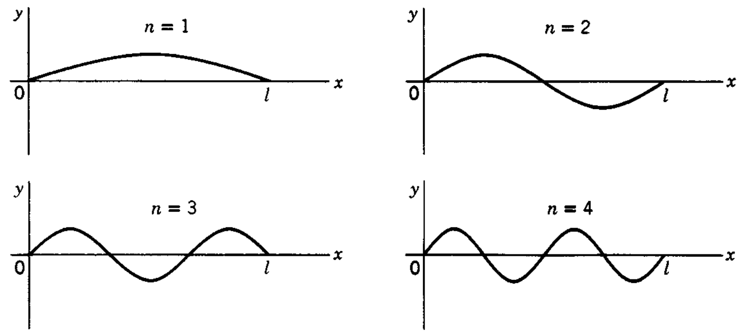


Figure 4.2

a point x on the string; for this point $\sin(n\pi x/l)$ is some number, say A. Then the displacement of this point at time t is [from (4.12)]

$$y = A\sin\frac{n\pi vt}{l}. \tag{4.14}$$

As time passes, this point of the string oscillates up and down with frequency ν_n given by $\omega_n = n\pi v/l = 2\pi\nu_n$ or $\nu_n = nv/(2l)$; the amplitude of the oscillation at this point is $A = \sin(n\pi x/l)$ (see Figure 4.2). Other points of the string oscillate with different amplitudes but the *same* frequency. This is the frequency of the musical note which the string is producing. (See Chapter 7, Section 10.) If $n = 1$ (see Figure 4.2), the frequency is $v/(2l)$; in music this tone is called the fundamental or first harmonic. If $n = 2$, the frequency is just twice that of the fundamental; this tone is called the first overtone or the second harmonic; etc. All the frequencies which this string can produce are multiples of the fundamental. These frequencies are called the *characteristic frequencies* of the string. (They are proportional to the *characteristic values* or *eigenvalues,* $k = n\pi/l$.) The corresponding ways in which the string may vibrate producing a pure tone of just one frequency [that is, with y given by (4.12) for one value of n] are called the *normal modes of vibration.* The first four normal modes are indicated in Figure 4.2. Any vibration is a combination of these normal modes [for example, (4.9) or (4.10)]. The solution (4.12) (for *one* n) describing one normal mode, is a *characteristic function* or *eigenfunction.*

The waves in Figure 4.2 are called standing waves. The d'Alembert solution of the wave equation (see Problem 1.2) represents traveling waves. Suppose we combine two traveling waves moving in opposite directions as follows:

$$\cos k(x - vt) - \cos k(x + vt) = 2\sin kx \sin kvt \tag{4.15}$$

(by a trigonometry formula). This is one of the solutions (4.5) so we see that this combination of two traveling waves produces a standing wave. Suppose these two traveling waves are moving along a string which is fastened at $x = 0$ and at $x = l$. First consider the wave $\cos k(x + vt)$ which is moving in the negative x direction toward $x = 0$. When it reaches $x = 0$, it will be reflected, and the combination of the incident and reflected waves must equal zero at $x = 0$ for all t. We see that this is true in (4.15), so the wave $\cos k(x - vt)$ is the reflection of $-\cos k(x + vt)$. Now consider $\cos k(x - vt)$ traveling toward $x = l$. When it reaches $x = l$ and is reflected, we can verify (Problem 10) that, if $k = n\pi/l$, then the reflection at $x = l$ is $-\cos\frac{n\pi}{l}(x + vt)$. We can think of a wave traveling back and forth between $x = 0$

and $x = l$, being reflected at each end. The net result as we see from (4.15) is a standing wave.

So far we have been considering problems in which a string is pinned at both ends. We could, instead, have a "free" end; this means free to move up and down along $x = 0$ or $x = l$, say by allowing the end to slide along a frictionless track. The mathematical condition for this is $\partial y/\partial x = 0$ at the free end (compare the condition for an insulated face in Section 3). If the $x = 0$ end is free, we choose the solution containing $\cos kx$ (since $\frac{\partial}{\partial x}\cos kx = -k\sin kx = 0$ at $x = 0$). Then, if the string is pinned at $x = l$, we want $\cos kl = 0$, so $kl = (n + \frac{1}{2})\pi$. Thus the basis functions when the $x = 0$ end is free, the $x = l$ end is pinned, and the initial string velocity is zero, are

$$y = \cos\frac{\left(n+\frac{1}{2}\right)\pi x}{l}\cos\frac{\left(n+\frac{1}{2}\right)\pi vt}{l}. \tag{4.16}$$

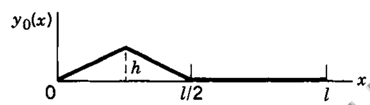
For a discussion of these functions, see Chapter 7, Section 11 and Problem 11.11.

PROBLEMS, SECTION 4

As in Sections 2 and 3, use a computer to plot your answers.

1. Complete the plucked string problem to get equation (4.9).

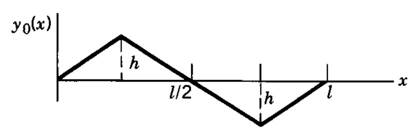
2. A string of length l has a zero initial velocity and a displacement $y_0(x)$ as shown. (This initial displacement might be caused by stopping the string at the center and plucking half of it.) Find the displacement as a function of x and t.



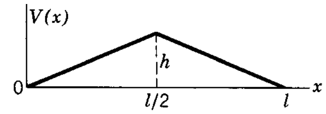
Answer: $y = \dfrac{8h}{\pi^2}\displaystyle\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}$, where $B_n = (2\sin n\pi/4 - \sin n\pi/2)/n^2$.

3. Solve Problem 2 if the initial displacement is:

4. Solve Problem 2 if the initial displacement is:

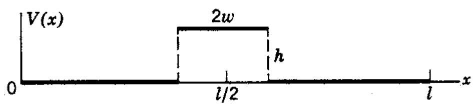


5. A string of length l is initially stretched straight; its ends are fixed for all t. At time $t = 0$, its points are given the *velocity* $V(x) = (\partial y/\partial t)_{t=0}$ as indicated in the diagram (for example, by hitting the string). Determine the shape of the string at time t, that is, find the displacement y as a function of x and t in the form of a series similar to (4.9). *Warning:* What basis functions do you need here?



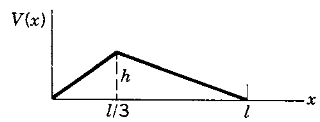
Answer: $y = \dfrac{8hl}{\pi^3 v}\left(\sin\frac{\pi x}{l}\sin\frac{\pi vt}{l} - \frac{1}{3^3}\sin\frac{3\pi x}{l}\sin\frac{3\pi vt}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l}\sin\frac{5\pi vt}{l} - \cdots\right)$.

6. Do Problem 5 if the initial velocity $V(x) = (\partial y/\partial t)_{t=0}$ is as shown.



Answer: $y = \dfrac{4hl}{\pi^2 v}\left(\sin\dfrac{\pi w}{l}\sin\dfrac{\pi x}{l}\sin\dfrac{\pi vt}{l} - \dfrac{1}{9}\sin\dfrac{3\pi w}{l}\sin\dfrac{3\pi x}{l}\sin\dfrac{3\pi vt}{l} + \cdots\right).$

7. Solve Problem 5 if the initial velocity is:



8. Solve Problem 5 if the initial velocity is

$$V(x) = \begin{cases} \sin 2\pi x/l, & 0 < x < l/2, \\ 0, & l/2 < x < l. \end{cases}$$

9. In each of the Problems 1 to 8, find the frequency of the most important harmonic.

10. Verify that, if $k = \frac{n\pi}{l}$, then the sum of the two traveling waves in equation (4.15) is zero at $x = l$, for all t.

11. Verify (4.16) and find a similar formula for a string pinned at $x = 0$ and free at $x = l$. Solve Problems 2, 3, and 4, for a string with a free end (a) at $x = 0$; (b) at $x = l$.

12. In Sections 2, 3, 4, we have solved a number of physics problems which led to the expansion of a given $f(x)$ in a Fourier sine series. Look at (2.9) and (2.25), temperature in a plate; (3.12), heat flow; (3.26), wave function for a particle in a box; (4.7) and (4.10), displacement of a vibrating string plucked or struck. If we have expanded a given $f(x)$ in a Fourier sine series on $(0, l)$, we can immediately write the corresponding solutions for these six different physics problems on the same interval. Do this for $f(x) = x - x^2$ on $(0, 1)$, that is with $l = 1$. Make computer plots of your results.

13. Do Problem 12 for $f(x) = 1 - \cos 2x$ on $(0, \pi)$.

14. Do Problem 12 for $f(x) = x - x^3$ on $(0, 1)$.

▶ 5. STEADY-STATE TEMPERATURE IN A CYLINDER

Consider the following problem. Find the steady-state temperature distribution u in a semi-infinite solid cylinder (Figure 5.1) of radius a if the base is held at 100° and the curved sides at 0°. This sounds very much like the problem of the temperature distribution in a semi-infinite plate. However, it is not convenient here to use the solutions in rectangular coordinates, because the boundary condition $u = 0$ is given for $r = a$ rather than for constant values of x or y. The natural variables for this problem are the cylindrical coordinates r, θ, z. The temperature u inside the cylinder satisfies Laplace's equation since there are no sources of heat there.

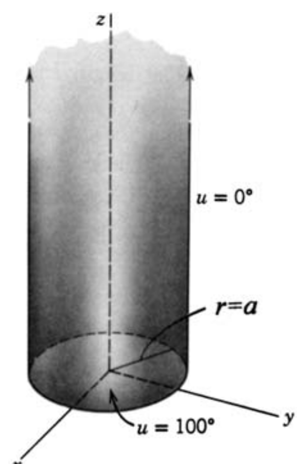


Figure 5.1

Laplace's equation in cylindrical coordinates is (see Chapter 10, Section 9)

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{5.1}$$

To separate the variables, we assume a solution of the form

$$u = R(r)\Theta(\theta)Z(z). \tag{5.2}$$

Substitute (5.2) into (5.1) and divide by $R\Theta Z$ to get

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0. \tag{5.3}$$

The last term is a function only of z, while the other two terms do not contain z. Therefore the last term is a constant and the *sum* of the first two terms is minus the same constant. Notice that neither of the first two terms is constant alone since both contain r.

> In order to say that a term is constant, we must be sure that:
> (a) it is a function of only one variable, and
> (b) that variable does not appear elsewhere in the equation.

Thus we have

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = K^2, \qquad Z = \begin{cases} e^{Kz}, \\ e^{-Kz}. \end{cases} \tag{5.4}$$

Since we want the temperature u to tend to zero as z tends to infinity, we call the separation constant $+K^2$ ($K > 0$) and then use only the e^{-Kz} solution. Next write (5.3) with the last term replaced by K^2—see (5.4).

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} + K^2 = 0.$$

We can separate the variables by multiplying by r^2.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + K^2 r^2 = 0. \tag{5.5}$$

In (5.5) the second term is a function of θ only, and the other terms are independent of θ. Thus we have

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -n^2, \qquad \Theta = \begin{cases} \sin n\theta, \\ \cos n\theta. \end{cases} \tag{5.6}$$

Here we must use $-n^2$ as the separation constant and then require n to be an integer for the following reason. When we locate a point using polar coordinates, we can choose the angle as θ or as $\theta + 2m\pi$ where m is any integer. But regardless of the value of m, there is *one* physical point and *one* temperature there. The mathematical formula for the temperature at the point must give the same value at

θ as at $\theta+2m\pi$, that is, the temperature must be a periodic function of θ with period 2π. This is true only if the Θ solutions are sines and cosines instead of exponentials (hence the negative separation constant) and the constant n is an integer (to give period 2π). The solutions of (5.6) when $n = 0$ are θ and constant. Since θ is not periodic, we can use only the constant solution which is already contained in the $\cos n\theta$ solution when $n = 0$.

Finally, the r equation is

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 + K^2r^2 = 0$$

or

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(K^2r^2 - n^2\right)R = 0. \tag{5.7}$$

This is a Bessel equation with solutions $J_n(Kr)$ and $N_n(Kr)$ [see Chapter 12, equation (16.5)]. Since the base of the cylinder contains the origin, we can use only the J_n and not the N_n solutions since N_n becomes infinite at the origin. Hence we have

$$R(r) = J_n(Kr). \tag{5.8}$$

We can find the possible values of K from the condition that the temperature is zero on the curved surface of the cylinder. Thus $u = 0$ when $r = a$ (for all θ and z) or $R(r) = 0$ when $r = a$. So from (5.8) we see that $J_n(Ka) = 0$, that is, the possible values of Ka are the zeros of J_n. If we define $k = Ka$, or $K = k/a$, then

$$R(r) = J_n(kr/a) \qquad \text{and} \qquad Z(z) = e^{-kz/a}. \tag{5.9}$$

Thus the solutions for u are

$$u = \begin{cases} J_n(kr/a)\,\sin n\theta\; e^{-kz/a}, \\ J_n(kr/a)\,\cos n\theta\; e^{-kz/a}, \end{cases} \tag{5.10}$$

where k is a zero of J_n.

For our problem, the base of the cylinder is held at a constant temperature of $100°$. If we turn the cylinder through any angle the boundary conditions are not changed; thus the solution does not depend on the angle θ. This means that we use $\cos n\theta$ with $n = 0$ in (5.10). The possible values of k are the zeros of J_0; call these zeros k_m, where $m = 1, 2, 3, \cdots$. Thus we have the basis functions for the problem and write the solution in terms of them:

$$u = \sum_{m=1}^{\infty} c_m J_0(k_m r/a) e^{-k_m z/a}. \tag{5.11}$$

When $z = 0$, we want $u = 100$, that is,

$$u_{z=0} = \sum_{m=1}^{\infty} c_m J_0(k_m r/a) = 100. \tag{5.12}$$

This should remind you of a Fourier series; here we want to expand 100 in a series of Bessel functions instead of a series of sines or cosines. We proved [see Chapter 12, equation (19.11)] that the functions $J_0(k_m r/a)$ are orthogonal on $(0, a)$ with respect to the weight function r. We can then find the coefficients c_m in (5.12) by the same method used in finding the coefficients in a Fourier sine or cosine series. (In fact, series like (5.12) are often called Fourier-Bessel series.) Multiply (5.12) by $rJ_0(k_\mu r/a), \mu = 1, 2, 3, \cdots$, and integrate term by term from $r = 0$ to $r = a$. Because of the orthogonality [see Chapter 12, equation (19.11)], all terms of the series drop out except the term with $m = \mu$, and we have

$$c_\mu \int_0^a r\,[J_0(k_\mu r/a)]^2\,dr = \int_0^a 100 r J_0(k_\mu r/a)\,dr. \tag{5.13}$$

For each value of $\mu = 1, 2, 3, \cdots$, equation (5.13) gives one of the coefficients in (5.11) and (5.12); thus any c_m in (5.11) is given by (5.13) with μ replaced by m.

We need to evaluate the integrals in (5.13). Equation (19.11) of Chapter 12 gives (for $p = 0, \alpha = \beta = k_m$)

$$\int_0^a r\,[J_0(k_m r/a)]^2\,dr = \frac{a^2}{2} J_1^2(k_m). \tag{5.14}$$

By equation (15.1) of Chapter 12

$$\frac{d}{dx}\,[xJ_1(x)] = xJ_0(x).$$

If we put $x = k_m r/a$ in this formula, we get

$$\frac{a}{k_m}\frac{d}{dr}\,[(k_m r/a)J_1(k_m r/a)] = (k_m r/a)J_0(k_m r/a).$$

Cancelling one k_m/a factor and integrating from 0 to a, we have

$$\int_0^a rJ_0(k_m r/a)\,dr = \frac{a}{k_m} rJ_1(k_m r/a)\Big|_0^a = \frac{a^2}{k_m} J_1(k_m). \tag{5.15}$$

Now we write (5.13) for c_m, substitute the values of the integrals from (5.14) and (5.15), and solve for c_m. The result is

$$c_m = \frac{100a^2 J_1(k_m)}{k_m} \cdot \frac{2}{a^2 J_1^2(k_m)} = \frac{200}{k_m J_1(k_m)}. \tag{5.16}$$

The solution of our problem is now (5.11) with the values of c_m given by (5.16). The numerical value of the temperature at any point can be found by computing a few terms of the series (Problem 1). The values of the zeros and of the Bessel functions can be found either from your computer or from tables. *Warning:* Remember that k_m is a zero of J_0, not of J_1.

Suppose the given temperature of the base of the cylinder is more complicated than just a constant value, say $f(r, \theta)$, some function of r and θ. Down to (5.10) we proceed as before. But now the series solution is more complicated than (5.11) since we must include all J_n's instead of just J_0. We need a double subscript on the numbers k which are the zeros of the Bessel functions; by k_{mn} we shall mean the

mth positive zero of J_n, where $n = 0, 1, 2, \cdots$ and $m = 1, 2, 3, \cdots$. The temperature u is a double infinite series, summed over the indices m, n of all zeros of all the J_n's:

$$u = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} J_n(k_{mn}r/a)(A_{mn}\cos n\theta + B_{mn}\sin n\theta)e^{-k_{mn}z/a}. \tag{5.17}$$

At $z = 0$, we want $u = f(r, \theta)$. Thus we write

$$u_{z=0} = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} J_n(k_{mn}r/a)(A_{mn}\cos n\theta + B_{mn}\sin n\theta) = f(r, \theta). \tag{5.18}$$

To determine the coefficients A_{mn}, multiply this equation by $J_\nu(k_{\mu\nu}r/a)\cos\nu\theta$ and integrate over the whole base of the cylinder, (0 to 2π for θ, 0 to a for r). Because of the orthogonality of the functions $\sin n\theta$ and $\cos n\theta$ on $(0, 2\pi)$, all the B_{mn} terms drop out, and only the A_{mn} terms for $n = \nu$ remain. Because of the orthogonality of the functions $J_n(k_{mn}r/a)$ (one n, all m), only the one term $A_{\mu\nu}$ remains. Thus we have

$$\begin{aligned}\int_0^a\int_0^{2\pi} & f(r,\theta)J_\nu(k_{\mu\nu}r/a)\cos\nu\theta\, r\,dr\,d\theta \\ &= A_{\mu\nu}\int_0^a\int_0^{2\pi} J_\nu^2(k_{\mu\nu}r/a)\cos^2\nu\theta\, r\,dr\,d\theta = A_{\mu\nu}\cdot\frac{a^2}{2}J_{\nu+1}^2(k_{\mu\nu})\cdot\pi.\end{aligned} \tag{5.19}$$

[The r integral is given by (19.11) of Chapter 12, and the θ integral by Chapter 7, Section 4]. Notice how the weight function r in the Bessel function integral arises here as part of the polar coordinate area element. Similarly, we can find

$$B_{\mu\nu} = \frac{2}{\pi a^2 J_{\nu+1}^2(k_{\mu\nu})}\int_0^a\int_0^{2\pi} f(r,\theta)J_\nu(k_{\mu\nu}r/a)\sin\nu\theta\, r\,dr\,d\theta. \tag{5.20}$$

By substituting the values of the A and B coefficients from (5.19) and (5.20) into (5.17), we find the solution to the problem.

▶ PROBLEMS, SECTION 5

1. (a) Compute numerically the coefficients (5.16) of the first three terms of the series (5.11) for the steady-state temperature in a solid semi-infinite cylinder when $u = 0$ at $r = 1$, and $u = 100$ at $z = 0$. Find u at $r = \frac{1}{2}, z = 1$.

(b) In part (a), if $u = 0$ at $r = 10$ and $u = 100$ at $z = 0$, find u at $r = 5, z = 10$. What is the relation between parts (a) and (b)? *Hint:* Suppose in part (a) that the length units for r and z are centimeters. Consider the identical physics problem but with distances measured in millimeters, and compare part (b). Note that in equation (5.10), r/a and z/a are just measurements as multiples of the radius a.

2. (a) Find the steady-state temperature distribution in a solid semi-infinite cylinder if the boundary temperatures are $u = 0$ at $r = 1$ and $u = y = r\sin\theta$ at $z = 0$. *Hints:* In (5.10) you want the solution containing $\sin\theta$; therefore you want the functions J_1. You will need to integrate $r^2 J_1$; follow the text method of integrating rJ_0 just before (5.15).

(b) Do part (a) if the cylinder radius is $r = a$.

$$\textit{Answer:} \quad u = \sum_{m=1}^{\infty} \frac{2a}{k_m J_2(k_m)} J_1(k_m r/a) e^{-k_m z/a} \sin\theta, \qquad k_m = \text{zeros of } J_1.$$

If $a = 2$, find u when $r = 1, z = 1, \theta = \pi/2$.

3. (a) Find the steady-state temperature distribution in a solid cylinder of height 10 and radius 1 if the top and curved surface are held at 0° and the base at 100°. *Hint*: See Section 2.

(b) Generalize part (a) to a cylinder of height H and radius a.

4. A flat circular plate of radius a is initially at temperature 100°. From time $t = 0$ on, the circumference of the plate is held at 0°. Find the time-dependent temperature distribution $u(r, \theta, t)$. *Hint:* Separate variables in equation (3.1) in polar coordinates.

5. Do Problem 4 if the initial temperature distribution is $u(r, \theta, t = 0) = 100r \sin\theta$.

6. Consider Problem 4 if the initial temperature distribution is given as some function $f(r, \theta)$. The solution is, in general, a double infinite series similar to (5.17). Find formulas for the coefficients in the series.

7. Find the steady-state temperature distribution in a solid cylinder of height 20 and radius 3 if the flat ends are held at 0° and the curved surface at 100°. *Hints:* Use $-K^2$ in (5.4). Also see Chapter 12, Sections 17 and 20.

8. Water at 100° is flowing through a long pipe of radius 1 rapidly enough so that we may assume that the temperature is 100° at all points. At $t = 0$, the water is turned off and the surface of the pipe is maintained at 40° from then on (neglect the wall thickness of the pipe). Find the temperature distribution in the water as a function of r and t. Note that you need only consider a cross section of the pipe.

$$\textit{Answer:} \quad u = 40 + \sum_{m=1}^{\infty} \frac{120}{k_m J_1(k_m)} J_0(k_m r) e^{-(\alpha k_m)^2 t}, \qquad \text{where } J_0(k_m) = 0.$$

9. Find the steady-state distribution of temperature in a cube of side 10 if the temperature is 100° on the face $z = 0$ and 0° on the other five faces. *Hint:* Separate Laplace's equation in three dimensions in rectangular coordinates, and follow the methods of Section 2. You will want to expand 100 in the double Fourier series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{l} \sin \frac{m\pi y}{l}.$$

The coefficients a_{nm} are determined by using the orthogonality of the functions $\sin(n\pi x/l) \sin(m\pi y/l)$ over the square, that is,

$$\int_0^l \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi y}{l} \sin \frac{p\pi x}{l} \sin \frac{q\pi y}{l}\, dx\, dy = 0 \qquad \text{unless} \quad \begin{cases} n = p, \\ m = q. \end{cases}$$

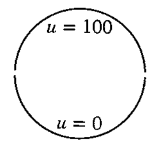
10. A cube is originally at 100°. From $t = 0$ on, the faces are held at 0°. Find the time-dependent temperature distribution. *Hint:* This problem leads to a triple Fourier series; see the double Fourier series in Problem 9 and generalize it to three dimensions.

11. The following two $R(r)$ equations arise in various separation of variables problems in polar, cylindrical, or spherical coordinates:

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2R,$$
$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R.$$

There are various ways of solving them: They are a standard kind of equation (often called Euler or Cauchy equations—see Chapter 8, Section 7d); you could use power series methods; given the fact that the solutions are just powers of r, it is easy to find the powers. Choose any method you like, and solve the two equations for future reference. Consider the case $n = 0$ separately. Is this necessary for $l = 0$?

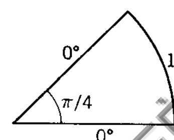
12. Separate Laplace's equation in two dimensions in polar coordinates [equation (5.1) without the z term] and solve the r and θ equations. (See Problem 11.) Remember that for the θ equation, only periodic solutions are of interest. Use your results to solve the problem of the steady-state temperature in a circular plate if the upper semicircular boundary is held at 100° and the lower at 0°.



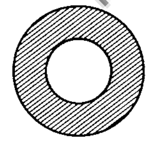
Comment: Another physical problem whose mathematical solution is identical with this temperature problem is this: Find the electrostatic potential inside a capacitor formed by two half-cylinders, insulated from each other and maintained at potentials 0 and 100.

Answer: $u = 50 + \dfrac{200}{\pi}\displaystyle\sum_{\text{odd } n}\left(\frac{r}{a}\right)^n\frac{\sin n\theta}{n}.$

13. Find the steady-state distribution of temperature in the sector of a circular plate of radius 10 and angle $\pi/4$ if the temperature is maintained at 0° along the radii and at 100° along the curved edge. *Hint:* See Problem 12.



14. Find the steady state temperature distribution in a circular annulus (shaded area) of inner radius 1 and outer radius 2 if the inner circle is held at 0° and the outer circle has half its circumference at 0° and half at 100°. *Hint:* Don't forget the r solutions corresponding to $k = 0$.



15. Solve Problem 14 if the temperatures of the two circles are interchanged.

▶ 6. VIBRATION OF A CIRCULAR MEMBRANE

A circular membrane (for example, a drumhead) is attached to a rigid support along its circumference. Find the characteristic vibration frequencies and the corresponding normal modes of vibration.

Take the (x, y) plane to be the plane of the circular support and take the origin at its center. Let $z(x, y, t)$ be the displacement of the membrane from the (x, y) plane. Then z satisfies the wave equation

$$\nabla^2 z = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}. \tag{6.1}$$

Putting

$$z = F(x, y)T(t), \tag{6.2}$$

we separate (6.1) into a space equation (Helmholtz) and a time equation (see Problem 3.10 and Section 3). We get the two equations

$$\nabla^2 F + K^2 F = 0 \qquad \text{and} \qquad \ddot{T} + K^2 v^2 T = 0. \tag{6.3}$$

Because the membrane is circular we write ∇^2 in polar coordinates (see Chapter 10, Section 9); then the F equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + K^2 F = 0. \tag{6.4}$$

When we put

$$F = R(r)\Theta(\theta), \tag{6.5}$$

(6.4) becomes (5.5), and the separated equations and their solutions are just (5.6), (5.7), and (5.8). The solutions of the time equation (6.3) are $\sin Kvt$ and $\cos Kvt$. Thus the solutions for z are $z = R(r)\Theta(\theta)T(t)$, where $R(r) = J_n(Kr), \Theta(\theta) = \{\sin n\theta, \cos n\theta\}$ and $T(t) = \{\sin Kvt, \cos Kvt\}$. Just as in Section 5, n must be an integer. To find possible values of K, we use the fact that the membrane is attached to a rigid frame at $r = a$, so we must have $z = 0$ at $r = a$ for all values of θ and t. Thus $J_n(Ka) = 0$ so the possible values of Ka are the zeros of J_n. As in Section 5, let $k = Ka$, that is, $K = k/a$. Then the possible values of k for each J_n are k_{mn}, the zeros of J_n. We can now write the solutions for z as

$$z = J_n(kr/a)\begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}\begin{Bmatrix} \sin kvt/a \\ coskvt/a \end{Bmatrix}. \tag{6.6}$$

For a given initial displacement or velocity of the membrane, we could find z as a double series as we found (5.17) in the cylinder temperature problem. However, here we shall do something different, namely investigate the separate normal modes of vibration and their frequencies. Recall that for the vibrating string (Section 4), each n gives a different frequency and a corresponding normal mode of vibration (Figure 4.2). The frequencies of the string are $\nu = nv/(2l)$; all frequencies are integral multiples of the frequency $\nu_1 = v/(2l)$ of the fundamental. For the circular membrane, the frequencies are [from (6.6)]

$$\nu = \frac{\omega}{2\pi} = \frac{kv}{2\pi a}.$$

The possible values of k are the zeros k_{mn} of the Bessel functions. Each value of k_{mn} gives a frequency $\nu_{mn} = k_{mn}v/(2\pi a)$, so we have a doubly infinite set of characteristic frequencies and the corresponding normal modes of vibration. All these frequencies are different, and they are not integral multiples of the fundamental as is true for the string. This is why a drum is less musical than a violin. From your computer or tables you can find several k_{mn} values (Problem 2) and find the frequencies as (nonintegral) multiples of the fundamental (which corresponds to k_{10}, the first zero of J_0). Let us sketch a few graphs (Figure 6.1) of the normal vibration modes corresponding to those in Figure 4.2 for the string, and write the corresponding formulas (eigenfunctions) for the displacement z given in (6.6). (For simplicity, we have used just the $\cos n\theta \cos kvt/a$ solutions in Figure 6.1.) In the fundamental mode of vibration corresponding to k_{10}, the membrane vibrates as a whole. In the

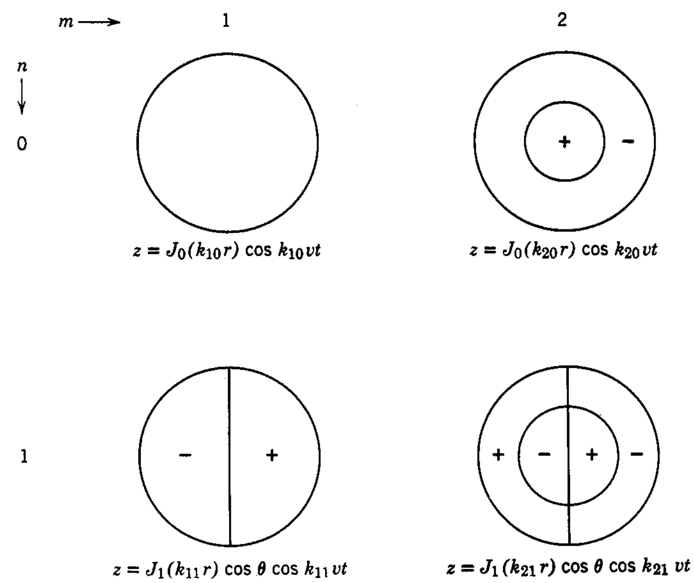


Figure 6.1

k_{20} mode, it vibrates in two parts as shown, the + part vibrating up while the − part vibrates down, and vice versa, with the circle between them at rest. We can show that there is such a circle (called a nodal line) and find its radius. Since $k_{20} > k_{10}$, the circle $r = ak_{10}/k_{20}$ is a circle of radius less than a. Hence it is a circle on the membrane. For this value of r, $J_0(k_{20}r/a) = J_0(k_{20}k_{10}/k_{20})) = J_0(k_{10}) = 0$, so points on this circle are at rest. For the k_{11} mode, $\cos\theta = 0$ when $\theta = \pm\pi/2$ and is positive or negative as shown. Continuing in this way you can sketch any normal mode (Problem 1).

It is difficult experimentally to obtain pure normal modes of a vibrating object. However, a complicated vibration will have nodal lines of some kind and it is easy to observe these. Fine sand sprinkled on the vibrating object will collect along the nodal lines (where there is no vibration) so that you can see them clearly—but see Am. J. Phys. **72**, 1345–1346, (2004). [For experimental work on the vibrating circular membrane, see Am. J. Phys. **35**, 1029–1031, (1967); Am. J. Phys. **40**, 186–188, (1972); Am. J. Phys. **59**, 376–377, (1991). Also see Problem 1(b).]

▶ PROBLEMS, SECTION 6

1. (a) Continue Figure 6.1 to show the fundamental modes of vibration of a circular membrane for $n = 0, 1, 2$, and $m = 1, 2, 3$. As in Figure 6.1, write the formula for the displacement z under each sketch.

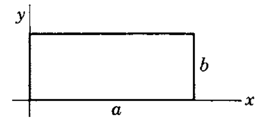
(b) Use a computer to set up animations of the various modes of vibration of a circular membrane. [This has been discussed in a number of places. See, for example, Am. J. Phys. **67**, 534–537, (1999).]

2. Find, from computer or tables, the first three zeros k_{mn} of each of the Bessel functions J_0, J_1, J_2, and J_3. Find the first six frequencies of a vibrating circular membrane as (non-integral) multiples of the fundamental frequency.

3. Separate the wave equation in two-dimensional rectangular coordinates x, y. Consider a rectangular membrane as shown, rigidly attached to supports along its sides.

Show that its characteristic frequencies are

$$\nu_{nm} = (v/2)\sqrt{(n/a)^2 + (m/b)^2},$$



where n and m are positive integers, and sketch the normal modes of vibration corresponding to the first few frequencies. That is, indicate the nodal lines as we did for the circular membrane in Figure 6.1 and Problem 1.

Next suppose the membrane is square. Show that in this case there may be two or more normal modes of vibration corresponding to a single frequency. (*Hint for one example:* $7^2 + 1^1 = 1^2 + 7^2 = 5^2 + 5^2$.) This is an example of what is called *degeneracy*; we say that there is degeneracy when several different solutions of the wave equation (eigenfunctions) correspond to the same frequency (eigenvalue). Sketch several normal modes giving rise to the same frequency. *Comment:* Compare Chapter 3, Section 11, where an eigenvalue of a matrix is called degenerate if several eigenvectors correspond to it.

4. Find the characteristic frequencies for sound vibration in a rectangular box (say a room) of sides a, b, c. *Hint:* Separate the wave equation in three dimensions in rectangular coordinates. This problem is like Problem 3 but for three dimensions instead of two. Discuss degeneracy (see Problem 3).

5. A square membrane of side l is distorted into the shape

$$f(x, y) = xy(l - x)(l - y)$$

and released. Express its shape at subsequent times as an infinite series. *Hint:* Use a double Fourier series as in Problem 5.9.

6. Let $V = 0$ in the Schrödinger equation (3.22) and separate variables in 2-dimensional rectangular coordinates. Solve the problem of a particle in a 2-dimensional square box, $0 < x < l, 0 < y < l$. This means to find solutions of the Schrödinger equation which are 0 for $x = 0, x = l, y = 0, y = l$, that is, on the boundary of the box, and to find the corresponding energy eigenvalues. *Comments:* If we extend the idea of a "particle in a box" (see Section 3, Example 3) to two or three dimensions, the box in 2D might be a square (as in this problem) or a circle (Problem 8); in 3D it might be a cube (Problem 7.17) or a sphere (Problem 7.19). In all cases, the mathematical problem is to find solutions of the Schrödinger equation with $V = 0$ inside the box and $\Psi = 0$ on the boundary of the box, and to find the corresponding energy eigenvalues. In quantum mechanics, Ψ describes a particle trapped inside the box and the energy eigenvalues are the possible values of the energy of the particle.

7. In your Problem 6 solutions, find some examples of degeneracy. (See Problem 3. Degeneracy means that several eigenfunctions correspond to the same energy eigenvalue.)

8. Do Problem 6 in polar coordinates to find the eigenfunctions and energy eigenvalues of a particle in a circular box $r < a$. You want $\Psi = 0$ when $r = a$.

7. STEADY-STATE TEMPERATURE IN A SPHERE

Find the steady-state temperature inside a sphere of radius a when the surface of the upper half is held at 100° and the surface of the lower half at 0°.

Inside the sphere, the temperature u satisfies Laplace's equation. In spherical coordinates this is (see Chapter 10, Section 9)

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \tag{7.1}$$

We separate this equation following our standard procedure. Substitute

$$u = R(r)\Theta(\theta)\Phi(\phi) \tag{7.2}$$

into (7.1) and multiply by $r^2/R\Theta\Phi$ to get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0. \tag{7.3}$$

If we multiply (7.3) by $\sin^2\theta$, the last term becomes a function of ϕ only and the other terms do not contain ϕ. Thus we obtain the ϕ equation and its solutions:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2, \qquad \Phi = \begin{cases}\sin m\phi,\\ \cos m\phi.\end{cases} \tag{7.4}$$

The separation constant must be negative and m an integer to make Φ a periodic function of ϕ [see the discussion after (5.6)].

Equation (7.3) can now be written as

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0. \tag{7.5}$$

The first term is a function of r and the last two terms are functions of θ, so we have two equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = k, \tag{7.6}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + k\Theta = 0. \tag{7.7}$$

If you compare (7.7) with the equation of Problem 10.2 in Chapter 12, you will see that (7.7) is the equation for the associated Legendre functions if $k = l(l+1)$. Recall that l must be an integer in order for the solution of Legendre's equation to be finite at $x = \cos\theta = \pm 1$, that is, at $\theta = 0$ or π; the same statement is true for the equation for the associated Legendre functions. The corresponding result for (7.7) is that k must be a product of two successive integers; it is then convenient to replace k by $l(l+1)$, where l is an integer. The solutions of (7.7) are then the associated Legendre functions (see Problem 10.2, Chapter 12)

$$\Theta = P_l^m(\cos\theta). \tag{7.8}$$

In (7.6), we put $k = l(l+1)$; you can then easily verify (Problem 5.11) that the solutions of (7.6) are

$$R = \begin{cases} r^l,\\ r^{-l-1}.\end{cases} \tag{7.9}$$

Since we are interested in the interior of the sphere, we discard the solutions r^{-l-1} because they become infinite at the origin. If we were discussing a problem (say about water flow or electrostatic potential) outside the sphere, we would use the r^{-l-1} solutions and discard the solutions r^l because they become infinite at infinity.

The basis functions for our problem are then

$$u = r^l P_l^m(\cos\theta)\begin{cases}\sin m\phi,\\ \cos m\phi.\end{cases} \tag{7.10}$$

[The functions $P_l^m(\cos\theta)\sin m\phi$ and $P_l^m(\cos\theta)\cos m\phi$ are called *spherical harmonics*and are often denoted by $Y_l^m(\theta,\phi)$; also see Problem 16.] If the surface temperature at $r = a$ were given as a function of θ and ϕ, we would have a double series (summed on l and m). For the given surface temperatures in our problem (100° on the top hemisphere and 0° on the lower hemisphere), the temperature is independent of ϕ; thus in (7.10) we must have $m = 0, \cos m\phi = 1$. The solutions (7.10) then reduce to $r^l P_l(\cos\theta)$. We write the solution of the problem as a series of these basis functions:

$$u = \sum_{l=0}^{\infty} c_l r^l P_l(\cos\theta). \tag{7.11}$$

We determine the coefficients c_l by using the given temperatures when $r = a$; that is, we must have

$$\begin{aligned} u_{r=a} &= \sum_{l=0}^{\infty} c_l a^l P_l(\cos\theta) \\ &= \begin{cases} 100, & 0 < \theta < \frac{\pi}{2}, \quad \text{that is,} \quad 0 < \cos\theta < 1,\\ 0, & \frac{\pi}{2} < \theta < \pi, \quad \text{that is,} \quad -1 < \cos\theta < 0,\end{cases}\end{aligned} \tag{7.12}$$

or, with $x = \cos\theta$,

$$u_{r=a} = \sum_{l=0}^{\infty} c_l a^l P_l(x) = 100 f(x) \tag{7.13}$$

where

$$f(x) = \begin{cases} 0, & -1 < x < 0,\\ 1, & 0 < x < 1.\end{cases}$$

(Note that here x just stands for $\cos\theta$ and is not the coordinate x.) In Section 9 of Chapter 12, we expanded this $f(x)$ in a series of Legendre polynomials and obtained:

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \cdots. \tag{7.14}$$

The coefficients c_l in (7.13) are just these coefficients times $100/a^l$. Substituting the c's into (7.11), we get the final solution:

$$\begin{aligned} u = 100\Big[&\tfrac{1}{2}P_0(\cos\theta) + \tfrac{3}{4}\tfrac{r}{a}P_1(\cos\theta) - \tfrac{7}{16}\left(\tfrac{r}{a}\right)^3 P_3(\cos\theta) \\ &+ \tfrac{11}{32}\left(\tfrac{r}{a}\right)^5 P_5(\cos\theta) + \ldots\Big].\end{aligned} \tag{7.15}$$

We can do variations of this problem. Notice that we have not even mentioned so far what temperature scale we are using (Celsius, Fahrenheit, absolute, etc.). This is a very easy adjustment to make once we have a solution in any one scale. To see why, observe that if u is a solution of Laplace's equation $\nabla^2 u = 0$ or of the heat flow equation $\nabla^2 u = (1/\alpha^2)(\partial u/\partial t)$, then $u + C$ and Cu are also solutions for any constant C. If we add, say, 50° to the solution (7.15), we have the temperature distribution inside a sphere with the top half of the surface at 150° and the lower half at 50°. If we multiply the solution (7.15) by 2, we find the temperature distribution with given surface temperatures of 200° and 0°, and so on.

The temperature of the equatorial plane $\theta = \pi/2$ or $\cos\theta = 0$ as given by equations (7.11) to (7.15) is halfway between the top and bottom surface temperatures, because Legendre series, like Fourier series, converge to the midpoint of a jump in the function which was expanded to get the series. To solve the problem of the temperature in a hemisphere given the temperatures of the curved surface and of the equatorial plane, we need only imagine the lower hemisphere in place and at the proper temperature to give the desired average on the equatorial plane. When the temperature of the equatorial plane is 0°, this amounts to defining the function $f(x)$ in (7.13) on $(-1, 0)$ to make it an odd function.

▶ PROBLEMS, SECTION 7

Find the steady-state temperature distribution inside a sphere of radius 1 when the surface temperatures are as given in Problems 1 to 10.

1. $35\cos^4\theta$

2. $\cos\theta - \cos^3\theta$

3. $\cos\theta - 3\sin^2\theta$

4. $5\cos^3\theta - 3\sin^2\theta$

5. $|\cos\theta|$

6. $\pi/2 - \theta$. See Chapter 12, Problem 9.4.

7. $\begin{cases} \cos\theta, & 0 < \theta < \pi/2, \quad \text{that is, upper hemisphere,} \\ 0, & \pi/2 < \theta < \pi, \quad \text{that is, lower hemisphere.} \end{cases}$

8. $\begin{cases} 100°, & 0 < \theta < \pi/3, \\ 0°, & \text{otherwise.} \end{cases}$ *Hint:* See Problem 9.8 of Chapter 12.

9. $3\sin\theta\cos\theta\sin\phi$. *Hint:* See equation (7.10) and Chapter 12, equation (10.6).

10. $\sin^2\theta\cos\theta\cos 2\phi - \cos\theta$. (See Problem 9.)

11. Find the steady-state temperature distribution inside a hemisphere if the spherical surface is held at 100° and the equatorial plane at 0°. *Hint:* See the last paragraph of this section above.

12. Do Problem 11 if the curved surface is held at $\cos^2\theta$ and the equatorial plane at zero. *Careful:* The answer does *not* involve P_2; read the last sentence of this section.

13. Find the electrostatic potential outside a conducting sphere of radius a placed in an originally uniform electric field, and maintained at zero potential. *Hint:* Let the original field $\mathbf{E}$ be in the negative z direction so that $\mathbf{E} = -E_0\mathbf{k}$. Then since $\mathbf{E} = -\boldsymbol{\nabla}\Phi$, where Φ is the potential, we have $\Phi = E_0 z = E_0 r\cos\theta$ (Verify this!) for the original potential. You then want a solution of Laplace's equation $\nabla^2 u = 0$ which is zero at $r = a$ and becomes $u \sim \Phi$ for large r (that is, far away from the sphere). Select the solutions of Laplace's equation in spherical coordinates which have the right θ and ϕ dependence (there are just two such solutions) and find the combination which reduces to zero for $r = a$.

14. Find the steady-state temperature distribution in a spherical shell of inner radius 1 and outer radius 2 if the inner surface is held at 0° and the outer surface has its upper half at 100° and its lower half at 0°. *Hint:* $r = 0$ is not in the region of interest, so the solutions r^{-l-1} in (7.9) should be included. Replace $c_l r^l$ in (7.11) by $(c_l r^l + b_l r^{-l-1})$.

15. A sphere initially at 0° has its surface kept at 100° from $t = 0$ on (for example, a frozen potato in boiling water!). Find the time-dependent temperature distribution. *Hint:* Subtract 100° from all temperatures and solve the problem; then add the 100° to the answer. Can you justify this procedure? Show that the Legendre function required for this problem is P_0 and the r solution is $(1/\sqrt{r})J_{1/2}$ or j_0 [see (17.4) in Chapter 12]. Since spherical Bessel functions can be expressed in terms of elementary functions, the series in this problem can be thought of as either a Bessel series or a Fourier series. Show that the results are identical.

16. Separate the wave equation in spherical coordinates, and show that the θ, ϕ solutions are the spherical harmonics $Y_l^m(\theta, \phi) = P_l^m(\cos\theta)e^{\pm im\phi}$ and the r solutions are the spherical Bessel functions $j_l(kr)$ and $y_l(kr)$ [Chapter 12, equations (17.4)].

17. Do Problem 6.6 in 3 dimensional rectangular coordinates. That is, solve the "particle in a box" problem for a cube.

18. Separate the time-independent Schrödinger equation (3.22) in spherical coordinates assuming that $V = V(r)$ is independent of θ and ϕ. (If V depends only on r, then we are dealing with central forces, for example, electrostatic or gravitational forces.) *Hints:* You may find it helpful to replace the mass m in the Schrödinger equation by M when you are working in spherical coordinates to avoid confusion with the letter m in the spherical harmonics (7.10). Follow the separation of (7.1) but with the extra term $[V(r) - E]\Psi$. Show that the θ, ϕ solutions are spherical harmonics as in (7.10) and Problem 16. Show that the r equation with $k = l(l+1)$ is [compare (7.6)]

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2Mr^2}{\hbar^2}[V(r) - E) = l(l+1).$$

19. Find the eigenfunctions and energy eigenvalues for a "particle in a spherical box" $r < a$. *Hints:* See Problem 6.6. Write the R equation from Problem 18 with $V = 0$, and compare Chapter 12, Problem 17.6, with $y = R, x = \beta r$ where $\beta = \sqrt{2ME/\hbar^2}$, and $n = l$.

20. Write the Schrödinger equation (3.22) if ψ is a function of x, and $V = \frac{1}{2}m\omega^2x^2$ (this is a one-dimensional harmonic oscillator). Find the solutions $\psi_n(x)$ and the energy eigenvalues E_n. *Hints:* In Chapter 12, equation (22.1) and the first equation in (22.11), replace x by αx where $\alpha = \sqrt{m\omega/\hbar}$. (Don't forget appropriate factors of α for the x's in the denominators of $D = d/dx$ and $\psi'' = d^2\psi/dx^2$.) Compare your results for equation (22.1) with the Schrödinger equation you wrote above to see that they are identical if $E_n = (n + \frac{1}{2})\hbar\omega$. Write the solutions $\psi_n(x)$ of the Schrödinger equation using Chapter 12, equations (22.11) and (22.12).

21. Separate the Schrödinger equation (3.22) in rectangular coordinates in 3 dimensions assuming that $V = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$. (This is a 3-dimensional harmonic oscillator). Observe that each of the separated equations is of the form of the one-dimensional oscillator equation in Problem 20. Thus write the solutions $\psi_n(x, y, z)$ for the 3-dimensional problem, where $n = n_x + n_y + n_z$. Find the energy eigenvalues E_n and their degree of degeneracy (see Problem 6.7 and Chapter 15, Problem 4.21).

22. Find the energy eigenvalues and eigenfunctions for the hydrogen atom. The potential energy is $V(r) = -e^2/r$ in Gaussian units, where e is the charge of the electron and r is in spherical coordinates. Since V is a function of r only, you know from Problem 18 that the eigenfunctions are $R(r)$ times the spherical harmonics $Y_l^m(\theta, \phi)$, so you only have to find $R(r)$. Substitute $V(r)$ into the R equation in Problem 18 and make the following simplifications: Let $x = 2r/\alpha$, $y = rR$; show that then

$$r = \alpha x/2, \quad R(r) = \frac{2}{\alpha x}y(x), \quad \frac{d}{dr} = \frac{2}{\alpha}\frac{d}{dx}, \quad \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{2}{\alpha}xy''.$$

Let $\alpha^2 = -2ME/\hbar^2$ (note that for a bound state, E is negative, so α^2 is positive) and $\lambda = Me^2\alpha/\hbar^2$, to get the first equation in Problem 22.26 of Chapter 12. Do this problem to find $y(x)$, and the result that λ is an integer, say n. [*Caution:* **not** the same n as in equation (22.26)]. Hence find the possible values of α (these are the radii of the Bohr orbits), and the energy eigenvalues. You should have found α proportional to n; let $\alpha = na$, where a is the value of α when $n = 1$, that is, the radius of the first Bohr orbit. Write the solutions $R(r)$ by substituting back $y = rR$, and $x = 2r/(na)$, and find E_n from α.

8. POISSON'S EQUATION

We are going to derive Poisson's equation (1.2) for a simple problem whose answer we know in advance. Using our known solution, we shall be able to see a method of solving more difficult problems.

Recall from Chapter 6, Section 8, that the gravitational field is conservative, that is, $\operatorname{curl}\mathbf{F} = 0$, and there is a potential function V such that $\mathbf{F} = -\nabla V$. If we consider the gravitational field at a point P due to a point mass m a distance r away, we have

$$V = -\frac{Gm}{r} \quad \text{and} \quad \mathbf{F} = -\frac{Gm}{r^2}\mathbf{u} \tag{8.1}$$

where $\mathbf{u}$ is a unit vector along r toward P. It is straightforward to show that $\operatorname{div}\mathbf{F} = 0$ and V satisfies Laplace's equation (Problem 1), that is,

$$\nabla \cdot \mathbf{F} = -\nabla \cdot \nabla V = -\nabla^2 V = 0. \tag{8.2}$$

Now suppose there are many masses m_i at distances r_i from P. The total potential at P is the sum of the potentials due to the individual m_i, that is,

$$V = \sum_i V_i = -\sum_i \frac{Gm_i}{r_i}$$

and the total gravitational field at P is the vector sum of the fields $\mathbf{F}_i$, that is,

$$\mathbf{F} = -\sum_i \nabla V_i = -\nabla V.$$

Note that we are taking it for granted that none of the masses m_i are *at* P, that is, that no r_i is zero. Since

$$\nabla \cdot \mathbf{F}_i = -\nabla^2 V_i = 0,$$

we have also

$$\nabla \cdot \mathbf{F} = -\nabla^2 V = 0.$$

Instead of a number of masses m_i, we can consider a continuous distribution of mass inside a volume τ (Figure 8.1). Let ρ be the mass density of the distribution; then the mass in an element $d\tau$ is $\rho\, d\tau$. The gravitational potential at P due to this mass $\rho d\tau$ is $-(G\rho/r)\, d\tau$. Then the total gravitational potential at P due to the whole mass distribution is the triple integral over the volume τ:

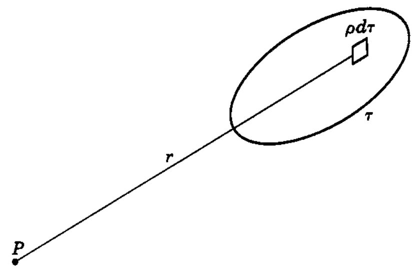


Figure 8.1

$$V = -\iiint\limits_{\text{volume } \tau} \frac{G\rho\, d\tau}{r}. \tag{8.3}$$

As before, the contribution to V at P due to each bit of mass satisfies Laplace's equation and therefore V satisfies Laplace's equation. Also the total field $\mathbf{F}$ at P is the vector sum of the fields due to the elements of mass, and as before we have

$$\nabla \cdot \mathbf{F} = -\nabla^2 V = 0.$$

Again note that we are implicitly assuming that none of the mass distribution coincides with P, that is, that $r \neq 0$, which means that point P is not a point of the region τ.

Now let us investigate what happens if P *is* a point of τ. Can we find V from (8.3) and does V satisfy Laplace's equation? Let S be a small sphere of radius a about P; imagine all the mass removed from inside S (Figure 8.2). Then our previous discussion holds at points inside S since these points are not in the mass distribution. If $\mathbf{F}'$ and V' are the new field and potential (with the matter inside S removed), then $\nabla \cdot \mathbf{F}' = -\nabla^2 V' = 0$ at points inside S. Now restore the mass inside S; let $\mathbf{F}$ and V represent the field and potential due to the whole distribution and let $\mathbf{F}_S$ and V_S represent the field and potential due to just the mass inside S. Then $\mathbf{F} = \mathbf{F}' + \mathbf{F}_S$ and at points inside S

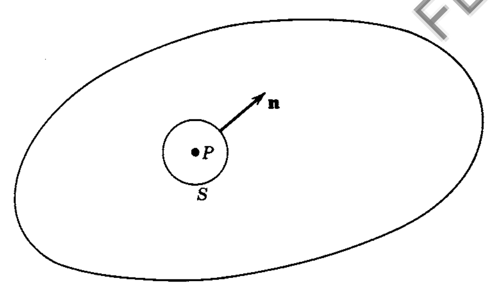


Figure 8.2

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}' + \nabla \cdot \mathbf{F}_S = \nabla \cdot \mathbf{F}_S \tag{8.4}$$

since $\nabla \cdot \mathbf{F}' = 0$ inside S.

By the divergence theorem (see Figure 8.2 and Chapter 6, Section 10)

$$\iiint\limits_{\text{volume of } S} \nabla \cdot \mathbf{F}_S\, d\tau = \iint\limits_{\text{surface of } S} \mathbf{F}_S \cdot \mathbf{n}\, d\sigma. \tag{8.5}$$

If we let the radius a of S tend to zero, the density ρ of matter inside S tends to its value at P; thus for small a, S contains a total mass M approximately equal to

$\frac{4}{3}\pi a^3\rho$, where ρ is evaluated at P. The gravitational field at the surface of S due to this mass is of magnitude

$$F_s = \frac{GM}{a^2} = G\frac{4}{3}\pi a\rho$$

directed *toward* P. Thus in (8.5), $\mathbf{F}_S \cdot \mathbf{n} = -\frac{4}{3}G\pi a\rho$ because $\mathbf{F}_S$ and $\mathbf{n}$ are antiparallel. Since F_S is constant over the surface S, the right-hand side of (8.5) is $\mathbf{F}_S \cdot \mathbf{n}$ times the area of the sphere. The left-hand side is, for small a, approximately the value of $\boldsymbol{\nabla} \cdot \mathbf{F}_S$ at P times the volume of S. Then we have

$$(\boldsymbol{\nabla} \cdot \mathbf{F}_S)(\frac{4}{3}\pi a^3) = (-\frac{4}{3}G\pi a\rho)(4\pi a^2)$$

or

$$\boldsymbol{\nabla} \cdot \mathbf{F}_S = -4\pi G\rho \qquad \text{at } P. \tag{8.6}$$

Since

$$\boldsymbol{\nabla} \cdot \mathbf{F}_S = \boldsymbol{\nabla} \cdot \mathbf{F} = -\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} V = -\nabla^2 V,$$

we have

$$\nabla^2 V = 4\pi G\rho. \tag{8.7}$$

This is Poisson's equation; we see that the gravitational potential in a region containing matter satisfies Poisson's equation as claimed in (1.2). Note that if $\rho = 0$, (8.7) becomes (8.2) as it should.

Next we must consider whether our formula (8.3) for V is valid when P is a point of the mass distribution. The integral appears to diverge at $r = 0$, but this is not really so as we see most easily by using spherical coordinates. Then (8.3) becomes

$$V = -\iiint_{\text{volume } \tau} \frac{G\rho}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

and we see that there is no trouble when $r = 0$. Thus (8.3) is valid in general and gives a solution for (8.7).

Using the notation of (1.2) for Poisson's equation [that is, replacing $4\pi G\rho$ by f and V by u in (8.7) and (8.3)] we can write

$$u = -\frac{1}{4\pi}\iiint \frac{f\,d\tau}{r} \qquad \text{is a solution of} \quad \nabla^2 u = f. \tag{8.8}$$

In the more detailed notation needed when we use this solution in a problem, (8.8) becomes (see Figure 8.3):

$$u(x,y,z) = -\frac{1}{4\pi}\iiint \frac{f(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\,dx'\,dy'\,dz' \tag{8.9}$$

is a solution of

$$\nabla^2 u(x,y,z) = f(x,y,z)$$

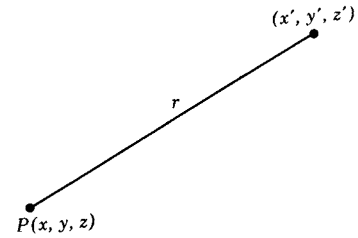


Figure 8.3

In (8.9) and Figure 8.3, the point (x, y, z) is the point at which we are calculating the potential u; the point (x', y', z') is a point in the mass distribution over which we integrate; r in (8.8) is the distance between these two points and is written out in full in (8.9).

Equations (8.8) or (8.9) actually give a very special solution of Poisson's equation. Recall that it is customary to take the zero point for gravitational (and electrostatic) potential energy at infinity, and this is what we have done. Thus (8.8) or (8.9) gives a solution of Poisson's equation which tends to zero at infinity. In another problem this may not be what we want. For example, suppose we have an electrostatic charge distribution near a grounded plane. The electrostatic potential satisfies Poisson's equation, but here we want a solution which is zero on the grounded plane rather than at infinity. To see how we might find such a solution, observe that if u is a solution of Poisson's equation, and w is any solution of Laplace's equation ($\nabla^2 w = 0$), then

$$\nabla^2(u + w) = \nabla^2 u + \nabla^2 w = \nabla^2 u = f; \tag{8.10}$$

thus $u + w$ is a solution of Poisson's equation. Then we can add to the solution (8.9) any solution of Laplace's equation; the combination must be adjusted to fit the given boundary conditions just as we have done in the problems in previous paragraphs.

Example 1. Let us do the following simple problem to illustrate this process. In Figure 8.4, a point charge q at $(0, 0, a)$ is outside a grounded sphere of radius R and center at the origin. Our problem is to find the electrostatic potential V at points outside the sphere. The potential V and the charge density ρ are related by Poisson's equation

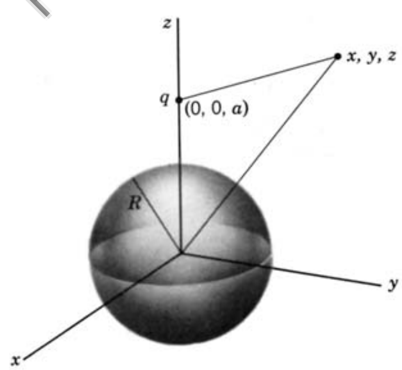


FIgure 8.4

$$\nabla^2 V = -4\pi\rho \qquad \text{(in Gaussian units).} \tag{8.11}$$

The potential at (x, y, z) due to a given charge distribution ρ is given by (8.8) or

(8.9) with $f = -4\pi\rho$:

$$V(x,y,z) = -\frac{1}{4\pi}\iiint \frac{-4\pi\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\,dx'\,dy'\,dz'. \tag{8.12}$$

For a given space-charge distribution, we would next evaluate this integral. For the single point charge q, we have $(x',y',z') = (0,0,a)$ and we replace $\iiint \rho\,dx'\,dy'\,dz'$ (which is simply the total charge) by q to obtain

$$V = \frac{q}{\sqrt{x^2+y^2+(z-a)^2}}. \tag{8.13}$$

[We could, of course, simply have written down (8.13) without using (8.8); (8.13) is just the electrostatic formula corresponding to the gravitational formula (8.1) with which we started.]

Now we want to add to (8.13) a solution of Laplace's equation such that the combination is zero on the given sphere (Figure 8.4). It will be convenient to change to spherical coordinates and to use solutions of Laplace's equation in spherical coordinates. [Note a change in the meaning of r from now on. We have been using r to mean the distance from q at (x',y',z') to (x,y,z); from now on we want to use it to mean the distance from $(0,0,0)$ to (x,y,z). See, for example, Figures 8.3 and 8.4.] Writing V_q for V in (8.13) (to distinguish it from our final answer which will be a sum of V_q and a solution of Laplace's equation) and changing to spherical coordinates, we get

$$V_q = \frac{q}{\sqrt{r^2-2ar\cos\theta+a^2}}. \tag{8.14}$$

The solutions of Laplace's equation in spherical coordinates are (Section 7):

$$\left\{\begin{matrix} r^l \\ r^{-l-1}\end{matrix}\right\} P_l^m(\cos\theta) \left\{\begin{matrix}\sin m\phi \\ \cos m\phi\end{matrix}\right\}. \tag{8.15}$$

Since we are interested in the region outside the sphere, we want r solutions which do not become infinite at infinity; thus we use r^{-l-1} and discard the r^l solutions. Because the physical problem is symmetric about the z axis, we look for solutions independent of ϕ; that is, we choose $m = 0, \cos m\phi = 1$. Then the basis functions for our problem are $r^{-l-1}P_l(\cos\theta)$ and we try to find a solution of the form

$$V = V_q + \sum_l c_l r^{-l-1} P_l(\cos\theta). \tag{8.16}$$

We must satisfy the boundary condition $V = 0$ when $r = R$. This gives

$$V_{r=R} = \frac{q}{\sqrt{R^2-2aR\cos\theta+a^2}} + \sum_l c_l R^{-l-1}P_l(\cos\theta) = 0. \tag{8.17}$$

Thus we want to expand V_q in a Legendre series. Since V_q is essentially the generating function for Legendre polynomials, this is very easy. Comparing (8.17) and the formulas of Chapter 12, Section 5 [(5.1) and (5.2), or more simply, (5.12) and (5.17)], we find

$$\frac{q}{\sqrt{R^2-2aR\cos\theta+a^2}} = q\sum_l \frac{R^l P_l(\cos\theta)}{a^{l+1}}. \tag{8.18}$$

Thus the coefficients c_l in (8.17) are given by

$$c_l R^{-l-1} = -\frac{qR^l}{a^{l+1}} \qquad \text{or} \qquad c_l = -\frac{qR^{2l+1}}{a^{l+1}}. \tag{8.19}$$

Substituting (8.19) into (8.16), we obtain the final solution for V:

$$V = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} - q\sum_l \frac{R^{2l+1}r^{-l-1}P_l(\cos\theta)}{a^{l+1}}. \tag{8.20}$$

Since the second term in (8.20) is of the same general form as (8.18), we can simplify (8.20) by summing the series to get (Problem 2)

$$V = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} - \frac{(R/a)q}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a)\cos\theta}}. \tag{8.21}$$

Formula (8.21) has a very interesting physical interpretation. The second term is the potential of a charge $-(R/a)q$ at the point $(0, 0, R^2/a)$; thus we could replace the grounded sphere by this charge and have the same potential for $r > R$. This result can be shown also by elementary analytic geometry and is known as the "method of images." For problems with simple geometry (involving planes, spheres, circular cylinders), it may offer a simpler method of solution than the one we have discussed; however, our purpose was to illustrate the more general method.

Use of Green Functions In Chapter 8, Section 12, we used Green functions to solve ordinary differential equations with a nonzero right-hand side. Here we consider the use of Green functions to solve a corresponding partial differential equation in three dimensions, namely Poisson's equation

$$\nabla^2 u = f(\mathbf{r}) = f(x, y, z). \tag{8.22}$$

Suppose that we have a solution of Poisson's equation when the right hand side is a 3-dimensional δ function (see Chapter 8, Sections 11 and 12):

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z'). \tag{8.23}$$

The three-dimensional δ function has the property that

$$\iiint f(x', y', z')\delta(\mathbf{r} - \mathbf{r}')\, d\tau' = f(x, y, z) \tag{8.24}$$

if the volume of integration includes the point (x, y, z) (and the integral is zero otherwise). Recall that the right-hand side of Poisson's equation is proportional to the mass density or the charge density. The volume integral of the density gives the total mass or total charge. Since $\iiint \delta(\mathbf{r} - \mathbf{r}')\, d\tau' = 1$, the right-hand side of (8.23) corresponds to a point mass or point charge. That is, the Green function in (8.23) is the potential due to a point source. Just as we showed in Chapter 8, Section 12, that (12.4) is a solution of (12.1), we find here that a solution of (8.22) is given by (see Problem 6)

$$u(\mathbf{r}) = \iiint G(\mathbf{r}, \mathbf{r}')f(\mathbf{r}')\, d\tau'. \tag{8.25}$$

In equation (8.9) we found that a solution of (8.22) is

$$u(\mathbf{r}) = -\frac{1}{4\pi}\iiint \frac{f(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'. \tag{8.26}$$

Comparing (8.25) and (8.26), we conclude that a solution of (8.23) is

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{8.27}$$

Now (8.26) and (8.27) give solutions which are zero at infinity; usually we want solutions which are zero on some given surface (for example, zero electrostatic potential on a grounded sphere or plane). In order to obtain such a solution, we add to (8.27) a solution $F(\mathbf{r},\mathbf{r}')$ of Laplace's equation chosen so that the new Green function

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} + F(\mathbf{r},\mathbf{r}') \tag{8.28}$$

satisfies the desired zero boundary conditions. Then (8.25) with $G(\mathbf{r},\mathbf{r}')$ as in (8.28) gives a solution of (8.22) which is zero of the boundary. For example, in equation (8.21), V is the potential outside the grounded sphere $r=R$ due to a point charge at $r=a>R$. Rewriting that result in our present notation gives the Green function (8.28) which satisfies (8.23) and is zero on the sphere $r=R$, namely (Problem 7)

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{R/r'}{4\pi|\mathbf{r}-R^2\mathbf{r}'/r'^2|}. \tag{8.29}$$

(Also see Problems 8 and 9.)

▶ PROBLEMS, SECTION 8

1. Show that the gravitational potential $V=-Gm/r$ satisfies Laplace's equation, that is, show that $\nabla^2(1/r)=0$ where $r^2=x^2+y^2+z^2, r\neq 0$.

2. Using the formulas of Chapter 12, Section 5, sum the series in (8.20) to get (8.21).

3. Do the problem in Example 1 for the case of a charge q inside a grounded sphere to obtain the potential V inside the sphere. Sum the series solution and state the image method of solving this problem.

4. Do the two-dimensional analogue of the problem in Example 1. A "point charge" in a plane means physically a uniform charge along an infinite line perpendicular to the plane; a "circle" means an infinitely long circular cylinder perpendicular to the plane. However, since all cross sections of the parallel line and cylinder are the same, the problem is a two-dimensional one. *Hint:* The potential must satisfy Laplace's equation in charge-free regions. What are the solutions of the two-dimensional Laplace equation?

5. Find the method of images for problem 4.

6. Substitute (8.25) into (8.22) and use (8.23) and (8.24) to show that (8.25) is a solution of (8.22).

7. Verify that the Green function in (8.29) is zero when $r=R$. Also verify that the point at which the second term becomes infinite is *inside* the sphere, so outside the sphere this term satisfies Laplace's equation as required. Thus write a triple integral for the solution of (8.22) for $r>R$ which is zero on the sphere $r=R$.

8. Show that the Green function (8.28) which is zero on the plane $z = 0$ is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2} + \frac{1}{4\pi}\left[(x-x')^2 + (y-y')^2 + (z+z')^2\right]^{-1/2}.$$

Hence write a triple integral for the solution of (8.22) for $z > 0$ which is zero for $z = 0$.

9. Show that our results can be extended to find the following solution of (8.22) which satisfies given nonzero boundary conditions:

$$u(\mathbf{r}) = \iiint G(\mathbf{r}, \mathbf{r}')f(\mathbf{r}')\,d\tau' + \iint u(\mathbf{r}')\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'}\,d\sigma'$$

where $G(r, r')$ is the Green function (8.28) which is zero on the surface σ, and $\partial G/\partial n' = \nabla G \cdot \mathbf{n}'$ is the normal derivative of G (see Chapter 6, Section 6). *Hints:* In Green's second identity (Chapter 6, Problem 10.16) let $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r}, \mathbf{r}')$, and use (8.22) and (8.23) to find $\nabla^2\phi$ and $\nabla^2\psi$. *Comment:* Although we derived the divergence theorem and so Green's identities only for bounded regions in Chapter 6, they are valid for unbounded regions if the functions involved tend to zero sufficiently rapidly.

9. INTEGRAL TRANSFORM SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Laplace Transform Solutions We have seen (Chapter 8, Section 9) that taking the Laplace transform of an ordinary differential equation converts it into an algebraic equation. Taking the Laplace transform of a partial differential equation reduces the number of independent variables by one, and so converts a two-variable partial differential equation into an ordinary differential equation. To illustrate this, we solve the following problem.

Example 1. A semi-infinite bar (extending from $x = 0$ to $x = \infty$), with insulated sides, is initially at the uniform temperature $u = 0°$. At $t = 0$, the end at $x = 0$ is brought to $u = 100°$ and held there. Find the temperature distribution in the bar as a function of x and t.

The differential equation satisfied by u is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}. \tag{9.1}$$

We are going to take the t Laplace transform of (9.1); the variable x will just be a parameter in this process. Let U be the Laplace transform of u, that is,

$$U(x, p) = \int_0^\infty u(x, t)e^{-pt}\,dt. \tag{9.2}$$

By Chapter 8, equation (9.1) we have

$$L\left(\frac{\partial u}{\partial t}\right) = pU - u_{t=0} = pU$$

since $u = 0$ when $t = 0$. Also

$$L\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2}L(u) = \frac{\partial^2 U}{\partial x^2}$$

(remember that x is just a parameter here; we are taking a t Laplace transform). The transform of (9.1) is then

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{\alpha^2} pU. \tag{9.3}$$

Now if we think of p as a constant and x as the variable, this is an ordinary differential equation for U as a function of x. Its solutions are

$$U = \begin{cases} e^{(\sqrt{p}/\alpha)x}, \\ e^{-(\sqrt{p}/\alpha)x}. \end{cases} \tag{9.4}$$

To find the correct combination of these solutions to fit our problem, we need the Laplace transforms of the boundary conditions on u since these give the conditions on U. Using $L1$ (see Laplace Transform Table, page 469) to find the transforms, we have

$$\begin{array}{llllll} u = 100 & \text{at} & x = 0, & U = L(100) = \frac{100}{p} & \text{at} & x = 0; \\ u \to 0 & \text{as} & x \to \infty, & U \to L(0) = 0 & \text{as} & x \to \infty. \end{array} \tag{9.5}$$

Since $U \to 0$ as $x \to \infty$, we see that we must use the solution $e^{-(\sqrt{p}/\alpha)x}$ from (9.4) and discard the positive exponential solution. We determine the constant multiple of this solution which fits our problem from the condition that $U = 100/p$ at $x = 0$. Thus we find that the U solution satisfying the given boundary conditions is

$$U = \frac{100}{p} e^{-(\sqrt{p}/\alpha)x}. \tag{9.6}$$

We find u by looking up the inverse transform of (9.6); it is, by $L22$

$$u = 100 \left[1 - \operatorname{erf} \frac{x}{2\alpha\sqrt{t}}\right] \tag{9.7}$$

and this is the solution of the problem.

Fourier Transform Solutions In the examples in Sections 2, 3 and 4, we expanded a given function in a Fourier series. This was possible because the function was to be represented by a series over a finite interval. We could then take that interval as the period for the Fourier series. If we are dealing with a function which is given over an infinite interval (and not periodic), then instead of representing it by a Fourier series we represent it by a Fourier integral (Chapter 7, Section 12). Let us do this for a specific problem.

▶ **Example 2.** An infinite metal plate (Figure 9.1) covering the first quadrant has the edge along the y axis held at 0°, and the edge along the x axis held at

$$u(x, 0) = \begin{cases} 100^\circ, & 0 < x < 1, \\ 0^\circ, & x > 1. \end{cases} \tag{9.8}$$

Find the steady-state temperature distribution as a function of x and y.

The differential equation and its solutions are the same as in the semi-infinite plate problem discussed in Section 2, equations (2.1), (2.6), and (2.7). As in that problem, we assume $u \to 0$ as $y \to \infty$, and use only the e^{-ky} terms. Since $u = 0$ when $x = 0$, we use only the sine solutions. The basis functions we want are then $u = e^{-ky} \sin kx$. We do not have any requirement here which determines k as we did in Section 2. We must then allow all k's and try to find a solution in the form of an integral over k. Instead of coefficients b_n in a series, we have a coefficient function $B(k)$ to determine. Remember that $k > 0$ since e^{-ky} must tend to zero as $y \to \infty$. Thus we try to find a solution of the form

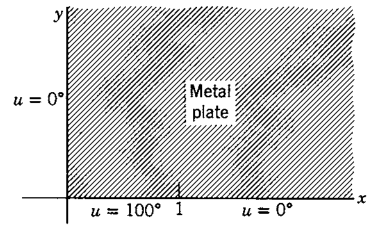


Figure 9.1

(9.9) $$u(x, y) = \int_0^\infty B(k) e^{-ky} \sin kx \, dk.$$

When $y = 0$, we have

(9.10) $$u(x, 0) = \int_0^\infty B(k) \sin kx \, dk.$$

This is the first of equations (12.14) in Chapter 7, if we identify k with α, $u(x, 0)$ with $f_s(x)$, and $B(k)$ with $\sqrt{2/\pi}\, g_s(\alpha)$. Thus the given temperature on the x axis is a Fourier sine transform of the desired coefficient function, so $B(k)$ can be found as the inverse transform. Using the second of equations (12.14) in Chapter 7, we get

(9.11) $$B(k) = \sqrt{\frac{2}{\pi}} g_s(k) = \frac{2}{\pi} \int_0^\infty f_s(x) \sin kx \, dx = \frac{2}{\pi} \int_0^\infty u(x, 0) \sin kx \, dx.$$

For the given $u(x, 0)$ in (9.8), we find

(9.12) $$B(k) = \frac{2}{\pi} \int_0^1 100 \sin kx \, dx = -\frac{200}{\pi} \frac{\cos kx}{k} \bigg|_0^1 = \frac{200}{\pi k}(1 - \cos k).$$

Finding $B(k)$ corresponds to evaluating the coefficients in a Fourier series. Substituting (9.12) into (9.9), we get the solution to our problem in the form of an integral instead of a series:

(9.13) $$u(x, y) = \frac{200}{\pi} \int_0^\infty \frac{1 - \cos k}{k} e^{-ky} \sin kx \, dk.$$

An integral can, of course, be evaluated numerically just as a convergent series can be approximated by calculating a few terms. However, (9.13) can be integrated; a convenient way to do it is to recognize that it is a Laplace transform of $f(k) = [(1 - \cos k) \sin kx]/k$, where x is just a parameter and y corresponds to p and k to t. From $L19$ and $L20$

(9.14) $$u(x, y) = \frac{200}{\pi} \left[\arctan \frac{x}{y} - \frac{1}{2} \arctan \frac{x + 1}{y} - \frac{1}{2} \arctan \frac{x - 1}{y} \right].$$

This can also be written in polar coordinates as (Problem 1)

(9.15) $$u = \frac{100}{\pi} \left(\frac{\pi}{2} - \arctan \frac{r^2 - \cos 2\theta}{\sin 2\theta} \right).$$

▶ PROBLEMS, SECTION 9

1. Verify that (9.15) follows from (9.14). *Hint:* Use the formulas for $\tan(\alpha \pm \beta), \tan 2\alpha$, etc., to condense (9.14) and then change to polar coordinates. You may find

$$u = \frac{100}{\pi} \arctan \frac{\sin 2\theta}{r^2 - \cos 2\theta}.$$

Show that if you use principal values of the arc tangent, this formula does not give the correct boundary conditions on the x axis, whereas (9.15) does.

2. A metal plate covering the first quadrant has the edge which is along the y axis insulated and the edge which is along the x axis held at

$$u(x,0) = \begin{cases} 100(2-x), & \text{for } 0 < x < 2, \\ 0, & \text{for } x > 2. \end{cases}$$

Find the steady-state temperature distribution as a function of x and y. *Hint:* Follow the procedure of Example 2, but use a cosine transform (because $\partial u/\partial x = 0$ for $x = 0$). Leave your answer as an integral like (9.13).

3. Consider the heat flow problem of Section 3. Solve this by Laplace transforms (with respect to t) by starting as in Example 1. You should get

$$\frac{\partial^2 U}{\partial x^2} - \frac{p}{\alpha^2} U = -\frac{100}{\alpha^2 l} x \qquad \text{and } U(0,p) = U(l,p) = 0.$$

Solve this differential equation to get

$$U(x,p) = -\frac{100 \sinh (p^{1/2}/\alpha)x}{p \sinh (p^{1/2}/\alpha)l} + \frac{100}{pl} x.$$

Assume the following expansion, and find u by looking up the inverse Laplace transforms of the individual terms of U:

$$\frac{\sinh (p^{1/2}/\alpha)x}{p \sinh (p^{1/2}/\alpha)l} = \frac{x}{pl} - \frac{2}{\pi}\left[\frac{\sin(\pi x/l)}{p + (\pi^2\alpha^2/l^2)} - \frac{\sin(2\pi x/l)}{2[p + (4\pi^2\alpha^2/l^2)]} + \frac{\sin(3\pi x/l)}{3[p + (9\pi^2\alpha^2/l^2)]} \cdots\right].$$

Your answer should be (3.15).

4. A semi-infinite bar is initially at temperature $100°$ for $0 < x < 1$, and $0°$ for $x > 1$. Starting at $t = 0$, the end $x = 0$ is maintained at $0°$ and the sides are insulated. Find the temperature in the bar at time t, as follows. Separate variables in the heat flow equation and get elementary solutions $e^{-\alpha^2 k^2 t} \sin kx$ and $e^{-\alpha^2 k^2 t} \cos kx$. Discard the cosines since $u = 0$ at $x = 0$. Look for a solution

$$u(x,t) = \int_0^\infty B(k) e^{-\alpha^2 k^2 t} \sin kx \, dk.$$

and proceed as in Example 2. Leave your answer as an integral.

5. A long wire occupying the x axis is initially at rest. The end $x = 0$ is oscillated up and down so that

$$y(0,t) = 2 \sin 3t, \qquad t > 0.$$

Find the displacement $y(x,t)$. The initial and boundary conditions are $y(0,t) = 2\sin 3t$, $y(x,0) = 0$, $\partial y/\partial t|_{t=0} = 0$. Take Laplace transforms of these conditions and of the wave equation with respect to t as in Example 1. Solve the resulting differential equation to get

$$Y(x,p) = \frac{6e^{-(p/v)x}}{p^2 + 9}.$$

Use $L3$ and $L28$ to find

$$y(x,t) = \begin{cases} 2\sin 3\left(t - \frac{x}{v}\right), & x < vt, \\ 0, & x > vt. \end{cases}$$

6. Continue the problem of Example 2 in the following way: Instead of using the explicit form of $B(k)$ from (9.12), leave it as an integral and write (9.13) in the form

$$u(x,y) = \frac{200}{\pi} \int_0^\infty e^{-ky} \sin kx \, dk \int_0^1 \sin kt \, dt.$$

Change the order of integration and evaluate the integral with respect to k first. (*Hint:* Write the product of sines as a difference of cosines.) Now do the t integration and get (9.14).

7. Continue with Problem 4 as in Problem 6.

10. MISCELLANEOUS PROBLEMS

1. Find the steady-state temperature distribution in a rectangular plate covering the area $0 < x < 1$, $0 < y < 2$, if $T = 0$ for $x = 0$, $x = 1$, $y = 2$, and $T = 1 - x$ for $y = 0$.

2. Solve Problem 1 if $T = 0$ for $x = 0$, $x = 1$, $y = 0$, and $T = 1 - x$ for $y = 2$. *Hint:* Use $\sinh ky$ as the y solution; then $T = 0$ when $y = 0$ as required.

3. Solve Problem 1 if the sides $x = 0$ and $x = 1$ are insulated (see Problems 2.14 and 2.15), and $T = 0$ for $y = 2$, $T = 1 - x$ for $y = 0$.

4. Find the steady-state temperature distribution in a plate with the boundary temperatures $T = 30°$ for $x = 0$ and $y = 3$; $T = 20°$ for $y = 0$ and $x = 5$. *Hint:* Subtract $20°$ from all temperatures and solve the problem; then add $20°$. (Also see Problem 2.)

5. A bar of length l is initially at $0°$. From $t = 0$ on, the ends are held at $20°$. Find $u(x,t)$ for $t > 0$.

6. Do Problem 5 if the $x = 0$ end is insulated and the $x = l$ end held at $20°$ for $t > 0$. (See Problem 3.9.)

7. Solve Problem 2 if the sides $x = 0$ and $x = 1$ are insulated.

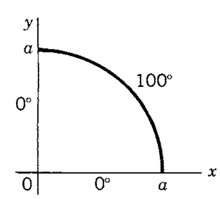
8. A slab of thickness 10 cm has its two faces at $10°$ and $20°$. At $t = 0$, the face temperatures are interchanged. Find $u(x,t)$ for $t > 0$.

9. A string of length l has initial displacement $y_0 = x(l - x)$. Find the displacement as a function of x and t.

10. Solve Problem 5.7 if half the curved surface of the cylinder is held at $100°$ and the other half at $-100°$ with the ends at $0°$.

11. The series in Problem 5.12 can be summed (see Problem 2.6). Show that

$$u = 50 + \frac{100}{\pi} \arctan \frac{2ar\sin\theta}{a^2 - r^2}.$$



12. A plate in the shape of a quarter circle has boundary temperatures as shown. Find the interior steady-state temperature $u(r,\theta)$. (See Problem 5.12.)

13. Sum the series in Problem 12 to get

$$u = \frac{200}{\pi} \arctan \frac{2a^2r^2 \sin 2\theta}{a^4 - r^4}.$$

Hint: See Problem 2.6.

14. A long cylinder has been cut into quarter cylinders which are insulated from each other; alternate quarter cylinders are held at potentials $+100$ and -100. Find the electrostatic potential inside the cylinder. *Hints:* Do you see a relation to Problem 12 above? Also see Problem 5.12.

15. Repeat Problems 12 and 13 for a plate in the shape of a circular sector of angle 30° and radius 10 if the boundary temperatures are 0° on the straight sides and 100° on the circular arc. Can you then state and solve a problem like 14?

16. Consider the normal modes of vibration for a square membrane of side π (see Problem 6.3). Sketch the $2, 1$ and $1, 2$ modes. Show that the line $y = x$ is a nodal line for the combination $\sin x \sin 2y - \sin 2x \sin y$ of these two modes. Thus find a vibration frequency of a membrane in the shape of a 45° right triangle.

17. Sketch some of the normal modes of vibration for a semicircular drumhead and find the characteristic vibration frequencies as multiples of the fundamental for the corresponding circular drumhead.

18. Repeat Problem 17 for a membrane in the shape of a circular sector of angle 60°.

19. A long conducting cylinder is placed parallel to the z axis in an originally uniform electric field in the negative x direction. The cylinder is held at zero potential. Find the potential in the region outside the cylinder. *Hints:* See Problem 7.13. You want solutions of Laplace's equation in polar coordinates (Problem 5.12).

20. Use Problem 7.16 to find the characteristic vibration frequencies of sound in a spherical cavity.

21. The surface temperature of a sphere of radius 1 is held at $u = \sin^2 \theta + \cos^3 \theta$. Find the interior temperature $u(r, \theta, \phi)$.

22. Find the interior temperature in a hemisphere if the curved surface is held at $u = \cos \theta$ and the equatorial plane at $u = 1$.

23. Find the steady-state temperature in the region between two spheres $r = 1$ and $r = 2$ if the surface of the outer sphere has its upper half held at 100° and its lower half at -100° and these temperatures are reversed for the inner sphere. *Hint:* See Problem 7.14. Here you will need to find two Legendre series (when $r = 1$ and when $r = 2$) and solve for a_l and b_l.

24. Find the general solution for the steady-state temperature in Figure 2.2 if the boundary temperatures are the constants $T = A$, $T = B$, etc., on the four sides, and the rectangle covers the area $0 < x < a$, $0 < y < b$. *Hints:* You can subtract, say, A from all four temperatures, solve the problem, and then add A back again. Thus a solution with one side at $T = 0$ and the other three at given temperatures solves the general problem. You have previously solved problems (Section 2) with temperatures C and D given. For B, see Problem 2.

25. The Klein-Gordon equation is $\nabla^2 u = (1/v^2)\partial^2 u/\partial t^2 + \lambda^2 u$. This equation is of interest in quantum mechanics, but it also has a simpler application. It describes, for example, the vibration of a stretched string which is embedded in an elastic medium. Separate the one-dimensional Klein-Gordon equation and find the characteristic frequencies of such a string.

Answer: $\nu_n = \dfrac{v}{2}\sqrt{(n/l)^2 + (\lambda/\pi)^2}$.

26. Find the characteristic frequencies of a circular membrane which satisfies the Klein-Gordon equation (Problem 25). *Hint:* Separate the equation in two dimensions in polar coordinates.

27. Do Problem 26 for a rectangular membrane.

28. Find the steady-state temperature in a semi-infinite plate covering the region $x > 0$, $0 \le y \le 1$, if the edges along the x axis and y axis are insulated (see Problem 2.14) and the top edge is held at

$$u(x, 1) = \begin{cases} 100°, & 0 < x < 1, \\ 0°, & x > 1. \end{cases}$$

Hint: Look for a solution as a Fourier integral. Leave your answer as an integral (just as we usually give answers as series.)

CHAPTER 14

Functions of a Complex Variable

1. INTRODUCTION

In Chapter 2 we discussed plotting complex numbers $z = x + iy$ in the complex plane (see Figure 1.1) and finding values of the elementary functions of z such as roots, trigonometric functions, logarithms, etc. Now we want to discuss the calculus of functions of z, differentiation, integration, power series, etc. As you know from such topics as differential equations, Fourier series and integrals, mechanics, electricity, etc., it is often very convenient to use complex expressions. The basic facts and theorems about functions of a complex variable not only simplify many calculations but often lead to a better understanding of a problem and consequently to a more efficient method of solution. We are going to state some of the basic definitions and theorems of the subject (omitting the longer proofs), and show some of their uses.

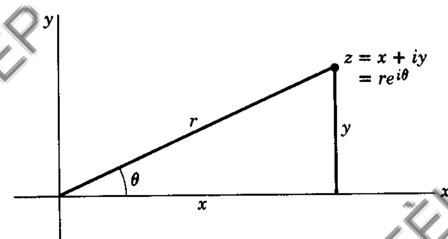


Figure 1.1

As in Chapter 2, the value of a function of z for a given z is a complex number.

Example. Consider a simple function of z, namely $f(z) = z^2$. We may write

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = u(x, y) + iv(x, y),$$

where $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

In Chapter 2, we observed that a complex number $z = x+iy$ is equivalent to a pair of real numbers x, y. Here we see that a function of z is equivalent to a pair of real functions, $u(x, y)$ and $v(x, y)$, of the real variables x and y. In general, we write

$$(1.1) \qquad f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where it is understood that u and v are real functions of the real variables x and y.

Recall that functions are customarily *single-valued*, that is, $f(z)$ has just one (complex) value for each z. Does this mean that we cannot define a function by a formula such as $\ln z$ or $\arctan z$? By Chapter 2, we have

$$\ln z = \ln|z| + i(\theta + 2n\pi),$$

where $\tan\theta = y/x$. For each z, $\ln z$ has an infinite set of values. But if θ is allowed a range of only 2π , then $\ln z$ has one value for each z and this single-valued function is called a *branch* of $\ln z$. Thus in using formulas such as $\sqrt{z}$, $\ln z$, $\arctan z$, to define functions, we always discuss a single branch at a time so that we have a single-valued function. (As a matter of terminology, however, you should know that the whole collection of branches is sometimes called a "multiple-valued function.")

PROBLEMS, SECTION 1

Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the following functions.

1. z^3 **2.** z **3.** $\bar{z}$

4. $|z|$ **5.** $\operatorname{Re} z$ **6.** e^z

7. $\cosh z$ **8.** $\sin z$ **9.** $\dfrac{1}{z}$

10. $\dfrac{2z+3}{z+2}$ **11.** $\dfrac{2z-i}{iz+2}$ **12.** $\dfrac{z}{z^2+1}$

13. $\ln|z|$ **14.** $z^2\bar{z}$ **15.** $\overline{e^z}$

16. $z^2 - \bar{z}^2$ **17.** $\cos\bar{z}$ **18.** $\sqrt{z}$

19. $\ln z$ (Use $0 < \theta < 2\pi$.) **20.** $(1+2i)z^2 + (i-1)z + 3$

21. e^{iz} (*Careful:* $\cos z$ and $\sin z$ are *not* u and v.)

2. ANALYTIC FUNCTIONS

Definition The derivative of $f(z)$ is defined (just as it is for a function of a real variable) by the equation

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}, \tag{2.1}$$

where $\Delta f = f(z + \Delta z) - f(z)$ and $\Delta z = \Delta x + i\Delta y$.

Definition: A function $f(z)$ is *analytic* (or *regular* or *holomorphic* or *monogenic*) in a region* of the complex plane if it has a (unique) derivative at every point of the region. The statement "$f(z)$ is analytic *at a point* $z = a$" means that $f(z)$ has a derivative at every point inside some small circle about $z = a$.

*Isolated points and curves are not regions; a region must be two-dimensional.

Let us consider what it means for $f(z)$ to have a derivative. First think about a function $f(x)$ of a real variable x; it is possible for the limit of $\Delta f/\Delta x$ to have two values at a point x_0, as shown in Figure 2.1—one value when we approach x_0 from the left and a different value when we approach x_0 from the right. When we say that $f(x)$ has a derivative at $x = x_0$, we mean that these two values are equal. However, for a function $f(z)$ of a complex variable z, there are an infinite number of ways we can approach a point z_0; a few ways are shown in Figure 2.2. When we say that $f(z)$ has a derivative at $z = z_0$, we mean that $f'(z)$ [as defined by (2.1)] has the same value no matter how we approach z_0. This is an amazingly stringent requirement and we might well wonder whether there *are* any analytic functions. On the other hand, it is hard to imagine making any progress in calculus unless we can find derivatives!

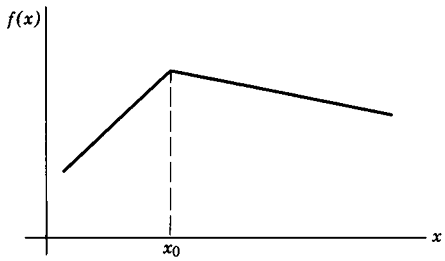


Figure 2.1

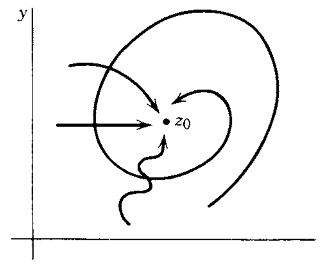


Figure 2.2

Let us immediately reassure ourselves that there *are* analytic functions by using the definition (2.1) to find the derivatives of some simple functions.

▶ **Example 1.** Show that $(d/dz)(z^2) = 2z$. By (2.1) we have

$$\begin{aligned}\frac{d}{dz}(z^2) &= \lim_{\Delta z\to 0}\frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0}\frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z}\\ &= \lim_{\Delta z\to 0}(2z+\Delta z) = 2z.\end{aligned}$$

We see that the result is independent of *how* Δz tends to zero; thus z^2 is an analytic function. By the same method it follows that $(d/dz)(z^n) = nz^{n-1}$ if n is a positive integer (Problem 30).

Observe that the definition (2.1) of a derivative is of exactly the same form as the corresponding definition for a function of a real variable. Because of this similarity, many familiar formulas can be proved by the same methods used in the real case, as we have just discovered in differentiating z^2. You can easily show (Problems 25 to 28) that derivatives of sums, products, and quotients follow the familiar rules and that the chain rule holds [if $f = f(g)$ and $g = g(z)$, then $df/dz = (df/dg)(dg/dz)$]. Then derivatives of rational functions of z follow the familiar real-variable formulas. If we assume the definitions and theorems of Chapters 1 and 2, we can see that the derivatives of the other elementary functions also follow the familiar formulas; for example, $(d/dz)(\sin z) = \cos z$, etc. (Problems 29 to 33).

Now you may be wondering what is new here since all our results so far seem to be just the same as for functions of a real variable. The reason for this is that we have been discussing only functions $f(z)$ that *have* derivatives. Comparing Figures 2.1 and 2.2, we pointed out the essential difference between finding $(d/dx)f(x)$

and finding $(d/dz)f(z)$, namely that there are an infinite number of ways we can approach z_0 in Figure 2.2.

Example 2. Find $(d/dz)(|z|^2)$. Note that $|x|^2 = x^2$, and its derivative is $2x$. If $|z|^2$ has a derivative, it is given by (2.1), that is, by

$$\lim_{\Delta z\to 0} = \lim_{\Delta z\to 0}\frac{|z+\Delta z|^2 - |z|^2}{\Delta z}.$$

The numerator of this fraction is always real (because absolute values are real—recall $|z| = \sqrt{x^2+y^2} = r$.) Consider the denominator $\Delta z = \Delta x + i\Delta y$. As we approach z_0 in Figure 2.2 (that is, let $\Delta z \to 0$), Δz has different values depending on our method of approach. For example, if we come in along a horizontal line, then $\Delta y = 0$ and $\Delta z = \Delta x$; along a vertical line $\Delta x = 0$ so $\Delta z = i\Delta y$, and along other directions Δz is some complex number; in general, Δz is neither real nor pure imaginary. Since the numerator of $\Delta f/\Delta z$ is real and the denominator may be real or imaginary (in general, complex), we see that $\lim_{\Delta z\to 0}\Delta f/\Delta z$ has different values for different directions of approach to z_0, that is, $|z|^2$ is not analytic.

Now we have seen examples of both analytic and nonanalytic functions, but we still do not know how to tell whether a function has a derivative [except to appeal to (2.1)]. The following theorems answer this question.

Theorem I (which we shall prove). If $f(z) = u(x,y) + iv(x,y)$ is analytic in a region, then in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.2}$$

These equations are called the *Cauchy-Riemann conditions.*

Proof. Remembering that $f = f(z)$, where $z = x + iy$, we find by the rules of partial differentiation (see Problem 28 and also Chapter 4)

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}\cdot 1,\\ \frac{\partial f}{\partial y} &= \frac{df}{dz}\frac{\partial z}{\partial y} = \frac{df}{dz}\cdot i.\end{aligned} \tag{2.3}$$

Since $f = u(x,y) + iv(x,y)$ by (1.1), we also have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \quad\text{and}\quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}. \tag{2.4}$$

Notice that if f has a derivative with respect to z, then it also has partial derivatives with respect to x and y by (2.3). Since a complex function has a derivative with respect to a real variable if and only if its real and imaginary parts do [see (1.1)], then by (2.4) u and v also have partial derivatives with respect to x and y. Combining (2.3) and (2.4) we have

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad\text{and}\quad \frac{df}{dz} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Since we assumed that df/dz exists and is unique (this is what analytic means), these two expressions for df/dz must be equal. Taking real and imaginary parts, we get the Cauchy-Riemann equations (2.2).

Theorem II (which we state without proof). If $u(x, y)$ and $v(x, y)$ and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann conditions in a region, then $f(z)$ is analytic at all points inside the region (not necessarily on the boundary).

Although we shall not prove this (see texts on complex variables), we can make it plausible by showing that it is true when we approach z_0 along any straight line.

▶ **Example 3.** Find df/dz assuming that we approach z_0 along a straight line of slope m, and show that df/dz does not depend on m if u and v satisfy (2.2). The equation of the straight line of slope m through the point $z_0 = x_0 + iy_0$ is

$$y - y_0 = m(x - x_0)$$

and along this line we have $dy/dx = m$. Then we find

$$\begin{aligned}\frac{df}{dz} = \frac{du + i\,dv}{dx + i\,dy} &= \frac{\dfrac{\partial u}{\partial x}\,dx + \dfrac{\partial u}{\partial y}\,dy + i\left(\dfrac{\partial v}{\partial x}\,dx + \dfrac{\partial v}{\partial y}\,dy\right)}{dx + i\,dy} \\ &= \frac{\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\,m + i\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial y}\,m\right)}{1 + im}.\end{aligned}$$

Using the Cauchy-Riemann equations (2.2), we get

$$\begin{aligned}\frac{df}{dz} &= \frac{\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial x}\,m + i\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\,m\right)}{1 + im} \\ &= \frac{\dfrac{\partial u}{\partial x}(1 + im) + i\,\dfrac{\partial v}{\partial x}(1 + im)}{1 + im} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}.\end{aligned}$$

Thus df/dz has the same value for approach along *any* straight line. The theorem states that it also has the same value for approach along *any curve*.

Some definitions:

A *regular point* of $f(z)$ is a point at which $f(z)$ is analytic.

A *singular point* or *singularity* of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an *isolated* singular point if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

Theorem III (which we state without proof). If $f(z)$ is analytic in a region (R in Figure 2.3), then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges *inside* the circle about z_0 that extends to the nearest singular point (C in Figure 2.3).

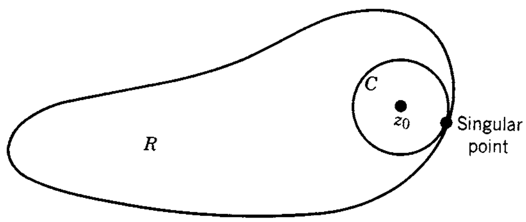


Figure 2.3

Notice again what a strong condition it is on $f(z)$ to say that it has a derivative. It is quite possible for a function of a real variable $f(x)$ to have a first derivative but not higher derivatives. But if $f(z)$ has a first derivative with respect to z, then it has derivatives of all orders, and all these derivatives are analytic functions.

This theorem also explains a fact about power series which may have puzzled you. The function $f(x) = 1/(1 + x^2)$ does not have anything peculiar about its behavior at $x = \pm 1$. Yet if we expand it in a power series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \tag{2.5}$$

we see that the series converges only for $|x| < 1$. We can see why this happens if we consider instead

$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots . \tag{2.6}$$

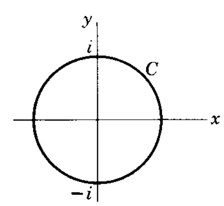


Figure 2.4

When $z = \pm i$, $f(z)$ and its derivatives become infinite; that is, $f(z)$ is not analytic in any region containing $z = \pm i$. The point z_0 of the theorem is the origin and the circle C (bounding the disk of convergence of the series) passes through the nearest singular points $\pm i$ (Figure 2.4). Since a power series in z always converges inside its disk of convergence and diverges outside (Chapter 2, Problem 6.14), we see that (2.5) [which is (2.6) for $y = 0$] converges for $|x| < 1$ and diverges for $|x| > 1$. This simple example shows an important reason for studying functions of a complex variable; our study of $f(z)$ gives us insights about the corresponding $f(x)$. Formulas involving not only the elementary functions but also Γ functions, Bessel functions, and many others are more easily derived and understood by considering them as functions of z.

A function $\phi(x, y)$ which satisfies Laplace's equation in two dimensions, namely, $\nabla^2\phi = \partial^2\phi/dx^2 + \partial^2\phi/\partial y^2$, is called a *harmonic* function. A great many physical problems lead to Laplace's equation, and consequently we are very much interested in finding solutions of it. (See Section 10 and Chapter 13.) The following theorem should then give you a clue as to one reason why the theory of functions of a complex variable is important in applications.

Theorem IV. Part 1 (to be proved in Problem 44). If $f(z) = u + iv$ is analytic in a region, then u and v satisfy Laplace's equation in the region (that is, u and v are harmonic functions).

Part 2 (which we state without proof). Any function u (or v) satisfying Laplace's equation in a simply-connected region, is the real or imaginary part of an analytic function $f(z)$.

Thus we can find solutions of Laplace's equation simply by taking the real or imaginary parts of an analytic function of z. It is also often possible, starting with a simple function which satisfies Laplace's equation, to find the explicit function $f(z)$ of which it is, say, the real part.

▶ **Example 4.** Consider the function $u(x, y) = x^2 - y^2$. We find that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0,$$

that is, u satisfies Laplace's equation (or u is a harmonic function). Let us find the function $v(x, y)$ such that $u + iv$ is an analytic function of z. By the Cauchy-Riemann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x.$$

Integrating partially with respect to y, we get

$$v(x, y) = 2xy + g(x),$$

where $g(x)$ is a function of x to be found. Differentiating partially with respect to x and again using the Cauchy-Riemann equations, we have

$$\frac{\partial v}{\partial x} = 2y + g'(x) = -\frac{\partial u}{\partial y} = 2y.$$

Thus we find

$$g'(x) = 0, \qquad \text{or} \qquad g = \text{const.}$$

Then

$$f(z) = u + iv = x^2 - y^2 + 2ixy + \text{const.} = z^2 + \text{const.}$$

The pair of functions u, v are called *conjugate harmonic functions*. (Also see Problem 64.)

▶ PROBLEMS, SECTION 2

1 to **21**. Use the Cauchy-Riemann conditions to find out whether the functions in Problems 1.1 to 1.21 are analytic. Similarly, find out whether the following functions are analytic.

22. $y + ix$ **23.** $\dfrac{x - iy}{x^2 + y^2}$ **24.** $\dfrac{y - ix}{x^2 + y^2}$

Using the definition (2.1) of $(d/dz)f(z)$, show that the following familiar formulas hold. *Hint:* Use the same methods as for functions of a real variable.

25. $\dfrac{d}{dz}[Af(z)+Bg(z)] = A\dfrac{df}{dz} + B\dfrac{dg}{dz}$.

26. $\dfrac{d}{dz}[f(z)g(z)] = f(z)\dfrac{dg}{dz} + g(z)\dfrac{df}{dz}$.

27. $\dfrac{d}{dz}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{gf' - fg'}{g^2}, \quad g(z) \neq 0.$

28. $\dfrac{d}{dz}f[g(z)] = \dfrac{df}{dg}\dfrac{dg}{dz}$. (See hint below.)

Problem 28 is the chain rule for the derivative of a function of a function. *Hint:* Assume that df/dg and dg/dz exist, and write equations like (3.5) of Chapter 4 for Δf and Δg; substitute Δg into Δf, divide by Δz, and take limits.

29. $\dfrac{d}{dz}(z^3) = 3z^2$.

30. $\dfrac{d}{dz}(z^n) = nz^{n-1}$.

31. $\dfrac{d}{dz}\ln z = \dfrac{1}{z}, \quad z \neq 0.$ *Hint:* Expand $\ln\left(1+\dfrac{\Delta z}{z}\right)$ in series.

32. Using the definition of e^z by its power series [(8.1) of Chapter 2], and the theorem (Chapters 1 and 2) that power series may be differentiated term by term (within the disk of convergence), and the result of Problem 30, show that $(d/dz)(e^z) = e^z$.

33. Using the definitions of $\sin z$ and $\cos z$ [Chapter 2, equation (11.4)], find their derivatives. Then using Problem 27, find $(d/dz)(\cot z)$, $z \neq n\pi$.

Using series you know from Chapter 1, write the power series (about the origin) of the following functions. Use Theorem III to find the disk of convergence of each series. What you are looking for is the point (anywhere in the complex plane) nearest the origin, at which the function does not have a derivative. Then the disk of convergence has center at the origin and extends to that point. The series converges *inside* the disk.

34. $\ln(1-z)$

35. $\cos z$

36. $\sqrt{1+z^2}$

37. $\tanh z$

38. $\dfrac{1}{2i+z}$

39. $\dfrac{z}{z^2+9}$

40. $(1-z)^{-1}$

41. e^{iz}

42. $\sinh z$

43. In Chapter 12, equations (5.1) and (5.2), we expanded the function $\phi(x,h)$ in a series of powers of h. Use Theorem III (see instructions for Problems 34 to 42 above) to show that the series for $\phi(x,h)$ converges for $|h| < 1$ and $-1 \le x \le 1$. Here h is the variable and x is a parameter; you should find the (complex) value of h which makes Φ infinite, and show that the absolute value of this complex number is 1 (independent of x when $x^2 \le 1$). This proves that the series for real h converges for $|h| < 1$.

44. Prove Theorem IV, Part 1. *Hint:* Recall the equality of the second cross partial derivatives; see Chapter 4, end of Section1.

45. Let $f(z) = u + iv$ be an analytic function, and let $\mathbf{F}$ be the vector $\mathbf{F} = v\mathbf{i} + u\mathbf{j}$. Show that the equations $\operatorname{div}\mathbf{F} = 0$ and $\operatorname{curl}\mathbf{F} = 0$ are equivalent to the Cauchy-Riemann equations.

46. Find the Cauchy-Riemann equations in polar coordinates. *Hint*: Write $z = re^{i\theta}$ and $f(z) = u(r,\theta) + iv(r,\theta)$. Follow the method of equations (2.3) and (2.4).

47. Using your results in Problem 46 and the method of Problem 44, show that u and v satisfy Laplace's equation in polar coordinates (see Chapter 10, Section 9) if $f(z) = u + iv$ is analytic.

Using polar coordinates (Problem 46), find out whether the following functions satisfy the Cauchy-Riemann equations.

48. $\sqrt{z}$ **49.** $|z|$ **50.** $\ln z$

51. z^n **52.** $|z|^2$ **53.** $|z|^{1/2}e^{i\theta/2}$

Show that the following functions are harmonic, that is, that they satisfy Laplace's equation, and find for each a function $f(z)$ of which the given function is the real part. Show that the function $v(x, y)$ (which you find) also satisfies Laplace's equation.

54. y **55.** $3x^2y - y^3$ **56.** xy **57.** $x + y$

58. $\cosh y \cos x$ **59.** $e^x \cos y$ **60.** $\ln(x^2 + y^2)$

61. $\dfrac{x}{x^2 + y^2}$ **62.** $e^{-y} \sin x$ **63.** $\dfrac{y}{(1 - x)^2 + y^2}$

64. It can be shown that, if $u(x, y)$ is a harmonic function which is defined at $z_0 = x_0 + iy_0$, then an analytic function of which $u(x, y)$ is the real part is given by

$$f(z) = 2u\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) + \text{const.}$$

[See Struble, Quart. Appl. Math., 37 (1979), 79-81.] Use this formula to find $f(z)$ in Problems 54 to 63. *Hint:* If $u(0, 0)$ is defined, take $z_0 = 0$.

3. CONTOUR INTEGRALS

Theorem V Cauchy's theorem (see discussion below). Let C be a simple† closed curve with a continuously turning tangent except possibly at a finite number of points (that is, we allow a finite number of corners, but otherwise the curve must be "smooth"). If $f(z)$ is analytic on and inside C, then

$$\oint_{\text{around } C} f(z)\, dz = 0. \tag{3.1}$$

(This is a line integral as in vector analysis; it is called a *contour integral* in the theory of complex variables.)

Proof. We shall prove Cauchy's theorem assuming that $f'(z)$ is continuous. (With more effort it is possible to prove it without this assumption, and then show that if $f'(z)$ exists in a region, it is, in fact, continuous there. See also Theorem III which we stated without proof; it is usually proved using the results of Cauchy's theorem.)

$$\begin{aligned} \oint_C f(z)\, dz &= \oint_C (u + iv)(dx + i\, dy) \\ &= \oint_C (u\, dx - v\, dy) + i \oint_C (v\, dx + u\, dy). \end{aligned} \tag{3.2}$$

†A simple curve is one which does not cross itself.

Green's theorem in the plane (Chapter 6, Section 9) says that if $P(x,y)$, $Q(x,y)$, and their partial derivatives are continuous in a simply-connected region R, then

$$\oint_C P\,dx + Q\,dy = \iint\limits_{\substack{\text{area}\\ \text{inside } C}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy, \tag{3.3}$$

where C is a simple closed curve lying entirely in R. The curve C is traversed in a direction so that the area inclosed is always to the left; the area integral is over the area inside C. Applying (3.3) to the first integral in (3.2), we get

$$\oint_C (u\,dx - v\,dy) = \iint\limits_{\substack{\text{area}\\ \text{inside } C}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy. \tag{3.4}$$

Since we are assuming that $f'(z)$ is continuous, then u and v and their derivatives are continuous; by the Cauchy-Riemann equations the integrand on the right of (3.4) is zero at every point of the area of integration, so the integral is equal to zero. In the same way the second integral in (3.2) is zero; thus (3.1) is proved.

Theorem VI Cauchy's integral formula (which we shall prove). If $f(z)$ is analytic on and inside a simple closed curve C, the value of $f(z)$ at a point $z = a$ inside C is given by the following contour integral along C:

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz.$$

Proof. Let a be a fixed point inside the simple closed curve C and consider the function

$$\phi(z) = \frac{f(z)}{z-a}, \tag{3.5}$$

where $f(z)$ is analytic on and inside C. Let C' be a small circle (inside C) with center at a and radius ρ. Make a cut between C and C' along AB (Figure 3.1); two cuts are shown to make the picture clear, but later we shall make them coincide. We are now going to integrate along the path shown in Figure 3.1 (in the direction shown by the arrows) from A, around C, to B, around C', and back to A. Notice that the area between the curves C and C' is always to the left of the path of integration and is inclosed by it. In this area between C and C', the function $\phi(z)$ is analytic; we have cut out a small disk about the point $z = a$ at which $\phi(z)$ is not analytic. Cauchy's theorem then applies to the integral along the combined path consisting of C counterclockwise, C' clockwise, and the two cuts. The two integrals, in opposite directions along the cuts, cancel when the cuts are made to coincide. Thus we have

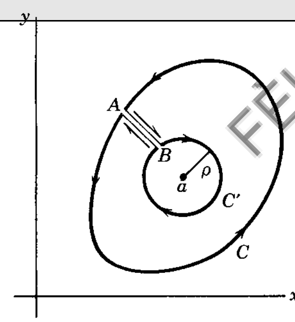


Figure 3.1

$$\oint\limits_{\substack{C\text{ counter-}\\ \text{clockwise}}} \phi(z)\,dz + \oint\limits_{C'\text{ clockwise}} \phi(z)\,dz = 0 \quad \text{or}$$

$$\oint_C \phi(z)\,dz = \oint_{C'} \phi(z)\,dz \qquad \text{where both are counterclockwise.} \tag{3.6}$$

Along the circle C', $z = a + \rho e^{i\theta}$, $dz = \rho i e^{i\theta}\, d\theta$, and (3.6) becomes

$$\oint_C \phi(z)\, dz = \oint_{C'} \phi(z)\, dz = \oint_{C'} \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}} \rho i e^{i\theta}\, d\theta = \int_0^{2\pi} f(z) i \, d\theta. \tag{3.7}$$

Since our calculation is valid for any (sufficiently small) value of ρ, we shall let $\rho \to 0$ (that is, $z \to a$) to simplify the formula. Because $f(z)$ is continuous at $z = a$ (it is analytic inside C), $\lim_{z\to a} f(z) = f(a)$. Then (3.7) becomes

$$\oint_C \phi(z)\, dz = \oint_C \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} f(z) i \, d\theta = \int_0^{2\pi} f(a) i\, d\theta = 2\pi i f(a) \tag{3.8}$$

or

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz, \qquad a \text{ inside } C. \tag{3.9}$$

This is Cauchy's integral formula. Note carefully that the point a is inside C; if a were outside C, then $\phi(z)$ would be analytic everywhere inside C and the integral would be zero by Cauchy's theorem. A useful way to look at (3.9) is this: If the values of $f(z)$ are given on the boundary of a region (curve C), then (3.9) gives the value of $f(z)$ at any point a inside C. With this interpretation you will find Cauchy's integral formula written with a replaced by z, and z replaced by some different dummy integration variable, say w:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw, \qquad z \text{ inside } C. \tag{3.10}$$

For some important uses of this theorem, see Problems 11.3 and 11.36 to 11.38.

▶ PROBLEMS, SECTION 3

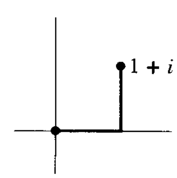
Evaluate the following line integrals in the complex plane by direct integration, that is, as in Chapter 6, Section 8, *not* using theorems from this chapter. (If you see that a theorem applies, use it to check your result.)

1. $\int_i^{i+1} z\, dz$ along a straight line parallel to the x axis.

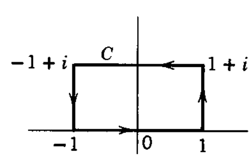
2. $\int_0^{1+i} (z^2 - z)\, dz$

(a) along the line $y = x$;

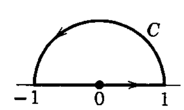
(b) along the indicated broken line.



3. $\oint_C z^2\, dz$ along the indicated paths:



(a)

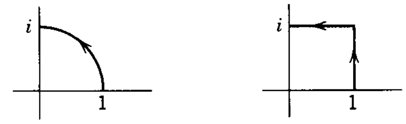


(b)

4. $\int dz/(1-z^2)$ along the whole positive imaginary axis, that is, the y axis; this is frequently written as $\int_0^{i\infty} dz/(1-z^2)$.

5. $\int e^{-z}$ along the positive part of the line $y = \pi$; this is frequently written as $\int_{i\pi}^{\infty+i\pi} e^{-z}\,dz$.

6. $\int_1^i z\,dz$ along the indicated paths:

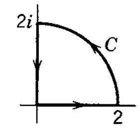


(a) (b)

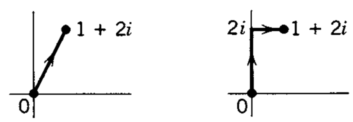
7. $\displaystyle\int \frac{dz}{8i+z^2}$ along the line $y = x$ from 0 to ∞.

8. $\displaystyle\int_{2\pi}^{2\pi+i\infty} e^{2iz}\,dz$ 9. $\displaystyle\int_{1+2i}^{\infty+2i} \frac{dz}{(x-2i)^2}$ 10. $\displaystyle\int_2^{2+i\infty} ze^{iz}\,dz$

11. Evaluate $\oint_C (\bar{z}-3)\,dz$ where C is the indicated closed curve along the first quadrant part of the circle $|z| = 2$, and the indicated parts of the x and y axes. *Hint:* Don't try to use Cauchy's theorem! (Why not? *Further hint:* See Problem 2.3.)



12. $\int_0^{1+2i} |z|^2\,dz$ along the indicated paths:



(a) (b)

13. In Chapter 6, Section 11, we showed that a necessary condition for $\int_a^b \mathbf{F}\cdot d\mathbf{r}$ to be independent of the path of integration, that is, for $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around a simple closed curve C to be zero, was curl $\mathbf{F} = 0$, or in two dimensions, $\partial F_y/\partial x = \partial F_x/\partial y$. By considering (3.2), show that the corresponding condition for $\oint_C f(z)\,dz$ to be zero is that the Cauchy-Riemann conditions hold.

14. In finding complex Fourier series in Chapter 7, we showed that

$$\int_0^{2\pi} e^{inx}e^{-imx}\,dx = 0, \qquad n \neq m.$$

Show this by applying Cauchy's theorem to

$$\oint_C z^{n-m-1}\,dz, \qquad n > m,$$

where C is the circle $|z| = 1$. (Note that although we take $n > m$ to make z^{n-m-1} analytic at $z = 0$, an identical proof using z^{m-n-1} with $n < m$ completes the proof for all $n \neq m$.)

15. If $f(z)$ is analytic on and inside the circle $|z| = 1$, show that $\int_0^{2\pi} e^{i\theta} f(e^{i\theta})\,d\theta = 0$.

16. If $f(z)$ is analytic in the disk $|z| \le 2$, evaluate $\int_0^{2\pi} e^{2i\theta} f(e^{i\theta})\,d\theta$.

Use Cauchy's theorem or integral formula to evaluate the integrals in Problems 17 to 20.

17. $\displaystyle\oint_C \frac{\sin z\,dz}{2z-\pi}$ where C is the circle (a) $|z| = 1$, (b) $|z| = 2$.

18. $\displaystyle\oint_C \frac{\sin 2z\,dz}{6z-\pi}$ where C is the circle $|z| = 3$.

19. $\displaystyle\oint \frac{e^{3z}\,dz}{z-\ln 2}$ if C is the square with vertices $\pm 1 \pm i$.

20. $\displaystyle\oint_C \frac{\cosh z\,dz}{2\ln 2 - z}$ if C is the circle (a) $|z| = 1$, (b) $|z| = 2$.

21. Differentiate Cauchy's formula (3.9) or (3.10) to get

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z)^2} \qquad \text{or} \quad f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2}.$$

By differentiating n times, obtain

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z)^{n+1}} \qquad \text{or} \quad f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}.$$

Use Problem 21 to evaluate the following integrals.

22. $\displaystyle\oint_C \frac{\sin 2z\,dz}{(6z-\pi)^3}$ where C is the circle $|z| = 3$.

23. $\displaystyle\oint_C \frac{e^{3z}\,dz}{(z-\ln 2)^4}$ where C is the square in Problem 19.

24. $\displaystyle\oint_C \frac{\cosh z\,dz}{(2\ln 2 - z)^5}$ where C is the circle $|z| = 2$.

4. LAURENT SERIES

Theorem VII Laurent's theorem [equation (4.1)] (which we shall state without proof). Let C_1 and C_2 be two circles with center at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \tag{4.1}$$

convergent in R. Such a series is called a *Laurent series*. The "b" series in (4.1) is called the *principal part* of the Laurent series.

▶ **Example 1.** Consider the Laurent series

$$f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots + \left(\frac{z}{2}\right)^n + \cdots + \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \cdots + \frac{(-1)^n}{z^n} + \cdots\right). \tag{4.2}$$

Let us see where this series converges. First consider the series of positive powers; by the ratio test (see Chapters 1 and 2), this series converges for $|z/2| < 1$, that is, for $|z| < 2$. Similarly, the series of negative powers converges for $|1/z| < 1$, that is, $|z| > 1$. Then both series converge (and so the Laurent series converges) for $|z|$ between 1 and 2, that is, in a ring between two circles of radii 1 and 2.

We expect this result in general. The "a" series is a power series, and a power series converges *inside* some circle (say C_2 in Figure 4.1). The "b" series is a series of inverse powers of z, and so converges for $|1/z| <$ some constant; thus the "b" series converges *outside* some circle (say C_1 in Figure 4.1). Then a Laurent series converges between two circles (if it converges at all). (Note that the inner circle may be a point and the outer circle may have infinite radius).

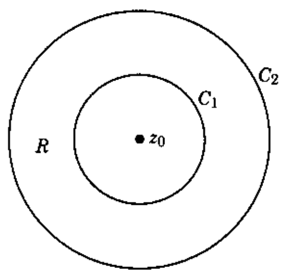


Figure 4.1

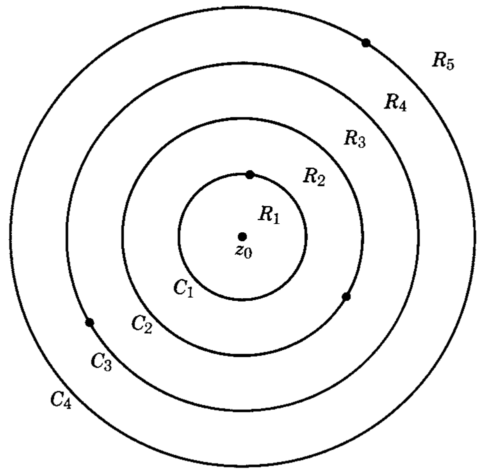


Figure 4.2

The formulas for the coefficients in (4.1) are (Problem 5.2)

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}, \qquad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}, \tag{4.3}$$

where C is any simple closed curve surrounding z_0 and lying in R. However, this is not usually the easiest way to find a Laurent series. Like power series about a point, the Laurent series (about z_0) for a function in a given annular ring (about z_0) where the function is analytic, is unique, and we can find it by any method we choose. (See examples below.) *Warning:* If $f(z)$ has several isolated singularities (Figure 4.2), there are several annular rings, R_1, R_2, $\cdots$, in which $f(z)$ is analytic; then there are several different Laurent series for $f(z)$, one for each ring. The Laurent series which we usually want is the one that converges near z_0. If you have any doubt about the ring of convergence of a Laurent series, you can find out by testing the "a" series and the "b" series separately.

Example 2. The function from which we obtained (4.2) was

$$f(z) = \frac{12}{z(2-z)(1+z)}. \tag{4.4}$$

This function has three singular points, at $z = 0$, $z = 2$, and $z = -1$. Thus there are two circles C_1 and C_2 about $z_0 = 0$ in Figure 4.2, and three Laurent series about $z_0 = 0$, one series valid in each of the three regions R_1 $(0 < |z| < 1)$, $R_2 (1 < |z| < 2)$, and $R_3 (|z| > 2)$. To find these series we first write $f(z)$ in the following form using partial fractions (Problem 2):

$$f(z) = \frac{4}{z}\left(\frac{1}{1+z} + \frac{1}{2-z}\right). \tag{4.5}$$

Now, for $0 < |z| < 1$, we expand each of the fractions in the parenthesis in (4.5) in powers of z. This gives (Problem 2):

(4.6) $$f(z) = -3 + 9z/2 - 15z^2/4 + 33z^3/8 + \cdots + 6/z.$$

This is the Laurent series for $f(z)$ which is valid in the region $0 < |z| < 1$. To obtain the series valid in the region $|z| > 2$, we write the fractions in (4.5) as

(4.7) $$\frac{1}{1+z} = \frac{1}{z}\frac{1}{1+1/z}, \qquad \frac{1}{2-z} = -\frac{1}{z}\frac{1}{1-2/z}$$

and expand each fraction in powers of $1/z$. This gives the Laurent series valid for $|z| > 2$ (problem 2):

(4.8) $$f(z) = -(12/z^3)(1 + 1/z + 3/z^2 + 5/z^3 + 11/z^4 + \cdots).$$

Finally, to obtain (4.2), we expand the fraction $1/(2-z)$ in powers of z, and the fraction $1/(1+z)$ in powers of $1/z$; this gives a Laurent series which converges for $1 < |z| < 2$. Thus the Laurent series (4.6), (4.2) and (4.8) all represent $f(z)$ in (4.4), but in three different regions.

Let z_0 in Figure 4.2 be either a regular point or an isolated singular point and assume that there are no other singular points inside C_1. Let $f(z)$ be expanded in the Laurent series about z_0 which converges inside C_1 (except possibly at z_0); we say that we have expanded $f(z)$ in the Laurent series which converges near z_0. Then we have the following definitions.

Definitions:

If all the b's are zero, $f(z)$ is analytic at $z = z_0$, and we call z_0 a *regular point.* (See Problem 4.1)

If $b_n \neq 0$, but all the b's after b_n are zero, $f(z)$ is said to have a *pole of order n* at $z = z_0$. If $n = 1$, we say that $f(z)$ has a *simple pole.*

If there are an infinite number of b's different from zero, $f(z)$ has an *essential singularity* at $z = z_0$.

The coefficient b_1 of $1/(z - z_0)$ is called the *residue* of $f(z)$ at $z = z_0$.

▶ **Example 3.**

(a) $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

is analytic at $z = 0$; the residue of e^z at $z = 0$ is 0.

(b) $$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} + \cdots$$

has a pole of order 3 at $z = 0$; the residue of $\frac{e^z}{z^3}$ at $z = 0$ is $\frac{1}{2!}$.

(c) $$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

has an essential singularity at $z = 0$; the residue of $e^{1/z}$ at $z = 0$ is 1.

Most of the functions we shall consider will be analytic except for poles—such functions are called *meromorphic* functions. If $f(z)$ has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$. A three-dimensional graph with $|f(z)|$ plotted vertically over a horizontal complex plane would look like a tapered pole near $z = z_0$. We can often see that a function has a pole and find the order of the pole without finding the Laurent series.

Example 4.

(a) $$\frac{z+3}{z^2(z-1)^3(z+1)}$$

has a pole of order 2 at $z = 0$, a pole of order 3 at $z = 1$, and a simple pole at $z = -1$.

(b) $$\frac{\sin^2 z}{z^3} \qquad \text{has a simple pole at } z = 0.$$

To see that these results are correct, consider finding the Laurent series for $f(z) = g(z)/(z - z_0)^n$. We write $g(z) = a_0 + a_1(z - z_0) + \cdots$; then the Laurent series for $f(z)$ starts with the term $(z - z_0)^{-n}$ unless $a_0 = 0$, that is unless $g(z_0) = 0$. Then the order of the pole of $f(z)$ is n unless some factors cancel. In Example 4b, the $\sin z$ series starts with z, so $\sin^2 z$ has a factor z^2; thus $(\sin^2 z)/z^3$ has a simple pole at $z = 0$.

PROBLEMS, SECTION 4

1. Show that the sum of a power series which converges inside a circle C is an analytic function inside C. *Hint:* See Chapter 2, Section 7, and Chapter 1, Section 11, and the definition of an analytic function.

2. Show that equation (4.4) can be written as (4.5). Then expand each of the fractions in the parenthesis in (4.5) in powers of z and in powers of $1/z$ [see equation (4.7)] and combine the series to obtain (4.6), (4.8), and (4.2).

For each of the following functions find the first few terms of each of the Laurent series about the origin, that is, one series for each annular ring between singular points. Find the residue of each function at the origin. (*Warning:* To find the residue, you must use the Laurent series which converges near the origin.) *Hints:* See Problem 2. Use partial fractions as in equations (4.5) and (4.7). Expand a term $1/(z - a)$ in powers of z to get a series convergent for $|z| < a$, and in powers of $1/z$ to get a series convergent for $|z| > a$.

3. $\dfrac{1}{z(z-1)(z-2)}$ **4.** $\dfrac{1}{z(z-1)(z-2)^2}$ **5.** $\dfrac{z-1}{z^3(z-2)}$

6. $\dfrac{1}{z^2(1+z)^2}$ **7.** $\dfrac{2-z}{1-z^2}$ **8.** $\dfrac{30}{(1+z)(z-2)(3+z)}$

For each of the following functions, say whether the indicated point is regular, an essential singularity, or a pole, and if a pole of what order it is.

9. (a) $\dfrac{\sin z}{z}, \quad z = 0$ (b) $\dfrac{\cos z}{z^3}, \quad z = 0$

(c) $\dfrac{z^3 - 1}{(z-1)^3}, \quad z = 1$ (d) $\dfrac{e^z}{z-1}, \quad z = 1$

10. (a) $\dfrac{e^z - 1}{z^2 + 4}, \quad z = 2i$ (b) $\tan^2 z, \quad z = \pi/2$

(c) $\dfrac{1 - \cos z}{z^4}, \quad z = 0$ (d) $\cos\left(\dfrac{\pi}{z - \pi}\right), \quad z = \pi$

11. (a) $\dfrac{e^z - 1 - z}{z^2}, \quad z = 0$ (b) $\dfrac{\sin z}{z^3}, \quad z = 0$

(c) $\dfrac{z^2 - 1}{(z-1)^2}, \quad z = 1$ (d) $\dfrac{\cos z}{(z - \pi/2)^4}, \quad z = \pi/2$

12. (a) $\dfrac{\sin z - z}{z^6}, \quad z = 0$ (b) $\dfrac{z^2 - 1}{(z^2+1)^2}, \quad z = i$

(c) $ze^{1/z}, \quad z = 0$ (d) $\Gamma(z), \quad z = 0$ [See Chapter 11, equation (4.1)]

▶ 5. THE RESIDUE THEOREM

Let z_0 be an isolated singular point of $f(z)$. We are going to find the value of $\oint_C f(z)\,dz$ around a simple closed curve C surrounding z_0 but inclosing no other singularities. Let $f(z)$ be expanded in the Laurent series (4.1) about $z = z_0$ that converges near $z = z_0$. By Cauchy's theorem (V), the integral of the "a" series is zero since this part is analytic. To evaluate the integrals of the terms in the "b" series in (4.1), we replace the integrals around C by integrals around a circle C' with center at z_0 and radius ρ as in (3.6), (3.7), and Figure 3.1. Along C', $z = z_0 + \rho e^{i\theta}$; calculating the integral of the b_1 term in (4.1), we find

$$\text{(5.1)} \qquad \oint_C \frac{b_1 dz}{(z - z_0)} = b_1 \int_0^{2\pi} \frac{\rho i e^{i\theta}\, d\theta}{\rho e^{i\theta}} = 2\pi i b_1.$$

It is straightforward to show (Problem 1) that the integrals of all the other b_n terms are zero. Then $\oint_C f(z)\,dz = 2\pi i b_1$, or since b_1 is called the residue of $f(z)$ at $z = z_0$, we can say

$$\oint_C f(z)\,dz = 2\pi i \cdot \text{residue of } f(z) \text{ at the singular point inside } C.$$

The only term of the Laurent series which has survived the integration process is the b_1 term; you can see the reason for the term "residue." If there are several isolated singularities inside C, say at z_0, z_1, z_2, $\cdots$, we draw small circles about each as shown in Figure 5.1 so that $f(z)$ is analytic in the region between C and the circles. Then, introducing cuts as in Figure 3.1, we find that the integral around C counterclockwise, plus the integrals around the circles clockwise, is zero (since the

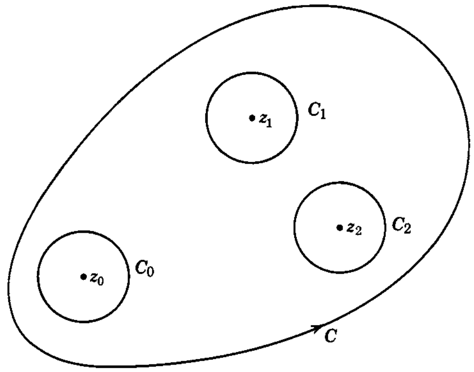


Figure 5.1

integrals along the cuts cancel), or the integral along C is the sum of the integrals around the circles (all counterclockwise). But by (5.1), the integral around each circle is $2\pi i$ times the residue of $f(z)$ at the singular point inside. Thus we have the *residue theorem*:

(5.2) $$\oint_C f(z)\,dz = 2\pi i \cdot \text{sum of the residues of } f(z) \text{ inside } C,$$

where the integral around C is in the counterclockwise direction.

The residue theorem is useful in evaluating many definite integrals; we shall consider this in Section 7. But first, in Section 6, we need to develop some techniques for finding residues.

PROBLEMS, SECTION 5

1. If C is a circle of radius ρ about z_0, show that

$$\oint_C \frac{dz}{(z-z_0)^n} = 2\pi i \qquad \text{if } n = 1,$$

but for any other integral value of n, positive or negative, the integral is zero. *Hint:* Use the fact that $z = z_0 + \rho e^{i\theta}$ on C.

2. Verify the formulas (4.3) for the coefficients in a Laurent series. *Hint:* To get a_n, divide equation (4.1) by $(z-z_0)^{n+1}$ and use the results of Problem 1 to evaluate the integrals of the terms of the series. Use a similar method to find b_n.

3. Obtain Cauchy's integral formula (3.9) from the residue theorem (5.2).

6. METHODS OF FINDING RESIDUES

A. Laurent Series If it is easy to write down the Laurent series for $f(z)$ about $z = z_0$ that is valid near z_0, then the residue is just the coefficient b_1 of the term $1/(z-z_0)$. *Caution*: Be sure you have the expansion about $z = z_0$; the series you have memorized for e^z, $\sin z$, etc., are expansions about $z = 0$ and so can be used only for finding residues at the origin (see Section 4, Example 3). Here is another example: Given $f(z) = e^z/(z-1)$, find the residue, $R(1)$, of $f(z)$ at $z = 1$. We want to expand e^z in powers of $z-1$; we write

$$\frac{e^z}{z-1} = \frac{e \cdot e^{z-1}}{z-1} = \frac{e}{z-1}\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right] \quad = \frac{e}{z-1} + e + \cdots .$$

Then the residue is the coefficient of $1/(z-1)$, that is, $R(1) = e$.

B. Simple Pole If $f(z)$ has a simple pole at $z = z_0$, we find the residue by multiplying $f(z)$ by $(z - z_0)$ and evaluating the result at $z = z_0$ (Problem 10).

▶ **Example 1.** Find $R(-\frac{1}{2})$ and $R(5)$ for

$$f(z) = \frac{z}{(2z+1)(5-z)}.$$

Multiply $f(z)$ by $(z + \frac{1}{2})$, [*Caution*: not by $(2z+1)$], and evaluate the result at $z = -\frac{1}{2}$. We find

$$(z+\tfrac{1}{2})f(z) = (z+\tfrac{1}{2})\frac{z}{(2z+1)(5-z)} = \frac{z}{2(5-z)},$$

$$R(-\tfrac{1}{2}) = \frac{-\frac{1}{2}}{2(5+\frac{1}{2})} = -\frac{1}{22}.$$

Similarly,

$$(z-5)f(z) = (z-5)\frac{z}{(2z+1)(5-z)} = -\frac{z}{2z+1},$$

$$R(5) = -\frac{5}{11}.$$

▶ **Example 2.** Find $R(0)$ for $f(z) = (\cos z)/z$. Since $zf(z) = \cos z$, we have

$$R(0) = (\cos z)_{z=0} = \cos 0 = 1.$$

To use this method, we may in some problems have to evaluate an indeterminate form, so in general we write

$$(6.1) \qquad R(z_0) = \lim_{z \to z_0} (z - z_0) f(z) \qquad \text{when } z_0 \text{ is a simple pole.}$$

▶ **Example 3.** Find the residue of $\cot z$ at $z = 0$. By (6.1)

$$R(0) = \lim_{z \to 0} \frac{z \cos z}{\sin z} = \cos 0 \cdot \lim_{z \to 0} \frac{z}{\sin z} = 1 \cdot 1 = 1.$$

If, as often happens, $f(z)$ can be written as $g(z)/h(z)$, where $g(z)$ is analytic and not zero at z_0 and $h(z_0) = 0$, then (6.1) becomes

$$R(z_0) = \lim_{z \to z_0} \frac{(z-z_0)g(z)}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{z - z_0}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{1}{h'(z)} = \frac{g(z_0)}{h'(z_0)}$$

by L'Hôpital's rule or the definition of $h'(z)$ (Problem 11).

Thus we have

$$(6.2) \qquad R(z_0) = \frac{g(z_0)}{h'(z_0)} \quad \text{if} \begin{cases} f(z) = g(z)/h(z), \text{ and} \\ g(z_0) = \text{finite const.} \neq 0, \text{ and} \\ h(z_0) = 0, \; h'(z_0) \neq 0. \end{cases}$$

Often (6.2) gives the most convenient way of finding the residue at a simple pole.

Example 4. Find the residue of $(\sin z)/(1 - z^4)$ at $z = i$. By (6.2) we have

$$R(i) = \left.\frac{\sin z}{-4z^3}\right|_{z=i} = \frac{\sin i}{-4i^3} = \frac{e^{-1} - e}{(2i)(4i)} = \frac{1}{8}(e - e^{-1}) = \frac{1}{4}\sinh 1.$$

Now you may ask how you know, without finding the Laurent series, that a function has a simple pole. Perhaps the simplest answer is that if the limit obtained using (6.1) is some constant (not 0 or ∞), then $f(z)$ *does* have a simple pole and the constant is the residue. [If the limit $= 0$, the function is analytic and the residue $= 0$; if the limit is infinite, the pole is of higher order.] However, you can often recognize the order of a pole in advance. [See end of Section 4 for the simple case in which $(z - z_0)^n$ is a factor of the denominator.] Suppose $f(z)$ is written in the form $g(z)/h(z)$, where $g(z)$ and $h(z)$ are analytic. Then you can think of $g(z)$ and $h(z)$ as power series in $(z - z_0)$. If the denominator has the factor $(z - z_0)$ to *one* higher power than the numerator, then $f(z)$ has a simple pole at z_0. For example,

$$z \cot^2 z = \frac{z \cos^2 z}{\sin^2 z} = \frac{z(1 - z^2/2 + \cdots)^2}{(z - z^3/3! + \cdots)^2} = \frac{z(1 + \cdots)}{z^2(1 + \cdots)}$$

has a simple pole at $z = 0$. By the same method we can see whether a function has a pole of any order

C. Multiple Poles When $f(z)$ has a pole of order n, we can use the following method of finding residues.

Multiply $f(z)$ by $(z - z_0)^m$, where m is an integer greater than or equal to the order n of the pole, differentiate the result $m - 1$ times, divide by $(m - 1)!$, and evaluate the resulting expression at $z = z_0$.

It is easy to prove that this rule is correct (Problem 12) by using the Laurent series (4.1) for $f(z)$ and showing that the result of the outlined process is b_1.

▶ **Example 5.** Find the residue of $f(z) = (z\sin z)/(z-\pi)^3$ at $z = \pi$.

We take $m = 3$ to eliminate the denominator before differentiating; this is an allowed choice for m because the order of the pole of $f(z)$ at π is not greater than 3 since $z\sin z$ is finite at π. (The pole is actually of order 2, but we do not need this fact.) Then following the rule stated, we get

$$R(\pi) = \frac{1}{2!}\frac{d^2}{dz^2}(z\sin z)\bigg|_{z=\pi} = \frac{1}{2}[-z\sin z + 2\cos z]_{z=\pi} = -1.$$

(To compute the derivative quickly, use Leibniz' rule for differentiating a product; see Chapter 12, Section 3.)

Much of this work can be done by computer. However, remember that the point of doing these problems is to gain skill in using the ideas and techniques of complex variable theory. So a good study method is to do the problems as outlined above and then check your results by computer.

▶ PROBLEMS, SECTION 6

Find the Laurent series for the following functions about the indicated points; hence find the residue of the function at the point. (Be sure you have the Laurent series which converges near the point.)

1. $\dfrac{1}{z(z+1)}$, $z = 0$ **2.** $\dfrac{1}{z(z-1)}$, $z = 1$ **3.** $\dfrac{\sin z}{z^4}$, $z = 0$

4. $\dfrac{\cosh z}{z^2}$, $z = 0$ **5.** $\dfrac{e^z}{z^2-1}$, $z = 1$ **6.** $\sin\dfrac{1}{z}$, $z = 0$

7. $\dfrac{\sin \pi z}{4z^2-1}$, $z = \dfrac{1}{2}$ **8.** $\dfrac{1+\cos z}{(z-\pi)^2}$, $z = \pi$ **9.** $\dfrac{1}{z^2-5z+6}$, $z = 2$

10. Show that rule B is correct by applying it to (4.1).

11. Derive (6.2) by using the limit definition of the derivative $h'(z_0)$ instead of using L'Hôpital's rule. Remember that $h(z_0) = 0$ because we are assuming that $f(z)$ has a simple pole at z_0.

12. Prove rule C for finding the residue at a multiple pole, by applying it to (4.1). Note that the rule is valid for $n = 1$ (simple pole) although we seldom use it for that case.

13. Prove rule C by using (3.9). *Hints:* If $f(z)$ has a pole of order n at $z = a$, then $f(z) = g(z)/(z-a)^n$ with $g(z)$ analytic at $z = a$. By (3.9)

$$\int_C \frac{g(z)}{(z-a)}\,dz = 2\pi i g(a)$$

with C a contour inclosing a but no other singularities. Differentiate this equation $(n-1)$ times with respect to a. (Or, use Problem 3.21.)

Find the residues of the following functions at the indicated points. Try to select the easiest of the methods outlined above. Check your results by computer.

14. $\dfrac{1}{(3z+2)(2-z)}$ at $z = -\frac{2}{3}$; at $z = 2$ **15.** $\dfrac{1}{(1-2z)(5z-4)}$ at $z = \frac{1}{2}$; at $z = \frac{4}{5}$

16. $\dfrac{z-2}{z(1-z)}$ at $z = 0$; at $z = 1$ **17.** $\dfrac{z+2}{4z^2-1}$ at $z = \dfrac{1}{2}$; at $z = -\dfrac{1}{2}$

18. $\dfrac{z+2}{z^2+9}$ at $z = 3i$

19. $\dfrac{\sin^2 z}{2z-\pi}$ at $z = \pi/2$

20. $\dfrac{z}{1-z^4}$ at $z = i$

21. $\dfrac{z^2}{z^4+16}$ at $z = \sqrt{2}(1+i)$

22. $\dfrac{e^{2z}}{1+e^z}$ at $z = i\pi$

23. $\dfrac{e^{iz}}{9z^2+4}$ at $z = \dfrac{2i}{3}$

24. $\dfrac{1-\cos 2z}{z^3}$ at $z = 0$

25. $\dfrac{e^{2z}-1}{z^2}$ at $z = 0$

26. $\dfrac{e^{2\pi i z}}{1-z^3}$ at $z = e^{2\pi i/3}$

27. $\dfrac{\cos z}{1-2\sin z}$ at $z = \pi/6$

28. $\dfrac{z+2}{(z^2+9)(z^2+1)}$ at $z = 3i$

29. $\dfrac{e^{2z}}{4\cosh z - 5}$ at $z = \ln 2$

30. $\dfrac{\cosh z - 1}{z^7}$ at $z = 0$

31. $\dfrac{e^{3z}-3z-1}{z^4}$ at $z = 0$

32. $\dfrac{e^{iz}}{(z^2+4)^2}$ at $z = 2i$

33. $\dfrac{1+\cos z}{(\pi-z)^3}$ at $z = \pi$

34. $\dfrac{z-2}{z^2(1-2z)^2}$ at $z = 0$ and at $z = \dfrac{1}{2}$

35. $\dfrac{z}{(z^2+1)^2}$ at $z = i$

14′ to 35′ Use the residue theorem to evaluate the contour integrals of each of the functions in Problems 14 to 35 around a circle of radius $\frac{3}{2}$ and center at the origin. Check carefully to see which singular points are inside the circle. You may use your results in the previous problems as far as they go, but you may have to compute some more residues.

36. For complex z, $J_p(z)$ can be defined by the series (12.9) in Chapter 12. Use this definition to find the Laurent series about $z = 0$ for $z^{-3}J_0(z)$. Find the residue of the function at $z = 0$.

37. The gamma function $\Gamma(z)$ is analytic except for poles at $z = x = 0, -1, -2, -3 \cdots$ (all the negative integers). Find the residues at these poles. *Hints:* See Example 1 above and Chapter 11, Equation (4.1).

7. EVALUATION OF DEFINITE INTEGRALS BY USE OF THE RESIDUE THEOREM

We are going to use (5.2) and the techniques of Section 6 to evaluate several different types of definite integrals. The methods are best shown by examples.

Example 1. Find $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$.

If we make the change of variable $z = e^{i\theta}$, then as θ goes from 0 to 2π, z traverses the unit circle $|z| = 1$ (Figure 7.1) in the counterclockwise direction, and we have a contour integral. We shall evaluate this integral by the residue theorem. If $z = e^{i\theta}$, we have

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta \quad \text{or} \quad d\theta = \frac{1}{iz}\,dz,$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}.$$

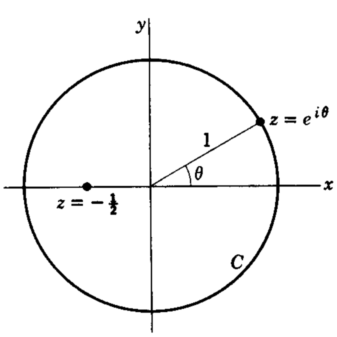


Figure 7.1

Making these substitutions in I, we get

$$I = \oint_C \frac{\frac{1}{iz}\,dz}{5 + 2(z + 1/z)} = \frac{1}{i}\oint_C \frac{dz}{5z + 2z^2 + 2}$$
$$= \frac{1}{i}\oint_C \frac{dz}{(2z+1)(z+2)},$$

where C is the unit circle. The integrand has poles at $z = -\frac{1}{2}$ and $z = -2$; only $z = -\frac{1}{2}$ is inside the contour C. The residue of $1/[(2z+1)(z+2)]$ at $z = -\frac{1}{2}$ is

$$R(-\tfrac{1}{2}) = \lim_{z\to -1/2}(z + \tfrac{1}{2}) \cdot \frac{1}{(2z+1)(z+2)} = \left.\frac{1}{2(z+2)}\right|_{z=-1/2} = \tfrac{1}{3}.$$

Then by the residue theorem

$$I = \frac{1}{i}\,2\pi i R(-\tfrac{1}{2}) = 2\pi \cdot \tfrac{1}{3} = \frac{2\pi}{3}.$$

This method can be used to evaluate the integral of any rational function of $\sin\theta$ and $\cos\theta$ between 0 and 2π, provided the denominator is never zero for any value of θ. You can also find an integral from 0 to π if the integrand is even, since the integral from 0 to 2π of an even periodic function is twice the integral from 0 to π of the same function. (See Chapter 7, Section 9 for discussion of even and odd functions.)

▶ **Example 2.** Evaluate $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Here we could easily find the indefinite integral and so evaluate I by elementary methods. However, we shall do this simple problem by contour integration to illustrate a method which is useful for more complicated problems.

This time we are not going to make a change of variable in I. We are going to start with a different integral and show how to find I from it. We consider

$$\oint_C \frac{dz}{1+z^2},$$

where C is the closed boundary of the semicircle shown in Figure 7.2. For any $\rho > 1$, the semicircle incloses the singular point $z = i$ and no others; the residue of the integrand at $z = i$ is

$$R(i) = \lim_{z\to i}(z - i)\frac{1}{(z-i)(z+i)} = \frac{1}{2i}.$$

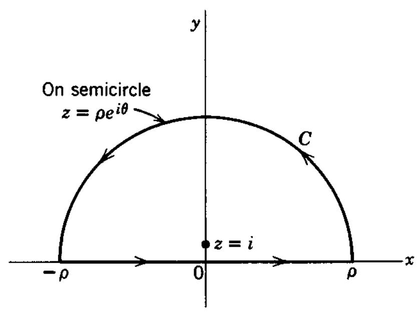


Figure 7.2

Then the value of the contour integral is $2\pi i(1/2i) = \pi$. Let us write the integral in two parts: (1) an integral along the x axis from $-\rho$ to ρ; for this part $z = x$; (2) an integral along the semicircle, where $z = \rho e^{i\theta}$. Then we have

$$\int_C \frac{dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{dx}{1+x^2} + \int_0^{\pi} \frac{\rho i e^{i\theta}\,d\theta}{1+\rho^2 e^{2i\theta}}. \tag{7.1}$$

We know that the value of the contour integral is π no matter how large ρ becomes since there are no other singular points besides $z = i$ in the upper half-plane. Let $\rho \to \infty$; then the second integral on the right in (7.1) tends to zero since the numerator contains ρ and the denominator ρ^2. Thus the first term on the right tends to π (the value of the contour integral) as $\rho \to \infty$, and we have

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

This method can be used to evaluate any integral of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

if $P(x)$ and $Q(x)$ are polynomials with the degree of Q at least two greater than the degree of P, and if $Q(z)$ has no real zeros (that is, zeros on the x axis). If the integrand $P(x)/Q(x)$ is an even function, then we can also find the integral from 0 to ∞.

▶ **Example 3.** Evaluate $I = \displaystyle\int_0^{\infty} \frac{\cos x\,dx}{1+x^2}$.

We consider the contour integral

$$\oint_C \frac{e^{iz}\,dz}{1+z^2},$$

where C is the same semicircular contour as in Example 2. The singular point inclosed is again $z = i$, and the residue there is

$$\lim_{z\to i}(z-i)\frac{e^{iz}}{(z-i)(z+i)} = \frac{e^{-1}}{2i} = \frac{1}{2ie}.$$

The value of the contour integral is $2\pi i(1/2ie) = \pi/e$. As in Example 2 we write the contour integral as a sum of two integrals:

$$(7.2)\qquad \oint_C \frac{e^{iz}\,dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{e^{ix}\,dx}{1+x^2} + \int\limits_{\substack{\text{along upper half}\\ \text{of } z=\rho e^{i\theta}}} \frac{e^{iz}\,dz}{1+z^2}.$$

As before, we want to show that the second integral on the right of (7.2) tends to zero as $\rho \to \infty$. This integral is the same as the corresponding integral in (7.1) except for the e^{iz} factor. Now

$$|e^{iz}| = |e^{ix-y}| = |e^{ix}||e^{-y}| = e^{-y} \le 1$$

since $y \ge 0$ on the contour we are considering. Since $|e^{iz}| \le 1$, this factor does not change the proof given in Example 2 that the integral along the semicircle tends to zero as the radius $\rho \to \infty$. We have then

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \frac{\pi}{e},$$

or taking the real part of both sides of this equation,

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{1+x^2} = \frac{\pi}{e}.$$

Since the integrand $(\cos x)/(1+x^2)$ is an even function, the integral from 0 to ∞ is half the integral from $-\infty$ to ∞. Hence we have

$$I = \int_0^{\infty} \frac{\cos x\,dx}{1+x^2} = \frac{\pi}{2e}.$$

Observe that the same proof would work if we replaced e^{iz} by e^{imz} $(m > 0)$ in the above integrals. At the point where we said $e^{-y} \le 1$ (since $y \ge 0$) we would then want $e^{-my} \le 1$ for $y \ge 0$, which is true if $m > 0$. [For $m < 0$, we *could* use a semicircle in the lower half-plane $(y < 0)$; then we would have $e^{my} \le 1$ for $y \le 0$. This is an unnecessary complication, however, in evaluating integrals containing $\sin mx$ or $\cos mx$ since we *can* then choose m to be positive.] Although we have assumed here that (as in Example 2) $Q(x)$ is of degree at least 2 higher than $P(x)$, a more detailed proof (see books on complex variables) shows that degree at least one higher is enough to make the integral

$$\int \frac{P(z)}{Q(z)} e^{imz}\,dz$$

around the semicircle tend to zero as $\rho \to \infty$. Thus

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx}\,dx = 2\pi i \cdot \text{sum of the residues of } \frac{P(z)}{Q(z)} e^{imz}$$

in the upper half-plane if all the following requirements are met:

1. $P(x)$ and $Q(x)$ are polynomials, and
2. $Q(x)$ has no real zeros, and
3. the degree of $Q(x)$ is at least 1 greater than the degree of $P(x)$, and $m > 0$.

By taking real and imaginary parts, we then find the integrals

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx\,dx, \qquad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx\,dx.$$

▶ **Example 4.** Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$.

Here we remove the restriction of Examples 2 and 3 that $Q(x)$ has no real zeros. As in Example 3, we consider

$$\int \frac{e^{iz}}{z}\,dz.$$

To avoid the singular point at $z = 0$, we integrate around the contour shown in Figure 7.3. We then let the radius r shrink to zero so that in effect we are integrating straight through the simple pole at the origin. We are going to show (later in this section and Problem 21) that the net result of integrating in the counterclockwise direction around a closed contour which passes straight‡ through one or more simple poles is $2\pi i\cdot$ (sum of the residues at interior points plus one-half the sum of the residues at the simple poles on the boundary). (*Warning:* this rule does not hold in general for a multiple pole on a boundary.) You might expect this result. If a pole is inside a contour, it contributes $2\pi i\cdot$ residue, to the integral; if it is outside, it contributes nothing; if it is *on* the straight line boundary, its contribution is just halfway between zero and $2\pi i\cdot$ residue. [See Am. J. Phys. **52**, 276 (1984).] Using this fact, and observing that, as in Example 3, the integral along the large semicircle tends to zero as R tends to infinity, we have

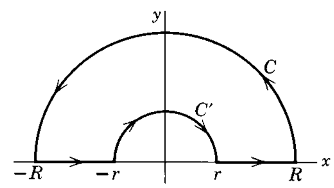


Figure 7.3

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx = 2\pi i\cdot\frac{1}{2}\left(\text{residue of }\frac{e^{iz}}{z}\text{ at }z=0\right) = 2\pi i\cdot\frac{1}{2}\cdot 1 = i\pi.$$

Taking the imaginary parts of both sides, we get

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \pi.$$

To show more carefully that our result is correct, let us return to the contour of Figure 7.3. Since e^{iz}/z is analytic inside this contour, the integral around the whole contour is zero. As we have said, the integral along C tends to zero as $R\to\infty$ by the theorem at the end of Example 3. Along the small semicircle C', we have

$$z = re^{i\theta},\quad dz = re^{i\theta}i\,d\theta,\quad \frac{dz}{z} = i\,d\theta,$$
$$\int_{C'}\frac{e^{iz}\,dz}{z} = \int_{C'} e^{iz}i\,d\theta.$$

As $r\to 0$, $z\to 0$, $e^{iz}\to 1$, and the integral (along C' in the direction indicated in Figure 7.3) tends to

$$\int_{\pi}^{0} i\,d\theta = -i\pi.$$

Then we have as $R\to\infty$, and $r\to 0$,

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx - i\pi = 0$$

or

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx = i\pi$$

as before. Taking real and imaginary parts of this equation (and using Euler's formula $e^{ix} = \cos x + i\sin x$), we get

$$\int_{-\infty}^{\infty}\frac{\cos x}{x}\,dx = 0,\qquad \int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \pi.$$

‡By "straight" we mean that the contour curve has a tangent at the pole, that is, it does not turn a corner there.

Since $(\sin x)/x$ is an even function, we have

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

[For another way of evaluating this integral, see Chapter 7, equation (12.19).]

Principal value Now consider the cosine integral.

$$\int_0^\infty \frac{\cos x}{x}\,dx$$

is a divergent integral since the integrand $(\cos x)/x$ is approximately $1/x$ near $x = 0$. The value zero which we found for $I = \int_{-\infty}^\infty (\cos x)/x\,dx$ is called the *principal value* (or Cauchy principal value) of I. To see what this means, consider a simpler integral, namely

$$\int_0^5 \frac{dx}{x-3}.$$

The integrand becomes infinite at $x = 3$, and both $\int_0^3 dx/(x-3)$ and $\int_3^5 dx/(x-3)$ are divergent. Suppose we cut out a small symmetric interval about $x = 3$, and integrate from 0 to $3 - r$ and from $3 + r$ to 5. We find

$$\int_0^{3-r} \frac{dx}{x-3} = \ln|x-3|\Big|_0^{3-r} = \ln r - \ln 3,$$
$$\int_{3+r}^5 \frac{dx}{x-3} = \ln 2 - \ln r.$$

The sum of these two integrals is

$$\ln 2 - \ln 3 = \ln\frac{2}{3};$$

this sum is independent of r. Thus, if we let $r \to 0$, we get the result $\ln\frac{2}{3}$ which is called the principal value of

$$\int_0^5 \frac{dx}{x-3} \qquad \left(\text{often written } PV\int_0^5 \frac{dx}{x-3} = \ln\frac{2}{3}\right).$$

The terms $\ln r$ and $-\ln r$ have been allowed to cancel each other; graphically an infinite area above the x axis and a corresponding infinite area below the x axis have been canceled. In computing the contour integral we integrated along the x axis from $-\infty$ up to $-r$, and from $+r$ to $+\infty$, and then let $r \to 0$; this is just the process we described for finding principal values, so the result we found for the improper integral $\int_{-\infty}^\infty (\cos x)/x\,dx$, namely zero, was the principal value of this integral.

▶ **Example 5.** Evaluate

$$\int_0^\infty \frac{r^{p-1}}{1+r}\,dr, \qquad 0 < p < 1,$$

and use the result to prove (5.4) of Chapter 11.

We first find

$$(7.3)\qquad \oint \frac{z^{p-1}}{1+z}\,dz, \qquad 0<p<1, \qquad \text{around } C \text{ in Figure 7.4.}$$

Before we can evaluate this integral, we must ask what z^{p-1} means, since for each z there may be more than one value of z^{p-1}. (See discussion of branches at the end of Section 1.) For example, consider the case $p = \frac{1}{2}$; then $z^{p-1} = z^{-1/2}$. Recall from Chapter 2, Section 10, that there are two square roots of any complex number. At a point where $\theta = \pi/4$, say, we have

$$z = re^{i\pi/4}, \qquad z^{-1/2} = r^{-1/2}e^{-i\pi/8}.$$

But if θ increases by 2π (we think of following a circle around the origin and back to our starting point), we have

$$z = re^{i(\pi/4+2\pi)}, \qquad z^{-1/2} = r^{-1/2}e^{-i(\pi/8+\pi)} = -r^{-1/2}e^{-i\pi/8}.$$

Similarly, for any starting point (with $r \neq 0$), we find that $z^{-1/2}$ or z^{p-1} comes back to a different value (different branch) when θ increases by 2π and we return to our starting point. If we want to use the formula z^{p-1} to define a (single-valued) function, we must decide on some interval of length 2π for θ (that is, we must select one branch of z^{p-1}). Let us agree to restrict θ to the values of 0 to 2π in evaluating the contour integral (7.3). We may imagine an artificial barrier or cut (which we agree not to cross) along the positive x axis; this is called a *branch cut*. (See Example 3, Section 9.) A point which we cannot encircle (on an arbitrarily small circle) without crossing a branch cut (thus changing to another branch) is called a *branch point*; the origin is a branch point here.

In Figure 7.4, then, $\theta = 0$ along AB (upper side of the positive x axis); when we follow C around to DE, θ increases by 2π, so $\theta = 2\pi$ on the lower side of the positive x axis. Note that the contour in Figure 7.4 never takes us outside the 0 to 2π interval, so the factor z^{p-1} in (7.3) is a single-valued function. The integrand in (7.3), namely $z^{p-1}/(1+z)$, is now an analytic function inside the closed curve C in Figure 7.4 except for the pole at $z = -1 = e^{i\pi}$. The residue there is $(e^{i\pi})^{p-1} = -e^{i\pi p}$. Then we have

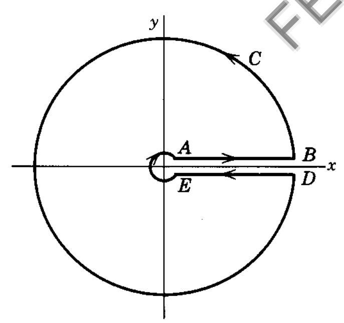


Figure 7.4

$$(7.4)\qquad \oint_C \frac{z^{p-1}}{1+z}\,dz = -2\pi i e^{i\pi p}, \qquad 0<p<1.$$

Along either of the two circles in Figure 7.4 we have $z = re^{i\theta}$ and the integral is

$$\int \frac{r^{p-1}e^{i(p-1)\theta}}{1+re^{i\theta}} rie^{i\theta}\,d\theta = i\int \frac{r^p e^{ip\theta}}{1+re^{i\theta}}\,d\theta.$$

This integral tends to zero if $r \to 0$ or if $r \to \infty$. (Verify this; note that the denominator is approximately 1 for small r, and approximately $re^{i\theta}$ for large r.) Thus the integrals along the circular parts of the contour tend to zero as the little circle shrinks to a point and the large circle expands indefinitely. We are left with the two integrals along the positive x axis with AB now extending from 0 to ∞ and DE from ∞ to 0. Along AB we agreed to have $\theta = 0$, so $z = re^{i\cdot 0} = r$, and this integral is

$$\int_{r=0}^{\infty} \frac{r^{p-1}}{1+r}\,dr.$$

Along DE, we have $\theta = 2\pi$, so $z = re^{2\pi i}$ and this integral is

$$\int_{r=\infty}^{0} \frac{(re^{2\pi i})^{p-1}}{1+re^{2\pi i}} e^{2\pi i}\,dr = -\int_0^{\infty} \frac{r^{p-1}e^{2\pi ip}}{1+r}\,dr.$$

Adding the AB and DE integrals, we get

$$(1-e^{2\pi ip})\int_0^{\infty} \frac{r^{p-1}}{1+r}\,dr = -2\pi i e^{i\pi p}$$

by (7.4). Then the desired integral is

$$\int_0^{\infty} \frac{r^{p-1}}{1+r}\,dr = \frac{-2\pi i e^{i\pi p}}{1-e^{2\pi ip}} = \frac{\pi\cdot 2i}{e^{i\pi p}-e^{-i\pi p}} = \frac{\pi}{\sin \pi p}. \tag{7.5}$$

Let us use (7.5) to obtain (5.4) of Chapter 11. Putting $q = 1-p$ in (6.5) and (7.1) of Chapter 11, we have

$$\begin{aligned} B(p, 1-p) &= \int_0^{\infty} \frac{y^{p-1}}{1+y}\,dy \qquad \text{and} \\ B(p, 1-p) &= \Gamma(p)\Gamma(1-p) \qquad \Gamma(1) = 1. \end{aligned} \tag{7.6}$$

Combining (7.5) and (7.6) gives (5.4) of Chapter 11, namely

$$\Gamma(p)\Gamma(1-p) = B(p, 1-p) = \int_0^{\infty} \frac{y^{p-1}}{1+y}\,dy = \frac{\pi}{\sin \pi p}.$$

Argument Principle Since $w = f(z)$ is a complex number for each z, we can write $w = Re^{i\Theta}$ (just as we write $z = re^{i\theta}$) where $R = |w|$ and Θ is the angle of w [or we could call it the angle of $f(z)$]. As z changes, $w = f(z)$ also changes and so R and Θ vary as we go from point to point in the complex (x, y) plane. We want to show that

(a) if $f(z)$ is analytic on and inside a simple closed curve C and $f(z) \neq 0$ on C, then the number of zeros of $f(z)$ inside C is equal to $(1/2\pi)\cdot$ (change in the angle of $f(z)$ as we traverse the curve C);

(b) if $f(z)$ has a finite number of poles inside C, but otherwise meets the requirements stated,[§] then the change in the angle of $f(z)$ around C is equal to $(2\pi)\cdot$ (the number of zeros minus the number of poles).

(Just as we say that a quadratic equation with equal roots has *two* equal roots, so here we mean that a zero of order n counts as n zeros and a pole of order n counts as n poles.)

To show (a) and (b) we consider

$$\oint_C \frac{f'(z)}{f(z)}\,dz.$$

[§] A function which is analytic except for poles is called *meromorphic*.

By the residue theorem, the integral is equal to $2\pi i\cdot$(sum of the residues at singularities inside C). It is straightforward to show (Problem 42) that the residue of $F(z) = f'(z)/f(z)$ at a zero of $f(z)$ of order n is n, and the residue of $F(z)$ at a pole of $f(z)$ of order p is $-p$. Then if N is the number of zeros and P the number of poles of $f(z)$ inside C, the integral is $2\pi i(N - P)$. Now by direct integration, we have

$$\oint_C \frac{f'(z)}{f(z)}\,dz = \ln f(z)\big|_C = \ln Re^{i\Theta}\big|_C = \operatorname{Ln} R\big|_C + i\Theta\big|_C, \tag{7.7}$$

where $R = |f(z)|$ and Θ is the angle of $f(z)$. Recall from Chapter 2, Section 13, that $\operatorname{Ln} R$ means the ordinary real logarithm (to the base e) of the positive number R, and is single-valued; $\ln f(z)$ is multiple-valued because Θ is multiple-valued. Then if we integrate from a point A on C all the way around the curve and back to A, $\operatorname{Ln} R$ has the same value at A both at the beginning and at the end, so the term $\operatorname{Ln} R|_C$ is $\operatorname{Ln} R$ at A minus $\operatorname{Ln} R$ at A; this is zero. The same result may not be true for Θ; that is, the angle may have changed as we go from point A all the way around C and back to A. (Think, for example, of the angle of z as we go from $z = 1$ around the unit circle and back to $z = 1$; the angle of z has increased from 0 to 2π.) Collecting our results, we have

$$\begin{aligned} N - P &= \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i} i\Theta_C \\ &= \frac{1}{2\pi}\cdot(\text{change in the angle of } f(z) \text{ around } C), \end{aligned} \tag{7.8}$$

where N is the number of zeros and P the number of poles of $f(z)$ inside C, with poles of order n counted as n poles and similarly for zeros of order n. Equation (7.8) is known as the *argument principle* (recall from Chapter 2 that *argument* means *angle*).

This principle is often used to find out how many zeros (or poles) a given function has in a given region. (Locating the zeros of a function has important applications to determining the stability of linear systems such as electric circuits and servomechanisms.

Example 6. Let us show that $f(z) = z^3 + 4z + 1$ has exactly one zero in the first quadrant. The closed curve C in (7.8) is, for this problem, the contour OPQ in Figure 7.5, where PQ is a large quarter circle. We first observe that on the x axis, $x^3 + 4x + 1 > 0$ for $x > 0$, and on the y axis, $(iy)^3 + 4iy + 1 \neq 0$ for any y (since its real part, namely 1, $\neq 0$). Then $f(z) \neq 0$ on OP or OQ. Also $f(z) \neq 0$ on PQ if we choose a circle large enough to inclose all zeros. We now want to find the change in the angle Θ of $f(z) = Re^{i\Theta}$ as we go around C. Along OP, $z = x$; then $f(z) = f(x)$ is real and so $\Theta = 0$. Along PQ, $z = re^{i\theta}$, with r constant and very large. For very large r, the z^3 term in $f(z)$ far outweighs the other terms, and we have $f(z) \cong z^3 = r^3 e^{3i\theta}$. As θ goes from 0 to $\pi/2$ along PQ,

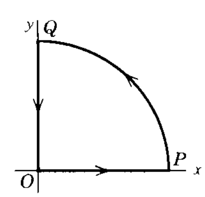


Figure 7.5

$\Theta = 3\theta$ goes from 0 to $3\pi/2$. On QO, $z = iy$, $f(z) = -iy^3 + 4iy + 1$; then

$$\tan\Theta = \frac{\text{imaginary part of } f(z)}{\text{real part of } f(z)} = \frac{4y - y^3}{1}.$$

For very large y (that is, at Q), we had $\Theta \cong 3\pi/2$ (for $y = \infty$, we would have $\tan\theta = -\infty$, and Θ would be exactly $3\pi/2$). Now as y decreases along QO, the value of $\tan\Theta = 4y - y^3$ decreases in magnitude but remains negative until it becomes 0 at $y = 2$. This means that Θ changes from $3\pi/2$ to 2π. Between $y = 2$ and $y = 0$, the tangent becomes positive, but then decreases to zero again without becoming infinite. This means that the angle Θ increases beyond 2π but not as far as $2\pi+\pi/2$, and then decreases again to 2π. Thus the total change in Θ around C is 2π, and by (7.8), the number of zeros of $f(z)$ in the first quadrant is $(1/2\pi)\cdot 2\pi = 1$. If we realize that (for a polynomial with real coefficients) the zeros off the real axis always occur in conjugate pairs, we see that there must also be one zero for z in the fourth quadrant, and the third zero must be on the negative x axis.

Bromwich integral (Inverse Laplace Transform) In Chapter 8 (Section 8ff), we found inverse Laplace transforms from a table (pages 469–471), or by computer, but we had no general formula for the inverse transform. By analogy with Fourier transforms (Chapter 7, Section 12), where we have similar integrals for the direct and inverse transforms, we might reasonably wonder whether an inverse Laplace transform could be given by an integral. To discuss this, we repeat here for convenience the definitions of Laplace and Fourier transforms.

(7.9) $$L(f) = \int_0^\infty f(t)e^{-pt}\,dt = F(p)$$

(7.10) $$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha$$
$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx$$

If we compare the Laplace transform (7.9) with the Fourier transform [$g(\alpha)$ in (7.10)], we observe that if p were imaginary, the integrals would be almost the same. This suggests that we should consider complex p, and that the integral we want for the inverse Laplace transform might be an integral in the complex p plane (that is, a contour integral). Let's investigate this idea.

In the definition (7.9) of the Laplace transform of $f(t)$, let p be complex, say $p = z = x+iy$. (Note that this possibility has already been considered in Chapter 8.) Then (7.9) becomes

(7.11) $$F(p) = F(z) = F(x+iy) = \int_0^\infty e^{-pt}f(t)\,dt$$
$$= \int_0^\infty e^{-(x+iy)t}f(t)\,dt = \int_0^\infty e^{-xt}f(t)e^{-iyt}\,dt, \quad x = \operatorname{Re} p > k.$$

[Recall (Chapter 8, Section 8) that we must have some restriction on $\operatorname{Re} p$ to make the integral converge at infinity. The restriction depends on what the function $f(t)$ is, but is always of the form $\operatorname{Re} p > k$, for *some* real k, as you can see in the table of Laplace transforms, pages 469–471.] Now (7.11) is of the form of a Fourier transform. To see this, compare (7.11) with (7.10) making the following correspondences: $e^{-iyt}\,dt$ corresponds to $e^{-i\alpha x}\,dx$, that is, y corresponds to α and t to x [the x in (7.11) is just a constant parameter in this discussion]; the function

$$\phi(t) = \begin{cases} e^{-xt} f(t), & t > 0, \\ 0, & t < 0, \end{cases} \tag{7.12}$$

corresponds to $f(x)$ in (7.10) and $F(p) = F(x+iy)$ corresponds to $g(\alpha)$; and finally, we recall that the $1/(2\pi)$ factor may be in either integral in (7.10). Then assuming that $\phi(t)$ satisfies the required conditions for a function to have a Fourier transform [see Chapter 7, Section 12: Dirichlet conditions, and $\int_{-\infty}^{\infty} |\phi(t)|\,dt$ finite], we can write the inverse transform to get

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x+iy)e^{iyt}\,dy. \tag{7.13}$$

Using the definition (7.12) of $\phi(t)$, we find

$$f(t) = e^{xt}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} F(x+iy)e^{iyt}\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x+iy)e^{(x+iy)t}\,dy \tag{7.14}$$

for $t > 0$. Since x is constant, say $x = c$, we have $dz = d(x+iy) = i\,dy$, and we can write (7.14) as

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)e^{zt}\,dz, \qquad t > 0, \tag{7.15}$$

where the notation means (see Problem 3.4) that we integrate along a vertical line $x = c$ in the z plane. [This can be *any* vertical line on which $x = c > k$ as required by the restriction on $\operatorname{Re} p$ in (7.11).] The integral (7.15) for the inverse Laplace transform is known as the *Bromwich integral*.

We would like to use contour integration and the residue theorem to evaluate $f(t)$ in (7.15) for a given $F(p)$ [which we call $F(z)$ since we consider complex p]. In Examples 2 and 3, we have evaluated integrals along a straight line (the x axis) by considering the contour made up of the x axis and a large semicircle inclosing the upper half plane. If we rotate this contour 90°, we have a contour consisting of a vertical straight line and a semicircle inclosing a left half-plane (that is, the area to the left of $x = c$). Let's use this contour to evaluate (7.15). We restrict $F(z)$ to be of the form $P(z)/Q(z)$ with $P(z)$ and $Q(z)$ polynomials, and $Q(z)$ of degree at least one higher than $P(z)$ (compare the conditions in Example 3). Then it can be shown that, as in Examples 2 and 3, the integral along the semicircle tends to zero as the radius tends to infinity. Thus the integral along the straight line is equal to $2\pi i$ times the sum of the residues of $F(z)e^{zt}$ at its poles, or, cancelling the factor $2\pi i$ in (7.15),

$$f(t) = \text{sum of residues of } F(z)e^{zt} \text{ at all poles.} \tag{7.16}$$

We must include *all* poles in (7.16); to see this we can argue as follows. We know that (7.15) is true for any value of $c > k$. Suppose we use a value of c which is large enough so that all poles lie to the left of $x = c$; then we know that our answer is correct. Turning the argument around, we can say that since we would get a different answer if we did not take $x = c$ to the right of all poles, we *must* integrate along a vertical line such that all poles of $F(z)e^{zt}$ are included in the contour to the left of the line.

▶ **Example 7.** Find the inverse transform of $F(p) = \dfrac{5}{(p+2)(p^2+1)}$.

We first find the poles of $F(z)e^{zt}$ and factor the denominator to get

$$F(z)e^{zt} = \frac{5e^{zt}}{(z+2)(z+i)(z-i)}.$$

Evaluating the residues at the three simple poles (Section 6, method B), we find

residue at $z = -2$	is	$\dfrac{5e^{-2t}}{5} = e^{-2t}$
residue at $z = i$	is	$\dfrac{5e^{it}}{(2+i)(2i)}$
residue at $z = -i$	is	$\dfrac{5e^{-it}}{(2-i)(-2i)}$

Then by (7.16) we have

$$f(t) = e^{-2t} + \frac{5e^{it}(2-i) - 5e^{-it}(2+i)}{(2+i)(2-i)(2i)} = e^{-2t} + 2\sin t - \cos t.$$

Dispersion relations Consider $\displaystyle\int \frac{f(z)}{z-a}\,dz$ around the upper half plane as in Problem 21. Let a be real. Let $f(z)$ be analytic for $y \geq 0$, and $\to 0$ rapidly enough at ∞ so that the integral around the semicircle in the upper half plane $\to 0$ as the radius of the semicircle $\to \infty$. Then by Example 4 and Problem (21b) we get

$$PV \int_{-\infty}^{\infty} \frac{f(x)}{x-a}\,dx = i\pi f(a). \tag{7.17}$$

Now we write $f(x) = u(x) + iv(x)$, and take real and imaginary parts of (7.17):

$$PV \int_{-\infty}^{\infty} \frac{u(x)}{x-a}\,dx = -\pi v(a), \qquad PV \int_{-\infty}^{\infty} \frac{v(x)}{x-a}\,dx = \pi u(a). \tag{7.18}$$

These (and similar integrals relating the real and imaginary parts of a function satisfying the given conditions) are called *dispersion relations*. From them, you can find the *Kramers-Kronig relations* (see Problem 66) named for the two people who developed similar relations involving the complex index of refraction for light. (Light traveling through a material medium is both refracted and absorbed. The real part of the complex index of refraction is related to refraction and the imaginary part to absorption.) These formulas have widespread applications, to optics, electricity, solid state, elementary particle theory, quantum mechanics, etc.

The integrals in (7.18) are called *Hilbert transforms*, and (7.18) may be stated in the form: $u(x)$ and $v(x)$ are Hilbert transforms of each other. Compare Fourier transforms (Chapter 7, Section 12), or a Laplace transform and the corresponding Bromwich integral. In each case two functions have the property that each is given by an integral involving the other. This is what an integral transform means, and there are other integral transforms which you may discover in tables or computer.

PROBLEMS, SECTION 7

The values of the following integrals are known and can be found in integral tables or by computer. Your goal in evaluating them is to learn about contour integration by applying the methods discussed in the examples above. Then check your answers by computer.

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$

2. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$

3. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$

4. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{5 + 3\cos\theta}$

5. $\displaystyle\int_0^{\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2} \quad (0 \le r < 1)$

6. $\displaystyle\int_0^{\pi} \frac{d\theta}{(2 + \cos\theta)^2}$

7. $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5 + 4\cos\theta}$

8. $\displaystyle\int_0^{\pi} \frac{\sin^2\theta\, d\theta}{13 - 12\cos\theta}$

9. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + \sin\theta\cos\alpha} \quad (\alpha = \text{const.})$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5}$

11. $\displaystyle\int_0^{\infty} \frac{dx}{(4x^2 + 1)^3}$

12. $\displaystyle\int_0^{\infty} \frac{x^2\, dx}{x^4 + 16}$

13. $\displaystyle\int_0^{\infty} \frac{x^2\, dx}{(x^2 + 4)(x^2 + 9)}$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2 + 4x + 5}$

15. $\displaystyle\int_0^{\infty} \frac{\cos 2x\, dx}{9x^2 + 4}$

16. $\displaystyle\int_0^{\infty} \frac{x\sin x\, dx}{9x^2 + 4}$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x\, dx}{x^2 + 4x + 5}$

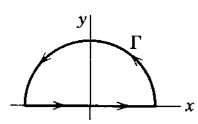
18. $\displaystyle\int_0^{\infty} \frac{\cos \pi x\, dx}{1 + x^2 + x^4}$

19. $\displaystyle\int_0^{\infty} \frac{\cos 2x\, dx}{(4x^2 + 9)^2}$

20. $\displaystyle\int_0^{\infty} \frac{\cos x\, dx}{(1 + 9x^2)^2}$

21. In Example 4 we stated a rule for evaluating a contour integral when the contour passes through simple poles. We proved that the result was correct for

$$PV \int_\Gamma \frac{e^{iz}}{z}\, dz$$



around the contour Γ shown here.
(a) By following the same method (integrating around C' of Figure 7.3 and letting $r \to 0$) show that the result is correct if we replace e^{iz} by any $f(z)$ which is analytic at $z = 0$.
(b) Repeat the proof in (a) for

$$PV \int_\Gamma \frac{f(z)}{(z - a)}\, dz, \qquad a \text{ real}$$

(that is, a pole on the x axis), with $f(z)$ analytic at $z = a$.

Using the rule of Example 4 (also see problem 21), evaluate the following integrals. Find principal values if necessary.

22. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+1)}$

23. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)(2-x)}$

24. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin \pi x}{1-x^2}\,dx$

25. $\displaystyle\int_{0}^{\infty} \frac{x\sin x}{9x^2-\pi^2}\,dx$

26. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x-1)^4-1}$

27. $\displaystyle\int_{0}^{\infty} \frac{\cos \pi x}{1-4x^2}\,dx$

28. $\displaystyle\int_{0}^{\infty} \frac{dx}{1-x^4}$

29. $\displaystyle\int_{0}^{\infty} \frac{\sin ax}{x}\,dx$

30. (a) By the method of Example 2 evaluate $\displaystyle\int_{0}^{\infty} \frac{dx}{1+x^4}$.

(b) Evaluate the same integral by using tables or computer to get the indefinite integral; unless you are very careful you may get zero. Explain why.

(c) Make the change of variables $u = x^4$ in the integral in (a) and evaluate the u integral using (7.5).

31. Use the method of Problem 30(c) to evaluate $\displaystyle\int_{0}^{\infty} \frac{dx}{1+x^6}$.

32. Use the method of Problem 30(c) and the contour and method of Example 5 to evaluate $\displaystyle\int_{0}^{\infty} \frac{dx}{(1+x^4)^2}$.

Evaluate the following integrals by the method of Example 5.

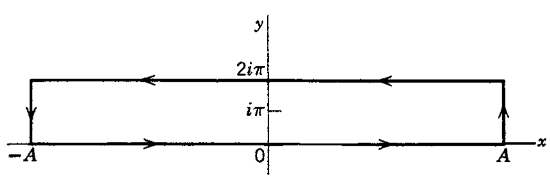
33. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}\,dx}{1+x^2}$

34. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}\,dx}{(1+x)^2}$

35. $\displaystyle\int_{0}^{\infty} \frac{x^{1/3}\,dx}{(1+x)(2+x)}$

36. $\displaystyle\int_{0}^{\infty} \frac{\ln x}{x^{3/4}(1+x)}\,dx$

37.



(a) Show that

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\,dx = \frac{\pi}{\sin \pi p}$$

for $0 < p < 1$. *Hint:* Find $\int e^{pz}\,dz/(1+e^z)$ around the rectangular contour shown. Show that the integrals along the vertical sides tend to zero as $A \to \infty$. Note that the integral along the upper side is a multiple of the integral along the x axis.

(b) Make the change of variable $y = e^x$ in the x integral of part (a), and using (6.5) of Chapter 11, show that this integral is the beta function, $B(p, 1-p)$. Then using (7.1) of Chapter 11, show that $\Gamma(p)\Gamma(1-p) = \pi/\sin \pi p$.

38. Using the same contour and method as in Problem 37a evaluate

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^x}\,dx, \qquad 0<p<1.$$

Hint: The only difference between this problem and Problem 37a is that you now have two simple poles on the contour instead of a pole inside. Use the rule of Example 4.

39. Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{2\pi x/3}}{\cosh \pi x}\,dx.$$

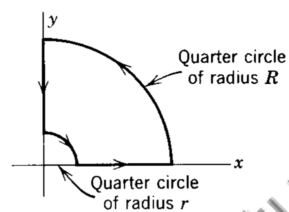
Hint: Use a rectangle as in Problem 37a but of height 1 instead of 2π. Note that there is a pole at $i/2$.

40. Evaluate

$$\int_0^{\infty} \frac{x\,dx}{\sinh x}.$$

Hint: First find the $-\infty$ to ∞ integral. Use a rectangle of height π and note the simple pole at $i\pi$ on the contour.

41. The Fresnel integrals, $\int_0^u \sin u^2\,du$ and $\int_0^u \cos u^2\,du$, are important in optics. For the case of infinite upper limits, evaluate these integrals as follows: Make the change of variable $x = u^2$; to evaluate the resulting integrals, find $\oint z^{-1/2} e^{iz}\,dz$ around the contour shown. Let $r \to 0$ and $R \to \infty$ and show that the integrals along these quarter-circles tend to zero. Recognize the integral along the y axis as a Γ function and so evaluate it. Hence evaluate the integral along the x axis; the real and imaginary parts of this integral are the integrals you are trying to find.



42. If $F(z) = f'(z)/f(z)$,

(a) show that the residue of $F(z)$ at an nth order zero of $f(z)$, is n. *Hint:* If $f(z)$ has a zero of order n at $z = a$, then

$$f(z) = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \cdots.$$

(b) Also show that the residue of $F(z)$ at a pole of order p of $f(z)$, is $-p$. *Hint:* See the definition of a pole of order p at the end of Section 4.

43. By using theorem (7.8), show that $z^3 + z^2 + 9 = 0$ has exactly one root in the first quadrant. Hence show that it has one root in the fourth quadrant and one on the negative real axis. *Hint*: See Example 6.

44. The *fundamental theorem of algebra* says that every equation of the form $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$, $a_n \neq 0$, $n \geq 1$, has at least one root. From this it follows that an nth degree equation has n roots. Prove this by using the argument principle. *Hint:* Follow the increase in the angle of $f(z)$ around a very large circle $z = re^{i\theta}$; for sufficiently large r, all roots are inclosed, and $f(z)$ is approximately $a_n z^n$.

As in Problem 43 find out in which quadrants the roots of the following equations lie:

45. $z^3 + z^2 + z + 4 = 0$

46. $z^3 + 3z^2 + 4z + 2 = 0$

47. $z^3 + 4z^2 + 12 = 0$

48. $z^4 - z^3 + 6z^2 - 3z + 5 = 0$

49. $z^4 - 4z^3 + 11z^2 - 14z + 10 = 0$ **50.** $z^4 + z^3 + 4z^2 + 2z + 3 = 0$

51. Use (7.8) to evaluate

$$\oint_C \frac{f'(z)}{f(z)}\,dz, \qquad \text{where} \quad f(z) = \frac{z^3(z+1)^2 \sin z}{(z^2+1)^2(z-3)},$$

around the circle $|z| = 2$; around $|z| = \frac{1}{2}$.

52. Use (7.8) to evaluate $\displaystyle\oint \frac{z^3\,dz}{1+2z^4}$ around $|z| = 1$.

53. Use (7.8) to evaluate $\displaystyle\oint \frac{z^3+4z}{z^4+8z^2+16}\,dz$ around the circle $|z-2i| = 2$.

54. Use (7.8) to evaluate

$$\oint_C \frac{\sec^2(z/4)\,dz}{1-\tan(z/4)},$$

where C is the rectangle formed by the lines $y = \pm 1$, $x = \pm\frac{5}{2}\pi$.

Find the inverse Laplace transform of the following functions by using (7.16).

55. $\dfrac{p^3}{p^4+4}$ *Hint:* Use (6.2). **56.** $\dfrac{1}{p^4-1}$

57. $\dfrac{p+1}{p(p^2+1)}$ **58.** $\dfrac{p^3}{p^4-16}$ **59.** $\dfrac{3p^2}{p^3+8}$

60. $\dfrac{1}{p^2(p+1)}$ **61.** $\dfrac{p^5}{p^6-64}$ **62.** $\dfrac{(p-1)^2}{p(p+1)^2}$

63. $\dfrac{p}{p^4-1}$ **64.** $\dfrac{p^2}{(p^2-1)(p^2-4)}$ **65.** $\dfrac{p}{(p+1)(p^2+4)}$

66. In equation (7.18), let $u(x)$ be an even function and $v(x)$ be an odd function.

(a) If $f(x) = u(x)+iv(x)$, show that these conditions are equivalent to the equation $f^*(x) = f(-x)$.

(b) Show that

$$\pi u(a) = PV\int_0^\infty \frac{2xv(x)}{x^2-a^2}\,dx, \qquad \pi v(a) = -PV\int_0^\infty \frac{2au(x)}{x^2-a^2}\,dx.$$

These are the Kramers-Kronig relations. *Hint:* To find $u(a)$, write the integral for $u(a)$ in (7.18) as an integral from $-\infty$ to 0 plus an integral from 0 to ∞. Then in the $-\infty$ to 0 integral, replace x by $-x$ to get an integral from 0 to ∞, and use $v(-x) = -v(x)$. Add the two 0 to ∞ integrals and simplify. Similarly find $v(a)$.

▶ 8. THE POINT AT INFINITY; RESIDUES AT INFINITY

It is often useful to think of the complex plane as corresponding to the surface of a sphere in the following way. In Figure 8.1, the sphere is tangent to the plane at the origin O. Let O be the south pole of the sphere, and N be the north pole of the sphere. If a line through N intersects the sphere at P and the plane at Q, we say that the point P on the sphere and the point Q on the plane are corresponding points. Then we have a one-to-one correspondence between points on the sphere (except N) and points of the plane (at finite distances from O). Imagine point Q moving farther and farther out away from O; then P moves nearer and nearer to N. If $z = x + iy$ is the complex coordinate of Q, then as Q moves out farther and farther from O, we would say $z \to \infty$. It is customary to say that the point N corresponds to the *point at infinity* in the complex plane. Observe that straight lines through the origin in the plane correspond to meridians of the sphere. The meridians all pass through both the north pole and the south pole. Corresponding to this, straight lines through the origin in the complex plane pass through the point at infinity. Circles in the complex plane with center at O correspond to parallels of latitude on the sphere. This mapping of the complex plane onto a sphere (or the mapping of the sphere onto a tangent plane) is called a *stereographic projection*.

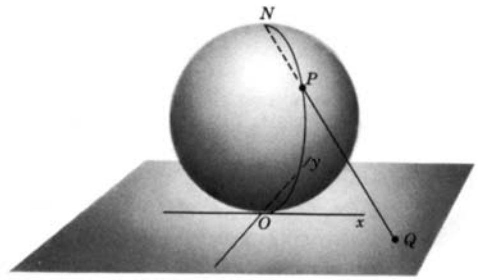


Figure 8.1

To investigate the behavior of a function at infinity, we replace z by $1/z$ and consider how the new function behaves at the origin. We then say that infinity is a regular point, a pole, etc., of the original function, depending on what the new function does at the origin. For example, consider z^2 at infinity; $1/z^2$ has a pole of order 2 at the origin, so z^2 has a pole of order 2 at infinity. Or consider $e^{1/z}$; since e^z is analytic at $z = 0$, $e^{1/z}$ is analytic at ∞.

Next we want to see how to find the residue of a function at ∞. To do this, we want to replace z by $1/z$ and work around the origin. In order to keep our notation straight, let us use two variables, namely Z which takes on values near ∞, and $z = 1/Z$ which takes on values near 0. The residue of a function at ∞ is defined so that the residue theorem holds, that is

$$\oint_C f(Z)\,dZ = 2\pi i \cdot (\text{residue of } f(Z) \text{ at } Z = \infty) \tag{8.1}$$

if C is a closed path around the point at ∞ but inclosing no other singular points. Now what does it mean to integrate "around ∞"? Recall that we have agreed to traverse contours so that the area inclosed always lies to our left. The area we wish to "inclose" is the area "around ∞"; if C is a circle, this area would lie *outside* the circle in our usual terminology. Figure 8.1 may clarify this. Imagine a small circle about the north pole; the area inside this circle (that is, the area including N) corresponds to points in the plane which are outside a large circle C. We must go around C in the clockwise direction in order to have the area "around ∞" to our left. Note that if $Z = Re^{i\Theta}$, then in going clockwise around C, we are going in the direction of *decreasing* Θ. Let us make the following change of variable in the integral (8.1):

$$Z = \frac{1}{z}, \qquad dZ = -\frac{1}{z^2}\,dz.$$

If $Z = Re^{i\Theta}$ traverses a circle C of radius R in the direction of decreasing Θ, then $z = 1/Z = (1/R)e^{-i\Theta} = re^{i\theta}$ traverses a circle C' of radius $r = 1/R$ in the counterclockwise direction (that is, $\theta = -\Theta$ increases as Θ decreases). Thus (8.1) becomes

$$\oint_{C'} -\frac{1}{z^2} f\left(\frac{1}{z}\right) dz = 2\pi i \cdot \text{residue of } f(Z) \text{ at } Z = \infty. \tag{8.2}$$

The integral in (8.2) is an integral about the origin and so can be evaluated by calculating the residue of $(-1/z^2)f(1/z)$ at the origin. (There are no other singular points of $f(1/z)$ inside C' because we assumed that there were no singular points of $f(Z)$ outside C except perhaps ∞.) Thus we have

$$(\text{residue of } f(Z) \text{ at } Z = \infty) = -\left(\text{residue of } \frac{1}{z^2} f\left(\frac{1}{z}\right) \text{ at } z = 0\right) \tag{8.3}$$

and we can use the methods we already know for computing residues at the origin.

Note that a function may be analytic at ∞ and still have a residue there.

▶ **Example.** $f(Z) = 1/Z$ is analytic at ∞ because z is analytic at the origin. But the residue of $f(Z) = 1/Z$ at $Z = \infty$ is

$$-\left(\text{residue of } \frac{1}{z^2} \cdot z \text{ at } z = 0\right) = -1.$$

▶ PROBLEMS, SECTION 8

1. Let $f(z)$ be expanded in the Laurent series that is valid for all z *outside* some circle, that is, $|z| > M$ (see Section 4). This series is called the Laurent series "about infinity." Show that the result of integrating the Laurent series term by term around a very large circle (of radius $> M$) in the positive direction, is $2\pi i b_1$ (just as in the original proof of the residue theorem in Section 5). Remember that the integral "around ∞ " is taken in the negative direction, and is equal to $2\pi i\cdot$(residue at ∞). Conclude that $R(\infty) = -b_1$. *Caution*: In using this method of computing $R(\infty)$, be sure you have the Laurent series that converges for all sufficiently large z.

2. (a) Show that if $f(z)$ tends to a finite limit as z tends to infinity, then the residue of $f(z)$ at infinity is $\lim_{z\to\infty} z^2 f'(z)$.

(b) Also show that if $f(z)$ tends to zero as z tends to infinity, then the residue of $f(z)$ at infinity is $-\lim_{z\to\infty} zf(z)$.

Find out whether infinity is a regular point, an essential singularity, or a pole (and if a pole, of what order) for each of the following functions. Using Problem 1, or Problem 2, or (8.3), find the residue of each function at infinity. Check your results by computer.

3. $\dfrac{z}{z^2+1}$ **4.** $\dfrac{2z+3}{(z+2)^2}$ **5.** $\sin\dfrac{1}{z}$ **6.** $\dfrac{z^2+5}{z}$

7. $\dfrac{4z^3+2z+3}{z^2}$ **8.** $\dfrac{z^2+2}{3z^2}$ **9.** $\dfrac{z^2-1}{z^2+1}$ **10.** $\dfrac{1+z}{1-z}$

11. $\tan\dfrac{1}{z}$ **12.** $\ln\dfrac{z+1}{z-1}$

13. Give another proof of the fundamental theorem of algebra (see Problem 7.44) as follows. Let $I = \oint f'(z)/f(z)\,dz$ about infinity, that is, in the negative direction around a very large circle C. Use the argument principle (7.8), and also evaluate I by finding the residue of $f'(z)/f(z)$ at infinity; thus show that $f(z)$ has n zeros inside C.

Evaluate the following integrals by computing residues at infinity. Check your answers by computing residues at all the finite poles. (It is understood that $\oint$ means in the positive direction.)

14. $\displaystyle\oint \frac{1-z^2}{1+z^2}\frac{dz}{z}$ around $|z|=2$. **15.** $\displaystyle\oint \frac{z^2\,dz}{(2z+1)(z^2+9)}$ around $|z|=5$.

16. Observe that in Problems 14 and 15 the sum of the residues at finite points plus the residue at infinity is zero. Prove that this is always true for a function which has a finite number of singularities.

9. MAPPING

We often find it useful to sketch a graph of a given function $y = f(x)$ of a real variable x. Imagine trying to make a similar sketch for a function $w = f(z)$ of a complex variable z. We need a plane to plot values of z and another plane to plot values of $w = f(z)$, that is, we need a four-dimensional space. Lacking this, we must resort to a different method. Imagine trying to "graph" $y = f(x)$ using only two straight lines, but not a plane. A "graph" of $y = x^2$ might look like Figure 9.1. Given a point on the x axis, we can locate a corresponding point $y = f(x)$ on the y axis and label the two points with the same letter to indicate this correspondence. (Note that to finish our "graph," we really need a second positive y axis to hold the y points corresponding to negative values of x.)

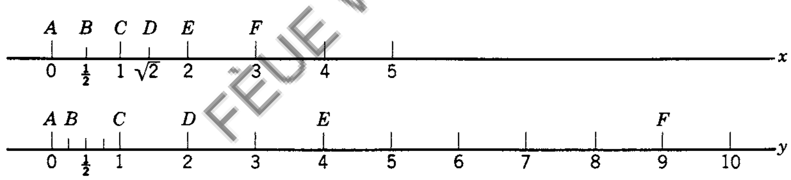


Figure 9.1

Now consider a similar method of representing a function of a complex variable $w = f(z)$. We use a z plane and a w plane; a given point in the z plane (that is, a value of z) determines a corresponding value of w, that is, a point in the w plane. The pair of points, one z and one w, are called *images* of each other. Although we *could* label pairs of corresponding z and w points (as we did corresponding x and y points in Figure 9.1), it is usually more interesting to sketch corresponding curves or regions in the two planes. The correspondence between a point (or curve or region) in the z plane, and the image point (or curve or region) in the w plane, is called a *mapping* or a *transformation*.

Example 1. Consider the function $w = i + ze^{i\pi/4}$, and let us map the grid of coordinate lines $x =$ const., $y =$ const. (z plane in Figure 9.2) into the w plane. You may be able to see at once that this transformation amounts to a rotation of the grid through an angle of $\pi/4$ (since $ze^{i\pi/4} = re^{i(\theta+\pi/4)}$) plus a translation i (the image of $z = 0$ is $w = i$), giving the result shown in the w plane in Figure 9.2. Alternatively,

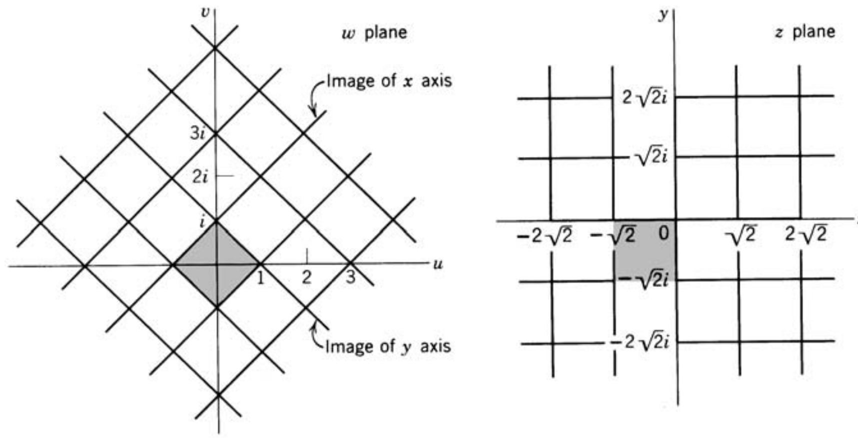


Figure 9.2

we can compute u and v as follows:

$$\begin{aligned} w &= i + ze^{i\pi/4} = i + (x+iy)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\ &= i + (x+iy)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{x-y}{\sqrt{2}} + i\left(1 + \frac{x+y}{\sqrt{2}}\right). \end{aligned}$$

Since $w = u + iv$, we have

$$u = \frac{x-y}{\sqrt{2}}, \qquad v = 1 + \frac{x+y}{\sqrt{2}}. \tag{9.1}$$

Then (eliminating x and y in turn), we have

$$u - v = -1 - y\sqrt{2}, \qquad u + v = 1 + x\sqrt{2}. \tag{9.2}$$

The image of the x axis ($y = 0$) is, from the first equation in (9.2), $u - v = -1$; the image of the y axis ($x = 0$) is, from the second equation in (9.2), $u + v = 1$. Plotting these lines in the w plane, and also plotting the images of $x = \pm\sqrt{2}$, $x = \pm 2\sqrt{2}$, $y = \pm\sqrt{2}$, $y = \pm 2\sqrt{2}$ [using the equations (9.2)], we get Figure 9.2. (Verify that the shaded squares are images of each other.)

If the elimination [to get (9.2)] is not easy, we can use equations (9.1) directly. Suppose that we want the image of $y = 0$. With $y = 0$, equations (9.1) become $u = x/\sqrt{2}$, $v = 1 + x/\sqrt{2}$; these are a pair of parametric equations for a curve in the (u, v) plane, with x as the parameter. Similarly, to find the image of $x =$ const., we substitute the value of x into (9.1); we then have a pair of parametric equations with y as the parameter.

Note that we could just as easily have found the images in the z plane of the lines $u =$ const., and $v =$ const. For example, letting $u = 0$ in (9.1), we get $x - y = 0$; the image of the v axis ($u = 0$) is the 45° line in the (x, y) plane. (We might have guessed that going back to the z plane would involve a rotation through $-45°$.) In any given problem, we may start with simple curves (or regions) in either the z plane or the w plane, and find their images in the other plane.

Example 2. Let us map the coordinate grid $u = \text{const.}$, $v = \text{const.}$, into the z plane by the function $w = z^2$. We have

$$w = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy,$$
$$u = x^2 - y^2, \qquad v = 2xy. \tag{9.3}$$

Then the images of $u = \text{const.}$ are hyperbolas $x^2 - y^2 = \text{const.}$, and the images of $v = \text{const.}$ are also hyperbolas $xy = \text{const.}$ (Figure 9.3). Alternatively, we could map the lines $x = \text{const.}$, $y = \text{const.}$ into the w plane (Problem 1); this gives two sets of parabolas in the (u, v) plane. Accurate graphs can be obtained by computer.

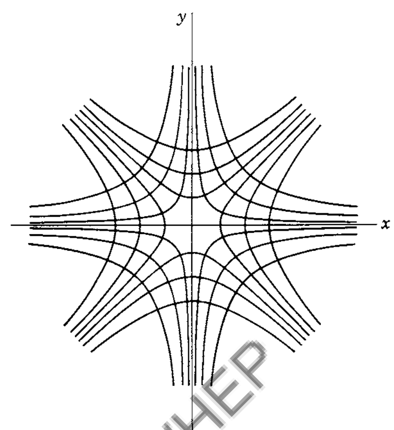


Figure 9.3

Example 3. Let us consider still another useful way of discussing the mapping by $w = z^2$. Using polar coordinates, we have

$$z = re^{i\theta}, \qquad w = z^2 = r^2 e^{2i\theta}. \tag{9.4}$$

Consider the region inside the circle $r = 1$ in the (x, y) plane. If $r = 1$ in (9.4), we have $z = e^{i\theta}$, $w = e^{2i\theta}$. The angle of w is twice the angle of z; thus the first-quadrant part of the area inside the circle $r = 1$ in the z plane maps into a semicircular area in the w plane as indicated by the shading in Figure 9.4. The second quadrant of the z plane disk (θ between $\pi/2$ and π) maps into the lower half of the disk in the w plane (angle of w between π and 2π) as indicated. We have now used up the whole area of the disk in the w plane and only half of the z plane disk. (Compare Figure 9.1 and the comment about a second y axis.) In order to have a one-to-one correspondence between points in the z plane and their images in the w plane, we draw a second w plane (w plane II in Figure 9.4) to contain the images of points in the lower half of the z plane. (Convince yourself that the two lower quarter-disks in the z plane and their images in w plane II are correctly indicated by the shading.) We agree that as we reach the angle 2π in w plane I, we go over to w plane II, and as we reach the angle 4π in w plane II, we go back to w plane I. The two w planes joined in this way are called a *Riemann surface*; each plane is called a *sheet* of the Riemann surface. Note that the line along which the sheets of the Riemann surface are joined (positive real axis here) is a branch cut, and the origin is a branch point (see Example 5, Section 7). Here the branch cut and Riemann surface are in

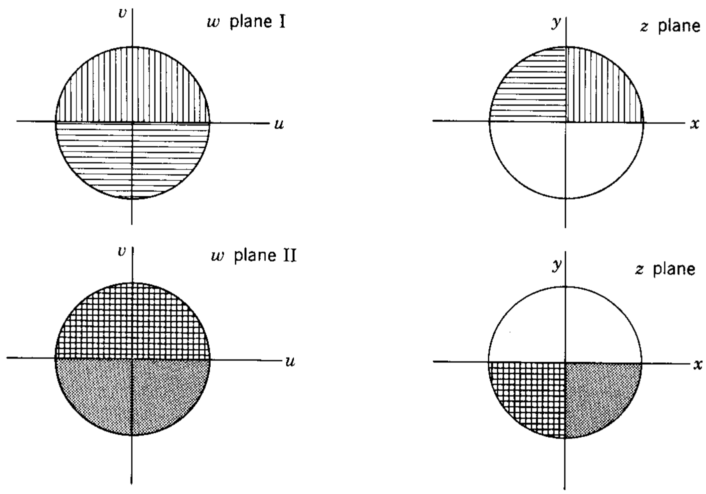


Figure 9.4

the w plane because $z = \sqrt{w}$ has two branches; for $w = \sqrt{z}$, the Riemann surface would be in the z plane (as in Section 7). It is not necessary to take the branch cut along the positive x axis; we can select any 2π interval for one branch of $\sqrt{w}$, for example from $-\pi$ to π instead of 0 to 2π. A Riemann surface may have many sheets; for example for $w = z^5$, there are 5 sheets, and for $w = \ln z$, there are an infinite number. For more detail, see complex variables texts.

Conformal Mapping We have been discussing mappings or transformations. We used the term *transformation* in Chapters 3 and 10, meaning a change of variables or a change of coordinate system or a change of basis; let us see the connection between the two discussions. In Chapter 10 we used only one plane [the (x, y) plane]; we located a point in the (x, y) plane by giving its rectangular coordinates (x, y), or its polar coordinates (r, θ), or some other coordinates (u, v). In polar coordinates, the circles $r =$ const., and the rays $\theta =$ const., were sketched in the (x, y) plane. Similarly, for any coordinate system (u, v) (in Chapter 10, see Section 8 and the Section 8 problems), we sketched the curves $u =$ const., $v =$ const., in the (x, y) plane. In the complex variable language we are now using, this amounts to mapping the w plane lines $u =$ const., $v =$ const., into the z plane. In Chapter 10 we were particularly interested in transformations to orthogonal curvilinear coordinates. Let us see that any analytic function $w = f(z) = u + iv$ gives us a transformation to an orthogonal coordinate system (u, v). We have

$$\begin{aligned} dz &= dx + i\,dy, & dw &= du + i\,dv, \\ |dz|^2 &= dx^2 + dy^2, & |dw|^2 &= du^2 + dv^2. \end{aligned} \tag{9.5}$$

Then the square of the arc length element in the (x, y) plane is

$$ds^2 = dx^2 + dy^2 = |dz|^2 = \left|\frac{dz}{dw}\right|^2 |dw|^2 = \left|\frac{dz}{dw}\right|^2 (du^2 + dv^2). \tag{9.6}$$

Since there is no $du\,dv$ term in ds^2, the (u, v) coordinate system is orthogonal (Chapter 10, Section 8). By this we mean that if we obtain $u(x, y)$ and $v(x, y)$ from

$f(z) = u + iv$ and plot the curves $u(x, y) =$ const., $v(x, y) =$ const. in the (x, y) plane, we have two sets of mutually orthogonal curves. These are the coordinate curves for the (u, v) coordinate systems as in Chapter 10. If we solve the equations $u = u(x, y)$, $v = v(x, y)$ for x and y in terms of u and v, we have the transformation equations from the variables x, y to the variables u, v as in Problems 8.6 to 8.9 of Chapter 10, and by (9.6) we know that the coordinate system (u, v) is an orthogonal system [if $f(z)$ is analytic]. We see an example of this in Figure 9.3 (two orthogonal sets of hyperbolas). Note from (9.6) that the two scale factors in a (u, v) coordinate system obtained this way are equal.

Although we used only one plane in Chapter 10, for complex variables we find it useful to consider both the z plane [that is the (x, y) plane] and the w plane [that is, the (u, v) plane]. In the (x, y) plane, the arc length element ds is given by $ds^2 = dx^2 + dy^2$. Similarly, in the (u, v) plane, the arc length element (which we shall call dS) is given by $dS^2 = du^2 + dv^2$. From (9.5) we see that $ds = |dz|$ and $dS = |dw|$. Then the ratio of dS to ds is $|dw/dz|$. Consider a point z (and its image w) at which $w(z)$ is analytic and dw/dz is not zero. If we stay near z, the value of dw/dz is almost constant, and the ratio dS/ds is nearly constant. This says that if we consider a small area in the z plane ($ABCD$ in Figure 9.5) and its image ($A'B'C'D'$ in Figure 9.5) in the w plane, then

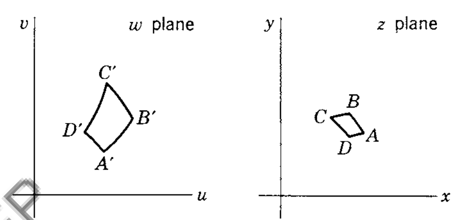


Figure 9.5

$$(9.7) \qquad \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA} = \frac{dS}{ds} = \left|\frac{dw}{dz}\right|,$$

that is, the two small areas are similar figures (since corresponding sides are proportional). Because of this property of any mapping by an analytic function, we call the mapping or transformation *conformal* (same form or shape). Corresponding angles are equal ($A = A'$, etc.) and the net result of the transformation is to magnify (or minify) and rotate each infinitesimal area. Note that the conformal property is a local one; since the value of dw/dz changes from point to point, each tiny bit of a figure is magnified and rotated by a different amount, and so a large figure will not have the same shape after mapping. Also note that we do not have conformality in the neighborhood of a point where $dw/dz = 0$; for example, in Figure 9.4 a tiny quarter-circle about the origin in the z plane maps into a tiny semicircle in the w plane.

PROBLEMS, SECTION 9

In these problems you should be able to make rough sketches by hand, but for accurate graphs use a computer.

1. Solve equations (9.3) for x and y in terms of u and v. Use your equations to sketch the images in the w plane of the z plane lines $x =$ const. (for several values of x) and similarly of $y =$ const.

For each of the following functions $w = f(z) = u + iv$, find u and v as functions of x and y. Sketch the graphs in the (x, y) plane of the images of $u = \text{const.}$ and $v = \text{const.}$ for several values of u and several values of v as was done for $w = z^2$ in Figure 9.3. The curves $u = \text{const.}$ should be orthogonal to the curves $v = \text{const.}$

2. $w = \dfrac{z+1}{2i}$ **3.** $w = \dfrac{1}{z}$ **4.** $w = e^z$ **5.** $w = \dfrac{z-i}{z+i}$

6. $w = \sqrt{z}$. *Hint:* This is equivalent to $w^2 = z$; find x and y in terms of u and v and then solve the pair of equations for u and v in terms of x and y. Note that this is really the same problem as Problem 1 with the z and w planes interchanged.

7. $w = \sin z$ **8.** $w = \cosh z$

Describe the Riemann surface for

9. $w = z^3$ **10.** $w = \sqrt{z}$ **11.** $w = \ln z$

12. If $w = f(z) = u(x, y) + iv(x, y)$, $f(z)$ analytic, defines a transformation from the variables x, y to the variables u, v, show that the Jacobian of the transformation (Chapter 5, Section 4) is $\partial(u, v)/\partial(x, y) = |f'(z)|^2$. *Hint:* To simplify the determinant, use the Cauchy-Riemann equations and the equations (Section 2) used in obtaining them.

13. Verify the matrix equation

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \mathrm{J} \begin{pmatrix} dx \\ dy \end{pmatrix},$$

where J is a matrix whose determinant is the Jacobian in Problem 12. Multiply the matrix equation by its transpose and use Problem 12 to obtain $dS/ds = |dw/dz|$ as in (9.7).

14. We have discussed the fact that a conformal transformation magnifies and rotates an infinitesimal geometrical figure. We showed that $|dw/dz|$ is the magnification factor. Show that the angle of dw/dz is the rotation angle. *Hint:* Consider the rotation and magnification of an arc $dz = dx + idy$ (of length ds and angle arc tan dy/dx) which is required to obtain the image of dz, namely dw.

15. Compare the directional derivative $d\phi/ds$ (Chapter 6, Section 6) at a point and in the direction given by dz in the z plane, and the directional derivative $d\phi/dS$ in the direction in the w plane given by the image dw of dz. Hence show that the rate of change of T in a given direction in the z plane is proportional to the corresponding rate of change of T in the image direction in the w plane. (See Section 10, Example 2.) Show that the proportionality constant is $|dw/dz|$. *Hint:* See equations (9.6) and (9.7).

▶ 10. SOME APPLICATIONS OF CONFORMAL MAPPING

Many different physical problems require solution of Laplace's equation. We are going to show how to solve a few such problems by conformal mapping. Much of this work can be done by computer. But before you can use the computer, you need to know the basic theory behind the use of conformal mapping. Our purpose in this section is to learn this background. First we consider a very simple problem for which we know the answer from elementary physics.

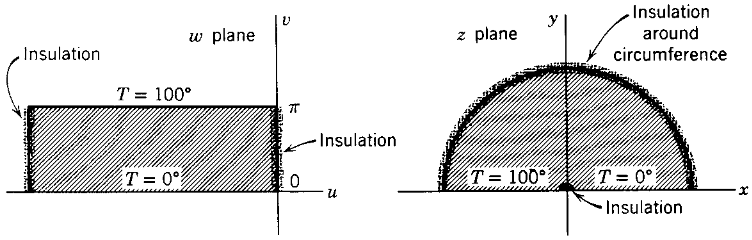


Figure 10.1

▶ **Example 1.** In Figure 10.1, the shaded area in the (u, v) plane represents a rectangular plate. The ends and faces of the plate are insulated, the bottom edge is held at temperature $T = 0°$, and the top edge at $T = 100°$. Then we know from elementary physics that the temperature increases linearly from the bottom edge ($v = 0$) to the top edge ($v = \pi$), that is, $T = (100/\pi)v$ at any point of the plate. Let us also derive this answer by a more advanced method. It is known from the theory of heat that the temperature T of a body satisfies Laplace's equation in regions where there is no source of heat. (See Chapter 13, Section 2.) In our problem we want a solution of Laplace's equation which satisfies the *boundary conditions*, that is, $T = 100°$ when $v = \pi$, $T = 0°$ when $v = 0$, and $\partial T/\partial u = 0$ on the ends (see Chapter 13, Problem 2.14). You should verify that $T = 100v/\pi$ satisfies $\partial^2 T/\partial u^2 + \partial^2 T/\partial v^2 = 0$, and satisfies all the boundary conditions. Also note that an easy way to know that v satisfies Laplace's equation is to observe that it is the imaginary part of $w = u + iv$, and use Theorem IV of Section 2 which says that the real and imaginary parts of an analytic function of a complex variable satisfy Laplace's equation.

Now let us use our results to solve a harder problem.

▶ **Example 2.** Consider the mapping of the rectangle in the w plane into the z plane by the function $w = \ln z$ (Figure 10.1, z plane). We have

$$(10.1) \qquad \begin{aligned} w &= \ln z = \ln(re^{i\theta}) = \ln r + i\theta = u + iv, \\ u &= \ln r, \qquad v = \theta. \end{aligned}$$

Then $v = 0$ maps into $\theta = 0$, that is, the positive x axis; $v = \pi$ maps into $\theta = \pi$, that is, the negative x axis (z plane, Figure 10.1). The insulated end of the rectangle at $u = 0$ maps into $\ln r = 0$ or $r = 1$; the left-hand end of the rectangle maps into a small semicircle about the origin which we can think of as a bit of insulation at the origin separating the 0° and 100° parts of the x axis. (If the left-hand end of the rectangle is at $u = -\infty$, we have $\ln r = -\infty$, $r = 0$, and the image is just the origin; for finite negative u, the image is a semicircle with $r < 1$.) We can now solve the problem indicated by the picture in the z plane of Figure 10.1. A semicircular plate has its faces and its curved boundary insulated, and has half its flat boundary at 0° and the other half at 100° (with a bit of insulation at the center). Find the temperature T at any point of the plate. To solve this problem we need only transform our solution in the (u, v) plane to the variables x, y by using (10.1). Thus we find

$$(10.2) \qquad T = \frac{100}{\pi} V = \frac{100}{\pi}\theta = \frac{100}{\pi}\arctan\frac{y}{x}, \qquad 0 \le \theta \le \pi.$$

It is not hard to justify our method; we need to show that our solution satisfies Laplace's equation and that it satisfies the boundary conditions. It is straightforward to show (Problem 1) that if a function $\phi(u, v)$ satisfies Laplace's equation $\partial^2\phi/\partial u^2 + \partial^2\phi/\partial v^2 = 0$, then the function of x and y obtained by substituting $u = u(x, y)$, $v = v(x, y)$ in ϕ satisfies Laplace's equation in x and y, where u and v are the real and imaginary parts of an analytic function $w = f(z)$. Thus we know that (10.2) satisfies Laplace's equation (or in this case you can easily verify the fact directly). We must also know that the transformed T satisfies the boundary conditions; this is where conformal mapping is so useful. Observe in Figure 10.1 that we had a transformation which took the boundaries of a simple region (a rectangle) for which we knew the solution of the temperature problem, into the boundaries of a more complicated region for which we wanted the solution. This is the basic method of conformal mapping—to transform from a simple region where you know the answer to a given problem, to the region in which you want the solution. The temperature at any (x, y) point is the same as the temperature at the (u, v) image point, since we obtain the temperature as a function of x and y by the same substitution $u = u(x, y)$, $v = v(x, y)$ that we use to obtain image points. Thus the temperatures on the boundaries of the transformed region are the same as the temperatures on the corresponding boundaries of the simpler (u, v) region. Similarly, isothermals (curves of constant temperature) transform into isothermals; in this problem the (u, v) isothermals are the lines $v =$ const., and so the (x, y) isothermals are $\theta =$ const. You can show that the rate of change of T in a direction perpendicular to a boundary in the (u, v) plane is proportional to the corresponding rate of change of T in a direction perpendicular to the image boundary in the (x, y) plane (Problem 9.15). Thus insulated boundaries (across which the rate of change of T is zero) map into insulated boundaries. The lines (or curves) perpendicular to the isothermals give the direction of flow of heat; in Figure 10.1 heat flows along the lines $u =$ const. in the w plane, and along the circles $r =$ const. (which are the images of $u =$ const.) in the z plane.

Using the same mapping function $w = \ln z$, we can solve a number of other physics problems. Observe first that if we think of Figure 10.1 as representing a cross section of a three-dimensional problem (with all parallel cross sections identical), then (10.2) gives the solution of the three-dimensional problem also. In Figure 10.1 the (u, v) diagram would be the cross section of a slab with faces at $T = 100°$ and $T = 0°$ and all other surfaces insulated (or extending to infinity); the (x, y) diagram would similarly represent half a cylinder. Now let us do a three-dimensional problem in electrostatics.

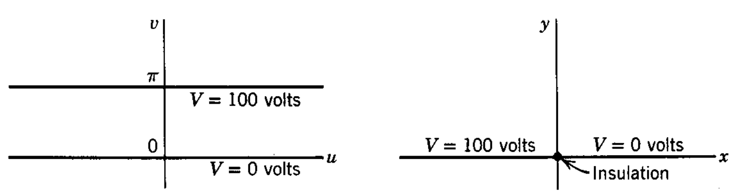


Figure 10.2

▶ **Example 3.** In Figure 10.2 the (u, v) diagram represents (the cross section of) two infinite parallel plates, one at potential $V = 0$ volts and one at potential $V = 100$ volts. The (x, y) diagram represents (the cross section of) one plane with its right-hand half at potential $V = 0$ volts and its left-hand half at $V = 100$ volts. From electricity we know that the electrostatic potential V satisfies Laplace's equation in regions where there is no free charge (see Chapter 13, Section 1). You should convince yourself that the mapping by (10.1) gives the result shown in Figure 10.2, and that the potential is given by

$$V = \frac{100}{\pi} v = \frac{100}{\pi} \theta = \frac{100}{\pi} \arctan \frac{y}{x}, \qquad 0 \le \theta \le \pi$$

as in (10.2). The equipotentials ($V =$ const.) in the (x, y) plane are the lines $\theta =$ const. Recall that the electric field is given by $\mathbf{E} = -\boldsymbol{\nabla} V$, and that the gradient of V is perpendicular to $V =$ const. (Chapter 6, Section 6). Then the direction of the electric field at any point is perpendicular to the equipotential through that point. Thus if we sketch the curves $r =$ const. which are perpendicular to the equipotentials $\theta =$ const., then the tangent to a circle at a point gives the direction of the electric field $\mathbf{E}$ at that point. Note the correspondence between the isothermals of the temperature problem and the equipotentials here, and between the lines of electric flux (curves tangent to $\mathbf{E}$) and the lines or curves along which heat flows.

We can also solve problems in hydrodynamics (see Chapter 6, Section 10) by conformal mapping. We consider a two-dimensional flow of water by which we mean either that we think of the water as flowing in a thin sheet over the (x, y) [or (u, v)] plane, or if it has depth, the flow is the same in all planes parallel to the (x, y) [or (u, v)] plane. Although it is convenient to talk about water, what we actually require is an irrotational flow (see Chapter 6, Section 11) of a nonviscous incompressible fluid. For then (see Problem 2) the velocity $\mathbf{V}$ of the liquid is given by $\mathbf{V} = \boldsymbol{\nabla}\Phi$, where Φ (called the *velocity potential*) satisfies Laplace's equation. Water approximately meets these requirements.

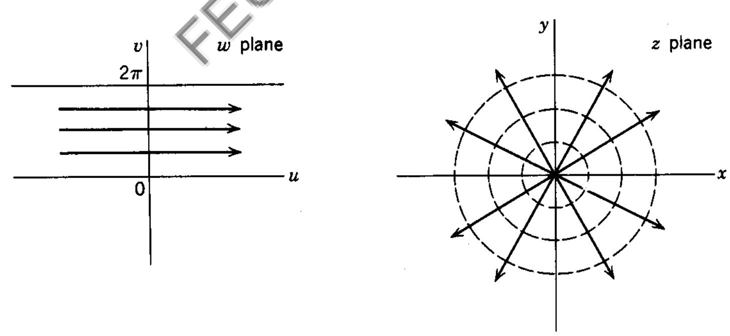


Figure 10.3

Example 4. Figure 10.3 shows two flow patterns related by the same transformation we have used in the heat problem and the electrostatics problem, namely $w = \ln z$. In the w plane of Figure 10.3, we picture water flowing in the u direction at constant speed V_0 down a channel between $v = 0$ and $v = 2\pi$. (Note that v is the imaginary part of $w = u + iv$ and has nothing to do with velocity.) The velocity potential is $\Phi = V_0 u$; for then the velocity, $\mathbf{V} = \boldsymbol{\nabla}\Phi$, has components $\partial\Phi/\partial u = V_0$ in the u direction and $\partial\Phi/\partial v = 0$ in the v direction as we have assumed. The function $\Phi + i\Psi = V_0 w = V_0(u + iv)$ is called the *complex potential*; the function Ψ (conjugate to Φ; see Section 2) is called the *stream function*. The lines $\Psi =$ const. (that is,

$v =$ const. in the w plane) are the lines along which the water flows and are called *streamlines*. Observe that the lines $\Phi =$ const. and the lines $\Psi =$ const. are mutually perpendicular sets of lines. The water flows across lines of constant Φ and along streamlines (constant Ψ); boundaries of the channel ($v = 0$ and $v = 2\pi$) must then be streamlines. The water comes from the left (Figure 10.3, w plane) and goes off to the right; we say that there is a *source* at the left and a *sink* at the right.

Now consider the mapping of the w plane flow of Figure 10.3 into the z plane by the function $w = \ln z$. The complex potential is

$$\Phi + i\Psi = V_0 w = V_0 \ln z = V_0(\ln r + i\theta).$$

The streamlines are $\Psi =$ const., or $\theta =$ const., that is, radial lines; the curves $\Phi =$ const. are circles $r =$ const. and are perpendicular to the streamlines. The velocity is given by

$$\mathbf{V} = \boldsymbol{\nabla}\Phi = V_0\boldsymbol{\nabla}(\ln r) = V_0\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\ln r = \mathbf{e}_r\frac{V_0}{r}.$$

What we are describing, then, is the flow of water from a source at the origin out along radial lines. Since the same amount of water crosses a small circle (about the origin) or a large one, the velocity of the water decreases with r as we have found ($|\mathbf{V}| = V_0/r$).

We can obtain another flow pattern from any given one by interchanging the equipotentials and the streamlines. In Figure 10.3, z plane, this new flow would have the circles $r =$ const. as streamlines and would correspond to a whirlpool motion of the water about the origin (called a *vortex*). There are still other applications of this diagram. The circles $r =$ const. give the direction of the magnetic field about a long current-carrying wire perpendicular to the (x, y) plane and passing through the origin. The radial lines $\theta =$ const. give the direction of the electric field about a similar long wire with a static charge on it. The radial lines give the direction of heat flow from a small hot object at the origin, and the circles $r =$ const. are then the isothermals. By starting with problems like these to which we know the answers and using various conformal transformations, we can solve many other physics problems involving fluid flow, electricity, heat, and so on. Some examples are outlined in the problems and you will find many more in books on complex variables.

▶ **Example 5.** Let us consider one somewhat more complicated example of the use of conformal mapping. We shall be able to solve two interesting physics problems in this example: (1) to find the flow pattern for water flowing out of the end of a straight channel into the open, and (2) to find the edge effect (fringing) at the ends of a parallel-plate capacitor.

We consider the mapping function

$$\begin{aligned} z &= w + e^w = u + iv + e^u e^{iv} = u + iv + e^u(\cos v + i \sin v), \\ x &= u + e^u \cos v, \qquad y = v + e^u \sin v. \end{aligned} \tag{10.3}$$

In Figure 10.4, w plane, we picture a parallel flow of water at constant velocity in the region between the lines DEF and GHI; this is just like the flow of Figure 10.3,

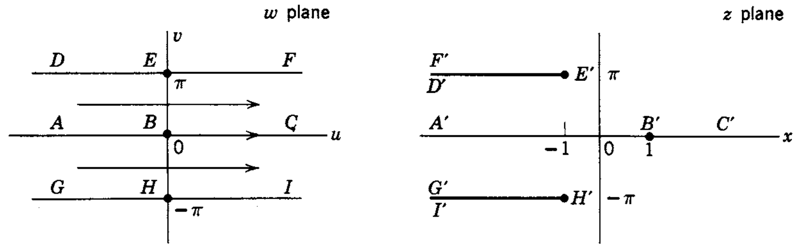


Figure 10.4

w plane. Now let us map the w plane streamlines into the z plane using (10.3). On the u axis, $v = 0$; putting $v = 0$ in (10.3), we find $y = 0$, and $x = u + e^u$. Thus the u axis maps into the x axis ($y = 0$) with $u = -\infty$ corresponding to $x = -\infty$, $u = 0$ corresponding to $x = 1$, and $u = +\infty$ corresponding to $x = +\infty$ as shown in Figure 10.4 (line ABC maps into $A'B'C'$). Now on DEF, $v = \pi$; substituting $v = \pi$ into (10.3), we find $y = \pi$, $x = u + e^u \cos\pi = u - e^u$. However, the image of $v = \pi$ is not the entire line $y = \pi$. To see this consider $x = u - e^u$. We find the maximum value of x for $dx/du = 1 - e^u = 0$, $d^2x/du^2 = -e^u < 0$. These equations are satisfied for $u = 0$, $x = -1$. The point E ($u = 0$, $v = \pi$) maps into the point E' ($x = -1$, $y = \pi$). Thus DE in the w plane maps into the part of the line $y = \pi$ in the z plane up to $x = -1$ with $u = -\infty$ corresponding to $x = -\infty$ and $u = 0$ corresponding to $x = -1$. To see how to map EF, we realize that x has its largest value at $u = 0$ and so decreases as u increases; for very large positive u, $x = u - e^u$ is negative and of large absolute value since $e^u \gg u$. Thus the positive part of $v = \pi$ (EF) maps into the same line segment ($y = \pi$, $x \le -1$) that we obtained for the mapping of the negative part (DE), but this time the line segment ($E'F'$, z plane) is traversed backward. It is as if the line $y = \pi$ were broken at $x = -1$ and bent back upon itself through an angle of 180°. By a parallel discussion of the line GHI, we find that it maps as shown in Figure 10.4 into $G'H'I'$. Other streamlines in the w plane are given by $v =$ const. for any v between $-\pi$ and π. If we substitute $v =$ const. into the x and y equations in (10.3), we have parametric equations (with u as the parameter) for the streamlines in the z plane. For any value of v, these streamlines can be plotted in the z plane: some of them are shown by the solid curves in Figure 10.5. Think of $D'E'$ and $G'H'$ as boundaries of a channel (in the z plane) down which water flows coming from $x = -\infty$. The boundaries stop at $x = -1$ and the water flows out of the channel spreading over the whole plane, including spreading back along the outsides ($E'F'$ and $H'I'$) of the channel boundaries. This is correct according to our mapping, for the boundary streamline DEF mapped into the broken-and-folded-back line $D'E'F'$, and similarly for GHI to $G'H'I'$.

For the electrical application, let DEF and GHI represent (the cross section of) a large parallel plate capacitor. Then the lines $v =$ const. are the equipotentials and the lines $u =$ const. give the direction of the electric field $\mathbf{E}$. The image in the z plane represents (a cross section of) the end of a parallel plate capacitor. The images of the equipotentials $v =$ const. are the equipotentials in the z plane (same as the streamlines, shown as solid curves in Figure 10.5). The images of the lines $u =$ const. (shown as dotted curves in Figure 10.5) give the direction of the electric field at the end of a parallel-plate capacitor. Well inside the plates the $\mathbf{E}$ lines are vertical, but at the end they bulge out; this effect is known as *fringing*.

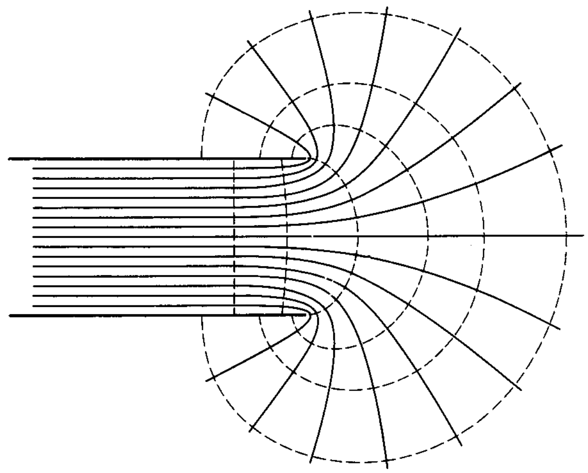


Figure 10.5

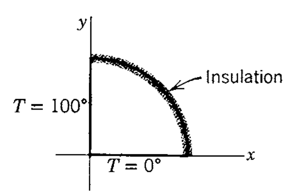
▶ PROBLEMS, SECTION 10

1. Prove the theorem stated just after (10.2) as follows. Let $\phi(u,v)$ be a harmonic function (that is, ϕ satisfies $\partial^2\phi/\partial u^2 + \partial^2\phi/\partial v^2 = 0$). Show that there is then an analytic function $g(w) = \phi(u,v) + i\psi(u,v)$ (see Section 2). Let $w = f(z) = u + iv$ be another analytic function (this is the mapping function). Show that the function $h(z) = g(f(z))$ is analytic. *Hint:* Show that $h(z)$ has a derivative. (How do you find the derivative of a function of a function, for example, $\ln \sin z$?) Then (by Section 2) the real part of $h(z)$ is harmonic. Show that this real part is $\phi(u(x,y), v(x,y))$.

2. A fluid flow is called irrotational if $\nabla \times \mathbf{V} = 0$ where $\mathbf{V}$ = velocity of fluid (Chapter 6, Section 11); then $\mathbf{V} = \nabla\Phi$. Use Problem 10.15 of Chapter 6 to show that if the fluid is incompressible, the Φ satisfies Laplace's equation. (*Caution:* In Chapter 6, we used $\mathbf{V} = \mathbf{v}\rho$, with $\mathbf{v}$ = velocity; here $\mathbf{V}$ = velocity.)

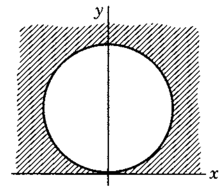
3. Assuming from electricity the equations $\nabla \cdot \mathbf{D} = \rho$, $\mathbf{E} = -\nabla V$, $\mathbf{D} = \epsilon\mathbf{E}$ (ϵ = const.), show that in regions where the free charge density ρ is zero, V satisfies Laplace's equation.

4. Let a flat plate in the shape of a quarter-circle, as shown, have its faces and curved boundary insulated, and its two straight edges held at 0° and 100°. Find the temperature distribution $T(x,y)$ in the plate, and the equations of the isothermals. *Hint:* Use the mapping function $w = \ln z$ as in Figure 10.1; what w plane line maps into the y axis?



5. Consider a capacitor made of two very large perpendicular plates. (Let the positive x and y axes in the diagram of Problem 4 represent a cross section of the capacitor.) Let one plate (x axis) be held at potential $V = 0$, and the other plate (y axis) be held at potential $V = 100$ volts. Find the potential $V(x,y)$ for $x > 0$, $y > 0$, and the equations of the equipotentials. *Hint:* This problem is mathematically the same as Problem 4.

6. Let the figure represent (the cross section of) a hot cylinder (say $T = 100°$) lying on a cold plane (say $T = 0°$). (Separate the two by a bit of insulation.) Find the temperature in the shaded region. Alternatively, let the cylinder and the plane be held at two different electric potentials (with insulation between), and find the electric potential in the shaded region. Find and sketch some of the isothermals (equipotentials) and some of the curves (perpendicular to the isothermals) along which heat flows (lines of flux for the electric case). *Hint:* Use the mapping function $w = 1/z$, and consider the image of the w plane region between $v = 0$ and $v = -1$.



7. Use the mapping function $w = z^2$ to find the streamlines for the flow of water around the inside of a right-angle boundary. Find the velocity potential Φ, the stream function Ψ, and the velocity $\mathbf{V} = \nabla\Phi$.

8. Observe that the magnitude of the velocity in Problem 7 can be obtained from $V = V_0|dw/dz|$. Show that this result holds in general as follows. Let $w = f(z)$ be an analytic mapping function such that the lines $v =$ const. map into the streamlines of the flow you want to consider in the z plane. Then

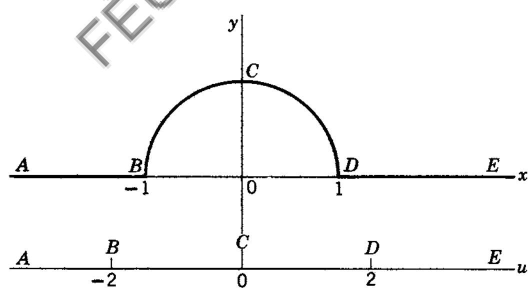
$$V_0 w = V_0(u + iv) = \Phi(x, y) + i\Psi(x, y).$$

Show that

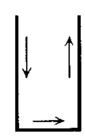
$$V_0 \frac{dw}{dz} = \frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y} = V_x - iV_y$$

(this expression is called the *complex velocity*). Hence show that $V = V_0|dw/dz|$.

9. Find and sketch the streamlines for the flow of water over a semicircular hump (say a half-buried log at the bottom of a stream) as shown. *Hint:* Use the mapping function $w = z + z^{-1}$. Show that the u axis maps into the contour $ABCDE$ with the correspondence shown.



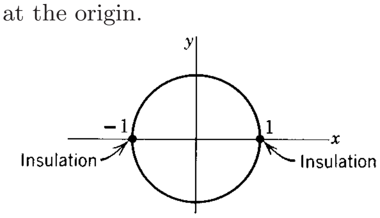
10. Find and sketch the streamlines for the indicated flow of water inside a rectangular boundary. *Hint:* Consider $w = \sin z$; map the u axis into the boundary of the rectangle.



11. For $w = \ln[(z + 1)/(z - 1)]$, show that the images of $u =$ const. and $v =$ const. are two orthogonal sets of circles. Find centers and radii of five or six circles of each set and sketch them. Include the circle with center at the origin.

Use the results of Problem 11 to solve the following physics problems.

12. The figure represents the cross section of a long cylinder (assume it infinitely long) cut in half, with the top half and the bottom half insulated



from each other. Let the surface of the top half be held at temperature $T = 30°$ and the surface of the bottom half at $T = 10°$. Find the temperature $T(x, y)$ inside the cylinder. *Hint:* Show that the line $v = \pi/2$ maps into the lower half of the circle $|z| = 1$, and the line $v = 3\pi/2$ maps into the upper half of the circle.

13. Let the figure in Problem 12 represent (the cross section of) a capacitor with the lower half at potential V_1 and the upper half at potential V_2. Find the potential $V(x, y)$ between the plates (that is, inside the circle). *Hint:* This is almost like Problem 12. Observe that in the text and in Problem 12, the w-plane temperature is of the form Av, with A constant; here you need the potential of the form $Av + B$, A and B constants.

14. In the figure in Problem 12, let $z = -1$ be a source and $z = +1$ a sink, and let the water flow inside the circular boundary. Find $\mathbf{\Phi}$, $\mathbf{\Psi}$, and $\mathbf{V}$. Sketch the streamlines.

15. In Problem 14, the streamlines were the images of $v =$ const. Consider the flow (over the whole plane, that is, with no boundaries) with streamlines $u =$ const. This flow may be described as two vortices rotating in opposite directions. Sketch a number of streamlines indicating the direction of the velocity with arrows. Since a boundary is a streamline, a flow is not disturbed by inserting a boundary along a streamline. Insert two circular boundaries corresponding to $u = a$ and $u = -a$. Show that the velocity through the narrow neck (say at $z = 0$) is greater than the velocity elsewhere (say at $z = i$). You can simplify your calculation of the velocity by showing that the result in Problem 8 holds here also.

16. Two long parallel cylinders form a capacitor. (Let their cross sections be the images of $u = a$ and $u = -a$.) If they are held at potentials V_0 and $-V_0$, find the potential $V(x, y)$ at points between them. given that the charge (per unit length) on a cylinder is $q = V_0/(2a)$, show that the capacitance (per unit length), that is, $q/(2V_0)$, is given by $1/(4\cosh^{-1} d/(2r))$, where d is the distance between the centers of the two cylinders, and their radii are r.

17. Other problems to consider using the mapping function of Problem 11: (a) a capacitor consisting of two long cylinders one inside the other, but not concentric; (b) the magnetic field in a plane perpendicular to two long parallel wires carrying equal but opposite currents; (c) the electric field in a plane perpendicular to two long parallel wires, one charged positive and the other negative; (d) other flow problems obtained by inserting boundaries along streamlines.

▶ 11. MISCELLANEOUS PROBLEMS

In Problems 1 and 2, verify that the given function is harmonic, and find a function $f(z)$ of which it is the real part. *Hint:* Use Problem 2.64. For Problem 2, see Chapter 2, Section 17, Problem 19.

1. $\ln\sqrt{(1+x)^2 + y^2}$ **2.** $\arctan \dfrac{y}{x+1}$

3. Liouville's theorem: Suppose $f(z)$ is analytic for all z (except ∞), and bounded [that is, $|f(z)| \le M$ for all z and some M]. Prove that $f(z)$ is a constant. *Hints:* If $f'(z) = 0$, then $f(z) =$ const. To show this, write $f'(z)$ as in Problem 3.21 where C is a circle of radius R and center z, that is, $w = z + Re^{i\theta}$. Show that $|f'(z)| \le M/R$, and let $R \to \infty$.

4. Use Liouville's theorem (Problem 3) to prove the fundamental theorem of algebra (see Problem 7.44). *Hint:* Let $P(z)$ be a polynomial of degree ≥ 1; then $f(z) = 1/P(z)$ is a bounded analytic function in a region not containing any zeros of $P(z)$. Disprove the assumption that $P(z)$ has no zeros anywhere.

In Problems 5 to 8, find the residues of the given function at all poles. Take $z = re^{i\theta}$, $0 \le \theta < 2\pi$.

5. $\dfrac{z^{1/3}}{1+z^2}$ **6.** $\dfrac{\sqrt{z}}{1+8z^3}$ **7.** $\dfrac{\ln z}{1+z^2}$ **8.** $\dfrac{\ln z}{(2z-1)^2}$

In Problems 9 to 10, use Laurent series to find the residues of the given functions at the origin.

9. $\dfrac{\sin z^2}{z^7}$ **10.** $\dfrac{\ln(1-z)}{\sin^2 z}$

11. Find the Laurent series of $f(z) = e^z/(1-z)$ for $|z| < 1$ and $|z| > 1$. *Hints:* For $|z| < 1$, multiply two power series; you should find $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n = \sum_{k=0}^{n} 1/k!$. For $|z| > 1$, use (4.3) where C is a circle $|z| = a$ with $a > 1$. Evaluate the integrals by finding residues at 1 and 0. You should find $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ where all $b_n = -e$ and $a_n = -e + \sum_{k=0}^{n} 1/k!$.

12. Let $f(z)$ be the branch of $\sqrt{z^2-1}$ which is positive for large positive real values of z. Expand the square root in powers of $1/z$ to obtain the Laurent series of $f(z)$ about ∞. Thus by Problem 8.1 find the residue of $f(z)$ at ∞. Check your result by using equation (8.2).

In Problems 13 and 14, find the residues at the given points.

13. (a) $\dfrac{\cos z}{(2z-\pi)^4}$ at $\dfrac{\pi}{2}$ (b) $\dfrac{2z^2+3z}{z-1}$ at ∞

(c) $\dfrac{z^3}{1+32z^5}$ at $z = -\frac{1}{2}$ (d) $\csc(2z-3)$ at $z = \frac{3}{2}$

14. (a) $\dfrac{\ln(1+2z)}{z^2}$ at 0 (b) $\dfrac{1}{z}\sin(2z+5)$ at ∞

(c) $\dfrac{z^3}{4z^4+1}$ at $\frac{1}{2}(1+i)$ (d) $\dfrac{z\sin 2z}{(z+\pi)^2}$ at $-\pi$

In Problem 15 to 20, evaluate the integrals by contour integration.

15. $\displaystyle\int_0^{\pi} \frac{\cos\theta\, d\theta}{5-4\cos\theta}$ **16.** $\displaystyle\int_0^{2\pi} \frac{\sin\theta\, d\theta}{5+3\sin\theta}$

17. $\displaystyle\int_0^{\infty} \frac{\cos x\, dx}{(4x^2+1)(x^2+9)}$ **18.** $\displaystyle\int_0^{\infty} \frac{x\sin(\pi x/2)}{x^4+4}\, dx$

19. $\displaystyle PV\int_{-\infty}^{\infty} \frac{\sin x\, dx}{(3x-\pi)(x^2+\pi^2)}$ **20.** $\displaystyle PV\int_{-\infty}^{\infty} \frac{\cos x\, dx}{x(1-x)(x^2+1)}$

Verify the formulas in Problem 21 to 27 by contour integration or as indicated. Assume $a > 0$, $m > 0$.

21. $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \qquad |b| < a$

22. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}, \qquad |b| < a$

Hint: You can do this directly by contour integration, but it is easier to differentiate Problem 21 with respect to a.

23. $\displaystyle\int_0^{2\pi} \frac{\sin\theta\, d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{\cos\theta\, d\theta}{a+b\cos\theta} = \frac{2\pi}{b}\left(1 - \frac{a}{\sqrt{a^2-b^2}}\right), \qquad |b| < a$

24. $\displaystyle\int_0^\infty \frac{\cos mx\,dx}{x^2+a^2} = \frac{\pi}{2a}e^{-ma}$

25. $\displaystyle PV\int_0^\infty \frac{\cos mx\,dx}{x^2-a^2} = -\frac{\pi}{2a}\sin ma$

26. $\displaystyle\int_0^\infty \frac{x\sin mx\,dx}{x^2+a^2} = \frac{\pi}{2}e^{-ma}$

27. $\displaystyle PV\int_0^\infty \frac{x\sin mx\,dx}{x^2-a^2} = \frac{\pi}{2}\cos ma$

Hint for Problems 26 and 27: Differentiate Problems 24 and 25 with respect to m.

28. Evaluate $\displaystyle\int_0^\infty \frac{\sqrt{x}\,\ln x\,dx}{(1+x)^2}$ by using the contour of Figure 7.4.
Hint: Along DE, $z = re^{2\pi i}$ so $\ln z = \ln r + 2\pi i$.

29. Evaluate $\displaystyle\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx$ by using the contour of Figure 7.3. *Comment:* Note that your work also shows that $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$.

30. Show that

$$PV\int_0^\infty \frac{\cos(\ln x)}{x^2+1}\,dx = \frac{\pi}{2\cosh(\pi/2)}$$

by integrating $e^{i\ln z}/(z^2-1)$ around a contour like Figure 7.3 but rotated 90° clockwise so the straight side is along the y axis.

As in Section 7, find out how many roots the equations in Problem 31 to 34 have in each quadrant.

31. $z^4 + 3z + 5 = 0$

32. $z^3 + 2z^2 + 5z + 6 = 0$

33. $z^6 + z^3 + 9z + 64 = 0$
(no real roots)

34. $z^8 + 5z^3 + 3z + 4 = 0$
(2 negative real roots)

35. Show that the Cauchy-Riemann equations [see (2.2) and Problem 2.46] in a general orthogonal curvilinear coordinate system [see Chapter 10, Sections 8 and 9] are

$$\frac{1}{h_1}\frac{\partial u}{\partial x_1} = \frac{1}{h_2}\frac{\partial v}{\partial x_2}, \qquad \frac{1}{h_1}\frac{\partial v}{\partial x_1} = -\frac{1}{h_2}\frac{\partial u}{\partial x_2}$$

where, as in Chapter 10, the variables are x_1, x_2 and the scale factors are h_1, h_2. *Hint:* Consider the directional derivatives (Chapter 6, Section 6) in two perpendicular directions. (Compare Problem 2.46.) Also show that u and v satisfy Laplace's equation, Chapter 10, equation (9.10) (drop the x_3 term and set $h_3 = 1$).

36. Show that a harmonic function $u(x,y)$ is equal at every point a to its average value on any circle centered at a [and lying in the region where $f(z) = u(x,y) + iv(x,y)$ is analytic]. *Hint:* In (3.9), let $z = a + re^{i\theta}$ (that is, C is a circle with center at a), and show that the average value of $f(z)$ on the circle is $f(a)$ (see Chapter 7, Section 4 for discussion of the average of a function). Take real and imaginary parts of $f(a) = [u(x,y) + iv(x,y)]_{z=a}$.

37. A (nonconstant) harmonic function takes its maximum value and its minimum value on the boundary of any region (not at an interior point). Thus, for example, the electrostatic potential V in a region containing no free charge takes on its largest and smallest values on the boundary of the region; similarly, the temperature T of a body containing no sources of heat takes its largest and smallest values on the surface of the body. Prove this fact (for two-dimensional regions) as follows: Suppose that it is claimed that $u(x,y)$ takes its maximum value at some interior point a; this means that, at all points of some small disk about a, the values of $u(x,y)$ are no larger than at a. Show by Problem 36 that such a claim leads to a contradiction (unless u = const.). Similarly prove that $u(x,y)$ cannot take its minimum value at an interior point.

38. Show that a Dirichlet problem (see Chapter 13, Section 3) for Laplace's equation in a finite region has a unique solution; that is, two solutions u_1 and u_2 with the same boundary values are identical. *Hint:* Consider $u_2 - u_1$ and use Problem 37. [Also see Chapter 13, discussion following equation (2.17).]

39. Use the following sequence of mappings to find the steady state temperature $T(x, y)$ in the semi-infinite strip $y \geq 0$, $0 \leq x \leq \pi$ if $T(x, 0) = 100°$, $T(0, y) = T(\pi, y) = 0$, and $T(x, y) \to 0$ as $y \to \infty$. (See Chapter 13, Section 2 and Problem 2.6.)

(a) Use $w = (z' - 1)/(z' + 1)$ to map the half plane $v \geq 0$ on the upper half plane $y' > 0$, with the positive u axis corresponding to the two rays $x' > 1$ and $x' < -1$, and the negative u axis corresponding to the interval $-1 \leq x \leq 1$ of the x' axis.

(b) Use $z' = -\cos z$ to map the half-strip $0 < x < \pi$, $y > 0$ on the z' half plane described in (a). The interval $-1 \leq x' < 1$, $y' = 0$ corresponds to the base $0 < x < \pi$, $y = 0$ of the strip.

Comments: The temperature problem in the (u, v) plane is like the problems shown in the z plane of Figures 10.1 and 10.2, and so is given by $T = (100/\pi) \arctan(v/u)$. In the z plane you will find

$$T(x, y) = \frac{100}{\pi} \arctan \frac{2 \sin x \sinh y}{\sinh^2 y - \sin^2 x}.$$

Put $\tan \alpha = \dfrac{\sin x}{\sinh y}$ and use the formula for $\tan 2\alpha$ to get

$$T(x, y) = \frac{200}{\pi} \arctan \frac{\sin x}{\sinh y}.$$

Note that this is the same answer as in Chapter 13, Problem 2.6, if we replace 10 by π.

40. Use $L13$ of the Laplace transform table to find the Laplace transform of $\sin at \sinh at$. Verify your result by finding its inverse transform using the Bromwich integral.

41. Evaluate by contour integration $\displaystyle\int_0^\infty \frac{\cos^2(\alpha\pi/2)}{(1 - \alpha^2)^2}\, d\alpha$.

Hint: $\cos^2(\alpha\pi/2) = (1 + \cos \alpha\pi)/2$. Evaluate $\displaystyle\oint \frac{1 + e^{i\pi z}}{(z - 1)^2(z + 1)^2}\, dz$ around the upper half plane; note that the poles are actually simple poles (see Section 7, Example 4).

CHAPTER 15

Probability and Statistics

1. INTRODUCTION

The theory of probability has many applications in the physical sciences. It is of basic importance in quantum mechanics where results may be expressed in terms of probabilities (see Chapter 13, Schrödinger equation). It is needed whenever we are dealing with large numbers of particles or variables where it is impossible or impractical to have complete information about each one, such as in kinetic theory and statistical mechanics and a great variety of engineering problems. Statistics is the part of probability theory which deals with the interpretation of sets of data. You need statistical terms and methods every time you make a set of laboratory measurements. In this chapter, we shall discuss some of the basic ideas of probability and statistics which are most useful in applications.

The word "probably" is frequently used in everyday life. We say "The test will probably be hard," "It will probably snow today," "We will probably win this game," and so on. Such statements always imply a state of partial ignorance about the outcome of some event; we do not say "probably" about something whose outcome we know. The theory of probability tries to express more precisely just what our state of ignorance is. We say that the probability of getting a head in one toss of a coin is $\frac{1}{2}$, and similarly for a tail. We mean by this that there are two possible outcomes of the experiment (if we do not consider the possibility of the coin's standing on edge) and that we have no reason to expect one outcome more than the other; therefore we assign equal probabilities to the two possible outcomes. (See end of Section 2 for further discussion of this.)

Consider the following problem. You and I each toss a coin and look at our own coins but not each other's. The question is "What is the probability that both coins show heads?" Suppose you see that your coin shows tails; you say that the probability that both coins are heads is zero because you *know* that yours is tails. On the other hand, suppose I see that my coin is heads; then I say that the probability of both heads is $\frac{1}{2}$ because I don't know whether your coin shows heads or tails. Now suppose neither of us looks at either coin, but a third person looks at both coins and gives us the information that at least one is heads. Without this information, there are four possibilities, namely

$$hh \qquad tt \qquad th \qquad ht \tag{1.1}$$

to each of which we would ordinarily assign the probability $\frac{1}{4}$ (see end of Section 2, and Section 3). The information "at least one head" rules out tt, but gives no new information about the other three cases. Since hh, th, ht were equally likely before, we still consider them equally likely and say that the probability of hh is $\frac{1}{3}$.

Notice in the above discussion that the answer to a probability problem depends on the state of knowledge (or ignorance) of the person giving the answer. Notice also that in order to find the probability of an event, we consider all the different equally likely outcomes which are possible according to our information. We say that these are mutually exclusive (for example, if a coin is heads it cannot be tails), collectively exhaustive (we must consider *all* possibilities), and equally likely (we have no information which makes us expect one result more than another so we assume the same probability for each one of the set of outcomes). Let us now formalize this notion of probability as a definition (also see Section 2).

(1.2) If there are several equally likely, mutually exclusive, and collectively exhaustive outcomes of an experiment, the probability of an event E is

$$p = \frac{\text{number of outcomes favorable to } E}{\text{total number of outcomes}}.$$

Example 1. Find the probability that a single card drawn from a shuffled deck of cards will be either a diamond or a king (or both).

There are 52 different possible outcomes of the drawing; since the deck is shuffled, we assume all cards equally likely. Of the 52 cards, 16 are favorable (13 diamonds and the 3 other kings); therefore by (1.2) the desired probability is $\frac{16}{52} = \frac{4}{13}$.

Example 2. A three-digit number (that is, a number from 100–999) is selected "at random." ("At random" means that we assume all numbers to have the same probability of being selected.) What is the probability that all three digits are the same?

There are 900 three-digit numbers; 9 of them (namely 111, 222, $\cdots$, 999) have all three digits the same. Hence the desired probability is $\frac{9}{900} = \frac{1}{100}$.

PROBLEMS, SECTION 1

1. If you select a three-digit number at random, what is the probability that the units digit is 7? What is the probability that the hundreds digit is 7?

2. Three coins are tossed; what is the probability that two are heads and one tails? That the first two are heads and the third tails? If at least two are heads, what is the probability that all are heads?

3. In a box there are 2 white, 3 black, and 4 red balls. If a ball is drawn at random, what is the probability that it is black? That it is *not* red?

4. A single card is drawn at random from a shuffled deck. What is the probability that it is red? That it is the ace of hearts? That it is either a three or a five? That it is either an ace or red or both?

5. Given a family of two children (assume boys and girls equally likely, that is, probability 1/2 for each), what is the probability that both are boys? That at least one is a girl? Given that at least one is a girl, what is the probability that both are girls? Given that the first two are girls, what is the probability that an expected third child will be a boy?

6. A trick deck of cards is printed with the hearts and diamonds black, and the spades and clubs red. A card is chosen at random from this deck (after it is shuffled). Find the probability that it is either a red card or the queen of hearts. That it is either a red face card or a club. That it is either a red ace or a diamond.

7. A letter is selected at random from the alphabet. What is the probability that it is one of the letters in the word "probability?" What is the probability that it occurs in the first half of the alphabet? What is the probability that it is a letter after x?

8. An integer N is chosen at random with $1 \leq N \leq 100$. What is the probability that N is divisible by 11? That $N > 90$? That $N \leq 3$? That N is a perfect square?

9. You are trying to find instrument A in a laboratory. Unfortunately, someone has put both instruments A and another kind (which we shall call B) away in identical unmarked boxes mixed at random on a shelf. You know that the laboratory has 3 A's and 7 B's. If you take down one box, what is the probability that you get an A? If it is a B and you put it on the table and take down another box, what is the probability that you get an A this time?

10. A shopping mall has four entrances, one on the North, one on the South, and two on the East. If you enter at random, shop and then exit at random, what is the probability that you enter and exit on the same side of the mall?

▶ 2. SAMPLE SPACE

It is frequently convenient to make a list of the possible outcomes of an experiment [as we did in (1.1)]. Such a set of all possible mutually exclusive outcomes is called a *sample space*; each individual outcome is called a *point* of the sample space. There are many different sample spaces for any given problem. For example, instead of (1.1), we could say that a set of all mutually exclusive outcomes of two tosses of a coin is

(2.1) 2 heads, 1 head, no heads.

Still another sample space for the same problem is

(2.2) no heads, at least 1 head.

(Can you list some more examples?) On the other hand, the set of outcomes

2 heads, at least 1 head, exactly 1 tail.

cannot be used as a sample space, because these outcomes are not mutually exclusive. "At least 1 head" includes "2 heads" and also includes "exactly 1 tail" (which means also "exactly 1 head").

In order to use a sample space to solve problems, we need to have the probabilities corresponding to the different points in the sample space. We usually assign probability 1/4 to each of the outcomes listed in (1.1). (See end of Section 2 and Section 3.) We call such a list of equally likely outcomes a *uniform* sample space. Now suppose the outcomes are not equally likely. Satisfy yourself that the probabilities associated with the points of (2.1) and (2.2) are as follows.

For (2.1): $\begin{array}{ccc} 2h & 1h & \text{no } h \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}$ and for (2.2): $\begin{array}{cc} \text{no } h & \text{at least 1 } h \\ \frac{1}{4} & \frac{3}{4} \end{array}$

The sample spaces (2.1) and (2.2) with different probabilities associated with different points are called *nonuniform* sample spaces. For some problems, there may be both uniform and nonuniform sample spaces; for example, (1.1) is a uniform sample space, and (2.1) and (2.2) are nonuniform sample spaces for a toss of two coins. But sometimes there *is* no uniform sample space; for example, consider a weighted coin which has a probability $\frac{1}{3}$ for heads and $\frac{2}{3}$ for tails. In such cases, we cannot use the definition (1.2) of probability, and we need the following more general definition.

Definition of Probability. Given any sample space (uniform or not) and the probabilities associated with the points, we find the probability of an event by adding the probabilities associated with all the sample points favorable to the event.

For a given nonuniform sample space, we must use this definition since (1.2) does not apply. If the given sample space is uniform, or if there is an underlying uniform sample space [for example, (1.1) is the uniform space underlying (2.1) and (2.2)], then this definition is consistent with the definition (1.2) by equally likely cases (Problems 15 and 16), and we may use either definition. As an example, let us find from (2.1) the probability of at least one head; this is the probability of one head plus the probability of two heads or $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. We get the same result from the uniform sample space (1.1) using either (1.2) or the definition above.

If we can easily construct several sample spaces for a given problem, we must choose an appropriate one for the question we want to answer. Suppose we ask the question: In two tosses of a coin, what is the probability that both are heads? From either (1.1) or (2.1) we find the answer $\frac{1}{4}$; (2.2) is not an appropriate sample space to use in answering this question. (Why not?) To find the probability of both tails, we could use any of the three listed sample spaces, and to find the probability that the first toss gave a head and the second a tail, we could use only (1.1) since the other sample spaces do not give enough information. Let us now consider some less trivial examples.

Example 1. A coin is tossed three times. A uniform sample space for this problem contains eight points,

$$\begin{array}{cccc} hhh & hth & ttt & tht \\ hht & thh & tth & htt \end{array} \tag{2.3}$$

and we attach probability $\frac{1}{8}$ to each. Now let us use this sample space to answer some questions.

What is the probability of at least two tails in succession? By actual count, we see that there are three such cases, so the probability is $\frac{3}{8}$.

What is the probability that two consecutive coins fall the same? Again by actual count, this is true in six cases, so the probability is $\frac{6}{8}$ or $\frac{3}{4}$.

If we know that there was at least one tail, what is the probability of all tails? The point *hhh* is now ruled out; we have a new sample space consisting of seven points. Since the new information (at least one tail) tells us nothing new about these seven outcomes, we consider them equally probable, each with probability $\frac{1}{7}$. Thus the probability of all tails when all heads is ruled out is $\frac{1}{7}$.

(See problems 11 and 12 for further discussion of this example.)

▶ **Example 2.** Let two dice be thrown; the first die can show any number from 1 to 6 and similarly for the second die. Then there are 36 possible outcomes or points in a uniform sample space for this problem; with each point we associate the probability $\frac{1}{36}$. We can indicate a 3 on the first die and a 2 on the second die by the symbol 3,2. Then the sample space is as shown in (2.4). (Ignore the circling of some points and the letters a and b right now; they are for use in the problems below.)

$$\begin{array}{cccccccc}
 & 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 & \\
 & 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 & \\
 & 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 & \\
a & 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 & b \\
 & 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 & \\
 & 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 &
\end{array} \tag{2.4}$$

Let us now ask some questions and use the sample space (2.4) to answer them.

(a) What is the probability that the sum of the numbers on the dice will be 5? The sample space points circled and marked a in (2.4) give all the cases for which the sum is 5. There are four of these sample points; therefore the probability that the sum is 5 is $\frac{4}{36}$ or $\frac{1}{9}$.

(b) What is the probability that the sum on the dice is divisible by 5? This means a sum of 5 or 10; the four points circled and marked a in (2.4) correspond to a sum of 5, and the three points circled and marked b correspond to a sum of 10. Thus there are seven points in the sample space corresponding to a sum divisible by 5, so the probability of a sum divisible by 5 is $\frac{7}{36}$ (7 favorable cases out of 36 possible cases, or 7 times the probability $\frac{1}{36}$ of each of the favorable sample points).

(c) Set up a sample space in which the points correspond to the possible sums of the two numbers on the dice, and find the probabilities associated with the points of this nonuniform sample space. The possible sums range from 2 (that is, $1+1$) to 12 (that is, $6+6$). From (2.4) we see that the points corresponding to any given sum lie on a diagonal (parallel to the diagonal elements marked a or b). There is one point corresponding to the sum 2; there are two points giving the sum 3, three points for sum 4, etc. Thus we have:

(2.5)

Sample Space	2	3	4	5	6	7	8	9	10	11	12
Associated probabilities	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(d) What is the most probable sum in a toss of two dice? Although we can answer this from the sample space (2.4) (Try it!), it is easier from (2.5). We see that the sum 7 has the largest probability, namely $\frac{6}{36} = \frac{1}{6}$.

(e) What is the probability that the sum on the dice is greater than or equal to 9? Using (2.5), we add the probabilities associated with the sums 9, 10, 11, and 12. Thus the desired probability is

$$\frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18}.$$

So far we have been talking as if it were perfectly obvious and unquestionable that heads and tails are equally likely in the toss of a coin. If you have felt skeptical about this, you are perfectly right. It is *not* obvious; it is not even necessarily true, as a bent or weighted coin would show. We must distinguish here between the mathematical theory of probability and its application to a problem about the physical world. Mathematical probability (like all of mathematics) starts with a set of assumptions and shows that *if* the assumptions are true, *then* various results follow. The basic assumptions in a mathematical probability problem are the probabilities associated with the points of the sample space. Thus in a coin tossing problem, we *assume* that for each toss the probability of heads and the probability of tails are both $\frac{1}{2}$, and then we show that the probability of both heads in two tosses is $\frac{1}{4}$. (See Section 3.) The question of whether the assumptions are correct is not a mathematical one. Here we must ask what physical problem we are trying to solve. If we are dealing with a weighted coin, and if we know or can somehow estimate experimentally the probability p of heads (and so $1 - p$ of tails), then the mathematical theory starts with these values instead of $\frac{1}{2}, \frac{1}{2}$. In the absence of any information as to whether heads or tails is more likely, we often make the "natural" or "intuitive" assumption that the probabilities are both $\frac{1}{2}$. The only possible answer to the question of whether this is correct or not lies in experiment. If the results predicted on the basis of our assumptions agree with experiment, then the assumptions are good; otherwise we must revise the assumptions. (See Section 4, Example 5.)

In this chapter we shall consider mainly the mathematical methods of calculating the probabilities of complicated happenings if we are given the probabilities associated with the points of the sample space. For simplicity, we shall often assume these probabilities to be the "natural" ones; the mathematical theory we develop applies, however, if we replace these "natural" probabilities ($\frac{1}{2}$, $\frac{1}{2}$ in the coin toss problem, etc.) by any set of non-negative fractions whose sum is 1.

PROBLEMS, SECTION 2

1 to **10.** Set up an appropriate sample space for each of Problems 1.1 to 1.10 and use it to solve the problem. Use either a uniform or nonuniform sample space or try both.

11. Set up several nonuniform sample spaces for the problem of three tosses of a coin (Example 1, above).

12. Use the sample space of Example 1 above, or one or more of your sample spaces in Problem 11, to answer the following questions.

(a) If there were more heads than tails, what is the probability of one tail?

(b) If two heads did not appear in succession, what is the probability of all tails?

(c) If the coins did not all fall alike, what is the probability that two in succession were alike?

(d) If N_t = number of tails and N_h = number of heads, what is the probability that $|N_h - N_t| = 1$?

(e) If there was at least one head, what is the probability of exactly two heads?

13. A student claims in Problem 1.5 that if one child is a girl, the probability that both are girls is $\frac{1}{2}$. Use appropriate sample spaces to show what is wrong with the following argument: It doesn't matter whether the girl is the older child or the younger; in either case the probability is $\frac{1}{2}$ that the other child is a girl.

14. Two dice are thrown. Use the sample space (2.4) to answer the following questions.

(a) What is the probability of being able to form a two-digit number greater than 33 with the two numbers on the dice? (Note that the sample point 1, 4 yields the two-digit number 41 which is greater than 33, etc.)

(b) Repeat part (a) for the probability of being able to form a two-digit number greater than or equal to 42.

(c) Can you find a two-digit number (or numbers) such that the probability of being able to form a larger number is the same as the probability of being able to form a smaller number? [See note, part (a).]

15. Use both the sample space (2.4) and the sample space (2.5) to answer the following questions about a toss of two dice.

(a) What is the probability that the sum is ≥ 4?

(b) What is the probability that the sum is even?

(c) What is the probability that the sum is divisible by 3?

(d) If the sum is odd, what is the probability that it is equal to 7?

(e) What is the probability that the product of the numbers on the two dice is 12?

16. Given an nonuniform sample space and the probabilities associated with the points, we defined the probability of an event A as the sum of the probabilities associated with the sample points favorable to A. [You used this definition in Problem 15 with the sample space (2.5).] Show that this definition is consistent with the definition by equally likely cases if there is also a uniform sample space for the problem (as there was in Problem 15). *Hint:* Let the uniform sample space have N points each with the probability N^{-1}. Let the nonuniform sample space have $n < N$ points, the first point corresponding to N_1 points of the uniform space, the second to N_2 points, etc. What is

$$N_1 + N_2 + \cdots + N_n?$$

What are $p_1, p_2, \ldots,$ the probabilities associated with the first, second, etc., points of the nonuniform space? What is $p_1 + p_2 + \cdots + p_n$? Now consider an event for which several points, say i, j, k, of the nonuniform sample space are favorable. Then using the nonuniform sample space, we have, by definition of the probability p of the event, $p = p_i + p_j + p_k$. Write this in terms of the N's and show that the result is the same as that obtained by equally likely cases using the uniform space. Refer to Problem 15 as a specific example if you need to.

17. Two dice are thrown. Given the information that the number on the first die is even, and the number on the second is < 4, set up an appropriate sample space and answer the following questions.

(a) What are the possible sums and their probabilities?

(b) What is the most probable sum?

(c) What is the probability that the sum is even?

18. Are the following correct nonuniform sample spaces for a throw of two dice? If so, find the probabilities of the given sample points. If not show what is wrong. *Suggestion:* Copy sample space (2.4) and circle on it the regions corresponding to the points of the proposed nonuniform spaces.

(a) First die shows an even number.
First die shows an odd number.

(b) Sum of two numbers on dice is even.
First die is even and second odd.
First die is odd and second even.

(c) First die shows a number ≤ 3.
At least one die shows a number > 3.

19. Consider the set of all permutations of the numbers 1, 2, 3. If you select a permutation at random, what is the probability that the number 2 is in the middle position? In the first position? Do your answers suggest a simple way of answering the same questions for the set of all permutations of the numbers 1 to 7?

3. PROBABILITY THEOREMS

It is not always easy to make direct use of our definitions to calculate probabilities. Definition (1.2) asks us to find a uniform sample space for a problem, that is, a set of all possible *equally likely*, mutually exclusive outcomes of an experiment, and then determine how many of these are favorable to a given event. The definition in Section 2 similarly requires a sample space, that is, a list of the possible outcomes and their probabilities. Such lists may be prohibitively long; we want to consider some theorems which will shorten our work.

Suppose there are 5 black balls and 10 white balls in a box; we draw one ball "at random" (this means we are assuming that each ball has probability $\frac{1}{15}$ of being drawn), and then without replacing the first ball, we draw another. Let us ask for the probability that the first ball is white and the second one is black. The probability of drawing a white ball the first time is $\frac{10}{15}$ (10 of the 15 balls are white). The probability of *then* drawing a black ball is $\frac{5}{14}$ since there are 14 balls left and 5 of them are black. We are going to show that the probability of drawing first a white ball and then (without replacement) a black is the product $\frac{10}{15} \cdot \frac{5}{14}$. We reason in the following way, using a uniform sample space. Imagine that the balls are numbered 1 to 15. The symbol 5,3 will mean that ball 5 was drawn the first time and ball 3 the second time. In such pairs of two (different) numbers representing a drawing of two balls in succession, there are 15 choices for the first number and 14 for the second (the first ball was not replaced). Thus the uniform sample space representing all possible drawings consists of a rectangular array of symbols (like 5,3) with 15 columns (for the 15 different choices for the first number) and 14 rows (for the 14 choices for the second number). Thus there are $15 \cdot 14$ points in the sample space. [See also (4.1)]. How many of these sample points correspond to drawing first a white ball and then a black ball? Ten numbers correspond to white balls and the other five to black balls. Thus to obtain a sample point corresponding to drawing first a white and then a black ball, we can choose the first number in 10 ways and then the second number in 5 ways, and so choose the sample point in $10 \cdot 5$ ways; that is, there are $10 \cdot 5$ sample points favorable to the desired drawing. Then by the definition (1.2), the desired probability is $(10 \cdot 5)/(15 \cdot 14)$ as claimed.

Let us state in general the theorem we have just illustrated. We are interested in two successive events A and B. Let $P(A)$ be the probability that A will happen, $P(AB)$ be the probability that both A and B will happen, and $P_A(B)$ be the probability that B will happen if know that A has happened. Then

$$P(AB) = P(A) \cdot P_A(B) \tag{3.1}$$

or in words, the probability of the compound event "*A and B*" is the product of the probability that A will happen times the probability that B will happen if A does. Using the idea of a uniform sample space, we can prove (3.1) by following the method in the ball drawing problem. Let N be the total number of sample points in a uniform sample space, $N(A)$ and $N(B)$ be the numbers of sample points corresponding to the events A and B respectively, and $N(AB)$ be the number of sample points corresponding to the compound event *A and B*. It is useful to picture the sample space geometrically (Figure 3.1) as an array of N points [compare with sample space (2.4)]. We can then circle all points which correspond to A's happening and mark this region A; it contains $N(A)$ points. Similarly, we can circle the $N(B)$ points which correspond to B's happening and call this region B. The overlapping region we call AB; it is part of both A and B and contains $N(AB)$ points which correspond to the compound event *A and B*. Then by the definition (1.2):

$$P(AB) = \frac{N(AB)}{N},$$
$$P(A) = \frac{N(A)}{N}, \tag{3.2}$$
$$P_A(B) = \frac{N(AB)}{N(A)}.$$

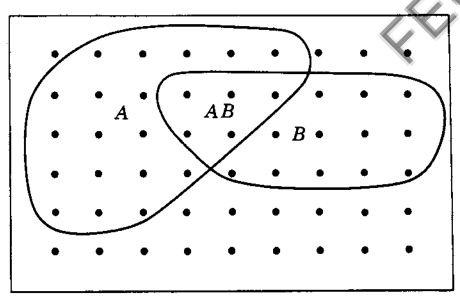


Figure 3.1

Perhaps this last formula for $P_A(B)$ needs some discussion. Recall from Section 2, Example 1, the uniform sample space (2.3) for three tosses of a coin. To find the probability of all tails given that there was at least one tail, we reduced our sample space to seven points (eliminating *hhh*). We then assumed that the seven points of the new sample space had the same relative probability as before the deletion of the point *hhh*; thus each of the seven points had probability $\frac{1}{7}$. (This is no more and no less "obvious" than the original assumption that the eight points had equal probability; it is an additional assumption which we make in the absence of any information to the contrary; see end of Section 2.) Now let us look at the third equation of (3.2). $N(A)$ is the number of sample points corresponding to event A; the N points in the original sample apace all had the same probability so we now assume that when we cross off all the points corresponding to A's *not* happening, the remaining $N(A)$ points also have equal probability. Thus we have a new uniform sample space consisting of $N(A)$ points. $N(AB)$ of these $N(A)$ points correspond to the event B (assuming A). Thus by (1.2), the probability of "B if A" is $N(AB)/N(A)$. From the three equations (3.2), we then have (3.1). In a similar way we can show that

$$P(BA) = P(B) \cdot P_B(A) = P(AB) \tag{3.3}$$

(see Problem 1). [We have proved (3.1) assuming a uniform sample space. This assumption is not necessary; (3.1) is true whether or not we can construct a uniform sample space; see Problem 2.]

Suppose, now, in our example of 5 black and 10 white balls in a box, we draw a ball and replace it and then draw a second ball. The probability of a black ball on the second drawing is then $\frac{5}{15} = \frac{1}{3}$; this is exactly the same result we would get if we had not drawn and replaced the first ball. In the notation of the last paragraph

$$P(B) = P_A(B), \qquad A \text{ and } B \text{ independent.} \tag{3.4}$$

When (3.4) is true, we say that the event B is *independent* of event A and (3.1) becomes

$$\boxed{P(AB) = P(A) \cdot P(B), \qquad A \text{ and } B \text{ independent.}} \tag{3.5}$$

Because of the symmetry of (3.5), we may simply say that A and B are independent if (3.5) is true. (Also see Problem 7.)

Example 1. (a) In three tosses of a coin, what is the probability that all three are heads? We found $p = \frac{1}{8}$ for this problem in Section 2 by seeing that one sample point out of eight corresponds to all heads. Now we can do the problem more simply by saying that the probability of heads on each toss is $\frac{1}{2}$, the tosses are independent, and therefore

$$p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

(b) If we should want the probability of all heads when a coin is tossed ten times, the sample space would be unwieldy; instead of using the sample space, we can say that since the tosses are independent, the desired probability is $p = (\frac{1}{2})^{10}$.

(c) To find the probability of at least one tail in ten tosses, we see that this event corresponds to all the rest of the sample space except the "all heads" point. Since the sum of the probabilities of all the sample points is 1, the desired probability is $1 - (\frac{1}{2})^{10}$.

In Figure 3.1 or Figure 3.2 the region AB corresponds to the happening of *both* A and B. The whole region consisting of points in A or B or both corresponds to the happening of *either* A *or* B *or* both. We write $P(AB)$ for the probability that both A and B occur. We shall write $P(A + B)$ for the probability that either or both occur. Then we can prove that

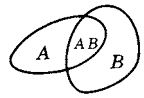


Figure 3.2

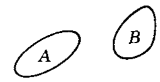


Figure 3.3

$$(3.6) \qquad P(A+B) = P(A) + P(B) - P(AB).$$

To see why this is true, consider Figure 3.2. To find $P(A+B)$ we add the probabilities of all the sample points in the region consisting of A or B or both. But if we add $P(A)$ and $P(B)$, we have included the probabilities of all the sample points in AB twice [once in $P(A)$ and once in $P(B)$]. Thus we must subtract $P(AB)$, which is the sum of the probabilities of all the sample points in AB. This is just what (3.6) says.

If the sample space diagram is like the one in Figure 3.3, so that $P(AB) = 0$, we say that A and B are mutually exclusive. Then (3.6) becomes

$$(3.7) \qquad P(A+B) = P(A) + P(B), \qquad A \text{ and } B \text{ mutually exclusive.}$$

▶ **Example 2.** Two students are working separately on the same problem. If the first student has probability $\frac{1}{2}$ of solving it and the second student has probability $\frac{3}{4}$ of solving it, what is the probability that at least one of them solves it?

Let A be the event "first student succeeds," and B be the event "second student succeeds." Then $P(AB) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$ (assume A and B independent since the students work separately). Then by (3.6) the probability that one or the other or both students solve the problem is

$$P(A+B) = \frac{1}{2} + \frac{3}{4} - \frac{3}{8} = \frac{7}{8}.$$

Conditional Probability; Bayes' Formula If we are asked for the probability of event B assuming that event A occurs [that is, $P_A(B)$], it is often useful to find it from (3.1):

$$(3.8) \qquad P_A(B) = \frac{P(AB)}{P(A)}.$$

Equation (3.8) is called Bayes' formula. In any conditional probability problem to which the answer is not immediately obvious, you should consider whether you can easily find $P(A)$ and $P(AB)$; if so, the conditional probability $P_A(B)$ is given by (3.8).

Example 3. A preliminary test is customarily given to the students at the beginning of a certain course. The following data are accumulated after several years:

(a) 95% of the students pass the course, 5% fail.

(b) 96% of the students who pass the course also passed the preliminary test.

(c) 25% of the students who fail the course passed the preliminary test.

What is the probability that a student who has failed the preliminary test will pass the course?

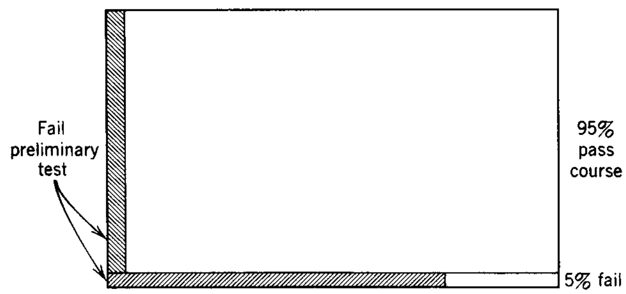


Figure 3.4

Let A be the event "fails preliminary test" and B be the event "Passes course." The probability we want is then $P_A(B)$ in (3.8), so we need $P(AB)$ and $P(A)$. $P(AB)$ is the probability that the student both fails the preliminary test and passes the course; this is $P(AB) = (0.95)(0.04) = 0.038$. (See Figure 3.4; 95% of the students passed the course and of these 4% had failed the preliminary test.) We also want $P(A)$, the probability that a students fails the preliminary test; this event corresponds to the shaded area in Figure 3.4. Thus $P(A)$ is the sum of the probabilities of the two events "passes course after failing test," "fails course after failing test." Then

$$P(A) = (0.095)(0.04) + (0.05)(0.75) = 0.0755$$

(See Figure 3.4; of the 95% of students who passed the course, 4% failed the preliminary test; of the 5% of the students who failed the course, 75% failed the preliminary test since we are given that 25% passed.) By (3.8) we have

$$P_A(B) = \frac{P(AB)}{P(A)} = \frac{0.038}{0.0755} = 50\%,$$

that is, half of the students who fail the preliminary test succeed in passing the course.

Note that in Figure 3.4, the shaded area corresponds to event A (fails preliminary test). We are interested in event B (passes course) given event A. Thus instead of the original sample space (whole rectangle in Figure 3.4) we consider a smaller sample space (shaded area in Figure 3.4). We then want to know what part of this sample space corresponds to event B (passes course). This fraction is $P(AB)/P(A)$ which we computed.

▶ PROBLEMS, SECTION 3

1. (a) Set up a sample space for the 5 black and 10 white balls in a box discussed above assuming the first ball is not replaced. *Suggestions:* Number the balls, say 1 to 5 for black and 6 to 15 for white. Then the sample points form an array something like (2.4), but the point 3,3 for example is not allowed. (Why? What other points are not allowed?) You might find it helpful to write the numbers for black balls and the numbers for white balls in different colors.

(b) Let A be the event "first ball is white" and B be the event "second ball is black." Circle the region of your sample space containing points favorable to A and mark this region A. Similarly, circle and mark region B. Count the number of sample points in A and in B; these are $N(A)$ and $N(B)$. The region AB is the region inside both A and B; the number of points in this region is $N(AB)$. Use the numbers you have found to verify (3.2) and (3.1). Also find $P(B)$ and $P_B(A)$ and verify (3.3) numerically.

(c) Use Figure 3.1 and the ideas of part (b) to prove (3.3) in general.

2. Prove (3.1) for a nonuniform sample space. *Hints:* Remember that the probability of an event is the sum of the probabilities of the sample points favorable to it. Using Figure 3.1, let the points in A but not in AB have probabilities $p_1, p_2, \ldots, p_n$, the points in AB have probabilities $p_{n+1}, p_{n+2}, \ldots, p_{n+k}$, and the points in B but not in AB have probabilities $p_{n+k+1}, p_{n+k+2}, \ldots, p_{n+k+l}$. Find each of the probabilities in (3.1) in terms of the p's and show that you then have an identity.

3. What is the probability of getting the sequence *hhhttt* in six tosses of a coin? If you know the first three are heads, what is the probability that the last three are tails?

4. (a) A weighted coin has probability of $\frac{2}{3}$ of showing heads and $\frac{1}{3}$ of showing tails. Find the probabilities of *hh*, *ht*, *th* and *tt* in two tosses of the coin. Set up the sample space and the associated probabilities. Do the probabilities add to 1 as they should? What is the probability of at least one head? What is the probability of two heads if you know there was at least one head?

(b) For the coin in (a), set up the sample space for three tosses, find the associated probabilities, and use it to answer the questions in Problem 2.12.

5. What is the probability that a number n, $1 \le n \le 99$, is divisible by *both* 6 and 10? By *either* 6 *or* 10 or both?

6. A card is selected from a shuffled deck. What is the probability that it is either a king or a club? That it is both a king and a club?

7. (a) Note that (3.4) assumes $P(A) \neq 0$ since $P_A(B)$ is meaningless if $P(A) = 0$. Assuming both $P(A) \neq 0$ and $P(B) \neq 0$, show that if (3.4) is true, then $P(A) = P_B(A)$; that is if B is independent of A, then A is independent of B. If either $P(A)$ or $P(B)$ is zero, then we use (3.5) to define independence.

(b) When is an event E independent of itself? When is E independent of "not E"?

8. Show that

$$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

Hint: Start with Figure 3.2 and sketch in a region C overlapping some of the points of each of the regions A, B, and AB.

9. Two cards are drawn at random from a shuffled deck and laid aside without being examined. Then a third card is drawn. Show that the probability that the third card is a spade is $\frac{1}{4}$ just as it was for the first card. *Hint:* Consider all the (mutually exclusive) possibilities (two discarded cards spades, third card spade or not spade, etc.).

10. (a) Three typed letters and their envelopes are piled on a desk. If someone puts the letters into the envelopes at random (one letter in each), what is the probability that each letter gets into its own envelope? Call the envelopes A, B, C, and the corresponding letters a, b, c, and set up the sample space. Note that "a in C, b in B, c in A" is *one* point in the sample space.

(b) What is the probability that at least one letter gets into its own envelope? *Hint:* What is the probability that no letter gets into its own envelope?

(c) Let A mean that a got into envelope A, and so on. Find the probability $P(A)$ that a got into A. Find $P(B)$ and $P(C)$. Find the probability $P(A+B)$ that either a or b or both got into their correct envelopes, and the probability $P(AB)$ that both got into their correct envelopes. Verify equation (3.6).

11. In paying a bill by mail, you want to put your check and the bill (with a return address printed on it) into a window envelope so that the address shows right side up and is not blocked by the check. If you put check and bill at random into the envelope, what is the probability that the address shows correctly?

12. (a) A loaded die has probabilities $\frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}, \frac{6}{21}$, of showing 1, 2, 3, 4, 5, 6. What is the probability of throwing two 3's in succession?

(b) What is the probability of throwing a 4 the first time and not a 4 the second time with a die loaded as in (a)?

(c) If two dice loaded as in (a) are thrown, and we know that the sum of the numbers on the faces is greater than or equal to 10, what is the probability that both are 5's?

(d) How many times must we throw a die loaded as in (a) to have probability greater than $\frac{1}{2}$ of getting an ace?

(e) A die, loaded as in (a), is thrown twice. What is the probability that the number on the die is even the first time > 4 the second time?

13. (a) A candy vending machine is out of order. The probability that you get a candy bar (with or without return of your money) is $\frac{1}{2}$, the probability that you get your money back (with or without candy) is $\frac{1}{3}$, and the probability that you get both the candy and your money back is $\frac{1}{12}$. What is the probability that you get nothing at all? *Suggestion:* Sketch a geometric diagram similar to Figure 3.1, indicate regions representing the various possibilities and their probabilities; then set up a four-point sample space and the associated probabilities of the points.

(b) Suppose you try again to get a candy bar as in part (a). Set up the 16-point sample space corresponding to the possible results of your two attempts to buy a candy bar, and find the probability that you get two candy bars (and no money back); that you get no candy and lose your money both times; that you just get your money back both times.

14. A basketball player succeeds in making a basket 3 tries out of 4. How many tries are necessary in order to have probability > 0.99 of at least one basket?

15. Use Bayes' formula (3.8) to repeat these simple problems previously done by using a reduced sample space.

(a) In a family of two children, what is the probability that both are girls if at least one is a girl?

(b) What is the probability of all heads in three tosses of a coin if you know that at least one is a head?

16. Suppose you have 3 nickels and 4 dimes in your right pocket and 2 nickels and a quarter in your left pocket. You pick a pocket at random and from it select a coin at random. If it is a nickel, what is the probability that it came from your right pocket?

17. (a) There are 3 red and 5 black balls in one box and 6 red and 4 white balls in another. If you pick a box at random, and then pick a ball from it at random, what is the probability that it is red? Black? White? That it is either red or white?

(b) Suppose the first ball selected is red and is not replaced before a second ball is drawn. What is the probability that the second ball is red also?

(c) If both balls are red, what is the probability that they both came from the same box?

18. Two cards are drawn at random from a shuffled deck.

(a) What is the probability that at least one is a heart?

(b) If you know that at least one is a heart, what is the probability that both are hearts?

19. Suppose it is known that 1% of the population have a certain kind of cancer. It is also known that a test for this kind of cancer is positive in 99% of the people who have it but is also positive in 2% of the people who do not have it. What is the probability that a person who tests positive has cancer of this type?

20. Some transistors of two different kinds (call them N and P) are stored in two boxes. You know that there are 6 N's in one box and that 2 N's and 3 P's got mixed in the other box, but you don't know which box is which. You select a box and a transistor from it at random and find that it is an N; what is the probability that it came from the box with the 6 N's? From the other box? If another transistor is picked from the same box as the first, what is the probability that it is also an N?

21. Two people are taking turns tossing a pair of coins; the first person to toss two alike wins. What are the probabilities of winning for the first player and for the second player? *Hint:* Although there are an infinite number of possibilities here (win on first turn, second turn, third turn, etc.), the sum of the probabilities is a geometric series which can be summed; see Chapter 1 if necessary.

22. Repeat Problem 21 if the players toss a pair of dice trying to get a double (that is, both dice showing the same number).

23. A thick coin has probability $\frac{3}{7}$ of falling heads, $\frac{3}{7}$ of falling tails, and $\frac{1}{7}$ of standing on edge. Show that if it is tossed repeatedly it has probability 1 of eventually standing on edge.

▶ 4. METHODS OF COUNTING

Let us digress for a bit to review some ideas and formulas we need in computing probabilities in more complicated problems.

Let us ask how many two-digit numbers have either 5 or 7 for the tens digit and either 3, 4, or 6 for the units digit. The answer becomes obvious if we arrange the possible numbers in a rectangle

$$\begin{matrix} 53 & 54 & 56 \\ 73 & 74 & 76 \end{matrix}$$

with two rows corresponding to the two choices of the tens digit and three columns corresponding to the three choices of the units digit. This is an example of the *fundamental principle of counting*:

(4.1) If one thing can be done N_1 ways, and after that a second thing can be done in N_2 ways, the two things can be done in succession in that order in $N_1 \cdot N_2$ ways. This can be extended to doing any number of things one after the other, the first N_1 ways, the second N_2 ways, the third N_3 ways, etc. Then the total number of ways to perform the succession of acts is the product $N_1 N_2 N_3 \cdots$.

Now consider a set of n things lined up in a row; we ask how many ways we can arrange (permute) them. This result is called the number of *permutations* of n things n at a time, and is denoted by ${}_nP_n$ or $P(n,n)$ or P_n^n. To find this number, we think of seating n people in a row of n chairs. We can place anyone in the first chair, that is, we have n possible ways of filling the first chair. Once we have selected someone for the first chair, there are $(n-1)$ choices left for the second chair, then $(n-2)$ choices for the third chair, and so on. Thus by the fundamental principle, there are $n(n-1)(n-2)\cdots 2\cdot 1 = n!$ ways of arranging the n people in the row of n chairs. The number of permutations of n things n at a time is

$$P(n,n) = n!. \tag{4.2}$$

Next suppose there are n people but only $r < n$ chairs and we ask how many ways we can select groups of r people and seat them in the r chairs. The result is called the number of permutations of n things r at a time and is denoted by ${}_nP_r$ or $P(n,r)$ or P_r^n. Arguing as before, we find that there are n ways to fill the first chair, $(n-1)$ ways to fill the second chair, $(n-2)$ ways for the third [note that we could write $(n-2)$ as $(n-3+1)$], etc., and finally $(n-r+1)$ ways of filling chair r. Thus we have for the number of permutations of n things r at a time

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1).$$

By multiplying and dividing by $(n-r)!$ we can write this as

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)\frac{(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}. \tag{4.3}$$

So far we have been talking about arranging things in a definite order. Suppose, instead that we ask how many committees of r people can be chosen from a group of n people $(n \geq r)$. Here the order of the people in the committee is not considered; the committee made up of people A, B, C, is the same as the committee made up of people B, A, C. We call the number of such committees of r people which we can select from n people, the number of *combinations* or *selections* of n things r at a time, and denote this number by ${}_nC_r$ or $C(n,r)$ or $\binom{n}{r}$. To find $C(n,r)$, we go back to the problem of selecting r people from a group of n and seating them in r chairs; we found that the number of ways of doing this is $P(n,r)$ as given in

(4.3). We can perform this job by first selecting r people from the total n and then arranging the r people in r chairs. The selection of r people can be done in $C(n,r)$ ways (this is the number we are trying to find), and after r people are selected, they can be arranged in r chairs in $P(r,r)$ ways by (4.2). By the fundamental principle (4.1), the total number of ways $P(n,r)$ of selecting and seating r people out of n is the product $C(n,r) \cdot P(r,r)$. Thus we have

$$P(n,r) = C(n,r) \cdot P(r,r). \tag{4.4}$$

We can solve this equation to find the value $C(n,r)$ which we wanted. Substituting the values of $P(n,r)$ and $P(r,r)$ from (4.3) and (4.2) into (4.4) and solving for $C(n,r)$, we find for the number of combinations of n things r at a time

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!r!} = \binom{n}{r}. \tag{4.5}$$

Each time we select r people to be seated, we leave $n-r$ people without chairs. Then there are exactly the same number of combinations of n things $n-r$ at a time as there are combinations of n things r at a time. Hence we write

$$C(n,n-r) = C(n,r) = \frac{n!}{(n-r)!r!}. \tag{4.6}$$

We can also obtain (4.6) from (4.5) by replacing r by $(n-r)$.

▶ **Example 1.** A club consists of 50 members. In how many ways can a president, vice-president, secretary, and treasurer be chosen? In how many ways can a committee of 4 members be chosen?

In the selection of officers, we must not only select 4 people, but decide which one is president, etc.; we could think of seating the 4 people in chairs labeled president, vice-president, etc. Thus the number of ways of selecting the officers is

$$P(50,4) = \frac{50!}{(50-4)!} = \frac{50!}{46!} = 50 \cdot 49 \cdot 48 \cdot 47.$$

The committee members, however, are all equivalent (we are neglecting the possibility that one is named chairman), so the number of ways of selecting committees of 4 people is

$$C(50,4) = \frac{50!}{46!4!} = \frac{50 \cdot 49 \cdot 48 \cdot 47}{24}.$$

▶ **Example 2.** Find the coefficient of x^8 in the binomial expansion of $(1+x)^{15}$.

Think of multiplying out

$$(1+x)(1+x)(1+x)\cdots(1+x), \qquad \text{(with 15 factors)}.$$

We obtain a term in x^8 each time we multiply 1's from seven of the parentheses by x's from eight of the parentheses. The number of ways of selecting 8 parentheses out of 15 is

$$C(15,8) = \frac{15!}{8!7!}.$$

This is the desired coefficient of x^8.

Generalizing this example, we see that in the expansion of $(a+b)^n$, the coefficient of $a^{n-r}b^r$ is $C(n,r)$, usually written $\binom{n}{r}$ when used in connection with a binomial expansion (see Chapter 1, Section 13C). Thus the expressions $C(n,r)$ are just the binomial coefficients, and we can write

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r. \tag{4.7}$$

Example 3. A basic problem in statistical mechanics is this: Given N balls, and n boxes, in how many ways can the balls be put into the boxes so that there will be given numbers of balls in the boxes, say N_1 balls in the first box, N_2 balls in the second box, N_3 in the third, $\cdots$, N_n in the nth, and what is the probability that this given distribution will occur when the balls are put into the boxes? In statistical mechanics the "balls" may be molecules, electrons, photons, etc., and each "box" corresponds to a small range of values of position and momentum of a particle. We can state many other problems in this same language of putting balls into boxes. For example, in tossing a coin, we can equate heads with box 1, and tails with box 2; in tossing a die, there are six "boxes." In putting letters into envelopes, the letters are the balls, and the envelopes are the boxes. In dealing cards, the cards are the balls and the players who receive them are the boxes. In an alpha scattering experiment, the alpha particles are the balls, and the boxes are elements of area on the detecting screen which the particles hit after they are scattered. (Also see Problems 14 and 21 and Feller, pp. 10–11.)

Let us do a special case of this problem in which we have 15 balls and 6 boxes, and the numbers of balls we are to put into the various boxes are:

Number of balls:	3	1	4	2	3	2
In box number:	1	2	3	4	5	6

We first ask how many ways we can select 3 balls to go in the first box from the 15 balls; this is $C(15,3)$. (Note that the order of the balls in the boxes is not considered; this is like the committee problem in Example 1.) Now we have 12 balls left, of which we are to select 1 for box 2; we can do this in $C(12,1)$ ways. We can then select the 4 balls for box 3 from the remaining 11 balls in $C(11,4)$ ways, the 2 balls for box 4 in $C(7,2)$ ways, the 3 balls for box 5 in $C(5,3)$ ways, and finally the balls for box 6 in $C(2,2)$ ways (verify that this is 1). By the fundamental principle, the total number of ways of putting the required numbers of balls into the boxes is

$$\begin{aligned} &C(15,3)\cdot C(12,1)\cdot C(11,4)\cdot C(7,2)\cdot C(5,3)\cdot C(2,2) \\ &\qquad = \frac{15!}{3!\cdot 12!}\cdot\frac{12!}{1!\cdot 11!}\cdot\frac{11!}{4!\cdot 7!}\cdot\frac{7!}{2!\cdot 5!}\cdot\frac{5!}{3!\cdot 2!}\cdot\frac{2!}{2!\cdot 0!} \\ &\qquad = \frac{15!}{3!\cdot 1!\cdot 4!\cdot 2!\cdot 3!\cdot 2!}. \end{aligned}$$

(Remember from Chapters 1 and 11 that $0! = 1$.)

Next we want the probability of this particular distribution. Let us assume that the balls are distributed "at random" into the boxes; by this we mean that a ball has the same probability (namely $\frac{1}{6}$) of being put into any one box as into any other box. We can put the first ball into any one of the 6 boxes, the second ball into any one of the 6 boxes, and so on. Thus by the fundamental principle, the total number of ways of distributing the 15 balls into the 6 boxes is $6 \cdot 6 \cdot 6 \cdot 6 \cdots 6 = 6^{15}$ and we are assuming that these distributions are equally probable. Then the probability that, when 15 balls are distributed "at random" into 6 boxes, there will be 3 balls in box 1, 1 in box 2, etc., as given, is, by (1.2) (favorable cases ÷ total)

$$\frac{15!}{3! \cdot 1! \cdot 4! \cdot 2! \cdot 3! \cdot 2!} \div 6^{15}.$$

▶ **Example 4.** In Example 3, we assumed that the 6^{15} possible distributions of 15 balls into 6 boxes were equally likely. This seems very reasonable if we think of putting the balls into the boxes by tossing a die for each ball; if the die shows 1 we put the ball into box 1, etc. However, we can think of situations to which this method and result do not apply. For example, suppose we are putting letters into envelopes or seating people in chairs; then we may reasonably require only one letter per envelope, not more than one person per chair, that is, one ball (or none) per box. Consider the problem of seating 4 people in 6 chairs, that is of putting 4 balls into 6 boxes. If we number the chairs from 1 to 6 and let each person choose a chair by tossing a die, we may have two or more people choosing the same chair. The result 6^4 (which the method of Example 3 gives for the problem of 4 balls in 6 boxes) then does not apply to this problem. However, let us consider the uniform sample space of 6^4 points and select from it the points corresponding to our restriction (one ball or none per box). The new sample space contains $C(6,4) \cdot 4!$ points (number of ways of selecting the 4 chairs to be occupied times the number of ways of then arranging 4 people in 4 chairs). Since these points were equally probable in the original (uniform) sample space, we still consider them equally probable. Now let us ask for the probability that the first two chairs are vacant when the 4 people are seated. The number of sample points corresponding to this event is 4! (the number of ways of arranging the 4 people in the last 4 chairs). Thus the desired probability is

$$\frac{4!}{C(6,4) \cdot 4!} = \frac{1}{C(6,4)}.$$

We can now see an easier way of doing problems of this kind. The factor 4!, which canceled in the probability calculation, was the number of rearrangements of the 4 people among the 4 occupied chairs. Since this is the same for any given set of 4 chairs, we can lump together all the sample points corresponding to each given set of 4 chairs, and have a smaller (still uniform) sample space of $C(6,4)$ points. Each point now corresponds to a given set of 4 occupied chairs; the quantity $C(6,4)$ is just the number of ways of picking 4 occupied chairs out of 6. The probability that the first two chairs are vacant when 4 people are seated is $1/C(6,4)$ since there is only one way to select 4 occupied chairs leaving the first two chairs vacant.

Another useful way of looking at this problem is to consider a set of 4 *identical* balls to be put into 6 boxes. Since the balls are identical, the 4! arrangements of the 4 balls in 4 given boxes all look alike. We can say that there are $C(6,4)$ *distinguishable* arrangements of the 4 identical balls in 6 boxes (one ball or none per box). Since all these arrangements are equally probable, the probability of any one arrangement (say the first two boxes empty) is $1/C(6,4)$ as we found previously.

Example 5. In Example 4 we found the same answer for the probability that two particular boxes were empty whether or not we considered the balls distinguishable. This was true because the allowed distinguishable arrangements were equally probable. Without the restriction of one ball or none per box, all distinguishable arrangements are not equally probable according to the methods of Examples 3 and 4. For example, the probability of all balls in box 1 is $1/6^4$; compare this with the probability of no balls in the first 2 boxes and one ball in each of the other 4 boxes, which is $4! \div 6^4 = \frac{1}{54}$. We see that the concentrated arrangements (all or several balls in one box) are less probable than the more uniform arrangements.

Now we are going to try to imagine a situation in which *all* distinguishable arrangements *are* equally probable. Suppose the 6 boxes are benches in a waiting room and the 4 balls are people who are going to come in and sit on the benches. Then if the people are friends, there will be a certain tendency for them to sit together and the probabilities we have been calculating will not apply—the probabilities of the concentrated arrangements will increase. Consider the following mathematical model. (This is a modification of Pólya's urn model.) We have 6 boxes labeled 1 to 6, and 4 balls. From 6 cards labeled 1 to 6 we draw one at random and place a ball in the box numbered the same as the card drawn. We then replace the card and also add another card of the same number so that there are now 7 cards, two with the number first drawn. We now select a card at random from these 7, put a ball in the corresponding box and again replace the card adding a duplicate to make 8 cards. We repeat this process two more times (until all balls are distributed). Then the probability that all balls are in box 1 is $\frac{1}{6}\cdot\frac{2}{7}\cdot\frac{3}{8}\cdot\frac{4}{9}$. The probability of one ball in each of the first 4 boxes is $\frac{1}{6}\cdot\frac{1}{7}\cdot\frac{1}{8}\cdot\frac{1}{9}\cdot 4!$ (here $\frac{1}{6}\cdot\frac{1}{7}\cdot\frac{1}{8}\cdot\frac{1}{9}$ is the probability that the first ball is in box 1, the second in box 2, etc.; we must add to this the probability that the first ball is in box 3, the second in box 1, etc.; there are 4! such possibilities all giving one ball in each of the first 4 boxes). We see that the distributions "all balls in box 1" and "one ball in each of the first 4 boxes" are equally probable. Further calculation (Problem 20) shows that all distinguishable arrangements are equally probable.

To find the number of distinguishable arrangements, consider the following picture of the 4 balls in the 6 boxes.

	o		o o		o	
Box number:	1	2	3	4	5	6
Number of balls:	1	0	2	0	1	0

The lines mean the sides of the boxes and the circles are the balls; note that it requires 7 lines to picture the 6 boxes. This picture shows one of many possible arrangements of the 4 balls in 6 boxes. In any such picture there must be a line at the beginning and at the end, but the rest of the lines (5 of them) and the 4 circles can be arranged in any order. You should convince yourself that every arrangement of the balls in the boxes can be so pictured. Then the number of such distinguishable arrangements is just the number of ways we can select 4 positions for the 4 circles out of 9 positions for the 5 lines and 4 circles. Thus there are $C(9,4)$ equally likely arrangements in this problem.

We see then that putting balls in boxes is not quite as simple as we thought; we must say *how* we propose to distribute them and even before that we must think what practical problem we are trying to solve; this is what determines the sample space and the probabilities to be associated with the sample points. Unfortunately, it may not always be clear what the sample space probabilities should be; then the best we can do is to try various assumptions. In statistical mechanics it is found that certain particles (for example, the molecules of a gas) are correctly described if we assume that they behave like the balls of Example 3 (all 6^{15} arrangements equally likely); we then say that they obey Maxwell-Boltzmann statistics. Other particles (for example, electrons) behave like the people to be seated in Example 4 (one particle or none per box); we say that such particles obey Fermi-Dirac statistics. Finally some particles (for example, photons) act something like the friends who want to sit near each other (all distinguishable arrangements of identical particles are equally likely); we say that these particles obey Bose-Einstein statistics. For the problem of 4 particles in 6 boxes, there are then 6^4 equally likely arrangements for Maxwell-Boltzmann particles, $C(6, 4)$ for Fermi-Dirac particles, and $C(9, 4)$ for Bose-Einstein particles. (See Problems 15 to 20.)

▶ PROBLEMS, SECTION 4

1. (a) There are 10 chairs in a row and 8 people to be seated. In how many ways can this be done?

(b) There are 10 questions on a test and you are to do 8 of them. In how many ways can you choose them?

(c) In part (a) what is the probability that the first two chairs in the row are vacant?

(d) In part (b), what is the probability that you omit the first two problems in the test?

(e) Explain why the answer to parts (a) and (b) are different, but the answers to (c) and (d) are the same.

2. In the expansion of $(a + b)^n$ (see Example 2), let $a = b = 1$, and interpret the terms of the expansion to show that the total number of combinations of n things taken 1, 2, 3, $\cdots$, n at a time, is $2^n - 1$.

3. A bank allows one person to have only one savings account insured to \$100,000. However, a larger family may have accounts for each individual, and also accounts in the names of any 2 people, any 3 and so on. How many accounts are possible for a family of 2? Of 3? Of 5? Of n? *Hint:* See Problem 2.

4. Five cards are dealt from a shuffled deck. What is the probability that they are all of the same suit? That they are all diamond? That they are all face cards? That the five cards are a sequence in the same suit (for example, 3, 4, 5, 6, 7 of hearts)?

5. A bit (meaning binary digit) is 0 or 1. An ordered array of eight bits (such as 01101001) is a byte. How many different bytes are there? If you select a byte at random, what is the probability that you select 11000010? What is the probability that you select a byte containing three 1's and five 0's?

6. A so-called 7-way lamp has three 60-watt bulbs which may be turned on one or two or all three at a time, and a large bulb which may be turned to 100 watts, 200 watts or 300 watts. How many different light intensities can the lamp be set to give if the completely off position is not included? (The answer is *not* 7.)

7. What is the probability that the 2 and 3 of clubs are next to each other in a shuffled deck? *Hint:* Imagine the two cards accidentally stuck together and shuffled as one card.

8. Two cards are drawn from a shuffled deck. What is the probability that both are aces? If you know that at least one is an ace, what is the probability that both are aces? If you know that one is the ace of spades, what is the probability that both are aces?

9. Two cards are drawn from a shuffled deck. What is the probability that both are red? If at least one is red, what is the probability that both are red? If at least one is a red ace, what is the probability that both are red? If exactly one is a red ace, what is the probability that both are red?

10. What is the probability that you and a friend have different birthdays? (For simplicity, let a year have 365 days.) What is the probability that three people have three different birthdays? Show that the probability that n people have n different birthdays is

$$p = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right)\cdots\left(1 - \frac{n-1}{365}\right).$$

Estimate this for $n \ll 365$ by calculating $\ln p$ [recall that $\ln(1+x)$ is approximately x for $x \ll 1$]. Find the smallest (integral) n for which $p < \frac{1}{2}$. Hence, show that for a group of 23 people or more, the probability is greater than $\frac{1}{2}$ that two of them have the same birthday. (Try it with a group of friends or a list of people such as the presidents of the United States.)

11. The following game was being played on a busy street: Observe the last two digits on each license plate. What is the probability of observing at least two cars with the same last two digits among the first 5 cars? 10 cars? 15 cars? How many cars must you observe in order for the probability to be greater than $\frac{1}{2}$ of observing two with the same last two digits?

12. Consider Problem 10 for different months of birth. What is the smallest number of people for which the probability is greater than $\frac{1}{2}$ that two of them were born in the same month?

13. Generalize Example 3 to show that the number of ways of putting N balls in n boxes with N_1 in box 1, N_2 in box 2, etc., is

$$\left(\frac{N!}{N_1! \cdot N_2! \cdot N_3! \cdots N_n!}\right).$$

14. (a) Find the probability that in two tosses of a coin, one is heads and one tails. That in six tosses of a die, all six of the faces show up. That in 12 tosses of a 12-sided die, all 12 faces show up. That in n tosses of an n-sided die, all n faces show up.

(b) The last problem in part (a) is equivalent to finding the probability that, when n balls are distributed at random into n boxes, each box contains exactly one ball. Show that for large n, this is approximately $e^{-n}\sqrt{2\pi n}$.

15. Set up the uniform sample spaces for the problem of putting 2 particles in 3 boxes: for Maxwell-Boltzmann particles, for Fermi-Dirac particles, and for Bose-Einstein particles. See Example 5. (You should find 9 sample points for MB, 3 for FD, and 6 for BE.)

16. Do Problem 15 for 2 particles in 2 boxes. Using the model discussed in Example 5, find the probability of each of the three sample points in the Bose-Einstein case. (You should find that each has probability $\frac{1}{3}$, that is, they are equally probable.)

17. Find the number of ways of putting 2 particles in 4 boxes according to the three kinds of statistics.

18. Find the number of ways of putting 3 particles in 5 boxes according to the three kinds of statistics.

19. (a) Following the methods of Examples 3, 4, and 5, show that the number of equally likely ways of putting N particles in n boxes, $n > N$, is n^N for Maxwell-Boltzmann particles, $C(n, N)$ for Fermi-Dirac particles, and $C(n-1+N, N)$ for Bose-Einstein particles.

(b) Show that if n is much larger than N (think , for example, of $n = 10^6$, $N = 10$), then both the Bose-Einstein and the Fermi-Dirac results in part (a) contain products of N numbers, each number approximately equal to n. Thus show that for $n \gg N$, both the BE and the FD results are approximately equal to $n^N/N!$, which is $1/N!$ times the MB result.

20. (a) In Example 5, a mathematical model is discussed which claims to give a distribution of identical balls into boxes in such a way that all distinguishable arrangements are equally probable (Bose-Einstein statistics). Prove this by showing that the probability of a distribution of N balls into n boxes (according to this model) with N_1 balls in the first box, N_2 in the second, $\cdots$, N_n in the nth, is $1/C(n-1+N, N)$ for any set of numbers N_i such that $\sum_{i=1}^{n} N_i = N$.

(b) Show that the model in (a) leads to Maxwell-Boltzmann statistics if the drawn card is replaced (but no extra card added) and to Fermi-Dirac statistics if the drawn card is not replaced. *Hint:* Calculate in each case the number of possible arrangements of the balls in the boxes. First do the problem of 4 particles in 6 boxes as in the example, and then do N particles in n boxes ($n > N$) to get the results in Problem 19.

21. The following problem arises in quantum mechanics (see Chapter 13, Problem 7.21). Find the number of ordered triples of nonnegative integers a, b, c whose sum $a+b+c$ is a given positive integer n. (For example, if $n = 2$, we could have $(a, b, c) = (2, 0, 0)$ or (0, 2, 0) or (0, 0, 2) or (0, 1, 1) or (1, 0, 1) or (1, 1, 0).) *Hint:* Show that this is the same as the number of distinguishable distributions of n identical balls in 3 boxes, and follow the method of the diagram in Example 5.

22. Suppose 13 people want to schedule a regular meeting one evening a week. What is the probability that there is an evening when everyone is free if each person is already busy one evening a week?

23. Do Problem 22 if one person is busy 3 evenings, one is busy 2 evenings, two are each busy one evening, and the rest are free every evening.

▶ 5. RANDOM VARIABLES

In the problem of tossing two dice (Example 2, Section 2), we may be more interested in the value of the sum of the numbers on the two dice than we are in the individual numbers. Let us call this sum x; then for each point of the sample space in (2.4), x has a value. For example, for the point 2,1, we have $x = 2+1 = 3$; for the point 6,2, we have $x = 8$, etc. Such a variable, x, which has a definite value for each sample point, is called a *random variable*. We can easily construct many more examples of random variables for the sample space (2.4); here are a few (Can you construct some more?):

$$x = \text{number on first die minus number on second;}$$
$$x = \text{number on second die;}$$
$$x = \text{probability } p \text{ associated with the sample point;}$$
$$x = \begin{cases} 1 \text{ if the sum is 7 or 11,} \\ 0 \text{ otherwise.} \end{cases}$$

For each of these random variables x, we could set up a table listing all the sample points in (2.4) and, next to each sample point, the corresponding value of x. This table may remind you of the tables of values we could use in plotting the graph of a function. In analytical geometry or in a physics problem, knowing x as a function of t means that for any given t we can find the corresponding value of x. In probability the sample point corresponds to the independent variable t; given the sample point, we can find the corresponding value of the random variable x if we are given a description of x (for example, $x =$ the sum of numbers on dice). The "description" corresponds to the formula $x(t)$ that we use in plotting a graph in analytic geometry. Thus we may say that a *random variable x is a function defined on a sample space.*

Probability Functions Let us consider further the random variable $x =$ "sum of numbers on dice" for a toss of two dice [sample space (2.4)]. We note that there are several sample points for which $x = 5$, namely the points marked a in (2.4). Similarly, there are several sample points for most of the other values of x. It is then convenient to lump together all the sample points corresponding to a given value of x, and consider a new sample space in which each point corresponds to one value of x; this is the sample space (2.5). The probability associated with each point of the new sample space is obtained as in Section 2, by adding the probabilities associated with all the points in the original sample space corresponding to the particular value of x. Each value of x, say x_i, has a probability p_i of occurrence; we may write $p_i = f(x_i) =$ probability that $x = x_i$, and call the function $f(x)$ the *probability function* for the random variable x. In (2.5) we have listed on the first line the values of x and on the second line the values of $f(x)$. [In this problem, x and $f(x)$ take on only a finite number of discrete values; in some later problems they will take on a continuous set of values.] We could also exhibit these values graphically (Figure 5.1).

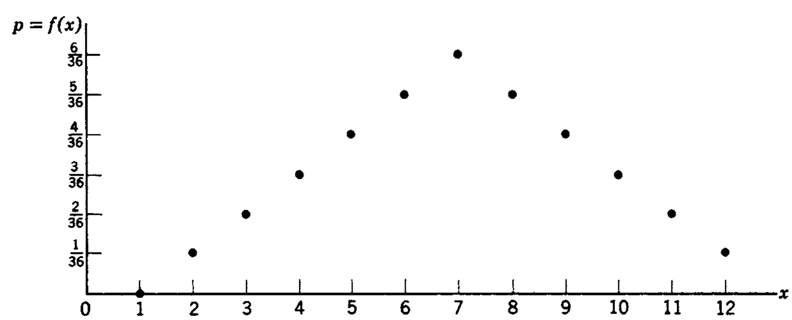


Figure 5.1

Now that we have the table of values (2.5) or the graph (Figure 5.1) to describe the random variable x and its probability function $f(x)$, we can dispense with the original sample space (2.4). But since we used (2.4) in defining what is meant by a random variable, let us now give another definition using (2.5) or Figure 5.1. We can say that *x is a random variable if it takes various values x_i with probabilities $p_i = f(x_i)$*. This definition may explain the name random variable; x is called a variable since it takes various values. A random (or stochastic) process is one whose outcome is not known in advance. The way the two dice fall is such an unknown outcome, so the value of x is unknown in advance, and we call x a *random* variable.

You may note that at first we thought of x as a dependent variable or function with the sample point as the independent variable. Although we didn't say much about it, there was also a value of the probability p attached to each sample point, that is p and x were both functions of the sample point. In the last paragraph, we have thought of x as an independent variable with p as a function of x. This is quite analogous to having both x and p given as functions of t and eliminating t to obtain p as a function of x. We have here eliminated the sample point from the forefront of our discussion in order to consider directly the probability function $p = f(x)$.

▶ **Example 1.** Let x = number of heads when three coins are tossed. The uniform sample space is (2.3) and we could write the value of x for each sample point in (2.3). Instead, let us go immediately to a table of x and $p = f(x)$. [Can you verify this table by using (2.3), or otherwise?]

(5.1)

x	0	1	2	3
$p = f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Other terms used for the probability function $p = f(x)$ are: *probability density function*, *frequency function*, or *probability distribution* (*caution*: **not** distribution function, which means the *cumulative distribution* as we will discuss later; see Figure 5.2). The origins of these terms will become clearer as we go on (Sections 6 and 7) but we can get some idea of the terms frequency and distribution from (5.1). Suppose we toss three coins repeatedly; we might reasonably expect to get three heads in about $\frac{1}{8}$ of the tosses, two heads in about $\frac{3}{8}$ of the tosses, etc. That is, each value of $p = f(x)$ is proportional to the *frequency* of occurrence of that value of x—hence the term *frequency function* (see also Section 7). Again in (5.1), imagine four boxes labeled $x = 0, 1, 2, 3$, and put a marble into the appropriate box for each toss of three coins. Then $p = f(x)$ indicates approximately how the marbles are distributed into the boxes after many tosses—hence the term *distribution*.

Mean Value; Standard Deviation The probability function $f(x)$ of a random variable x gives us detailed information about it, but for many purposes we want a simpler description. Suppose, for example, that x represents experimental measurements of the length of a rod, and that we have a large number N of measurements x_i. We might reasonably take $p_i = f(x_i)$ proportional to the number of times N_i we obtained the value x_i, that is $p_i = N_i/N$. We are especially interested in two numbers, namely a mean or average value of all our measurements, and some number which indicates how widely the original set of values spreads out about that average. Let us define two such quantities which are customarily used to describe a random variable. To calculate the average of a set of N numbers, we add them and divide by N. Instead of adding the large number of measurements, we can multiply each measurement by the number of times it occurs and add the results. This gives for the average of the measurements, the value

$$\frac{1}{N} \cdot \sum_i N_i x_i = \sum_i p_i x_i.$$

By analogy with this calculation, we now define the *average* or *mean value* μ *of a random variable* x whose probability function is $f(x)$ by the equation

$$\mu = \text{average of } x = \sum_i x_i p_i = \sum_i x_i f(x_i). \tag{5.2}$$

To obtain a measure of the spread or dispersion of our measurements, we might first list how much each measurement differs from the average. Some of these deviations are positive and some are negative; if we average them, we get zero (Problem 10). Instead, let us square each deviation and average the squares. We define the *variance* of a random variable x by the equation

$$\mathrm{Var}(x) = \sum_i (x_i - \mu)^2 f(x_i). \tag{5.3}$$

(The variance is sometimes called the dispersion.) If nearly all the measurements x_i are very close to μ, then $\mathrm{Var}(x)$ is small; if the measurements are widely spread, $\mathrm{Var}(x)$ is large. Thus we have a number which indicates the spread of the measurements; this is what we wanted. The square root of $\mathrm{Var}(x)$, called the *standard deviation* of x, is often used instead of $\mathrm{Var}(x)$:

$$\sigma_x = \text{standard deviation of } x = \sqrt{\mathrm{Var}(x)}. \tag{5.4}$$

Example 2. For the data in (5.1) we can compute:

By (5.2), $\mu = \text{average of } x = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = \frac{3}{2}$.

By (5.3), $\mathrm{Var}(x) = \left(0 - \frac{3}{2}\right)^2 \cdot \frac{1}{8} + \left(1 - \frac{3}{2}\right)^2 \cdot \frac{3}{8} + \left(2 - \frac{3}{2}\right)^2 \cdot \frac{3}{8} + \left(3 - \frac{3}{2}\right)^2 \cdot \frac{1}{8}$
$= \frac{9}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{3}{8} + \frac{1}{4} \cdot \frac{3}{8} + \frac{9}{4} \cdot \frac{1}{8} = \frac{3}{4}$.

By (5.4), $\sigma_x = \text{standard deviation of } x = \sqrt{\mathrm{Var}(x)} = \frac{1}{2}\sqrt{3}$.

The mean or average value of a random variable x is also called its *expectation* or its *expected value* or (especially in quantum mechanics) its *expectation value*. Instead of μ, the symbols $\overline{x}$ or $E(x)$ or $\langle x \rangle$ may be used to denote the mean value of x.

$$\overline{x} = E(x) = \langle x \rangle = \mu = \sum_i x_i f(x_i). \tag{5.5}$$

The term expectation comes from games of chance.

▶ **Example 3.** Suppose you will be paid \$5 if a die shows a 5, \$2 if it shows a 2 or a 3, and nothing otherwise. Let x represent your gain in playing the game. Then the possible values of x and the corresponding probabilities are $x = 5$ with $p = \frac{1}{6}$, $x = 2$ with $p = \frac{1}{3}$, and $x = 0$ with $p = \frac{1}{2}$. We find for the average or expectation of x:

$$E(x) = \sum x_i p_i = \$5 \cdot \frac{1}{6} + \$2 \cdot \frac{1}{3} + \$0 \cdot \frac{1}{2} = \$1.50.$$

If you play the game many times, this is a reasonable estimate of your average gain per game; this is what your expectation means. It is also a reasonable amount to pay as a fee for each game you play. The term *expected value* (which means the same as *expectation* or *average*) may be somewhat confusing and misleading if you try to interpret "expected" in an everyday sense. Note that the expected value (\$1.50) of x is not one of the possible values of x, so you cannot ever "expect" to have $x = \$1.50$. If you think of expected value as a technical term meaning the same as average, then there is no difficulty. Of course, in some cases, it makes reasonable sense with its everyday meaning; for example, if a coin is tossed n times, the expected number of heads is $n/2$ (Problem 11) and it is true that we may reasonably "expect" a fair approximation to this result (see Section 7).

Cumulative Distribution Functions So far we have been using the probability function $f(x)$ which gives the probability $p_i = f(x_i)$ that x is exactly x_i. In some problems we may be more interested in the probability that x is less than some particular value. For example, in an election we would like to know the probability that less than half the votes would be cast for the opposing candidate, that is, that our candidate would win. In an experiment on radioactivity, we would like to know the probability that the background radiation always remains below a certain level. Given the probability function $f(x)$, we can obtain the probability that x is less than or equal to a certain value x_i by adding all the probabilities of values of x less than or equal to x_i. For example, consider the sum of the numbers on two dice; the probability function $p = f(x)$ is plotted in Figure 5.1. The probability that x is, say, less than or equal to 4 is the sum of the probabilities that x is 2 or 3 or 4, that is, $\frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{1}{6}$. Similarly, we could find the probability that x is less than or equal to any given number. The resulting function of x is plotted in Figure 5.2. Such a function $F(x)$ is called a *cumulative distribution function*; we can write

$$F(x_i) = (\text{probability that } x \le x_i) = \sum_{x_j \le x_i} f(x_j). \tag{5.6}$$

Note carefully that, although the probability function $f(x)$ may be referred to as a *probability distribution*, the term *distribution function* means the *cumulative distribution* $F(x)$.

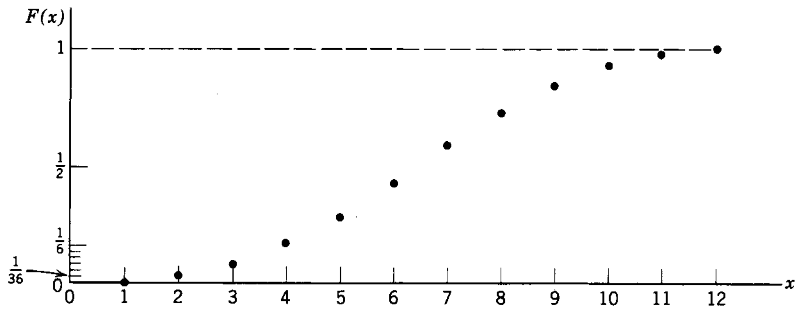


Figure 5.2

PROBLEMS, SECTION 5

Set up sample spaces for Problems 1 to 7 and list next to each sample point the value of the indicated random variable x, and the probability associated with the sample point. Make a table of the different values x_i of x and the corresponding probabilities $p_i = f(x_i)$. Compute the mean, the variance, and the standard deviation for x. Find and plot the cumulative distribution function $F(x)$.

1. Three coins are tossed; $x =$ number of heads minus number of tails.

2. Two dice are thrown; $x =$ sum of the numbers on the dice.

3. A coin is tossed repeatedly; $x =$ number of the toss at which a head first appears.

4. Suppose that Martian dice are 4-sided (tetrahedra) with points labeled 1 to 4. When a pair of these dice is tossed, let x be the product of the two numbers at the tops of the dice if the product is odd; otherwise $x = 0$.

5. A random variable x takes the values 0, 1, 2, 3, with probabilities $\frac{5}{12}$, $\frac{1}{3}$, $\frac{1}{12}$, $\frac{1}{6}$.

6. A card is drawn from a shuffled deck. Let $x = 10$ if it is an ace or a face card; $x = -1$ if it is a 2; and $x = 0$ otherwise.

7. A weighted coin with probability p of coming down heads is tossed three times; $x =$ number of heads minus number of tails.

8. Would you pay \$10 per throw of two dice if you were to receive a number of dollars equal to the product of the numbers on the dice? *Hint:* What is your expectation? If it is more than \$10, then the game would be favorable for you.

9. Show that the expectation of the sum of two random variables defined over the same sample space is the sum of the expectations. *Hint:* Let $p_1, p_2, \cdots, p_n$ be the probabilities associated with the n sample points; let $x_1, x_2, \cdots, x_n$, and $y_1, y_2, \cdots, y_n$, be the values of the random variables x and y for the n sample points. Write out $E(x)$, $E(y)$, and $E(x + y)$.

10. Let μ be the average of the random variable x. Then the quantities $(x_i - \mu)$ are the deviations of x from its average. Show that the average of these deviations is zero. *Hint:* Remember that the sum of all the p_i must equal 1.

11. Show that the expected number of heads in a single toss of a coin is $\frac{1}{2}$. Show in two ways that the expected number of heads in two tosses of a coin is 1:

(a) Let $x =$ number of heads in two tosses and find $\overline{x}$.

(b) Let $x =$ number of heads in toss 1 and $y =$ number of heads in toss 2; find the average of $x + y$ by Problem 9. Use this method to show that the expected number of heads in n tosses of a coin is $\frac{1}{2}n$.

12. Use Problem 9 to find the expected value of the sum of the numbers on the dice in Problem 2.

13. Show that adding a constant K to a random variable increases the average by K but does not change the variance. Show that multiplying a random variable by K multiplies both the average and the standard deviation by K.

14. As in Problem 11, show that the expected number of 5's in n tosses of a die is $n/6$.

15. Use Problem 9 to find $\overline{x}$ in Problem 7.

16. Show that $\sigma^2 = E(x^2) - \mu^2$. *Hint:* Write the definition of σ^2 from (5.3) and (5.4) and use Problems 9 and 13.

17. Use Problem 16 to find σ in Problems 2, 6, and 7.

6. CONTINUOUS DISTRIBUTIONS

In Section 5, we discussed random variables x which took a discrete set of values x_i. It is not hard to think of cases in which a random variable takes a continuous set of values.

Example 1. Consider a particle moving back and forth along the x axis from $x = 0$ to $x = l$, rebounding elastically at the turning points so that its speed is constant. (This could be a simple-minded model of an alpha particle in a radioactive nucleus, or of a gas molecule bouncing back and forth between the walls of a container.) Let the position x of the particle be the random variable; then x takes a continuous set of values from $x = 0$ to $x = l$. Now suppose that, following Section 5, we ask for the probability that the particle is *at* a particular point x; this probability must be the same, say k, for all points (because the speed is constant). In Section 5, with a finite number of points, we would say $k = 1/N$. In the continuous case, there are an infinite number of points so we would find $k = 0$, that is, the probability that the particle is *at* a given point) must be zero. But this is not a very useful result. Let us instead divide $(0, l)$ into small intervals dx; since the particle has constant speed, the time it spends in each dx is proportional to the length of dx. In fact, since the particle spends the fraction $(dx)/l$ of its time in a given interval dx, the probability of finding it in dx is just $(dx)/l$.

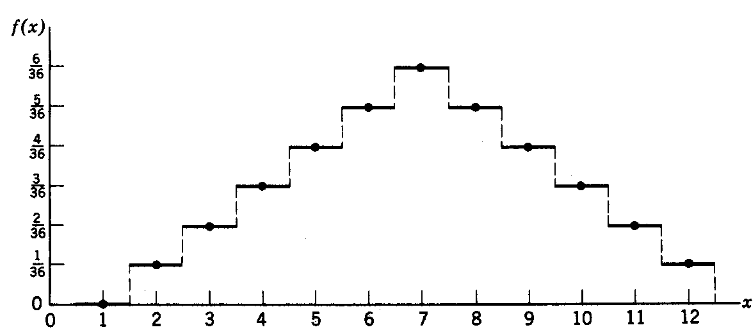


Figure 6.1

Comparison of Discrete and Continuous Probability Functions To see how to define a probability function for the continuous case and to correlate this discussion with the discrete case, let us return for a moment to Figure 5.1. There we plotted a vertical *distance* to represent the probability $p = f(x)$ of each value of x. Instead of a dot (as in Figure 5.1) to indicate p for each x, let us now draw a horizontal line segment of length 1 centered on each dot, as in Figure 6.1. Then the *area* under the horizontal line segment at a particular x_i is $f(x_i) \cdot 1 = f(x_i) = p_i$ (since the length of each horizontal line segment is 1), and we could use this *area* instead of the ordinate as a measure of the probability. Such a graph is called a *histogram*.

Example 2. Now let us apply this area idea to Example 1. Consider Figure 6.2. We have plotted the function

$$f(x) = \begin{cases} 1/l, & 0 < x < l, \\ 0, & x < 0 \quad \text{and} \quad x > l. \end{cases}$$

If we consider any interval x to $x + dx$ on $(0, l)$, the area under the curve $f(x) = 1/l$ for this interval is $(1/l)\,dx$ or $f(x)\,dx$, and this is just the probability that the particle is in this interval. The probability that the particle is in some longer subinterval of $(0, l)$, say (a, b), is $(b - a)/l$ or $\int_a^b f(x)\,dx$, that is, the area under the curve from a to b. If the interval (a, b) is outside $(0, l)$, then $\int_a^b f(x)\,dx = 0$ since $f(x)$ is zero, and again this is the correct value of the probability of finding the particle on the given interval.

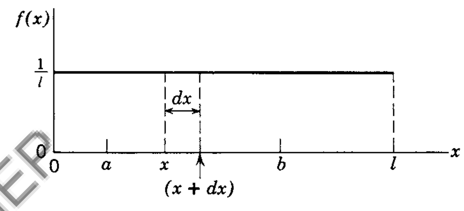


Figure 6.2

When $f(x)$ is constant over an interval (as in Figure 6.2), we say that x is *uniformly* distributed on that interval. Let us consider an example in which $f(x)$ is not constant.

Example 3. This time suppose the particle of Example 1 is sliding up and down an inclined plane (no friction) rebounding elastically (no energy loss) against a spring at the bottom and reaching zero speed at height $y = h$ (Figure 6.3). The total energy, namely $\frac{1}{2}mv^2 + mgy$ is constant and equal to mgh since $v = 0$ at $y = h$. Thus we have

$$v^2 = \frac{2}{m}(mgh - mgy) = 2g(h - y). \tag{6.1}$$

The probability of finding the particle within an interval dy at a given height y is proportional to the time dt spent in that interval. From $v = ds/dt$, we have $dt = (ds)/v$; from Figure 6.3, we find $ds = (dy)\csc\alpha$. Combining these with (6.1) we have

$$dt = \frac{ds}{v} = \frac{(dy)\csc\alpha}{\sqrt{2g}\sqrt{h-y}}$$

Since the probability $f(y)\,dy$ of finding the particle in the interval dy at height y is proportional to dt, we can drop the constant factor $(\csc\alpha)/\sqrt{2g}$, and say that

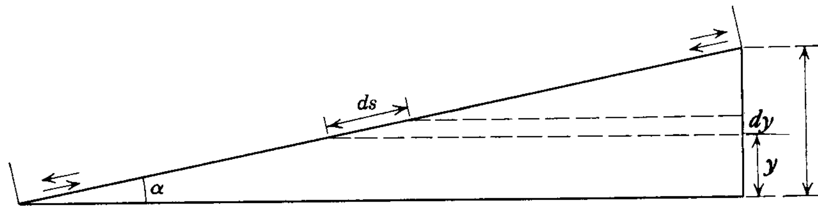


Figure 6.3

$f(y)\,dy$ is proportional to $dy/\sqrt{h-y}$. In order to find $f(y)$, we must multiply by a constant factor which makes the total probability $\int_0^h f(y)\,dy$ equal to 1 since this is the probability that the particle is *somewhere*. You can easily verify that

$$f(y)\,dy = \frac{1}{2\sqrt{h}}\frac{dy}{\sqrt{h-y}} \quad \text{or} \quad f(y) = \frac{1}{2\sqrt{h(h-y)}}$$

A graph of $f(y)$ is plotted in Figure 6.4. Note that although $f(y)$ becomes infinite at $y = h$, the area under the $f(y)$ curve for any interval is finite; this area represents the probability that the particle is in that height interval.

Figure 6.4

We can now extend the definitions of mean (expectation), variance, standard deviation, and cumulative distribution function to the continuous case. Let $f(x)$ be a probability density function; remember that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ just as $\sum_{i=1}^{n} p_i = 1$. The average of a random variable x with probability density function $f(x)$ is

$$\mu = \overline{x} = E(x) = \langle x \rangle = \int_{-\infty}^{\infty} x f(x)\,dx. \tag{6.2}$$

(In writing the limits $-\infty, \infty$ here, we assume that $f(x)$ is defined to be zero on intervals where the probability is zero.) Note that (6.2) is a natural extension of the sum in (5.5). Having found the mean of x, we now define the variance as in Section 5 as the average of $(x-\mu)^2$, that is,

$$\text{(6.3)} \qquad \operatorname{Var}(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \sigma_x^2.$$

As before, the standard deviation σ_x is the square root of the variance. Finally, the cumulative distribution function $F(x)$ gives for each x the probability that the random variable is less than or equal to that x. But this probability is just the area under the $f(x)$ curve from $-\infty$ up to the point x. Also, of course, the integral of $f(x)$ from $-\infty$ to ∞ must $=1$ since that is the total probability for all values of x. Thus we have

$$\text{(6.4)} \qquad F(x) = \int_{-\infty}^{x} f(u)\,du, \qquad \int_{-\infty}^{\infty} f(x)\,dx = F(\infty) = 1.$$

Example 4. For the problem in Example 3, we find:

By (6.2), $\mu_y = \int_0^h y f(y)\,dy = \frac{1}{2\sqrt{h}}\int_0^h y\frac{1}{\sqrt{h-y}}\,dy = \frac{2}{3}h.$

By (6.3), $\operatorname{Var}(y) = \int_0^h (y-\mu_y)^2 f(y)\,dy = \int_0^h \left(y-\frac{2}{3}h\right)^2 \frac{1}{\sqrt{h-y}}\,dy = \frac{4h^2}{45},$

so standard deviation $\sigma_y = \sqrt{\operatorname{Var}(y)} = 2h/\sqrt{45}.$

By (6.4), cumulative distribution function $F(y) = \int_0^y f(u)\,du$

$$= \frac{1}{2\sqrt{h}}\int_0^y \frac{du}{\sqrt{h-u}}.$$

Why "density function"? In Section 5, we mentioned that the probability function $f(x)$ is often called the *probability density*. We can now explain why. Consider (6.2). If $f(x)$ represents the density (mass per unit length) of a thin rod, then the center of mass of the rod is given by [see Chapter 5, (3.3)]

$$\text{(6.5)} \qquad \overline{x} = \int x f(x)\,dx \Big/ \int f(x)\,dx,$$

where the integrals are over the length of the rod, or from $-\infty$ to ∞ as in (6.2) with $f(x)=0$ outside the rod. But in (6.2), $\int f(x)\,dx$ is the total probability that x has *some* value, and so this integral is equal to 1. Then (6.5) and (6.2) are really the same; we see that it is reasonable to call $f(x)$ a density, and also that the mean of x corresponds to the center of mass of a linear mass distribution of density $f(x)$. In a similar way, we can interpret (6.3) as giving the moment of inertia of the mass distribution about the center of mass (see Chapter 5, Section 3).

Joint Distributions We can easily generalize the ideas and formulas above to two (or more) dimensions. Suppose we have two random variables x and y; we define their joint probability density function $f(x,y)$ so that $f(x_i,y_j)\,dx\,dy$ is the probability that the point (x,y) is in an element of area $dx\,dy$ at $x=x_i$, $y=y_j$. Then the probability that the point (x,y) is in a given region of the (x,y) plane, is the integral of $f(x,y)$ over that area. The average or expected values of x and y, the variances and standard deviations of x and y, and the covariance of x, y (see Problems 13 to 16) are given by

$$\begin{aligned}
\overline{x} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy,\\
\overline{y} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\,dx\,dy,\\
\text{Var}(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\overline{x})^2 f(x,y)\,dx\,dy = \sigma_x^2,\\
\text{Var}(y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\overline{y})^2 f(x,y)\,dx\,dy = \sigma_y^2,\\
\text{Cov}(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\overline{x})(y-\overline{y}) f(x,y)\,dx\,dy.
\end{aligned} \tag{6.6}$$

You should see that these are generalizations of (6.2) and (6.3); that (6.6) can be interpreted as giving the coordinates of the center of mass and the moments of inertia of a two-dimensional mass distribution; and that similar formulas can be written for three (or more) random variables (that is, in three or more dimensions). Also note that the formulas in (6.6) could be written in terms of polar coordinates (see Problems 6 to 9).

We have discussed a number of probability distributions both discrete and continuous, and you will find others in the problems. We will discuss three very important named distributions (binomial, normal, and Poisson) in the following sections. Learning about these and related graphs, formulas, and terminology should make it possible for you to cope with any of the many other named distributions you find in texts, reference books, and computer programs.

▶ PROBLEMS, SECTION 6

1. (a) Find the probability density function $f(x)$ for the position x of a particle which is executing simple harmonic motion on $(-a,a)$ along the x axis. (See Chapter 7, Section 2, for a discussion of simple harmonic motion.) *Hint:* The value of x at time t is $x=a\cos\omega t$. Find the velocity dx/dt; then the probability of finding the particle in a given dx is proportional to the time it spends there which is inversely proportional to its speed there. Don't forget that the total probability of finding the particle *somewhere* must be 1.

(b) Sketch the probability density function $f(x)$ found in part (a) and also the cumulative distribution function $F(x)$ [see equation (6.4)].

(c) Find the average and the standard deviation of x in part (a).

2. It is shown in the kinetic theory of gases that the probability for the distance a molecule travels between collisions to be between x and $x+dx$, is proportional to $e^{-x/\lambda}\,dx$, where λ is a constant. Show that the average distance between collisions (called the "mean free path") is λ. Find the probability of a free path of length $\geq 2\lambda$.

3. A ball is thrown straight up and falls straight back down. Find the probability density function $f(h)$ so that $f(h)\,dh$ is the probability of finding it between height h and $h + dh$. *Hint:* Look at Example 3.

4. In Problem 1 we found the probability density function for a classical harmonic oscillator. In quantum mechanics, the probability density function for a harmonic oscillator (in the ground state) is proportional to $e^{-\alpha^2 x^2}$, where α is a constant and x takes values from $-\infty$ to ∞. Find $f(x)$ and the average and standard deviation of x. (In quantum mechanics, the standard deviation of x is called the uncertainty in position and is written Δx.)

5. The probability for a radioactive particle to decay between time t and time $t + dt$ is proportional to $e^{-\lambda t}$. Find the density function $f(t)$ and the cumulative distribution function $F(t)$. Find the expected lifetime (called the mean life) of the radioactive particle. Compare the mean life and the so-called "half life" which is defined as the value of t when $e^{-\lambda t} = 1/2$.

6. A circular garden bed of radius 1 m is to be planted so that N seeds are uniformly distributed over the circular area. Then we can talk about the number n of seeds in some particular area A, or we can call n/N the probability for any one particular seed to be in the area A. Find the probability $F(r)$ that a seed (that is, some particular seed) is within r of the center. (*Hint:* What is F(1)?) Find $f(r)\,dr$, the probability for a seed to be between r and $r + dr$ from the center. Find $\overline{r}$ and σ.

7. (a) Repeat Problem 6 where the "circular" area is now on the curved surface of the earth, say all points at distance s from Chicago (measured along a great circle on the earth's surface) with $s \leq \pi R/3$ where R = radius of the earth. The seeds could be replaced by, say, radioactive fallout particles (assuming these to be uniformly distributed over the surface of the earth). Find $F(s)$ and $f(s)$.

(b) Also find $F(s)$ and $f(s)$ if $s \leq 1 \ll R$ (say $s \leq 1$ mile where $R = 4000$ miles). Do your answers then reduce to those in Problem 6?

8. Given that a particle is inside a sphere of radius 1, and that it has equal probabilities of being found in any two volume elements of the same size, find the cumulative distribution function $F(r)$ for the spherical coordinate r, and from it find the density function $f(r)$. *Hint:* $F(r)$ is the probability that the particle is inside a sphere of radius r. Find $\overline{r}$ and σ.

9. A hydrogen atom consists of a proton and an electron. According to the Bohr theory, the electron revolves about the proton in a circle of radius a ($a = 5 \cdot 10^{-9}$cm for the ground state). According to quantum mechanics, the electron may be at any distance r (from 0 to ∞) from the proton; for the ground state, the probability that the electron is in a volume element dV, at a distance r to $r + dr$ from the proton, is proportional to $e^{-2r/a}dV$, where a is the Bohr radius. Write dV in spherical coordinates (see Chapter 5, Section 4) and find the density function $f(r)$ so that $f(r)\,dr$ is the probability that the electron is at a distance between r and $r + dr$ from the proton. (Remember that the probability for the electron to be somewhere must be 1.) Computer plot $f(r)$ and show that its maximum value is at $r = a$; we then say that the most probable value of r is a. Also show that the average value of r^{-1} is a^{-1}.

10. Do Problem 5.10 for a continuous distribution.

11. Do Problem 5.13 for a continuous distribution.

12. Do Problem 5.16 for a continuous distribution.

13. Given a joint distribution function $f(x, y)$ as in (6.6), show that $E(x + y) = E(x) + E(y)$ and $\text{Var}(x + y) = \text{Var}(x) + \text{Var}(y) + 2\,\text{Cov}(x, y)$.

14. Recall that two events A and B are called independent if $p(AB) = p(A)p(B)$. Similarly two random variables x and y are called independent if the joint probability function $f(x, y) = g(x)h(y)$. Show that if x and y are independent, then the expectation or average of xy is $E(xy) = E(x)E(y) = \mu_x\mu_y$.

15. Show that the covariance of two independent (see Problem 14) random variables is zero, and so by Problem 13, the variance of the sum of two independent random variables is equal to the sum of their variances.

16. By Problem 15, if x and y are independent, then $\text{Cov}(x, y) = 0$. The converse is not always true, that is, if $\text{Cov}(x, y) = 0$, it is not necessarily true that the joint distribution function is of the form $f(x, y) = g(x)h(y)$. For example, suppose $f(x, y) = (3y^2 + \cos x)/4$ on the rectangle $-\pi/2 < x < \pi/2, -1 < y < 1$, and $f(x, y) = 0$ elsewhere. Show that $\text{Cov}(x, y) = 0$, but x and y are not independent, that is, $f(x, y)$ is not of the form $g(x)h(y)$. Can you construct some more examples?

▶ 7. BINOMIAL DISTRIBUTION

▶ **Example 1.** Let a coin be tossed 5 times; what is the probability of exactly 3 heads out of the 5 tosses? We can represent any sequence of 5 tosses by a symbol such as $thhth$. The probability of this particular sequence (or any other particular sequence) is $(\frac{1}{2})^5$ since the tosses are independent (see Example 1 of Section 3). The number of such sequences containing 3 heads and 2 tails is the number of ways we can select 3 positions out of 5 for heads (or 2 for tails), namely $C(5, 3)$. Hence, the probability of exactly 3 heads in 5 tosses of a coin is $C(5, 3)(\frac{1}{2})^5$. Suppose a coin is tossed repeatedly, say n times; let x be the number of heads in the n tosses. We want to find the probability density function $p = f(x)$ which gives the probability of exactly x heads in n tosses. By generalizing the case of 3 heads in 5 tosses, we see that

$$f(x) = C(n, x)(\tfrac{1}{2})^n \tag{7.1}$$

▶ **Example 2.** Let us do a similar problem with a die, asking this time for the probability of exactly 3 aces in 5 tosses of the die. If A means ace and N not ace, the probability of a particular sequence such as $ANNAA$ is $\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$ since the probability of A is $\frac{1}{6}$, the probability of N is $\frac{5}{6}$, and the tosses are independent. The number of such sequences containing 3 A's and 2 N's is $C(5, 3)$; thus the probability of exactly 3 aces in 5 tosses of a die is $C(5, 3)(\frac{1}{6})^3(\frac{5}{6})^2$. Generalizing this, we find that the probability of exactly x aces in n tosses of a die is

$$f(x) = C(n, x)(\tfrac{1}{6})^x(\tfrac{5}{6})^{n-x}. \tag{7.2}$$

Bernoulli Trials In the two examples we have just done, we have been concerned with repeated independent trials, each trial having two possible outcomes (h or t, A or N) of given probability. There are many examples of such problems; let's consider a few. A manufactured item is good or defective; given the probability of a defect we want the probability of x defectives out of n items. An archer has probability p of hitting a target; we ask for the probability of x hits out of n tries. Each atom of a radioactive substance has probability p of emitting an alpha particle during the next minute; we are to find the probability that x alpha particles will be emitted in the next minute from the n atoms in the sample. A particle moves back and forth along the x axis in unit jumps; it has, at each step, equal probabilities of

Graphs of the binomial distribution, $f(x) = C(n, x)p^x q^{n-x}$

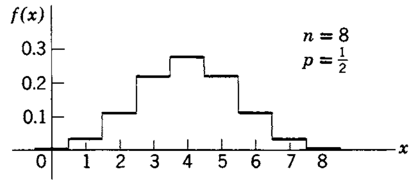


Figure 7.1

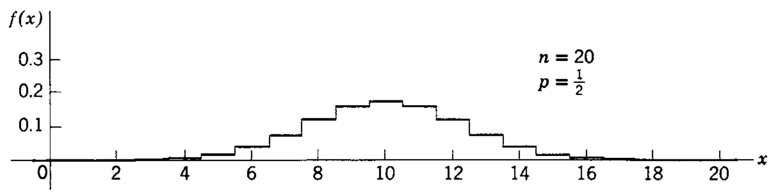


Figure 7.2

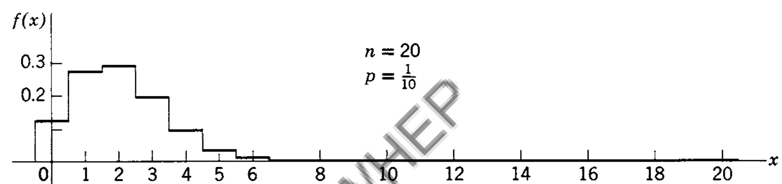


Figure 7.3

jumping forward or backward. (This motion is called a *random walk*; it can be used as a model of a diffusion process.) We want to know the probability that, after n jumps, the particle is at a distance

$$d = \text{number } x \text{ of positive jumps} - \text{number } (n-x) \text{ of negative jumps},$$

from its starting point; this probability is the probability of x positive jumps out of a total of n jumps.

In all these problems, something is tried repeatedly. At each trial there are two possible outcomes of probabilities p (usually called the probability of "success") and

Binomial distribution graphs of $nf(x)$ plotted against x/n

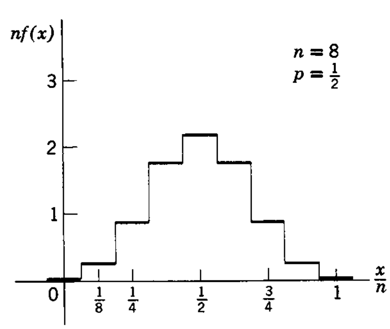


Figure 7.4

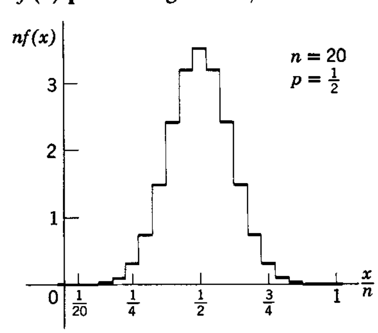


Figure 7.5

$q = 1 - p$ (where q = probability of "failure"). Such repeated independent trials with constant probabilities p and q are called *Bernoulli trials*.

Binomial Probability Functions Let us generalize (7.1) and (7.2) to obtain a formula which applies to any similar problem, namely the probability $f(x)$ of exactly x successes in n Bernoulli trials. Reasoning as we did to obtain (7.1) and (7.2), we find that

$$f(x) = C(n, x)p^x q^{n-x}. \tag{7.3}$$

We might also ask for the probability of *not more than* x successes in n trials. This is the sum of the probabilities of $0, 1, 2, \cdots, x$ successes, that is, it is the cumulative distribution function $F(x)$ for the random variable x whose probability density function is (7.3) [see (5.6)]. We can write

$$\begin{aligned} F(x) &= f(0) + f(1) + \cdots + f(x) \\ &= C(n,0)p^0q^n + C(n,1)p^1q^{n-1} + \cdots + C(n,x)p^xq^{n-x} \\ &= \sum_{u=0}^{x} C(n,u)p^uq^{n-u} = \sum_{u=0}^{x} \binom{n}{u} p^uq^{n-u}. \end{aligned} \tag{7.4}$$

Observe that (7.3) is one term of the binomial expansion of $(p + q)^n$ and (7.4) is a sum of several terms of this expansion (see Section 4, Example 2). For this reason, the functions $f(x)$ in (7.1), (7.2), or (7.3) are called *binomial probability* (or *density*) *functions* or *binomial distributions*, and the function $F(x)$ in (7.4) is called a *binomial cumulative distribution function*.

We shall find it very useful to computer plot graphs of the binomial density function $f(x)$ for various values of p and n. (See Figures 7.1 to 7.5 and Problems 1 to 8.) Instead of a point at $y = f(x)$ for each x, we plot a horizontal line segment of length 1 centered on each x as in Figure 6.1; the probabilities are then represented by *areas* under the broken line, rather than by ordinates. From Figures 7.1 to 7.3 and similar graphs, we can draw a number of conclusions. The most probable value of x [corresponding to the largest value of $f(x)$] is approximately $x = np$ (Problems 10 and 11); for example for $p = \frac{1}{2}$, the most probable value of x is $\frac{1}{2}n$ for even n; for odd n, there are two consecutive values of x, namely $\frac{1}{2}(n \pm 1)$, for which the probability is largest. The graphs for $p = \frac{1}{2}$ are symmetric about $x = \frac{1}{2}n$. For $p \neq \frac{1}{2}$, the curve is asymmetric, favoring small x values for small p and large x values for large p. As n increases, the graph of $f(x)$ becomes wider and flatter (the total area under the graph must remain 1). The probability of the most probable value of x decreases with n. For example, the most probable number of heads in 8 tosses of a coin is 4 with probability 0.27; the most probable number of heads in 20 tosses is 10 with probability 0.17; for 10^6 tosses, the probability of exactly 500,000 heads is less than 10^{-3}.

Let us redraw Figures 7.1 and 7.2 plotting $nf(x)$ against the relative number of successes x/n (Figures 7.4 and 7.5). Since this change of scale (ordinate times n, abscissa divided by n) leaves the area unchanged, we can still use the area to represent probability. Note that now the curves become narrower and taller as n

increases. This means that values of the ratio x/n tend to cluster about their most probable value, namely $np/n = p$. For example, if we toss a coin repeatedly, the difference "number of heads $-\frac{1}{2}$ number of tosses" is apt to be large and to increase with n (Figures 7.1 and 7.2), but the ratio "number of heads $\div$ number of tosses" is apt to be closer and closer to $\frac{1}{2}$ as n increases (Figures 7.4 and 7.5). It is for this reason that we can use experimentally determined values of x/n as a reasonable estimate of p.

Chebyshev's Inequality This is a simple but very general result which we will find useful. We consider a random variable x with probability function $f(x)$, and let μ be the mean value and σ the standard deviation of x. We are going to prove that if we select any number t, the probability that x differs from its mean value μ by more than t, is less than σ^2/t^2. This means that x is unlikely to differ from μ by more than a few standard deviations; for example, if t is twice the standard deviation σ, we find that the probability for x to differ from μ by more than 2σ is less than $\sigma^2/t^2 = \sigma^2/(2\sigma)^2 = \frac{1}{4}$. The proof is simple. By definition of σ, we have

$$\sigma^2 = \sum (x-\mu)^2 f(x)$$

where the sum is over all x. Then if we sum just over the values of x for which $|x-\mu| \geq t$, we get less than σ^2:

$$\sigma^2 > \sum_{|x-\mu|\geq t} (x-\mu)^2 f(x). \tag{7.5}$$

If we replace each $x-\mu$ by the number t in (7.5), the sum is decreased, so we have

$$\sigma^2 > \sum_{|x-\mu|\geq t} t^2 f(x) = t^2 \sum_{|x-\mu|\geq t} f(x) \qquad \text{or} \qquad \sum_{|x-\mu|\geq t} f(x) < \frac{\sigma^2}{t^2}. \tag{7.6}$$

But $\sum_{|x-\mu|\geq t} f(x)$ is just the sum of all probabilities of x values which differ from μ by more than t, and (7.6) says that this probability is less than σ^2/t^2, as we claimed.

Laws of Large Numbers Statements and proofs which make more precise our general comments about the effect of large n are known as *laws of large numbers*. Let us state and prove one such law. We apply Chebyshev's inequality to a random variable whose probability function is the binomial distribution (7.3). From Problems 9 and 13 we have $\mu = np$ and $\sigma = \sqrt{npq}$. Then by Chebyshev's inequality,

$$(\text{probability of } |x - np| \geq t) \qquad \text{is less than} \quad npq/t^2. \tag{7.7}$$

Let us choose the arbitrary value of t in (7.7) proportional to n, that is, $t = n\epsilon$ where ϵ is now arbitrary. Then (7.7) becomes

$$(\text{probability of } |x - np| \geq n\epsilon) \qquad \text{is less than} \quad npq/n^2\epsilon^2, \tag{7.8}$$

or, when we divide the first inequality by n,

$$\left(\text{probability of } \left|\frac{x}{n} - p\right| \geq \epsilon\right) \qquad \text{is less than} \quad \frac{pq}{n\epsilon^2}. \tag{7.9}$$

Recall that x/n is the relative number of successes; we intuitively expect x/n to be near p for large n. Now (7.9) says that, if ϵ is any small number, the probability is less than $pq/(n\epsilon^2)$ for x/n to differ from p by ϵ; that is, as n tends to infinity, this probability tends to zero. (Note, however, that x/n need not tend to p.) This is one form of the law of large numbers and it justifies our intuitive ideas.

▶ PROBLEMS, SECTION 7

For the values of n indicated in Problems 1 to 4:

(a) Write the probability density function $f(x)$ for the probability of x heads in n tosses of a coin and computer plot a graph of $f(x)$ as in Figures 7.1 and 7.2. Also computer plot a graph of the corresponding cumulative distribution function $F(x)$.

(b) Computer plot a graph of $nf(x)$ as a function of x/n as in Figures 7.4 and 7.5.

(c) Use your graphs and other calculations if necessary to answer these questions: What is the probability of exactly 7 heads? Of at most 7 heads? [Hint: Consider $F(x)$.] Of at least 7 heads? What is the most probable number of heads? The expected number of heads?

1. $n = 7$ **2.** $n = 12$ **3.** $n = 15$ **4.** $n = 18$

5. Write the formula for the binomial density function $f(x)$ for the case $n = 6, p = 1/6$, representing the probability of, say, x aces in 6 throws of a die. Computer plot $f(x)$ as in Figure (7.3). Also plot the cumulative distribution function $F(x)$. What is the probability of at least 2 aces out of 6 tosses of a die? *Hint:* Can you read the probability of at most one ace from one of your graphs?

For the given values of n and p in Problems 6 to 8, computer plot graphs of the binomial density function for the probability of x successes in n Bernoulli trials with probability p of success.

6. $n = 6$, $p = 5/6$ (Compare Problem 5)

7. $n = 50$, $p = 1/5$ **8.** $n = 50$, $p = 4/5$

9. Use the second method of Problem 5.11 to show that the expected number of successes in n Bernoulli trials with probability p of success is $\overline{x} = np$. *Hint:* What is the expected number of successes in one trial?

10. Show that the most probable number of heads in n tosses of a coin is $\frac{1}{2}n$ for even n [that is, $f(x)$ in (7.1) has its largest value for $x = n/2$] and that for odd n, there are two equal "largest" values of $f(x)$, namely for $x = \frac{1}{2}(n+1)$ and $x = \frac{1}{2}(n-1)$. *Hint:* Simplify the fraction $f(x+1)/f(x)$, and then find the values of x for which it is greater than 1 [that is, $f(x+1) > f(x)$], and less than or equal to 1 [that is, $f(x+1) \le f(x)$]. Remember that x must be an integer.

11. Use the method of Problem 10 to show that for the binomial distribution (7.3), the most probable value of x is approximately np (actually within 1 of this value).

12. Let x = number of heads in one toss of a coin. What are the possible values of x and their probabilities? What is μ_x? Hence show that $\mathrm{Var}(x)$ = [average of $(x-\mu_x)^2$] $= \frac{1}{4}$, so the standard deviation is $\frac{1}{2}$. Now use the result from Problem 6.15 "variance of a sum of independent random variables = sum of their variances" to show that if x = number of heads in n tosses of a coin, $\mathrm{Var}(x) = \frac{1}{4}n$ and the standard deviation $\sigma_x = \frac{1}{2}\sqrt{n}$.

13. Generalize Problem 12 to show that for the general binomial distribution (7.3), $\mathrm{Var}(x) = npq$, and $\sigma = \sqrt{npq}$.

8. THE NORMAL OR GAUSSIAN DISTRIBUTION

The graph of the *normal* or *Gaussian distribution* is the bell-shaped curve you may know as the normal error curve (Figure 8.1). The normal distribution is used a great deal because, as we shall see, it is not only of interest in itself (see Problems 2 and 3), but also other distributions become almost normal when n (the number of trials or measurements) becomes large (see Figures 8.2 and 8.3).

The probability density function $f(x)$ and the cumulative distribution function $F(x)$ for the normal or Gaussian distribution are given by

(8.1)
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)},$$
$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(t-\mu)^2/(2\sigma^2)}\, dt.$$
Normal distribution

It is straightforward to show (Problem 1) that if x is a random variable with probability density $f(x)$ in (8.1), then the mean of x is μ and the standard deviation is σ. Also we can show that the integral of $f(x)$ from $-\infty$ to ∞ is equal to 1 as it must be for a probability function. Then the probability that a normally distributed random variable x lies between x_1 and x_2 is the area under the $f(x)$ curve between x_1 and x_2 which is

(8.2)
$$F(x_2) - F(x_1) = \text{probability that } x_1 \le x \le x_2.$$

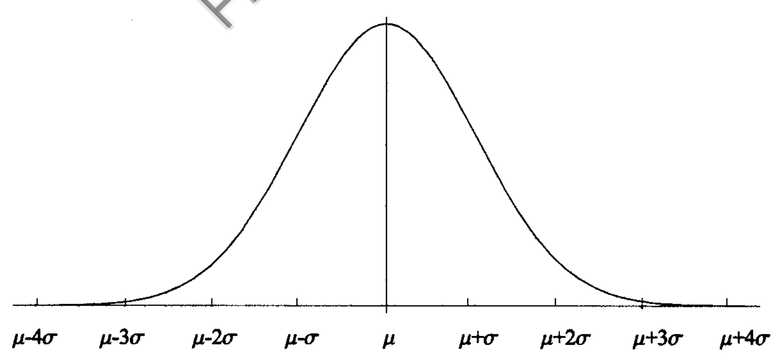


Figure 8.1

A normal density function graph (Figure 8.1) has its peak at $x = \mu$ and is symmetric with respect to the line $x = \mu$. Since the area from $-\infty$ to ∞ is 1, the area from $-\infty$ to μ is $\frac{1}{2}$ (that is, $F(\mu) = \frac{1}{2}$), and similarly the area from μ to ∞ is $\frac{1}{2}$. A change in μ merely translates the graph with no change in shape. An increase in σ widens and flattens the graph so that the area remains 1, and similarly a decrease in σ makes the graph taller and narrower. (Problems 4 to 6). The area from $\mu - \sigma$ to $\mu + \sigma$ is 0.6827, that is, the probability that x differs from its mean value by 1 standard deviation or less, is just over 68%. The probability that $|x - \mu| \le 2\sigma$

is over 95% and the probability that $|x-\mu| \le 3\sigma$ is over 99.7%. Note that these probabilities are independent of the values of μ and σ (Problem 7).

Normal Approximation to the Binomial Distribution As an example of approximating another distribution by a normal distribution, let's consider the binomial distribution (7.3). For large n and large np, we can use Stirling's formula (Chapter 11, Section 11) to approximate the factorials in $C(n,x)$ in (7.3) and make other approximations to find

$$f(x) = C(n,x)p^x q^{n-x} \sim \frac{1}{\sqrt{2\pi npq}} e^{-(x-np)^2/(2npq)}. \tag{8.3}$$

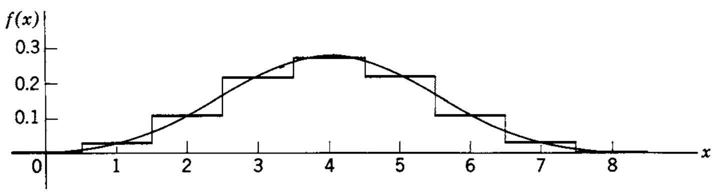


Figure 8.2 **Binomial distribution for $n = 8$, $p = \frac{1}{2}$, and the normal approximation.**

The sign $\sim$ means (as in Chapter 11, Section 11) that the ratio of the exact binomial distribution (7.3) and the right-hand side of (8.3) tends to 1 as $n \to \infty$. An outline of a derivation of (8.3) is given in Problem 8, but you may be more impressed by doing some computer plotting of graphs like Figures 8.2 and 8.3 (Problems 9 and 10). Although we have said that equation (8.3) gives an approximation valid for large n, the agreement is quite good even for fairly small values of n. Figure 8.2 shows this for the case $n = 8$. The binomial distribution $f(x)$ is defined only for integral x; you should compare the values of $f(x)$ with the values of the approximating normal curve at integral values of x. When n is very large (Figure 8.3), a graph of the exact binomial distribution is very close to the normal approximation (Problem 9).

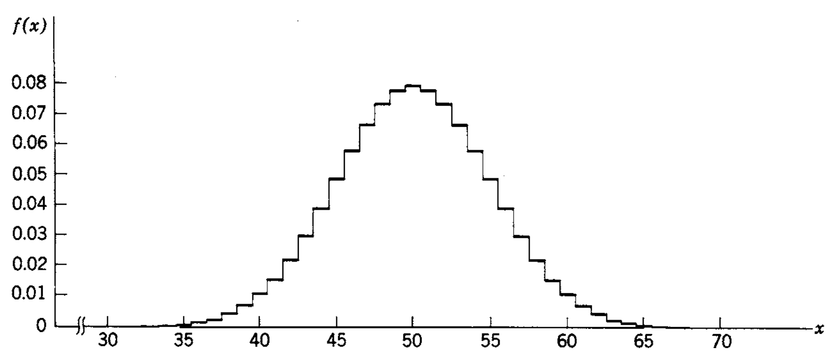


Figure 8.3 **Binomial distribution for $n = 100$, $p = \frac{1}{2}$.**

In (8.3), the left-hand side is the exact binomial distribution and the right-hand side is a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$ as we see by comparing (8.3) and (8.1). Recall from Problems 7.9 and 7.13 that the mean value

μ and standard deviation σ for a random variable whose probability function is the binomial distribution (7.3) are also $\mu = np$ and $\sigma = \sqrt{npq}$.

(8.4) For the binomial distribution and its normal approximation,
$$\mu = np, \qquad \sigma = \sqrt{npq}\,.$$

We can expect this in general; whatever the μ and σ are for a given distribution, the normal approximation will have the same μ and σ.

Example 1. Find the probability of exactly 52 heads in 100 tosses of a coin using the binomial distribution and using the normal approximation.

See Figure (8.3) which is a plot of the binomial probability density function with $n = 100, p = \frac{1}{2}$. We find by computer for $x = 52$, binomial $f(52) = 0.07353$, which you could also read approximately from Figure (8.3).

For the normal approximation, we find from (8.4), $\mu = np = 100 \cdot \frac{1}{2} = 50$, $\sigma = \sqrt{npq} = \sqrt{100 \cdot \frac{1}{2} \cdot \frac{1}{2}} = 5$. Then for the normal approximation with $\mu = 50$, $\sigma = 5$, we find by computer for $x = 52$, normal $f(52) = 0.07365$.

Example 2. Find the probability $P(45, 55)$ of between 45 and 55 heads in 100 tosses of a coin, that is $45 \le x \le 55$.

As in Example 1, for the binomial distribution we have $n = 100, p = \frac{1}{2}$. The cumulative binomial distribution function $F(x)$ in (7.4) gives $P(45, 55)$ as a sum of terms; we want the sum of the 11 terms with $x = 45, 46, \cdots 55$. By computer, we can find $F(55)$, the binomial cumulative distribution function with $x = 55$, which is the probability of 55 heads or less, and then find and subtract $F(44)$, the probability of 44 heads or less. Thus we find $P(45, 55) =$ binomial $F(55) -$ binomial $F(44) = 0.72875$.

For the normal approximation, we find by computer from (8.2), $P(45, 55) =$ normal $F(55) -$ normal $F(45) = 0.68269$. We can get a better approximation by integrating from 44.5 to 55.5; this corresponds more closely to the appropriate area under the exact binomial graph in Figure 8.3 by including the whole steps at $x = 45$ and $x = 55$. This gives $P(44.5, 55.5) =$ normal $F(55.5) -$ normal $F(44.5) = 0.72867$.

Standard Normal Distribution This is just the normal distribution in (8.1) for the special case $\mu = 0$ and $\sigma = 1$. The density function is often denoted by $\phi(z)$, and the corresponding cumulative distribution function by $\Phi(z)$:

(8.5)
$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$
Standard normal distribution

The cumulative distribution function $\Phi(z)$ is related to the error function (see Chapter 11, Section 9).

It is sometimes convenient to write the functions in (8.1) in terms of $\phi(z)$ and $\Phi(z)$. We can do this by making the change of variables $z = (x - \mu)/\sigma$. The result is (Problem 21)

$$\text{(8.6)} \qquad \begin{aligned} f(x) &= \frac{1}{\sigma}\phi(z), \\ F(x) &= \Phi(z), \end{aligned} \qquad \text{where } z = \frac{(x-\mu)}{\sigma}.$$

The functions $\phi(z)$ and $\Phi(z)$ [or sometimes $\Phi(z) - \frac{1}{2}$] are tabulated so you can use either tables or computer to do problems.

▶ **Example 3.** Find the number r such that the area under the normal distribution curve $y = f(x)$ from $\mu - r$ to $\mu + r$ is equal to $1/2$.

Look at Figure 8.1 and recall that the area from $-\infty$ to ∞ is 1 and that the graph is symmetric about $x = \mu$. Then the integral from $-\infty$ to $\mu - r$ and the integral from $\mu + r$ to ∞ are equal to each other and so each is equal to $1/4$. Thus the integral from $-\infty$ to $\mu + r$ must be $3/4$, that is $F(\mu + r) = 3/4$. By (8.6) this is $\Phi(z) = 3/4$ where $z = (\mu + r - \mu)/\sigma = r/\sigma$. By computer or tables we find that if $\Phi(z) = 3/4$, then $z = 0.6745$. Thus $r = 0.6745\sigma$.

▶ **Example 4.** You have taken a test (academic like the SAT, or medical like a bone density test) and a report gives your z-score as 1.14. What percent of your peers scored higher than you?

If we call the actual test scores x, and their average is μ and standard deviation σ, then the term z-score means the value of $z = (x-\mu)/\sigma$ as in (8.6). (In words, the z-score is the difference between x and its average, measured in units of the standard deviation.) Now we want the area $1 - F(x) = 1 - \Phi(z)$ by (8.6). By computer (or tables) we find $\Phi(1.14) = 0.87$; then $1 - 0.87 = 0.13$, so 13% of your peers scored higher than you. If your z-score is negative, then you are below average—bad if it's a physics test, good if it's your cholesterol! For example, if $z = -0.25$, then $\Phi(z) = 0.40$, so 60% of your peers scored higher than you.

▶ **Example 5.** Suppose that boxes of a certain kind of cereal have an average weight of 16 ounces and it is known that 70% of the boxes weigh within 1 ounce of the average. What is the probability that the box you buy weighs less than 14 ounces?

If x represents the weight of a box, then we are given that the probability of $15 < x < 17$ is 0.7. Assuming a normal distribution, the area under the $f(x)$ curve up to $x = \mu = 16$ is $\frac{1}{2}$ and the area from $x = 16$ to $x = 17$ is half of 0.7 (by symmetry; see Figure 8.1). Thus $F(17) = 0.5 + 0.35 = 0.85$. We want to find the probability that $x < 14$; this is $F(14)$. Using (8.6), $x = 17$ gives $z = (17 - 16)/\sigma = 1/\sigma$, and similarly $x = 14$ gives $z = -2/\sigma$. So we are given $\Phi(1/\sigma) = 0.85$, and we want to find $\Phi(-2/\sigma)$. By computer (or tables) we find that if $\Phi(1/\sigma) = 0.85$, then $1/\sigma = 1.0364$, so $2/\sigma = 2.0728$, and $\Phi(-2/\sigma) = 0.019$. So there is almost a 2% chance that we would get a box weighing less than 14 ounces.

Note that in Examples 4 and 5 we assumed a normal distribution with no obvious justification. It is a very interesting and useful fact that such an assumption is reasonable if the number of measurements is very large. We will discuss this further at the end of Section 10.

PROBLEMS, SECTION 8

1. Verify that for a random variable x with normal density function $f(x)$ as in (8.1), the mean value of x is μ , the standard deviation is σ, and the integral of $f(x)$ from $-\infty$ to ∞ is 1 as it must be for a probability function. *Hint:* Write and evaluate the integrals $\int_{-\infty}^{\infty} f(x)\,dx$, $\int_{-\infty}^{\infty} xf(x)\,dx$, $\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$. See equations (6.2), (6.3), and (6.4).

2. Do Problem 6.4 by comparing e^{-ax^2} with $f(x)$ in (8.1).

3. The probability density function for the x component of the velocity of a molecule of an ideal gas is proportional to $e^{-mv^2/(2kT)}$ where v is the x component of the velocity, m is the mass of the molecule, T is the temperature of the gas and k is the Boltzmann constant. By comparing this with (8.1), find the mean and standard deviation of v, and write the probability density function $f(v)$.

4. Computer plot on the same axes the normal probability density functions with $\mu = 0$, $\sigma = 1$, and with $\mu = 3$, $\sigma = 1$ to note that they are identical except for a translation.

5. Computer plot on the same axes the normal density functions with $\mu = 0$ and $\sigma = 1$, 2, and 5. Label each curve with its σ.

6. Do Problem 5 for $\sigma = \frac{1}{6}$, $\frac{1}{3}$, 1.

7. By computer find the value of the normal cumulative distribution function at $\mu+\sigma$, $\mu+2\sigma$, $\mu+3\sigma$, and satisfy yourself that these are independent of your choices for μ and σ. Find the probabilities that x is within 1, 2, or 3 standard deviations of its mean value μ to verify the results stated in the paragraph following (8.2). *Hint:* See Figure (8.1). The probability that x is within 1 standard deviation of its mean value is the area from $\mu-\sigma$ to $\mu+\sigma$; this is twice the area from μ to $\mu+\sigma$. Subtract $\frac{1}{2}$ (that is the area from $-\infty$ to μ) from your value of $F(\mu+\sigma)$ and then double the result.

8. Carry through the following details of a derivation of (8.3). Start with (7.3); we want an approximation to (7.3) for large n. First approximate the factorials in $C(n,x)$ by Stirling's formula (Chapter 11, Section 11) and simplify to get

$$f(x) \sim \left(\frac{np}{x}\right)^x \left(\frac{nq}{n-x}\right)^{n-x} \sqrt{\frac{n}{2\pi x(n-x)}}.$$

Show that if $\delta = x - np$, then $x = np+\delta$ and $n-x = nq-\delta$. Make these substitutions for x and $n-x$ in the approximate $f(x)$. To evaluate the first two factors in $f(x)$ (ignore the square root for now): Take the logarithm of the first two factors; show that

$$\ln\frac{np}{x} = -\ln\left(1+\frac{\delta}{np}\right)$$

and a similar formula for $\ln[nq/(n-x)]$; expand the logarithms in a series of powers of $\delta/(np)$, collect terms and simplify to get

$$\ln\left(\frac{np}{x}\right)^x \left(\frac{nq}{n-x}\right)^{n-x} \sim -\frac{\delta^2}{2npq}\left(1+\text{powers of }\frac{\delta}{n}\right).$$

Hence

$$\left(\frac{np}{x}\right)^x \left(\frac{nq}{n-x}\right)^{n-x} \sim e^{-\delta^2/(2npq)}$$

for large n. [We really want δ/n small, that is, x near enough to its average value np so that $\delta/n = (x - np)/n$ is small. This means that our approximation is valid for the central part of the graph (see Figures 7.1 to 7.3) around $x = np$ where $f(x)$ is large. Since $f(x)$ is negligibly small anyway for x far from np, we ignore the fact that our approximation may not be good there. For more detail on this point, see Feller, p. 192]. Returning to the square root factor in $f(x)$, approximate x by np and $n - x$ by nq (assuming $\delta \ll np$ or nq) and obtain (8.3).

9. Computer plot a graph like Figure 8.3 of the binomial distribution with $n = 1000$, $p = \frac{1}{2}$, and observe that you have practically the corresponding normal approximation.

10. Computer plot graphs like Figure 8.2 but with $p \neq \frac{1}{2}$ to see that as n increases, the normal approximation becomes good (at least in the region around $x = \mu$ where the probabilities are large) even though the binomial graph is not symmetric (see Figure 7.3).

As in Examples 1 and 2, use (a) the binomial distribution; (b) the corresponding normal approximation, to find the probabilities of each of the following:

11. Exactly 50 heads in 100 tosses of a coin.

12. Exactly 120 aces in 720 tosses of a die.

13. Between 100 and 140 aces in 720 tosses of a die.

14. Between 499,000 and 501,000 heads in 10^6 tosses of a coin.

15. Exactly 195 tails in 400 tosses of a coin.

16. Between 195 and 205 tails in 400 tosses of a coin.

17. Exactly 31 4's in 180 tosses of a die.

18. Between 29 and 33 4's in 180 tosses of a die.

19. Exactly 21 successes in 100 Bernoulli trials with probability $\frac{1}{5}$ of success.

20. Between 17 and 21 successes in 100 Bernoulli trials with probability $\frac{1}{5}$ of success.

21. Verify equations (8.6). *Hints:* In $F(x)$, let $u = (t - \mu)/\sigma$; note that $dt = \sigma du$. What is u when $t = -\infty$? When $t = x$? Remember that by definition $z = (x - \mu)/\sigma$.

22. Using (8.6), do Problem 7.

23. Using (8.6), find h such that 90% of the area under a normal $f(x)$ lies between $\mu - h$ and $\mu + h$. Repeat for 95%. *Hint:* See Example 3.

24. Write out a proof of Chebyshev's inequality (see end of Section 7) for the case of a continuous probability function $f(x)$.

25. An instructor who grades "on the curve" computes the mean and standard deviation of the grades, and then, assuming a normal distribution with this μ and σ, sets the border lines between the grades at: C from $\mu - \frac{1}{2}\sigma$ to $\mu + \frac{1}{2}\sigma$, B from $\mu + \frac{1}{2}\sigma$ to $\mu + \frac{3}{2}\sigma$, A from $\mu + \frac{3}{2}\sigma$ up, etc. Find the percentages of the students receiving the various grades. Where should the border lines be set to give the percentages: A and F, 10%; B and D, 20%; C, 40%?

9. THE POISSON DISTRIBUTION

The Poisson distribution is useful in a variety of problems in which the probability of some occurrence is small and constant. (See Example 1 and Problems 3 to 9.) It is also a good approximation to the binomial distribution when p is so small that np is small even though n is large (see Example 2).

Let's derive the Poisson distribution by considering the following experiment. Suppose we observe and count the number of particles emitted per unit time by a radioactive substance. We assume that our period of observation is much less than the half-life of the substance, so that the average counting rate does not decrease during the experiment. Then the probability that one particle is emitted during a small time interval Δt is $\mu\Delta t$, $\mu =$const., if Δt is short enough so that the probability of two particles during Δt is negligible. We want to find the probability $P_n(t)$ of observing exactly n counts during a time interval t. The probability $P_n(t+\Delta t)$ is the probability of observing n counts in the time interval $t+\Delta t$. For $n>0$, this is the sum of the probabilities of the two mutually exclusive events, "n particles in t, none in Δt" and "$(n-1)$ particles in t, one in Δt"; in symbols,

$$P_n(t+\Delta t) = P_n(t)P_0(\Delta t) + P_{n-1}(t)P_1(\Delta t). \tag{9.1}$$

Now $P_1(\Delta t)$ is the probability of one particle in Δt; this, by assumption, is $\mu\Delta t$. Then the probability of no particles in Δt is $1-P_1(\Delta t) = 1-\mu\Delta t$. Substituting these values into (9.1), we get

$$P_n(t+\Delta t) = P_n(t)(1-\mu\Delta t) + P_{n-1}(t)\mu\Delta t, \tag{9.2}$$

or,

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = \mu P_{n-1}(t) - \mu P_n(t). \tag{9.3}$$

Letting $\Delta t \to 0$, we have

$$\frac{dP_n(t)}{dt} = \mu P_{n-1}(t) - \mu P_n(t). \tag{9.4}$$

For $n=0$, (9.1) simplifies since the only possible event is "no particles in t, no particles in Δt," and (9.4) becomes, for $n=0$,

$$\frac{dP_0(t)}{dt} = -\mu P_0(t). \tag{9.5}$$

Then, since $P_0(0) =$ "probability that no particle is emitted during a zero time interval" $=1$, integration of (9.5) gives

$$P_0 = e^{-\mu t}. \tag{9.6}$$

Substituting (9.6) into (9.4) with $n=1$ gives a differential equation for $P_1(t)$; its solution (Problem 1) is $P_1(t) = \mu t e^{-\mu t}$. Solving (9.4) successively (Problem 1) for $P_2, P_3, \cdots, P_n$, we obtain

$$P_n(t) = \frac{(\mu t)^n}{n!}e^{-\mu t}. \tag{9.7}$$

Putting $t = 1$, we get for the probability of exactly n counts per unit time

(9.8) $$P_n = \frac{\mu^n}{n!}\,e^{-\mu}.$$ Poisson distribution

The probability density function (9.8) is called the *Poisson distribution* or the *Poisson probability density function*. You can show (Problem 2) that for the random variable n, the mean (that is the average number of counts per unit time) is μ, and the variance is also μ so the standard deviation is $\sqrt{\mu}$.

▶ **Example 1.** The number of particles emitted each minute by a radioactive source is recorded for a period of 10 hours; a total of 1800 counts are registered. During how many 1-minute intervals should we expect to observe no particles; exactly one; etc.?

The average number of counts per minute is $1800/(10\cdot 60) = 3$ counts per minute; this is the value of μ. Then by (9.8), the probability of n counts per minute is

$$P_n = \frac{3^n}{n!}e^{-3}.$$

A graph of this probability function is shown in Figure 9.1. For $n = 0$, we find $P_0 = e^{-3} = 0.05$; then we should expect to observe no particles in about 5% of the 600 1-minute intervals, that is, during 30 1-minute intervals. Similarly we could compute the expected number of 1-minute intervals during which 1, 2, $\cdots$, particles would be observed.

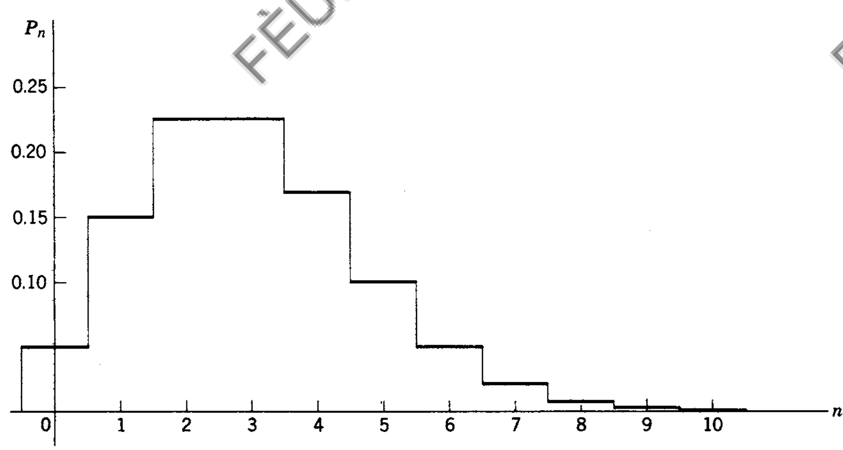


Figure 9.1 Poisson distribution $\mu = 3$.

Poisson Approximation of the Binomial Distribution In Section 8, we discussed the fact that the binomial distribution can be approximated by the normal distribution for large n and large np. If p is very small so that np is very much less than n (say, for example, $p = 10^{-3}, n = 2000, np = 2$), the normal approximation is not good. In this case you can show (Problem 10) that the Poisson distribution gives a good approximation to the binomial distribution (7.3), that is, that

$$(9.9)\qquad C(n,x)p^x q^{n-x} \sim \frac{(np)^x e^{-np}}{x!}, \qquad \text{Large } n, \text{ small } p.$$

[The exact meaning of (9.9) is that, for any fixed x, the ratio of the two sides approaches 1 as $n \to \infty$ and $p \to 0$ with np remaining constant.]

Example 2. If 1500 people each select a number at random between 1 and 500, what is the probability that 2 people selected the number 29?

The answer is given by the binomial distribution (7.3) with $n = 1500$, $p = 1/500$, $x = 2$. This is

$$C(n,x)p^x q^{n-x} = \frac{1500!}{2!1498!}\left(\frac{1}{500}\right)^2\left(\frac{499}{500}\right)^{998} = 0.2241.$$

(Or from your computer: the binomial probability density function with $n = 1500$, $p = 1/500$, $x = 2$, is 0.2241 to four decimal places.). A simpler formula from (9.9) is the Poisson approximation with $\mu = np = 3$, $x = 2$, namely $\mu^x e^{-x}/x! = 3^2 e^{-2}/2! = 0.2240$. (Or from your computer, the Poisson probability density function with $\mu = 3$, $x = 2$, is 0.2240 to four decimal places.) It is interesting to computer plot on the same axes the binomial distribution with $n = 1500$, $p = 1/500$, and the Poisson distribution with $\mu = 3$ as in Figure 9.1 to discover that they are almost identical (Problem 12).

Approximations by the Normal Distribution We have commented that many distributions can be approximated by the normal distribution when n and $\mu = np$ are both large, and have shown this for the binomial distribution in (8.1). The Poisson distribution when μ is large is also fairly well approximated by the normal distribution as in (9.10).

$$(9.10)\qquad \frac{\mu^x e^{-\mu}}{x!} \cong \frac{1}{\sqrt{2\pi\mu}} e^{-(x-\mu)^2/(2\mu)}, \qquad \mu \text{ large}.$$

Note that the normal distribution in (9.10) has the same mean and variance as the Poisson distribution it is approximating (see Problem 2 for the Poisson mean and variance). It is useful to computer plot on the same axes graphs of the Poisson distribution and their normal approximations (Problem 13).

PROBLEMS, SECTION 9

1. Solve the sequence of differential equations (9.4) for successive n values [as started in (9.5) and (9.6)] to obtain (9.7).

2. Show that the average value of a random variable n whose probability function is the Poisson distribution (9.8) is the number μ in (9.8). Also show that the standard deviation of the random variable is $\sqrt{\mu}$. *Hint:* Write the infinite series for e^x, differentiate it and multiply by x to get $xe^x = \sum(nx^n/n!)$; put $x = \mu$. To find σ^2 differentiate the xe^x series again, etc.

3. In an alpha-particle counting experiment the number of alpha particles is recorded each minute for 50 hours. The total number of particles is 6000. In how many 1-minute intervals would you expect no particles? Exactly n particles, for $n = 1$, 2, 3, 4, 5? Plot the Poisson distribution.

4. Suppose you receive an average of 4 phone calls per day. What is the probability that on a given day you receive no phone calls? Just one call? Exactly 4 calls?

5. Suppose that you have 5 exams during the 5 days of exam week. Find the probability that on a given day you have no exams; just 1 exam; 2 exams; 3 exams.

6. If you receive, on the average, 5 email messages per day, in how many days out of a 365-day year would you expect to receive exactly 5 messages? Fewer than 5? Exactly 10? More than 10? Just 1? None at all?

7. In a club with 500 members, what is the probability that exactly two people have birthdays on July 4?

8. If there are 100 misprints in a magazine of 40 pages, on how many pages would you expect to find no misprints? Two misprints? Five misprints?

9. If there are, on the average, 7 defects in a new car, what is the probability that your new car has only 2 defects? That it has 6 or 7? That it has more than 10?

10. Derive equation (9.9) as follows: In $C(n, x)$, show that $n!/(n-x)! \sim n^x$ for fixed x and large n [write $n!/(n-x)!$ as a product of x factors, divide by n^x, and show that the limit is 1 as $n \to \infty$]. Then write $q^{n-x} = (1-p)^{n-x}$ as $(1-p)^n(1-p)^{-x} = (1-np/n)^n(1-p)^{-x}$; evaluate the limit of the first factor as $n \to \infty$, np fixed; the limit of the second factor as $p \to 0$ is 1. Collect your results to obtain equation (9.9).

11. Suppose 520 people each have a shuffled deck of cards and draw one card from the deck. What is the probability that exactly 13 of the 520 cards will be aces of spades? Write the binomial formula and approximate it. Which is best, the normal or the Poisson approximation? Although you only need values at one x to answer the question, you might like to computer plot on the same axes graphs of the three distributions for the given n and p.

12. Computer plot on the same axes graphs of the binomial distribution in Example 2 and the Poisson and normal approximations.

13. Computer plot on the same axes a graph of the Poisson distribution and the corresponding normal approximation for the cases $\mu = 1$, 5, 10, 20, 30.

▶ 10. STATISTICS AND EXPERIMENTAL MEASUREMENTS

Statistics uses probability theory to consider sets of data and draw reasonable conclusions from them. So far in this chapter, we have been discussing problems for which we could write down a density function formula (normal, Poisson, etc.). Suppose that, instead, we have only a table of data, say a set of laboratory measurements of some physical quantity. Presumably, if we spent more time, we could enlarge this table of data as much as we liked. We can then imagine an infinite set of measurements of which we have only a sample. The infinite set is called the *parent population* or *universe*. What we would really like to know is the probability function for the parent population, or at least the average value μ (often thought of as the "true" value of the quantity being measured) and the standard deviation σ of the parent population. We must content ourselves with the best estimates we can make of these quantities using our available sample, that is, the set of measurements which we have made.

Estimate of Population Average As a quick estimate of μ we might take the median of our measurements x_i (a value such that there are equal numbers of larger and smaller measurements), or the mode (the measurement we obtained the most times, that is the most probable measurement). The most frequently used estimate of μ is, however, the arithmetic mean (or average) of the measurements, that is the sample mean $\overline{x} = (1/n)\sum_{i=1}^{n} x_i$. Thus we have

$$\text{(10.1)} \qquad \text{Estimate of population mean is } \mu \simeq \overline{x} = (1/n)\sum_{i=1}^{n} x_i.$$

For a large set of measurements we can justify this choice as follows (also see Problem 1). Assuming that the parent population for our measurements has probability density function $f(x)$ with expected value μ and standard deviation σ, it is easy to show (Problem 2) that the expected value of $\overline{x}$ is μ and the standard deviation of $\overline{x}$ is $\sigma/\sqrt{n}$. Now Chebyshev's inequality (end of Section 7) says that a random variable is unlikely to differ from its expected value by more than a few standard deviations. For our problem this says that $\overline{x}$ is unlikely to differ from μ by more than a few multiples of $\sigma/\sqrt{n}$, which becomes small as n increases. Thus $\overline{x}$ becomes an increasingly good estimate of μ as we increase the number n of measurements. Note that this just says mathematically what you would assume from experience, that the average of a large number of measurements is more likely to be accurate than the average of a small number. For example, two measurements might both be too large, but it's unlikely that 20 would all be too large.

Estimate of Population Variance Our first guess for an estimate of σ^2 might be $s^2 = (1/n)\sum_{i=1}^{n}(x_i - \overline{x})^2$, but we would be wrong. To see what is reasonable, we find the expected value of s^2 assuming that our measurements are from a population with mean μ and variance σ^2. The result is (Problem 3), $E(s^2) = [(n-1)/n]\sigma^2$. We conclude that a reasonable estimate of σ^2 is $\frac{n}{n-1}s^2$.

$$\text{(10.2)} \qquad \text{Estimate of population variance is } \sigma^2 \simeq \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2.$$

(*Caution:* The term "sample variance" is used in various references—texts, reference books, computer programs—to mean either our s^2 or our estimate of σ^2, so check the definition carefully in any reference you use. We shall avoid using the term.)

The quantity σ which we have just estimated is the standard deviation for the parent population whose probability function we call $f(x)$. Consider just a single measurement x. The function $f(x)$ (if we knew it) would give us the probabilities of the different possible values of x, the population mean μ would tell us approximately the value we are apt to find for x, and the standard deviation σ would tell us roughly the spread of x values about μ. Since σ tells us something about a single measurement, it is often called the *standard deviation of a single measurement.*

Standard Deviation of the Mean; Standard Error Instead of a single measurement, let us consider $\overline{x}$, the average (mean) of a set of n measurements. (The mean, $\overline{x}$, will be what we will use or report as the result of an experiment.) Just as we originally imagined obtaining the probability function $f(x)$ by making a large number of single measurements, so we can imagine obtaining a probability function $g(\overline{x})$ by making a large number of *sets* of n measurements with each set giving us a value of $\overline{x}$. The function $g(\overline{x})$ (if we knew it) would give us the probability of different values of $\overline{x}$. We have seen (Problem 2) that $\mathrm{Var}(\overline{x}) = \sigma^2/n$, so the *standard deviation of the mean* (that is, of $\overline{x}$) is

$$\sigma_m = \sqrt{\mathrm{Var}(\overline{x})} = \frac{\sigma}{\sqrt{n}}. \tag{10.3}$$

The quantity σ_m is also called the *standard error*; it gives us an estimate of the spread of values of $\overline{x}$ about μ. We see that the new probability function $g(\overline{x})$ must be much more peaked than $f(x)$ about the value μ because the standard deviation $\sigma/\sqrt{n}$ is much smaller than σ. Collecting formulas (10.2) and (10.3), we have

$$\sigma_m \cong \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n(n-1)}} \tag{10.4}$$

▶ **Example 1.** To illustrate our discussion, let's consider the following set of measurements: $\{7.2, 7.1, 6.7, 7.0, 6.8, 7.0, 6.9, 7.4, 7.0, 6.9\}$. [Note that, to show methods but minimize computation, we consider unrealistically small sets of measurements.]

From (10.1) we find $\quad \mu \simeq \overline{x} = \dfrac{1}{10}\displaystyle\sum_{i=1}^{10} x_i = \dfrac{70}{10} = 7.0.$

From (10.2) we find $\quad \sigma^2 \simeq \dfrac{1}{9}\displaystyle\sum_{i=1}^{10}(x_i - 7)^2 = \dfrac{0.36}{9} = 0.04, \sigma \simeq 0.2.$

From (10.4), the standard error is $\quad \sigma_m \simeq \sqrt{\dfrac{0.36}{10 \cdot 9}} = 0.0632.$

Combination of Measurements We have discussed how we can use a set of measurements x_i to estimate μ (the population average) by $\overline{x}$ (the sample average) and to estimate the standard error $\sigma_{mx} = \sqrt{\mathrm{Var}(\overline{x})}$ [equation (10.4)]. Now suppose we have done this for two quantities, x and y, and we want to use a known formula $w = w(x, y)$ to estimate a value for w and the standard error in w. First we consider the simple example $w = x + y$. Then, by Problem 6.13,

$$E(w) = E(x) + E(y) = \mu_x + \mu_y \tag{10.5}$$

where μ_x and μ_y are population averages. As discussed above, we estimate μ_x and μ_y by $\overline{x}$ and $\overline{y}$ and conclude that a reasonable estimate of w is

$$\overline{w} = \overline{x} + \overline{y}. \tag{10.6}$$

Now let us assume that x and y are independently measured quantities. Then by Problem 6.15,

$$\text{(10.7)} \qquad \begin{aligned} \operatorname{Var}(\overline{w}) &= \operatorname{Var}(\overline{x}) + \operatorname{Var}(\overline{y}) = \sigma_{mx}^2 + \sigma_{my}^2, \\ \sigma_{mw} &= \sqrt{\sigma_{mx}^2 + \sigma_{my}^2}. \end{aligned}$$

Next consider the case $w = 4 - 2x + 3y$. As in equations (10.5) and (10.6), we find $\overline{w} = 4 - 2\overline{x} + 3\overline{y}$. Now by Problem 5.13, we have $\operatorname{Var}(x + K) = \operatorname{Var}(x)$, and $\operatorname{Var}(Kx) = K^2 \operatorname{Var}(x)$, where K is a constant. Thus,

$$\text{(10.8)} \qquad \begin{aligned} \operatorname{Var}(\overline{w}) &= \operatorname{Var}(4 - 2\overline{x} + 3\overline{y}) = \operatorname{Var}(-2\overline{x} + 3\overline{y}) \\ &= (-2)^2 \operatorname{Var}(\overline{x}) + (3)^2 \operatorname{Var}(\overline{y}) = 4\sigma_{mx}^2 + 9\sigma_{my}^2, \end{aligned}$$

$$\text{(10.9)} \qquad \sigma_{mw} = \sqrt{4\sigma_{mx}^2 + 9\sigma_{my}^2}.$$

We can now see how to find $\overline{w}$ and σ_{mw} for any function $w(x, y)$ which can be approximated by the linear terms of its Taylor series about the point (μ_x, μ_y), namely (see Chapter 4, Section 2)

$$\text{(10.10)} \qquad w(x, y) \cong w(\mu_x, \mu_y) + \left(\frac{\partial w}{\partial x}\right)(x - \mu_x) + \left(\frac{\partial w}{\partial y}\right)(y - \mu_y)$$

where the partial derivatives are evaluated at $x = \mu_x$, $y = \mu_y$, and so are constants. [Practically speaking, this means that the first partial derivatives should not be near zero—we can't expect good results near a maximum or minimum of w—and the higher derivatives should not be large, that is, w should be "smooth" near the point (μ_x, μ_y).] Assuming (10.10), and remembering that $w(\mu_x, \mu_y)$ and the partial derivatives are constants, we find

$$\text{(10.11)} \qquad \begin{aligned} E[w(x, y)] &\cong w(\mu_x, \mu_y) + \left(\frac{\partial w}{\partial x}\right)[E(x) - \mu_x] + \left(\frac{\partial w}{\partial y}\right)[E(y) - \mu_y] \\ &= w(\mu_x, \mu_y). \end{aligned}$$

Since we have agreed to estimate μ_x and μ_y by $\overline{x}$ and $\overline{y}$, we conclude that a reasonable estimate of w is

$$\text{(10.12)} \qquad \overline{w} = w(\overline{x}, \overline{y}).$$

(This may look obvious, but see Problem 7.)

Then, putting $x = \overline{x}, y = \overline{y}$ in (10.10) and remembering the comment just before (10.11), we find as in (10.8)

$$\begin{aligned} \operatorname{Var}(\overline{w}) &= \operatorname{Var}[w(\overline{x}, \overline{y})] \\ &= \operatorname{Var}\left[w(\mu_x, \mu_y) + \left(\frac{\partial w}{\partial x}\right)(\overline{x} - \mu_x) + \left(\frac{\partial w}{\partial y}\right)(\overline{y} - \mu_y)\right] \\ &= \left(\frac{\partial w}{\partial x}\right)^2 \sigma_{mx}^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_{my}^2, \end{aligned}$$

$$\text{(10.13)} \qquad \sigma_{mw} = \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 \sigma_{mx}^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_{my}^2}.$$

We can use (10.12) and (10.13) to estimate the value of a given function w of two measured quantities x and y and to find the standard error in w.

▶ **Example 2.** From Example 1 we have $\overline{x} = 7$ and $\sigma_{mx} = 0.0632$. Suppose we have also found from measurements that $\overline{y} = 5$ and $\sigma_{my} = 0.0591$. If $w = x/y$, find $\overline{w}$ and σ_{mw}. From (10.12) we have $\overline{w} = \overline{x}/\overline{y} = 7/5 = 1.4$. From (10.13) we find

$$\sigma_{mw} = \sqrt{\left(\frac{1}{\overline{y}}\right)^2 \sigma_{mx}^2 + \left(\frac{-\overline{x}}{\overline{y}^2}\right)^2 \sigma_{my}^2} = \sqrt{\left(\frac{1}{5}\right)^2 (0.0632)^2 + \left(\frac{-7}{25}\right)^2 (0.0591)^2}$$
$$= 0.0208.$$

Central Limit Theorem So far we have not assumed any special form (such as normal, etc.) for the density function $f(x)$ of the parent population, so that our results for computation of approximate values of μ, σ, and σ_m from a set of measurements apply whether or not the parent distribution is normal. (And, in fact, it may not be; for example, Poisson distributions are quite common.) You will find, however, that most discussions of experimental errors are based on an assumed normal distribution. Let us discuss the justification for this. We have seen above that we can think of the sample average $\overline{x}$ as a random variable with average μ and standard deviation $\sigma/\sqrt{n}$. We have said that we might think of a density function $g(\overline{x})$ for $\overline{x}$ and that it would be more strongly peaked about μ than the density function $f(x)$ for a single measurement, but we have not said anything so far about the form of $g(\overline{x})$. There is a basic theorem in probability (which we shall quote without proof) which gives us some information about the probability function for $\overline{x}$. The *central limit theorem* says that no matter what the parent probability function $f(x)$ is (provided μ and σ exist), the probability function for $\overline{x}$ is approximately the normal distribution with standard deviation $\sigma/\sqrt{n}$ if n is large.

Confidence Intervals, Probable Error If we assume that the probability function for $\overline{x}$ is normal (a reasonable assumption if n is large), then we can give a more specific meaning to σ_m (standard deviation of the mean) than our vague statement that it gives us an estimate of the spread of $\overline{x}$ values about μ. Since the probability for a normally distributed random variable to have values between $\mu - \sigma$ and $\mu + \sigma$ is 0.6827 (see Section 8 and Problem 8.7), we can say that the probability is about 68% for a measurement of $\overline{x}$ to lie between $\mu - \sigma_m$ and $\mu + \sigma_m$. This interval is called the 68% *confidence interval*. Similarly we can find an interval $\mu \pm r$ such that the probability is $\frac{1}{2}$ that a new measurement would fall in this interval (and so also the probability is $\frac{1}{2}$ that it would fall outside!), that is, a 50% confidence interval. From Section 8, Example 3, this is $r = 0.6745\sigma_m$. The number r is called the *probable error*. When we have found σ_m as in Examples 1 and 2, we just have to multiply it by 0.6745 to find the corresponding probable error. Similarly we can find the error corresponding to other choices of confidence interval (see Problem 4).

▶ PROBLEMS, SECTION 10

1. Let $m_1, m_2, \cdots, m_n$ be a set of measurements, and define the values of x_i by $x_1 = m_1 - a, x_2 = m_2 - a, \cdots, x_n = m_n - a$, where a is some number (as yet unspecified, but the same for all x_i). Show that in order to minimize $\sum_{i=1}^n x_i^2$, we should choose $a = (1/n)\sum_{i=1}^n m_i$. *Hint:* Differentiate $\sum_{i=1}^n x_i^2$ with respect to a. You have shown that the arithmetic mean is the "best" average in the least squares sense, that is, that if the sum of the squares of the deviations of the measurements from their

"average" is a minimum, the "average" is the arithmetic mean (rather than, say, the median or mode).

2. Let $x_1, x_2, \cdots, x_n$ be independent random variables, each with density function $f(x)$, expected value μ, and variance σ^2. Define the sample mean by $\overline{x} = \sum_{i=1}^{n} x_i$. Show that $E(\overline{x}) = \mu$, and $\mathrm{Var}(\overline{x}) = \sigma^2/n$. (See Problems 5.9, 5.13, and 6.15.)

3. Define s by the equation $s^2 = (1/n)\sum_{i=1}^{n}(x_i - \overline{x})^2$. Show that the expected value of s^2 is $[(n-1)/n]\sigma^2$. *Hints:* Write

$$\begin{aligned}(x_i - \overline{x})^2 &= [(x_i - \mu) - (\overline{x} - \mu)]^2 \\ &= (x_i - \mu)^2 - 2(x_i - \mu)(\overline{x} - \mu) + (\overline{x} - \mu)^2.\end{aligned}$$

Find the average value of the first term from the definition of σ^2 and the average value of the third term from Problem 2. To find the average value of the middle term write

$$(\overline{x} - \mu) = \left(\frac{x_1 + x_2 + \cdots + x_n}{n} - \mu\right) = \frac{1}{n}[(x_1 - \mu) + (x_2 - \mu) + \cdots + (x_n - \mu)].$$

Show by Problem 6.14 that

$$E[(x_i - \mu)(x_j - \mu)] = E(x_i - \mu)E(x_j - \mu) = 0 \qquad \text{for } i \neq j,$$

and evaluate $E[(x_i - \mu)^2]$ (same as the first term). Collect terms to find

$$E(s^2) = \frac{n-1}{n}\sigma^2.$$

4. Assuming a normal distribution, find the limits $\mu \pm h$ for a 90% confidence interval; for a 95% confidence interval; for a 99% confidence interval. What percent confidence interval is $\mu \pm 1.3\sigma$? *Hints:* See Section 8, Example 3, and Problems 8.7, 8.22, and 8.23.

5. Show that if $w = xy$ or $w = x/y$, then (10.14) gives the convenient formula for relative error

$$\frac{r_w}{w} = \sqrt{\left(\frac{r_x}{x}\right)^2 + \left(\frac{r_y}{y}\right)^2}.$$

6. By expanding $w(x, y, z)$ in a three-variable power series similar to (10.10), show that

$$r_w = \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 r_x^2 + \left(\frac{\partial w}{\partial y}\right)^2 r_y^2 + \left(\frac{\partial w}{\partial z}\right)^2 r_z^2}.$$

7. Equation (10.12) is only an approximation (but usually satisfactory). Show, however, that if you keep the second order terms in (10.10), then

$$\overline{w} = w(\overline{x}, \overline{y}) + \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)\sigma_x^2 + \frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2}\right)\sigma_y^2.$$

8. The following measurements of x and y have been made.

$x: 5.1, 4.9, 5.0, 5.2, 4.9, 5.0, 4.8, 5.1$

$y: 1.03, 1.05, 0.96, 1.00, 1.02, 0.95, 0.99, 1.01, 1.00, 0.99$

Find the mean value and the probable error of x, y, $x + y$, xy, $x^3 \sin y$, and $\ln x$. *Hint*: See Examples 1 and 2 and the last paragraph of this section.

9. Given the measurements

$$x : 98, 101, 102, 100, 99$$
$$y : 21.2, 20.8, 18.1, 20.3, 19.6, 20.4, 19.5, 20.1$$

find the mean value and probable error of $x - y$, x/y, x^2y^3, and $y \ln x$.

10. Given the measurements

$$x : 5.8, 6.1, 6.4, 5.9, 5.7, 6.2, 5.9$$
$$y : 2.7, 3.0, 2.9, 3.3, 3.1$$

find the mean value and probable error of $2x - y$, $y^2 - x$, e^y, and x/y^2.

▶ 11. MISCELLANEOUS PROBLEMS

1. (a) Suppose you have two quarters and a dime in your left pocket and two dimes and three quarters in your right pocket. You select a pocket at random and from it a coin at random. What is the probability that it is a dime?

(b) Let x be the amount of money you select. Find $E(x)$.

(c) Suppose you selected a dime in (a). What is the probability that it came from your right pocket?

(d) Suppose you do not replace the dime, but select another coin which is also a dime. What is the probability that this second coin came from your right pocket?

2. (a) Suppose that Martian dice are regular tetrahedra with vertices labeled 1 to 4. Two such dice are tossed and the sum of the numbers showing is even. Let x be this sum. Set up the sample space for x and the associated probabilities.

(b) Find $E(x)$ and σ_x.

(c) Find the probability of exactly fifteen 2's in 48 tosses of a Martian die using the binomial distribution.

(d) Approximate (c) using the normal distribution.

(e) Approximate (c) using the Poisson distribution.

3. There are 3 red and 2 white balls in one box and 4 red and 5 white in the second box. You select a box at random and from it pick a ball at random. If the ball is red, what is the probability that it came from the second box?

4. If 4 letters are put at random into 4 envelopes, what is the probability that at least one letter gets into the correct envelope?

5. Two decks of cards are "matched," that is, the order of the cards in the decks is compared by turning the cards over one by one from the two decks simultaneously; a "match" means that the two cards are identical. Show that the probability of at least one match is nearly $1 - 1/e$.

6. Find the number of ways of putting 2 particles in 5 boxes according to the different kinds of statistics.

7. Suppose a coin is tossed three times. Let x be a random variable whose value is 1 if the number of heads is divisible by 3, and 0 otherwise. Set up the sample space for x and the associated probabilities. Find $\overline{x}$ and σ.

8. (a) A weighted coin has probability $\frac{2}{3}$ of coming up heads and probability $\frac{1}{3}$ of coming up tails. The coin is tossed twice. Let x = number of heads. Set up the sample space for x and the associated probabilities.

(b) Find $\overline{x}$ and σ.

(c) If in (a) you know that there was at least one tail, what is the probability that both were tails?

9. (a) One box contains one die and another box contains two dice. You select a box at random and take out and toss whatever is in it (that is, toss *both* dice if you have picked box 2). Let x = number of 3's showing. Set up the sample space and associated probabilities for x.

(b) What is the probability of at least one 3?

(c) If at least one 3 turns up, what is the probability that you picked the first box?

(d) Find $\overline{x}$ and σ.

Do Problems 10 to 12 using both the binomial distribution and the normal approximation.

10. A true coin is tossed 10^4 times.

(a) Find the probability of getting exactly 5000 heads.

(b) Find the probability of between 4900 and 5075 heads.

11. A die is thrown 720 times.

(a) Find the probability that 3 comes up exactly 125 times.

(b) Find the probability that 3 comes up between 115 and 130 times.

12. Consider a biased coin with probability 1/3 of heads and 2/3 of tails and suppose it is tossed 450 times.

(a) Find the probability of getting exactly 320 tails.

(b) Find the probability of getting between 300 and 320 tails.

13. A radioactive source emits 1800 α particles during an observation lasting 10 hours. In how many one minute intervals do you expect no α's? 5α's?

14. Suppose a 200-page book has, on the average, one misprint every 10 pages. On about how many pages would you expect to find 2 misprints?

In Problems 15 and 16, find the binomial probability for the given problem, and then compare the normal and the Poisson approximations.

15. Out of 1095 people, what is the probability that exactly 2 were born on Jan. 1? Assume 365 days in a year.

16. Find the probability of x successes in 100 Bernoulli trials with probability $p = 1/5$ of success (a) if $x = 25$; (b) if $x = 21$.

17. Given the measurements

$$x : 2.3, 2.1, 1.8, 1.7, 2.1$$
$$y : 1.0, 1.1, 0.9$$

find the mean value and the probable error for $x - y$, xy, and x/y^3.

18. Given the measurements

$$x : 5.7, 4.5, 4.8, 5.1, 4.9$$
$$y : 61.5, 60.1, 59.7, 60.3, 58.4$$

find the mean value and the probable error for $x + y$, y/x, and x^2.

References

This list includes the details of references cited in the text, plus a few other books you might find useful.

Abramowitz, Milton, and Irene A. Stegun, editors, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series, 55, U. S. Government Printing Office, Washington, D. C., 1964.

Arfken, George B., and Hans J. Weber, *Mathematical Methods for Physicists*, Academic Press, fifth edition, 2001.

Boyce, William E., and Richard C. DiPrima, *Introduction to Differential Equations*, Wiley, 1970.

Butkov, Eugene, *Mathematical Physics*, Addison-Wesley, 1968.

Callen, Herbert B., *Thermodynamics and an Introduction to Thermostatistics*, Wiley, second edition, 1985.

Cantrell, C. D., *Modern Mathematical Methods for Physicists and Engineers*, Cambridge University Press, 2000.

Chow, Tai L., *Mathematical Methods for Physicists: A Concise Introduction*, Cambridge University Press, 2000.

Courant, Richard, and Herbert Robbins, *What Is Mathematics?*, Oxford University Press, second edition revised by Ian Stewart, 1996.

CRC Standard Mathematical Tables, CRC Press, any recent edition.

Feller, William, *An Introduction to Probability Theory and Its Applications*, Wiley, second edition, 1966.

Folland, G. B., *Fourier analysis and its applications*, Brooks/Cole, 1992.

Goldstein, Herbert, Charles P. Poole, and John L. Safko, *Classical Mechanics*, Addison Wesley, third edition, 2002.

Griffiths, David J., *Introduction to Electrodynamics*, Prentice Hall, third edition, 1999.

Griffiths, David J., *Introduction to Quantum Mechanics*, Prentice Hall, second edition, 2004.

Hassani, Sadri, *Mathematical Methods: For Students of Physics and Related Fields*, Springer, 2000.

Jackson, John David, *Classical Electrodynamics*, Wiley, third edition, 1999.

Jahnke, E., and F. Emde, *Tables of Higher Functions*, McGraw-Hill, sixth edition revised by Friedrich Lösch, 1960.

Jeffreys, Harold, *Cartesian Tensors*, Cambridge University Press, 1965 reprint.

Jordan, D. W., and Peter Smith, *Mathematical Techniques: An Introduction for the Engineering, Physical, and Mathematical Sciences*, Oxford University Press, third edition, 2002.

Kittel, Charles, *Elementary Statistical Physics*, Dover edition, 2004.

Kreyszig, Erwin, *Advanced Engineering Mathematics*, Wiley, eighth edition, 1999.

Lighthill, M. J., *Introduction to Fourier Analysis and Generalised Functions*, Cambridge University Press, 1958.

Lyons, Louis, *All You Wanted To Know About Mathematics but Were Afraid To Ask: Mathematics for Science Students*, two volumes, Cambridge University Press, 1995–1998.

Mathews, Jon, and R. L. Walker, *Mathematical Methods of Physics*, Benjamin, second edition, 1970.

McQuarrie, Donald A., *Mathematical Methods for Scientists and Engineers*, University Science Books, 2003.

Morse, Philip M., and Herman Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, 1953.

NBS Tables. See Abramowitz and Stegun.

Parratt, Lyman G., *Probability and Experimental Errors in Science*, Dover edition, 1971.

Relton, F. E., *Applied Bessel Functions*, Dover edition, 1965.

Riley, K. F., M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering: A Comprehensive Guide*, Cambridge University Press, second edition, 2002.

Schey, H. M., *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus*, Norton, fourth edition, 2004.

Snieder, Roel, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, second edition, 2004.

Strang, Gilbert, *Linear Algebra and Its Applications*, Harcourt, Brace, Jovanovich, third edition, 1988.

Weinstock, Robert, *Calculus of Variations, with Applications to Physics and Engineering*, Dover edition, 1974.

Weisstein, Eric W., *CRC Concise Encyclopedia of Mathematics*, Chapman & Hall /CRC, second edition, 2003.

Woan, Graham, *Cambridge Handbook of Physics Formulas*, Cambridge University Press, reprinted 2003 with corrections.

Young, Hugh D., *Statistical Treatment of Experimental Data*, McGraw-Hill, 1962.

Answers to Selected Problems

Chapter 1

1.1	0.0173 yd; 0.104 yd (compared to a total of 5 yd)								
1.3	$\frac{5}{9}$	1.5	$\frac{7}{12}$	1.9	$\frac{6}{7}$	1.11	$\frac{19}{28}$	1.15	1
2.1	1	2.4	∞	2.7	e^2	2.9	1		

4.2 $a_n = \dfrac{1}{5^{n-1}} \to 0$; $S_n = \dfrac{5}{4}\left(1 - \dfrac{1}{5^n}\right) \to \dfrac{5}{4}$; $R_n = \dfrac{1}{4 \cdot 5^{n-1}} \to 0$

4.4 $a_n = \dfrac{1}{3^n} \to 0$; $S_n = \dfrac{1}{2}\left(1 - \dfrac{1}{3^n}\right) \to \dfrac{1}{2}$; $R_n = \dfrac{1}{2 \cdot 3^n} \to 0$

4.6 $a_n = \dfrac{1}{n(n+1)} \to 0$; $S_n = 1 - \dfrac{1}{n+1} \to 1$; $R_n = \dfrac{1}{n+1} \to 0$

5.2	Test further	5.4	D	5.5	D
5.6	Test further	5.8	Test further	5.9	D

6.5 b	D	6.7	D	6.9	C	6.10	C	6.14	D
6.18	D	6.20	C	6.22	C	6.23	D	6.24	D
6.26	C	6.29	D	6.31	D	6.32	D	6.35	C
6.36	D								
7.1	C	7.2	D	7.4	C	7.6	D	7.8	C
9.2	D	9.3	C	9.7	D	9.8	C	9.9	D
9.10	D	9.12	C	9.13	C	9.15	D	9.16	C
9.20	C	9.21	C	9.22	(b) D				

10.1	$\lvert x\rvert < 1$	10.3	$\lvert x\rvert \le 1$	10.4	$\lvert x\rvert \le \sqrt{2}$
10.5	All x	10.9	$\lvert x\rvert < 1$	10.10	$\lvert x\rvert \le 1$
10.11	$-5 \le x < 5$	10.13	$-1 < x \le 1$	10.15	$-1 < x < 5$
10.17	$-2 < x \le 0$	10.18	$-\frac{3}{4} \le x \le -\frac{1}{4}$	10.20	All x
10.21	$0 \le x \le 1$	10.22	No x	10.24	$\lvert x\rvert < \frac{1}{2}\sqrt{5}$
10.25	$n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6}$				

13.4 $\binom{-1/2}{0} = 1$; $\binom{-1/2}{n} = \dfrac{(-1)^n(2n-1)!!}{(2n)!!}$

13.6 $\sum_0^\infty \binom{1/2}{n} x^{n+1}$ (see Example 2)

13.8 $\sum_0^\infty \binom{-1/2}{n} (-x^2)^n$ (see Problem 13.4)

13.11 $\displaystyle\sum_0^\infty \frac{(-1)^n x^n}{(2n+1)!}$

13.14 $\displaystyle\sum_0^\infty \frac{x^{2n+1}}{2n+1}$

13.15 $\displaystyle\sum_0^\infty \binom{-1/2}{n}(-1)^n \frac{x^{2n+1}}{2n+1}$

13.17 $\displaystyle 2\sum_{\text{odd } n} \frac{x^n}{n}$

13.21 $x^2 + 2x^4/3 + 17x^6/45 \cdots$

13.22 $1 + 2x + 5x^2/2 + 8x^3/3 + 65x^4/24 \cdots$

13.25 $1 - x + x^2/3 - x^4/45 \cdots$

13.27 $1 + x + x^2/2 - x^4/8 - x^5/15 \cdots$

13.28 $x - x^2/2 + x^3/6 - x^5/12 \cdots$

13.29 $1 + x/2 - 3x^2/8 + 17x^3/48 \cdots$

13.34 $x - x^2 + x^3 - 13x^4/12 + 5x^5/4 \cdots$

13.35 $1 + x^2/3! + 7x^4/(3\cdot 5!) + 31x^6/(3\cdot 7!) \cdots$

13.41 $e^3[1 + (x-3) + (x-3)^2/2! + (x-3)^3/3! \cdots]$

13.44 $5 + (x-25)/10 - (x-25)^2/10^3 + (x-25)^3/(5\times 10^4) \cdots$

14.8 For $x < 0$, error < 0.001; for $x > 0$, error < 0.002.

15.1 $-x^4/24 - x^5/30 \cdots \cong -3.376 \times 10^{-16}$

15.3 $x^5/15 - 2x^7/45 \cdots \cong 6.667 \times 10^{-17}$

15.6 12

15.8 1/2

15.10 -1

15.12 1/3

15.14 $t - \frac{t^3}{3}$, error $< 10^{-6}$

15.17 $\cos(\pi/2) = 0$

15.19 $\sqrt{2}$

15.20 (b) $5e$

15.21 (b) 0.937548

15.22 (b) 1.202057

15.23 (a) 1/2 (c) 1/3

15.24 (a) $-\pi$ (d) 0 (f) 0

15.27 (a) $1 - \frac{v}{c} = 1.3 \times 10^{-5}$, or $v = 0.999987c$

(d) $1 - \frac{v}{c} = 1.3 \times 10^{-11}$

15.28 $mc^2 + \frac{1}{2}mv^2$

15.29 (b) $\dfrac{F}{W} = \dfrac{x}{l} + \dfrac{x^3}{2l^3} + \dfrac{3x^5}{8l^5} \cdots$

15.30 (b) $T = \dfrac{1}{2}\dfrac{F}{\theta}\left(1 + \dfrac{\theta^2}{6} + 7\dfrac{\theta^4}{360} \cdots\right)$

15.31 (a) finite (b) infinite

16.6 C

16.7 D

16.9 $-1 \le x < 1$

16.10 $-4 < x < 4$

16.13 $-5 < x \le 1$

16.15 $-x^2/6 - x^4/180 - x^6/2835 \cdots$

16.16 $1 - x/2 + 3x^2/8 - 11x^3/48 + 19x^4/128 \cdots$

16.19 $-(x-\pi) + (x-\pi)^3/3! - (x-\pi)^5/5! \cdots$

16.20 $2 + \dfrac{x-8}{12} - \dfrac{(x-8)^2}{2^5\cdot 3^2} + \dfrac{5(x-8)^3}{2^8\cdot 3^4} \cdots$

16.26 $-1/3$

16.28 1

16.31 (b) 2.66×10^{86} terms. For $N = 15$, $1.6905 < S < 1.6952$

Chapter 2

	x	y	r	θ	
4.1	1	1	$\sqrt{2}$	$\pi/4$	See Fig. 5.1
4.2	-1	1	$\sqrt{2}$	$3\pi/4$	See Fig. 9.6
4.3	1	$-\sqrt{3}$	2	$-\pi/3$	
4.5	0	2	2	$\pi/2$	See Fig. 5.2
4.7	-1	0	1	π	See Fig. 9.2
4.9	-2	2	$2\sqrt{2}$	$3\pi/4$	
4.11	$\sqrt{3}$	1	2	$\pi/6$	See Fig. 9.1
4.14	$\sqrt{2}$	$\sqrt{2}$	2	$\pi/4$	
4.15	-1	0	1	$-\pi$ or π	See Fig. 9.2
4.17	1	-1	$\sqrt{2}$	$-\pi/4$	See Fig. 9.5
4.20	-2.39	-6.58	7	-110° $= -1.92$ radians	
5.2	$-1/2$	$-1/2$	$1/\sqrt{2}$	$-3\pi/4$ or $5\pi/4$	
5.4	0	2	2	$\pi/2$	
5.6	-1	0	1	π	
5.8	1.6	-2.7	3.14	-59.3°	
5.10	$-25/17$	$19/17$	$\sqrt{58/17}$	142.8°	
5.12	2.65	1.41	3	28°	
5.14	1.27	-2.5	2.8	-1.1 radians $= -63^\circ$	
5.16	1.53	-1.29	2	-40°	
5.17	-7.35	-10.9	13.1	-124°	
5.18	-0.94	-0.36	1	201° or -159°	

5.19 $(2+3i)/13$; $(x-yi)/(x^2+y^2)$
5.21 $(1+i)/6$; $(x+1-yi)/[(x+1)^2+y^2]$
5.23 $(-6-3i)/5$; $(1-x^2-y^2+2yi)/[(1-x)^2+y^2]$
5.26 1 5.30 $3/2$ 5.31 1
5.32 169 5.34 1 5.35 $x=-4,\ y=3$
5.36 $x=-1/2,\ y=3$ 5.39 $x=y=$ any real number
5.42 $x=-1/7,\ y=-10/7$ 5.43 $(x,y)=(0,0)$, or $(1,1)$, or $(-1,1)$
5.45 $x=0$, any real y; or $y=0$, any real x
5.46 $y=-x$ 5.48 $x=36/13,\ y=2/13$
5.49 $y=0,\ x=1/2$
5.53 Circle (Find center and radius)
5.55 Straight line (What is its equation?)
5.56 Part of a straight line (Describe it.)
5.57 Hyperbola (What is its equation?)
5.60 Circle (Find center and radius)
5.62 Ellipse (Find its equation; where are the foci?)
5.63 Two straight lines (What lines?)
5.68 $v=2,\ a=4$

6.2 D 6.3 C 6.4 D
6.5 D 6.10 C 6.12 C

7.1 All z
7.3 All z
7.6 $|z| < 1/3$
7.7 All z
7.10 $|z| < 1$
7.12 $|z| < 4$
7.14 $|z - 2i| < 1$
7.16 $|z + (i - 3)| < 1/\sqrt{2}$

8.3 See Problem 17.30

9.3 $-9i$
9.4 $-e(1 + i\sqrt{3})/2$
9.6 1
9.7 $3e^2$
9.8 $-\sqrt{3} + i$
9.10 -2
9.11 $-1 - i$
9.13 $-4 + 4i$
9.14 64
9.17 $-(1 + i)/4$
9.19 16
9.20 i
9.21 1
9.24 $4i$
9.26 $(1 + i\sqrt{3})/2$
9.29 1
9.32 $3e^2$
9.34 $4/e$
9.35 21
9.38 $1/\sqrt{2}$

10.3 $\pm 1, \ \pm i$
10.4 $\pm 2, \ \pm 2i$
10.7 $\pm\sqrt{2}, \ \pm i\sqrt{2}, \ \pm 1 \pm i$
10.9 $1, \ 0.309 \pm 0.951i, \ -0.809 \pm 0.588i$
10.16 $\pm i, \ (\pm\sqrt{3} \pm i)/2$
10.17 $-1, \ 0.809 \pm 0.588i, \ -0.309 \pm 0.951i$
10.18 $\pm(1 + i)/\sqrt{2}$
10.21 $\pm(\sqrt{3} + i)$
10.22 $r = \sqrt{2}$, $\theta = 45° + 120°n$: $1 + i, \ -1.366 + 0.366i, \ 0.366 - 1.366i$
10.24 $\pm(\sqrt{3} + i)/2, \ \pm(1 - i\sqrt{3})/2, \ \pm(0.259 + 0966i), \ \pm(0.966 - 0.259i)$
10.25 $0.758(1 + i), -0.487 + 0.955i, -1.059 - 0.168i, -0.168 - 1.059i,$
$0.955 - 0.487i$

11.3 $3(1 - i)/\sqrt{2}$
11.5 $1 + i$
11.8 $-41/9$
11.9 $4i/3$

12.25 $\sin x \cosh y - i \cos x \sinh y, \ \sqrt{\sin^2 x + \sinh^2 y}$
12.26 $\cosh 2 \cos 3 - i \sinh 2 \sin 3 = -3.72 - 0.51i, \ 3.76$
12.28 $\tanh 1 = 0.762$
12.30 $-i$
12.32 $-4i/3$
12.33 $i \tanh 1 = 0.762i$
12.35 $-\cosh 2 = -3.76$

14.2 $-i\pi/2$ or $3i\pi/2$
14.3 $\mathrm{Ln}\,2 + i\pi/6$
14.5 $\mathrm{Ln}\,2 + 5i\pi/4$
14.6 $-i\pi/4$ or $7i\pi/4$
14.8 $-1, \ (1 \pm i\sqrt{3})/2$
14.10 $e^{-\pi^2/4}$
14.11 $\cos(\mathrm{Ln}\,2) + i \sin(\mathrm{Ln}\,2) = 0.769 + 0.639i$
14.14 $0.3198i - 0.2657$
14.15 $e^{-\pi \sinh 1} = 0.0249$
14.18 -1
14.20 1
14.23 $e^{\pi/2} = 4.81$

15.2 $\pi/2 + n\pi + (i\,\mathrm{Ln}\,3)/2$
15.3 $i(\pm\pi/3 + 2n\pi)$
15.4 $i(2n\pi + \pi/6), \ i(2n\pi + 5\pi/6)$
15.5 $\pm[\pi/2 + 2n\pi - i\,\mathrm{Ln}(3 + \sqrt{8})]$
15.8 $\pi/2 + 2n\pi \pm i\,\mathrm{Ln}\,3$
15.9 $i(\pi/3 + n\pi)$
15.12 $i(2n\pi \pm \pi/6)$
15.14 $2n\pi + i\,\mathrm{Ln}\,2, \ (2n + 1)\pi - i\,\mathrm{Ln}\,2$
15.15 $n\pi + 3\pi/8 + (i/4)\,\mathrm{Ln}\,2$

16.3 $|z| = \sqrt{2}$; motion around a circle of radius $\sqrt{2}$, at constant speed $v = \sqrt{2}$, constant acceleration $a = \sqrt{2}$.

16.5 $v = |z_1 - z_2|$; $a = 0$
16.6 (a) Series: $3 - 2i$; parallel: $5 + i$
(b) Series: $2(1 + i\sqrt{3})$; parallel: $i\sqrt{3}$
16.8 $[R - i(\omega CR^2 + \omega^3 L^2 C - \omega L)]/[(\omega CR)^2 + (\omega^2 LC - 1)^2]$; this simplifies to $L/(RC)$ at resonance.
16.9 (b) $\omega = 1/\sqrt{LC}$
16.12 $(1 + r^4 - 2r^2 \cos\theta)^{-1}$

17.2 $(\sqrt{3} + i)/2$
17.4 $i \cosh 1 = 1.54i$
17.6 $-e^{-\pi^2} = -5.17 \times 10^{-5}$
17.7 $e^{\pi/2} = 4.81$
17.9 $\pi/2 \pm 2n\pi$
17.11 i
17.13 $x = 0,\ y = 4$
17.15 $|z| < 1/e$
17.26 1
17.27 (c) $e^{-2(x-t)^2}$
17.28 $1 + (a^2 + b^2)^2 (2ab)^{-2} \sinh^2 b$
17.30 $e^x \cos x = \sum_{n=0}^{\infty} \left(2^{n/2} x^n / n!\right) \cos n\pi/4$
$e^x \sin x = \sum_{n=0}^{\infty} \left(2^{n/2} x^n / n!\right) \sin n\pi/4$

Chapter 3

2.4 $\begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 \end{pmatrix}$, $x = \frac{1}{2}(z + 1)$, $y = 1$

2.8 $\begin{pmatrix} 1 & -1 & 0 & -11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $x = y - 11$, $z = 7$

2.9 $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, inconsistent, no solution

2.12 $\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $x = -2$, $y = 1$, $z = 1$

2.17 $R = 2$

3.1 -11
3.5 -544
3.12 16
3.16 $A = -(K + ik)/(K - ik)$, $|A| = 1$

4.12 $\arccos(-1/\sqrt{2}) = 3\pi/4$
4.14 (a) $\arccos(1/3) = 70.5°$
4.14 (c) $\arccos\sqrt{2/3} = 35.3°$
4.15 (b) $8\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$
4.18 $2\mathbf{i} - 8\mathbf{j} - 3\mathbf{k}$
4.19 $\mathbf{i} + \mathbf{j} + \mathbf{k}$
4.22 Law of cosines
4.24 $A^2 B^2$

5.1 $\mathbf{r} = (2\mathbf{i} - 3\mathbf{j}) + (4\mathbf{i} + 3\mathbf{j})t$ [Note that $2\mathbf{i} - 3\mathbf{j}$ may be replaced by *any* point on the line; $4\mathbf{i} + 3\mathbf{j}$ may be replaced by *any* vector along the line. Thus, for example, $\mathbf{r} = 6\mathbf{i} - (8\mathbf{i} + 6\mathbf{j})t$ is just as good an answer, and similarly for all such problems.]
5.4 $\mathbf{r} = \mathbf{i} + (2\mathbf{i} + \mathbf{j})t$
5.6 $(x - 1)/1 = (y + 1)/(-2) = (z + 5)/2$, or $\mathbf{r} = \mathbf{i} - \mathbf{j} - 5\mathbf{k} + (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})t$
5.8 $x/3 = (z - 4)/(-5)$, $y = -2$; or $\mathbf{r} = -2\mathbf{j} + 4\mathbf{k} + (3\mathbf{i} - 5\mathbf{k})t$
5.9 $x = -1$, $z = 7$; or $\mathbf{r} = -\mathbf{i} + 7\mathbf{k} + \mathbf{j}t$

5.11 $(x-4)/1 = (z-3)/(-2)$, $y=-1$; or $\mathbf{r} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k} + (\mathbf{i} - 2\mathbf{k})t$
5.12 $(x-5)/5 = (y+4)/(-2) = (z-2)/1$; or $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} + (5\mathbf{i} - 2\mathbf{j} + \mathbf{k})t$
5.14 $36x - 3y - 22z = 23$ 5.16 $5x - 2y + z = 35$
5.18 $x + 6y + 7z + 5 = 0$ 5.20 $x - 4y - z + 5 = 0$
5.21 $\cos\theta = 25/(7\sqrt{30}) = 0.652$, $\theta = 49.3°$
5.22 $\cos\theta = 2/\sqrt{6}$, $\theta = 35.3°$
5.24 $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + (\mathbf{j} + 2\mathbf{k})t$, $d = 2\sqrt{6/5}$
5.25 $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + (4\mathbf{i} + 9\mathbf{j} - \mathbf{k})t$, $d = (3\sqrt{3})/7$
5.29 $2/\sqrt{6}$ 5.31 $5/7$ 5.33 $\sqrt{43/15}$
5.34 $\sqrt{11/10}$ 5.36 3 5.38 $\arccos\sqrt{21/22} = 12.3°$
5.39 Intersect at $(3,2,0)$; $\cos\theta = 5/\sqrt{60}$, $\theta = 49.8°$
5.42 $1/\sqrt{5}$ 5.43 $20/\sqrt{21}$ 5.45 $d = \sqrt{2}$, $t = -1$

6.2 $\mathrm{AB} = \begin{pmatrix} -2 & -2 \\ 1 & 2 \end{pmatrix}$, $\mathrm{BA} = \begin{pmatrix} -6 & 17 \\ -2 & 6 \end{pmatrix}$, $\mathrm{A}+\mathrm{B} = \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix}$, $\mathrm{A}-\mathrm{B} = \begin{pmatrix} 3 & -9 \\ -1 & 1 \end{pmatrix}$, $\mathrm{A}^2 = \begin{pmatrix} 9 & -25 \\ -5 & 14 \end{pmatrix}$, $\mathrm{B}^2 = \begin{pmatrix} 1 & 4 \\ 0 & 4 \end{pmatrix}$, $5\mathrm{A} = \begin{pmatrix} 10 & -25 \\ -5 & 15 \end{pmatrix}$, $3\mathrm{B} = \begin{pmatrix} -3 & 12 \\ 0 & 6 \end{pmatrix}$, $\det(5\mathrm{A}) = 5^2 \det \mathrm{A}$ for a 2×2 matrix
6.4 You should have found BA, C^2, CB, C^3, $\mathrm{C}^2\mathrm{B}$, and CBA; all others are meaningless. $\mathrm{C}^2\mathrm{B} = \begin{pmatrix} 32 & 12 \\ 53 & 7 \\ -13 & -9 \end{pmatrix}$, $\mathrm{CBA} = \begin{pmatrix} 36 & 46 & 14 & -36 \\ 40 & 22 & 1 & 91 \\ -8 & -2 & 1 & -29 \end{pmatrix}$
6.13 $\begin{pmatrix} 5/3 & -3 \\ -1 & 2 \end{pmatrix}$ 6.15 $-\dfrac{1}{2}\begin{pmatrix} 4 & 5 & 8 \\ -2 & -2 & -2 \\ 2 & 3 & 4 \end{pmatrix}$
6.19 $\mathrm{A}^{-1} = \dfrac{1}{7}\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$, $(x,y) = (5,0)$
6.22 $\mathrm{A}^{-1} = \dfrac{1}{12}\begin{pmatrix} 4 & 4 & 0 \\ -7 & -1 & 3 \\ 1 & -5 & 3 \end{pmatrix}$, $(x,y,z) = (1,-1,2)$
6.30 $\sin k\mathrm{A} = \mathrm{A}\sin k = \begin{pmatrix} 0 & \sin k \\ \sin k & 0 \end{pmatrix}$, $\cos k\mathrm{A} = \mathrm{I}\cos k = \begin{pmatrix} \cos k & 0 \\ 0 & \cos k \end{pmatrix}$, $e^{k\mathrm{A}} = \begin{pmatrix} \cosh k & \sinh k \\ \sinh k & \cosh k \end{pmatrix}$, $e^{ik\mathrm{A}} = \begin{pmatrix} \cos k & i\sin k \\ i\sin k & \cos k \end{pmatrix}$

7.1 Not linear 7.4 Linear 7.6 Not linear
7.8 Not linear 7.11 Not linear 7.12 Linear
7.14 Not linear 7.15 Linear
7.22 $D = 1$, rotation $\theta = -45°$ 7.24 $D = -1$, reflection line $x + y = 0$
7.26 $D = -1$, reflection line $x = 2y$
7.30 $\mathrm{R} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathrm{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$,
R is a 90° rotation about the z axis, S is a 90° rotation about the x axis.
7.32 180° rotation about $\mathbf{i} - \mathbf{k}$
7.35 Reflection through the (x,y) plane and 90° rotation about the z axis.

8.1 In terms of basis $\mathbf{u} = \frac{1}{9}(9,0,7)$, $\mathbf{v} = \frac{1}{9}(0,-9,13)$, the vectors are: $\mathbf{u} - 4\mathbf{v}$, $5\mathbf{u} - 2\mathbf{v}$, $2\mathbf{u} + \mathbf{v}$, $3\mathbf{u} + 6\mathbf{v}$.

8.3 Basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$

8.6 $\mathbf{V} = 3\mathbf{A} - \mathbf{B}$

8.19 $x = y = z = w = 0$

8.20 $x = -z$, $y = z$

8.23 For $\lambda = 3$, $x = 2y$; for $\lambda = 8$, $x = -2y$

8.25 For $\lambda = 2$: $x = 0$, $y = -3z$; for $\lambda = -3$: $x = -5y$, $z = 3y$; for $\lambda = 4$: $z = 3y$, $x = 2y$

8.26 $\mathbf{r} = (3,1,0) + (-1,1,1)z$

9.4 $\mathrm{A}^\dagger = \begin{pmatrix} 0 & i & 3 \\ -2i & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $\quad \mathrm{A}^{-1} = \dfrac{1}{6}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & i \\ -6 & 6i & -2 \end{pmatrix}$

9.14 $\mathrm{C^T B A^T}$, $\quad \mathrm{C^{-1} M^{-1} C}$, $\quad \mathrm{H}$

10.1 (b) $d = 8$

10.2 The number of basis vectors given is the dimension of the space. We list one possible basis; other bases consist of the same number of independent linear combinations of the vectors given.
(b) $(1,0,0,5,0,1)$, $(0,1,0,0,6,4)$, $(0,0,1,0,-3,0)$

10.3 (a) Label the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$. Then $\cos(\mathbf{A},\mathbf{B}) = 1/\sqrt{15}$, $\cos(\mathbf{A},\mathbf{C}) = \sqrt{2}/3$, $\cos(\mathbf{B},\mathbf{D}) = \sqrt{17/690}$.

10.4 (b) $\mathbf{e}_1 = (0,0,0,1)$, $\mathbf{e}_2 = (1,0,0,0)$, $\mathbf{e}_3 = (0,1,1,0)/\sqrt{2}$

10.5 (b) $\|\mathbf{A}\| = 7$, $\|\mathbf{B}\| = \sqrt{60}$, $\quad$ |Inner product of $\mathbf{A}$ and $\mathbf{B}$| $= \sqrt{5}$

11.5 $\theta = 1.1 = 63.4°$

11.11 $\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$, not orthogonal

In the following answers, for each eigenvalue, the components of a corresponding eigenvector are listed in parentheses.

11.12	4	$(1,1)$
	-1	$(3,-2)$
11.15	1	$(0,0,1)$
	-1	$(1,-1,0)$
	5	$(1,1,0)$
11.18	4	$(2,1,3)$
	2	$(0,-3,1)$
	-3	$(5,-1,-3)$
11.20	3	$(0,-1,2)$
	4	$(1,2,1)$
	-2	$(-5,2,1)$
11.22	-4	$(-4,1,1)$
	5	$(1,2,2)$
	-2	$(0,-1,1)$
11.23	18	$(2,2,-1)$
	9	$(1,-1,0)$
	9	$(1,1,4)$

The two eigenvectors corresponding to the eigenvalue 9 may be any two vectors orthogonal to $(2,2,-1)$ and orthogonal to each other.

11.26	4	$(1,1,1)$
	1	$(1,-1,0)$
	1	$(1,1,-2)$

11.27 $\mathrm{D} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathrm{C} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

11.29 $\mathrm{D} = \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathrm{C} = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

11.31 $\mathrm{D} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathrm{C} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

11.41 $\lambda = 1,\ 3$; $\quad \mathrm{U} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

11.44 $\lambda = 3,\ -7;\quad \mathrm{U} = \dfrac{1}{5\sqrt{2}}\begin{pmatrix} 5 & -3-4i \\ 3-4i & 5 \end{pmatrix}$

11.52 $60°$ rotation about $-\mathbf{i}\sqrt{2} + \mathbf{k}$ and reflection through the plane $z = x\sqrt{2}$

11.53 $180°$ rotation about $\mathbf{i} + \mathbf{j} + \mathbf{k}$

11.56 $45°$ rotation about $\mathbf{j} - \mathbf{k}$

11.58 $\mathrm{M}^{10} = \dfrac{1}{5}\begin{pmatrix} 1 + 4\cdot 6^{10} & 2 - 2\cdot 6^{10} \\ 2 - 2\cdot 6^{10} & 4 + 6^{10} \end{pmatrix}$

11.59 $e^{\mathrm{M}} = e^3\begin{pmatrix} \cosh 1 & -\sinh 1 \\ -\sinh 1 & \cosh 1 \end{pmatrix}$

12.2 $3x'^2 - 2y'^2 = 24$

12.3 $10x'^2 = 35$

12.6 $3x'^2 + \sqrt{3}\,y'^2 - \sqrt{3}\,z'^2 = 12$

12.15 $y = 2x$ with $\omega = \sqrt{3k/m}$; $x = -2y$ with $\omega = \sqrt{8k/m}$

12.17 $x = -2y$ with $\omega = \sqrt{2k/m}$; $3x = 2y$ with $\omega = \sqrt{2k/(3m)}$

12.19 $y = -x$ with $\omega = \sqrt{3k/m}$; $y = 2x$ with $\omega = \sqrt{3k/(2m)}$

12.22 $y = -x$ with $\omega = \sqrt{2k/m}$; $y = 3x$ with $\omega = \sqrt{2k/(3m)}$

13.6 The cyclic group

13.11 The four matrices of the symmetry group of the rectangle are:

$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathrm{P} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathrm{P}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathrm{I}$

This group is isomorphic to the 4's group.

13.21 SO(2) is Abelian; SO(3) is not Abelian.

14.3 $x,\ \cos x,\ x\cos x,\ e^x\cos x$

14.5 $1,\ x + x^3,\ x^2,\ x^4,\ x^5$

14.6 Not a vector space

14.8 $1,\ x^2,\ x^4,\ x^6$

15.3 (a) $(x-4)/1 = (y+1)/(-2) = (z-2)/(-2)$; or $\mathbf{r} = (4,-1,2) + (1,-2,-2)t$

(b) $x - 5y + 3z = 0$

(c) $5/7$

(d) $5\sqrt{2}/3 = 2.36$

(e) $\arcsin 19/21 = 64.8°$

15.5 (a) $y = 7$, $(x-2)/3 = (z+1)/4$; or $\mathbf{r} = (2,7,-1) + (3,0,4)t$

(b) $x - 4y - 9z = 0$

(c) $\arcsin(\frac{33}{70}\sqrt{2}) = 41.8°$

(d) $12/\sqrt{98} = 1.21$

(e) $\sqrt{29}/5 = 1.08$

15.7 You should have found all except $\mathrm{A^TB^T}$, $\mathrm{BA^T}$, ABC, $\mathrm{AB^TC}$, $\mathrm{B^{-1}C}$, and $\mathrm{CB^T}$, which are meaningless.

$\mathrm{B^TAC} = \begin{pmatrix} 2 & 2 \\ 1-3i & 1 \\ -1-5i & -1 \end{pmatrix}$, $\quad \mathrm{C^{-1}A} = \begin{pmatrix} 0 & -i \\ 1 & -1 \end{pmatrix}$

15.9 $\dfrac{1}{f} = (n-1)\left[\dfrac{1}{R_1} - \dfrac{1}{R_2} + \dfrac{(n-1)d}{nR_1R_2}\right]$

15.13 Area $= \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = 7/2$

15.14 $x'' = -x$, $y'' = -y$, $180°$ rotation

15.15 $x'' = -y$, $y'' = x$; $90°$ rotation of vectors or $-90°$ rotation of axes

15.18

1	$(1,1)$
-2	$(0,1)$

15.20

1	$(1,1)$
9	$(1,-1)$

15.22

1	$(1,0,1)$
4	$(0,1,0)$
5	$(1,0,-1)$

15.24

2	$(0,4,3)$
7	$(5,-3,4)$
-3	$(5,3,-4)$

15.27 $3x'^2 - y'^2 - 5z'^2 = 15$, $d = \sqrt{5}$

15.29 $3x'^2 + 6y'^2 - 4z'^2 = 54$, $d = 3$

Chapter 4

1.1 $\partial u/\partial x = 2xy^2/(x^2+y^2)^2$, $\partial u/\partial y = -2x^2y/(x^2+y^2)^2$
1.3 $\partial z/\partial u = u/(u^2+v^2+w^2)$
1.4 At $(0,0)$, both $=0$; at $(-2/3,\ 2/3)$, both $=-4$
1.7 $2x$ 1.9 $2x(1+2\tan^2\theta)$ 1.11 $2y$
1.13 $4r^2\tan\theta$ 1.15 $r^2\sin 2\theta$ 1.17 $4r$
1.19 0 1.21 $-4x\csc^2\theta$ 1.23 $2r\sin 2\theta$
1.8′ $-2r^4/x^3$ 1.10′ $2y+4y^3/x^2$ 1.12′ $2y\sec^2\theta$
1.14′ $2y^2\sec^2\theta\tan\theta$ 1.16′ $2r\tan^2\theta$ 1.18′ $-2ry^4/(r^2-y^2)^2$
1.20′ $4x(\tan\theta\sec^2\theta)(\tan^2\theta+\sec^2\theta)$
1.22′ $-8r^3/x^3$ 1.24′ $-8y^3/x^3$

2.1 $y+y^3/6-x^2y/2+x^4y/24-x^2y^3/12+y^5/120\cdots$
2.3 $x-x^2/2-xy+x^3/3+x^2y/2+xy^2\cdots$
2.5 $1+xy/2-x^2y^2/8+x^3y^3/16-5x^4y^4/128\cdots$
2.8 $e^x\cos y = 1+x+(x^2-y^2)/2+(x^3-3xy^2)/6\cdots$

4.2 2.5×10^{-13} 4.4 12.2 4.6 9%
4.8 5% 4.10 4.28 nt 4.11 3.95
4.15 8×10^{23}

5.1 $e^{-y}\sinh t + z\sin t$ 5.3 $2r(q^2-p^2)$
5.7 $(1-2b-e^{2a})\cos(a-b)$

6.2 $y'=1$, $y''=0$ 6.3 $y'=4(\ln 2-1)/(2\ln 2-1)$
6.5 $2x+11y-24=0$ 6.6 $1800/11^3$
6.10 $x+y=0$ 6.11 $y''=4$

7.1 $dx/dy = z-y+\tan(y+z)$, $d^2x/dy^2 = \frac{1}{2}\sec^3(y+z)+\frac{1}{2}\sec(y+z)-2$
7.4 $\partial w/\partial u = -2(rv+s)w$, $\partial w/\partial v = -2(ru+2s)w$
7.7 $(\partial y/\partial\theta)_r = x$, $(\partial y/\partial\theta)_x = r^2/x$, $(\partial\theta/\partial y)_x = x/r^2$
7.8 $\partial x/\partial s = -19/13$, $\partial x/\partial t = -21/13$, $\partial y/\partial s = 24/13$, $\partial y/\partial t = 6/13$
7.10 $\partial x/\partial s = 1/6$, $\partial x/\partial t = 13/6$, $\partial y/\partial s = 7/6$, $\partial y/\partial t = -11/6$
7.13 $(\partial p/\partial q)_m = -p/q$, $(\partial p/\partial q)_a = 1/(a\cos p - 1)$,
$(\partial p/\partial q)_b = 1-b\sin q$, $(\partial b/\partial a)_p = (\sin p)(b\sin q-1)/\cos q$,
$(\partial a/\partial q)_m = [q+p(a\cos p-1)]/(q\sin p)$
7.15 $(\partial x/\partial u)_v = (2yv^2-x^2)/(2yv+2xu)$, $(\partial x/\partial u)_y = (x^2u+y^2v)/(y^2-2xu^2)$
7.17 $(\partial p/\partial s)_t = -9/7$, $(\partial p/\partial s)_q = 3/2$
7.19 $(\partial x/\partial z)_s = 7/2$, $(\partial x/\partial z)_r = 4$, $(\partial x/\partial z)_y = 3$

8.3 $(-1,2)$ is a minimum point 8.4 $(-1,-2)$ is a saddle point
8.8 $\theta=\pi/3$; bend up 8 cm on each side
8.9 $l=w=2h$ 8.11 $\theta=30^\circ$, $x=y\sqrt{3}=z/2$
8.13 $(4/3,5/3)$ 8.16 $m=5/2$, $b=1/3$

9.2 $r:l:s=\sqrt{5}:(1+\sqrt{5}):3$ 9.4 $4/\sqrt{3}$ by $6/\sqrt{3}$ by $10/\sqrt{3}$
9.6 $V=1/3$ 9.8 $(8/13,12/13)$
9.12 Let legs of right triangle be a and b, height of prism $=h$; then $a=b$, $h=(2-\sqrt{2})a$.

10.2 4, 2

10.4 $d = 1$

10.6 $d = 2$

10.7 $\frac{1}{2}\sqrt{11}$

10.10 (a) $\max T = \frac{1}{2}$, $\min T = -\frac{1}{2}$
(b) $\max T = 1$, $\min T = -\frac{1}{2}$
(c) $\max T = 1$, $\min T = -\frac{1}{2}$

10.12 Largest sum $= 180°$
Smallest sum $= 3 \arccos(1/\sqrt{3})$
$= 164.2°$

10.13 Largest sum $= 3 \arcsin(1/\sqrt{3}) = 105.8°$, smallest sum $= 90°$

11.1 $z = f(y + 2x) + g(y + 3x)$

11.6 $d^2y/dz^2 + dy/dz - 5y = 0$

11.11 $H = p\dot{q} - L$

12.1 $\frac{1}{2}x^{-1/2} \sin x$

12.3 $dz/dx = -\sin(\cos x) \tan x - \sin(\sin x) \cot x$

12.4 $\frac{1}{2} \sin 2$

12.7 $(\partial u/\partial x)_y = -e^4$, $(\partial u/\partial y)_x = e^4/\ln 2$, $(\partial y/\partial x)_u = \ln 2$

12.10 $dy/dx = (e^x - 1)/x$

12.12 $(2x + 1)/\ln(x + x^2) - 2/\ln(2x)$

12.14 $\pi/(4y^3)$

13.2 (a) and (b) $d = 4/\sqrt{13}$

13.4 $-\csc\theta \cot\theta$

13.5 $-6x$, $2x^2 \tan\theta \sec^2\theta$, $4x \tan\theta \sec^2\theta$

13.9 $dz/dt = 1 + (t/z)(2 - x - y)$, $z \neq 0$

13.10 $[x \ln x - (y^2/x)]x^y$ where $x = r\cos\theta$, $y = r\sin\theta$

13.13 -1

13.14 $(\partial w/\partial x)_y = (\partial f/\partial x)_{s,t} + 2(\partial f/\partial s)_{x,t} + 2(\partial f/\partial t)_{x,s} = f_1 + 2f_2 + 2f_3$

13.18 $\sqrt{26/3}$

13.21 $T(2) = 4$, $T(5) = -5$

13.23 $t \cot t$

13.25 $-e^x/x$

13.29 $dt = 3.9$

Chapter 5

2.1	3	2.3	4	2.5	$\frac{1}{4}e^2 - \frac{5}{12}$	2.7	5/3	2.9	6
2.11	36	2.13	7/4	2.15	3/2	2.17	$\frac{1}{2}\ln 2$	2.19	32
2.21	131/6	2.23	9/8	2.25	3/2	2.27	32/5	2.29	2
2.31	6	2.33	16/3	2.36	1/6	2.37	7/6	2.39	70
2.41	5	2.43	9/2	2.45	$46k/15$	2.47	16/3	2.49	1/3

3.2 (b) $Ml^2/12$ (c) $Ml^2/3$

3.3 (a) $M = 140$ (b) $\bar{x} = 130/21$ (c) $I_m = 6.92M$
(d) $I = 150M/7$

3.5 (a) $Ma^2/3$ (b) $Ma^2/12$ (c) $2Ma^2/3$

3.7 (a) $M = 9$ (b) $(\bar{x}, \bar{y}) = (2, 4/3)$
(c) $I_x = 2M$, $I_y = 9M/2$ (d) $I_m = 13M/18$

3.9 (a) 1/6 (b) $(1/4, 1/4, 1/4)$ (c) $M = 1/24$, $\bar{z} = 2/5$

3.11 (a) $M = (5\sqrt{5} - 1)/6 = 1.7$
(b) $\bar{x} = 0$, $\bar{y} = (313 + 15\sqrt{5})/620 = 0.56$

3.14 $V = 2\pi^2 a^2 b$, $A = 4\pi^2 ab$, where a = radius of revolving circle, and b = distance to axis from center of this circle.

3.15 For area, $(\bar{x}, \bar{y}) = (0, \frac{4}{3}r/\pi)$, for arc, $(\bar{x}, \bar{y}) = (0, 2r/\pi)$

3.18 $s = [3\sqrt{2} + \ln(1+\sqrt{2}\,)]/2$
3.20 $13\pi/3$
3.21 $s\bar{x} = [51\sqrt{2} - \ln(1+\sqrt{2}\,)]/32$, $s\bar{y} = 13/6$, s as in Problem 3.18
3.23 $(149/130, 0, 0)$
3.25 I/M has the same numerical value as $\bar{x}$ in Problem 3.21
3.26 $2M/3$ 3.27 $149M/130$ 3.29 2 3.30 $32/5$

4.1 (b) $\bar{x} = \bar{y} = \frac{4}{3}a/\pi$ (c) $I = Ma^2/4$ (e) $\bar{x} = \bar{y} = 2a/\pi$
4.2 (c) $\bar{y} = \frac{4}{3}a/\pi$
(d) $I_x = Ma^2/4$, $I_y = 5Ma^2/4$, $I_z = 3Ma^2/2$
(e) $\bar{y} = 2a/\pi$
(f) $\bar{x} = 6a/5$, $I_x = 48Ma^2/175$, $I_y = 288Ma^2/175$, $I_z = 48Ma^2/25$
(g) $A = (\frac{2}{3}\pi - \frac{1}{2}\sqrt{3}\,)a^2$
4.4 (b) $(0, 0, a/2)$ (c) $2Ma^2/3$ (e) $(0, 0, 3a/8)$
4.5 $7\pi/3$
4.11 12π
4.12 (c) $M = (16\rho/9)(3\pi - 4) = 9.64\rho$
$I = (128\rho/15^2)(15\pi - 26) = 12.02\rho = 1.25M$
4.14 $\pi(1 - e^{-1})/4$ 4.16 $u^2 + v^2$ 4.19 $\pi/4$
4.22 $12(1 + 36\pi^2)^{1/2}$ 4.24 $\rho G\pi a/2$ 4.26 (a) $\frac{7}{5}Ma^2$
4.27 $2\pi ah$ (where h = distance between parallel planes)

5.1 $\frac{9}{5}\pi\sqrt{30}$ 5.3 $\pi(37^{3/2} - 1)/6$
5.5 8π for each nappe 5.6 4
5.8 $\frac{3}{16}\sqrt{6} + \frac{9}{16}\ln(\sqrt{2} + \sqrt{3}\,)$ 5.9 $\pi\sqrt{2}$
5.12 $M = \frac{1}{6}\sqrt{3}$, $(\bar{x}, \bar{y}, \bar{z}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ 5.14 $M = \frac{1}{2}\pi - \frac{4}{3}$
5.16 $\bar{x} = 0$, $\bar{y} = 1$, $\bar{z} = [32/(9\pi)]\sqrt{2/5} = 0.716$

6.2 $45(2 + \sqrt{2}\,)/112$ 6.3 $15\pi/8$
6.4 (a) $\frac{1}{2}MR^2$ (b) $\frac{3}{2}MR^2$ 6.6 (a) $(4\pi - 3\sqrt{3}\,)/6$
6.7 $(8\pi - 3\sqrt{3}\,)(4\pi - 3\sqrt{3}\,)^{-1}M$ 6.8 (b) $27/20$
6.10 (a) $(\bar{x}, \bar{y}) = (\pi/2, \pi/8)$ 6.10 (c) $3M/8$
6.12 $(abc)^2/6$ 6.14 $16a^3/3$
6.15 $I_x = \frac{8}{15}Ma^2$, $I_y = \frac{7}{15}Ma^2$ 6.16 $\bar{x} = \bar{y} = 2a/5$
6.18 $(0, 0, 5h/6)$
6.19 $I_x = I_y = 20Mh^2/21$, $I_z = 10Mh^2/21$, $I_m = 65Mh^2/252$
6.21 $\pi G\rho h(2 - \sqrt{2}\,)$ 6.24 $(0, 0, 2c/3)$
6.26 $\frac{1}{2}\sinh 1$ 6.27 $e^2 - e - 1$

Chapter 6

3.1 $(\mathbf{A} \cdot \mathbf{B})\mathbf{C} = 6\mathbf{C}$, $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -8$,
$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = -4(\mathbf{i} + 2\mathbf{k})$
3.3 -5
3.6 $\mathbf{v} = (2/\sqrt{6}\,)(\mathbf{A} \times \mathbf{B}) = (2/\sqrt{6}\,)(\mathbf{i} - 7\mathbf{j} - 3\mathbf{k})$,
$\mathbf{r} \times \mathbf{F} = (\mathbf{A} - \mathbf{C}) \times \mathbf{B} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$,
$\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F}) = [(\mathbf{A} - \mathbf{C}) \times \mathbf{B}] \cdot \mathbf{C}/|\mathbf{C}| = 8/\sqrt{26}$
3.7 (a) $11\mathbf{i} + 3\mathbf{j} - 13\mathbf{k}$, (b) 3, (c) 17

3.9 $-9\mathbf{i} - 23\mathbf{j} + \mathbf{k}$, $1/\sqrt{21}$
3.15 $\mathbf{u}_1 \cdot \mathbf{u} = -\mathbf{u}_3 \cdot \mathbf{u}$, $n_1\mathbf{u}_1 \times \mathbf{u} = n_2\mathbf{u}_2 \times \mathbf{u}$
3.17 $\mathbf{a} = (\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} - \omega^2\mathbf{r}$; for $\mathbf{r} \perp \boldsymbol{\omega}$, $\mathbf{a} = -\omega^2\mathbf{r}$, $|\mathbf{a}| = v^2/r$.
3.19 (a) $16\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$ (b) $8/\sqrt{6}$
3.20 (b) 12

4.2 (a) $t = 2$
(b) $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, $|\mathbf{v}| = 2\sqrt{14}$
(c) $(x-4)/4 = (y+4)/(-2) = (z-8)/6$, $2x - y + 3z = 36$
4.5 $|d\mathbf{r}/dt| = \sqrt{2}$; $|d^2\mathbf{r}/dt^2| = 1$; path is a helix.
4.8 $d\mathbf{r}/dt = \mathbf{e}_r(dr/dt) + \mathbf{e}_\theta(r\,d\theta/dt)$;
$d^2\mathbf{r}/dt^2 = \mathbf{e}_r[d^2r/dt^2 - r(d\theta/dt)^2] + \mathbf{e}_\theta[r\,d^2\theta/dt^2 + 2(dr/dt)(d\theta/dt)]$

6.2 $-\mathbf{i}$
6.4 $\pi e/(3\sqrt{5}\,)$
6.6 $6x + 8y - z = 25$, $(x-3)/6 = (y-4)/8 = (z-25)/(-1)$
6.9 (a) $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ (b) $5/\sqrt{6}$
(c) $\mathbf{r} = (1, 1, 1) + (2, -2, -1)t$
6.12 (a) $2\sqrt{5}$, $-2\mathbf{i} + \mathbf{j}$ (b) $3\mathbf{i} + 2\mathbf{j}$ (c) $\sqrt{10}$
6.14 (b) Down, at the rate $11\sqrt{2}$
6.17 $\mathbf{e}_r$ 6.19 $\mathbf{j}$

7.1 $\boldsymbol{\nabla} \cdot \mathbf{r} = 3$, $\boldsymbol{\nabla} \times \mathbf{r} = 0$
7.2 $\boldsymbol{\nabla} \cdot \mathbf{r} = 2$, $\boldsymbol{\nabla} \times \mathbf{r} = 0$
7.4 $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$, $\boldsymbol{\nabla} \times \mathbf{V} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$
7.6 $\boldsymbol{\nabla} \cdot \mathbf{V} = 5xy$, $\boldsymbol{\nabla} \times \mathbf{V} = \mathbf{i}xz - \mathbf{j}yz + \mathbf{k}(y^2 - x^2)$
7.7 $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$, $\boldsymbol{\nabla} \times \mathbf{V} = \mathbf{i}x - \mathbf{j}y - \mathbf{k}x\cos y$
7.10 0 7.11 $-(x^2 + y^2)/(x^2 - y^2)^{3/2}$
7.13 $2xy$ 7.14 0
7.16 $2(x^2 + y^2 + z^2)^{-1}$ 7.19 $2/r$

8.1 $-11/3$ 8.2 (a) -4π (b) -16 (c) -8
8.3 (a) $5/3$ (b) 1 (c) $2/3$ 8.4 (a) 3 (b) $8/3$
8.7 (b) 0 (d) 2π 8.8 $yz - x$
8.9 $3xy - x^3yz - z^2$ 8.11 $-y\sin^2 x$
8.14 $-\arcsin xy$ 8.18 (a) $\pi + \pi^2/2$ (b) $\pi^2/2$

9.2 40 9.4 $-3/2$ 9.7 πab
9.8 24π 9.10 -20 9.11 2

10.2 3 10.4 36π 10.5 $4\pi \cdot 5^2$ 10.7 48π 10.9 16π
10.12 $\phi = \begin{cases} 0, & r \le R_1; \\ (k/2\pi\epsilon_0)\ln(R_1/r), & R_1 \le r \le R_2; \\ (k/2\pi\epsilon_0)\ln(R_1/R_2), & r \ge R_2. \end{cases}$

11.2 $2ab^2$ 11.3 0 11.4 -12
11.5 36 11.6 45π 11.7 0
11.10 -6π 11.12 18π 11.15 $-2\pi\sqrt{2}$
11.18 $\mathbf{A} = (xz - yz^2 - y^2/2)\mathbf{i} + (x^2/2 - x^2z + yz^2/2 - yz)\mathbf{j} + \boldsymbol{\nabla}u$, any u
11.20 $\mathbf{A} = \mathbf{i}\sin zx + \mathbf{j}\cos zx + \mathbf{k}e^{zy} + \boldsymbol{\nabla}u$, any u

12.1 $(\sin\theta\cos\theta)\mathbf{C}$
12.7 (a) $9\mathbf{i}+5\mathbf{j}-3\mathbf{k}$ (b) $29/3$ 12.9 24
12.11 (a) $\operatorname{grad}\phi=-3y\mathbf{i}-3x\mathbf{j}+2z\mathbf{k}$ (b) $-\sqrt{3}$
(c) $2x+y-2z+2=0$, $\mathbf{r}=(1,2,3)+(2,1,-2)t$
12.13 (a) $6\mathbf{i}-\mathbf{j}-4\mathbf{k}$ (b) $53^{-1/2}(6\mathbf{i}-\mathbf{j}-4\mathbf{k})$ (c) same as (a)
(d) $53^{1/2}$ (e) $53^{1/2}$
12.18 Not conservative (a) $1/2$ (b) $4/3$
12.21 4 12.23 192π 12.25 -18π
12.27 4 12.29 10 12.31 $29/3$

Chapter 7

		Amplitude	Period	Frequency	Velocity Amplitude
2.2		2	$\pi/2$	$2/\pi$	8
2.3		$1/2$	2	$1/2$	$\pi/2$
2.6	$s=6\cos(\pi/8)\sin(2t)$	$6\cos(\pi/8)=5.54$	π	$1/\pi$	$12\cos(\pi/8)=11.1$
2.8		2	4π	$1/(4\pi)$	1
2.10		4	π	$1/\pi$	8
2.11	q	3	$1/60$	60	
	I	360π	$1/60$	60	

2.13 $A=$ maximum value of θ, $\omega=\sqrt{g/l}$
2.16 $t\cong 4.91\cong 281°$
2.19 $A=1$, $T=4$, $f=1/4$, $v=1/4$, $\lambda=1$
2.21 $y=20\sin\frac{1}{2}\pi(x-6t)$, $\partial y/\partial t=-60\pi\cos\frac{1}{2}\pi(x-6t)$
2.23 $y=\sin 880\pi((x/350)-t)$
2.25 $y=10\sin[\pi(x-3\cdot 10^8t)/250]$

3.6 $\sin(2x+\frac{1}{3}\pi)$

4.5 $\pi^{-1}+\frac{1}{2}$ 4.6 $2/\pi$ 4.8 0
4.11 $1/2$ 4.14 (a) $2\pi/3$ (b) π 4.15 (a) $3/2$

$x\to$	-2π	$-\pi$	$-\pi/2$	0	$\pi/2$	π	2π
6.2	$1/2$	0	0	$1/2$	$1/2$	0	$1/2$
6.4	-1	0	-1	-1	0	0	-1
6.6	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
6.8	1	1	$1-\frac{1}{2}\pi$	1	$1+\frac{1}{2}\pi$	1	1
6.10	π	0	$\pi/2$	π	$\pi/2$	0	π

7.1 $f(x) = \dfrac{1}{2} + \dfrac{i}{\pi} \sum_{\substack{-\infty \\ \text{odd } n}}^{\infty} \dfrac{1}{n} e^{inx}$

7.2 $f(x) = \dfrac{1}{4} + \dfrac{1}{2\pi}\Big[(1-i)e^{ix} + (1+i)e^{-ix} - i(e^{2ix} - e^{-2ix}) - \dfrac{1+i}{3}e^{3ix} - \dfrac{1-i}{3}e^{-3ix} + \dfrac{1-i}{5}e^{5ix} + \dfrac{1+i}{5}e^{-5ix} \cdots\Big]$

7.7 $f(x) = \dfrac{\pi}{4} - \sum_{\substack{-\infty \\ \text{odd } n}}^{\infty} \left(\dfrac{1}{n^2\pi} + \dfrac{i}{2n}\right) e^{inx} + \sum_{\substack{-\infty \\ \text{even } n\neq 0}}^{\infty} \dfrac{i}{2n} e^{inx}$

7.11 $f(x) = \dfrac{1}{\pi} + \dfrac{e^{ix} - e^{-ix}}{4i} - \dfrac{1}{\pi} \sum_{\substack{-\infty \\ \text{even } n\neq 0}}^{\infty} \dfrac{e^{inx}}{n^2 - 1}$

8.2 $f(x) = \dfrac{1}{4} + \dfrac{1}{\pi}\left(\cos\dfrac{\pi x}{l} - \dfrac{1}{3}\cos\dfrac{3\pi x}{l} + \dfrac{1}{5}\cos\dfrac{5\pi x}{l} \cdots\right) + \dfrac{1}{\pi}\left(\sin\dfrac{\pi x}{l} + \dfrac{2}{2}\sin\dfrac{2\pi x}{l} + \dfrac{1}{3}\sin\dfrac{3\pi x}{l} + \dfrac{1}{5}\sin\dfrac{5\pi x}{l} + \dfrac{2}{6}\sin\dfrac{6\pi x}{l} \cdots\right)$

8.6 $f(x) = \dfrac{1}{2} + \dfrac{4}{\pi} \sum \dfrac{1}{n} \sin\dfrac{n\pi x}{l} \qquad (n = 2, 6, 10, \cdots)$

8.11 (a) $f(x) = \dfrac{\pi^2}{3} + 4\sum_{1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx$

(b) $f(x) = \dfrac{4\pi^2}{3} + 2\sum_{-\infty}^{\infty} \left(\dfrac{1}{n^2} + \dfrac{i\pi}{n}\right) e^{inx}, \quad n \neq 0$

8.14 (a) $f(x) = \dfrac{8}{\pi} \sum_{1}^{\infty} \dfrac{n(-1)^{n+1}}{4n^2 - 1} \sin 2n\pi x$

(b) $f(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi} \sum_{1}^{\infty} \dfrac{\cos 2n\pi x}{4n^2 - 1} = -\dfrac{2}{\pi} \sum_{-\infty}^{\infty} \dfrac{1}{4n^2 - 1} e^{2in\pi x}$

8.19 $f(x) = \dfrac{1}{8} - \dfrac{1}{\pi^2} \sum_{\text{odd } n=1}^{\infty} \dfrac{1}{n^2} \cos 2n\pi x + \dfrac{1}{2\pi} \sum_{1}^{\infty} \dfrac{(-1)^{n+1}}{n} \sin 2n\pi x$

8.20 $f(x) = \dfrac{2}{3} - \dfrac{9}{8\pi^2}\left[\cos\dfrac{2\pi x}{3} + \dfrac{1}{2^2}\cos\dfrac{4\pi x}{3} + \dfrac{1}{4^2}\cos\dfrac{8\pi x}{3} + \cdots\right] - \left(\dfrac{3\sqrt{3}}{8\pi^2} + \dfrac{1}{\pi}\right)\sin\dfrac{2\pi x}{3} + \left(\dfrac{3\sqrt{3}}{32\pi^2} - \dfrac{1}{2\pi}\right)\sin\dfrac{4\pi x}{3} - \dfrac{1}{3\pi}\sin\dfrac{6\pi x}{3} - \left(\dfrac{3\sqrt{3}}{128\pi^2} + \dfrac{1}{4\pi}\right)\sin\dfrac{8\pi x}{3} \cdots$

9.2 (a) $\frac{1}{2}\ln|1 - x^2| + \frac{1}{2}\ln|(1-x)/(1+x)|$

9.5 $f(x) = \dfrac{4}{\pi} \sum_{\text{odd } n=1}^{\infty} \dfrac{1}{n} \sin nx$

9.19 $f_c(x) = f_p(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\sum_1^\infty \dfrac{(-1)^n \cos 2nx}{4n^2-1}$

$f_s(x) = \dfrac{2}{\pi}\left(\sin x + \sin 3x + \frac{1}{3}\sin 5x + \frac{1}{3}\sin 7x + \frac{1}{5}\sin 9x + \frac{1}{5}\sin 11x \cdots\right)$

9.20 $f_c(x) = \dfrac{1}{3} + \dfrac{4}{\pi^2}\sum_1^\infty \dfrac{(-1)^n}{n^2}\cos n\pi x$

$f_s(x) = \dfrac{2}{\pi}\sum_1^\infty \dfrac{(-1)^{n+1}}{n}\sin n\pi x - \dfrac{8}{\pi^3}\sum_{\text{odd } n=1}^\infty \dfrac{1}{n^3}\sin n\pi x$

$f_p(x) = \dfrac{1}{3} + \dfrac{1}{\pi^2}\sum_1^\infty \dfrac{1}{n^2}\cos 2n\pi x - \dfrac{1}{\pi}\sum_1^\infty \dfrac{1}{n}\sin 2n\pi x$

9.22 $f_c(x) = 15 - \dfrac{20}{\pi}\left(\cos\dfrac{\pi x}{20} - \dfrac{1}{3}\cos\dfrac{3\pi x}{20} + \dfrac{1}{5}\cos\dfrac{5\pi x}{20}\cdots\right)$

$f_s(x) = \dfrac{20}{\pi}\left(3\sin\dfrac{\pi x}{20} - \dfrac{2}{2}\sin\dfrac{2\pi x}{20} + \dfrac{3}{3}\sin\dfrac{3\pi x}{20} + \dfrac{3}{5}\sin\dfrac{5\pi x}{20} - \dfrac{2}{6}\sin\dfrac{6\pi x}{20}\cdots\right)$

$f_p(x) = 15 - \dfrac{20}{\pi}\sum_{\text{odd } n=1}^\infty \dfrac{1}{n}\sin\dfrac{n\pi x}{10}$

9.23 $f(x,0) = \dfrac{8h}{\pi^2}\left(\sin\dfrac{\pi x}{l} - \dfrac{1}{3^2}\sin\dfrac{3\pi x}{l} + \dfrac{1}{5^2}\sin\dfrac{5\pi x}{l}\cdots\right)$

10.1 Relative intensities $= 1:0:0:0:\frac{1}{25}:0:\frac{1}{49}:0:0:0$

10.3 Relative intensities $= 1:25:\frac{1}{9}:0:\frac{1}{25}:\frac{25}{9}:\frac{1}{49}:0:\frac{1}{81}:1$

10.5 $I(t) = \dfrac{5}{\pi}\left[1 - 2\sum_{\text{even } n=2}^\infty \dfrac{1}{n^2-1}\cos 120n\pi t\right] + \dfrac{5}{2}\sin 120\pi t$

10.6 $V(t) = 50 - \dfrac{400}{\pi^2}\sum_{\text{odd } n=1}^\infty \dfrac{1}{n^2}\cos 120n\pi t$

10.7 $I(t) = -\dfrac{20}{\pi}\sum_1^\infty \dfrac{(-1)^n}{n}\sin 120n\pi t$

10.10 $V(t) = 75 - \dfrac{200}{\pi^2}\sum_{\text{odd } n=1}^\infty \dfrac{1}{n^2}\cos 120n\pi t - \dfrac{100}{\pi}\sum_1^\infty \dfrac{1}{n}\sin 120n\pi t$

Relative intensities $= 1.4 : 0.25 : 0.12 : 0.06 : 0.04$

11.5 $\pi^2/8$ 11.7 $\pi^2/6$ 11.9 $\dfrac{\pi^2}{16} - \dfrac{1}{2}$

12.2 $f_s(x) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{1-\cos\alpha}{\alpha}\sin\alpha x\, d\alpha$

12.4 $f(x) = \displaystyle\int_{-\infty}^\infty \dfrac{\sin\alpha\pi - \sin(\alpha\pi/2)}{\alpha\pi}e^{i\alpha x}\, d\alpha$

12.6 $f(x) = \displaystyle\int_{-\infty}^\infty \dfrac{\sin\alpha - \alpha\cos\alpha}{i\pi\alpha^2}e^{i\alpha x}\, d\alpha$

12.8 $f(x) = \displaystyle\int_{-\infty}^\infty \dfrac{(i\alpha+1)e^{-i\alpha}-1}{2\pi\alpha^2}e^{i\alpha x}\, d\alpha$

12.10 $f(x) = 2\int_{-\infty}^{\infty} \frac{\alpha a - \sin \alpha a}{i\pi\alpha^2} e^{i\alpha x}\, d\alpha$

12.11 $f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\alpha\pi/2)}{1-\alpha^2} e^{i\alpha x}\, d\alpha$

12.13 $f_c(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin \alpha\pi - \sin(\alpha\pi/2)}{\alpha} \cos \alpha x\, d\alpha$

12.16 $f_c(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos(\alpha\pi/2)}{1-\alpha^2} \cos \alpha x\, d\alpha$

12.18 $f_s(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin \alpha - \alpha \cos \alpha}{\alpha^2} \sin \alpha x\, d\alpha$

12.19 $f_s(x) = \frac{4}{\pi}\int_0^{\infty} \frac{\alpha a - \sin \alpha a}{\alpha^2} \sin \alpha x\, d\alpha$

12.21 $g(\alpha) = \sigma(2\pi)^{-1/2} e^{-\alpha^2\sigma^2/2}$

12.25 (a) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1+e^{-i\alpha\pi}}{1-\alpha^2} e^{i\alpha x}\, d\alpha$

12.28 (a) $f_c(x) = \frac{4}{\pi}\int_0^{\infty} \frac{\cos 3\alpha \sin \alpha}{\alpha} \cos \alpha x\, d\alpha$

(b) $f_s(x) = \frac{4}{\pi}\int_0^{\infty} \frac{\sin 3\alpha \sin \alpha}{\alpha} \sin \alpha x\, d\alpha$

12.30 (a) $f_c(x) = \frac{1}{\pi}\int_0^{\infty} \frac{1-\cos 2\alpha}{\alpha^2} \cos \alpha x\, d\alpha$

(b) $f_s(x) = \frac{1}{\pi}\int_0^{\infty} \frac{2\alpha - \sin 2\alpha}{\alpha^2} \sin \alpha x\, d\alpha$

13.7 $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2}\cos nx$

13.8 (b) 1

13.10 (d) -1, $-1/2$, -2, -1

13.14 (a) $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_1^{\infty} \frac{\cos n\pi x}{n^2}$

(b) $\pi^4/90$

13.15 $-\pi/4$

13.23 $\pi^2/8$

Chapter 8

1.5 $x = -A\omega^{-2}\sin \omega t + v_0 t + x_0$

1.7 $x = (c/F)[(m^2c^2 + F^2t^2)^{1/2} - mc]$

2.2 $(1-x^2)^{1/2} + (1-y^2)^{1/2} = C$, $C = \sqrt{3}$

2.3 $\ln y = A(\csc x - \cot x)$, $A = \sqrt{3}$

2.6 $2y^2 + 1 = A(x^2-1)^2$, $A = 1$

2.7 $y^2 = 8 + e^{K-x^2}$, $K = 1$

2.9 $ye^y = ae^x$, $a = 1$

2.13 $y \equiv 1$, $y \equiv -1$, $x \equiv 1$, $x \equiv -1$

2.19 (a) $I/I_0 = e^{-0.5} = 0.6$ for $s = 50$ ft

Half value thickness $= (\ln 2)/\mu = 69.3$ ft

(b) Half life $T = (\ln 2)/\lambda$

2.20 (c) $\tau = RC$, $\tau = L/R$. Corresponding quantities are a, $\lambda = (\ln 2)/T$, μ, $1/\tau$.

2.22 $N = N_0e^{Kt} - (R/K)(e^{Kt} - 1)$ where N_0 = number of bacteria at $t = 0$, KN = rate of increase, R = removal rate.

2.23 $T = 100[1 - (\ln r)/(\ln 2)]$

2.26 (a) k = weight divided by terminal speed
(b) $t = g^{-1} \cdot$ (terminal speed) $\cdot$ ($\ln 100$); typical terminal speeds are 0.02 to 0.1 cm/sec, so t is of the order of 10^{-4} sec.

2.27 $t = 10(\ln \frac{5}{13})/(\ln \frac{3}{13}) = 6.6$ min

2.29 $t = 100 \ln \frac{9}{4} = 81.1$ min

2.31 $ay = bx$

2.33 $x^2 + ny^2 = C$

2.35 $x(y-1) = C$

3.1 $y = \frac{1}{2}e^x + Ce^{-x}$

3.3 $y = (\frac{1}{2}x^2 + C)e^{-x^2}$

3.6 $y = (x+C)/(x+\sqrt{x^2+1}\,)$

3.8 $y = \frac{1}{2}\ln x + C/\ln x$

3.9 $y(1-x^2)^{1/2} = x^2 + C$

3.11 $y = 2(\sin x - 1) + Ce^{-\sin x}$

3.13 $x = \frac{1}{2}e^y + Ce^{-y}$

3.14 $x = y^{2/3} + Cy^{-1/3}$

3.15 $S = (10^7/2)[(1+3t/10^4) + (1+3t/10^4)^{-1/3}]$, where S = number of pounds of salt, and t is in hours.

3.17 $I = Ae^{-t/(RC)} - V_0\omega C(\sin\omega t - \omega RC\cos\omega t)/(1+\omega^2R^2C^2)$

3.21 $N_n = c_1e^{-\lambda_1 t} + c_2e^{-\lambda_2 t} + \cdots$ where
$$c_1 = \frac{\lambda_1\lambda_2\cdots\lambda_{n-1}N_0}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\dots(\lambda_n-\lambda_1)},\quad c_2 = \frac{\lambda_1\lambda_2\cdots\lambda_{n-1}N_0}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\cdots(\lambda_n-\lambda_2)},$$
etc. (all λ's different)

3.22 $y = x + 1 + Ke^x$

4.1 $y^{1/3} = x - 3 + Ce^{-x/3}$

4.4 $x^2e^{3y} + e^x - \frac{1}{3}y^3 = C$

4.5 $x^2 - y^2 + 2x(y+1) = C$

4.7 $x = y(\ln x + C)$

4.9 $y^2 = Ce^{-x^2/y^2}$

4.11 $\tan\frac{1}{2}(x+y) = x + C$

4.13 $y^2 = -\sin^2 x + C\sin^4 x$

4.16 $y^2 = C(C \pm 2x)$

4.18 $x^2 + (y-k)^2 = k^2$

4.19 $r = Ae^{-\theta}$, $r = Be^{\theta}$

5.1 $y = Ae^x + Be^{-2x}$

5.3 $y = Ae^{3ix} + Be^{-3ix}$ or other forms as in (5.24)

5.5 $y = (Ax+B)e^x$

5.7 $y = Ae^{3x} + Be^{2x}$

5.9 $y = Ae^{2x}\sin(3x+\gamma)$

5.11 $y = (A+Bx)e^{-3x/2}$

5.20 $y = Ae^{-x} + Be^{ix}$

5.22 $y = Ae^x + Be^{-3x} + Ce^{-5x}$

5.24 $y = Ae^{-x} + Be^{x/2}\sin(\frac{1}{2}x\sqrt{3}+\gamma)$

5.26 $y = Ae^{5x} + (Bx+C)e^{-x}$

5.28 $y = e^x(A\sin x + B\cos x) + e^{-x}(C\sin x + D\cos x)$

5.29 $y = (A+Bx)e^{-x} + Ce^{2x} + De^{-2x} + E\sin(2x+\gamma)$

5.35 $T = 2\pi\sqrt{R/g} \cong 85$ min.

6.1 $y = Ae^{2x} + Be^{-2x} - \frac{5}{2}$

6.3 $y = Ae^x + Be^{-2x} + \frac{1}{4}e^{2x}$

6.5 $y = Ae^{ix} + Be^{-ix} + e^x$

6.7 $y = Ae^{-x} + Be^{2x} + xe^{2x}$

6.9 $y = (Ax + B + x^2)e^{-x}$

6.11 $y = e^{-x}(A\sin 3x + B\cos 3x) + 8\sin 4x - 6\cos 4x$

6.13 $y = (Ax+B)e^x - \sin x$

6.15 $y = e^{-6x/5}[A\sin(8x/5) + B\cos(8x/5)] - 5\cos 2x$

6.17 $y = A\sin 4x + B\cos 4x + 2x\sin 4x$

6.18 $y = e^{-x}(A\sin 4x + B\cos 4x) + 2e^{-4x}\cos 5x$

6.20 $y = Ae^{-2x}\sin(2x+\gamma) + 4e^{-x/2}\sin(5x/2)$

6.22 $y = A + Be^{-x/2} + x^2 - 4x$

6.24 $y = (A + Bx + 2x^3)e^{3x}$

6.26 $y = A\sin x + B\cos x - 2x^2\cos x + 2x\sin x$

6.33 $y = A\sin(x+\gamma) + x^3 - 6x - 1 + x\sin x + (3-2x)e^x$

6.34 $y = Ae^{3x} + Be^{2x} + e^x + x$

6.37 $y = (A + Bx)e^x + 2x^2e^x + (3 - x)e^{2x} + x + 1$

6.41 $$y = e^{-x}(A\cos x + B\sin x) + \tfrac{1}{4}\pi + \sum_{\text{odd } n=1}^{\infty} [4(n^2 - 2)\cos nx - 8n\sin nx]/[\pi n^2(n^2 + 4)]$$

7.1 (a) $y \equiv 5$ (b) $y = 2/(x+1)$
(c) $y = \tan(\frac{\pi}{4} - \frac{x}{2}) = \sec x - \tan x$ (d) $y = 2\tanh x$

7.4 $x^2 + (y - b)^2 = a^2$, or $y = C$

7.11 $x = (1 - 3t)^{1/3}$

7.12 $t = \int_1^x u^2(1 - u^4)^{-1/2}\,du$

7.16 (a) $y = Ax + Bx^{-3}$

7.16 (c) $y = (A + B\ln x)/x^3$

7.18 $y = Ax + Bx^{-1} + \frac{1}{2}\left(x + x^{-1}\right)\ln x$

7.20 $y = x^2(A + B\ln x) + x^2(\ln x)^3$

7.22 $y = A\cos\ln x + B\sin\ln x + x$

7.25 $x^{-1} - 1$

7.27 x^3e^x

7.29 $xe^{1/x}$

8.8 $e^{-2t} - te^{-2t}$

8.10 $\frac{1}{3}e^t\sin 3t + 2e^t\cos 3t$

8.12 $3\cosh 5t + 2\sinh 5t$

8.21 $2b(p + a)/[(p + a)^2 + b^2]^2$

8.23 $y = te^{-2t}(\cos t - \sin t)$

8.25 $e^{-p\pi/2}/(p^2 + 1)$

9.3 $y = e^{-2t}(4t + \frac{1}{2}t^2)$

9.4 $y = \cos t + \frac{1}{2}(\sin t - t\cos t)$

9.7 $y = 1 - e^{2t}$

9.9 $y = (t + 2)\sin 4t$

9.11 $y = te^{2t}$

9.12 $y = \frac{1}{2}(t^2e^{-t} + 3e^t - e^{-t})$

9.13 $y = \sinh 2t$

9.17 $y = 2$

9.19 $y = e^{2t}$

9.21 $y = e^{3t} + 2e^{-2t}\sin t$

9.23 $y = \sin t + 2\cos t - 2e^{-t}\cos 2t$

9.25 $y = (3 + t)e^{-2t}\sin t$

9.27 $\begin{cases} y = t + \frac{1}{4}(1 - e^{4t}) \\ z = \frac{1}{3} + e^{4t} \end{cases}$

9.28 $\begin{cases} y = t\cos t - 1 \\ z = \cos t + t\sin t \end{cases}$

9.30 $\begin{cases} y = t - \sin 2t \\ z = \cos 2t \end{cases}$

9.32 $\begin{cases} y = \sin 2t \\ z = \cos 2t - 1 \end{cases}$

9.36 $\arctan(2/3)$

9.38 $4/5$

9.40 1

9.42 $\pi/4$

10.3 $\frac{1}{2}t\sinh t$

10.5 $[b(b - a)te^{-bt} + a(e^{-bt} - e^{-at})]/(b - a)^2$

10.7 $(a\cosh bt - b\sinh bt - ae^{-at})/(a^2 - b^2)$

10.9 $(2t^2 - 2t + 1 - e^{-2t})/4$

10.12 $(b^2 - a^2)^{-1}(b^{-2}\cos bt - a^{-2}\cos at) + a^{-2}b^{-2}$

10.13 $\frac{1}{2}(e^{-t} + \sin t - \cos t)$

10.15 $\frac{1}{14}e^{3t} + \frac{1}{35}e^{-4t} - \frac{1}{10}e^t$

10.17 $y = \begin{cases} (\cosh at - 1)/a^2, & t > 0 \\ 0, & t < 0 \end{cases}$

11.7 $y = \begin{cases} (t - t_0)e^{-(t-t_0)}, & t > t_0 \\ 0, & t < t_0 \end{cases}$

11.9 $y = \begin{cases} \frac{1}{3}e^{-(t-t_0)}\sin 3(t-t_0), & t > t_0 \\ 0, & t < t_0 \end{cases}$

11.11 $y = \begin{cases} \frac{1}{2}[\sinh(t-t_0) - \sin(t-t_0)], & t > t_0 \\ 0, & t < t_0 \end{cases}$

11.13 (b) $3\delta(x+5) - 4\delta(x-10)$

11.15 (b) 0 (d) $\cosh 1$

11.21 (b) $\phi(|a|)/(2|a|)$ (c) 1/2

11.23 (a) $\delta(x+5)\delta(y-5)\delta(z)$, $\delta(r-5\sqrt{2})\delta(\theta-\frac{3\pi}{4})\delta(z)/r$, $\delta(r-5\sqrt{2})\delta(\theta-\frac{\pi}{2})\delta(\phi-\frac{3\pi}{4})/(r\sin\theta)$
(c) $\delta(x+2)\delta(y)\delta(z-2\sqrt{3})$, $\delta(r-2)\delta(\theta-\pi)\delta(z-2\sqrt{3})/r$, $\delta(r-4)\delta(\theta-\frac{\pi}{6})\delta(\phi-\pi)/(r\sin\theta)$

11.25 (c) $G''(x) = \delta(x) + 5\delta'(x)$

12.2 $y = (\sin\omega t - \omega t\cos\omega t)/(2\omega^2)$

12.7 $y = [a(\cosh at - e^{-t}) - \sinh at]/[a(a^2-1)]$

12.11 $y = -\frac{1}{3}\sin 2x$

12.13 $y = \begin{cases} x - \sqrt{2}\sin x, & x < \pi/4 \\ \frac{1}{2}\pi - x - \sqrt{2}\cos x, & x > \pi/4 \end{cases}$

12.16 $y = -x\ln x - x - x(\ln x)^2/2$

12.18 $y = x^2/2 + x^4/6$

13.1 $y = -\frac{1}{3}x^{-2} + Cx$ 13.3 $y = A + Be^{-x}\sin(x+\gamma)$

13.5 $x^2 + y^2 - y\sin^2 x = C$ 13.7 $3x^2y^3 + 1 = Ax^3$

13.8 $y = x(A + B\ln x) + \frac{1}{2}x(\ln x)^2$ 13.10 $u - \ln u + \ln v + v^{-1} = C$

13.13 $y = Ae^{-2x}\sin(x+\gamma) + e^{3x}$ 13.15 $y = (A+Bx)e^{2x} + 3x^2e^{2x}$

13.18 $x = (y+C)e^{-\sin y}$

13.20 $y = Ae^x\sin(2x+\gamma) + x + \frac{2}{5} + e^x(1 - x\cos 2x)$

13.22 $y = (A+Bx)e^{2x} + C\sin(3x+\gamma)$

13.24 $y^2 = ax^2 + b$ 13.26 $y = x^2 + x$

13.28 $y^2 + 4(x-1)^2 = 9$ 13.30 $y = g/3$, $v = 7g/12$, $a = 5g/12$

13.32 1:23 p.m.

13.33 In both (a) and (b), the temperature of the mixture at time t is given by the formula $T_a(1-e^{-kt}) + (n+n')^{-1}(nT_0 + n'T_0')e^{-kt}$.

13.38 $\frac{1}{2}\ln[(a^2+p^2)/p^2]$ 13.41 $\frac{1}{4}(\tanh 1 - \operatorname{sech}^2 1) = 0.0854$

13.43 $(\sin at + at\cos at)/(2a)$

13.46 For e^{-x}: $g_s(\alpha) = (2/\pi)^{1/2}\alpha/(1+\alpha^2)$, $g_c(\alpha) = (2/\pi)^{1/2}/(1+\alpha^2)$

13.47 $y = A\sin t + B\cos t + \sin t\ln(\sec t + \tan t) - 1$

Chapter 9

2.1 Parabola 2.2 Circle

2.3 $ax = \sinh(ay+b)$ 2.6 $x + a = \frac{4}{3}(y^{1/2} - 2b)(b + y^{1/2})^{1/2}$

3.1 $dx/dy = C/\sqrt{y^3 - C^2}$ 3.3 $x^4y'^2 = C^2(1 + x^2y'^2)^3$

3.6 $x = ay^{3/2} - \frac{1}{2}y^2 + b$ 3.7 $y = K\sinh(x+C)$

3.9 $\cot\theta = A\cos(\phi-\alpha)$ 3.12 $(x-a)^2 + y^2 = C^2$

3.15 $r\cos(\theta+\alpha)=C$ or, in rectangular coordinates, the straight line $x\cos\alpha - y\sin\alpha = C$

3.18 See Problem 3.9

4.6 Cycloid

5.2 $\begin{cases} m(\ddot r - r\dot\theta^2) = -\partial V/\partial r \\ m(r\ddot\theta + 2\dot r\dot\theta) = -(1/r)(\partial V/\partial\theta) \\ m\ddot z = -\partial V/\partial z \end{cases}$

Comment: These equations are in the form $m\mathbf{a} = \mathbf{F}$; recall from Chapter 6, equation (6.7), the polar coordinate form for $\mathbf{F} = -\boldsymbol{\nabla} V$.

5.4 $l\ddot\theta + g\sin\theta = 0$

5.6 $\begin{cases} a\ddot\theta - a\sin\theta\cos\theta\ \dot\phi^2 - g\sin\theta = 0 \\ (d/dt)(\sin^2\theta\ \dot\phi) = 0 \end{cases}$

5.8 $L = \frac{1}{2}m(2\dot r^2 + r^2\dot\theta^2) - mgr$
$2\ddot r - r\dot\theta^2 + g = 0,\ (d/dt)(r^2\dot\theta) = 0$

5.11 $L = \frac{1}{2}(m + Ia^{-2})\dot z^2 - mgz$
$(ma^2 + I)\ddot z + mga^2 = 0$

5.12 $L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - \left[\frac{1}{2}k(r-r_0)^2 - mgr\cos\theta\right]$
$\ddot r - r\dot\theta^2 + \frac{k}{m}(r - r_0) - g\cos\theta = 0,\quad (d/dt)(r^2\dot\theta) + gr\sin\theta = 0$

5.14 $L = M\dot x^2 + Mgx\sin\alpha,\quad 2M\ddot x - Mg\sin\alpha = 0$

5.16 $L = \frac{1}{2}m(l + a\theta)^2\dot\theta^2 - mg[a\sin\theta - (l + a\theta)\cos\theta]$
$(l + a\theta)\ddot\theta + a\dot\theta^2 + g\sin\theta = 0$

5.19 $x = y$ with $\omega = \sqrt{g/l}$; $x = -y$ with $\omega = \sqrt{3g/l}$

5.21 $2\ddot\theta + \ddot\phi\cos(\theta-\phi) + \dot\phi^2\sin(\theta-\phi) + \frac{2g}{l}\sin\theta = 0$
$\ddot\phi + \ddot\theta\cos(\theta-\phi) - \dot\theta^2\sin(\theta-\phi) + \frac{g}{l}\sin\phi = 0$

5.23 $\phi = 2\theta$ with $\omega = \sqrt{2g/(3l)}$; $\phi = -2\theta$ with $\omega = \sqrt{2g/l}$

6.1 Catenary 6.3 Circular cylinder 6.5 Circle

8.4 $dr/d\theta = Kr\sqrt{r^4 - K^2}$

8.6 $(x-a)^2 + (y+1)^2 = C^2$

8.8 Intersection of $r = 1 + \cos\theta$ with $z = a + b\sin(\theta/2)$

8.10 Intersection of $y = x^2$ with $az = b[2x\sqrt{4x^2+1} + \sinh^{-1}2x] + c$

8.12 $e^y\cos(x-a) = K$

8.16 Hyperbola: $r^2\cos(2\theta+\alpha) = K$ or $(x^2 - y^2)\cos\alpha - 2xy\sin\alpha = K$

8.17 $K\ln r = \cosh(K\theta + C)$

8.18 Parabola: $(x - y - C)^2 = 4K^2(x + y - K^2)$

8.20 $m(\ddot r - r\dot\theta^2) + Kr^{-2} = 0,\ r^2\dot\theta = \text{const.}$

8.22 $r^{-1}m(r^2\ddot\theta + 2r\dot r\dot\theta - r^2\sin\theta\cos\theta\ \dot\phi^2) = -r^{-1}(\partial V/\partial\theta) = F_\theta = ma_\theta$,
$a_\theta = r\ddot\theta + 2\dot r\dot\theta - r\sin\theta\cos\theta\ \dot\phi^2$

8.27 $dr/d\theta = r\sqrt{K^2(1+\lambda r)^2 - 1}$

Chapter 10

4.6 $I = \begin{pmatrix} 9 & 0 & -3 \\ 0 & 6 & 0 \\ -3 & 0 & 9 \end{pmatrix}$; principal moments: $(6, 6, 12)$; principal axes along the vectors $(1, 0, -1)$ and any two orthogonal vectors in the plane $z = x$, say $(0, 1, 0)$ and $(1, 0, 1)$.

5.6 (a) 3 (c) 2 (e) -1

6.15 (c) vector

8.1 $h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta$

$d\mathbf{s} = \mathbf{e}_r\,dr + \mathbf{e}_\theta\,r\,d\theta + \mathbf{e}_\phi\,r\sin\theta\,d\phi$

$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

$\mathbf{a}_r = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta = \mathbf{e}_r$

$\mathbf{a}_\theta = \mathbf{i}r\cos\theta\cos\phi + \mathbf{j}\,r\cos\theta\sin\phi - \mathbf{k}r\sin\theta = r\mathbf{e}_\theta$

$\mathbf{a}_\phi = -\mathbf{i}r\sin\theta\sin\phi + \mathbf{j}\,r\sin\theta\cos\phi = r\sin\theta\,\mathbf{e}_\phi$

8.3 $d\mathbf{s}/dt = \mathbf{e}_r\dot{r} + \mathbf{e}_\theta\,r\dot{\theta} + \mathbf{e}_\phi\,r\sin\theta\,\dot{\phi}$

$$d^2\mathbf{s}/dt^2 = \mathbf{e}_r(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2) + \mathbf{e}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2) + \mathbf{e}_\phi(r\sin\theta\,\ddot{\phi} + 2r\cos\theta\,\dot{\theta}\dot{\phi} + 2\sin\theta\,\dot{r}\dot{\phi})$$

8.5 $\mathbf{V} = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta - \mathbf{e}_\phi\,r\sin\theta$

8.6 $h_u = h_v = (u^2+v^2)^{1/2}, \quad h_z = 1$

$d\mathbf{s} = (u^2+v^2)^{1/2}(\mathbf{e}_u\,du + \mathbf{e}_v\,dv) + \mathbf{e}_z\,dz$

$dV = (u^2+v^2)\,du\,dv\,dz$

$\mathbf{a}_u = \mathbf{i}u + \mathbf{j}v = (u^2+v^2)^{1/2}\mathbf{e}_u$

$\mathbf{a}_v = -\mathbf{i}v + \mathbf{j}u = (u^2+v^2)^{1/2}\mathbf{e}_v$

$\mathbf{a}_z = \mathbf{k} = \mathbf{e}_z$

8.9 $h_u = h_v = a(\cosh u + \cos v)^{-1}$

$d\mathbf{s} = a(\cosh u + \cos v)^{-1}(\mathbf{e}_u\,du + \mathbf{e}_v\,dv)$

$dA = a^2(\cosh u + \cos v)^{-2}\,du\,dv$

$\mathbf{a}_u = (h_u^2/a)[\mathbf{i}(1+\cos v\cosh u) - \mathbf{j}\sin v\sinh u] = h_u\mathbf{e}_u$

$\mathbf{a}_v = (h_v^2/a)[\mathbf{i}\sinh u\sin v + \mathbf{j}(1+\cos v\cosh u)] = h_v\mathbf{e}_v$

8.11 $d\mathbf{s}/dt = (u^2+v^2)^{1/2}(\mathbf{e}_u\dot{u} + \mathbf{e}_v\dot{v}) + \mathbf{e}_z\dot{z}$

$$d^2\mathbf{s}/dt^2 = \mathbf{e}_u(u^2+v^2)^{-1/2}[(u^2+v^2)\ddot{u} + u(\dot{u}^2 - \dot{v}^2) + 2v\dot{u}\dot{v}] + \mathbf{e}_v(u^2+v^2)^{-1/2}[(u^2+v^2)\ddot{v} + v(\dot{v}^2 - \dot{u}^2) + 2u\dot{u}\dot{v}] + \mathbf{e}_z\ddot{z}$$

8.14 $d\mathbf{s}/dt = a(\cosh u + \cos v)^{-1}(\mathbf{e}_u\dot{u} + \mathbf{e}_v\dot{v})$

$$d^2\mathbf{s}/dt^2 = \mathbf{e}_u a(\cosh u + \cos v)^{-2}[(\cosh u + \cos v)\ddot{u} + (\dot{v}^2 - \dot{u}^2)\sinh u + 2\dot{u}\dot{v}\sin v] + \mathbf{e}_v a(\cosh u + \cos v)^{-2}[(\cosh u + \cos v)\ddot{v} + (\dot{v}^2 - \dot{u}^2)\sin v - 2\dot{u}\dot{v}\sinh u]$$

9.10 Let $h = h_u = h_v = (u^2+v^2)^{1/2}$ represent the u and v scale factors.

$$\boldsymbol{\nabla} U = h^{-1}\left(\mathbf{e}_u\frac{\partial U}{\partial u} + \mathbf{e}_v\frac{\partial U}{\partial v}\right) + \mathbf{k}\frac{\partial U}{\partial z}$$

$$\boldsymbol{\nabla}\cdot\mathbf{V} = h^{-2}\left[\frac{\partial}{\partial u}(hV_u) + \frac{\partial}{\partial v}(hV_v)\right] + \frac{\partial V_z}{\partial z}$$

$$\nabla^2 U = h^{-2}\left(\frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2}\right) + \frac{\partial^2 U}{\partial z^2}$$

$$\boldsymbol{\nabla}\times\mathbf{V} = \left(h^{-1}\frac{\partial V_z}{\partial v} - \frac{\partial V_v}{\partial z}\right)\mathbf{e}_u + \left(\frac{\partial V_u}{\partial z} - h^{-1}\frac{\partial V_z}{\partial u}\right)\mathbf{e}_v + h^{-2}\left[\frac{\partial}{\partial u}(hV_v) - \frac{\partial}{\partial v}(hV_u)\right]\mathbf{e}_z$$

9.13 Same as 9.10 if $h = a(\cosh u + \cos v)^{-1}$ and terms involving either z derivatives or V_z are omitted. Note, however, that $\nabla \times \mathbf{V}$ has *only* a z component if $\mathbf{V} = \mathbf{e}_u V_u + \mathbf{e}_v V_v$ where V_u and V_v are functions of u and v.

9.15 $h_u = 1$, $h_v = u/\sqrt{1-v^2}$
$\mathbf{e}_u = \mathbf{i}v + \mathbf{j}\sqrt{1-v^2}$, $\mathbf{e}_v = \mathbf{i}\sqrt{1-v^2} - \mathbf{j}v$
$m[\ddot{u} - u\dot{v}^2/(1-v^2)] = -\partial V/\partial u = F_u$
$m[(u\ddot{v} + 2\dot{u}\dot{v})/(1-v^2)^{1/2} + uv\dot{v}^2/(1-v^2)^{3/2}] = -h_v^{-1}\partial V/\partial v = F_v$

9.16 r^{-1}, 0, 0, $r^{-1}\mathbf{e}_z$

9.19 $2\mathbf{e}_\phi$, $\mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta$, 3

9.21 $2r^{-1}$, 6, $2r^{-4}$, $-k^2 e^{ikr\cos\theta}$

Chapter 11

3.3 $9/10$

3.7 8

3.9 $\Gamma(5/4)$

3.11 1

3.14 $-\Gamma(4/3)$

3.17 $\Gamma(p)$

7.1 $\frac{1}{2}B(\frac{5}{2},\frac{1}{2}) = 3\pi/16$

7.3 $\frac{1}{3}B(\frac{1}{3},\frac{1}{2})$

7.5 $B(3,3) = 1/30$

7.7 $\frac{1}{2}B(\frac{1}{4},\frac{1}{2})$

7.11 $2B(\frac{2}{3},\frac{4}{3})/B(\frac{1}{3},\frac{4}{3})$

7.13 $I_y/M = 8B(\frac{4}{3},\frac{4}{3})/B(\frac{5}{3},\frac{1}{3})$

8.1 $B(\frac{1}{2},\frac{1}{4})\sqrt{2l/g} = 7.4163\sqrt{l/g}$
(Compare $2\pi\sqrt{l/g}$)

8.3 $t = \pi\sqrt{a/g}$

10.2 $\Gamma(p,x) \sim x^{p-1}e^{-x}[1 + (p-1)x^{-1} + (p-1)(p-2)x^{-2}\cdots]$

10.5 (a) $E_1(x) = \Gamma(0,x)$

10.6 (b) $\mathrm{Ei}(x)$

11.5 1

12.1 $K = F(\pi/2,k) = (\pi/2)\{1 + (\frac{1}{2})^2k^2 + [(1\cdot 3)/(2\cdot 4)]^2k^4\cdots\}$
$E = E(\pi/2,k) = (\pi/2)\{1 - (\frac{1}{2})^2k^2 - [1/(2\cdot 4)]^2\cdot 3k^4 - [(1\cdot 3)/(2\cdot 4\cdot 6)]^2\cdot 5k^6\cdots\}$

Caution: For the following answers, see the warning about elliptic integral notation just after equations (12.3) and in Example 1.

12.5 $E(1/3) \cong 1.526$

12.6 $\frac{1}{3}F(\frac{\pi}{3},\frac{1}{3}) \cong 0.355$

12.7 $5E(\frac{5\pi}{4},\frac{1}{5}) \cong 19.46$

12.10 $\frac{1}{2}F(\frac{\pi}{4},\frac{1}{2}) \cong 0.402$

12.11 $F(\frac{3\pi}{8},\frac{3}{\sqrt{10}}) + K(\frac{3}{\sqrt{10}}) \cong 4.097$

12.13 $3E(\frac{\pi}{6},\frac{2}{3}) + 3E(\arcsin\frac{3}{4},\frac{2}{3}) \cong 3.96$

12.16 $2\sqrt{2}\,E(1/\sqrt{2}\,) \cong 3.820$

12.23 $T = 8\sqrt{\frac{a}{5g}}\,K(1/\sqrt{5}\,)$; for small vibrations, $T \cong 2\pi\sqrt{\frac{2a}{3g}}$

13.8 $\frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(1)$

13.10 $\sqrt{2}\,K(1/\sqrt{2}\,) \cong 2.622$

13.11 $\frac{1}{5}F(\arcsin\frac{3}{4},\frac{4}{5}) \cong 0.1834$

13.13 $-\,\mathrm{sn}\,u\,\mathrm{dn}\,u$

13.15 $\Gamma(7/2) = 15\sqrt{\pi}/8$

13.17 $\frac{1}{2}B(\frac{5}{4},\frac{7}{4}) = 3\pi\sqrt{2}/64$

13.19 $\frac{1}{2}\sqrt{\pi}\,\mathrm{erfc}\,5$

13.21 $5^4B(\frac{2}{3},\frac{13}{3}) = (\frac{5}{3})^5(\frac{14\pi}{\sqrt{3}})$

13.24 $-2^{55}\sqrt{\pi}/109!!$

Chapter 12

1.2 $y = a_0 e^{x^3}$

1.3 $y = a_1 x$

1.7 $y = Ax + Bx^3$

1.9 $y = a_0(1 - x^2) + a_1 x$

2.4 $Q_0 = \frac{1}{2}\ln\frac{1+x}{1-x}$, $Q_1 = \frac{x}{2}\ln\frac{1+x}{1-x} - 1$

3.3 $(30 - x^2)\sin x + 12x\cos x$

3.5 $(x^2 - 200x + 9900)e^{-x}$

5.3 $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, $P_3(x) = (5x^3 - 3x)/2$, $P_4(x) = (35x^4 - 30x^2 + 3)/8$, $P_5(x) = (63x^5 - 70x^3 + 15x)/8$, $P_6(x) = (231x^6 - 315x^4 + 105x^2 - 5)/16$

5.9 $2P_2 + P_1$

5.11 $\frac{2}{5}(P_1 - P_3)$

5.12 $\frac{8}{5}P_4 + 4P_2 - 3P_1 + \frac{12}{5}P_0$

8.2 $N = \sqrt{\frac{2}{5}}$, $\sqrt{\frac{5}{2}}\,P_2(x)$

8.4 $N = \pi^{1/4}$, $\pi^{-1/4}e^{-x^2/2}$

9.1 $\frac{3}{2}P_1 - \frac{7}{8}P_3 + \frac{11}{16}P_5 \cdots$

9.4 $\frac{1}{8}\pi(3P_1 + \frac{7}{16}P_3 + \frac{11}{64}P_5 \cdots)$

9.6 $P_0 + \frac{3}{8}P_1 - \frac{20}{9}P_2 \cdots$

9.8 $\frac{1}{2}(1-a)P_0 + \frac{3}{4}(1-a^2)P_1 + \frac{5}{4}a(1-a^2)P_2 + \frac{7}{16}(1-a^2)(5a^2-1)P_3 \cdots$

9.11 $\frac{8}{5}P_4 + 4P_2 - 3P_1 + \frac{12}{5}P_0$

9.12 $\frac{2}{5}(P_1 - P_3)$

9.14 $\frac{1}{2}P_0 + \frac{5}{8}P_2 = \frac{3}{16}(5x^2 + 1)$

10.5 $\frac{1}{2}(\sin\theta)(35\cos^3\theta - 15\cos\theta)$

11.2 $y = Ax^{-3} + Bx^3$

11.4 $y = Ax^{-2} + Bx^3$

11.6 $y = Ae^{-x} + Bx^{2/3}[1 - 3x/5 + (3x)^2/(5\cdot 8) - (3x)^3/(5\cdot 8\cdot 11) + \cdots]$

11.8 $y = A(x^{-1} - 1) + Bx^2(1 - x + 3x^2/5 - 4x^3/15 + 2x^4/21 + \cdots)$

11.10 $y = A[1 + 2x - (2x)^2/2! + (2x)^3/(3\cdot 3!) - (2x)^4/(3\cdot 5\cdot 4!) + \cdots]$
$+ Bx^{3/2}[1 - 2x/5 + (2x)^2/(5\cdot 7\cdot 2!) - (2x)^3/(5\cdot 7\cdot 9\cdot 3!) + \cdots]$

11.11 $y = Ax^{1/6}[1 + 3x^2/2^5 + 3^2x^4/(5\cdot 2^{10}) + \cdots]$
$+ Bx^{-1/6}[x + 3x^3/2^6 + 3^2x^5/(7\cdot 2^{11}) + \cdots]$

16.1 $y = x^{-3/2}Z_{1/2}(x)$

16.3 $y = x^{-1/2}Z_1(4x^{1/2})$

16.5 $y = xZ_0(2x)$

16.7 $y = x^{-1}Z_{1/2}(x^2/2)$

16.9 $y = x^{1/3}Z_{2/3}(4\sqrt{x}\,)$

16.11 $y = x^{-2}Z_2(x)$

16.15 $y = Z_2(5x)$

16.17 $y = Z_0(3x)$

17.7 (a) $y = x^{1/2}I_1(2x^{1/2})$. Note that the factor i does not need to be included, since *any* multiple of y is a solution.

18.11 1.7 m for steel.

20.1 1/6

20.3 $4/\pi$

20.5 1/2

20.7 $h_n^{(1)}(x) \sim x^{-1}e^{i[x-(n+1)\pi/2]}$

20.9 $h_n^{(1)}(ix) \sim -i^{-n}x^{-1}e^{-x}$

21.1 $y = Ax + B\left(x \sinh^{-1} x - \sqrt{x^2+1}\right)$
21.2 $y = A(1+x) + Bxe^{1/x}$
21.5 $y = A(x-1) + B[(x-1)\ln x - 4]$
21.7 $y = A\frac{x}{1-x} + B[\frac{x}{1-x}\ln x + \frac{1+x}{2}]$
21.8 $y = A(x^2+2x) + B[(x^2+2x)\ln x + 1 + 5x - x^3/6 + x^4/72 + \cdots]$

22.4 $H_0(x) = 1$ $\quad H_3(x) = 8x^3 - 12x$
$H_1(x) = 2x$ $\quad H_4(x) = 16x^4 - 48x^2 + 12$
$H_2(x) = 4x^2 - 2$ $\quad H_5(x) = 32x^5 - 160x^3 + 120x$

22.13 $L_0(x) = 1$
$L_1(x) = 1 - x$
$L_2(x) = \frac{1}{2}(2 - 4x + x^2)$
$L_3(x) = \frac{1}{6}(6 - 18x + 9x^2 - x^3)$
$L_4(x) = \frac{1}{24}(24 - 96x + 72x^2 - 16x^3 + x^4)$
$L_5(x) = \frac{1}{120}(120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5)$

Note: The factor $1/n!$ is omitted in most quantum mechanics books but is included as here in most reference books.

Chapter 13

2.12
$$T = \sum_{\text{odd } n} \frac{400}{n\pi \sinh 3n\pi} \sinh \frac{n\pi}{10}(30-y) \sin \frac{n\pi x}{10} + \sum_{\text{odd } n} \frac{400}{n\pi \sinh(n\pi/3)} \sinh \frac{n\pi}{30}(10-x) \sin \frac{n\pi y}{30}$$

2.14 For $f(x) = x - 5$: $T = -\frac{40}{\pi^2}\sum_{\text{odd } n} \frac{1}{n^2}\cos\frac{n\pi x}{10} e^{-n\pi y/10}$

For $f(x) = x$: add 5 to the answer just given.

3.9
$$u = 100 - \frac{400}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-[(2n+1)\pi\alpha/4]^2 t} \cos\left(\frac{2n+1}{4}\pi x\right)$$

3.11 $E_n = n^2\hbar^2/(2m)$; $\Psi(x,t) = \frac{4}{\pi}\sum_{\text{odd } n} \frac{\sin nx}{n} e^{-iE_n t/\hbar}$

4.8
$$y = \frac{4l}{\pi^2 v}\left[\frac{1}{3}\sin\frac{\pi x}{l}\sin\frac{\pi vt}{l} + \frac{\pi}{16}\sin\frac{2\pi x}{l}\sin\frac{2\pi vt}{l} - \sum_{n=3}^{\infty}\frac{\sin n\pi/2}{n(n^2-4)}\sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}\right]$$

4.9 Problem 2: $n = 2$, $\nu = v/l$
Problem 3: $n = 3$, $\nu = \frac{3}{2}v/l$ and $n = 4$, $\nu = 2v/l$ have nearly equal intensity.
Problem 5: $n = 1$, $\nu = \frac{1}{2}v/l$

5.1 (a) $u \cong 9.76$

5.4 $u = 200\sum_{m=1}^{\infty}\frac{1}{k_m J_1(k_m)} J_0(k_m r/a) e^{-(k_m\alpha/a)^2 t}$, $\quad k_m =$ zeros of J_0

5.10 $u = \dfrac{6400}{\pi^3} \displaystyle\sum_{\text{odd } n} \sum_{\text{odd } m} \sum_{\text{odd } p} \dfrac{1}{nmp} \sin \dfrac{n\pi x}{l} \sin \dfrac{m\pi y}{l} \sin \dfrac{p\pi z}{l} e^{-(\alpha\pi/l)^2(n^2+m^2+p^2)t}$

5.11 $R = r^n,\ r^{-n},\ n \neq 0$; $R = \ln r$, const., $n = 0$.
$R = r^l,\ r^{-l-1}$.

5.13 $u = \dfrac{400}{\pi} \displaystyle\sum_{\text{odd } n} \dfrac{1}{n} \left(\dfrac{r}{10}\right)^{4n} \sin 4n\theta$

5.14 $u = \dfrac{50 \ln r}{\ln 2} + \dfrac{200}{\pi} \displaystyle\sum_{\text{odd } n} \dfrac{r^n - r^{-n}}{n(2^n - 2^{-n})} \sin n\theta$

6.5 $z = \dfrac{64 l^4}{\pi^6} \displaystyle\sum_{\text{odd } m} \sum_{\text{odd } n} \dfrac{1}{n^3 m^3} \sin \dfrac{n\pi x}{l} \sin \dfrac{m\pi y}{l} \cos \dfrac{\pi v (m^2 + n^2)^{1/2} t}{l}$

6.8 $\Psi_{mn} = J_n(k_{mn} r) \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix} e^{-iE_{mn}t/\hbar}, \quad E_{mn} = \dfrac{\hbar^2 k_{mn}^2}{2ma^2}$

7.2 $u = \frac{2}{5} r P_1(\cos\theta) - \frac{2}{5} r^3 P_3(\cos\theta)$
7.5 $u = \frac{1}{2} P_0(\cos\theta) + \frac{5}{8} r^2 P_2(\cos\theta) - \frac{3}{16} r^4 P_4(\cos\theta) \cdots$
7.6 $u = \frac{1}{8}\pi[3rP_1(\cos\theta) + \frac{7}{16} r^3 P_3(\cos\theta) + \frac{11}{64} r^5 P_5(\cos\theta) \cdots]$
7.8 $u = 25[P_0(\cos\theta) + \frac{9}{4} r P_1(\cos\theta) + \frac{15}{8} r^2 P_2(\cos\theta) + \frac{21}{64} r^3 P_3(\cos\theta) \cdots]$
7.10 $u = \frac{1}{15} r^3 P_3^2(\cos\theta)\cos 2\phi - rP_1(\cos\theta)$
7.12 $u = \frac{3}{4} r P_1(\cos\theta) + \frac{7}{24} r^3 P_3(\cos\theta) - \frac{11}{192} r^5 P_5(\cos\theta) \cdots$
7.13 $u = E_0(r - a^3/r^2) P_1(\cos\theta)$

7.15 $u = 100 + \dfrac{200a}{\pi r} \displaystyle\sum_1^\infty \dfrac{(-1)^n}{n} \sin \dfrac{n\pi r}{a} e^{-(\alpha n\pi/a)^2 t}$

$$= 100 + 200 \sum_1^\infty (-1)^n j_0(n\pi r/a) e^{-(\alpha n\pi/a)^2 t}$$

7.19 $\Psi(r,\theta,\phi) = j_l(\beta r) P_l^m(\cos\theta) e^{\pm im\phi} e^{-iEt/\hbar}$, where

$$\beta = \sqrt{2ME/\hbar^2}, \quad \beta a = \text{ zeros of} j_l, \quad E = \frac{\hbar^2}{2Ma^2}(\text{ zeros of } j_l)^2$$

7.20 $\psi_n(x) = e^{-\alpha^2 x^2/2} H_n(\alpha x), \quad \alpha = \sqrt{m\omega/\hbar}$
7.21 Degree of degeneracy of E_n is $C(n+2, n) = (n+2)(n+1)/2,\ n = 0$ to ∞.
7.22 $\Psi(r,\theta,\phi) = R(r) Y_l^m(\theta,\phi),\ R(r) = r^l e^{-r/(na)} L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right),\ E_n = -\frac{Me^4}{2\hbar^2 n^2}$

8.4 Let $K =$ line charge per unit length. Then

$$V = -K\ln(r^2 + a^2 - 2ra\cos\theta) + K\ln a^2 - K\ln R^2$$
$$+ K\ln[r^2 + (R^2/a)^2 - 2(R^2/a) r \cos\theta]$$

8.5 K at $(a, 0)$, $-K$ at $(R^2/a, 0)$

9.2 $u = 200\pi^{-1} \int_0^\infty k^{-2}(1 - \cos 2k) e^{-ky} \cos kx\, dk$
9.7 $u(x,t) = 100\,\mathrm{erf}[x/(2\alpha t^{1/2})] - 50\,\mathrm{erf}[(x-1)/(2\alpha t^{1/2})] - 50\,\mathrm{erf}[(x+1)/(2\alpha t^{1/2})]$

10.3 $T = \dfrac{1}{4}(2 - y) + \dfrac{4}{\pi^2} \displaystyle\sum_{\text{odd } n} \dfrac{1}{n^2 \sinh 2n\pi} \sinh n\pi(2-y) \cos n\pi x$

10.4 $T = 20 + \dfrac{40}{\pi}\sum_{\text{odd } n}\dfrac{1}{n\sinh(3n\pi/5)}\sinh\dfrac{n\pi y}{5}\sin\dfrac{n\pi x}{5}$
$\quad + \dfrac{40}{\pi}\sum_{\text{odd } n}\dfrac{1}{n\sinh(5n\pi/3)}\sinh\dfrac{n\pi(5-x)}{3}\sin\dfrac{n\pi y}{3}$

10.6 $u = 20 - \dfrac{80}{\pi}\sum_0^\infty \dfrac{(-1)^n}{2n+1}e^{-[(2n+1)\pi\alpha/(2l)]^2 t}\cos\left(\dfrac{2n+1}{2l}\pi x\right)$

10.8 $u = 20 - x - \dfrac{40}{\pi}\sum_{\text{even } n}\dfrac{1}{n}e^{-(n\pi\alpha/10)^2 t}\sin\dfrac{n\pi x}{10}$

10.10 $u = \dfrac{1600}{\pi^2}\sum_{\text{odd } n}\sum_{\text{odd } m}\dfrac{1}{nmI_n(3m\pi/20)}I_n\left(\dfrac{m\pi r}{20}\right)\sin n\theta\sin\dfrac{m\pi z}{20}$

10.16 $v\sqrt{5}/(2\pi)$

10.18 ν_{mn}, $n = 3, 6, \cdots$; the lowest frequencies are:
$\nu_{13} = 2.65\,\nu_{10}$, $\nu_{23} = 4.06\,\nu_{10}$, $\nu_{16} = 4.13\,\nu_{10}$, $\nu_{33} = 5.4\,\nu_{10}$

10.20 $\nu = v\lambda_l/(2\pi a)$ where λ_l = zeros of j_l, a = radius of sphere, v = speed of sound

10.22 $u = 1 - \frac{1}{2}rP_1(\cos\theta) + \frac{7}{8}r^3P_3(\cos\theta) - \frac{11}{16}r^5P_5(\cos\theta)\cdots$

10.26 $\nu = [v/(2\pi)][(k_{mn}/a)^2 + \lambda^2]^{1/2}$ where k_{mn} is a zero of J_n

Chapter 14

1.1 $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$

1.3 $u = x$, $v = -y$

1.4 $u = (x^2+y^2)^{1/2}$, $v = 0$

1.7 $u = \cos y\cosh x$, $v = \sin y\sinh x$

1.9 $u = x/(x^2+y^2)$, $v = -y/(x^2+y^2)$

1.11 $u = 3x/[x^2+(y-2)^2]$, $v = (-2x^2-2y^2+5y-2)/[x^2+(y-2)^2]$

1.13 $u = \ln(x^2+y^2)^{1/2}$, $v = 0$

1.17 $u = \cos x\cosh y$, $v = \sin x\sinh y$

1.18 $u = \pm 2^{-1/2}[(x^2+y^2)^{1/2}+x]^{1/2}$, $v = \pm 2^{-1/2}[(x^2+y^2)^{1/2}-x]^{1/2}$, where the $\pm$ signs are chosen so that uv has the sign of y.

1.19 $u = \ln(x^2+y^2)^{1/2}$, $v = \arctan(y/x)$
[The angle is in the quadrant of the point (x, y).]

In 2.1–2.23, A = analytic, N = not analytic

2.1	A	2.3	N	2.4	N	2.7	A
2.9	A, $z \neq 0$	2.11	A, $z \neq 2i$	2.13	N	2.17	N
2.18	A, $z \neq 0$	2.19	A, $z \neq 0$	2.23	A, $z \neq 0$		

2.34 $-z - \frac{1}{2}z^2 - \frac{1}{3}z^3\cdots$, $|z| < 1$

2.38 $-\frac{1}{2}i + \frac{1}{4}z + \frac{1}{8}iz^2 - \frac{1}{16}z^3\cdots$, $|z| < 2$

2.42 $z + z^3/3! + z^5/5!\cdots$, all z

2.48	Yes, $z \neq 0$	2.52	No	2.53	Yes, $z \neq 0$
2.54	$-iz$	2.56	$-iz^2/2$	2.59	e^z
2.60	$2\ln z$	2.63	$-i/(1-z)$		

3.1	$\frac{1}{2}+i$	3.3	0	3.5	-1
3.7	$\pi(1-i)/8$	3.9	1	3.12	(a) $\frac{5}{3}(1+2i)$
3.17	(a) 0 (b) $i\pi$	3.19	$16i\pi$	3.23	$72i\pi$

4.4 For $0 < |z| < 1$: $-\frac{1}{4}z^{-1} - \frac{1}{2} - \frac{11}{16}z - \frac{13}{16}z^2\cdots$; $R(0) = -\frac{1}{4}$
For $1 < |z| < 2$: $\cdots + z^{-3} + z^{-2} + \frac{3}{4}z^{-1} + \frac{1}{2} + \frac{5}{16}z + \frac{3}{16}z^2\cdots$
For $|z| > 2$: $z^{-4} + 5z^{-5} + 17z^{-6} + 49z^{-7}\cdots$

4.6 For $0 < |z| < 1$: $z^{-2} - 2z^{-1} + 3 - 4z + 5z^2 \cdots$; $R(0) = -2$
For $|z| > 1$: $z^{-4} - 2z^{-5} + 3z^{-6} \cdots$
4.8 For $|z| < 1$: $-5 + \frac{25}{6}z - \frac{175}{36}z^2 \cdots$; $R(0) = 0$
For $1 < |z| < 2$: $-5(\cdots + z^{-3} - z^{-2} + z^{-1} + \frac{1}{6}z + \frac{1}{36}z^2 + \frac{7}{216}z^3 \cdots)$
For $2 < |z| < 3$: $\cdots + 3z^{-3} + 9z^{-2} - 3z^{-1} + 1 - \frac{1}{3}z + \frac{1}{9}z^2 - \frac{1}{27}z^3 \cdots$
For $|z| > 3$: $30(z^{-3} - 2z^{-4} + 9z^{-5} \cdots)$

4.9	(a) Regular	(b) Pole of order 3
4.10	(b) Pole of order 2	(d) Essential singularity
4.11	(c) Simple pole	(d) Pole of order 3
4.12	(b) Pole of order 2	(d) Pole of order 1

6.1 $z^{-1} - 1 + z - z^2 \cdots$; $R = 1$
6.3 $z^{-3} - z^{-1}/3! + z/5! \cdots$; $R = -\frac{1}{6}$
6.5 $\frac{1}{2}e[(z-1)^{-1} + \frac{1}{2} + \frac{1}{4}(z-1)\cdots]$; $R = \frac{1}{2}e$
6.7 $\frac{1}{4}[(z-\frac{1}{2})^{-1} - 1 + (1 - \pi^2/2)(z - \frac{1}{2})\cdots]$; $R = \frac{1}{4}$
6.9 $-[(z-2)^{-1} + 1 + (z-2) + (z-2)^2 \cdots]$; $R = -1$

6.14	$R(-2/3) = 1/8$, $R(2) = -1/8$	6.16	$R(0) = -2$, $R(1) = 1$
6.18	$R(3i) = \frac{1}{2} - \frac{1}{3}i$	6.19	$R(\pi/2) = 1/2$
6.21	$R[\sqrt{2}\,(1+i)] = \sqrt{2}\,(1-i)/16$	6.22	$R(i\pi) = -1$
6.27	$R(\pi/6) = -1/2$	6.28	$R(3i) = -\frac{1}{16} + \frac{1}{24}i$
6.31	$R(0) = 9/2$	6.33	$R(\pi) = -1/2$
6.35	$R(i) = 0$	6.14′	$\pi i/4$
6.16′	$-2\pi i$	6.18′	0
6.19′	0	6.27′	$-\pi i$
6.28′	$\pi i/4$	6.31′	$9\pi i$
6.33′	0	6.35′	0

7.1	$\pi/6$	7.3	$2\pi/3$
7.5	$\pi/(1-r^2)$	7.7	$\pi/6$
7.9	$2\pi/\lvert\sin\alpha\rvert$	7.11	$3\pi/32$
7.13	$\pi/10$	7.15	$\pi e^{-4/3}/12$
7.17	$(\pi/e)(\cos 2 + 2\sin 2)$	7.19	$\pi e^{-3}/54$
7.23	$\pi/8$	7.24	π
7.26	$-\pi/2$	7.28	$\pi/4$
7.30	$\pi/(2\sqrt{2})$	7.32	$\frac{3}{16}\pi\sqrt{2}$
7.33	$\pi\sqrt{2}/2$	7.36	$-\pi^2\sqrt{2}$
7.39	2	7.41	$(2\pi)^{1/2}/4$

7.45 One negative real, one each in quadrants I and IV
7.48 Two each in quadrants I and IV
7.50 Two each in quadrants II and III

7.52	πi	7.54	$8\pi i$
7.55	$\cosh t \cos t$	7.57	$1 + \sin t - \cos t$
7.60	$t + e^{-t} - 1$	7.61	$\left(\cosh 2t + 2\cosh t \cos t\sqrt{3}\right)/3$
7.63	$(\cosh t - \cos t)/2$	7.65	$(\cos 2t + 2\sin 2t - e^{-t})/5$

8.3	Regular, $R = -1$	8.5	Regular, $R = -1$
8.7	Simple pole, $R = -2$	8.9	Regular, $R = 0$
8.11	Regular, $R = -1$	8.14	$-2\pi i$

9.3 $u = x/(x^2+y^2)$, $v = -y/(x^2+y^2)$
9.4 $u = e^x \cos y$, $v = e^x \sin y$
9.7 $u = \sin x \cosh y$, $v = \cos x \sinh y$

10.6 $T = 100y/(x^2+y^2)$; isothermals $y/(x^2+y^2) = \text{const.}$; flow lines $x/(x^2+y^2) = \text{const.}$
10.9 Streamlines $y - y/(x^2+y^2) = \text{const.}$
10.12 $T = (20/\pi)\arctan[2y/(1-x^2-y^2)]$, arc tan between $\pi/2$ and $3\pi/2$
10.14 $\Phi = \frac{1}{2}V_0 \ln\{[(x+1)^2+y^2]/[(x-1)^2+y^2]\}$
$\Psi = V_0 \arctan\{2y/[1-x^2-y^2]\}$, arc tan between $\pi/2$ and $3\pi/2$
$V_x = 2V_0(1-x^2+y^2)/[(1-x^2+y^2)^2+4x^2y^2]$,
$V_y = -4V_0xy/[(1-x^2+y^2)^2+4x^2y^2]$

11.2 $-i\ln(1+z)$ 11.5 $R(i) = \frac{1}{4}(1-i\sqrt{3})$, $R(-i) = -\frac{1}{2}$
11.8 $R(1/2) = 1/2$ 11.10 -1 11.12 $1/2$
11.14 (a) 2 (b) $-\sin 5$ (c) $1/16$ (d) -2π 11.16 $-\pi/6$
11.18 $\frac{1}{4}\pi e^{-\pi/2}$ 11.20 $\frac{1}{2}\pi(e^{-1}+\sin 1)$
11.29 $\pi^3/8$. Caution: $-\pi^3/8$ is wrong.
11.32 One negative real, one each in quadrants II and III
11.34 Two each in quadrants I and IV, one each in II and III
11.41 $\pi^2/8$

Chapter 15

1.2 3/8, 1/8, 1/4 1.5 1/4, 3/4, 1/3, 1/2
1.6 27/52, 16/52, 15/52 1.8 9/100, 1/10, 3/100, 1/10

2.12 (a) 3/4 (b) 1/5 (c) 2/3 (d) 3/4 (e) 3/7
2.14 (a) 3/4 (b) 25/36 (c) 37, 38, 39, 40
2.17 (a) 3 to 9 with $p(5) = p(7) = 2/9$; others, $p = 1/9$. (c) 1/3

3.4 (a) 8/9, 1/2 (b) 3/5, 1/11, 2/3, 2/3, 6/13 3.5 1/33, 2/9
3.12 (a) 1/49 (b) 68/441 (c) 25/169 (d) 15 times (e) 44/147
3.14 $n > 3.3$, so 4 tries are needed. 3.16 9/23
3.17 (a) 39/80, 5/16, 1/5, 11/16 (b) 374/819 (c) 185/374
3.20 5/7, 2/7, 11/14 3.21 2/3, 1/3

4.1 (a) $P(10,8)$ (b) $C(10,8)$ (c) 1/45
4.4 1.98×10^{-3}, 4.95×10^{-4}, 3.05×10^{-4}, 1.39×10^{-5} 4.7 1/26
4.8 1/221, 1/33, 1/17 4.11 0.097, 0.37, 0.67; 13
4.17 MB: 16, FD: 6, BE: 10

5.1 $\mu = 0$, $\sigma = \sqrt{3}$ 5.3 $\mu = 2$, $\sigma = \sqrt{2}$
5.5 $\mu = 1$, $\sigma = \sqrt{7/6}$ 5.7 $\mu = 3(2p-1)$, $\sigma = 2\sqrt{3p(1-p)}$

6.1 (c) $\bar{x} = 0$, $\sigma = 2^{-1/2}a$ 6.4 $\bar{x} = 0$, $\sigma = (2^{1/2}\alpha)^{-1}$
6.5 $f(t) = \lambda e^{-\lambda t}$, $F(t) = 1 - e^{-\lambda t}$, $\bar{t} = 1/\lambda$, half life $= \bar{t}\ln 2$

6.7 (a) $F(s) = 2[1 - \cos(s/R)]$, $f(s) = (2/R)\sin(s/R)$
(b) $F(s) = [1 - \cos(s/R)]/[1 - \cos(1/R)] \cong s^2$,
$f(s) = R^{-1}[1 - \cos(1/R)]^{-1}\sin(s/R) \cong 2s$

	n	Exactly $7\ h$	At most $7\ h$	At least $7\ h$	Most probable number of h	Expected number of h
7.1	7	0.0078	1	0.0078	3 or 4	7/2
7.2	12	0.193	0.806	0.387	6	6

In the following answers, the first number is the binomial result and the second number is the normal approximation using whole steps at the ends as in Example 2.

8.12 0.03987, 0.03989
8.14 0.9546, 0.9546
8.17 0.0770, 0.0782
8.18 0.372, 0.376
8.20 0.462, 0.455

9.3

Number of particles	0	1	2	3	4	5
Number of intervals	406	812	812	541	271	108

9.5 $P_0 = 0.37$, $P_1 = 0.37$, $P_2 = 0.18$, $P_3 = 0.06$
9.8 3, 10, 3
9.11 Normal: 0.08, Poisson: 0.0729, (binomial: 0.0732)

10.8 $\bar{x} = 5$, $\bar{y} = 1$, $s_x = 0.122$, $s_y = 0.029$,
$\sigma_x = 0.131$, $\sigma_y = 0.030$, $\sigma_{mx} = 0.046$, $\sigma_{my} = 0.0095$,
$r_x = 0.031$, $r_y = 0.0064$,
$\overline{x+y} = 6$ with $r = 0.03$, $\overline{xy} = 5$ with $r = 0.04$,
$\overline{x^3 \sin y} = 105$ with $r = 2.00$, $\overline{\ln x} = 1.61$ with $r = 0.006$
10.10 $\bar{x} = 6$ with $r = 0.062$, $\bar{y} = 3$ with $r = 0.067$,
$\overline{e^y} = 20$ with $r = 1.3$, $\overline{x/y^2} = 0.67$ with $r = 0.03$

11.3 20/47
11.7 $\bar{x} = 1/4$, $\sigma = \sqrt{3}/4$
11.9 (d) $\bar{x} = 1/4$, $\sigma = \sqrt{31}/12$
11.13 30, 60
11.17 $\bar{x} = 2$ with $r = 0.073$, $\bar{y} = 1$ with $r = 0.039$, $\overline{x-y} = 1$ with $r = 0.08$,
$\overline{xy} = 2$ with $r = 0.11$, $\overline{x/y^3} = 2$ with $r = 0.25$

Index