\,2} &= R^2 \end{aligned} \tag{1.34}$$

The first constraint tells us that the motion of the string must be perpendicular to the string itself. In other words, the physical modes of the string are transverse oscillations. There is no longitudinal mode. We'll also see this again in Section 2.2.

From the second constraint, we can understand the meaning of the constant $R$: it is related to the length of the string when $\dot{\vec{x}} = 0$,

$$\int d\sigma \, \sqrt{(d\vec{x}/d\sigma)^2} = 2\pi R \ .$$

Of course, if we have a stretched string with $\dot{\vec{x}} = 0$ at one moment of time, then it won't stay like that for long. It will contract under its own tension. As this happens, the second constraint equation relates the length of the string to the instantaneous velocity of the string.



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### 1.4 Mode Expansions

Let's look at the equations of motion and constraints more closely. The equations of motion (1.31) are easily solved. We introduce lightcone coordinates on the worldsheet,

$$\sigma^{\pm} = \tau \pm \sigma \ ,$$

in terms of which the equations of motion simply read

$$\partial_+ \partial_- X^\mu = 0$$

The most general solution is,

$$X^\mu(\sigma, \tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

for arbitrary functions $X^\mu_L$ and $X^\mu_R$. These describe left-moving and right-moving waves respectively. Of course the solution must still obey both the constraints (1.33) as well as the periodicity condition,

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau) \ . \tag{1.35}$$

The most general, periodic solution can be expanded in Fourier modes,

$$\begin{aligned} X^\mu_L(\sigma^+) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \, \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \tilde{\alpha}^\mu_n \, e^{-in\sigma^+} \ , \\ X^\mu_R(\sigma^-) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \alpha^\mu_n \, e^{-in\sigma^-} \ . \end{aligned} \tag{1.36}$$

This mode expansion will be very important when we come to the quantum theory. Let's make a few simple comments here.

- Various normalizations in this expression, such as the $\alpha'$ and factor of $1/n$ have been chosen for later convenience.

- $X_L$ and $X_R$ do not individually satisfy the periodicity condition (1.35) due to the terms linear in $\sigma^\pm$. However, the sum of them is invariant under $\sigma \to \sigma + 2\pi$ as required.

- The variables $x^\mu$ and $p^\mu$ are the position and momentum of the center of mass of the string. This can be checked, for example, by studying the Noether currents arising from the spacetime translation symmetry $X^\mu \to X^\mu + c^\mu$. One finds that the conserved charge is indeed $p^\mu$.

- Reality of $X^\mu$ requires that the coefficients of the Fourier modes, $\alpha^\mu_n$ and $\tilde{\alpha}^\mu_n$, obey

$$\alpha^\mu_n = (\alpha^\mu_{-n})^\star \quad , \quad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^\star \ . \tag{1.37}$$

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### 1.4.1 The Constraints Revisited

We still have to impose the two constraints (1.33). In the worldsheet lightcone coordinates $\sigma^\pm$, these become,

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \; . \tag{1.38}$$

These equations give constraints on the momenta $p^\mu$ and the Fourier modes $\alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$. To see what these are, let's look at

$$\begin{aligned} \partial_- X^\mu = \partial_- X_R^\mu &= \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu \, e^{-in\sigma^-} \\ &= \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^-} \end{aligned}$$

where in the second line the sum is over all $n \in \mathbf{Z}$ and we have defined $\alpha_0^\mu$ to be

$$\alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu \; .$$

The constraint (1.38) can then be written as

$$\begin{aligned} (\partial_- X)^2 &= \frac{\alpha'}{2} \sum_{m,p} \alpha_m \cdot \alpha_p \, e^{-i(m+p)\sigma^-} \\ &= \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} \, e^{-in\sigma^-} \\ &\equiv \alpha' \sum_n L_n \, e^{-in\sigma^-} = 0 \; . \end{aligned}$$

where we have defined the sum of oscillator modes,

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m \; . \tag{1.39}$$

We can also do the same for the left-moving modes, where we again define an analogous sum of operator modes,

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m \; . \tag{1.40}$$

with the zero mode defined to be,

$$\tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu \; .$$

The fact that $\tilde{\alpha}_0^\mu = \alpha_0^\mu$ looks innocuous but is a key point to remember when we come to quantize the string. The $L_n$ and $\tilde{L}_n$ are the Fourier modes of the constraints. Any classical solution of the string of the form (1.36) must further obey the infinite number of constraints,

$$L_n = \tilde{L}_n = 0 \qquad n \in \mathbf{Z} \; .$$

We'll meet these objects $L_n$ and $\tilde{L}_n$ again in a more general context when we come to discuss conformal field theory.

The constraints arising from $L_0$ and $\tilde{L}_0$ have a rather special interpretation. This is because they include the square of the spacetime momentum $p^\mu$. But, the square of the spacetime momentum is an important quantity in Minkowski space: it is the square of the rest mass of a particle,

$$p_\mu p^\mu = -M^2 \; .$$

So the $L_0$ and $\tilde{L}_0$ constraints tell us the effective mass of a string in terms of the excited oscillator modes, namely

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} \tag{1.41}$$

Because both $\alpha_0^\mu$ and $\tilde{\alpha}_0^\mu$ are equal to $\sqrt{\alpha'/2}\, p^\mu$, we have two expressions for the invariant mass: one in terms of right-moving oscillators $\alpha_n^\mu$ and one in terms of left-moving oscillators $\tilde{\alpha}_n^\mu$. And these two terms must be equal to each other. This is known as *level matching*. It will play an important role in the next section where we turn to the quantum theory.



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# 2. The Quantum String

Our goal in this section is to quantize the string. We have seen that the string action involves a gauge symmetry and whenever we wish to quantize a gauge theory we're presented with a number of different ways in which we can proceed. If we're working in the canonical formalism, this usually boils down to one of two choices:

- We could first quantize the system and then subsequently impose the constraints that arise from gauge fixing as operator equations on the physical states of the system. For example, in QED this is the Gupta-Bleuler method of quantization that we use in Lorentz gauge. In string theory it consists of treating all fields $X^\mu$, including time $X^0$, as operators and imposing the constraint equations (1.33) on the states. This is usually called covariant quantization.

- The alternative method is to first solve all of the constraints of the system to determine the space of physically distinct classical solutions. We then quantize these physical solutions. For example, in QED, this is the way we proceed in Coulomb gauge. Later in this chapter, we will see a simple way to solve the constraints of the free string.

Of course, if we do everything correctly, the two methods should agree. Usually, each presents a slightly different challenge and offers a different viewpoint.

In these lectures, we'll take a brief look at the first method of covariant quantization. However, at the slightest sign of difficulties, we'll bail! It will be useful enough to see where the problems lie. We'll then push forward with the second method described above which is known as lightcone quantization in string theory. Although we'll succeed in pushing quantization through to the end, our derivations will be a little cheap and unsatisfactory in places. In Section 5 we'll return to all these issues, armed with more sophisticated techniques from conformal field theory.

## 2.1 A Lightning Look at Covariant Quantization

We wish to quantize $D$ free scalar fields $X^\mu$ whose dynamics is governed by the action (1.30). We subsequently wish to impose the constraints

$$\dot{X} \cdot X' = \dot{X}^2 + X'^2 = 0 \ . \tag{2.1}$$

The first step is easy. We promote $X^\mu$ and their conjugate momenta $\Pi_\mu = (1/2\pi\alpha')\dot{X}_\mu$ to operator valued fields obeying the canonical equal-time commutation relations,

$$\begin{aligned} [X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] &= i\delta(\sigma-\sigma')\,\delta^\mu{}_\nu \ , \\ [X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] &= [\Pi_\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = 0 \ . \end{aligned}$$

We translate these into commutation relations for the Fourier modes $x^\mu$, $p^\mu$, $\alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$. Using the mode expansion (1.36) we find

$$[x^\mu, p_\nu] = i\delta^\mu{}_\nu \quad \text{and} \quad [\alpha_n^\mu, \alpha_m^\nu] = [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\,\eta^{\mu\nu}\delta_{n+m,0} \ , \tag{2.2}$$

with all others zero. The commutation relations for $x^\mu$ and $p^\mu$ are expected for operators governing the position and momentum of the center of mass of the string. The commutation relations of $\alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$ are those of harmonic oscillator creation and annihilation operators in disguise. And the disguise isn't that good. We just need to define (ignoring the $\mu$ index for now)

$$a_n = \frac{\alpha_n}{\sqrt{n}} \quad , \quad a_n^\dagger = \frac{\alpha_{-n}}{\sqrt{n}} \qquad\qquad n > 0 \tag{2.3}$$

Then (2.2) gives the familiar $[a_n, a_m^\dagger] = \delta_{mn}$. So each scalar field gives rise to two infinite towers of creation and annihilation operators, with $\alpha_n$ acting as a rescaled annihilation operator for $n > 0$ and as a creation operator for $n < 0$. There are two towers because we have right-moving modes $\alpha_n$ and left-moving modes $\tilde{\alpha}_n$.

With these commutation relations in hand we can now start building the Fock space of our theory. We introduce a vacuum state of the string $|0\rangle$, defined to obey

$$\alpha_n^\mu\,|0\rangle = \tilde{\alpha}_n^\mu\,|0\rangle = 0 \qquad \text{for} \quad n > 0 \tag{2.4}$$

The vacuum state of string theory has a different interpretation from the analogous object in field theory. This is not the vacuum state of spacetime. It is instead the vacuum state of a single string. This is reflected in the fact that the operators $x^\mu$ and $p^\mu$ give extra structure to the vacuum. The true ground state of the string is $|0\rangle$, tensored with a spatial wavefunction $\Psi(x)$. Alternatively, if we work in momentum space, the vacuum carries another quantum number, $p^\mu$, which is the eigenvalue of the momentum operator. We should therefore write the vacuum as $|0;p\rangle$, which still obeys (2.4), but now also

$$\hat{p}^\mu\,|0;p\rangle = p^\mu|0;p\rangle \tag{2.5}$$

where (for the only time in these lecture notes) we've put a hat on the momentum operator $\hat{p}^\mu$ on the left-hand side of this equation to distinguish it from the eigenvalue $p^\mu$ on the right-hand side.

We can now start to build up the Fock space by acting with creation operators $\alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$ with $n < 0$. A generic state comes from acting with any number of these creation operators on the vacuum,

$$(\alpha_{-1}^{\mu_1})^{n_{\mu_1}}(\alpha_{-2}^{\mu_2})^{n_{\mu_2}}\ldots(\tilde{\alpha}_{-1}^{\nu_1})^{n_{\nu_1}}(\tilde{\alpha}_{-2}^{\nu_2})^{n_{\nu_2}}\ldots|0;p\rangle$$

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Each state in the Fock space is a different excited state of the string. Each has the interpretation of a different species of particle in spacetime. We'll see exactly what particles they are shortly. But for now, notice that because there's an infinite number of ways to excite a string there are an infinite number of different species of particles in this theory.

### 2.1.1 Ghosts

There's a problem with the Fock space that we've constructed: it doesn't have positive norm. The reason for this is that one of the scalar fields, $X^0$, comes with the wrong sign kinetic term in the action (1.30). From the perspective of the commutation relations, this issue raises its head in presence of the spacetime Minkowski metric in the expression

$$[\alpha_n^\mu, \alpha_m^{\nu\,\dagger}] = n\,\eta^{\mu\nu}\,\delta_{n,m}\ .$$

This gives rise to the offending negative norm states, which come with an odd number of timelike oscillators excited, for example

$$\langle p'; 0|\alpha_1^0\alpha_{-1}^0|0;p\rangle \sim -\delta^D(p-p')$$

This is the first problem that arises in the covariant approach to quantization. States with negative norm are referred to as *ghosts*. To make sense of the theory, we have to make sure that they can't be produced in any physical processes. Of course, this problem is familiar from attempts to quantize QED in Lorentz gauge. In that case, gauge symmetry rides to the rescue since the ghosts are removed by imposing the gauge fixing constraint. We must hope that the same happens in string theory.

### 2.1.2 Constraints

Although we won't push through with this programme at the present time, let us briefly look at what kind of constraints we have in string theory. In terms of Fourier modes, the classical constraints can be written as $L_n = \tilde{L}_n = 0$, where

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m$$

and similar for $\tilde{L}_n$. As in the Gupta-Bleuler quantization of QED, we don't impose all of these as operator equations on the Hilbert space. Instead we only require that the operators $L_n$ and $\tilde{L}_n$ have vanishing matrix elements when sandwiched between two physical states $|\text{phys}\rangle$ and $|\text{phys}'\rangle$,

$$\langle \text{phys}'|L_n|\text{phys}\rangle = \langle \text{phys}'|\tilde{L}_n|\text{phys}\rangle = 0$$

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Because $L_n^\dagger = L_{-n}$, it is therefore sufficient to require

$$L_n|\text{phys}\rangle = \tilde{L}_n|\text{phys}\rangle = 0 \quad \text{for} \;\; n > 0 \tag{2.6}$$

However, we still haven't explained how to impose the constraints $L_0$ and $\tilde{L}_0$. And these present a problem that doesn't arise in the case of QED. The problem is that, unlike for $L_n$ with $n \neq 0$, the operator $L_0$ is not uniquely defined when we pass to the quantum theory. There is an operator ordering ambiguity arising from the commutation relations (2.2). Commuting the $\alpha_n^\mu$ operators past each other in $L_0$ gives rise to extra constant terms.

**Question:** How do we know what order to put the $\alpha_n^\mu$ operators in the quantum operator $L_0$? Or the $\tilde{\alpha}_n^\mu$ operators in $\tilde{L}_0$?

**Answer:** We don't! Yet. Naively it looks as if each different choice will define a different theory when we impose the constraints. To make this ambiguity manifest, for now let's just pick a choice of ordering. We define the quantum operators to be normal ordered, with the annihilation operators $\alpha_n^i$, $n > 0$, moved to the right,

$$L_0 = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \frac{1}{2}\,\alpha_0^2 \quad , \quad \tilde{L}_0 = \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m + \frac{1}{2}\,\tilde{\alpha}_0^2$$

Then the ambiguity rears its head in the different constraint equations that we could impose, namely

$$(L_0 - a)|\text{phys}\rangle = (\tilde{L}_0 - a)|\text{phys}\rangle = 0 \tag{2.7}$$

for some constant $a$.

As we saw classically, the operators $L_0$ and $\tilde{L}_0$ play an important role in determining the spectrum of the string because they include a term quadratic in the momentum $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}\,p^\mu$. Combining the expression (1.41) with our constraint equation for $L_0$ and $\tilde{L}_0$, we find the spectrum of the string is given by,

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m\right) = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m\right)$$

We learn therefore that the undetermined constant $a$ has a direct physical effect: it changes the mass spectrum of the string. In the quantum theory, the sums over $\alpha_n^\mu$ modes are related to the number operators for the harmonic oscillator: they count the number of excited modes of the string. The level matching in the quantum theory tells us that the number of left-moving modes must equal the number of right-moving modes.

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Ultimately, we will find that the need to decouple the ghosts forces us to make a unique choice for the constant $a$. (Spoiler alert: it turns out to be $a = 1$). In fact, the requirement that there are no ghosts is much stronger than this. It also restricts the number of scalar fields that we have in the theory. (Another spoiler: $D = 26$). If you're interested in how this works in covariant formulation then you can read about it in the book by Green, Schwarz and Witten. Instead, we'll show how to quantize the string and derive these values for $a$ and $D$ in lightcone gauge. However, after a trip through the world of conformal field theory, we'll come back to these ideas in a context which is closer to the covariant approach.

## 2.2 Lightcone Quantization

We will now take the second path described at the beginning of this section. We will try to find a parameterization of all classical solutions of the string. This is equivalent to finding the classical phase space of the theory. We do this by solving the constraints (2.1) in the classical theory, leaving behind only the physical degrees of freedom.

Recall that we fixed the gauge to set the worldsheet metric to

$$g_{\alpha\beta} = \eta_{\alpha\beta} .$$

However, this isn't the end of our gauge freedom. There still remain gauge transformations which preserve this choice of metric. In particular, any coordinate transformation $\sigma \to \tilde{\sigma}(\sigma)$ which changes the metric by

$$\eta_{\alpha\beta} \to \Omega^2(\sigma)\eta_{\alpha\beta} , \tag{2.8}$$

can be undone by a Weyl transformation. What are these coordinate transformations? It's simplest to answer this using lightcone coordinates on the worldsheet,

$$\sigma^{\pm} = \tau \pm \sigma , \tag{2.9}$$

where the flat metric on the worldsheet takes the form,

$$ds^2 = -d\sigma^+ d\sigma^-$$

In these coordinates, it's clear that any transformation of the form

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \quad , \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-) , \tag{2.10}$$

simply multiplies the flat metric by an overall factor (2.8) and so can be undone by a compensating Weyl transformation. Some quick comments on this surviving gauge symmetry:

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- Recall that in Section 1.3.2 we used the argument that 3 gauge invariances (2 reparameterizations + 1 Weyl) could be used to fix 3 components of the worldsheet metric $g_{\alpha\beta}$. What happened to this argument? Why do we still have some gauge symmetry left? The reason is that $\tilde{\sigma}^{\pm}$ are functions of just a single variable, not two. So we did fix nearly all our gauge symmetries. What is left is a set of measure zero amongst the full gauge symmetry that we started with.

- The remaining reparameterization invariance (2.10) has an important physical implication. Recall that the solutions to the equations of motion are of the form $X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$ which looks like $2D$ functions worth of solutions. Of course, we still have the constraints which, in terms of $\sigma^\pm$, read

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \ , \tag{2.11}$$

which seems to bring the number down to $2(D-1)$ functions. But the reparameterization invariance (2.10) tells us that even some of these are fake since we can always change what we mean by $\sigma^\pm$. The physical solutions of the string are therefore actually described by $2(D-2)$ functions. But this counting has a nice interpretation: the degrees of freedom describe the *transverse* fluctuations of the string.

- The above comment reaches the same conclusion as the discussion in Section 1.3.2. There, in an attempt to get some feel for the constraints, we claimed that we could go to static gauge $X^0 = R\tau$ for some dimensionful parameter $R$. It is easy to check that this is simple to do using reparameterizations of the form (2.10). However, to solve the string constraints in full, it turns out that static gauge is not that useful. Rather we will use something called "lightcone gauge".

### 2.2.1 Lightcone Gauge

We would like to gauge fix the remaining reparameterization invariance (2.10). The best way to do this is called lightcone gauge. In counterpoint to the worldsheet lightcone coordinates (2.9), we introduce the spacetime lightcone coordinates,

$$X^\pm = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1}) \ . \tag{2.12}$$

Note that this choice picks out a particular time direction and a particular spatial direction. It means that any calculations that we do involving $X^\pm$ will not be manifestly Lorentz invariant. You might think that we needn't really worry about this. We could try to make the following argument: "The equations may not *look* Lorentz invariant

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but, since we started from a Lorentz invariant theory, at the end of the day any physical process is guaranteed to obey this symmetry". Right?! Well, unfortunately not. One of the more interesting and subtle aspects of quantum field theory is the possibility of anomalies: these are symmetries of the classical theory that do not survive the journey of quantization. When we come to the quantum theory, if our equations don't look Lorentz invariant then there's a real possibility that it's because the underlying physics actually isn't Lorentz invariant. Later we will need to spend some time figuring out under what circumstances our quantum theory keeps the classical Lorentz symmetry.

In lightcone coordinates, the spacetime Minkowski metric reads

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} dX^i dX^i$$

This means that indices are raised and lowered with $A_+ = -A^-$ and $A_- = -A^+$ and $A_i = A^i$. The product of spacetime vectors reads $A \cdot B = -A^+B^- - A^-B^+ + A^iB^i$.

Let's look at the solution to the equation of motion for $X^+$. It reads,

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-) \; .$$

We now gauge fix. We use our freedom of reparameterization invariance to choose coordinates such that

$$X_L^+ = \tfrac{1}{2}\,x^+ + \tfrac{1}{2}\,\alpha' p^+ \sigma^+ \;\;, \qquad X_R^+ = \tfrac{1}{2}\,x^+ + \tfrac{1}{2}\,\alpha' p^+ \sigma^- \; .$$

You might think that we could go further and eliminate $p^+$ and $x^+$ but this isn't possible because we don't quite have the full freedom of reparameterization invariance since all functions should remain periodic in $\sigma$. The upshot of this choice of gauge is that

$$X^+ = x^+ + \alpha' p^+ \tau \; . \qquad (2.13)$$

This is *lightcone gauge*. Notice that, as long as $p^+ \neq 0$, we can always shift $x^+$ by a shift in $\tau$.

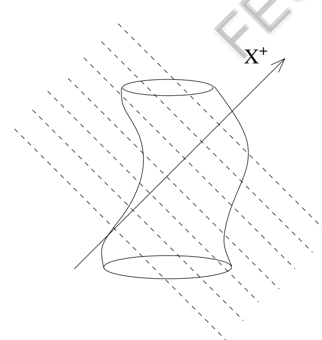


**Figure 9:**

There's something a little disconcerting about the choice (2.13). We've identified a timelike worldsheet coordinate with a null spacetime coordinate. Nonetheless, as you can see from the figure, it seems to be a good parameterization of the worldsheet. One could imagine that the parameterization might break if the string is actually massless and travels in the $X^-$ direction, with $p^+ = 0$. But otherwise, all should be fine.

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**Solving for $X^-$**

The choice (2.13) does the job of fixing the reparameterization invariance (2.10). As we will now see, it also renders the constraint equations trivial. The first thing that we have to worry about is the possibility of extra constraints arising from this new choice of gauge fixing. This can be checked by looking at the equation of motion for $X^+$,

$$\partial_+\partial_- X^- = 0$$

But we can solve this by the usual ansatz,

$$X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-)\ .$$

We're still left with all the other constraints (2.11). Here we see the real benefit of working in lightcone gauge (which is actually what makes quantization possible at all): $X^-$ is completely determined by these constraints. For example, the first of these reads

$$2\partial_+ X^- \partial_+ X^+ = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \tag{2.14}$$

which, using (2.13), simply becomes

$$\partial_+ X_L^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i\ . \tag{2.15}$$

Similarly,

$$\partial_- X_R^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i\ . \tag{2.16}$$

So, up to an integration constant, the function $X^-(\sigma^+, \sigma^-)$ is completely determined in terms of the other fields. If we write the usual mode expansion for $X^-_{L/R}$

$$X_L^-(\sigma^+) = \tfrac{1}{2}x^- + \tfrac{1}{2}\alpha' p^-\, \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \tilde{\alpha}_n^-\, e^{-in\sigma^+}\ ,$$

$$X_R^-(\sigma^-) = \tfrac{1}{2}x^- + \tfrac{1}{2}\alpha' p^-\, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \alpha_n^-\, e^{-in\sigma^-}\ .$$

then $x^-$ is the undetermined integration constant, while $p^-$, $\alpha_n^-$ and $\tilde{\alpha}_n^-$ are all fixed by the constraints (2.15) and (2.16). For example, the oscillator modes $\alpha_n^-$ are given by,

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \alpha_m^i\ , \tag{2.17}$$

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A special case of this is the $\alpha_0^- = \sqrt{\alpha'/2}\, p^-$ equation, which reads

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\tfrac{1}{2}\,\alpha' p^i p^i + \sum_{n\neq 0}\alpha_n^i \alpha_{-n}^i\right) . \tag{2.18}$$

We also get another equation for $p^-$ from the $\tilde{\alpha}_0^-$ equation arising from (2.15)

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\tfrac{1}{2}\,\alpha' p^i p^i + \sum_{n\neq 0}\tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i\right) . \tag{2.19}$$

From these two equations, we can reconstruct the old, classical, level matching conditions (1.41). But now with a difference:

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i . \tag{2.20}$$

The difference is that now the sum is over oscillators $\alpha^i$ and $\tilde{\alpha}^i$ only, with $i = 1, \ldots, D-2$. We'll refer to these as *transverse* oscillators. Note that the string isn't necessarily living in the $X^0$-$X^{D-1}$ plane, so these aren't literally the transverse excitations of the string. Nonetheless, if we specify the $\alpha^i$ then all other oscillator modes are determined. In this sense, they are the physical excitation of the string.

Let's summarize the state of play so far. The most general classical solution is described in terms of $2(D-2)$ transverse oscillator modes $\alpha_n^i$ and $\tilde{\alpha}_n^i$, together with a number of zero modes describing the center of mass and momentum of the string: $x^i, p^i, p^+$ and $x^-$. But $x^+$ can be absorbed by a shift of $\tau$ in (2.13) and $p^-$ is constrained to obey (2.18) and (2.19). In fact, $p^-$ can be thought of as (proportional to) the lightcone Hamiltonian. Indeed, we know that $p^-$ generates translations in $x^+$, but this is equivalent to shifts in $\tau$.

### 2.2.2 Quantization

Having identified the physical degrees of freedom, let's now quantize. We want to impose commutation relations. Some of these are easy:

$$\begin{gathered}[x^i, p^j] = i\delta^{ij} \quad , \quad [x^-, p^+] = -i \\ [\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0} .\end{gathered} \tag{2.21}$$

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all of which follow from the commutation relations (2.2) that we saw in covariant quantization[1].

What to do with $x^+$ and $p^-$? We could implement $p^-$ as the Hamiltonian acting on states. In fact, it will prove slightly more elegant (but equivalent) if we promote both $x^+$ and $p^-$ to operators with the expected commutation relation,

$$[x^+, p^-] = -i \; . \tag{2.22}$$

This is morally equivalent to writing $[t, H] = -i$ in non-relativistic quantum mechanics, which is true on a formal level. In the present context, it means that we can once again choose states to be eigenstates of $p^\mu$, with $\mu = 0, \ldots, D$, but the constraints (2.18) and (2.19) must still be imposed as operator equations on the physical states. We'll come to this shortly.

The Hilbert space of states is very similar to that described in covariant quantization: we define a vacuum state, $|0;p\rangle$ such that

$$\hat{p}^\mu |0;p\rangle = p^\mu |0;p\rangle \quad , \quad \alpha^i_n |0;p\rangle = \tilde{\alpha}^i_n |0;p\rangle = 0 \quad \text{for} \;\; n > 0 \tag{2.23}$$

and we build a Fock space by acting with the creation operators $\alpha^i_{-n}$ and $\tilde{\alpha}^i_{-n}$ with $n > 0$. The difference with the covariant quantization is that we only act with transverse oscillators which carry a spatial index $i = 1, \ldots, D-2$. For this reason, the Hilbert space is, by construction, positive definite. We don't have to worry about ghosts.

[1]**Mea Culpa:** We're not really supposed to do this. The whole point of the approach that we're taking is to quantize just the physical degrees of freedom. The resulting commutation relations are not, in general, inherited from the larger theory that we started with simply by closing our eyes and forgetting about all the other fields that we've gauge fixed. We can see the problem by looking at (2.17), where $\alpha^-_n$ is determined in terms of $\alpha^i_n$. This means that the commutation relations for $\alpha^i_n$ might be infected by those of $\alpha^-_n$ which could potentially give rise to extra terms. The correct procedure to deal with this is to figure out the Poisson bracket structure of the physical degrees of freedom in the classical theory. Or, in fancier language, the symplectic form on the phase space which schematically looks like

$$\omega \sim \int d\sigma \;\; - d\dot{X}^+ \wedge dX^- - d\dot{X}^- \wedge dX^+ + 2d\dot{X}^i \wedge dX^i \; ,$$

The reason that the commutation relations (2.21) do not get infected is because the $\alpha^-$ terms in the symplectic form come multiplying $X^+$. Yet $X^+$ is given in (2.13). It has no oscillator modes. That means that the symplectic form doesn't pick up the Fourier modes of $X^-$ and so doesn't receive any corrections from $\alpha^-_n$. The upshot of this is that the naive commutation relations (2.21) are actually right.

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**The Constraints**

Because $p^-$ is not an independent variable in our theory, we must impose the constraints (2.18) and (2.19) by hand as operator equations which define the physical states. In the classical theory, we saw that these constraints are equivalent to mass-shell conditions (2.20).

But there's a problem when we go to the quantum theory. It's the same problem that we saw in covariant quantization: there's an ordering ambiguity in the sum over oscillator modes on the right-hand side of (2.20). If we choose all operators to be normal ordered then this ambiguity reveals itself in an overall constant, $a$, which we have not yet determined. The final result for the mass of states in lightcone gauge is:

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - a\right)$$

Since we'll use this formula quite a lot in what follows, it's useful to introduce quantities related to the number operators of the harmonic oscillator,

$$N = \sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n \quad , \quad \tilde{N} = \sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \ . \tag{2.24}$$

These are not quite number operators because of the factor of $1/\sqrt{n}$ in (2.3). The value of $N$ and $\tilde{N}$ is often called the level. Which, if nothing else, means that the name "level matching" makes sense. We now have

$$M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a) \ . \tag{2.25}$$

How are we going to fix $a$? Later in the course we'll see the correct way to do it. For now, I'm just going to give you a quick and dirty derivation.

**The Casimir Energy**

> "I told him that the sum of an infinite no. of terms of the series: $1+2+3+4+\ldots = -\frac{1}{12}$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. "
>
> *Ramanujan, in a letter to G.H.Hardy.*

What follows is a heuristic derivation of the normal ordering constant $a$. Suppose that we didn't notice that there was any ordering ambiguity and instead took the naive classical result directly over to the quantum theory, that is

$$\frac{1}{2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_{n<0}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n \ .$$

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where we've left the sum over $i = 1, \ldots, D-2$ implicit. We'll now try to put this in normal ordered form, with the annihilation operators $\alpha_n^i$ with $n > 0$ on the right-hand side. It's the first term that needs changing. We get

$$\frac{1}{2}\sum_{n<0}\left[\alpha_n^i\alpha_{-n}^i - n(D-2)\right] + \frac{1}{2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i = \sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0} n \quad .$$

The final term clearly diverges. But it at least seems to have a physical interpretation: it is the sum of zero point energies of an infinite number of harmonic oscillators. In fact, we came across exactly the same type of term in the course on quantum field theory where we learnt that, despite the divergence, one can still extract interesting physics from this. This is the physics of the Casimir force.

Let's recall the steps that we took to derive the Casimir force. Firstly, we introduced an ultra-violet cut-off $\epsilon \ll 1$, probably muttering some words about no physical plates being able to withstand very high energy quanta. Unfortunately, those words are no longer available to us in string theory, but let's proceed regardless. We replace the divergent sum over integers by the expression,

$$\begin{aligned}\sum_{n=1}^{\infty} n \longrightarrow \sum_{n=1}^{\infty} n e^{-\epsilon n} &= -\frac{\partial}{\partial\epsilon}\sum_{n=1}^{\infty} e^{-\epsilon n} \\ &= -\frac{\partial}{\partial\epsilon}(1-e^{-\epsilon})^{-1} \\ &= \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon)\end{aligned}$$

Obviously the $1/\epsilon^2$ piece diverges as $\epsilon \to 0$. This term should be renormalized away. In fact, this is necessary to preserve the Weyl invariance of the Polyakov action since it contributes to a cosmological constant on the worldsheet. After this renormalization, we're left with the wonderful answer, first intuited by Ramanujan

$$\sum_{n=1}^{\infty} n = -\frac{1}{12} \ .$$

While heuristic, this argument does predict the correct physical Casimir energy measured in one-dimensional systems. For example, this effect is seen in simulations of quantum spin chains.

What does this mean for our string? It means that we should take the unknown constant $a$ in the mass formula (2.25) to be,

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right) \ . \tag{2.26}$$

This is the formula that we will use to determine the spectrum of the string.

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### Zeta Function Regularization

I appreciate that the preceding argument is not totally convincing. We could spend some time making it more robust at this stage, but it's best if we wait until later in the course when we will have the tools of conformal field theory at our disposal. We will eventually revisit this issue and provide a respectable derivation of the Casimir energy in Section 4.4.1. For now I merely offer an even less convincing argument, known as zeta-function regularization.

The zeta-function is defined, for $\mathrm{Re}(s) > 1$, by the sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \ .$$

But $\zeta(s)$ has a unique analytic continuation to all values of $s$. In particular,

$$\zeta(-1) = -\frac{1}{12} \ .$$

Good? Good. This argument is famously unconvincing the first time you meet it! But it's actually a very useful trick for getting the right answer.

## 2.3 The String Spectrum

Finally, we're in a position to analyze the spectrum of a single, free string.

### 2.3.1 The Tachyon

Let's start with the ground state $|0;p\rangle$ defined in (2.23). With no oscillators excited, the mass formula (2.26) gives

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6} \ . \tag{2.27}$$

But that's a little odd. It's a negative mass-squared. Such particles are called *tachyons*.

In fact, tachyons aren't quite as pathological as you might think. If you've heard of these objects before, it's probably in the context of special relativity where they're strange beasts which always travel faster than the speed of light. But that's not the right interpretation. Rather we should think more in the language of quantum field theory. Suppose that we have a field in spacetime — let's call it $T(X)$ — whose quanta will give rise to this particle. The mass-squared of the particle is simply the quadratic term in the action, or

$$M^2 = \left.\frac{\partial^2 V(T)}{\partial T^2}\right|_{T=0}$$

So the negative mass-squared in (2.27) is telling us that we're expanding around a maximum of the potential for the tachyon field as shown in the figure. Note that from this perspective, the Higgs field in the standard model at $H = 0$ is also a tachyon.

The fact that string theory turns out to sit at an unstable point in the tachyon field is unfortunate. The natural question is whether the potential has a good minimum elsewhere, as shown in the figure to the right. No one knows the answer to this! Naive attempts to understand this don't work. We know that around $T = 0$, the leading order contribution to the potential is negative and quadratic. But there are further terms that we can compute using techniques that we'll describe in Section 6. An expansion of the tachyon potential around $T = 0$ looks like

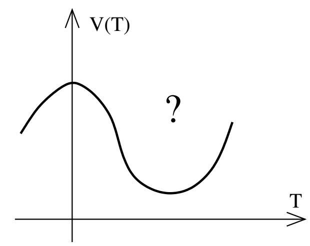


**Figure 11:**

$$V(T) = \frac{1}{2}M^2T^2 + c_3T^3 + c_4T^4 + \ldots$$

It turns out that the $T^3$ term in the potential does give rise to a minimum. But the $T^4$ term destabilizes it again. Moreover, the $T$ field starts to mix with other scalar fields in the theory that we will come across soon. The ultimate fate of the tachyon in the bosonic string is not yet understood.

The tachyon is a problem for the bosonic string. It may well be that this theory makes no sense — or, at the very least, has no time-independent stable solutions. Or perhaps we just haven't worked out how to correctly deal with the tachyon. Either way, the problem does not arise when we introduce fermions on the worldsheet and study the superstring. This will involve several further technicalities which we won't get into in this course. Instead, our time will be put to better use if we continue to study the bosonic string since all the lessons that we learn will carry over directly to the superstring. However, one should be aware that the problem of the unstable vacuum will continue to haunt us throughout this course.

Although we won't describe it in detail, at several times along our journey we'll make an aside about how calculations work out for the superstring.

### 2.3.2 The First Excited States

We now look at the first excited states. If we act with a creation operator $\alpha^j_{-1}$, then the level matching condition (2.25) tells us that we also need to act with a $\tilde{\alpha}^i_{-1}$ operator. This gives us $(D-2)^2$ particle states,

$$\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0;p\rangle\ , \tag{2.28}$$

each of which has mass

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right) .$$

But now we seem to have a problem. Our states have space indices $i, j = 1, \ldots, D-2$. The operators $\alpha^i$ and $\tilde{\alpha}^i$ each transform in the vector representation of $SO(D-2) \subset SO(1, D-1)$ which is manifest in lightcone gauge. But ultimately we want these states to fit into some representation of the full Lorentz $SO(1, D-1)$ group. That looks as if it's going to be hard to arrange. This is the first manifestation of the comment that we made after equation (2.12): it's tricky to see Lorentz invariance in lightcone gauge.

To proceed, let's recall Wigner's classification of representations of the Poincaré group. We start by looking at massive particles in $\mathbf{R}^{1,D-1}$. After going to the rest frame of the particle by setting $p^\mu = (p, 0, \ldots, 0)$, we can watch how any internal indices transform under the little group $SO(D-1)$ of spatial rotations. The upshot of this is that any massive particle must form a representation of $SO(D-1)$. But the particles described by (2.28) have $(D-2)^2$ states. There's no way to package these states into a representation of $SO(D-1)$ and this means that there's no way that the first excited states of the string can form a massive representation of the $D$-dimensional Poincaré group.

It looks like we're in trouble. Thankfully, there's a way out. If the states are massless, then we can't go to the rest frame. The best that we can do is choose a spacetime momentum for the particle of the form $p^\mu = (p, 0, \ldots, 0, p)$. In this case, the particles fill out a representation of the little group $SO(D-2)$. This means that massless particles get away with having fewer internal states than massive particles. For example, in four dimensions the photon has two polarization states, but a massive spin-1 particle must have three.

The first excited states (2.28) happily sit in a representation of $SO(D-2)$. We learn that if we want the quantum theory to preserve the $SO(1, D-1)$ Lorentz symmetry that we started with, then these states will have to be massless. And this is only the case if the dimension of spacetime is

$$D = 26 .$$

This is our first derivation of the critical dimension of the bosonic string.

Moreover, we've found that our theory contains a bunch of massless particles. And massless particles are interesting because they give rise to long range forces. Let's look



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more closely at what massless particles the string has given us. The states (2.28) transform in the $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$. These decompose into three irreducible representations:

$$\text{traceless symmetric} \ \oplus \ \text{anti-symmetric} \ \oplus \ \text{singlet (=trace)}$$

To each of these modes, we associate a massless field in spacetime such that the string oscillation can be identified with a quantum of these fields. The fields are:

$$G_{\mu\nu}(X) \quad , \quad B_{\mu\nu}(X) \quad , \quad \Phi(X) \tag{2.29}$$

Of these, the first is the most interesting and we shall have more to say momentarily. The second is an anti-symmetric tensor field which is usually called the anti-symmetric tensor field. It also goes by the names of the "Kalb-Ramond field" or, in the language of differential geometry, the "2-form". The scalar field is called the *dilaton*. These three massless fields are common to all string theories. We'll learn more about the role these fields play later in the course.

The particle in the symmetric traceless representation of $SO(24)$ is particularly interesting. This is a massless spin 2 particle. However, there are general arguments, due originally to Feynman and Weinberg, that *any* theory of interacting massless spin two particles must be equivalent to general relativity[2]. We should therefore identify the field $G_{\mu\nu}(X)$ with the metric of spacetime. Let's pause briefly to review the thrust of these arguments.

**Why Massless Spin 2 = General Relativity**

Let's call the spacetime metric $G_{\mu\nu}(X)$. We can expand around flat space by writing

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(X) \ .$$

Then the Einstein-Hilbert action has an expansion in powers of $h$. If we truncate to quadratic order, we simply have a free theory which we may merrily quantize in the usual canonical fashion: we promote $h_{\mu\nu}$ to an operator and introduce the associated creation and annihilation operators $a_{\mu\nu}$ and $a^\dagger_{\mu\nu}$. This way of looking at gravity is anathema to those raised in the geometrical world of general relativity. But from a particle physics language it is very standard: it is simply the quantization of a massless spin 2 field, $h_{\mu\nu}$.

[2] A very readable description of this can be found in the first few chapters of the Feynman Lectures on Gravitation.



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However, even on this simple level, there is a problem due to the indefinite signature of the spacetime Minkowski metric. The canonical quantization relations of the creation and annihilation operators are schematically of the form,

$$[a_{\mu\nu}, a^\dagger_{\rho\sigma}] \sim \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}$$

But this will lead to a Hilbert space with negative norm states coming from acting with time-like creation operators. For example, the one-graviton state of the form,

$$a^\dagger_{0i}|0\rangle \tag{2.30}$$

suffers from a negative norm. This should be becoming familiar by now: it is the usual problem that we run into if we try to covariantly quantize a gauge theory. And, indeed, general relativity is a gauge theory. The gauge transformations are diffeomorphisms. We would hope that this saves the theory of quantum gravity from these negative norm states.

Let's look a little more closely at what the gauge symmetry looks like for small fluctuations $h_{\mu\nu}$. We've butchered the Einstein-Hilbert action and left only terms quadratic in $h$. Including all the index contractions, we find

$$S_{EH} = \frac{M_{pl}^2}{2}\int d^4x\;\left[\partial_\mu h^\rho{}_\rho \partial_\nu h^{\mu\nu} - \partial^\rho h^{\mu\nu}\partial_\mu h_{\rho\nu} + \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{2}\partial_\mu h^\nu{}_\nu \partial^\mu h^\rho{}_\rho\right] + \ldots$$

One can check that this truncated action is invariant under the gauge symmetry,

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{2.31}$$

for any function $\xi_\mu(X)$. The gauge symmetry is the remnant of diffeomorphism invariance, restricted to small deviations away from flat space. With this gauge invariance in hand one can show that, just like QED, the negative norm states decouple from all physical processes.

To summarize, theories of massless spin 2 fields only make sense if there is a gauge symmetry to remove the negative norm states. In general relativity, this gauge symmetry descends from diffeomorphism invariance. The argument of Feynman and Weinberg now runs this logic in reverse. It goes as follows: suppose that we have a massless, spin 2 particle. Then, at the linearized level, it must be invariant under the gauge symmetry (2.31) in order to eliminate the negative norm states. Moreover, this symmetry must survive when interaction terms are introduced. But the only way to do this is to ensure that the resulting theory obeys diffeomorpism invariance. That means the theory of any interacting, massless spin 2 particle is Einstein gravity, perhaps supplemented by higher derivative terms.



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We haven't yet shown that string theory includes interactions for $h_{\mu\nu}$ but we will come to this later in the course. More importantly, we will also explicitly see how Einstein's field equations arise directly in string theory.

#### A Comment on Spacetime Gauge Invariance

We've surreptitiously put $\mu, \nu = 0, \ldots, 25$ indices on the spacetime fields, rather than $i, j = 1, \ldots, 24$. The reason we're allowed to do this is because both $G_{\mu\nu}$ and $B_{\mu\nu}$ enjoy a spacetime gauge symmetry which allows us to eliminate appropriate modes. Indeed, this is exactly the gauge symmetry (2.31) that entered the discussion above. It isn't possible to see these spacetime gauge symmetries from the lightcone formalism of the string since, by construction, we find only the physical states (although, by consistency alone, the gauge symmetries must be there). One of the main advantages of pushing through with the covariant calculation is that it does allow us to see how the spacetime gauge symmetry emerges from the string worldsheet. Details can be found in Green, Schwarz and Witten. We'll also briefly return to this issue in Section 5.

### 2.3.3 Higher Excited States

We rescued the Lorentz invariance of the first excited states by choosing $D = 26$ to ensure that they are massless. But now we've used this trick once, we still have to worry about all the other excited states. These also carry indices that take the range $i, j = 1, \ldots, D-2 = 24$ and, from the mass formula (2.26), they will all be massive and so must form representations of $SO(D-1)$. It looks like we're in trouble again.

Let's examine the string at level $N = \tilde{N} = 2$. In the right-moving sector, we now have two different states: $\alpha^i_{-1}\alpha^j_{-1}|0\rangle$ and $\alpha^i_{-2}|0\rangle$. The same is true for the left-moving sector, meaning that the total set of states at level 2 is (in notation that is hopefully obvious, but probably technically wrong)

$$(\alpha^i_{-1}\alpha^j_{-1} \oplus \alpha^i_{-2}) \otimes (\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1} \oplus \tilde{\alpha}^i_{-2})|0; p\rangle \ .$$

These states have mass $M^2 = 4/\alpha'$. How many states do we have? In the left-moving sector, we have,

$$\tfrac{1}{2}(D-2)(D-1) + (D-2) = \tfrac{1}{2}D(D-1) - 1 \ .$$

But, remarkably, that does fit nicely into a representation of $SO(D-1)$, namely the traceless symmetric tensor representation.

In fact, one can show that all excited states of the string fit nicely into $SO(D-1)$ representations. The only consistency requirement that we need for Lorentz invariance is to fix up the first excited states: $D = 26$.

Note that if we are interested in a fundamental theory of quantum gravity, then all these excited states will have masses close to the Planck scale so are unlikely to be observable in particle physics experiments. Nonetheless, as we shall see when we come to discuss scattering amplitudes, it is the presence of this infinite tower of states that tames the ultra-violet behaviour of gravity.

## 2.4 Lorentz Invariance Revisited

The previous discussion allowed to us to derive both the critical dimension and the spectrum of string theory in the quickest fashion. But the derivation creaks a little in places. The calculation of the Casimir energy is unsatisfactory the first time one sees it. Similarly, the explanation of the need for massless particles at the first excited level is correct, but seems rather cheap considering the huge importance that we're placing on the result.

As I've mentioned a few times already, we'll shortly do better and gain some physical insight into these issues, in particular the critical dimension. But here I would just like to briefly sketch how one can be a little more rigorous within the framework of lightcone quantization. The question, as we've seen, is whether one preserves spacetime Lorentz symmetry when we quantize in lightcone gauge. We can examine this more closely.

Firstly, let's go back to the action for free scalar fields (1.30) before we imposed lightcone gauge fixing. Here the full Poincaré symmetry was manifest: it appears as a global symmetry on the worldsheet,

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{2.32}$$

But recall that in field theory, global symmetries give rise to Noether currents and their associated conserved charges. What are the Noether currents associated to this Poincaré transformation? We can start with the translations $X^\mu \to X^\mu + c^\mu$. A quick computation shows that the current is,

$$P^\alpha_\mu = T\partial^\alpha X_\mu \tag{2.33}$$

which is indeed a conserved current since $\partial_\alpha P^\alpha_\mu = 0$ is simply the equation of motion. Similarly, we can compute the $\frac{1}{2}D(D-1)$ currents associated to Lorentz transformations. They are,

$$J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu$$

It's not hard to check that $\partial_\alpha J^\alpha_{\mu\nu} = 0$ when the equations of motion are obeyed.

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The conserved charges arising from this current are given by $M_{\mu\nu} = \int d\sigma J^\tau_{\mu\nu}$. Using the mode expansion (1.36) for $X^\mu$, these can be written as

$$\begin{aligned}\mathcal{M}^{\mu\nu} &= (p^\mu x^\nu - p^\nu x^\mu) - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^\nu_{-n}\alpha^\mu_n - \alpha^\mu_{-n}\alpha^\nu_n\right) - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_n - \tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n\right)\\ &\equiv l^{\mu\nu} + S^{\mu\nu} + \tilde{S}^{\mu\nu}\end{aligned}$$

The first piece, $l^{\mu\nu}$, is the orbital angular momentum of the string while the remaining pieces $S^{\mu\nu}$ and $\tilde{S}^{\mu\nu}$ tell us the angular momentum due to excited oscillator modes. Classically, these obey the Poisson brackets of the Lorentz algebra. Moreover, if we quantize in the covariant approach, the corresponding operators obey the commutation relations of the Lorentz Lie algebra, namely

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}$$

However, things aren't so easy in lightcone gauge. Lorentz invariance is not guaranteed and, in general, is not there. The right way to go about looking for it is to make sure that the Lorentz algebra above is reproduced by the generators $\mathcal{M}^{\mu\nu}$. It turns out that the smoking gun lies in the commutation relation,

$$[\mathcal{M}^{i-}, \mathcal{M}^{j-}] = 0$$

Does this equation hold in lightcone gauge? The problem is that it involves the operators $p^-$ and $\alpha^-_n$, both of which are fixed by (2.17) and (2.18) in terms of the other operators. So the task is to compute this commutation relation $[\mathcal{M}^{i-}, \mathcal{M}^{j-}]$, given the commutation relations (2.21) for the physical degrees of freedom, and check that it vanishes. To do this, we re-instate the ordering ambiguity $a$ and the number of spacetime dimension $D$ as arbitrary variables and proceed.

The part involving orbital angular momenta $l^{i-}$ is fairly straightforward. (Actually, there's a small subtlety because we must first make sure that the operator $l^{\mu\nu}$ is Hermitian by replacing $x^\mu p^\nu$ with $\frac{1}{2}(x^\mu p^\nu + p^\nu x^\mu)$). The real difficulty comes from computing the commutation relations $[S^{i-}, S^{j-}]$. This is messy[3]. After a tedious computation, one finds,

$$[\mathcal{M}^{i-}, \mathcal{M}^{j-}] = \frac{2}{(p^+)^2}\sum_{n>0}\left(\left[\frac{D-2}{24} - 1\right]n + \frac{1}{n}\left[a - \frac{D-2}{24}\right]\right)(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) + (\alpha \leftrightarrow \tilde{\alpha})$$

[3]The original, classic, paper where lightcone quantization was first implemented is Goddard, Goldstone, Rebbi and Thorn "*Quantum Dynamics of a Massless Relativistic String*", Nucl. Phys. B56 (1973). A pedestrian walkthrough of this calculation can be found in the lecture notes by Gleb Arutyunov. A link is given on the course webpage.

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The right-hand side does not, in general, vanish. We learn that the relativistic string can only be quantized in flat Minkowski space if we pick,

$$D = 26 \quad \text{and} \quad a = 1 \; .$$

### 2.5 A Nod to the Superstring

We won't provide details of the superstring in this course, but will pause occasionally to make some pertinent comments. Although what follows is nothing more than a list of facts, it will hopefully be helpful in orienting you when you do come to study this material.

The key difference between the bosonic string and the superstring is the addition of fermionic modes on its worldsheet. The resulting worldsheet theory is supersymmetric. (At least in the so-called Neveu-Schwarz-Ramond formalism). Hence the name "superstring". Applying the kind of quantization procedure we've discussed in this section, one finds the following results:

- The critical dimension of the superstring is $D = 10$.
- There is no tachyon in the spectrum.
- The massless bosonic fields $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$ are all part of the spectrum of the superstring. In this context, $B_{\mu\nu}$ is sometimes referred to as the Neveu-Schwarz 2-form. There are also massless spacetime fermions, as well as further massless bosonic fields. As we now discuss, the exact form of these extra bosonic fields depends on exactly what superstring theory we consider.

While the bosonic string is unique, there are a number of discrete choices that one can make when adding fermions to the worldsheet. This gives rise to a handful of different perturbative superstring theories. (Although later developments reveal that they are actually all part of the same framework which sometimes goes by the name of *M-theory*). The most important of these discrete options is whether we add fermions in both the left-moving and right-moving sectors of the string, or whether we choose the fermions to move only in one direction, usually taken to be right-moving. This gives rise to two different classes of string theory.

- Type II strings have both left and right-moving worldsheet fermions. The resulting spacetime theory in $D = 10$ dimensions has $\mathcal{N} = 2$ supersymmetry, which means 32 supercharges.
- Heterotic strings have just right-moving fermions. The resulting spacetime theory has $\mathcal{N} = 1$ supersymmetry, or 16 supercharges.

In each of these cases, there is then one further discrete choice that we can make. This leaves us with four superstring theories. In each case, the massless bosonic fields include $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$ together with a number of extra fields. These are:

- **Type IIA**: In the type II theories, the extra massless bosonic excitations of the string are referred to as *Ramond-Ramond* fields. For Type IIA, they are a 1-form $C_\mu$ and a 3-form $C_{\mu\nu\rho}$. Each of these is to be thought of as a gauge field. The gauge invariant information lies in the field strengths which take the form $F = dC$.

- **Type IIB**: The Ramond-Ramond gauge fields consist of a scalar $C$, a 2-form $C_{\mu\nu}$ and a 4-form $C_{\mu\nu\rho\sigma}$. The 4-form is restricted to have a self-dual field strength: $F_5 = {}^\star F_5$. (Actually, this statement is almost true...we'll look a little closer at this in Section 7.3.3).

- **Heterotic** $SO(32)$: The heterotic strings do not have Ramond-Ramond fields. Instead, each comes with a non-Abelian gauge field in spacetime. The heterotic strings are named after the gauge group. For example, the Heterotic $SO(32)$ string gives rise to an $SO(32)$ Yang-Mills theory in ten dimensions.

- **Heterotic** $E_8 \times E_8$: The clue is in the name. This string gives rise to an $E_8 \times E_8$ Yang-Mills field in ten-dimensions.

It is sometimes said that there are five perturbative superstring theories in ten dimensions. Here we've only mentioned four. The remaining theory is called Type I and includes open strings moving in flat ten dimensional space as well as closed strings. We'll mention it in passing in the following section.

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# 3. Open Strings and D-Branes

In this section we discuss the dynamics of open strings. Clearly their distinguishing feature is the existence of two end points. Our goal is to understand the effect of these end points. The spatial coordinate of the string is parameterized by

$$\sigma \in [0, \pi] \ .$$

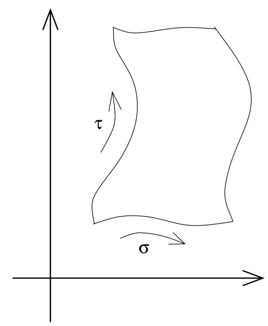


**Figure 12:**

The dynamics of a generic point on a string is governed by local physics. This means that a generic point has no idea if it is part of a closed string or an open string. The dynamics of an open string must therefore still be described by the Polyakov action. But this must now be supplemented by something else: boundary conditions to tell us how the end points move. To see this, let's look at the Polyakov action in conformal gauge

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \ \partial_\alpha X \cdot \partial^\alpha X \ .$$

As usual, we derive the equations of motion by finding the extrema of the action. This involves an integration by parts. Let's consider the string evolving from some initial configuration at $\tau = \tau_i$ to some final configuration at $\tau = \tau_f$:

$$\begin{aligned}\delta S &= -\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \ \ \partial_\alpha X \cdot \partial^\alpha \delta X \\ &= \frac{1}{2\pi\alpha'}\int d^2\sigma \ (\partial^\alpha\partial_\alpha X)\cdot\delta X + \text{total derivative}\end{aligned}$$

For an open string the total derivative picks up the boundary contributions

$$\frac{1}{2\pi\alpha'}\left[\int_0^\pi d\sigma \ \dot{X}\cdot\delta X\right]_{\tau=\tau_i}^{\tau=\tau_f} - \frac{1}{2\pi\alpha'}\left[\int_{\tau_i}^{\tau_f} d\tau \ X'\cdot\delta X\right]_{\sigma=0}^{\sigma=\pi}$$

The first term is the kind that we always get when using the principle of least action. The equations of motion are derived by requiring that $\delta X^\mu = 0$ at $\tau = \tau_i$ and $\tau_f$ and so it vanishes. However, the second term is novel. In order for it too to vanish, we require

$$\partial_\sigma X^\mu \, \delta X_\mu = 0 \qquad \text{at } \sigma = 0, \pi$$

There are two different types of boundary conditions that we can impose to satisfy this:

- Neumann boundary conditions.

$$\partial_\sigma X^\mu = 0 \qquad \text{at } \sigma = 0, \pi \tag{3.1}$$

Because there is no restriction on $\delta X^\mu$, this condition allows the end of the string to move freely. To see the consequences of this, it's useful to repeat what we did for the closed string and work in static gauge with $X^0 \equiv t = R\tau$, for some dimensionful constant $R$. Then, as in equations (1.34), the constraints read

$$\dot{\vec{x}} \cdot \vec{x}' = 0 \quad \text{and} \quad \dot{\vec{x}}^{\,2} + \vec{x}'^{\,2} = R^2$$

But at the end points of the string, $\vec{x}' = 0$. So the second equation tells us that $|d\vec{x}/dt| = 1$. Or, in other words, the end point of the string moves at the speed of light.

- Dirichlet boundary conditions

$$\delta X^\mu = 0 \qquad \text{at } \sigma = 0, \pi \tag{3.2}$$

This means that the end points of the string lie at some constant position, $X^\mu = c^\mu$, in space.

At first sight, Dirichlet boundary conditions may seem a little odd. Why on earth would the strings be fixed at some point $c^\mu$? What is special about that point? Historically people were pretty hung up about this and Dirichlet boundary conditions were rarely considered until the mid-1990s. Then everything changed due to an insight of Polchinski...

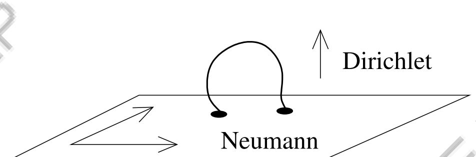


**Figure 13:**

Let's consider Dirichlet boundary conditions for some coordinates and Neumann for the others. This means that at both end points of the string, we have

$$\begin{aligned} \partial_\sigma X^a &= 0 \qquad \text{for } \; a = 0, \ldots, p \\ X^I &= c^I \qquad \text{for } \; I = p+1, \ldots, D-1 \end{aligned} \tag{3.3}$$

This fixes the end-points of the string to lie in a $(p+1)$-dimensional hypersurface in spacetime such that the $SO(1, D-1)$ Lorentz group is broken to,

$$SO(1, D-1) \to SO(1, p) \times SO(D-p-1) \ .$$

This hypersurface is called a *D-brane* or, when we want to specify its dimension, a D$p$-brane. Here D stands for Dirichlet, while $p$ is the number of spatial dimensions of the brane. So, in this language, a D0-brane is a particle; a D1-brane is itself a string; a D2-brane a membrane and so on. The brane sits at specific positions $c^I$ in the transverse space. But what is the interpretation of this hypersurface?

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It turns out that the D-brane hypersurface should be thought of as a new, dynamical object in its own right. This is a conceptual leap that is far from obvious. Indeed, it took decades for people to fully appreciate this fact. String theory is not just a theory of strings: it also contains higher dimensional branes. In Section 7.5 we will see how these D-branes develop a life of their own. Some comments:

- We've defined D-branes that are infinite in space. However, we could just as well define finite D-branes by specifying closed surfaces on which the string can end.
- There are many situations where we want to describe strings that have Neumann boundary conditions in all directions, meaning that the string is free to move throughout spacetime. It's best to understand this in terms of a space-filling D-brane. No Dirichlet conditions means D-branes are everywhere!
- The D$p$-brane described above always has Neumann boundary conditions in the $X^0$ direction. What would it mean to have Dirichlet conditions for $X^0$? Obviously this is a little weird since the object is now localized at a fixed point in time. But there is an interpretation of such an object: it is an *instanton*. This "D-instanton" is usually referred to as a D($-1$)-brane. It is related to tunneling effects in the quantum theory.

### Mode Expansion

We take the usual mode expansion for the string, with $X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ and

$$X^\mu_L(\sigma^+) = \tfrac{1}{2}x^\mu + \alpha' p^\mu\,\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}^\mu_n\,e^{-in\sigma^+}\ ,$$
$$X^\mu_R(\sigma^-) = \tfrac{1}{2}x^\mu + \alpha' p^\mu\,\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\,e^{-in\sigma^-}\ . \tag{3.4}$$

The boundary conditions impose relations on the modes of the string. They are easily checked to be:

- Neumann boundary conditions, $\partial_\sigma X^a = 0$, at the end points require that
$$\alpha^a_n = \tilde{\alpha}^a_n \tag{3.5}$$
- Dirichlet boundary conditions, $X^I = c^I$, at the end points require that
$$x^I = c^I \quad , \quad p^I = 0 \quad , \quad \alpha^I_n = -\tilde{\alpha}^I_n$$

So for both boundary conditions, we only have one set of oscillators, say $\alpha_n$. The $\tilde{\alpha}_n$ are then determined by the boundary conditions.

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It's worth pointing out that there is a factor of 2 difference in the $p^\mu$ term between the open string (3.4) and the closed string (1.36). This is to ensure that $p^\mu$ for the open string retains the interpretation of the spacetime momentum of the string when $\sigma \in [0, \pi]$. To see this, one needs to check the Noether current associated to translations of $X^\mu$ on the worldsheet: it was given in (2.33). The conserved charge is then

$$P^\mu = \int_0^\pi d\sigma\, (P^\tau)^\mu = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma\, \dot{X}^\mu = p^\mu$$

as advertised. Note that we've needed to use the Neumann conditions (3.5) to ensure that the Fourier modes don't contribute to this integral.

### 3.1 Quantization

To quantize, we promote the fields $x^a$ and $p^a$ and $\alpha_n^\mu$ to operators. The other elements in the mode expansion are fixed by the boundary conditions. An obvious, but important, point is that the position and momentum degrees of freedom, $x^a$ and $p^a$, have a spacetime index that takes values $a = 0, \ldots, p$. This means that the spatial wavefunctions only depend on the coordinates of the brane not the whole spacetime. Said another, quantizing an open string gives rise to states which are restricted to lie on the brane.

To determine the spectrum, it is again simplest to work in lightcone gauge. The spacetime lightcone coordinate is chosen to lie within the brane,

$$X^\pm = \sqrt{\frac{1}{2}}\,(X^0 \pm X^p)$$

Quantization now proceeds in the same manner as for the closed string until we arrive at the mass formula for states which is a sum over the transverse modes of the string.

$$M^2 = \frac{1}{\alpha'}\left(\sum_{i=1}^{p-1}\sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{i=p+1}^{D-1}\sum_{n>0} \alpha_{-n}^i \alpha_n^i - a\right)$$

The first sum is over modes parallel to the brane, the second over modes perpendicular to the brane. It's worth commenting on the differences with the closed string formula. Firstly, there is an overall factor of 4 difference. This can be traced to the lack of the factor of 1/2 in front of $p^\mu$ in the mode expansion that we discussed above. Secondly, there is a sum only over $\alpha$ modes. The $\tilde{\alpha}$ modes are not independent because of the boundary conditions.

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### Open and Closed

In the mass formula, we have once again left the normal ordering constant $a$ ambiguous. As in the closed string case, requiring the Lorentz symmetry of the quantum theory — this time the reduced symmetry $SO(1,p) \times SO(D-p-1)$ — forces us to choose

$$D = 26 \quad \text{and} \quad a = 1 \ .$$

These are the same values that we found for the closed string. This reflects an important fact: the open string and closed string are not different theories. They are both different states inside the same theory.

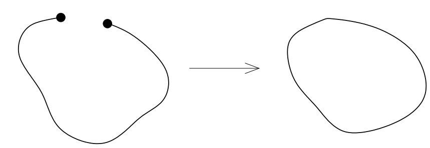

**Figure 14:**

More precisely, theories of open strings necessarily contain closed strings. This is because, once we consider interactions, an open string can join to form a closed string as shown in the figure. We'll look at interactions in Section 6. The question of whether this works the other way — meaning whether closed string theories require open strings — is a little more involved and is cleanest to state in the context of the superstring. For type II superstrings, the open strings and D-branes are necessary ingredients. For heterotic superstrings, there appear to be no open strings and no D-branes. For the bosonic theory, it seems likely that the open strings are a necessary ingredient although I don't know of a killer argument. But since we're not sure whether the theory exists due to the presence of the tachyon, the point is probably moot. In the remainder of these lectures, we'll view the bosonic string in the same manner as the type II string and assume that the theory includes both closed strings and open strings with their associated D-branes.

### 3.1.1 The Ground State

The ground state is defined by

$$\alpha^i_n |0;p\rangle = 0 \qquad\qquad n > 0$$

The spatial index now runs over $i = 1, \ldots, p-1, p+1, \ldots, D-1$. The ground state has mass

$$M^2 = -\frac{1}{\alpha'}$$

It is again tachyonic. Its mass is half that of the closed string tachyon. As we commented above, this time the tachyon is confined to the brane. In contrast to the closed string tachyon, the open string tachyon is now fairly well understood and its potential

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is of the form shown in the figure. The interpretation is that the brane is unstable. It will decay, much like a resonance state in field theory. It does this by dissolving into closed string modes. The end point of this process – corresponding to the minimum at $T > 0$ in the figure – is simply a state with no D-brane. The difference between the value of the potential at the minimum and at $T = 0$ is the tension of the D-brane.

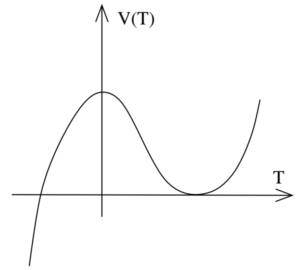


**Figure 15:**

Notice that although there is a minimum of the potential at $T > 0$, it is not a global minimum. The potential seems to drop off without bound to the left. This is still not well understood. There are suggestions that it is related in some way to the closed string tachyon.

### 3.1.2 First Excited States: A World of Light

The first excited states are massless. They fall into two classes:

- Oscillators longitudinal to the brane,

$$\alpha^a_{-1}|0;p\rangle \qquad a = 1, \ldots, p-1$$

  The spacetime indices $a$ lie within the brane so this state transforms under the $SO(1,p)$ Lorentz group. It is a spin 1 particle on the brane or, in other words, it is a photon. We introduce a gauge field $A_a$ with $a = 0, \ldots, p$ lying on the brane whose quanta are identified with this photon.

- Oscillators transverse to the brane,

$$\alpha^I_{-1}|0;p\rangle \qquad I = p+1, \ldots, D-1$$

  These states are scalars under the $SO(1,p)$ Lorentz group of the brane. They can be thought of as arising from scalar fields $\phi^I$ living on the brane. These scalars have a nice interpretation: they are fluctuations of the brane in the transverse directions. This is our first hint that the D-brane is a dynamical object. Note that although the $\phi^I$ are scalar fields under the $SO(1,p)$ Lorentz group of the brane, they do transform as a vector under the $SO(D-p-1)$ rotation group transverse to the brane. This appears as a global symmetry on the brane worldvolume.

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### 3.1.3 Higher Excited States and Regge Trajectories

At level $N$, the mass of the string state is

$$M^2 = \frac{1}{\alpha'}(N-1)$$

The maximal spin of these states arises from the symmetric tensor. It is

$$J_{max} = N = \alpha' M^2 + 1$$

Plotting the spin vs. the mass-squared, we find straight lines. These are usually called *Regge trajectories*. (Or sometimes Chew-Frautschi trajectories). They are seen in Nature in both the spectrum of mesons and baryons. Some examples involving $\rho$-mesons are shown in the figure. These stringy Regge trajectories suggest a naive cartoon picture of mesons as two rotating quarks connected by a confining flux tube.

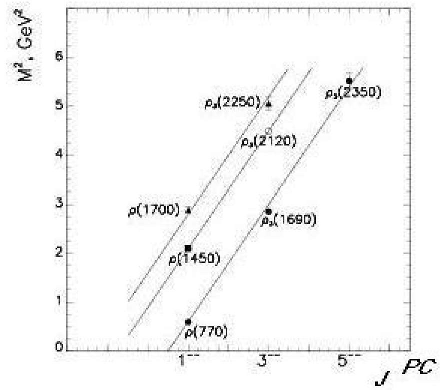


**Figure 16:**

The value of the string tension required to match the hadron spectrum of QCD is $T \sim 1$ GeV. This relationship between the strong interaction and the open string was one of the original motivations for the development of string theory and it is from here that the parameter $\alpha'$ gets its (admittedly rarely used) name "Regge slope". In these enlightened modern times, the connection between the open string and quarks lives on in the AdS/CFT correspondence.

### 3.1.4 Another Nod to the Superstring

Just as supersymmetry eliminates the closed string tachyon, so it removes the open string tachyon. Open strings are an ingredient of the type II string theories. The possible D-branes are

- Type IIA string theory has stable D$p$-branes with $p$ even.
- Type IIB string theory has stable D$p$-branes with $p$ odd.

The most important reason that D-branes are stable in the type II string theories is that they are charged under the Ramond-Ramond fields. (This was actually Polchinski's insight that made people take D-branes seriously). However, type II string theories also contain unstable branes, with $p$ odd in type IIA and $p$ even in type IIB.

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The fifth string theory (which was actually the first to be discovered) is called Type I. Unlike the other string theories, it contains both open and closed strings moving in flat ten-dimensional Lorentz-invariant spacetime. It can be thought of as the Type IIB theory with a bunch of space-filling D9-branes, together with something called an orientifold plane. You can read about this in Polchinski.

As we mentioned above, the heterotic string doesn't have (finite energy) D-branes. This is due to an inconsistency in any attempt to reflect left-moving modes into right-moving modes.

### 3.2 Brane Dynamics: The Dirac Action

We have introduced D-branes as fixed boundary conditions for the open string. However, we've already seen a hint that these objects are dynamical in their own right, since the massless scalar excitations $\phi^I$ have a natural interpretation as transverse fluctuations of the brane. Indeed, if a theory includes both open strings and closed strings, then the D-branes have to be dynamical because there can be no rigid objects in a theory of gravity. The dynamical nature of D-branes will become clearer as the course progresses.

But any dynamical object should have an action which describes how it moves. Moreover, after our discussion in Section 1, we already know what this is! On grounds of Lorentz invariance and reparameterization invariance alone, the action must be a higher dimensional extension of the Nambu-Goto action. This is

$$S_{Dp} = -T_p \int d^{p+1}\xi \; \sqrt{-\det\gamma} \tag{3.6}$$

where $T_p$ is the tension of the D$p$-brane which we will determine later, while $\xi^a$, $a = 0, \ldots, p$, are the worldvolume coordinates of the brane. $\gamma_{ab}$ is the pull back of the spacetime metric onto the worldvolume,

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu} \; .$$

This is called the *Dirac action*. It was first written down by Dirac for a membrane some time before Nambu and Goto rediscovered it in the context of the string.

To make contact with the fields $\phi^I$, we can use the reparameterization invariance of the Dirac action to go to static gauge. For an infinite, flat D$p$-brane we can choose

$$X^a = \xi^a \qquad a = 0, \ldots, p \; .$$

The dynamical transverse coordinates are then identified with the fluctuations $\phi^I$ through

$$X^I(\xi) = 2\pi\alpha'\,\phi^I(\xi) \qquad I = p+1, \ldots, D-1$$

However, the Dirac action can't be the whole story. It describes the transverse fluctuations of the D-brane, but has nothing to say about the $U(1)$ gauge field $A_\mu$ which lives on the D-brane. There must be some action which describes how this gauge field moves as well. We will return to this in Section 7.

**What's Special About Strings?**

We could try to quantize the Dirac action (3.6) for a D-brane in the same manner that we quantized the action for the string. Is this possible? The answer, at present, is no. There appear to be both technical and conceptual obstacles . The technical issue is just that it's hard. Weyl invariance was one of our chief weapons in attacking the string, but it doesn't hold for higher dimensional objects.

The conceptual issue is that quantizing a membrane, or higher dimensional object, would not give rise to a discrete spectrum of states which have the interpretation of particles. In this way, they appear to be fundamentally different from the string.

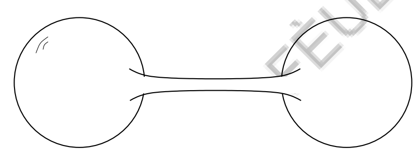

**Figure 17:**

Let's get some intuition for why this is the case. The energy of a string is proportional to its length. This ensures that strings behave more or less like familiar elastic bands. What about D2-branes? Now the energy is proportional to the area. In the back of your mind, you might be thinking of a rubber-like sheet. But membranes, and higher dimensional objects, governed by the Dirac action don't behave as household rubber sheets. They are more flexible. This is because a membrane can form many different shapes with the same area. For example, a tubular membrane of length $L$ and radius $1/L$ has the same area for all values of $L$; short and stubby, or long and thin. This means that long thin spikes can develop on a membrane at no extra cost of energy. In particular, objects connected by long thin tubes have the same energy, regardless of their separation. After quantization, this property gives rise to a continuous spectrum of states. A quantum membrane, or higher dimensional object, does not have the single particle interpretation that we saw for the string. The expectation is that the quantum membrane should describe multi-particle states.

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### 3.3 Multiple Branes: A World of Glue

Consider two parallel D$p$-branes. An open string now has options. It could either end on the same brane, or stretch between the two branes. Let's consider the string that stretches between the two. It obeys

$$X^I(0,\tau) = c^I \quad \text{and} \quad X^I(\pi,\tau) = d^I$$

where $c^I$ and $d^I$ are the positions of the two branes. In terms of the mode expansion, this requires

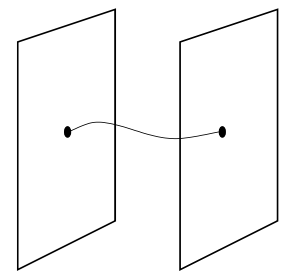

**Figure 18:**

$$X^I = c^I + \frac{(d^I - c^I)\sigma}{\pi} + \text{oscillator modes}$$

The classical constraints then read

$$\partial_+ X \cdot \partial_+ X = \alpha'^2 p^2 + \frac{|\vec{d} - \vec{c}|^2}{4\pi^2} + \text{oscillator modes} = 0$$

which means the classical mass-shell condition is

$$M^2 = \frac{|\vec{d} - \vec{c}|^2}{(2\pi\alpha')^2} + \text{oscillator modes}$$

The extra term has an obvious interpretation: it is the mass of a classical string stretched between the two branes. The quantization of this string proceeds as before. After we include the normal ordering constant, the ground state of this string is only tachyonic if $|\vec{d} - \vec{c}|^2 < 4\pi^2\alpha'$. Or in other words, the ground state is tachyonic if the branes approach to a sub-stringy distance.

There is an obvious generalization of this to the case of $N$ parallel branes. Each end point of the string has $N$ possible places on which to end. We can label each end point with a number $m, n = 1, \ldots, N$ which tell us which brane it ends on. This label is sometimes referred to as a *Chan-Paton factor*.

Consider now the situation where all branes lie at the same position in spacetime. Each end point can lie on one of $N$ different branes, giving $N^2$ possibilities in total. Each of these strings has the mass spectrum of an open string, meaning that there are now $N^2$ different particles of each type. It's natural to arrange the associated fields to sit inside $N \times N$ Hermitian matrices. We then have the open string tachyon $T^m_{\ n}$ and the massless fields

$$(\phi^I)^m_{\ n} \quad , \quad (A_a)^m_{\ n} \tag{3.7}$$

Here the components of the matrix tell us which string the field came from. Diagonal components arise from strings which have both ends on the same brane.

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The gauge field $A_a$ is particularly interesting. Written in this way, it looks like a $U(N)$ gauge connection. We will later see that this is indeed the case. One can show that as $N$ branes coincide, the $U(1)^N$ gauge symmetry of the branes is enhanced to $U(N)$. The scalar fields $\phi^I$ transform in the adjoint of this symmetry.

# 4. Introducing Conformal Field Theory

The purpose of this section is to get comfortable with the basic language of two dimensional conformal field theory[4]. This is a topic which has many applications outside of string theory, most notably in statistical physics where it offers a description of critical phenomena. Moreover, it turns out that conformal field theories in two dimensions provide rare examples of interacting, yet exactly solvable, quantum field theories. In recent years, attention has focussed on conformal field theories in higher dimensions due to their role in the AdS/CFT correspondence.

A *conformal transformation* is a change of coordinates $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ such that the metric changes by

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma) \tag{4.1}$$

A *conformal field theory* (CFT) is a field theory which is invariant under these transformations. This means that the physics of the theory looks the same at all length scales. Conformal field theories care about angles, but not about distances.

A transformation of the form (4.1) has a different interpretation depending on whether we are considering a fixed background metric $g_{\alpha\beta}$, or a dynamical background metric. When the metric is dynamical, the transformation is a diffeomorphism; this is a gauge symmetry. When the background is fixed, the transformation should be thought of as an honest, physical symmetry, taking the point $\sigma^\alpha$ to point $\tilde{\sigma}^\alpha$. This is now a global symmetry with the corresponding conserved currents.

In the context of string theory in the Polyakov formalism, the metric is dynamical and the transformations (4.1) are residual gauge transformations: diffeomorphisms which can be undone by a Weyl transformation.

In contrast, in this section we will be primarily interested in theories defined on fixed backgrounds. Apart from a few noticeable exceptions, we will usually take this background to be flat. This is the situation that we are used to when studying quantum field theory.

[4]Much of the material covered in this section was first described in the ground breaking paper by Belavin, Polyakov and Zamalodchikov, "*Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*", Nucl. Phys. B241 (1984). The application to string theory was explained by Friedan, Martinec and Shenker in "*Conformal Invariance, Supersymmetry and String Theory*", Nucl. Phys. B271 (1986). The canonical reference for learning conformal field theory is the excellent review by Ginsparg. A link can be found on the course webpage.



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Of course, we can alternate between thinking of theories as defined on fixed or fluctuating backgrounds. Any theory of 2d gravity which enjoys both diffeomorphism and Weyl invariance will reduce to a conformally invariant theory when the background metric is fixed. Similarly, any conformally invariant theory can be coupled to 2d gravity where it will give rise to a classical theory which enjoys both diffeomorphism and Weyl invariance. Notice the caveat "classical"! In some sense, the whole point of this course is to understand when this last statement also holds at the quantum level.

Even though conformal field theories are a subset of quantum field theories, the language used to describe them is a little different. This is partly out of necessity. Invariance under the transformation (4.1) can only hold if the theory has no preferred length scale. But this means that there can be nothing in the theory like a mass or a Compton wavelength. In other words, conformal field theories only support massless excitations. The questions that we ask are not those of particles and S-matrices. Instead we will be concerned with correlation functions and the behaviour of different operators under conformal transformations.

### 4.0.1 Euclidean Space

Although we're ultimately interested in Minkowski signature worldsheets, it will be much simpler and elegant if we work instead with Euclidean worldsheets. There's no funny business here — everything we do could also be formulated in Minkowski space.

The Euclidean worldsheet coordinates are $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$ and it will prove useful to form the complex coordinates,

$$z = \sigma^1 + i\sigma^2 \quad \text{and} \quad \bar{z} = \sigma^1 - i\sigma^2$$

which are the Euclidean analogue of the lightcone coordinates. Motivated by this analogy, it is common to refer to holomorphic functions as "left-moving" and anti-holomorphic functions as "right-moving".

The holomorphic derivatives are

$$\partial_z \equiv \partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \partial_{\bar{z}} \equiv \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$$

These obey $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial}z = 0$. We will usually work in flat Euclidean space, with metric

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz\, d\bar{z} \tag{4.2}$$

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In components, this flat metric reads

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad \text{and} \quad g_{z\bar{z}} = \frac{1}{2}$$

With this convention, the measure factor is $dzd\bar{z} = 2d\sigma^1 d\sigma^2$. We define the delta-function such that $\int d^2z\, \delta(z,\bar{z}) = 1$. Notice that because we also have $\int d^2\sigma\, \delta(\sigma) = 1$, this means that there is a factor of 2 difference between the two delta functions. Vectors naturally have their indices up: $v^z = (v^1 + iv^2)$ and $v^{\bar{z}} = (v^1 - iv^2)$. When indices are down, the vectors are $v_z = \frac{1}{2}(v^1 - iv^2)$ and $v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2)$.

### 4.0.2 The Holomorphy of Conformal Transformations

In the complex Euclidean coordinates $z$ and $\bar{z}$, conformal transformations of flat space are simple: they are any holomorphic change of coordinates,

$$z \to z' = f(z) \quad \text{and} \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z})$$

Under this transformation, $ds^2 = dzd\bar{z} \to |df/dz|^2\, dzd\bar{z}$, which indeed takes the form (4.1). Note that we have an infinite number of conformal transformations — in fact, a whole functions worth $f(z)$. This is special to conformal field theories in two dimensions. In higher dimensions, the space of conformal transformations is a finite dimensional group. For theories defined on $\mathbf{R}^{p,q}$, the conformal group is $SO(p+1, q+1)$ when $p + q > 2$.

A couple of particularly simple and important examples of 2d conformal transformations are

- $z \to z + a$: This is a translation.
- $z \to \zeta z$: This is a rotation for $|\zeta| = 1$ and a scale transformation (also known as a *dilatation*) for real $\zeta \neq 1$.

For many purposes, it's simplest to treat $z$ and $\bar{z}$ as independent variables. In doing this, we're really extending the worldsheet from $\mathbf{R}^2$ to $\mathbf{C}^2$. This will allow us to make use of various theorems from complex methods. However, at the end of the day we should remember that we're really sitting on the real slice $\mathbf{R}^2 \subset \mathbf{C}^2$ defined by $\bar{z} = z^\star$.

## 4.1 Classical Aspects

We start by deriving some properties of classical theories which are invariant under conformal transformations (4.1).

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#### 4.1.1 The Stress-Energy Tensor

One of the most important objects in any field theory is the *stress-energy tensor* (also known as the energy-momentum tensor). This is defined in the usual way as the matrix of conserved currents which arise from translational invariance,

$$\delta\sigma^\alpha = \epsilon^\alpha \ .$$

In flat spacetime, a translation is a special case of a conformal transformation.

There's a cute way to derive the stress-energy tensor in any theory. Suppose for the moment that we are in flat space $g_{\alpha\beta} = \eta_{\alpha\beta}$. Recall that we can usually derive conserved currents by promoting the constant parameter $\epsilon$ that appears in the symmetry to a function of the spacetime coordinates. The change in the action must then be of the form,

$$\delta S = \int d^2\sigma \ J^\alpha \, \partial_\alpha \epsilon \tag{4.3}$$

for some function of the fields, $J^\alpha$. This ensures that the variation of the action vanishes when $\epsilon$ is constant, which is of course the definition of a symmetry. But when the equations of motion are satisfied, we must have $\delta S = 0$ for all variations $\epsilon(\sigma)$, not just constant $\epsilon$. This means that when the equations of motion are obeyed, $J^\alpha$ must satisfy

$$\partial_\alpha J^\alpha = 0$$

The function $J^\alpha$ is our conserved current.

Let's see how this works for translational invariance. If we promote $\epsilon$ to a function of the worldsheet variables, the change of the action must be of the form (4.3). But what is $J^\alpha$? At this point we do the cute thing. Consider the same theory, but now coupled to a dynamical background metric $g_{\alpha\beta}(\sigma)$. In other words, coupled to gravity. Then we could view the transformation

$$\delta\sigma^\alpha = \epsilon^\alpha(\sigma)$$

as a diffeomorphism and we know that the theory is invariant as long as we make the corresponding change to the metric

$$\delta g_{\alpha\beta} = \partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha \ .$$

This means that if we just make the transformation of the coordinates in our original theory, then the change in the action must be the opposite of what we get if we just transform the metric. (Because doing both together leaves the action invariant). So we have

$$\delta S = -\int d^2\sigma \; \frac{\partial S}{\partial g_{\alpha\beta}} \, \delta g_{\alpha\beta} = -2 \int d^2\sigma \; \frac{\partial S}{\partial g_{\alpha\beta}} \, \partial_\alpha \epsilon_\beta$$

Note that $\partial S/\partial g_{\alpha\beta}$ in this expression is really a functional derivatives but we won't be careful about using notation to indicate this. We now have the conserved current arising from translational invariance. We will add a normalization constant which is standard in string theory (although not necessarily in other areas) and define the stress-energy tensor to be

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\partial S}{\partial g^{\alpha\beta}} \tag{4.4}$$

If we have a flat worldsheet, we evaluate $T_{\alpha\beta}$ on $g_{\alpha\beta} = \delta_{\alpha\beta}$ and the resulting expression obeys $\partial^\alpha T_{\alpha\beta} = 0$. If we're working on a curved worldsheet, then the energy-momentum tensor is covariantly conserved, $\nabla^\alpha T_{\alpha\beta} = 0$.

**The Stress-Energy Tensor is Traceless**

In conformal theories, $T_{\alpha\beta}$ has a very important property: its trace vanishes. To see this, let's vary the action with respect to a scale transformation which is a special case of a conformal transformation,

$$\delta g_{\alpha\beta} = \epsilon g_{\alpha\beta} \tag{4.5}$$

Then we have

$$\delta S = \int d^2\sigma \; \frac{\partial S}{\partial g_{\alpha\beta}} \, \delta g_{\alpha\beta} = -\frac{1}{4\pi} \int d^2\sigma \sqrt{g} \, \epsilon \, T^\alpha_{\ \alpha}$$

But this must vanish in a conformal theory because scaling transformations are a symmetry. So

$$T^\alpha_{\ \alpha} = 0$$

This is the key feature of a conformal field theory in any dimension. Many theories have this feature at the classical level, including Maxwell theory and Yang-Mills theory in four-dimensions. However, it is much harder to preserve at the quantum level. (The weight of the world rests on the fact that Yang-Mills theory fails to be conformal at the quantum level). Technically the difficulty arises due to the need to introduce a scale when regulating the theories. Here we will be interested in two-dimensional theories



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