6.5 A race

6.5.1 General model

Firms compete with each other to develop new technologies; authors compete with each other to write books and film scripts about momentous current events; scientists compete with each other to make discoveries. In each case the winner enjoys a significant advantage over the losers, and each competitor can, at a cost, increase her pace of activity. How do the presence of competitors and size of the prize affect the pace of activity? How does the identity of the winner of the race depend on the each competitor's initial distance from the finish line?

We can model a race as an extensive game with perfect information in which the players alternately choose how many "steps" to take. Here I study a simple example of such a game, with two players.

Player i is initially $k_i > 0$ steps from the finish line, for $i = 1, 2$. On each of her turns, a player can either not take any steps (at a cost of 0), or can take one step, at a cost of $c(1)$, or two steps, at a cost of $c(2)$. The first player to reach the finish line wins a prize, worth $v_i > 0$ to player i; the losing player's payoff is 0. To make the game finite, I assume that if, on successive turns, neither player takes any step, the game ends and neither player obtains the prize.

I denote the game in which player i moves first by $G_i(k_1, k_2)$. The game $G_1(k_1, k_2)$ is defined precisely as follows.

Players The two parties.

Terminal histories The set of sequences of the form $(x^1, y^1, x^2, y^2, \ldots, x^T)$ or $(x^1, y^1, x^2, y^2, \ldots, y^T)$ for some integer T, where each x^t (the number of steps taken by player 1 on her tth turn) and each y^t (the number of steps taken by player 2 on her tth turn) is 0, 1, or 2, there are never two successive 0's except possibly at the end of a sequence, and either $x^1 + \cdots + x^T = k_1$ and $y^1 + \cdots + y^T < k_2$ (player 1 reaches the finish line first), or $x^1 + \cdots + x^T < k_1$ and $y^1 + \cdots + y^T = k_2$ (player 2 reaches the finish line first).

Player function $P(\varnothing) = 1$, $P(x^1) = 2$ for all x^1, $P(x^1, y^1) = 1$ for all (x^1, y^1), $P(x^1, y^1, x^2) = 2$ for all (x^1, y^1, x^2), and so on.

Preferences For a terminal history in which player i loses, her payoff is the negative of the sum of the costs of all her moves; for a terminal history in which she wins it is v_i minus the sum of these costs.

6.5.2 Subgame perfect equilibria of an example

A simple example illustrates the features of the subgame perfect equilibria of this game. Suppose that both v_1 and v_2 are between 6 and 7 (their exact values do not affect the equilibria), the cost $c(1)$ of a single step is 1, and the cost $c(2)$ of two steps is 4. (Given that $c(2) > 2c(1)$, each player, in the absence of a competitor, would like to take one step at a time.)

The game has a finite horizon, so we may use backward induction to find its subgame perfect equilibria. Each of its subgames is either a game $G_i(m_1, m_2)$ with $i = 1$ or $i = 2$ and $0 < m_1 \leq k_1$ and $0 < m_2 \leq k_2$, or, if the last player to move before the subgame took no steps, a game that differs from $G_i(m_1, m_2)$ only in that it ends if player i initially takes no steps (i.e. the only terminal history starting with 0 consists only of 0).

First consider the very simplest game, $G_1(1, 1)$, in which each player is initially one step from the finish line. If player 1 takes one step, she wins; if she does not move then player 2 optimally takes one step (if she does not, the game ends) and wins. We conclude that the game has a unique subgame perfect equilibrium, in which player 1 initially takes one step and wins.

A similar argument applies to the game $G_1(1, 2)$. If player 1 does not move then player 2 has the option of taking one or two steps. If she takes one step then play moves to a subgame identical $G_1(1, 1)$, in which we have just concluded that player 1 wins. Thus player 2 takes two steps, and wins, if player 1 does not move at the start of $G_1(1, 2)$. We conclude that the game has a unique subgame perfect equilibrium, in which player 1 initially takes one step and wins.

Now consider player 1's options in the game $G_1(2, 1)$.

Player 1 takes two steps: She wins, and obtains a payoff of at least $6 - 4 = 2$ (her valuation is more than 6, and the cost of two steps is 4).

Player 1 take one step: Play moves to a subgame identical to $G_2(1, 1)$; we know that in the equilibrium of this subgame player 2 initially takes one step and wins.

Player 1 does not move: Play moves to a subgame in which player 2 is the first-mover and is one step from the finish line, and, if player 2 does not move, the game ends. In an equilibrium of this subgame player 2 takes one step and wins.

We conclude that the game $G_1(2, 1)$ has a unique subgame perfect equilibrium, in which player 1 initially takes two steps and wins.

I have spelled out the details of the analysis of these cases to show how we use the result for the game $G_1(1, 1)$ to find the equilibria of the games $G_1(1, 2)$ and $G_1(2, 1)$. In general, the equilibria of the games $G_i(k_1, k_2)$ for all values of k_1 and k_2 up to $\overline{k}$ tell us the consequences of player 1's taking one or two steps in the game $G_1(\overline{k} + 1, \overline{k})$.

? EXERCISE 195.1 (The race $G_1(2, 2)$) Show that the game $G_1(2, 2)$ has a unique subgame perfect equilibrium outcome, in which player 1 initially takes two steps, and wins.

So far we have concluded that in any game in which each player is initially at most two steps from the finish line, the first-mover takes enough steps to reach the finish line, and wins.

Now suppose that player 1 is at most two steps from the finish line, but player 2 is three steps away. Suppose that player 1 takes only *one* step (even if she is initially two steps from the finish line). Then if player 2 takes either one or two steps, play moves to a subgame in which player 1 (the first-mover) wins. Thus player 2 is better off not moving (and not incurring any cost), in which case player 1 takes one step on her next turn, and wins. (Player 1 prefers to move one step at a time than to move two steps initially, because the former costs her 2 whereas the latter costs her 4.) We conclude that the outcome of a subgame perfect equilibrium in the game $G_1(2,3)$ is that player 1 takes one step on her first turn, then player 2 does not move, and then player 1 takes another step, and wins.

By a similar argument, in a subgame perfect equilibrium of any game in which player 1 is at most two steps from the finish line and player 2 is three or more steps away, player 1 moves one step at a time, and player 2 does not move; player 1 wins. Symmetrically, in a subgame perfect equilibrium of any game in which player 1 is three or more steps from the finish line and player 2 is at most two steps away, player 1 does not move, and player 2 moves one step at a time, and wins.

Our conclusions so far are illustrated in Figure 197.1. In this figure, player 1 moves to the left, and player 2 moves down. The values of (k_1, k_2) for which the subgame perfect equilibrium outcome has been determined so far are labeled. The label "1" means that, regardless of who moves first, in a subgame perfect equilibrium player 1 moves one step on each turn, and player 2 does not move; player 1 wins. Similarly, the label "2" means that, regardless of who moves first, player 2 moves one step on each turn, and player 1 does not move; player 2 wins. The label "f" means that the first player to move takes enough steps to reach the finish line, and wins.

Now consider the game $G_1(3,3)$. If player 1 takes one step, we reach the game $G_2(2,3)$. From Figure 197.1 we see that in the subgame perfect equilibrium of this game player 1 wins, and does so by taking one step at a time (the point $(2,3)$ is labeled "1"). If player 1 takes two steps, we reach the game $G_2(1,3)$, in which player 1 also wins. Player 1 prefers not to take two steps unless she has to, so in the subgame perfect equilibrium of $G_1(3,3)$ she takes one step at a time, and wins, and player 2 does not move. Similarly, in a subgame perfect equilibrium of $G_2(3,3)$, player 2 takes one step at a time, and wins, and player 1 does not move.

A similar argument applies to each of the games $G_i(3,4)$, $G_i(4,3)$, and $G_i(4,4)$ for $i = 1, 2$. The argument differs only if the first-mover is four steps from the finish line, in which case she initially takes two steps in order to reach a game in which she wins. (If she initially takes only one step, the other player wins.)

Now consider the game $G_i(3,5)$ for $i = 1, 2$. By taking one step in $G_1(3,5)$, player 1 reaches a game in which she wins by taking one step at a time. The cost of her taking three steps is less than v_1, so in a subgame perfect equilibrium of $G_1(3,5)$ she takes one step at a time, and wins, and player 2 does not move.

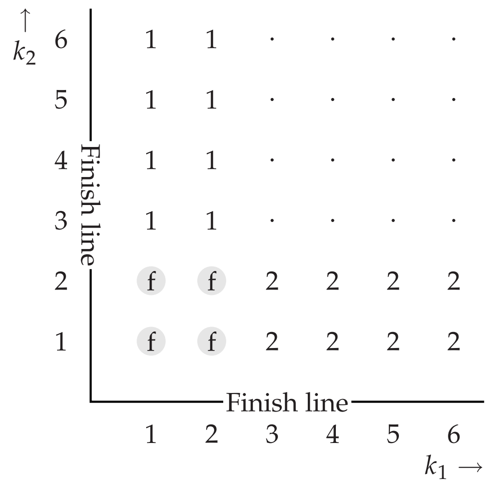

Figure 197.1 The subgame perfect equilibrium outcomes of the race $G_i(k_1, k_2)$. Player 1 moves to the left, and player 2 moves down. The values of (k_1, k_2) for which the subgame perfect equilibrium outcome has been determined so far are labeled; dots represent cases that have not yet been studied. The labels are explained in the text.

If player 2 takes either one or two steps in $G_2(3,5)$, she reaches a game (either $G_1(3,4)$ or $G_1(3,3)$) in which player 1 wins. Thus whatever she does, she loses, so that in a subgame perfect equilibrium she does not move and player 1 moves one step at a time. We conclude that in a subgame perfect equilibrium of both $G_1(3,5)$ and $G_2(3,5)$, player 1 takes one step on each turn and player 2 does not move; player 1 wins.

A similar argument applies to any game in which one player is initially three or four steps from the finish line, and the other player is five or more steps from the finish line. We have now made arguments to justify the labeling in Figure 198.1. In this figure the labels have the same meaning as in the previous figure, except that "f" means that the first player to move takes enough steps to reach the finish line *or* to reach the closest point labeled with her name, whichever is closer.

A feature of the subgame perfect equilibrium of the game $G_1(4,4)$ is noteworthy. Suppose that, as planned, player 1 takes two steps, but then player 2 deviates from her equilibrium strategy and takes two steps (rather than not moving). According to our analysis, player 1 should take two steps, to reach the finish line. If she does so, her payoff is negative (less than $7 - 4 - 4 = -1$). Nevertheless she should definitely take the two steps: if she does not, her payoff is even smaller (-4), because player 2 wins. The point is that the cost of her first move is "sunk"; her decision after player 2 deviates must be based on her options from that point on.

The analysis of the games in which each player is initially either 5 or 6 steps from the finish line involves arguments similar to those used in the previous cases, with one amendment. A player who is initially 6 steps from the finish line is better

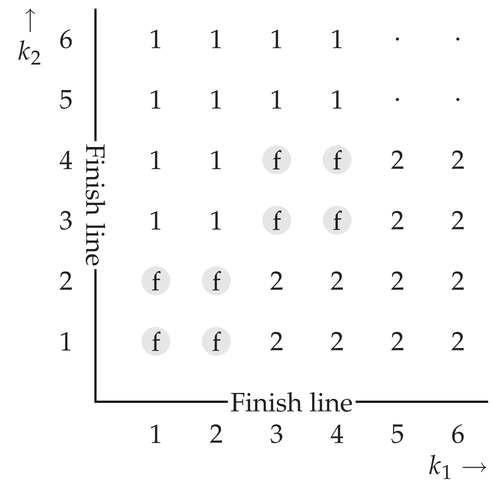

Figure 198.1 The subgame perfect equilibrium outcomes of the race $G_i(k_1, k_2)$. Player 1 moves to the left, and player 2 moves down. The values of (k_1, k_2) for which the subgame perfect equilibrium outcome has been determined so far are labeled; dots represent cases that have not yet been studied. The labels are explained in the text.

off not moving at all (and obtaining the payoff 0) than she is moving two steps on any turn (and obtaining a negative payoff). An implication is that in the game $G_1(6,5)$, for example, player 1 does not move: if she takes only one step then player 2 becomes the first-mover and, by taking a single step, moves the play to a game that she wins. We conclude that the first-mover wins in the games $G_i(5,5)$ and $G_i(6,6)$, whereas player 2 wins in $G_i(6,5)$ and player 1 wins in $G_i(5,6)$, for $i = 1, 2$.

A player who is initially more than six steps from the finish line obtains a negative payoff if she moves, even if she wins, so in any subgame perfect equilibrium she does not move. Thus our analysis of the game is complete. The subgame perfect equilibrium outcomes are indicated in Figure 199.1, which shows also the steps taken in the equilibrium of each game when player 1 is the first-mover.

? EXERCISE 198.1 (A race in which the players' valuations of the prize differ) Find the subgame perfect equilibrium outcome of the game in which player 1's valuation of the prize is between 6 and 7, and player 2's valuation is between 4 and 5.

In both of the following exercises, inductive arguments on the length of the game, like the one for $G_i(k_1, k_2)$, can be used.

? EXERCISE 198.2 (Removing stones) Two people take turns removing stones from a pile of n stones. Each person may, on each of her turns, remove either one stone or two stones. The person who takes the last stone is the winner; she gets \$1 from her opponent. Find the subgame perfect equilibria of the games that model this situation for $n = 1$ and $n = 2$. Find the winner in each subgame perfect

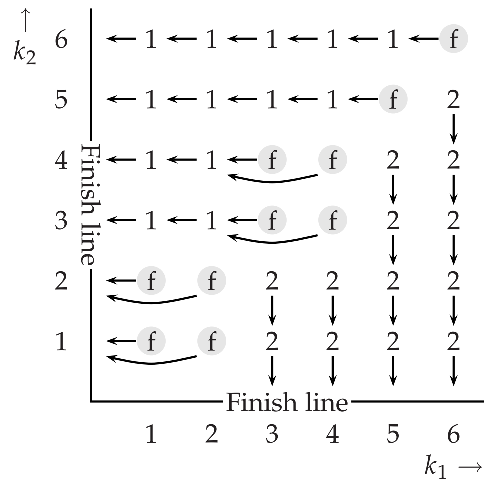

Figure 199.1 The subgame perfect equilibrium outcomes of the race $G_i(k_1, k_2)$. Player 1 moves to the left, and player 2 moves down. The arrows indicate the steps taken in the subgame perfect equilibrium outcome of the games in which player 1 moves first. The labels are explained in the text.

equilibrium for $n = 3$, using the fact that the subgame following player 1's removal of one stone is the game for $n = 2$ in which player 2 is the first-mover, and the subgame following player 1's removal of two stones is the game for $n = 1$ in which player 2 is the first mover. Use the same technique to find the winner in each subgame perfect equilibrium for $n = 4$, and, if you can, for an arbitrary value of n.

?? EXERCISE 199.1 (Hungry lions) The members of a hierarchical group of hungry lions face a piece of prey. If lion 1 does not eat the prey, the game ends. If it eats the prey, it becomes fat and slow, and lion 2 can eat it. If lion 2 does not eat lion 1, the game ends; if it eats lion 1 then it may be eaten by lion 3, and so on. Each lion prefers to eat than to be hungry, but prefers to be hungry than to be eaten. Find the subgame perfect equilibrium (equilibria?) of the extensive game that models this situation for any number n of lions.

6.5.3 General lessons

Each player's equilibrium strategy involves a "threat" to speed up if the other player deviates. Consider, for example, the game $G_1(3, 3)$. Player 1's equilibrium strategy calls for her to take one step at a time, and player 2's equilibrium strategy calls for her not to move. Thus along the equilibrium path player 1's debt climbs to 3 (the cost of her three single steps) before she reaches the finish line.

Now suppose that after player 1 takes her first step, player 2 deviates and takes a step. In this case, player 1's strategy calls for her to take two steps. If she does so, her debt climbs to 5. If at no stage can her debt exceed 3 (its maximal level on the equilibrium path) then her strategy cannot embody such threats.

The general point is that a limit on the debt a player can accumulate may affect the outcome even if it exceeds the player's debt along the equilibrium path in the absence of any limits. You are asked to study an example in the next exercise.

? EXERCISE 200.1 (A race with a liquidity constraint) Find the subgame perfect equilibrium of the variant of the game $G_1(3,3)$ in which player 1's debt may never exceed 3.

In the subgame perfect equilibrium of every game $G_i(k_1, k_2)$, only one player moves; her opponent "gives up". This property of equilibrium holds in more general games. What added ingredient might lead to an equilibrium in which both players are active? A player's uncertainty about the other's characteristics would seem to be such an ingredient: if a player does not know the cost of its opponent's moves, it may assign a positive probability less than one to its winning, at least until it has accumulated some evidence of its opponent's behavior, and while it is optimistic it may be active even though its rival is also active. To build such considerations into the model we need to generalize the model of an extensive game to encompass imperfect information, as we do in Chapter 10.

Another feature of the subgame perfect equilibrium of $G_i(k_1, k_2)$ that holds in more general games is that the presence of a competitor has little effect on the speed of the player who moves. A lone player would move one step at a time. When there are two players, for most starting points the one that moves does so at the same leisurely pace. Only for a small number of starting points, in all of which the players' initial distances from the starting line are similar, does the presence of a competitor induce the active player to hasten its progress, and then only in the first period.

Notes

The first experiment on the ultimatum game is reported in Güth, Schmittberger, and Schwarze (1982). Grout (1984) is an early analysis of a holdup game. The model in Section 6.3 is due to von Stackelberg (1934). The vote-buying game in Section 6.4 is taken from Groseclose and Snyder (1996). The model of a race in Section 6.5 is a simplification suggested by Vijay Krishna of a model of Harris and Vickers (1985).

For more discussion of the experimental evidence on the ultimatum game (discussed in the box on page 181), see Roth (1995). Bolton and Ockenfels (2000) study the implications of assuming that players are equity-conscious, and relate these implications to the experimental outcomes in various games. The explanation of the experimental results in terms of rules of thumb is discussed by Aumann (1997, 7–8). The problem of fair division, an example of which is given in Exercise 183.2, is studied in detail by Brams and Taylor (1996), who trace the idea of divide-and-choose back to antiquity (p. 10). I have been unable to find the origin of the idea in Exercise 199.1; Barton Lipman suggested the formulation in the exercise.

7 Extensive Games with Perfect Information: Extensions and Discussion

Prerequisite: Chapter 5.

7.1 Allowing for simultaneous moves

7.1.1 Definition

THE model of an extensive game with perfect information (Definition 153.1) assumes that after every sequences of events, a single decision-maker takes an action, knowing every decision-maker's previous actions. I now describe a more general model that allows us to study situations in which, after some sequences of events, the members of a group of decision-makers choose their actions "simultaneously", each member knowing every decision-maker's *previous* actions, but not the contemporaneous actions of the other members of the group.

In the more general model, a terminal history is a sequence of *lists* of actions, each list specifying the actions of a set of players. (A game in which each set contains a single player is an extensive game with perfect information as defined previously.) For example, consider a situation in which player 1 chooses either C or D, then players 2 and 3 simultaneously take actions, each choosing either E or F. In the extensive game that models this situation, $(C, (E, E))$ is a terminal history, in which first player 1 chooses C, and then players 2 and 3 both choose E. In the general model, the player function assigns a *set* of players to each nonterminal history. In the example just described, this set consists of the single player 1 for the initial history, and consists of players 2 and 3 for the history C.

An extensive game with perfect information (Definition 153.1) does not specify explicitly the sets of actions available to the players. However, we may derive the set of actions of the player who moves after any nonterminal history from the set of terminal histories and the player function (see (154.1)). When we allow simultaneous moves, the players' sets of actions are conveniently specified in the definition of a game. In the example of the previous paragraph, for instance, we specify the game by giving the eight possible terminal histories (C or D followed by one of the four pairs (E, E), (E, F), (F, E), and (F, F)), the player function defined by $P(\varnothing) = 1$ and $P(C) = P(D) = \{2, 3\}$, the sets of actions $\{C, D\}$ for player 1 at the start of the game and $\{E, F\}$ for both player 2 and player 3 after the histories C and D, and each player's preferences over terminal histories.

In any game, the set of terminal histories, player function, and sets of actions for the players must be consistent: the list of actions that follows a subhistory of any terminal history must be a list of actions of the players assigned by the player function to that subhistory. In the game described above, for example, the list of actions following the subhistory C of the terminal history $(C, (E, E))$ is (E, E), which is a pair of actions for the players (2 and 3) assigned by the player function to the history C.

Precisely, an extensive game with perfect information and simultaneous moves is defined as follows.

▶ DEFINITION 202.1 An **extensive game with perfect information and simultaneous moves** consists of

- a set of **players**
- a set of sequences (**terminal histories**) with the property that no sequence is a proper subhistory of any other sequence
- a function (the **player function**) that assigns a set of players to every sequence that is a proper subhistory of some terminal history
- for each proper subhistory h of each terminal history and each player i that is a member of the set of players assigned to h by the player function, a set $A_i(h)$ (the set of **actions** available to player i after the history h)
- for each player, **preferences** over the set of terminal histories

such that the set of terminal histories, player function, and sets of actions are consistent in the sense that h is a terminal history if and only if either (*i*) h takes the form $(a^1, \ldots, a^k)$ for some integer k, the player function is not defined at h, and for every $\ell = 0, \ldots, k-1$, $a^{\ell+1}$ is a list of actions of the players assigned by the player function to $(a^1, \ldots, a^\ell)$ (the empty history if $\ell = 0$), or (*ii*) h takes the form $(a^1, a^2, \ldots)$ and for every $\ell = 0, 1, \ldots$, $a^{\ell+1}$ is a list of actions of the players assigned by the player function to $(a^1, \ldots, a^\ell)$ (the empty history if $\ell = 0$).

This definition encompasses both extensive games with perfect information as in Definition 153.1 and, in a sense, strategic games. An extensive game with perfect information is an extensive game with perfect information and simultaneous moves in which the set of players assigned to each history consists of exactly one member. (The definition of an extensive game with perfect information and simultaneous moves includes the players' actions, whereas the definition of an extensive game with perfect information does not. However, actions may be derived from the terminal histories and player function of the latter.)

For any strategic game there is an extensive game with perfect information and simultaneous moves in which every terminal history has length one that models the same situation. In this extensive game, the set of terminal histories is the set of action profiles in the strategic game, the player function assigns the set of all players to the initial history, and the single set $A_i(\varnothing)$ of actions of each player i is the set of actions of player i in the strategic game.

◆ EXAMPLE 203.1 (Variant of *BoS*) First, person 1 decides whether to stay home and read a book or to attend a concert. If she reads a book, the game ends. If she decides to attend a concert then, as in *BoS*, she and person 2 independently choose whether to sample the aural delights of Bach or Stravinsky, not knowing the other person's choice. Both people prefer to attend the concert of their favorite composer in the company of the other person to the outcome in which person 1 stays home and reads a book, and prefer this outcome to attending the concert of their less preferred composer in the company of the other person; the worst outcome for both people is that they attend different concerts.

The following extensive game with perfect information and simultaneous moves models this situation.

Players The two people (1 and 2).

Terminal histories $Book$, $(Concert, (B, B))$, $(Concert, (B, S))$, $(Concert, (S, B))$, $(Concert, (S, S))$.

Player function $P(\varnothing) = 1$ and $P(Concert) = \{1, 2\}$.

Actions The set of player 1's actions at the initial history $\varnothing$ is $A_1(\varnothing) = \{Concert, Book\}$ and the set of her actions after the history *Concert* is $A_1(Concert) = \{B, S\}$; the set of player 2's actions after the history *Concert* is $A_2(Concert) = \{B, S\}$.

Preferences Player 1 prefers $(Concert, (B, B))$ to *Book* to $(Concert, (S, S))$ to $(Concert, (B, S))$, which she regards as indifferent to $(Concert, (S, B))$. Player 2 prefers $(Concert, (S, S))$ to *Book* to $(Concert, (B, B))$ to $(Concert, (B, S))$, which she regards as indifferent to $(Concert, (S, B))$.

This game is illustrated in Figure 204.1, in which I represent the simultaneous choices between B and S in the way that I previously represented a strategic game. (Only a game in which all the simultaneous moves occur at the end of terminal histories may be represented in a diagram like this one. For most other games no convenient diagrammatic representation exists.)

7.1.2 Strategies and Nash equilibrium

As in a game without simultaneous moves, a player's strategy specifies the action she chooses for every history after which it is her turn to move. Definition 157.1 requires only minor rewording to allow for the possibility that players may move simultaneously.

▶ DEFINITION 203.2 A **strategy** of player i in an extensive game with perfect information and simultaneous moves is a function that assigns to each history h after

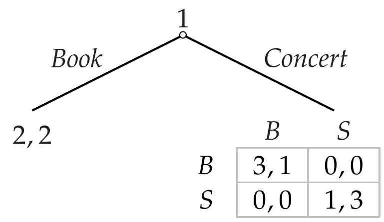

Figure 204.1 The variant of *BoS* described in Example 204.1.

which i is one of the players whose turn it is to move (i.e. i is a member of $P(h)$, where P is the player function of the game) an action in $A_i(h)$ (the set of actions available to player i after h).

The definition of a *Nash equilibrium* of an extensive game with perfect information and simultaneous moves is exactly the same as the definition for a game with no simultaneous moves (Definition 159.2): a Nash equilibrium is a strategy profile with the property that no player can induce a better outcome for herself by changing her strategy, given the other players' strategies. Also as before, the *strategic form* of a game is the strategic game in which the players' actions are their strategies in the extensive game (see Section 5.4), and a strategy profile is a Nash equilibrium of the extensive game if and only if it is a Nash equilibrium of the strategic form of the game.

◆ EXAMPLE 204.1 (Nash equilibria of a variant of *BoS*) In the game in Example 203.1, a strategy of player 1 specifies her actions at the start of the game and after the history *Concert*; a strategy of player 2 specifies her action after the history *Concert*. Thus player 1 has four strategies, $(Concert, B)$, $(Concert, S)$, $(Book, B)$, and $(Book, S)$, and player 2 has two strategies, B and S. (Remember that a player's strategy is more than a plan of action; it specifies an action for *every* history after which the player moves, even histories that it precludes. For example, player 1's strategy specifies her action after the history *Concert* even if it specifies that she choose *Book* at the beginning of the game.)

The strategic form of the game is given in Figure 204.2. We see that the game has three pure Nash equilibria: $((Concert, B), B)$, $((Book, B), S)$, and $((Book, S), S)$.

	B	S
$(Concert, B)$	3, 1	0, 0
$(Concert, S)$	0, 0	1, 3
$(Book, B)$	2, 2	2, 2
$(Book, S)$	2, 2	2, 2

Figure 204.2 The strategic form of the game in Example 203.1.

Every extensive game has a unique strategic form. However, some strategic games are the strategic forms of more than one extensive game. Consider, for example, the strategic game in Figure 205.1. This game is the strategic form of the extensive game with perfect information and simultaneous moves in which the two players choose their actions simultaneously; it is also the strategic form of the entry game in Figure 154.1.

	L	R
T	$1,2$	$1,2$
B	$0,0$	$2,0$

Figure 205.1 A strategic game that is the strategic form of more than one extensive game.

7.1.3 Subgame perfect equilibrium

As for a game in which one player moves after each history, the subgame following the history h of an extensive game with perfect information and simultaneous moves is the extensive game "starting at h". (The formal definition is a variant of Definition 162.1.)

For instance, the game in Example 203.1 has two subgames: the whole game, and the game in which the players engage after player 1 chooses *Concert*. In the second subgame, the terminal histories are (B, B), (B, S), (S, B), and (S, S), the player function assigns the set $\{1, 2\}$ consisting of both players to the initial history (the only nonterminal history), the set of actions of each player at the initial history is $\{B, S\}$, and the players' preferences are represented by the payoffs in the table in Figure 204.1. (This subgame models the same situation as *BoS*.)

A subgame perfect equilibrium is defined as before: a *subgame perfect equilibrium* of an extensive game with perfect information and simultaneous moves is a strategy profile with the property that in no subgame can any player increase her payoff by choosing a different strategy, given the other players' strategies. The formal definition differs from the definition of a subgame perfect equilibrium of a game without simultaneous moves (164.1) only in that the meaning of "it is player i's turn to move" is that i is a member of $P(h)$, rather than $P(h) = i$.

To find the set of subgame perfect equilibria of an extensive game with perfect information and simultaneous moves that has a finite horizon, we can, as before, use backward induction. The only wrinkle is that some (perhaps all) of the situations we need to analyze are not single-person decision problems, as they are in the absence of simultaneous moves, but problems in which several players choose actions simultaneously. We cannot simply find an optimal action for the player whose turn it is to move at the start of each subgame, given the players' behavior in the remainder of the game. We need to find a *list* of actions for the players who move at the start of each subgame, with the property that each player's action is optimal given the other players' simultaneous actions and the players' behavior in the remainder of the game. That is, the argument we need to make is the same as the one we make when finding a Nash equilibrium of a strategic game. This argument may use any of the techniques discussed in Chapter 2: it may check each action profile in turn, it may construct and study the players' best response functions, or it may show directly that an action profile we have obtained by a combination of intuition and trial and error is an equilibrium.

◆ EXAMPLE 206.1 (Subgame perfect equilibria of a variant of *BoS*) Consider the game in Figure 204.1. Backward induction proceeds as follows.

- In the subgame that follows the history *Concert*, there are two Nash equilibria (in pure strategies), namely (S, S) and (B, B), as we found in Section 2.7.2.
- If the outcome in the subgame that follows *Concert* is (S, S) then the optimal choice of player 1 at the start of the game is *Book*.
- If the outcome in the subgame that follows *Concert* is (B, B) then the optimal choice of player 1 at the start of the game is *Concert*.

We conclude that the game has two subgame perfect equilibria: $((Book, S), S)$ and $((Concert, B), B)$.

Every finite extensive game with perfect information has a (pure) subgame perfect equilibrium (Proposition 171.1). The same is not true of a finite extensive game with perfect information and simultaneous moves because, as we know, a finite strategic game (which corresponds to an extensive game with perfect information and simultaneous moves of length one) may not possess a pure strategy Nash equilibrium. (Consider *Matching pennies* (Example 17.1).) If you have studied Chapter 4, you know that some strategic games that lack a pure strategy Nash equilibrium have a "mixed strategy Nash equilibrium", in which each player randomizes. The same is true of extensive games with perfect information and simultaneous moves. However, in this chapter I restrict attention almost exclusively to pure strategy equilibria; the only occasion on which mixed strategy Nash equilibrium appears is Exercise 208.1.

? EXERCISE 206.2 (Extensive game with simultaneous moves) Find the subgame perfect equilibria of the following game. First player 1 chooses either A or B. After either choice, she and player 2 simultaneously choose actions. If player 1 initially chooses A then she and player 2 subsequently each choose either C or D; if player 1 chooses B initially then she and player 2 subsequently each choose either E or F. Among the terminal histories, player 1 prefers $(A, (C, C))$ to $(B, (E, E))$ to $(A, (D, D))$ to $(B, (F, F))$, and prefers all these to $(A, (C, D))$, $(A, (D, C))$, $(B, (E, F))$, and $(B, (F, E))$, between which she is indifferent. Player 2 prefers $(A, (D, D))$ to $(B, (F, F))$ to $(A, (C, C))$ to $(B, (E, E))$, and prefers all these to $(A, (C, D))$, $(A, (D, C))$, $(B, (E, F))$, and $(B, (F, E))$, between which she is indifferent.

? EXERCISE 206.3 (Two-period *Prisoner's Dilemma*) Two people simultaneously choose actions; each person chooses either Q or F (as in the *Prisoner's Dilemma*). Then they simultaneously choose actions again, once again each choosing either Q or F. Each person's preferences are represented by the payoff function that assigns to the terminal history $((W, X), (Y, Z))$ (where each component is either Q or F)

a payoff equal to the sum of the person's payoffs to (W, X) and to (Y, Z) in the *Prisoner's Dilemma* given in Figure 13.1. Specify this situation as an extensive game with perfect information and simultaneous moves and find its subgame perfect equilibria.

? EXERCISE 207.1 (Timing claims on an investment) An amount of money is accumulating; in period t $(= 1, 2, \ldots, T)$ its size is $\$2t$. In each period two people simultaneously decide whether to claim the money. If only one person does so, she gets all the money; if both people do so, they split the money equally; and if neither person does so, both people have the opportunity to do so in the next period. If neither person claims the money in period T, each person obtains $\$T$. Each person cares only about the amount of money she obtains. Formulate this situation as an extensive game with perfect information and simultaneous moves, and find its subgame perfect equilibria). (Start by considering the cases $T = 1$ and $T = 2$.)

? EXERCISE 207.2 (A market game) A seller owns one indivisible unit of a good, which she does not value. Several potential buyers, each of whom attaches the same positive value v to the good, simultaneously offer prices they are willing to pay for the good. After receiving the offers, the seller decides which, if any, to accept. If she does not accept any offer, then no transaction takes place, and all payoffs are 0. Otherwise, the buyer whose offer the seller accepts pays the amount p she offered and receives the good; the payoff of the seller is p, the payoff of the buyer who obtained the good is $v - p$, and the payoff of every other buyer is 0. Model this situation as an extensive game with perfect information and simultaneous moves and find its subgame perfect equilibria. (Use a combination of intuition and trial and error to find a strategy profile that appears to be an equilibrium, then argue directly that it is. The incentives in the game are closely related to those in Bertrand's oligopoly game (see Exercise 66.1), with the roles of buyers and sellers reversed.) Show, in particular, that in every subgame perfect equilibrium every buyer's payoff is zero.

MORE EXPERIMENTAL EVIDENCE ON SUBGAME PERFECT EQUILIBRIUM

Experiments conducted in 1989 and 1990 among college students (mainly taking economics classes) show that the subgame perfect equilibria of the game in Exercise 207.2 correspond closely to experimental outcomes (Roth, Prasnikar, Okuno-Fujiwara, and Zamir 1991), in contrast to the subgame perfect equilibrium of the ultimatum game (see the box on page 181).

In experiments conducted at four locations (Jerusalem, Ljubljana, Pittsburgh, and Tokyo), nine "buyers" simultaneously bid for the rough equivalent (in terms of local purchasing power) of US\$10, held by a "seller". Each experiment involved a group of 20 participants, which was divided into two markets, each with one seller and nine buyers. Each participant was involved in ten rounds of the market; in each round the sellers and buyers were assigned anew, and in any given round no participant knew who, among the other participants, were sellers and buyers, and who was involved in her market. In every session of the experiment the maximum proposed price was accepted by the seller, and by the seventh round of every experiment the highest bid was at least (the equivalent of) US$9.95.

Experiments involving the ultimatum game, run in the same locations using a similar design, yielded results similar to those of previous experiments (see the box on page 181): proposers kept considerably less than 100% of the pie, and nontrivial offers were rejected.

The box on page 181 discusses two explanations for the experimental results in the ultimatum game. Both explanations are consistent with the results in the market game. One explanation is that people are concerned not only with their own monetary payoffs, but also with other people's payoffs. At least some specifications of such preferences do not affect the subgame perfect equilibria of a market game with many buyers, which still all yield every buyer the payoff of zero. (When there are many buyers, even a seller who cares about the other players' payoffs accepts the highest price offered, because accepting a lower price has little impact on the distribution of monetary payoffs, all but two of which remain zero.) Thus such preferences are consistent with both sets of experimental outcomes. Another explanation is that people incorrectly recognize the ultimatum game as one in which the rule of thumb "don't be a sucker" is advantageously invoked, and thus reject a poor offer, "punishing" the person who makes such an offer. In the market game, the players treated poorly in the subgame perfect equilibrium are the buyers, who have no opportunity to punish any other player, because they move first. Thus the rule of thumb is not relevant in this game, so that this explanation is also consistent with both sets of experimental outcomes.

In the next exercise you are asked to investigate subgame perfect equilibria in which some players use mixed strategies (discussed in Chapter 4).

?? EXERCISE 208.1 (Price competition) Extend the model in Exercise 125.2 by having the sellers simultaneously choose their prices before the buyers simultaneously choose which seller to approach. Assume that each seller's preferences are represented by the expected value of a Bernoulli payoff function in which the payoff to not trading is 0 the payoff to trading at the price p is p. Formulate this model precisely as an extensive game with perfect information and simultaneous moves. Show that for every $p \geq \frac{1}{2}$ the game has a subgame perfect equilibrium in which each seller announces the price p. (You may use the fact that if seller j's price is at least $\frac{1}{2}$, seller i's payoff in the mixed strategy equilibrium of the subgame in which the buyers choose which seller to approach is decreasing in her price p_i when $p_i > p_j$.)

7.2 Illustration: entry into a monopolized industry

7.2.1 General model

An industry is currently monopolized by a single firm (the "incumbent"). A second firm (the "challenger") is considering entry, which entails a positive cost f in addition to its production cost. If the challenger stays out then its profit is zero, whereas if it enters, the firms simultaneously choose outputs (as in Cournot's model of duopoly (Section 3.1)). The cost to firm i of producing q_i units of output is $C_i(q_i)$. If the firms' total output is Q then the market price is $P_d(Q)$. (As in Section 6.3, I add a subscript to P to avoid a clash with the player function of the game.)

We can model this situation as the following extensive game with perfect information and simultaneous moves, illustrated in Figure 209.1.

Players The two firms: the incumbent (firm 1) and the challenger (firm 2).

Terminal histories $(In, (q_1, q_2))$ for any pair (q_1, q_2) of outputs (nonnegative numbers), and (Out, q_1) for any output q_1.

Player function $P(\varnothing) = \{2\}$, $P(In) = \{1, 2\}$, and $P(Out) = \{1\}$.

Actions $A_2(\varnothing) = \{In, Out\}$; $A_1(In)$, $A_1(Out)$, and $A_2(In)$ are all equal to the set of possible outputs (nonnegative numbers).

Preferences Each firm's preferences are represented by its profit, which for a terminal history $(In, (q_1, q_2))$ is $q_1 P_d(q_1 + q_2) - C_1(q_1)$ for the incumbent and $q_2 P_d(q_1 + q_2) - C_2(q_2) - f$ for the challenger, and for a terminal history (Out, q_1) is $q_1 P_d(q_1) - C_1(q_1)$ for the incumbent and 0 for the challenger.

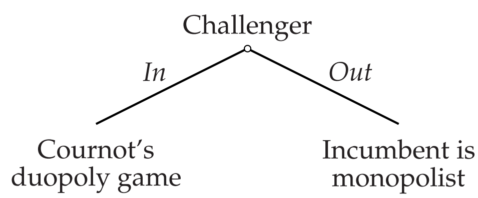

Figure 209.1 An entry game.

7.2.2 Example

Suppose that $C_i(q_i) = cq_i$ for all q_i ("unit cost" is constant, equal to c), and the inverse demand function is linear where it is positive, given by $P_d(Q) = \alpha - Q$ for $Q \leq \alpha$, as in Section 3.1.3. To find the subgame perfect equilibria, first consider the subgame that follows the history In. The strategic form of this subgame is the same as the example of Cournot's duopoly game studied in Section 3.1.3, except that the payoff of the challenger is reduced by f (the fixed cost of entry) regardless of the challenger's output. Thus the subgame has a unique Nash equilibrium, in which the output of each firm is $\frac{1}{3}(\alpha - c)$; the incumbent's profit is $\frac{1}{9}(\alpha - c)^2$, and the challenger's profit is $\frac{1}{9}(\alpha - c)^2 - f$.

Now consider the subgame that follows the history *Out*. In this subgame the incumbent chooses an output. The incumbent's profit when it chooses the output q_1 is $q_1(\alpha - q_1) - cq_1 = q_1(\alpha - c - q_1)$. This function is a quadratic that increases and then decreases as q_1 increases, and is zero when $q_1 = 0$ and when $q_1 = \alpha - c$. Thus the function is maximized when $q_1 = \frac{1}{2}(\alpha - c)$. We conclude that in any subgame perfect equilibrium the incumbent chooses $q_1 = \frac{1}{2}(\alpha - c)$ in the subgame following the history *Out*.

Finally, consider the challenger's action at the start of the game. If the challenger stays out then its profit is 0, whereas if it enters then, given the actions chosen in the resulting subgame, its profit is $\frac{1}{9}(\alpha - c)^2 - f$. Thus in any subgame perfect equilibrium the challenger enters if $\frac{1}{9}(\alpha - c)^2 > f$ and stays out if $\frac{1}{9}(\alpha - c)^2 < f$. If $\frac{1}{9}(\alpha - c)^2 = f$ then the game has two subgame perfect equilibria, in one of which the challenger enters and in the other of which it does not.

In summary, the set of subgame perfect equilibria depend on the value of f. In all equilibria the incumbent's strategy is to produce $\frac{1}{3}(\alpha - c)$ if the challenger enters and $\frac{1}{2}(\alpha - c)$ if it does not, and the challenger's strategy involves its producing $\frac{1}{3}(\alpha - c)$ if it enters.

- If $f < \frac{1}{9}(\alpha - c)^2$ there is a unique subgame perfect equilibrium, in which the challenger enters. The outcome is that the challenger enters and each firm produces the output $\frac{1}{3}(\alpha - c)$.
- If $f > \frac{1}{9}(\alpha - c)^2$ there is a unique subgame perfect equilibrium, in which the challenger stays out. The outcome is that the challenger stays out and the incumbent produces $\frac{1}{2}(\alpha - c)$.
- If $f = \frac{1}{9}(\alpha - c)^2$ the game has two subgame perfect equilibria: the one for the case $f < \frac{1}{9}(\alpha - c)^2$ and the one for the case $f > \frac{1}{9}(\alpha - c)^2$.

Why, if f is small, does the game have no subgame perfect equilibrium in which the incumbent floods the market if the challenger enters, so that the challenger optimally stays out and the incumbent obtains a profit higher than its profit if the challenger enters? Because the action this strategy prescribes after the history in which the challenger enters is not the incumbent's action in a Nash equilibrium of the subgame: the subgame has a unique Nash equilibrium, in which each firm produces $\frac{1}{3}(\alpha - c)$. Put differently, the incumbent's "threat" to flood the market if the challenger enters is not credible.

? EXERCISE 210.1 (Bertrand's duopoly game with entry) Find the subgame perfect equilibria of the variant of the game studied in this section in which the post-entry competition is a game in which each firm chooses a price, as in the example of Bertrand's duopoly game studied in Section 3.2.2, rather than an output.

7.3 Illustration: electoral competition with strategic voters

The voters in Hotelling's model of electoral competition (Section 3.3) are not players in the game: each citizen is assumed simply to vote for the candidate whose position she most prefers. How do the conclusions of the model change if we assume that each citizen *chooses* the candidate for whom to vote?

Consider the extensive game in which the candidates first simultaneously choose actions, then the citizens simultaneously choose how to vote. As in the variant of Hotelling's game considered on page 72, assume that each candidate may either choose a position (as in Hotelling's original model) or choose to stay out of the race, an option she is assumed to rank between losing and tying for first place with all the other candidates.

Players The candidates and the citizens.

Terminal histories All sequences (x, v) where x is a list of the candidates' actions, each component of which is either a position (a number) or *Out*, and v is a list of voting decisions for the citizens (i.e. a list of candidates, one for each citizen).

Player function $P(\varnothing)$ is the set of all the candidates, and $P(x)$, for any list x of positions for the candidates, is the set of all citizens.

Actions The set of actions available to each candidate at the start of the game consists of *Out* and the set of possible positions. The set of actions available to each citizen after a history x is the set of candidates.

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins outright, k to every terminal history in which she ties for first place with $n-k$ other candidates (for $1 \leq k \leq n-1$), 0 to every terminal history in which she stays out of the race, and -1 to every terminal history in which she loses, where n is the number of candidates. Each citizen's preferences are represented by a payoff function that assigns to each terminal history the average distance from the citizen's favorite position of the set of winning candidates in that history.

First consider the game in which there are two candidates (and an arbitrary number of citizens). Every subgame following choices of positions by the candidates has many Nash equilibria (as you know if you solved Exercise 47.1). For example, any action profile in which *all* citizens vote for the same candidate is a Nash equilibrium. (A citizen's switching her vote to another candidate has no effect on the outcome.)

This plethora of Nash equilibria allows us to construct, for *every* pair of positions, a subgame perfect equilibrium in which the candidates choose those positions! Consider the strategy profile in which the candidates choose the positions x_1 and x_2, and

- all citizens vote for candidate 1 after a history (x_1', x_2') in which $x_1' = x_1$
- all citizens vote for candidate 2 after a history (x_1', x_2') in which $x_1' \neq x_1$.

The outcome is that the candidates choose the positions x_1 and x_2 and candidate 1 wins. The strategy profile is a subgame perfect equilibrium because for every history (x_1, x_2) the profile of the citizens' actions is a Nash equilibrium, and neither candidate can induce an outcome she prefers by deviating: a deviation by candidate 1 to a position different from x_1 leads her to lose, and a deviation by candidate 2 has no effect on the outcome.

However, most of the Nash equilibria of the voting subgames are fragile (as you know if you solved Exercise 47.1): a citizen's voting for her less preferred candidate is weakly dominated (Definition 45.1) by her voting for her favorite candidate. (A citizen who switches from voting for her less preferred candidate to voting for her favorite candidate either does not affect the outcome (if her favorite candidate was three or more votes behind) or causes her favorite candidate either to tie for first place rather than lose, or to win rather than tie.) Thus in the only Nash equilibrium of a voting subgame in which no citizen uses a weakly dominated action, each citizen votes for the candidate whose position is closest to her favorite position.

Hotelling's model (Section 3.3) *assumes* that each citizen votes for the candidate whose position is closest to her favorite position; in its unique Nash equilibrium, each candidate's position is the median of the citizens' favorite positions. Combining this result with the result of the previous paragraph, we conclude that the game we are studying has only one subgame perfect equilibrium in which no player's strategy is weakly dominated: each candidate chooses the median of the citizens' favorite positions, and for every pair of the candidates' positions, each citizen votes for her favorite candidate.

In the game with three or more candidates, not only do many of the voting subgames have many Nash equilibria, with a variety of outcomes, but restricting to voting strategies that are not weakly dominated does not dramatically affect the set of equilibria: a citizen's only weakly dominated strategy is a vote for her least preferred candidate (see Exercise 47.2).

However, the set of equilibrium outcomes is dramatically restricted by the assumption that each candidate prefers to stay out of the race than to enter and lose, as the next two exercises show. The result in the first exercise is that the game has a subgame perfect equilibrium in which no citizen's strategy is weakly dominated and every candidate enters and chooses as her position the median of the citizens' favorite positions. The result in the second exercise is that under an assumption that makes the citizens averse to ties and an assumption that there exist citizens with extreme preferences, in *every* subgame perfect equilibrium all candidates who enter do so at the median of the citizens' favorite positions. The additional assumptions about the citizens' preferences are much stronger that necessary; they are designed to make the argument relatively easy.

? EXERCISE 212.1 (Electoral competition with strategic voters) Assume that there are $n \geq 3$ candidates and q citizens, where $q \geq 2n$ is odd (so that the median of the voters' favorite positions is well-defined) and divisible by n. Show that the game has a subgame perfect equilibrium in which no citizen's strategy is weakly dominated and every candidate enters the race and chooses the median of the citizens' favorite positions. (You may use the fact that every voting subgame has a (pure) Nash equilibrium in which no citizen's action is weakly dominated.)

?? EXERCISE 213.1 (Electoral competition with strategic voters) Consider the variant of the game in this section in which (*i*) the set of possible positions is the set of numbers x with $0 \leq x \leq 1$, (*ii*) the favorite position of at least one citizen is 0 and the favorite position of at least one citizen is 1, and (*iii*) each citizen's preferences are represented by a payoff function that assigns to each terminal history the distance from the citizen's favorite position to the position of the candidate in the set of winners whose position is *furthest* from her favorite position. Under the other assumptions of the previous exercise, show that in every subgame perfect equilibrium in which no citizen's action is weakly dominated, the position chosen by every candidate who enters is the median of the citizens' favorite positions. To do so, first show that in any equilibrium each candidate that enters is in the set of winners. Then show that in any Nash equilibrium of any voting subgame in which there are more than two candidates and not all candidates' positions are the same, some candidate loses. (Argue that if all candidates tie for first place, some citizen can increase her payoff by changing her vote.) Finally, show that in any subgame perfect equilibrium in which either only two candidates enter, or all candidates who enter choose the same position, every entering candidates chooses the median of the citizens' favorite positions.

7.4 Illustration: committee decision-making

How does the procedure used by a committee affect the decision it makes? One approach to this question models a decision-making procedure as an extensive game with perfect information and simultaneous moves in which a sequence of ballots are taken, in each of which the committee members vote simultaneously, and the result of each ballot determines the choices on the next ballot, or, eventually, the decision to be made.

Fix a set of committee members and a set of *alternatives* over which each member has strict preferences (no member is indifferent between any two alternatives). Assume that the number of committee members is odd, to avoid ties in votes. If there are two alternatives, the simplest committee procedure is that in which the members vote simultaneously for one of the alternatives. (We may interpret the game in Section 2.9.3 as a model of this procedure.) In the procedure illustrated in Figure 214.1, there are three alternatives, x, y, and z. The committee first votes whether to choose x (option "a") or to eliminate it from consideration (option "b"). If it votes to eliminate x, it subsequently votes between y and z.

In these procedures, each vote is between two options. Such procedures are

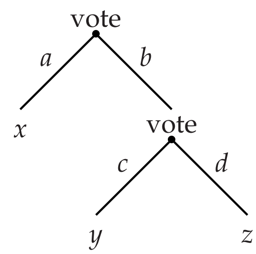

Figure 214.1 A voting procedure, or "binary agenda".

called *binary agendas*. We may define a binary agenda with the aid of an auxiliary one-player extensive game with perfect information in which the set $A(h)$ of actions following any nonterminal history h contains two elements, and the number of terminal histories is at least the number of alternatives. We associate with every terminal history h of this auxiliary game an alternative $\alpha(h)$ in such a way that each alternative is associated with at least one terminal history.

In the binary agenda associated with the auxiliary game G, all players vote simultaneously whenever the player in G takes an action. The options on the ballot following the nonterminal history in which a majority of committee members choose option a^1 at the start of the game, then option a^2, and so on, are the members of the set $A(a^1, \ldots, a^k)$ of actions of the player in G after the history $(a^1, \ldots, a^k)$. The alternative selected after the terminal history in which the majority choices are $a^1, \ldots, a^k$ is the alternative $\alpha(a^1, \ldots, a^k)$ associated with $(a^1, \ldots, a^k)$ in G. For example, in the auxiliary one-person game that defines the structure of the agenda in Figure 214.1, the single player first chooses a or b; if she chooses a the game ends, whereas if she chooses b, she then chooses between c and d. The alternative x is associated with the terminal history a, y is associated with (b, c), and z is associated with (b, d).

Precisely, the **binary agenda** associated with the auxiliary game G is the extensive game with perfect information and simultaneous moves defined as follows.

Players The set of committee members.

Terminal histories A sequence $(v^1, \ldots, v^k)$ of action profiles (in which each v^j is a list of the players' votes) is a terminal history if and only if there is a terminal history $(a^1, \ldots, a^k)$ of G such that for every $j = 0, \ldots, k-1$, every element of v^{j+1} is a member of $A(a^1, \ldots, a^j)$ ($A(\varnothing)$ if $j = 0$) and a majority of the players' actions in v^{j+1} are equal to a^{j+1}.

Player function For every nonterminal history h, $P(h)$ is the set of all players.

Actions For every player i and every nonterminal history $(v^1, \ldots, v^j)$, player i's set of actions is $A(a^1, \ldots, a^j)$, where $(a^1, \ldots, a^j)$ is the history of G in which, for all ℓ, a^ℓ is the action chosen by the majority of players in v^ℓ.

Preferences The rank each player assigns to the terminal history $(v^1, \ldots, v^k)$ is equal to the rank she assigns to the alternative $\alpha(a^1, \ldots, a^k)$ associated with the terminal history $(a^1, \ldots, a^k)$ of G in which, for all j, a^j is the action chosen by a majority of players in v^j.

Every binary agenda, like every voting subgame of the model in the previous section, has many subgame perfect equilibria. In fact, in any binary agenda, *every* alternative is the outcome of some subgame perfect equilibrium, because if, in every vote, every player votes for the same option, no player can affect the outcome by changing her strategy. However, if we restrict attention to weakly undominated strategies, we greatly reduce the set of equilibria. As we saw before (Section 2.9.3), in a ballot with two options, a player's action of voting for the option she prefers weakly dominates the action of voting for the other option. Thus in a subgame perfect equilibrium of a binary agenda in which every player's vote on every ballot is weakly undominated, on each ballot every player votes for the option that leads, ultimately (given the outcomes of the later ballots), to the alternative she prefers. The alternative associated with the terminal history generated by such a subgame perfect equilibrium is said to be the outcome of *sophisticated voting*.

Which alternatives are the outcomes of sophisticated voting in binary agendas? Say that alternative x *beats* alternative y if a majority of committee members prefer x to y. An alternative that beats every other alternative is called a *Condorcet winner*. For any preferences, there is either one Condorcet winner or no Condorcet winner (see Exercise 74.1).

First suppose that the players' preferences are such that some alternative, say x^*, is a Condorcet winner. I claim that x^* is the outcome of sophisticated voting in *every* binary agenda. The argument, using backward induction, is simple. First consider a subgame of length 1 in which one option leads to the alternative x^*. In this subgame a majority of the players vote for the option that leads to x^*, because a majority prefers x^* to every other alternative, and each player's only weakly undominated strategy is to vote for the option that leads to the alternative she prefers. Thus in at least one subgame of length 2, at least one option leads ultimately to the decision x^* (given the players' votes in the subgames of length 1). In this subgame, by the same argument as before, the winning option leads to x^*. Continuing backwards, we conclude that at least one option on the first ballot leads ultimately to x^*, and that consequently the winning option on this ballot leads to x^*.

Thus if the players' preferences are such that a Condorcet winner exists, the agenda does not matter: the outcome of sophisticated voting is always the Condorcet winner. If the players' preferences are such that no alternative is a Condorcet winner, the outcome of sophisticated voting depends on the agenda. Consider, for example, a committee with three members facing three alternatives. Suppose that one member prefers x to y to z, another prefers y to z to x, and the third prefers z to x to y. For these preferences, no alternative is a Condorcet winner. The outcome of sophisticated voting in the binary agenda in Figure 214.1 is the alternative x. (Use backward induction: y beats z, and x beats y.) If the positions of x

and y are interchanged then the outcome is y, and if the positions of x and z are interchanged then the outcome is z. Thus in this case, for *every* alternative there is a binary agenda for which that alternative is the outcome of sophisticated voting.

Which alternatives are the outcomes of sophisticated voting in binary agendas when no alternative is a Condorcet winner? Consider a committee with arbitrary preferences (not necessarily the ones considered in the previous paragraph), using the agenda in Figure 214.1. In order for x to be the outcome of sophisticated voting it must beat the winner of y and z. It may not beat both y and z directly, but it must beat them both at least "indirectly": either x beats y beats x, or x beats z beats y. Similarly, if y or z is the outcome of sophisticated voting then it must beat both of the other alternatives at least indirectly.

Precisely, say that alternative x *indirectly beats* alternative y if for some $k \geq 1$ there are alternatives $u_1, \ldots, u_k$ such that x beats u_1, u_j beats u_{j+1} for $j = 1, \ldots, k-1$, and u_k beats y. The set of alternatives x such that x beats every other alternative either directly or indirectly is called the *top cycle set*. (Note that if alternative x beats any alternative indirectly, it beats at least one alternative directly.) If there is a Condorcet winner, then the top cycle set consists of this single alternative. If there is no Condorcet winner, then the top cycle set contains more than one alternative.

? EXERCISE 216.1 (Top cycle set) A committee has three members.

a. Suppose that there are three alternatives, x, y, and z, and that one member prefers x to y to z, another prefers y to z to x, and the third prefers z to x to y. Find the top cycle set.

b. Suppose that there are four alternatives, w, x, y, and z, and that one member prefers w to z to x to y, one member prefers y to w to z to x, and one member prefers x to y to w to z. Find the top cycle set. Show, in particular, that z is in the top cycle set even though *all* committee members prefer w.

Rephrasing my conclusion for the agenda in Figure 214.1, if an alternative is the outcome of sophisticated voting, then it is in the top cycle set. The argument for this conclusion extends to any binary agenda. In every subgame, the outcome of sophisticated voting must beat the alternative that will be selected if it is rejected. Thus by backward induction, the outcome of sophisticated voting in the whole game must beat every other alternative either directly or indirectly: the outcome of sophisticated voting in any binary agenda is in the top cycle set.

Now consider a converse question: for any given alternative x in the top cycle set, is there a binary agenda for which x is the outcome of sophisticated voting? The answer is affirmative. The idea behind the construction of an appropriate agenda is illustrated by a simple example. Suppose that there are three alternatives, x, y, and z, and x beats y beats z. Then the agenda in Figure 214.1 is one for which x is the outcome of sophisticated voting. Now suppose there are two additional alternatives, u and w, and x beats u beats w. Then we can construct a larger agenda in which x is the outcome of sophisticated voting by replacing the alternative x in Figure 214.1 with a subgame in which a vote is taken for or against

x, and, if x is rejected, a vote is subsequently taken between u and w. If there are other chains through which x beats other alternatives, we can similarly add further subgames.

? EXERCISE 217.1 (Designing agendas) A committee has three members; there are five alternatives. One member prefers x to y to v to w to z, another prefers z to x to v to w to y, and the third prefers y to z to w to v to x. Find the top cycle set, and for each alternative a in the set design a binary agenda for which a is the outcome of sophisticated voting. Convince yourself that for no binary agenda is the outcome of sophisticated voting outside the top cycle set.

? EXERCISE 217.2 (An agenda that yields an undesirable outcome) Design a binary agenda for the committee in Exercise 216.1 for which the outcome of sophisticated voting is z (which is worse for all committee members than w).

In summary, (*i*) for any binary agenda, the alternative generated by the subgame perfect equilibrium in which no citizen's action in any ballot is weakly dominated is in the top cycle set, and (*ii*) for every alternative in the top cycle set, there is a binary agenda for which that alternative is generated by the subgame perfect equilibrium in which no citizen's action in any ballot is weakly dominated. In particular, the extent to which the procedure used by a committee affects its decision depends on the nature of the members' preferences. At one extreme, for preferences such that some alternative is a Condorcet winner, the agenda is irrelevant. At another extreme, for preferences for which every alternative is in the top cycle set, the agenda is instrumental in determining the decision. Further, for some preferences there are agendas for which the subgame perfect equilibrium yields an alternative that is unambiguously undesirable in the sense that there is another alternative that *all* committee members prefer.

7.5 Illustration: exit from a declining industry

An industry currently consists of two firms, one with a large capacity, and one with a small capacity. Demand for the firms' output is declining steadily over time. When will the firms leave the industry? Which firm will leave first? Do the firms' financial resources affect the outcome? The analysis of a model that answers these questions illustrates a use of backward induction more sophisticated than that in the previous sections of this chapter.

7.5.1 A model

Take time to be a discrete variable, starting in period 1. Denote by $P_t(Q)$ the market price in period t when the firms' total output is Q, and assume that this price is declining over time: for every value of Q, we have $P_{t+1}(Q) < P_t(Q)$ for all $t \geq 1$. (See Figure 219.1.) We are interested in the firms' decisions to exit, rather than their decisions of how much to produce in the event they stay in the market, so we assume that firm i's only decision is whether to produce some fixed output, denoted k_i, or to produce no output. (You may think of k_i as firm i's capacity.) Once a firm stops production, it cannot start up again. Assume that $k_2 < k_1$ (firm 2 is smaller than firm 1) and that each firm's cost of producing q units of output is cq.

The following extensive game with simultaneous moves models this situation.

Players The two firms.

Terminal histories All sequences $(X^1, \ldots, X^t)$ for some $t \geq 1$, where $X^s = (Stay, Stay)$ for $1 \leq s \leq t-1$ and $X^t = (Exit, Exit)$ (both firms exit in period t), or $X^s = (Stay, Stay)$ for all s with $1 \leq s \leq r-1$ for some r, $X^r = (Stay, Exit)$ or $(Exit, Stay)$, $X^s = Stay$ for all s with $r+1 \leq s \leq t-1$, and $X^t = Exit$ (one firm exits in period r and the other exits in period t), and all infinite sequences $(X^1, X^2, \ldots)$ where $X^r = (Stay, Stay)$ for all r (neither firm ever exits).

Player function $P(h) = \{1, 2\}$ after any history h in which neither firm has exited; $P(h) = 1$ after any history h in which only firm 2 has exited; and $P(h) = 2$ after any history h in which only firm 1 has exited.

Actions Whenever a firm moves, its set of actions is $\{Stay, Exit\}$.

Preferences Each firm's preferences are represented by a payoff function that associates with each terminal history the firm's total profit, where the profit of firm i $(= 1, 2)$ in period t is $(P_t(k_i) - c)k_i$ if the other firm has exited and $(P_t(k_1 + k_2) - c)k_i$ if the other firm has not exited.

7.5.2 Subgame perfect equilibrium

In a period in which $P_t(k_i) < c$, firm i makes a loss even if it is the only firm remaining (the market price for its output is less than its unit cost). Denote by t_i the last period in which firm i is profitable if it is the only firm in the market. That is, t_i is the largest value of t for which $P_t(k_i) \geq c$. (Refer to Figure 219.1.) Because $k_1 > k_2$, we have $t_1 \leq t_2$: the time at which the large firm becomes unprofitable as a loner is no later than the time at which the small firm becomes unprofitable as a loner.

The game has an infinite horizon, but after period t_i firm i's profit is negative even if it is the only firm remaining in the market. Thus if firm i is in the market in any period after t_i, it chooses *Exit* in that period in every subgame perfect equilibrium. In particular, both firms choose *Exit* in every period after t_2. We can use backward induction from period t_2 to find the firms' subgame perfect equilibrium actions in earlier periods.

If firm 1 (the larger firm) is in the market in any period from t_1 on, it should exit, whether or not firm 2 is still operating. As a consequence, if firm 2 is still

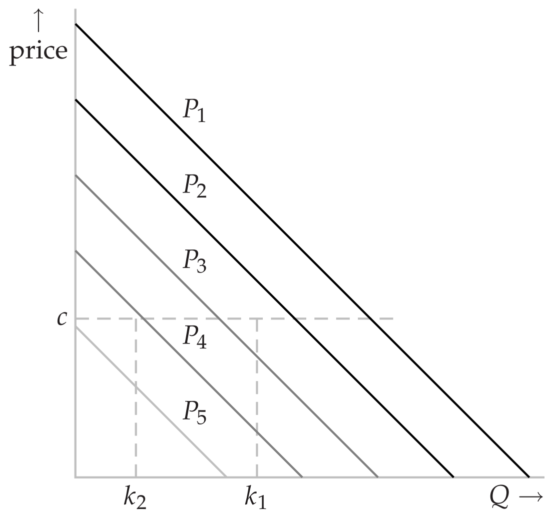

Figure 219.1 The inverse demand curves in a declining industry. In this example, t_1 (the last period in which firm 1 is profitable if it is the only firm in the market) is 2, and t_2 is 4.

operating in any period from $t_1 + 1$ to t_2 it should stay: firm 1 will exit in any such period, and in its absence firm 2's profit is positive.

So far we have concluded that in every subgame perfect equilibrium, firm 1's strategy is to exit in every period from $t_1 + 1$ on if it has not already done so, and firm 2's strategy is to exit in every period from $t_2 + 1$ on if it has not already done so.

Now consider period t_1, the last period in which firm 1's profit is positive if firm 2 is absent. If firm 2 exits, its profit from then on is zero. If it stays and firm 1 exits then it earns a profit from period t_1 to period t_2, after which it leaves. If both firms stay, firm 2 sustains a loss in period t_1 but earns a profit in the subsequent periods up to t_2, because in every subgame perfect equilibrium firm 1 exits in period $t_1 + 1$. Thus if firm 2's one-period loss in period t_1 when firm 1 stays in that period is less than the sum of its profits from period $t_1 + 1$ on, then *regardless of whether firm 1 stays or exits in period* t_1, firm 2 stays in every subgame perfect equilibrium. In period $t_1 + 1$, when firm 1 is absent from the industry, the price is relatively high, so that the assumption that firm 2's one-period loss is less than its subsequent multi-period profit is valid for a significant range of parameters. From now on, I assume that this condition holds.

We conclude that in every subgame perfect equilibrium firm 2 stays in period t_1, so that firm 1 optimally exits. (It definitely exits in the next period, and if it stays in period t_1 it makes a loss, because firm 2 stays.)

Now continue to work backwards. If firm 2 stays in period $t_1 - 1$ it earns a profit in periods t_1 through t_2, because in every subgame perfect equilibrium firm 1 exits in period t_1. It may make a loss in period $t_1 - 1$ (if firm 1 stays in that period), but this loss is less than the loss it makes in period t_1 in the company of firm 1, which we have assumed is outweighed by its subsequent profit. Thus regardless of firm 1's action in period $t_1 - 1$, firm 2's best action is to stay in that period. If $t_2 < t_1 - 1$ then firm 1 makes a loss in period $t_1 - 1$ in the company of firm 2, and so should exit.

The same logic applies to all periods back to the first period in which the firms cannot profitably co-exist in the industry: in every such period, in every subgame perfect equilibrium firm 1 exits if it has not already done so. Denote by t_0 the last period in which both firms can profitably co-exist in the industry: that is, t_0 is the largest value of t for which $P_t(k_1 + k_2) \geq c$.

We conclude that if firm 2's loss in period t_1 when both firms are active is less than the sum of its profits in periods $t_1 + 1$ through t_2 when it alone is active, then the game has a unique subgame perfect equilibrium, in which the large firm exits in period $t_0 + 1$, the first period in which both firms cannot profitably co-exist in the industry, and the small firm continues operating until period t_2, after which it alone becomes unprofitable.

? EXERCISE 220.1 (Exit from a declining industry) Assume that $c = 10$, $k_1 = 40$, $k_2 = 20$, and $P_t(Q) = 100 - t - Q$ for all values of t and Q for which $100 - t - Q > 0$, otherwise $P_t(Q) = 0$. Find the values of t_1 and t_2 and check whether firm 2's loss in period t_1 when both firms are active is less than the sum of its profits in periods $t_1 + 1$ through t_2 when it alone is active.

7.5.3 The effect of a constraint on firm 2's debt

When the firms follow their subgame perfect equilibrium strategies, each firm's profit is nonnegative in every period. However, the equilibrium depends on firm 2's ability to go into debt. Firm 2's strategy calls for it to stay in the market if firm 1, contrary to its strategy, does not exit in the first period in which the market cannot profitably sustain both firms. This feature of firm 2's strategy is essential to the equilibrium. If such a deviation by firm 1 induces firm 2 to exit, then firm 1's strategy of exiting may not be optimal, and the equilibrium may consequently fall apart.

Consider an extreme case, in which firm 2 can never go into debt. We can incorporate this assumption into the model by making firm 2's payoff a large negative number for any terminal history in which its profit in any period is negative. (The size of firm 2's profit depends on the contemporaneous action of firm 1, so we cannot easily incorporate the assumption by modifying the choices available to firm 2.) Consider a history in which firm 1 stays in the market after the last period in which the market can profitably sustain both firms. After such a history firm 2's best action is no longer to stay: if it does so it profit is negative, whereas if it exits its profit is zero. Thus if firm 1 deviates from its equilibrium strategy in the absence of a borrowing constraint for firm 2, and stays in the first period in which it is supposed to exit, then firm 2 optimally exits, and firm 1 reaps positive profits for several periods, as the lone firm in the market. Consequently in this case firm 2 exits first; firm 1 stays in the market until period t_1.

How much debt does firm 2 need to be able to bear in order that the game has a subgame perfect equilibrium in which firm 1 exits in period t_0 and firm 2 stays until period t_2? Suppose that firm 2 can sustain losses from period $t_0 + 1$ through period $t_0 + k$, but no longer, when both firms stay in the market. In order for firm 1 to optimally exit in period $t_0 + 1$, the consequence of its staying in the market must be that firm 2 also stays. Suppose that firm 2's strategy is to stay through period $t_0 + k$, but no longer, if firm 1 does so. Which strategy is best for firm 1 in the subgame starting in period $t_0 + 1$? If it exits, its payoff is zero. If it stays through period $t_0 + k$, its payoff is negative (it makes a loss in every period). If it stays beyond period $t_0 + k$ (when firm 2 exits), it should stay until period t_1, when its payoff is the sum of profits that are negative from period $t_0 + 1$ through period $t_0 + k$ and then positive through period t_1. (See Figure 221.1.) If this payoff is positive it should stay through period t_1; otherwise it should exit immediately.

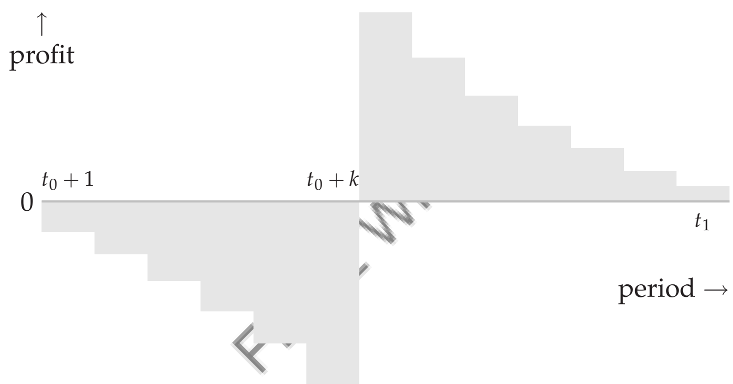

Figure 221.1 Firm 1's profits starting in period $t_0 + 1$ when firm 2 stays in the market until period $t_1 + k$ and firm 1 stays until period t_1.

We conclude that in order for firm 1 to exit in period $t_0 + 1$, the period $t_0 + k$ until which firm 2 can sustain losses must be large enough that firm 1's total profit from period $t_0 + 1$ through period t_1 if it shares the market with firm 2 until period $t_0 + k$, then has the market to itself, is nonpositive. This value of k determines the debt that firm 2 must be able to accumulate: the requisite debt equals its total loss when it remains in the market with firm 1 from period $t_0 + 1$ through period $t_0 + k$.

? EXERCISE 221.1 (Effect of borrowing constraint of firms' exit decisions in declining industry) Under the assumptions of Exercise 220.1, how much debt does firm 2 need to be able to bear in order for the subgame perfect equilibrium outcome in the absence of a debt constraint to remain a subgame perfect equilibrium outcome?

7.6 Allowing for exogenous uncertainty

7.6.1 General model

The model of an extensive game with perfect information (with or without simultaneous moves) does not allow random events to occur during the course of play. However, we can easily extend the model to cover such situations. The definition of an **extensive game with perfect information and chance moves** is a variant of the definition of an extensive game with perfect information (153.1) in which

- the player function assigns "chance", rather than a set of players, to some histories
- the probabilities that chance uses after any such history are specified
- the players' preferences are defined over the set of lotteries over terminal histories (rather than simply over the set of terminal histories).

(We may similarly add chance moves to an extensive game with perfect information and simultaneous moves by modifying Definition 202.1.) To keep the analysis simple, assume that the random event after any given history is independent of the random event after any other history. (That is, the realization of any random event is not affected by the realization of any other random event.)

The definition of a player's strategy remains the same as before. The outcome of a strategy profile is now a probability distribution over terminal histories. The definition of subgame perfect equilibrium remains the same as before.

◆ EXAMPLE 222.1 (Extensive game with chance moves) Consider a situation involving two players in which player 1 first chooses A or B. If she chooses A the game ends, with (Bernoulli) payoffs $(1, 1)$. If she chooses B then with probability $\frac{1}{2}$ the game ends, with payoffs $(3, 0)$, and with probability $\frac{1}{2}$ player 2 gets to choose between C, which yields payoffs $(0, 1)$ and D, which yields payoffs $(1, 0)$. An extensive game with perfect information and chance moves that models this situation is shown in Figure 223.1. The label c denotes chance; the number beside each action of chance is the probability with which that action is chosen.

We may use backward induction to find the subgame perfect equilibria of this game. In any equilibrium, player 2 chooses C. Now consider the consequences of player 1's actions. If she chooses A then she obtains the payoff 1. If she chooses B then she obtains 3 with probability $\frac{1}{2}$ and 0 with probability $\frac{1}{2}$, yielding an expected payoff of $\frac{3}{2}$. Thus the game has a unique subgame perfect equilibrium, in which player 1 choose B and player 2 chooses C.

? EXERCISE 222.2 (Variant of ultimatum game with equity-conscious players) Consider a variant of the game in Exercise 181.1 in which $\beta_1 = 0$, and the person 2 whom person 1 faces is drawn randomly from a population in which the fraction p have $\beta_2 = 0$ and the remaining fraction $1 - p$ have $\beta_2 = 1$. When making her

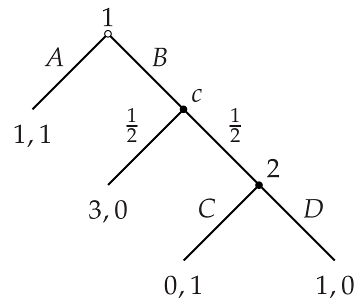

Figure 223.1 An extensive game with perfect information and chance moves. The label c denotes chance; the number beside each action of chance is the probability with which that action is chosen.

offer, person 1 knows only that her opponent's characteristic is $\beta_2 = 0$ with probability p and $\beta_2 = 1$ with probability $1 - p$. Model this situation as an extensive game with perfect information and chance moves in which person 1 makes an offer, then chance determines the type of person 2, and finally person 2 accepts or rejects person 1's offer. opponent Find the subgame perfect equilibria of this game. (Use the fact that if $\beta_2 = 0$, then in any subgame perfect equilibrium of the game in Exercise 181.1 person 2 accepts all offers $x > 0$, rejects all offers $x < 0$, and may accept or reject the offer 0, and if $\beta_2 = 1$ then she accepts all offers $x > \frac{1}{3}$, may accept or reject the offer $\frac{1}{3}$, and rejects all offers $x < \frac{1}{3}$.) Are there any values of p for which an offer is rejected in equilibrium?

? EXERCISE 223.1 (Sequential duel) In a sequential duel, two people alternately have the opportunity to shoot each other; each has an infinite supply of bullets. On each of her turns, a person may shoot, or refrain from doing so. Each of person i's shots hits (and kills) its intended target with probability p_i (independently of whether any other shots hit their targets). (If you prefer to think about a less violent situation, interpret the players as political candidates who alternately may launch attacks, which may not be successful, against each other.) Each person cares only about her probability of survival (not about the other person's survival). Model this situation as an extensive game with perfect information and chance moves. Show that the strategy pairs in which neither person ever shoots and in which each person always shoots are both subgame perfect equilibria. (Note that the game does not have a finite horizon, so backward induction cannot be used.)

?? EXERCISE 223.2 (Sequential truel) Each of persons A, B, and C has a gun containing a single bullet. Each person, as long as she is alive, may shoot at any surviving person. First A can shoot, then B (if still alive), then C (if still alive). (As in the previous exercise, you may interpret the players as political candidates. In this exercise, each candidate has a budget sufficient to launch a negative campaign to discredit exactly one of its rivals.) Denote by p_i the probability that player i hits her intended target; assume that $0 < p_i < 1$. Assume that each player wishes to maximize her probability of survival; among outcomes in which her survival probability is the same, she wants the danger posed by any other survivors to be as small as possible. (The last assumption is intended to capture the idea that there is some chance that further rounds of shooting may occur, though the possibility of such rounds is not incorporated explicitly into the game.) Model this situation as an extensive game with perfect information and chance moves. (Draw a diagram. Note that the subgames following histories in which A misses her intended target are the same.) Find the subgame perfect equilibria of the game. (Consider only cases in which p_A, p_B, and p_C are all different.) Explain the logic behind A's equilibrium action. Show that "weakness is strength" for C: she is better off if $p_C < p_B$ than if $p_C > p_B$.

Now consider the variant in which each player, on her turn, has the additional option of shooting into the air. Find the subgame perfect equilibria of this game when $p_A < p_B$. Explain the logic behind A's equilibrium action.

?? EXERCISE 224.1 (Cohesion in legislatures) The following pair of games is designed to study the implications of different legislative procedures for the cohesion of a governing coalition. In both games a legislature consists of three members. Initially a governing coalition, consisting of two of the legislators, is given. There are two periods. At the start of each period a member of the governing coalition is randomly chosen (i.e. each legislator is chosen with probability $\frac{1}{2}$) to propose a bill, which is a partition of one unit of payoff between the three legislators. Then the legislators simultaneously cast votes; each legislator votes either for or against the bill. If two or more legislators vote for the bill, it is accepted. Otherwise the course of events differs between the two games. In a game that models the current US legislature, rejection of a bill in period t leads to a given partition d^t of the pie, where $0 < d_i^t < \frac{1}{2}$ for $i = 1, 2, 3$; the governing coalition (the set from which the proposer of a bill is drawn) remains the same in period 2 following a rejection in period 1. In a game that models the current UK legislature, rejection of a bill brings down the government; a new governing coalition is determined randomly, and no legislator receives any payoff in that period. Specify each game precisely and find its subgame perfect equilibrium outcomes. Study the degree to which the governing coalition is cohesive (i.e. all its members vote in the same way).

7.6.2 Using chance moves to model mistakes

A game with chance moves may be used to model the possibility that players make mistakes. Suppose, for example, that two people simultaneously choose actions. Each person may choose either A or B. Absent the possibility of mistakes, suppose that the situation is modeled by the strategic game in Figure 225.1, in which the numbers in the boxes are Bernoulli payoffs. This game has two Nash equilibria, (A, A) and (B, B).

Now suppose that each person may make a mistake. With probability $1 - p_i > \frac{1}{2}$ the action chosen by person i is the one she intends, and with probability $p_i < \frac{1}{2}$ it is her other action. We can model this situation as the following extensive game

	A	B
A	$1,1$	$0,0$
B	$0,0$	$0,0$

Figure 225.1 The players' Bernoulli payoffs to the four pairs of actions in the game studied in Section 7.6.2.

with perfect information, simultaneous moves, and chance moves.

Players The two people.

Terminal histories All sequences of the form $((W, X), Y, Z)$, where W, X, Y, and Z are all either A or B; in the history $((W, X), Y, Z)$ player 1 chooses W, player 2 chooses X, and then chance chooses Y for player 1 and Z for player 2.

Player function $P(\varnothing) = \{1, 2\}$ (both players move simultaneously at the start of the game), and $P(W, X) = P((W, X), Y) = \{c\}$ (chance moves twice after the players have acted, first selecting player 1's action and then player 2's action).

Actions The set of actions available to each player at the start of the game, and to chance at each of its moves, is $\{A, B\}$.

Chance probabilities After any history (W, X), chance chooses W with probability $1 - p_1$ and player 1's other action with probability p_1. After any history $((W, X), Y)$, chance chooses X with probability $1 - p_2$ and player 2's other action with probability p_2.

Preferences Each player's preferences are represented by the expected value of a Bernoulli payoff function that assigns 1 to any history $((W, X), A, A)$ (in which chance chooses the action A for each player), and 0 to any other history.

The players in this game move simultaneously, so that the subgame perfect equilibria of the game are its Nash equilibria. To find the Nash equilibria we construct the strategic form of the game. Suppose that each player chooses the action A. Then the outcome is (A, A) with probability $(1 - p_1)(1 - p_2)$ (the probability that neither player makes a mistake). Thus each player's expected payoff is $(1 - p_1)(1 - p_2)$. Similarly, if player 1 chooses A and player 2 chooses B then the outcome is (A, A) with probability $(1 - p_1)p_2$ (the probability that player 1 does not make a mistake, whereas player 2 does). Making similar computations for the other two cases yields the strategic form in Figure 226.1.

For $p_1 = p_2 = 0$, this game is the same as the original game (Figure 225.1); it has two Nash equilibria, (A, A) and (B, B). If at least one of the probabilities is positive then only (A, A) is a Nash equilibrium: if $p_i > 0$ then $(1 - p_j)p_i > p_j p_i$

	A	B
A	$(1-p_1)(1-p_2), (1-p_1)(1-p_2)$	$(1-p_1)p_2, (1-p_1)p_2$
B	$p_1(1-p_2), p_1(1-p_2)$	p_1p_2, p_1p_2

Figure 226.1 The strategic form of the extensive game with chance moves that models the situation in which with probability p_i each player i in the game in Figure 225.1 chooses an action different from the one she intends.

(given that each probability is less than $\frac{1}{2}$). That is, only the equilibrium (A, A) of the original game is robust to the possibility that the players make small mistakes.

In the original game each player's action B is weakly dominated (Definition 45.1). Introducing the possibility of mistakes captures the fragility of the equilibrium (B, B): B is optimal for a player only if she is absolutely certain that the other player will choose B also. The slightest chance that the other player will choose A is enough to make A unambiguously the best choice.

We may use the idea that an equilibrium should survive when the players may make small mistakes to discriminate among the Nash equilibria of any strategic game. For two-player games we are led to the set of Nash equilibria in which no player's action is weakly dominated, but for games with more than two players we are led to a smaller set of equilibria, as the following exercise shows.

? EXERCISE 226.1 (Nash equilibria when players may make mistakes) Consider the three-player game in Figure 226.2. Show that (A, A, A) is a Nash equilibrium in which no player's action is weakly dominated. Now modify the game by assuming that the outcome of any player i's choosing an action X is that X occurs with probability $1 - p_i$ and the player's other action occurs with probability $p_i > 0$. Show that (A, A, A) is not a Nash equilibrium of the modified game when $p_i < \frac{1}{2}$ for $i = 1, 2, 3$.

	A	B
A	$1,1,1$	$0,0,1$
B	$1,1,1$	$1,0,1$

A

	A	B
A	$0,1,0$	$1,0,0$
B	$1,1,0$	$0,0,0$

B

Figure 226.2 A three-player strategic game in which each player has two actions. Player 1 chooses a row, player 2 chooses a column, and player 3 chooses a table.

7.7 Discussion: subgame perfect equilibrium and backward induction

Some of the situations we have studied do not fit well into the idealized setting for the steady state interpretation of a subgame perfect equilibrium discussed in Section 5.5.4, in which each player repeatedly engages in the same game with a variety of randomly selected opponents. In some cases an alternative interpretation fits better: each player deduces her optimal strategy from an analysis of the other players' best actions, given her knowledge of their preferences. Here I discuss a difficulty with this interpretation.

Consider the game in Figure 227.1, in which player 1 moves both before and after player 2. This game has a unique subgame perfect equilibrium, in which player 1's strategy is (B, F) and player 2's strategy is C. Consider player 2's analysis of the game. If she deduces that the only rational action for player 1 at the start of the game is B, then what should she conclude if player 1 chooses A? It seems that she must conclude that something has "gone wrong": perhaps player 1 has made a "mistake", or she misunderstands player 1's preferences, or player 1 is not rational. If she is convinced that player 1 simply made a mistake, then her analysis of the rest of the game should not be affected. However, if player 1's move induces her to doubt player 1's motivation, she may need to reconsider her analysis of the rest of the game. Suppose, for example, that A and E model similar actions; specifically, suppose that they both correspond to player 1's moving left, whereas B and F both involve her moving right. Then player 1's choice of A at the start of the game may make player 2 wonder whether player 1 confuses left and right, and therefore may choose E after the history (A, C). If so, player 2 should choose D rather than C after player 1 chooses A, giving player 1 an incentive to choose A rather than B at the start of the game.

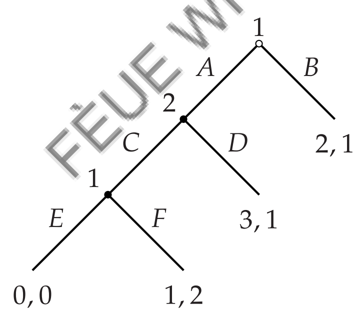

Figure 227.1 An extensive game in which player 1 moves both before and after player 2.

The next two examples are richer games that more strikingly manifest the difficulty with the alternative interpretation of subgame perfect equilibrium. The first example is an extension of the entry game in Figure 154.1.

◆ EXAMPLE 227.1 (Chain-store game) A chain-store operates in K markets. In each market a single challenger must decide whether to compete with it. The challengers make their decisions sequentially. If any challenger enters, the chain-store may acquiesce to its presence (A) or fight it (F). Thus in each period k the outcome is either *Out* (challenger k does not enter), (In, A) (challenger k enters and the chain-store acquiesces), or (In, F) (challenger k enters and is fought). When taking an action, any challenger knows all the actions previously chosen. The profits of challenger k and the chain-store in market k are shown in Figure 228.1 (cf. Figure 154.1); the chain-store's profit in the whole game is the sum of its profits in the

K markets.

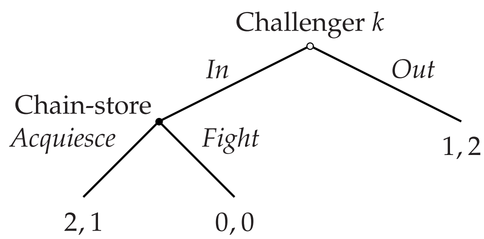

Figure 228.1 The structure of the players' choices in market k in the chain-store game. The first number in each pair is challenger k's profit and the second number is the chain-store's profit.

We can model this situation as the following extensive game with perfect information.

Players The chain-store and the K challengers.

Terminal histories The set of all sequences $(e_1, \ldots, e_K)$, where each e_j is either *Out*, (In, A), or (In, F).

Player function The chain-store is assigned to every history that ends with *In*, challenger 1 is assigned to the initial history, and challenger k (for $k = 2, \ldots, K$) is assigned to every history $(e_1, \ldots, e_{k-1})$, where each e_j is either *Out*, (In, A), or (In, F).

Preferences Each player's preferences are represented by its profits.

This game has a finite horizon, so we may find its subgame perfect equilibria by using backward induction. Every subgame at the start of which challenger K moves resembles the game in Figure 228.1 for $k = K$; it differs only in that the chain-store's profit after each of the three terminal histories is greater by an amount equal to its profit in the previous $K - 1$ markets. Thus in a subgame perfect equilibrium challenger K chooses *In* and the incumbent chooses A in market K.

Now consider the subgame faced by challenger $K - 1$. We know that the outcome in market K is independent of the actions of challenger $K - 1$ and the chain-store in market $K - 1$: whatever they do, challenger K enters and the chain-store acquiesces to its entry. Thus the chain-store should choose its action in market $K - 1$ on the basis of its payoffs in that market alone. We conclude that the chain-store's optimal action in market $K - 1$ is A, and challenger $K - 1$'s optimal action is *In*.

We have now concluded that in any subgame perfect equilibrium, the outcome in each of the last two markets is (In, A), regardless of the history. Continuing to work backwards to the start of the game we see that the game has a unique subgame perfect equilibrium, in which every challenger enters and the chain-store always acquiesces to entry.

? EXERCISE 228.1 (Nash equilibria of chain-store game) Find the set of Nash equilibrium outcomes of the game for an arbitrary value of K. (First think about the case $K = 1$, then generalize your analysis.)

? EXERCISE 229.1 (Subgame perfect equilibrium of chain-store game) Consider the following strategy pair in the game for $K = 100$. For $k = 1, \ldots, 90$, challenger k stays out after any history in which every previous challenger that entered was fought (or no challenger entered), and otherwise enters; challengers 91 through 100 enter. The chain-store fights every challenger up to challenger 90 that enters after a history in which it fought every challenger that entered (or no challenger entered), acquiesces to any of these challengers that enters after any other history, and acquiesces to challengers 91 through 100 regardless of the history. Find the players' payoffs in this strategy pair. Show that the strategy pair is not a subgame perfect equilibrium: find a player who can increase her payoff in some subgame. How much can the deviant increase its payoff?

Suppose that $K = 100$. You are in charge of challenger 21. You observe, contrary to the subgame perfect equilibrium, that every previous challenger entered and that the chain-store fought each one. What should you do? According to the subgame perfect equilibrium, the chain-store will acquiesce to your entry. But should you really regard the chain-store's 19 previous decisions as "mistakes"? You might instead read some logic into the chain-store's *deliberately* fighting the first 20 entrants: if, by doing so, it persuades more than 20 of the remaining challengers to stay out, then its profit will be higher than it is in the subgame perfect equilibrium. That is, you may imagine that the chain-store's aggressive behavior in the earlier markets is an attempt to establish a reputation for being a fighter, which, if successful, will make it better off. By such reasoning you may conclude that your best strategy is to stay out.

Thus, a deviation from the subgame perfect equilibrium by the chain-store in which it engages in a long series of fights may not be dismissed by challengers as a series of mistakes, but rather may cause them to doubt the chain-store's future behavior. This doubt may lead a challenger who is followed by enough future challengers to stay out.

◆ EXAMPLE 229.2 (Centipede game) The two-player game in Figure 230.1 is known as a "centipede game" because of its shape. (The game, like the arthropod, may have fewer than 100 legs.) The players move alternately; on each move a player can stop the game (S) or continue (C). On any move, a player is better off stopping the game than continuing if the other player stops immediately afterwards, but is worse off stopping than continuing if the other player continues, regardless of the subsequent actions. After k periods, the game ends.

This game has a finite horizon, so we may find its subgame perfect equilibria by using backward induction. The last player to move prefers to stop the game than to continue. Given this player's action, the player who moves before her also prefers to stop the game than to continue. Working backwards, we conclude that the game has a unique subgame perfect equilibrium, in which each player's strategy is to stop the game whenever it is her turn to move. The outcome is that player 1 stops the game immediately.

? EXERCISE 229.3 (Nash equilibria of the centipede game) Show that the outcome

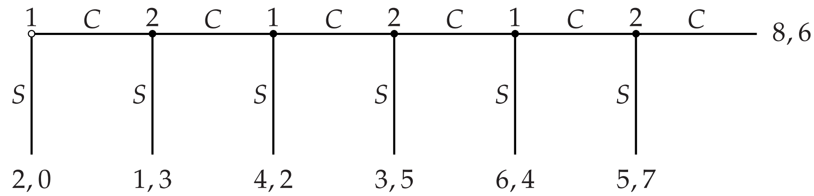

Figure 230.1 A 6-period centipede game.

of every Nash equilibrium of this game is the same as the outcome of the unique subgame perfect equilibrium (i.e. player 1 stops the game immediately).

The logic that in the only steady state player 1 stops the game immediately is unassailable. Yet this pattern of behavior is intuitively unappealing, especially if the number k of periods is large. The optimality of player 1's choosing to stop the game depends on her believing that if she continues, then player 2 will stop the game in period 2. Further, player 2's decision to stop the game in period 2 depends on her believing that if she continues then player 1 will stop the game in period 3. Each decision to stop the game is based on similar considerations. Consider a player who has to choose an action in period 21 of a 100-period game, after each player has continued in the first 20 periods. Is she likely to consider the first 20 decisions—half of which were hers—"mistakes"? Or will these decisions induce her to doubt that the other player will stop the game in the next period? These questions have no easy answers; some experimental evidence is discussed in the accompanying box.

Experimental evidence on the centipede game

In experiments conducted in the USA in 1989, each of 58 student subjects played a game with the monetary payoffs (in US$) shown in Figure 231.1 (McKelvey and Palfrey 1992). Each subject played the game 9 or 10 times, facing a different opponent each time; in each play of the game, each subject had previously played the same number of games. Each subject knew in advance how many times she would play the game, and knew that she would not play against the same opponent more than once. If each subject cared only about her own monetary payoff, the game induced by the experiment was a 6-period centipede.

The fraction of plays of the game that ended in each period is shown in Figure 231.2. Results are broken down according to the players' experience (first 5 rounds, last 5 rounds). The game ended earlier when the participants were experienced, but even among experienced participants the outcomes are far from the Nash equilibrium outcome, in which the game ends in period 1.

Ten plays of the game may not be enough to achieve convergence to a steady state. But putting aside this limitation of the data, and supposing that convergence

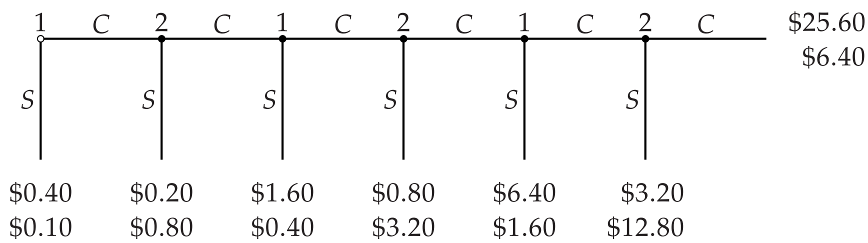

Figure 231.1 The game in McKelvey and Palfrey's (1992) experiment. The payoff of player 1 is written above the payoff of player 2.

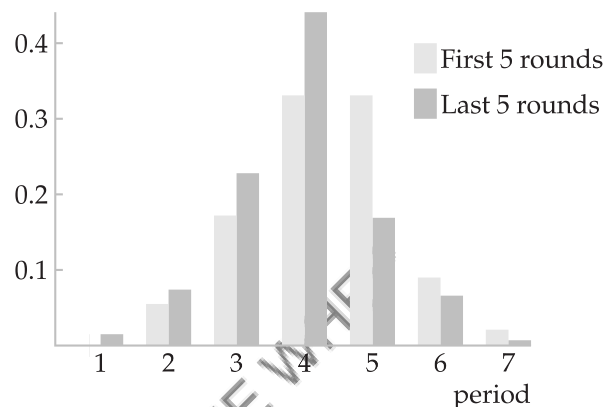

Figure 231.2 Fraction of games ending in each period of McKelvey and Palfrey's experiments on the six-period centipede game. (A game is counted as ending in period 7 if the last player to move chose *C*.) Computed from McKelvey and Palfrey (1992, Table IIIA).

was in fact achieved at the end of 10 rounds, how far does the observed behavior differ from a Nash equilibrium (maintaining the assumption that each player cares only about her own monetary payoff)?

The theory of Nash equilibrium has two components: each player optimizes, given her beliefs about the other players, and these beliefs are correct. Some decisions in McKelvey and Palfrey's experiment were patently suboptimal, regardless of the subjects' beliefs: a few subjects in the role of player 2 chose to continue in period 6, obtaining \$6.40 with certainty instead of \$12.80 with certainty. To assess the departure of the other decisions from optimality we need to assign the subjects beliefs (which were not directly observed). An assumption consistent with the steady state interpretation of Nash equilibrium is that a player's belief is based on her observations of the other players' actions. Even in round 10 of the experiment each player had only 9 observations on which to base her belief, and could have used these data in various ways. But suppose that, somehow, at the end of round 4, each player correctly inferred the distribution of her opponents' strategies in the next 5 rounds. What strategy should she subsequently have used? From Palfrey and McKelvey (1992, Table IIIB) we may deduce that the optimal strategy of player 1 stops in period 5 and that of player 2 stops in period 6. That is, each player's best response to the empirical distribution of the other players' strategies differs dramatically from her subgame perfect equilibrium strategy. Other assumptions about the subjects' beliefs rationalize other strategies; the data seem too limited to conclude that the subjects were not optimizing given beliefs they might reasonably have held, given their experience. That is, the experimental data are not strongly inconsistent with the theory of Nash equilibrium as a steady state.

Are the data inconsistent with the theory that rational players, even those with no experience playing the game, will deduce their opponents' rational actions from an analysis of the game using backward induction? This theory predicts that the first player immediately stops the game, so certainly the data are inconsistent with it. How inconsistent? One way to approach this question is to consider the implications of each player's thinking that the others are *likely* to be rational, but are not *certainly* so. If, in any period, player 1 thinks that the probability that player 2 will stop the game in the next period is less than $\frac{6}{7}$, continuing yields a higher expected payoff than stopping. Given the limited time the subjects had to analyze the game (and the likelihood that they had never before thought about any related game), even those who understood the implications of backward induction may reasonably have entertained the relatively small doubt about the other players' cognitive abilities required to make stopping the game immediately an unattractive option. Or, alternatively, a player confident of her opponents' logical abilities may have doubted her opponents' assessment of *her own* analytical skills. If player 1 believes that player 2 thinks that the probability that player 1 will continue in period 3 is greater than $\frac{1}{7}$, then she should continue in period 1, because player 2 will continue in period 2. That is, relatively minor departures from the theory yield outcomes close to those observed.

Notes

The idea of regarding games with simultaneous moves as games with perfect information is due to Dubey and Kaneko (1984).

The model in Section 7.3 was first studied by Ledyard (1981, 1984). The approach to voting in committees in Section 7.4 was initiated by Farquharson (1969). (The publication of Farquharson's book was delayed; the book was completed in 1958.) The top cycle set was first defined by Ward (1961) (who called it the "majority set"). The characterization of the outcomes of sophisticated voting in binary agendas in terms of the top cycle set is due to Miller (1977) (who calls the top cycle set the "Condorcet set") and McKelvey and Niemi (1978). Miller (1995) surveys the field. The model in Section 7.5 is taken from Nalebuff and Ghemawat (1985); the idea is closely related to that of Benoît (1984, Section 1) (see Exercise 172.2). My discussion draws on an unpublished exposition of the model by Vijay Krishna. The idea of discriminating among Nash equilibria by considering the possibility that players make mistakes, briefly discussed in Section 7.6.2, is due to Selten (1975). The chain-store game in Example 227.1 is due to Selten (1978). The centipede game in Example 229.2 is due to Rosenthal (1981).

The experimental results discussed in the box on page 207 are due to Roth, Prasnikar, Okuno-Fujiwara, and Zamir (1991). The subgame perfect equilibria of a variant of the market game in which each player's payoff depends on the other players' monetary payoffs are analyzed by Bolton and Ockenfels (2000). The model in Exercise 208.1 is taken from Peters (1984). The results in Exercises 212.1 and 213.1 are due to Feddersen, Sened, and Wright (1990). The game in Exercise 223.2 is a simplification of an example due to Shubik (1954); the main idea appears in Phillips (1937, 159) and Kinnaird (1946, 246), both of which consist mainly of puzzles previously published in newspapers. Exercise 224.1 is based on Diermeier and Feddersen (1996). The experiment discussed in the box on page 230 is reported in McKelvey and Palfrey (1992).

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8 Coalitional Games and the Core

Prerequisite: Chapter 1.

8.1 Coalitional games

A COALITIONAL GAME is a model of interacting decision-makers that focuses on the behavior of groups of players. It associates a set of actions with every group of players, not only with individual players, like the models of a strategic game (Definition 11.1) and extensive game (Definition 153.1)). We call each group of players a *coalition*, and the coalition of *all* the players the *grand coalition*.

An outcome of a coalitional game consists of a partition of the set of players into groups, together with an action for each group in the partition. (See Section 17.3 if you are not familiar with the notion of a "partition" of a set.) At one extreme, each group in the partition may consist of a single player, who acts on her own; at another extreme, the partition may consist of a single group containing all the players. The most general model of a coalitional game allows players to care about the action chosen by each group in the partition that defines the outcome. I discuss only the widely-studied class of games in which each player cares only about the action chosen by the member of the partition to which she belongs. In such games, each player's preferences rank the actions of all possible groups of players that contain her.

▶ DEFINITION 235.1 (*Coalitional game*) A **coalitional game** consists of

- a set of **players**
- for each coalition, a set of **actions**
- for each player, **preferences** over the set of all actions of all coalitions of which she is a member.

I usually denote the grand coalition (the set of all the players) by N and an arbitrary coalition by S. As before, we may conveniently specify a player's preferences by giving a payoff function that represents them.

In several of the examples that I present, each coalition controls some quantity of a good, which may be distributed among its members. Each action of a coalition S in such a game is a distribution among the members of S of the good that S controls, which I refer to as an S-**allocation** of the good. I refer to an N-allocation simply as an **allocation**.

Note that the definition of a coalitional game does not relate the actions of a coalition to the actions of the members of the coalition. The coalition's actions are simply taken as given; they are not derived from the individual players' actions.

A coalitional game is designed to model situations in which players can beneficially form groups, rather than acting individually. Most of the theory is oriented to situations in which the incentive to coalesce is extreme, in the sense that there is no disadvantage to the formation of the *single* group consisting of all the players. In considering the action that this single group takes in such a situation, we need to consider the possibility that smaller groups break away on their own; but when looking for "equilibria" we can restrict attention to outcomes in which all the players coalesce. Such situations are modeled as games in which the grand coalition can achieve outcomes at least as desirable for every player as those achievable by any partition of the players into subgroups. We call such games "cohesive", defined precisely as follows.

▶ DEFINITION 236.1 (*Cohesive coalitional game*) A coalitional game is **cohesive** if, for every partition $\{S_1, \ldots, S_k\}$ of the set of all players and every combination $(a_{S_1}, \ldots, a_{S_k})$ of actions, one for every coalition in the partition, the grand coalition N has an action that is at least as desirable for every player i as the action a_{S_j} of the member S_j of the partition to she player i belongs.

The concepts I subsequently describe may be applied to any game, cohesive and not, but have attractive interpretations only for cohesive games.

◆ EXAMPLE 236.2 (Two-player unanimity game) Two people can together produce one unit of output, which they may share in any way they wish. Neither person by herself can produce any output. Each person cares only about the amount of output she receives, and prefers more to less. The following coalitional game models this situation.

Players The two people (players 1 and 2).

Actions Each player by herself has a single action, which yields her no output. The set of actions of the coalition $\{1, 2\}$ of both players is the set of all pairs (x_1, x_2) of nonnegative numbers such that $x_1 + x_2 = 1$ (the set of divisions of one unit of output between the two players).

Preferences Each player's preferences are represented by the amount of output she obtains.

The possible partitions of the set of players are $\{\{1,2\}\}$, consisting of the single coalition of both players, and $\{\{1\},\{2\}\}$, in which each player acts alone. The latter has only one combination of actions available to it, which produces not output. Thus the game is cohesive.

In the next example the opportunities for producing output are richer and the participants are not all symmetric.

◆ EXAMPLE 237.1 (Landowner and workers) A landowner's estate, when used by k workers, produces the output $f(k+1)$ of food, where f is a increasing function for which $f(0) = 0$. The total number of workers is m. The landowner and each worker care only about the amount of output she receives, and prefer more to less. The following coalitional game models this situation.

Players The landowner and the m workers.

Actions A coalition consisting solely of workers has a single action in which no member receives any output. The set of actions of a coalition S consisting of the landowner and k workers is the set of all S-allocations of the output $f(k+1)$ among the members of S.

Preferences Each player's preferences are represented by the amount of output she obtains.

This game is cohesive because the grand coalition produces more output than any other coalition, and, for any partition of the set of all the players, only one coalition produces any output.

◆ EXAMPLE 237.2 (Three-player majority game) Three people have access to one unit of output. Any majority—two or three people—may control the allocation of this output. Each person cares only about the amount of output she obtains.

We may model this situation as the following coalitional game.

Players The three people.

Actions Each coalition consisting of a single player has a single action, which yields the player no output. The set of actions of each coalition S with two or three players is the set of S-allocations of one unit of output.

Preferences Each player's preferences are represented by the amount of output she obtains.

This game is cohesive because every partition of the set of players contains at most one majority coalition, and for every action of such a coalition there is an action of the grand coalition that yields each player as least as much output.

In these examples the set of actions of each coalition S is the set of S-allocations of the output that S can obtain, and each player's preferences are represented by the amount of output she obtains. Thus we can summarize each coalition's set of actions by a single number, equal to the total output it can obtain, and can interpret this number as the total "payoff" that may be distributed among the members of the coalition. A coalitional game in which the set of payoff distributions resulting from each coalition's actions may be represented in this way is said to have **transferable payoff**.

We refer to the total payoff of any coalition S in a game with transferable payoff as the **worth** of S, and denote it $v(S)$. Such a game is thus specified by its set of players N and its worth function.

For the two-player unanimity game, for example, we have $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = 0$, and $v(\{1, 2\}) = 1$. For the landlord–worker game we have $N = \{1, \ldots, m+1\}$ (where 1 is the landowner and $2, \ldots, m$ are the workers) and

$$v(S) = \begin{cases} 0 & \text{if 1 is not a member of } S \\ f(k) & \text{if } S \text{ consists of 1 and } k \text{ workers.} \end{cases}$$

For the three-player majority game we have $N = \{1, 2, 3\}$, $v(\{i\}) = 0$ for $i = 1$, 2, 3, and $v(S) = 1$ for every other coalition S.

In the next two examples, payoff is not transferable.

◆ EXAMPLE 238.1 (House allocation) Each member of a group of n people has a single house. Any subgroup may reallocate its members' houses in any way it wishes (one house to each person). (Time-sharing and other devices to evade the indivisibility of a house are prohibited.) The values assigned to houses vary among the people; each person cares only about the house she obtains. The following coalitional game models this situation.

Players The n people.

Actions The set of actions of a coalition S is the set of all assignments to members of S of the houses originally owned by members of S.

Preferences Each player prefers one outcome to another according to the house she is assigned.

This game is cohesive because any allocation of the houses that can be achieved by the coalitions in any partition of the set of players can also be achieved by the set of all players. It does not have transferable payoff. For example, a coalition of players 1 and 2 can achieve only the two payoff distributions (v_1, w_2) and (v_2, w_1), where v_i is the payoff to player 1 of the house owned by player i and w_i is the payoff to player 2 of the house owned by player i.

◆ EXAMPLE 238.2 (Marriage market) A group of men and a group of women may be matched in pairs. Each person cares about her partner. A *matching* of the members of a coalition S is a partition of the members of S into male-female pairs and singles. The following coalitional game models this situation.

Players The set of all the men and all the women.

Actions The set of actions of a coalition S is the set of all matchings of the members of S.

Preferences Each player prefers one outcome to another according to the partner she is assigned.

This game is cohesive because the matching of the members of the grand coalition induced by any collection of actions of the coalitions in a partition can be achieved by some action of the grand coalition.

8.2 The core

Which action may we expect the grand coalition to choose? We seek an action compatible with the pressures imposed by the opportunities of each coalition, rather than simply those of individual players as in the models of a strategic game (Chapter 2) and an extensive game (Chapter 5). We define an action of the grand coalition to be "stable" if no coalition can break away and choose an action that all its members prefer. The set of all stable actions of the grand coalition is called the *core*, defined precisely as follows.

▶ DEFINITION 239.1 (*Core*) The **core** of a coalitional game is the set of actions a_N of the grand coalition N such that no coalition has an action that all its members prefer to a_N.

If a coalition S has an action that all its members prefer to some action a_N of the grand coalition, we say that S can **improve upon** a_N. Thus we may alternatively define the core to be the set of all actions of the grand coalition upon which no coalition can improve.

Note that the core is defined as a *set* of actions, so it always *exists*; a game cannot fail to have a core, though it may be the empty set, in which case no action of the grand coalition is immune to deviations.

We have restricted attention to games in which, when evaluating an outcome, each player cares only about the action chosen by the coalition in the partition of which she is a member. Thus the members of a coalition do not need to speculate about the remaining players' behavior when considering a deviation. Consequently an interpretation of the core does not require us to assume that the players are experienced; the concept makes sense even for naïve players with no experience in the game. (By contrast, the main interpretations of Nash equilibrium and subgame perfect equilibrium require the players to have experience playing the game.)

In a game with transferable payoff, a coalition S can improve upon an action a_N of the grand coalition if and only if its worth $v(S)$ (i.e. the total payoff it can achieve by itself) exceeds the total payoff of its members in a_N. That is, a_N is in the core if and only if for every coalition S the total payoff $x_S(a_N)$ it yields the members of S is at least $v(S)$:

$$x_S(a_N) \geq v(S) \text{ for every coalition } S.$$

To find the core of a coalitional game we need to find the set of all actions of the grand coalition upon which no coalition can improve. In the next example, no coalition can improve upon any action of the grand coalition, so the core consists of *all* actions of the grand coalition.

◆ EXAMPLE 240.1 (Two-player unanimity game) Consider the two-player unanimity game in Example 236.2. An action of the grand coalition is a pair (x_1, x_2) with $x_1 + x_2 = 1$ and $x_i \geq 0$ for $i = 1, 2$ (a division of the one unit of output between the two players). I claim that the core consists of *all* possible divisions:

$$\{(x_1, x_2) : x_1 + x_2 = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, 2\}.$$

Any such division is in the core because if a single player deviates she obtains no output, and if the grand coalition chooses a different division then one player is worse off.

In this example no coalition has any action that imposes any restriction on the action of the grand coalition. In most other games the coalitions' opportunities constrain the actions of the grand coalition.

One way to find the core is to check each action of the grand coalition in turn. For each action and each coalition S, we impose the condition that S cannot make all its members better off; an action is a member of the core if and only if it satisfies these conditions.

Consider, for example, a variant of the two-player unanimity game in which player 1, by herself, can obtain p units of output, and player 2, by herself, can obtain q units of output. Then the condition that the coalition consisting of player 1 not be able to improve upon the action (x_1, x_2) of the grand coalition is $x_1 \geq p$, and the condition that the coalition consisting of player 2 not be able to improve upon this action is $x_2 \geq q$. As in the original game, the coalition of both players cannot improve upon any action (x_1, x_2), so the core is

$$\{(x_1, x_2) : x_1 + x_2 = 1, x_1 \geq p, \text{ and } x_2 \geq q\}.$$

(An implication is that if $p + q > 1$—in which case the game is not cohesive—the core is empty.)

An example of the landowner–worker game further illustrates this method of finding the core.

◆ EXAMPLE 240.2 (Landowner–worker game with two workers) Consider the game in Example 237.1 in which there are two workers ($k = 2$). Let (x_1, x_2, x_3) be an action of the grand coalition. That is, let (x_1, x_2, x_3) be an allocation of the output $f(3)$ among the three players. The only coalitions that can obtain a positive amount of output are that consisting of the landowner (player 1), which can obtain the output $f(1)$, those consisting of the landowner and a worker, which can obtain $f(2)$, and the grand coalition. Thus (x_1, x_2, x_3) is in the core if and only if

$$\begin{aligned} x_1 &\geq f(1) \\ x_2 &\geq 0 \\ x_3 &\geq 0 \\ x_1 + x_2 &\geq f(2) \\ x_1 + x_3 &\geq f(2) \\ x_1 + x_2 + x_3 &= f(3), \end{aligned}$$

where the last condition ensures that (x_1, x_2, x_3) is an allocation of $f(3)$.

From the last condition we have $x_1 = f(3) - x_2 - x_3$, so that we may rewrite the conditions as

$$\begin{aligned} 0 \leq x_2 &\leq f(3) - f(2) \\ 0 \leq x_3 &\leq f(3) - f(2) \\ x_2 + x_3 &\leq f(3) - f(1) \\ x_1 + x_2 + x_3 &= f(3). \end{aligned}$$

That is, in an action in the core, each worker obtains at most the extra output $f(3) - f(2)$ produced by the third player, and the workers together obtain at most the extra output $f(3) - f(1)$ produced by the second and third players together.

? EXERCISE 241.1 (Three-player majority game) Show that the core of the three-player majority game (Example 237.2) has an empty core. Find the core of the variant of this game in which player 1 has three votes (and player 2 and player 3 each has one vote, as in the original game).

The next example introduces a class of games that model the market for an economic good.

◆ EXAMPLE 241.2 (Market with one owner and two buyers) A person holds one indivisible unit of a good and each of two (potential) buyers has a large amount of money. The owner values money but not the good; each buyer values both money and the good and regards the good as equivalent to one unit of money. Each coalition may assign the good (if owned by one of its members) to any of its members and allocate its members' money in any way it wishes among its members.

We may model this situation as the following coalitional game.

Players The owner and the two buyers.

Actions The set of actions of each coalition S is the set of S-allocations of the money and good (if any) owned by S.

Preferences The owner's preferences are represented by the amount of money she obtains; each buyer's preferences are represented by the amount of the good (either 0 or 1) she obtains plus the amount of money she holds.

I claim that for any action in the core, the owner does not keep the good. Let a_N be an action of the grand coalition in which the owner keeps the good, and let m_i be the amount of money transferred from potential buyer i to the owner in this action. (Transfers of money from the buyers to the owner when the owner keeps the good may not sound sensible, but they are feasible, so that we need to consider them.) Consider the alternative action a'_N of the grand coalition in which the good is allocated to buyer 1, who transfers $m_1 + 2\epsilon$ money to the owner, and buyer 2 transfers $m_2 - \epsilon$ money to the owner, where $0 < \epsilon < \frac{1}{2}$. We see that all the players' payoffs are higher in a'_N than they are in a_N. (The owner's payoff is ϵ

higher, buyer 1's payoff is $1-2\epsilon$ higher, and buyer 2's payoff is ϵ higher.) Thus a_N is not in the core.

Consider an action a_N in the core in which buyer 1 obtains the good. I claim that in a_N buyer 1 pays one unit of money to the owner and buyer 2 pays no money to the owner. If buyer 2 pays a positive amount she can improve upon a_N by acting by herself (and making no payment). If buyer 1 pays more than one unit of money to the owner she too can improve upon a_N by acting by herself. Finally, suppose buyer 1 pays $m_1 < 1$ to the owner. Then the owner and buyer 2 can improve upon a_N by allocating the good to buyer 2 and transferring $\frac{1}{2}(1+m_1)$ units of money from buyer 2 to the owner, yielding the owner a payoff greater than m_1 and buyer 2 a positive payoff.

We conclude that the core contains exactly two actions, in each of which the good is allocated to a buyer and one unit of the buyer's money is allocated to the owner. That is, the good is sold to a buyer at the price of 1, yielding the buyer who obtains the good the same payoff that she obtains if she does not trade. This extreme outcome is a result of the competition between the buyers for the good: any outcome in which the owner trades with buyer i at a price less than 1 can be improved upon by the coalition consisting of the owner and the *other* buyer, who is willing to pay a little more for the good than does buyer i.

? EXERCISE 242.1 (Market with one owner and two heterogeneous buyers) Consider the variant of the game in the previous example in which buyer 1's valuation of the good is 1 and buyer 2's valuation is $v < 1$ (i.e. buyer 2 is indifferent between owning the good and owning v units of money). Find the core the game that models this situation.

In the next exercise, the grand coalition has finitely many actions; one way of finding the core is to check each one in turn.

? EXERCISE 242.2 (Vote trading) A legislature with three members decides, by majority vote, the fate of three bills, A, B, and C. Each legislator's preferences are represented by the sum of the values she attaches to the bills that pass. The value attached by each legislator to each bill is indicated in Figure 242.1. For example, if bills A and B pass and C fails, then the three legislators' payoffs are 1, 3, and 0 respectively. Each majority coalition can achieve the passage of any set of bills, whereas each minority is powerless.

	A	B	C
Legislator 1	2	−1	1
Legislator 2	1	2	−1
Legislator 3	−1	1	2

Figure 242.1 The legislators' payoffs to the three bills in Exercise 242.2.

a. Find the core of the coalitional game that models this situation.

b. Find the core of the game in which the values the legislators attach to the payoff of each bill differ from those in Figure 242.1 only in that legislator 3 values the passage of bill C at 0.

c. Find the core of the game in which the values the legislators attach to the payoff of each bill differ from those in Figure 242.1 only in that each 1 is replaced by -1.

8.3 Illustration: ownership and the distribution of wealth

In economies dominated by agriculture, the distribution and institutions of land ownership differ widely. By studying the cores of coalitional games that model various institutions, we can gain an understanding of the implications of these institutions for the distribution of wealth.

A group of $n \geq 3$ people may work land to produce food. Denote the output of food when k people work all the land by $f(k)$. Assume that f is an increasing function, $f(0) = 0$, and the output produced by an additional person decreases as the number of workers increases: $f(k) - f(k-1)$ is decreasing in k. An example of such a function f is shown in Figure 243.1. In all the games that I study the set of players is the set of the n people and each person cares only about the amount of food she obtains.

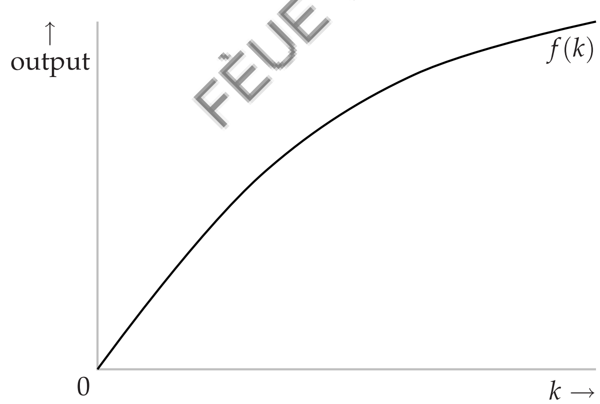

Figure 243.1 The output of food as a function of the number k of workers, under the assumption that the output of an additional worker decreases as the number of workers increases.

8.3.1 Single landowner and landless workers

First suppose that the land is owned by a single person, the *landowner*. I refer to the other people as *workers*. In this case we obtain the game in Example 237.1. In this game the action a_N of the grand coalition in which the landowner obtains all the output $f(n)$ is in the core: all coalitions that can produce any output include the landowner, and none of these coalitions has any action that makes her better off than she is in a_N.

Are the workers completely powerless, or does the core contain actions in which they receive some output? The workers need the landowner to produce any output, but the landowner also needs the workers to produce more than $f(1)$, so there is reason to think that stable actions of the grand coalition exist in which the workers receive some output. Take the landowner to be player 1, and consider the action a_N of the grand coalition in which each player i obtains the output x_i, where $x_1 + \cdots + x_n = f(n)$. Under what conditions on $(x_1, \ldots, x_n)$ is a_N in the core? Because of my assumption on the shape of the function f, the coalitions most capable of profitably deviating from a_N consist of the landowner and every worker but one. Such a coalition can, by itself, produce $f(n-1)$, and may distribute this output in any way among its members. Thus for a deviation by such a coalition not to be profitable, the sum of x_1 and any collection of $n-2$ other x_i's must be at least $f(n-1)$. That is, $(x_1 + \cdots + x_n) - x_j \geq f(n-1)$ for every $j = 2, \ldots, n$. Because $x_1 + \cdots + x_n = f(n)$, we conclude that $x_j \leq f(n) - f(n-1)$ for every player j with $j \geq 2$ (i.e. every worker). That is, if a_N is in the core then $0 \leq x_j \leq f(n) - f(n-1)$ for every player $j \geq 2$. If fact, every such action is in the core, as you are asked to verify in the following exercise.

? EXERCISE 244.1 (Core of landowner–worker game) Check that no coalition can improve upon any action of the grand coalition in which the output received by every worker is nonnegative and at most $f(n) - f(n-1)$. (Use the fact that the form of f implies that $f(n) - f(k) \geq (n-k)(f(n) - f(n-1))$ for every $k \leq n$.)

We conclude that the core of the game is the set of all actions of the grand coalition in which the output x_i obtained by each worker i satisfies $0 \leq x_i \leq f(n) - f(n-1)$ and the output obtained by the landowner is the difference between $f(n)$ and the sum of the workers' shares. In economic jargon, $f(n) - f(n-1)$ is a worker's "marginal product". Thus in any action in the core, each worker obtains at most her marginal product.

The workers' shares of output are driven down to at most $f(n) - f(n-1)$ by competition between coalitions consisting of the landowner and workers. If the output received by any worker exceeds $f(n) - f(n-1)$ then the other workers, in cahoots with the landowner, can deviate and increase their share of output. That is, each worker's share of output is limited by her comrades' attempts to obtain more output.

The fact that each worker's share of output is held down by inter-worker competition suggests that if the workers were to agree not to join deviating coalitions *except as a group* then they might be better off. You are asked to check this idea in the following exercise.

? EXERCISE 244.2 (Unionized workers in landowner–worker game) Formulate as a coalitional game the variant of the landowner–worker game in which any group of fewer than $n-1$ workers refuses to work with the landowner, and find its core.

The core of the original game is closely related to the outcomes predicted by the economic notion of "competitive equilibrium". Suppose that the landowner believes she can hire any number of workers at the fixed wage w (given as an amount of output), and every worker believes that she can obtain employment at this wage. If $w \geq 0$ then every worker wishes to work, and if $w \leq f(n) - f(n-1)$ the landowner wishes to employ all $n-1$ workers. (Reducing the number of workers by one reduces the output by $f(n) - f(n-1)$; further reducing the number of workers reduces the output by successively larger amounts, given the shape of f.) If $w > f(n) - f(n-1)$ then the landowner wishes to employ fewer than $n-1$ workers, because the wage exceeds the increase in the total output that results when the $(n-1)$th worker is employed. Thus the demand for workers is equal to the supply if and only if $0 \leq w \leq f(n) - f(n-1)$; every such wage w is a "competitive equilibrium".

A different assumption about the form of f yields a different conclusion about the core. Suppose that each additional worker produces *more* additional output than the previous one. An example of a function f with this form is shown in Figure 245.1. Under this assumption the economy has no competitive equilibrium: for any wage, the landowner wishes to employ an indefinitely large number of workers. The next exercise asks you to study the core of the induced coalitional game.

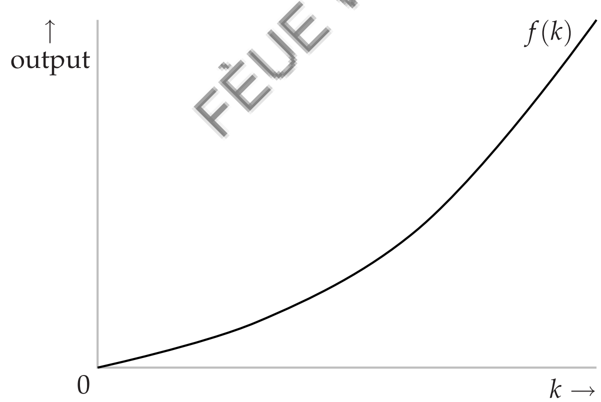

Figure 245.1 The output of food as a function of the number k of workers, under the assumption that the output of an additional worker increases as the number of workers increases.

? EXERCISE 245.1 (Landowner–worker game with increasing marginal products) Consider the variant of the landowner–worker game in which each additional worker produces more additional output than the previous one. (That is, $f(k)/k < f(k+1)/(k+1)$ for all k.) Show that the core of this game contains the action of the grand coalition in which each player obtains an equal share of the total output.

8.3.2 Small landowners

Suppose that the land is distributed equally between all n people, rather than being concentrated in the hands of a single landowner. Assume that a group of k people who pool their land and work together produce $(k/n)f(n)$ units of output. (The output produced by half the people working half the land, for example, is half the output produced by all the people working all the land.)

The following specification of the set of actions available to each coalition models this situation.

Actions The set of actions of a coalition S consisting of k players is the set of all S-allocations of the output $(k/n)f(n)$ between the members of S.

As you might expect, one action in the core of this game is that in which every player obtains an equal share of the total output—that is, $f(n)/n$ units. Under this action, the total amount received by each coalition is precisely the total amount the coalition produces. In fact, no other action is in the core. In any other action, some player receives less than $f(n)/n$, and hence can improve upon the action alone (obtaining $f(n)/n$ for herself). That is, the core consists of the single action in which every player obtains $f(n)/n$ units of output.

8.3.3 Collective ownership

Suppose that the land is owned collectively and the distribution of output is determined by majority voting. Assume that any majority may distribute the output in any way it wishes; any majority may, in particular, take all the output for itself. In this case the set of actions available to each coalition are given as follows.

Actions The set of actions of a coalition S consisting of more than $n/2$ players is the set of all S-allocations of the output $f(n)$ between the members of S. The set of actions of a coalition S consisting of at most $n/2$ players is the single S-allocation in which no player in S receives any output.

The core of the coalitional game defined by this assumption is empty. For every action of the grand coalition, at least one player obtains a positive amount of output. But if player i obtains a positive amount of output then the coalition of the remaining players, which is a majority, may improve upon the action, distributing the output $f(n)$ among its members (so that player i gets nothing). Thus every action of the grand coalition may be improved upon by some coalition; no distribution of output is "stable".

The core of this game is empty because of the extreme power of every majority coalition. If any majority coalition may control how the land is used, but every player owns a "share" that entitles her to the fraction $1/n$ of the output, then a majority coalition with k members can lay claim to only the fraction k/n of the total output, and a stable distribution of output may exist. This alternative ownership institution, which tempers the power of majority coalitions, does not have interesting implications in the model in this section because the control of land use vested in a majority coalition is inconsequential—only one sensible pattern of use exists (all the players work!). If choices exist—if, for example, different crops may be grown, and people differ in their preferences for these crops—then collective ownership in which each player is entitled to an equal share of the output may yield a different outcome from individual ownership.

8.4 Illustration: exchanging homogeneous horses

Markets may be modeled as coalitional games in which the set of actions of each coalition S is the set of S-allocations of the good initially owned by the members of S. The core of such a game is the set of allocations of the goods available in the economy that are robust to the trading opportunities of all possible groups of participants: if a_N is in the core then no group of agents can secede from the economy, trade among themselves, and produce an outcome they all prefer to a_N.

In this section I describe a simple example of a market, in which there is money and a single homogeneous good (all units of which are identical). In the next section I describe a market in which there is a single *heterogeneous* good. In both case the core makes a very precise prediction about the outcome.

8.4.1 Model

Some people own one unit of an indivisible good, whereas others possess only money. Some non-owners value a unit of the good more highly than some owners, so that mutually beneficial trades exist. Which allocation of goods and money will result?

We may address this question with the help of a coalitional game that generalizes the one in Example 241.2. I refer to the goods as "horses" (following the literature on the model, which takes off from an analysis by Eugen von Böhm-Bawerk (1851–1914)). Call each person who owns a horse simply an *owner*, and every other person a *nonowner*. Assume that all horses are identical, and that no one wishes to own more than one. People value a horse differently; denote player i's valuation by v_i. Assume that there are at least two owners and two nonowners, and that some owner's valuation is less than some nonowner's valuation (i.e. for some owner i and nonowner j have $v_i < v_j$), so that some trade is mutually desirable. Assume also, to avoid some special cases, that some nonowner's valuation is less than some owner's valuation (i.e. for some nonowner i and owner j we have $v_i < v_j$) and that no two players have the same valuation. Further assume that every person has enough money to fully compensate the owner who values a horse most highly, so that no one's behavior is constrained by her cash balance.

As to preferences, assume that each person cares only about the amount of money she has and whether or not she has a horse. (In particular, no one cares about any other person's holdings.) Specifically, assume that each player i's preferences are represented by the payoff function

$$\begin{cases} v_i + r & \text{if she has a horse and \$}r\text{ more money than she had originally} \\ r & \text{if she has no horse and \$}r\text{ more money than she had originally.} \end{cases}$$

(This assumption does not mean that people do not value the money they have initially. Equivalently we could represent player i's preferences by the functions $v_i + r + m_i$ if she has a horse and $r + m_i$ if she does not, where m_i is the amount of money she has initially.)

The following coalitional game models, which I call a **horse trading game**, models the situation.

Players The group of people (owners and nonowners).

Actions The set of actions of each coalition S is the set of S-allocations of the horses and the total amount of money owned by S in which each player obtains at most one horse.

Preferences Each player's preferences are represented by the payoff function described above.

This game incorporates no restriction on the way in which a coalition may distribute its money and horses. In particular, players are not restricted to bilateral trades of money for horses. A coalition of two owners and two nonowners, for example, may, if it wishes, allocate each of the owners' horses to a nonowner and transfer money from *both* nonowners to only one owner, or from one nonowner to the other.

8.4.2 The core

Number the owners in ascending order and the nonowners in descending order of the valuations they attach to a horse. Figure 249.1 illustrates the valuations, ordered in this way. (This diagram should be familiar—perhaps it is a little too familiar—if you have studied economics.) Denote owner i's valuation σ_i and nonowner i's valuation β_i. Denote by k^* the largest number i such that $\beta_i > \sigma_i$ (so that among the owners and nonowners whose indices are k^* or less, every nonowner's valuation is greater than every owner's valuation).

Let a_N be an action in the core. Denote by L^* the set of owners who have no horse in a_N (the set of *sellers*) and by B^* the set of nonowners who have a horse in a_N (the set of *buyers*). These two sets must have the same number of members (by the law of conservation of horses). Denote by r_i the amount of money received by owner i and by p_j the amount paid by nonowner j in a_N.

I claim that $p_j = 0$ for every nonowner j not in B^*. (That is, no nonowner who does not acquire a horse either pays or receives any money.)

- If $p_j > 0$ for some nonowner j not in B^* then her payoff is negative, and she can unilaterally improve upon a_N by retaining her original money.

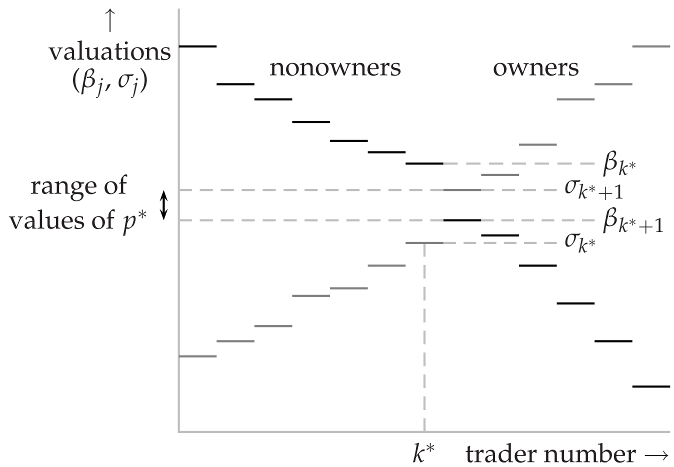

Figure 249.1 An example of the players' valuations in a market with an indivisible good. The buyers' valuations are given in black, and the sellers' in gray.

- If $p_j < 0$ for some nonowner j not in B^* then the coalition of all players *other than* j has p_j less money than it owned initially, and the same number of horses. Thus this coalition can improve upon a_N by assigning horses in the same way as they are assigned in a_N and giving each of its members $p_j/(n-1)$ more units of money than she gets in a_N (where n is the total number of players).

By a similar argument, $r_i = 0$ for every owner not in L^* (an owner who does not sell her horse neither pays nor receives any money.)

I now argue that in a_N every seller (member of L^*) receives the same amount of money, every buyer (member of B^*) pays the same amount of money, and these amounts are equal: $r_i = p_j$ for every seller i and buyer j. That is, all trades occur at the same price.

Suppose that $r_i < p_j$ for seller i and buyer j. I argue that the coalition $\{i, j\}$ can improve upon a_N: i can sell her horse to j at a price between r_i and p_j. Under a_N, seller i's payoff is r_i and buyer j's payoff is $\beta_j - p_j$. If i sells her horse to j at the price $\frac{1}{2}(r_i + p_j)$ then her payoff is $\frac{1}{2}(r_i + p_j) > r_i$ and j's payoff is $\beta_j - \frac{1}{2}(r_i + p_j) > \beta_j - p_j$, so that both i and j are better off than they are in a_N. Thus $r_i \geq p_j$ for every seller i and every buyer j.

Now, the sum of all the amounts r_i received by sellers is equal to the sum of all the amounts p_j paid by buyers (by the law of conservation of money), and L^* and B^* have the same number of members. Thus we have $r_i = p_j$ for every seller i in L^* and buyer j in B^*.

In summary,

> for every action a_N in the core there exists p^* such that $r_i = p_i = p^*$ for every owner i in L^* and every nonowner j in B^*, and $r_i = p_i = 0$ for

every owner not in L^* and every nonowner j not in B^*.

I now argue that the common price p^* at which all trades take place lies in a narrow range.

In a_N, every owner i whose valuation of a horse is less than p^* must sell her horse: if she did not then the coalition consisting of herself and any nonowner j who buys a horse in a_N could improve upon a_N by taking the action in which j buys i's horse at a price between the owner's valuation and p^*. Also, no owner whose valuation exceeds p^* trades, because her payoff from doing so is negative. Similarly, every nonowner whose valuation is greater than p^* buys a horse, and no nonowner whose valuation is less than p^* does so.

? EXERCISE 250.1 (Range of prices in horse market) Show that the requirement that the number of owners who sell their horses must equal the number of nonowners who buy horses, together with the arguments above, implies that the common trading price p^* is at least σ_{k^*}, at least β_{k^*+1}, at most β_{k^*}, and at most σ_{k^*+1}. That is, $p^* \geq \max\{\sigma_{k^*}, \beta_{k^*+1}\}$ and $p^* \leq \min\{\beta_{k^*}, \sigma_{k^*+1}\}$.

Finally, I argue that in any action in the core a player whose valuation is equal to p^* trades. Suppose nonowner i's valuation is equal to p^*. Then owner i's valuation is less than p^* and owner $i+1$'s valuation is greater than p^* (given my assumption that no two players have the same valuation), so that exactly i owners trade. Thus exactly i nonowners must trade, implying that nonowner i trades. Symmetrically, a owner whose valuation is equal to p^* trades.

In summary, in every action in the core of a horse trading game,

- every nonowner pays the same price for a horse
- the common price is at least $\max\{\sigma_{k^*}, \beta_{k^*+1}\}$ and at most $\min\{\beta_{k^*}, \sigma_{k^*+1}\}$
- every owner whose valuation is at most the price trades her horse
- every nonowner whose valuation is at least the price obtains a horse.

(250.2)

The action satisfying these conditions for the price p^* yields the payoffs

$$\begin{cases} \max\{v_i, p^*\} & \text{for every owner } i \\ \max\{v_i, p^*\} - p^* & \text{for every nonowner } i. \end{cases}$$

The core does not impose any additional restrictions on the actions of the grand coalition: every action that satisfies these conditions is in the core. To establish this result, I need to show that for any action a_N that satisfies the conditions, no coalition has an action that is better for all its members. When a coalition deviates, which of its actions has the best chance of improving upon a_N? The optimal action definitely assigns the coalition's horses to the members who value a horse most highly. (If $v_i < v_j$ then the transfer of a horse from i to j, accompanied by the transfer from j to i of an amount of money between v_i and v_j makes both i and j better off.) No transfer of money makes anyone better off without making someone worse off, so in order for a coalition to improve upon a_N there must be some distribution of the total amount of money it owns that, given the optimal distribution of horses, makes all its members better off than they are in a_N. For every distribution of a coalition's money the total payoff of the members of the coalition is the same. Thus a coalition can improve upon a_N if and only if the total payoff of its members under a_N is less than its total payoff when it assigns its horses optimally.

Consider an arbitrary coalition S. Denote by ℓ the total number of owners in S, by b the total number of nonowners in S, and by S^* the set of ℓ members of S whose valuations are highest. Then S's total payoff when it assigns its horses optimally is

$$\sum_{i \in S^*} v_i,$$

whereas its total payoff under a_N is

$$\sum_{i \in S} \max\{v_i, p^*\} - bp^* = \sum_{i \in S^*} \max\{v_i, p^*\} + \sum_{i \in S \setminus S^*} \max\{v_i, p^*\} - bp^*,$$

where $S \setminus S^*$ is the set of members of S *not* in S^*. The former is never higher than the latter because $S \setminus S^*$ has b members, so that $\sum_{i \in S \setminus S^*} \max\{v_i, p^*\} - bp^* \geq 0$.

In summary, the core of a horse trading game is the set of actions of the grand coalition that satisfies the four conditions in (250.2).

? EXERCISE 251.1 (Horse trading game with single seller) Find the core of the variant of the horse trading game in which there is a single owner, whose valuation is less than the highest valuation of the nonowners.

If you have studied economics you know that this outcome is the same as the "competitive equilibrium". The theories differ, however. The theory of competitive equilibrium *assumes* that all trades take place at the same price. It defines an equilibrium price to be one at which "demand" (the total number of nonowners whose valuations exceed the price) is equal to "supply" (the total number of owners whose valuations are less than the price). This equilibrium may be justified by the argument that if demand exceeds supply then the price will tend to rise, and if supply exceeds demand it will tend to fall. Thus in this theory, "market pressures" generate an equilibrium price; no agent in the market chooses a price.

By contrast, the coalitional game we have studied models the players' actions explicitly; each group may exchange its horses and money in any way it wishes. The core is the set of actions of all players that survives the pressures imposed by the trading opportunities of each possible group of players. A uniform price is not assumed, but is shown to be a necessary property of any action in the core.

? EXERCISE 251.2 (Horse trading game with large seller) Consider the variant of the horse trading game in which there is a single owner who has two horses. Assume that the owner's payoff is $\sigma_1 + r$ if she keeps one of her horses and $2\sigma_1 + r$ if she keeps both of them, where r is the amount of money she receives. Assume that there are at least two nonowners, both of whose values of a horse exceed σ_1. Find the core of this game. (Do all trades take place at the same price, as they do in a competitive equilibrium?)

8.5 Illustration: exchanging heterogeneous houses

8.5.1 Model

Each member of a group of n people owns an indivisible good—call it a house. Houses, unlike the horses of the previous section, differ. Any subgroup may reallocate its members' houses in any way it wishes (one house to each person). (Time-sharing and other devices to evade the indivisibility of a house are prohibited.) Each person cares only about the house she obtains, and has a strict ranking of the houses (she is not indifferent between any two houses).

Which assignments of houses to people are stable? You may think that without imposing any restrictions on the nature or diversity of preferences, this question is hard to answer, and that for some sufficiently conflicting configurations of preferences no assignment is stable. If so, you are wrong on both counts, at least as far as the core is concerned; remarkably, for *any* preferences, a slight variant of the core yields a *unique* stable outcome.

The following coalitional game, which I call a **house exchange game**, models the situation.

Players The n people.

Actions The set of actions of a coalition S is the set of all assignments to members of S of the houses originally owned by members of S.

Preferences Each player prefers one outcome to another according to the house she is assigned.

8.5.2 The top trading cycle procedure and the core

One property of an action in the core is immediate: any player who initially owns her favorite house obtains that house in any assignment in the core, because every player has the option of simply keeping the house she initially owns.

This property allows us to completely analyze the simplest nontrivial example of the game, with two people. Denote the person who initially owns player i's favorite house by $o(i)$.

- If at least one person initially owns her favorite house (i.e. if $o(1) = 1$ or $o(2) = 2$), then the core contains the single assignment in which each person keeps the house she owns.

- If each person prefers the house owned by the other person (i.e. if $o(1) = 2$ and $o(2) = 1$), then the core contains the single assignment in which the two people exchange houses.

In the second case we say that "12 is a 2-cycle". When there are more players, longer cycles are possible. For example, if there are three or more players and $o(i) = j$, $o(j) = k$, and $o(k) = i$, then we say that "ijk is a 3-cycle". (If $o(i) = i$, we can think of i as a "1-cycle".)

The case in which there are three people raises some new possibilities.

- If at least two people initially own their favorite houses, then the core contains the single assignment in which each person keeps the house she initially owns.

- If exactly one person, say player i, initially owns her favorite house, then in any assignment in the core, that person keeps her house. Whether the other two people exchange their houses depends on their preferences over these houses, ignoring player i's house (which has already been assigned); the analysis is the same as that for the two-player game.

- If no person initially owns her favorite house, there are two cases.
  - If there is a 2-cycle (i.e. if there exist persons i and j such that j initially owns i's favorite house and i initially owns j's favorite house), then the only assignment in the core is that in which i and j swap houses and the remaining player keeps the house she owns initially.
  - Otherwise, suppose that $o(i) = j$. Then $o(j) = k$, where k is the third player (otherwise ij is a 2-cycle), and $o(k) = i$ (otherwise kj is a 2-cycle.) That is, ijk is a 3-cycle. Consider the assignment in which i gets j's house, j gets k's house, and k gets i's house. Every player is assigned her favorite house, so the assignment is in the core. (This argument does not show that the core contains no other assignments.)

This construction of an assignment in the core can be extended to games with any number of players. First we look for cycles among the houses at the top of the players' rankings, and assign to each member of each cycle her favorite house. (If there are at most three players, only one cycle containing more than one player may exist, but if there are more players, many cycles may exist.) Then we eliminate from consideration the players involved in these cycles and the houses they are allocated, look for any cycles at the top of the remains of the players' rankings, and assign to each member of each of these cycles her favorite house among those remaining. We continue in the same manner until all players are assigned houses. This procedure is called the *top trading cycle procedure*.

To illustrate the procedure, consider the game with four players whose preferences satisfy the specification in Figure 254.1. In this figure, h_i denotes the house owned by player i and the players' rankings are listed from best to worst, starting at the top (player 3 prefers player 1's house to player 2's house to player 4's house, for example). Hyphens indicate irrelevant parts of the rankings. We see that 12 is a 2-cycle, so at the first step players 1 and 2 are assigned their favorite houses (h_2 and h_1 respectively). After eliminating these players and their houses, 34 becomes a 2-cycle, so that player 3 is assigned h_4 and player 4 is assigned h_3. If player 3's ranking of h_3 and h_4 were reversed then at the second stage 3 would be a one-cycle, so that player 3 would be assigned h_3, and then at the third stage player 4 would be assigned h_4.

Player 1	Player 2	Player 3	Player 4
h_2	h_1	h_1	h_3
-	-	h_2	h_2
-	-	h_4	h_4
-	-	h_3	-

Figure 254.1 A partial specification of the players' preferences in a game with four players, illustrating the top trading cycle procedure. Each player's ranking is given from best to worst, reading from top to bottom. Hyphens indicate irrelevant parts of the rankings.

? EXERCISE 254.1 (House assignment with identical preferences) Find all the assignments in the core of the n-player game in which every player ranks the houses in the same way.

I now argue that

> for any (strict preferences), the core of a house exchange game contains the assignment induced by the top trading cycle procedure.

The following argument establishes this result. Every player assigned a house in the first round receives her favorite house, so that no coalition containing such a player can make all its members better off than they are in a_N. Now consider a coalition that contains players assigned houses in the second round, but no players assigned houses in the first round. Such a coalition does not own any of the houses assigned on the first round, so that its members who were assigned in the second round obtain their favorite houses *among the houses it owns*. Thus such a coalition has no action that makes all its members better off than they are in a_N. A similar argument applies to coalitions containing players assigned in later rounds.

8.5.3 The strong core

I remarked that my analysis of a three-player game does not establish the existence of a *unique* assignment in the core. Indeed, consider the preferences in Figure 255.1. We see that 123 is a 3-cycle, so that the top cycle trading procedure generates the assignment in which each player receives her favorite house.

Player 1	Player 2	Player 3
h_3	h_1	h_2
h_2	h_2	h_3
h_1	h_3	h_1

Figure 255.1 The players' preferences in a game with three players. Each player's ranking is given from best to worst, reading from top to bottom.

I claim that the alternative assignment a'_N, in which player 1 obtains h_2, player 2 obtains h_1, and player 3 obtains h_3 is also in the core. Player 2 obtains her favorite house, so no coalition containing her can improve upon a'_N. Neither player 1 nor player 3 alone can improve upon a'_N because player 1 prefers h_2 to h_1 and player 3 obtains the house she owns. The only remaining coalition is $\{1,3\}$, which owns h_1 and h_3. If it deviates and assigns h_1 to player 1 then she is worse off than she is in a'_N, and if it deviates and assigns h_1 to player 3 then she is worse off than she is in a'_N. Thus no coalition can improve upon a'_N.

Although no coalition S can achieve any S-allocation that makes all of its members better off than they are in a'_N, the coalition N of all three players *can* make two of its members (players 1 and 3) better off, while keeping the remaining member (player 2) with the same house. That is, it can "weakly" improve upon a'_N.

This example suggests that if we modify the definition of the core so that actions upon which any coalition can weakly improve are eliminated, we might reduce the core to a single assignment.

Define the *strong core* of any game to be the set of actions a_N of the grand coalition N such that no coalition S has an action a_S that some of its members prefer to a_N and all of its members regard to be at least as good as a_N.

The argument I have given shows that the action a'_N is not in the strong core of the game in which the players' preferences are given in Figure 255.1, though it is in the core. In fact,

> for any (strict) preferences, the strong core of a house exchange game consists of the single assignment defined by the top cycle trading procedure.

I omit details of the argument for this result. The result shows that the (strong) core is a highly successful solution for house exchange games; for *any* (strict) preferences, it pinpoints a *single* stable assignment, which is the outcome of a simple, intuitively appealing, procedure.

Unfortunately, the strengthening of the definition of the core has a side effect: if we depart from the assumption that all preferences are strict, and allow players to be indifferent between houses, then the core may be empty. The next exercise gives an example.

? EXERCISE 255.1 (Emptiness of the strong core when preferences are not strict) Suppose that some players are indifferent between some pairs of houses. Specifically, suppose there are three players, whose preferences are given in Figure 256.1. Find the core and show that the strong core is empty.

Player 1	Player 2	Player 3
h_2	h_1, h_3	h_2
h_1, h_3	h_2	h_1, h_3

Figure 256.1 The players' preferences in the game in Exercise 255.1. A cell containing two houses indicates indifference between these two houses.

8.6 Illustration: voting

A group of people chooses a policy by majority voting. How does the chosen policy depend on their preferences? In Chapter 2 we studied a strategic game that models this situation and found that the notion of Nash equilibrium admits a very wide range of stable outcomes. In a Nash equilibrium no single player, by changing her vote, can improve the outcome for herself, but a group of players, by coordinating their votes, may be able to do so. By modeling the situation as a coalitional game and using the notion of the core to isolate stable outcomes, we can find the implications of group deviations for the outcome.

To model voting as a coalitional game, the specification I have given of such a game needs to be slightly modified. Recall that an outcome of a coalitional game is a partition of the set of players and an aciton for each coalition in the partition. So far I have assumed that each player cares only about the action chosen by the coalition in the partition to which she belongs. This assumption means that the payoff of a coalition that deviates from an outcome is determined independently of the action of any other coalition; when deviating, a coalition does not have to consider the action that any other coalition takes. In the situation I now present, a different constellation of conditions has the same implication: only coalitions containing a majority of the players have more than one possible action, and every player cares only about the action chosen by the majority coalition (of which there is at most one) in the outcome partition. In brief, any majority may choose an action that affects everyone, and every minority is powerless.

Precisely, assume that there is an odd number of players, each of whom has preferences over a set of *policies* and prefers the outcome x to the outcome y if and only if either there are majority coalitions in the partitions associated with both x and y and she prefers the action chosen by the majority coalition in x to the action chosen by the majority coalition in y, or there is a majority coalition in x by not in y. (If there is a majority coalition in neither x nor y, she is indifferent between x and y.) The set of actions available to any coalition containing a majority of the players is the set of all policies; every other coalition has a single action.

The definition of the core of this variant of a coalitional game is the natural variant of Definition 239.1: the set of actions a_N of the grand coalition N such that no majority coalition has an action that all its members prefer to a_N.

Suppose that the policy x is in the core of this game. Then no policy is preferred to x by a coalition consisting of a majority of the players. Equivalently, for every policy $y \neq x$, the set of players who either prefer x to y or regard x and y to be equally good is a majority. If we assume that every player's preferences are strict—no player is indifferent between any two policies—then for every policy $y \neq x$, the set of players who prefer x to y is a majority. That is, x is a Condorcet winner (see Exercise 74.1). For any preferences, there is at most one Condorcet winner, so we have established that

> if every player's preferences are strict, the core of a majority voting game is empty if there is no Condorcet winner, and otherwise is the set consisting of the single Condorcet winner.

How does the existence and character of a Condorcet winner depend on the players' preferences? First suppose that a policy is a number. Assume that each player i has a favorite policy x_i^*, and that her preferences are *single-peaked*: if x and x' are policies for which $x < x' < x_i^*$ or $x_i^* < x' < x$ then she prefers x' to x. Then the median of the players' favorite positions is the Condorcet winner, as you are asked to show in the next exercise, and hence the unique member of the core of the voting game. (The median is well-defined because the number of players is odd.)

? EXERCISE 257.1 (Median voter theorem) Show that when the policy space is one-dimensional and the players' preferences are single-peaked the unique Condorcet winner is the median of the players' favorite positions. (This result is known as the *median voter theorem*.)

A one-dimensional space captures some policy choices, but in other situations a higher dimensional space is needed. For example, a government has to choose the amounts to spend on health care and defense, and not all citizens' preferences are aligned on these issues. Unfortunately, for most configurations of the players' preferences, a Condorcet winner does not exist in a policy space of two or more dimensions, so that the core is empty.

To see why this claim is plausible, suppose the policy space is two-dimensional and there are three players. Place the players' favorite positions at three arbitrary points, like x_1^*, x_2^*, and x_3^* in Figure 258.1. Assume that each player i's distaste for a position x different from her favorite position x_i^* is exactly the distance between x and x_i^*, so that for any value of r she is indifferent between all policies on the circle with radius r centered at x_i^*.

Now choose any policy and ask if it is a Condorcet winner. The policy x in the figure is not, because any policy in the shaded area is preferred to x by players 1 and 2, who constitute a majority. The policy x is also beaten in a majority vote by any policy in either of the other lens-shaped areas defined by the intersection of the circles centered at x_1^*, x_2^*, and x_3^*. Is there any policy for which no such lens-shaped

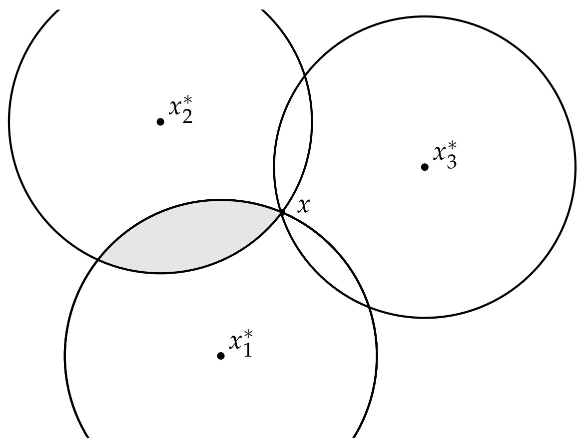

Figure 258.1 A two-dimensional policy space with three players. The point x_i^* is the favorite position of player i for $i = 1, 2, 3$. Every policy in the shaded lens is preferred by players 1 and 2 to x.

area is created? By checking a few other policies you can convince yourself that there is no such policy. That is, no policy is a Condorcet winner, so that the core of the game is empty.

For *some* configurations of the players' favorite positions a Condorcet winner exists. For example, if the positions lie on a straight line then the middle one is a Condorcet winner. But only very special configurations yield a Condorcet winner—in general there is none, so that the core is empty, and our analysis suggests that no policy is stable under majority rule when the policy space is multidimensional.

In some situations in which policies are determined by a vote, a decision requires a positive vote by more than a simple majority. For example, some jury verdicts in the USA require unanimity, and changes in some organizations' and countries' constitutions require a two-thirds majority. To study the implications of these alternative voting rules, fix q with $n/2 \leq q \leq n$ and consider a variant of the majority-rule game that I call the *q-rule game*, in which the only coalitions that can choose policies are those containing at least q players. Roughly, the larger is the value of q, the larger is the core. You are invited to explore some examples in the next exercise.

? EXERCISE 258.1 (Cores of q-rule games)

a. Suppose that the set of policies is one-dimensional and that each player's preferences are single-peaked. Find the core of the q-rule game for any value of q with $n/2 \leq q \leq n$.

b. Find the core of the q-rule game when $q = 3$ in the example in Figure 258.1 (with a two-dimensional policy space and three players).

8.7 Illustration: matching

Applicants must be matched with universities, workers with firms, and football players with teams. Do stable matchings exist? If so, what are their properties, and which institutions generate them?

In this section I analyze a model of two-sided one-to-one matching: each party on one side must be matched with exactly *one* party on the other side. Most of the main ideas that emerge apply also to many-to-one matching problems.

The model I analyze is sometimes referred to as one of "marriage", though of course it captures only one dimension of matrimony. Some of the language I use is taken from this interpretation of the model.

8.7.1 Model

I refer to the two sides as X's and Y's. Each X may be matched with at most one Y, and each Y may be matched with at most one X; staying single is an option for each individual. A *matching* of any set of individuals thus splits the set into pairs, each consisting of an X and a Y, and single individuals. I denote the partner of any player i under the matching μ by $\mu(i)$. If i and j are matched, we thus have $\mu(i) = j$ and $\mu(j) = i$; if i is single then $\mu(i) = i$. Each person cares only about her partner, not about anyone else's partner. Assume that every person's preferences are *strict*: no person is indifferent between any two partners. I refer to the set of partners that i prefers to the option of remaining single as the set of i's *acceptable* partners. The following coalitional game, which I refer to as a **two-sided one-to-one matching game**, models this situation.

Players The set of all X's and all Y's.

Actions The set of actions of a coalition S is the set of all matchings of the members of S.

Preferences Each player prefers one outcome to another according to the partner she is assigned.

An example of possible preferences is given in Figure 260.1. For instance, player x_1 ranks y_2 first, then y_1, and finds y_3 unacceptable.

8.7.2 The core and the deferred acceptance procedure

A matching in the core of a two-sided one-to-one matching game has the property that no group of players may, by rearranging themselves, produce a matching that they all like better. I claim that when looking for matchings in the core, we may restrict attention to coalitions consisting either of a single individual or of one X and one Y. Precisely, a matching is in the core if and only if

a. each player prefers her partner to being single

X's			Y's		
x_1	x_2	x_3	y_1	y_2	y_3
y_2	y_1	y_1	x_1	x_2	x_1
y_1	y_2	y_2	x_3	x_1	x_3
	y_3		x_2	x_3	x_2

Figure 260.1 An example of the players' preferences in a two-sided one-to-one matching game. Each column gives one player's ranking (from best to worst) of all the players of the other type that she finds acceptable.

b. for no pair (i, j) consisting of an X and a Y is it the case that i prefers j to $\mu(i)$ and j prefers i to $\mu(j)$.

The following argument establishes this claim. First, any matching μ that does not satisfy the conditions is not in the core: if (*a*) is violated then some player can improve upon μ by staying single, and if (*b*) is violated then some pair of players can improve upon μ by matching with each other. Second, suppose that μ is not in the core. Then for some coalition S there is a matching μ' of its members for which every member i prefers $\mu'(i)$ to $\mu(i)$. If S consists of a single individual, then (*a*) is violated. Otherwise suppose that i is a member of S, and let $j = \mu'(i)$, so that $i = \mu'(j)$. Then i prefers j to $\mu(i)$ and j prefers i to $\mu(j)$. Thus (*b*) is violated.

In the game in which the players' preferences are those given in Figure 260.1, for example, the matching μ in which $\mu(x_1) = y_1$, $\mu(x_2) = y_2$, $\mu(x_3) = x_3$, and $\mu(y_3) = y_3$ (i.e. x_3 and y_3 stay single) is in the core, by the following argument. No single player can improve upon it, because every matched player's partner is acceptable to her. Now consider pairs of players. No pair containing x_3 or y_3 can improve upon the matching, because x_1 and x_2 are matched with partners they prefer to y_3, and y_1 and y_2 are matched with partners they prefer to x_3. A matched pair cannot improve upon the matching either, so the only pairs to consider are $\{x_1, y_2\}$ and $\{x_2, y_1\}$. The first cannot improve upon μ because y_2 prefers x_2, with whom she is matched, to x_1; the second cannot upon μ because y_1 prefers x_1, with whom she is matched, to x_2.

How may matchings in the core be found? As in the case of the market for houses studied in Section 8.5, one member of the core is generated by an intuitively appealing procedure. (In contrast to the core of the house market, however, the core of a two-sided one-to-one matching game may contain more than one action, as we shall see.)

The procedure comes in two flavors, one in which proposals are made by X's, and one in which they are made by Y's. The *deferred acceptance procedure with proposals by X's* is defined as follows. Initially, each X proposes to her favorite Y, and each Y either rejects all the proposals she receives, if none is from an X acceptable to her, or rejects all but the best proposal (according to her preferences). Each proposal that is not rejected results in a tentative match between an X and a Y. If every offer is accepted, the process ends, and the tentative matches become definite. Otherwise, there is a second stage in which each X whose proposal was rejected in the first stage proposes to the Y she ranks second, and each Y chooses among the set of X's who proposed to her *and* the one with whom she was tentatively matched in the first stage, rejecting all but her favorite among these X's. Again, if every offer is accepted, the process ends, and the tentative matches become definite, whereas if some offer is rejected, there is another round of proposals.

Precisely, each stage has two steps, as follows.

1. Each X (*a*) whose offer was rejected at the previous stage and (*b*) for whom some Y is acceptable, proposes to her top-ranked Y out of those who have not previously rejected an offer from her.

2. Each Y rejects the proposal of any X who is unacceptable to her, and is "engaged" to the X she likes best in the set consisting of all those who proposed to her and the one to whom she was previously engaged.

The procedure stops when the proposal of no X is rejected or when every X whose offer was rejected has run out of acceptable Y's.

Consider, for example, the preferences in Figure 260.1. The progress of the procedure is shown in Figure 261.1, in which "$\rightarrow$" stands for "proposes to". First x_1 proposes to y_2 and both x_2 and x_3 propose to y_1; y_1 rejects x_2's proposal. Then x_2 proposes to y_2, so that y_2 may choose between x_2 and x_1 (with whom she was tentatively matched at the first stage). Player y_2 chooses x_2, and rejects x_1, who then proposes to y_1. Player y_1 now chooses between x_1 and x_3 (with whom she was tentatively matched at the first stage), and rejects x_3. Finally, x_3 proposes to y_2, who rejects her offer. The final matching is thus (x_1, y_1), (x_2, y_2), x_3 (alone), and y_3 (alone).

	Stage 1		Stage 2		Stage 3		Stage 4	
x_1:	$\rightarrow y_2$			reject	$\rightarrow y_1$			
x_2:	$\rightarrow y_1$	reject	$\rightarrow y_2$					
x_3:	$\rightarrow y_1$					reject	$\rightarrow y_2$	reject

Figure 261.1 The progress of the deferred acceptance procedure with proposals by X's when the players' preferences are those given in Figure 260.1. Each row gives the proposals of one X.

For any preferences, the procedure eventually stops, because there are finitely many players. To show that the matching μ it produces is in the core we need to consider deviations by coalitions of only one or two players, by an earlier argument.

- No single player may improve upon μ because no X ever proposes to an unacceptable Y, and every Y always rejects every unacceptable X.

- Consider a coalition $\{i, j\}$ of two players, where i is an X and j is a Y. If i prefers j to $\mu(i)$, she must have proposed to j, and been rejected, before proposing to $\mu(i)$. The fact that j rejected her proposal means that j obtained a more desirable proposal. Thus j prefers $\mu(j)$ to i, so that $\{i, j\}$ cannot improve upon μ.

The analogous procedure in which proposals are made by Y's generates a matching in the core, by the same argument. For some preferences the matchings produced by the two procedures are the same, whereas for others they are different.

? EXERCISE 262.1 (Deferred acceptance procedure with proposals by Y's) Find the matching produced by the deferred acceptance procedure with proposals by Y's for the preferences given in Figure 260.1.

In particular, the core may contain more than one matching. It can be shown that the matching generated by the deferred acceptance procedure with proposals by X's yields each X her most preferred partner among all her partners in matchings in the core, and yields each Y her least preferred partner among all her partners in matchings in the core. Similarly, the matching generated by the deferred acceptance procedure with proposals by Y's yields each Y her most preferred partner among all her partners in matchings in the core, and yields each X her least preferred partner among all her partners in matchings in the core.

? EXERCISE 262.2 (Example of deferred acceptance procedure) Find the matchings produced by the deferred acceptance procedure both with proposals by X's and with proposals by Y's for the preferences given in Figure 262.1. Verify the results in the previous paragraph. (Argue that the only matchings in the core are the two generated by the procedures.)

x_1	x_2	x_3	y_1	y_2	y_3
y_1	y_1	y_1	x_1	x_1	x_1
y_2	y_2	y_3	x_2	x_3	x_2
y_3	y_3	y_2	x_3	x_2	x_3

Figure 262.1 The players' preferences in the game in Exercise 262.2.

In summary, every two-sided one-to-one matching game has a nonempty core, which contains the matching generated by each deferred acceptance procedure. The matching generated by the procedure is the best one in the core for the side making proposals, and the worst one in the core for the other side.

8.7.3 Variants

Strategic behavior So far, I have considered the deferred acceptance procedures only as algorithms that an administrator who *knows* the participants' preferences may use to find matchings in the core. Suppose the participants' preferences are *not* known. We may use the tools developed in Chapter 2 to study whether the participants' interests are served by revealing their true preferences. Consider the strategic game in which each player names a ranking of her possible partners and the outcome is the matching produced by the deferred acceptance procedure with proposals by X's, given the announced rankings. One can show that in this game each X's naming her true ranking is a dominant action, and although the equilibrium actions of Y's may *not* be their true rankings, the equilibrium of the game is in the core of the coalitional game defined by the players' *true* rankings.

? EXERCISE 263.1 (Strategic behavior under a deferred acceptance procedure) Consider the preferences in Figure 263.1. Find the matchings produced by the deferred acceptance procedures, and show that the core contains no other matchings. Consider the strategic game described in the previous paragraph that is induced by the procedure with proposals by X's. Take as given that each X's naming her true ranking is a dominant strategy. Show that the game has a Nash equilibrium in which y_1 names the ranking (x_1, x_2, x_3) and every other player names her true ranking.

x_1	x_2	x_3	y_1	y_2	y_3
y_2	y_1	y_1	x_1	x_3	x_1
y_1	y_3	y_2	x_3	x_1	x_3
y_3	y_2	y_3	x_2	x_2	x_2

Figure 263.1 The players' preferences in the game in Exercise 263.1.

Other matching problems I motivated the topic of matching by citing the problems of matching applicants with universities, workers with firms, and football players with teams. All these problems are many-to-one rather than one-to-one. Under mild assumptions about the players' preferences, the results I have presented for one-to-one matching games hold, with minor changes, for many-to-one matching games. In particular, the strong core (defined on page 255) is nonempty, and a variant of the deferred acceptance procedure generates matchings in it.

At this point you may suspect that the nonemptiness of the core in matching games is a very general result. If so, the next exercise shows that your suspicion is incorrect—at least, if "very general" includes the "roommate problem".

? EXERCISE 263.2 (Empty core in roommate problem) An even number of people have to be split into pairs; any person may be matched with any other person. (The matching problem is "one-sided".) Consider an example in which there are four people, i, j, k, and ℓ. Show that if the preferences of i, j, and k are those given in Figure 264.1 then for any preferences of ℓ the core is empty. (Notice that ℓ is the least favorite roommate of every other player.)

i	j	k
j	k	i
k	i	j
ℓ	ℓ	ℓ

Figure 264.1 The preferences of players i, j, and k in the game in Exercise 263.2.

EXERCISE 264.1 (Spatial preferences in roommate problem) An even number of people have to be split into pairs. Each person's characteristic is a number; no two characteristics are the same. Each person would like to have a roommate whose characteristic is as close as possible to her own, and prefers to be matched with the most remote partner to remaining single. Find the set of matchings in the core.

MATCHING DOCTORS WITH HOSPITALS

Around 1900, newly-trained doctors in the USA were first given the option of working as "interns" (now called "residents") in hospitals, where they gain experience in clinical medicine. Initially, hospitals advertised positions, for which newly-trained doctors applied. The number of positions exceeded the supply of doctors, and the competition between hospitals for interns led the date at which agreements were finalized to retreat. By 1944, student doctors were finalizing agreements two full years before their internships were to begin. Making agreements at such an early date was undesirable for hospitals, who at that point lacked extensive information about the students.

The American Association of Medical Colleges attempted to solve the problem by having its members agree not to release any information about students before the end of their third year (of a four-year program). This change prevented hospitals from making earlier appointments, but in doing so brought to the fore the problem of coordinating offers and acceptances. Hospitals wanted their first-choice students to accept quickly, but students wanted to delay as much as possible, hoping to receive better offers. In 1945, hospitals agreed to give students 10 days to consider offers. But there was pressure to reduce this period. In 1949 a 12-hour period was rejected by the American Hospital Association as too long; it was agreed that all offers be made at 12:01AM on November 15, and hospitals could insist on a response within any period. Forcing students to make decisions without having a chance to collect offers from hospitals whose first-choice students had rejected them obviously led to inefficient matches.

These difficulties with efficiently matching doctors with hospitals led to the design of a centralized matching procedure that combines hospitals' rankings of students and students' rankings of hospitals to produce an assignment of students to hospitals. It can be shown that this procedure, designed ten years before Gale and Shapley's work on the deferred acceptance procedure, generates a matching in the core for any stated preferences! It differs from the natural generalization of Gale and Shapley's deferred acceptance procedure to a many-to-one matching problem, but generates precisely the same matching, namely the one in the core that is best for the hospitals. (Gale and Shapley, and the designers of the student–hospital matching procedure were not aware of each other's work until the mid-1970s, when a physician heard Gale speak on his work.)

In the early years of operation of the procedure, over 95% of students and hospitals participated. In the mid-1970s the participation rate fell to around 85%. Many nonparticipants were married couples both members of which wished to obtain positions in the same city. The matching procedure contained a mechanism for dealing with married couples, but, unlike the mechanism for single students, it could lead to a matching upon which some couple could improve. The difficulty is serious: when couples exist who restrict themselves to accept positions in the same city, for some preferences the core of the resulting game is empty—no matching is stable.

Further problems arose. In the 1990s, associations of medical students began to argue that changes were needed because the procedure was favorable to hospitals, and possibilities for strategic behavior on the part of students existed. The game theorist Alvin E. Roth was retained by the "National Resident Matching Program" to design a new procedure to generate stable matchings that are as favorable as possible to applicants. The new procedure was first used in 1998; it matches around 20,000 new doctors with hospitals each year.

8.8 Discussion: other solution concepts

In replacing the requirement of a Nash equilibrium that no individual player may profitably deviate with the requirement that no *group* of players may profitably deviate, the notion of the core makes an assumption that is unnecessary when interpreting a Nash equilibrium. A single player who deviates from an action profile in a strategic game can be *sure* of her deviant action, because she unilaterally chooses it. But a member of a group of players that chooses a deviant action must assume that no subgroup of her comrades will deviate further, or, at least, she will remain better off if they do.

Consider, for example, the three-player majority game (Example 237.2 and Exercise 241.1). The action $(\frac{1}{2}, \frac{1}{2}, 0)$ of the grand coalition in this game is not in the core because, for example, the coalition consisting of players 1 and 3 can take an action that gives player 1 an amount x with $\frac{1}{2} < x < 1$ and player 3 the amount $1 - x$, which leads to the payoff profile $(x, 0, 1 - x)$. But this profile itself is not stable—the coalition consisting of players 2 and 3, for example, has an action that generates the payoff profile $(0, y, 1 - y)$, where $0 < y < x$, in which both of them are better off than they are in $(x, 0, 1 - x)$. The fact that player 3 will be tempted by an offer of player 2 to deviate from $(x, 0, 1-x)$ may dampen player 1's enthusiasm for joining player 3 in the deviation from $(\frac{1}{2}, \frac{1}{2}, 0)$. For similar reasons, player 2 may be reluctant to join in a deviation from this action.

Several solution concepts that take into account these considerations have been suggested. None has so far had anything like the success of the core in illuminating social and economic phenomena, however.

Notes

The notion of a coalitional game is due to von Neumann and Morgenstern (1944). Shapley and Shubik (1953), Luce and Raiffa (1957, 234–235), and Aumann and Peleg (1960) generalized von Numann and Morgenstern's notion. The notion of the core was introduced in the early 1950s by Gillies as a tool to study another solution concept (his work is published in Gillies 1959); Shapley and Shubik developed it as a solution concept.

Edgeworth (1881, 35–39) pointed out a connection between the competitive equilibria of a market model and the set of outcomes we now call the core. von Neumann and Morgensten (1944, 583–584) first suggested modeling markets as coalitional games; Shubik (1959a) recognized the game-theoretic content of Edgeworth's arguments and, together with Shapley (1959), developed the analysis. Section 8.3 is based on Shapley and Shubik (1967). The core of the market studied in Section 8.4 was first studied by Shapley and Shubik (1971/72). My discussion owes a debt to Moulin (1995, Section 2.3).

Voting behavior in committees was first studied formally by Black (1958) (written in the mid-1940s), Black and Newing (1951), and Arrow (1951). Black used the core as the solution (before it had been defined generally) and established the median voter theorem (Exercise 257.1). He also noticed that in policy spaces of dimension greater than 1 a Condorcet winner is not likely to exist, a result extended by Plott (1967) and refined by Banks (1995) and others, who find conditions relating the number of voters, the dimension of the policy space, and the value of q for which the core of the q-rule game is generally empty; see Austen-Smith and Banks (1999, Secton 6.1) for details.

The model and result on the nonemptiness of the core in Section 8.5 are due to Shapley and Scarf (1974), who credit David Gale with the top trading cycle procedure. The result that the strong core contains a single action is due to Roth and Postlewaite (1977). The model is discussed in detail by Moulin (1995, Section 3.2).

The model and main results in Section 8.7 are due to Gale and Shapley (1962). The result about the strategic properties of the deferred acceptance procedures at the end of the section is a combination of results due to Dubins and Freedman (1981) and Roth (1982), and to Roth (1984a). Exercise 263.1 is based on an example in Moulin (1995, 113 and 116). Exercise 263.2 is taken from Gale and Shapley (1962, Example 3). For a comprehensive presentation of results on two-sided matching, see Roth and Sotomayor (1990). The box on page 264 is based on Roth (1984b), Roth and Sotomayor (1990, Section 5.4), and Roth and Peranson (1999).

9 Bayesian Games

Prerequisite: Chapter 2 and Section 4.1.3; Section 9.8 requires Chapter 4.

9.1 Introduction

An assumption underlying the notion of Nash equilibrium is that each player holds the correct belief about the other players' actions. To do so, a player must know the game she is playing; in particular, she must know the other players' preferences. In many situations the participants are not perfectly informed about their opponents' characteristics: bargainers may not know each others' valuations of the object of negotiation, firms may not know each others' cost functions, combatants may not know each others' strengths, and jurors may not know their colleagues' interpretations of the evidence in a trial. In some situations, a participant may be well informed about her opponents' characteristics, but may not know how well these opponents are informed about her own characteristics. In this chapter I describe the model of a "Bayesian game", which generalizes the notion of a strategic game to allows us to analyze any situation in which each player is imperfectly informed about some aspect of her environment relevant to her choice of an action.

9.2 Motivational examples

I start with two examples that illustrate the main ideas in the model of a Bayesian game. I define the notion of Nash equilibrium separately for each game. In the next section I define the general model of a Bayesian game and the notion of Nash equilibrium for such a game.

◆ Example 271.1 (Variant of *BoS* with imperfect information) Consider a variant of the situation modeled by *BoS* (Figure 16.1) in which player 1 is unsure whether player 2 prefers to go out with her or prefers to avoid her, whereas player 2, as before, knows player 1's preferences. Specifically, suppose player 1 thinks that with probability $\frac{1}{2}$ player 2 wants to got out with her, and with probability $\frac{1}{2}$ player 2 wants to avoid her. (Presumably this assessment comes from player 1's experience: half of the time she is involved in this situation she faces a player who wants to go out with her, and half of the time she faces a player who wants to avoid her.) That is, player 1 thinks that with probability $\frac{1}{2}$ she is playing the game on the left of Figure 272.1 and with probability $\frac{1}{2}$ she is playing the game on the right. Because probabilities are involved, an analysis of the situation requires us to know the players' preferences over lotteries, even if we are interested only in pure strategy equilibria; thus the numbers in the tables are Bernoulli payoffs.

1

prob. $\frac{1}{2}$

2

	B	S
B	$2,1$	$0,0$
S	$0,0$	$1,2$

2 wishes to meet 1

prob. $\frac{1}{2}$

2

	B	S
B	$2,0$	$0,2$
S	$0,1$	$1,0$

2 wishes to avoid 1

Figure 272.1 A variant of *BoS* in which player 1 is unsure whether player 2 wants to meet her or to avoid her. The frame labeled 2 enclosing each table indicates that player 2 knows the relevant table. The frame labeled 1 enclosing both tables indicates that player 1 does not know the relevant table; the probabilities she assigns to the two tables are printed on the frame.

We can think of there being two *states*, one in which the players' Bernoulli payoffs are given in the left table and one in which these payoffs are given in the right table. Player 2 knows the state—she knows whether she wishes to meet or avoid player 2—whereas player 1 does not; player 1 assigns probability $\frac{1}{2}$ to each state.

The notion of Nash equilibrium for a strategic game models a steady state in which each player's beliefs about the other players' actions are correct, and each player acts optimally, given her beliefs. We wish to generalize this notion to the current situation.

From player 1's point of view, player 2 has two possible *types*, one whose preferences are given in the left table of Figure 272.1, and one whose preferences are given in the right table. Player 1 does not know player 2's type, so to choose an action rationally she needs to form a belief about the action of each type. Given these beliefs and her belief about the likelihood of each type, she can calculate her expected payoff to each of her actions. For example, if she thinks that the type who wishes to meet her will choose B and the type who wishes to avoid her will choose S, then she thinks that B will yield her a payoff of 2 with probability $\frac{1}{2}$ and a payoff of 0 with probability $\frac{1}{2}$, so that her expected payoff is $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$, and S will yield her an expected payoff of $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$. Similar calculations for the other combinations of actions for the two types of player 2 yield the expected payoffs in Figure 273.1. Each column of the table is a pair of actions for the two types of player 2, the first member of each pair being the action of the type who wishes to meet player 1 and the second member being the action of the type who wishes to avoid player 1.

	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Figure 273.1 The expected payoffs of player 1 for the four possible pairs of actions of the two types of player 2 in Example 271.1.

For this situation we define a pure strategy *Nash equilibrium* to be a triple of actions, one for player 1 and one for each type of player 2, with the property that

- the action of player 1 is optimal, given the actions of the two types of player 2 (and player 1's belief about the state)
- the action of each type of player 2 is optimal, given the action of player 1.

That is, we treat the two types of player 2 as separate players, and analyze the situation as a three-player strategic game in which player 1's payoffs as a function of the actions of the two other players (i.e. the two types of player 2) are given in Figure 273.1, and the payoff of each type of player 2 is independent of the actions of the other type and depends on the action of player 1 as given in the tables in Figure 272.1 (the left table for the type who wishes to meet player 1, and the right table for the type who wishes to avoid player 1). In a Nash equilibrium, player 1's action is a best response in Figure 273.1 to the pair of actions of the two types of player 2, the action of the type of player 2 who wishes to meet player 1 is a best response in the left table of Figure 272.1 to the action of player 1, and the action of the type of player 2 who wishes to avoid player 1 is a best response in the right table of Figure 272.1 to the action of player 1.

Why should player 2, who knows whether she wants to meet or avoid player 1, have to plan what to do in both cases? She does not have to do so! But we, as analysts, need to consider what she does in both cases, because player 1, who does not know player 2's type, needs to think about the action each type would take; we would like to impose the condition that player 1's beliefs are correct, in the sense that for each type of player 2 they specify a best response to player 1's equilibrium action.

I claim that $(B,(B,S))$, where the first component is the action of player 1 and the other component is the pair of actions of the two types of player 2, is a Nash equilibrium. Given that the actions of the two types of player 2 are (B,S), player 1's action B is optimal, from Figure 273.1; given that player 1 chooses B, B is optimal for the type who wishes to meet player 2 and S is optimal for the type who wishes to avoid player 2, from Figure 272.1. Suppose that in fact player 2 wishes to meet player 1. Then we interpret the equilibrium as follows. Both player 1 and player 2 chooses B; player 1, who does not know if player 2 wants to meet her or avoid her, believes that if player 2 wishes to meet her she will choose B, and if she wishes to avoid her she will choose S.

? EXERCISE 274.1 (Equilibria of a variant of *BoS* with imperfect information) Show that there is no pure strategy Nash equilibrium of this game in which player 1 chooses S. If you have studied mixed strategy Nash equilibrium (Chapter 4), find the mixed strategy Nash equilibria of the game. (First check whether there is an equilibrium in which both types of player 2 use pure strategies, then look for equilibria in which one or both of these types randomize.)

We can interpret the actions of the two types of player 2 to reflect player 2's intentions in the hypothetical situation *before* she knows the state. We can tell the following story. Initially player 2 does not know the state; she is informed of the state by a *signal* that depends on the state. Before receiving this signal, she plans an action for each possible signal. After receiving the signal she carries out her planned action for that signal. We can tell a similar story for player 1. To be consistent with her not knowing the state when she takes an action, her signal must be uninformative: it must be the same in each state. Given her signal, she is unsure of the state; when choosing an action she takes into account her belief about the likelihood of each state, given her signal. The framework of states, beliefs, and signals is unnecessarily baroque in this simple example, but comes into its own in the analysis of more complex situations.

◆ EXAMPLE 274.2 (Variant of *BoS* with imperfect information) Consider another variant of the situation modeled by *BoS*, in which neither player knows whether the other wants to go out with her. Specifically, suppose that player 1 thinks that with probability $\frac{1}{2}$ player 2 wants to go out with her, and with probability $\frac{1}{2}$ player 2 wants to avoid her, and player 2 thinks that with probability $\frac{2}{3}$ player 1 wants to go out with her and with probability $\frac{1}{3}$ player 1 wants to avoid her. As before, assume that each player knows her own preferences.

We can model this situation by introducing four states, one for each of the possible configurations of preferences. I refer to these states as yy (each player wants to go out with the other), yn (player 1 wants to go out with player 2, but player 2 wants to avoid player 1), ny, and nn.

The fact that player 1 does not know player 2's preferences means that she cannot distinguish between states yy and yn, or between states ny and nn. Similarly, player 2 cannot distinguish between states yy and ny, and between states yn and nn. We can model the players' information by assuming that each player receives a *signal* before choosing an action. Player 1 receives the same signal, say y_1, in states yy and yn, and a different signal, say n_1, in states ny and nn; player 2 receives the same signal, say y_2, in states yy and ny, and a different signal, say n_2, in states yn and nn. After player 1 receives the signal y_1, she is referred to as *type* y_1 of player 1 (who wishes to go out with player 2); after she receives the signal n_1 she is referred to as *type* n_1 of player 1 (who wishes to avoid player 2). Similarly, player 2 has two *types*, y_2 and n_2.

Type y_1 of player 1 believes that the probability of each of the states yy and yn is $\frac{1}{2}$; type n_1 of player 1 believes that the probability of each of the states ny and nn is $\frac{1}{2}$. Similarly, type y_2 of player 2 believes that the probability of state yy is $\frac{2}{3}$ and that of state ny is $\frac{1}{3}$; type n_2 of player 2 believes that the probability of state yn is $\frac{2}{3}$ and that of state nn is $\frac{1}{3}$. This model of the situation is illustrated in Figure 275.1.

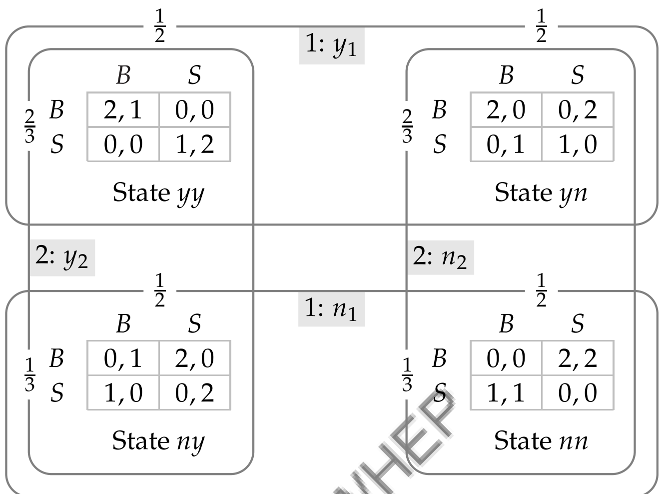

Figure 275.1 A variant of *BoS* in which each player is unsure of the other player's preferences. The frame labeled i: x encloses the states that generate the signal x for player i; the numbers printed over this frame next to each table are the probabilities that type x of player i assigns to each state that she regards to be possible.

As in the previous example, to study the equilibria of this model we consider the players' plans of action before they receive their signals. That is, each player plans an action for each of the two possible signals she may receive. We may think of there being four players: the two types of player 1 and the two types of player 2. A *Nash equilibrium* consists of four actions, one for each of these players, such that the action of each type of each original player is optimal, given her belief about the state after observing her signal, and given the actions of each type of the other original player.

Consider the payoffs of type y_1 of player 1. She believes that with probability $\frac{1}{2}$ she faces type y_2 of player 2, and with probability $\frac{1}{2}$ she faces type n_2. Suppose that type y_2 of player 2 chooses B and type n_2 chooses S. Then if type y_1 of player 1 chooses B, her expected payoff is $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$, and if she chooses S, her expected payoff is $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$. Her expected payoffs for all four pairs of actions of the two types of player 2 are given in Figure 276.1.

? EXERCISE 275.1 (Expected payoffs in a variant of *BoS* with imperfect information) Construct tables like the one in Figure 276.1 for type n_1 of player 1, and for types y_2 and n_2 of player 2.

I claim that $((B, B), (B, S))$ and $((S, B), (S, S))$ are Nash equilibria of the game, where in each case the first component gives the actions of the two types of player 1

	(B, B)	(B, S)	(S, B)	(S, S)
B	2	1	1	0
S	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Figure 276.1 The expected payoffs of type y_1 of player 1 in Example 274.2. Each row corresponds to a pair of actions for the two types of player 2; the action of type y_2 is listed first, that of type n_2 second.

and the second component gives the actions of the two types of player 2. Using Figure 276.1 you may verify that B is a best response of type y_1 of player 1 to the pair (B, S) of actions of player 2, and S is a best response to the pair of actions (S, S). You may use your answer to Exercise 275.1 to verify that in each of the claimed Nash equilibria the action of type n_1 of player 1 and the action of each type of player 2 is a best response to the other players' actions.

In each of these examples a Nash equilibrium is a list of actions, one for each type of each player, such that the action of each type of each player is a best response to the actions of all the types of the other player, given the player's beliefs about the state after she observes her signal. The actions planned by the various types of player i are not relevant to the decision problem of any type of player i, but there is no harm in taking them, as well as the actions of the types of the *other* player, as given when player i is choosing an action. Thus we may define a Nash equilibrium in each example to be a Nash equilibrium of the strategic game in which the set of players is the set of all types of all players in the original situation.

In the next section I define the general notion of a Bayesian game, and the notion of Nash equilibrium in such a game. These definitions require significant theoretical development. If you find the theory in the next section heavy-going, you may be able to skim the section and then study the subsequent illustrations, relying on the intuition developed in the examples in this section, and returning to the theory only as necessary for clarification.

9.3 General definitions

9.3.1 Bayesian games

A strategic game with imperfect information is called a "Bayesian game". (The reason for this nomenclature will become apparent.) As in a strategic game, the decision-makers are called *players*, and each player is endowed with a set of *actions*.

A key component in the specification of the imperfect information is the set of *states*. Each state is a complete description of one collection of the players' relevant characteristics, including both their preferences and their information. For every collection of characteristics that some player believes to be possible, there must be a state. For instance, suppose in Example 271.1 that player 2 wishes to meet player 1. In this case, the reason for including in the model the state in which player 2 wishes to avoid player 1 is that player 1 believes such a preference to be possible.

At the start of the game a state is realized. The players do not observe this state. Rather, each player receives a *signal* that may give her some information about the state. Denote the signal player i receives in state ω by $\tau_i(\omega)$. The function τ_i is called player i's *signal function*. (Note that the signal is a *deterministic* function of the state: for each state a definite signal is received.) The states that generate any given signal t_i are said to be *consistent* with t_i. The sizes of the sets of states consistent with each of player i's signals reflect the quality of player i's information. If, for example, $\tau_i(\omega)$ is different for each value of ω, then player i knows, given her signal, the state that has occurred; after receiving her signal, she is perfectly informed about all the players' relevant characteristics. At the other extreme, if $\tau_i(\omega)$ is the same for all states, then player i's signal conveys no information about the state. If $\tau_i(\omega)$ is constant over some subsets of the set of states, but is not the same for all states, then player i's signal conveys partial information. For example, if there are three states, ω_1, ω_2, and ω_3, and $\tau_i(\omega_1) \neq \tau_i(\omega_2) = \tau_i(\omega_3)$, then when the state is ω_1 player i knows that it is ω_1, whereas when it is either ω_2 or ω_3 she knows only that it is one of these two states.

We refer to player i in the event that she receives the signal t_i as *type* t_i of player i. Each type of each player holds a *belief* about the likelihood of the states consistent with her signal. If, for example, $t_i = \tau_i(\omega_1) = \tau_i(\omega_2)$, then type t_i of player i assigns probabilities to ω_1 and ω_2. (A player who receives a signal consistent with only one state naturally assigns probability 1 to that state.)

Each player may care about the actions chosen by the other players, as in a strategic game with perfect information, and also about the state. The players may be uncertain about the state, so we need to specify their preferences regarding probability distributions over pairs (a, ω) consisting of an action profile a and a state ω. I assume that each player's preferences over such probability distributions are represented by the expected value of a *Bernoulli payoff function*. Thus I specify each player i's preferences by giving a Bernoulli payoff function u_i over pairs (a, ω). (Note that in both Example 271.1 and Example 274.2, both players care only about the other player's action, not independently about the state.)

In summary, a Bayesian game is defined as follows.

▶ DEFINITION 277.1 A **Bayesian game** consists of

- a set of **players**
- a set of **states**

and for each player

- a set of **actions**
- a set of **signals** that she may receive and a **signal function** that associates a signal with each state

- for each signal that she may receive, a **belief** about the states consistent with the signal (a probability distribution over the set of states with which the signal is associated)
- a **Bernoulli payoff function** over pairs (a, ω), where a is an action profile and ω is a state, the expected value of which represents the player's preferences among lotteries over the set of such pairs.

The eponymous Thomas Bayes (1702–61) first showed how probabilities should be changed in the light of new information. His formula (discussed in Section 17.7.5) is needed when working with a variant of Definition 277.1 in which each player is endowed with a "prior" belief about the states, from which the belief of each of her types is derived. For the purposes of this chapter, the belief of each type of each player is more conveniently taken as a primitive, rather than being derived from a prior belief.

The game in Example 271.1 fits into this general definition as follows.

Players The pair of people.

States The set of states is $\{meet, avoid\}$.

Actions The set of actions of each player is $\{B, S\}$.

Signals Player 1 may receive a single signal, say z; her signal function τ_1 satisfies $\tau_1(meet) = \tau_1(avoid) = z$. Player 2 receives one of two signals, say m and v; her signal function τ_2 satisfies $\tau_2(meet) = m$ and $\tau_2(avoid) = v$.

Beliefs Player 1 assigns probability $\frac{1}{2}$ to each state after receiving the signal z. Player 2 assigns probability 1 to the state *meet* after receiving the signal m, and probability 1 to the state *avoid* after receiving the signal v.

Payoffs The payoffs $u_i(a, meet)$ of each player i for all possible action pairs are given in the left panel of Figure 272.1, and the payoffs $u_i(a, avoid)$ are given in the right panel.

Similarly, the game in Example 274.2 fits into the definition as follows.

Players The pair of people.

States The set of states is $\{yy, yn, ny, nn\}$.

Actions The set of actions of each player is $\{B, S\}$.

Signals Player 1 receives one of two signals, y_1 and n_1; her signal function τ_1 satisfies $\tau_1(yy) = \tau_1(yn) = y_1$ and $\tau_1(ny) = \tau_1(nn) = n_1$. Player 2 receives one of two signals, y_2 and n_2; her signal function τ_2 satisfies $\tau_2(yy) = \tau_2(ny) = y_2$ and $\tau_2(yn) = \tau_2(nn) = n_2$.

Beliefs Player 1 assigns probability $\frac{1}{2}$ to each of the states yy and yn after receiving the signal y_1 and probability $\frac{1}{2}$ to each of the states ny and nn after receiving the signal n_1. Player 2 assigns probability $\frac{2}{3}$ to the state yy and probability $\frac{1}{3}$ to the state ny after receiving the signal y_2, and probability $\frac{2}{3}$ to the state yn and probability $\frac{1}{3}$ to the state nn after receiving the signal n_2.

Payoffs The payoffs $u_i(a, \omega)$ of each player i for all possible action pairs and states are given in Figure 275.1.

9.3.2 Nash equilibrium

In a strategic game, each player chooses an action. In a Bayesian game, each player chooses a collection of actions, one for each signal she may receive. That is, in a Bayesian game each type of each player chooses an action. In a Nash equilibrium of such a game, the action chosen by each type of each player is optimal, given the actions chosen by every type of every other player. (In a steady state, each player's experience teaches her these actions.) Any given type of player i is not affected by the actions chosen by the other types of player i, so there is no harm in thinking that player i takes as given these actions, as well as those of the other players. Thus we may define a Nash equilibrium of a Bayesian game to be a Nash equilibrium of a strategic game in which each player is one type of one of the players in the Bayesian game. What is each player's payoff function in this strategic game?

Consider type t_i of player i. For each state ω she knows every other player's type (i.e. she knows the signal received by every other player). This information, together with her belief about the states, allows her to calculate her expected payoff for each of her actions and each collection of actions for the various types of the other players. For instance, in Example 271.1, player 1's belief is that the probability of each state is $\frac{1}{2}$, and she knows that player 2 is type m in the state *meet* and type v in the state *avoid*. Thus if type m of player 2 chooses B and type v of player 2 chooses S, player 1 thinks that if she chooses B then her expected payoff is

$$\tfrac{1}{2}u_1(B, B, \mathit{meet}) + \tfrac{1}{2}u_1(B, S, \mathit{avoid}),$$

where u_1 is her payoff function in the Bayesian game. (In general her payoff may depend on the state, though in this example it does not.) The top box of the second column in Figure 273.1 gives this payoff; the other boxes give player 1's payoffs for her other action and the other combinations of actions for the two types of player 2.

In a general game, denote the probability assigned by the belief of type t_i of player i to state ω by $\Pr(\omega|t_i)$. Denote the action taken by each type t_j of each player j by $a(j, t_j)$. Player j's signal in state ω is $\tau_j(\omega)$, so her action in state ω is $a(j, \tau_j(\omega))$. For each state ω, denote by $\hat{a}(\omega)$ the action profile in which each player j chooses the action $a(j, \tau_j(\omega))$. Then the expected payoff of type t_i of player i when she chooses the action a_i is

$$\sum_{\omega \in \Omega} \Pr(\omega \mid t_i) u_i((a_i, \hat{a}_{-i}(\omega)), \omega), \qquad (279.1)$$

where Ω is the set of states and $(a_i, \hat{a}_{-i}(\omega))$ is the action profile in which player i chooses the action a_i and every other player j chooses $\hat{a}_j(\omega)$. (Note that this expected payoff does not depend on the actions of any other types of player i, but only on the actions of the various types of the *other* players.)

We may now define precisely a Nash equilibrium of a Bayesian game.

▶ DEFINITION 280.1 A **Nash equilibrium of a Bayesian game** is a Nash equilibrium of the strategic game (with vNM preferences) defined as follows.

Players The set of all pairs (i, t_i) where i is a player in the Bayesian game and t_i is one of the signals that i may receive.

Actions The set of actions of each player (i, t_i) is the set of actions of player i in the Bayesian game.

Preferences The Bernoulli payoff function of each player (i, t_i) is given by (279.1).

? EXERCISE 280.2 (A fight with imperfect information about strengths) Two people are involved in a dispute. Person 1 does not know whether person 2 is strong or weak; she assigns probability α to person 2's being strong. Person 2 is fully informed. Each person can either fight or yield. Each person's preferences are represented by the expected value of a Bernoulli payoff function that assigns the payoff of 0 if she yields (regardless of the other person's action) and a payoff of 1 if she fights and her opponent yields; if both people fight then their payoffs are $(-1, 1)$ if person 2 is strong and $(1, -1)$ if person 2 is weak. Formulate this situation as a Bayesian game and find its Nash equilibria if $\alpha < \frac{1}{2}$ and if $\alpha > \frac{1}{2}$.

? EXERCISE 280.3 (An exchange game) Each of two individuals receives a ticket on which there is an integer from 1 to m indicating the size of a prize she may receive. The individuals' tickets are assigned randomly and independently; the probability of an individual's receiving each possible number is positive. Each individual is given the option to exchange her prize for the other individual's prize; the individuals are given this option simultaneously. If both individuals wish to exchange then the prizes are exchanged; otherwise each individual receives her own prize. Each individual's objective is to maximize her expected monetary payoff. Model this situation as a Bayesian game and show that in any Nash equilibrium the highest prize that either individual is willing to exchange is the smallest possible prize.

? EXERCISE 280.4 (Adverse selection) Firm A (the "acquirer") is considering taking over firm T (the "target"). It does not know firm T's value; it believes that this value, when firm T is controlled by its own management, is at least \$0 and at most \$100, and assigns equal probability to each of the 101 dollar values in this range. Firm T will be worth 50% more under firm A's management than it is under its own management. Suppose that firm A bids y to take over firm T, and firm T is worth x (under its own management). Then if T accepts A's offer, A's payoff is $\frac{3}{2}x - y$ and T's payoff is y; if T rejects A's offer, A's payoff is 0 and T's payoff is

x. Model this situation as a Bayesian game in which firm A chooses how much to offer and firm T decides the lowest offer to accept. Find the Nash equilibrium (equilibria?) of this game. Explain why the logic behind the equilibrium is called *adverse selection*.

9.4 Two examples concerning information

The notion of a Bayesian game may be used to study how information patterns affect the outcome of strategic interaction. Here are two examples.

9.4.1 More information may hurt

A decision-maker in a single-person decision problem cannot be worse off if she has more information: if she wishes, she can ignore the information. In a game the same is not true: if a player has more information and the other players know that she has more information then she may be worse off.

Consider, for example, the two-player Bayesian game in Figure 281.1, where $0 < \epsilon < \frac{1}{2}$. In this game there are two states, and neither player knows the state. Player 2's unique best response to every strategy of player 1 is L (which yields the expected payoff $2 - 2(1-\epsilon)p$, whereas M and R both yield $\frac{3}{2} - \frac{3}{2}(1-\epsilon)p$, where p is the probability player 1 assigns to T), and player 1's unique best response to L is B. Thus (B, L) is the unique Nash equilibrium of the game, yielding each player a payoff of 2.

Player 1: $\frac{1}{2}$ (State ω_1), $\frac{1}{2}$ (State ω_2)

Player 2: $\frac{1}{2}$ (State ω_1), $\frac{1}{2}$ (State ω_2)

State ω_1

	L	M	R
T	$1, 2\epsilon$	$1, 0$	$1, 3\epsilon$
B	$2, 2$	$0, 0$	$0, 3$

State ω_2

	L	M	R
T	$1, 2\epsilon$	$1, 3\epsilon$	$1, 0$
B	$2, 2$	$0, 3$	$0, 0$

Figure 281.1 The first Bayesian game considered in Section 9.4.1.

Now consider the variant of this game in which player 2 is informed of the state: player 2's signal function τ_2 satisfies $\tau_2(\omega_1) \neq \tau_2(\omega_2)$. In this game $(T, (R, M))$ is the unique Nash equilibrium. (Each type of player 2 has a strictly dominant action, to which T is player 1's unique best response.)

Player 2's payoff in the unique Nash equilibrium of the original game is 2, whereas her payoff in the unique Nash equilibrium of the game in which she knows the state is 3ϵ in each state. Thus she is worse off when she knows the state than when she does not.

Player 2's action R is good only in state ω_1 whereas her action M is good only in state ω_2. When she does not know the state she optimally chooses L, which is better than the average of R and M whatever player 1 does. Her choice induces player 1 to choose B. When player 2 is fully informed she optimally tailors her action to the state, which induces player 1 to choose T. There is no steady state in which she ignores her information and chooses L because this action leads player 1 to choose B, making R better for player 2 in state ω_1 and M better in state ω_2.

9.4.2 Infection

The notion of a Bayesian game may be used to model not only situations in which players are uncertain about each others' preferences, but also situations in which they are uncertain about each others' *knowledge*. Consider, for example, the Bayesian game in Figure 282.1.

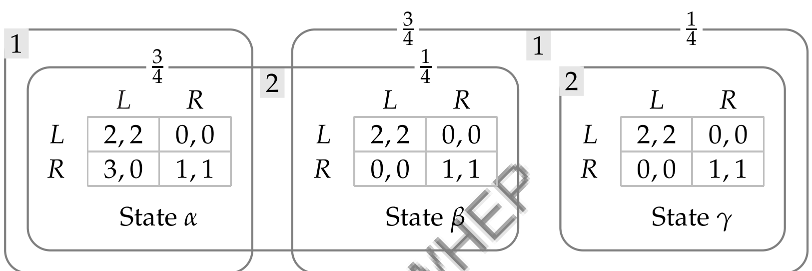

Figure 282.1 The first Bayesian game in Section 9.4.2. In the unique Nash equilibrium of this game, each type of each player chooses R.

Notice that player 2's preferences are the same in all three states, and player 1's preferences are the same in states β and γ. In particular, in state γ, each player knows the other player's preferences, and player 2 knows that player 1 knows her preferences. The shortcoming in the players' information in state γ is that player 1 does not know that player 2 knows her preferences: player 1 knows only that the state is either β or γ, and in state β player 2 does not know whether the state is α or β, and hence does not know player 1's preferences (because player 1's preferences in these two states differ).

This imperfection in player 1's knowledge of player 2's information significantly affects the equilibria of the game. If information were perfect in state γ, then both (L, L) and (R, R) would be Nash equilibria. However, the whole game has a *unique* Nash equilibrium, in which the outcome in state γ is (R, R), as you are asked to show in the next exercise. The argument shows that the incentives faced by player 1 in state α "infect" the remainder of the game.

? EXERCISE 282.1 (Infection) Show that the Bayesian game in Figure 282.1 has a unique Nash equilibrium, in which each player chooses R regardless of her signal. (Start by considering player 1's action in state α. Next consider player 2's action when she gets the signal that the state is α or β. Then consider player 1's action when she gets the signal that the state is β or γ. Finally consider player 2's action in state γ.)

Now extend the game as in Figure 283.1. Consider state δ. In this state, player 2 knows player 1's preferences (because she knows that the state is either γ or δ, and in both states player 1's preferences are the same). What player 2 does not know is whether player 1 knows that player 2 knows player 1's preferences. The reason is that player 2 does not know whether the state is γ or δ; and in state γ player 1 does not know that player 2 knows her preferences, because she does not know whether the state is β or γ, and in state β player 2 (who does not know whether the state is α or β) does not know her preferences. Thus the level of the shortcoming in the players' information is higher than it is in the game in Figure 282.1. Nevertheless, the incentives faced by player 1 in state α again "infect" the remainder of the game, and in the only Nash equilibrium every type of each player chooses R.

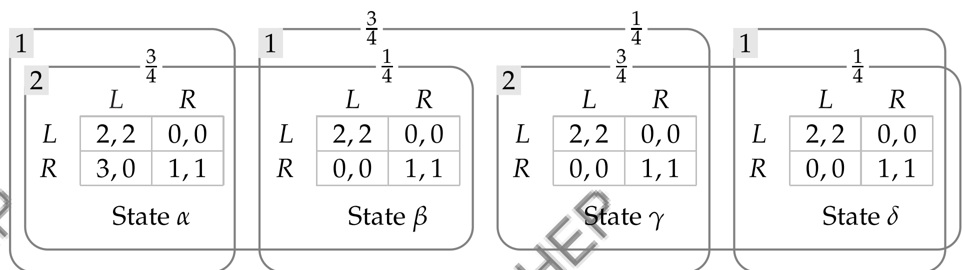

Figure 283.1 The second Bayesian game in Section 9.4.2.

The game may be further extended. As it is extended, the level of the imperfection in the players' information in the last state increases. When the number of states is large, the players' information in the last state is only very slightly imperfect. Nevertheless, the incentives of player 1 in state α still cause the game to have a unique Nash equilibrium, in which every type of each player chooses R.

In each of these examples, the equilibrium induces an outcome in every state that is worse for both players than another outcome (namely (L, L)); in all states but the first, the alternative outcome is a Nash equilibrium in the game with perfect information. For some other specifications of the payoffs in state α and the players' beliefs, the game has a unique equilibrium in which the "good" outcome (L, L) occurs in every state; the point is only that one of the two Nash equilibria are selected, not that the "bad" equilibrium is necessarily selected. (Modify the payoffs of player 1 in state α so that L strictly dominates R, and change the beliefs to assign probability $\frac{1}{2}$ to each state compatible with each signal.)

9.5 Illustration: Cournot's duopoly game with imperfect information

9.5.1 Imperfect information about cost

Two firms compete in selling a good; one firm does not know the other firm's cost function. How does the imperfect information affect the firms' behavior?

Assume that both firms can produce the good at constant unit cost. Assume also that they both know that firm 1's unit cost is c, but only firm 2 knows its own unit cost; firm 1 believes that firm 2's cost is c_L with probability θ and c_H with probability $1-\theta$, where $0 < \theta < 1$ and $c_L < c_H$.

We may model this situation as a Bayesian game that is a variant of Cournot's game (Section 3.1).

Players Firm 1 and firm 2.

States $\{L, H\}$.

Actions Each firm's set of actions is the set of its possible outputs (nonnegative numbers).

Signals Firm 1's signal function τ_1 satisfies $\tau_1(H) = \tau_2(L)$ (its signal is the same in both states); firm 2's signal function τ_2 satisfies $\tau_2(H) \neq \tau_2(L)$ (its signal is perfectly informative of the state).

Beliefs The single type of firm 1 assigns probability θ to state L and probability $1-\theta$ to state H. Each type of firm 2 assigns probability 1 to the single state consistent with its signal.

Payoff functions The firms' Bernoulli payoffs are their profits; if the actions chosen are (q_1, q_2) and the state is I (either L or H) then firm 1's profit is $q_1(P(q_1+q_2)-c)$ and firm 2's profit is $q_2(P(q_1+q_2)-c_I)$, where $P(q_1+q_2)$ is the market price when the firms' outputs are q_1 and q_2.

The information structure in this game is similar to that in Example 271.1; it is illustrated in Figure 284.1.

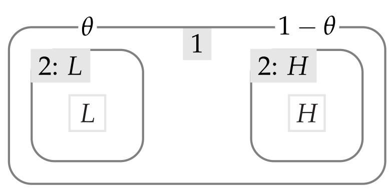

Figure 284.1 The information structure for the model in the variant of Cournot's model in Section 9.5.1, in which firm 1 does not know firm 2's cost. The frame labeled 2: x, for $x = L$ and $x = H$, encloses the state that generates the signal x for firm 2.

A Nash equilibrium of this game is a triple (q_1^*, q_L^*, q_H^*), where q_1^* is the output of firm 1, q_L^* is the output of type L of firm 2 (i.e. firm 2 when it receives the signal $\tau_2(L)$), and q_H^* is the output of type H of firm 2 (i.e. firm 2 when it receives the signal $\tau_2(H)$), such that

- q_1^* maximizes firm 1's profit given the output q_L^* of type L of firm 2 and the output q_H^* of type H of firm 2
- q_L^* maximizes the profit of type L of firm 2 given the output q_1^* of firm 1

- q_H^* maximizes the profit of type H of firm 2 given the output q_1^* of firm 1.

To find an equilibrium, we first find the firms' best response functions. Given firm 1's posterior beliefs, its best response $b_1(q_L, q_H)$ to (q_L, q_H) solves

$$\max_{q_1} \left[\theta(P(q_1 + q_L) - c)q_1 + (1 - \theta)(P(q_1 + q_H) - c)q_1\right].$$

Firm 2's best response $b_L(q_1)$ to q_1 when its cost is c_L solves

$$\max_{q_L} \left[(P(q_1 + q_L) - c_L)q_L\right],$$

and its best response $b_H(q_1)$ to q_1 when its cost is c_H solves

$$\max_{q_H} \left[(P(q_1 + q_H) - c_H)q_H\right].$$

A Nash equilibrium is a triple (q_1^*, q_L^*, q_H^*) such that

$$q_1^* = b_1(q_L^*, q_H^*),\ q_L^* = b_L(q_1^*), \text{ and } q_H^* = b_H(q_1^*).$$

? EXERCISE 285.1 (Cournot's duopoly game with imperfect information) Consider the game when the inverse demand function is given by $P(Q) = \alpha - Q$ for $Q \le \alpha$ and $P(Q) = 0$ for $Q > \alpha$ (see (54.2)). For values of c_H and c_L close enough that there is a Nash equilibrium in which all outputs are positive, find this equilibrium. Compare this equilibrium with the Nash equilibrium of the game in which firm 1 knows that firm 2's unit cost is c_L, and with the Nash equilibrium of the game in which firm 1 knows that firm 2's unit cost is c_H.

9.5.2 Imperfect information about both cost and information

Now suppose that firm 2 does not know whether firm 1 knows its cost. That is, suppose that one circumstance that firm 2 believes to be possible is that firm 1 knows it cost (although in fact it does not). Because firm 2 thinks this circumstance to be possible, we need *four* states to model the situation, which I call $L0$, $H0$, $L1$, and $H1$, with the following interpretations.

$L0$: firm 2's cost is low and firm 1 does not know whether it is low or high

$H0$: firm 2's cost is high and firm 1 does not know whether it is low or high

$L1$: firm 2's cost is low and firm 1 knows it is low

$H1$: firm 2's cost is high and firm 1 knows it is high.

Firm 1 receives one of three possible signals, 0, L, and H. The states $L0$ and $H0$ generate the signal 0 (firm 1 does not know firm 2's cost), the state $L1$ generates the signal L (firm 1 knows firm 2's cost is low), and the state $H1$ generates the signal H (firm 1 knows firm 2's cost is high). Firm 2 receives one of two possible signals, L, in states $L0$ and $L1$, and H, in states $H0$ and $H1$. Denote by θ (as before) the probability assigned by type 0 of firm 1 to firm 2's cost being c_L, and by π the probability assigned by each type of firm 2 to firm 1's knowing firm 2's cost. (The case $\pi = 0$ is equivalent to the one considered in the previous section.) A Bayesian game that models the situation is defined as follows.

Players Firm 1 and firm 2.

States $\{L0, L1, H0, H1\}$, where the first letter in the name of the state indicates firm 2's cost and the second letter indicates whether (1) or not (0) firm 1 knows firm 2's cost.

Actions Each firm's set of actions is the set of its possible outputs (nonnegative numbers).

Signals Firm 1 gets one of the signals 0, L, and H, and her signal function τ_1 satisfies $\tau_1(L0) = \tau_1(H0) = 0$, $\tau_1(L1) = L$, and $\tau_1(H1) = H$. Firm 2 gets the signal L or H and her signal function τ_2 satisfies $\tau_2(L0) = \tau_2(L1) = L$ and $\tau_2(H0) = \tau_2(H1) = H$.

Beliefs Firm 1: type 0 assigns probability θ to state $L0$ and probability $1 - \theta$ to state $H0$; type L assigns probability 1 to state $L1$; type H assigns probability 1 to state H. Firm 2: type L assigns probability π to state $L1$ and probability $1 - \pi$ to state $L0$; type H assigns probability π to state $H1$ and probability $1 - \pi$ to state $H0$.

Payoff functions The firms' Bernoulli payoffs are their profits; if the actions chosen are (q_1, q_2), then firm 1's profit is $q_1(P(q_1 + q_2) - c)$ and firm 2's profit is $q_2(P(q_1 + q_2) - c_L)$ in states $L0$ and $L1$, and $q_2(P(q_1 + q_2) - c_L)$ in states $H0$ and $H1$.

The information structure in this game is illustrated in Figure 287.1. You are asked to investigate its Nash equilibria in the following exercise.

? EXERCISE 286.1 (Cournot's duopoly game with imperfect information) Write down the maximization problems that determine the best response function each type of each player. (Denote by q_0, q_ℓ, and q_h the outputs of types 0, ℓ, and h of firm 1, and by q_L and q_H the outputs of types L and H of firm 2.) Now suppose that the inverse demand function is given by $P(Q) = \alpha - Q$ for $Q \le \alpha$ and $P(Q) = 0$ for $Q > \alpha$. For values of c_H and c_L close enough that there is a Nash equilibrium in which all outputs are positive, find this equilibrium. Check that when $\pi = 0$ the equilibrium output of type 0 of firm 1 is equal to the equilibrium output of firm 1 you found in Exercise 285.1, and that the equilibrium outputs of the two types of firm 2 are the same as the ones you found in that exercise. Check also that when $\pi = 1$ the equilibrium outputs of type ℓ of firm 1 and type L of firm 2 are the same as the equilibrium outputs when there is perfect information and the costs are c and c_L, and that the equilibrium outputs of type h of firm 1 and type H of firm 2 are the same as the equilibrium outputs when there is perfect information and the

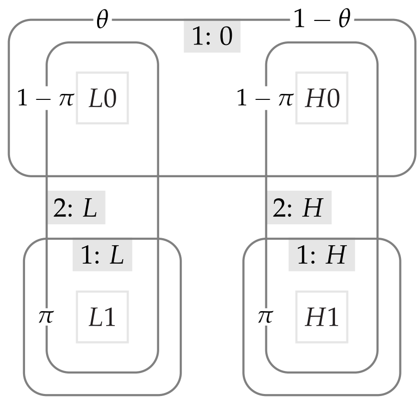

Figure 287.1 The information structure for the model in Section 9.5.2, in which firm 2 does not know whether firm 1 knows its cost. The frame labeled i: x encloses the states that generates the signal x for firm i.

costs are c and c_H. Show that for $0 < \pi < 1$, the equilibrium outputs of types L and H of firm 2 lie between their values when $\pi = 0$ and when $\pi = 1$.

9.6 Illustration: providing a public good

Suppose that a public good is provided to a group of people if at least one person is willing to pay the cost of the good (as in the model of crime-reporting in Section 4.8). Assume that the people differ in their valuations of the good, and each person knows only her own valuation. Who, if anyone, will pay the cost?

Denote the number of individuals by n, the cost of the good by $c > 0$, and individual i's payoff if the good is provided by v_i. If the good is not provided then each individual's payoff is 0. Each individual i knows her own valuation v_i. She does not know anyone else's valuation, but knows that all valuations are at least $\underline{v}$ and at most $\overline{v}$, where $0 \leq \underline{v} < c < \overline{v}$. She believes that the probability that any one individual's valuation is at most v is $F(v)$, independent of all other individuals' valuations, where F is a continuous increasing function. The fact that F is increasing means that the individual does not assign zero probability to any range of values between $\underline{v}$ and $\overline{v}$; the fact that it is continuous means that she does not assign positive probability to any single valuation. (An example of the function F is shown in Figure 288.1.)

The following mechanism determines whether the good is provided. All n individuals simultaneously submit envelopes; the envelope of any individual i may contain either a contribution of c or nothing (no intermediate contributions are allowed). If all individuals submit 0 then the good is not provided and each individual's payoff is 0. If at least one individual submits c then the good is provided, each individual i who submits c obtains the payoff $v_i - c$, and each individual i who submits 0 obtains the payoff v_i. (The pure strategy Nash equilibria of a vari-

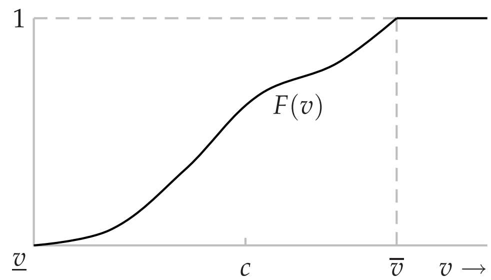

Figure 288.1 An example of the function F for the model in Section 9.6. For each value of v, $F(v)$ is the probability that any given individual's valuation is at most v.

ant of this model, in which more than one contribution is needed to provide the good, are considered in Exercise 31.1.)

We can formulate this situation as a Bayesian game as follows.

Players The set of n individuals.

States The set of all profiles $(v_1, \ldots, v_n)$ of valuations, where $0 \leq v_i \leq \overline{v}$ for all i.

Actions Each player's set of actions is $\{0, c\}$.

Signals The set of signals that each player may observe is the set of possible valuations. The signal function τ_i of each player i is given by $\tau_i(v_1, \ldots, v_n) = v_i$ (each player knows her own valuation).

Beliefs Each type of player i assigns probability $F(v_1)F(v_2)\cdots F(v_{i-1})F(v_{i+1})\cdots F(v_n)$ to the event that the valuation of every other player j is at most v_j.

Payoff functions Player i's Bernoulli payoff in state $(v_1, \ldots, v_n)$ is

$$\begin{cases} 0 & \text{if no one contributes} \\ v_i & \text{if } i \text{ does not contribute but some other player does} \\ v_i - c & \text{if } i \text{ contributes.} \end{cases}$$

? EXERCISE 288.1 (Nash equilibria of game of contributing to a public good) Find conditions under which for each value of i this game has a pure strategy Nash equilibrium in which each type v_i of player i with $v_i \geq c$ contributes, whereas every other type of player i, and all types of every other player, do not contribute.

In addition to the Nash equilibria identified in this exercise, the game has a symmetric Nash equilibrium in which every player contributes if and only if her valuation exceeds some critical amount v^*. For such a strategy profile to be an equilibrium, a player whose valuation is less than v^* must optimally not contribute, and a player whose valuation is at least v^* must optimally contribute. Consider player i. Suppose that every other player contributes if and only if her valuation is at least v^*. The probability that at least one of the other players contributes is the probability that at least one of the other players' valuations is at least v^*, which is $1 - (F(v^*))^{n-1}$. (Note that $(F(v^*))^{n-1}$ is the probability that all the other valuations are at most v^*.) Thus if player i's valuation is v_i, her expected payoff is $(1 - (F(v^*))^{n-1})v_i$ if she does not contribute and $v_i - c$ if she does contribute. Hence the conditions for player i to optimally not contribute when $v_i < v^*$ and optimally contribute when $v_i \geq v^*$ are $(1 - (F(v^*))^{n-1})v_i \geq v_i - c$ if $v_i < v^*$, and $(1 - (F(v^*))^{n-1})v_i \leq v_i - c$ if $v_i \geq v^*$, or equivalently

$$\begin{aligned} v_i(F(v^*))^{n-1} &\leq c \quad \text{if } v_i < v^* \\ v_i(F(v^*))^{n-1} &\geq c \quad \text{if } v_i \geq v^*. \end{aligned} \tag{289.1}$$

If these inequalities are satisfied then

$$v^*(F(v^*))^{n-1} = c. \tag{289.2}$$

Conversely, if v^* satisfies (289.2) then it satisfies the two equations in (289.1). Thus the game has a Nash equilibrium in which every player contributes whenever her valuation is at least v^* if and only if v^* satisfies (289.2).

Note that because $F(v) = 1$ only if $v \geq \overline{v}$, and $\overline{v} > c$, we have $v^* > c$. That is, every player's cutoff for contributing exceeds the cost of the public good. When at least one player's valuation exceeds c, all players are better off if the public good is provided and the high-valuation player contributes than if the good is not provided. But in the equilibrium, the good is provided only if at least one player's valuation exceeds v^*, which exceeds c.

As the number of individuals increases, is the good more or less likely to be provided in this equilibrium? The probability that the good is provided is the probability that at least one player's valuation is at least v^*, which is equal to $1 - (F(v^*))^n$. (Note that $(F(v^*))^n$ is the probability that every player's valuation is less than v^*.) From (289.2) this probability is equal to $1 - cF(v^*)/v^*$. How does v^* vary with n? As n increases, for any given value of v^* the value of $(F(v^*))^{n-1}$ decreases, and thus the value of $v^*(F(v^*))^{n-1}$ decreases. Thus to maintain the equality (289.2), the value of v^* must increase as n increases. We conclude that as n increases the change in the probability that the good is provided depends on the change in $F(v^*)/v^*$ as v^* increases: the probability increases if $F(v^*)/v^*$ is a decreasing function of v^*, whereas it decreases if $F(v^*)/v^*$ is an increasing function of v^*. If F is uniform and $\underline{v} > 0$, for example, $F(v^*)/v^*$ is a decreasing function of v^*, so that the probability that the good is provided increases as the population size increases.

The notion of a Bayesian game may be used to model a situation in which each player is uncertain of the number of other players. In the next exercise you are asked to study another variant of the crime-reporting model of Section 4.8 in which each of the two players does not know whether she is the only witness or whether there is another witness (in which case she knows that witness's valuation). (The exercise requires a knowledge of mixed strategy Nash equilibrium (Chapter 4).)

? EXERCISE 290.1 (Reporting a crime with an unknown number of witnesses) Consider the variant of the model of Section 4.8 in which each of two players does not know whether she is the only witness, or whether there is another witness. Denote by π the probability each player assigns to being the sole witness. Model this situation as a Bayesian game with three states: one in which player 1 is the only witness, one in which player 2 is the only witness, and one in which both players are witnesses. Find a condition on π under which the game has a pure Nash equilibrium in which each player chooses *Call* (given the signal that she is a witness). When the condition is violated, find the symmetric mixed strategy Nash equilibrium of the game, and check that when $\pi = 0$ this equilibrium coincides with the one found in Section 4.8 for $n = 2$.

9.7 Illustration: auctions

9.7.1 Introduction

In the analysis of auctions in Section 3.5, every bidder knows every other bidder's valuation of the object for sale. Here I use the notion of a Bayesian game to analyze auctions in which bidders are not perfectly informed about each others' valuations.

Assume that a single object is for sale, and that each bidder independently receives some information—a "signal"—about the value of the object to her. If each bidder's signal is simply her valuation of the object, as assumed in Section 3.5, we say that the bidders' valuations are *private*. If each bidder's valuation depends on other bidders' signals as well as her own, we say that the valuations are *common*.

The assumption of private values is appropriate, for example, for a work of art whose beauty rather than resale value interests the buyers. Each bidder knows her valuation of the object, but not that of any other bidder; the other bidders' valuations have no bearing on her valuation. The assumption of common values is appropriate, for example, for an oil tract containing unknown reserves on which each bidder has conducted a test. Each bidder i's test result gives her some information about the size of the reserves, and hence her valuation of these reserves, but the other bidders' test results, if known to bidder i, would typically improve this information.

As in the analysis of auctions in which the bidders are perfectly informed about each others' valuations, I study models in which bids for a single object are submitted simultaneously (bids are *sealed*), and the participant who submits the highest bid obtains the object. As before I consider both *first-price* auctions, in which the winner pays the price she bid, and *second-price* auctions, in which the winner pays the highest of the remaining bids.

(In Section 3.5 I argue that the first-price rule models an open descending ("Dutch") auction, and the second-price rule models an open ascending ("English") auction. Note that the argument that the second-price rule corresponds to an open ascending auction depends upon the bidders' valuations being private. If a bidder is uncertain of her valuation, which is related to that of other bidders, then in an open ascending auction she may obtain information about her valuation from other participants' bids, information not available in a sealed-bid auction.)

I first consider the case in which the bidders' valuations are private, then the case in which they are common.

9.7.2 *Independent private values*

In the case in which the bidders' valuations are private, the assumptions about these valuations are similar to those in the previous section (on the provision of a public good). Each bidder knows that all other bidders' valuations are at least $\underline{v}$, where $\underline{v} \geq 0$, and at most $\overline{v}$. She believes that the probability that any given bidder's valuation is at most v is $F(v)$, independent of all other bidders' valuations, where F is a continuous increasing function (as in Figure 288.1).

The preferences of a bidder whose valuation is v are represented by the expected value of the Bernoulli payoff function that assigns 0 to the outcome in which she does not win the object and $v - p$ to the outcome in which she wins the object and pays the price p. (That is, each bidder is risk neutral.) I assume that the expected payoff of a bidder whose bid is tied for first place is $(v - p)/m$, where m is the number of tied winning bids. (The assumption about the outcome when bids are tied for first place has mainly "technical" significance; in Section 3.5, it was convenient to make an assumption different from the one here.)

Denote by $P(b)$ the price paid by the winner of the auction when the profile of bids is b. For a first-price auction $P(b)$ is the winning bid (the largest b_i), whereas for a second-price auction it is the highest bid made by a bidder different from the winner. Given the appropriate specification of P, the following Bayesian game models **first- and second-price auctions with independent private valuations** (and imperfect information about valuations).

Players The set of bidders, say $1, \ldots, n$.

States The set of all profiles $(v_1, \ldots, v_n)$ of valuations, where $\underline{v} \leq v_i \leq \overline{v}$ for all i.

Actions Each player's set of actions is the set of possible bids (nonnegative numbers).

Signals The set of signals that each player may observe is the set of possible valuations. The signal function τ_i of each player i is given by $\tau_i(v_1, \ldots, v_n) = v_i$ (each player knows her own valuation).

Beliefs Each type of player i assigns probability $F(v_1)F(v_2)\cdots F(v_{i-1})F(v_{i+1})\cdots F(v_n)$ to the event that the valuation of every other player j is at most v_j.

Payoff functions Player i's Bernoulli payoff in state $(v_1, \ldots, v_n)$ is 0 if her bid b_i is not the highest bid, and $(v_i - P(b))/m$ if no bid is higher than b_i and m

bids (including b_i) are equal to b_i:

$$u_i(b,(v_1,\ldots,v_n)) = \begin{cases} (v_i - P(b))/m & \text{if } b_j \le b_i \text{ for all } j \ne i \text{ and} \\ & \quad b_j = b_i \text{ for } m \text{ players} \\ 0 & \text{if } b_j > b_i \text{ for some } j \ne i. \end{cases}$$

Nash equilibrium in a second-price sealed-bid auction As in a second-price sealed-bid auction in which every bidder knows every other bidder's valuation,

> *in a second-price sealed-bid auction with imperfect information about valuations, a player's bid equal to her valuation weakly dominates all her other bids.*

Precisely, consider some type v_i of some player i, and let b_i be a bid not equal to v_i. Then for all bids by all types of all the other players, the expected payoff of type v_i of player i is at least as high when she bids v_i as it is when she bids b_i, and for some bids by the various types of the other players, her expected payoff is greater when she bids v_i than it is when she bids b_i.

The argument for this result is similar to the argument in Section 3.5.2 in the case in which the players know each others' valuations. The main difference between the arguments arises because in the case in which the players do not know each others' valuations, any given bids for every type of every player but i leave player i uncertain about the highest of the remaining bids, because she is uncertain of the other players' types. (The difference in the tie-breaking rules between the two cases also necessitates a small change in the argument.) In the next exercise you are asked to fill in the details.

? EXERCISE 292.1 (Weak domination in second-price sealed-bid action) Show that for each type v_i of each player i in a second-price sealed-bid auction with imperfect information about valuations the bid v_i weakly dominates all other bids.

We conclude, in particular, that a second-price sealed-bid auction with imperfect information about valuations has a Nash equilibrium in which every type of every player bids her valuation. The game has also other equilibria, some of which you are asked to find in the next exercise.

? EXERCISE 292.2 (Nash equilibria of a second-price sealed-bid auction) For every player i, find a Nash equilibrium of a second-price sealed-bid auction in which player i wins. (Think about the Nash equilibria when the players know each others' valuations, studied in Section 3.5.)

Nash equilibrium in a first-price sealed-bid auction As when the players are perfectly informed about each others' valuations, the bid of v_i by type v_i of player i weakly dominates any bid greater than v_i, but does not weakly dominate bids less than v_i, and is itself weakly dominated by any such lower bid. (If type v_i of player i bids v_i, her payoff is certainly 0 (either she wins and pays her valuation, or she loses), whereas if she bids less than v_i, she may win and obtain a positive payoff.)

These facts suggest that the game may have a Nash equilibrium in which each player bids less than her valuation. An analysis of the game for an arbitrary distribution F of valuations requires calculus, and is relegated to an appendix (Section 9.9). Here I consider the case in which there are two bidders and each player's valuation is distributed "uniformly" between 0 and 1. This assumption on the distribution of valuations means that the fraction of valuations less than v is exactly v, so that $F(v) = v$ for all v with $0 \leq v \leq 1$.

Denote by $\beta_i(v)$ the bid of type v of player i. I claim that if there are two bidders and the distribution of valuations is uniform between 0 and 1, the game has a (symmetric) Nash equilibrium in which the function β_i is the same for both players, with $\beta_i(v) = \frac{1}{2}v$ for all v. That is, each type of each player bids exactly half her valuation.

To verify this claim, suppose that each type of player 2 bids in this way. Then as far as player 1 is concerned, player 2's bids are distributed uniformly between 0 and $\frac{1}{2}$. Thus if player 1 bids more than $\frac{1}{2}$ she surely wins, whereas if she bids $b_1 \leq \frac{1}{2}$ the probability that she wins is the probability that player 2's valuation is less than $2b_1$ (in which case player 2 bids less than b_1), which is $2b_1$. Consequently her payoff as a function of her bid b_1 is

$$\begin{cases} 2b_1(v_1 - b_1) & \text{if } 0 \leq b_1 \leq \frac{1}{2} \\ v_1 - b_1 & \text{if } b_1 > \frac{1}{2}. \end{cases}$$

This function is shown in Figure 293.1. Its maximizer is $\frac{1}{2}v_1$ (see Exercise 446.1), so that player 1's optimal bid is half her valuation. Both players are identical, so this argument shows also that given $\beta_1(v) = \frac{1}{2}v$, player 2's optimal bid is half her valuation. Thus, as claimed, the game has a Nash equilibrium in which each type of each player bids half her valuation.

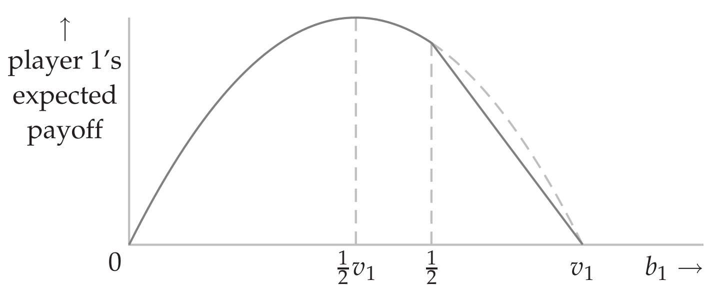

Figure 293.1 Player 1's expected payoff as a function of its bid in a first-price sealed-bid auction in which there are two bidders and the valuations are uniformly distributed from 0 to 1, given that player 2 bids $\frac{1}{2}v_2$.

When the number n of bidders exceeds two, a similar analysis shows that the game has a (symmetric) Nash equilibrium in which every player bids the fraction $1 - 1/n$ of her valuation: $\beta_i(v) = (1 - 1/n)v$ for every player i and every valuation v. (You are asked to verify a claim more general than this one in Exercise 295.1.)

In this example—and, it turns out, for any distribution F satisfying the conditions in Section 9.7.2—the players' common bidding function in a symmetric Nash equilibrium may be given an illuminating interpretation. Choose $n-1$ valuations randomly and independently, each according to the cumulative distribution function F. The highest of these $n-1$ valuations is a "random variable": its value depends on the $n-1$ valuations that were chosen. Denote it by $\mathbf{X}$. Fix a valuation v. Some values of $\mathbf{X}$ are less than v; others are greater than v. Consider the distribution of $\mathbf{X}$ in those cases in which it is less than v. The expected value of this distribution is denoted $\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$: the expected value of $\mathbf{X}$ conditional on $\mathbf{X}$ being less than v. We may prove the following result. (A proof is given in the appendix, Section 9.9.)

> *For a distribution of valuations satisfying the conditions in Section 9.7.2, a first-price sealed-bid auction with imperfect information about valuations has a (symmetric) Nash equilibrium in which each type v of each player bids $\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$, the expected value of the highest of the other players' bids conditional on v being higher than all the other valuations.*

Put differently, each bidder asks the following question: Over all the cases in which my valuation is the highest, what is the expectation of the highest of the other players' valuations? This expectation is the amount she bids.

In the case considered above in which F is uniform from 0 to 1 and $n = 2$, we may verify that indeed the equilibrium we found may be expressed in this way. For any valuation v of player 1, the cases in which player 2's valuation is less than v are distributed uniformly from 0 to v, so that the expected value of player 2's valuation conditional on its being less than v is $\frac{1}{2}v$, which is equal to the equilibrium bidding function that we found.

Comparing equilibria of first- and second-price auctions At the end of Section 3.5.3 we saw that first- and second-price auctions are "revenue equivalent" when the players know each others' valuations: their distinguished equilibria yield the same outcome. The same is true when the players are uncertain of each others' valuations.

Consider the equilibrium of a second-price auction in which every player bids her valuation. In this equilibrium, the expected price paid by a bidder with valuation v who wins is the expectation of the highest of the other $n-1$ valuations, conditional on this maximum being less than v, or, in the notation above, $\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$. We have just seen that a first-price auction has a symmetric Nash equilibrium in which this amount is precisely the bid of a player with valuation v, and hence the amount paid by such a player. Thus in the equilibria of both auctions the expected price paid by a winning bidder is the same. In both cases, the player with the highest valuation submits the winning bid, so both auctions yield the same revenue for the auctioneer:

> *if each bidder is risk neutral and the distribution of valuations satisfies the conditions in Section 9.7.2, then the Nash equilibrium of a second-price sealed-bid*

auction with independent private valuations (and imperfect information about valuations) in which each player bids her valuation yields the same revenue as the symmetric Nash equilibrium of the corresponding first-price sealed-bid auction.

This result depends on the assumption that each player's preferences are represented by the expected value of a risk neutral Bernoulli payoff function. The next exercise asks you to study an example in which each player is risk averse. (See page 101 for a discussion of risk neutrality and risk aversion.)

?? EXERCISE 295.1 (Auctions with risk averse bidders) Consider a variant of the Bayesian game defined in Section 9.7.2 in which the players are risk averse. Specifically, suppose each of the n players' preferences are represented by the expected value of the Bernoulli payoff function $x^{1/m}$, where x is the player's monetary payoff and $m > 1$. Suppose also that each player's valuation is distributed uniformly between 0 and 1, as in the example in Section 9.7.2. Show that the Bayesian game that models a first-price sealed-bid auction under these assumptions has a (symmetric) Nash equilibrium in which each type v_i of each player i bids $(1 - 1/[m(n-1)+1])v_i$. (You need to use the mathematical fact that the solution of the problem $\max_b[b^k(v - b)^\ell]$ is $kv/(k+\ell)$.) Compare the auctioneer's revenue in this equilibrium with her revenue in the symmetric Nash equilibrium of a second-price sealed-bid auction in which each player bids her valuation. (Note that the equilibrium of the second-price auction does not depend on the players' payoff functions.)

9.7.3 Common valuations

In an auction with common valuations, each player's valuation depends on the other players' signals as well as her own. (As before, I assume that the players' signals are independent.) I denote the function that gives player i's valuation by g_i, and assume that it is increasing in all the signals. Given the appropriate specification of the function P that determines the price $P(b)$ paid by the winner as a function of the profile b of bids, the following Bayesian game models **first- and second-price auctions with common valuations** (and imperfect information about valuations).

Players The set of bidders, say $\{1, \ldots, n\}$.

States The set of all profiles $(t_1, \ldots, t_n)$ of signals that the players may receive.

Actions Each player's set of actions is the set of possible bids (nonnegative numbers).

Signals The signal function τ_i of each player i is given by $\tau_i(t_1, \ldots, t_n) = t_i$ (each player observes her own signal).

Beliefs Each type of each player believes that the signal of every type of every other player is independent of all the other players' signals.

Payoff functions Player i's Bernoulli payoff in state $(t_1, \ldots, t_n)$ is 0 if her bid b_i is not the highest bid, and $(g_i(t_1, \ldots, t_n) - P(b))/m$ if no bid is higher than b_i and m bids (including b_i) are equal to b_i:

$$u_i(b, (t_1, \ldots, t_n)) = \begin{cases} (g_i(t_1, \ldots, t_n) - P(b))/m & \text{if } b_j \leq b_i \text{ for all } j \neq i \text{ and} \\ & \quad b_j = b_i \text{ for } m \text{ players} \\ 0 & \text{if } b_j > b_i \text{ for some } j \neq i. \end{cases}$$

Nash equilibrium in a second-price sealed-bid auction The main ideas in the analysis of sealed-bid common value auctions are illustrated by an example in which there are two bidders, each bidder's signal is uniformly distributed from 0 to 1, and the valuation of each bidder i is given by $v_i = \alpha t_i + \gamma t_j$, where j is the other player and $\alpha \geq \gamma \geq 0$. The case in which $\alpha = 1$ and $\gamma = 0$ is exactly the one studied in Section 9.7.2: in this case, the bidders' valuations are private. If $\alpha = \gamma$ then for any given signals, each bidder's valuation is the same—a case of "pure common valuations". If, for example, the signal t_i is the number of barrels of oil in a tract, then the expected valuation of a bidder i who knows the signals t_i and t_j is $p \cdot \frac{1}{2}(t_i + t_j)$, where p is the monetary worth of a barrel of oil. Our assumption, of course, is that a bidder does *not* know any other player's signal. However, a key point in the analysis of common value auctions is that the other players' bids contain *some* information about the other players' signals—information that may profitably be used.

I claim that under these assumptions a second-price sealed-bid auction has a Nash equilibrium in which each type t_i of each player i bids $(\alpha + \gamma)t_i$.

To verify this claim, suppose that each type of player 2 bids in this way and type t_1 of player 1 bids b_1. To determine the expected payoff of type t_1 of player 1, we need to find the probability with which she wins, and both the expected price she pays and the expected value of player 2's signal if she wins.

Probability that player 1 wins: Given that player 2's bidding function is $(\alpha + \gamma)t_2$, player 1's bid of b_1 wins only if $b_1 \geq (\alpha + \gamma)t_2$, or if $t_2 \leq b_1/(\alpha + \gamma)$. Now, t_2 is distributed uniformly from 0 to 1, so the probability that it is at most $b_1/(\alpha + \gamma)$ is $b_1/(\alpha + \gamma)$. Thus a bid of b_1 by player 1 wins with probability $b_1/(\alpha + \gamma)$.

Expected price player 1 pays if she wins: The price she pays is equal to player 2's bid, which, *conditional on its being less than* b_1, is distributed uniformly from 0 to b_1. Thus the expected value of player 2's bid, *given that it is less than* b_1, is $\frac{1}{2}b_1$.

Expected value of player 2's signal if player 1 wins: Player 2's bid, given her signal t_2, is $(\alpha + \gamma)t_2$, so that the expected value of signals that yield a bid of less than b_1 is $\frac{1}{2}b_1/(\alpha + \gamma)$ (because of the uniformity of the distribution of t_2).

Now, player 1's expected payoff if she bids b_1 is the difference between her expected valuation, given her signal t_1 and the fact that she wins, and the expected price she pays, multiplied by her probability of winning. Combining the calculations above, player 1's expected payoff if she bids b_1 is thus $(\alpha t_1 + \frac{1}{2}\gamma b_1/(\alpha+\gamma) - \frac{1}{2}b_1)b_1/(\alpha+\gamma)$, or

$$\frac{\alpha}{2(\alpha+\gamma)^2} \cdot (2(\alpha+\gamma)t_1 - b_1)b_1.$$

This function is maximized at $b_1 = (\alpha+\gamma)t_1$. That is, if each type t_2 of player 2 bids $(\alpha+\gamma)t_2$, any type t_1 of player 1 optimally bids $(\alpha+\gamma)t_1$. Symmetrically, if each type t_1 of player 1 bids $(\alpha+\gamma)t_1$, any type t_2 of player 2 optimally bids $(\alpha+\gamma)t_2$. Hence, as claimed, the game has a Nash equilibrium in which each type t_i of each player i bids $(\alpha+\gamma)t_i$.

? EXERCISE 297.1 (Asymmetric Nash equilibria of second-price sealed-bid common value auctions) Show that when $\alpha = \gamma = 1$, for *any* value of $\lambda > 0$ the game studied above has an (asymmetric) Nash equilibrium in which each type t_1 of player 1 bids $(1+\lambda)t_1$ and each type t_2 of player 2 bids $(1+1/\lambda)t_2$.

Note that when player 1 calculates her expected value of the object, she finds the expected value of player 2's signal *given that her bid wins*. If her bid is low then she is unlikely to be the winner, but if she *is* the winner, player 2's signal must be low, and so she should impute a low value to the object. She should not base her bid simply on an estimate of the valuation derived from her own signal and the (unconditional) expectation of the other player's signal. If she does so, then over all the cases in which she wins, she more likely than not overvalues the object. A bidder who incorrectly behaves in this way is said to suffer the *winner's curse*. (Bidders in real auctions know this problem: when a contractor gives you a quotation to renovate your house, she does not base her price simply on an unbiased estimate out how much it will cost her to do the job, but takes into account that you will select her only if her competitors' estimates are all be higher than hers, in which case her estimate may be suspiciously low.)

Nash equilibrium in a first-price sealed-bid auction I claim that under the assumptions on the players' signals and valuations in the previous section, a first-price sealed-bid auction has a Nash equilibrium in which each type t_i of each player i bids $\frac{1}{2}(\alpha+\gamma)t_i$. This claim may be verified by arguments like those in the previous section. In the next exercise, you are asked to supply the details.

? EXERCISE 297.2 (First-price sealed-bid auction with common valuations) Verify that under the assumptions on signals and valuations in the previous section, a first-price sealed-bid auction has a Nash equilibrium in which the bid of each type t_i of each player i is $\frac{1}{2}(\alpha+\gamma)t_i$.

Comparing equilibria of first- and second-price auctions We see that the revenue equivalence of first- and second-price auctions that holds when valuations are private hold also for the symmetric equilibria of the examples above in which the valuations are common. That is, the expected price paid by a player of any given type is the same in the symmetric equilibrium of the first-price auction as it is in the symmetric equilibrium of the second-price auction: in each case type t_i of player i pays $\frac{1}{2}(\alpha + \gamma)t_i$ if she wins, and wins with the same probability.

In fact, the revenue equivalence principle holds much more generally. Whenever each bidder is risk neutral and independently receives a signal that the same distribution, which satisfies the conditions on the distribution of valuations in Section 9.7.2, the expected payment of a bidder of any given type is the same in the symmetric Nash equilibrium of a second-price sealed-bid auction revenue-equivalent as it is in the symmetric Nash equilibrium of a first-price sealed-bid auction. Further, this revenue equivalence is not restricted to first- and second-price auctions; a general result, encompassing a wider range of auction forms, is stated at the end of the appendix (Section 9.9).

AUCTIONS OF THE RADIO SPECTRUM

In the 1990s several countries started auctioning the right to use parts of the radio spectrum used for wireless communication (by mobile telephones, for example). Spectrum licenses in the USA were originally allocated on the basis of hearings by the Federal Communications Commission (FCC). This procedure was time-consuming, and a large backlog developed, prompting a switch to lotteries. Licenses awarded by the lotteries could be re-sold at high prices, attracting many participants. In one case that drew attention, the winner of a license to run cellular telephones in Cape Cod sold it to Southwestern Bell for US$41.5 million (*New York Times*, May 30, 1991, p. A1). In the early 1990s, the US government was persuaded that auctioning licenses would allocate them more efficiently and might raise nontrivial revenue.

For each interval of the spectrum, many licenses were available, each covering a geographic area. A buyer's valuation of a license could be expected to depend on the other licenses it owned, so many interdependent goods were for sale. In designing an auction mechanism, the FCC had many choices: for example, the bidding could be open, or it could be sealed, with the price equal to either the highest bid or the second-highest bid; the licenses could be sold sequentially, or simultaneously, in which case participants could submit bids for individual licenses, or for combinations of licenses. Experts in auction theory were consulted on the design of the mechanism. John McMillan (who advised the FCC), writes that "When theorists met the policy-makers, concepts like Bayes-Nash equilibrium, incentive-compatibility constraints, and order-statistic theorems came to be discussed in the corridors of power" (1994, 146). No theoretical analysis fitted the environment of the auction well, but the experts appealed to some principles from the existing theory, the results of laboratory experiments, and experience in auctions held in New Zealand and Australia in the early 1990s in making their recommendations. The mechanism adopted in 1994 was an open ascending auction for which bids were accepted simultaneously for all licenses in each round. Experts argued that the open (as opposed to sealed-bid) format and the simultaneity of the auctions promoted an efficient outcome because at each stage the bidders could see their rivals' previous bids for all licenses.

The FCC has conducted several auctions, starting with "narrowband" licenses (each covering a sliver of the spectrum, used by paging services) and continuing with "broadband" licenses (used for voice and data communications). These auctions have provided more employment for game theorists, many of whom have advised the companies bidding for licenses. In response to growing congestion of the airwaves and the expectation that a significant part of the rapidly growing Internet traffic will move to wireless devices, in 2000 the US president Bill Clinton ordered further auctions of large parts of the spectrum (*New York Times*, October 14, 2000). Whether the auctions that have been held have allocated licenses efficiently is hard to tell, though it appears that the winners were able to obtain the sets of licenses they wanted. Certainly the auctions have been successful in generating revenue: the first four generated over US$18 billion.

9.8 Illustration: juries

9.8.1 Model

In a trial, jurors are presented with evidence concerning the guilt or innocence of a defendant. They may interpret the evidence differently. On the basis of her interpretation, each juror votes either to convict or acquit the defendant. Assume that a unanimous verdict is required for conviction: the defendant is convicted if and only if every juror votes to convict her. (This rule is used in the USA and Canada, for example.) What can we say about the chances of an innocent defendant's being convicted and a guilty defendant's being acquitted?

In deciding how to vote, each juror must consider the costs of convicting an innocent person and of acquitting a guilty person. She must consider also the likely effect of her vote on the outcome, which depends on the other jurors' votes. For example, a juror who thinks that at least one of her colleagues is likely to vote for acquittal may act differently from one who is sure that all her colleagues will vote for conviction. Thus an answer to the question requires us to consider the strategic interaction between the jurors, which we may model as a Bayesian game.

Assume that each juror comes to the trial with the belief that the defendant is guilty with probability π (the same for every juror), a belief modified by the evidence presented. We model the possibility that jurors interpret the evidence differently by assuming that for each of the defendant's true statuses (guilty and innocent), each juror interprets the evidence to point to guilt with positive probability, and to innocence with positive probability, and that the jurors' interpretations are independent (no juror's interpretation depends on any other juror's interpretation). I assume that the probabilities are the same for all jurors, and denote the probability of any given juror's interpreting the evidence to point to guilt when the defendant is guilty by p, and the probability of her interpreting the evidence to point to innocence when the defendant is innocent by q. I assume also that a juror is more likely than not to interpret the evidence correctly, so that $p > \frac{1}{2}$ and $q > \frac{1}{2}$, and hence in particular $p > 1 - q$.

Each juror wishes to convict a guilty defendant and acquit an innocent one. She is indifferent between these two outcomes, and prefers each of them to one in which an innocent defendant is convicted or a guilty defendant is acquitted. Assume specifically that each juror's Bernoulli payoffs are:

$$\begin{cases} 0 & \text{if guilty defendant convicted or innocent defendant acquitted} \\ -z & \text{if innocent defendant convicted} \\ -(1-z) & \text{if guilty defendant acquitted.} \end{cases} \tag{300.1}$$

The parameter z may be given an appealing interpretation. Denote by r the probability a juror assigns to the defendant's being guilty, given all her information. Then her expected payoff if the defendant is acquitted is $-r(1-z) + (1-r)\cdot 0 = -r(1-z)$ and her expected payoff if the defendant is convicted is $r \cdot 0 - (1-r)z = -(1-r)z$. Thus she prefers the defendant to be acquitted if $-r(1-z) > -(1-r)z$, or $r < z$, and convicted if $r > z$. That is, z is equal to the probability of guilt required for the juror to want the defendant to be convicted. Put differently, for any juror

$$\begin{array}{l} \text{acquittal is at least as good as conviction if and only if} \\ \Pr(\text{defendant is guilty, given juror's information}) \leq z. \end{array} \tag{300.2}$$

We may now formulate a Bayesian game that models the situation. The players are the jurors, and each player's action is a vote to convict (C) or to acquit (Q). We need one state for each configuration of the players' preferences and information. Each player's preferences depend on whether the defendant is guilty or innocent, and each player's information consists of her interpretation of the evidence. Thus we define a state to be a list $(X, s_1, \ldots, s_n)$, where X denotes the defendant's true status, guilty (G) or innocent (I), and s_i represents player i's interpretation of the evidence, which may point to guilt (g) or innocence (b). (I do not use i for "innocence" because I use it to index the players; b stands for "blameless".) The signal that each player i receives is her interpretation of the evidence, s_i. In any state in which $X = G$ (i.e. the defendant is guilty), each player assigns the probability p to any other player's receiving the signal g, and the probability $1 - p$ to her receiving the signal b, independently of all other players' signals. Similarly, in any state in which $X = I$ (i.e. the defendant is innocent), each player assigns the probability q to any other player's receiving the signal b, and the probability $1 - q$ to her receiving the signal g, independently of all other players' signals.

Each player cares about the verdict, which depends on the players' actions, and the defendant's true status. Given the assumption that unanimity is required to convict the defendant, only the action profile $(C, \ldots, C)$ leads to conviction. Thus (300.1) implies that player i's payoff function in the Bayesian game is defined as follows.

$$u_i(a,\omega) = \begin{cases} 0 & \text{if } a \neq (C,\ldots,C) \text{ and } \omega_1 = I \text{ or} \\ & \quad \text{if } a = (C,\ldots,C) \text{ and } \omega_1 = G \\ -z & \text{if } a = (C,\ldots,C) \text{ and } \omega_1 = I \\ -(1-z) & \text{if } a \neq (C,\ldots,C) \text{ and } \omega_1 = G, \end{cases} \tag{301.1}$$

where ω_1 is the first component of the state, giving the defendant's true status.

In summary, the following Bayesian game models the situation.

Players A set of n jurors.

States The set of states is the set of all lists $(X, s_1, \ldots, s_n)$ where $X \in \{G, I\}$ and $s_j \in \{g, b\}$ for every juror j, where $X = G$ if the defendant is guilty, $X = I$ if she is innocent, $s_j = g$ if player j receives the signal that she is guilty, and $s_j = b$ if player j receives the signal that she is innocent.

Actions The set of actions of each player is $\{C, Q\}$, where C means vote to convict, and Q means vote to acquit.

Signals The set of signals that each player may receive is $\{g, b\}$ and player j's signal function is defined by $\tau_j(X, s_1, \ldots, s_n) = s_j$ (each juror is informed only of her own signal).

Beliefs Type g of any player i believes that the state is $(G, s_1, \ldots, s_n)$ with probability $\pi p^{k-1}(1-p)^{n-k}$ and $(I, s_1, \ldots, s_n)$ with probability $(1-\pi)(1-q)^{k-1}q^{n-k}$, where k is the number of players j (including i) for whom $s_j = g$ in each case. Type b of any player i believes that the state is $(G, s_1, \ldots, s_n)$ with probability $\pi p^k(1-p)^{n-k-1}$ and $(I, s_1, \ldots, s_n)$ with probability $(1-\pi)(1-q)^{k-1}q^{n-k-1}$, where k is the number of players j for whom $s_j = g$ in each case.

Payoff functions The Bernoulli payoff function of each player i is given in (301.1).

9.8.2 Nash equilibrium

One juror Start by considering the very simplest case, in which there is a single juror. Suppose that her signal is b. To determine whether she prefers conviction or acquittal we need to find the probability she assigns to the defendant's being guilty, given her signal. We can find this probability, denoted $\Pr(G \mid b)$, by using Bayes' Rule (see Section 17.7.5, in particular (454.2)), as follows.

$$\begin{aligned} \Pr(G \mid b) &= \frac{\Pr(b \mid G)\Pr(G)}{\Pr(b \mid G)\Pr(G) + \Pr(b \mid I)\Pr(I)} \\ &= \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)} \end{aligned}$$

Thus by (300.2), acquittal yields an expected payoff at least as high as does conviction if and only if

$$z \geq \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}.$$

That is, after getting the signal that the defendant is innocent, the juror chooses acquittal as long as z is not too small—as long as she is too concerned about acquitting a guilty defendant. If her signal is g then a similar calculation leads to the conclusion that conviction yields an expected payoff at least as high as does acquittal if

$$z \leq \frac{p\pi}{p\pi + (1-q)(1-\pi)}.$$

Thus if

$$\frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)} \leq z \leq \frac{p\pi}{p\pi + (1-q)(1-\pi)} \tag{302.1}$$

then the juror optimally acts according to her signal, acquitting the defendant when her signal is b and convicting her when it is g. (A bit of algebra shows that the term on the left of (302.1) is less than the term on the right, given $p > 1 - q$.)

Two jurors Now suppose there are two jurors. Are there values for z such that the game has a Nash equilibrium in which each juror votes according to her signal? Suppose that juror 2 acts in this way: type b votes to acquit, and type g votes to convict. Consider type b of juror 1. If juror 2's signal is b, juror 1's vote has no effect on the outcome, because juror 2 votes to acquit and unanimity is required for conviction. Thus when deciding how to vote, juror 1 should ignore the possibility that juror 2's signal is b, and assume it is g. That is, juror 1 should take as evidence her signal and the fact that juror 2's signal is g. Hence, given (300.2), for type b of juror 1 acquittal is at least as good as conviction if the probability that the defendant is guilty, given juror 1's signal is b and juror 2's signal is g, is at most z. This probability is

$$\begin{aligned}\Pr(G \mid b, g) &= \frac{\Pr(b, g \mid G)\Pr(G)}{\Pr(b, g \mid G)\Pr(G) + \Pr(b, g \mid I)\Pr(I)} \\ &= \frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)}.\end{aligned}$$

Thus type b of juror 1 optimally votes for acquittal if

$$z \geq \frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)}.$$

By a similar argument, for type g of juror 1 conviction is at least as good as acquittal if

$$z \leq \frac{p^2\pi}{p^2\pi + (1-q)^2(1-\pi)}.$$

Thus when there are two jurors, the game has a Nash equilibrium in which each juror acts according to her signal, voting to acquit the defendant when her signal is b and to convict her when it is g, if

$$\frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)} \leq z \leq \frac{p^2\pi}{p^2\pi + (1-q)^2(1-\pi)}. \tag{302.2}$$

Consider the expressions on the left of (302.1) and (302.2). Divide the numerator and denominator of the expression on the left of (302.1) by $1 - p$ and the numerator and denominator of the expression on the left of (302.2) by $(1 - p)p$. Then, given $p > 1 - q$, we see that the expression on the left of (302.2) is greater than the expression on the left of (302.1). That is, the lowest value of z for which an equilibrium exists in which each juror votes according to her signal is higher when there are two jurors than when there is only one juror. Why? Because a juror who receives the signal b, knowing that her vote makes a difference only if the other juror votes to convict, makes her decision on the assumption that the other juror's signal is g, and so is less worried about convicting an innocent defendant than is a single juror in isolation.

Many jurors Now suppose the number of jurors is arbitrary, equal to n. Suppose that every juror other than juror 1 votes to acquit when her signal is b and to convict when her signal is g. Consider type b of juror 1. As in the case of two jurors, juror 1's vote has no effect on the outcome unless every other juror's signal is g. Thus when deciding how to vote, juror 1 should assume that all the other signals are g. Hence, given (300.2), for type b of juror 1 acquittal is at least as good as conviction if the probability that the defendant is guilty, given juror 1's signal is b and every other juror's signal is g, is at most z. This probability is

$$\begin{aligned}\Pr(G \mid b, g, \ldots, g) &= \frac{\Pr(b, g, \ldots, g \mid G)\Pr(G)}{\Pr(b, g, \ldots, g \mid G)\Pr(G) + \Pr(b, g, \ldots, g \mid I)\Pr(I)} \\ &= \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)}.\end{aligned}$$

Thus type b of juror 1 optimally votes for acquittal if

$$\begin{aligned}z &\geq \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)} \\ &= \frac{1}{1 + \dfrac{q}{1-p}\left(\dfrac{1-q}{p}\right)^{n-1}\dfrac{1-\pi}{\pi}}.\end{aligned}$$

Now, given that $p > 1 - q$, the denominator decreases to 1 as n increases. Thus the lower bound on z for which type b of juror 1 votes for acquittal approaches 1 as n increases. (You may check that if $p = q = 0.8$, $\pi = 0.5$, and $n = 12$, the lower bound on z exceeds 0.999999.) In particular, in a large jury, if jurors care even slightly about acquitting a guilty defendant then a juror who interprets the evidence to point to innocence will nevertheless vote for conviction. The reason is that the vote of a juror who interprets the evidence to point to innocence makes a difference to the outcome only if every other juror interprets the evidence to point to guilt, in which case the probability that the defendant is in fact guilty is very high.

We conclude that the model of a large jury in which the jurors are concerned about acquitting a guilty defendant has no Nash equilibrium in which every juror votes according to her signal. What *are* its equilibria? You are asked to find the conditions for two equilibria in the next exercise.

EXERCISE 304.1 (Signal-independent equilibria in a model of a jury) Find conditions under which the game, for an arbitrary number of jurors, has a Nash equilibrium in which every juror votes for acquittal regardless of her signal, and conditions under which every juror votes for conviction regardless of her signal.

Under some conditions on z the game has in addition a symmetric mixed strategy Nash equilibrium in which each type g juror votes for conviction, and each type b juror votes for acquittal and conviction each with positive probability. Denote by β the mixed strategy of each juror of type b. As before, a juror's vote affects the outcome only if all other jurors vote for conviction, so when choosing an action a juror should assume that all other jurors vote for conviction.

Each type b juror must be indifferent between voting for conviction and voting for acquittal, because she takes each action with positive probability. By (300.2) we thus need the mixed strategy β to be such that the probability that the defendant is guilty, given that all other jurors vote for conviction, is equal to z. Now, the probability of any given juror's voting for conviction is $p + (1-p)\beta(C)$ if the defendant is guilty and $1 - q + q\beta(C)$ if she is innocent. Thus

$$\begin{aligned}
&\Pr(G \mid \text{signal } b \text{ and } n-1 \text{ votes for } C)\\
&\quad = \frac{\Pr(b \mid G)(\Pr(\text{vote for } C \mid G))^{n-1}\Pr(G)}{\Pr(b \mid G)(\Pr(\text{vote for } C \mid G))^{n-1}\Pr(G) + \Pr(b \mid I)(\Pr(\text{vote for } C \mid I))^{n-1}\Pr(I)}\\
&\quad = \frac{(1-p)(p+(1-p)\beta(C))^{n-1}\pi}{(1-p)(p+(1-p)\beta(C))^{n-1}\pi + q(1-q+q\beta(C))^{n-1}(1-\pi)}.
\end{aligned}$$

The condition that this probability equals z implies

$$(1-p)(p+(1-p)\beta(C))^{n-1}\pi(1-z) = q(1-q+q\beta(C))^{n-1}(1-\pi)z \qquad (304.2)$$

and hence

$$\beta(C) = \frac{pX - (1-q)}{q - (1-p)X},$$

where $X = [\pi(1-p)(1-z)/((1-\pi)qz)]^{1/(n-1)}$. For a range of parameter values, $0 \le \beta(C) \le 1$, so that $\beta(C)$ is indeed a probability. Notice that when n is large, X is close to 1, and hence $\beta(C)$ is close to 1: a juror who interprets the evidence as pointing to innocence very likely nonetheless votes for conviction.

Each type g juror votes for conviction, and so must get an expected payoff at least as high from conviction as from acquittal. From an analysis like that for each type b juror, this condition is

$$p(p+(1-p)\beta(C))^{n-1}\pi(1-z) \ge (1-q)(1-q+q\beta(C))^{n-1}(1-\pi)z.$$

Given $p > \frac{1}{2}$ and $q > \frac{1}{2}$, this condition follows from (304.2).

An interesting property of this equilibrium is that the probability that an innocent defendant is convicted *increases* as n increases: the larger the jury, the *more* likely an innocent defendant is to be convicted. (The proof of this result is not simple.)

Variants The key point behind the results is that under unanimity rule a juror's vote makes a difference to the outcome only if every other juror votes for conviction. Consequently, a juror, when deciding how to vote, rationally assesses the defendant's probability of guilt under the assumption that every other juror votes for conviction. The fact that this implication of unanimity rule drives the results suggests that the Nash equilibria might be quite different if less than unanimity were required for conviction. The analysis of such rules is difficult, but indeed the Nash equilibria they generate differ significantly from the Nash equilibria under unanimity rule. In particular, the analog of the mixed strategy Nash equilibria considered above generate a probability that an innocent defendant is convicted that approaches zero as the jury size increases, as Feddersen and Pesendorfer (1998) show.

The idea behind the equilibria of the model in the next exercise is related to the ideas in this section, though the model is different.

? EXERCISE 305.1 (Swing voter's curse) Whether candidate 1 or candidate 2 is elected depends on the votes of two citizens. The economy may be in one of two states, A and B. The citizens agree that candidate 1 is best if the state is A and candidate 2 is best if the state is B. Each citizen's preferences are represented by the expected value of a Bernoulli payoff function that assigns a payoff of 1 if the best candidate for the state wins (obtains more votes than the other candidate), a payoff of 0 if the other candidate wins, and payoff of $\frac{1}{2}$ if the candidates tie. Citizen 1 is informed of the state, whereas citizen 2 believes it is A with probability 0.9 and B with probability 0.1. Each citizen may either vote for candidate 1, vote for candidate 2, or not vote.

a. Formulate this situation as a Bayesian game. (Construct the table of payoffs for each state.)

b. Show that the game has exactly two pure Nash equilibria, in one of which citizen 2 does not vote and in the other of which she votes for 1.

c. Show that one of the player's actions in the second of these equilibria is weakly dominated.

d. Why is the "swing voter's curse" an appropriate name for the determinant of citizen 2's decision in the second equilibrium?

9.9 Appendix: Analysis of auctions for an arbitrary distribution of valuations

9.9.1 First-price sealed-bid auctions

In this section I construct a symmetric equilibrium of a first-price sealed-bid auction for an arbitrary distribution F of valuations that satisfies the assumptions in Section 9.7.2. (Unlike the remainder of the book, the section uses calculus.)

The method I use to find the equilibrium is the same as the one used previously: first I find conditions satisfied by the players' best response functions, then impose the equilibrium condition that the bid of each type of each player be a best response to the bids of each type of every other player.

As before, denote the bid of type v_i of player i (i.e. player i when her valuation is v_i) by $\beta_i(v_i)$. In a symmetric equilibrium we have $\beta_i = \beta$ for every player i. A reasonable guess is that in an equilibrium the common bidding function β is increasing: bidders with higher valuations bid more. I start by making this assumption. After finding a possible equilibrium, I check that in fact the bidding function has this property.

Each player is uncertain about the other players' valuations, and hence is uncertain about the bids they will make, even though she knows the bidding function β. Denote by $G_\beta(b)$ the probability that, given β, any given player's bid is at most b. Under my assumption that β is increasing, a player's bid is at most b if and only if her valuation is at most $\beta^{-1}(b)$ (where β^{-1} is the inverse of β). Thus

$$G_\beta(b) = \Pr\{v \leq \beta^{-1}(b)\} = F(\beta^{-1}(b)).$$

Now, the expected payoff of a player with valuation v who bids b when all other players act according to the bidding function β is

$$(v - b)\Pr\{\text{Highest bid is } b\}. \qquad (306.1)$$

The probability $\Pr\{\text{Highest bid is } b\}$ is equal to the probability that all the valuations of the other $n-1$ bidders are less than b, which is $(G_\beta(b))^{n-1}$. Thus the expected payoff in (306.1) is

$$(v - b)(G_\beta(b))^{n-1}. \qquad (306.2)$$

Consider the best response function of each type of an arbitrary player. Denote the optimal bid by a player with valuation v, given that all the other players use the bidding function β, by $B_v(\beta)$. This bid maximizes the expected payoff in (306.2), and thus satisfies the condition that the derivative of this payoff with respect to b is zero:

$$-(G_\beta(B_v(\beta)))^{n-1} + (v - B_v(\beta))(n-1)(G_\beta(B_v(\beta)))^{n-2}G'_\beta(B_v(\beta)) = 0. \qquad (306.3)$$

For $(\beta^*, \ldots, \beta^*)$ to be a Nash equilibrium, we need

$$B_v(\beta^*) = \beta^*(v) \text{ for all } v.$$

That is, for every valuation v, the best response of a player with valuation v when every other player acts according to β^* must be precisely $\beta^*(v)$.

Now, from the definition of G_β we have $G_{\beta^*}(B_v(\beta^*)) = F(\beta^{*-1}(\beta^*(v)) = F(v)$, and, for any β,

$$G'_\beta(b) = F'(\beta^{-1}(b))(\beta^{-1})'(b) = \frac{F'(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}.$$

Hence $G'_{\beta^*}(\beta^*(v)) = F'(v)/\beta^{*\prime}(v)$. Thus we deduce from (306.3) that an equilibrium bidding function β^* satisfies

$$-(F(v))^{n-1} + (v - \beta^*(v))(n-1)(F(v))^{n-2}F'(v)/\beta^{*\prime}(v) = 0,$$

or

$$\beta^{*\prime}(v)(F(v))^{n-1} + (n-1)\beta^*(v)(F(v))^{n-2}F'(v) = (n-1)v(F(v))^{n-2}F'(v).$$

We may solve this differential equation by noting that the left-hand side is precisely the derivative with respect to v of $\beta^*(v)(F(v))^{n-1}$. Thus integrating both sides we obtain

$$\begin{aligned}\beta^*(v)(F(v))^{n-1} &= \int_{\underline{v}}^{v} (n-1)x(F(x))^{n-2}F'(x)\,dx \\ &= v(F(v))^{n-1} - \int_{\underline{v}}^{v} (F(x))^{n-1}\,dx\end{aligned}$$

(using integration by parts to obtain the second line). Hence

$$\beta^*(v) = v - \frac{\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{n-1}}. \qquad (307.1)$$

?? EXERCISE 307.2 (Properties of the bidding function in a first-price auction) Show that the bidding function defined in (307.1) is increasing in v for $v > \underline{v}$. Show also that a bidder with the lowest possible valuation bids her valuation, whereas a bidder with any other valuation bids less than her valuation: $\beta^*(\underline{v}) = \underline{v}$ and $\beta^*(v) < v$ for all $v > \underline{v}$ (use L'Hôpital's rule).

? EXERCISE 307.3 (Example of Nash equilibrium in a first-price auction) Verify that for the distribution F uniform from 0 to 1 the bidding function defined by (307.1) is $(1 - 1/n)v$.

The alternative expression for the Nash equilibrium bidding function discussed in the text may be derived as follows. As before, denote by $\mathbf{X}$ the random variable equal to the highest of $n-1$ independent valuations, each with cumulative distribution function F. The cumulative distribution function of $\mathbf{X}$ is H defined by $H(x) = (F(x))^{n-1}$. Thus the expected value of $\mathbf{X}$, conditional on its being less than v, is

$$\begin{aligned}\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v] &= \frac{\int_{\underline{v}}^{v} xH'(x)\,dx}{H(v)} \\ &= \frac{\int_{\underline{v}}^{v}(n-1)x(F(x))^{n-2}F'(x)\,dx}{(F(v))^{n-1}},\end{aligned}$$

which is precisely $\beta^*(v)$. (Integrating the numerator by parts.) That is, $\beta^*(v) = \mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$.

9.9.2 Revenue equivalence of auctions

I argued in the text that the expected price paid by the winner of a first-price auction is the same as the expected price paid by the winner of a second-price auction. A much more general result may be established.

Suppose that n risk neutral bidders are involved in a sealed-bid auction in which the price is an arbitrary function of the bids (not necessarily the highest, or second highest). Each player's bid affects the probability p that she wins and the expected amount $e(p)$ that she pays. Thus we can think of each bidder's choosing a value of p, and can formulate the problem of a bidder with valuation v as

$$\max_p (p \cdot v - e(p)).$$

Denote the solution of this problem by $p^*(v)$. Assuming that e is differentiable, the first-order condition for this problem implies that

$$v = e'(p^*(v)) \text{ for all } v.$$

Integrating both sides of this equation we have

$$e(p^*(v)) = e(p^*(\underline{v})) + \int_{\underline{v}}^{v} x\, dp^*(x). \tag{308.1}$$

Now consider an equilibrium with the property that the object is sold to the bidder with the highest valuation, so that $p^*(v) = \Pr\{\mathbf{X} < v\}$, and the expected payoff $e(p^*(\underline{v})) = 0$ of a bidder with the lowest possible valuation is zero. In any such equilibrium, (308.1) implies that the expected payment $e(p^*(v))$ of a bidder with any given valuation v is independent of the price-determination rule in the auction, equal to $\Pr(\mathbf{X} < v)\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$.

This result generalizes the earlier observation that the expected payments of bidders in the Nash equilibria of first- and second-price auctions in which the bidders' valuations are independent and private are the same. It is a special case of the more general *revenue equivalence principle*, which applies to a class of common value auctions, as well as private value auctions, and may be stated as follows.

> *Suppose that each bidder (i) is risk neutral, (ii) independently receives a signal from the same distribution, which satisfies the conditions on the distribution of valuations in Section 9.7.2, and (iii) has a valuation that may depend on all the bidders' signals. Consider auction mechanisms in the symmetric Nash equilibria of which the object is sold to the bidder with the highest signal and the expected payoff of a bidder with the lowest possible valuation is zero. In the symmetric Nash equilibrium of any such mechanism the expected payment of a bidder of any given type is the same, and hence the auctioneer's expected revenue is the same.*

Notes

The notion of a general Bayesian game was defined and studied by Harsanyi (1967/68). The formulation I describe here is taken (with a minor change) from Osborne and Rubinstein (1994, Section 2.6).

The origin of the observation that more information may hurt (Section 9.4.1) is unclear. The idea of "infection" in Section 9.4.2 was first studied by Rubinstein (1989). The game in Figure 282.1 is a variant suggested by Eddie Dekel of the one analyzed by Morris, Rob, and Shin (1995).

Games modeling voluntary contributions to a public good were first considered by Olson (1965, Section I.D), and have been subsequently much studied. The model in Section 9.6 is a variant of one in an unpublished paper of William F. Samuelson dated 1984.

Vickrey (1961) initiated the study of auctions described in Section 9.7. First-price common value auctions (Section 9.7.3) were first studied by Wilson (1967, 1969, 1977). The "winner's curse" appears to have been first articulated by Capen, Clapp, and Campbell (1971). The general revenue equivalence principle at the end of Section 9.9.2 is due to Myerson (1981) and Riley and Samuelson (1981); their results are generalized by Bulow and Klemperer (1996, Lemma 3). The equilibria in Exercise 297.1 are described by Milgrom (1981, Theorem 6.3). The literature is surveyed by Klemperer (1999). The box on spectrum auctions on page 298 is based on McMillan (1994), Cramton (1995, 1997, 1998), and McAfee and McMillan (1996).

Section 9.8 is based on Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1996).

Exercise 280.2 was suggested by Ariel Rubinstein. Exercise 280.3 is based on Brams, Kilgour, and Davis (1993). A model of adverse selection was first studied by Akerlof (1974); the model in Exercise 280.4 is taken from Samuelson and Bazerman (1985). Exercise 305.1 is based on Feddersen and Pesendorfer (1996).

11 Strictly Competitive Games and Maxminimization

Prerequisite: Chapters 2 and 4.

11.1 Introduction

THE NOTION of Nash equilibrium (studied in Chapters 2, 3, and 4) models a steady state. The idea is that each player, through her experience playing the game against various opponents, knows the actions that the other players in the game will take, and chooses her action in light of this knowledge.

In this chapter and the next, we study the likely outcome of a game from a different angle. We consider the implications of each player's forming a belief about the other players' actions not from her experience, but from her analysis of the game.

In this chapter we focus on two-player strictly competitive games, in which the players' interests are diametrically opposed. In such games a simple decision-making procedure leads each player to choose a Nash equilibrium action.

11.2 Definitions and examples

You are confronted with a game for the first time; you have no idea what actions your opponents will take. How should you choose your action? A conservative criterion entails your working under the assumption that whatever you do, your opponents will take the worst possible action for you. For each of your actions, you look at all the outcomes that can occur, as the other players choose different actions, and find the one that gives you the lowest payoff. Then you choose the action for which this lowest payoff is largest. This procedure for choosing an action is called **maxminimization**.

Many of the interesting examples of this procedure involve mixed strategies, so from the beginning I define the concepts for a strategic game with vNM preferences (Definition 103.1), though the ideas do not depend upon the players' randomizing. Let U_i be an expected payoff function that represents player i's preferences on lotteries over action profiles in a strategic game. For any given mixed strategy α_i of player i, the lowest payoff that she obtains, for any possible vector α_{-i}

of mixed strategies of the other players, is

$$\min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}).$$

A maxminimizing mixed strategy for player i is a mixed strategy that maximizes this minimal payoff.

▶ DEFINITION 336.1 A **maxminimizing mixed strategy** for player i in a strategic game (with vNM payoffs) is a mixed strategy α_i^* that solves the problem

$$\max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}),$$

where U_i is player i's vNM payoff function.

In words, a maxminimizing strategy for player i maximizes her payoff under the (pessimistic) assumption that whatever she does the other players will act in such a way as to minimize her expected payoff.

A different way of looking at a maxminimizing strategy is useful. Say that a mixed strategy α_i **guarantees** player i the payoff $\overline{u}_i$ if, no matter what mixed strategies α_{-i} the other players use, i's payoff is at least $\overline{u}_i$:

$$u_i(\alpha_i, \alpha_{-i}) \geq \overline{u}_i \text{ for every list } \alpha_{-i} \text{ of the other players' mixed strategies.}$$

A maxminimizing mixed strategy maximizes the payoff that a player can guarantee: if α_i^* is a maxminimizer then

$$\min_{\alpha_{-i}} u_i(\alpha_i^*, \alpha_{-i}) \geq \min_{\alpha_{-i}} u_i(\alpha_i, \alpha_{-i}) \text{ for every mixed strategy } \alpha_i \text{ of player } i.$$

◆ EXAMPLE 336.2 (Maxminimizers in a bargaining game) Consider the game in Exercise 36.2, restricting attention to pure strategies (actions). If you demand any amount x up to \$5 then your payoff is x regardless of the other player's action. If you demand \$6 then you may get \$6 (if the other player demands \$4 or less, or \$7 or more), but you may get only \$5 (if the other player demands \$5 or \$6). If you demand $x \geq \$7$ then you may get x (if the other player demands at most $\$(10 - x)$), but you may get only $\$(11 - x)$ (if the other player demands $x - 1$). For each amount that you can demand, the smallest amount that you may get is given in Figure 337.1. Maxminimization in this game thus leads each player to demand either \$5 or \$6 (for both of which the worst possible outcome is that the player receives \$5).

Why should you assume that the other players will take actions that minimize your payoff? In some games such an assumption is not sensible. But if you have only one opponent and her interests in the game are diametrically opposed to yours—in which case we call the game *strictly competitive*—then the assumption may be reasonable. In fact, it turns out that in such games there is a very close relationship between the outcome that occurs if each player maxminimizes and

Amount demanded	0	1	2	3	4	5	6	7	8	9	10
Smallest amount obtained	0	1	2	3	4	5	5	4	3	2	1

Figure 337.1 The lowest payoffs that a player receives in the game in Exercise 36.2 for each of her possible actions, as the other player's action varies.

the Nash equilibrium outcome. Another reason that you may be attracted to a maxminimizing action is that such an action maximizes the payoff that you can guarantee: there is no other action that yields a higher payoff no matter what the other players do.

In the game in Example 336.2 we restricted attention to pure strategies. The following example shows that a player may be able to guarantee a higher payoff by using a mixed strategy, and illustrates how a maxminimizing mixed strategy may be found.

EXAMPLE 337.1 (Example of maxminimizers) Consider the game in Figure 337.2. If player 1 chooses T then the worse that can happen is that player 2 chooses R; if player 1 chooses B then the worst that can happen is that player 2 chooses L. In both cases player 1's payoff is -1, so that if player 1 is restricted to choose either T or B then there is nothing to choose between them; both guarantee her a payoff of -1.

	L	R
T	$2, -2$	$-1, 1$
B	$-1, 1$	$1, -1$

Figure 337.2 The game in Example 337.1.

However, player 1 can do better if she randomizes between T and B. Let p be the probability she assigns to T. To find her maxminimizing mixed strategy it is helpful to refer to Figure 338.1. The upward-sloping line indicates player 1's expected payoff, as p varies, if player 2 chooses the action L; the downward-sloping line indicates player 1's expected payoff, as p varies, if player 2 chooses R. Player 1's expected payoff if player 2 randomizes lies between the two lines; in particular it lies above the lower line. Thus for each value of p, the lower of the two lines indicates the lowest payoff that player 1 can obtain if she chooses that value of p. That is, the lowest payoff that player 1 can obtain for each value of p is indicated by the heavy inverted V; the maxminimizing mixed strategy of player 1 is thus $p = \frac{2}{5}$, which yields her a payoff of $\frac{1}{5}$.

The maxminimizing mixed strategy of player 1 in this example has the property that it yields player 1 the same payoff whether player 2 chooses L or R. Note that the indifference here is different from that in a Nash equilibrium, in which player 1's mixed strategy yields *player 2* the same payoff to each of her actions.

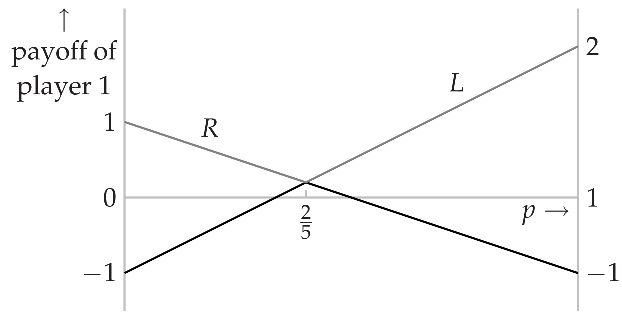

Figure 338.1 The expected payoff of player 1 in the game in Figure 337.2 for each of player 2's actions, as a function of the probability p that player 1 assigns to T.

What is the relation between Nash equilibrium strategies and maxminimizers? In the next section I show that for the class of strictly competitive games the relation is very close. In an *arbitrary* game, whether strictly competitive or not, a player's Nash equilibrium payoff is *at least* her maxminimized payoff.

■ LEMMA 338.1 *The payoff of each player in any Nash equilibrium of a strategic game is at least equal to her maxminimized payoff.*

Proof. Let (α_1^*, α_2^*) be a Nash equilibrium. Consider player 1. First note that by the definition of a Nash equilibrium,

$$U_1(\alpha_1^*, \alpha_2^*) \geq U_1(\alpha_1, \alpha_2^*) \text{ for every mixed strategy } \alpha_1 \text{ of player 1,}$$

so that

$$U_1(\alpha_1^*, \alpha_2^*) \geq \min_{\alpha_2} U_1(\alpha_1, \alpha_2) \text{ for every mixed strategy } \alpha_1 \text{ of player 1.}$$

Since the inequality holds for every mixed strategy α_1 of player 1, we conclude that

$$U_1(\alpha_1^*, \alpha_2^*) \geq \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2),$$

as required. □

? EXERCISE 338.2 (Nash equilibrium payoffs and maxminimized payoffs) Give an example of a game with a unique Nash equilibrium in which each player's Nash equilibrium payoff exceeds her maxminimized payoff.

11.3 Strictly competitive games

A strictly competitive game is a strategic game in which there are two players, whose preferences are diametrically opposed: whenever one player prefers some outcome a to another outcome b, the other players prefers b to a. Assume for convenience that the players' names are "1" and "2". If we restrict attention to pure strategies then we have the following definition.

▷ DEFINITION 339.1 (*Strictly competitive strategic game with ordinal preferences*) A strategic game with ordinal preferences is **strictly competitive** if it has two players and

$$(a_1, a_2) \succsim_1 (b_1, b_2) \text{ if and only if } (b_1, b_2) \succsim_2 (a_1, a_2),$$

where (a_1, a_2) and (b_1, b_2) are pairs of actions.

Note that it follows from this definition that in a strictly competitive game we have $(a_1, a_2) \sim_1 (b_1, b_2)$ if and only if $(a_1, a_2) \sim_2 (b_1, b_2)$ (since $(a_1, a_2) \sim_2 (b_1, b_2)$ implies both $(a_1, a_2) \succsim_1 (b_1, b_2)$ and $(b_1, b_2) \succsim_1 (a_1, a_2)$) and $(a_1, a_2) \succ_1 (b_1, b_2)$ if and only if $(b_1, b_2) \succ_2 (a_1, a_2)$.

Note also that there are payoff functions representing the players' preferences in a strictly competitive game with the property that the sum of the players' payoffs is zero for every action profile. (For example, we can assign payoffs as follows: 0 to both players for the worst outcome for player 1, 1 to player 1 and -1 to player 2 for the next worst outcome for player 1, and so on.) For this reason a strictly competitive game is sometimes referred to as a **zerosum** game.

The *Prisoner's Dilemma* (Figure 13.1) is not strictly competitive since both players prefer $(Quiet, Quiet)$ to $(Fink, Fink)$. *BoS* (Figure 16.1) is not strictly competitive either, since (for example) both players prefer (B, B) to (S, B). *Matching Pennies* (Figure 17.1), on the other hand, *is* strictly competitive: player 1's preference ordering over the four outcomes is precisely the reverse of player 2's. The game in Figure 339.1 is also strictly competitive: player 1's preference ordering is $(B, R) \succ_1 (T, L) \succ_1 (B, L) \succ_1 (T, R)$, the reverse of player 2's ordering $(T, R) \succ_1 (B, L) \succ_1 (T, L) \succ_1 (B, R)$.

	L	R
T	$2, 1$	$0, 5$
B	$1, 3$	$5, 0$

Figure 339.1 A strategic game. If attention is restricted to pure strategies then the game is strictly competitive. If mixed strategies are considered, however, it is not.

If we consider mixed strategies, then the appropriate definition of a strictly competitive game is the following.

▷ DEFINITION 339.2 (*Strictly competitive strategic game with vNM preferences*) A strategic game with vNM preferences is **strictly competitive** if it has two players and

$$U_1(\alpha_1, \alpha_2) \geq U_1(\beta_1, \beta_2) \text{ if and only if } U_2(\beta_1, \beta_2) \geq U_2(\alpha_1, \alpha_2),$$

where (α_1, α_2) and (β_1, β_2) are pairs of mixed strategies and U_i is player i's expected payoff as a function of the pair of mixed strategies (her vNM payoff function).

As for the case of games in which we restrict attention to pure strategies, there are payoff functions representing the players' preferences in a strictly competitive game with the property that the sum of the players' payoffs is zero for every action profile. To see this, let u_i, for each player i, represent i's preferences in a strictly competitive game. Denote by $\overline{a}$ and $\underline{a}$ the best and worst outcomes respectively for player 1. Now choose another representation v_i with the property that $v_1(\overline{a}) = 1$ and $v_1(\underline{a}) = 0$, and $v_2(\overline{a}) = -1$ and $v_2(\underline{a}) = 0$. (Why is it possible to do this?) Let a be any outcome and let $p = u_1(a)$. Then $u_1(a) = pu_1(\overline{a}) + (1-p)u_1(\underline{a})$. But since the game is strictly competitive we have $u_2(a) = pu_2(\underline{a}) + (1-p)u_2(\overline{a}) = -p$. Hence $u_1(a) + u_2(a) = 0$. Thus if player 2's preferences are not represented by the payoff function $-u_1$ then we know that the game is not strictly competitive.

Any game that is strictly competitive when we allow mixed strategies is clearly strictly competitive when we restrict attention to pure strategies, but the converse is not true. Consider, for example, the game in Figure 339.1, interpreting the numbers in the boxes as vNM payoffs. In this game player 1 is indifferent between the outcome (T, L) and the lottery in which (T, R) occurs with probability $\frac{3}{5}$ and (B, R) occurs with probability $\frac{2}{5}$ (since $\frac{3}{5} \cdot 0 + \frac{2}{5} \cdot 5 = 2$), but player 2 is not indifferent between these two outcomes (her payoff to (T, L) is 1, while her expected payoff to the lottery is $\frac{3}{5} \cdot 5 + \frac{2}{5} \cdot 0 = 3$).

? EXERCISE 340.1 (Determining strict competitiveness) Are either of the two games in Figure 340.1 strictly competitive (*a*) if we restrict attention to pure strategies and (*b*) if we allow mixed strategies?

	L	R
U	$1, -1$	$3, -5$
D	$2, -3$	$1, -1$

	L	R
U	$1, -1$	$3, -6$
D	$2, -3$	$1, -1$

Figure 340.1 The games in Exercise 340.1.

We saw above that in any game a player's Nash equilibrium payoff is at least her maxminimized payoff. I now show that for a strictly competitive game that possesses a Nash equilibrium, the two payoffs are the same: a pair of actions is a Nash equilibrium if and only if the action of each player is a maxminimizer. Denote player i's vNM payoff function by U_i and assume, without loss of generality, that $U_2 = -U_1$.

Though the proof may look complicated, the ideas it entails are very simple; the arguments involve no more than the manipulation of inequalities. The following fact is used in the argument. The maximum of any function f is equal to the negative of the minimum of $-f$: $\max_x f(x) = -\min_x(-f(x))$. It follows that

$$\begin{aligned}\max_{\alpha_2}\min_{\alpha_1} U_2(\alpha_1,\alpha_2) &= \max_{\alpha_2}\min_{\alpha_1}(-U_1(\alpha_1,\alpha_2)) \\ &= \max_{\alpha_2}(-\max_{\alpha_1} U_1(\alpha_1,\alpha_2))\end{aligned}$$

so that

$$\max_{\alpha_2}\min_{\alpha_1} U_2(\alpha_1,\alpha_2) = -\min_{\alpha_2}\max_{\alpha_1} U_1(\alpha_1,\alpha_2). \tag{340.2}$$

■ PROPOSITION 341.1 (Nash equilibrium strategies and maxminimizers of strictly competitive games) *Consider a strictly competitive strategic game with vNM preferences. Denote the vNM payoff function of each player i by U_i.*

a. *If (α_1^*, α_2^*) is a Nash equilibrium then α_1^* is a maxminimizer for player 1, α_2^* is a maxminimizer for player 2, and* $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha_1^*, \alpha_2^*)$.

b. *If α_1^* is a maxminimizer for player 1, α_2^* is a maxminimizer for player 2, and* $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$ *(and thus, in particular, if the game has a Nash equilibrium (see part a)), then (α_1^*, α_2^*) is a Nash equilibrium.*

Proof. I first prove part a. By the definition of Nash equilibrium we have

$$U_2(\alpha_1^*, \alpha_2^*) \geq U_2(\alpha_1^*, \alpha_2) \text{ for every mixed strategy } \alpha_2 \text{ of player 2}$$

or, since $U_2 = -U_1$,

$$U_1(\alpha_1^*, \alpha_2^*) \leq U_1(\alpha_1^*, \alpha_2) \text{ for every mixed strategy } \alpha_2 \text{ of player 2}.$$

Hence

$$U_1(\alpha_1^*, \alpha_2^*) = \min_{\alpha_2} U_1(\alpha_1^*, \alpha_2).$$

Now, the function on the right hand side of this equality is evaluated at the specific strategy α_1^*, so that its value is not more than the maximum as we vary α_1, namely $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2)$. Thus we conclude that

$$U_1(\alpha_1^*, \alpha_2^*) \leq \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2).$$

Now, from Lemma 338.1 we have the opposite inequality: a player's Nash equilibrium payoff is at least her maxminimized payoff. Thus $U_1(\alpha_1^*, \alpha_2^*) = \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2)$, so that α_1^* is a maxminimizer for player 1.

An analogous argument for player 2 establishes that α_2^* is a maxminimizer for player 2 and $U_2(\alpha_1^*, \alpha_2^*) = \max_{\alpha_2} \min_{\alpha_1} U_2(\alpha_1, \alpha_2)$. From (340.2) we deduce that $U_1(\alpha_1^*, \alpha_2^*) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$, completing the proof of part *a*.

To prove part *b*, let

$$v^* = \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2).$$

From (340.2) we have $\max_{\alpha_2} \min_{\alpha_1} U_2(\alpha_1, \alpha_2) = -v^*$. Since α_1^* is a maxminimizer for player 1 we have $U_1(\alpha_1^*, \alpha_2) \geq v^*$ for every mixed strategy a_2 of player 2; since α_2^* is a maxminimizer for player 2 we have $U_2(\alpha_1, \alpha_2^*) \geq -v^*$ for every mixed strategy α_1 of player 1. Letting $\alpha_2 = \alpha_2^*$ and $\alpha_1 = \alpha_1^*$ in these two inequalities we obtain $U_1(\alpha_1^*, \alpha_2^*) \geq v^*$ and $U_2(\alpha_1^*, \alpha_2^*) \geq -v^*$, or $U_1(\alpha_1^*, \alpha_2^*) \leq v^*$, so that $U_1(\alpha_1^*, \alpha_2^*) = v^*$. Thus

$$U_1(\alpha_1^*, \alpha_2) \geq U_1(\alpha_1^*, \alpha_2^*) \text{ for every mixed strategy } \alpha_2 \text{ of player 2,}$$

or

$$U_2(\alpha_1^*, \alpha_2) \leq U_2(\alpha_1^*, \alpha_2^*) \text{ for every mixed strategy } \alpha_2 \text{ of player 2.}$$

Similarly,

$$U_2(\alpha_1, \alpha_2^*) \geq U_2(\alpha_1^*, \alpha_2^*) \text{ for every mixed strategy } \alpha_1 \text{ of player 1,}$$

or

$$U_1(\alpha_1, \alpha_2^*) \leq U_1(\alpha_1^*, \alpha_2^*) \text{ for every mixed strategy } \alpha_1 \text{ of player 1,}$$

so that (α_1^*, α_2^*) is a Nash equilibrium of the game. □

This result is of interest not only because it shows the close relation between the Nash equilibria and maxminimizers in a strictly competitive game, but also because it reveals properties of Nash equilibria in a strictly competitive game that are independent of the notion of maxminimization.

First, part *a* of the result implies that the Nash equilibrium payoff of each player in a strictly competitive game is unique.

■ COROLLARY 342.1 *Every Nash equilibrium of a strictly competitive game yields the same pair of payoffs.*

As we have seen, this property of Nash equilibria is not necessarily satisfied in games that are not strictly competitive (consider *BoS* (Figure 16.1), for example).

Second, the result implies that a Nash equilibrium of a strictly competitive game can be found by solving the problem $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2)$. Further, if we know player 1's equilibrium payoff then any mixed strategy that yields this payoff when player 2 uses any of her pure strategies solves the maxminimization problem, and hence is an equilibrium mixed strategy of player 1. This fact is sometimes useful when calculating the mixed strategy equilibria of a game when we know the equilibrium payoffs before we have found the equilibrium strategies (see, for example, Exercise 344.2).

Third, suppose that (α_1, α_2) and (α_1', α_2') are Nash equilibria of a strictly competitive game. Then by part *a* of the result the strategies α_1 and α_1' are maxminimizers for player 1, the strategies α_2 and α_2' are maxminimizers for player 2, and

$$\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha_1, \alpha_2) = U_1(\alpha_1', \alpha_2').$$

But then by part *b* of the result both (α_1, α_2') and (α_1', α_2) are Nash equilibria of the game. That is, the result implies that Nash equilibria of a strictly competitive game have the following property.

■ COROLLARY 342.2 *The Nash equilibria of a strictly competitive game are interchangeable: if (α_1, α_2) and (α_1', α_2') are Nash equilibria then so are (α_1, α_2') and (α_1', α_2).*

The game *BoS* shows that the Nash equilibria of a game that is not strictly competitive are not necessarily interchangeable.

Part *a* of Proposition 341.1 shows that for any strictly competitive game that has a Nash equilibrium we have

$$\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2).$$

Note that the inequality

$$\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) \leq \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$$

holds more generally: for any α_1' we have $U_1(\alpha_1', \alpha_2) \leq \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$ for all α_2, so that $\min_{\alpha_2} U_1(\alpha_1', \alpha_2) \leq \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$. Thus in *any* game (whether or not it is strictly competitive) the payoff that player 1 can guarantee herself is *at most* the amount that player 2 can hold her down to.

If $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$ then we say that this payoff, the equilibrium payoff of player 1, is the **value** of the game. An implication of Proposition 341.1 is that any equilibrium strategy of player 1 *guarantees* that her payoff is *at least* v^*, and any equilibrium strategy of player 2 *guarantees* that player 1's payoff is *at most* v^*.

■ COROLLARY 343.1 *Any Nash equilibrium strategy of player 1 in a strictly competitive game guarantees that her payoff is at least the value of the game, and any Nash equilibrium strategy of player 2 guarantees that player 1's payoff is at most the value.*

Proof. For $i = 1, 2$, let α_i^* be an equilibrium strategy of player i and let v^* be the value of the game. By Proposition 341.1*a*, α_1^* is a maxminimizer, so that it guarantees that player 1's payoff is at least v^*:

$$U_1(\alpha_1^*, \alpha_2) \geq \min_{\alpha_2} U_1(\alpha_1^*, \alpha_2) = \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1^*, \alpha_2) = v^*.$$

Similarly, any equilibrium strategy of player 2 guarantees that her payoff is at least her equilibrium payoff $-v^*$; or, equivalently, any equilibrium strategy of player 2 guarantees that player 1's payoff is at most v^*. □

In a game that is not strictly competitive a player's equilibrium strategy does not in general have these properties, as the following exercise shows.

? EXERCISE 343.2 (Maxminimizers in *BoS*) For the game *BoS* (Figure 16.1) find the maxminimizer of each player. Show for each equilibrium, the strategy of neither player guarantees her equilibrium payoff.

? EXERCISE 343.3 (Increasing payoffs and eliminating actions in strictly competitive games) Let G be a strictly competitive game that has a Nash equilibrium.

a. Show that if some of player 1's payoffs in G are increased in such a way that the resulting game G' is strictly competitive then G' has no equilibrium in which player 1 is worse off than she was in an equilibrium of G. (Note that G' may have no equilibrium at all.)

b. Show that the game that results if player 1 is prohibited from using one of her actions in G does not have an equilibrium in which player 1's payoff is higher than it is in an equilibrium of G.

c. Give examples to show that neither of the above properties necessarily holds for a game that is not strictly competitive.

? EXERCISE 344.1 (Equilibrium in strictly competitive games) Either prove or give a counterexample to the claim that if the equilibrium payoff of player 1 in a strictly competitive game is v then any strategy pair that gives player 1 a payoff of v is an equilibrium.

? EXERCISE 344.2 (Guessing Morra) In the two-player game "Guessing Morra", each player simultaneously holds up one or two fingers and also guesses the total shown. If exactly one player guesses correctly then the other player pays her the amount of her guess (in \$, say). If either both players guess correctly or neither does so then no payments are made.

a. Specify this situation as a strategic game.

b. Use the symmetry of the game to show that the unique equilibrium payoff of each player is 0.

c. Find the mixed strategies of player 1 that guarantee that her payoff is at least 0, and hence find all the mixed strategy equilibria of the game.

? EXERCISE 344.3 (O'Neill's game) Consider the game in Figure 344.1.

a. Find a completely mixed Nash equilibrium in which each player assigns the same probability to the actions 1, 2, and 3.

b. Use the facts that in a strictly competitive game the players' equilibrium payoffs are unique and each player's equilibrium strategy guarantees her payoff is at least her equilibrium payoff to show that the equilibrium you found in part *a* is the only equilibrium of the game.

	1	2	3	J
1	$-1, 1$	$1, -1$	$1, -1$	$-1, 1$
2	$1, -1$	$-1, 1$	$1, -1$	$-1, 1$
3	$1, -1$	$1, -1$	$-1, 1$	$-1, 1$
J	$-1, 1$	$-1, 1$	$-1, 1$	$1, -1$

Figure 344.1 The game in Exercise 344.3.

MAXMINIMIZATION: SOME HISTORY

The theory of maxminimization in general strictly competitive games was developed by John von Neumann in the late 1920's. However, the idea of maxminimization in the context of a specific game appeared two centuries earlier. In 1713 or 1714

Pierre Rémond de Montmort, a Frenchman who "devoted himself to religion, philosophy, and mathematics" (Todhunter (1865, p. 78)) published *Essay d'analyse sur les jeux de hazard* (Analytical essay on games of chance), in which he reported correspondence with Nikolaus Bernoulli (a member of the Swiss family of scientists and mathematicians). Among the correspondence is a letter in which Montmort describes a letter (dated November 13, 1713) he received from "M. de Waldegrave" (probably Baron Waldegrave of Chewton, a British noble born and educated in France). Montmort, Bernoulli, and Waldegrave had been corresponding about the two-player card game *le Her* ("the gentleman").

This two player game uses an ordinary deck of cards. Each player is first dealt a single card, which she alone sees. The object is to hold a card with a higher value than your opponent, with the ace counted as 1 and the jack, queen, and king counted as 11, 12, and 13 respectively. After each player has received her card, player 1 can, if she wishes, exchange her card with that of player 2, who must make the exchange unless she holds a king, in which case she is automatically the winner. Then, whether or not player 1 exchanges her card, player 2 has the option of exchanging hers for a card randomly selected from the remaining cards in the deck; if the randomly selected card is a king she automatically loses, and otherwise she makes the exchange. Finally, the players compare their cards and the one whose card has the higher value wins; if both cards have the same value then player 2 wins.

We can view this situation as a strategic game in which an action for player 1 is a rule that says, for each possible card that she may receive, whether she *keeps* or *exchanges* the card. For example, one possible action is to *exchange* any card with value up to 5 and to *keep* any card with higher value; another possible action is to *exchange* any even card and to *keep* any odd card. Since there are 13 different values of cards, player 1 has 2^{13} actions. If player 1 exchanges her card then player 2 knows both cards being held, and she should clearly exchange with a random card from the deck if and only if the card she hold would otherwise lose. If player 1 does not exchange her card then player 2's decision of whether to exchange or not is not as clear. As for player 1 at the start of the game, an action of player 2 is a rule that says, for each possible card that she holds, whether to *keep* or *exchange* the card. Like player 1, player 2 has 2^{13} actions.

Montmort, Bernoulli, and Waldegrave had argued that the only actions that could possibly be optimal are "*exchange* up to 6 and *keep* 7 and over" or "*exchange* up to 7 and *keep* 8 and over" for player 1, and "*exchange* up to 7 and *keep* 8 and over" or "*exchange* up to 8 and *keep* 9 and over" for player 2. When the players are restricted to use only these actions the game is equivalent to

$0, \quad 0$	$5, -5$
$3, -3$	$0, \quad 0$

The three scholars had corresponded about which of these actions is best. As you can see, the best action for each player depends on the other player's action, and the game has no pure strategy Nash equilibrium. Waldegrave made the key conceptual leap of considering the possibility that the players randomize. He observed that if player 1 uses the mixed strategy $(\frac{3}{8}, \frac{5}{8})$ then her payoff is the same regardless of player 2's action, and guarantees her a payoff of $\frac{15}{8}$, and that if player 2 uses the mixed strategy $(\frac{5}{8}, \frac{3}{8})$ then she ensures that player 1's payoff is no more than $\frac{15}{8}$.

That is, Waldegrave found the maxminimizers for each player and appreciated their significance; Montmort wrote to Bernoulli that "it seems to me that [Waldegrave's letter] exhausts everything that one can say on [the players' behavior in *le Her*]".

The decision criterion of maxminimization seems to be conservative. In particular, in any game, a player's Nash equilibrium payoff is at least her maxminimized payoff. We have seen that in strictly competitive games the two are equal, and the notions of Nash equilibrium and maxminimizing yield the same predictions. In some games that are not strictly competitive the two payoffs are also equal. The next example gives such an example, in which the notions of Nash equilibrium and maxminimization do not yield the same outcome and, from a decision-theoretic viewpoint, a maxminimizer seems preferable to a Nash equilibrium strategy.

◆ EXAMPLE 346.1 (Maxminimizers *vs.* Nash equilibrium actions) The game in Figure 346.1 has a unique Nash equilibrium, in which player 1's strategy is $(\frac{1}{4}, \frac{3}{4})$ and player 2's strategy is $(\frac{2}{3}, \frac{1}{3})$. In this equilibrium player 1's payoff is 4.

	L	R
T	$6,0$	$0,6$
B	$3,2$	$6,0$

Figure 346.1 A strategic game.

Now consider the maxminimizer for player 1. Player 1's payoff as a function of the probability that she assigns to T is shown in Figure 347.1. We see that the maxminimizer for player 1 is $(\frac{1}{3}, \frac{2}{3})$, and this strategy guarantees player 1 a payoff of 4.

Thus in this game player 1's maxminimizer guarantees that she obtain her payoff in the unique equilibrium, while her equilibrium strategy does not. If player 1 is certain that player 2 will adhere to the equilibrium then her equilibrium strategy yields her equilibrium payoff of 4, but if player 2 chooses a different strategy then player 1's payoff may be less than 4 (it also may be greater than 4). Player 1's maxminimizer, on the other hand, guarantees a payoff of 4 regardless of player 2's behavior.

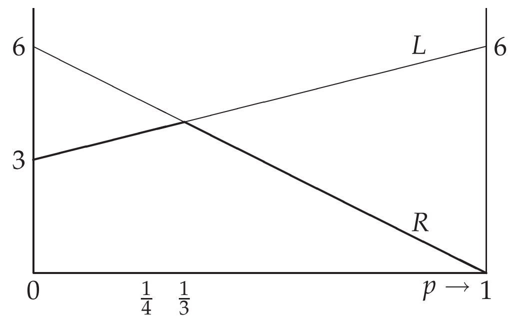

Figure 347.1 The expected payoff of player 1 in the game in Figure 346.1 for each of player 2's actions, as a function of the probability p that player 1 assigns to T.

Testing the theory of Nash equilibrium in strictly competitive games

The theory of maxminimization makes a sharp prediction about the outcome of a strictly competitive game. Does human behavior correspond to this prediction?

In designing an experiment, we face the problem in a general game of inducing the appropriate preferences. We can avoid this problem by working with games with only two outcomes. In such games the players' preferences are represented by the monetary payoffs of the outcomes, so that we do not need to control for subjects' risk attitudes.

O'Neill (1987) conducted an experiment with such a game. He confronted people with the game in Exercise 344.3, in which each player's equilibrium strategy is $(0.2, 0.2, 0.2, 0.4)$. In order to collect a large amount of data he had each of 25 pairs of people play the game 105 times. This design raises two issues. First, when players confront each other repeatedly, strategic possibilities that are absent from a one-shot game emerge: each player may condition her current action on her opponent's past actions. However, an analysis that takes into account these strategic options leads to the conclusion that, for the game used in the experiment, the players will eschew them. Second, in the experiment, each subject faced more than two possible outcomes. However, under the hypothesis that each player's preferences are separable between the different trials, these preferences in any trial are still represented by the expected monetary payoffs.

Each subject was given US\$2.50 in cash at the start of the game, was paid US\$0.05 for every win, and paid her opponent US\$0.05 for every loss. On average, subjects in the role of player 1 chose the actions with probabilities $(0.221, 0.215, 0.203, 0.362)$ and subjects in the role of player 2 chose them with probabilities $(0.226, 0.179, 0.169, 0.426)$. These observed frequencies seem fairly close to those predicted by the theory of maxminimization. But how can we measure closeness? A standard statistical test (χ^2) asks the question: if each player used exactly her equilibrium strategy, what is the probability of the observed frequencies deviating at least as much from the predicted ones? Applying this test to the aggregate data on the frequencies of the 16 possible outcomes of the game leads to minmax behavior being decisively rejected (the probability of a deviation from the prediction at least as large as that observed is less than 1 in 1,000). Other tests on O'Neill's data also reject the minmax hypothesis (Brown and Rosenthal (1990)).

In a variant of O'Neill's experiment, with considerably higher stakes and a somewhat more complicated game, the evidence also does not support maxminimization, although maxminimization explains the data better than two alternative theories (Rapoport and Boebel (1992)). (Of course, it's relatively easy to design a theory that works well in *one particular* game; in order to "understand" behavior we want a theory that works well in a large class of games.) In summary, the evidence so far tends not to support the theory of maxminimization, although no other theory is systematically superior.

Notes

The material in the box on page 344 is based on Todhunter (1865) and Kuhn (1968). Guilbaud (1961) rediscovered Monmort's report of Waldegrave's work.

12 Rationalizability

Prerequisite: Chapters 2 and 4.

12.1 Introduction

WHAT outcomes in a strategic game are consistent with the players' analyses of each others' rational behavior? The main solution notion we have studied so far, Nash equilibrium, is not designed to address this question, but rather models a steady state in which each player has learned the other players' actions from her long experience playing the game. In this chapter I discuss an approach to the question that considers players who carefully study a game, deducing their opponents' rational actions from their knowledge of their opponents' preferences and analyses of their opponents' reasoning about their rational actions.

Suppose that we model each player's decision problem as follows. She forms a probabilistic belief about the other players' actions, and chooses her action (or mixed strategy) to maximize her expected payoff given this probabilistic belief. We say that a player who behaves in this way is *rational*. Precisely, suppose that player i's preferences are represented by the expected value of the Bernoulli payoff function u_i. Denote by μ_i her probabilistic *belief* about the other players' actions: $\mu_i(a_{-i})$ is the probability she assigns to the collection a_{-i} of the other players' actions. Denote by $U_i(\alpha_i, a_{-i})$ player i's expected payoff when she uses the mixed strategy α_i and the other players' actions are given by a_{-i}.

► DEFINITION 349.1 A **belief** of player i about the other players' actions is a probability distribution over A_{-i}. Player i is **rational** if she chooses her mixed strategy α_i to solve the problem

$$\max_{\alpha_i} \sum_{a_{-i}} \mu_i(a_{-i}) U_i(\alpha_i, a_{-i}),$$

where μ_i is her belief about the other players' actions.

Suppose that each player's belief is correct—that is, the probability that it assigns to each collection of actions of the other players is the probability implied by their mixed strategies. Then a solution of each player's maximization problem is her Nash equilibrium strategy. That is, if each player's belief about the other players' behavior is correct then her equilibrium action is optimal for her. (Note, however, that some nonequilibrium actions may be optimal too.)

The assumption that each player's belief about the other players is correct is not very appealing if we imagine a player confronting a game in which she has little or no experience. In such a case the most that we might reasonably assume is that she knows (or at least assumes) that the other players are rational—that is, that the other players, like her, have beliefs and choose their actions to maximize their expected payoffs given these beliefs.

To think about the consequences of this assumption, consider a variant of the game in Exercise 36.2.

◆ EXAMPLE 350.1 (Rationalizable actions in a bargaining game) Two players split \$4 using the following procedure. Each announces an integral number of dollars. If the sum of the amounts named is at most \$4 then each player receives the amount she names. If the sum of the amounts named exceeds \$4 and both players name the same amount then each receives \$2. If the sum of the amounts named exceeds \$4 and the players name different amounts then the player who names the smaller amount receives that amount plus a small amount proportional to the difference between the amounts, and the other player receives the balance of the \$4. (That is, there is a small penalty for making a demand that is "excessive" relative to that of the other player.) In summary, the payoff of each player i is given by

$$\begin{cases} a_i & \text{if } a_1 + a_2 \leq 4 \\ 2 & \text{if } a_1 + a_2 > 4 \text{ and } a_i = a_j \\ 4 - a_j - (a_i - a_j)\epsilon & \text{if } a_1 + a_2 > 4 \text{ and } a_i > a_j, \\ a_i + (a_j - a_i)\epsilon & \text{if } a_1 + a_2 > 4 \text{ and } a_i < a_j, \end{cases}$$

where $\epsilon > 0$ is a small amount (less than 30 cents); the payoffs are shown in Figure 350.1.

	0	1	2	3	4
0	$0, 0$	$0, 1$	$0, 2$	$0, 3$	$0, 4$
1	$1, 0$	$1, 1$	$1, 2$	$1, 3$	$1 + 3\epsilon, 3 - 3\epsilon$
2	$2, 0$	$2, 1$	$2, 2$	$2 + \epsilon, 2 - \epsilon$	$2 + 2\epsilon, 2 - 2\epsilon$
3	$3, 0$	$3, 1$	$2 - \epsilon, 2 + \epsilon$	$2, 2$	$3 + \epsilon, 1 - \epsilon$
4	$4, 0$	$3 - 3\epsilon, 1 + 3\epsilon$	$2 - 2\epsilon, 2 + 2\epsilon$	$1 - \epsilon, 3 + \epsilon$	$2, 2$

Figure 350.1 The players' payoffs in the game in Example 350.1.

Suppose that you, as a player in this game, hold a probabilistic belief about your opponent's action and choose an action that maximizes your expected payoff given this belief. I claim that, whatever your belief, you will not demand \$0. Why? Because if you do so then you receive \$0 *whatever* amount the other player names, while if instead you name \$1 then you receive at least \$1 *whatever* amount the other player names. Thus for *no* belief about the other player's behavior is it optimal for you to demand \$0. Without considering whether your belief about the other player's behavior is consistent with her being rational, we can conclude that if you maximize your payoff given some belief about the other player then you will not demand \$0. We say that a demand of \$0 is a *never best response*.

By a similar argument we can conclude that you will not demand \$1, whatever your belief. But you might demand \$2. Why? Because you might believe, for example, that the other player is sure to demand \$2 (that is, you might assign probability 1 to the other player's demanding \$2), in which case your best action is to demand \$2 (if you demand more than \$2 then you obtain less than \$2, since you pay a small penalty for making an excessive demand).

Is there any belief under which it is optimal for you to demand \$3? Yes: if you are sure that the other player will demand \$1 then it is optimal to demand \$3 (if you demand less then the sum of the demands will be less than \$4 and you will receive what you demand, while if you demand more then the sum of the demands will exceed \$4 and you will receive \$3 minus a small penalty). Similarly, if you are sure that the other player will demand \$0 then it is optimal for you to demand \$4.

In summary, any demand of at least \$2 is consistent with your choosing an action to maximize your expected payoff given some belief, while any smaller demand is not. Or, more succinctly,

> the only demands consistent with your being rational are \$2, \$3, and \$4.

Now take the argument one step further. Suppose that you work under the assumption that your adversary is rational. Then you can conclude that she will not demand less than \$2: for *any* belief that *she* holds about *you*, it is not optimal for her to demand less than \$2 (just as it is not optimal for you to demand less than \$2 if you are rational). But if she demands at least \$2 then it is not optimal for you to demand \$4, whatever belief you hold about her demand: you are better off demanding \$2 or \$3 than you are demanding \$4, whether you think your adversary will demand \$2, \$3, or \$4. On the other hand, the demands of \$2 and \$3 are both optimal for some belief that assigns positive probability only to your adversary demanding \$2, \$3, or \$4: if you are sure that the other player will demand \$4, for example, it is optimal for you to demand \$3.

We have now argued that only the demands \$2 and \$3 are consistent with your choosing an action to maximize your expected payoff given some belief about the other player's actions that is consistent with her being rational in the sense that for each action to which it assigns positive probability there is a belief that *she* can hold about *your* behavior that makes that action optimal for her:

> only the demands \$2 and \$3 are consistent with your being rational *and* your assuming that the other player is rational.

We can take the argument yet another step. What if you assume not only that your opponent is rational but that she assumes that you are rational? Then each of the actions to which each of her beliefs about you assigns positive probability should in turn be justified by a possible belief of yours about her. The only demands consistent with your rationality are those at least equal to \$2, as we saw above. Thus if she assumes that you are rational then each of her beliefs about you must assign positive probability only to demands of at least \$2. But then, by the last argument above, the belief that you hold must assign positive probability only to demands of \$2 or \$3. Finally, referring to Figure 350.1 you can see that if you hold such a belief you will not demand \$3: a demand of \$2 generates a higher payoff for you, whether your opponent demands \$2 or \$3. To summarize:

only the demand of \$2 is consistent with your rationality, your assuming that your opponent is rational, and your assuming that your opponent assumes that you are rational.

The line of reasoning can be taken further: we can consider the consequence of your assuming that your opponent assumes that you assume that she is rational. However, such reasoning eliminates no more actions: a demand of \$2 survives every additional level, since a demand of \$2 is optimal for a player who is sure that her opponent will demand \$2. (That is, $(\$2, \$2)$ is a Nash equilibrium of the game.)

In summary, in this game we conclude that

- if you are rational you will demand either \$2, \$3, or \$4
- if you assume that your opponent is rational you will demand either \$2 or \$3
- if you assume that your opponent assumes that you are rational then you will demand \$2.

The general structure of this argument is illustrated in Figure 353.1. (I restrict the informal discussion, though not the definitions and results, to two-player games.) The rectangles represent the sets A_i and A_j of players i and j in the game. Assume that the action a_i^* is consistent with player i's acting rationally. Then there is a belief of player i about player j's actions under which a_i^* is optimal. Let μ_i^1 be one such belief, and let the set of actions to which this belief assigns positive probability be the shaded set on the right, which I denote X_j^1. In the example, if $a_1^* = \$0$ or \$1 then there is no such belief. If $a_1^* = \$2$, \$3, or \$4 there are such beliefs; if $a_1^* = \$4$, for example, then all such beliefs assign relatively high probability to \$0.

Now further assume that a_i^* is consistent with player i's assuming that player j is rational. Then for some belief of player i about player j's actions that makes a_i^* optimal—say μ_i^1—each action in X_j^1 (the set of actions to which μ_i^1 assigns positive probability) must be optimal for player j under some belief about player i's action. For the two actions a_j' and a_j'' in X_j^1 the beliefs $\mu_j^2(a_j')$ and $\mu_j^2(a_j'')$ under which the actions are optimal are indicated in the figure, together with the sets of actions of player i to which they assign positive probability. The shaded set on the left is the set of actions of player i to which some belief $\mu_j^2(a_j)$ of player j for a_j in X_j^1 assigns positive probability. (The action a_i^* may or may not be a member of X_i^2; in the figure it is not.)

Note that we do not require that for *every* belief of player i under which a_i^* is optimal the actions of player j to which that belief assigns positive probability be

Figure 353.1 An illustration of the argument that an action is rationalizable.

optimal given some belief of player j about player i; rather, we require only that *there exists* a belief of player i under which a_i^* is optimal with this property. In the *Prisoner's Dilemma*, for example, the belief of player 1 that assigns probability 1 to player 2's choosing *Fink* has the properties that if player 1 holds this belief then it is optimal for her to choose *Fink*, and there is some belief of player 2 under which the action that player 1's belief assigns positive probability is optimal for player 2. It is also optimal for player 1 to choose *Fink* if she holds a belief that assigns positive probability to player 2's choosing *Quiet*. However, such a belief cannot play the role of μ_1^1 in the argument above, since there is no belief of player 2 under which the action *Quiet* of player 2, to which the belief assigns positive probability, is optimal. That is, if we start off by letting μ_1^1 be a belief of player 1 that assigns positive probability to both *Fink* and *Quiet* then we get stuck at the next round: there is no belief that justifies *Quiet*. On the other hand, if we start off by letting μ_1^1 be the belief of player 1 that assigns probability 1 to player 2's choosing *Fink* then we can continue the argument.

The next step of the argument requires that every action a_i in X_i^2 be optimal for player i given some belief $\mu_i^3(a_i)$ about player j; denote the set of actions to which $\mu_i^3(a_i)$ assigns positive probability for some a_i in X_i^2 by X_j^3. Subsequent steps are similar: at each step every action in X_k^t has to be optimal for some belief about the other player and the set of actions of the other player (say ℓ) to at least one of these beliefs in this set assigns positive probability is the new set X_ℓ^{t+1}.

If we can continue the process indefinitely then we say that the action a_i^* is *rationalizable*. If we cannot—that is, if there is a stage t at which some action in the set X_k^t is not justified by *any* belief of player k—then a_i^* is not rationalizable.

Under what circumstances can we continue the argument indefinitely? Certainly we can do so if there are sets Z_1 and Z_2 of actions of player 1 and player 2 respectively such that Z_i contains a_i^*, every action in Z_1 is a best response to a belief of player 1 on Z_2 (i.e. a belief that assigns positive probability only to actions in Z_2), and every action in Z_2 is a best response to a belief of player 2 on Z_1. Conversely, suppose that it can be continued indefinitely. For player i let Z_i be the union of $\{a_i^*\}$ with the union of the sets X_i^t for all even values of t and let Z_j be the union of the sets X_j^t for all odd values of t. Then for $i = 1, 2$, every action in Z_i is a best response to a belief on Z_j. Thus we can define an action to be rationalizable as follows, where Z_{-i} denotes the set of all collections a_{-i} of actions for the players other than i for which $a_j \in Z_j$ for all j.

DEFINITION 354.1 The action a_i^* of player i in a strategic game is **rationalizable** if for each player j there exists a set Z_j of actions such that

- Z_i contains a_i^*
- for every player j, every action a_j in Z_j is a best response to a belief of player j on Z_{-j}.

Suppose that a^* is a pure strategy Nash equilibrium. Then for each player i the action a_i^* is a best response to a belief that assigns probability one to the other players' choosing a_{-i}^*. Setting $Z_i = \{a_i^*\}$ for each i, we see that a^* is rationalizable. In fact, we have the following stronger result.

PROPOSITION 354.2 *Every action used with positive probability in some mixed strategy Nash equilibrium is rationalizable.*

Proof. For each player i, let Z_i be the set of actions to which player i's equilibrium mixed strategy assigns positive probability. Then every action in Z_i is a best response to the belief of player i that coincides with the probability distribution over the other players' actions that is generated by their mixed strategies (which by definition assigns positive probability only to collections of actions in Z_{-i}). Hence every action in Z_i is rationalizable. □

In many games, actions not used with positive probability in some Nash equilibrium are rationalizable. Consider, for example, the game in Figure 355.1, which has a unique Nash equilibrium (M, C).

EXERCISE 354.3 (Mixed strategy equilibrium of game in Figure 355.1) Show that the game in Figure 355.1 has no nondegenerate mixed strategy equilibrium.

Each action of each player is a best response to some action of the other player (for example, T is a best response of player 1 to R, M is a best response to C, and B is a best response to L). Thus, setting $Z_1 = \{T, M, B\}$ and $Z_2 = \{L, C, R\}$ we see that every action of each player is rationalizable. In particular the actions T and B of player 1 are rationalizable, even though they are not used with positive probability in any Nash equilibrium. The argument for player 1's choosing T, for example, is that player 2 might choose R, which is rational for her if she thinks player 1 will choose B, and it is reasonable for player 2 to so think since B is optimal for player 1 if she thinks that player 2 will choose L, which in turn is rational for player 2 if she thinks that player 1 will choose T, and so on.

	L	C	R
T	$0,7$	$2,5$	$7,0$
M	$5,2$	$3,3$	$5,2$
B	$7,0$	$2,5$	$0,7$

Figure 355.1 A game in which the actions T and B or player 1 and L and R of player 2 are not used with positive probability in any Nash equilibrium, but are rationalizable.

Even in games in which every rationalizable action is used with positive probability in some Nash equilibrium, the predictions of the notion of rationalizability are weaker than those of Nash equilibrium. The reason is that the notion of Nash equilibrium makes a prediction about the *profile* of chosen actions, while the notion of rationalizability makes a prediction about the actions chosen by each player. In a game with more than one Nash equilibrium these two predictions may differ. Consider, for example, the game in Figure 355.2. The notion of Nash equilibrium predicts that the outcome will be either (T, L) or (B, R) in this game, while the notion of rationalizability does not restrict the outcome at all: both T and B are rationalizable for player 1 and both L and R are rationalizable for player 2, so the outcome could be any of the four possible pairs of actions.

	L	R
T	$2,2$	$1,0$
B	$0,1$	$1,1$

Figure 355.2 A game with two Nash equilibria, (T, L) and (B, R).

12.2 Iterated elimination of strictly dominated actions

The notion of rationalizability, in requiring that a player act rationally, starts by restricting attention to actions that are best responses to some belief. That is, it eliminates from consideration actions that are not best responses to any belief: *never best responses*.

▶ DEFINITION 355.1 A player's action is a **never best response** if it is not a best response to any belief about the other players' actions.

Another criterion that we might use to eliminate an action from consideration is *domination*. Define an action a_i of player i to be *strictly dominated* if there is a mixed strategy of player i that yields her a higher payoff than does a_i *regardless* of the other players' behavior.

▶ DEFINITION 355.2 An action a_i of player i in a strategic game is **strictly dominated** if there is a mixed strategy α_i of player i for which

$$U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i}) \text{ for all } a_{-i}.$$

(As before, $U_i(\alpha_i, a_{-i})$ is the expected payoff of player i when she uses the mixed strategy α_i and the collection of actions chosen by the other players is a_{-i}.)

In the *Prisoner's Dilemma*, for example, the action *Quiet* is strictly dominated by the action *Fink*: whichever action the other player chooses, *Fink* yields a higher payoff than does *Quiet*. In the game in Figure 356.1, no action of either player is strictly dominated by another action, but the action R of player 2 is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to L and probability $\frac{1}{2}$ to C: the action R yields player 2 a payoff of 1 regardless of how player 1 behaves, while the mixed strategy yields her a payoff of $\frac{3}{2}$ regardless of how player 1 behaves. (The action R is dominated by other mixed strategies too: a mixed strategy that assigns probability q to L and probability $1 - q$ to C yields the payoff $3q$ if player 1 chooses T and $3(1 - q)$ if player 1 chooses B, and hence strictly dominates R whenever $3q > 1$ and $3(1 - q) > 1$, or whenever $\frac{1}{3} < q < \frac{2}{3}$.)

	L	C	R
T	$1, 3$	$0, 0$	$1, 1$
B	$0, 0$	$1, 3$	$0, 1$

Figure 356.1 A strategic game in which the action R of player 2 is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to each of the actions L and C.

If an action is strictly dominated then it is a never best response by the following argument. Suppose that a_i^* is strictly dominated by the mixed strategy α_i and let μ_i be a belief of player i about the other players' actions. Then since $U_i(\alpha_i, a_{-i}) > u_i(a_i^*, a_{-i})$ for all a_{-i} we have

$$\sum_{a_{-i}} \mu_i(a_{-i}) U_i(\alpha_i, a_{-i}) > \sum_{a_{-i}} \mu_i(a_{-i}) u_i(a_i^*, a_{-i}).$$

Hence a_i^* is not a best response to μ_i; since μ_i is arbitrary, a_i^* is a never best response. In fact, the converse is also true: if an action is a never best response then it is strictly dominated. Although it is easy to convince oneself that this result is reasonable, the proof is not trivial. In summary, we have the following.

■ LEMMA 356.1 *A player's action in a finite strategic game is a never best response if and only if it is strictly dominated.*

Now reconsider the argument behind the rationalizability of an action of player i. First we argued that player i will not use a never best response, or equivalently, a strictly dominated action. Then we argued that if she works under the assumption that her opponent is rational then her belief should not assign positive probability to any action of her opponent that is a never best response. That is, she should not choose an action that is strictly dominated in the game that results when we eliminate all her opponent's strictly dominated actions. At the next step we argued that if player i works under the assumption that her opponent assumes that she is rational then she will assume that the action chosen by her opponent is a best response to some belief that assigns positive probability to actions of player i that are best responses to beliefs of player i. That is, in this case player i will assume that her opponent's action is not strictly dominated in the game that results when all of player i's strictly dominated actions are eliminated. Thus player i will choose an action that is not strictly dominated in the game that results when first all of player i's strictly dominated actions are eliminated, then all of player j's strictly dominated actions are eliminated.

We see that each step in the argument is equivalent to one more round of elimination of strictly dominated strategies in the game; the actions that remain no matter how many rounds of elimination we perform are the rationalizable actions. That is, rationalizability is equivalent to *iterative elimination of strictly dominated actions*.

In fact, we do not have to remove *all* the strictly dominated actions of one of the players at each stage: the set of action profiles that remain if we keep eliminating strictly dominated actions until we are left with a game in which no action of any player is strictly dominated does not depend on the order in which we perform the elimination or the number of actions that we eliminate at each stage; the surviving set is always the set of rationalizable action profiles. We now state this result precisely.

DEFINITION 357.1 Suppose that for each player i in a strategic game and each $t = 1, \ldots, T$ there is a set X_i^t of actions of player i such that

- $X_i^1 = A_i$ (we start with the set of all possible actions).
- X_i^{t+1} is a subset of X_i^t for each $t = 1, \ldots, T-1$ (at each stage we may eliminate some actions).
- For each $t = 0, \ldots, T-1$ every action of player i in X_i^t that is not in X_i^{t+1} is strictly dominated in the game in which the set of actions of each player j is X_j^t (we eliminate only strictly dominated actions)
- No action in X_i^T is strictly dominated in the game in which the set of actions of each player j is X_j^T (at the end of the process no action of any player is strictly dominated).

Then the set of action profiles a such that $a_i \in X_i^T$ for every player i **survives iterated elimination of strictly dominated actions**.

Then we can show the following.

■ PROPOSITION 357.2 *For any finite strategic game, there is a unique set of action profiles that survives iterated elimination of strictly dominated actions, and this set coincides with the set of profiles of rationalizable actions.*

◆ EXAMPLE 357.3 (Rationalizable actions in an extension of *BoS*) Consider the game in Figure 358.1. The action B of player 2 is strictly dominated by *Book*. In the game obtained by eliminating B for player 2 the action B of player 1 is strictly dominated. Finally, in the game obtained by eliminating B for player 1 the action

	B	*S*	*Book*
B	3, 1	0, 0	−1, 2
S	0, 0	1, 3	0, 2

Figure 358.1 Bach, Stravinsky, or a book.

Book for player 2 is strictly dominated. We conclude that the only rationalizable action for each player is *S*.

? EXERCISE 358.1 (Finding rationalizable actions) Find the set of rationalizable actions of each player in the game in Figure 358.2.

	L	*C*	*R*
T	2, 1	1, 4	0, 3
B	1, 8	0, 2	1, 3

Figure 358.2 The game in Exercise 358.1

? EXERCISE 358.2 (Rationalizable actions in Guessing Morra) Find the rationalizable actions of each player in the game *Guessing Morra* (Exercise 344.2).

? EXERCISE 358.3 (Rationalizable actions in a public good game) (More difficult, but also more interesting.) Show the following results for the variant of the game in Exercise 42.1in which contributions are restricted to be nonnegative.

a. Any contribution of more than $w_i/2$ is strictly dominated for player i.

b. If $n = 3$ and $w_1 = w_2 = w_3 = w$ then every contribution of at most $w/2$ is rationalizable. [Show that every such contribution is a best response to a belief that assigns probability one to each of the other players' contributing some amount at most equal to $w/2$.]

c. If $n = 3$ and $w_1 = w_2 < \frac{1}{3}w_3$ then the unique rationalizable contribution of players 1 and 2 is 0 and the unique rationalizable contribution of player 3 is w_3. [Eliminate strictly dominated actions iteratively. After eliminating a contribution of more than $w_i/2$ for each player i (by part *a*), you can eliminate small contributions by player 3; subsequently you can eliminate any positive contribution by players 1 and 2.]

? EXERCISE 358.4 (Rationalizable actions in Hotelling's spatial model) Consider a variant of the game in Section 3.3 in which there are two players, the distribution of the citizens' favorite positions is uniform [not needed?, but makes things easier to talk about?], and each player is restricted to choose a position of the form ℓ/m for some integer ℓ between 0 and m, where m is even (or to stay out of the competition). Show that the unique rationalizable action of each player is the position $\frac{1}{2}$.

12.3 Iterated elimination of weakly dominated actions

A strictly dominated action is clearly unattractive to a rational player. Now consider an action a_i that is *weakly dominated* in the sense that there is another action that yields *at least as high* a payoff as does a_i whatever the other players choose and yields a higher payoff than does a_i for some choice of the other players. In the game in Figure 359.1, for example, the action T of player 1 weakly (though not strictly) dominates B.

	L	R
T	$1,1$	$0,0$
B	$0,0$	$0,0$

Figure 359.1 A game in which the action B for player 1 and the action R for player 2 are weakly, but not strictly, dominated.

▶ DEFINITION 359.1 The action a_i of player i in a strategic game is **weakly dominated** if there is a mixed strategy α_i of player i such that

$$U_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

and

$$U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i}) \text{ for some } a_{-i} \in A_{-i}.$$

A weakly dominated action that is not strictly dominated, unlike a strictly dominated one, is not an unambiguously poor choice: by Lemma 356.1 such an action is a best response to *some* belief. For example, in the game in Figure 359.1, if player 1 is *sure* that player 2 will choose R then B is an optimal choice for her. However, the rationale for choosing a weakly dominated action is very weak: there is no advantage to a player's choosing a weakly dominated action, whatever her belief. For example, if player 1 in the game in Figure 359.1 has the the slightest suspicion that player 2 might choose L then T is better than B, and even if player 2 chooses R, T is no worse than B.

If we argue that it is unreasonable for a player to choose a weakly dominated action then we can argue also that each player should work under the assumption that her opponents will not choose weakly dominated actions, and they will assume that she does not do so, and so on. Thus, as in the case of strictly dominated actions, we can argue that weakly dominated actions should be removed *iteratively* from the game. That is, first we should mark actions of player 1 that are weakly dominated; then, without removing these actions of player 1, mark actions of player 2 that are weakly dominated, and proceed similarly with the other players. Then we should remove all the marked actions, and again mark weakly dominated actions for every player. Once again, having marked weakly dominated actions for every player, we should remove all the actions and go through the process again. We should repeat the process until no more actions can be eliminated for any player. This procedure, however, is less compelling than the iterative removal of strictly dominated actions since the set of actions that survive may depend on whether we remove *all* the weakly dominated actions at each round, or only some of them, as the two-player game in Figure 360.1 shows. The sequence in which we first eliminate L (weakly dominated by C) and then T (weakly dominated by B) leads to an outcome in which player 1 chooses B and the payoff profile is $(1, 2)$. On the other hand, the sequence in which we first eliminate R (weakly dominated by C) and then B (weakly dominated by T) leads to an outcome in which player 1 chooses T and the payoff profile is $(1, 1)$.

	L	C	R
T	$1,1$	$1,1$	$0,0$
B	$0,0$	$1,2$	$1,2$

Figure 360.1 A two-player game in which the set of actions that survive iterated elimination of weakly dominated actions depends on the order in which actions are eliminated.

◆ EXAMPLE 360.1 (A card game) A set of n cards consists of one with "1" on one side and "2" on the other side, one with "2" on one side and "3" on the other side, and so on. A card is selected at random; player 1 sees one side (determined randomly) and player 2 sees the other side. Each player can either *veto* the card, or *accept* it. If at least one player vetoes a card, the players tie; if both players accept it, the one who sees the higher number wins (and the other player loses).

We can model this situation as a strategic game in which a player's action is the set of numbers she accepts. If $n = 2$, for example, each player has 8 actions: $\varnothing$ (accept no number), $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. A player's payoff is her probability of winning minus her probability of losing. If $n = 2$ and player 1's strategy is $\{3\}$ and player 2's action is $\{2, 3\}$, for example, then if the card $1 - -2$ is selected one player vetoes it, while if the card $2 - -3$ is selected player 1 vetoes it if she sees "2" and both players accept it if player 1 sees "3", in which case player 1 wins. Thus player 1's payoff is $\frac{1}{4}$ and player 2's payoff is $-\frac{1}{4}$.

I claim that only the pairs of actions in which each player either accepts only $n + 1$ or does not accept any number survive iterated elimination of weakly dominated actions.

I first argue that any action a_i that accepts 1 is weakly dominated by the action a_i' that differs only in that it vetoes 1. Given any action of the other player, a_i and a_i' lead to possibly different outcomes only if the player sees the number 1, in which case a_i either loses (if the other player's action accepts 2) or ties, while a_i' is guaranteed to tie.

Now eliminate all actions of each player that accept 1. I now argue that any action a_i that accepts 2 is weakly dominated by the action a_i' that differs only in that it vetoes 2. Given any action of the other player, a_i and a_i' lead to possibly different outcomes only if the player sees the number 2, in which case a_i never wins, because all remaining actions of the other player veto 1. Thus a_i either loses (if the other player's action accepts 3) or ties, while a_i' is guaranteed to tie.

Continuing the argument, we eliminate all actions that accept any number up to n. The only pairs of actions that remain are those in which each player either accepts only $n+1$ or accepts no number. These two actions yield the same payoffs, given the other player's remaining actions (all payoffs are 0), so neither action can be eliminated.

Now consider the special case in which *all* weakly dominated actions of each player are eliminated at each step. If all the players are indifferent between all action profiles that survive when we perform such iterated elimination then we say that the game is *dominance solvable*. candidates,

? EXERCISE 361.1 (Dominance solvability) Find the set of Nash equilibria (mixed as well as pure) of the game in Figure 361.1. Show that the game is dominance solvable; find the pair of payoffs that survives. Find an order of elimination such that more than one outcome survives.

	L	C	R
T	$2,2$	$0,2$	$0,1$
M	$2,0$	$1,1$	$0,2$
B	$1,0$	$2,0$	$0,0$

Figure 361.1 The game for Exercise 361.1.

? EXERCISE 361.2 (Dominance solvability) Show that the variant of the game in Example 350.1 in which $\epsilon = 0$ is dominance solvable and find the set of surviving outcomes.

? EXERCISE 361.3 (Dominance solvability in Bertrand's duopoly game) Consider the variant of Bertrand's duopoly game in Exercise 65.2, in which each firm is restricted to choose prices that are integral numbers of cents. Assume that the profit function $(p-c)D(p)$ has a single local maximum. Show that the game is dominance solvable and find the set of surviving outcomes.

Notes

[Highly incomplete.]

The notion of rationalizability is due to Bernheim (1984) and Pearce (1984). Example 360.1 is taken from Littlewood (1953, 4). (Whether Littlewood is the originator or not is unclear. He presents the situation as a good example of "mathematics with minimum 'raw material' ".)

13 Evolutionary equilibrium

Monomorphic pure strategy equilibrium · Mixed strategies and polymorphic equilibrium · Asymmetric equilibria · Extensive games · Illustrations: sibling behavior; nesting behavior of wasps. *Prerequisite*: Chapters 2, 4, and 5.

13.1 Introduction

According to the Darwinian theory of evolution the modes of behavior that survive are those that are most successful in producing offspring. In an environment in which organisms interact, the reproductive success of a mode of behavior may depend on the modes of behavior followed by all the organisms in the population. For example, if all organisms act aggressively, then an organism may be able to survive only if it is aggressive; if all organisms are passive, then an organism's reproductive success may be greater if the organism acts passively than if it acts aggressively. Game theory provides tools with which to study evolution in such an environment.

In the games studied in this chapter, the players are representatives from an evolving population of organisms (humans, animals, plants, bacteria, ...). Each player's payoffs measure the increments in the player's biological *fitness*, or reproductive success (e.g. expected number of healthy offspring), associated with the possible outcomes, rather than indicating the player's subjective feelings about the outcomes. Each player's actions are modes of behavior that the player is programmed to follow.

The players do not make conscious choices. Rather, each player's mode of behavior comes from one of two sources: with high probability it is inherited from the player's parent (or parents), and with low (but positive) probability it is assigned to the player as the result of a mutation. For most of the models in this chapter, inheritance is conceived very simply: each player has a single parent, and, unless it is a mutant, simply takes the same action as does its parent. This model of inheritance captures the essential features of both genetic inheritance and social inheritance: players either follow the programs encoded in their genes, which come from their parents, or learn how to behave by imitating their parents. The distinction between genetic and social evolution may be significant if we wish to change society, but is insignificant for most of the models considered in this chapter.

We choose each player's set of actions to consist of all the modes of behavior that will, eventually, be generated by mutation (that is, we assume that for each action a, mutation eventually produces an organism that follows a). If, given the modes of behavior of all other organisms, the increment to biological fitness associated with the action a exceeds that associated with the action a' for some player, then adherents of a reproduce faster than adherents of a', and hence come to dominate the population. Very roughly, adherents of actions that are not best responses to the environment are eventually overwhelmed by adherents of better actions. The population from which each player in the game is drawn is subject to the same selective pressure, so this argument suggests that outcomes that are evolutionarily stable are related to Nash equilibria of the game. In this chapter we study the relation precisely.

The theory finds many applications in which the organisms are animals or plants. However, human behavior also can sometimes insightfully be modeled as the outcome of an evolutionary process: some human action, at least, appear to be more the result of inherited behavior than the outcome of reasoned choice.

13.2 Monomorphic pure strategy equilibrium

13.2.1 Introduction

Members of a single large population of organisms are repeatedly randomly matched in pairs. The set of possible modes of behavior of each member of any pair is the same, and the consequence of an interaction for an organism depends only on the actions of the organism and its opponent, not on its name. As an example, think of a population of identical animals, pairs of which periodically are engaged in conflicts (over prey, for example). The actions available to each animal may correspond to various degrees of aggression, and the outcome for each animal depends only on its degree of aggression and that of its opponent. Each organism produces offspring (reproduction is asexual), to each of whom, with high probability, it passes on its mode of behavior; with low probability, each offspring is a mutant that adopts some other mode of behavior.

We can model the interaction between each pair of organisms as a symmetric strategic game (Definition 48.1) in which the payoff $u(a, a')$ of an organism that takes the action a when its opponent takes the action a' measures its expected number of offspring. We assume that the adherents of each mode of behavior multiply at a rate proportional to their payoff, and look for a configuration of modes of behavior in the population that is stable in the sense that in the event that the population contains a small fraction of mutants taking the same action, every mutant obtains an expected payoff lower than that of any nonmutant. (We ignore the case in which mutants taking different actions are present in the population at the same time.)

In this section I restrict attention to situations in which all organisms (except those thrown up by mutation) follow the *same* mode of behavior, which has no random component. That is, I consider only *monomorphic pure strategy* equilibria ("monomorphic" = "one form").

13.2.2 Examples

To get an idea of the implications of evolutionary stability, consider two examples. First suppose that the game between each pair of organisms is the one in the left panel of Figure 281.1. Suppose that every organism normally takes the action X. If

	X	Y
X	$2,2$	$0,0$
Y	$0,0$	$1,1$

	X	Y
X	$2,2$	$0,0$
Y	$0,0$	$0,0$

Figure 281.1 Two strategic games, illustrating the idea of an evolutionarily stable strategy.

the population contains the small fraction ϵ of mutants who take the action Y, then a normal organism has as its opponent another normal organism with probability $1-\epsilon$ and a mutant with probability ϵ. (The population is large, so that we can treat the fraction of mutants in the *rest* of the population as equal to the fraction of mutants in the entire population.) Thus the expected payoff of a normal organism is

$$2 \cdot (1-\epsilon) + 0 \cdot \epsilon = 2(1-\epsilon).$$

Similarly, the expected payoff of a mutant is

$$0 \cdot (1-\epsilon) + 1 \cdot \epsilon = \epsilon.$$

If ϵ is small enough then the first payoff exceeds the second, so that the entry of a *small* fraction of mutants leads to a situation in which the expected payoff (fitness) of every mutant is lower than the payoff of every normal organism. We conclude that the action X is evolutionarily stable.

Now suppose that every organism normally takes the action Y. Then, making a similar calculation, the expected payoff of a normal organism is $1-\epsilon$, while the expected payoff of a mutant is 2ϵ. Mutants who meet each other obtain a payoff higher than that of normal organisms who meet each other. But when ϵ is small mutants are usually paired with normal organisms, in which case their expected payoff is 0 and, as in the previous case, the first payoff exceeds the second, so that the action Y is evolutionarily stable. The value of ϵ for which a normal organism does better than a mutant is smaller in this case than it is in the case that the normal action is X. However, in both cases, if ϵ is *sufficiently* small then mutants cannot invade. Since we wish to capture the idea that mutation is extremely rare, relative to normal behavior, we are satisfied with this existence of *some* value of ϵ that prevents invasion by mutants; we do not attach significance to the size of the critical value of ϵ.

Now consider the game in the right panel of Figure 281.1. By an argument like those above, the action X is evolutionarily stable. Is the action Y also evolutionarily stable? In a population containing the fraction ϵ of mutants choosing X, the expected payoff of a normal organism is 0 (it obtains 0 whether its opponent is normal or a mutant) while the expected payoff of a mutant is 2ϵ (it obtains 2 against another mutant and 0 against a normal organism). Thus the action Y is *not* evolutionarily stable: for *any* value of ϵ the expected payoff of a mutant exceeds that of a normal organism.

In both games, both (X, X) and (Y, Y) are Nash equilibria, but while X is an evolutionarily stable action in both games, Y is evolutionarily stable only in the left game. What is the essential difference between the games? If the normal action is Y then in the left game a mutant who chooses X is worse off than a normal organism in encounters with normal organisms, while in the right game a mutant that chooses X obtains the *same* expected payoff as does a normal organism in encounters with normal organisms. In the left game, there is always a value of ϵ small enough that the gain (relative to the payoff of a normal organism) that a mutant obtains with probability ϵ when it faces another mutant does not cancel out the loss it obtains with probability $1 - \epsilon$ when it faces a normal organism. In the right game, however, a mutant loses nothing relative to a normal organism, so no matter how small ϵ is, a mutant is better off than a normal organism. That is, the essential difference between the games is that $u(X, Y) < u(Y, Y)$ in the left game, but $u(X, Y) = u(Y, Y)$ in the right game.

13.2.3 General definitions

Consider now an arbitrary symmetric strategic game in which each player has finitely many actions. Under what circumstances is the action a^* evolutionarily stable?

Suppose that a small group of mutants choosing the action b different from a^* enters the population. The notion of stability that we consider requires that each such mutant obtain an expected payoff less than that of each normal organism, so that the mutants die out. (If the mutants obtained a payoff higher than that of the normal organisms then they would eventually come to dominate the population; if they obtained the same payoff as that of the normal organisms then they would neither multiply nor decline. Our notion of stability excludes the latter case: it is a strong notion that requires that mutants be driven out of the population.)

Denote the fraction of mutants in the population by ϵ. First consider a mutant, which adopts the action b. In a random encounter, the probability that it faces an organism that adopts the action a^* is approximately $1 - \epsilon$ (the population is large, so that the fraction in the *rest* of the population is close to the fraction in the entire population), while the probability that it faces a mutant, which adopts b, is approximately ϵ. Thus its expected payoff is

$$(1 - \epsilon)u(b, a^*) + \epsilon u(b, b).$$

Similarly, the expected payoff of an organism that adopts the action a^* is

$$(1 - \epsilon)u(a^*, a^*) + \epsilon u(a^*, b).$$

In order that any mutation be driven out of the population, we need the expected payoff of any mutant to be less than the expected payoff of a normal organism:

$$(1-\epsilon)u(a^*,a^*) + \epsilon u(a^*,b) > (1-\epsilon)u(b,a^*) + \epsilon u(b,b) \text{ for all } b \neq a^*. \qquad (283.1)$$

To capture the idea that mutation is extremely rare, the notion of evolutionary stability requires only that there is *some* (small) number $\bar{\epsilon}$ such that the inequality holds whenever $\epsilon < \bar{\epsilon}$. That is, we can make the following definition:

> The action a^* is *evolutionarily stable* if there exists $\bar{\epsilon} > 0$ such that a^* satisfies (283.1) for all $\epsilon < \bar{\epsilon}$.

Intuitively, the larger is $\bar{\epsilon}$, the "more stable" is the action a^*, since larger mutations are resisted. However, in the current discussion we do not attach any significance to the value of $\bar{\epsilon}$; in order that a^* be evolutionarily stable we require only that there is *some* size for $\bar{\epsilon}$ such that all smaller mutations are resisted.

The condition in this definition of evolutionary stability is a little awkward to work with, since whenever we apply it we need to check whether we can find a suitable value of $\bar{\epsilon}$. I now reformulate the condition in a way that avoids the variable $\bar{\epsilon}$.

I first claim that

> if there exists $\bar{\epsilon} > 0$ such that a^* satisfies (283.1) for all $\epsilon < \bar{\epsilon}$ then (a^*,a^*) is a Nash equilibrium.

To reach this conclusion, suppose that (a^*,a^*) is not a Nash equilibrium. Then there exists an action b such that $u(b,a^*) > u(a^*,a^*)$. Hence (283.1) is strictly violated when $\epsilon = 0$, and thus remains violated for all sufficiently small positive values of ϵ. (If $w < x$ and y and z are any numbers, then $(1-\epsilon)w + \epsilon y < (1-\epsilon)x + \epsilon z$ whenever ϵ is small enough.) Thus there is no $\bar{\epsilon}$ such that the inequality holds whenever $\epsilon < \bar{\epsilon}$. Our conclusion is that a *necessary* condition for an action a^* to be evolutionarily stable is that (a^*,a^*) be a Nash equilibrium.

Similar considerations lead to the conclusion that

> if (a^*,a^*) is a *strict* Nash equilibrium then there exists $\bar{\epsilon} > 0$ such that a^* satisfies (283.1) for all $\epsilon < \bar{\epsilon}$.

The argument is that if (a^*,a^*) is a strict Nash equilibrium then $u(b,a^*) < u(a^*,a^*)$ for all b, so that the strict inequality in (283.1) is satisfied for $\epsilon = 0$; hence it is also satisfied for sufficiently small positive values of ϵ. That is, we conclude that a *sufficient* condition for a^* to be evolutionarily stable is that (a^*,a^*) be a strict Nash equilibrium.

What happens if (a^*,a^*) is a Nash equilibrium, but is not strict? Suppose that $b \neq a^*$ is a best response to a^*: $u(b,a^*) = u(a^*,a^*)$. Then (283.1) reduces to the condition $u(a^*,b) > u(b,b)$, so that a^* is evolutionarily stable if and only if this condition is satisfied.

We conclude that *necessary and sufficient* conditions for the action a^* to be evolutionarily stable are that (i) (a^*, a^*) is a Nash equilibrium, and (ii) $u(a^*, b) > u(b, b)$ for every $b \neq a^*$ that is a best response to a^*. Intuitively, in order that mutant behavior die out it must be that (i) no mutant does better than a^* in encounters with organisms using a^* and (ii) any mutant that does as well as a^* in such encounters must do worse than a^* in encounters with mutants.

To summarize, the definition of evolutionary stability given above is equivalent to the following definition (which is much easier to work with).

► DEFINITION 284.1 An action a^* of a player in a symmetric two-player game is **evolutionarily stable** with respect to mutants using pure strategies if

- (a^*, a^*) is a Nash equilibrium, and
- $u(b, b) < u(a^*, b)$ for every best response b to a^* for which $b \neq a^*$,

where u is each player's payoff function.

As I argued above, if (a^*, a^*) is a *strict* Nash equilibrium then a^* is evolutionarily stable. This fact follows from the definition, since if (a^*, a^*) is a strict Nash equilibrium then the only best response to a^* is a^*, so that the second condition in the definition is vacuously satisfied.

Note that the inequality in the second condition is strict. If it were an equality then we would include as stable situations in which mutants neither multiply nor die out, but reproduce at the same rate as the normal population.

13.2.4 Examples

Both of the symmetric pure Nash equilibria of the left game in Figure 281.1 are strict, so that both X and Y are evolutionarily stable (confirming our previous analysis). In the right game in Figure 281.1, (X, X) and (Y, Y) are symmetric pure Nash equilibria also. But in this case (X, X) is strict while (Y, Y) is not. Further, since $u(X, X) > u(Y, X)$, the second condition in the definition of evolutionary stability is not satisfied by Y. Thus in this game only X is evolutionarily stable (again confirming our previous analysis).

The *Prisoner's Dilemma* (Figure 13.1) has a unique symmetric Nash equilibrium (D, D), and this Nash equilibrium is strict. Thus the action D is the only evolutionarily stable action. The game *BoS* (Figure 16.1) has no symmetric pure Nash equilibrium, and hence no evolutionarily stable action. (I consider mixed strategies in the next section.)

The following game, which generalizes the ideas on the game in Exercise 28.3, presents a richer range of possibilities for evolutionarily stable actions.

◆ EXAMPLE 284.2 (Hawk–Dove) Two animals of the same species compete for a resource (e.g. food, or a good nesting site) whose value (in units of "fitness") is $v > 0$. (That is, v measures the increase in the expected number of offspring brought by control of the resource.) Each animal can be either *aggressive* or *passive*. If both

EVOLUTIONARY GAME THEORY: SOME HISTORY

In his book *The Descent of Man*, Charles Darwin gave a game-theoretic argument that in sexually-reproducing species, the only evolutionarily stable sex ratio is 50:50 (1871, Vol. I, 316). Darwin's argument is game-theoretic in appealing to the fact that the number of an animal's descendants depends on the "behavior" of the other members of the population (the sex ratio of their offspring; see Exercise 303.1). Coming as it did 50 years before the language and methods of game theory began to develop, however, it is not couched in game-theoretic terms. In the late 1960s, two decades after the appearance of von Neumann and Morgenstern's (1944) seminal book, Hamilton (1967) proposed an explicitly game theoretic model of sex ratio evolution that applies to situations more general than that considered by Darwin.

But the key figure in the application of game theory to evolutionary biology is John Maynard Smith. Maynard Smith (1972a) and Maynard Smith and Price (1973) propose the notion of an evolutionarily stable strategy, and Maynard Smith's subsequent research develops the field in many directions. (Maynard Smith gives significant credit to Price: he writes that he would probably not have had the idea of using game theory had he not seen unpublished work by Price; "[u]nfortunately", he writes, "Dr. Price is better at having ideas than at publishing them" (1972b, vii).)

In the last two decades evolutionary game theory has blossomed. Biological models abound, and the methods of the theory have made their way into economics.

animals are aggressive they fight until one is seriously injured; the winner obtains the resource without sustaining any injury, while the loser suffers a loss of c. Each animal is equally likely to win, so each animal's expected payoff is $\frac{1}{2}v + \frac{1}{2}(-c)$. If both animals are passive then each obtains the resource with probability $\frac{1}{2}$, without a fight. Finally, if one animal is aggressive while the other is passive then the aggressor obtains the resource without a fight. The game is shown in Figure 285.1.

	A	P
A	$\frac{1}{2}(v-c), \frac{1}{2}(v-c)$	$v, 0$
P	$0, v$	$\frac{1}{2}v, \frac{1}{2}v$

Figure 285.1 The game *Hawk–Dove*.

If $v > c$ then the game has a unique Nash equilibrium (A, A), which is strict, so that A is the unique evolutionarily stable action.

If $v = c$ then also the game has a unique Nash equilibrium (A, A). But in this case the equilibrium is not strict: against an opponent that chooses A, a player obtains the same payoff whether it chooses A or P. However, the second condition in Definition 284.1 is satisfied: $v/2 = u(P, P) < u(A, P) = v$. Thus A is the unique evolutionarily stable action in this case also.

In both of these cases, a population of passive players can be invaded by aggressive players: an aggressive mutant does better than a passive player when its opponent is passive, and at least as well as a passive player when its opponent is aggressive.

If $v < c$ then the game has no symmetric Nash equilibrium in pure strategies: neither (A, A) nor (P, P) is a Nash equilibrium. Thus in this case the game has no evolutionarily stable action. (The game has only *asymmetric* Nash equilibria in this case.)

? EXERCISE 286.1 (Evolutionary stability and weak domination) Let a^* be an evolutionarily stable action. Does a^* necessarily weakly dominate every other action? Is it possible that some other action weakly dominates a^*?

? EXERCISE 286.2 (Example of evolutionarily stable actions) Pairs of members of a single population engage in the following game. Each player has three actions, corresponding to demands of 1, 2, or 3 units of payoff. If both players in a pair make the same demand, each player obtains her demand. Otherwise the player who demands less obtains the amount demanded by her opponent, while the player who demands more obtains $a\delta$, where a is her demand and δ is a number less than $\frac{1}{3}$. Find the set of pure strategy symmetric Nash equilibria of the game, and the set of pure evolutionarily stable strategies. What happens if each player has n actions, corresponding to demands of 1, 2, ..., n units of payoff (and $\delta < 1/n$)?

To gain an understanding of the outcome that evolutionary pressure might induce in games that have no evolutionarily stable action (e.g. *BoS*, and *Hawk–Dove* when $v < c$) we can take several routes. One is to consider mixed strategies as well as pure strategies; another is to allow for the possibility of several types of behavior coexisting in the population; a third is to consider interpretations of the asymmetric equilibria. I begin by discussing the first two approaches; in the following section I consider the third approach.

13.3 Mixed strategies and polymorphic equilibrium

13.3.1 Definition

So far we have considered only situations in which both "normal" organisms and mutants use pure strategies. If we assume that mixed strategies, as well as pure strategies, are passed on from parents to offspring, and may be thrown up by mutation, then an argument analogous to the one in the previous section leads to the conclusion that an evolutionarily stable mixed strategy satisfies conditions like those in Definition 284.1. Precisely, we can define an evolutionarily stable (mixed) strategy, known briefly as an ESS, as follows.

► DEFINITION 286.3 An **evolutionarily stable strategy** (**ESS**) in a symmetric two-player game is a mixed strategy α^* such that

- (α^*, α^*) is a Nash equilibrium
- $U(\beta, \beta) < U(\alpha^*, \beta)$ for every best response β to α^* for which $\beta \neq \alpha^*$,

where $U(\alpha, \alpha')$ is the expected payoff of a player using the mixed strategy α when its opponent uses the mixed strategy α'.

(If you do not believe that animals can randomize, you may be persuaded by an argument of Maynard Smith:

> "If it were selectively advantageous, a randomising device could surely evolve, either as an entirely neuronal process or by dependence on functionally irrelevant external stimuli. Perhaps the one undoubted example of a mixed ESS is the production of equal numbers of X and Y gametes by the heterogametic sex: if the gonads can do it, why not the brain?" (1982, 76).

Or you may be convinced by the evidence presented by Brockman et al. (1979) indicating that certain wasps pursue mixed strategies. (For a discussion of Brockman et al.'s model, see Section 13.6.))

13.3.2 Pure strategies and mixed strategies

Of course, Definition 286.3 does not preclude the use of pure strategies: every pure strategy is a special case of a mixed strategy. Suppose that a^* is an evolutionarily stable action in the sense of the definition in the previous section (284.1), and let α^* be the mixed strategy that assigns probability 1 to the action a^*. Since a^* is evolutionarily stable, (a^*, a^*) is a Nash equilibrium, so (α^*, α^*) is a mixed strategy Nash equilibrium (see Proposition 116.2). Is α^* necessarily an ESS (in the sense of the definition just given)? No: the second condition in the definition of an ESS may be violated. That is, a pure strategy may be immune to invasion by mutants that follow *pure* strategies, but may not be immune to invasion by mutants that follow some *mixed* strategy. Stated briefly, though a pure strategy Nash equilibrium is a mixed strategy Nash equilibrium, an action that is evolutionarily stable in the sense of Definition 284.1 is *not* necessarily an ESS in the sense of Definition 286.3.

	X	Y	Z
X	$2,2$	$1,2$	$1,2$
Y	$2,1$	$0,0$	$3,3$
Z	$2,1$	$3,3$	$0,0$

Figure 287.1 A game illustrating the difference between Definitions 284.1 and 286.3. The action X is an evolutionarily stable action in the sense of the first definition, but not in the sense of the second.

The game in Figure 287.1 illustrates this point. In studying this game, it may help to think of pairs of players working on a project. Two type X's work well together, and both a type Y and a type Z work well with an X, although the X

suffers a bit in each case. However, two type Y's are a disaster working together, as are two type Z's; but a Y and a Z make a great combination.

The action X is evolutionarily stable in the sense of Definition 284.1: (X, X) is a Nash equilibrium, and the two actions Y and Z different from X that are best responses to X satisfy $u(Y,Y) = 0 < 1 = u(X,Y)$ and $u(Z,Z) = 0 < 1 = u(X,Z)$. However, the action X is *not* an ESS in the sense of Definition 286.3. Precisely, the mixed strategy α^* that assigns probability 1 to X is not an ESS. To establish this claim we need only find a mixed strategy β that is a best response to α^* and satisfies $U(\beta,\beta) \geq U(\alpha^*,\beta)$ (in which case a mutant that uses the mixed strategy β will not die out of the population). Let β be the mixed strategy that assigns probability $\frac{1}{2}$ to Y and probability $\frac{1}{2}$ to Z. Since both Y and Z are best responses to X, so is β. Further, $U(\alpha^*,\beta) = 1 < \frac{3}{2} = U(\beta,\beta)$ (when both players use β the outcome is (Y,Y) with probability $\frac{1}{4}$, (Y,Z) with probability $\frac{1}{4}$, (Z,Y) with probability $\frac{1}{4}$, and (Z,Z) with probability $\frac{1}{4}$). Thus α^* is not a mixed strategy ESS: even though a population of adherents to α^* cannot be invaded by any mutant using a pure strategy, it *can* be invaded by mutants using the mixed strategy β. The point is that Y types do poorly against each other and so do Z types, but the match of a Y and a Z is very productive. Thus if all mutants either invariantly choose Y or invariantly choose Z then they fare badly when they meet each other; but if all mutants follow the mixed strategy that chooses Y and Z with equal probability then with probability $\frac{1}{2}$ two mutants that are matched are of different types, and are very productive.

13.3.3 Strict equilibria

We saw in the previous section that a *strict* pure Nash equilibrium is evolutionarily stable. Any strict Nash equilibrium is also an ESS, since the second condition in Definition 286.3 is then vacuously satisfied. However, this fact is of no help when we consider truly mixed strategies, since no mixed strategy Nash equilibrium in which positive probability is assigned to two or more actions is strict. Why not? Since if (α^*, α^*) is a mixed strategy equilibrium then, as we saw in Chapter 4, every action to which α^* assigns positive probability is a best response to α^*, and so too is any mixed strategy that assigns positive probability to the same pure strategies as does α^* (Proposition 111.1). Thus the second condition in the definition of an ESS is never vacuously satisfied for any mixed strategy equilibrium (α^*, α^*) that is not pure: when considering the possibility that a mixed equilibrium strategy is an ESS, at a minimum we need to check that $U(\beta,\beta) < U(\alpha^*,\beta)$ for every mixed strategy β that assigns positive probability to the same set of actions as does α^*.

13.3.4 Polymorphic steady states

A mixed strategy ESS corresponds to a monomorphic steady state in which each organism randomly chooses an action in each play of the game, according to the probabilities in the mixed strategy. Alternatively, it corresponds to a *polymorphic* steady state, in which a variety of pure strategies is in use in the population, the fraction of the population using each pure strategy being given by the probability the mixed strategy assigns to that pure strategy. (Cf. one of the interpretations of a mixed strategy equilibrium discussed in Section 4.1.) In Section 13.2.3 I argue that, in the case of a monomorphic steady state in which each player's strategy is pure, the two conditions in the definition of an ESS are equivalent to the requirement that any mutant die out. The same argument applies also to the case of a monomorphic steady state in which every player's strategy is mixed, but does not apply directly to the case of a polymorphic steady state. However, a different argument, based on similar ideas, shows that in this case too the conditions in the definition of an ESS are necessary and sufficient for the stability of a steady state (see Hammerstein and Selten (1994, 948–951)): mutations that change the fractions of the population using each pure strategy generate changes in payoffs that cause the fractions to return to their equilibrium values.

13.3.5 Examples

◆ EXAMPLE 289.1 (Bach or Stravinsky?) The members of a single population are randomly matched in pairs, and play *BoS*, with payoffs given in Figure 289.1. This

	L	D
L	$0,0$	$2,1$
D	$1,2$	$0,0$

Figure 289.1 The game *BoS*.

game has no symmetric pure strategy equilibrium. It has a unique symmetric mixed strategy equilibrium, in which the strategy α^* of each player assigns probability $\frac{2}{3}$ to L. As for any mixed strategy equilibrium, any mixed strategy that assigns positive probabilities to the same pure strategies as does α^* are best responses to α^*. Let $\beta = (p, 1-p)$ be such a mixed strategy. In order that α^* be an ESS we need $U(\beta,\beta) < U(\alpha^*,\beta)$ whenever $\beta \neq \alpha^*$. The payoffs in the game are low when the players choose the same action, so it seems possible that this condition is satisfied. To check the condition precisely, we need to find $U(\beta,\beta)$ and $U(\alpha^*,\beta)$. If both players use the strategy β then the outcome is (L,L) with probability p^2, (L,D) and (D,L) each with probability $p(1-p)$, and (D,D) with probability $(1-p)^2$. Thus $U(\beta,\beta) = 3p(1-p)$. Similarly, $U(\alpha^*,\beta) = \frac{4}{3} - p$. Thus for α^* to be an ESS we need

$$3p(1-p) < \tfrac{4}{3} - p$$

for all $p \neq \frac{2}{3}$. This inequality is equivalent to $(p-\frac{2}{3})^2 > 0$, so the strategy $\alpha^* = (\frac{2}{3}, \frac{1}{3})$ is an ESS.

◆ EXAMPLE 289.2 (A coordination game) The members of a single population are randomly matched in pairs, and play the game in Figure 290.1. In this game both (X,X) and (Y,Y) are strict pure Nash equilibria (as we noted previously), so that

	X	Y
X	$2,2$	$0,0$
Y	$0,0$	$1,1$

Figure 290.1 The game in Example 289.2.

both X and Y are ESSs. The game also has a symmetric mixed strategy equilibrium (α^*, α^*), in which $\alpha^* = (\frac{1}{3}, \frac{2}{3})$. Since every mixed strategy $\beta = (p, 1-p)$ is a best response to α^*, we need $U(\beta, \beta) < U(\alpha^*, \beta)$ whenever $\beta \neq \alpha^*$ in order that α^* be an ESS. In this game the players are better off choosing the same action as each other than they are choosing different actions, so it is plausible that this condition is not satisfied. The β that seems most likely to violate the condition is the pure strategy X (i.e. $\beta = (1, 0)$). In this case we have $U(\beta, \beta) = 2$ and $U(\alpha^*, \beta) = \frac{2}{3}$, so indeed the condition is violated. Thus the game has no mixed strategy ESS.

The intuition for this result is that a mutant that uses the pure strategy X is better off than a normal organism that uses the mixed strategy $(\frac{1}{3}, \frac{2}{3})$ both when it encounters a mutant, and when it encounters a normal organism. Thus such mutants will invade a population of organisms using the mixed strategy $(\frac{1}{3}, \frac{2}{3})$. (In fact, a mutant following *any* strategy different from α^* invades the population, as you can easily verify.)

◆ EXAMPLE 290.1 (Mixed strategies in Hawk–Dove) Consider again the game *Hawk–Dove* (Example 284.2). If $v > c$ then the only symmetric Nash equilibrium is the strict pure equilibrium (A, A), so that the only ESS is A.

If $v \leq c$ the game has a unique symmetric mixed strategy equilibrium, in which the strategy of each player is $(v/c, 1-v/c)$. To see whether this strategy is an ESS we need to check the second condition in the definition of an ESS. Let $\beta = (p, 1-p)$ be any mixed strategy. We need to determine whether $U(\beta, \beta) < U(\alpha^*, \beta)$ for $\beta \neq \alpha^*$, where $\alpha^* = (v/c, 1 - v/c)$. If each player uses the strategy β then the outcome is (A, A) with probability p^2, (A, P) and (P, A) each with probability $p(1-p)$, and (P, P) with probability $(1-p)^2$. Thus

$$U(\beta, \beta) = p^2 \cdot \tfrac{1}{2}(v - c) + p(1-p) \cdot v + p(1-p) \cdot 0 + (1-p)^2 \cdot \tfrac{1}{2}v.$$

Similarly, if a player uses the strategy α^* and its opponent uses the strategy β then its expected payoff is

$$U(\alpha^*, \beta) = (v/c)p \cdot \tfrac{1}{2}(v - c) + (v/c)(1-p) \cdot v + (1 - v/c)(1-p) \cdot \tfrac{1}{2}v.$$

Upon simplification we find that $U(\alpha^*, \beta) - U(\beta, \beta) = \frac{1}{2}c(v/c - p)^2$, which is positive if $p \neq v/c$. Thus $U(\beta, \beta) < U(\alpha^*, \beta)$ for any $\beta \neq \alpha^*$. We conclude that if $v \leq c$ then the game has a unique ESS, namely the mixed strategy $\alpha^* = (v/c, 1 - v/c)$.

To summarize, if injury is not costly ($c \leq v$) then only aggression survives. In this case, a passive mutant is doomed: it is worse off than an aggressive organism in encounters with other mutants and does no better than an aggressive organism in encounters with aggressive organisms. If injury costs more than the value of the resource ($c > v$) then aggression is not universal in an ESS. A population containing exclusively aggressive organisms is not evolutionarily stable in this case, since passive mutants do better than aggressive organisms against aggressive opponent. Nor is a population containing exclusively passive organisms evolutionarily stable, since aggressive pays against a passive opponent. The only ESS is a mixed strategy, which may be interpreted as corresponding to a situation in which the fraction v/c of organisms are aggressive and the fraction $1 - v/c$ are passive. As the cost of injury increases the fraction of aggressive organisms declines; the incidence of fights decreases, and an increasing number of encounters end without a fight (the dispute is settled "conventionally", in the language of biologists).

? EXERCISE 291.1 (Hawk–Dove–Retaliator) Consider the variant of *Hawk–Dove* in which a third strategy is available: "retaliator", which fights only if the opponent does so. Assume that a retaliator has a slight advantage over a passive animal against a passive opponent. The game is shown in Figure 291.1; assume $\delta < \frac{1}{2}v$. Find the ESSs.

	A	P	R
A	$\frac{1}{2}(v-c), \frac{1}{2}(v-c)$	$v, 0$	$\frac{1}{2}(v-c), \frac{1}{2}(v-c)$
P	$0, v$	$\frac{1}{2}v, \frac{1}{2}v$	$\frac{1}{2}v-\delta, \frac{1}{2}v+\delta$
R	$\frac{1}{2}(v-c), \frac{1}{2}(v-c)$	$\frac{1}{2}v+\delta, \frac{1}{2}v-\delta$	$\frac{1}{2}v, \frac{1}{2}v$

Figure 291.1 The game *Hawk–Dove–Retaliator*.

? EXERCISE 291.2 (Variant of *BoS*) Find all the ESSs, in pure and mixed strategies, of the game

	A	B	C
A	$0, 0$	$3, 1$	$0, 0$
B	$1, 3$	$0, 0$	$0, 0$
C	$0, 0$	$0, 0$	$1, 1$

? EXERCISE 291.3 (Bargaining) Pairs of players bargain over the division of a pie of size 10. The members of a pair simultaneously make demands; the possible demands are the nonnegative even integers up to 10. If the demands sum to 10 then each player receives her demand; if the demands sum to less than 10 then each player receives her demand plus half of the pie that remains after both demands have been satisfied; if the demands sum to more than 10 then no player receives any payoff. Show that the game has an ESS that assigns positive probability only to the demands 2 and 8 and also has an ESS that assigns positive probability only to the demands 4 and 6.

The next example reexamines the *War of attrition*, studied previously in Section 3.4 (pure equilibria). The game entered the literature as a model of animal conflicts. The actions of each player are the lengths of time the animal displays; the animal that displays longest wins.

◆ EXAMPLE 292.1 (War of attrition) Consider the *War of attrition* introduced in Section 3.4. If $v_1 = v_2$ then the game is symmetric. We found that even in this case the game has no symmetric pure strategy equilibrium. The only symmetric equilibrium is a mixed strategy equilibrium, in which each player's mixed strategy has the probability distribution function

$$F(t) = 1 - e^{-t/v},$$

where v is the common valuation.

Is this equilibrium strategy an ESS? Since the strategy assigns positive probability to every interval of actions, *every* strategy is a best response to it. Thus it is an ESS if and only if $U(G, G) < U(F, G)$ for every strategy $G \neq F$. To show this inequality is difficult. Here I show only that the inequality holds whenever G is a *pure* strategy. Let G be the pure strategy that assigns probability 1 to the action a. Then $U(G, G) = \frac{1}{2}v - a$ and

$$U(F, G) = \int_0^a (-s)F'(s)ds + (1 - F(a))(v - a) = v(2e^{-a/v} - 1)$$

(substituting for F and performing the integrations). Thus

$$U(F, G) - U(G, G) = 2ve^{-a/v} - \tfrac{3}{2}v + a,$$

which is positive for all values of a (find the minimum (by setting the derivative equal to zero) and show it is positive). Thus no mutant using a pure strategy can invade a population of players using the strategy F.

13.3.6 Games that have no ESS

Every game we have studied so far possesses an ESS. But there are games that do not. A very simple example is the trivial game shown in Figure 292.1. Let α be

	X	Y
X	$1, 1$	$1, 1$
Y	$1, 1$	$1, 1$

Figure 292.1

any mixed strategy. Then the strategy pair (α, α) is a Nash equilibrium. However, since $U(X, X) = 1 = U(\alpha, X)$, the mixed strategy α does not satisfy the second condition in the definition of an ESS. In a population in which all players use α, a mutant who uses X reproduces at the same rate as the other players (its fitness is the same), and thus does not die out. At the same time, such a mutant does not come to dominate the population. Thus, although the game has no ESS, every mixed strategy is neutrally stable.

However, we can easily give an example of a game in which there is not even any mixed strategy that is neutrally stable. Consider, for example, the game in Figure 293.1 with $\gamma > 0$. (If γ were zero then the game would be *Rock, paper, scissors* (Exercise 125.2).) This game has a unique symmetric Nash equilibrium, in

	A	B	C
A	$\gamma, \ \gamma$	$-1, \ 1$	$1, -1$
B	$1, -1$	$\gamma, \ \gamma$	$-1, \ 1$
C	$-1, \ 1$	$1, -1$	$\gamma, \ \gamma$

Figure 293.1 A game that has no ESS. In the unique symmetric Nash equilibrium of this game each player's mixed strategy is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; this strategy is not an ESS.

which each player's mixed strategy is $\alpha^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. To see that this strategy is not an ESS, let a be a pure strategy. Every pure strategy is a best response to α^* and $U(a, a) = \gamma > \gamma/3 = U(\alpha^*, a)$, strictly violating the second requirement for an ESS. Thus the game not only lacks an ESS; since the violation of the second requirement of an ESS is strict, it also lacks a neutrally stable strategy. The only candidate for a stable strategy is the unique symmetric mixed equilibrium strategy, but if all members of the population use this strategy then a mutant using any of the three pure strategy invades the population. Put differently, the notion of evolutionary stability—even in a weak form—makes no prediction about the outcome of this game.

13.4 Asymmetric equilibria

13.4.1 Introduction

So far we have studied the case of a homogeneous population, in which all organisms are identical, so that only symmetric equilibria are relevant: the players' roles are the same, so that a player cannot condition its behavior on whether it is player 1 or player 2. If the population is heterogeneous—if the players differ by size, by weight, by their current ownership status, or by any other observable characteristic—then even if the differences among players do not affect the payoffs, asymmetric equilibria may be relevant. I restrict attention to an example that illustrates some of the main ideas.

13.4.2 Example: Hawk–Dove

Consider a variant of *Hawk–Dove* (Example 284.2), in which the resource being contested is a nesting site, and one animal is the (current) owner while the other is an intruder. An individual will sometimes be an owner and sometimes be an intruder; its strategy specifies its action in each case. Thus we can describe the situation as a (symmetric) strategic game in which each player has *four* strategies: AA, AP, PA, and PP, where XY means that the player uses X when it is an owner and Y when it is an intruder. Since in each encounter there is one owner and one intruder, it is natural to assume that the probability that any given animal has each role is $\frac{1}{2}$.

Assume that the value of the nesting site may be different for the owner and the intruder; denote it by V for the owner and by v for the intruder. Assume also that $v < c$ and $V < c$, where c (as before) measures the loss suffered by a loser. (Recall that in the case $v < c$ there is no symmetric pure strategy equilibrium in the original version of the game.) Then in an encounter between an animal using the strategy AA and an animal using the strategy AP, for example, with probability $\frac{1}{2}$ the first animal is the owner and the second is the intruder, and the owner obtains the payoff V (the pair of actions chosen in the interaction being (A, P)), and with probability $\frac{1}{2}$ the first animal is the intruder and the second is the owner, and the intruder obtains the payoff $\frac{1}{2}(v-c)$ (the pair of actions chosen in the interaction being (A, A)). Thus in this case the expected payoff of the first animal is $\frac{1}{2}V + \frac{1}{4}(v-c) = \frac{1}{4}(2V + v - c)$. The payoffs to all strategy pairs are given in Figure 294.1; for convenience they are multiplied by four, and player 1's payoff is displayed above, not beside, player 2's.

	AA	AP	PA	PP
AA	$V + v - 2c$ $V + v - 2c$	$2V + v - c$ $V - c$	$V + 2v - c$ $v - c$	$2V + 2v$ 0
AP	$V - c$ $2V + v - c$	$2V$ $2V$	$V + v - c$ $V + v - c$	$2V + v$ V
PA	$v - c$ $V + 2v - c$	$V + v - c$ $V + v - c$	$2v$ $2v$	$V + 2v$ v
PP	0 $2V + 2v$	V $2V + v$	v $V + 2v$	$V + v$ $V + v$

Figure 294.1 A variant of *Hawk–Dove*, in which one player in each encounter is an owner and the other is an intruder. The payoffs are multiplied by four and player 1's is shown above, not beside, player 2's (for convenience in presentation). The strategy XY means take the action X when an owner and the action Y when an intruder.

The strategy pairs (AP, AP) and (PA, PA) are symmetric pure strategy equilibria of the game. Both of these equilibria are strict, so both AP and PA are ESSs (*regardless* of the relative sizes of v and V).

Now consider the possibility that the game has a mixed strategy ESS, say α^*. Then (α^*, α^*) is a mixed strategy equilibrium. I now argue that α^* does not assign positive probability to either of the actions AP or PA. If α^* assigns positive probability to AP then AP is a best response to α^* (since (α^*, α^*) is a Nash equilibrium), so that for α^* to be an ESS we need $U(AP,AP) < U(\alpha^*,AP)$. But this inequality contradicts the fact that (AP,AP) is a Nash equilibrium. Hence α^* does not assign positive probability to AP. An analogous argument shows that α^* does not assign positive probability to PA. In the following exercise you are asked to show that the game has no symmetric mixed strategy equilibrium (α^*, α^*) in which α^* assigns positive probability only to the actions AA and PP. We conclude that the game has no mixed ESS.

? EXERCISE 295.1 (Nash equilibrium in an asymmetric variant of *Hawk–Dove*) Let β be a mixed strategy that assigns positive probability only to the actions AA and PP in the game in Figure 294.1. Show that in order that AA and PP yield a player the same expected payoff when her opponent uses the strategy β, we need β to assign probability $(V+v)/2c$ to AA. Show further that when her opponent uses this strategy β, a player obtains a higher expected payoff from the action AP than she does from the action AA, so that (β, β) is not a Nash equilibrium.

? EXERCISE 295.2 (ESSs and mixed strategy equilibria) Generalize the argument that no ESS in the game in Figure 294.1 assigns positive probability to AP or to PA, to show the following result. Let (α^*, α^*) be a mixed strategy equilibrium; denote the set of actions to which α^* assigns positive probability by A^*. Then the only strategy assigning positive probability to every action in A^* that can be an ESS is α^*.

In summary, this analysis of *Hawk–Dove* for the case in which $v < c$ and $V < c$ leads to the conclusion that there are two evolutionarily stable strategies. In one, a player is aggressive when it is an owner and passive when it is an intruder, and in the other a player is passive when it is an owner and aggressive when it is an intruder. In both cases the dispute is resolved without a fight. The first strategy, in which an intruder concedes to an owner without a fight, is known as the *bourgeois strategy*; the second, in which the owner concedes to an intruder, is known as the *paradoxical strategy*. There are many examples in nature of the bourgeois strategy. The paradoxical strategy gets its name from the fact that it leads the members of a population to constantly change roles: whenever there is an encounter, the intruder becomes the owner, and the owner becomes a potential intruder. One example of this convention is described in the box on p. 295.

EXPLAINING THE OUTCOMES OF CONTESTS IN NATURE

[Note: this box is rough.] *Hawk–Dove* and its variants give us insights into the way in which animal conflicts are resolved. Before the development of evolutionary game theory, one explanation of the observation that conflicts are often settled without a fight was that it is not in the interest of a species for its members to be killed or injured. This theory is not explicit about how evolution could generate a situation in which individual members of a species act in a way that benefits the species as a whole. Further, by no means all animal conflicts are resolved peacefully, and the theory has nothing to say about the conditions under which peaceful resolution is likely to be the norm. As we have seen, a game theoretic analysis in which the unit of analysis is the individual member of the species suggests that in a symmetric contest the relation between the value of the resource under contention and the cost of an escalated contest determines the incidence of escalation. In an asymmetric contest the theory predicts that no escalation will occur, regardless of the value of the resource and the cost of injury. In particular, the convention that the owner always wins (the *bourgeois strategy*) is evolutionarily stable. (Classical theory appealed to an unexplained bias towards the owner, or to behavior that, in the context of the game theoretic models, is not rational.)

Biologists have studied behavior in many species in order to determine whether the predictions of the theory correspond to observed outcomes. Maynard Smith motivated his models by facts about conflicts between baboons. An example of more recent work concerns the behavior of the funnel web spider *Agelenopsis aperta* in New Mexico. Spiders differ in weight and web sites differ greatly in their desirability (some offer much more prey). At an average web site a confrontation usually ends without a fight. If the weights of the owner and intruder are similar, the dispute is usually settled in favor of the owner; if the weights are significantly different then the heavier spider wins. Hammerstein and Riechert (1988) estimate from field observations the fitness associated with various events and conclude that the ESS yields good predictions.

? Exercise 296.1 (Variant of BoS) Members of a population are randomly matched and play the game BoS. Each player in any given match can condition her action on whether she was the first to suggest getting together. Assume that for any given player the probability of being the first is one half. Find the ESSs of this game.

13.5 Variation on a theme: sibling behavior

The models of the previous sections are simple examples illustrating the main ideas of evolutionary game theory. In this section and the next I describe more detailed models that illustrate how these ideas may be applied in specific contexts.

Consider the interaction of siblings. The models in the previous sections assume that each player is equally likely to encounter any other player in the population. If we wish to study siblings' behavior toward each other we need to modify this assumption. I retain the other assumptions made previously: players interact pairwise, and payoffs measure fitness (reproductive success). I restrict attention throughout to pure strategies.

13.5.1 Asexual reproduction

The analysis in the previous sections rests on a simple model of reproduction, in which each organism, on its own, produces offspring. Before elaborating upon this model, consider its implications for the evolution of intrasibling behavior. Suppose that every organism in the population originally uses the action a^* when interacting with its siblings, obtaining the payoff $u(a^*, a^*)$. If a mutant using the action b appears, then, assuming that it has *some* offspring, all these offspring inherit the same behavior (ignoring further mutations). Thus the payoff (fitness) of each of these offspring in its interactions with its siblings is $u(b,b)$. All the descendants of any of these offspring also obtain the payoff $u(b,b)$ in their interaction with each other, so that the mutant behavior b invades the population if and only if $u(b,b) > u(a^*,a^*)$; it is driven out of the population if and only if $u(b,b) < u(a^*,a^*)$. We conclude that

> if an action a^* is evolutionarily stable then $u(a^*,a^*) \geq u(b,b)$ for every action b; if $u(a^*,a^*) > u(b,b)$ for every action b then a^* is evolutionarily stable.

When studying the behavior of one member of a population in interactions with another arbitrary member of the population, we found that a necessary condition for an action a^* to be evolutionarily stable is that (a^*,a^*) be a Nash equilibrium of the game. In intrasibling interaction, however, no such requirement appears: only actions a^* for which $u(a^*,a^*)$ is as high as possible can be evolutionarily stable. For example, if the game the siblings play is the *Prisoner's Dilemma* (Figure 297.1), then the only evolutionarily stable action is C; if this game is played between unrelated organisms then the only evolutionarily stable action is D.

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Figure 297.1 The *Prisoner's Dilemma.*

We can think of an evolutionarily stable action as follows. A player assumes that whatever action it takes, its sibling will take the same action. An action is evolutionarily stable if, under this assumption, the action maximizes the player's payoff. An important assumption in reaching this conclusion is that reproduction is asexual. As Bergstrom (1995, 61) succinctly puts it, "Careful observers of human siblings will not be surprised to find that in sexually reproducing species, equilibrium behavior is not so perfectly cooperative".

13.5.2 Sexual reproduction

This model, like the ones in the previous sections, incorporates an unrefined model of reproduction and inheritance. We have assumed that each organism by itself produces offspring, which inherit their parent's behavior. For species (like humans) in which offspring are the result of two animals mating, this assumption is only a rough approximation. I now describe a model in which each player has two parents. We need to specify how behavior is inherited: what behavior does the offspring of parents with different modes of behavior inherit?

The model I describe goes back to the level of individual genes in order to answer this question. Each animal carries two genes. Each offspring of a pair of animals inherits one randomly chosen gene from each of its parents; the pair of genes that it carries is its *genotype*. Denote by a the action that an animal of genotype xx

(i.e. with two x genes) is programmed to take and denote by b the action that an animal of genotype XX is programmed to take. Suppose that $a \neq b$, and that an animal of genotype xx mates with an animal of genotype XX. All the offspring have genotype Xx, and there are two possibilities for the action taken by these offspring: a and b. If the offspring are programmed to take the action b, we say that X is *dominant* and x is *recessive*, and if they are programmed to take the action a then X is recessive and x is dominant.

Assume that all mating is monogamous: all siblings share the same two parents. Reproductive success depends on both parents' characteristics; it simplifies the discussion to assume that animals differ not in their fecundity, but in their chance of surviving to adulthood (the age at which they start reproducing).

Under what circumstances is a population of animals of genotype xx, each choosing the action a^*, evolutionarily stable? Genes are now the basic unit of analysis, from which behavior is derived, so we need to consider whether any mutant gene, say X, can invade the population. That is, we need to consider the consequences of an animal of genotype Xx being produced. There are two cases to consider: X may be dominant or recessive.

Invasion by dominant genes

First consider the case in which X is dominant. Denote the action taken by animals of genotype XX and Xx by b, and assume that $b \neq a^*$. (If $b = a^*$ then the mutation is inconsequential for behavior.) Since almost all animals have genotype xx, almost every mutant (of genotype Xx) mates with an animal of genotype xx. Each of the offspring of such a pair inherits an x gene from her xx parent, and a second gene from her genotype Xx parent that is x with probability $\frac{1}{2}$ and X with probability $\frac{1}{2}$. Thus each offspring has genotype xx with probability $\frac{1}{2}$ and genotype Xx with probability $\frac{1}{2}$.

We now need to compare the payoffs of mutants and normal animals. We are assuming that the mutation is rare, so every mutant Xx has one Xx parent and one xx parent. Thus in its random matchings with its siblings, such a mutant faces an Xx with probability $\frac{1}{2}$ and an xx with probability $\frac{1}{2}$. Hence its expected payoff is

$$\tfrac{1}{2}u(b, a^*) + \tfrac{1}{2}u(b, b).$$

Normal xx animals are present both in ("normal") families with two xx parents and in families with one xx parent and one Xx parent; the vast majority are in normal families. Thus to determine whether Xx's come to dominate the population we need to consider only the payoff (survival probability) of an Xx relative to that of an xx in a normal family. All the siblings of an xx in a normal family have genotype xx, and hence obtain the payoff

$$u(a^*, a^*).$$

We conclude that no dominant mutant gene can invade the population if

$$\tfrac{1}{2}u(b, a^*) + \tfrac{1}{2}u(b, b) < u(a^*, a^*) \text{ for every action } b.$$

Conversely, a dominant mutant gene *can* invade if the inequality is reversed for any action b.

If we define the function v by

$$v(b, a) = \tfrac{1}{2}u(b, a) + \tfrac{1}{2}u(b, b),$$

then, noting that $v(a, a) = u(a, a)$ for any action a, we can rewrite the sufficient condition for a^* to be evolutionarily stable as

$$v(b, a^*) < v(a^*, a^*) \text{ for every action } b.$$

That is, (a^*, a^*) is a strict Nash equilibrium *of the game with payoff function* v. If the inequality is reversed for any action b then a^* is not evolutionarily stable, so that a necessary condition for a^* to be evolutionarily stable is that (a^*, a^*) be a Nash equilibrium of the game with payoff function v.

In summary, a sufficient condition for a^* to be evolutionarily stable is that (a^*, a^*) be a strict Nash equilibrium of the game with payoff function v, in which a player's payoff is the average of its payoff in the original game and the payoff it obtains if its sibling mimics its behavior; a necessary condition is that (a^*, a^*) be a Nash equilibrium of this game.

Invasion by recessive genes

Now consider the case in which X is recessive. An animal of genotype Xx choose the same action a^* as does an animal of genotype xx in this case. In a family in which one parent has genotype xx and the other has genotype Xx, half the offspring have genotype xx and half have genotype Xx, and hence all take the action a^* and receive the payoff $u(a^*, a^*)$ in interactions with each other. Thus on this account the X gene neither invades the population nor is eliminated from it. To determine the fate of mutants, we need to consider the outcome of the interaction between siblings in families that constitute an even smaller fraction of the population.

The next smallest group of families are those in which the genotype of both parents is Xx, in which case one fourth of the offspring have genotype XX. Suppose that an animal of genotype XX takes the action $b \neq a^*$. If, in interactions with its siblings, such an animal is more successful than animals of genotypes xx or Xx then the mutant gene X, though starting from a very small base, can invade the population.

In families with two Xx parents, the genotypes of the offspring are distributed as follows: one fourth are xx, one half are Xx, and one fourth are XX. To find the expected payoff of an X gene in the offspring of such families, we need to consider each possible pair of siblings in turn. The analysis is somewhat complicated; I omit the details. The conclusion is that the expected payoff to an X gene is

$$\tfrac{1}{8}u(b, b) + \tfrac{1}{8}u(a^*, b) + \tfrac{3}{8}u(b, a^*) + \tfrac{3}{8}u(a^*, a^*).$$

The expected payoff of the "normal" gene x (which initially dominates the population, in families in which both parents are xx) is $u(a^*, a^*)$, so the mutant gene cannot invade the population if

$$\tfrac{1}{8}u(b,b) + \tfrac{1}{8}u(a^*,b) + \tfrac{3}{8}u(b,a^*) + \tfrac{3}{8}u(a^*,a^*) < u(a^*,a^*).$$

or

$$\tfrac{1}{5}u(b,b) + \tfrac{1}{5}u(a^*,b) + \tfrac{3}{5}u(b,a^*) < u(a^*,a^*).$$

If we define the function w by

$$w(a,b) = \tfrac{1}{5}u(a,a) + \tfrac{1}{5}u(b,a) + \tfrac{3}{5}u(a,b),$$

then the sufficient condition for evolutionary stability can be rewritten as

$$w(b,a^*) < w(a^*,a^*) \text{ for every action } b.$$

That is, (a^*,a^*) is a strict Nash equilibrium *of the game with payoff function* w. As before, a necessary condition for a^* to be evolutionarily stable is that (a^*,a^*) be a Nash equilibrium of this game.

Evolutionary stability

In order that a^* be evolutionarily stable, it must resist invasion by both dominant *and* recessive genes. Thus we have the following conclusion.

> If (a^*,a^*) is a strict Nash equilibrium of the game with payoff function v *and* a strict Nash equilibrium of the game with payoff function w then a population of players of genotype xx, choosing a^*, is evolutionarily stable. If (a^*,a^*) is not a Nash equilibrium of both these games then a^* is not evolutionarily stable.

Consider the implications for the *Prisoner's Dilemma*. The evolutionarily stable action depends on the relative magnitudes of the payoffs corresponding to each outcome. First consider the case of the payoff function in the left of Figure 300.1. In the middle and right figures the games with payoff functions v and w are shown.

	C	D
C	$5,5$	$0,6$
D	$6,0$	$2,2$

u

	C	D
C	$5,5$	$\frac{5}{2},4$
D	$4,\frac{5}{2}$	$2,2$

v

	C	D
C	$5,5$	$\frac{11}{5},4$
D	$4,\frac{11}{5}$	$2,2$

w

Figure 300.1 A *Prisoner's Dilemma*. On the left is the basic game, with payoff function u. In the middle is the game with payoff function v, and on the right is the game with payoff function w.

We see that (C,C) is a Nash equilibrium for both of the payoff functions v and w, and (D,D) is not a Nash equilibrium for either one. Hence in this case C is the only evolutionarily stable strategy in the game between siblings.

Now consider the case of the payoff function in the left of Figure 301.1. We see that (D,D) is a Nash equilibrium for both of the payoff function v and w, while

	C	D
C	$3,3$	$0,6$
D	$6,0$	$2,2$

u

	C	D
C	$3,3$	$\frac{3}{2},4$
D	$4,\frac{3}{2}$	$2,2$

v

	C	D
C	$3,3$	$\frac{9}{5},4$
D	$4,\frac{9}{5}$	$2,2$

w

Figure 301.1 A version of the *Prisoner's Dilemma.* On the left is the basic game, with payoff function u. In the middle is the game with payoff function v, and on the right is the game with payoff function w.

(C, C) is not a Nash equilibrium of either game. Hence in this case D is the only evolutionarily stable strategy in the game between siblings.

Thus in the *Prisoner's Dilemma*, whether or not siblings in a sexually reproducing species are cooperative or not depends on the gain to be had from being uncooperative. When this gain is small, the cooperative outcome is evolutionarily stable. Even though purely selfish behavior fails to sustain cooperation, the genetic similarity of siblings causes cooperative behavior to be evolutionarily stable. When the gain is large enough, however, the relatedness of siblings is not enough to overcome the pressure to defect, and the only evolutionarily stable outcome is joint defection.

? EXERCISE 301.1 (A coordination game between siblings) Consider the game in Figure 301.2. For what values of $x > 1$ is X the unique evolutionarily stable action when the game is played between siblings?

	X	Y
X	x,x	$0,0$
Y	$0,0$	$1,1$

Figure 301.2 The game in Exercise 301.1.

13.6 Variation on a theme: nesting behavior of wasps

In all the situations I have analyzed so far, the players interact in pairs. In many situations the result of a player's action depends on the behavior of all the other players, not only on the action of one of these players; pairwise interactions cannot be identified. In this section I consider such a situation; the analysis illustrates how the methods of the previous sections can be generalized.

Female great golden digger wasps (*Sphex ichneumoneus*) lay their eggs in burrows, which must be stocked with katydids for the larva to feed on when they hatch. In a simple model, each wasp decides, when ready to lay an egg, whether to dig a burrow or to invade an existing burrow. A wasp that invades a burrow fights with the occupant, losing with probability π. If invading is less prevalent than digging then not all diggers are invaded, so that while digging takes time, it offers the possibility of laying an egg without a fight. The higher the proportion of invaders, the worse off is a wasp that digs its own burrow, since it is more likely to be invaded.

Each wasp's fitness is measured by the number of eggs it lays. Assuming that the length of a wasp's life is independent of its behavior, we can work with payoffs equal to the number of eggs laid per unit time. Let T_d be the time it takes for a wasp to build a burrow and stock it with katydids; let T_i be the time spent on a nest by an invader (T_i is not zero, since fighting takes time) and assume $T_i < T_d$. Assume that all wasps lay the same number of eggs in a nest, and choose the units in which eggs are measured so that this number is 1.

Suppose that the fraction of the population that digs is p and the fraction that invades is $1-p$. In order to determine the probability that a digger is invaded, we need to take into account the fact that since invading takes less time than digging, an invader can invade more than one nest in the time that it takes a digger to dig. If invading takes half the time of digging, for example, and there are only half as many invaders as there are diggers in the population, then *all* diggers will be invaded—the probability of a digger being invaded is 1. In general, a digger can invade T_d/T_i burrows during a time period of length T_d. For every digger there are $(1-p)/p$ invaders, so the probability that a digger is invaded is $q = [(1-p)/p]T_d/T_i$, or $q = (1-p)T_d/(pT_i)$, assuming that this number is at most 1.

A wasp that digs its own burrow thus faces the following lottery: with probability $1-q$ it is not invaded, with probability $q\pi$ it is invaded and wins the fight, and with probability $q(1-\pi)$ it is invaded and loses the fight (in which case assume that the whole time T_d is wasted). Thus the payoff—the expected number of eggs laid per unit time—of such a wasp is

$$(1-q+q\pi)/T_d.$$

Similarly the expected number of eggs laid per unit time by an invader is $(1-\pi)/T_i$.

If $1/T_d \geq (1-\pi)/T_i$ there is an equilibrium in which every wasp digs its own burrow: the expected payoff to digging is at least the expected payoff to invading, given that $q = 0$. Clearly there is no equilibrium in which all wasps invade—for then there are no nests to invade! The remaining possibility is that there is an equilibrium in which diggers and invaders coexist in the population. In such an equilibrium the expected payoffs to the two activities must be equal, or $(1-q+q\pi)/T_d = (1-\pi)/T_i$. Substituting $(1-p)T_d/(pT_i)$ for q we find that $p = (1-\pi)T_d/T_i$. Looking back at the definition of q, we find that if the parameters π, T_i, and T_d satisfy $\pi T_i \leq (1-\pi)T_d$ then $q \leq 1$ for this value of p, so that we do indeed have an equilibrium.

Are these equilibria evolutionarily stable? First consider the equilibrium in which every wasp digs its own burrow. If $1/T_d > (1-\pi)/T_i$—that is, if the condition for the equilibrium to exist is satisfied *strictly*—then mutants that invade obtain a smaller payoff than the normal wasps that dig, and hence die out. Thus in this case the equilibrium is stable. (I do not consider the unlikely case that $1/T_d = (1-\pi)/T_i$.)

Now consider the equilibrium in which diggers and invaders coexist in the population. Suppose that there is a small mutation that increases slightly the fraction of diggers in the population. That is, p rises slightly. Then q, the probability of being invaded, falls, and the expected payoff to digging increases; the expected payoff to invading does not change. Thus a slight increase in p leads to an increase in the relative attractiveness of digging; diggers prosper relative to invaders, further increasing the value of p. We conclude that the equilibrium is not evolutionarily stable.

The polymorphic equilibrium I have analyzed can alternatively be interpreted as a mixed strategy equilibrium, in which each individual wasp randomizes between digging and invading, choosing to dig with probability p. In the populations that Brockmann et al. (1979) observe, digging and invading do coexist, and in fact individual wasps pursue mixed strategies—sometimes they dig and sometimes they invade. This evidence raises the question of how the model could be modified so that the mixed strategy equilibrium is evolutionarily stable. Brockman et al. suggest two such variants. In one case, for example, they assume that a wasp who digs a nest is better off if she is invaded and wins the fight than she is if she is not invaded (the invader may have helped to stock the nest with katydids before it got into a fight with the digger). The data Brockman et al. collected in one site generates a value of p that fits their observations very well; the data from another site does not fit well.

The following exercise illustrates another application of the main ideas of evolutionary game theory.

? EXERCISE 303.1 (*Darwin's theory of the sex ratio*) A population of males and females mate pairwise to produce offspring. Suppose that each offspring is male with probability p and female with probability $1-p$. Then there is a steady state in which the fraction p of the population is male and the fraction $1-p$ is female. If $p \neq \frac{1}{2}$ then males and females have different numbers of offspring (on average). Is such an equilibrium evolutionarily stable? Denote the number of children born to each female by n, so that the number of children born to each male is $(p/(1-p))n$. Suppose a mutation occurs that produces boys and girls each with probability $\frac{1}{2}$. Assume for simplicity that the mutant trait is dominant: if one partner in a couple has it, then all the offspring of the couple have it. Assume also that the number of children produced by a female with the trait is n, the same as for "normal" members of the population. Since both normal and mutant females produce the same number of children, it might seem that the fitness of a mutant is the same as that of a normal organism. But compare the number of *grand*children of mutants and normal organisms. How many female offspring does a normal organism produce? How many male offspring? Use your answers to find the number of grandchildren born to each mutant and to each normal organism. Does the mutant invade the population? Which value (values?) of p is evolutionarily stable?

Notes

[Incomplete.]

The main ideas in this chapter are due to Maynard Smith.

The chapter draws on the expositions of Hammerstein and Selten (1994) and van Damme (1987, Chapter 9).

Darwin's theory of sex ratio evolution (see the box on page 285) was independently discovered by Ronald A. Fisher (1930, 141–143), and is often referred to as "Fisher's theory". In the second edition of Darwin's book (1874, 256), he retracted his theory for reasons that are not apparent, and Fisher appears to have been aware only of the retraction, not of the original theory. Bulmer (1994, 207–208) appears to have been the first to notice that "Fisher's theory" was given by Darwin.

Hawk–Dove (Example 284.2) is due to Maynard Smith and Price (1973).

The discussion in Section 13.4 is based on van Damme (1987, Section 9.5).

Exercise 295.2 is a slightly less general version of Lemma 9.2.4 of van Damme (1987).

The material in Section 13.5 is taken from Bergstrom (1995).

The model in Section 13.6 is taken from Brockmann, Grafen, and Dawkins (1979), simplified along the lines of Bergstrom and Varian (1987, 324–327).

14 Repeated games: The Prisoner's Dilemma

Prerequisite: Chapters 5 and 7.

14.1 The main idea

MANY of the strategic interactions in which we are involved are ongoing: we repeatedly interact with the same people. In many such interactions we have the opportunity to "take advantage" of our co-players, but do not. We look after our neighbors' house while they're away, even if it is time-consuming for us to do so; we may give money to friends who are temporarily in need. The theory of repeated games provides a framework that we can use to study such behavior.

The basic idea in the theory is that a player may be deterred from exploiting her short-term advantage by the "threat" of "punishment" that reduces her long-term payoff. Suppose, for example, that two people are involved repeatedly in an interaction for which the short-term incentives are captured by the *Prisoner's Dilemma* (see Section 2.2), with payoffs as in Figure 389.1. Think of C as "cooperation" and D as "defection".

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Figure 389.1 The *Prisoner's Dilemma*.

As we know, the *Prisoner's Dilemma* has a unique Nash equilibrium, in which each player chooses D. Now suppose that a player adopts the following long-term strategy: choose C so long as the other player chooses C; if in any period the other player chooses D, then choose D in *every subsequent period*. What should the other player do? If she chooses C in every period then the outcome is (C, C) in every period and she obtains a payoff of 2 in every period. If she switches to D in some period then she obtains a payoff of 3 in that period and a payoff of 1 in every subsequent period. She may value the present more highly than the future—she may be *impatient*—but as long as the value she attaches to future payoffs is not too small compared with the value she attaches to her current payoff, the stream of payoffs $(3, 1, 1, \ldots)$ is worse for her than the stream $(2, 2, 2, \ldots)$, so that she is better off choosing C in every period.

This argument shows that if a player is sufficiently patient, the strategy that chooses C after every history is a best response to the strategy that starts off choosing C and "punishes" any defection by switching to D. Clearly another best response is this same punishment strategy: if your opponent is using this punishment strategy then the outcome is the same if you use the strategy that chooses C after every history, or the same punishment strategy as your opponent is using. In both cases, the outcome in every period is (C, C) (the other player never defects, so if you use the punishment strategy you are never induced to switch to punishment). Thus the strategy pair in which both players use the punishment strategy is a Nash equilibrium of the game: neither player can do better by adopting another long-term strategy.

The conclusion that the repeated *Prisoner's Dilemma* has a Nash equilibrium in which the outcome is (C, C) in every period accords with our intuition that in long-term relationships there is scope for mutually supportive strategies that do not relentlessly exploit short-term gain. However, this strategy pair is not the only Nash equilibrium of the game. Another Nash equilibrium is the strategy pair in which each player chooses D after every history: if one player adopts this strategy then the other player can do no better than to adopt the strategy herself, regardless of how she values the future, since whatever she does has no effect on the other player's behavior.

This analysis leaves open many questions.

- We have seen that the outcome in which (C, C) occurs in every period is supported as a Nash equilibrium if the players are sufficiently patient. Exactly how patient do they have to be?

- We have seen also that the outcome in which (D, D) occurs in every period is supported as a Nash equilibrium. What other outcomes are supported?

- We saw in Chapter 5 that Nash equilibria of extensive games are not always intuitively appealing, since the actions they prescribe after histories that result from deviations may not be optimal. The notion of subgame perfect equilibrium, which requires actions to be optimal after every possible history, not only those that are reached if the players adhere to their strategies, may be more appealing. Is the strategy pair in which each player uses the punishment strategy I have described a subgame perfect equilibrium? That is, is it optimal for each player to punish the other player for deviating? If not, is there any other strategy pair that supports desirable outcomes and is a subgame perfect equilibrium?

- The punishment strategy studied above is rather severe; in switching permanently to D in response to a deviation it leaves no room for error. Are there any Nash equilibria or subgame perfect equilibria in which the players' strategies punish deviations less severely?

- The arguments above are restricted to the *Prisoner's Dilemma*. To what other games do they apply?

I now formulate the model of a repeated game more precisely in order to answer these questions.

14.2 Preferences

14.2.1 Discounting

The outcome of a repeated game is a sequence of outcomes of a strategic game. How does each player evaluate such sequences? I assume that she associates a payoff with each outcome of the strategic game, and evaluates each sequence of outcomes by the **discounted sum** of the associated sequence of payoffs. More precisely, each player i has a payoff function u_i for the strategic game and a discount factor δ between 0 and 1 such that she evaluates the sequence $(a^1, a^2, \ldots, a^T)$ of outcomes of the strategic game by the sum

$$u_i(a^1) + \delta u_i(a^2) + \delta^2 u_i(a^3) + \cdots + \delta^{T-1} u_i(a^T) = \sum_{t=1}^{T} \delta^{t-1} u_i(a^t).$$

(Note that in this expression superscripts are used for two purposes: a^t is the action profile in period t, while δ^t is the discount factor δ raised to the power t.) I assume throughout that all players have the same discount factor δ. A player whose discount factor is close to zero cares very little about the future—she is very impatient; a player whose discount factor is close to one is very patient.

Why should a person value future payoffs less than current ones? Possibly she is simply impatient. Or, possibly, her underlying preferences do not display impatience, but in comparing streams of outcomes she takes into account the positive probability with which she may die in any given period.[1] Or, if the outcome in each period involves the payment to her of some amount of money, possibly impatience is induced by the fact that she can borrow and lend at a positive interest rate. For example, suppose her underlying preferences over streams of monetary payoffs do not display impatience. Then if she can borrow and lend at the interest rate r she is indifferent between the sequence $(\$100, \$100, 0, 0, \ldots)$ of amounts of money

[1] Alternatively, the hazard of death may have favored those who reproduce early, leading to the evolution of people who are "impatient".

and the sequence ($\$100 + \$100/(1+r), 0, 0, \ldots$), since by lending $\$100/(1+r)$ of the amount she obtains in the first period she obtains \$100 in the second period. In fact, under these assumptions her preferences are represented precisely by the discounted sum of her payoffs with a discount factor of $1/(1+r)$: any stream can be obtained from any other stream with the same discounted sum by borrowing and lending. (If you win one of the North American lotteries that promises \$1m you will quickly learn about discounted values: you will receive a stream of 20 yearly payments each of \$50,000, which at an interest rate of 7% is equivalent to receiving about \$567,000 as a lump sum.)

Obviously the assumption that everyone's preferences over sequences of outcomes are represented by a discounted sum of payoffs is restrictive: people's preferences do not *necessarily* take this form. However, a discounted sum captures simply the idea that people may value the present more highly than the future and appears not to obscure any other feature of preferences significant to the problem we are considering.

14.2.2 Equivalent payoff functions

When we considered preferences over atemporal outcomes and atemporal lotteries, we found that many payoff functions represent the same preferences. Specifically, if u is a payoff function that represents a person's preferences over deterministic outcomes, then any increasing function of u also represents her preferences. If u is a Bernoulli payoff function whose expected value represents a person's preferences over lotteries, then the expected value of any increasing affine function of u also represents her preferences.

Consider the same question for preferences over sequences of outcomes. Suppose that a person's preferences are represented by the discounted sum of payoffs with payoff function u and discount factor δ. Then if the two sequences of outcomes $(x^1, x^2, \ldots)$ and $(y^1, y^2, \ldots)$ are indifferent, we have

$$\sum_{t=0}^{\infty} \delta^{t-1} u(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} u(y^t).$$

Now let v be an increasing affine function of u: $v(x) = \alpha + \beta u(x)$ with $\beta > 0$. Then

$$\sum_{t=0}^{\infty} \delta^{t-1} v(x^t) = \sum_{t=0}^{\infty} \delta^{t-1}[\alpha + \beta u(x^t)] = \sum_{t=0}^{\infty} \delta^{t-1}\alpha + \beta \sum_{t=0}^{\infty} \delta^{t-1} u(x^t)$$

and similarly

$$\sum_{t=0}^{\infty} \delta^{t-1} v(y^t) = \sum_{t=0}^{\infty} \delta^{t-1}[\alpha + \beta u(y^t)] = \sum_{t=0}^{\infty} \delta^{t-1}\alpha + \beta \sum_{t=0}^{\infty} \delta^{t-1} u(y^t),$$

so that

$$\sum_{t=0}^{\infty} \delta^{t-1} v(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} v(y^t).$$

Thus the person's preferences are represented also by the discounted sum of payoffs with payoff function v and discount factor δ. That is, if a person's preferences are represented by the discounted sum of payoffs with payoff function u and discount factor δ then they are also represented by the discounted sum of payoffs with payoff function $\alpha + \beta u$ and discount factor δ, for any α and any $\beta > 0$.

In fact, as in the case of payoff representations of preferences over lotteries (see Lemma 145.1), the converse is also true: if preferences over a stream of outcomes are represented by the discounted sum of payoffs with payoff function u and discount factor δ, and also by the discounted sum of payoffs with payoff function v and discount factor δ, then v must be an increasing affine function of u.

■ LEMMA 393.1 (Equivalence of payoff functions under discounting) *Suppose there are at least three possible outcomes. The discounted sum of payoffs with the payoff function u and discount factor δ represents the same preferences over streams of payoffs as the discounted sum of payoffs with the payoff function v and discount factor δ if and only if there exist α and $\beta > 0$ such that $u(x) = \alpha + \beta v(x)$ for all x.*

The significance of this result is that the payoffs in the strategic games that generate the repeated games we now study are no longer simply ordinal, even if we restrict attention to deterministic outcomes. For example, the players' preferences in the repeated game based on a *Prisoner's Dilemma* with the payoffs given in Figure 389.1 are different from the players' preferences in the repeated game based on the variant of this game in which the payoff pairs $(0, 3)$ and $(3, 0)$ are replaced by $(0, 5)$ and $(5, 0)$. (When the discount factor is close enough to 1, for instance, each player prefers the sequence of outcomes $((C, C), (C, C))$ to the sequence of outcomes $((D, C), (C, D))$ in the first case, but not in the second case.) Thus I refer to *a* repeated *Prisoner's Dilemma*, rather than *the* repeated *Prisoner's Dilemma*. More generally, throughout the remainder of this chapter I define strategic games in terms of payoff functions rather than preferences: a **strategic game** consists of a set of players, and, for each player, a set of actions and a payoff function.

If a player's preferences over streams $(w^1, w^2, \ldots)$ of payoffs are represented by the discounted sum $\sum_{t=1}^{\infty} \delta^{t-1} w^t$ of these payoffs, where $\delta < 1$, then they are also represented by the **discounted average** $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} w^t$ of these payoffs (since this discounted average is simply a constant times the discounted sum). The discounted average has the advantage that its values are directly comparable to the payoffs in a single period. Specifically, for any discount factor δ between 0 and 1 the constant stream of payoffs $(c, c, \ldots)$ has discounted average $(1 - \delta)(c + \delta c + \delta^2 c + \cdots) = c$ (see (449.2)). For this reason I subsequently work with the discounted average rather than the discounted sum.

14.3 Infinitely repeated games

I start by studying a model of a repeated interaction in which play may continue indefinitely—there is no fixed final period. In many situations play cannot continue indefinitely. But the assumption that it can may nevertheless capture well the players' perceptions. The players may be aware that play cannot go on forever, but, especially if the termination date is very far in the future, may ignore this fact in their strategic reasoning. (I consider a model in which there is a definite final period in Section 15.3.)

A repeated game is an extensive game with perfect information and simultaneous moves. A history is a sequence of action profiles in the strategic game. After every nonterminal history, *every* player i chooses an action from the set of actions available to her in the strategic game.

▶ DEFINITION 394.1 Let G be a strategic game. Denote the set of players by N and the set of actions and payoff function of each player i by A_i and u_i respectively. The **infinitely repeated game of** G for the discount factor δ is the extensive game with perfect information and simultaneous moves in which

- the set of players is N
- the set of terminal histories is the set of infinite sequences $(a^1, a^2, \ldots)$ of action profiles in G
- the player function assigns the set of all players to every proper subhistory of every terminal history
- the set of actions available to player i after any history is A_i
- each player i evaluates each terminal history $(a^1, a^2, \ldots)$ according to its discounted average $(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a^t)$.

14.4 Strategies

A player's strategy in an extensive game specifies her action after all possible histories after which it is her turn to move, including histories that are inconsistent with her strategy (Definition 203.2). Thus a strategy of player i in an infinitely repeated game of the strategic game G specifies an action of player i (a member of A_i) for every sequence $(a^1, \ldots, a^T)$ of outcomes of G.

For example, if player i's strategy s_i is the one discussed at the beginning of this chapter, it is defined as follows: $s_i(\varnothing) = C$ and

$$s_i(a^1, \ldots, a^t) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1, \ldots, t \\ D & \text{otherwise.} \end{cases} \tag{394.2}$$

That is, player i chooses C at the start of the game (after the initial history $\varnothing$) and after any history in which every previous action of player j was C; she chooses D after every other history. We refer to this strategy as a *grim trigger strategy*, since it is a mode of behavior in which a defection by the other player triggers relentless ("grim") punishment.

We can think of the strategy as having two *states*: one, call it $\mathcal{C}$, in which C is chosen, and another, call it $\mathcal{D}$, in which D is chosen. Initially the state is $\mathcal{C}$; if the other player chooses D in any period then the state changes to $\mathcal{D}$, where it stays

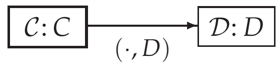

Figure 395.1 A grim trigger strategy for an infinitely repeated *Prisoner's Dilemma*.

forever. Figure 395.1 gives a natural representation of the strategy when we think of it in these terms. The box with a bold outline is the initial state, $\mathcal{C}$, in which the player chooses the action C. If the other player chooses D (indicated by the $(\cdot, D)$ under the arrow) then the state changes to $\mathcal{D}$, in which the player chooses D. If the other player does not choose D (i.e. chooses C) then the state remains $\mathcal{C}$. (The convention in the diagrams is that the state remains the same unless an event occurs that is a label for one of the arrows emanating from the state.) Once $\mathcal{D}$ is reached it is never left: there is no arrow leaving the box for state $\mathcal{D}$.

Any strategy can be represented in a diagram like Figure 395.1. In many cases, such a diagram is easier to interpret than a symbolic specification of the action taken after each history like (394.2). Note that since a player's strategy must specify her action after all histories, including those that do not occur if she follows her strategy, the diagram that represents a strategy must include, for every state, a transition for each of the possible outcomes in the game. In particular, if in some state the strategy calls for the player to choose the action B, then there must be one transition from the state for each of the cases in which the player chooses an action *different* from B. Figure 395.1 obscures this fact, since the event that triggers a change in the player's action is an action of her opponent; none of her own actions trigger a change in the state, so that the (null) transitions that her own actions induce are not indicated explicitly in the diagram.

A strategy that entails less draconian punishment is shown in Figure 395.2. This strategy punishes deviations for only three periods: it responds to a deviation by choosing the action D for three periods, then reverting to C, no matter how the other player behaved during her punishment.

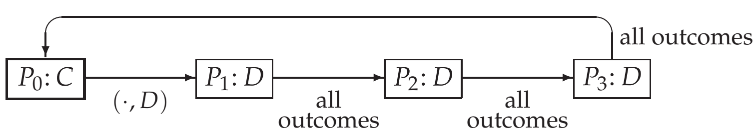

Figure 395.2 A strategy in an infinitely repeated *Prisoner's Dilemma* that punishes deviations for three periods.

In the strategy *tit-for-tat* the length of the punishment depends on the behavior of the player being punished. If she continues to choose D then *tit-for-tat* continues to do so; if she reverts to C then *tit-for-tat* reverts to C also. The strategy can be given a very compact description: do whatever the other player did in the previous period. It is illustrated in Figure 396.1.

? EXERCISE 395.1 (Strategies in the infinitely repeated *Prisoner's dilemma*) Represent each of the following strategies s in an infinitely repeated *Prisoner's Dilemma* in a

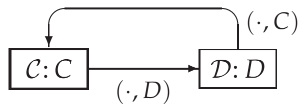

Figure 396.1 The strategy *tit-for-tat* in an infinitely repeated *Prisoner's Dilemma.*

diagram like Figure 395.1.

a. Choose C in period 1, and after any history in which the other player chose C in every period except, possibly, the previous period; choose D after any other history. (That is, punishment is grim, but its initiation is delayed by one period.)

b. Choose C in period 1 and after any history in which the other player chose D in at most one period; choose D after any other history. (That is, punishment is grim, but a single lapse is forgiven.)

c. (*Pavlov*, or *win-stay, lost-shift*) Choose C in period 1 and after any history in which the outcome in the last period is either (C, C) or (D, D); choose D after any other history. (That is, choose the same action again if the outcome was relatively good for you, and switch actions if it was not.)

14.5 Some Nash equilibria of the infinitely repeated Prisoner's Dilemma

If one player chooses D after every history in an infinitely repeated *Prisoner's Dilemma* then it is clearly optimal for the other player to do the same (since (D, D) is a Nash equilibrium of the *Prisoner's Dilemma*). The argument at the start of the chapter suggests that an infinitely repeated *Prisoner's Dilemma* has other, less dismal, equilibria, so long as the players are sufficiently patient—for example, the strategy pair in which each player uses the grim trigger strategy defined in Figure 395.1. I now make this argument precise. Throughout I consider the infinitely repeated *Prisoner's Dilemma* in which each player's discount factor is δ and the one-shot payoffs are given in Figure 389.1.

14.5.1 Grim trigger strategies

Suppose that player 1 adopts the grim trigger strategy. If player 2 does so then the outcome is (C, C) in every period and she obtains the stream of payoffs $(2, 2, \ldots)$, whose discounted average is 2. If she adopts a strategy that generates a different sequence of outcomes then there is one period (at least) in which she chooses D. In all subsequent periods player 1 chooses D (player 2's choice of D triggers the grim punishment), so the best deviation for player 2 chooses D in every subsequent period (since D is her unique best response to D). Further, if she can increase her payoff by deviating then she can do so by deviating to D in the first period. If she does so she obtains the stream of payoffs $(3, 1, 1, \ldots)$ (she gains one unit of payoff in the first period, then loses one unit in every subsequent period), whose discounted average is

$$(1-\delta)[3+\delta+\delta^2+\delta^3+\cdots] = 3(1-\delta)+\delta.$$

Thus she cannot increase her payoff by deviating if and only if

$$2 \geq 3(1-\delta)+\delta,$$

or $\delta \geq \frac{1}{2}$. We conclude that if $\delta \geq \frac{1}{2}$ then the strategy pair in which each player's strategy is the grim trigger strategy defined in Figure 395.1 is a Nash equilibrium of the infinitely repeated *Prisoner's Dilemma* with one-shot payoffs as in Figure 389.1.

14.5.2 Limited punishment

Now consider a generalization of the limited punishment strategy in Figure 395.2 in which a player who chooses D is punished for k periods. (The strategy in Figure 395.2 has $k = 3$; the grim punishment strategy corresponds to $k = \infty$.) If one player adopts this strategy, is it optimal for the other to do so? Suppose that player 1 does so. As in the argument for the grim trigger strategy, if player 2 can increase her payoff by deviating then she can increase her payoff by deviating in the first period. So suppose she chooses D in the first period. Then player 1 chooses D in each of the next k periods, regardless of player 2's choices, so player 2 also should choose D in these periods. In the $(k+1)$st period after the deviation player 1 switches back to C (regardless of player 2's behavior in the previous period), and player 2 faces precisely the same situation that she faced at the beginning of the game. Thus if her deviation increases her payoff, it increases her payoff during the first $k+1$ periods. If she adheres to her strategy then her discounted average payoff during these periods is

$$(1-\delta)[2+2\delta+2\delta^2+\cdots+2\delta^k] = 2(1-\delta^{k+1})$$

(see (449.1)), whereas if she deviates as described above then her payoff during these periods is

$$(1-\delta)[3+\delta+\delta^2+\cdots+\delta^k] = 3(1-\delta)+\delta(1-\delta^k).$$

Thus she cannot increase her payoff by deviating if and only if

$$2(1-\delta^{k+1}) \geq 3(1-\delta)+\delta(1-\delta^k),$$

or $\delta^{k+1} - 2\delta + 1 \leq 0$. If $k = 1$ then no value of δ less than 1 satisfies the inequality: one period of punishment is not severe enough to discourage a deviation, however patient the players are. If $k = 2$ then the inequality is satisfied for $\delta \geq 0.62$, and if $k = 3$ it is satisfied for $\delta \geq 0.55$. As k increases the lower bound on δ approaches $\frac{1}{2}$, the lower bound for the grim strategy.

We conclude that the strategy pair in which each player punishes the other for k periods in the event of a deviation is a Nash equilibrium of the infinitely repeated game so long as $k \geq 2$ and δ is large enough; the larger is k, the smaller is the lower bound on δ. Thus short punishment is effective in sustaining the mutually desirable outcome (C, C) only if the players are very patient.

14.5.3 Tit-for-tat

Now consider the conditions under which the strategy pair in which each player uses the strategy *tit-for-tat* is a Nash equilibrium. Suppose that player 1 adheres to this strategy. Then, as above, if player 2 can gain by deviating then she can gain by choosing D in the first period. If she does so, then player 1 chooses D in the second period, and continues to choose D until player 2 reverts to C. Thus player 2 has two options: she can revert to C, in which case in the next period she faces the same situation as she did at the start of the game, or she can continue to choose D, in which case player 1 will continue to do so too. We conclude that if player 2 can increase her payoff by deviating then she can do so either by alternating between D and C or by choosing D in every period. If she alternates between D and C then her stream of payoffs is $(3, 0, 3, 0, \ldots)$, with a discounted average of $(1-\delta) \cdot 3/(1-\delta^2) = 3/(1+\delta)$, while if she chooses D in every period her stream of payoffs is $(3, 1, 1, \ldots)$, with a discounted average of $3(1-\delta) + \delta = 3 - 2\delta$. Since her discounted average payoff to adhering to the strategy *tit-for-tat* is 2, we conclude that *tit-for-tat* is a best response to *tit-for-tat* if and only if

$$2 \geq \frac{3}{1+\delta} \text{ and } 2 \geq 3 - 2\delta.$$

Both of these conditions are equivalent to $\delta \geq \frac{1}{2}$.

Thus if $\delta \geq \frac{1}{2}$ then the strategy pair in which the strategy of each player is *tit-for-tat* is a Nash equilibrium of the infinitely repeated *Prisoner's Dilemma* with payoffs as in Figure 389.1.

? EXERCISE 398.1 (Nash equilibria of the infinitely repeated *Prisoner's Dilemma*) For each of the three strategies s in Exercise 395.1 determine the values of δ, if any, for which the strategy pair (s, s) is a Nash equilibrium of an infinitely repeated *Prisoner's Dilemma* with discount factor δ and the one-shot payoffs given in Figure 389.1. For each strategy s for which there is no value of δ such that (s, s) is a Nash equilibrium of this game, determine whether there are any payoffs for the *Prisoner's Dilemma* such that for some δ the strategy pair (s, s) is a Nash equilibrium of the infinitely repeated game with discount factor δ.

14.6 Nash equilibrium payoffs of the infinitely repeated Prisoner's Dilemma when the players are patient

All the Nash equilibria of the infinitely repeated *Prisoner's Dilemma* that I have discussed so far generate either the outcome (C, C) in every period or the outcome

(D, D) in every period. The first outcome path yields the discounted average payoff of 2 to each player, while the second outcome path yields the discounted average payoff of 1 to each player. What other discounted average payoffs are consistent with Nash equilibrium? It turns out that this question is hard to answer for an arbitrary discount factor. The question is relatively straightforward to answer, however, in the case that the discount factor is close to 1 (the players are very patient). Before tackling it, we need to determine the set of discounted average pairs of payoffs that are *feasible*—i.e. can be achieved by outcome paths.

14.6.1 Feasible discounted average payoffs

If the outcome is (X, Y) in every period then the discounted average payoff is $(u_1(X, Y), u_2(X, Y))$, for any X and Y. Thus $(2, 2)$, $(3, 0)$, $(0, 3)$, and $(1, 1)$ can all be achieved as pairs of discounted average payoffs.

Now consider the path in which the outcome alternates between (C, C) and (C, D). Along this path player 1's payoff alternates between 2 and 0 while player 2's alternates between 2 and 3. Thus the players' average payoffs along the path are 1 and $\frac{5}{2}$ respectively. Since player 1 receives more of her payoff in the first period of each two-period cycle than in the second period (in fact, she obtains nothing in the second period), her *discounted* average payoff exceeds 1, whatever the discount factor. But if the discount factor is close to 1 then her discounted average payoff is close to 1: the fact that more payoff is obtained in the first period of each two-period cycle is insignificant if the discount factor is close to 1. Similarly, since player 2 receives most of her payoff in the second period of each two-period cycle, her discounted average payoff is less than $\frac{5}{2}$, whatever the discount factor, but is close to $\frac{5}{2}$ when the discount factor is close to 1. Thus $(1, \frac{5}{2})$ can approximately be achieved as a pair of discounted average payoffs when the discount factor is close to 1.

This argument can be extended to any outcome path in which a sequence of outcomes is repeated. If the discount factor is close to 1 then a player's discounted average payoff on such a path is close to her average payoff in the sequence. For example, the outcome path that consists of repetitions of the sequence $((C, C), (D, C), (D, C))$ yields player 1 a discounted average payoff close to $\frac{1}{3}(2 + 3 + 3) = \frac{8}{3}$ and player 2 a discounted average payoff close to $\frac{1}{3}(2 + 0 + 0) = \frac{2}{3}$.

We conclude that the average of the payoffs to any sequence of outcomes can approximately be achieved as the discounted average payoff if the discount factor is close to 1. Further, if the discount factor is close to 1 then only such discounted average payoffs can be achieved. Thus if the discount factor is close to 1, the set of *feasible* discounted average payoff pairs in the infinitely repeated game is approximately the set of all pairs of weighted averages of payoffs in the component game. The same argument applies to any strategic game, and for convenience I make the following definition.

▶ DEFINITION 399.1 The set of **feasible** payoff profiles of a strategic game is the set of all weighted averages of payoff profiles in the game.

This definition is standard. Note, however, that the name "feasible" is a little misleading, in the sense that a feasible payoff profile is *not* in general achievable in the game, but only (approximately) as a discounted average payoff profile in the infinitely repeated game.

It is useful to represent the set of feasible payoff pairs in the *Prisoner's Dilemma* geometrically. Suppose that (x_1, x_2) and (y_1, y_2) are in the set. Now fix integers k and m with $m > k$ and consider the outcome path that consists of k repetitions of the cycle of outcomes that yields (x_1, x_2) followed by $m - k$ repetitions of the cycle that yields (y_1, y_2), and continues indefinitely with repetitions of this whole cycle. The average payoff pair on this outcome path is $(k/m)(x_1, x_2) + (1 - k/m)(y_1, y_2)$. This point lies on the straight line joining (x_1, x_2) and (y_1, y_2). As we vary k and m essentially all points on this straight line are achieved. (Precisely, every point that is a weighted average of (x_1, x_2) and (y_1, y_2) with rational weights are achieved.) We conclude that the set of feasible discounted average payoffs is the parallelogram in Figure 400.1.

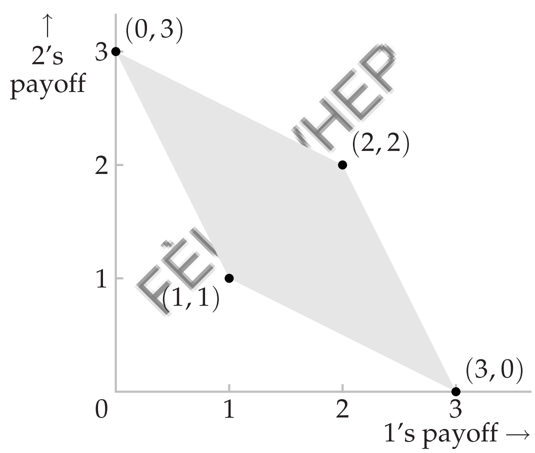

Figure 400.1 The set of feasible payoffs in the *Prisoner's Dilemma* with payoffs as in Figure 389.1. Any pair of payoffs in this set can approximately be achieved as a pair of discounted average payoffs in the infinitely repeated game when the discount factor is close to 1.

14.6.2 Nash equilibrium discounted average payoffs

We have seen that the feasible payoff pairs $(2, 2)$ and $(1, 1)$ can be achieved as discounted average payoffs pairs in Nash equilibria. Which other feasible payoff pairs can be achieved in Nash equilibria? By choosing D in every period, each player can obtain a payoff of at least 1 in each period, and hence a discounted average payoff of at least 1. Thus no pair of payoffs in which either player's payoff is less than 1 is the discounted average payoff pair of a Nash equilibrium.

I claim further that every feasible pair of payoffs in which each player's payoff is greater than 1 is close to a pair of payoffs that is the discounted average payoff pair of a Nash equilibrium when the discount factor is close enough to 1. For any feasible pair (x_1, x_2) of payoffs there is a finite sequence $(a^1, \ldots, a^k)$ of outcomes for which each player i's average payoff is x_i, so that her *discounted* average payoff can be made as close as we want to x_i by taking the discount factor close enough to 1.

Now consider the outcome path of the infinitely repeated games that consists of repetitions of the sequence $(a^1, \ldots, a^k)$; denote this outcome path by $(b^1, b^2, \ldots)$. (That is, $b^1 = b^{k+1} = b^{2k+1} = \ldots = a^1$, $b^2 = b^{k+2} = b^{2k+2} = \ldots = a^2$, and so on.) I now construct a strategy profile that yields this outcome path and, for a large enough discount factor, is a Nash equilibrium. In each period, each player's strategy chooses the action specified for her by the path so long as the other player did so in every previous period, and otherwise chooses the "punishment" action D. Precisely, player i's strategy s_i chooses the action b_i^1 in the first period and the action

$$s_i(h^1, \ldots, h^{t-1}) = \begin{cases} b_i^t & \text{if } h_j^r = b_j^r \text{ for } r = 1, \ldots, t-1 \\ D & \text{otherwise,} \end{cases}$$

after any other history $(h^1, \ldots, h^{t-1})$, where j is the other player. If every player adheres to this strategy then the outcome in each period t is b^t, so that the average payoff of each player i is x_i. Thus the discounted average payoff of each player i can be made arbitrarily close to x_i by choosing the discount factor to be close enough to 1.

If $x_i > 1$ for each player i then the strategy profile is a Nash equilibrium by the following argument. First note that since for each player i we have $x_i > 1$, for each player i there is an integer, say t_i, for which $u_i(a^{t_i}) > 1$. Now suppose that one of the players, say i, deviates from the path $(b^1, b^2, \ldots)$ in some period. In every subsequent period player j chooses D, so that player i's payoff is at most 1. In particular, in every period in which the outcome was supposed to be a^{t_i}, player i obtains the payoff 1 rather than $u_i(a^{t_i}) > 1$. If the discount factor is close enough to 1 then the discounted value of these future losses more than outweigh any gain that player i may have pocketed in the period in which she deviated. Hence for a discount factor close enough to 1, each player i is better off adhering to the strategy s_i than she is deviating, so that (s, s) is a Nash equilibrium. Further, by taking the discount factor close enough to 1 we can ensure that the discounted average payoff pair of the outcome path that (s, s) generates is arbitrarily close to (x_1, x_2).

In summary, we have proved the following result for the infinitely repeated *Prisoner's Dilemma* generated by the one-shot game with payoffs given in Figure 389.1:

- for any discount factor, each player's payoff in every discounted average payoff pair generated by a Nash equilibrium of the infinitely repeated game is at least 1
- for every feasible pair (x_1, x_2) of payoffs in the game for which $x_i > 1$ for each player i, there is a pair (y_1, y_2) close to (x_1, x_2) such that for a discount

> factor close enough to 1 there is a Nash equilibrium of the infinitely repeated game in which the pair of discounted average payoffs is (y_1, y_2).

(This result is a special case of a result I state, precisely, later; see Proposition 413.1.)

You may wonder why the second part of this statement is not simpler: why do I not claim that any outcome path in which every player's discounted average payoff exceeds 1 can be generated by a Nash equilibrium? The reason is simple: this claim is not true! Consider, for example, the outcome path $((C, C), (D, D), (D, D), \ldots)$ in which the outcome in every period but the first is (D, D). For any discount factor less than 1 each player's discounted average payoff exceeds 1 on this path, but no Nash equilibrium generates the path: a player who deviates to D in the first period obtains a higher payoff in the first period and at least the *same* payoff in every subsequent period, however her opponent behaves.

The set in Figure 402.1 illustrates the set of discounted average payoffs generated by Nash equilibria. For every point (x_1, x_2) in the set, by choosing the discount factor close enough to 1 we can ensure that there is a point (y_1, y_2) as close as we want to (x_1, x_2) that is the pair of discounted average payoffs of the infinitely repeated game. The diagram makes it clear how large the set of Nash equilibrium payoffs of the repeated game is: even though the one-shot game has a unique Nash equilibrium, and hence a unique pair of Nash equilibrium payoffs, the repeated game has a large set of Nash equilibria, with payoffs that vary from dismal to jointly maximal.

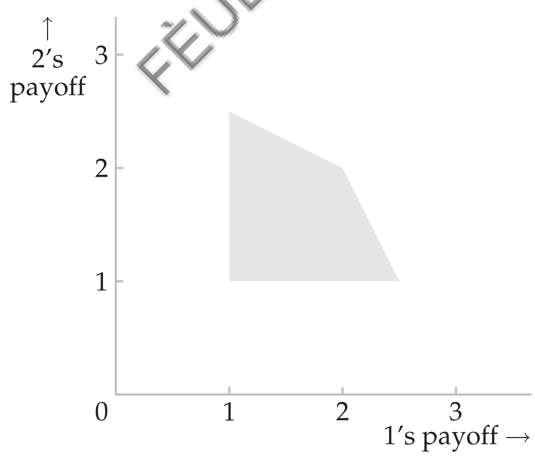

Figure 402.1 The approximate set of Nash equilibrium discounted average payoffs for the infinitely repeated *Prisoner's Dilemma* with one-shot payoffs as in Figure 389.1 when the discount factor is close to 1.

14.7 Subgame perfect equilibria and the one-deviation property

We saw in Section ????? that a strategy profile in a finite horizon extensive game is a subgame perfect equilibrium if and only if it satisfies the *one-deviation property*: no player can increase her payoff by changing her action at the start of any subgame in which she is the first mover, *given* the other player's strategies *and* the rest of her own strategy. I now argue that the same is true in an infinitely repeated game, a fact that can greatly simplify the process of determining whether or not a strategy profile is a subgame perfect equilibrium.

As in the case of a finite horizon game, if a strategy profile is a subgame perfect equilibrium then certainly it satisfies the one-deviation property, since no player must be able to increase her payoff by *any* change in her strategy. What we need to show is the converse: if a strategy profile is not a subgame perfect equilibrium then there is some subgame in which the first-mover can increase her payoff by changing *only* her initial action.

Let s be a strategy profile that is not a subgame perfect equilibrium. Specifically, suppose that in the subgame following the nonterminal history h, player i can increase her payoff by using the strategy s_i' rather than s_i. Now, since payoffs in the distant future are worth very little, there is some period T such that any strategy that coincides with s_i' through period T is better than any strategy that coincides with s_i through period T: T can be chosen to be sufficiently large that the first strategy yields a higher discounted average payoff than the second one even if the first strategy induces the best possible outcome for player i in every period after T, and the second strategy induces the worst possible outcome in every such period. In particular, the strategy s_i'' that coincides with s_i' through period T and with s_i after period T is better for player i than s_i.

But now by the same argument as for finite horizon games (Proposition ????), we can find a strategy for player i and a subgame such that in the subgame the strategy differs from s_i only in its first action and yields a payoff higher than that yielded by s_i (given that the other players adhere to s_{-i}). A more precise statement of the result and proof follows.

■ PROPOSITION 403.1 (One-deviation property of subgame perfect equilibria of infinitely repeated games) *A strategy profile in an infinitely repeated game is a subgame perfect equilibrium if and only if no player can gain by changing her action after any history, given both the strategies of the other players and the remainder of her own strategy.*

Proof. If the strategy profile s is a subgame perfect equilibrium then no player can gain by any deviation, so that if some player can gain by a one-period deviation then s is definitely not a subgame perfect equilibrium.

I now need to show that if s is not a subgame perfect equilibrium then in the subgame that follows some history h, some player, say i, can gain by a one-period deviation from s_i. Without loss of generality, assume that h is the initial history.

Now, since payoffs in the sufficiently distant future have an arbitrarily small value from today's point of view, there is some period T such that the payoff to any strategy that follows s_i' through period T exceeds the payoff to any strategy that follows by s_i through period T (given that the other players adhere to s_{-i}). (The integer T can be chosen to be sufficiently large that the first strategy yields a higher discounted average payoff than the second one even if the first strategy induces the best possible outcome for player i in every period after T, and the second strategy induces the worst possible outcome in every such period.) In particular, the strategy s_i'' that coincides with s_i' through period T and with s_i subsequently is better for player i than the strategy s_i.

Now, s_i and s_i'' differ only in the actions they prescribe after finitely many histories, so we can apply the argument in the proof of Proposition ??? to find a strategy of player i and a history such that in the subgame that follows the history, the strategy differs from s_i only in the action it prescribes initially, and player i is better off following the strategy than following s_i.

Thus we have shown that if s is not a subgame perfect equilibrium then some player can increase her payoff by making a one-period deviation after some history. □

14.8 Some subgame perfect equilibria of the infinitely repeated Prisoner's Dilemma

The notion of Nash equilibrium requires only that each player's strategy be optimal in the whole game, given the other players' strategies; after histories that do not occur if the players follow their strategies, the actions specified by a player's Nash equilibrium strategy may not be optimal. In some cases we can think of the actions prescribed by a strategy for histories that will not occur if the players follow their strategies as "threats"; the notion of Nash equilibrium does not require that it be optimal for a player to carry out these threats if called upon to do so. In the previous chapter we studied the notion of subgame perfect equilibrium, which does impose such a requirement: a strategy profile is a subgame perfect equilibrium if every player's strategy is optimal not only in the whole game, but after *every* history (including histories that do not occur if the players adhere to their strategies).

Are the Nash equilibria we considered in the previous section subgame perfect equilibria of the infinitely repeated *Prisoner's Dilemma* with payoffs as in Figure 389.1? Clearly the Nash equilibrium in which each player chooses D after every history is a subgame perfect equilibrium: whatever happens, each player chooses D, so it is optimal for the other player to do likewise. Now consider the other Nash equilibria we studied.

14.8.1 Grim trigger strategies

Suppose that the outcome in the first period is (C, D). Is it optimal for each player to subsequently adhere to the grim trigger strategy, given that the other player does so? In particular, is it optimal for player 1 to carry out the punishment that the grim trigger strategy prescribes? If both players adhere to the strategy then player 1 chooses D in every subsequent period while player 2 chooses C in period 2 and then D subsequently, so that the sequence of outcomes in the subgame following the history (C, D) is $((D, C), (D, D), (D, D), \ldots)$, yielding player 1 a discounted average payoff of

$$3(1-\delta)+\delta = 3-2\delta.$$

If player 1 refrains from punishing player 2 for her lapse, and simply chooses C in every subsequent period, then the outcome in period 2 and subsequently is (C, C), so that the sequence of outcomes in the game yields player 1 a discounted average payoff of 2. If $\delta > \frac{1}{2}$ then $2 > 3 - 2\delta$, so that player 1 prefers not to punish player 2 for a deviation, and hence the strategy pair in which each player uses the grim trigger strategy is not a subgame perfect equilibrium.

In fact, the strategy pair in which each player uses the grim trigger strategy is not a subgame perfect equilibrium for *any* value of δ, for the following reason. If player 1 adheres to the grim trigger strategy, then in the subgame following the outcome (C, D), player 2 prefers to choose D in period 2 and subsequently, regardless of the value of δ (since the outcome is then (D, D) in every period, rather than (D, C) in the first period of the subgame and (D, D) subsequently).

In summary, the strategy pair in which both players use the grim trigger strategy defined in Figure 395.1 is not a subgame perfect equilibrium of the infinitely repeated game for any value of the discount factor: after the history (C, D) player 1 has no incentive to punish player 2, and player 2 prefers to choose D in every subsequent period if she is going to be punished, rather than choosing C in the second period of the game and then D subsequently.

However, a small modification of the grim trigger strategy fixes both of these problems. Consider the variant of the grim trigger strategy in which a player chooses D after any history in which *either* player chose D in some period. This strategy is illustrated in Figure 405.1. If both players adopt this strategy then in the subgame following a deviation, the miscreant chooses D in every period, so that her opponent is better off "punishing" her by choosing D than she is by choosing C. Further, a player's behavior during her punishment is optimal—she chooses D in every period. The point is that (D, D) is a Nash equilibrium of a *Prisoner's Dilemma*, so that neither player has any quarrel with the prescription of the modified grim trigger strategy that she choose D after any history in which some player chose D. The fact that the strategy specifies that a player choose D after any history in which she deviated means that it is optimal for the other player to punish her, and since she is punished it is optimal for her to choose D. Effectively, a player's strategy "punishes" her opponent—by choosing D—if her opponent does not "punish" her for deviating.

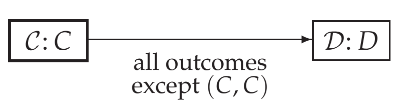

Figure 405.1 A variant of the grim strategy in an infinitely repeated *Prisoner's Dilemma*.

14.8.2 Limited punishment

The pair of strategies (s, s) in which s is the limited punishment strategy studied in Section 14.5.2 is not a subgame perfect equilibrium of the infinitely repeated *Prisoner's Dilemma* for the same reason that a pair of grim trigger strategies is not a subgame perfect equilibrium. However, as in the case of grim trigger strategies, we can modify the limited punishment strategy in order to obtain a subgame perfect equilibrium. Specifically, we need the transition from state P_0 to state P_1 in Figure 395.2 to occur whenever *either* player chooses D (not just if the *other* player chooses D). A player using this modified strategy chooses D during her punishment, which both is optimal for her and makes the other player's choice to punish optimal. When the punishment ends she, like her punisher, reverts to C.

? EXERCISE 406.1 (Lengths of punishment in subgame perfect equilibrium) Is there any subgame perfect equilibrium of an infinitely repeated *Prisoner's Dilemma* (with payoffs as in Figure 389.1), for any value of δ, in which each player's strategy involves limited punishment, but the lengths of the punishment are different for each player? If so, describe such a subgame perfect equilibrium; if not, argue why not.

14.8.3 Tit-for-tat

The behavior in a subgame of a player who uses the strategy *tit-for-tat* depends only on the last outcome in the history that preceded the subgame. Thus to examine whether the strategy pair in which both players use the strategy *tit-for-tat* is a subgame perfect equilibrium we need to consider four types of subgame, following histories in which the last outcome is (C, C), (C, D), (D, C), and (D, D).

The optimality of *tit-for-tat* in a subgame following a history ending in (C, C), given that the other player uses *tit-for-tat*, is covered by our analysis of Nash equilibrium: if $\delta \geq \frac{1}{2}$ then *tit-for-tat* is a best response to *tit-for-tat* in such a subgame.

In studying subgames following histories ending in other outcomes, I appeal to the fact that a strategy profile is a subgame perfect equilibrium if and only if it satisfies the one-deviation property (Propostion ???).

Consider the subgame following a history ending in the outcome (C, D). Suppose that player 2 adheres to *tit-for-tat*. If player 1 also adheres to *tit-for-tat* then the outcome alternates between (D, C) and (C, D), and player 1's discounted average payoff in the subgame is

$$(1 - \delta)(3 + 3\delta^2 + \cdots) = \frac{3}{1 + \delta}.$$

If player 1 instead chooses C in the first period of the subgame, and subsequently adheres to *tit-for-tat*, then the outcome is (C, C) in every period of the subgame, so that player 1's discounted average payoff is 2. Thus in order that *tit-for-tat* be optimal in such a subgame we need

$$\frac{3}{1 + \delta} \geq 2, \text{ or } \delta \leq \tfrac{1}{2}.$$

In the subgame following a history ending with the outcome (D,C), the outcome alternates between (C,D) and (D,C) if both players adhere to *tit-for-tat*, yielding player 1 a discounted average payoff of $3\delta/(1+\delta)$ (the first outcome is (C,D), rather than (D,C) as in the previous case). If player 1 deviates to D in the first period, and then adheres to *tit-for-tat* then the outcome is (D,D) in every period, yielding player 1 a discounted average payoff of 1. Thus for *tit-for-tat* to be optimal for player 1 we need

$$\frac{3\delta}{1+\delta} \geq 1, \text{ or } \delta \geq \tfrac{1}{2}.$$

Finally, in a subgame following a history ending with the outcome (D,D), the outcome is (D,D) in every period if both players adhere to *tit-for-tat*, yielding player 1 a discounted average payoff of 1. If player 1 deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (C,D) and (D,C), yielding player 1 a discounted average payoff of $3\delta/(1+\delta)$. Thus *tit-for-tat* is optimal for player 1 only if $\delta \leq \frac{1}{2}$.

We conclude that (*tit-for-tat*,*tit-for-tat*) is a subgame perfect equilibrium of the infinitely repeated *Prisoner's Dilemma* with payoffs as in Figure 389.1 if and only if $\delta = \frac{1}{2}$. In fact, the existence of any value of the discount factor for which (*tit-for-tat*,*tit-for-tat*) is a subgame perfect equilibrium depends on the specific payoffs I have assumed for the component game: this strategy pair is a subgame perfect equilibrium of an infinitely repeated *Prisoner's Dilemma* only if the payoffs of the component game are rather special, as you are asked to show in the following exercise.

? EXERCISE 407.1 (*Tit-for-tat* as a subgame perfect equilibrium in the infinitely repeated *Prisoner's Dilemma*) Consider the infinitely repeated *Prisoner's Dilemma* in which the payoffs of the component game are those given in Figure 407.1. Show that (*tit-for-tat*,*tit-for-tat*) is a subgame perfect equilibrium if and only if $y - x = 1$ and $\delta = 1/x$. (Use the fact that subgame perfect equilibria have the one-deviation property.)

	C	D
C	x, x	$0, y$
D	$y, 0$	$1, 1$

Figure 407.1 The component game for the infinitely repeated *Prisoner's Dilemma* considered in Exercise 407.1.

14.8.4 *Subgame perfect equilibrium payoffs of the infinitely repeated* Prisoner's Dilemma *when the players are patient*

In Section 14.6 we saw that every pair (x_1, x_2) in which $x_i > 1$ is close to a pair of discounted average payoffs to some Nash equilibrium of the infinitely repeated

Prisoner's Dilemma with payoffs as in Figure 389.1 when the players are sufficiently patient. Since every subgame perfect equilibrium is a Nash equilibrium, the set of subgame perfect equilibrium payoff pairs is a subset of the set of Nash equilibrium payoff pairs. I now argue that, in fact, the two sets are the same. The strategy pair that I used in the argument of Section 14.6 is not a subgame perfect equilibrium, but can be modified, along the lines we considered in the previous section, to turn it into such an equilibrium.

Let (x_1, x_2) be a pair of feasible payoffs in the *Prisoner's Dilemma* for which $x_i > 1$ for each player i. Let $(a^1, \ldots, a^k)$ be a sequence of outcomes of the game for which each player i's average payoff is x_i, and let $(b^1, b^2, \ldots)$ be the outcome path of the infinitely repeated game that consists of repetitions of the sequence $(a^1, \ldots, a^k)$. I claim that the strategy pair in which each player follows the path $(b^1, b^2, \ldots)$ so long as *both* she and the other player have done so in the past, and otherwise chooses D, is a subgame perfect equilibrium. If one player deviates then subsequent to her deviation she continues to choose D, making it optimal for her opponent to "punish" her by choosing D. Precisely, the strategy s_i of player i chooses the action b_i^1 in the first period and the action

$$s_i(h^1, \ldots, h^{t-1}) = \begin{cases} b_i^t & \text{if } h^r = b^r \text{ for } r = 1, \ldots, t-1 \\ D & \text{otherwise,} \end{cases}$$

after any other history $(h^1, \ldots, h^{t-1})$.

I claim that (s, s) is a subgame perfect equilibrium of the infinitely repeated game. There are two types of subgame to consider. First, consider a history in which the outcome was b^r in every period r. The argument that if one player acts according to s in the subgame that follows such a history then it is optimal for the other to do so is the same as the argument that the strategy pair defined in Section 14.6 is a Nash equilibrium. Briefly, if both players adhere to the strategy s in the subgame, the outcome is b^t in every period t, yielding each player i a discounted average payoff close to x_i when the discount factor is close to 1. If one player deviates from s, then she may gain in the period in which she deviates, but her deviation will trigger her opponent to choose D in every subsequent period, so that given $x_i > 1$ for each i, her deviation makes her worse off if her discount factor is close enough to 1.

Now consider a history in which the outcome was different from b^r in some period r. If, in the subgame following this history, the players both use the strategy s, then they both choose D regardless of the outcomes in the subgame. Since the strategy pair in which both players always choose D regardless of history is a Nash equilibrium of the infinitely repeated game, the strategy pair that (s, s) induces in such a subgame is a Nash equilibrium.

We conclude that the strategy pair (s, s) is a subgame perfect equilibrium. The point is that after any deviation the players' strategies lead them to choose Nash equilibrium actions of the component game in every subsequent period, so that neither player has any incentive to deviate.

Since no player's discounted average payoff can be less than 1 in any Nash equilibrium of the infinitely repeated game, we conclude that the set of discounted average payoffs possible in subgame perfect equilibria is exactly the same as the set of discounted average payoffs possible in Nash equilibria:

- for any discount factor, each player's payoff in every discounted average payoff pair generated by a subgame perfect equilibrium of the infinitely repeated game is at least 1
- for every pair (x_1, x_2) of feasible payoffs in the game for which $x_i > 1$ for each player i, there is a pair (y_1, y_2) close to (x_1, x_2) such that for a discount factor close enough to 1 there is a subgame perfect equilibrium of the infinitely repeated game in which the pair of discounted average payoffs is (y_1, y_2).

Notes

15 Repeated games: General Results

Prerequisite: Chapter 14

15.1 Nash equilibria of general infinitely repeated games

THE IDEA behind the analysis of an infinitely repeated *Prisoner's Dilemma* applies to any infinitely repeated game: every feasible payoff profile in the one shot game in which each player's payoff exceeds some minimum is close (at least) to the discounted average payoff profile of a Nash equilibrium in which a deviation triggers each player to begin an indefinite "punishment" of the deviant.

For the *Prisoner's Dilemma* the minimum payoff of player i that is supported by a Nash equilibrium is $u_i(D, D)$. The significance of this payoff is that player j can ensure (by choosing D) that player i's payoff does not exceed $u_i(D, D)$, and there is no lower payoff with this property. That is, $u_i(D, D)$ is the lowest payoff that player j can force upon player i.

How can we find this minimum payoff in an arbitrary strategic game? Suppose that the deviant is player i. For any collection a_{-i} of the other players' actions, player i's highest possible payoff is her payoff when she chooses a best response to a_{-i}, namely

$$\max_{a_i \in A_i} u_i(a_i, a_{-i}).$$

As a_{-i} varies, this maximal payoff varies. We seek a collection a_{-i} of "punishment" actions that make this maximum as small as possible. That is, we seek a solution to the problem

$$\min_{a_{-i} \in A_{-i}} \left(\max_{a_i \in A_i} u_i(a_i, a_{-i}) \right).$$

This payoff is known, not surprisingly, as player i's *minmax* payoff.

▶ DEFINITION 411.1 Player i's **minmax** payoff in a strategic game in which the action set and payoff function of each player i are A_i and u_i respectively is

$$\min_{a_{-i} \in A_{-i}} \left(\max_{a_i \in A_i} u_i(a_i, a_{-i}) \right). \tag{411.2}$$

(Note that I am restricting attention to pure strategies in the strategic game; a player's minmax payoff is different if we consider mixed strategies.)

For example, in the *Prisoner's Dilemma* with the payoffs in Figure 389.1, each player's minmax payoff is 1; in *BoS* (Example 16.2) each player's minmax payoff is also 1.

? EXERCISE 412.1 (Minmax payoffs) Find each player's minmax payoff in each of the following games.

a. The game of dividing money in Exercise 36.2.

b. Cournot's oligopoly game (Section 3.1) when $C_i(0) = 0$ for each firm i and $P(Q) = 0$ for some sufficiently large value of Q.

c. Hotelling's model of electoral competition (Section 3.3) when (*i*) there are two candidates and (*ii*) there are three candidates, under the assumptions that the set of possible positions is the interval $[0, 1]$, the distribution of the candidates' ideal positions has a unique median, a tie results in each candidate's winning with probability $\frac{1}{2}$, and each candidate's payoff is her probability of winning.

Whatever the other players' strategies, any player can obtain at least her minmax payoff in every period, and hence a discounted average payoff at least equal to her minmax payoff, by choosing in each period a best response to the other players' actions. More precisely, player i can ensure that her payoff in every period is at least her minmax payoff by using a strategy that, after every history h, chooses a best response to $s_{-i}(h)$, the collection of actions prescribed for the other players' strategies after the history h. Thus in no Nash equilibrium of the infinitely repeated game is player i's discounted average payoff less than her minmax payoff.

We saw that in the *Prisoner's Dilemma*, a converse of this result holds: for every feasible payoff profile x in the game in which x_i exceeds player i's minmax payoff for $i = 1, 2$, for a discount factor sufficiently close to 1 there is a Nash equilibrium of the infinitely repeated game in which the discounted average payoff of player i is close to x_i for $i = 1, 2$.

An analogous result holds in general. The simplest case to consider is that in which x is a payoff profile of the game. Let x be the payoff profile generated by the action profile a; assume that each x_i exceeds player i's minmax payoff. For each player i, let p_{-i} be a collection of actions for the players other than i that holds player i down to her minmax payoff. (That is, p_{-i} is a solution of the minimization problem (411.2).) Define a strategy for each player as follows. In each period, the strategy of each player i chooses a_i as long as every other player j chose a_j in every previous period, and otherwise chooses the action $(p_{-j})_i$, where j is the player who deviated in the first period in which exactly one player deviated. Precisely, let H^* be the set of histories in which there is at least one period in which exactly one player j chose an action different from a_j. Refer to such a player as a *lone deviant*. The strategy of player i is defined by $s_i(\varnothing) = a_i$ (her action at the start of the game is a_i) and

$$s_i(h) = \begin{cases} a_i & \text{if } h \text{ is not in } H^* \\ (p_{-j})_i & \text{if } h \in H^* \text{ and } j \text{ is the first lone deviant in } h. \end{cases}$$

The strategy profile s is a Nash equilibrium by the following argument. If player i adheres to s_i then, given that every other player j adheres to s_j, her payoff is x_i in every period. If player i deviates from s_i, while every other player j adheres to s_j, then she may gain in the period in which she deviates, but she loses in every subsequent period, obtaining at most her minmax payoff, rather than x_i. Thus for a discount factor close enough to 1, s_i is a best response to s_{-i} for every player i, so that s is a Nash equilibrium.

(Note that the strategies I have defined do not react when more than one player deviates in any one period. They do not need to, since the notion of Nash equilibrium requires only that no *single* player has an incentive to deviate.)

This argument can be extended to deal with the case in which x is a feasible payoff profile that is not the payoff profile of a single action profile in the component game, along the same lines as the argument in the case of the *Prisoner's Dilemma* in the previous section. The result we obtain is known as a "folk theorem", since the basic form of the result was known long before it was written down precisely.[1]

■ PROPOSITION 413.1 (Nash folk theorem) *Let G be a strategic game.*

- *For any discount factor δ with $0 < \delta < 1$, the discounted average payoff of every player in any Nash equilibrium of the infinitely repeated game of G is at least her minmax payoff.*
- *Let w be a feasible payoff profile of G for which each player's payoff exceeds her minmax payoff. Then for all $\epsilon > 0$ there exists $\overline{\delta} < 1$ such that if the discount factor exceeds $\overline{\delta}$ then the infinitely repeated game of G has a Nash equilibrium whose discounted average payoff profile w' satisfies $|w - w'| < \epsilon$.*

? EXERCISE 413.2 (Nash equilibrium payoffs in infinitely repeated games) For the infinitely repeated games for which each of the following strategic games is the component game, find the set of discounted average payoffs to Nash equilibria of these infinitely repeated games when the discount factor is close to 1. (Parts *b* and *c* of Exercise 412.1 are relevant to parts *b* and *c*.)

a. *BoS* (Example 16.2).

b. Cournot's oligopoly game (Section 3.1) when there are two firms, $C_i(q_i) = q_i$ for all q_i for each firm i, and $P(Q) = \max\{0, \alpha - \beta Q\}$.

c. Hotelling's model of electoral competition (Section 3.3) when there are two candidates, under the assumptions that the set of possible positions is the interval $[0, 1]$, the distribution of the citizens' ideal positions has a unique median, a tie results in each candidate's winning with probability $\frac{1}{2}$, and each candidate's payoff is her probability of winning.

[1] If $x = (x_1, \ldots, x_n)$ is a vector then $|x|$ is the norm of x, namely $(x_1^2 + \cdots + x_n^2)^{1/2}$. If x and y are vectors and $|x - y|$ is small then the components of x and y are close to each other.

The strategies in the Nash equilibrium used to prove Proposition 413.1 are grim trigger strategies: any transgression leads to interminable punishment. As in the case of the *Prisoner's Dilemma*, less draconian punishment is sufficient to deter deviations; grim trigger strategies are simply easy to work with. The punishment embedded in a strategy has only to be severe enough that any deviation ultimately results in a net loss for its perpetrator.

? EXERCISE 414.1 (Repeated Bertrand duopoly) Consider Bertrand's model of duopoly (Section 3.2) in the case that each firm's unit cost is constant, equal to c. Let $\Pi(p) = (p - c)D(p)$ for any price p, and assume that Π is continuous and is uniquely maximized at the price p^m (the "monopoly price").

a. Let s be the strategy for the infinitely repeated game that charges p^m in the first period and subsequently as long as the other firm continues to charge p^m, and punishes any deviation from p^m by the other firm by choosing the price c for k periods, then reverting to p^m. Given any value of δ, for what values of k is the strategy pair (s, s) a Nash equilibrium of the infinitely repeated game?

b. Let s be the strategy for the infinitely repeated game defined as follows:

- in the first period charge the price p^m
- in every subsequent period charge the lowest of all the prices charged by the other firm in all previous periods.

Is the strategy pair (s, s) a Nash equilibrium of the infinitely repeated game for any discount factor less than 1?

15.2 Subgame perfect equilibria of general infinitely repeated games

The *Prisoner's Dilemma* has a feature that makes it easy to construct a subgame perfect equilibrium of the infinitely repeated game to prove the result in the previous section: it has a Nash equilibrium in which each player's payoff is her minmax payoff. In any game, each player's payoff is at least her minmax payoff, but in general there is no Nash equilibrium in which the payoffs are exactly the minmax payoffs. It may be clear how to generalize the arguments above to define a subgame perfect equilibrium of any infinitely repeated game in which both players' discounted average payoffs exceed their payoffs in some Nash equilibrium of the component game. However, it is not clear whether there are subgame perfect equilibrium payoff pairs in which the players' payoffs are between their minmax payoffs and their payoffs in the worst Nash equilibrium of the component game.

Consider the game in Figure 415.1. Each player's minmax payoff is 1: by choosing C, each player can ensure that the other player's payoff does not exceed 1, and there is no action that ensures that the other player's payoff is less than 1. In the unique Nash equilibrium (A, A), on the other hand, each player's payoff is 4. Payoffs between 1 and 4 cannot be achieved by strategies that react to deviations by choosing A, since one player's choosing A allows the other to obtain a payoff of 4 (by choosing A also), which exceeds her payoff if she does not deviate.

	A	B	C
A	$4,4$	$3,0$	$1,0$
B	$0,3$	$2,2$	$1,0$
C	$0,1$	$0,1$	$0,0$

Figure 415.1 A strategic game with a unique Nash equilibrium in which each player's payoff exceeds her minmax payoff.

Nevertheless, such payoffs can be achieved in subgame perfect equilibria. The punishments built into the players' strategies in these equilibria need to be carefully designed. A deviation cannot lead to the indefinite play of (C, C), since each player has an incentive to deviate from this action pair. In order to make it worthwhile for a player to punish her opponent for deviating, she must be made worse off if she fails to punish than if she does so. We can achieve this effect by designing strategies that punish deviations for a limited amount of time—enough to wipe out the gain from a deviation—so long as both players act as they are supposed to during the punishment, but are extended whenever one of the players misbehaves.

Specifically, consider the strategy s shown in Figure 415.2 for a player in the game in Figure 415.1. This strategy starts a two-period punishment after a de-

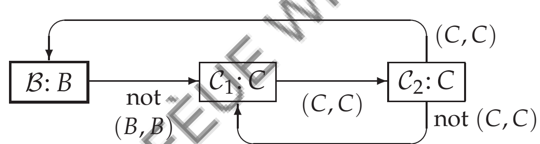

Figure 415.2 A subgame perfect equilibrium strategy for a player in the infinitely repeated game for which the component game is that given in Figure 415.1.

viation from the outcome (B, B). If both players choose the action C during the punishment phase then after two periods they both revert to choosing B. If, however, one of them does not choose C in the first period of the punishment then the punishment starts again: the transition from the first punishment state $\mathcal{C}_1$ to the second punishment state $\mathcal{C}_2$ does not occur unless both players choose C after a deviation from (B, B). Further, if there is a deviation from C in the second period of the punishment then there is a transition back to $\mathcal{C}_1$: the punishment starts again. Thus built into the strategy is punishment for a player who does not carry out a punishment.

I claim that if the discount factor is close enough to 1 then the strategy pair in which both players use this strategy is a subgame perfect equilibrium of the infinitely repeated game. The players' behavior in period t is determined only by the current state, so we need to consider only three cases. Suppose that player 2 adheres to the strategy, and in each case consider whether player 1 can increase her payoff by deviating at the start of the subgame, holding the rest of her strategy fixed.

State $\mathcal{B}$: If player 1 adheres to the strategy her payoffs in the next three periods are $(2, 2, 2)$, while if she deviates they are at most $(3, 0, 0)$; in both cases her payoff is subsequently 2. Thus adhering to the strategy is optimal if $2 + 2\delta + 2\delta^2 \geq 3$, or $\delta \geq \frac{1}{2}(\sqrt{3} - 1)$.

State $\mathcal{C}_1$: If player 1 adheres to the strategy her payoffs in the next three periods are $(0, 0, 2)$, while if she deviates they are at most $(1, 0, 0)$; in both cases, her payoff is subsequently 2. Thus adhering to the strategy is optimal if $2\delta^2 \geq 1$, or $\delta \geq \frac{1}{2}\sqrt{2}$.

State $\mathcal{C}_2$: If player 1 adheres to the strategy her payoffs in the next three periods are $(0, 2, 2)$, while if she deviates they are at most $(1, 0, 0)$; in both cases, her payoff is subsequently 2. Thus adhering to the strategy is optimal if $2\delta + 2\delta^2 \geq 1$, or certainly if $2\delta^2 \geq 1$, as required by the previous case.

We conclude, using the fact that a strategy profile is a subgame perfect equilibrium if and only if it satisfies the one-deviation property, that if $\delta \geq \frac{1}{2}\sqrt{2}$ then (s, s) is a subgame perfect equilibrium.

The idea behind this example can be extended to any two-player game. Consider an outcome a of such a game for which both players' payoffs exceed their minmax payoffs. I construct a subgame perfect equilibrium in which the outcome is a in every period. Let p_j be an action of player i that holds player j down to her minmax payoff (a "punishment" for player j), and let $p = (p_2, p_1)$ (each player punishes the other). Let s_i be a strategy of player i of the form shown in Figure 416.1, for some value of k. This strategy starts off choosing a_i, and continues to choose a_i so long as the outcome is a; otherwise, it chooses the action p_j that holds player j to her minmax payoff. Once punishment begins, it continues for k periods as long as both players choose their punishment actions. If any player deviates from her assigned punishment action then the punishments are re-started (from each state $\mathcal{P}_\ell$ there is a transition to state $\mathcal{P}_1$ if the outcome in the previous period is not p).

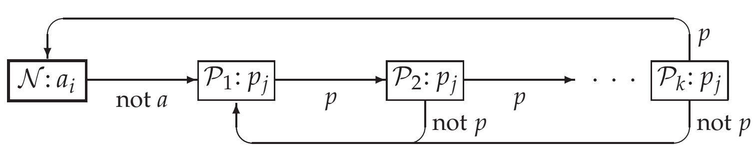

Figure 416.1 A subgame perfect equilibrium strategy for player i in a two-player infinitely repeated game. The outcome p is that in which each player's action is one that holds the other player down to her minmax payoff.

I claim that we can find $\underline{\delta}$ and $k(\delta)$ such that if $\delta > \underline{\delta}$ then the strategy pair (s_1, s_2) is a subgame perfect equilibrium of the infinitely repeated game. Suppose that player j adheres to s_j. If player i adheres to s_i in state $\mathcal{N}$ then her discounted average payoff is $u_i(a)$. If she deviates, she obtains at most her maximal payoff in the game, say $\overline{u}_i$, in the period of her deviation, then $u_i(p)$ for k periods, and subsequently $u_i(a)$ in the future. Thus her discounted average payoff from the deviation is at most

$$(1-\delta)[\overline{u}_i + \delta u_i(p) + \cdots + \delta^k u_i(p)] + \delta^{k+1}u_i(a) = \\ (1-\delta)\overline{u}_i + \delta(1-\delta^k)u_i(p) + \delta^{k+1}u_i(a).$$

In order for her not to want to deviate it is thus sufficient that

$$u_i(a) \geq (1-\delta)\overline{u}_i + \delta(1-\delta^k)u_i(p) + \delta^{k+1}u_i(a). \quad (417.1)$$

If player i adheres to s_i in any state $\mathcal{P}_\ell$ then she obtains $u_i(p)$ for at most k periods, then $u_i(a)$ in every subsequent period, which yields a discounted average payoff of at least

$$(1-\delta^k)u_i(p) + \delta^k u_i(a)$$

(since $u_i(p)$ is at most player i's minmax payoff and $u_i(a)$ exceeds this minmax payoff). If she deviates from s_i, she obtains at most her minmax payoff in the period of her deviation, then $u_i(p)$ for k periods, then $u_i(a)$ in the future, which yields a discounted average payoff of at most

$$(1-\delta)m_i + \delta(1-\delta^k)u_i(p) + \delta^{k+1}u_i(a),$$

where m_i is her minmax payoff. Thus in order that she not want to deviate it is sufficient that

$$(1-\delta^k)u_i(p) + \delta^k u_i(a) \geq (1-\delta)m_i + \delta(1-\delta^k)u_i(p) + \delta^{k+1}u_i(a)$$

or

$$(1-\delta^k)u_i(p) + \delta^k u_i(a) \geq m_i. \quad (417.2)$$

Thus if for each value of δ sufficiently close to 1 we can find $k(\delta)$ such that $(\delta, k(\delta))$ satisfies (417.1) and (417.2) then the strategy pair (s_1, s_2) is a subgame perfect equilibrium. [Need to make this argument.]

This argument shows that for any outcome of the component game in which each player's payoff exceeds her minmax payoff there is a subgame perfect equilibrium that yields this outcome path. More generally, for any two-player strategic game and any feasible payoff pair (x_1, x_2) in which each player's payoff exceeds her minmax payoff, we can construct a Nash equilibrium strategy pair that generates an outcome path for which the discounted average payoff of each player i is x_i. A precise statement of this result follows.

■ PROPOSITION 417.3 (Subgame perfect folk theorem for two-player games) *Let G be a two-player strategic game.*

- *For any discount factor δ with $0 < \delta < 1$, the discounted average payoff of every player in any subgame perfect equilibrium of the infinitely repeated game of G is at least her minmax payoff.*

- *Let w be a feasible payoff profile of G for which each player's payoff exceeds her minmax payoff. Then for all $\epsilon > 0$ there exists $\overline{\delta} < 1$ such that if the discount factor exceeds $\overline{\delta}$ then the infinitely repeated game of G has a subgame perfect equilibrium whose discounted average payoff profile w' satisfies $|w - w'| < \epsilon$.*

The conclusion of this result does not hold for all multi-player games.

AXELROD'S EXPERIMENTS

In the late 1970s, Robert Axelrod (a political scientist at the University of Michigan) invited some economists, psychologists, mathematicians, and sociologists familiar with the repeated *Prisoner's Dilemma* to submit strategies (written in computer code) for a finitely repeated *Prisoner's Dilemma* with payoffs of $(3,3)$ for (C,C), $(5,0)$ for (D,C), $(0,5)$ for (C,D), and $(1,1)$ for (D,D). He received 14 entries, which he pitted against each other, and against a strategy that randomly chooses C and D each with probability $\frac{1}{2}$, in 200-fold repetitions of the game. Each strategy was paired against each other five times. (Strategies could involve random choices, so a pair of strategies could generate different outcomes when paired repeatedly.) The strategy with the highest payoff was *tit-for-tat* (submitted by Anatol Rapoport, then a member of the Psychology Department of the University of Toronto). (See Axelrod (1980a, 1984).)

Axelrod, intrigued by the result, subsequently ran a second tournament. He invited the participants in the first tournament to compete again, and also recruited entrants by advertising in journals read by microcomputer users (a relatively small crowd in the early 1980s); contestants were informed of the results of the first round. Sixty-two strategies were submitted. The contest was run slightly differently from the previous one: the length of each game was determined probabilistically. Again *tit-for-tat* (again submitted by Anatol Rapoport) won. (See Axelrod (1980b, 1984).

Using the strategies submitted in his second tournament, Axelrod simulated an environment in which strategies that do well reproduce faster than other strategies. He repeatedly matched the strategies against each other, increasing the number of representatives of strategies that achieved high payoffs. A strategy that obtained a high payoff initially might, under these conditions, obtain a low one later on if the opponents against which it did well become much less numerous relative to the others. Axelrod found that after a large number of "generations" *tit-for-tat* had the most representatives in the population.

However, *tit-for-tat*'s supremacy has been subsequently shown to be fragile. [Discussion to be added.]

Axelrod's simulations are limited by the set of strategies that were submitted to him. Other simulations have included all strategies of a particular type. One type of strategy that has been examined is the class of "reactive strategies", in which a player's action in any period depends only on the other player's action in the previous period (Nowak and Sigmund (1992)). In evolutionary simulations in which the initial population consists of randomly selected reactive strategies, the strategy that chooses D in every period, regardless of the history, is found to come to dominate. However, if *tit-for-tat* is included in the set of strategies initially in the population, a strategy known as *generous tit-for-tat*, which differs from *tit-for-tat* only in that after its opponent chooses D it chooses D with probability $\frac{1}{3}$ (given the payoffs for the *Prisoner's Dilemma* used by Axelrod), XXXXXXXXXXXXX.

The results are different when the larger class of strategies in which the action chosen in any period depends on *both* actions chosen in the previous period is studied. In this case the strategy *Pavlov* (also known as *win–stay, lose–shift*; see Exercise 398.1), which chooses C when the outcome in the previous period was either (C, C) or (D, D) and otherwise chooses D, tends to come to dominate the population.

In summary, simulations show that a variety of strategies may emerge as "winners" in the repeated *Prisoner's Dilemma*; Axelrod's conclusions about the robustness of *tit-for-tat* appear to have been premature.

Given these results, it is natural to ask if the theory of evolutionary games (Chapter 13) can offer insights into the strategies that might be expected to survive. Unfortunately, the existing results are negative: depending on how one defines an evolutionarily stable strategy (ESS) in an extensive game, an infinitely repeated *Prisoner's Dilemma* either has no ESS, or the only ESS is the strategy that chooses D in every period regardless of history, or every feasible pair of payoffs can be sustained by some pair of ESSs (Kim (1994)).

Reciprocal altruism among sticklebacks

The idea that a population of animals repeatedly involved in a conflict with the structure of a *Prisoner's Dilemma* might evolve a mode of behavior involving reciprocal altruism (as in the strategy *tit-for-tat*), was suggested by Trivers (1971) and led biologists to look for examples of such behavior.

One much-discussed example involves predator inspection by sticklebacks. Sticklebacks often approach a predator in pairs, the members of a pair taking turns to be the first to move forward a few centimeters. (It is advantageous for them to approach the predator closely, since they thereby obtain more information about it.) The process can be modeled as a repeated *Prisoner's Dilemma*, in which moving forward is analogous to cooperating and holding back is like defecting. Milinski (1987) reports an experiment in which he put a stickleback into one compartment of a tank and a cichlid, which resembles a perch, a common predator of sticklebacks, in another compartment, separated by glass. In one condition he placed a mirror along one side of the tank (a "cooperating mirror"), so that as the stickleback approached the predator is had the impression that there was another stickleback mimicking its actions, as if following the strategy *tit-for-tat*. In a second condition he placed the mirror at an angle (a "defecting mirror"), so that a stickleback that approached the cichlid had the impression that there was another stickleback that was increasingly holding back. He found that the stickleback approached the cichlid much more closely with a cooperating mirror than with a defecting mirror. With a defecting mirror, the apparent second stickleback held back when the real one moved forward, and disappeared entirely when the real stickleback moved into the front half of the tank—that is, it tended to defect. Milinski interpreted the behavior of the real stickleback as consistent with its following the strategy *tit-for-tat*. (The same behavior was subsequently observed in guppies approaching a pumpkinseed sunfish (Dugatkin (1988, 1991)).)

Other explanations have been offered for the observed behavior of the fish. For example, one stickleback might simply be attracted to another, since sticklebacks shoal, or a stickleback might be bolder if in the company of another one, since its chances of being captured by the predator are lower (Lazarus and Metcalfe (1990)). Milinski (1990) argues that neither of these alternative theories fits the evidence; in Milinski (1993) he suggests that further evidence indicates that the strategy that his sticklebacks follow may not be *tit-for-tat* but rather *Pavlov* (see Exercise 398.1).

15.3 Finitely repeated games

To be written.

Notes

Early discussions of the notion of a repeated game and the ideas behind the Nash folk theorem (Proposition 413.1) appear in Luce and Raiffa (1957, pp. 97–105 (especially p. 102) and Appendix 8), Shubik (1959b, Ch. 10 (especially p. 226)), and Friedman (1971). Proposition 417.3 (a perfect folk theorem) is due to Fudenberg and Maskin (1986); related results were established earlier (see Aumann and Shapley (1994), Rubinstein (1994), and Rubinstein (1979)).

17 Appendix: Mathematics

17.1 Introduction

THIS CHAPTER presents informal definitions and discussions of the mathematical concepts used in the text. Much of the material should be familiar to you, though a few concepts may be new.

17.2 Numbers

I take the concept of a **number** as basic; 3, -7.4, $\frac{1}{2}$, and $\sqrt{2}$ are all numbers. The whole numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ are called **integers**. Let x be a number. If $x > 0$ then x is **positive**; if $x \geq 0$ then x is **nonnegative**; if $x < 0$ then x is **negative**; and if $x \leq 0$ then x is **nonpositive**. Note that 0 is both nonnegative and nonpositive, but neither positive nor negative.

When working with sums of numbers, a shorthand that uses the symbol $\sum$ (a large uppercase Greek sigma) is handy. Instead of writing $x_1 + x_2 + x_3 + x_4$, for example, where x_1, x_2, x_3, and x_4 are numbers, we can write

$$\sum_{i=1}^{4} x_i.$$

This expression is read as "the sum from $i = 1$ to $i = 4$ of x_i". The name we give the indexing variable is arbitrary; we frequently use i or j, but can alternatively use any other letter. If the number of items in the sum is a variable, say n, the notation is even more useful. Instead of writing $x_k + \cdots + x_n$, which leaves in doubt the variables indicated by the ellipsis, we can write

$$\sum_{i=k}^{n} x_i,$$

which has a precise meaning: first set $i = k$ and take x_i; then increase i by one and add the new x_i; continue increasing i by one at a time and adding x_i to the sum at each step, until $i = n$.

17.3 Sets

A **set** is a collection of objects. If we can count the members of a set, and, when we do so, we eventually exhaust the members of the set, then the set is **finite**. We can specify a finite set by listing the names of its members within braces: {Paris, Venice, Havana} is a set of (beautiful) cities, for example. Neither the order in which the members of the set are listed nor the number of times each one appears has any significance: {Paris, Venice, Havana} is the same set as {Venice, Paris, Havana}, which is the same set as {Paris, Venice, Paris, Havana} (and has three members).

The symbol $\in$ is used to denote set membership: for example, Havana $\in$ {Paris, Venice, Havana}. We read the statement "$a \in A$" as "a is in A".

If every member of the set B is a member of the set A, we say that B is a **subset** of A. For example, the set {Paris} consisting of the single city Paris is a subset of the set {Paris, Venice, Havana}, since Paris is a member of this set. The set {Paris, Havana} is also a subset of {Paris, Venice, Havana}, since both Paris and Havana are members of the set. Further, the set {Paris, Venice, Havana} is a subset of itself: saying that A is a subset of B does *not* rule out the possibility that A and B are equal.

A **partition** of a set A is a collection $\{A_1, \ldots, A_k\}$ of subsets of A such that every member of A is in exactly one of the sets A_j. The set {Paris, Venice, Havana}, for example, has five partitions: {{Paris}, {Venice}, {Havana}}, {{Paris, Venice}, {Havana}}, {{Paris, Havana}, {Venice}}, {{Paris}, {Venice, Havana}, and {{Paris, Venice, Havana}}.

Some sets are not finite. We can divide such sets into two groups. The members of some sets can be counted, but if we count them then we go on counting forever. The set of positive integers is a set of this type. The members of other sets cannot be counted. For example, the set of all numbers between 0 and 1 cannot be counted. (Of course, one can arbitrarily choose one number in this set, then arbitrarily choose another number, and so on. But there is no systematic way of counting all the numbers.) We say that both types of sets have **infinitely many** members.

A set with infinitely many members obviously cannot be described by listing all its members! One way to describe such a set is to state a property that characterizes its members. For example, if a person's set of actions is a set of numbers A then we can describe the subset of her actions that exceed 1 as

$$\{a \in A : a > 1\}.$$

We read this as "the set of a in A such that a exceeds 1". If the set from which the objects come—in this case, the set A—is the set of all numbers, I do not include it explicitly. Thus

$$\{p: 0 \leq p \leq 1\}$$

is the set of all nonnegative numbers that are at most 1.

Sometimes we wish to calculate the sum of the numbers x_i for every i in some set S. If S is a set of consecutive numbers of the form $\{1, \ldots, k\}$ then we can write this sum as

$$\sum_{i=1}^{k} x_i,$$

as described at the end of the previous section. If S is not a set of consecutive numbers then we can use a variant of the previous notation to denote the sum

$$\sum_{i \in S} x_i,$$

which means "the sum of all values of x_i for i in the set S".

For example, if S is the set of cities {Paris, Venice, Havana} and the population of city i is x_i then the total population of the cities in S is

$$\sum_{i \in S} x_i.$$

17.4 Functions

A **function** is a rule defining a relationship between two variables. We usually specify a function by giving the formula that defines it. For example, the function, say f, that associates with every number twice that number is defined by $f(x) = 2x$ for each number x; the function, say g, that associates with every number its square is defined by $g(x) = x^2$.

If the variables that a function relates are both numbers then the function can be represented in a graph, like the one in Figure 446.1. We usually put the independent variable (denoted x in the examples above) on the horizontal axis, and the value of the function, $f(x)$, on the vertical axis. To read the graph, find a value of x on the horizontal axis, go vertically up to the graph, then horizontally to the vertical axis; the number on this axis is the value $f(x)$ of the function at x.

Two classes of functions figure prominently in the examples in this book. A function f defining a relationship between two numbers is **affine** if it takes the form $f(x) = ax + b$, where a and b are constants. For example, the functions $-3x + 1$ and $4x$ are both affine. (Sometimes such functions are called "linear", rather than "affine"; I follow the convention that a linear function is an affine function for which $b = 0$.) The graph of a general affine function $ax + b$ is a straight line with slope a that goes through the points $(0, b)$ and $(-b/a, 0)$ (since $a \cdot 0 + b = b$ and $a \cdot (-b/a) + b = 0$). In particular, if $a > 0$ then the slope is positive and if $a < 0$ then the slope is negative. An example is given in Figure 446.2.

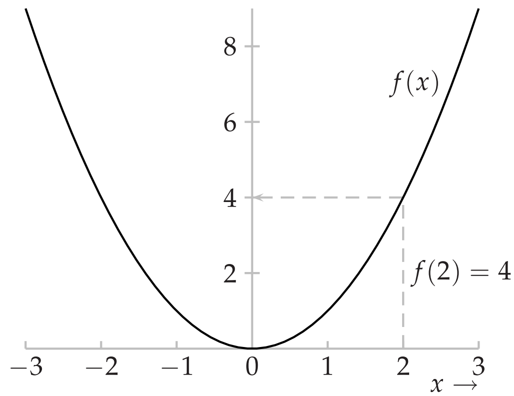

Figure 446.1 The graph of the function f defined by $f(x) = x^2$, for $-3 \leq x \leq 3$.

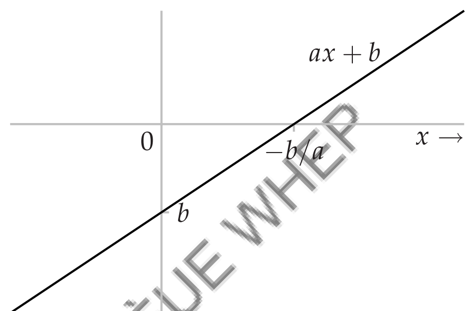

Figure 446.2 The graph of the affine function $ax + b$ (with $a > 0$).

A function f defining a relationship between two numbers is **quadratic** if it takes the form $f(x) = ax^2 + bx + c$, where a, b, and c are constants. If $a > 0$ then the graph of a quadratic function is U-shaped, as in the left-hand panel of Figure 447.1; if $a < 0$ then the shape of the graph is an inverted U, as in the right-hand panel of Figure 447.1.

In both cases the graph is symmetric about a vertical line through the extremum of the function (the minimum when the graph of the function is U-shaped and the maximum when it is an inverted U). Thus if we know the points x_0 and x_1 at which the graph of the function intersects some horizontal line (e.g. the horizontal axis) then we know that its extremum occurs at the midpoint of x_0 and x_1, namely $\frac{1}{2}(x_0 + x_1)$.

We can write the quadratic function $ax^2 + bx + c$ as $x(ax + b) + c$. Doing so allows us to see that the value of the function is c when $x = 0$ and when $x = -b/a$. That is, the function crosses the horizontal line of height c when $x = 0$ and when $x = -b/a$, so that its maximum (if $a < 0$) or minimum (if $a > 0$) occurs at $-\frac{1}{2}b/a$ (the midpoint of 0 and $-b/a$).

? EXERCISE 446.1 (Maximizer of quadratic function) Find the maximizer of the func-

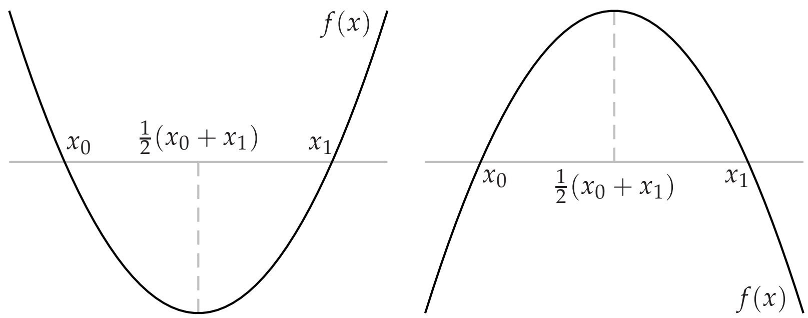

Figure 447.1 The graphs of two quadratic functions. In both cases the function takes the form $ax^2 + bx + c$; in the left panel $a > 0$, while in the right panel $a < 0$.

tion $x(\alpha - x)$, where α is a constant.

The graphs of the functions in Figures 446.1 and 447.1 do not have any jumps in them: for every point x, by choosing x' close enough to x we can ensure that the values $f(x)$ and $f(x')$ of the function at x and x' are as close as we wish. A function that has this property is **continuous**. The graph of a continuous function may be very steep, but does not have any holes in it. For example, the function whose graph is shown in the left panel of Figure 447.2 is continuous, while the function whose graph is shown in the right panel is not continuous. In graphs of discontinuous functions I use the convention that a small disk indicates a point that is included and a small circle indicates a point that is excluded.

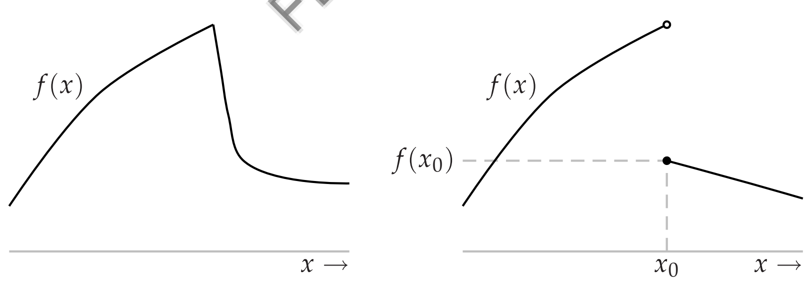

Figure 447.2 The function in the left panel is continuous, while the function in the right panel is not. The small disk indicates a point that is included in the graph, while the small circle indicates a point that is excluded.

For all the functions I have described so far, for each value of x the value $f(x)$ of the function is a single number. In this book we sometimes need to work with functions whose values are sets rather than points. Suppose, for example, that we need a function that assigns to each starting point x in some city the best route from x to city hall. For some values of x there may be a single best route, but for other values of x there are quite possibly several routes that are equally good. At these latter points, the value of our function would be the *set* of all the optimal routes. Since we should like our function to assign the same "type" of object to every value of x, we would take all the values to be sets; if the single route A is optimal from the starting point x then we take the value of the function to be the set $\{A\}$ consisting of the single route A.

We can specify a set-valued function, like a point-valued function, by giving its graph. I indicate values of the function that are sets of points by shading in gray; I indicate boundaries that are included by drawing lines along them. For example, for the function in Figure 448.1, $f(x_1) = \{y : y_2 < y \leq y_3\}$ and $f(x_2) = \{y : y = y_0 \text{ or } y_1 < y \leq y_4\}$.

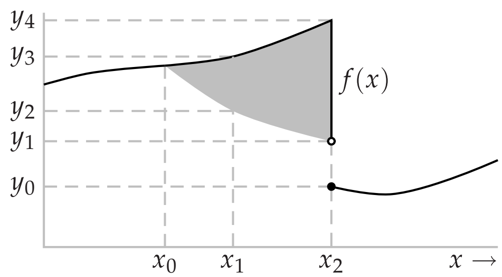

Figure 448.1 The graph of a set-valued function. For $x_0 < x \leq x_2$ the set $f(x)$ consists of more than one point. We have $f(x_1) = \{y : y_2 < y \leq y_3\}$ and $f(x_2) = \{y : y = y_0 \text{ or } y_1 < y \leq y_4\}$.

17.5 Profiles

Frequently in this book we wish to associate an object with each member of a set of players. For example, we often need to refer to the action chosen by each player. We can describe the correspondence between players and actions by specifying the function that associates each player with the action she takes. For example, if the players are Ernesto, whose action is R, and Hilda, whose action is S, then the correspondence between players and actions is described by the function a defined by $a(\text{Ernesto}) = R$ and $a(\text{Hilda}) = S$. We can alternatively present the function a by writing $(a_{\text{Ernesto}}, a_{\text{Hilda}}) = (R, S)$. We call such a function a a **profile**. The order in which we write the elements is irrelevant: we can alternatively write the profile above as $(a_{\text{Hilda}}, a_{\text{Ernesto}}) = (S, R)$.

In most of the book I sacrifice color for convenience and name the players 1, 2, 3, and so on. Doing so allows me to write a profile of actions as a list like (R, S), without saying explicitly which action belongs to which player: the convention is that the first action is that of player 1, the second is that of player 2, and so on. When the number of players is arbitrary, equal to say n, I follow convention and write an action profile as $(a_1, \ldots, a_n)$, where the ellipsis stands for the actions of players 2 through $n - 1$.

I frequently need to refer to the action profile that differs from $(a_1, \ldots, a_n)$ only in that the action of player i is b_i (say) rather than a_i. I denote this variant of $(a_1, \ldots, a_n)$ by (b_i, a_{-i}). The $-i$ subscript on a stands for "except i": every player except i chooses her component of a. If $(a_1, a_2, a_3) = (T, L, M)$ and $b_2 = R$, for example, then $(b_2, a_{-2}) = (T, R, M)$.

17.6 Sequences

A **sequence** is an ordered list. In this book the sequences consist of events that unfold over time; the first element of a sequence is an event that occurs before the second element of the sequence, and so on. A sequence that continues indefinitely is **infinite**; one that ends eventually is **finite**.

In Chapters 14 and 15 the formula for the sum of a sequence of numbers of the form $a, ar, ar^2, ar^3, \ldots$ is useful. For a finite sequence we have

$$a + ar + ar^2 + \cdots + ar^T = \frac{a(1 - r^{T+1})}{1 - r} \tag{449.1}$$

if $r \neq 1$ and $r \neq -1$. (Note that the exponent of r in the numerator of the formula is the number of terms in the sequence.) For an infinite sequence we have

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r} \tag{449.2}$$

if $-1 < r < 1$.

? EXERCISE 449.3 (Sums of sequences) Find the sums $1 + \delta^2 + \delta^4 + \cdots$ and $1 + 2\delta + \delta^2 + 2\delta^3 + \cdots$, where δ is a constant with $0 < \delta < 1$. (Split the second sum into two parts.)

17.7 Probability

17.7.1 Basic concepts

We may sometimes conveniently model events as "random". Rather than modeling the causes of such an event, we assume that if the event occurs many times then sometimes it takes one value, sometimes another value, in no regular pattern. We refer to the proportion of times it takes any given value as the **probability** of its taking that value.

A simple example is the outcome of a coin toss. We could model this outcome as depending on the initial position of the coin, the speed and direction in which it is tossed, the nature of the air currents, and so on. But it is simpler, and for many purposes satisfactory, to model the outcome as being a head with probability $\frac{1}{2}$ and a tail with probability $\frac{1}{2}$. Given the sensitivity of the outcome to tiny changes in the initial position of the coin and the speed and direction in which it is tossed, and the inability of a person to precisely control these factors, the probabilistic theory is likely to work very well over many tosses: if the coin is tossed a large number n of times, then the number of heads is likely to be close to $n/2$.

We refer to an assignment of probabilities to events as a **probability distribution**. If, for example, there are three possible events, A, B, and C, then one probability distribution assigns probability $\frac{1}{3}$ to A, probability $\frac{1}{2}$ to B, and probability $\frac{1}{6}$ to C. In any probability distribution the sum of the probabilities of all possible events is 1 (on any given occasion, *one* of the events must occur), and each probability is nonnegative and at most 1. Saying that an event occurs with positive probability is equivalent to saying that there is some chance that it may occur; saying that an event occurs with probability zero is equivalent to saying that it will never occur. Similarly, saying that an event occurs with probability less than one is equivalent to saying that there is some chance that it may not occur; saying that an event occurs with probability one is equivalent to saying that it is certain to occur. We sometimes denote the probability of an event E by $\Pr(E)$.

If the events E and F cannot both occur, then the probability that *either E or F* occurs is the sum $\Pr(E) + \Pr(F)$. For example, suppose we model the outcome of the toss of a die as random, with the probability of each side equal to $\frac{1}{6}$. Then the probability that the side is either 3 or 4 is $\Pr(3) + \Pr(4) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

17.7.2 Independence

Two events E and F are **independent** if the probability $\Pr(E \text{ and } F)$ that they both occur is the product $\Pr(E)\Pr(F)$ of the probabilities that each occurs. Events may sensibly be modeled as independent if the occurrence of one has no bearing on the occurrence of the other. For example, the outcome of an election may sensibly be modeled as independent of the outcome of a coin toss, but not independent of the weather on the polling day (which may affect the candidates' supporters differently). In a strategic game, we model the players' choices of actions as independent: the probability that player 1 chooses action a_1 and player 2 chooses action a_2 is assumed to be the product of the probability that player 1 chooses a_1 and the probability that player 2 chooses a_2.

17.7.3 Lotteries and expected values

The material in this section is used only in Chapter 4 (Mixed strategy equilibrium), Section 7.6 (Extensive games with perfect information, simultaneous moves, and chance moves), Chapter 9 (Bayesian Games), Chapter 10 (Extensive games with imperfect information), Chapter 11 (Strictly competitive games and maxminimization), and Chapter 12 (Rationalizability).

Consider a decision-maker who faces a situation in which there are probabilistic elements. Each action that she chooses induces a probability distribution over outcomes. If you make an offer for an item in a classified advertisement, for example, then given the behavior of other potential buyers, your offer may be accepted with probability $\frac{1}{3}$ and rejected with probability $\frac{2}{3}$. We refer to a probability distribution over outcomes as a **lottery** over outcomes.

If the outcomes of a lottery are numerical (for example, amounts of money), we may be interested in their average value—the value we should expect to get if we found the total of the values on a large number n of trials and divided by n. For the lottery that yields the amount x_i with probability p_i, for $i = 1, \ldots, n$, this average value is

$$p_1 x_1 + \cdots + p_n x_n$$

or, more compactly, $\sum_{i=1}^{n} p_n x_n$. It is called the **expected value** of the lottery. A lottery that yields \$12 with probability $\frac{1}{3}$, \$4 with probability $\frac{1}{2}$, and \$6 with probability $\frac{1}{6}$, for example, has an expected value of $\frac{1}{3} \cdot 12 + \frac{1}{2} \cdot 4 + \frac{1}{6} \cdot 6 = 7$. On no single occasion does the lottery yield \$7, but over a large number of occasions the average amount that it yields is likely to be close to \$7 (the more likely, the larger the number of occasions).

17.7.4 Cumulative probability distributions

The material in this section is used only in Section 4.11 (Mixed strategy equilibrium in games in which each player has a continuum of actions) and Chapter 9 (Bayesian Games).

If the events in our model are associated with numbers, we can describe the probabilities assigned to them by giving the **cumulative probability distribution**, which assigns to each number x the total of the probabilities of all numbers at most equal to x. The cumulative probability distribution of the number of dots on the exposed side of a die, for example, is the function F for which $F(1) = \frac{1}{6}$, $F(2) = \frac{1}{3}$, $F(3) = \frac{1}{2}$, and so on. Given a cumulative probability distribution we can recover the probabilities of the events by calculating the differences between values of F: the probability of x is $F(x) - F(x')$, where x' is the next smaller event.

When the number of events is finite, we can represent the assignment of probabilities to events either by a probability distribution or by a cumulative probability distribution. When the number of events is infinite, we can usefully represent the probabilities only by a cumulative probability distribution, because the probability of any single event is typically zero. If the set of events is the set of numbers from $\underline{a}$ to $\overline{a}$ then a cumulative probability distribution is a nondecreasing function, say F, for which $F(x) = 0$ if $x < \underline{a}$ (the probability of a number less than $\underline{a}$ is 0) and $F(\overline{a}) = 1$ (the probability of a number at most equal to $\overline{a}$ is 1). The number $F(x)$ is the probability of an event at most equal to x.

For example, if $\underline{a} = 0$ and $\overline{a} = 1$ then the function $F(x) = x$ is a cumulative probability distribution. This distribution represents uniform randomization over the interval (sets of the same size have the same probability). Another cumulative probability distribution is given by the function $F(x) = x^2$. In this distribution the probabilities of sets of numbers close to 0 are lower than the probabilities of sets of numbers close to 1.

17.7.5 Conditional probability and Bayes' rule

The material in this section is used only in Section 9.8 (Juries) and Chapter 10 (Extensive Games with Imperfect Information).

We sometimes use the notion of probability to refer to the character of a person's belief, in a situation in which there is no possibility of an event's being repeated. For example, a jury in a civil case is asked to determine whether the probability of a person's being guilty is greater than or less than one half; you may form a belief about the probability of your carrying a particular gene or of your getting into graduate school. In some cases these beliefs may be tightly linked to numerical evidence. If, for example, the only information you have about the prevalence of a particular gene is that it is carried by 10% of the population, then it is reasonable for you to believe that your probability of carrying the gene is 0.1. In other cases beliefs may be at most loosely linked to numerical evidence. The evidence presented to a jury, for example, is likely to be qualitative, and open to alternative interpretations.

Whatever the basis for probabilistic beliefs, however, the theory of probability gives a specific rule for how they should be modified in the light of new probabilistic evidence. In this context in which a belief is changed by evidence, the initial belief is called the **prior belief** and the belief modified by the evidence is called the **posterior belief**.

Suppose that 10% of the population carries the gene X, so that in the absence of any other information your prior belief is that you carry the gene with probability 0.1. An imperfect test for the presence of X is available. The test is positive in 90% of subjects who carry X and in 20% of subjects who do not carry X. The test on you is positive. What should be your posterior belief about your carrying X? The probabilities are illustrated in Figure 452.1.

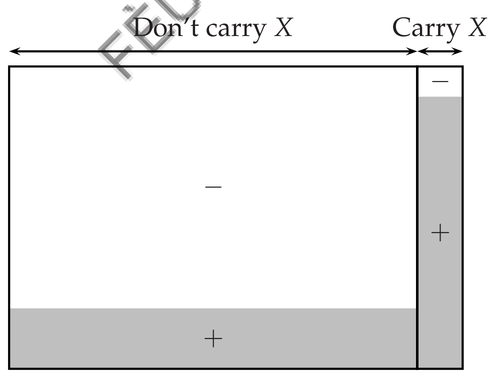

Figure 452.1 The outer box represents the set of people. People to the right of the vertical line carry gene X, while people to the left of this line do not. People in the shaded areas test positive for the gene.

Consider a random group of 100 people from the population. Of these, on average 10 carry X and 90 do not. If all these 100 people were tested, then, on average, 9 of the 10 (90%) who carry X and 18 of the 90 (20%) who do not carry X would test positive. These sets are represented by the shaded areas in Figure 452.1. Of all the people who test positive, what fraction of them carry the gene? That is, what fraction of the total shaded area in Figure 452.1 is the shaded area to the right of the vertical line? Of the 100 people, a total of $9 + 18 = 27$ test positive, and one-third of these (9/27) carry the gene. Thus after testing positive, your posterior belief that you carry the gene is $\frac{1}{3}$: the positive test raises the probability you assign to your carrying X from $\frac{1}{10}$ to $\frac{1}{3}$.

To generalize the analysis in this example, we introduce the concept of conditional probability. Let E and F be two events that may be related; assume that $\Pr(F) > 0$. Suppose that F is true. Define the **probability** $\Pr(E \mid F)$ **of** E **conditional on** F by

$$\Pr(E \mid F) = \frac{\Pr(E \text{ and } F)}{\Pr(F)}. \tag{453.1}$$

This number makes sense as the probability that E is true *given that F is true*. One way to see that it makes sense is to consider Figure 452.1 again. Let E be the event that you carry X and let F be the event that you test positive. If you test positive then we know you lie in the shaded area. Given you lie in this area, what is the probability $\Pr(E \mid F)$ that you lie to the right of the vertical line? This probability is the ratio of the shaded area to the right of the vertical line—the probability $\Pr(E \text{ and } F)$ that you carry the gene and test positive—to the total shaded area—the probability $\Pr(F)$ that you test positive.

If the events E and F are independent then

$$\Pr(E \mid F) = \Pr(E) \text{ and } \Pr(F) > 0$$

or, alternatively,

$$\Pr(F \mid E) = \Pr(F) \text{ and } \Pr(E) > 0.$$

These conditions express directly the idea that the occurrence of one event has no bearing on the occurrence of the other event.

In using the expression for conditional probability to find the posterior belief in this case, we needed to calculate $\Pr(E \text{ and } F)$ and $\Pr(F)$, which were not given directly as data in the problem. The data we were given were the prior belief $\Pr(E)$, the probability $\Pr(F \mid E)$ of a person who carries the gene testing positive, and the probability $\Pr(F \mid \text{not } E)$ of a person who does not carry the gene testing positive.

Bayes' rule expresses the conditional probability $\Pr(E \mid F)$ directly in terms of $\Pr(E)$, $\Pr(F \mid E)$, and $\Pr(F \mid \text{ not } E)$:

$$\Pr(E \mid F) = \frac{\Pr(E)\Pr(F \mid E)}{\Pr(E)\Pr(F \mid E) + \Pr(\text{not } E)\Pr(F \mid \text{not } E)}. \tag{453.2}$$

(The probability $\Pr(\text{not } E)$ is of course equal to $1 - \Pr(E)$; recall that I have assumed that $\Pr(F) > 0$.) This formula follows from the definition of conditional probability (453.1) and the properties of probabilities. First, interchanging E and F in (453.1) we deduce $\Pr(E)\Pr(F \mid E) = \Pr(E \text{ and } F)$. Thus the numerator of (453.2) is equal to $\Pr(E \text{ and } F)$. Second, again using (453.1) we see that the denominator of (453.2) is equal to $\Pr(E \text{ and } F) + \Pr((\text{not } E) \text{ and } F)$. Now, either the event E or the event not E occurs, but not both. Thus $\Pr(E \text{ and } F) + \Pr((\text{not } E) \text{ and } F) = \Pr(F)$. (The probability that either "it rains and you carry an umbrella" or "it rains and you do not carry an umbrella" is equal to the probability that "it rains"!)

? EXERCISE 454.1 (Bayes' rule) Consider a generalization of the example of testing positive for a gene in which the fraction p of the population carry the gene. Verify that as p decreases, the posterior probability that you carry X given that you test positive decreases. What value does this posterior probability take when p is 0.001? What value does the posterior probability take when p is 0.001 and the test is positive for 99% of those who carry X and is negative for 99% of those who do not carry X? (Are you surprised?)

In the cases I have described so far, the event about which we form a belief takes two possible values (E, or not E). In a more general setting, this event may take many values. For example, we may form a belief about the quality of an item—a variable that may take many values—on its price. In general, let F be an event and let $E_1, \ldots, E_n$ be a collection of events, exactly one of which must occur. (In the example above, F is the event that you test positive, $n = 2$, E_1 is the event you carry the gene, and E_2 is the event you do not carry the gene.) Then the probability of E_k conditional on F is

$$\Pr(E_k \mid F) = \frac{\Pr(F \mid E_k)\Pr(E_k)}{\sum_{j=1}^{n}\Pr(F \mid E_j)\Pr(E_j)}. \tag{454.2}$$

This general formula is known as **Bayes' rule**, after Thomas Bayes (1702–61). In the context in which we use this rule in a Bayesian game to find the probability of a state given the observed signal, the events $E_1, \ldots, E_n$ are the states and the event F is a signal. Thus every probability $\Pr(F \mid E_k)$ is either one or zero, depending on whether the state E_k generates the signal F or not.

17.8 Proofs

This book focuses on concepts, but contains precise arguments, and, in some cases, proofs of results. The results are given three names: Lemma, Proposition, and Corollary. These names have no formal significance—they do not have any implications for the type of logic used—but are intended to convey the role of the result in the analysis. Lemmas are results whose importance lies mainly in their being steps on the way to proving further results. Propositions are the main results. Corollaries are more or less direct implications of the main results.

A result consists of a series of statements of the form "if A is true then B is true". Frequently the series contains only one such statement, which may not be explicitly rendered as "if A then B". For example, "all prime numbers are odd" is a result; it can be transformed into the "if ... then" form: "if a number is prime then it is odd". A result that makes the two claims "if A is true then B is true" and "if B is true then A is true" is sometimes stated compactly as "A is true if and only if B is true".

A proof of the result "if A then B" is a series of arguments that lead from A to B, each of which follows from a known fact (including an earlier member of the series). Except for the proofs of very simple results, most proofs are not, and should not sensibly be, "complete". To spell out how each step follows from the basic principles of mathematics would make a proof extremely long and very difficult to read. Some facts must be taken for granted; judging which to put in and which to leave out is an art. A good proof convinces readers that the result is true and gives them some understanding of *why* it is true (the features of A that are significant, and those that are not significant).

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