18.3.3 PRACTICAL CALCULATION OF MEAN DEVIATION

In calculating a mean deviation, the following short-cuts usually turn out to be useful, especially for larger collections of values:

(a) If a constant, k, is subtracted from each of the values x_i (i = 1,2,3...n), and also we use the "fictitious" arithmetic mean, $\overline{x}-k$, in the formula, then the mean deviation is unaffected.

Proof:

$$\frac{1}{n}\sum_{i=1}^{n} |x_i - \overline{x}| = \frac{1}{n}\sum_{i=1}^{n} |(x_i - k) - (\overline{x} - k)|.$$

(b) If we divide each of the values x_i (i = 1,2,3,...n) by a positive constant, l, and also we use the "fictitious" arithmetic mean $\frac{\overline{x}}{l}$, then the mean deviation will be divided by l.

Proof:

$$\frac{1}{ln}\sum_{i=1}^{n} |x_i - \overline{x}| = \frac{1}{n}\sum_{i=1}^{n} \left|\frac{x_i}{l} - \frac{\overline{x}}{l}\right|.$$

Summary
If we code the data using both a subtraction by k and a division by l, the value obtained from the mean deviation formula needs to multiplied by l to give the correct value.

18.3.4 THE ROOT MEAN SQUARE (OR STANDARD) DEVIATION

A more common method of measuring dispersion, which ensures that negative deviations from the arithmetic mean do not tend to cancel out positive deviations, is to determine the arithmetic mean of their squares, and then take the square root.

DEFINITION
The **"root mean square deviation"** (*or "standard deviaton"*) is defined by the formula

$$\text{R.M.S.D.} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{x})^2}.$$

Notes:

(i) The root mean square deviation is usually denoted by the symbol, σ.

(ii) The quantity σ^2 is called the **"variance"**.

18.3.5 PRACTICAL CALCULATION OF THE STANDARD DEVIATION

In calculating a standard deviation, the following short-cuts usually turn out to be useful, especially for larger collections of values:

(a) If a constant, k, is subtracted from each of the values x_i (i = 1,2,3...n), and also we use the "fictitious" arithmetic mean, $\overline{x} - k$, in the formula, then σ is unaffected.

Proof:

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}[(x_i - k) - (\overline{x} - k)]^2}.$$

(b) If we divide each of the values x_i (i = 1,2,3,...n) by a constant, l, and also we use the "fictitious" arithmetic mean $\frac{\overline{x}}{l}$, then σ will be divided by l.

Proof:

$$\frac{1}{l}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{l} - \frac{\overline{x}}{l}\right)^2}.$$

Summary
If we code the data using both a subtraction by k and a division by l, the value obtained from the standard deviation formula needs to multiplied by l to give the correct value, σ.

(c) For the calculation of the standard deviation, whether by coding or not, a more convenient formula may be obtained by expanding out the expression $(x_i - \overline{x})^2$ as follows:

$$\sigma^2 = \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2\overline{x}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \overline{x}^2\right].$$

That is,

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\ x_i^2 - 2\overline{x}^2 + \overline{x}^2,$$

which gives the formula

$$\sigma = \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n}\ x_i^2\right) - \overline{x}^2}.$$

Note:
In advanced statistical work, the above formulae for standard deviation are used only for descriptive problems in which we know every member of a collection of observations.

For inference problems, it may be shown that the standard deviation of a sample is always smaller than that of a total population; and the basic formula used for a sample is

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\ (x_i - \overline{x})^2}.$$

18.3.6 OTHER MEASURES OF DISPERSION

We mention here, briefly, two other measures of dispersion:

(i) The Range

This is the difference between the highest and the smallest members of a collection of values.

(ii) The Coefficient of Variation

This is a quantity which expresses the standard deviation as a percentage of the arithmetic mean. It is given by the formula

$$\text{C.V.} = \frac{\sigma}{\overline{x}} \times 100.$$

EXAMPLE

The following grouped frequency distribution table shows the diameter of 98 rivets:

Class Intvl.	Cls. Mid Pt. x_i	Freq. f_i	Cum. Freq.	$(x_i - 6.61) \div 0.02 = {x_i}'$	$f_i {x_i}'$	${x_i}'^2$	$f_i {x_i}'^2$	$f_i \lvert {x_i}' - \overline{x}' \rvert$
$6.60 - 6.62$	6.61	1	1	0	0	0	0	0.58
$6.62 - 6.64$	6.63	4	5	1	4	1	4	2.40
$6.64 - 6.66$	6.65	6	11	2	12	4	24	3.72
$6.66 - 6.68$	6.67	12	23	3	36	9	108	7.68
$6.68 - 6.70$	6.69	5	28	4	20	16	80	3.30
$6.70 - 6.72$	6.71	10	38	5	50	25	250	6.80
$6.72 - 6.74$	6.73	17	55	6	102	36	612	11.90
$6.74 - 6.76$	6.75	10	65	7	70	49	490	7.20
$6.76 - 6.78$	6.77	14	79	8	112	64	896	10.36
$6.78 - 6.80$	6.79	9	88	9	81	81	729	6.84
$6.80 - 6.82$	6.81	7	95	10	70	100	700	5.46
$6.82 - 6.84$	6.83	2	97	11	22	121	242	1.60
$6.84 - 6.86$	6.85	1	98	12	12	144	144	0.82
Totals		98			591		4279	68.66

Estimate the arithmetic mean, the standard deviation and the mean (absolute) deviation of these diameters.

Solution

$$\text{Fictitious arithmetic mean} = \frac{591}{98} \simeq 6.03$$

$$\text{Actual arithmetic mean} = 6.03 \times 0.02 + 6.61 \simeq 6.73$$

$$\text{Fictitious standard deviation} = \sqrt{\frac{4279}{98} - 6.03^2} \simeq 2.70$$

$$\text{Actual standard deviation} = 2.70 \times 0.02 \simeq 0.054$$

$$\text{Fictitious mean deviation} = \frac{68.66}{98} \simeq 0.70$$

$$\text{Actual mean deviation} \simeq 0.70 \times 0.02 \simeq 0.014$$

18.3.7 EXERCISES

1. For the collection of numbers

$$6.5,\ 8.3,\ 4.7,\ 9.2,\ 11.3,\ 8.5,\ 9.5,\ 9.2$$

   calculate (correct to two places of decimals) the arithmetic mean, the standard deviation and the mean (absolute) deviation.

2. Estimate the arithmetic mean, the standard deviation and the mean (absolute) deviation for the following grouped frequency distribution table:

Class Interval	10 – 30	30 – 50	50 – 70	70 – 90	90 – 110	110 – 130
Frequency	5	8	12	18	3	2

3. The following table shows the registered speeds of 100 speedometers at 30m.p.h.

Regd. Speed	Frequency
27.5 – 28.5	2
28.5 – 29.5	9
29.5 – 30.5	17
30.5 – 31.5	26
31.5 – 32.5	24
32.5 – 33.5	16
33.5 – 34.5	5
34.5 – 35.5	1

   Estimate the arithmetic mean, the standard deviation and the coefficient of variation.

18.3.8 ANSWERS TO EXERCISES

1. Arithmetic mean $\simeq 8.40$, standard deviation $\simeq 1.88$, mean deviation $\simeq 1.43$
2. Arithmetic mean $\simeq 65$, standard deviation $\simeq 24.66$, mean deviation $\simeq 20.21$
3. Arithmetic mean $\simeq 31.4$, standard deviation $\simeq 1.44$, coefficient of variation $\simeq 4.61$.

2004

"JUST THE MATHS"

UNIT NUMBER

18.4

STATISTICS 4
(The principle of least squares)

by

A.J.Hobson

UNIT 18.4 - STATISTICS 4

THE PRINCIPLE OF LEAST SQUARES

18.4.1 THE NORMAL EQUATIONS

Suppose two variables, x and y, are known to obey a **"straight line law"** of the form $y = a + bx$, where a and b are constants to be found.

Suppose also that, in an experiment to test this law, we obtain n pairs of values, (x_i, y_i), where i = 1,2,3,...,n.

If the values x_i are **assigned** values, they are likely to be free from error, whereas the **observed** values, y_i will be subject to experimental error.

The principle underlying the straight line of **"best fit"** is that, in its most likely position, the sum of the squares of the y-deviations, from the line, of all observed points is a minimum.

Using partial differentiation, it may be shown that

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \quad - - - (1)$$

and

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \quad - - - (2).$$

The statements (1) and (2) are two simultaneous equations which may be solved for a and b.

They are called the **"normal equations"**.

A simpler notation for the normal equations is

$$\Sigma\ y = na + b\Sigma\ x$$

and

$$\Sigma xy = a\Sigma x + b\Sigma\ x^2.$$

By eliminating a and b in turn, we obtain the solutions

$$a = \frac{\Sigma\ x^2.\Sigma\ y - \Sigma x.\Sigma\ xy}{n\Sigma\ x^2 - (\Sigma\ x)^2} \quad \text{and} \quad b = \frac{n\Sigma\ xy - \Sigma\ x.\Sigma\ y}{n\Sigma\ x^2 - (\Sigma\ x)^2}.$$

With these values of a and b, the straight line $y = a + bx$ is called the **"regression line of y on x"**.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

x	y	xy	x^2
45	6.53	293.85	2025
42	6.30	264.60	1764
56	9.52	533.12	3136
48	7.50	360.00	2304
42	6.99	293.58	1764
35	5.90	206.50	1225
58	9.49	550.42	3364
40	6.20	248.00	1600
39	6.55	255.45	1521
50	8.72	436.00	2500
455	73.70	3441.52	21203

The regression line of y on x thus has equation $y = a + bx$, where

$$a = \frac{(21203)(73.70) - (455)(3441.52)}{(10)(21203) - (455)^2} \simeq -0.645$$

and

$$b = \frac{(10)(3441.52) - (455)(73.70)}{(10)21203) - (455)^2} \simeq 0.176$$

Thus, $y = 0.176x - 0.645$

18.4.2. SIMPLIFIED CALCULATION OF REGRESSION LINES

A simpler method of determining the regression line of y on x for a given set of data, is to consider a temporary change of origin to the point $(\overline{x}, \overline{y})$, where $\overline{x}$ is the arithmetic mean of the values x_i and $\overline{y}$ is the arithmetic mean of the values y_i.

RESULT

The regression line of y on x contains the point $(\overline{x}, \overline{y})$.

Proof:

From the first of the normal equations,

$$\frac{\Sigma\ y}{n} = a + b\frac{\Sigma\ x}{n}.$$

That is,

$$\overline{y} = a + b\overline{x}.$$

A change of origin to the point $(\overline{x}, \overline{y})$, with new variables X and Y, is associated with the formulae

$$X = x - \overline{x} \quad \text{and} \quad Y = y - \overline{y}$$

and, in this system of reference, the regression line will pass through the origin.

Its equation is therefore

$$Y = BX,$$

where

$$B = \frac{n\Sigma\ XY - \Sigma\ X.\Sigma\ Y}{n\Sigma\ X^2 - (\Sigma\ X)^2}.$$

However,

$$\Sigma\ X = \Sigma\ (x - \overline{x}) = \Sigma\ x - \Sigma\ \overline{x} = n\overline{x} - n\overline{x} = 0$$

and

$$\Sigma\ Y = \Sigma\ (y - \overline{y}) = \Sigma\ y - \Sigma\ \overline{y} = n\overline{y} - n\overline{y} = 0.$$

Thus,

$$B = \frac{\Sigma\ XY}{\Sigma\ X^2}.$$

Note:
In a given problem, we make a table of values of x_i, y_i, X_i, Y_i, X_iY_i and X_i^2.

The regression line is then

$$y - \overline{y} = B(x - \overline{x}) \quad \text{or} \quad y = BX + (\overline{y} - B\overline{x});$$

though, there may be slight differences in the result obtained compared with that from the earlier method.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

The arithmetic mean of the x values is $\overline{x} = 45.5$

The arithmetic mean of the y values is $\overline{y} = 7.37$

This gives the following table:

x	y	$X = x - \overline{x}$	$Y = y - \overline{y}$	XY	X^2
45	6.53	-0.5	-0.84	0.42	0.25
42	6.30	-3.5	-1.07	3.745	12.25
56	9.52	10.5	2.15	22.575	110.25
48	7.50	2.5	0.13	0.325	6.25
42	6.99	-3.5	-0.38	1.33	12.25
35	5.90	-10.5	-1.47	15.435	110.25
58	9.49	12.5	2.12	26.5	156.25
40	6.20	-5.5	-1.17	6.435	30.25
39	6.55	-6.5	-0.82	5.33	42.25
50	8.72	4.5	1.35	6.075	20.25
455	73.70			88.17	500.5

Hence,

$$B = \frac{88.17}{500.5} \simeq 0.176$$

and so the regression line has equation

$$y = 0.176x + (7.37 - 0.176 \times 45.5).$$

That is,

$$y = 0.176x - 0.638$$

18.4.3 EXERCISES

1. For the following tables, determine the regression line of y on x, assuming that $y = a+bx$.

(a)

x	0	2	3	5	6
y	6	-1	-3	-10	-16

(b)

x	0	20	40	60	80
y	54	65	75	85	96

(c)

x	1	3	5	10	12
y	58	55	40	37	22

2. To determine the relation between the normal stress and the shear resistance of soil, a shear-box experiment was performed, giving the following results:

Normal Stress, x p.s.i.	11	13	15	17	19	21
Shear Stress, y p.s.i.	15.2	17.7	19.3	21.5	23.9	25.4

If $y = a + bx$, determine the regression line of y on x.

3. Fuel consumption, y miles per gallon, at speeds of x miles per hour, is given by the following table:

x	20	30	40	50	60	70	80	90
y	18.3	18.8	19.1	19.3	19.5	19.7	19.8	20.0

Assuming that

$$y = a + \frac{b}{x},$$

determine the most probable values of a and b.

18.4.4 ANSWERS TO EXERCISES

1. (a)

$$y = 6.46 - 3.52x;$$

(b)

$$y = 54.20 + 0.52x;$$

(c)

$$y = 60.78 - 2.97x.$$

2.

$$y = 4.09 + 1.03x.$$

3.

$$a \simeq -42 \quad \text{and} \quad b \simeq 20.$$

"JUST THE MATHS"

UNIT NUMBER

19.1

PROBABILITY 1
(Definitions and rules)

by

A.J.Hobson

UNIT 19.1 - PROBABILITY 1 - DEFINITIONS AND RULES

19.1.1 INTRODUCTION

To introduce the definition of probability, suppose 30 high-strength bolts became mixed with 25 ordinary bolts by mistake, all of the bolts being identical in appearance.

We would like to know how sure we can be that, in choosing a bolt, it will be a high-strength one. Phrases like "quite sure" or "fairly sure" are useless, mathematically, and we define a way of measuring the certainty.

We know that, in 55 simultaneous choices, 30 will be of high strength and 25 will be ordinary; so we say that, in one choice, there is a $\frac{30}{55}$ chance of success; that is, approximately, a 0.55 chance of success.

Obviously, in one single choice, we haven't any idea what the result will be; but experience has proved that, in a significant number of choices, just over half will most likely give a high-strength bolt.

Such predictions can be used, for example, to estimate the cost of mistakes on a production line.

DEFINITION 1.
The various occurrences which are possible in a statistical problem are called **"events"**.
If we are interested in one particular event, it is termed **"successful"** when it occurs and **"unsuccessful"** when it does not.

ILLUSTRATION

If, in a collection of 100 bolts, there are 30 high-strength, 25 ordinary and 45 low-strength, then we have three possible events according to which type is chosen.
We can make 100 **"trials"** and, in each trial, one of three events will occur.

DEFINITION 2.
If, in n possible trials, a successful event occurs s times, then the number $\frac{s}{n}$ is called the **"probability of success in a single trial"**. It is also known as the **"relative frequency of success"**.

ILLUSTRATIONS

1. From a bag containing 7 black balls and 4 white balls, the probability of drawing a white ball is $\frac{4}{11}$.

2. In tossing a perfectly balanced coin, the probability of obtaining a head is $\frac{1}{2}$.
3. In throwing a die, the probability of getting a six is $\frac{1}{6}$.
4. If 50 chocolates are identical in appearance, but consist of 15 soft-centres and 35 hard-centres, the probability of choosing a soft-centre is $\frac{15}{50} = 0.3$.

19.1.2 APPLICATION OF PROBABILITY TO GAMES OF CHANCE

If a competitor in a game of chance has a probability, p, of winning, and the prize money is £m, then £mp is considered to be a fair price for entry to the game.

The quantity mp is known as the **"expectation"** of the competitor.

19.1.3 EMPIRICAL PROBABILITY

So far, all the problems discussed on probability have been "descriptive"; that is, we know all the possible events, the number of successes and the number of failures. In other problems, called **"inference"** problems, it is necessary either (a) to take **"samples"** in order to infer facts about a total **"population'** (for example, a public census or an investigation of moon-rock); or (b) to rely on past experience (for example past records of heart deaths, road accidents, component failure).

If the probability of success, used in a problem, has been inferred by samples or previous experience, it is called **"empirical probability"**.

However, once the probability has been calculated, the calculations are carried out in the same way as for descriptive problems.

19.1.4 TYPES OF EVENT

DEFINITION 3.
If two or more events are such that not more than one of them can occur in a single trial, they are called "mutually exclusive".

ILLUSTRATION

Drawing an Ace or drawing a King from a pack of cards are mutually exclusive events; but drawing an Ace and drawing a Spade are not mutually exclusive events.

DEFINITION 4.
If two or more events are such that the probability of any one of them occurring is not affected by the occurrence of another, they are called **"independent"** events.

ILLUSTRATION

From a pack of 52 cards (that is, Jokers removed), the event of drawing and immediately replacing a red card will have a probability of $\frac{26}{52} = 0.5$; and the probability of this occurring a second time will be exactly the same. They are independent events.

However, two successive events of drawing a red card **without** replacing it are **not** independent. If the first card drawn is red, the probability that the second is red will be $\frac{25}{51}$; but, if the first card drawn is black, the probability that the second is red will be $\frac{26}{51}$

19.1.5 RULES OF PROBABILITY

1. If $p_1, p_2, p_3, ..., p_r$ are the separate probabilities of r mutually exclusive events, then the probability that some **one** of the r events will occur is

$$p_1 + p_2 + p_3 + ... + p_r.$$

ILLUSTRATION

Suppose a bag contains 100 balls of which 1 is red, 2 are blue and 3 are black. The probability of choosing any one of these three colours will be

$$0.06 = 0.01 + 0.02 + 0.03$$

However, the probability of drawing a spade or an ace from a pack of 52 cards will not be $\frac{13}{52} + \frac{4}{52} = \frac{17}{52}$ but $\frac{16}{52}$ since there are just 16 cards which are either a spade or an ace.

2. If $p_1, p_2, p_3, ..., p_r$ are the separate probabilities of r independent events, then the probability that **all** will occur in a single trial is

$$p_1.p_2.p_3......p_r.$$

ILLUSTRATION

Suppose there are three bags, each containing white, red and blue balls. Suppose also that the probabilities of drawing a white ball from the first bag, a red ball from the second bag and a blue ball from the third bag are respectively p_1, p_2 and p_3. The probability of making these three choices in succession is $p_1.p_2.p_3$ because they are independent events.

However, if three cards are drawn, without replacing, from a pack of 52 cards, the probability of drawing a 3, followed by an ace, followed by a red card will not be $\frac{4}{52}.\frac{4}{52}.\frac{26}{52}$.

19.1.6 CONDITIONAL PROBABILITIES

For dependent events, the multiplication rule requires a knowledge of the **new** probabilities of successive events in the trial, after the previous ones have been dealt with. These are called **"conditional probabilitities"**.

EXAMPLE

From a box, containing 6 white balls and 4 black balls, 3 balls are drawn at random without replacing them. What is the probability that there will be 2 white and 1 black ?

Solution

The cases to consider, together with their probabilities are as follows:

(a) White, White, Black..............Probability $= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720} = \frac{1}{6}$.

(b) Black, White, White..............Probability $= \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$.

(c) White, Black, White..............Probability $= \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$.

The probability of any one of these three outcomes is therefore

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

19.1.7 EXERCISES

1. A card is drawn at random from a deck of 52 playing cards. What is the probability that it is either an Ace or a picture card ?
2. If a die is rolled, what is the probability that the roll yields either a 3 or a 4 ?
3. In a single throw of two dice, what is the probability that a 9 or a doublet will be thrown ?
4. Ten balls, numbered 1 to 10, are placed in a bag. One ball is drawn and not replaced; and then a second ball is drawn. What are the probabilities that
   (a) the balls numbered 3 and 7 are drawn;
   (b) neither of these two balls are drawn ?
5. On a gaming machine, there are three reels with ten digits 0,1,2,3,4,5,6,7,8,9 plus a star on each reel. When a coin is inserted, and the machine started, the three reels revolve independently before coming to rest.

(a) What is the probabiity of getting a particular sequence of numbers ?
(b) What is the probability of getting three stars ?
(c) What is the probability of getting the same number on each reel ?

6. Three balls in succession are drawn, without replacement, from a bag containing 8 black, 8 white and 8 red balls. If a prize of £5 is awarded for drawing no black balls, what is the expectation ?
7. Three persons A,B and C take turns to throw three dice once. If the first one to throw a total of 11 is awarded a prize of £200, what are the expectations of A,B and C ?

19.1.8 ANSWERS TO EXERCISES

1. $\frac{4}{13}$.
2. $\frac{1}{3}$.
3. $\frac{5}{18}$.
4. (a)
$$\frac{1}{45};$$
(b)
$$\frac{28}{45}.$$
5. (a)
$$\frac{1}{11^3} \simeq 0.00075;$$
(b)
$$\frac{1}{11^3} \simeq 0.00075;$$
(c)
$$\frac{10}{11^3} \simeq 0.0075$$
6. The expectation is £1.38
7. For A,B and C, the expectations are £25, £21.88 and £19.14 respectively since the probabilities are $\frac{1}{8}$, $\frac{7}{8^2}$ and $\frac{7}{8^3}$ respectively.

"JUST THE MATHS"

UNIT NUMBER

19.2

PROBABILITY 2
(Permutations and combinations)

by

A.J.Hobson

UNIT 19.2 - PROBABILITY 2 - PERMUTATIONS AND COMBINATIONS

19.2.1 INTRODUCTION

In Unit 19.1, we saw that, in the type of problem known as "descriptive", we can work out the probability that an event will occur by counting up the total number of possible trials and the number of successful ones amongst them. But this can often be a tedious process without the results of the work which is included in the present Unit.

DEFINITION 1.
Each different arrangement of all or part of a set of objects is called a **"permutation"**.

DEFINITION 2.
Any set which can be made by using all or part of a given collection of objects, without regard to order, is called a **"combination"**.

EXAMPLES

1. Nine balls, numbered 1 to 9, are put into a bag, then emptied into a channel which guides them into a line of pockets. What is the probability of obtaining a particular nine digit number ?

   Solution

   We require the total number of arrangements of the nine digits and this is evaluated as follows:

   There are nine ways in which a digit can appear in the first pocket and, for each of these ways, there are then eight choices for the second pocket. Hence, the first two pockets can be filled in $9 \times 8 = 72$ ways.

   Continuing in this manner, the total number of arrangements will be

   $$9 \times 8 \times 7 \times 6 \times ...... \times 3 \times 2 \times \times 1 = 362880 = T, \text{ (say)}.$$

   This is the number of permutations of the ten digits and the required probability is therefore equal to $\frac{1}{T}$.

2. A box contains five components of identical appearance but different qualities. What is the probability of choosing a pair of components from the highest two qualities ?

   Solution

   Method 1.

   Let the components be A, B, C, D, E in order of descending quality.

The choices are
AB AC AD AE
BC BD BE
CD CE
DE,

giving ten choices.

These are the various combinations of five objects, two at a time; hence, the probability for AB = $\frac{1}{10}$.

Method 2.

We could also use the ideas of conditional probability as follows:

The probability of drawing A = $\frac{1}{5}$.
The probability of drawing B without replacing A = $\frac{1}{4}$.
The probability of drawing A and B **in either order** =

$$2 \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{10}.$$

19.2.2 RULES OF PERMUTATIONS AND COMBINATIONS

1. The number of permutations of all n objects in a set of n is

$$n(n-1)(n-2)....................3.2.1$$

   which is denoted for short by the symbol n! It is called **"n factorial"**.

   This rule was demonstrated in Example 1 of the previous section.

2. The number of permutations of n objects r at a time is given by

$$n(n-1)(n-2)....................(n-r+1) = \frac{n!}{(n-r)!}$$

   Explanation

   The first object can be chosen in any one of n different ways.
   For each of these, the second object can then be chosen in $n-1$ ways.
   For each of these, the third object can then be chosen in $n-2$ ways.
   ...
   ...
   ...
   For each of these, the r-th object can be chosen in $n-(r-1) = n-r+1$ ways.

Note:
In Example 2 of the previous section, the number of permutations of five components two at a time is given by

$$\frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5.4.3.2.1}{3.2.1} = 20.$$

This is double the number of choices we obtained for any two components out of five because, in a permutation, the order matters.

3. The number of combinations of n objects r at a time is given by

$$\frac{n!}{(n-r)!r!}$$

Explanation
This is very much the same problem as the number of permutations of n objects r at a time; but, as permutations, a particular set of objects will be counted $r!$ times. In the case of combinations, such a set will be counted only once, which reduces the number of possibilities by a factor of $r!$.

In Example 2 of the previous section, it is precisely the number of combinations of five objects two at a time which is being calculated. That is,

$$\frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5.4.3.2.1}{3.2.1.2.1} = 10,$$

as before.

Note:
A traditional notation for the number of permutations of n objects r at a time is ${}^nP_r$. That is,

$${}^nP_r = \frac{n!}{(n-r)!}$$

A traditional notation for the number of combinations of n objects r at a time is ${}^nC_r$. That is,

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

EXAMPLES

1. How many four digit numbers can be formed from the numbers 1,2,3,4,5,6,7,8,9 if no digit can be repeated ?

   Solution

   This is the number of permutations of 9 objects four at a time; that is,

   $$\frac{9!}{5!} = 9.8.7.6. = 3,024.$$

2. In how many ways can a team of nine people be selected from twelve ?

   Solution

   The required number is

   $${}^{12}C_9 = \frac{12!}{3!9!} = \frac{12.11.10}{3.2.1} = 220.$$

3. In how many ways can we select a group of three men and two women from five men and four women ?

   Solution

   The number of ways of selecting three men from five men is

   $${}^{5}C_3 = \frac{5!}{2!3!} = \frac{5.4.}{2.1} = 10.$$

   For each of these ways, the number of ways of selecting two women from four women is

   $${}^{4}C_2 - \frac{4!}{2!2!} = \frac{4.3}{2.1} = 6.$$

   The total number of ways is therefore $10 \times 6 = 60$.

4. What is the probability that one of four bridge players will obtain a thirteen card suit ?

   Solution

   The number of possible suits for each player is

   $$N = {}^{52}C_{13} = \frac{52!}{39!13!}$$

   The probabilty that any one of the four players will obtain a thirteen card suit is thus

   $$4 \times \frac{1}{N} = \frac{4.(39!)(13!)}{52!} \simeq 6.29 \times 10^{-10}.$$

5. A coin is tossed six times. Determine the probablity of obtaining exactly four heads.

   Solution

   In a single throw of the coin, the probability of a head (and of a tail) is $\frac{1}{2}$.

   Secondly, the probability that a particular four out of six throws will be heads, **and** the other two tails, will be

$$\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\left[\frac{1}{2}.\frac{1}{2}\right] = \frac{1}{2^6} = \frac{1}{64}.$$

   Finally, the number of choices of four throws from six throws is

$${}^6C_4 = \frac{6!}{2!4!} = 15.$$

   Hence, the required probability of exactly four heads is $\frac{15}{64}$.

19.2.3 PERMUTATIONS OF SETS WITH SOME OBJECTS ALIKE

INTRODUCTORY EXAMPLE

Suppose twelve switch buttons are to be arranged in a row, and there are two red buttons, three yellow and seven green. How many possible distinct patterns can be formed ?

Solution

If all twelve buttons were of a different colour, there would be 12! possible arrangements.
If we now colour two switches red, there will be only half the number of arrangements since every pair of positions previously held by them would have counted 2! times; that is, twice.
If we then colour another three switches yellow, the positions previously occupied by them would have counted 3! times; that is, 6 times, so we reduce the number of arrangements further by a factor of 6.
Similarly, by colouring another seven switches green, we reduce the number of arrangements further by a factor of 7!.

Hence, the final number of arrangements will be

$$\frac{12!}{2!3!7!} = 7920.$$

This example illustrates another standard rule that, if we have n objects of which r_1 are alike of one kind, r_2 are alike of another, r_3 are alike of another ....and r_k are alike of another, then the number of permutations of these n objects is given by

$$\frac{n!}{r_1!r_2!r_3!...r_k!}$$

19.2.4 EXERCISES

1. Evaluate the following:
   (a) ${}^6P_3$; (b) ${}^6C_3$; (c) ${}^7P_4$; (d) ${}^7C_4$.
2. Verify that ${}^nC_{n-r} = {}^nC_r$ in the following cases:
   (a) $n = 7$ and $r = 2$; (b) $n = 10$ and $r = 3$; (c) $n = 5$ and $r = 2$.
3. How many four digit numbers can be formed from the digits 1,3,5,7,8,9 if none of the digits appears more than once in each number ?
4. In how many ways can three identical jobs be filled by 12 different people ?
5. From a bag containing 7 black balls and 5 white balls, how many sets of 5 balls, of which 3 are black and two are white, can be drawn ?
6. In how many seating arrangements can 8 people be placed around a table if there are 3 who insist on sitting together ?
7. A commitee of 10 is to be selected from 6 lawyers 8 engineers and 5 doctors. If the committe is to consist of 4 lawyers, 3 engineers and 3 doctors, how many such committees are possible ?
8. An electrical engineer is faced with 8 brown wires and 9 blue wires. If he is to connect 4 brown wires and 3 blue wires to 7 numbered terminals, in how many ways can this be done ?
9. A railway coach has 12 seats facing backwards and 12 seats facing forwards, In how many ways can 10 passengers be seated if 2 refuse to face forwards and 4 refuse to face backwards ?

19.2.5 ANSWERS TO EXERCISES

1. (a) 120; (b) 20; (c) 840; (d) 35.
2. (a) 21; (b) 120; (c) 10.
3. $$^{6}P_4 = 360.$$
4. $$^{12}C_3 = 220.$$
5. $$^{7}C_3 \times {}^{5}C_2 = 350.$$
6. $$3! \times 8 \times 5! = 5760.$$
7. $$^{6}C_4 \times {}^{8}C_3 \times {}^{5}C_3 = 8400.$$
8. $$^{8}C_4 \times {}^{9}C_3 \times 7! = 29,635,200.$$
9. $$^{12}P_2 \times {}^{12}P_4 \times {}^{18}P_4 = 115,165,670,400.$$

"JUST THE MATHS"

UNIT NUMBER

19.3

PROBABILITY 3
(Random variables)

by

A.J.Hobson

UNIT 19.3 - PROBABILITY 3 - RANDOM VARIABLES

19.3.1 DEFINING RANDOM VARIABLES

(i) The kind of experiments discussed in the theory of probability are usually what are known as **"random experiments"**.

For example, in an experiment which involves throwing a die, suppose that the die is "unbiased". This means that it is just as likely to show one face as any other.

Similarly, drawing 6 numbers out of a possible 45 for a lottery is a random experiment, provided it is just as likely for one number to be drawn as any other.

In general, an experiment is a random experiment if there is more than one possible outcome (or event) and any one of those possible outcomes may occur. We assume that the outcomes are mutually exclusive (see Unit 19.1, section 19.1.4).

The probabilities of the possible outcomes of a random experiment form a collection called the **"probability distribution"** of the experiment. These probabilities need not be the same as one another.

The complete list of possible outcomes is called the **"sample space"** of the experiment.

(ii) In a random experiment, each of the possible outcomes may, for convenience, be associated with a certain numerical value called a **"random variable"**. This variable, which we shall call x in general, makes it possible to refer to an outcome without having to use a complete description of it.

In tossing a coin, for instance, we might associate a head with the number 1 and a tail with the number 0; then we could state the probabilities of either a head or a tail being obtained by means of the formulae

$$P(x = 1) = 0.5 \quad \text{and} \quad P(x = 1) = 0.5$$

Note:
There is no restriction on the way we define the values of x; it would have been just as correct to associate a head with -1 and a tail with 1. But it is customary to assign the values of random variables as logically as possible. For example, in discussing the probability that two 6's would be obtained in 5 throws of a dice, we could sensibly use $x = 1, 2, 3, 4, 5$ and 6, respectively, for the results that a 1,2,3,4,5 and 6 would be thrown.

(iii) It is usually necessary to distinguish between random variables which are **"discrete"** and those which are **"continuous"**.

Discrete random variables may take only certain specified values, while continuous random variables may take any value with in a certain specified range.

Examples of discrete random variables include those associated with the tossing of coins, the throwing of dice and numbers of defective components in a batch from a production line.

Examples of continuous variables include those associated with persons' height or weight, lifetimes of manufactured components and rainfall figures.

Note:
For a random variable, x, the associated probabilities form a function, $P(x)$, called the **"probability function"**.

19.3.2 PROBABILITY DISTRIBUTION AND PROBABILITY DENSITY FUNCTIONS

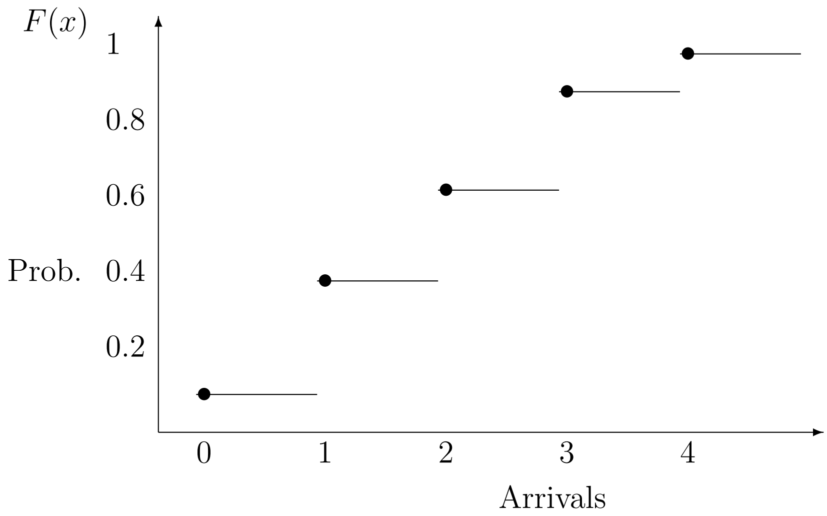
(a) Probability Distribution Functions

A **"probability distribution function"**, which will normally be denoted here by $F(x)$, is the relationship between a random variable, x, and the probability of obtaining any outcome for which the random variable can take values up to and including x. In other words, it is the probability, $P(\leq x)$, that the random variable for the outcome is less than or equal to x.

(i) Probability distribution functions for discrete variables

By way of illustration, suppose that the number of ships arriving at a container terminal during any one day can be 0,1,2,3 or 4, with respective probabilites 0.1, 0.3, 0.35, 0.2 and 0.05. The probabilites for outcomes other than those specified is taken to be zero.

The graph of the probability distribution function is as follows:



The value of the probability distribution function at a value, x, of the random variable is the sum of the probabilities to the left of, and including, x.

In view of the discontinuous nature of the graph, we have used "bullet" marks to indicate which end of each horizontal line belongs to the graph.

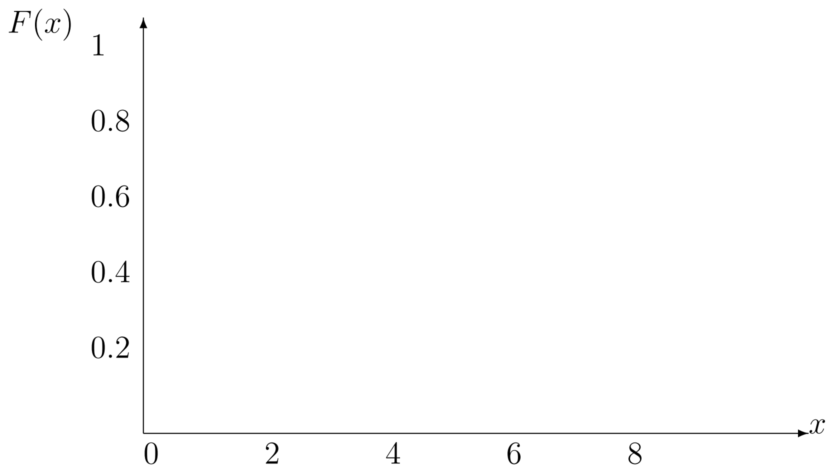
(ii) Probability distribution functions for continuous variables

For a continuous random variable, the probability distribution function is defined in a similar way as for a discrete variable. It measures (as before) the probability that the value of the random variable is less than or equal to x.

By way of illustration, we shall quote, here, the example of an **"exponential distribution"** in which it may be shown that the lifetime of a certain electronic component (in thousands of hours) is represented by a probability distribution function

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0. \end{cases}$$

The graph of the probability distribution function is as follows:



(b) Probability Density Functions

In the case of continuous random variables, a second function, $f(x)$ called the **"probability density function"** is defined to be the derivative, with respect to x, of the probability distribution function, $F(x)$.

That is,

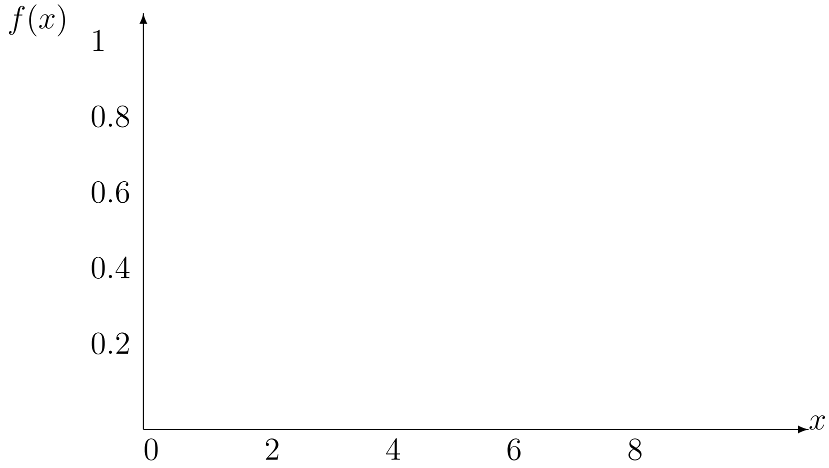
$$f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[F(x)].$$

The probability density function measures the **concentration** of possible values of x.

In the previous example, the probability density function is therefore given by

$$f(x) \equiv \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0 \end{cases}$$

The graph of the probability density function is as follows:



We may observe that most components have short lifetimes, while a small number can survive for much longer.

(c) Properties of probability distribution and probability density functions

The following properties are a consequence of the appropriate definitions:

(i)

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.$$

Proof:

It is impossible for a random variable to have a value less than $-\infty$ and it is certain to have a value less than ∞.

(ii) If $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.

Proof:

The outcomes of an experiment with random variable values up to and including x_2 includes those outcomes with random variable values up to and including x_1 so that $F(x_2)$ is as least as great as $F(x_1)$.

Note:
Results **(i)** and **(ii)** imply that, for any value of x, the probability distribution function is either constant or increasing between 0 and 1.

(iii) The probability that an outcome will have a random variable value, x, within the range $x_1 < x \leq x_2$ is given by the expression

$$F(x_2) - F(x_1).$$

Proof:

From, the outcomes of an experiment with random variable values up to and including x_2, suppose we exclude those outcomes with random variable values up to and including x_1. The residue will be those outcomes with random variable values which lie within the range $x_1 < x \leq x_2$.

Thus, the difference between the values of the probability distribution function at two particular points is the probability that the value of the random variable will either lie between those two points or will be equal to the higher of the two.

Note:
For a continuous random variable, this is also equal to the area under the graph of the probability density function between the two given points, by virtue of the definition that $f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[F(x)]$.

That is,

$$\int_{x_1}^{x_2} f(x)\ \mathrm{d}x.$$

(iv)

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1.$$

Proof:

The total area under the probability density function must be 1 since the random variable must have a value somewhere.

EXAMPLE

For the distribution of component lifetimes (in thousands of hours) given earlier by

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0, \end{cases}$$

determine the proportion of components which last longer than 3000 hours but less than or equal to 6000 hours.

Solution

The probability that components have lifetimes up to and including 3000 hours is given by

$$F(3) = 1 - e^{-\frac{3}{2}}.$$

The probability that components have lifetimes up to and including 600 hours is given by

$$F(6) = 1 - e^{-\frac{6}{2}} = 1 - e^{-3}.$$

The probability that components last longer than 3000 hours but less than or equal to 6000 hours is thus given by

$$F(6) - F(3) = e^{-\frac{3}{2}} - e^{-3} \simeq 0.173$$

The required proportion is thus approximately one in six.

19.3.3 EXERCISES

1. A coin is tossed three times and the random variable, x, represents the number of heads minus the number of tails.

   Construct a definition for the probability distribution function, $F(x)$,

   (a) if the coin is 'fair' (perfectly balanced);

   (b) if the coin is biased so that a head is twice as likely to occur as a tail.

2. Construct a definition for the probability distribution function, $F(x)$, for the sum, x, of numbers obtained when a pair of dice is tossed.

3. A certain assembly process is such that the probability of success at each attempt is 0.2. The probability, $P(x)$ that x independent attempts are needed to achieve success is given by

$$P(x) \equiv (0.2)(0.8)^{x-1} \quad x = 1,\ 2,\ 3, \ldots$$

Plot a grap of the probability distribution function, $F(x)$, and determine the probability that success will be achieved by

(a) less than four independent attempts;

(b) more than three but less than or equal to five independent attempts.

4. The probability density function of a random variable, x, is given by

$$f(x) \equiv \begin{cases} \frac{c}{\sqrt{x}} & \text{for } 0 < x < 4; \\ 0 & \text{elswhere,} \end{cases}$$

where c is a constant.

Determine

(a) the value of c;

(b) the probability distribution function, $F(x)$;

(c) the probability that $x > 1$.

5. The shelf life (in hours) of a certain perishable packaged food is a random variable, x, with probability density function, $f(x)$ given by

$$f(x) \equiv \begin{cases} 20000(x+100)^{-3} & \text{when } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probabilities that one of these packages will have a shelf life of

(a) at least 200 hours;

(b) at most 200 hours;

(c) more than 80 hours but less than or equal to 120 hours.

19.3.4 ANSWERS TO EXERCISES

1. (a)

$$F(x) \equiv \begin{cases} \frac{1}{8} & \text{for } -3 \leq x < -1; \\ \frac{1}{2} & \text{for } -1 \leq x < 1; \\ \frac{7}{8} & \text{for } 1 \leq x < 3; \\ 1 & \text{for } x \geq 3. \end{cases}$$

(b)

$$F(x) \equiv \begin{cases} \frac{1}{27} & \text{for } -3 \le x < -1; \\ \frac{7}{27} & \text{for } -1 \le x < 1; \\ \frac{19}{27} & \text{for } 1 \le x < 3; \\ 1 & \text{for } x \ge 3. \end{cases}$$

2.

$$F(x) \equiv \begin{cases} \frac{1}{36} & \text{for } 2 \le x < 3; \\ \frac{1}{12} & \text{for } 3 \le x < 4; \\ \frac{1}{6} & \text{for } 4 \le x < 5: \\ \frac{5}{18} & \text{for } 5 \le x < 6; \\ \frac{5}{12} & \text{for } 6 \le x < 7; \\ \frac{7}{12} & \text{for } 7 \le x < 8. \\ \frac{13}{18} & \text{for } 8 \le x < 9; \\ \frac{5}{6} & \text{for } 9 \le x < 10; \\ \frac{11}{12} & \text{for } 10 \le x < 11; \\ \frac{35}{36} & \text{for } 11 \le x < 12; \\ 1 & \text{for } x \ge 12. \end{cases}$$

3. (a) 0.488 (b) 0.3123

4. (a) $c = \frac{1}{4}$;
   (b)

$$F(x) = \begin{cases} 0 & \text{when } x < 0; \\ \frac{1}{2}\sqrt{x} & \text{when } 0 \le x \le 4; \\ 1 & \text{when } x > 4. \end{cases}$$

   (c) $\frac{1}{2}$

5. (a) $\frac{1}{9}$; (b) $\frac{3}{4}$; (c) 0.102

"JUST THE MATHS"

UNIT NUMBER

19.4

PROBABILITY 4
(Measures of location and dispersion)

by

A.J.Hobson

UNIT 19.4 - PROBABILITY 4
MEASURES OF LOCATION AND DISPERSION

19.4.1 COMMON TYPES OF MEASURE

We include, here three common measures of location (or central tendency), and one common measure of dispersion (or scatter), used in the discussion of probability distributions.

(a) The Mean

(i) For Discrete Random Variables

If the values $x_1,\ x_2,\ x_3,\ \ldots.\ ,\ x_n$ of a discrete random variable, x, have probabilities $P_1,\ P_2,\ P_3,\ \ldots.\ , P_n$, respectively, then P_i represents the expected frequency of x_i divided by the total number of possible outcomes. For example, if the probability of a certain value of x is 0.25, then there is a one in four chance of its occurring.

The arithmetic mean, μ, of the distribution may therefore be given by the formula

$$\mu = \sum_{i=1}^{n} x_i P_i.$$

(ii) For Continuous Random Variables

In this case, it is necessary to use the probability density function, $f(x)$, for the distribution which is the rate of increase of the probability distribution function, $F(x)$.

For a small interval, δx of x-values, the probability that any of these values occurs is approximately $f(x)\delta x$, which leads to the formula

$$\mu = \int_{-\infty}^{\infty} x f(x)\ \mathrm{d}x.$$

(b) The Median

(i) For Discrete Random Variables

The median provides an estimate of the middle value of x, taking into account the frequency at which each value occurs. More precisely, it is a value, m, of the random variable, x, for which

$$P(x \leq m) \geq \frac{1}{2} \ \textbf{and} \ P(x \geq m) \geq \frac{1}{2}.$$

The median for a discrete random variable may not be unique (see Example 1, on page 3).

(ii) For Continuous Random Variables

The median for a continuous random variable is a value of the random variable, x, for which there are equal chances of x being greater than or less than the median itself. More precisely, it may be defined as the value, m, for which $P(x \leq m) = F(m) = \frac{1}{2}$.

Note:
Other measures of location are sometimes used, such as **"quartiles"**, **"deciles"** and **"percentiles"**, which divide the range of x values into four, ten and one hundred equal parts, respectively. For example, the third quartile of a distribution function, $F(x)$, may be defined as a value, q_3, of the random variable, x, such that

$$F(q_3) = \frac{3}{4}.$$

(c) The Mode

The mode is a measure of the most likely value occurring of the random variable, x.

(i) For Discrete Random Variables

In this case, the mode is any value of x with the highest probability, and, again, it may not be unique (see Example 1, on page 3).

(ii) For Continuous Random Variables

In this case, we require a value of x for which the probability density function (measuring the concentration of x values) has a maximum.

(d) The Standard Deviation

The most common meaure of dispersion (or scatter) for a probability distribution is the **"standard deviation"**, σ.

(i) For Discrete Random Variables

In this case, the standard deviation is defined by the formula

$$\sigma = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 P(x)}.$$

(ii) For Continuous Random Variables

In this case, the standard deviation is defined by the formula

$$\sigma = \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, \mathrm{d}x},$$

where $f(x)$ denotes the probability density function.

Each measures the dispersion of the x values around the mean, μ.

Note:
σ^2 is known as the **"variance"** of the probability distribution.

EXAMPLES

1. Determine (a) the mean, (b) the median, (c) the mode and (d) the standard deviation for a simple toss of an unbiased die.

   Solution

   (a) The mean is given by

$$\mu = \sum_{i=1}^{6} i \times \frac{1}{6} = \frac{22}{6} = 3.5$$

   (b) Both 3 and 4 on the die fit the definition of a median, since

$$P(x \leq 3) = \frac{1}{2}, \quad P(x \geq 3) = \frac{2}{3}$$

   and

$$P(x \leq 4) = \frac{2}{3}, \quad P(x \geq 4) = \frac{1}{2}.$$

   (c) All six outcomes count as a mode since they all have a probability of $\frac{1}{6}$.

   (d) The standard deviation is given by

$$\sigma = \sqrt{\sum_{i=1}^{6} \frac{1}{6}(i-3.5)^2} \simeq 2.917$$

2. Determine (a) the mean, (b) the median, (c) the mode and (d) the standard deviation for the distribution function

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0. \end{cases}$$

Solution

First, we need the probability density function, $f(x)$, which is given by

$$f(x) \equiv \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ \\ 0 & \text{when } x < 0 \end{cases}$$

Hence,

(a)

$$\mu = \int_0^\infty \frac{1}{2} x e^{-\frac{x}{2}}\ \mathrm{d}x,$$

which, on integration by parts, gives

$$\mu = \left[-xe^{-\frac{x}{2}}\right]_0^\infty + \int_0^\infty e^{-\frac{x}{2}}\ \mathrm{d}x = \left[-2e^{-\frac{x}{2}}\right]_0^\infty = 2.$$

(b) The median is the value, m, for which

$$F(m) = \frac{1}{2}.$$

That is,

$$1 - e^{-\frac{m}{2}} = \frac{1}{2},$$

giving

$$-\frac{m}{2} = \ln\left[\frac{1}{2}\right];$$

and, hence, $m \simeq 1.386$.

(c) The mode is zero, since the maximum value of the probability density function occurs when $x = 0$.

(d) The standard deviation is given by

$$\sigma^2 = \int_0^\infty \frac{1}{2}(x-2)^2 e^{-\frac{x}{2}}\ \mathrm{d}x,$$

which, on integration by parts, gives

$$\sigma^2 = -\left[(x-2)^2 e^{-\frac{x}{2}}\right]_0^\infty + \int_0^\infty\ 2(x-2)e^{-\frac{x}{2}}\ \mathrm{d}x$$

$$= 4 - \left[4(x-2)e^{-\frac{x}{2}}\right]_0^\infty\ 4e^{-\frac{x}{2}}\ \mathrm{d}x = 4.$$

Thus $\sigma = 2$.

19.4.2 EXERCISES

1. A probability function, $P(x)$, is defined by the following table:

x	0	1	2	3	4	5	6
$P(x)$	0.17	0.29	0.27	0.16	0.07	0.03	0.01

   Determine (a) the mean, (b) the median, (c) the mode and (d) the standard deviaton of this distribution.

2. A certain assembly process is such that the probability of success at each attempt is 0.2. The probability, $P(x)$ that x independent attempts are needed to achieve success is given by

$$P(x) \equiv (0.2)(0.8)^{x-1} \quad x = 1,\ 2,\ 3, \ldots$$

   Determine (a) the mean, (b) the median and (c) the standard deviation of this distribution.

3. The running distance (in thousands of kilometres) which car owners achieve from a certain type of tyre is a random variable with probability density function, $f(x)$, where

$$f(x) \equiv \begin{cases} \frac{1}{30}e^{-x/30} & \text{when } x > 0; \\ 0 & \text{when } x \leq 0, \end{cases}$$

Determine

(a) the probability that one of these tyres will last at most 19000km;

(b) (i) the mean, (ii) the median and (iii) the standard deviation of the distribution.

4. The probability density function, $f(x)$, of a random variable, x, is defined by

$$f(x) \equiv \begin{cases} 30x^2(1-x)^2 & \text{when } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability that x will take a value within two standard deviations of its mean.

5. The probability density function, $f(x)$, of a random variable, x, is given by

$$f(x) \equiv \begin{cases} \frac{1}{12}xe^{-x^2/12} & \text{when } x > 0; \\ & \\ 0 & \text{otherwise.} \end{cases}$$

Determine (a) the median and (b) the mode for this distribution. Show also, (c), that the mean, μ, is given by

$$\mu = \int_0^\infty e^{-x^2/12}\, dx.$$

19.4.3 ANSWERS TO EXERCISES

1. (a) Mean = 1.8; (b) median = 2; (c) mode = 1; (d) standard deviation = 1.34
2. (a) Mean = 5; (b) median = 3; (c) standard deviation = 4.47
3. (a) 0.47; (b) (i) mean = 30; (ii) median = 20.8; (iii) standard deviation = 30.
4. 0.969
5. (a) Median = $\sqrt{12 \ln 2}$; (b) mode = $\sqrt{6}$.

"JUST THE MATHS"

UNIT NUMBER

19.5

PROBABILITY 5
(The binomial distribution)

by

A.J.Hobson

UNIT 19.5 - PROBABILITY 5 - THE BINOMIAL DISTRIBUTION

19.5.1 INTRODUCTION AND THEORY

In this Unit, we shall be concerned, firstly, with probability problems having only **two** events (which are mutually exclusive and independent), although many trials may be possible. For example, the pairs of events could be "up and down", "black and white", "good and bad", and, in general, "successful and unsuccessful".

Statement of the problem

Suppose that the probability of success in a single trial is unaffected when successive trials are carried out (that is, we have independent events). Then what is the probability that, in n successive trials, **exactly** r will be successful ?

General Analysis of the problem

Let us build up the solution in simple stages:

(a) If p is the probability of success in a single trial, then the probability of failure is $1-p = q$, say.

(b) In the following table, let S stand for success and let F stand for failure. The table shows the possible results of one, two or three trials and their corresponding probabilities:

NO. OF TRIALS	POSSIBLE RESULTS	RESPECTIVE PROBABILITIES
1	F,S	q, p
2	FF,FS,SF,SS	q^2, qp, pq, p^2
3	FFF,FFS,FSF,FSS, SFF,SFS,SSF,SSS	$q^3, q^2p, q^2p, qp^2,$ q^2p, qp^2, qp^2, p^3

(c) **Summary**

(i) In **one** trial, the probabilities that there will be exactly 0 or exactly 1 successes are the respective terms of the expression

$$q + p.$$

(ii) In **two** trials, the probabilities that there will be exactly 0, exactly 1 or exactly 2 successes are the respective terms of the expression

$$q^2 + 2qp + p^2; \quad \text{that is,} \quad (q + p)^2.$$

(iii) In **three** trials, the probabilities that there will be exactly 0, exactly 1, exactly 2 or exactly 3 successes are the respective terms of the expression

$$q^3 + 3q^2p + 3qp^2 + p^3; \quad \text{that is,} \quad (q+p)^3.$$

(iv) In **any number**, n, of trials, the probabilities that there will be exactly 0, exactly 1, exactly 2, exactly 3, ... or exactly n successes are the respective terms in the binomial expansion of the expression

$$(q+p)^n.$$

(d) **MAIN RESULT**:

The probability that, in n trials, there will be exactly r successes, is the term containing p^r in the binomial expansion of $(q+p)^n$.

It can be shown that this is the value of

$$^nC_r q^{n-r} p^r.$$

EXAMPLES

1. Determine the probability that, in 6 tosses of a coin, there will be exactly 4 heads.
   Solution
   $q = 0.5, \quad p = 0.5, \quad n = 6, \quad r = 4.$
   Hence, the required probability is given by
   $$^6C_4.(0.5)^2.(0.5)^4 = \frac{6!}{2!4!}.\frac{1}{4}.\frac{1}{16} = \frac{15}{64}.$$

2. Determine the probability of obtaining the most probable number of heads in 6 tosses of a coin.
   Solution
   The most probable number of heads is given by $\frac{1}{2} \times 6 = 3$.
   The probability of obtaining exactly 3 heads is given by
   $$^6C_3.(0.5)^3(0.5)^3 = \frac{6!}{3!3!}.\frac{1}{8}.\frac{1}{8} = \frac{20}{64} \simeq 0.31$$

3. Determine the probability of obtaining exactly 2 fives in 7 throws of a die.

   Solution

   $q = \frac{5}{6}, \quad p = \frac{1}{6}, \quad n = 7, \quad r = 2.$

   Hence, the required probability is given by

   $$^7C_2.\left(\frac{5}{6}\right)^5.\left(\frac{1}{6}\right)^2 = \frac{7!}{5!2!}.\left(\frac{5}{6}\right)^5.\left(\frac{1}{6}\right)^2 \simeq 0.234$$

4. Determine the probability of throwing at most 2 sixes in 6 throws of a die.

   Solution

   The phrase "at most 2 sixes" means exactly 0, or exactly 1, or exactly 2. Hence, we add together the first three terms in the expansion of $(q+p)^6$, where $q = \frac{5}{6}$ and $p = \frac{1}{6}$.

   It may be shown that

   $$(q+p)^6 = q^6 + 6q^5p + 15q^4p^2 + ...$$

   and, by substituting for q and p, the sum of the first three terms turns out to be

   $$\frac{21875}{23328} \simeq 0.938$$

5. It is known that 10% of certain components manufactured are defective. If a random sample of 12 such components is taken, what is the probability that at least 9 are defective ?

   Solution

   Firstly, we note how the information suggests that removal of components for examination does not affect the probability of 10%. This is reasonable, since our sample is almost certainly very small compared with all components in existence.

   Secondly, the probability of <u>success</u> in this example is 0.1, even though it refers to <u>defective</u> items, and hence the probability of failure is 0.9.

   Thirdly, using $p = 0.1, \quad q = 0.9, \quad n = 12$, we require the probabilities (added together) of exactly 9, 10, 11 or 12 defective items and these are the <u>last</u> four terms in the expansion of $(q+p)^n$.

   That is,

   $$^{12}C_9.(0.9)^3.(0.1)^9 + {}^{12}C_{10}.(0.9)^2.(0.1)^{10} + {}^{12}C_{11}.(0.9).(0.1)^{11} + (0.1)^{12} \simeq 1.658 \times 10^{-7}.$$

Note:

The use of the **"binomial distribution"** becomes very tedious when the number of trials is large and two other standard distributions (called the **"normal distribution"** and the **"Poisson distribution"**) can sometimes be used.

19.5.2 EXERCISES

1. A coin is tossed six times. What is the probability of getting exactly four heads ?
2. What is the probability of throwing at least four 7's in five throws of a pair of dice ?
3. In a roll of five dice, what is the probability of getting exactly four faces alike ?
4. If three dice are thrown, determine the probability that
   (a) all three will show the number 4;
   (b) all three will be alike;
   (c) two will show the number 4 and the third, something else;
   (d) all three will be different;
   (e) only two will be alike.
5. Hospital records show that 10% of the cases of a certain disease are fatal. If five patients suffer from this disease, determine the probability that
   (a) all will recover;
   (b) at least three will die;
   (c) exactly three will die;
   (d) a particular three will die and the others survive.
6. A particular student can solve, on average, half of the problems given to him. In order to pass the course, he is required to solve seven out of ten problems on an examination paper. What is the probability that he will pass ?

19.5.3 ANSWERS TO EXERCISES

1.

$$15 \times (0.5)^4 \times 0.5 \simeq 0.47$$

2.

$$\left(\frac{1}{6}\right)^5 + 5 \times \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right) \simeq 0.0032$$

3.

$$6 \times 5 \times \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right) \simeq 0.019$$

4. (a)

$$\left(\frac{1}{6}\right)^3 \simeq 0.0046;$$

(b)

$$3 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right) \simeq 0.069;$$

(c)

$$6 \times \left(\frac{1}{6}\right)^3 \simeq 0.028;$$

(d)

$$1 - 6\left[\left(\frac{1}{6}\right)^3 + 3 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)\right] \simeq 0.56;$$

(e)

$$6 \times 6 \times \left(\frac{1}{6}\right)^3 \simeq 0.168$$

5. (a)

$$(0.9)^5;$$

(b)

$$(0.1)^5 + 5 \times (0.9) \times (0.1)^4 + 10 \times (0.9)^2 \times (0.1)^3 \simeq 0.0086;$$

(c)

$$10 \times (0.9)^2 \times (0.1)^3 \simeq 0.0081;$$

(d)

$$\frac{10 \times (0.9)^2 \times (0.1)^3}{{}^5C_3} \simeq 0.00081$$

6.

$$(1 + 10 + 45 + 120) \times (0.5)^{10} \simeq 0.17$$

"JUST THE MATHS"

UNIT NUMBER

19.6

PROBABILITY 6
(Statistics for the binomial distribution)

by

A.J.Hobson

UNIT 19.6 - PROBABILITY 6

STATISTICS FOR THE BINOMIAL DISTRIBUTION

19.6.1 CONSTRUCTION OF HISTOGRAMS

Elementary discussion on the presentation of data, in the form of frequency tables, histograms etc., usually involves experiments which are actually carried out.

But we illustrate now how the binomial distribution may be used to estimate the results of a certain kind of experiment before it is performed.

EXAMPLE

For four coins, tossed 32 times, construct a histogram showing the expected number of occurrences of 0,1,2,3,4..... heads.

Solution

Firstly, in a single toss of the four coins, the probability of head (or tail) for each coin is $\frac{1}{2}$.

The terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^4$ give the probabilities of exactly 0,1,2,3 and 4 heads, respectively.

The expansion is

$$\left(\frac{1}{2}+\frac{1}{2}\right)^4 \equiv \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4.$$

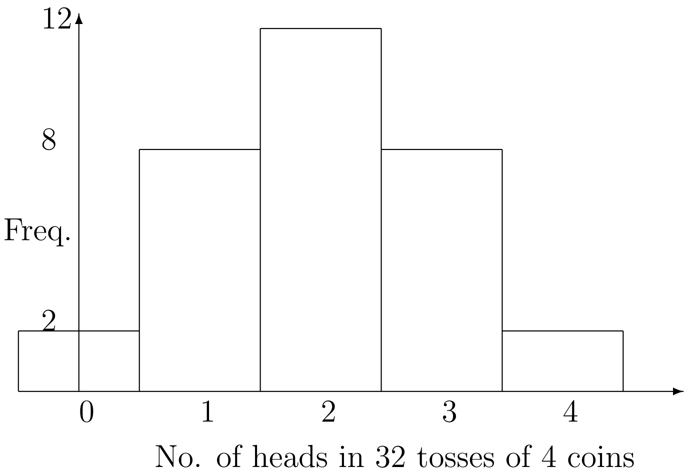
That is,

$$\left(\frac{1}{2}+\frac{1}{2}\right)^4 \equiv \left(\frac{1}{2}\right)^4(1+4+6+4+1),$$

showing that the probabilities of 0,1,2,3 and 4 heads in a single toss of four coins are $\frac{1}{16}$, $\frac{1}{4}$, $\frac{6}{16}$, $\frac{1}{4}$, and $\frac{1}{16}$, respectively.

Therefore, in 32 tosses of four coins, we may expect 0 heads, twice; 1 head, 8 times; 2 heads, 12 times; 3 heads, 8 times and 4 heads, twice.

The following histogram uses class-intervals for which each member is, in fact, situated at the mid-point:



Notes:

(i) The only reason that the above histogram is symmetrical in shape is that the probability of success and failure are equal to each other, so that the terms of the binomial expansion are, themselves, symmetrical.

(ii) Since the widths of the class-intervals in the above histogram are 1, the areas of the rectangles are equal to their heights. Thus, for example, the total area of the first three rectangles represents the expected number of times of obtaining <u>at most</u> 2 heads in 32 tosses of 4 coins.

19.6.2 MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

THEOREM

If p is the probability of success of an event in a single trial and q is the probability of its failure, then the binomial distribution, giving the expected frequencies of 0,1,2,3,.....n successes in n trials, has a mean of np and a standard deviation of $\sqrt{npq}$, irrespective of the number of times the experiment is to be carried out.

Proof:

(a) The Mean

From the binomial expansion formula,

$$(q+p)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{2!}q^{n-2}p^2 + \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \ldots + nqp^{n-1} + p^n.$$

Hence, if the n trials are made N times, the average number of successes is equal to the following expression, multiplied by N, then divided by N:

$$0 \times q^n + 1 \times nq^{n-1}p + 2 \times \frac{n(n-1)}{2!}q^{n-2}p^2 +$$

$$3 \times \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \; ..(n-1) \times nqp^{n-1} + np^n.$$

That is, the mean is

$$np\left(q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{2}q^{n-3}p^2 + \; \ldots + (n-1)qp^{n-2} + p^{n-1}\right)$$

$$= np(q+p)^{n-1} = np \;\text{ since }\; q+p = 1.$$

(b) The Standard Deviation

For the standard deviation, we observe that, if f_r is the frequency of r successes when the n trials are conducted N times, then

$$f_r = N\frac{n!}{(n-r)!r!}q^{n-r}p^r.$$

We use this, first, to establish a result for

$$\sum_{r=0}^{n} r^2 f_r.$$

For example,

$0^2 f_0 = 0.Nq^n = 0 = 0.f_0.$

$1^2 f_1 = 1.Nnq^{n-1}p = 1.f_1.$

$2^2 f_2 =$

$$2Nn(n-1)q^{n-2}p^2 = Nn(n-1)q^{n-2}p^2 + Nn(n-1)p^2q^{n-2}$$

$$= 2f_2 + Nn(n-1)p^2q^{n-2};$$

$3^2 f_3 =$

$$3N\frac{n(n-1)(n-2)}{2!}q^{n-3}p^3 = N\frac{n(n-1)(n-2)}{2!}q^{n-3}p^3 + Nn(n-1)p^2(n-2)q^{n-3}p$$

$$= 3f_3 + Nn(n-1)p^2(n-2)q^{n-3}p;$$

$4^2 f_4 =$

$$4N\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4 = N\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4 + Nn(n-1)p^2\frac{(n-2)(n-3)}{2!}q^{n-4}p^2$$

$$= 4f_4 + Nn(n-1)p^2\frac{(n-2)(n-3)}{2!}q^{n-4}p^2;$$

and, in general, when $r \geq 2$,

$r^2 f_r =$

$$N\frac{n(n-1)(n-2)....(n-r+1)}{(r-1)!} + Nn(n-1)p^2q^{n-r}p^r = rf_r + Nn(n-1)p^2\frac{(n-2)!}{(n-r)!(r-2)!}q^{n-r}p^{r-2}.$$

This result, together with those for $0^2.f_0$ and $1^2 f_1$, shows that

$$\sum_{r=0}^{n} r^2 f_r = \sum_{r=0}^{n} rf_r + Nn(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(n-r)!(r-2)!}q^{n-r}p^{r-2}.$$

That is,

$$\sum_{r=0}^{n} r^2 f_r = Nnp + Nn(n-1)p^2(q+p)^{n-2} = Nnp + Nn(n-1)p^2,$$

since $q + p = 1$.

It was also established, in Unit 18.3, that the standard deviation of a set, $x_1, x_2, x_3, \ldots x_m$, of m observations, with a mean value of $\overline{x}$, is given by the formula

$$\sigma^2 = \frac{1}{m}\sum_{i=1}^{m} x_i^2 - \overline{x}^2,$$

which, in the present case, may be written

$$\sigma^2 = \frac{1}{N}\sum_{r=0}^{n} r^2 f_r - \frac{1}{N^2}\left(\sum_{r=0}^{n} r f_r\right)^2.$$

Hence,

$$\sigma^2 = \frac{1}{N}\left(Nnp + Nn(n-1)p^2\right) - \frac{1}{N^2}(Nnp)^2,$$

which gives

$$\sigma^2 = np + n^2p^2 - np^2 - n^2p^2 = np(1-p) = npq;$$

and so,

$$\sigma = \sqrt{npq}.$$

ILLUSTRATION

For direct calculation of the mean and the standard deviation for the data in the previous coin-tossing problem, we may use the following table, in which x_i denotes numbers of heads and f_i denotes the corresponding expected frequencies:

x_i	f_i	$f_i x_i$	$f_i x_i^2$
0	2	0	0
1	8	8	8
2	12	24	48
3	8	24	72
4	2	8	32
Totals	32	64	160

The mean is given by

$$\overline{x} = \frac{64}{32} = 2 \quad \text{(obviously)},$$

which agrees with $np = 4 \times \frac{1}{2}$.

The standard deviation is given by

$$\sigma = \sqrt{\left[\frac{160}{32} - 2^2\right]} = 1,$$

which agrees with $\sqrt{npq} = \sqrt{4 \times \frac{1}{2} \times \frac{1}{2}}$.

Note:
If the experiment were carried out N times instead of 32 times, all values in the last three columns of the above table would be multiplied by a factor of $\frac{N}{32}$, which would then cancel out in the remaining calculations.

EXAMPLE

Three dice are rolled 216 times. Construct a binomial distribution and show the frequencies of occurrence for 0,1,2 and 3 sixes.

Evaluate the mean and the standard deviation of the distribution.

Solution

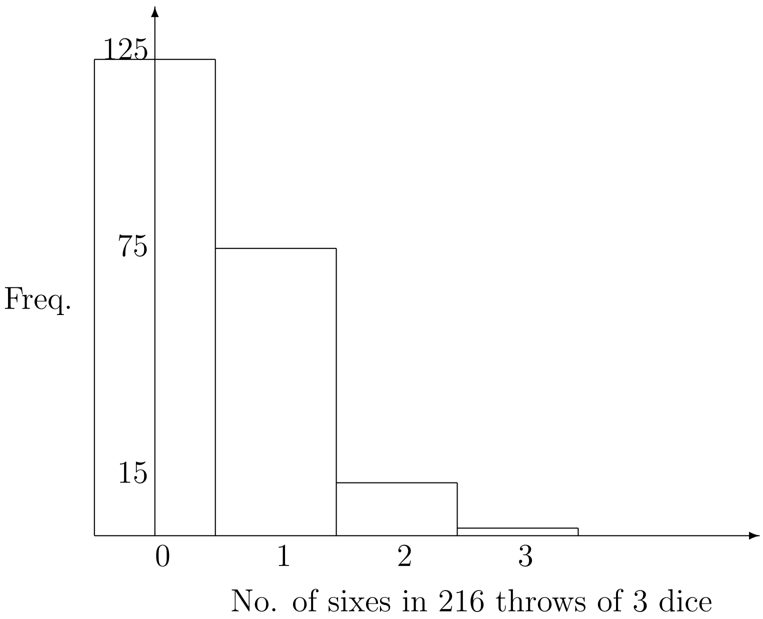
First of all, the probability of success in obtaining a six with a single throw of a die is $\frac{1}{6}$, and the corresponding probability of failure is $\frac{5}{6}$.

For a single throw of three dice, we require the expansion

$$\left(\frac{1}{6}+\frac{5}{6}\right)^3 \equiv \left(\frac{1}{6}\right)^3 + 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3,$$

showing that the probabilities of 0,1,2 and 3 sixes are $\frac{125}{216}$, $\frac{75}{216}$, $\frac{15}{216}$ and $\frac{1}{216}$, respectively.

Hence, in 216 throws of the three dice we may expect 0 sixes, 125 times; 1 six, 75 times; 2 sixes, 15 times and 3 sixes, once. The corresponding histogram is as follows:



From the previous Theorem, the mean value is

$$3 \times \frac{1}{6} = \frac{1}{2}$$

and the standard deviation is

$$\sqrt{3 \times \frac{1}{6} \times \frac{5}{6}} = \frac{\sqrt{15}}{6}.$$

19.6.3 EXERCISES

1. Four dice are rolled 81 times. If less than 5 on a die is considered to be a success, and everything else a failure,

   (a) draw the corresponding histogram for the expected frequencies of success;

   (b) determine the expected number of times of obtaining at least three successes among the four dice;

   (c) shade the area of the histogram which is a measure of the result in (c);

   (d) calculate the mean and the standard deviation of the frequency distribution in (a).

2. In a seed-viability test, 450 seeds were placed on a filter-paper in 90 rows of 5. The number of seeds that germinated in each row were counted, and the results were as follows:

No. of seeds germinating per row	0	1	2	3	4	5
Observed Frequency of rows	0	1	11	30	38	10

   If the germinating seeds were distributed, at random, among the rows, we would expect a binomial distribution with an index of $n = 5$.

   Determine

   (a) the average number of seeds germinating per row;

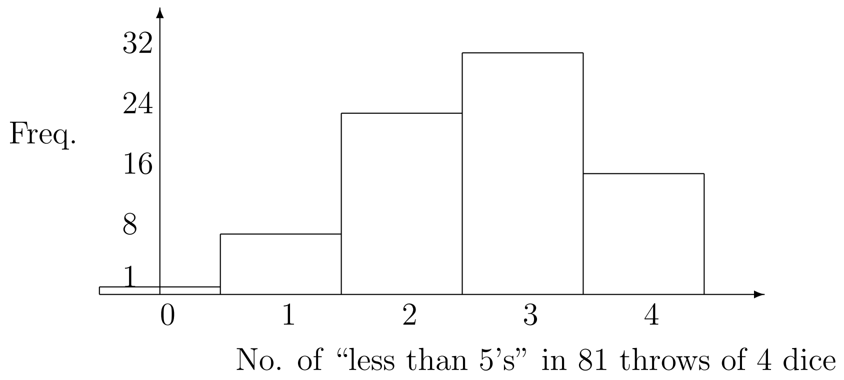
   (b) the probability of a single seed germinating;

   (c) the expected frequencies of rows for each number of seeds germinating;

   Draw the histogram for the expected frequencies and the histogram for the observed frequencies.

19.6.4 ANSWERS TO EXERCISES

1. (a) The histogram is as follows:



   (b) Expected frequency of 3 or 4 successes $= 32 + 16 = 48$;

   (c) Shade the last two rectangles on the right of the histogram;

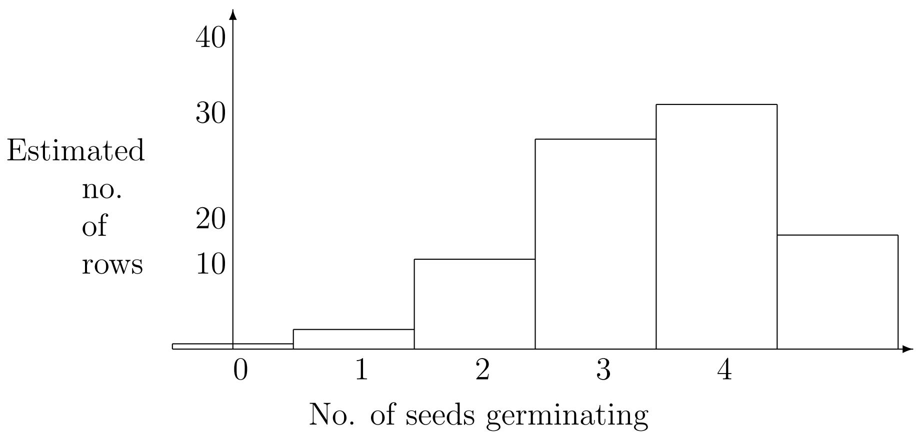
   (d) Mean $= 4 \times \frac{2}{3} = \frac{8}{3}$ and Standard Deviation $= \sqrt{4 \times \frac{2}{3} \times \frac{1}{3}} = \frac{2\sqrt{2}}{3}$.

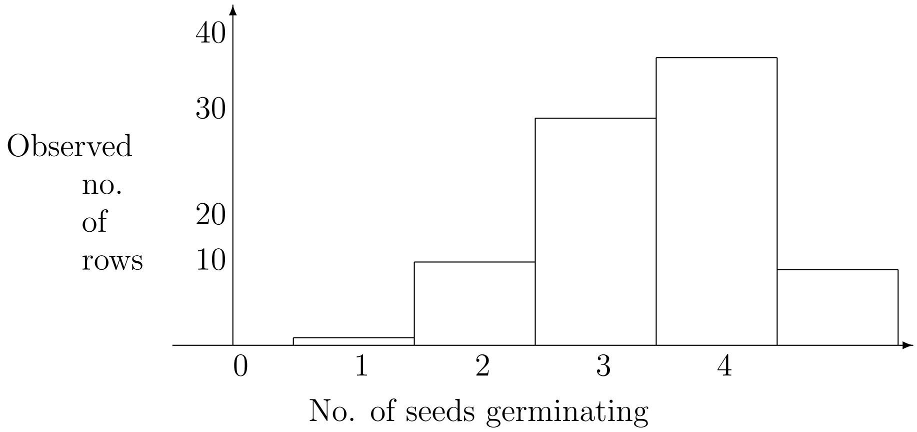
2. (a) Average number of seeds germinating per row is $\frac{325}{90} = 3.50$;

   (b) Probability of a single seed germinating is $\frac{3.5}{5} = \frac{7}{10}$;

   (c) Expected frequencies for 0,1,2,3,4,5 seeds are 0.2187, 2.5515, 11.9070, 27.7830, 32.4135, 15.1263 respectively.

   (d) The histograms are as follows:




40
30
20
10
Observed no. of rows
0
1
2
3
4
No. of seeds germinating

"JUST THE MATHS"

UNIT NUMBER

19.7

PROBABILITY 7
(The Poisson distribution)

by

A.J.Hobson

UNIT 19.7 - PROBABILITY 7

THE POISSON DISTRIBUTION

19.7.1 THE THEORY

We recall that, in a binomial distribution of n trials, the probability, P_r, that an event occurs exactly r times out of a possible n is given by

$$P_r = \frac{n!}{(n-r)!r!}p^r q^{n-r},$$

where p is the probability of success in a single trial and $q = 1 - p$ is the probability of failure.

Now suppose that n is very large compared with r and that p is very small compared with 1.

Then,

(a)

$$\frac{n!}{(n-r)!} = n(n-1)(n-2).....(n-r+1) \simeq n^r.$$

ILLUSTRATION

If $n = 120$ and $r = 3$, then

$$\frac{n!}{(n-r)!} = \frac{120!}{117!} = 120 \times 119 \times 118 \simeq 120^3.$$

(b)

$$q^r = (1-p)^r \simeq 1,$$

so that

$$q^{n-r} \simeq q^n = (1-p)^n.$$

We may deduce that

$$P_r \simeq \frac{n^r p^r (1-p)^n}{r!} = \frac{(np)^r (1-p)^n}{r!}$$

or

$$P_r \simeq \frac{(np)^r}{r!}\left[1 - np + \frac{n(n-1)}{2!}p^2 - \frac{n(n-1)(n-2)}{3!}p^3 + \ldots.\right]$$

$$\simeq \frac{(np)^r}{r!}\left[1 - np + \frac{(np)^2}{2!} - \frac{(np)^3}{3!} + \ldots.\right].$$

Hence,

$$P_r \simeq \frac{(np)^r}{r!}e^{-np}.$$

The number, np, in this formula is of special significance, being the average number of successes to be expected a single set of n trials.

If we denote np by μ, we obtain the **"Poisson distribution"** formula,

$$P_r \simeq \frac{\mu^r e^{-\mu}}{r!}.$$

Notes:

(i) Although the formula has been derived from the binomial distribution, as an approximation, it may also be used in its own right, in which case we drop the approximation sign.

(ii) The Poisson distribution is more use than the binomial distribution when n is a very large number, the binomial distribution requiring the tedious evaluation of its various coefficients.

(iii) The Poisson distribution is of particular use when the average frequency of occurrence of an event is known, but not the number of trials.

EXAMPLES

1. The number of cars passing over a toll-bridge during the time interval from 10a.m. until 11a.m. is 1,200.

   (a) Determine the probability that not more than 4 cars will pass during the time interval from 10.45a.m. until 10.46a.m.

   (b) Determine the probability that 5 or more cars pass during the same interval.

   Solution

   The number of cars which pass in 60 minutes is 1200, so that the average number of cars passing, per minute, is $20 = \mu$.

   (a) The probability that not more than 4 cars pass in a one-minute interval is the sum of the probabilities for 0,1,2,3 and 4 cars.

   That is,

$$\sum_{r=0}^{4} \frac{(20)^r e^{-20}}{r!} =$$

$$\left[\frac{(20)^0}{0!} + \frac{(20)^1}{1!} + \frac{(20)^2}{2!} + \frac{(20)^3}{3!} + \frac{(20)^4}{4!}\right] e^{-20} =$$

$$8221 e^{-20} \simeq 1.69 \times 10^{-5}.$$

   (b) The probability that 5 or more cars will pass in a one-minute interval is the probability of failure in (a). In other words,

$$1 - \sum_{r=0}^{4} \frac{(20)^r e^{-20}}{r!} =$$

$$1 - 8221 e^{-20} \simeq 0.99998$$

2. A company finds that, on average, there is a claim for damages which it must pay 7 times in every 10 years. It has expensive insurance to cover this situation.

   The premium has just been increased, and the firm is considering letting the insurance lapse for 12 months as it can afford to meet a single claim.

Assuming a Poisson distribution, what is the probability that there will be at least two claims during the year ?

Solution

Using $\mu = \frac{7}{10} = 0.7$, the probability that there will be at most one claim during the year is given by

$$P_0 + P_1 = e^{-0.7} + e^{-0.7}0.7 = e^{0.7}(1 + 0.7).$$

The probability that there will be at least two claims during the year is given by

$$1 - e^{-0.7}(1 + 0.7) \simeq 0.1558$$

3. There is a probability of 0.005 that a welding machine will produce a faulty joint when it is operated. The machine is used to weld 1000 rivets. Determine the probability that at least three of these are faulty.

Solution

First, we have $\mu = 0.005 \times 1000 = 5$.

Hence, the probability that at most two will be faulty is given by

$$P_0 + P_1 + P_2 = e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!}\right].$$

That is,

$$e^{-5}[1 + 5 + 12.5] \simeq 0.125$$

Hence, the probability that at least three will be faulty is approximately

$$1 - 0.125 = 0.875$$

19.7.2 EXERCISES

1. The probability that a glass fibre will shatter during an experiment is believed to follow a Poisson distribution.

   In a certain apparatus, it was found that, on average, 7 glass fibres shattered.

   Determine the probability that, during a single demonstration of the experiment,

   (a) two glass fibres will shatter;

   (b) at least one glass fibre will shatter.

2. A major airline operates 350 flights per day throughout the world. The probability that a flight will be delayed for more than one hour, for any reason, is 0.7%. If more than four flights suffer such delays in one day, the implications for route organisation and crewing become serious. Calling such a day a "bad day", determine the probabilities that

   (a) any particular day is a bad day;

   (b) at most two bad days occur in one week;

   (c) more than 50 bad days occur in a year of 365 days.

   State your answers correct to three significant figures.

3. It is known that 3% of bolts made by a certain machine are defective. If the bolts are packaged in boxes of 50, determine the probability that a given box will contain 4 defectives.

4. If 0.04% of cars break down while driving through a certain tunnel, determine the probability that at most 2 break down out of 2000 cars entering the tunnel on a given day. State your answer correct to three significant figures.

19.7.3 ANSWERS TO EXERCISES

1. (a) 2.28×10^{-3}; (b) 0.0676
2. (a) 0.102; (b) 0.128; (c) 0.011
3. 0.047 using $\mu = 1.5$
4. 0.952 using $\mu = 0.8$

"JUST THE MATHS"

UNIT NUMBER

19.8

PROBABILITY 8
(The normal distribution)

by

A.J.Hobson

UNIT 19.8 - PROBABILITY 8

THE NORMAL DISTRIBUTION

19.8.1 LIMITING POSITION OF A FREQUENCY POLYGON

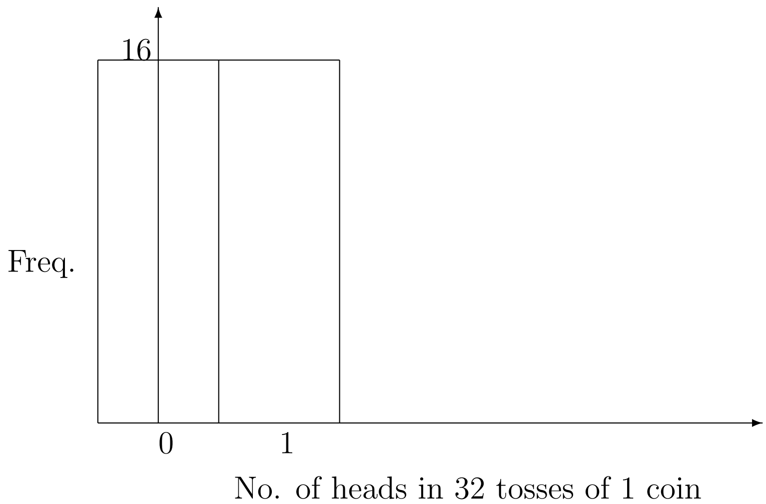
The distribution considered here is also appropriate to examples where the number of trials is large and, hence, the calculation of frequencies and probabilities, using the binomial distribution, would be inconvenient.

We shall introduce the **"normal distribution"** by considering the histograms of the binomial distribution for a toss of 32 coins as the number of coins increases.

The probability of obtaining a head is $\frac{1}{2}$ and the probability of obtaining a tail is also $\frac{1}{2}$.

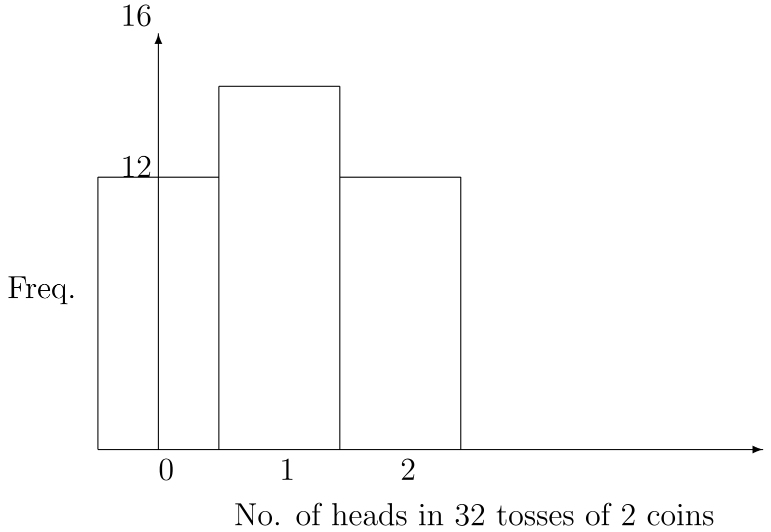
(i) One Coin

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^1 = 32\left(\frac{1}{2}+\frac{1}{2}\right) = 16+16.$$



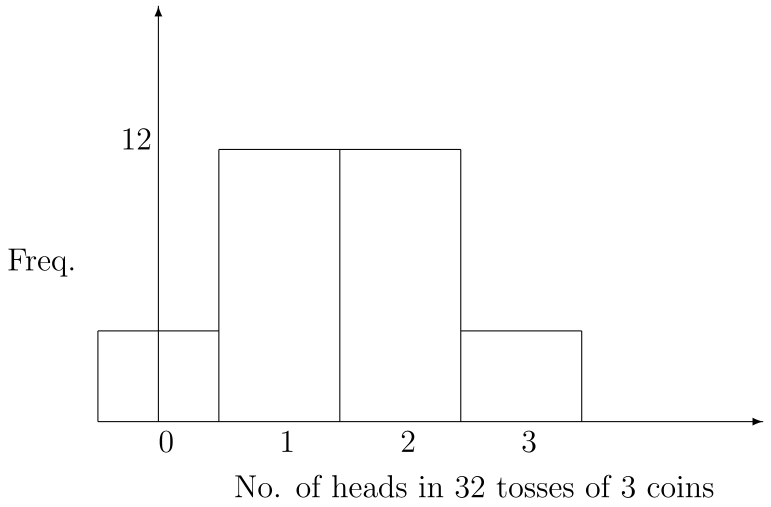
(ii) Two Coins

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^2 = 32\left(\left[\frac{1}{2}\right]^2 + 2\left[\frac{1}{2}\right]\left[\frac{1}{2}\right] + \left[\frac{1}{2}\right]^2\right) = 8+16+8.$$



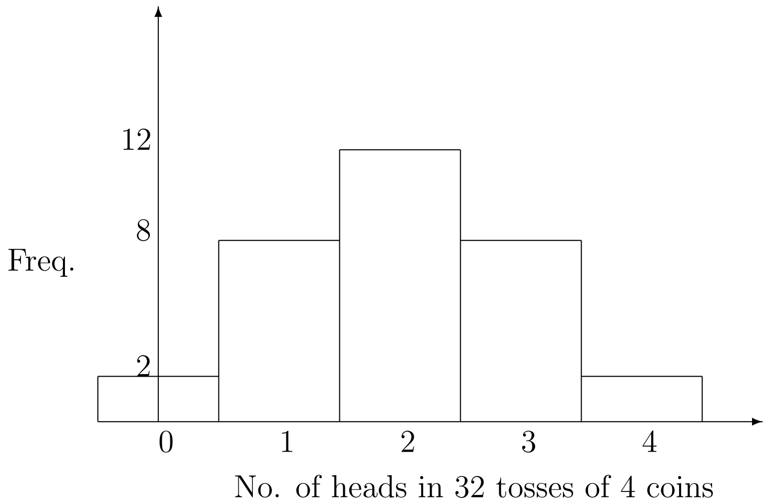
(iii) Three Coins

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^3 = 32\left(\left[\frac{1}{2}\right]^3 + 3\left[\frac{1}{2}\right]^2\left[\frac{1}{2}\right] + 3\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]^2 + \left[\frac{1}{2}\right]^3\right) = 4+12+12+4.$$



(iv) Four Coins

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^4 = 32\left(\left[\frac{1}{2}\right]^4 + 4\left[\frac{1}{2}\right]^3\left[\frac{1}{2}\right] + 6\left[\frac{1}{2}\right]^2\left[\frac{1}{2}\right]^2 + 4\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]^3 + \left[\frac{1}{2}\right]^4\right) = 2+8+12+8+2.$$



It is apparent that, as the number of coins increases, the frequency polygon (consisting of the straight lines joining the midpoints of the tops of each rectangle) approaches a symmetrical bell-shaped curve.

This, of course, is true only when the histogram itself is either symmetrical or nearly symmetrical.

DEFINITION

As the number of trials increases indefinitely, the limiting position of the frequency polygon is called the **"normal frequency curve"**.

THEOREM

In a binomial distribution for N samples of n trials each, where the probability of success in a single trial is p, it may be shown that, as n increases indefinitely, the frequency polygon approaches a smooth curve, called the **"normal curve"**, whose equation is

$$y = \frac{N}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\overline{x})^2}{2\sigma^2}}.$$

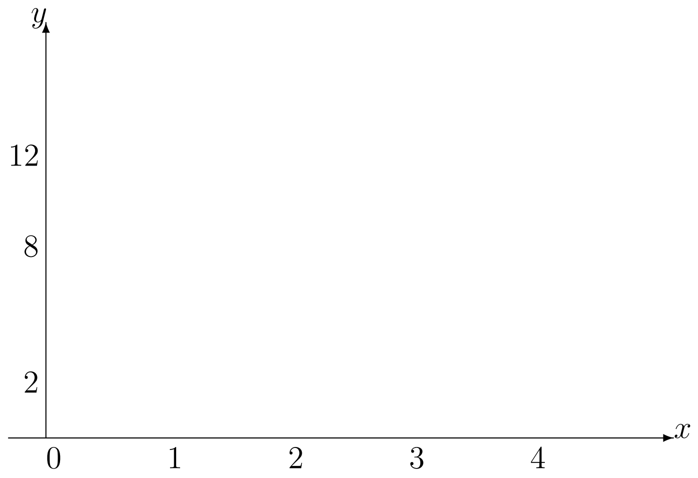
In this equation

$\overline{x}$ is the mean of the binomial distribution $= np$;

σ is the standard deviation of the binomial distribution $= \sqrt{np(1-p)}$;

y is the frequency of occurence of the value, x.

For example, the histogram for 32 tosses of 4 coins approximates to the following normal curve:



Notes:

(i) We omit the proof of the Theorem since it is beyond the scope of the present text.

(ii) The larger the value of n, the better is the level of approximation.

(iii) The normal curve is symmetrical about the straight line $x = \overline{x}$, since the value of y is the same at $x = \overline{x} \pm h$ for any number, h.

(iv) If the relative frequency (or probability) with which the value, x, occurs is denoted by P, then $P = y/N$ and the above relationship can be written

$$P = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}},$$

the graph of which is called the **"normal probability curve"**.

(v) Symmetrical curves are easier to deal with if the vertical axes of co-ordinates is the line of symmetry.
The normal probability curve can be simplified if we move the origin to the point $(\overline{x}, 0)$ and plot $P\sigma$ on the vertical axis instead of P.

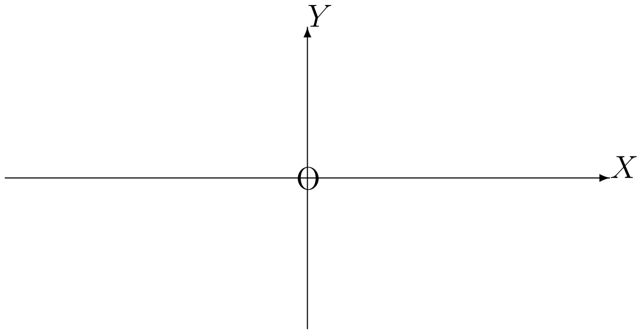
Letting $P\sigma = Y$ and $\frac{x-\overline{x}}{\sigma} = X$ the equation of the normal probability curve becomes

$$Y = \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}},$$

which represents the **"standard normal probability curve"**.

From any point on it, we may obtain the values of the original P and x values by using the formulae

$$x = \sigma X + \overline{x} \quad \text{and} \quad P = \frac{Y}{\sigma}.$$



(vi) If the probability of success, p, in a single trial is **not** equal to, or approximately equal to, $\frac{1}{2}$, then the distribution given by the normal frequency curve and the two subsequent curves will be a poor approximation and is seldom used for such cases.

19.8.2 AREA UNDER THE NORMAL CURVE

For the histogram of a binomial distribution, corresponding to values of x, suppose that $x = a$ and $x = b$ are the values of x at the base-centres of two particular rectangles, where $b > a$ and all rectangles have width 1.

Then, the area of the histogram from $x = a - \frac{1}{2}$ to $x = b + \frac{1}{2}$ represents the number of times which we can expect values of x, between $x = a$ and $x = b$ inclusive, to occur.

Consequently, for a large number of trials, we may use the area under the normal curve between $x = a - \frac{1}{2}$ and $x = b + \frac{1}{2}$.

In a similar way, the **probability** that x will lie between $x = a$ and $x = b$ is represented by the area under the normal probability curve from $x = a - \frac{1}{2}$ and $x = b + \frac{1}{2}$. We note that the total area under this curve must be 1, since it represents the probability that **any** value of x will occur (a certainty).

In order to make use of a <u>standard</u> normal probabilty curve for the same purpose, the conversion formulae from x to X and P to Y must be used.

Note:
Tables are commercially available for the area under a standard normal probability curve; and, in using such tables, the above conversions will usually be necessary. A sample table is given in an appendix at the end of this unit.

EXAMPLE

If 12 dice are thrown, determine the probability, using the normal probability curve approximation, that 7 or more dice will show a 5.

Solution

For this example, we use $p = \frac{1}{6}$, $q = \frac{5}{6}$, $n = 12$ and we need the area under the normal probability curve from $x = 6.5$ to $x = 12.5$.

The mean of the binomial distribution, in this case, is $\overline{x} = 12 \times \frac{1}{6} = 2$.

The standard deviation is $\sigma = \sqrt{2 \times \frac{1}{6} \times \frac{5}{6}} \simeq \sqrt{1.67} \simeq 1.29$.

The required area under the <u>standard</u> normal probability curve will be that lying between

$$X = \frac{6.5 - 2}{1.29} \simeq 3.49 \text{ and } X = \frac{12.5 - 2}{1.29} \simeq 8.14$$

In practice, we take the whole area to the right of $X = 3.49$ since the area beyond $X = 8.14$ is negligible.

Also, the total area to the right of $X = 0$ is 0.5; and, hence, the required area is 0.5 minus the area from $X = 0$ to $X = 3.49$.

From tables, the required area is $0.5 - 0.4998 = 0.0002$ and this is the probability that, when 12 dice are thrown, 7 or more will show a 5.

Note:
If we had required the probability that 7 or fewer dice show a 5, we would have needed the area under the normal probability curve from $x = -0.5$ to $x = 7.5$. This is equivalent to taking the whole of the area under the standard normal probability curve which lies to the left of

$$X = \frac{7.5 - 2}{1.29} \simeq 4.26$$

19.8.3 NORMAL DISTRIBUTION FOR CONTINUOUS VARIABLES

So far, the variable, x, considered in a Normal distribution has been "discrete"; that is, x has been able to take only the specific values 0,1,2,3.....etc.

Here, we consider the situation when x is a "continuous" variable; that is, it may take any value within a certain range appropriate to the problem under consideration.

For a large number of observations of a continuous variable, the corresponding histogram need not have rectangles of class-width 1 but of some other number, say c.

In this case, it may be shown that the normal curve approximation to the histogram has equation

$$y = \frac{Nc}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}}.$$

The smaller is the value of c, the larger is the number of rectangles and the better is the approximation supplied by the curve.

If we wished to calculate the number of x-values lying between $x = a$ and $x = b$ (where $b > a$), we would need to calculate the area of the histogram from $x = a$ to $x = b$ inclusive, then **divide by** c, since the base-width is no longer 1.

We conclude that the number of these x-values approximates to the area under the normal curve from $x = a$ to $x = b$; and similarly, the area under the normal probability curve, from $x = a$ to $x = b$ gives an estimate for the probability that values of x between $x = a$ and $x = b$ will occur.

EXAMPLE

Given a normal distribution of a continuous variable, x, with $N = 2000$, $\overline{x} = 20$ and $\sigma = 5$, determine

(a) the number of x-values lying between 12 and 22;

(b) the number of x-values larger than 30.

Solution

(a) The area under the normal probability curve between $x = 12$ and $x = 22$ is the area under the standard normal probability curve from

$$X = \frac{12 - 20}{5} = -1.6 \ \text{ to } \ X = \frac{22 - 20}{5} = 0.4$$

and, from tables, this is $0.4452 + 0.1554 = 0.6006$

Hence, the required number of values is approximately $0.6006 \times 2000 \simeq 1201$.

(b) The total area under the normal probability curve to the right of $x = 30$ is the area under the standard normal probability curve to the right of

$$X = \frac{30 - 20}{5} = 2$$

and, from tables, this is 0.0227

Hence, the required number of values is approximately $0.0227 \times 2000 \simeq 45$.

19.8.4 EXERCISES

1. Use a normal probability curve approximation to determine the probability that, in a toss of 9 coins, 3 to 6 heads are shown.
2. A coin is tossed 100 times. Use a normal probability curve approximation to determine the probability of obtaining
   (a) exactly 50 heads;
   (b) 60 or more heads.

3. Assume that one half of the people in a certain community are regular viewers of television. Of 100 investigators, each interviewing 10 individuals, how many would you expect to report that 3 people or fewer were regular viewers ? (Use a normal probability curve approximation).

4. A manufacturer knows that, on average, 2% of his products are defective. Using a normal probability curve approximation, determine the probability that a batch of 100 components will contain exactly 5 defectives.

5. If the average life of a certain make of storage battery is 30 months, with a standard deviation of 6 months, what percentage can be expected to last from 24 to 36 months, assuming that their lifetimes follow a normal distribution.

6. If the heights of 10,000 university students closely follow a normal distribution, with a mean of 69.0 inches and a standard deviation of 2.5 inches, how many of these students would you expect to be at least 6 feet in height ?

7. In a certain trade, the average wage is £7.20 per hour and the standard deviation is 90p. If the wages are assumed to follow a normal distribution, what percentage of the workers receive wages between £6.00 and £7.00 per hour ?

19.8.5 ANSWERS TO EXERCISES

1. $P \simeq 0.8614$
2. (a) 0.0796
   (b) 0.0287
3. 17.
4. 0.0305
5. 68.26
6. 1151.
7. 32.11

APPENDIX

AREA UNDER THE STANDARD NORMAL PROBABILITY CURVE

The area given is that from $X = 0$ to a given value, X_1

X_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1046	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4773	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4983	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4989	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4995	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998	0.4988
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.5000	0.5000	0.5000

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Mathematical Methods for Physics and Engineering

Third Edition

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Contents

I am the very Model for a Student Mathematical

I am the very model for a student mathematical;
I've information rational, and logical and practical.
I know the laws of algebra, and find them quite symmetrical,
And even know the meaning of 'a variate antithetical'.

I'm extremely well acquainted, with all things mathematical.
I understand equations, both the simple and quadratical.
About binomial theorems I'm teeming with a lot o'news,
With many cheerful facts about the square of the hypotenuse.

I'm very good at integral and differential calculus,
And solving paradoxes that so often seem to rankle us.
In short in matters rational, and logical and practical,
I am the very model for a student mathematical.

I know the singularities of equations differential,
And some of these are regular, but the rest are quite essential.
I quote the results of giants; with Euler, Newton, Gauss, Laplace,
And can calculate an orbit, given a centre, force and mass.

I can reconstruct equations, both canonical and formal,
And write all kinds of matrices, orthogonal, real and normal.
I show how to tackle problems that one has never met before,
By analogy or example, or with some clever metaphor.

I seldom use equivalence to help decide upon a class,
But often find an integral, using a contour o'er a pass.
In short in matters rational, and logical and practical,
I am the very model for a student mathematical.

When you have learnt just what is meant by 'Jacobian' and 'Abelian';
When you at sight can estimate, for the modal, mean and median;
When describing normal subgroups is much more than recitation;
When you understand precisely what is 'quantum excitation';

When you know enough statistics that you can recognise RV;
When you have learnt all advances that have been made in SVD;
And when you can spot the transform that solves some tricky PDE,
You will feel no better student has ever sat for a degree.

Your accumulated knowledge, whilst extensive and exemplary,
Will have only been brought down to the beginning of last century,
But still in matters rational, and logical and practical,
You'll be the very model of a student mathematical.

KFR, with apologies to W.S. Gilbert

Preface to the third edition

As is natural, in the four years since the publication of the second edition of this book we have somewhat modified our views on what should be included and how it should be presented. In this new edition, although the range of topics covered has been extended, there has been no significant shift in the general level of difficulty or in the degree of mathematical sophistication required. Further, we have aimed to preserve the same style of presentation as seems to have been well received in the first two editions. However, a significant change has been made to the format of the chapters, specifically to the way that the exercises, together with their hints and answers, have been treated; the details of the change are explained below.

The two major chapters that are new in this third edition are those dealing with 'special functions' and the applications of complex variables. The former presents a systematic account of those functions that appear to have arisen in a more or less haphazard way as a result of studying particular physical situations, and are deemed 'special' for that reason. The treatment presented here shows that, in fact, they are nearly all particular cases of the hypergeometric or confluent hypergeometric functions, and are special only in the sense that the parameters of the relevant function take simple or related values.

The second new chapter describes how the properties of complex variables can be used to tackle problems arising from the description of physical situations or from other seemingly unrelated areas of mathematics. To topics treated in earlier editions, such as the solution of Laplace's equation in two dimensions, the summation of series, the location of zeros of polynomials and the calculation of inverse Laplace transforms, has been added new material covering Airy integrals, saddle-point methods for contour integral evaluation, and the WKB approach to asymptotic forms.

Other new material includes a stand-alone chapter on the use of coordinate-free operators to establish valuable results in the field of quantum mechanics; amongst the physical topics covered are angular momentum and uncertainty principles. There are also significant additions to the treatment of numerical integration. In particular, Gaussian quadrature based on Legendre, Laguerre, Hermite and Chebyshev polynomials is discussed, and appropriate tables of points and weights are provided.

We now turn to the most obvious change to the format of the book, namely the way that the exercises, hints and answers are treated. The second edition of *Mathematical Methods for Physics and Engineering* carried more than twice as many exercises, based on its various chapters, as did the first. In its preface we discussed the general question of how such exercises should be treated but, in the end, decided to provide hints and outline answers to all problems, as in the first edition. This decision was an uneasy one as, on the one hand, it did not allow the exercises to be set as totally unaided homework that could be used for assessment purposes but, on the other, it did not give a full explanation of how to tackle a problem when a student needed explicit guidance or a model answer.

In order to allow both of these educationally desirable goals to be achieved, we have, in this third edition, completely changed the way in which this matter is handled. A large number of exercises have been included in the penultimate subsections of the appropriate, sometimes reorganised, chapters. Hints and outline answers are given, as previously, in the final subsections, *but only for the odd-numbered exercises*. This leaves all even-numbered exercises free to be set as unaided homework, as described below.

For the four hundred plus **odd-numbered** exercises, *complete* solutions are available, to both students and their teachers, in the form of a separate manual, *Student Solutions Manual for Mathematical Methods for Physics and Engineering* (Cambridge: Cambridge University Press, 2006); the hints and outline answers given in this main text are brief summaries of the model answers given in the manual. There, each original exercise is reproduced and followed by a fully worked solution. For those original exercises that make internal reference to this text or to other (even-numbered) exercises not included in the solutions manual, the questions have been reworded, usually by including additional information, so that the questions can stand alone.

In many cases, the solution given in the manual is even fuller than one that might be expected of a good student that has understood the material. This is because we have aimed to make the solutions instructional as well as utilitarian. To this end, we have included comments that are intended to show how the plan for the solution is fomulated and have given the justifications for particular intermediate steps (something not always done, even by the best of students). We have also tried to write each individual substituted formula in the form that best indicates how it was obtained, before simplifying it at the next or a subsequent stage. Where several lines of algebraic manipulation or calculus are needed to obtain a final result, they are normally included in full; this should enable the student to determine whether an incorrect answer is due to a misunderstanding of principles or to a technical error.

The remaining four hundred or so **even-numbered** exercises have no hints or answers, outlined or detailed, available for general access. They can therefore be used by instructors as a basis for setting unaided homework. Full solutions to these exercises, in the same general format as those appearing in the manual (though they may contain references to the main text or to other exercises), are available without charge to accredited teachers as downloadable pdf files on the password-protected website http://www.cambridge.org/9780521679718. Teachers wishing to have access to the website should contact solutions@cambridge.org for registration details.

In all new publications, errors and typographical mistakes are virtually unavoidable, and we would be grateful to any reader who brings instances to our attention. Retrospectively, we would like to record our thanks to Reinhard Gerndt, Paul Renteln and Joe Tenn for making us aware of some errors in the second edition. Finally, we are extremely grateful to Dave Green for his considerable and continuing advice concerning LaTeX.

Ken Riley, Michael Hobson,
Cambridge, 2006

Preface to the second edition

Since the publication of the first edition of this book, both through teaching the material it covers and as a result of receiving helpful comments from colleagues, we have become aware of the desirability of changes in a number of areas. The most important of these is that the mathematical preparation of current senior college and university entrants is now less thorough than it used to be. To match this, we decided to include a preliminary chapter covering areas such as polynomial equations, trigonometric identities, coordinate geometry, partial fractions, binomial expansions, necessary and sufficient condition and proof by induction and contradiction.

Whilst the general level of what is included in this second edition has not been raised, some areas have been expanded to take in topics we now feel were not adequately covered in the first. In particular, increased attention has been given to non-square sets of simultaneous linear equations and their associated matrices. We hope that this more extended treatment, together with the inclusion of singular value matrix decomposition, will make the material of more practical use to engineering students. In the same spirit, an elementary treatment of linear recurrence relations has been included. The topic of normal modes has been given a small chapter of its own, though the links to matrices on the one hand, and to representation theory on the other, have not been lost.

Elsewhere, the presentation of probability and statistics has been reorganised to give the two aspects more nearly equal weights. The early part of the probability chapter has been rewritten in order to present a more coherent development based on Boolean algebra, the fundamental axioms of probability theory and the properties of intersections and unions. Whilst this is somewhat more formal than previously, we think that it has not reduced the accessibility of these topics and hope that it has increased it. The scope of the chapter has been somewhat extended to include all physically important distributions and an introduction to cumulants.

Statistics now occupies a substantial chapter of its own, one that includes systematic discussions of estimators and their efficiency, sample distributions and t- and F-tests for comparing means and variances. Other new topics are applications of the chi-squared distribution, maximum-likelihood parameter estimation and least-squares fitting. In other chapters we have added material on the following topics: curvature, envelopes, curve-sketching, more refined numerical methods for differential equations and the elements of integration using Monte Carlo techniques.

Over the last four years we have received somewhat mixed feedback about the number of exercises at the ends of the various chapters. After consideration, we decided to increase the number substantially, partly to correspond to the additional topics covered in the text but mainly to give both students and their teachers a wider choice. There are now nearly 800 such exercises, many with several parts. An even more vexed question has been whether to provide hints and answers to all the exercises or just to 'the odd-numbered' ones, as is the normal practice for textbooks in the United States, thus making the remainder more suitable for setting as homework. In the end, we decided that hints and outline solutions should be provided for all the exercises, in order to facilitate independent study while leaving the details of the calculation as a task for the student.

In conclusion, we hope that this edition will be thought by its users to be 'heading in the right direction' and would like to place on record our thanks to all who have helped to bring about the changes and adjustments. Naturally, those colleagues who have noted errors or ambiguities in the first edition and brought them to our attention figure high on the list, as do the staff at The Cambridge University Press. In particular, we are grateful to Dave Green for continued LaTeX advice, Susan Parkinson for copy-editing the second edition with her usual keen eye for detail and flair for crafting coherent prose and Alison Woollatt for once again turning our basic LaTeX into a beautifully typeset book. Our thanks go to all of them, though of course we accept full responsibility for any remaining errors or ambiguities, of which, as with any new publication, there are bound to be some.

On a more personal note, KFR again wishes to thank his wife Penny for her unwavering support, not only in his academic and tutorial work, but also in their joint efforts to convert time at the bridge table into 'green points' on their record. MPH is once more indebted to his wife, Becky, and his mother, Pat, for their tireless support and encouragement above and beyond the call of duty. MPH dedicates his contribution to this book to the memory of his father, Ronald Leonard Hobson, whose gentle kindness, patient understanding and unbreakable spirit made all things seem possible.

Ken Riley, Michael Hobson
Cambridge, 2002

Preface to the first edition

A knowledge of mathematical methods is important for an increasing number of university and college courses, particularly in physics, engineering and chemistry, but also in more general science. Students embarking on such courses come from diverse mathematical backgrounds, and their core knowledge varies considerably. We have therefore decided to write a textbook that assumes knowledge only of material that can be expected to be familiar to all the current generation of students starting physical science courses at university. In the United Kingdom this corresponds to the standard of Mathematics A-level, whereas in the United States the material assumed is that which would normally be covered at junior college.

Starting from this level, the first six chapters cover a collection of topics with which the reader may already be familiar, but which are here extended and applied to typical problems encountered by first-year university students. They are aimed at providing a common base of general techniques used in the development of the remaining chapters. Students who have had additional preparation, such as Further Mathematics at A-level, will find much of this material straightforward.

Following these opening chapters, the remainder of the book is intended to cover at least that mathematical material which an undergraduate in the physical sciences might encounter up to the end of his or her course. The book is also appropriate for those beginning graduate study with a mathematical content, and naturally much of the material forms parts of courses for mathematics students. Furthermore, the text should provide a useful reference for research workers.

The general aim of the book is to present a topic in three stages. The first stage is a qualitative introduction, wherever possible from a physical point of view. The second is a more formal presentation, although we have deliberately avoided strictly mathematical questions such as the existence of limits, uniform convergence, the interchanging of integration and summation orders, etc. on the grounds that ‘this is the real world; it must behave reasonably’. Finally a worked example is presented, often drawn from familiar situations in physical science and engineering. These examples have generally been fully worked, since, in the authors’ experience, partially worked examples are unpopular with students. Only in a few cases, where trivial algebraic manipulation is involved, or where repetition of the main text would result, has an example been left as an exercise for the reader. Nevertheless, a number of exercises also appear at the end of each chapter, and these should give the reader ample opportunity to test his or her understanding. Hints and answers to these exercises are also provided.

With regard to the presentation of the mathematics, it has to be accepted that many equations (especially partial differential equations) can be written more compactly by using subscripts, e.g. u_{xy} for a second partial derivative, instead of the more familiar $\partial^2 u/\partial x \partial y$, and that this certainly saves typographical space. However, for many students, the labour of mentally unpacking such equations is sufficiently great that it is not possible to think of an equation’s physical interpretation at the same time. Consequently, wherever possible we have decided to write out such expressions in their more obvious but longer form.

During the writing of this book we have received much help and encouragement from various colleagues at the Cavendish Laboratory, Clare College, Trinity Hall and Peterhouse. In particular, we would like to thank Peter Scheuer, whose comments and general enthusiasm proved invaluable in the early stages. For reading sections of the manuscript, for pointing out misprints and for numerous useful comments, we thank many of our students and colleagues at the University of Cambridge. We are especially grateful to Chris Doran, John Huber, Garth Leder, Tom Körner and, not least, Mike Stobbs, who, sadly, died before the book was completed. We also extend our thanks to the University of Cambridge and the Cavendish teaching staff, whose examination questions and lecture hand-outs have collectively provided the basis for some of the examples included. Of course, any errors and ambiguities remaining are entirely the responsibility of the authors, and we would be most grateful to have them brought to our attention.

We are indebted to Dave Green for a great deal of advice concerning typesetting in LaTeX and to Andrew Lovatt for various other computing tips. Our thanks also go to Anja Visser and Graça Rocha for enduring many hours of (sometimes heated) debate. At Cambridge University Press, we are very grateful to our editor Adam Black for his help and patience and to Alison Woollatt for her expert typesetting of such a complicated text. We also thank our copy-editor Susan Parkinson for many useful suggestions that have undoubtedly improved the style of the book.

Finally, on a personal note, KFR wishes to thank his wife Penny, not only for a long and happy marriage, but also for her support and understanding during his recent illness – and when things have not gone too well at the bridge table! MPH is indebted both to Rebecca Morris and to his parents for their tireless support and patience, and for their unending supplies of tea. SJB is grateful to Anthony Gritten for numerous relaxing discussions about J. S. Bach, to Susannah Ticciati for her patience and understanding, and to Kate Isaak for her calming late-night e-mails from the USA.

Ken Riley, Michael Hobson and Stephen Bence
Cambridge, 1997

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Preliminary algebra

This opening chapter reviews the basic algebra of which a working knowledge is presumed in the rest of the book. Many students will be familiar with much, if not all, of it, but recent changes in what is studied during secondary education mean that it cannot be taken for granted that they will already have a mastery of all the topics presented here. The reader may assess which areas need further study or revision by attempting the exercises at the end of the chapter. The main areas covered are polynomial equations and the related topic of partial fractions, curve sketching, coordinate geometry, trigonometric identities and the notions of proof by induction or contradiction.

1.1 Simple functions and equations

It is normal practice when starting the mathematical investigation of a physical problem to assign an algebraic symbol to the quantity whose value is sought, either numerically or as an explicit algebraic expression. For the sake of definiteness, in this chapter we will use x to denote this quantity most of the time. Subsequent steps in the analysis involve applying a combination of known laws, consistency conditions and (possibly) given constraints to derive one or more equations satisfied by x. These equations may take many forms, ranging from a simple polynomial equation to, say, a partial differential equation with several boundary conditions. Some of the more complicated possibilities are treated in the later chapters of this book, but for the present we will be concerned with techniques for the solution of relatively straightforward algebraic equations.

1.1.1 Polynomials and polynomial equations

Firstly we consider the simplest type of equation, a *polynomial equation*, in which a *polynomial* expression in x, denoted by $f(x)$, is set equal to zero and thereby forms an equation which is satisfied by particular values of x, called the *roots* of the equation:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0. \tag{1.1}$$

Here n is an integer > 0, called the *degree* of both the polynomial and the equation, and the known coefficients $a_0, a_1, \ldots, a_n$ are real quantities with $a_n \neq 0$.

Equations such as (1.1) arise frequently in physical problems, the coefficients a_i being determined by the physical properties of the system under study. What is needed is to find some or all of the roots of (1.1), i.e. the x-values, α_k, that satisfy $f(\alpha_k) = 0$; here k is an index that, as we shall see later, can take up to n different values, i.e. $k = 1, 2, \ldots, n$. The roots of the polynomial equation can equally well be described as the zeros of the polynomial. When they are *real*, they correspond to the points at which a graph of $f(x)$ crosses the x-axis. Roots that are complex (see chapter 3) do not have such a graphical interpretation.

For polynomial equations containing powers of x greater than x^4 general methods do not exist for obtaining explicit expressions for the roots α_k. Even for $n = 3$ and $n = 4$ the prescriptions for obtaining the roots are sufficiently complicated that it is usually preferable to obtain exact or approximate values by other methods. Only for $n = 1$ and $n = 2$ can closed-form solutions be given. These results will be well known to the reader, but they are given here for the sake of completeness. For $n = 1$, (1.1) reduces to the *linear* equation

$$a_1 x + a_0 = 0; \tag{1.2}$$

the solution (root) is $\alpha_1 = -a_0/a_1$. For $n = 2$, (1.1) reduces to the *quadratic* equation

$$a_2 x^2 + a_1 x + a_0 = 0; \tag{1.3}$$

the two roots α_1 and α_2 are given by

$$\alpha_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}. \tag{1.4}$$

When discussing specifically quadratic equations, as opposed to more general polynomial equations, it is usual to write the equation in one of the two notations

$$ax^2 + bx + c = 0, \qquad ax^2 + 2bx + c = 0, \tag{1.5}$$

with respective explicit pairs of solutions

$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad \alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - ac}}{a}. \tag{1.6}$$

Of course, these two notations are entirely equivalent and the only important point is to associate each form of answer with the corresponding form of equation; most people keep to one form, to avoid any possible confusion.

If the value of the quantity appearing under the square root sign is positive then both roots are real; if it is negative then the roots form a complex conjugate pair, i.e. they are of the form $p \pm iq$ with p and q real (see chapter 3); if it has zero value then the two roots are equal and special considerations usually arise.

Thus linear and quadratic equations can be dealt with in a cut-and-dried way. We now turn to methods for obtaining partial information about the roots of higher-degree polynomial equations. In some circumstances the knowledge that an equation has a root lying in a certain range, or that it has no real roots at all, is all that is actually required. For example, in the design of electronic circuits it is necessary to know whether the current in a proposed circuit will break into spontaneous oscillation. To test this, it is sufficient to establish whether a certain polynomial equation, whose coefficients are determined by the physical parameters of the circuit, has a root with a positive real part (see chapter 3); complete determination of all the roots is not needed for this purpose. If the complete set of roots of a polynomial equation is required, it can usually be obtained to any desired accuracy by numerical methods such as those described in chapter 27.

There is no explicit step-by-step approach to finding the roots of a general polynomial equation such as (1.1). In most cases analytic methods yield only information *about* the roots, rather than their exact values. To explain the relevant techniques we will consider a particular example, 'thinking aloud' on paper and expanding on special points about methods and lines of reasoning. In more routine situations such comment would be absent and the whole process briefer and more tightly focussed.

Example: the cubic case

Let us investigate the roots of the equation

$$g(x) = 4x^3 + 3x^2 - 6x - 1 = 0 \tag{1.7}$$

or, in an alternative phrasing, investigate the zeros of $g(x)$. We note first of all that this is a *cubic* equation. It can be seen that for x large and positive $g(x)$ will be large and positive and, equally, that for x large and negative $g(x)$ will be large and negative. Therefore, intuitively (or, more formally, by continuity) $g(x)$ must cross the x-axis at least once and so $g(x) = 0$ must have at least one real root. Furthermore, it can be shown that if $f(x)$ is an nth-degree polynomial then the graph of $f(x)$ must cross the x-axis an even or odd number of times as x varies between $-\infty$ and $+\infty$, according to whether n itself is even or odd. Thus a polynomial of odd degree always has at least one real root, but one of even degree may have no real root. A small complication, discussed later in this section, occurs when repeated roots arise.

Having established that $g(x) = 0$ has at least one real root, we may ask how many real roots it *could* have. To answer this we need one of the fundamental theorems of algebra, mentioned above:

An nth-degree polynomial equation has exactly n roots.

It should be noted that this does not imply that there are n *real* roots (only that there are not more than n); some of the roots may be of the form $p + iq$.

To make the above theorem plausible and to see what is meant by repeated roots, let us suppose that the nth-degree polynomial equation $f(x) = 0$, (1.1), has r roots $\alpha_1, \alpha_2, \ldots, \alpha_r$, considered distinct for the moment. That is, we suppose that $f(\alpha_k) = 0$ for $k = 1, 2, \ldots, r$, so that $f(x)$ vanishes only when x is equal to one of the r values α_k. But the same can be said for the function

$$F(x) = A(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_r), \tag{1.8}$$

in which A is a non-zero constant; $F(x)$ can clearly be multiplied out to form a polynomial expression.

We now call upon a second fundamental result in algebra: that if two polynomial functions $f(x)$ and $F(x)$ have equal values for *all* values of x, then their coefficients are equal on a term-by-term basis. In other words, we can equate the coefficients of each and every power of x in the two expressions (1.8) and (1.1); in particular we can equate the coefficients of the highest power of x. From this we have $Ax^r \equiv a_n x^n$ and thus that $r = n$ and $A = a_n$. As r is both equal to n and to the number of roots of $f(x) = 0$, we conclude that the nth-degree polynomial $f(x) = 0$ has n roots. (Although this line of reasoning may make the theorem plausible, it does not constitute a proof since we have not shown that it is permissible to write $f(x)$ in the form of equation (1.8).)

We next note that the condition $f(\alpha_k) = 0$ for $k = 1, 2, \ldots, r$, could also be met if (1.8) were replaced by

$$F(x) = A(x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2} \cdots (x - \alpha_r)^{m_r}, \tag{1.9}$$

with $A = a_n$. In (1.9) the m_k are integers ≥ 1 and are known as the multiplicities of the roots, m_k being the multiplicity of α_k. Expanding the right-hand side (RHS) leads to a polynomial of degree $m_1 + m_2 + \cdots + m_r$. This sum must be equal to n. Thus, if any of the m_k is greater than unity then the number of *distinct* roots, r, is less than n; the total number of roots remains at n, but one or more of the α_k counts more than once. For example, the equation

$$F(x) = A(x - \alpha_1)^2(x - \alpha_2)^3(x - \alpha_3)(x - \alpha_4) = 0$$

has exactly seven roots, α_1 being a double root and α_2 a triple root, whilst α_3 and α_4 are unrepeated (*simple*) roots.

We can now say that our particular equation (1.7) has either one or three real roots but in the latter case it may be that not all the roots are distinct. To decide how many real roots the equation has, we need to anticipate two ideas from the

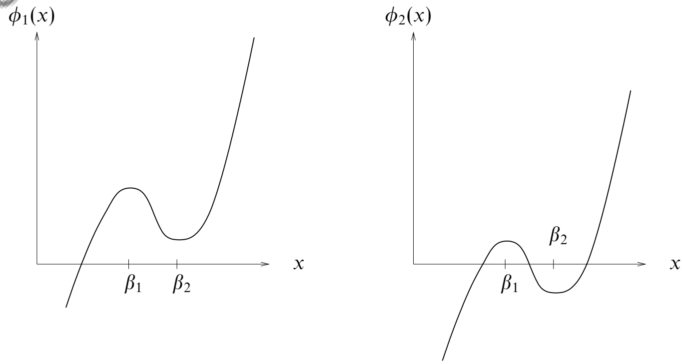

Figure 1.1 Two curves $\phi_1(x)$ and $\phi_2(x)$, both with zero derivatives at the same values of x, but with different numbers of real solutions to $\phi_i(x) = 0$.

next chapter. The first of these is the notion of the derivative of a function, and the second is a result known as Rolle's theorem.

The *derivative* $f'(x)$ of a function $f(x)$ measures the slope of the tangent to the graph of $f(x)$ at that value of x (see figure 2.1 in the next chapter). For the moment, the reader with no prior knowledge of calculus is asked to accept that the derivative of ax^n is nax^{n-1}, so that the derivative $g'(x)$ of the curve $g(x) = 4x^3 + 3x^2 - 6x - 1$ is given by $g'(x) = 12x^2 + 6x - 6$. Similar expressions for the derivatives of other polynomials are used later in this chapter.

Rolle's theorem states that if $f(x)$ has equal values at two different values of x then at some point between these two x-values its derivative is equal to zero; i.e. the tangent to its graph is parallel to the x-axis at that point (see figure 2.2).

Having briefly mentioned the derivative of a function and Rolle's theorem, we now use them to establish whether $g(x)$ has one or three real zeros. If $g(x) = 0$ does have three real roots α_k, i.e. $g(\alpha_k) = 0$ for $k = 1, 2, 3$, then it follows from Rolle's theorem that between any consecutive pair of them (say α_1 and α_2) there must be some real value of x at which $g'(x) = 0$. Similarly, there must be a further zero of $g'(x)$ lying between α_2 and α_3. Thus a *necessary* condition for three real roots of $g(x) = 0$ is that $g'(x) = 0$ itself has two real roots.

However, this condition on the number of roots of $g'(x) = 0$, whilst necessary, is not *sufficient* to guarantee three real roots of $g(x) = 0$. This can be seen by inspecting the cubic curves in figure 1.1. For each of the two functions $\phi_1(x)$ and $\phi_2(x)$, the derivative is equal to zero at both $x = \beta_1$ and $x = \beta_2$. Clearly, though, $\phi_2(x) = 0$ has three real roots whilst $\phi_1(x) = 0$ has only one. It is easy to see that the crucial difference is that $\phi_1(\beta_1)$ and $\phi_1(\beta_2)$ have the same sign, whilst $\phi_2(\beta_1)$ and $\phi_2(\beta_2)$ have opposite signs.

It will be apparent that for some equations, $\phi(x) = 0$ say, $\phi'(x)$ equals zero at a value of x for which $\phi(x)$ is also zero. Then the graph of $\phi(x)$ just touches the x-axis. When this happens the value of x so found is, in fact, a double real root of the polynomial equation (corresponding to one of the m_k in (1.9) having the value 2) and must be counted twice when determining the number of real roots.

Finally, then, we are in a position to decide the number of real roots of the equation

$$g(x) = 4x^3 + 3x^2 - 6x - 1 = 0.$$

The equation $g'(x) = 0$, with $g'(x) = 12x^2 + 6x - 6$, is a quadratic equation with explicit solutions[§]

$$\beta_{1,2} = \frac{-3 \pm \sqrt{9 + 72}}{12},$$

so that $\beta_1 = -1$ and $\beta_2 = \frac{1}{2}$. The corresponding values of $g(x)$ are $g(\beta_1) = 4$ and $g(\beta_2) = -\frac{11}{4}$, which are of opposite sign. This indicates that $4x^3 + 3x^2 - 6x - 1 = 0$ has three real roots, one lying in the range $-1 < x < \frac{1}{2}$ and the others one on each side of that range.

The techniques we have developed above have been used to tackle a cubic equation, but they can be applied to polynomial equations $f(x) = 0$ of degree greater than 3. However, much of the analysis centres around the equation $f'(x) = 0$ and this itself, being then a polynomial equation of degree 3 or more, either has no closed-form general solution or one that is complicated to evaluate. Thus the amount of information that can be obtained about the roots of $f(x) = 0$ is correspondingly reduced.

A more general case

To illustrate what can (and cannot) be done in the more general case we now investigate as far as possible the real roots of

$$f(x) = x^7 + 5x^6 + x^4 - x^3 + x^2 - 2 = 0.$$

The following points can be made.

(i) This is a seventh-degree polynomial equation; therefore the number of real roots is 1, 3, 5 or 7.

(ii) $f(0)$ is negative whilst $f(\infty) = +\infty$, so there must be at least one positive root.

§ The two roots β_1, β_2 are written as $\beta_{1,2}$. By convention β_1 refers to the upper symbol in $\pm$, β_2 to the lower symbol.

(iii) The equation $f'(x) = 0$ can be written as $x(7x^5 + 30x^4 + 4x^2 - 3x + 2) = 0$ and thus $x = 0$ is a root. The derivative of $f'(x)$, denoted by $f''(x)$, equals $42x^5 + 150x^4 + 12x^2 - 6x + 2$. That $f'(x)$ is zero whilst $f''(x)$ is positive at $x = 0$ indicates (subsection 2.1.8) that $f(x)$ has a minimum there. This, together with the facts that $f(0)$ is negative and $f(\infty) = \infty$, implies that the total number of real roots to the right of $x = 0$ must be odd. Since the total number of real roots must be odd, the number to the left must be even (0, 2, 4 or 6).

This is about all that can be deduced by *simple* analytic methods in this case, although some further progress can be made in the ways indicated in exercise 1.3.

There are, in fact, more sophisticated tests that examine the relative signs of successive terms in an equation such as (1.1), and in quantities derived from them, to place limits on the numbers and positions of roots. But they are not prerequisites for the remainder of this book and will not be pursued further here.

We conclude this section with a worked example which demonstrates that the practical application of the ideas developed so far can be both short and decisive.

►*For what values of k, if any, does*

$$f(x) = x^3 - 3x^2 + 6x + k = 0$$

have three real roots?

Firstly we study the equation $f'(x) = 0$, i.e. $3x^2 - 6x + 6 = 0$. This is a quadratic equation but, using (1.6), because $6^2 < 4 \times 3 \times 6$, it can have no real roots. Therefore, it follows immediately that $f(x)$ has no maximum or minimum; consequently $f(x) = 0$ cannot have more than one real root, whatever the value of k. ◄

1.1.2 Factorising polynomials

In the previous subsection we saw how a polynomial with r given distinct zeros α_k could be constructed as the product of factors containing those zeros:

$$\begin{aligned} f(x) &= a_n(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_r)^{m_r} \\ &= a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \end{aligned} \tag{1.10}$$

with $m_1 + m_2 + \cdots + m_r = n$, the degree of the polynomial. It will cause no loss of generality in what follows to suppose that all the zeros are simple, i.e. all $m_k = 1$ and $r = n$, and this we will do.

Sometimes it is desirable to be able to reverse this process, in particular when one exact zero has been found by some method and the remaining zeros are to be investigated. Suppose that we have located one zero, α; it is then possible to write (1.10) as

$$f(x) = (x - \alpha)f_1(x), \tag{1.11}$$

where $f_1(x)$ is a polynomial of degree $n-1$. How can we find $f_1(x)$? The procedure is much more complicated to describe in a general form than to carry out for an equation with given numerical coefficients a_i. If such manipulations are too complicated to be carried out mentally, they could be laid out along the lines of an algebraic 'long division' sum. However, a more compact form of calculation is as follows. Write $f_1(x)$ as

$$f_1(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_1 x + b_0.$$

Substitution of this form into (1.11) and subsequent comparison of the coefficients of x^p for $p = n, n-1, \ldots, 1, 0$ with those in the second line of (1.10) generates the series of equations

$$\begin{aligned} b_{n-1} &= a_n, \\ b_{n-2} - \alpha b_{n-1} &= a_{n-1}, \\ b_{n-3} - \alpha b_{n-2} &= a_{n-2}, \\ &\vdots \\ b_0 - \alpha b_1 &= a_1, \\ -\alpha b_0 &= a_0. \end{aligned}$$

These can be solved successively for the b_j, starting either from the top or from the bottom of the series. In either case the final equation used serves as a check; if it is not satisfied, at least one mistake has been made in the computation – or α is not a zero of $f(x) = 0$. We now illustrate this procedure with a worked example.

►*Determine by inspection the simple roots of the equation*

$$f(x) = 3x^4 - x^3 - 10x^2 - 2x + 4 = 0$$

and hence, by factorisation, find the rest of its roots.

From the pattern of coefficients it can be seen that $x = -1$ is a solution to the equation. We therefore write

$$f(x) = (x+1)(b_3x^3 + b_2x^2 + b_1x + b_0),$$

where

$$\begin{aligned} b_3 &= 3, \\ b_2 + b_3 &= -1, \\ b_1 + b_2 &= -10, \\ b_0 + b_1 &= -2, \\ b_0 &= 4. \end{aligned}$$

These equations give $b_3 = 3, b_2 = -4, b_1 = -6, b_0 = 4$ (check) and so

$$f(x) = (x+1)f_1(x) = (x+1)(3x^3 - 4x^2 - 6x + 4).$$

We now note that $f_1(x) = 0$ if x is set equal to 2. Thus $x - 2$ is a factor of $f_1(x)$, which therefore can be written as

$$f_1(x) = (x-2)f_2(x) = (x-2)(c_2x^2 + c_1x + c_0)$$

with

$$\begin{aligned} c_2 &= 3, \\ c_1 - 2c_2 &= -4, \\ c_0 - 2c_1 &= -6, \\ -2c_0 &= 4. \end{aligned}$$

These equations determine $f_2(x)$ as $3x^2 + 2x - 2$. Since $f_2(x) = 0$ is a quadratic equation, its solutions can be written explicitly as

$$x = \frac{-1 \pm \sqrt{1+6}}{3}.$$

Thus the four roots of $f(x) = 0$ are $-1, 2, \frac{1}{3}(-1 + \sqrt{7})$ and $\frac{1}{3}(-1 - \sqrt{7})$. ◀

1.1.3 Properties of roots

From the fact that a polynomial equation can be written in any of the alternative forms

$$\begin{aligned} f(x) &= a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0, \\ f(x) &= a_n(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_r)^{m_r} = 0, \\ f(x) &= a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = 0, \end{aligned}$$

it follows that it must be possible to express the coefficients a_i in terms of the roots α_k. To take the most obvious example, comparison of the constant terms (formally the coefficient of x^0) in the first and third expressions shows that

$$a_n(-\alpha_1)(-\alpha_2)\cdots(-\alpha_n) = a_0,$$

or, using the product notation,

$$\prod_{k=1}^{n} \alpha_k = (-1)^n \frac{a_0}{a_n}. \tag{1.12}$$

Only slightly less obvious is a result obtained by comparing the coefficients of x^{n-1} in the same two expressions of the polynomial:

$$\sum_{k=1}^{n} \alpha_k = -\frac{a_{n-1}}{a_n}. \tag{1.13}$$

Comparing the coefficients of other powers of x yields further results, though they are of less general use than the two just given. One such, which the reader may wish to derive, is

$$\sum_{j=1}^{n}\sum_{k>j}^{n} \alpha_j\alpha_k = \frac{a_{n-2}}{a_n}. \tag{1.14}$$

In the case of a quadratic equation these root properties are used sufficiently often that they are worth stating explicitly, as follows. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α_1 and α_2 then

$$\alpha_1 + \alpha_2 = -\frac{b}{a},$$
$$\alpha_1\alpha_2 = \frac{c}{a}.$$

If the alternative standard form for the quadratic is used, b is replaced by $2b$ in both the equation and the first of these results.

►*Find a cubic equation whose roots are* $-4, 3$ *and* 5.

From results (1.12) – (1.14) we can compute that, arbitrarily setting $a_3 = 1$,

$$-a_2 = \sum_{k=1}^{3} \alpha_k = 4, \qquad a_1 = \sum_{j=1}^{3}\sum_{k>j}^{3} \alpha_j\alpha_k = -17, \qquad a_0 = (-1)^3 \prod_{k=1}^{3} \alpha_k = 60.$$

Thus a possible cubic equation is $x^3 + (-4)x^2 + (-17)x + (60) = 0$. Of course, any multiple of $x^3 - 4x^2 - 17x + 60 = 0$ will do just as well. ◄

1.2 Trigonometric identities

So many of the applications of mathematics to physics and engineering are concerned with periodic, and in particular sinusoidal, behaviour that a sure and ready handling of the corresponding mathematical functions is an essential skill. Even situations with no obvious periodicity are often expressed in terms of periodic functions for the purposes of analysis. Later in this book whole chapters are devoted to developing the techniques involved, but as a necessary prerequisite we here establish (or remind the reader of) some standard identities with which he or she should be fully familiar, so that the manipulation of expressions containing sinusoids becomes automatic and reliable. So as to emphasise the angular nature of the argument of a sinusoid we will denote it in this section by θ rather than x.

1.2.1 Single-angle identities

We give without proof the basic identity satisfied by the sinusoidal functions $\sin\theta$ and $\cos\theta$, namely

$$\cos^2\theta + \sin^2\theta = 1. \tag{1.15}$$

If $\sin\theta$ and $\cos\theta$ have been defined geometrically in terms of the coordinates of a point on a circle, a reference to the name of Pythagoras will suffice to establish this result. If they have been defined by means of series (with θ expressed in radians) then the reader should refer to Euler's equation (3.23) on page 93, and note that $e^{i\theta}$ has unit modulus if θ is real.

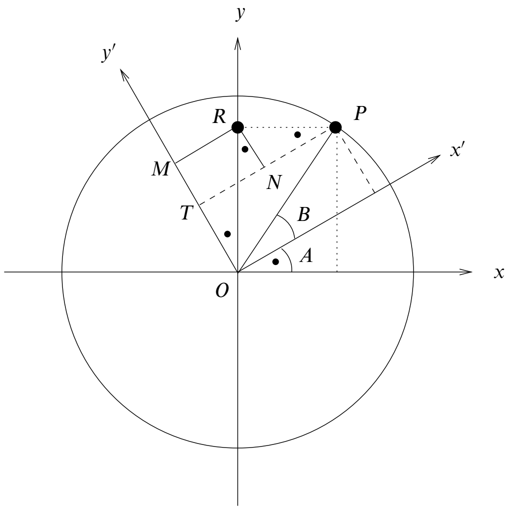

Figure 1.2 Illustration of the compound-angle identities. Refer to the main text for details.

Other standard single-angle formulae derived from (1.15) by dividing through by various powers of $\sin\theta$ and $\cos\theta$ are

$$1 + \tan^2\theta = \sec^2\theta, \tag{1.16}$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta. \tag{1.17}$$

1.2.2 Compound-angle identities

The basis for building expressions for the sinusoidal functions of compound angles are those for the sum and difference of just two angles, since all other cases can be built up from these, in principle. Later we will see that a study of complex numbers can provide a more efficient approach in some cases.

To prove the basic formulae for the sine and cosine of a compound angle $A+B$ in terms of the sines and cosines of A and B, we consider the construction shown in figure 1.2. It shows two sets of axes, Oxy and $Ox'y'$, with a common origin but rotated with respect to each other through an angle A. The point P lies on the unit circle centred on the common origin O and has coordinates $\cos(A+B), \sin(A+B)$ with respect to the axes Oxy and coordinates $\cos B, \sin B$ with respect to the axes $Ox'y'$.

Parallels to the axes Oxy (dotted lines) and $Ox'y'$ (broken lines) have been drawn through P. Further parallels (MR and RN) to the $Ox'y'$ axes have been drawn through R, the point $(0, \sin(A+B))$ in the Oxy system. That all the angles marked with the symbol $\bullet$ are equal to A follows from the simple geometry of right-angled triangles and crossing lines.

We now determine the coordinates of P in terms of lengths in the figure, expressing those lengths in terms of both sets of coordinates:

$$\begin{aligned}
\text{(i)}\ \cos B = x' &= TN + NP = MR + NP \\
&= OR \sin A + RP \cos A = \sin(A+B)\sin A + \cos(A+B)\cos A; \\
\text{(ii)}\ \sin B = y' &= OM - TM = OM - NR \\
&= OR \cos A - RP \sin A = \sin(A+B)\cos A - \cos(A+B)\sin A.
\end{aligned}$$

Now, if equation (i) is multiplied by $\sin A$ and added to equation (ii) multiplied by $\cos A$, the result is

$$\sin A \cos B + \cos A \sin B = \sin(A+B)(\sin^2 A + \cos^2 A) = \sin(A+B).$$

Similarly, if equation (ii) is multiplied by $\sin A$ and subtracted from equation (i) multiplied by $\cos A$, the result is

$$\cos A \cos B - \sin A \sin B = \cos(A+B)(\cos^2 A + \sin^2 A) = \cos(A+B).$$

Corresponding graphically based results can be derived for the sines and cosines of the difference of two angles; however, they are more easily obtained by setting B to $-B$ in the previous results and remembering that $\sin B$ becomes $-\sin B$ whilst $\cos B$ is unchanged. The four results may be summarised by

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{1.18}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B. \tag{1.19}$$

Standard results can be deduced from these by setting one of the two angles equal to π or to $\pi/2$:

$$\sin(\pi - \theta) = \sin\theta, \qquad \cos(\pi - \theta) = -\cos\theta, \tag{1.20}$$

$$\sin\left(\tfrac{1}{2}\pi - \theta\right) = \cos\theta, \qquad \cos\left(\tfrac{1}{2}\pi - \theta\right) = \sin\theta, \tag{1.21}$$

From these basic results many more can be derived. An immediate deduction, obtained by taking the ratio of the two equations (1.18) and (1.19) and then dividing both the numerator and denominator of this ratio by $\cos A \cos B$, is

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}. \tag{1.22}$$

One application of this result is a test for whether two lines on a graph are orthogonal (perpendicular); more generally, it determines the angle between them. The standard notation for a straight-line graph is $y = mx + c$, in which m is the slope of the graph and c is its intercept on the y-axis. It should be noted that the slope m is also the tangent of the angle the line makes with the x-axis.

Consequently the angle θ_{12} between two such straight-line graphs is equal to the difference in the angles they individually make with the x-axis, and the tangent of that angle is given by (1.22):

$$\tan\theta_{12} = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2}. \tag{1.23}$$

For the lines to be orthogonal we must have $\theta_{12} = \pi/2$, i.e. the final fraction on the RHS of the above equation must equal ∞, and so

$$m_1 m_2 = -1. \tag{1.24}$$

A kind of inversion of equations (1.18) and (1.19) enables the sum or difference of two sines or cosines to be expressed as the product of two sinusoids; the procedure is typified by the following. Adding together the expressions given by (1.18) for $\sin(A+B)$ and $\sin(A-B)$ yields

$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B.$$

If we now write $A + B = C$ and $A - B = D$, this becomes

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right). \tag{1.25}$$

In a similar way each of the following equations can be derived:

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right), \tag{1.26}$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right), \tag{1.27}$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right). \tag{1.28}$$

The minus sign on the right of the last of these equations should be noted; it may help to avoid overlooking this 'oddity' to recall that if $C > D$ then $\cos C < \cos D$.

1.2.3 Double- and half-angle identities

Double-angle and half-angle identities are needed so often in practical calculations that they should be committed to memory by any physical scientist. They can be obtained by setting B equal to A in results (1.18) and (1.19). When this is done, and use made of equation (1.15), the following results are obtained:

$$\sin 2\theta = 2\sin\theta\cos\theta, \tag{1.29}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta,\end{aligned} \tag{1.30}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}. \tag{1.31}$$

A further set of identities enables sinusoidal functions of θ to be expressed in terms of polynomial functions of a variable $t = \tan(\theta/2)$. They are not used in their primary role until the next chapter, but we give a derivation of them here for reference.

If $t = \tan(\theta/2)$, then it follows from (1.16) that $1+t^2 = \sec^2(\theta/2)$ and $\cos(\theta/2) = (1+t^2)^{-1/2}$, whilst $\sin(\theta/2) = t(1+t^2)^{-1/2}$. Now, using (1.29) and (1.30), we may write:

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2t}{1+t^2}, \tag{1.32}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{1-t^2}{1+t^2}, \tag{1.33}$$

$$\tan\theta = \frac{2t}{1-t^2}. \tag{1.34}$$

It can be further shown that the derivative of θ with respect to t takes the algebraic form $2/(1+t^2)$. This completes a package of results that enables expressions involving sinusoids, particularly when they appear as integrands, to be cast in more convenient algebraic forms. The proof of the derivative property and examples of use of the above results are given in subsection (2.2.7).

We conclude this section with a worked example which is of such a commonly occurring form that it might be considered a standard procedure.

►*Solve for θ the equation*

$$a\sin\theta + b\cos\theta = k,$$

where a, b and k are given real quantities.

To solve this equation we make use of result (1.18) by setting $a = K\cos\phi$ and $b = K\sin\phi$ for suitable values of K and ϕ. We then have

$$k = K\cos\phi\sin\theta + K\sin\phi\cos\theta = K\sin(\theta+\phi),$$

with

$$K^2 = a^2 + b^2 \qquad \text{and} \qquad \phi = \tan^{-1}\frac{b}{a}.$$

Whether ϕ lies in $0 \le \phi \le \pi$ or in $-\pi < \phi < 0$ has to be determined by the individual signs of a and b. The solution is thus

$$\theta = \sin^{-1}\left(\frac{k}{K}\right) - \phi,$$

with K and ϕ as given above. Notice that the inverse sine yields two values in the range 0 to 2π and that there is no real solution to the original equation if $|k| > |K| = (a^2+b^2)^{1/2}$. ◄

1.3 Coordinate geometry

We have already mentioned the standard form for a straight-line graph, namely

$$y = mx + c, \tag{1.35}$$

representing a linear relationship between the independent variable x and the dependent variable y. The slope m is equal to the tangent of the angle the line makes with the x-axis whilst c is the intercept on the y-axis.

An alternative form for the equation of a straight line is

$$ax + by + k = 0, \tag{1.36}$$

to which (1.35) is clearly connected by

$$m = -\frac{a}{b} \qquad \text{and} \qquad c = -\frac{k}{b}.$$

This form treats x and y on a more symmetrical basis, the intercepts on the two axes being $-k/a$ and $-k/b$ respectively.

A power relationship between two variables, i.e. one of the form $y = Ax^n$, can also be cast into straight-line form by taking the logarithms of both sides. Whilst it is normal in mathematical work to use natural logarithms (to base e, written $\ln x$), for practical investigations logarithms to base 10 are often employed. In either case the form is the same, but it needs to be remembered which has been used when recovering the value of A from fitted data. In the mathematical (base e) form, the power relationship becomes

$$\ln y = n \ln x + \ln A. \tag{1.37}$$

Now the slope gives the power n, whilst the intercept on the $\ln y$ axis is $\ln A$, which yields A, either by exponentiation or by taking antilogarithms.

The other standard coordinate forms of two-dimensional curves that students should know and recognise are those concerned with the *conic sections* – so called because they can all be obtained by taking suitable sections across a (double) cone. Because the conic sections can take many different orientations and scalings their general form is complex,

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0, \tag{1.38}$$

but each can be represented by one of four generic forms, an ellipse, a parabola, a hyperbola or, the degenerate form, a pair of straight lines. If they are reduced to their standard representations, in which axes of symmetry are made to coincide with the coordinate axes, the first three take the forms

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1 \qquad \text{(ellipse)}, \tag{1.39}$$

$$(y-\beta)^2 = 4a(x-\alpha) \qquad \text{(parabola)}, \tag{1.40}$$

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1 \qquad \text{(hyperbola)}. \tag{1.41}$$

Here, (α, β) gives the position of the 'centre' of the curve, usually taken as the origin $(0,0)$ when this does not conflict with any imposed conditions. The parabola equation given is that for a curve symmetric about a line parallel to the x-axis. For one symmetrical about a parallel to the y-axis the equation would read $(x-\alpha)^2 = 4a(y-\beta)$.

Of course, the circle is the special case of an ellipse in which $b = a$ and the equation takes the form

$$(x-\alpha)^2 + (y-\beta)^2 = a^2. \tag{1.42}$$

The distinguishing characteristic of this equation is that when it is expressed in the form (1.38) the coefficients of x^2 and y^2 are equal and that of xy is zero; this property is not changed by any reorientation or scaling and so acts to identify a general conic as a circle.

Definitions of the conic sections in terms of geometrical properties are also available; for example, a parabola can be defined as the locus of a point that is always at the same distance from a given straight line (the *directrix*) as it is from a given point (the *focus*). When these properties are expressed in Cartesian coordinates the above equations are obtained. For a circle, the defining property is that all points on the curve are a distance a from (α, β); (1.42) expresses this requirement very directly. In the following worked example we derive the equation for a parabola.

►*Find the equation of a parabola that has the line $x = -a$ as its directrix and the point $(a, 0)$ as its focus.*

Figure 1.3 shows the situation in Cartesian coordinates. Expressing the defining requirement that PN and PF are equal in length gives

$$(x+a) = [(x-a)^2 + y^2]^{1/2} \quad \Rightarrow \quad (x+a)^2 = (x-a)^2 + y^2$$

which, on expansion of the squared terms, immediately gives $y^2 = 4ax$. This is (1.40) with α and β both set equal to zero. ◄

Although the algebra is more complicated, the same method can be used to derive the equations for the ellipse and the hyperbola. In these cases the distance from the fixed point is a definite fraction, e, known as the *eccentricity*, of the distance from the fixed line. For an ellipse $0 < e < 1$, for a circle $e = 0$, and for a hyperbola $e > 1$. The parabola corresponds to the case $e = 1$.

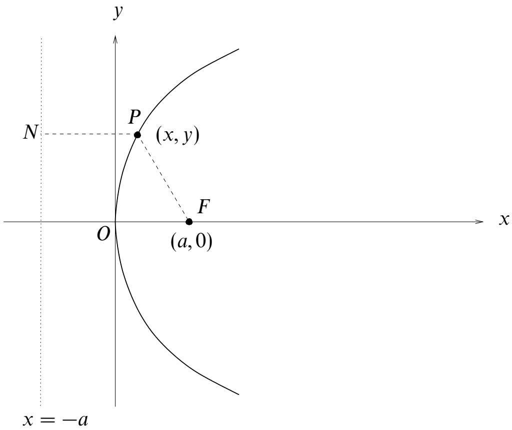

Figure 1.3 Construction of a parabola using the point $(a, 0)$ as the focus and the line $x = -a$ as the directrix.

The values of a and b (with $a \geq b$) in equation (1.39) for an ellipse are related to e through

$$e^2 = \frac{a^2 - b^2}{a^2}$$

and give the lengths of the semi-axes of the ellipse. If the ellipse is centred on the origin, i.e. $\alpha = \beta = 0$, then the focus is $(-ae, 0)$ and the directrix is the line $x = -a/e$.

For each conic section curve, although we have two variables, x and y, they are not independent, since if one is given then the other can be determined. However, determining y when x is given, say, involves solving a quadratic equation on each occasion, and so it is convenient to have *parametric* representations of the curves. A parametric representation allows each point on a curve to be associated with a unique value of a *single* parameter t. The simplest parametric representations for the conic sections are as given below, though that for the hyperbola uses hyperbolic functions, not formally introduced until chapter 3. That they do give valid parameterizations can be verified by substituting them into the standard forms (1.39)–(1.41); in each case the standard form is reduced to an algebraic or trigonometric identity.

$$\begin{aligned}
x &= \alpha + a\cos\phi, & y &= \beta + b\sin\phi & &\text{(ellipse)},\\
x &= \alpha + at^2, & y &= \beta + 2at & &\text{(parabola)},\\
x &= \alpha + a\cosh\phi, & y &= \beta + b\sinh\phi & &\text{(hyperbola)}.
\end{aligned}$$

As a final example illustrating several topics from this section we now prove the well-known result that the angle subtended by a diameter at any point on a circle is a right angle.

►*Taking the diameter to be the line joining $Q = (-a, 0)$ and $R = (a, 0)$ and the point P to be any point on the circle $x^2 + y^2 = a^2$, prove that angle QPR is a right angle.*

If P is the point (x, y), the slope of the line QP is

$$m_1 = \frac{y - 0}{x - (-a)} = \frac{y}{x + a}.$$

That of RP is

$$m_2 = \frac{y - 0}{x - (a)} = \frac{y}{x - a}.$$

Thus

$$m_1 m_2 = \frac{y^2}{x^2 - a^2}.$$

But, since P is on the circle, $y^2 = a^2 - x^2$ and consequently $m_1 m_2 = -1$. From result (1.24) this implies that QP and RP are orthogonal and that QPR is therefore a right angle. Note that this is true for *any* point P on the circle. ◄

1.4 Partial fractions

In subsequent chapters, and in particular when we come to study integration in chapter 2, we will need to express a function $f(x)$ that is the ratio of two polynomials in a more manageable form. To remove some potential complexity from our discussion we will assume that all the coefficients in the polynomials are real, although this is not an essential simplification.

The behaviour of $f(x)$ is crucially determined by the location of the zeros of its denominator, i.e. if $f(x)$ is written as $f(x) = g(x)/h(x)$ where both $g(x)$ and $h(x)$ are polynomials,§ then $f(x)$ changes extremely rapidly when x is close to those values α_i that are the roots of $h(x) = 0$. To make such behaviour explicit, we write $f(x)$ as a sum of terms such as $A/(x - \alpha)^n$, in which A is a constant, α is one of the α_i that satisfy $h(\alpha_i) = 0$ and n is a positive integer. Writing a function in this way is known as expressing it in *partial fractions*.

Suppose, for the sake of definiteness, that we wish to express the function

$$f(x) = \frac{4x + 2}{x^2 + 3x + 2}$$

§ It is assumed that the ratio has been reduced so that $g(x)$ and $h(x)$ do not contain any common factors, i.e. there is no value of x that makes both vanish at the same time. We may also assume without any loss of generality that the coefficient of the highest power of x in $h(x)$ has been made equal to unity, if necessary, by dividing both numerator and denominator by the coefficient of this highest power.

in partial fractions, i.e. to write it as

$$f(x) = \frac{g(x)}{h(x)} = \frac{4x+2}{x^2+3x+2} = \frac{A_1}{(x-\alpha_1)^{n_1}} + \frac{A_2}{(x-\alpha_2)^{n_2}} + \cdots . \tag{1.43}$$

The first question that arises is that of how many terms there should be on the right-hand side (RHS). Although some complications occur when $h(x)$ has repeated roots (these are considered below) it is clear that $f(x)$ only becomes infinite at the *two* values of x, α_1 and α_2, that make $h(x) = 0$. Consequently the RHS can only become infinite at the same two values of x and therefore contains only two partial fractions – these are the ones shown explicitly. This argument can be trivially extended (again temporarily ignoring the possibility of repeated roots of $h(x)$) to show that if $h(x)$ is a polynomial of degree n then there should be n terms on the RHS, each containing a different root α_i of the equation $h(\alpha_i) = 0$.

A second general question concerns the appropriate values of the n_i. This is answered by putting the RHS over a common denominator, which will clearly have to be the product $(x-\alpha_1)^{n_1}(x-\alpha_2)^{n_2}\cdots$. Comparison of the highest power of x in this new RHS with the same power in $h(x)$ shows that $n_1 + n_2 + \cdots = n$. This result holds whether or not $h(x) = 0$ has repeated roots and, although we do not give a rigorous proof, strongly suggests the following correct conclusions.

- The number of terms on the RHS is equal to the number of *distinct* roots of $h(x) = 0$, each term having a different root α_i in its denominator $(x-\alpha_i)^{n_i}$.
- If α_i is a multiple root of $h(x) = 0$ then the value to be assigned to n_i in (1.43) is that of m_i when $h(x)$ is written in the product form (1.9). Further, as discussed on p. 23, A_i has to be replaced by a polynomial of degree $m_i - 1$. This is also formally true for non-repeated roots, since then both m_i and n_i are equal to unity.

Returning to our specific example we note that the denominator $h(x)$ has zeros at $x = \alpha_1 = -1$ and $x = \alpha_2 = -2$; these x-values are the simple (non-repeated) roots of $h(x) = 0$. Thus the partial fraction expansion will be of the form

$$\frac{4x+2}{x^2+3x+2} = \frac{A_1}{x+1} + \frac{A_2}{x+2}. \tag{1.44}$$

We now list several methods available for determining the coefficients A_1 and A_2. We also remind the reader that, as with all the explicit examples and techniques described, these methods are to be considered as models for the handling of any ratio of polynomials, with or without characteristics that make it a special case.

(i) The RHS can be put over a common denominator, in this case $(x+1)(x+2)$, and then the coefficients of the various powers of x can be equated in the numerators on both sides of the equation. This leads to

$$4x + 2 = A_1(x + 2) + A_2(x + 1),$$
$$4 = A_1 + A_2 \qquad 2 = 2A_1 + A_2.$$

Solving the simultaneous equations for A_1 and A_2 gives $A_1 = -2$ and $A_2 = 6$.

(ii) A second method is to substitute two (or more generally n) different values of x into each side of (1.44) and so obtain two (or n) simultaneous equations for the two (or n) constants A_i. To justify this practical way of proceeding it is necessary, strictly speaking, to appeal to method (i) above, which establishes that there are unique values for A_1 and A_2 valid for all values of x. It is normally very convenient to take zero as one of the values of x, but of course any set will do. Suppose in the present case that we use the values $x = 0$ and $x = 1$ and substitute in (1.44). The resulting equations are

$$\frac{2}{2} = \frac{A_1}{1} + \frac{A_2}{2},$$
$$\frac{6}{6} = \frac{A_1}{2} + \frac{A_2}{3},$$

which on solution give $A_1 = -2$ and $A_2 = 6$, as before. The reader can easily verify that any other pair of values for x (except for a pair that includes α_1 or α_2) gives the same values for A_1 and A_2.

(iii) The very reason why method (ii) fails if x is chosen as one of the roots α_i of $h(x) = 0$ can be made the basis for determining the values of the A_i corresponding to non-multiple roots without having to solve simultaneous equations. The method is conceptually more difficult than the other methods presented here, and needs results from the theory of complex variables (chapter 24) to justify it. However, we give a practical 'cookbook' recipe for determining the coefficients.

(a) To determine the coefficient A_k, imagine the denominator $h(x)$ written as the product $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$, with any m-fold repeated root giving rise to m factors in parentheses.
(b) Now set x equal to α_k and evaluate the expression obtained after omitting the factor that reads $\alpha_k - \alpha_k$.
(c) Divide the value so obtained into $g(\alpha_k)$; the result is the required coefficient A_k.

For our specific example we find that in step (a) that $h(x) = (x + 1)(x + 2)$ and that in evaluating A_1 step (b) yields $-1 + 2$, i.e. 1. Since $g(-1) = 4(-1) + 2 = -2$, step (c) gives A_1 as $(-2)/(1)$, i.e in agreement with our other evaluations. In a similar way A_2 is evaluated as $(-6)/(-1) = 6$.

Thus any one of the methods listed above shows that

$$\frac{4x+2}{x^2+3x+2} = \frac{-2}{x+1} + \frac{6}{x+2}.$$

The best method to use in any particular circumstance will depend on the complexity, in terms of the degrees of the polynomials and the multiplicities of the roots of the denominator, of the function being considered and, to some extent, on the individual inclinations of the student; some prefer lengthy but straightforward solution of simultaneous equations, whilst others feel more at home carrying through shorter but more abstract calculations in their heads.

1.4.1 Complications and special cases

Having established the basic method for partial fractions, we now show, through further worked examples, how some complications are dealt with by extensions to the procedure. These extensions are introduced one at a time, but of course in any practical application more than one may be involved.

The degree of the numerator is greater than or equal to that of the denominator

Although we have not specifically mentioned the fact, it will be apparent from trying to apply method (i) of the previous subsection to such a case, that if the degree of the numerator (m) is not less than that of the denominator (n) then the ratio of two polynomials cannot be expressed in partial fractions.

To get round this difficulty it is necessary to start by dividing the denominator $h(x)$ into the numerator $g(x)$ to obtain a further polynomial, which we will denote by $s(x)$, together with a function $t(x)$ that *is* a ratio of two polynomials for which the degree of the numerator is less than that of the denominator. The function $t(x)$ *can* therefore be expanded in partial fractions. As a formula,

$$f(x) = \frac{g(x)}{h(x)} = s(x) + t(x) \equiv s(x) + \frac{r(x)}{h(x)}. \tag{1.45}$$

It is apparent that the polynomial $r(x)$ is the *remainder* obtained when $g(x)$ is divided by $h(x)$, and, in general, will be a polynomial of degree $n-1$. It is also clear that the polynomial $s(x)$ will be of degree $m-n$. Again, the actual division process can be set out as an algebraic long division sum but is probably more easily handled by writing (1.45) in the form

$$g(x) = s(x)h(x) + r(x) \tag{1.46}$$

or, more explicitly, as

$$g(x) = (s_{m-n}x^{m-n} + s_{m-n-1}x^{m-n-1} + \cdots + s_0)h(x) + (r_{n-1}x^{n-1} + r_{n-2}x^{n-2} + \cdots + r_0) \tag{1.47}$$

and then equating coefficients.

We illustrate this procedure with the following worked example.

►*Find the partial fraction decomposition of the function*

$$f(x) = \frac{x^3 + 3x^2 + 2x + 1}{x^2 - x - 6}.$$

Since the degree of the numerator is 3 and that of the denominator is 2, a preliminary long division is necessary. The polynomial $s(x)$ resulting from the division will have degree $3 - 2 = 1$ and the remainder $r(x)$ will be of degree $2 - 1 = 1$ (or less). Thus we write

$$x^3 + 3x^2 + 2x + 1 = (s_1 x + s_0)(x^2 - x - 6) + (r_1 x + r_0).$$

From equating the coefficients of the various powers of x on the two sides of the equation, starting with the highest, we now obtain the simultaneous equations

$$\begin{aligned} 1 &= s_1, \\ 3 &= s_0 - s_1, \\ 2 &= -s_0 - 6s_1 + r_1, \\ 1 &= -6s_0 + r_0. \end{aligned}$$

These are readily solved, in the given order, to yield $s_1 = 1$, $s_0 = 4$, $r_1 = 12$ and $r_0 = 25$. Thus $f(x)$ can be written as

$$f(x) = x + 4 + \frac{12x + 25}{x^2 - x - 6}.$$

The last term can now be decomposed into partial fractions as previously. The zeros of the denominator are at $x = 3$ and $x = -2$ and the application of any method from the previous subsection yields the respective constants as $A_1 = 12\frac{1}{5}$ and $A_2 = -\frac{1}{5}$. Thus the final partial fraction decomposition of $f(x)$ is

$$x + 4 + \frac{61}{5(x-3)} - \frac{1}{5(x+2)}. \blacktriangleleft$$

Factors of the form $a^2 + x^2$ in the denominator

We have so far assumed that the roots of $h(x) = 0$, needed for the factorisation of the denominator of $f(x)$, can always be found. In principle they always can but in some cases they are not real. Consider, for example, attempting to express in partial fractions a polynomial ratio whose denominator is $h(x) = x^3 - x^2 + 2x - 2$. Clearly $x = 1$ is a zero of $h(x)$, and so a first factorisation is $(x - 1)(x^2 + 2)$. However we cannot make any further progress because the factor $x^2 + 2$ cannot be expressed as $(x - \alpha)(x - \beta)$ for any real α and β.

Complex numbers are introduced later in this book (chapter 3) and, when the reader has studied them, he or she may wish to justify the procedure set out below. It can be shown to be equivalent to that already given, but the zeros of $h(x)$ are now allowed to be complex and terms that are complex conjugates of each other are combined to leave only real terms.

Since quadratic factors of the form a^2+x^2 that appear in $h(x)$ cannot be reduced to the product of two linear factors, partial fraction expansions including them need to have numerators in the corresponding terms that are not simply constants

A_i but linear functions of x, i.e. of the form $B_i x + C_i$. Thus, in the expansion, linear terms (first-degree polynomials) in the denominator have constants (zero-degree polynomials) in their numerators, whilst quadratic terms (second-degree polynomials) in the denominator have linear terms (first-degree polynomials) in their numerators. As a symbolic formula, the partial fraction expansion of

$$\frac{g(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_p)(x^2+a_1^2)(x^2+a_2^2)\cdots(x^2+a_q^2)}$$

should take the form

$$\frac{A_1}{x-\alpha_1}+\frac{A_2}{x-\alpha_2}+\cdots+\frac{A_p}{x-\alpha_p}+\frac{B_1x+C_1}{x^2+a_1^2}+\frac{B_2x+C_2}{x^2+a_2^2}+\cdots+\frac{B_qx+C_q}{x^2+a_q^2}.$$

Of course, the degree of $g(x)$ must be less than $p+2q$; if it is not, an initial division must be carried out as demonstrated earlier.

Repeated factors in the denominator

Consider trying (incorrectly) to expand

$$f(x)=\frac{x-4}{(x+1)(x-2)^2}$$

in partial fraction form as follows:

$$\frac{x-4}{(x+1)(x-2)^2}=\frac{A_1}{x+1}+\frac{A_2}{(x-2)^2}.$$

Multiplying both sides of this supposed equality by $(x+1)(x-2)^2$ produces an equation whose LHS is linear in x, whilst its RHS is quadratic. This is clearly wrong and so an expansion in the above form cannot be valid. The correction we must make is very similar to that needed in the previous subsection, namely that since $(x-2)^2$ is a quadratic polynomial the numerator of the term containing it must be a first-degree polynomial, and not simply a constant.

The correct form for the part of the expansion containing the doubly repeated root is therefore $(Bx+C)/(x-2)^2$. Using this form and either of methods (i) and (ii) for determining the constants gives the full partial fraction expansion as

$$\frac{x-4}{(x+1)(x-2)^2}=-\frac{5}{9(x+1)}+\frac{5x-16}{9(x-2)^2},$$

as the reader may verify.

Since any term of the form $(Bx+C)/(x-\alpha)^2$ can be written as

$$\frac{B(x-\alpha)+C+B\alpha}{(x-\alpha)^2}=\frac{B}{x-\alpha}+\frac{C+B\alpha}{(x-\alpha)^2},$$

and similarly for multiply repeated roots, an alternative form for the part of the partial fraction expansion containing a repeated root α is

$$\frac{D_1}{x-\alpha}+\frac{D_2}{(x-\alpha)^2}+\cdots+\frac{D_p}{(x-\alpha)^p}. \qquad (1.48)$$

In this form, all x-dependence has disappeared from the numerators but at the expense of $p-1$ additional terms; the total number of constants to be determined remains unchanged, as it must.

When describing possible methods of determining the constants in a partial fraction expansion, we noted that method (iii), p. 20, which avoids the need to solve simultaneous equations, is restricted to terms involving non-repeated roots. In fact, it can be applied in repeated-root situations, when the expansion is put in the form (1.48), but only to find the constant in the term involving the largest inverse power of $x-\alpha$, i.e. D_p in (1.48).

We conclude this section with a more protracted worked example that contains all three of the complications discussed.

►*Resolve the following expression $F(x)$ into partial fractions:*

$$F(x) = \frac{x^5 - 2x^4 - x^3 + 5x^2 - 46x + 100}{(x^2+6)(x-2)^2}.$$

We note that the degree of the denominator (4) is not greater than that of the numerator (5), and so we must start by dividing the latter by the former. It follows, from the difference in degrees and the coefficients of the highest powers in each, that the result will be a linear expression $s_1x + s_0$ with the coefficient s_1 equal to 1. Thus the numerator of $F(x)$ must be expressible as

$$(x + s_0)(x^4 - 4x^3 + 10x^2 - 24x + 24) + (r_3x^3 + r_2x^2 + r_1x + r_0),$$

where the second factor in parentheses is the denominator of $F(x)$ written as a polynomial. Equating the coefficients of x^4 gives $-2 = -4 + s_0$ and fixes s_0 as 2. Equating the coefficients of powers less than 4 gives equations involving the coefficients r_i as follows:

$$\begin{aligned}
-1 &= -8 + 10 + r_3,\\
5 &= -24 + 20 + r_2,\\
-46 &= 24 - 48 + r_1,\\
100 &= 48 + r_0.
\end{aligned}$$

Thus the remainder polynomial $r(x)$ can be constructed and $F(x)$ written as

$$F(x) = x + 2 + \frac{-3x^3 + 9x^2 - 22x + 52}{(x^2+6)(x-2)^2} \equiv x + 2 + f(x).$$

The polynomial ratio $f(x)$ can now be expressed in partial fraction form, noting that its denominator contains both a term of the form $x^2 + a^2$ and a repeated root. Thus

$$f(x) = \frac{Bx + C}{x^2+6} + \frac{D_1}{x-2} + \frac{D_2}{(x-2)^2}.$$

We could now put the RHS of this equation over the common denominator $(x^2+6)(x-2)^2$ and find B, C, D_1 and D_2 by equating coefficients of powers of x. It is quicker, however, to use methods (iii) and (ii). Method (iii) gives D_2 as $(-24 + 36 - 44 + 52)/(4 + 6) = 2$. We choose to evaluate the other coefficients by method (ii), and setting $x = 0$, $x = 1$ and

$x = -1$ gives respectively

$$\frac{52}{24} = \frac{C}{6} - \frac{D_1}{2} + \frac{2}{4},$$
$$\frac{36}{7} = \frac{B + C}{7} - D_1 + 2,$$
$$\frac{86}{63} = \frac{C - B}{7} - \frac{D_1}{3} + \frac{2}{9}.$$

These equations reduce to

$$\begin{aligned} 4C - 12D_1 &= 40, \\ B + C - 7D_1 &= 22, \\ -9B + 9C - 21D_1 &= 72, \end{aligned}$$

with solution $B = 0$, $C = 1$, $D_1 = -3$.

Thus, finally, we may rewrite the original expression $F(x)$ in partial fractions as

$$F(x) = x + 2 + \frac{1}{x^2 + 6} - \frac{3}{x - 2} + \frac{2}{(x - 2)^2}. \blacktriangleleft$$

1.5 Binomial expansion

Earlier in this chapter we were led to consider functions containing powers of the sum or difference of two terms, e.g. $(x - \alpha)^m$. Later in this book we will find numerous occasions on which we wish to write such a product of repeated factors as a polynomial in x or, more generally, as a sum of terms each of which contains powers of x and α separately, as opposed to a power of their sum or difference.

To make the discussion general and the result applicable to a wide variety of situations, we will consider the general expansion of $f(x) = (x + y)^n$, where x and y may stand for constants, variables or functions and, for the time being, n is a positive integer. It may not be obvious what form the general expansion takes but some idea can be obtained by carrying out the multiplication explicitly for small values of n. Thus we obtain successively

$$\begin{aligned} (x + y)^1 &= x + y, \\ (x + y)^2 &= (x + y)(x + y) = x^2 + 2xy + y^2, \\ (x + y)^3 &= (x + y)(x^2 + 2xy + y^2) = x^3 + 3x^2y + 3xy^2 + y^3, \\ (x + y)^4 &= (x + y)(x^3 + 3x^2y + 3xy^2 + y^3) = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4. \end{aligned}$$

This does not *establish* a general formula, but the regularity of the terms in the expansions and the suggestion of a pattern in the coefficients indicate that a general formula for power n will have $n + 1$ terms, that the powers of x and y in every term will add up to n and that the coefficients of the first and last terms will be unity whilst those of the second and penultimate terms will be n.

In fact, the general expression, the *binomial expansion* for power n, is given by

$$(x+y)^n = \sum_{k=0}^{k=n} {}^nC_k x^{n-k} y^k, \tag{1.49}$$

where ${}^nC_k$ is called the *binomial coefficient* and is expressed in terms of factorial functions by $n!/[k!(n-k)!]$. Clearly, simply to make such a statement does not constitute proof of its validity, but, as we will see in subsection 1.5.2, (1.49) can be *proved* using a method called induction. Before turning to that proof, we investigate some of the elementary properties of the binomial coefficients.

1.5.1 Binomial coefficients

As stated above, the binomial coefficients are defined by

$${}^nC_k \equiv \frac{n!}{k!(n-k)!} \equiv \binom{n}{k} \qquad \text{for } 0 \le k \le n, \tag{1.50}$$

where in the second identity we give a common alternative notation for ${}^nC_k$. Obvious properties include

(i) ${}^nC_0 = {}^nC_n = 1$,
(ii) ${}^nC_1 = {}^nC_{n-1} = n$,
(iii) ${}^nC_k = {}^nC_{n-k}$.

We note that, for any given n, the largest coefficient in the binomial expansion is the middle one ($k = n/2$) if n is even; the middle two coefficients ($k = \frac{1}{2}(n \pm 1)$) are equal largest if n is odd. Somewhat less obvious is the result

$$\begin{aligned}
{}^nC_k + {}^nC_{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= \frac{n![(n+1-k)+k]}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} = {}^{n+1}C_k.
\end{aligned} \tag{1.51}$$

An equivalent statement, in which k has been redefined as $k+1$, is

$${}^nC_k + {}^nC_{k+1} = {}^{n+1}C_{k+1}. \tag{1.52}$$

1.5.2 Proof of the binomial expansion

We are now in a position to *prove* the binomial expansion (1.49). In doing so, we introduce the reader to a procedure applicable to certain types of problems and known as the *method of induction*. The method is discussed much more fully in subsection 1.7.1.

We start by *assuming* that (1.49) is true for some positive integer $n = N$. We now proceed to show that this implies that it must also be true for $n = N+1$, as follows:

$$\begin{aligned}(x+y)^{N+1} &= (x+y)\sum_{k=0}^{N} {}^{N}C_k x^{N-k} y^k \\ &= \sum_{k=0}^{N} {}^{N}C_k x^{N+1-k} y^k + \sum_{k=0}^{N} {}^{N}C_k x^{N-k} y^{k+1} \\ &= \sum_{k=0}^{N} {}^{N}C_k x^{N+1-k} y^k + \sum_{j=1}^{N+1} {}^{N}C_{j-1} x^{(N+1)-j} y^j,\end{aligned}$$

where in the first line we have used the assumption and in the third line have moved the second summation index by unity, by writing $k+1 = j$. We now separate off the first term of the first sum, ${}^{N}C_0 x^{N+1}$, and write it as ${}^{N+1}C_0 x^{N+1}$; we can do this since, as noted in (i) following (1.50), ${}^{n}C_0 = 1$ for every n. Similarly, the last term of the second summation can be replaced by ${}^{N+1}C_{N+1} y^{N+1}$.

The remaining terms of each of the two summations are now written together, with the summation index denoted by k in both terms. Thus

$$\begin{aligned}(x+y)^{N+1} &= {}^{N+1}C_0 x^{N+1} + \sum_{k=1}^{N} \left({}^{N}C_k + {}^{N}C_{k-1}\right) x^{(N+1)-k} y^k + {}^{N+1}C_{N+1} y^{N+1} \\ &= {}^{N+1}C_0 x^{N+1} + \sum_{k=1}^{N} {}^{N+1}C_k x^{(N+1)-k} y^k + {}^{N+1}C_{N+1} y^{N+1} \\ &= \sum_{k=0}^{N+1} {}^{N+1}C_k x^{(N+1)-k} y^k.\end{aligned}$$

In going from the first to the second line we have used result (1.51). Now we observe that the final overall equation is just the original assumed result (1.49) but with $n = N+1$. Thus it has been shown that if the binomial expansion is *assumed* to be true for $n = N$, then it can be *proved* to be true for $n = N+1$. But it holds trivially for $n = 1$, and therefore for $n = 2$ also. By the same token it is valid for $n = 3, 4, \ldots$, and hence is established for all positive integers n.

1.6 Properties of binomial coefficients

1.6.1 Identities involving binomial coefficients

There are many identities involving the binomial coefficients that can be derived directly from their definition, and yet more that follow from their appearance in the binomial expansion. Only the most elementary ones, given earlier, are worth committing to memory but, as illustrations, we now derive two results involving sums of binomial coefficients.

The first is a further application of the method of induction. Consider the proposal that, for any $n \geq 1$ and $k \geq 0$,

$$\sum_{s=0}^{n-1} {}^{k+s}C_k = {}^{n+k}C_{k+1}. \tag{1.53}$$

Notice that here n, the number of terms in the sum, is the parameter that varies, k is a fixed parameter, whilst s is a summation index and does not appear on the RHS of the equation.

Now we suppose that the statement (1.53) about the value of the sum of the binomial coefficients ${}^kC_k, {}^{k+1}C_k, \ldots, {}^{k+n-1}C_k$ is true for $n = N$. We next write down a series with an extra term and determine the implications of the supposition for the new series:

$$\begin{aligned}\sum_{s=0}^{N+1-1} {}^{k+s}C_k &= \sum_{s=0}^{N-1} {}^{k+s}C_k + {}^{k+N}C_k \\ &= {}^{N+k}C_{k+1} + {}^{N+k}C_k \\ &= {}^{N+k+1}C_{k+1}.\end{aligned}$$

But this is just proposal (1.53) with n now set equal to $N + 1$. To obtain the last line, we have used (1.52), with n set equal to $N + k$.

It only remains to consider the case $n = 1$, when the summation only contains one term and (1.53) reduces to

$${}^kC_k = {}^{1+k}C_{k+1}.$$

This is trivially valid for any k since both sides are equal to unity, thus completing the proof of (1.53) for all positive integers n.

The second result, which gives a formula for combining terms from two sets of binomial coefficients in a particular way (a kind of 'convolution', for readers who are already familiar with this term), is derived by applying the binomial expansion directly to the identity

$$(x + y)^p (x + y)^q \equiv (x + y)^{p+q}.$$

Written in terms of binomial expansions, this reads

$$\sum_{s=0}^{p} {}^pC_s x^{p-s} y^s \sum_{t=0}^{q} {}^qC_t x^{q-t} y^t = \sum_{r=0}^{p+q} {}^{p+q}C_r x^{p+q-r} y^r.$$

We now equate coefficients of $x^{p+q-r} y^r$ on the two sides of the equation, noting that on the LHS all combinations of s and t such that $s + t = r$ contribute. This gives as an identity that

$$\sum_{t=0}^{r} {}^pC_{r-t} \, {}^qC_t = {}^{p+q}C_r = \sum_{t=0}^{r} {}^pC_t \, {}^qC_{r-t}. \tag{1.54}$$

We have specifically included the second equality to emphasise the symmetrical nature of the relationship with respect to p and q.

Further identities involving the coefficients can be obtained by giving x and y special values in the defining equation (1.49) for the expansion. If both are set equal to unity then we obtain (using the alternative notation so as to produce familiarity with it)

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n, \tag{1.55}$$

whilst setting $x = 1$ and $y = -1$ yields

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0. \tag{1.56}$$

1.6.2 Negative and non-integral values of n

Up till now we have restricted n in the binomial expansion to be a positive integer. Negative values can be accommodated, but only at the cost of an infinite series of terms rather than the finite one represented by (1.49). For reasons that are intuitively sensible and will be discussed in more detail in chapter 4, very often we require an expansion in which, at least ultimately, successive terms in the infinite series decrease in magnitude. For this reason, if $x > y$ we consider $(x + y)^{-m}$, where m itself is a positive integer, in the form

$$(x + y)^n = (x + y)^{-m} = x^{-m}\left(1 + \frac{y}{x}\right)^{-m}.$$

Since the ratio y/x is less than unity, terms containing higher powers of it will be small in magnitude, whilst raising the unit term to any power will not affect its magnitude. If $y > x$ the roles of the two must be interchanged.

We can now state, but will not explicitly prove, the form of the binomial expansion appropriate to negative values of n (n equal to $-m$):

$$(x + y)^n = (x + y)^{-m} = x^{-m} \sum_{k=0}^{\infty} {}^{-m}C_k \left(\frac{y}{x}\right)^k, \tag{1.57}$$

where the hitherto undefined quantity ${}^{-m}C_k$, which appears to involve factorials of negative numbers, is given by

$${}^{-m}C_k = (-1)^k \frac{m(m+1)\cdots(m+k-1)}{k!} = (-1)^k \frac{(m+k-1)!}{(m-1)!k!} = (-1)^k \,{}^{m+k-1}C_k. \tag{1.58}$$

The binomial coefficient on the extreme right of this equation has its normal meaning and is well defined since $m + k - 1 \geq k$.

Thus we have a definition of binomial coefficients for negative integer values of n in terms of those for positive n. The connection between the two may not be obvious, but they are both formed in the same way in terms of recurrence relations. Whatever the sign of n, the series of coefficients ${}^nC_k$ can be generated by starting with ${}^nC_0 = 1$ and using the recurrence relation

$$ {}^nC_{k+1} = \frac{n-k}{k+1}\,{}^nC_k. \qquad (1.59)$$

The difference is that for positive integer n the series terminates when $k = n$, whereas for negative n there is no such termination – in line with the infinite series of terms in the corresponding expansion.

Finally we note that, in fact, equation (1.59) generates the appropriate coefficients for all values of n, positive or negative, integer or non-integer, with the obvious exception of the case in which $x = -y$ and n is negative. For non-integer n the expansion does not terminate, even if n is positive.

1.7 Some particular methods of proof

Much of the mathematics used by physicists and engineers is concerned with obtaining a particular value, formula or function from a given set of data and stated conditions. However, just as it is essential in physics to formulate the basic laws and so be able to set boundaries on what can or cannot happen, so it is important in mathematics to be able to state general propositions about the outcomes that are or are not possible. To this end one attempts to establish theorems that state in as general a way as possible mathematical results that apply to particular types of situation. We conclude this introductory chapter by describing two methods that can sometimes be used to prove particular classes of theorems.

The two general methods of proof are known as proof by induction (which has already been met in this chapter) and proof by contradiction. They share the common characteristic that at an early stage in the proof an assumption is made that a particular (unproven) statement is true; the consequences of that assumption are then explored. In an inductive proof the conclusion is reached that the assumption is self-consistent and has other equally consistent but broader implications, which are then applied to establish the general validity of the assumption. A proof by contradiction, however, establishes an internal inconsistency and thus shows that the assumption is unsustainable; the natural consequence of this is that the negative of the assumption is established as true.

Later in this book use will be made of these methods of proof to explore new territory, e.g. to examine the properties of vector spaces, matrices and groups. However, at this stage we will draw our illustrative and test examples from earlier sections of this chapter and other topics in elementary algebra and number theory.

1.7.1 Proof by induction

The proof of the binomial expansion given in subsection 1.5.2 and the identity established in subsection 1.6.1 have already shown the way in which an inductive proof is carried through. They also indicated the main limitation of the method, namely that only an initially supposed result can be proved. Thus the method of induction is of no use for *deducing* a previously unknown result; a putative equation or result has to be arrived at by some other means, usually by noticing patterns or by trial and error using simple values of the variables involved. It will also be clear that propositions that can be proved by induction are limited to those containing a parameter that takes a range of integer values (usually infinite).

For a proposition involving a parameter n, the five steps in a proof using induction are as follows.

(i) Formulate the supposed result for general n.
(ii) Suppose (i) to be true for $n = N$ (or more generally for all values of $n \leq N$; see below), where N is restricted to lie in the stated range.
(iii) Show, using only proven results and supposition (ii), that proposition (i) is true for $n = N + 1$.
(iv) Demonstrate directly, and without any assumptions, that proposition (i) is true when n takes the lowest value in its range.
(v) It then follows from (iii) and (iv) that the proposition is valid for all values of n in the stated range.

(It should be noted that, although many proofs at stage (iii) require the validity of the proposition only for $n = N$, some require it for all n less than or equal to N – hence the form of inequality given in parentheses in the stage (ii) assumption.)

To illustrate further the method of induction, we now apply it to two worked examples; the first concerns the sum of the squares of the first n natural numbers.

►*Prove that the sum of the squares of the first n natural numbers is given by*

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1). \tag{1.60}$$

As previously we start by assuming the result is true for $n = N$. Then it follows that

$$\begin{aligned}\sum_{r=1}^{N+1} r^2 &= \sum_{r=1}^{N} r^2 + (N+1)^2 \\ &= \tfrac{1}{6}N(N+1)(2N+1) + (N+1)^2 \\ &= \tfrac{1}{6}(N+1)[N(2N+1) + 6N + 6] \\ &= \tfrac{1}{6}(N+1)[(2N+3)(N+2)] \\ &= \tfrac{1}{6}(N+1)[(N+1)+1][2(N+1)+1].\end{aligned}$$

This is precisely the original assumption, but with N replaced by $N+1$. To complete the proof we only have to verify (1.60) for $n=1$. This is trivially done and establishes the result for all positive n. The same and related results are obtained by a different method in subsection 4.2.5. ◄

Our second example is somewhat more complex and involves two nested proofs by induction: whilst trying to establish the main result by induction, we find that we are faced with a second proposition which itself requires an inductive proof.

►*Show that $Q(n) = n^4 + 2n^3 + 2n^2 + n$ is divisible by 6 (without remainder) for all positive integer values of n.*

Again we start by assuming the result is true for some particular value N of n, whilst noting that it is trivially true for $n = 0$. We next examine $Q(N+1)$, writing each of its terms as a binomial expansion:

$$\begin{aligned} Q(N+1) &= (N+1)^4 + 2(N+1)^3 + 2(N+1)^2 + (N+1) \\ &= (N^4 + 4N^3 + 6N^2 + 4N + 1) + 2(N^3 + 3N^2 + 3N + 1) \\ &\quad + 2(N^2 + 2N + 1) + (N+1) \\ &= (N^4 + 2N^3 + 2N^2 + N) + (4N^3 + 12N^2 + 14N + 6). \end{aligned}$$

Now, by our assumption, the group of terms within the first parentheses in the last line is divisible by 6 and clearly so are the terms $12N^2$ and 6 within the second parentheses. Thus it comes down to deciding whether $4N^3 + 14N$ is divisible by 6 – or equivalently, whether $R(N) = 2N^3 + 7N$ is divisible by 3.

To settle this latter question we try using a second inductive proof and assume that $R(N)$ *is* divisible by 3 for $N = M$, whilst again noting that the proposition is trivially true for $N = M = 0$. This time we examine $R(M+1)$:

$$\begin{aligned} R(M+1) &= 2(M+1)^3 + 7(M+1) \\ &= 2(M^3 + 3M^2 + 3M + 1) + 7(M+1) \\ &= (2M^3 + 7M) + 3(2M^2 + 2M + 3) \end{aligned}$$

By assumption, the first group of terms in the last line is divisible by 3 and the second group is patently so. We thus conclude that $R(N)$ is divisible by 3 for all $N \geq M$, and taking $M = 0$ shows that it is divisible by 3 for all N.

We can now return to the main proposition and conclude that since $R(N) = 2N^3 + 7N$ is divisible by 3, $4N^3 + 12N^2 + 14N + 6$ is divisible by 6. This in turn establishes that the divisibility of $Q(N+1)$ by 6 follows from the assumption that $Q(N)$ divides by 6. Since $Q(0)$ clearly divides by 6, the proposition in the question is established for all values of n. ◄

1.7.2 Proof by contradiction

The second general line of proof, but again one that is normally only useful when the result is already suspected, is proof by contradiction. The questions it can attempt to answer are only those that can be expressed in a proposition that is either true or false. Clearly, it could be argued that any mathematical result can be so expressed but, if the proposition is no more than a guess, the chances of success are negligible. Valid propositions containing even modest formulae are either the result of true inspiration or, much more normally, yet another reworking of an old chestnut!

The essence of the method is to exploit the fact that mathematics is required to be self-consistent, so that, for example, two calculations of the same quantity, starting from the same given data but proceeding by different methods, must give the same answer. Equally, it must not be possible to follow a line of reasoning and draw a conclusion that contradicts either the input data or any other conclusion based upon the same data.

It is this requirement on which the method of proof by contradiction is based. The crux of the method is to assume that the proposition to be proved is *not* true, and then use this incorrect assumption and 'watertight' reasoning to draw a conclusion that contradicts the assumption. The only way out of the self-contradiction is then to conclude that the assumption was indeed false and therefore that the proposition is true.

It must be emphasised that once a (false) contrary assumption has been made, every subsequent conclusion in the argument *must* follow of necessity. Proof by contradiction fails if at any stage we have to admit 'this may or may not be the case'. That is, each step in the argument must be a *necessary* consequence of results that precede it (taken together with the assumption), rather than simply a *possible* consequence.

It should also be added that if no contradiction can be found using sound reasoning based on the assumption then no conclusion can be drawn about either the proposition or its negative and some other approach must be tried.

We illustrate the general method with an example in which the mathematical reasoning is straightforward, so that attention can be focussed on the structure of the proof.

►*A rational number r is a fraction $r = p/q$ in which p and q are integers with q positive. Further, r is expressed in its lowest terms, any integer common factor of p and q having been divided out.*

Prove that the square root of an integer m cannot be a rational number, unless the square root itself is an integer.

We begin by supposing that the stated result is *not* true and that we *can* write an equation

$$\sqrt{m} = r = \frac{p}{q} \quad \text{for integers } m, p, q \text{ with} \quad q \neq 1.$$

It then follows that $p^2 = mq^2$. But, since r is expressed in its lowest terms, p and q, and hence p^2 and q^2, have no factors in common. However, m is an integer; this is only possible if $q = 1$ and $p^2 = m$. This conclusion contradicts the requirement that $q \neq 1$ and so leads to the conclusion that it was wrong to suppose that $\sqrt{m}$ can be expressed as a non-integer rational number. This completes the proof of the statement in the question. ◄

Our second worked example, also taken from elementary number theory, involves slightly more complicated mathematical reasoning but again exhibits the structure associated with this type of proof.

►*The prime integers p_i are labelled in ascending order, thus $p_1 = 1$, $p_2 = 2$, $p_5 = 7$, etc. Show that there is no largest prime number.*

Assume, on the contrary, that there is a largest prime and let it be p_N. Consider now the number q formed by multiplying together all the primes from p_1 to p_N and then adding one to the product, i.e.

$$q = p_1 p_2 \cdots p_N + 1.$$

By our assumption p_N is the largest prime, and so no number can have a prime factor greater than this. However, for every prime p_i, $i = 1, 2, \ldots, N$, the quotient q/p_i has the form $M_i + (1/p_i)$ with M_i an integer and $1/p_i$ non-integer. This means that q/p_i cannot be an integer and so p_i cannot be a divisor of q.

Since q is not divisible by any of the (assumed) finite set of primes, it must be itself a prime. As q is also clearly greater than p_N, we have a contradiction. This shows that our assumption that there is a largest prime integer must be false, and so it follows that there is no largest prime integer.

It should be noted that the given construction for q does not generate all the primes that actually exist (e.g. for $N = 3, q = 7$ rather than the next actual prime value of 5, is found), but this does not matter for the purposes of our proof by contradiction. ◄

1.7.3 Necessary and sufficient conditions

As the final topic in this introductory chapter, we consider briefly the notion of, and distinction between, necessary and sufficient conditions in the context of proving a mathematical proposition. In ordinary English the distinction is well defined, and that distinction is maintained in mathematics. However, in the authors' experience students tend to overlook it and assume (wrongly) that, having proved that the validity of proposition A implies the truth of proposition B, it follows by 'reversing the argument' that the validity of B automatically implies that of A.

As an example, let proposition A be that an integer N is divisible without remainder by 6, and proposition B be that N is divisible without remainder by 2. Clearly, if A is true then it follows that B is true, i.e. A is a sufficient condition for B; it is not however a necessary condition, as is trivially shown by taking N as 8. Conversely, the same value of N shows that whilst the validity of B is a necessary condition for A to hold, it is not sufficient.

An alternative terminology to 'necessary' and 'sufficient' often employed by mathematicians is that of 'if' and 'only if', particularly in the combination 'if and only if' which is usually written as IFF or denoted by a double-headed arrow $\Longleftrightarrow$. The equivalent statements can be summarised by

A if B	A is true if B is true *or*	$B \Longrightarrow A$,
	B is a sufficient condition for A	$B \Longrightarrow A$,
A only if B	A is true only if B is true *or*	$A \Longrightarrow B$,
	B is a necessary consequence of A	$A \Longrightarrow B$,

A IFF B	A is true if and only if B is true *or*	$B \iff A$,
	A and B necessarily imply each other	$B \iff A$.

Although at this stage in the book we are able to employ for illustrative purposes only simple and fairly obvious results, the following example is given as a model of how necessary and sufficient conditions should be proved. The essential point is that for the second part of the proof (whether it be the 'necessary' part or the 'sufficient' part) one needs to start again from scratch; more often than not, the lines of the second part of the proof will *not* be simply those of the first written in reverse order.

►*Prove that (A) a function $f(x)$ is a quadratic polynomial with zeros at $x = 2$ and $x = 3$ if and only if (B) the function $f(x)$ has the form $\lambda(x^2 - 5x + 6)$ with λ a non-zero constant.*

(1) Assume A, i.e. that $f(x)$ *is* a quadratic polynomial with zeros at $x = 2$ and $x = 3$. Let its form be $ax^2 + bx + c$ with $a \neq 0$. Then we have

$$4a + 2b + c = 0,$$
$$9a + 3b + c = 0,$$

and subtraction shows that $5a + b = 0$ and $b = -5a$. Substitution of this into the first of the above equations gives $c = -4a - 2b = -4a + 10a = 6a$. Thus, it follows that

$$f(x) = a(x^2 - 5x + 6) \quad \text{with} \quad a \neq 0,$$

and establishes the 'A only if B' part of the stated result.

(2) Now assume that $f(x)$ *has* the form $\lambda(x^2 - 5x + 6)$ with λ a non-zero constant. Firstly we note that $f(x)$ is a quadratic polynomial, and so it only remains to prove that its zeros occur at $x = 2$ and $x = 3$. Consider $f(x) = 0$, which, after dividing through by the non-zero constant λ, gives

$$x^2 - 5x + 6 = 0.$$

We proceed by using a technique known as *completing the square*, for the purposes of illustration, although the factorisation of the above equation should be clear to the reader. Thus we write

$$x^2 - 5x + (\tfrac{5}{2})^2 - (\tfrac{5}{2})^2 + 6 = 0,$$
$$(x - \tfrac{5}{2})^2 = \tfrac{1}{4},$$
$$x - \tfrac{5}{2} = \pm\tfrac{1}{2}.$$

The two roots of $f(x) = 0$ are therefore $x = 2$ and $x = 3$; these x-values give the zeros of $f(x)$. This establishes the second ('A if B') part of the result. Thus we have shown that the assumption of either condition implies the validity of the other and the proof is complete. ◄

It should be noted that the propositions have to be carefully and precisely formulated. If, for example, the word 'quadratic' were omitted from A, statement B would still be a sufficient condition for A but not a necessary one, since $f(x)$ could then be $x^3 - 4x^2 + x + 6$ and A would not require B. Omitting the constant λ from the stated form of $f(x)$ in B has the same effect. Conversely, if A were to state that $f(x) = 3(x - 2)(x - 3)$ then B would be a necessary condition for A but not a sufficient one.

1.8 Exercises

Polynomial equations

1.1 Continue the investigation of equation (1.7), namely

$$g(x) = 4x^3 + 3x^2 - 6x - 1,$$

as follows.

(a) Make a table of values of $g(x)$ for integer values of x between -2 and 2. Use it and the information derived in the text to draw a graph and so determine the roots of $g(x) = 0$ as accurately as possible.
(b) Find one accurate root of $g(x) = 0$ by inspection and hence determine precise values for the other two roots.
(c) Show that $f(x) = 4x^3 + 3x^2 - 6x - k = 0$ has only one real root unless $-5 \leq k \leq \frac{7}{4}$.

1.2 Determine how the number of real roots of the equation

$$g(x) = 4x^3 - 17x^2 + 10x + k = 0$$

depends upon k. Are there any cases for which the equation has exactly two distinct real roots?

1.3 Continue the analysis of the polynomial equation

$$f(x) = x^7 + 5x^6 + x^4 - x^3 + x^2 - 2 = 0,$$

investigated in subsection 1.1.1, as follows.

(a) By writing the fifth-degree polynomial appearing in the expression for $f'(x)$ in the form $7x^5 + 30x^4 + a(x-b)^2 + c$, show that there is in fact only one positive root of $f(x) = 0$.
(b) By evaluating $f(1)$, $f(0)$ and $f(-1)$, and by inspecting the form of $f(x)$ for negative values of x, determine what you can about the positions of the real roots of $f(x) = 0$.

1.4 Given that $x = 2$ is one root of

$$g(x) = 2x^4 + 4x^3 - 9x^2 - 11x - 6 = 0,$$

use factorisation to determine how many real roots it has.

1.5 Construct the quadratic equations that have the following pairs of roots: (a) $-6, -3$; (b) $0, 4$; (c) $2, 2$; (d) $3 + 2i, 3 - 2i$, where $i^2 = -1$.

1.6 Use the results of (i) equation (1.13), (ii) equation (1.12) and (iii) equation (1.14) to prove that if the roots of $3x^3 - x^2 - 10x + 8 = 0$ are α_1, α_2 and α_3 then

(a) $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} = 5/4$,
(b) $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 61/9$,
(c) $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = -125/27$.
(d) Convince yourself that eliminating (say) α_2 and α_3 from (i), (ii) and (iii) does *not* give a simple explicit way of finding α_1.

Trigonometric identities

1.7 Prove that

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

by considering

(a) the sum of the sines of $\pi/3$ and $\pi/6$,
(b) the sine of the sum of $\pi/3$ and $\pi/4$.

1.8 The following exercises are based on the half-angle formulae.

(a) Use the fact that $\sin(\pi/6) = 1/2$ to prove that $\tan(\pi/12) = 2 - \sqrt{3}$.
(b) Use the result of (a) to show further that $\tan(\pi/24) = q(2 - q)$ where $q^2 = 2 + \sqrt{3}$.

1.9 Find the real solutions of

(a) $3\sin\theta - 4\cos\theta = 2$,
(b) $4\sin\theta + 3\cos\theta = 6$,
(c) $12\sin\theta - 5\cos\theta = -6$.

1.10 If $s = \sin(\pi/8)$, prove that

$$8s^4 - 8s^2 + 1 = 0,$$

and hence show that $s = [(2 - \sqrt{2})/4]^{1/2}$.

1.11 Find all the solutions of

$$\sin\theta + \sin 4\theta = \sin 2\theta + \sin 3\theta$$

that lie in the range $-\pi < \theta \le \pi$. What is the multiplicity of the solution $\theta = 0$?

Coordinate geometry

1.12 Obtain in the form (1.38) the equations that describe the following:

(a) a circle of radius 5 with its centre at $(1, -1)$;
(b) the line $2x + 3y + 4 = 0$ and the line orthogonal to it which passes through $(1, 1)$;
(c) an ellipse of eccentricity 0.6 with centre $(1, 1)$ and its major axis of length 10 parallel to the y-axis.

1.13 Determine the forms of the conic sections described by the following equations:

(a) $x^2 + y^2 + 6x + 8y = 0$;
(b) $9x^2 - 4y^2 - 54x - 16y + 29 = 0$;
(c) $2x^2 + 2y^2 + 5xy - 4x + y - 6 = 0$;
(d) $x^2 + y^2 + 2xy - 8x + 8y = 0$.

1.14 For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with eccentricity e, the two points $(-ae, 0)$ and $(ae, 0)$ are known as its foci. Show that the sum of the distances from *any* point on the ellipse to the foci is $2a$. (The constancy of the sum of the distances from two fixed points can be used as an alternative defining property of an ellipse.)

Partial fractions

1.15 Resolve the following into partial fractions using the three methods given in section 1.4, verifying that the same decomposition is obtained by each method:

$$\text{(a)}\ \frac{2x + 1}{x^2 + 3x - 10}, \qquad \text{(b)}\ \frac{4}{x^2 - 3x}.$$

1.16 Express the following in partial fraction form:

$$\text{(a)}\ \frac{2x^3-5x+1}{x^2-2x-8}, \qquad \text{(b)}\ \frac{x^2+x-1}{x^2+x-2}.$$

1.17 Rearrange the following functions in partial fraction form:

$$\text{(a)}\ \frac{x-6}{x^3-x^2+4x-4}, \qquad \text{(b)}\ \frac{x^3+3x^2+x+19}{x^4+10x^2+9}.$$

1.18 Resolve the following into partial fractions in such a way that x does not appear in any numerator:

$$\text{(a)}\ \frac{2x^2+x+1}{(x-1)^2(x+3)}, \qquad \text{(b)}\ \frac{x^2-2}{x^3+8x^2+16x}, \qquad \text{(c)}\ \frac{x^3-x-1}{(x+3)^3(x+1)}.$$

Binomial expansion

1.19 Evaluate those of the following that are defined: (a) ${}^5C_3$, (b) ${}^3C_5$, (c) ${}^{-5}C_3$, (d) ${}^{-3}C_5$.

1.20 Use a binomial expansion to evaluate $1/\sqrt{4.2}$ to five places of decimals, and compare it with the accurate answer obtained using a calculator.

Proof by induction and contradiction

1.21 Prove by induction that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

1.22 Prove by induction that

$$1+r+r^2+\cdots+r^k+\cdots+r^n = \frac{1-r^{n+1}}{1-r}.$$

1.23 Prove that $3^{2n}+7$, where n is a non-negative integer, is divisible by 8.

1.24 If a sequence of terms, u_n, satisfies the recurrence relation $u_{n+1}=(1-x)u_n+nx$, with $u_1=0$, show, by induction, that, for $n\geq 1$,

$$u_n = \frac{1}{x}[nx-1+(1-x)^n].$$

1.25 Prove by induction that

$$\sum_{r=1}^{n} \frac{1}{2^r}\tan\left(\frac{\theta}{2^r}\right) = \frac{1}{2^n}\cot\left(\frac{\theta}{2^n}\right) - \cot\theta.$$

1.26 The quantities a_i in this exercise are all positive real numbers.

(a) Show that

$$a_1a_2 \leq \left(\frac{a_1+a_2}{2}\right)^2.$$

(b) Hence prove, by induction on m, that

$$a_1a_2\cdots a_p \leq \left(\frac{a_1+a_2+\cdots+a_p}{p}\right)^p,$$

where $p=2^m$ with m a positive integer. Note that each increase of m by unity doubles the number of factors in the product.

1.27 Establish the values of k for which the binomial coefficient ${}^pC_k$ is divisible by p when p is a prime number. Use your result and the method of induction to prove that $n^p - n$ is divisible by p for all integers n and all prime numbers p. Deduce that $n^5 - n$ is divisible by 30 for any integer n.

1.28 An arithmetic progression of integers a_n is one in which $a_n = a_0 + nd$, where a_0 and d are integers and n takes successive values $0, 1, 2, \ldots$.

(a) Show that if any one term of the progression is the cube of an integer then so are infinitely many others.
(b) Show that no cube of an integer can be expressed as $7n+5$ for some positive integer n.

1.29 Prove, by the method of contradiction, that the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0,$$

in which all the coefficients a_i are integers, cannot have a rational root, unless that root is an integer. Deduce that any integral root must be a divisor of a_0 and hence find all rational roots of

(a) $x^4 + 6x^3 + 4x^2 + 5x + 4 = 0$,
(b) $x^4 + 5x^3 + 2x^2 - 10x + 6 = 0$.

Necessary and sufficient conditions

1.30 Prove that the equation $ax^2 + bx + c = 0$, in which a, b and c are real and $a > 0$, has two real distinct solutions IFF $b^2 > 4ac$.

1.31 For the real variable x, show that a sufficient, but not necessary, condition for $f(x) = x(x+1)(2x+1)$ to be divisible by 6 is that x is an integer.

1.32 Given that at least one of a and b, and at least one of c and d, are non-zero, show that $ad = bc$ is both a necessary and sufficient condition for the equations

$$ax + by = 0,$$
$$cx + dy = 0,$$

to have a solution in which at least one of x and y is non-zero.

1.33 The coefficients a_i in the polynomial $Q(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x$ are all integers. Show that $Q(n)$ is divisible by 24 for all integers $n \geq 0$ if and only if all of the following conditions are satisfied:
(i) $2a_4 + a_3$ is divisible by 4;
(ii) $a_4 + a_2$ is divisible by 12;
(iii) $a_4 + a_3 + a_2 + a_1$ is divisible by 24.

1.9 Hints and answers

1.1 (b) The roots are $1, \frac{1}{8}(-7+\sqrt{33}) = -0.1569, \frac{1}{8}(-7-\sqrt{33}) = -1.593$. (c) -5 and $\frac{7}{4}$ are the values of k that make $f(-1)$ and $f(\frac{1}{2})$ equal to zero.

1.3 (a) $a = 4$, $b = \frac{3}{8}$ and $c = \frac{23}{16}$ are all positive. Therefore $f'(x) > 0$ for all $x > 0$.
(b) $f(1) = 5$, $f(0) = -2$ and $f(-1) = 5$, and so there is at least one root in each of the ranges $0 < x < 1$ and $-1 < x < 0$. $(x^7 + 5x^6) + (x^4 - x^3) + (x^2 - 2)$ is positive definite for $-5 < x < -\sqrt{2}$. There are therefore no roots in this range, but there must be one to the left of $x = -5$.

1.5 (a) $x^2 + 9x + 18 = 0$; (b) $x^2 - 4x = 0$; (c) $x^2 - 4x + 4 = 0$; (d) $x^2 - 6x + 13 = 0$.

1.7 (a) Use $\sin(\pi/4) = 1/\sqrt{2}$. (b) Use results (1.20) and (1.21).

1.9 (a) 1.339, -2.626. (b) No solution because $6^2 > 4^2 + 3^2$. (c) -0.0849, -2.276.

1.11 Show that the equation is equivalent to $\sin(5\theta/2)\sin(\theta)\sin(\theta/2) = 0$.
Solutions are $-4\pi/5, -2\pi/5, 0, 2\pi/5, 4\pi/5, \pi$. The solution $\theta = 0$ has multiplicity 3.

1.13 (a) A circle of radius 5 centred on $(-3, -4)$.
(b) A hyperbola with 'centre' $(3, -2)$ and 'semi-axes' 2 and 3.
(c) The expression factorises into two lines, $x + 2y - 3 = 0$ and $2x + y + 2 = 0$.
(d) Write the expression as $(x+y)^2 = 8(x-y)$ to see that it represents a parabola passing through the origin, with the line $x + y = 0$ as its axis of symmetry.

1.15 (a) $\dfrac{5}{7(x-2)} + \dfrac{9}{7(x+5)}$, (b) $-\dfrac{4}{3x} + \dfrac{4}{3(x-3)}$.

1.17 (a) $\dfrac{x+2}{x^2+4} - \dfrac{1}{x-1}$, (b) $\dfrac{x+1}{x^2+9} + \dfrac{2}{x^2+1}$.

1.19 (a) 10, (b) not defined, (c) -35, (d) -21.

1.21 Look for factors common to the $n = N$ sum and the additional $n = N + 1$ term, so as to reduce the sum for $n = N + 1$ to a single term.

1.23 Write 3^{2n} as $8m - 7$.

1.25 Use the half-angle formulae of equations (1.32) to (1.34) to relate functions of $\theta/2^k$ to those of $\theta/2^{k+1}$.

1.27 Divisible for $k = 1, 2, \ldots, p - 1$. Expand $(n + 1)^p$ as $n^p + \sum_1^{p-1} {}^pC_k n^k + 1$. Apply the stated result for $p = 5$. Note that $n^5 - n = n(n-1)(n+1)(n^2+1)$; the product of any three consecutive integers must divide by both 2 and 3.

1.29 By assuming $x = p/q$ with $q \neq 1$, show that a fraction $-p^n/q$ is equal to an integer $a_{n-1}p^{n-1} + \cdots + a_1 pq^{n-2} + a_0 q^{n-1}$. This is a contradiction, and is only resolved if $q = 1$ and the root is an integer.
(a) The only possible candidates are $\pm 1, \pm 2, \pm 4$. None is a root.
(b) The only possible candidates are $\pm 1, \pm 2, \pm 3, \pm 6$. Only -3 is a root.

1.31 $f(x)$ can be written as $x(x+1)(x+2) + x(x+1)(x-1)$. Each term consists of the product of three consecutive integers, of which one must therefore divide by 2 and (a different) one by 3. Thus each term separately divides by 6, and so therefore does $f(x)$. Note that if x is the root of $2x^3 + 3x^2 + x - 24 = 0$ that lies near the non-integer value $x = 1.826$, then $x(x+1)(2x+1) = 24$ and therefore divides by 6.

1.33 Note that, e.g., the condition for $6a_4 + a_3$ to be divisible by 4 is the same as the condition for $2a_4 + a_3$ to be divisible by 4.
For the necessary (only if) part of the proof set $n = 1, 2, 3$ and take integer combinations of the resulting equations.
For the sufficient (if) part of the proof use the stated conditions to prove the proposition by induction. Note that $n^3 - n$ is divisible by 6 and that $n^2 + n$ is even.

2

Preliminary calculus

This chapter is concerned with the formalism of probably the most widely used mathematical technique in the physical sciences, namely the calculus. The chapter divides into two sections. The first deals with the process of differentiation and the second with its inverse process, integration. The material covered is essential for the remainder of the book and serves as a reference. Readers who have previously studied these topics should ensure familiarity by looking at the worked examples in the main text and by attempting the exercises at the end of the chapter.

2.1 Differentiation

Differentiation is the process of determining how quickly or slowly a function varies, as the quantity on which it depends, its *argument*, is changed. More specifically it is the procedure for obtaining an expression (numerical or algebraic) for the rate of change of the function with respect to its argument. Familiar examples of rates of change include acceleration (the rate of change of velocity) and chemical reaction rate (the rate of change of chemical composition). Both acceleration and reaction rate give a measure of the change of a quantity with respect to time. However, differentiation may also be applied to changes with respect to other quantities, for example the change in pressure with respect to a change in temperature.

Although it will not be apparent from what we have said so far, differentiation is in fact a limiting process, that is, it deals only with the infinitesimal change in one quantity resulting from an infinitesimal change in another.

2.1.1 Differentiation from first principles

Let us consider a function $f(x)$ that depends on only one variable x, together with numerical constants, for example, $f(x) = 3x^2$ or $f(x) = \sin x$ or $f(x) = 2 + 3/x$.

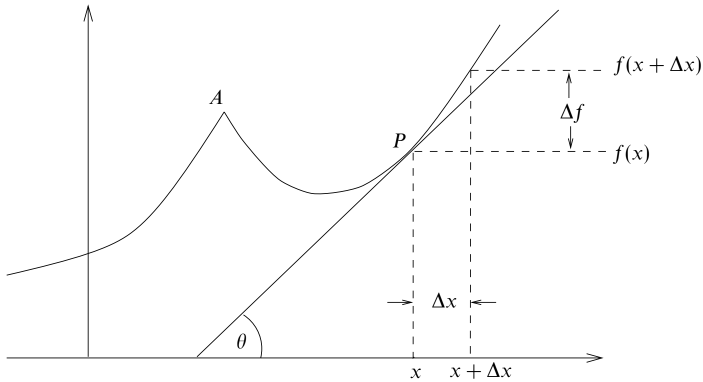

Figure 2.1 The graph of a function $f(x)$ showing that the gradient or slope of the function at P, given by $\tan\theta$, is approximately equal to $\Delta f/\Delta x$.

Figure 2.1 shows an example of such a function. Near any particular point, P, the value of the function changes by an amount Δf, say, as x changes by a small amount Δx. The slope of the tangent to the graph of $f(x)$ at P is then approximately $\Delta f/\Delta x$, and the change in the value of the function is $\Delta f = f(x+\Delta x) - f(x)$. In order to calculate the true value of the gradient, or *first derivative*, of the function at P, we must let Δx become infinitesimally small. We therefore define the first derivative of $f(x)$ as

$$f'(x) \equiv \frac{df(x)}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \tag{2.1}$$

provided that the limit exists. The limit will depend in almost all cases on the value of x. If the limit does exist at a point $x = a$ then the function is said to be differentiable at a; otherwise it is said to be non-differentiable at a. The formal concept of a limit and its existence or non-existence is discussed in chapter 4; for present purposes we will adopt an intuitive approach.

In the definition (2.1), we allow Δx to tend to zero from either positive or negative values and require the same limit to be obtained in both cases. A function that is differentiable at a is necessarily continuous at a (there must be no jump in the value of the function at a), though the converse is not necessarily true. This latter assertion is illustrated in figure 2.1: the function is continuous at the 'kink' A but the two limits of the gradient as Δx tends to zero from positive or negative values are different and so the function is not differentiable at A.

It should be clear from the above discussion that near the point P we may approximate the change in the value of the function, Δf, that results from a small change Δx in x by

$$\Delta f \approx \frac{df(x)}{dx}\Delta x. \tag{2.2}$$

As one would expect, the approximation improves as the value of Δx is reduced. In the limit in which the change Δx becomes infinitesimally small, we denote it by the *differential* dx, and (2.2) reads

$$df = \frac{df(x)}{dx}dx. \tag{2.3}$$

This *equality* relates the infinitesimal change in the function, df, to the infinitesimal change dx that causes it.

So far we have discussed only the first derivative of a function. However, we can also define the *second derivative* as the gradient of the gradient of a function. Again we use the definition (2.1) but now with $f(x)$ replaced by $f'(x)$. Hence the second derivative is defined by

$$f''(x) \equiv \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}, \tag{2.4}$$

provided that the limit exists. A physical example of a second derivative is the second derivative of the distance travelled by a particle with respect to time. Since the first derivative of distance travelled gives the particle's velocity, the second derivative gives its acceleration.

We can continue in this manner, the nth derivative of the function $f(x)$ being defined by

$$f^{(n)}(x) \equiv \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x+\Delta x) - f^{(n-1)}(x)}{\Delta x}. \tag{2.5}$$

It should be noted that with this notation $f'(x) \equiv f^{(1)}(x)$, $f''(x) \equiv f^{(2)}(x)$, etc., and that formally $f^{(0)}(x) \equiv f(x)$.

All this should be familiar to the reader, though perhaps not with such formal definitions. The following example shows the differentiation of $f(x) = x^2$ from first principles. In practice, however, it is desirable simply to remember the derivatives of standard functions; the techniques given in the remainder of this section can be applied to find more complicated derivatives.

►*Find from first principles the derivative with respect to x of $f(x) = x^2$.*

Using the definition (2.1),

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x).
\end{aligned}$$

As Δx tends to zero, $2x + \Delta x$ tends towards $2x$, hence

$$f'(x) = 2x. \blacktriangleleft$$

Derivatives of other functions can be obtained in the same way. The derivatives of some simple functions are listed below (note that a is a constant):

$$\frac{d}{dx}(x^n) = nx^{n-1}, \qquad \frac{d}{dx}(e^{ax}) = ae^{ax}, \qquad \frac{d}{dx}(\ln ax) = \frac{1}{x},$$

$$\frac{d}{dx}(\sin ax) = a\cos ax, \qquad \frac{d}{dx}(\cos ax) = -a\sin ax, \qquad \frac{d}{dx}(\sec ax) = a\sec ax \tan ax,$$

$$\frac{d}{dx}(\tan ax) = a\sec^2 ax, \qquad \frac{d}{dx}(\operatorname{cosec} ax) = -a \operatorname{cosec} ax \cot ax,$$

$$\frac{d}{dx}(\cot ax) = -a \operatorname{cosec}^2 ax, \qquad \frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}},$$

$$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}, \qquad \frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2 + x^2}.$$

Differentiation from first principles emphasises the definition of a derivative as the gradient of a function. However, for most practical purposes, returning to the definition (2.1) is time consuming and does not aid our understanding. Instead, as mentioned above, we employ a number of techniques, which use the derivatives listed above as 'building blocks', to evaluate the derivatives of more complicated functions than hitherto encountered. Subsections 2.1.2–2.1.7 develop the methods required.

2.1.2 Differentiation of products

As a first example of the differentiation of a more complicated function, we consider finding the derivative of a function $f(x)$ that can be written as the product of two other functions of x, namely $f(x) = u(x)v(x)$. For example, if $f(x) = x^3 \sin x$ then we might take $u(x) = x^3$ and $v(x) = \sin x$. Clearly the separation is not unique. (In the given example, possible alternative break-ups would be $u(x) = x^2$, $v(x) = x \sin x$, or even $u(x) = x^4 \tan x$, $v(x) = x^{-1} \cos x$.)

The purpose of the separation is to split the function into two (or more) parts, of which we know the derivatives (or at least we can evaluate these derivatives more easily than that of the whole). We would gain little, however, if we did not know the relationship between the derivative of f and those of u and v. Fortunately, they are very simply related, as we shall now show.

Since $f(x)$ is written as the product $u(x)v(x)$, it follows that

$$\begin{aligned} f(x + \Delta x) - f(x) &= u(x + \Delta x)v(x + \Delta x) - u(x)v(x) \\ &= u(x + \Delta x)[v(x + \Delta x) - v(x)] + [u(x + \Delta x) - u(x)]v(x). \end{aligned}$$

From the definition of a derivative (2.1),

$$\begin{aligned} \frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left\{ u(x + \Delta x) \left[\frac{v(x + \Delta x) - v(x)}{\Delta x} \right] + \left[\frac{u(x + \Delta x) - u(x)}{\Delta x} \right] v(x) \right\}. \end{aligned}$$

In the limit $\Delta x \to 0$, the factors in square brackets become dv/dx and du/dx (by the definitions of these quantities) and $u(x + \Delta x)$ simply becomes $u(x)$. Consequently we obtain

$$\frac{df}{dx} = \frac{d}{dx}[u(x)v(x)] = u(x)\frac{dv(x)}{dx} + \frac{du(x)}{dx}v(x). \tag{2.6}$$

In primed notation and without writing the argument x explicitly, (2.6) is stated concisely as

$$f' = (uv)' = uv' + u'v. \tag{2.7}$$

This is a general result obtained without making any assumptions about the specific forms f, u and v, other than that $f(x) = u(x)v(x)$. In words, the result reads as follows. *The derivative of the product of two functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first.*

►*Find the derivative with respect to x of $f(x) = x^3 \sin x$.*

Using the product rule, (2.6),

$$\begin{aligned} \frac{d}{dx}(x^3 \sin x) &= x^3 \frac{d}{dx}(\sin x) + \frac{d}{dx}(x^3) \sin x \\ &= x^3 \cos x + 3x^2 \sin x. \blacktriangleleft \end{aligned}$$

The product rule may readily be extended to the product of three or more functions. Considering the function

$$f(x) = u(x)v(x)w(x) \tag{2.8}$$

and using (2.6), we obtain, as before omitting the argument,

$$\frac{df}{dx} = u\frac{d}{dx}(vw) + \frac{du}{dx}vw.$$

Using (2.6) again to expand the first term on the RHS gives the complete result

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + u\frac{dv}{dx}w + \frac{du}{dx}vw \tag{2.9}$$

or

$$(uvw)' = uvw' + uv'w + u'vw. \tag{2.10}$$

It is readily apparent that this can be extended to products containing any number n of factors; the expression for the derivative will then consist of n terms with the prime appearing in successive terms on each of the n factors in turn. This is probably the easiest way to recall the product rule.

2.1.3 The chain rule

Products are just one type of complicated function that we may encounter in differentiation. Another is the function of a function, e.g. $f(x) = (3+x^2)^3 = u(x)^3$, where $u(x) = 3 + x^2$. If Δf, Δu and Δx are small finite quantities, it follows that

$$\frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta u}\frac{\Delta u}{\Delta x};$$

As the quantities become infinitesimally small we obtain

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}. \tag{2.11}$$

This is the *chain rule*, which we must apply when differentiating a function of a function.

►*Find the derivative with respect to x of $f(x) = (3 + x^2)^3$.*

Rewriting the function as $f(x) = u^3$, where $u(x) = 3 + x^2$, and applying (2.11) we find

$$\frac{df}{dx} = 3u^2\frac{du}{dx} = 3u^2\frac{d}{dx}(3 + x^2) = 3u^2 \times 2x = 6x(3 + x^2)^2. \blacktriangleleft$$

Similarly, the derivative with respect to x of $f(x) = 1/v(x)$ may be obtained by rewriting the function as $f(x) = v^{-1}$ and applying (2.11):

$$\frac{df}{dx} = -v^{-2}\frac{dv}{dx} = -\frac{1}{v^2}\frac{dv}{dx}. \tag{2.12}$$

The chain rule is also useful for calculating the derivative of a function f with respect to x when both x and f are written in terms of a variable (or parameter), say t.

►*Find the derivative with respect to x of $f(t) = 2at$, where $x = at^2$.*

We could of course substitute for t and then differentiate f as a function of x, but in this case it is quicker to use

$$\frac{df}{dx} = \frac{df}{dt}\frac{dt}{dx} = 2a\frac{1}{2at} = \frac{1}{t},$$

where we have used the fact that

$$\frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1}. \blacktriangleleft$$

2.1.4 Differentiation of quotients

Applying (2.6) for the derivative of a product to a function $f(x) = u(x)[1/v(x)]$, we may obtain the derivative of the quotient of two factors. Thus

$$f' = \left(\frac{u}{v}\right)' = u\left(\frac{1}{v}\right)' + u'\left(\frac{1}{v}\right) = u\left(-\frac{v'}{v^2}\right) + \frac{u'}{v},$$

where (2.12) has been used to evaluate $(1/v)'$. This can now be rearranged into the more convenient and memorisable form

$$f' = \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}. \tag{2.13}$$

This can be expressed in words as *the derivative of a quotient is equal to the bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared.*

►*Find the derivative with respect to x of $f(x) = \sin x/x$.*

Using (2.13) with $u(x) = \sin x$, $v(x) = x$ and hence $u'(x) = \cos x$, $v'(x) = 1$, we find

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2}. \blacktriangleleft$$

2.1.5 Implicit differentiation

So far we have only differentiated functions written in the form $y = f(x)$. However, we may not always be presented with a relationship in this simple form. As an example consider the relation $x^3 - 3xy + y^3 = 2$. In this case it is not possible to rearrange the equation to give y as a function of x. Nevertheless, by differentiating term by term with respect to x (*implicit differentiation*), we can find the derivative of y.

►*Find dy/dx if $x^3 - 3xy + y^3 = 2$.*

Differentiating each term in the equation with respect to x we obtain

$$\frac{d}{dx}(x^3) - \frac{d}{dx}(3xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(2),$$

$$\Rightarrow \quad 3x^2 - \left(3x\frac{dy}{dx} + 3y\right) + 3y^2\frac{dy}{dx} = 0,$$

where the derivative of $3xy$ has been found using the product rule. Hence, rearranging for dy/dx,

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}.$$

Note that dy/dx is a function of both x and y and cannot be expressed as a function of x only. ◄

2.1.6 Logarithmic differentiation

In circumstances in which the variable with respect to which we are differentiating is an exponent, taking logarithms and then differentiating implicitly is the simplest way to find the derivative.

►*Find the derivative with respect to x of $y = a^x$.*

To find the required derivative we first take logarithms and then differentiate implicitly:

$$\ln y = \ln a^x = x \ln a \qquad \Rightarrow \qquad \frac{1}{y}\frac{dy}{dx} = \ln a.$$

Now, rearranging and substituting for y, we find

$$\frac{dy}{dx} = y \ln a = a^x \ln a. \text{ ◄}$$

2.1.7 Leibnitz' theorem

We have discussed already how to find the derivative of a product of two or more functions. We now consider *Leibnitz' theorem*, which gives the corresponding results for the higher derivatives of products.

Consider again the function $f(x) = u(x)v(x)$. We know from the product rule that $f' = uv' + u'v$. Using the rule once more for each of the products, we obtain

$$\begin{aligned} f'' &= (uv'' + u'v') + (u'v' + u''v) \\ &= uv'' + 2u'v' + u''v. \end{aligned}$$

Similarly, differentiating twice more gives

$$\begin{aligned} f''' &= uv''' + 3u'v'' + 3u''v' + u'''v, \\ f^{(4)} &= uv^{(4)} + 4u'v''' + 6u''v'' + 4u'''v' + u^{(4)}v. \end{aligned}$$

The pattern emerging is clear and strongly suggests that the results generalise to

$$f^{(n)} = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} u^{(r)} v^{(n-r)} = \sum_{r=0}^{n} {}^{n}C_r u^{(r)} v^{(n-r)}, \tag{2.14}$$

where the fraction $n!/[r!(n-r)!]$ is identified with the binomial coefficient ${}^nC_r$ (see chapter 1). To *prove* that this is so, we use the method of induction as follows.

Assume that (2.14) is valid for n equal to some integer N. Then

$$\begin{aligned} f^{(N+1)} &= \sum_{r=0}^{N} {}^{N}C_r \frac{d}{dx}\left(u^{(r)} v^{(N-r)}\right) \\ &= \sum_{r=0}^{N} {}^{N}C_r [u^{(r)} v^{(N-r+1)} + u^{(r+1)} v^{(N-r)}] \\ &= \sum_{s=0}^{N} {}^{N}C_s u^{(s)} v^{(N+1-s)} + \sum_{s=1}^{N+1} {}^{N}C_{s-1} u^{(s)} v^{(N+1-s)}, \end{aligned}$$

where we have substituted summation index s for r in the first summation, and for $r+1$ in the second. Now, from our earlier discussion of binomial coefficients, equation (1.51), we have

$${}^{N}C_s + {}^{N}C_{s-1} = {}^{N+1}C_s$$

and so, after separating out the first term of the first summation and the last term of the second, obtain

$$f^{(N+1)} = {}^{N}C_0 u^{(0)} v^{(N+1)} + \sum_{s=1}^{N} {}^{N+1}C_s u^{(s)} v^{(N+1-s)} + {}^{N}C_N u^{(N+1)} v^{(0)}.$$

But ${}^{N}C_0 = 1 = {}^{N+1}C_0$ and ${}^{N}C_N = 1 = {}^{N+1}C_{N+1}$, and so we may write

$$\begin{aligned} f^{(N+1)} &= {}^{N+1}C_0 u^{(0)} v^{(N+1)} + \sum_{s=1}^{N} {}^{N+1}C_s u^{(s)} v^{(N+1-s)} + {}^{N+1}C_{N+1} u^{(N+1)} v^{(0)} \\ &= \sum_{s=0}^{N+1} {}^{N+1}C_s u^{(s)} v^{(N+1-s)}. \end{aligned}$$

This is just (2.14) with n set equal to $N+1$. Thus, assuming the validity of (2.14) for $n = N$ implies its validity for $n = N+1$. However, when $n = 1$ equation (2.14) is simply the product rule, and this we have already proved directly. These results taken together establish the validity of (2.14) for all n and prove Leibnitz' theorem.

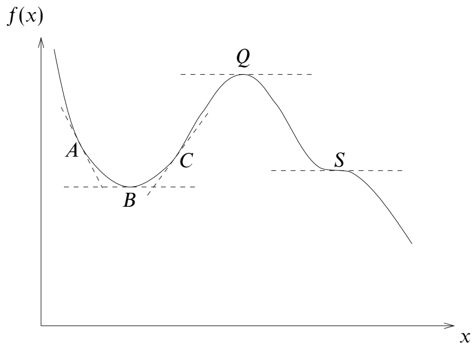

Figure 2.2 A graph of a function, $f(x)$, showing how differentiation corresponds to finding the gradient of the function at a particular point. Points B, Q and S are stationary points (see text).

►*Find the third derivative of the function* $f(x) = x^3 \sin x$.

Using (2.14) we immediately find

$$\begin{aligned} f'''(x) &= 6 \sin x + 3(6x) \cos x + 3(3x^2)(-\sin x) + x^3(-\cos x) \\ &= 3(2 - 3x^2) \sin x + x(18 - x^2) \cos x. \blacktriangleleft \end{aligned}$$

2.1.8 Special points of a function

We have interpreted the derivative of a function as the gradient of the function at the relevant point (figure 2.1). If the gradient is zero for some particular value of x then the function is said to have a *stationary point* there. Clearly, in graphical terms, this corresponds to a horizontal tangent to the graph.

Stationary points may be divided into three categories and an example of each is shown in figure 2.2. Point B is said to be a *minimum* since the function *increases* in value in both directions away from it. Point Q is said to be a *maximum* since the function *decreases* in both directions away from it. Note that B is not the overall minimum value of the function and Q is not the overall maximum; rather, they are a local minimum and a local maximum. Maxima and minima are known collectively as *turning points*.

The third type of stationary point is the *stationary point of inflection*, S. In this case the function falls in the positive x-direction and rises in the negative x-direction so that S is neither a maximum nor a minimum. Nevertheless, the gradient of the function is zero at S, i.e. the graph of the function is flat there, and this justifies our calling it a stationary point. Of course, a point at which the gradient of the function is zero but the function rises in the positive x-direction and falls in the negative x-direction is also a stationary point of inflection.

The above distinction between the three types of stationary point has been made rather descriptively. However, it is possible to define and distinguish stationary points mathematically. From their definition as points of zero gradient, all stationary points must be characterised by $df/dx = 0$. In the case of the minimum, B, the slope, i.e. df/dx, changes from negative at A to positive at C through zero at B. Thus df/dx is increasing and so the second derivative d^2f/dx^2 must be positive. Conversely, at the maximum, Q, we must have that d^2f/dx^2 is negative.

It is less obvious, but intuitively reasonable, that at S, d^2f/dx^2 is zero. This may be inferred from the following observations. To the left of S the curve is concave upwards so that df/dx is increasing with x and hence $d^2f/dx^2 > 0$. To the right of S, however, the curve is concave downwards so that df/dx is decreasing with x and hence $d^2f/dx^2 < 0$.

In summary, at a stationary point $df/dx = 0$ and

(i) for a minimum, $d^2f/dx^2 > 0$,

(ii) for a maximum, $d^2f/dx^2 < 0$,

(iii) for a stationary point of inflection, $d^2f/dx^2 = 0$ and d^2f/dx^2 changes sign through the point.

In case (iii), a stationary point of inflection, in order that d^2f/dx^2 changes sign through the point we normally require $d^3f/dx^3 \neq 0$ at that point. This simple rule can fail for some functions, however, and in general if the first non-vanishing derivative of $f(x)$ at the stationary point is $f^{(n)}$ then if n is even the point is a maximum or minimum and if n is odd the point is a stationary point of inflection. This may be seen from the Taylor expansion (see equation (4.17)) of the function about the stationary point, but it is not proved here.

►*Find the positions and natures of the stationary points of the function*

$$f(x) = 2x^3 - 3x^2 - 36x + 2.$$

The first criterion for a stationary point is that $df/dx = 0$, and hence we set

$$\frac{df}{dx} = 6x^2 - 6x - 36 = 0,$$

from which we obtain

$$(x-3)(x+2) = 0.$$

Hence the stationary points are at $x = 3$ and $x = -2$. To determine the nature of the stationary point we must evaluate d^2f/dx^2:

$$\frac{d^2f}{dx^2} = 12x - 6.$$

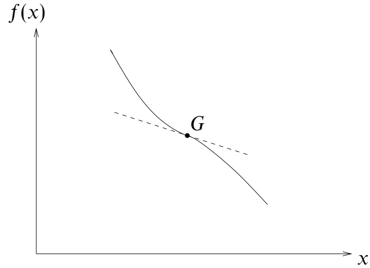

Figure 2.3 The graph of a function $f(x)$ that has a general point of inflection at the point G.

Now, we examine each stationary point in turn. For $x = 3$, $d^2f/dx^2 = 30$. Since this is positive, we conclude that $x = 3$ is a minimum. Similarly, for $x = -2$, $d^2f/dx^2 = -30$ and so $x = -2$ is a maximum. ◀

So far we have concentrated on stationary points, which are defined to have $df/dx = 0$. We have found that at a stationary point of inflection d^2f/dx^2 is also zero and changes sign. This naturally leads us to consider points at which d^2f/dx^2 is zero and changes sign but at which df/dx is *not*, in general, zero. Such points are called *general points of inflection* or simply *points of inflection*. Clearly, a stationary point of inflection is a special case for which df/dx is also zero. At a general point of inflection the graph of the function changes from being concave upwards to concave downwards (or vice versa), but the tangent to the curve at this point need not be horizontal. A typical example of a general point of inflection is shown in figure 2.3.

The determination of the stationary points of a function, together with the identification of its zeros, infinities and possible asymptotes, is usually sufficient to enable a graph of the function showing most of its significant features to be sketched. Some examples for the reader to try are included in the exercises at the end of this chapter.

2.1.9 Curvature of a function

In the previous section we saw that at a point of inflection of the function $f(x)$, the second derivative d^2f/dx^2 changes sign and passes through zero. The corresponding graph of f shows an inversion of its curvature at the point of inflection. We now develop a more quantitative measure of the curvature of a function (or its graph), which is applicable at general points and not just in the neighbourhood of a point of inflection.

As in figure 2.1, let θ be the angle made with the x-axis by the tangent at a

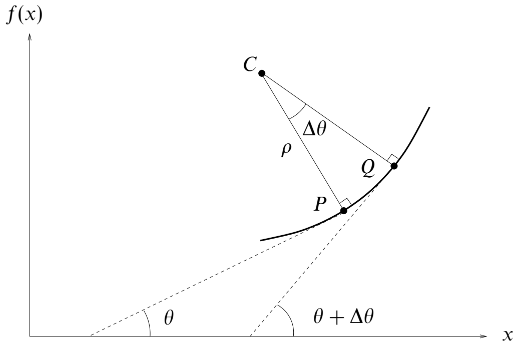

Figure 2.4 Two neighbouring tangents to the curve $f(x)$ whose slopes differ by $\Delta\theta$. The angular separation of the corresponding radii of the circle of curvature is also $\Delta\theta$.

point P on the curve $f = f(x)$, with $\tan\theta = df/dx$ evaluated at P. Now consider also the tangent at a neighbouring point Q on the curve, and suppose that it makes an angle $\theta + \Delta\theta$ with the x-axis, as illustrated in figure 2.4.

It follows that the corresponding normals at P and Q, which are perpendicular to the respective tangents, also intersect at an angle $\Delta\theta$. Furthermore, their point of intersection, C in the figure, will be the position of the centre of a circle that approximates the arc PQ, at least to the extent of having the same tangents at the extremities of the arc. This circle is called the *circle of curvature*.

For a finite arc PQ, the lengths of CP and CQ will not, in general, be equal, as they would be if $f = f(x)$ *were* in fact the equation of a circle. But, as Q is allowed to tend to P, i.e. as $\Delta\theta \to 0$, they do become equal, their common value being ρ, the radius of the circle, known as the *radius of curvature*. It follows immediately that the curve and the circle of curvature have a common tangent at P and lie on the same side of it. The reciprocal of the radius of curvature, ρ^{-1}, defines the *curvature* of the function $f(x)$ at the point P.

The radius of curvature can be defined more mathematically as follows. The length Δs of arc PQ is approximately equal to $\rho\Delta\theta$ and, in the limit $\Delta\theta \to 0$, this relationship defines ρ as

$$\rho = \lim_{\Delta\theta \to 0} \frac{\Delta s}{\Delta\theta} = \frac{ds}{d\theta}. \tag{2.15}$$

It should be noted that, as s increases, θ may increase or decrease according to whether the curve is locally concave upwards (i.e. shaped as if it were near a minimum in $f(x)$) or concave downwards. This is reflected in the sign of ρ, which therefore also indicates the position of the curve (and of the circle of curvature) relative to the common tangent, above or below. Thus a negative value of ρ indicates that the curve is locally concave downwards and that the tangent lies above the curve.

We next obtain an expression for ρ, not in terms of s and θ but in terms of x and $f(x)$. The expression, though somewhat cumbersome, follows from the defining equation (2.15), the defining property of θ that $\tan\theta = df/dx \equiv f'$ and the fact that the rate of change of arc length with x is given by

$$\frac{ds}{dx} = \left[1 + \left(\frac{df}{dx}\right)^2\right]^{1/2}. \tag{2.16}$$

This last result, simply quoted here, is proved more formally in subsection 2.2.13.

From the chain rule (2.11) it follows that

$$\rho = \frac{ds}{d\theta} = \frac{ds}{dx}\frac{dx}{d\theta}. \tag{2.17}$$

Differentiating both sides of $\tan\theta = df/dx$ with respect to x gives

$$\sec^2\theta \frac{d\theta}{dx} = \frac{d^2 f}{dx^2} \equiv f'',$$

from which, using $\sec^2\theta = 1 + \tan^2\theta = 1 + (f')^2$, we can obtain $dx/d\theta$ as

$$\frac{dx}{d\theta} = \frac{1+\tan^2\theta}{f''} = \frac{1+(f')^2}{f''}. \tag{2.18}$$

Substituting (2.16) and (2.18) into (2.17) then yields the final expression for ρ,

$$\rho = \frac{\left[1+(f')^2\right]^{3/2}}{f''}. \tag{2.19}$$

It should be noted that the quantity in brackets is always positive and that its three-halves root is also taken as positive. The sign of ρ is thus solely determined by that of d^2f/dx^2, in line with our previous discussion relating the sign to whether the curve is concave or convex upwards. If, as happens at a point of inflection, d^2f/dx^2 is zero then ρ is formally infinite and the curvature of $f(x)$ is zero. As d^2f/dx^2 changes sign on passing through zero, both the local tangent and the circle of curvature change from their initial positions to the opposite side of the curve.

►*Show that the radius of curvature at the point* (x, y) *on the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has magnitude $(a^4y^2 + b^4x^2)^{3/2}/(a^4b^4)$ *and the opposite sign to* y. *Check the special case* $b = a$, *for which the ellipse becomes a circle.*

Differentiating the equation of the ellipse with respect to x gives

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

and so

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

A second differentiation, using (2.13), then yields

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y - xy'}{y^2}\right) = -\frac{b^4}{a^2y^3}\left(\frac{y^2}{b^2} + \frac{x^2}{a^2}\right) = -\frac{b^4}{a^2y^3},$$

where we have used the fact that (x, y) lies on the ellipse. We note that d^2y/dx^2, and hence ρ, has the opposite sign to y^3 and hence to y. Substituting in (2.19) gives for the magnitude of the radius of curvature

$$|\rho| = \left|\frac{\left[1 + b^4x^2/(a^4y^2)\right]^{3/2}}{-b^4/(a^2y^3)}\right| = \frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4}.$$

For the special case $b = a$, $|\rho|$ reduces to $a^{-2}(y^2 + x^2)^{3/2}$ and, since $x^2 + y^2 = a^2$, this in turn gives $|\rho| = a$, as expected. ◄

The discussion in this section has been confined to the behaviour of curves that lie in one plane; examples of the application of curvature to the bending of loaded beams and to particle orbits under the influence of a central forces can be found in the exercises at the ends of later chapters. A more general treatment of curvature in three dimensions is given in section 10.3, where a vector approach is adopted.

2.1.10 Theorems of differentiation

Rolle's theorem

Rolle's theorem (figure 2.5) states that if a function $f(x)$ is continuous in the range $a \le x \le c$, is differentiable in the range $a < x < c$ and satisfies $f(a) = f(c)$ then for at least one point $x = b$, where $a < b < c$, $f'(b) = 0$. Thus Rolle's theorem states that for a well-behaved (continuous and differentiable) function that has the same value at two points either there is at least one stationary point between those points or the function is a constant between them. The validity of the theorem is immediately apparent from figure 2.5 and a full analytic proof will not be given. The theorem is used in deriving the mean value theorem, which we now discuss.

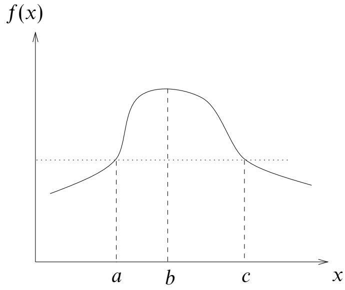

Figure 2.5 The graph of a function $f(x)$, showing that if $f(a) = f(c)$ then at one point at least between $x = a$ and $x = c$ the graph has zero gradient.

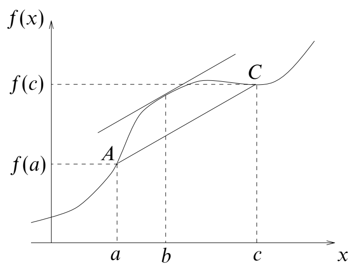

Figure 2.6 The graph of a function $f(x)$; at some point $x = b$ it has the same gradient as the line AC.

Mean value theorem

The mean value theorem (figure 2.6) states that if a function $f(x)$ is continuous in the range $a \leq x \leq c$ and differentiable in the range $a < x < c$ then

$$f'(b) = \frac{f(c) - f(a)}{c - a}, \tag{2.20}$$

for at least one value b where $a < b < c$. Thus the mean value theorem states that for a well-behaved function the gradient of the line joining two points on the curve is equal to the slope of the tangent to the curve for at least one intervening point.

The proof of the mean value theorem is found by examination of figure 2.6, as follows. The equation of the line AC is

$$g(x) = f(a) + (x - a)\frac{f(c) - f(a)}{c - a},$$

and hence the difference between the curve and the line is

$$h(x) = f(x) - g(x) = f(x) - f(a) - (x-a)\frac{f(c) - f(a)}{c-a}.$$

Since the curve and the line intersect at A and C, $h(x) = 0$ at both of these points. Hence, by an application of Rolle's theorem, $h'(x) = 0$ for at least one point b between A and C. Differentiating our expression for $h(x)$, we find

$$h'(x) = f'(x) - \frac{f(c) - f(a)}{c-a},$$

and hence at b, where $h'(x) = 0$,

$$f'(b) = \frac{f(c) - f(a)}{c-a}.$$

Applications of Rolle's theorem and the mean value theorem

Since the validity of Rolle's theorem is intuitively obvious, given the conditions imposed on $f(x)$, it will not be surprising that the problems that can be solved by applications of the theorem alone are relatively simple ones. Nevertheless we will illustrate it with the following example.

► *What semi-quantitative results can be deduced by applying Rolle's theorem to the following functions $f(x)$, with a and c chosen so that $f(a) = f(c) = 0$?* (i) $\sin x$, (ii) $\cos x$, (iii)$x^2 - 3x + 2$, (iv) $x^2 + 7x + 3$, (v) $2x^3 - 9x^2 - 24x + k$.

(i) If the consecutive values of x that make $\sin x = 0$ are $\alpha_1, \alpha_2, \ldots$ (actually $x = n\pi$, for any integer n) then Rolle's theorem implies that the derivative of $\sin x$, namely $\cos x$, has at least one zero lying between each pair of values α_i and α_{i+1}.

(ii) In an exactly similar way, we conclude that the derivative of $\cos x$, namely $-\sin x$, has at least one zero lying between consecutive pairs of zeros of $\cos x$. These two results taken together (but neither separately) imply that $\sin x$ and $\cos x$ have interleaving zeros.

(iii) For $f(x) = x^2 - 3x + 2$, $f(a) = f(c) = 0$ if a and c are taken as 1 and 2 respectively. Rolle's theorem then implies that $f'(x) = 2x - 3 = 0$ has a solution $x = b$ with b in the range $1 < b < 2$. This is obviously so, since $b = 3/2$.

(iv) With $f(x) = x^2 + 7x + 3$, the theorem tells us that if there are two roots of $x^2 + 7x + 3 = 0$ then they have the root of $f'(x) = 2x + 7 = 0$ lying between them. Thus if there are any (real) roots of $x^2 + 7x + 3 = 0$ then they lie one on either side of $x = -7/2$. The actual roots are $(-7 \pm \sqrt{37})/2$.

(v) If $f(x) = 2x^3 - 9x^2 - 24x + k$ then $f'(x) = 0$ is the equation $6x^2 - 18x - 24 = 0$, which has solutions $x = -1$ and $x = 4$. Consequently, if α_1 and α_2 are two different roots of $f(x) = 0$ then at least one of -1 and 4 must lie in the open interval α_1 to α_2. If, as is the case for a certain range of values of k, $f(x) = 0$ has three roots, α_1, α_2 and α_3, then $\alpha_1 < -1 < \alpha_2 < 4 < \alpha_3$.

In each case, as might be expected, the application of Rolle's theorem does no more than focus attention on particular ranges of values; it does not yield precise answers. ◄

Direct verification of the mean value theorem is straightforward when it is applied to simple functions. For example, if $f(x) = x^2$, it states that there is a value b in the interval $a < b < c$ such that

$$c^2 - a^2 = f(c) - f(a) = (c - a)f'(b) = (c - a)2b.$$

This is clearly so, since $b = (a + c)/2$ satisfies the relevant criteria.

As a slightly more complicated example we may consider a cubic equation, say $f(x) = x^3 + 2x^2 + 4x - 6 = 0$, between two specified values of x, say 1 and 2. In this case we need to verify that there is a value of x lying in the range $1 < x < 2$ that satisfies

$$18 - 1 = f(2) - f(1) = (2 - 1)f'(x) = 1(3x^2 + 4x + 4).$$

This is easily done, either by evaluating $3x^2+4x+4-17$ at $x = 1$ and at $x = 2$ and checking that the values have opposite signs or by solving $3x^2 + 4x + 4 - 17 = 0$ and showing that one of the roots lies in the stated interval.

The following applications of the mean value theorem establish some general inequalities for two common functions.

►*Determine inequalities satisfied by* $\ln x$ *and* $\sin x$ *for suitable ranges of the real variable* x.

Since for positive values of its argument the derivative of $\ln x$ is x^{-1}, the mean value theorem gives us

$$\frac{\ln c - \ln a}{c - a} = \frac{1}{b}$$

for some b in $0 < a < b < c$. Further, since $a < b < c$ implies that $c^{-1} < b^{-1} < a^{-1}$, we have

$$\frac{1}{c} < \frac{\ln c - \ln a}{c - a} < \frac{1}{a},$$

or, multiplying through by $c - a$ and writing $c/a = x$ where $x > 1$,

$$1 - \frac{1}{x} < \ln x < x - 1.$$

Applying the mean value theorem to $\sin x$ shows that

$$\frac{\sin c - \sin a}{c - a} = \cos b$$

for some b lying between a and c. If a and c are restricted to lie in the range $0 \leq a < c \leq \pi$, in which the cosine function is monotonically decreasing (i.e. there are no turning points), we can deduce that

$$\cos c < \frac{\sin c - \sin a}{c - a} < \cos a. \blacktriangleleft$$

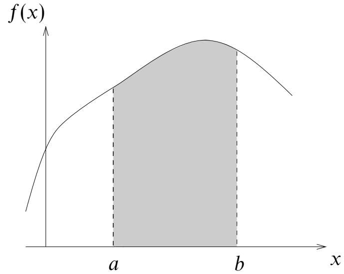

Figure 2.7 An integral as the area under a curve.

2.2 Integration

The notion of an integral as the area under a curve will be familiar to the reader. In figure 2.7, in which the solid line is a plot of a function $f(x)$, the shaded area represents the quantity denoted by

$$I = \int_a^b f(x)\,dx. \tag{2.21}$$

This expression is known as the *definite integral* of $f(x)$ between the *lower limit* $x = a$ and the *upper limit* $x = b$, and $f(x)$ is called the *integrand*.

2.2.1 Integration from first principles

The definition of an integral as the area under a curve is not a formal definition, but one that can be readily visualised. The formal definition of I involves subdividing the finite interval $a \le x \le b$ into a large number of subintervals, by defining intermediate points ξ_i such that $a = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = b$, and then forming the sum

$$S = \sum_{i=1}^{n} f(x_i)(\xi_i - \xi_{i-1}), \tag{2.22}$$

where x_i is an arbitrary point that lies in the range $\xi_{i-1} \le x_i \le \xi_i$ (see figure 2.8). If now n is allowed to tend to infinity in any way whatsoever, subject only to the restriction that the length of every subinterval ξ_{i-1} to ξ_i tends to zero, then S might, or might not, tend to a unique limit, I. If it does then the definite integral of $f(x)$ between a and b is defined as having the value I. If no unique limit exists the integral is undefined. For continuous functions and a finite interval $a \le x \le b$ the existence of a unique limit is assured and the integral is guaranteed to exist.

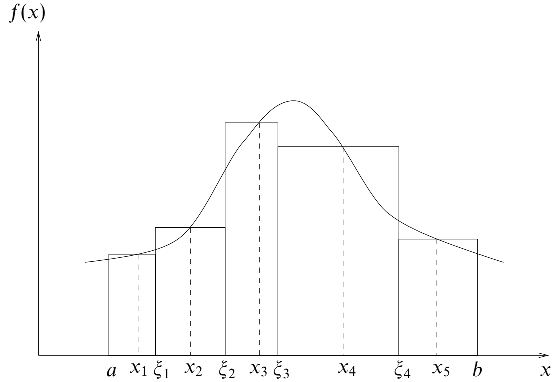

Figure 2.8 The evaluation of a definite integral by subdividing the interval $a \le x \le b$ into subintervals.

►*Evaluate from first principles the integral* $I = \int_0^b x^2\,dx$.

We first approximate the area under the curve $y = x^2$ between 0 and b by n rectangles of equal width h. If we take the value at the lower end of each subinterval (in the limit of an infinite number of subintervals we could equally well have chosen the value at the upper end) to give the height of the corresponding rectangle, then the area of the kth rectangle will be $(kh)^2h = k^2h^3$. The total area is thus

$$A = \sum_{k=0}^{n-1} k^2h^3 = (h^3)\tfrac{1}{6}n(n-1)(2n-1),$$

where we have used the expression for the sum of the squares of the natural numbers derived in subsection 1.7.1. Now $h = b/n$ and so

$$A = \left(\frac{b^3}{n^3}\right)\frac{n}{6}(n-1)(2n-1) = \frac{b^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right).$$

As $n \to \infty$, $A \to b^3/3$, which is thus the value I of the integral. ◄

Some straightforward properties of definite integrals that are almost self-evident are as follows:

$$\int_a^b 0\,dx = 0, \qquad \int_a^a f(x)\,dx = 0, \tag{2.23}$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx, \tag{2.24}$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \tag{2.25}$$

Combining (2.23) and (2.24) with c set equal to a shows that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{2.26}$$

2.2.2 Integration as the inverse of differentiation

The definite integral has been defined as the area under a curve between two fixed limits. Let us now consider the integral

$$F(x) = \int_a^x f(u)\,du \tag{2.27}$$

in which the lower limit a remains fixed but the upper limit x is now variable. It will be noticed that this is essentially a restatement of (2.21), but that the variable x in the integrand has been replaced by a new variable u. It is conventional to rename the *dummy variable* in the integrand in this way in order that the same variable does not appear in both the integrand and the integration limits.

It is apparent from (2.27) that $F(x)$ is a continuous function of x, but at first glance the definition of an integral as the area under a curve does not connect with our assertion that integration is the inverse process to differentiation. However, by considering the integral (2.27) and using the elementary property (2.24), we obtain

$$\begin{aligned} F(x+\Delta x) &= \int_a^{x+\Delta x} f(u)\,du \\ &= \int_a^x f(u)\,du + \int_x^{x+\Delta x} f(u)\,du \\ &= F(x) + \int_x^{x+\Delta x} f(u)\,du. \end{aligned}$$

Rearranging and dividing through by Δx yields

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{\Delta x}\int_x^{x+\Delta x} f(u)\,du.$$

Letting $\Delta x \to 0$ and using (2.1) we find that the LHS becomes dF/dx, whereas the RHS becomes $f(x)$. The latter conclusion follows because when Δx is small the value of the integral on the RHS is approximately $f(x)\Delta x$, and in the limit $\Delta x \to 0$ no approximation is involved. Thus

$$\frac{dF(x)}{dx} = f(x), \tag{2.28}$$

or, substituting for $F(x)$ from (2.27),

$$\frac{d}{dx}\left[\int_a^x f(u)\,du\right] = f(x).$$

From the last two equations it is clear that integration can be considered as the inverse of differentiation. However, we see from the above analysis that the lower limit a is arbitrary and so differentiation does not have a *unique* inverse. Any function $F(x)$ obeying (2.28) is called an *indefinite integral* of $f(x)$, though any two such functions can differ by at most an arbitrary additive constant. Since the lower limit is arbitrary, it is usual to write

$$F(x) = \int^{x} f(u)\,du \tag{2.29}$$

and explicitly include the arbitrary constant only when evaluating $F(x)$. The evaluation is conventionally written in the form

$$\int f(x)\,dx = F(x) + c \tag{2.30}$$

where c is called the *constant of integration*. It will be noticed that, in the absence of any integration limits, we use the same symbol for the arguments of both f and F. This can be confusing, but is sufficiently common practice that the reader needs to become familiar with it.

We also note that the definite integral of $f(x)$ between the fixed limits $x = a$ and $x = b$ can be written in terms of $F(x)$. From (2.27) we have

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{x_0}^{b} f(x)\,dx - \int_{x_0}^{a} f(x)\,dx \\ &= F(b) - F(a),\end{aligned} \tag{2.31}$$

where x_0 is *any* third fixed point. Using the notation $F'(x) = dF/dx$, we may rewrite (2.28) as $F'(x) = f(x)$, and so express (2.31) as

$$\int_a^b F'(x)\,dx = F(b) - F(a) \equiv [F]_a^b.$$

In contrast to differentiation, where repeated applications of the product rule and/or the chain rule will always give the required derivative, it is not always possible to find the integral of an arbitrary function. Indeed, in most real physical problems exact integration cannot be performed and we have to revert to numerical approximations. Despite this cautionary note, it is in fact possible to integrate many simple functions and the following subsections introduce the most common types. Many of the techniques will be familiar to the reader and so are summarised by example.

2.2.3 Integration by inspection

The simplest method of integrating a function is by inspection. Some of the more elementary functions have well-known integrals that should be remembered. The reader will notice that these integrals are precisely the inverses of the derivatives found near the end of subsection 2.1.1. A few are presented below, using the form given in (2.30):

$$\int a\,dx = ax + c, \qquad \int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c,$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c, \qquad \int \frac{a}{x}\,dx = a\ln x + c,$$

$$\int a\cos bx\,dx = \frac{a\sin bx}{b} + c, \qquad \int a\sin bx\,dx = \frac{-a\cos bx}{b} + c,$$

$$\int a\tan bx\,dx = \frac{-a\ln(\cos bx)}{b} + c, \qquad \int a\cos bx\sin^n bx\,dx = \frac{a\sin^{n+1} bx}{b(n+1)} + c,$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c, \qquad \int a\sin bx\cos^n bx\,dx = \frac{-a\cos^{n+1} bx}{b(n+1)} + c,$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c, \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c,$$

where the integrals that depend on n are valid for all $n \neq -1$ and where a and b are constants. In the two final results $|x| \leq a$.

2.2.4 Integration of sinusoidal functions

Integrals of the type $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$ may be found by using trigonometric expansions. Two methods are applicable, one for odd n and the other for even n. They are best illustrated by example.

►*Evaluate the integral* $I = \int \sin^5 x\,dx$.

Rewriting the integral as a product of $\sin x$ and an even power of $\sin x$, and then using the relation $\sin^2 x = 1 - \cos^2 x$ yields

$$\begin{aligned}
I &= \int \sin^4 x \sin x\,dx \\
&= \int (1-\cos^2 x)^2 \sin x\,dx \\
&= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx \\
&= \int (\sin x - 2\sin x\cos^2 x + \sin x\cos^4 x)\,dx \\
&= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + c,
\end{aligned}$$

where the integration has been carried out using the results of subsection 2.2.3. ◄

►*Evaluate the integral* $I = \int \cos^4 x\, dx$.

Rewriting the integral as a power of $\cos^2 x$ and then using the double-angle formula $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ yields

$$\begin{aligned} I &= \int (\cos^2 x)^2\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \int \tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x)\, dx. \end{aligned}$$

Using the double-angle formula again we may write $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$, and hence

$$\begin{aligned} I &= \int \left[\tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}(1 + \cos 4x)\right] dx \\ &= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x + c \\ &= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + c. \blacktriangleleft \end{aligned}$$

2.2.5 Logarithmic integration

Integrals for which the integrand may be written as a fraction in which the numerator is the derivative of the denominator may be evaluated using

$$\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + c. \tag{2.32}$$

This follows directly from the differentiation of a logarithm as a function of a function (see subsection 2.1.3).

►*Evaluate the integral*

$$I = \int \frac{6x^2 + 2\cos x}{x^3 + \sin x}\, dx.$$

We note first that the numerator can be factorised to give $2(3x^2 + \cos x)$, and then that the quantity in brackets is the derivative of the denominator. Hence

$$I = 2\int \frac{3x^2 + \cos x}{x^3 + \sin x}\, dx = 2\ln(x^3 + \sin x) + c. \blacktriangleleft$$

2.2.6 Integration using partial fractions

The method of partial fractions was discussed at some length in section 1.4, but in essence consists of the manipulation of a fraction (here the integrand) in such a way that it can be written as the sum of two or more simpler fractions. Again we illustrate the method by an example.

►*Evaluate the integral*

$$I = \int \frac{1}{x^2 + x}\, dx.$$

We note that the denominator factorises to give $x(x + 1)$. Hence

$$I = \int \frac{1}{x(x + 1)}\, dx.$$

We now separate the fraction into two partial fractions and integrate directly:

$$I = \int \left(\frac{1}{x} - \frac{1}{x + 1} \right) dx = \ln x - \ln(x + 1) + c = \ln \left(\frac{x}{x + 1} \right) + c. \blacktriangleleft$$

2.2.7 Integration by substitution

Sometimes it is possible to make a substitution of variables that turns a complicated integral into a simpler one, which can then be integrated by a standard method. There are many useful substitutions and knowing which to use is a matter of experience. We now present a few examples of particularly useful substitutions.

►*Evaluate the integral*

$$I = \int \frac{1}{\sqrt{1 - x^2}}\, dx.$$

Making the substitution $x = \sin u$, we note that $dx = \cos u\, du$, and hence

$$I = \int \frac{1}{\sqrt{1 - \sin^2 u}} \cos u\, du = \int \frac{1}{\sqrt{\cos^2 u}} \cos u\, du = \int du = u + c.$$

Now substituting back for u,

$$I = \sin^{-1} x + c.$$

This corresponds to one of the results given in subsection 2.2.3. ◄

Another particular example of integration by substitution is afforded by integrals of the form

$$I = \int \frac{1}{a + b \cos x}\, dx \qquad \text{or} \qquad I = \int \frac{1}{a + b \sin x}\, dx. \tag{2.33}$$

In these cases, making the substitution $t = \tan(x/2)$ yields integrals that can be solved more easily than the originals. Formulae expressing $\sin x$ and $\cos x$ in terms of t were derived in equations (1.32) and (1.33) (see p. 14), but before we can use them we must relate dx to dt as follows.

Since

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) = \frac{1+t^2}{2},$$

the required relationship is

$$dx = \frac{2}{1+t^2}\,dt. \tag{2.34}$$

►*Evaluate the integral*

$$I = \int \frac{2}{1+3\cos x}\,dx.$$

Rewriting $\cos x$ in terms of t and using (2.34) yields

$$\begin{aligned}
I &= \int \frac{2}{1+3\left[(1-t^2)(1+t^2)^{-1}\right]}\left(\frac{2}{1+t^2}\right)dt \\
&= \int \frac{2(1+t^2)}{1+t^2+3(1-t^2)}\left(\frac{2}{1+t^2}\right)dt \\
&= \int \frac{2}{2-t^2}\,dt = \int \frac{2}{(\sqrt{2}-t)(\sqrt{2}+t)}\,dt \\
&= \int \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}-t}+\frac{1}{\sqrt{2}+t}\right)dt \\
&= -\frac{1}{\sqrt{2}}\ln(\sqrt{2}-t)+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+t)+c \\
&= \frac{1}{\sqrt{2}}\ln\left[\frac{\sqrt{2}+\tan(x/2)}{\sqrt{2}-\tan(x/2)}\right]+c. \blacktriangleleft
\end{aligned}$$

Integrals of a similar form to (2.33), but involving $\sin 2x$, $\cos 2x$, $\tan 2x$, $\sin^2 x$, $\cos^2 x$ or $\tan^2 x$ instead of $\cos x$ and $\sin x$, should be evaluated by using the substitution $t = \tan x$. In this case

$$\sin x = \frac{t}{\sqrt{1+t^2}}, \qquad \cos x = \frac{1}{\sqrt{1+t^2}} \qquad \text{and} \qquad dx = \frac{dt}{1+t^2}. \tag{2.35}$$

A final example of the evaluation of integrals using substitution is the method of completing the square (cf. subsection 1.7.3).

►*Evaluate the integral*

$$I = \int \frac{1}{x^2 + 4x + 7}\, dx.$$

We can write the integral in the form

$$I = \int \frac{1}{(x+2)^2 + 3}\, dx.$$

Substituting $y = x + 2$, we find $dy = dx$ and hence

$$I = \int \frac{1}{y^2 + 3}\, dy,$$

Hence, by comparison with the table of standard integrals (see subsection 2.2.3)

$$I = \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{y}{\sqrt{3}}\right) + c = \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + c. \blacktriangleleft$$

2.2.8 Integration by parts

Integration by parts is the integration analogy of product differentiation. The principle is to break down a complicated function into two functions, at least one of which can be integrated by inspection. The method in fact relies on the result for the differentiation of a product. Recalling from (2.6) that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v,$$

where u and v are functions of x, we now integrate to find

$$uv = \int u\frac{dv}{dx}\, dx + \int \frac{du}{dx}\, v\, dx.$$

Rearranging into the standard form for integration by parts gives

$$\int u\frac{dv}{dx}\, dx = uv - \int \frac{du}{dx}\, v\, dx. \tag{2.36}$$

Integration by parts is often remembered for practical purposes in the form *the integral of a product of two functions is equal to* {*the first times the integral of the second*} *minus the integral of* {*the derivative of the first times the integral of the second*}. Here, u is 'the first' and dv/dx is 'the second'; clearly the integral v of 'the second' must be determinable by inspection.

►*Evaluate the integral* $I = \int x \sin x\, dx$.

In the notation given above, we identify x with u and $\sin x$ with dv/dx. Hence $v = -\cos x$ and $du/dx = 1$ and so using (2.36)

$$I = x(-\cos x) - \int (1)(-\cos x)\, dx = -x\cos x + \sin x + c. \blacktriangleleft$$

The separation of the functions is not always so apparent, as is illustrated by the following example.

►*Evaluate the integral* $I = \int x^3 e^{-x^2}\, dx$.

Firstly we rewrite the integral as

$$I = \int x^2 \left(xe^{-x^2} \right)\, dx.$$

Now, using the notation given above, we identify x^2 with u and xe^{-x^2} with dv/dx. Hence $v = -\frac{1}{2}e^{-x^2}$ and $du/dx = 2x$, so that

$$I = -\tfrac{1}{2}x^2 e^{-x^2} - \int (-x)e^{-x^2}\, dx = -\tfrac{1}{2}x^2 e^{-x^2} - \tfrac{1}{2}e^{-x^2} + c. \blacktriangleleft$$

A trick that is sometimes useful is to take '1' as one factor of the product, as is illustrated by the following example.

►*Evaluate the integral* $I = \int \ln x\, dx$.

Firstly we rewrite the integral as

$$I = \int (\ln x)\, 1\, dx.$$

Now, using the notation above, we identify $\ln x$ with u and 1 with dv/dx. Hence we have $v = x$ and $du/dx = 1/x$, and so

$$I = (\ln x)(x) - \int \left(\frac{1}{x} \right) x\, dx = x \ln x - x + c. \blacktriangleleft$$

It is sometimes necessary to integrate by parts more than once. In doing so, we may occasionally re-encounter the original integral I. In such cases we can obtain a linear algebraic equation for I that can be solved to obtain its value.

►*Evaluate the integral* $I = \int e^{ax} \cos bx\, dx$.

Integrating by parts, taking e^{ax} as the first function, we find

$$I = e^{ax} \left(\frac{\sin bx}{b} \right) - \int ae^{ax} \left(\frac{\sin bx}{b} \right) dx,$$

where, for convenience, we have omitted the constant of integration. Integrating by parts a second time,

$$I = e^{ax} \left(\frac{\sin bx}{b} \right) - ae^{ax} \left(\frac{-\cos bx}{b^2} \right) + \int a^2 e^{ax} \left(\frac{-\cos bx}{b^2} \right) dx.$$

Notice that the integral on the RHS is just $-a^2/b^2$ times the original integral I. Thus

$$I = e^{ax} \left(\frac{1}{b} \sin bx + \frac{a}{b^2} \cos bx \right) - \frac{a^2}{b^2} I.$$

Rearranging this expression to obtain I explicitly and including the constant of integration we find

$$I = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c. \tag{2.37}$$

Another method of evaluating this integral, using the exponential of a complex number, is given in section 3.6. ◄

2.2.9 Reduction formulae

Integration using reduction formulae is a process that involves first evaluating a simple integral and then, in stages, using it to find a more complicated integral.

► *Using integration by parts, find a relationship between I_n and I_{n-1} where*

$$I_n = \int_0^1 (1-x^3)^n\, dx$$

and n is any positive integer. Hence evaluate $I_2 = \int_0^1 (1-x^3)^2\, dx$.

Writing the integrand as a product and separating the integral into two we find

$$\begin{aligned} I_n &= \int_0^1 (1-x^3)(1-x^3)^{n-1}\, dx \\ &= \int_0^1 (1-x^3)^{n-1}\, dx - \int_0^1 x^3(1-x^3)^{n-1}\, dx. \end{aligned}$$

The first term on the RHS is clearly I_{n-1} and so, writing the integrand in the second term on the RHS as a product,

$$I_n = I_{n-1} - \int_0^1 (x)x^2(1-x^3)^{n-1}\, dx.$$

Integrating by parts we find

$$\begin{aligned} I_n &= I_{n-1} + \left[\frac{x}{3n}(1-x^3)^n\right]_0^1 - \int_0^1 \frac{1}{3n}(1-x^3)^n\, dx \\ &= I_{n-1} + 0 - \frac{1}{3n} I_n, \end{aligned}$$

which on rearranging gives

$$I_n = \frac{3n}{3n+1} I_{n-1}.$$

We now have a relation connecting successive integrals. Hence, if we can evaluate I_0, we can find I_1, I_2 etc. Evaluating I_0 is trivial:

$$I_0 = \int_0^1 (1-x^3)^0\, dx = \int_0^1 dx = [x]_0^1 = 1.$$

Hence

$$I_1 = \frac{(3\times 1)}{(3\times 1)+1} \times 1 = \frac{3}{4}, \qquad I_2 = \frac{(3\times 2)}{(3\times 2)+1} \times \frac{3}{4} = \frac{9}{14}.$$

Although the first few I_n could be evaluated by direct multiplication, this becomes tedious for integrals containing higher values of n; these are therefore best evaluated using the reduction formula. ◄

2.2.10 Infinite and improper integrals

The definition of an integral given previously does not allow for cases in which either of the limits of integration is infinite (an *infinite integral*) or for cases in which $f(x)$ is infinite in some part of the range (an *improper integral*), e.g. $f(x) = (2-x)^{-1/4}$ near the point $x = 2$. Nevertheless, modification of the definition of an integral gives infinite and improper integrals each a meaning.

In the case of an integral $I = \int_a^b f(x)\,dx$, the infinite integral, in which b tends to ∞, is defined by

$$I = \int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx = \lim_{b\to\infty} F(b) - F(a).$$

As previously, $F(x)$ is the indefinite integral of $f(x)$ and $\lim_{b\to\infty} F(b)$ means the limit (or value) that $F(b)$ approaches as $b \to \infty$; it is evaluated *after* calculating the integral. The formal concept of a limit will be introduced in chapter 4.

►*Evaluate the integral*

$$I = \int_0^\infty \frac{x}{(x^2+a^2)^2}\,dx.$$

Integrating, we find $F(x) = -\frac{1}{2}(x^2+a^2)^{-1} + c$ and so

$$I = \lim_{b\to\infty}\left[\frac{-1}{2(b^2+a^2)}\right] - \left(\frac{-1}{2a^2}\right) = \frac{1}{2a^2}. \blacktriangleleft$$

For the case of improper integrals, we adopt the approach of excluding the unbounded range from the integral. For example, if the integrand $f(x)$ is infinite at $x = c$ (say), $a \le c \le b$ then

$$\int_a^b f(x)\,dx = \lim_{\delta\to 0}\int_a^{c-\delta} f(x)\,dx + \lim_{\epsilon\to 0}\int_{c+\epsilon}^b f(x)\,dx.$$

►*Evaluate the integral* $I = \int_0^2 (2-x)^{-1/4}\,dx.$

Integrating directly,

$$I = \lim_{\epsilon\to 0}\left[-\tfrac{4}{3}(2-x)^{3/4}\right]_0^{2-\epsilon} = \lim_{\epsilon\to 0}\left[-\tfrac{4}{3}\epsilon^{3/4}\right] + \tfrac{4}{3}2^{3/4} = \left(\tfrac{4}{3}\right)2^{3/4}. \blacktriangleleft$$

2.2.11 Integration in plane polar coordinates

In plane polar coordinates ρ, ϕ, a curve is defined by its distance ρ from the origin as a function of the angle ϕ between the line joining a point on the curve to the origin and the x-axis, i.e. $\rho = \rho(\phi)$. The area of an element is given by

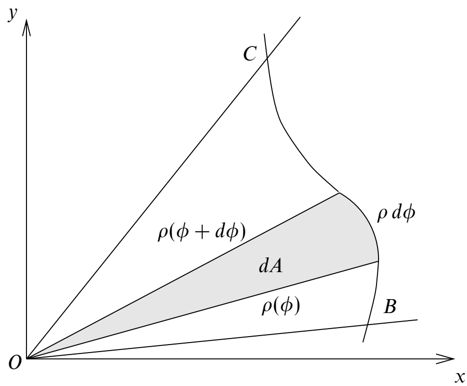

Figure 2.9 Finding the area of a sector OBC defined by the curve $\rho(\phi)$ and the radii OB, OC, at angles to the x-axis ϕ_1, ϕ_2 respectively.

$dA = \frac{1}{2}\rho^2\, d\phi$, as illustrated in figure 2.9, and hence the total area between two angles ϕ_1 and ϕ_2 is given by

$$A = \int_{\phi_1}^{\phi_2} \tfrac{1}{2}\rho^2\, d\phi. \tag{2.38}$$

An immediate observation is that the area of a circle of radius a is given by

$$A = \int_0^{2\pi} \tfrac{1}{2}a^2\, d\phi = \left[\tfrac{1}{2}a^2\phi\right]_0^{2\pi} = \pi a^2.$$

►*The equation in polar coordinates of an ellipse with semi-axes a and b is*

$$\frac{1}{\rho^2} = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}.$$

Find the area A of the ellipse.

Using (2.38) and symmetry, we have

$$A = \frac{1}{2}\int_0^{2\pi} \frac{a^2b^2}{b^2\cos^2\phi + a^2\sin^2\phi}\, d\phi = 2a^2b^2 \int_0^{\pi/2} \frac{1}{b^2\cos^2\phi + a^2\sin^2\phi}\, d\phi.$$

To evaluate this integral we write $t = \tan\phi$ and use (2.35):

$$A = 2a^2b^2 \int_0^{\infty} \frac{1}{b^2 + a^2t^2}\, dt = 2b^2 \int_0^{\infty} \frac{1}{(b/a)^2 + t^2}\, dt.$$

Finally, from the list of standard integrals (see subsection 2.2.3),

$$A = 2b^2 \left[\frac{1}{(b/a)} \tan^{-1}\frac{t}{(b/a)}\right]_0^{\infty} = 2ab\left(\frac{\pi}{2} - 0\right) = \pi ab. \blacktriangleleft$$

2.2.12 Integral inequalities

Consider the functions $f(x)$, $\phi_1(x)$ and $\phi_2(x)$ such that $\phi_1(x) \leq f(x) \leq \phi_2(x)$ for all x in the range $a \leq x \leq b$. It immediately follows that

$$\int_a^b \phi_1(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b \phi_2(x)\,dx, \tag{2.39}$$

which gives us a way of estimating an integral that is difficult to evaluate explicitly.

►*Show that the value of the integral*

$$I = \int_0^1 \frac{1}{(1+x^2+x^3)^{1/2}}\,dx$$

lies between 0.810 *and* 0.882.

We note that for x in the range $0 \leq x \leq 1$, $0 \leq x^3 \leq x^2$. Hence

$$(1+x^2)^{1/2} \leq (1+x^2+x^3)^{1/2} \leq (1+2x^2)^{1/2},$$

and so

$$\frac{1}{(1+x^2)^{1/2}} \geq \frac{1}{(1+x^2+x^3)^{1/2}} \geq \frac{1}{(1+2x^2)^{1/2}}.$$

Consequently,

$$\int_0^1 \frac{1}{(1+x^2)^{1/2}}\,dx \geq \int_0^1 \frac{1}{(1+x^2+x^3)^{1/2}}\,dx \geq \int_0^1 \frac{1}{(1+2x^2)^{1/2}}\,dx,$$

from which we obtain

$$\left[\ln(x+\sqrt{1+x^2})\right]_0^1 \geq I \geq \left[\tfrac{1}{\sqrt{2}}\ln\left(x+\sqrt{\tfrac{1}{2}+x^2}\right)\right]_0^1$$

$$0.8814 \geq I \geq 0.8105$$

$$0.882 \geq I \geq 0.810.$$

In the last line the calculated values have been rounded to three significant figures, one rounded up and the other rounded down so that the proved inequality cannot be unknowingly made invalid. ◄

2.2.13 Applications of integration

Mean value of a function

The mean value m of a function between two limits a and b is defined by

$$m = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.40}$$

The mean value may be thought of as the height of the rectangle that has the same area (over the same interval) as the area under the curve $f(x)$. This is illustrated in figure 2.10.

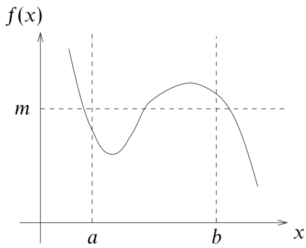

Figure 2.10 The mean value m of a function.

► *Find the mean value m of the function $f(x) = x^2$ between the limits $x = 2$ and $x = 4$.*

Using (2.40),

$$m = \frac{1}{4-2}\int_2^4 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_2^4 = \frac{1}{2}\left(\frac{4^3}{3} - \frac{2^3}{3}\right) = \frac{28}{3}. \blacktriangleleft$$

Finding the length of a curve

Finding the area between a curve and certain straight lines provides one example of the use of integration. Another is in finding the length of a curve. If a curve is defined by $y = f(x)$ then the distance along the curve, Δs, that corresponds to small changes Δx and Δy in x and y is given by

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}; \tag{2.41}$$

this follows directly from Pythagoras' theorem (see figure 2.11). Dividing (2.41) through by Δx and letting $\Delta x \to 0$ we obtain[§]

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Clearly the total length s of the curve between the points $x = a$ and $x = b$ is then given by integrating both sides of the equation:

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx. \tag{2.42}$$

[§] Instead of considering small changes Δx and Δy and letting these tend to zero, we could have derived (2.41) by considering infinitesimal changes dx and dy from the start. After writing $(ds)^2 = (dx)^2 + (dy)^2$, (2.41) may be deduced by using the formal device of dividing through by dx. Although not mathematically rigorous, this method is often used and generally leads to the correct result.

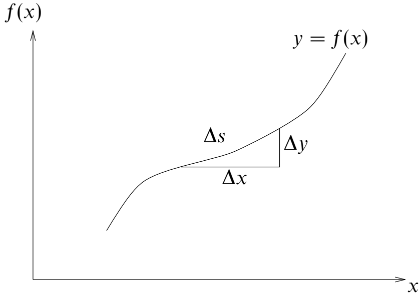

Figure 2.11 The distance moved along a curve, Δs, corresponding to the small changes Δx and Δy.

In plane polar coordinates,

$$ds = \sqrt{(dr)^2 + (r\,d\phi)^2} \qquad \Rightarrow \qquad s = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\phi}{dr}\right)^2}\, dr. \tag{2.43}$$

►*Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 2$.*

Using (2.42) and noting that $dy/dx = \frac{3}{2}\sqrt{x}$, the length s of the curve is given by

$$\begin{aligned} s &= \int_0^2 \sqrt{1 + \tfrac{9}{4}x}\, dx \\ &= \left[\tfrac{2}{3}\left(\tfrac{4}{9}\right)\left(1 + \tfrac{9}{4}x\right)^{3/2}\right]_0^2 = \tfrac{8}{27}\left[\left(1 + \tfrac{9}{4}x\right)^{3/2}\right]_0^2 \\ &= \tfrac{8}{27}\left[\left(\tfrac{11}{2}\right)^{3/2} - 1\right]. \end{aligned}$$ ◄

Surfaces of revolution

Consider the surface S formed by rotating the curve $y = f(x)$ about the x-axis (see figure 2.12). The surface area of the 'collar' formed by rotating an element of the curve, ds, about the x-axis is $2\pi y\, ds$, and hence the total surface area is

$$S = \int_a^b 2\pi y\, ds.$$

Since $(ds)^2 = (dx)^2 + (dy)^2$ from (2.41), the total surface area between the planes $x = a$ and $x = b$ is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{2.44}$$

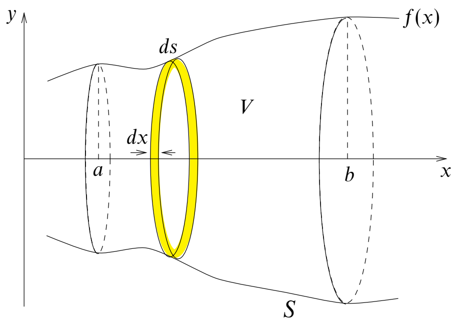

Figure 2.12 The surface and volume of revolution for the curve $y = f(x)$.

►*Find the surface area of a cone formed by rotating about the x-axis the line* $y = 2x$ *between* $x = 0$ *and* $x = h$.

Using (2.44), the surface area is given by

$$\begin{aligned} S &= \int_0^h (2\pi)2x\sqrt{1+\left[\frac{d}{dx}(2x)\right]^2}\,dx \\ &= \int_0^h 4\pi x\left(1+2^2\right)^{1/2}\,dx = \int_0^h 4\sqrt{5}\pi x\,dx \\ &= \left[2\sqrt{5}\pi x^2\right]_0^h = 2\sqrt{5}\pi(h^2-0) = 2\sqrt{5}\pi h^2. \blacktriangleleft \end{aligned}$$

We note that a surface of revolution may also be formed by rotating a line about the y-axis. In this case the surface area between $y = a$ and $y = b$ is

$$S = \int_a^b 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy. \tag{2.45}$$

Volumes of revolution

The volume V enclosed by rotating the curve $y = f(x)$ about the x-axis can also be found (see figure 2.12). The volume of the disc between x and $x + dx$ is given by $dV = \pi y^2\,dx$. Hence the total volume between $x = a$ and $x = b$ is

$$V = \int_a^b \pi y^2\,dx. \tag{2.46}$$

►*Find the volume of a cone enclosed by the surface formed by rotating about the x-axis the line $y = 2x$ between $x = 0$ and $x = h$.*

Using (2.46), the volume is given by

$$V = \int_0^h \pi(2x)^2\,dx = \int_0^h 4\pi x^2\,dx$$
$$= \left[\tfrac{4}{3}\pi x^3\right]_0^h = \tfrac{4}{3}\pi(h^3 - 0) = \tfrac{4}{3}\pi h^3. \blacktriangleleft$$

As before, it is also possible to form a volume of revolution by rotating a curve about the y-axis. In this case the volume enclosed between $y = a$ and $y = b$ is

$$V = \int_a^b \pi x^2\,dy. \tag{2.47}$$

2.3 Exercises

2.1 Obtain the following derivatives from first principles:

(a) the first derivative of $3x + 4$;
(b) the first, second and third derivatives of $x^2 + x$;
(c) the first derivative of $\sin x$.

2.2 Find from first principles the first derivative of $(x+3)^2$ and compare your answer with that obtained using the chain rule.

2.3 Find the first derivatives of

(a) $x^2 \exp x$, (b) $2 \sin x \cos x$, (c) $\sin 2x$, (d) $x \sin ax$,
(e) $(\exp ax)(\sin ax)\tan^{-1} ax$, (f) $\ln(x^a + x^{-a})$,
(g) $\ln(a^x + a^{-x})$, (h) x^x.

2.4 Find the first derivatives of

(a) $x/(a+x)^2$, (b) $x/(1-x)^{1/2}$, (c) $\tan x$, as $\sin x/\cos x$,
(d) $(3x^2 + 2x + 1)/(8x^2 - 4x + 2)$.

2.5 Use result (2.12) to find the first derivatives of

(a) $(2x+3)^{-3}$, (b) $\sec^2 x$, (c) $\mathrm{cosech}^3 3x$, (d) $1/\ln x$, (e) $1/[\sin^{-1}(x/a)]$.

2.6 Show that the function $y(x) = \exp(-|x|)$ defined by

$$y(x) = \begin{cases} \exp x & \text{for } x < 0, \\ 1 & \text{for } x = 0, \\ \exp(-x) & \text{for } x > 0, \end{cases}$$

is *not* differentiable at $x = 0$. Consider the limiting process for both $\Delta x > 0$ and $\Delta x < 0$.

2.7 Find dy/dx if $x = (t-2)/(t+2)$ and $y = 2t/(t+1)$ for $-\infty < t < \infty$. Show that it is always non-negative, and make use of this result in sketching the curve of y as a function of x.

2.8 If $2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$, show that $dy/dx = 16$ when $x = 1$.

2.9 Find the second derivative of $y(x) = \cos[(\pi/2) - ax]$. Now set $a = 1$ and verify that the result is the same as that obtained by first setting $a = 1$ and simplifying $y(x)$ before differentiating.

2.10 The function $y(x)$ is defined by $y(x) = (1 + x^m)^n$.

(a) Use the chain rule to show that the first derivative of y is $nmx^{m-1}(1+x^m)^{n-1}$.
(b) The binomial expansion (see section 1.5) of $(1 + z)^n$ is

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}z^r + \cdots.$$

Keeping only the terms of zeroth and first order in dx, apply this result twice to derive result (a) from first principles.
(c) Expand y in a series of powers of x before differentiating term by term. Show that the result is the series obtained by expanding the answer given for dy/dx in (a).

2.11 Show by differentiation and substitution that the differential equation

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 + 3)y = 0$$

has a solution of the form $y(x) = x^n \sin x$, and find the value of n.

2.12 Find the positions and natures of the stationary points of the following functions:

(a) $x^3 - 3x + 3$; (b) $x^3 - 3x^2 + 3x$; (c) $x^3 + 3x + 3$;
(d) $\sin ax$ with $a \neq 0$; (e) $x^5 + x^3$; (f) $x^5 - x^3$.

2.13 Show that the lowest value taken by the function $3x^4 + 4x^3 - 12x^2 + 6$ is -26.

2.14 By finding their stationary points and examining their general forms, determine the range of values that each of the following functions $y(x)$ can take. In each case make a sketch-graph incorporating the features you have identified.

(a) $y(x) = (x - 1)/(x^2 + 2x + 6)$.
(b) $y(x) = 1/(4 + 3x - x^2)$.
(c) $y(x) = (8 \sin x)/(15 + 8 \tan^2 x)$.

2.15 Show that $y(x) = xa^{2x} \exp x^2$ has no stationary points other than $x = 0$, if $\exp(-\sqrt{2}) < a < \exp(\sqrt{2})$.

2.16 The curve $4y^3 = a^2(x + 3y)$ can be parameterised as $x = a\cos 3\theta$, $y = a\cos\theta$.

(a) Obtain expressions for dy/dx (i) by implicit differentiation and (ii) in parameterised form. Verify that they are equivalent.
(b) Show that the only point of inflection occurs at the origin. Is it a stationary point of inflection?
(c) Use the information gained in (a) and (b) to sketch the curve, paying particular attention to its shape near the points $(-a, a/2)$ and $(a, -a/2)$ and to its slope at the 'end points' (a, a) and $(-a, -a)$.

2.17 The parametric equations for the motion of a charged particle released from rest in electric and magnetic fields at right angles to each other take the forms

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta).$$

Show that the tangent to the curve has slope $\cot(\theta/2)$. Use this result at a few calculated values of x and y to sketch the form of the particle's trajectory.

2.18 Show that the maximum curvature on the catenary $y(x) = a\cosh(x/a)$ is $1/a$. You will need some of the results about hyperbolic functions stated in subsection 3.7.6.

2.19 The curve whose equation is $x^{2/3} + y^{2/3} = a^{2/3}$ for positive x and y and which is completed by its symmetric reflections in both axes is known as an astroid. Sketch it and show that its radius of curvature in the first quadrant is $3(axy)^{1/3}$.

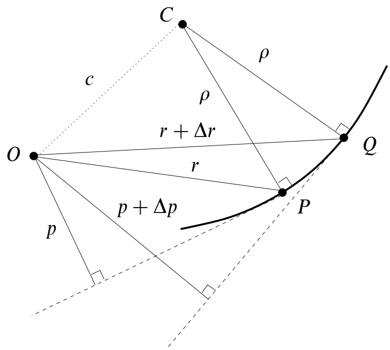

Figure 2.13 The coordinate system described in exercise 2.20.

2.20 A two-dimensional coordinate system useful for orbit problems is the tangential-polar coordinate system (figure 2.13). In this system a curve is defined by r, the distance from a fixed point O to a general point P of the curve, and p, the perpendicular distance from O to the tangent to the curve at P. By proceeding as indicated below, show that the radius of curvature, ρ, at P can be written in the form $\rho = r\,dr/dp$.

Consider two neighbouring points, P and Q, on the curve. The normals to the curve through those points meet at C, with (in the limit $Q \to P$) $CP = CQ = \rho$. Apply the cosine rule to triangles OPC and OQC to obtain two expressions for c^2, one in terms of r and p and the other in terms of $r + \Delta r$ and $p + \Delta p$. By equating them and letting $Q \to P$ deduce the stated result.

2.21 Use Leibnitz' theorem to find

(a) the second derivative of $\cos x \sin 2x$,
(b) the third derivative of $\sin x \ln x$,
(c) the fourth derivative of $(2x^3 + 3x^2 + x + 2) \exp 2x$.

2.22 If $y = \exp(-x^2)$, show that $dy/dx = -2xy$ and hence, by applying Leibnitz' theorem, prove that for $n \geq 1$

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0.$$

2.23 Use the properties of functions at their turning points to do the following:

(a) By considering its properties near $x = 1$, show that $f(x) = 5x^4 - 11x^3 + 26x^2 - 44x + 24$ takes negative values for some range of x.
(b) Show that $f(x) = \tan x - x$ cannot be negative for $0 \leq x < \pi/2$, and deduce that $g(x) = x^{-1} \sin x$ decreases monotonically in the same range.

2.24 Determine what can be learned from applying Rolle's theorem to the following functions $f(x)$: (a) e^x; (b) $x^2 + 6x$; (c) $2x^2 + 3x + 1$; (d) $2x^2 + 3x + 2$; (e) $2x^3 - 21x^2 + 60x + k$. (f) If $k = -45$ in (e), show that $x = 3$ is one root of $f(x) = 0$, find the other roots, and verify that the conclusions from (e) are satisfied.

2.25 By applying Rolle's theorem to $x^n \sin nx$, where n is an arbitrary positive integer, show that $\tan nx + x = 0$ has a solution α_1 with $0 < \alpha_1 < \pi/n$. Apply the theorem a second time to obtain the nonsensical result that there is a real α_2 in $0 < \alpha_2 < \pi/n$, such that $\cos^2(n\alpha_2) = -n$. Explain why this incorrect result arises.

2.26 Use the mean value theorem to establish bounds in the following cases.

(a) For $-\ln(1-y)$, by considering $\ln x$ in the range $0 < 1-y < x < 1$.
(b) For $e^y - 1$, by considering $e^x - 1$ in the range $0 < x < y$.

2.27 For the function $y(x) = x^2 \exp(-x)$ obtain a simple relationship between y and dy/dx and then, by applying Leibnitz' theorem, prove that

$$xy^{(n+1)} + (n + x - 2)y^{(n)} + ny^{(n-1)} = 0.$$

2.28 Use Rolle's theorem to deduce that, if the equation $f(x) = 0$ has a repeated root x_1, then x_1 is also a root of the equation $f'(x) = 0$.

(a) Apply this result to the 'standard' quadratic equation $ax^2 + bx + c = 0$, to show that a necessary condition for equal roots is $b^2 = 4ac$.
(b) Find all the roots of $f(x) = x^3 + 4x^2 - 3x - 18 = 0$, given that one of them is a repeated root.
(c) The equation $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2 = 0$ has a repeated integer root. How many real roots does it have altogether?

2.29 Show that the curve $x^3 + y^3 - 12x - 8y - 16 = 0$ touches the x-axis.

2.30 Find the following indefinite integrals:

(a) $\int (4 + x^2)^{-1}\, dx$; (b) $\int (8 + 2x - x^2)^{-1/2}\, dx$ for $2 \le x \le 4$;
(c) $\int (1 + \sin\theta)^{-1}\, d\theta$; (d) $\int (x\sqrt{1-x})^{-1}\, dx$ for $0 < x \le 1$.

2.31 Find the indefinite integrals J of the following ratios of polynomials:

(a) $(x + 3)/(x^2 + x - 2)$;
(b) $(x^3 + 5x^2 + 8x + 12)/(2x^2 + 10x + 12)$;
(c) $(3x^2 + 20x + 28)/(x^2 + 6x + 9)$;
(d) $x^3/(a^8 + x^8)$.

2.32 Express $x^2(ax + b)^{-1}$ as the sum of powers of x and another integrable term, and hence evaluate

$$\int_0^{b/a} \frac{x^2}{ax + b}\, dx.$$

2.33 Find the integral J of $(ax^2 + bx + c)^{-1}$, with $a \neq 0$, distinguishing between the cases (i) $b^2 > 4ac$, (ii) $b^2 < 4ac$ and (iii) $b^2 = 4ac$.

2.34 Use logarithmic integration to find the indefinite integrals J of the following:

(a) $\sin 2x/(1 + 4\sin^2 x)$;
(b) $e^x/(e^x - e^{-x})$;
(c) $(1 + x\ln x)/(x\ln x)$;
(d) $[x(x^n + a^n)]^{-1}$.

2.35 Find the derivative of $f(x) = (1 + \sin x)/\cos x$ and hence determine the indefinite integral J of $\sec x$.

2.36 Find the indefinite integrals, J, of the following functions involving sinusoids:

(a) $\cos^5 x - \cos^3 x$;
(b) $(1 - \cos x)/(1 + \cos x)$;
(c) $\cos x \sin x/(1 + \cos x)$;
(d) $\sec^2 x/(1 - \tan^2 x)$.

2.37 By making the substitution $x = a\cos^2\theta + b\sin^2\theta$, evaluate the definite integrals J between limits a and b $(> a)$ of the following functions:

(a) $[(x - a)(b - x)]^{-1/2}$;
(b) $[(x - a)(b - x)]^{1/2}$;

(c) $[(x-a)/(b-x)]^{1/2}$.

2.38 Determine whether the following integrals exist and, where they do, evaluate them:

(a) $\displaystyle\int_0^\infty \exp(-\lambda x)\,dx$; (b) $\displaystyle\int_{-\infty}^\infty \frac{x}{(x^2+a^2)^2}\,dx$;

(c) $\displaystyle\int_1^\infty \frac{1}{x+1}\,dx$; (d) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$;

(e) $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$; (f) $\displaystyle\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx$.

2.39 Use integration by parts to evaluate the following:

(a) $\displaystyle\int_0^y x^2\sin x\,dx$; (b) $\displaystyle\int_1^y x\ln x\,dx$;

(c) $\displaystyle\int_0^y \sin^{-1} x\,dx$; (d) $\displaystyle\int_1^y \ln(a^2+x^2)/x^2\,dx$.

2.40 Show, using the following methods, that the indefinite integral of $x^3/(x+1)^{1/2}$ is

$$J = \tfrac{2}{35}(5x^3 - 6x^2 + 8x - 16)(x+1)^{1/2} + c.$$

(a) Repeated integration by parts.
(b) Setting $x+1=u^2$ and determining dJ/du as $(dJ/dx)(dx/du)$.

2.41 The gamma function $\Gamma(n)$ is defined for all $n > -1$ by

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx.$$

Find a recurrence relation connecting $\Gamma(n+1)$ and $\Gamma(n)$.

(a) Deduce (i) the value of $\Gamma(n+1)$ when n is a non-negative integer, and (ii) the value of $\Gamma\left(\frac{7}{2}\right)$, given that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
(b) Now, taking factorial m for *any* m to be defined by $m! = \Gamma(m+1)$, evaluate $\left(-\frac{3}{2}\right)!$.

2.42 Define $J(m,n)$, for non-negative integers m and n, by the integral

$$J(m,n) = \int_0^{\pi/2} \cos^m\theta\,\sin^n\theta\,d\theta.$$

(a) Evaluate $J(0,0)$, $J(0,1)$, $J(1,0)$, $J(1,1)$, $J(m,1)$, $J(1,n)$.
(b) Using integration by parts, prove that, for m and n both > 1,

$$J(m,n) = \frac{m-1}{m+n}J(m-2,n) \quad\text{and}\quad J(m,n) = \frac{n-1}{m+n}J(m,n-2).$$

(c) Evaluate (i) $J(5,3)$, (ii) $J(6,5)$ and (iii) $J(4,8)$.

2.43 By integrating by parts twice, prove that I_n as defined in the first equality below for positive integers n has the value given in the second equality:

$$I_n = \int_0^{\pi/2} \sin n\theta\cos\theta\,d\theta = \frac{n - \sin(n\pi/2)}{n^2-1}.$$

2.44 Evaluate the following definite integrals:

(a) $\int_0^\infty xe^{-x}\,dx$; (b) $\int_0^1 \left[(x^3+1)/(x^4+4x+1)\right]\,dx$;
(c) $\int_0^{\pi/2}[a+(a-1)\cos\theta]^{-1}\,d\theta$ with $a > \frac{1}{2}$; (d) $\int_{-\infty}^\infty (x^2+6x+18)^{-1}\,dx$.

2.45 If J_r is the integral

$$\int_0^{\infty} x^r \exp(-x^2)\,dx$$

show that

(a) $J_{2r+1} = (r!)/2$,
(b) $J_{2r} = 2^{-r}(2r-1)(2r-3)\cdots(5)(3)(1)\,J_0$.

2.46 Find positive constants a, b such that $ax \le \sin x \le bx$ for $0 \le x \le \pi/2$. Use this inequality to find (to two significant figures) upper and lower bounds for the integral

$$I = \int_0^{\pi/2} (1+\sin x)^{1/2}\,dx.$$

Use the substitution $t = \tan(x/2)$ to evaluate I exactly.

2.47 By noting that for $0 \le \eta \le 1$, $\eta^{1/2} \ge \eta^{3/4} \ge \eta$, prove that

$$\frac{2}{3} \le \frac{1}{a^{5/2}} \int_0^a (a^2 - x^2)^{3/4}\,dx \le \frac{\pi}{4}.$$

2.48 Show that the total length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, which can be parameterised as $x = a\cos^3\theta$, $y = a\sin^3\theta$, is $6a$.

2.49 By noting that $\sinh x < \frac{1}{2}e^x < \cosh x$, and that $1 + z^2 < (1+z)^2$ for $z > 0$, show that, for $x > 0$, the length L of the curve $y = \frac{1}{2}e^x$ measured from the origin satisfies the inequalities $\sinh x < L < x + \sinh x$.

2.50 The equation of a cardioid in plane polar coordinates is

$$\rho = a(1 - \sin\phi).$$

Sketch the curve and find (i) its area, (ii) its total length, (iii) the surface area of the solid formed by rotating the cardioid about its axis of symmetry and (iv) the volume of the same solid.

2.4 Hints and answers

2.1 (a) 3; (b) $2x+1$, 2, 0; (c) $\cos x$.

2.3 Use: the product rule in (a), (b), (d) and (e)[3 factors]; the chain rule in (c), (f) and (g); logarithmic differentiation in (g) and (h).
(a) $(x^2+2x)\exp x$; (b) $2(\cos^2 x - \sin^2 x) = 2\cos 2x$;
(c) $2\cos 2x$; (d) $\sin ax + ax\cos ax$;
(e) $(a\exp ax)[(\sin ax + \cos ax)\tan^{-1} ax + (\sin ax)(1+a^2x^2)^{-1}]$;
(f) $[a(x^a - x^{-a})]/[x(x^a + x^{-a})]$; (g) $[(a^x - a^{-x})\ln a]/(a^x + a^{-x})$; (h) $(1+\ln x)x^x$.

2.5 (a) $-6(2x+3)^{-4}$; (b) $2\sec^2 x\tan x$; (c) $-9\,\text{cosech}^3 3x\coth 3x$;
(d) $-x^{-1}(\ln x)^{-2}$; (e) $-(a^2-x^2)^{-1/2}[\sin^{-1}(x/a)]^{-2}$.

2.7 Calculate dy/dt and dx/dt and divide one by the other. $(t+2)^2/[2(t+1)^2]$. Alternatively, eliminate t and find dy/dx by implicit differentiation.

2.9 $-\sin x$ in both cases.

2.11 The required conditions are $8n - 4 = 0$ and $4n^2 - 8n + 3 = 0$; both are satisfied by $n = \frac{1}{2}$.

2.13 The stationary points are the zeros of $12x^3 + 12x^2 - 24x$. The lowest stationary value is -26 at $x = -2$; other stationary values are 6 at $x = 0$ and 1 at $x = 1$.

2.15 Use logarithmic differentiation. Set $dy/dx = 0$, obtaining $2x^2 + 2x\ln a + 1 = 0$.

2.17 See figure 2.14.

2.19 $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$; $\dfrac{d^2y}{dx^2} = \dfrac{a^{2/3}}{3x^{4/3}y^{1/3}}$.

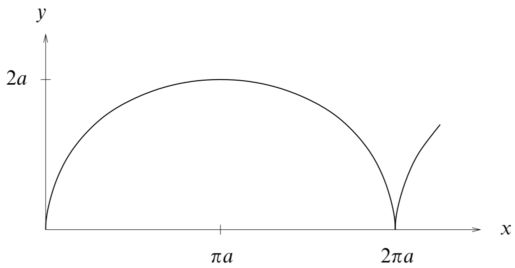

Figure 2.14 The solution to exercise 2.17.

2.21 (a) $2(2 - 9\cos^2 x)\sin x$; (b) $(2x^{-3} - 3x^{-1})\sin x - (3x^{-2} + \ln x)\cos x$; (c) $8(4x^3 + 30x^2 + 62x + 38)\exp 2x$.

2.23 (a) $f(1) = 0$ whilst $f'(1) \neq 0$, and so $f(x)$ must be negative in some region with $x = 1$ as an endpoint.
(b) $f'(x) = \tan^2 x > 0$ and $f(0) = 0$; $g'(x) = (-\cos x)(\tan x - x)/x^2$, which is never positive in the range.

2.25 The false result arises because $\tan nx$ is not differentiable at $x = \pi/(2n)$, which lies in the range $0 < x < \pi/n$, and so the conditions for applying Rolle's theorem are not satisfied.

2.27 The relationship is $x\,dy/dx = (2 - x)y$.

2.29 By implicit differentiation, $y'(x) = (3x^2 - 12)/(8 - 3y^2)$, giving $y'(\pm 2) = 0$. Since $y(2) = 4$ and $y(-2) = 0$, the curve touches the x-axis at the point $(-2, 0)$.

2.31 (a) Express in partial fractions; $J = \frac{1}{3}\ln[(x-1)^4/(x+2)] + c$.
(b) Divide the numerator by the denominator and express the remainder in partial fractions; $J = x^2/4 + 4\ln(x+2) - 3\ln(x+3) + c$.
(c) After division of the numerator by the denominator, the remainder can be expressed as $2(x+3)^{-1} - 5(x+3)^{-2}$; $J = 3x + 2\ln(x+3) + 5(x+3)^{-1} + c$.
(d) Set $x^4 = u$; $J = (4a^4)^{-1}\tan^{-1}(x^4/a^4) + c$.

2.33 Writing $b^2 - 4ac$ as $\Delta^2 > 0$, or $4ac - b^2$ as $\Delta'^2 > 0$:
(i) $\Delta^{-1}\ln[(2ax + b - \Delta)/(2ax + b + \Delta)] + k$;
(ii) $2\Delta'^{-1}\tan^{-1}[(2ax + b)/\Delta'] + k$;
(iii) $-2(2ax + b)^{-1} + k$.

2.35 $f'(x) = (1 + \sin x)/\cos^2 x = f(x)\sec x$; $J = \ln(f(x)) + c = \ln(\sec x + \tan x) + c$.

2.37 Note that $dx = 2(b - a)\cos\theta\sin\theta\,d\theta$.
(a) π; (b) $\pi(b-a)^2/8$; (c) $\pi(b-a)/2$.

2.39 (a) $(2 - y^2)\cos y + 2y\sin y - 2$; (b) $[(y^2\ln y)/2] + [(1 - y^2)/4]$;
(c) $y\sin^{-1} y + (1 - y^2)^{1/2} - 1$;
(d) $\ln(a^2 + 1) - (1/y)\ln(a^2 + y^2) + (2/a)[\tan^{-1}(y/a) - \tan^{-1}(1/a)]$.

2.41 $\Gamma(n+1) = n\Gamma(n)$; (a) (i) $n!$, (ii) $15\sqrt{\pi}/8$; (b) $-2\sqrt{\pi}$.

2.43 By integrating twice, recover a multiple of I_n.

2.45 $J_{2r+1} = rJ_{2r-1}$ and $2J_{2r} = (2r - 1)J_{2r-2}$.

2.47 Set $\eta = 1 - (x/a)^2$ throughout, and $x = a\sin\theta$ in one of the bounds.

2.49 $L = \int_0^x \left(1 + \frac{1}{4}\exp 2x\right)^{1/2} dx$.

3

Complex numbers and hyperbolic functions

This chapter is concerned with the representation and manipulation of complex numbers. Complex numbers pervade this book, underscoring their wide application in the mathematics of the physical sciences. The application of complex numbers to the description of physical systems is left until later chapters and only the basic tools are presented here.

3.1 The need for complex numbers

Although complex numbers occur in many branches of mathematics, they arise most directly out of solving polynomial equations. We examine a specific quadratic equation as an example.

Consider the quadratic equation

$$z^2 - 4z + 5 = 0. \tag{3.1}$$

Equation (3.1) has two solutions, z_1 and z_2, such that

$$(z - z_1)(z - z_2) = 0. \tag{3.2}$$

Using the familiar formula for the roots of a quadratic equation, (1.4), the solutions z_1 and z_2, written in brief as $z_{1,2}$, are

$$\begin{aligned} z_{1,2} &= \frac{4 \pm \sqrt{(-4)^2 - 4(1 \times 5)}}{2} \\ &= 2 \pm \frac{\sqrt{-4}}{2}. \end{aligned} \tag{3.3}$$

Both solutions contain the square root of a negative number. However, it is not true to say that there are no solutions to the quadratic equation. The *fundamental theorem of algebra* states that a quadratic equation will always have two solutions and these are in fact given by (3.3). The second term on the RHS of (3.3) is called an *imaginary* term since it contains the square root of a negative number;

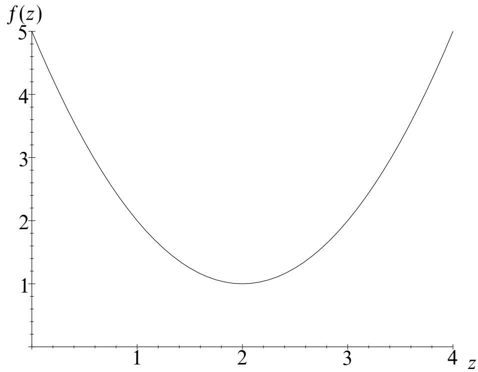

Figure 3.1 The function $f(z) = z^2 - 4z + 5$.

the first term is called a *real* term. The full solution is the sum of a real term and an imaginary term and is called a *complex number*. A plot of the function $f(z) = z^2 - 4z + 5$ is shown in figure 3.1. It will be seen that the plot does not intersect the z-axis, corresponding to the fact that the equation $f(z) = 0$ has no purely real solutions.

The choice of the symbol z for the quadratic variable was not arbitrary; the conventional representation of a complex number is z, where z is the sum of a real part x and i times an imaginary part y, i.e.

$$z = x + iy,$$

where i is used to denote the square root of -1. The real part x and the imaginary part y are usually denoted by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ respectively. We note at this point that some physical scientists, engineers in particular, use j instead of i. However, for consistency, we will use i throughout this book.

In our particular example, $\sqrt{-4} = 2\sqrt{-1} = 2i$, and hence the two solutions of (3.1) are

$$z_{1,2} = 2 \pm \frac{2i}{2} = 2 \pm i.$$

Thus, here $x = 2$ and $y = \pm 1$.

For compactness a complex number is sometimes written in the form

$$z = (x, y),$$

where the components of z may be thought of as coordinates in an xy-plot. Such a plot is called an *Argand diagram* and is a common representation of complex numbers; an example is shown in figure 3.2.

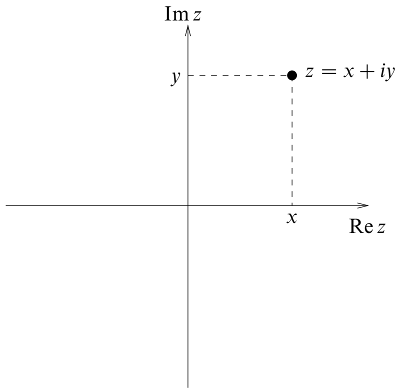

Figure 3.2 The Argand diagram.

Our particular example of a quadratic equation may be generalised readily to polynomials whose highest power (degree) is greater than 2, e.g. cubic equations (degree 3), quartic equations (degree 4) and so on. For a general polynomial $f(z)$, of degree n, the fundamental theorem of algebra states that the equation $f(z) = 0$ will have exactly n solutions. We will examine cases of higher-degree equations in subsection 3.4.3.

The remainder of this chapter deals with: the algebra and manipulation of complex numbers; their polar representation, which has advantages in many circumstances; complex exponentials and logarithms; the use of complex numbers in finding the roots of polynomial equations; and hyperbolic functions.

3.2 Manipulation of complex numbers

This section considers basic complex number manipulation. Some analogy may be drawn with vector manipulation (see chapter 7) but this section stands alone as an introduction.

3.2.1 Addition and subtraction

The addition of two complex numbers, z_1 and z_2, in general gives another complex number. The real components and the imaginary components are added separately and in a like manner to the familiar addition of real numbers:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

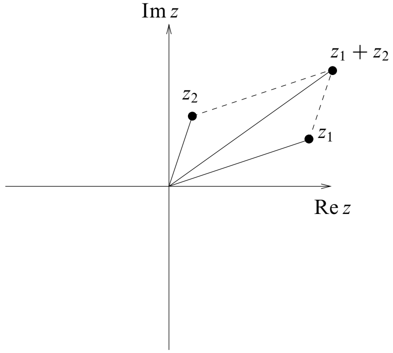

Figure 3.3 The addition of two complex numbers.

or in component notation

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

The Argand representation of the addition of two complex numbers is shown in figure 3.3.

By straightforward application of the commutativity and associativity of the real and imaginary parts separately, we can show that the addition of complex numbers is itself commutative and associative, i.e.

$$z_1 + z_2 = z_2 + z_1,$$
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$$

Thus it is immaterial in what order complex numbers are added.

►*Sum the complex numbers* $1 + 2i$, $3 - 4i$, $-2 + i$.

Summing the real terms we obtain

$$1 + 3 - 2 = 2,$$

and summing the imaginary terms we obtain

$$2i - 4i + i = -i.$$

Hence

$$(1 + 2i) + (3 - 4i) + (-2 + i) = 2 - i. \blacktriangleleft$$

The subtraction of complex numbers is very similar to their addition. As in the case of real numbers, if two identical complex numbers are subtracted then the result is zero.

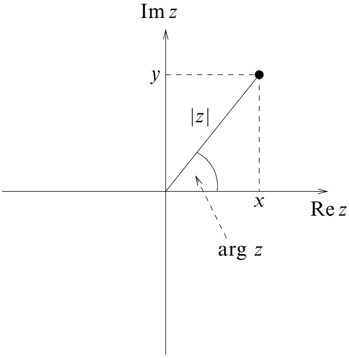

Figure 3.4 The modulus and argument of a complex number.

3.2.2 Modulus and argument

The modulus of the complex number z is denoted by $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2}. \tag{3.4}$$

Hence the modulus of the complex number is the distance of the corresponding point from the origin in the Argand diagram, as may be seen in figure 3.4.

The argument of the complex number z is denoted by $\arg z$ and is defined as

$$\arg z = \tan^{-1}\left(\frac{y}{x}\right). \tag{3.5}$$

It can be seen that $\arg z$ is the angle that the line joining the origin to z on the Argand diagram makes with the positive x-axis. The anticlockwise direction is taken to be positive by convention. The angle $\arg z$ is shown in figure 3.4. Account must be taken of the signs of x and y individually in determining in which quadrant $\arg z$ lies. Thus, for example, if x and y are both negative then $\arg z$ lies in the range $-\pi < \arg z < -\pi/2$ rather than in the first quadrant ($0 < \arg z < \pi/2$), though both cases give the same value for the ratio of y to x.

►*Find the modulus and the argument of the complex number $z = 2 - 3i$.*

Using (3.4), the modulus is given by

$$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}.$$

Using (3.5), the argument is given by

$$\arg z = \tan^{-1}\left(-\tfrac{3}{2}\right).$$

The two angles whose tangents equal -1.5 are -0.9828 rad and 2.1588 rad. Since $x = 2$ and $y = -3$, z clearly lies in the fourth quadrant; therefore $\arg z = -0.9828$ is the appropriate answer. ◄

3.2.3 Multiplication

Complex numbers may be multiplied together and in general give a complex number as the result. The product of two complex numbers z_1 and z_2 is found by multiplying them out in full and remembering that $i^2 = -1$, i.e.

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1 y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \end{aligned} \tag{3.6}$$

►*Multiply the complex numbers* $z_1 = 3 + 2i$ *and* $z_2 = -1 - 4i$.

By direct multiplication we find

$$\begin{aligned} z_1 z_2 &= (3 + 2i)(-1 - 4i) \\ &= -3 - 2i - 12i - 8i^2 \\ &= 5 - 14i. \blacktriangleleft \end{aligned} \tag{3.7}$$

The multiplication of complex numbers is both commutative and associative, i.e.

$$z_1 z_2 = z_2 z_1, \tag{3.8}$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3). \tag{3.9}$$

The product of two complex numbers also has the simple properties

$$|z_1 z_2| = |z_1||z_2|, \tag{3.10}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{3.11}$$

These relations are derived in subsection 3.3.1.

►*Verify that (3.10) holds for the product of* $z_1 = 3 + 2i$ *and* $z_2 = -1 - 4i$.

From (3.7)

$$|z_1 z_2| = |5 - 14i| = \sqrt{5^2 + (-14)^2} = \sqrt{221}.$$

We also find

$$\begin{aligned} |z_1| &= \sqrt{3^2 + 2^2} = \sqrt{13}, \\ |z_2| &= \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}, \end{aligned}$$

and hence

$$|z_1||z_2| = \sqrt{13}\sqrt{17} = \sqrt{221} = |z_1 z_2|. \blacktriangleleft$$

We now examine the effect on a complex number z of multiplying it by ± 1 and $\pm i$. These four multipliers have modulus unity and we can see immediately from (3.10) that multiplying z by another complex number of unit modulus gives a product with the same modulus as z. We can also see from (3.11) that if we

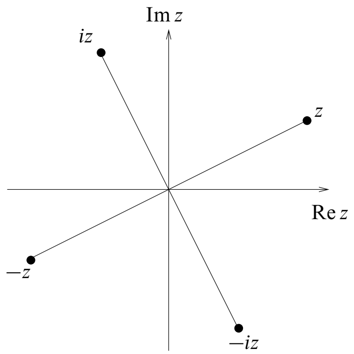

Figure 3.5 Multiplication of a complex number by ± 1 and $\pm i$.

multiply z by a complex number then the argument of the product is the sum of the argument of z and the argument of the multiplier. Hence multiplying z by unity (which has argument zero) leaves z unchanged in both modulus and argument, i.e. z is completely unaltered by the operation. Multiplying by -1 (which has argument π) leads to rotation, through an angle π, of the line joining the origin to z in the Argand diagram. Similarly, multiplication by i or $-i$ leads to corresponding rotations of $\pi/2$ or $-\pi/2$ respectively. This geometrical interpretation of multiplication is shown in figure 3.5.

►*Using the geometrical interpretation of multiplication by i, find the product* $i(1-i)$.

The complex number $1-i$ has argument $-\pi/4$ and modulus $\sqrt{2}$. Thus, using (3.10) and (3.11), its product with i has argument $+\pi/4$ and unchanged modulus $\sqrt{2}$. The complex number with modulus $\sqrt{2}$ and argument $+\pi/4$ is $1+i$ and so

$$i(1-i) = 1+i,$$

as is easily verified by direct multiplication. ◄

The division of two complex numbers is similar to their multiplication but requires the notion of the complex conjugate (see the following subsection) and so discussion is postponed until subsection 3.2.5.

3.2.4 Complex conjugate

If z has the convenient form $x+iy$ then the complex conjugate, denoted by z^*, may be found simply by changing the sign of the imaginary part, i.e. if $z = x+iy$ then $z^* = x-iy$. More generally, we may define the complex conjugate of z as the (complex) number having the same magnitude as z that when multiplied by z leaves a real result, i.e. there is no imaginary component in the product.

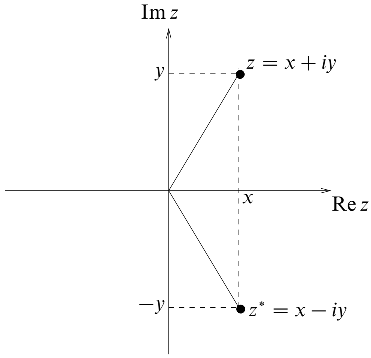

Figure 3.6 The complex conjugate as a mirror image in the real axis.

In the case where z can be written in the form $x + iy$ it is easily verified, by direct multiplication of the components, that the product zz^* gives a real result:

$$zz^* = (x + iy)(x - iy) = x^2 - ixy + ixy - i^2y^2 = x^2 + y^2 = |z|^2.$$

Complex conjugation corresponds to a reflection of z in the real axis of the Argand diagram, as may be seen in figure 3.6.

►*Find the complex conjugate of* $z = a + 2i + 3ib$.

The complex number is written in the standard form

$$z = a + i(2 + 3b);$$

then, replacing i by $-i$, we obtain

$$z^* = a - i(2 + 3b). \blacktriangleleft$$

In some cases, however, it may not be simple to rearrange the expression for z into the standard form $x + iy$. Nevertheless, given two complex numbers, z_1 and z_2, it is straightforward to show that the complex conjugate of their sum (or difference) is equal to the sum (or difference) of their complex conjugates, i.e. $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$. Similarly, it may be shown that the complex conjugate of the product (or quotient) of z_1 and z_2 is equal to the product (or quotient) of their complex conjugates, i.e. $(z_1z_2)^* = z_1^*z_2^*$ and $(z_1/z_2)^* = z_1^*/z_2^*$.

Using these results, it can be deduced that, no matter how complicated the expression, its complex conjugate may *always* be found by replacing every i by $-i$. To apply this rule, however, we must always ensure that all complex parts are first written out in full, so that no i's are hidden.

► *Find the complex conjugate of the complex number $z = w^{(3y+2ix)}$, where $w = x + 5i$.*

Although we do not discuss complex powers until section 3.5, the simple rule given above still enables us to find the complex conjugate of z.

In this case w itself contains real and imaginary components and so must be written out in full, i.e.

$$z = w^{3y+2ix} = (x + 5i)^{3y+2ix}.$$

Now we can replace each i by $-i$ to obtain

$$z^* = (x - 5i)^{(3y-2ix)}.$$

It can be shown that the product zz^* is real, as required. ◄

The following properties of the complex conjugate are easily proved and others may be derived from them. If $z = x + iy$ then

$$(z^*)^* = z, \tag{3.12}$$

$$z + z^* = 2\,\mathrm{Re}\,z = 2x, \tag{3.13}$$

$$z - z^* = 2i\,\mathrm{Im}\,z = 2iy, \tag{3.14}$$

$$\frac{z}{z^*} = \left(\frac{x^2 - y^2}{x^2 + y^2}\right) + i\left(\frac{2xy}{x^2 + y^2}\right). \tag{3.15}$$

The derivation of this last relation relies on the results of the following subsection.

3.2.5 Division

The division of two complex numbers z_1 and z_2 bears some similarity to their multiplication. Writing the quotient in component form we obtain

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}. \tag{3.16}$$

In order to separate the real and imaginary components of the quotient, we multiply both numerator and denominator by the complex conjugate of the denominator. By definition, this process will leave the denominator as a real quantity. Equation (3.16) gives

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}\\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.\end{aligned}$$

Hence we have separated the quotient into real and imaginary components, as required.

In the special case where $z_2 = z_1^*$, so that $x_2 = x_1$ and $y_2 = -y_1$, the general result reduces to (3.15).

►*Express z in the form $x + iy$, when*

$$z = \frac{3 - 2i}{-1 + 4i}.$$

Multiplying numerator and denominator by the complex conjugate of the denominator we obtain

$$z = \frac{(3 - 2i)(-1 - 4i)}{(-1 + 4i)(-1 - 4i)} = \frac{-11 - 10i}{17}$$
$$= -\frac{11}{17} - \frac{10}{17}i. \blacktriangleleft$$

In analogy to (3.10) and (3.11), which describe the multiplication of two complex numbers, the following relations apply to division:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{3.17}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{3.18}$$

The proof of these relations is left until subsection 3.3.1.

3.3 Polar representation of complex numbers

Although considering a complex number as the sum of a real and an imaginary part is often useful, sometimes the *polar representation* proves easier to manipulate. This makes use of the complex exponential function, which is defined by

$$e^z = \exp z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \tag{3.19}$$

Strictly speaking it is the function $\exp z$ that is defined by (3.19). The number e is the value of $\exp(1)$, i.e. it is just a number. However, it may be shown that e^z and $\exp z$ are equivalent when z is real and rational and mathematicians then *define* their equivalence for irrational and complex z. For the purposes of this book we will not concern ourselves further with this mathematical nicety but, rather, assume that (3.19) is valid for all z. We also note that, using (3.19), by multiplying together the appropriate series we may show that (see chapter 24)

$$e^{z_1}e^{z_2} = e^{z_1 + z_2}, \tag{3.20}$$

which is analogous to the familiar result for exponentials of real numbers.

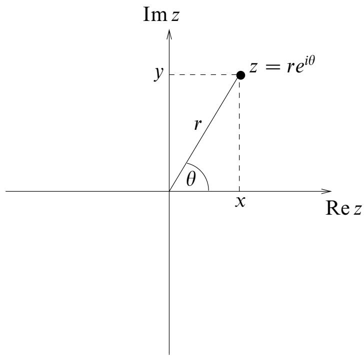

Figure 3.7 The polar representation of a complex number.

From (3.19), it immediately follows that for $z = i\theta$, θ real,

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots \tag{3.21}$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \tag{3.22}$$

and hence that

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3.23}$$

where the last equality follows from the series expansions of the sine and cosine functions (see subsection 4.6.3). This last relationship is called *Euler's equation.* It also follows from (3.23) that

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

for all n. From Euler's equation (3.23) and figure 3.7 we deduce that

$$\begin{aligned} re^{i\theta} &= r(\cos\theta + i\sin\theta) \\ &= x + iy. \end{aligned}$$

Thus a complex number may be represented in the polar form

$$z = re^{i\theta}. \tag{3.24}$$

Referring again to figure 3.7, we can identify r with $|z|$ and θ with arg z. The simplicity of the representation of the modulus and argument is one of the main reasons for using the polar representation. The angle θ lies conventionally in the range $-\pi < \theta \leq \pi$, but, since rotation by θ is the same as rotation by $2n\pi + \theta$, where n is any integer,

$$re^{i\theta} \equiv re^{i(\theta+2n\pi)}.$$

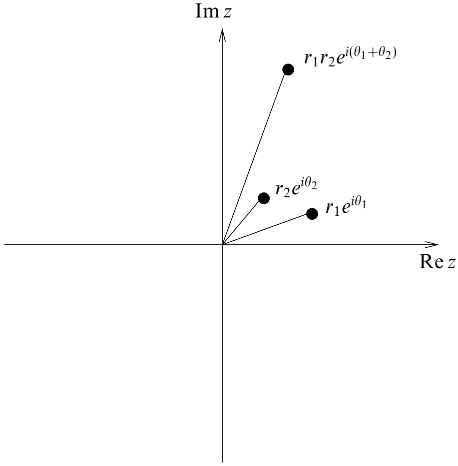

Figure 3.8 The multiplication of two complex numbers. In this case r_1 and r_2 are both greater than unity.

The algebra of the polar representation is different from that of the real and imaginary component representation, though, of course, the results are identical. Some operations prove much easier in the polar representation, others much more complicated. The best representation for a particular problem must be determined by the manipulation required.

3.3.1 Multiplication and division in polar form

Multiplication and division in polar form are particularly simple. The product of $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ is given by

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)}. \end{aligned} \tag{3.25}$$

The relations $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ follow immediately. An example of the multiplication of two complex numbers is shown in figure 3.8.

Division is equally simple in polar form; the quotient of z_1 and z_2 is given by

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}. \tag{3.26}$$

The relations $|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ are again

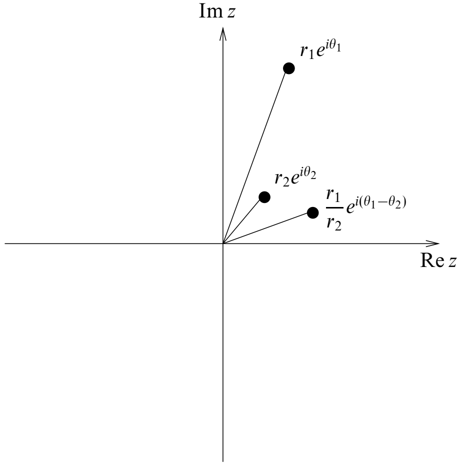

Figure 3.9 The division of two complex numbers. As in the previous figure, r_1 and r_2 are both greater than unity.

immediately apparent. The division of two complex numbers in polar form is shown in figure 3.9.

3.4 de Moivre's theorem

We now derive an extremely important theorem. Since $\left(e^{i\theta}\right)^n = e^{in\theta}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{3.27}$$

where the identity $e^{in\theta} = \cos n\theta + i\sin n\theta$ follows from the series definition of $e^{in\theta}$ (see (3.21)). This result is called *de Moivre's theorem* and is often used in the manipulation of complex numbers. The theorem is valid for all n whether real, imaginary or complex.

There are numerous applications of de Moivre's theorem but this section examines just three: proofs of trigonometric identities; finding the nth roots of unity; and solving polynomial equations with complex roots.

3.4.1 Trigonometric identities

The use of de Moivre's theorem in finding trigonometric identities is best illustrated by example. We consider the expression of a multiple-angle function in terms of a polynomial in the single-angle function, and its converse.

►*Express* $\sin 3\theta$ *and* $\cos 3\theta$ *in terms of powers of* $\cos\theta$ *and* $\sin\theta$.

Using de Moivre's theorem,

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta).\end{aligned} \tag{3.28}$$

We can equate the real and imaginary coefficients separately, i.e.

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta. \blacktriangleleft\end{aligned}$$

This method can clearly be applied to finding power expansions of $\cos n\theta$ and $\sin n\theta$ for any positive integer n.

The converse process uses the following properties of $z = e^{i\theta}$,

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \tag{3.30}$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta. \tag{3.31}$$

These equalities follow from simple applications of de Moivre's theorem, i.e.

$$\begin{aligned}z^n + \frac{1}{z^n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\ &= 2\cos n\theta\end{aligned}$$

and

$$\begin{aligned}z^n - \frac{1}{z^n} &= (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta \\ &= 2i\sin n\theta.\end{aligned}$$

In the particular case where $n = 1$,

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta, \tag{3.32}$$

$$z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta. \tag{3.33}$$

► *Find an expression for* $\cos^3\theta$ *in terms of* $\cos 3\theta$ *and* $\cos\theta$.

Using (3.32),

$$\begin{aligned}\cos^3\theta &= \frac{1}{2^3}\left(z+\frac{1}{z}\right)^3\\ &= \frac{1}{8}\left(z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right)\\ &= \frac{1}{8}\left(z^3+\frac{1}{z^3}\right)+\frac{3}{8}\left(z+\frac{1}{z}\right).\end{aligned}$$

Now using (3.30) and (3.32), we find

$$\cos^3\theta = \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta. \blacktriangleleft$$

This result happens to be a simple rearrangement of (3.29), but cases involving larger values of n are better handled using this direct method than by rearranging polynomial expansions of multiple-angle functions.

3.4.2 Finding the nth roots of unity

The equation $z^2 = 1$ has the familiar solutions $z = \pm 1$. However, now that we have introduced the concept of complex numbers we can solve the general equation $z^n = 1$. Recalling the fundamental theorem of algebra, we know that the equation has n solutions. In order to proceed we rewrite the equation as

$$z^n = e^{2ik\pi},$$

where k is any integer. Now taking the nth root of each side of the equation we find

$$z = e^{2ik\pi/n}.$$

Hence, the solutions of $z^n = 1$ are

$$z_{1,2,\ldots,n} = 1,\quad e^{2i\pi/n},\quad \ldots,\quad e^{2i(n-1)\pi/n},$$

corresponding to the values $0, 1, 2, \ldots, n-1$ for k. Larger integer values of k do not give new solutions, since the roots already listed are simply cyclically repeated for $k = n, n+1, n+2$, etc.

► *Find the solutions to the equation* $z^3 = 1$.

By applying the above method we find

$$z = e^{2ik\pi/3}.$$

Hence the three solutions are $z_1 = e^{0i} = 1$, $z_2 = e^{2i\pi/3}$, $z_3 = e^{4i\pi/3}$. We note that, as expected, the next solution, for which $k = 3$, gives $z_4 = e^{6i\pi/3} = 1 = z_1$, so that there are only three separate solutions. ◄

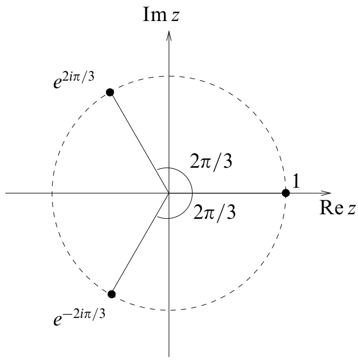

Figure 3.10 The solutions of $z^3 = 1$.

Not surprisingly, given that $|z^3| = |z|^3$ from (3.10), all the roots of unity have unit modulus, i.e. they all lie on a circle in the Argand diagram of unit radius. The three roots are shown in figure 3.10.

The cube roots of unity are often written 1, ω and ω^2. The properties $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$ are easily proved.

3.4.3 Solving polynomial equations

A third application of de Moivre's theorem is to the solution of polynomial equations. Complex equations in the form of a polynomial relationship must first be solved for z in a similar fashion to the method for finding the roots of real polynomial equations. Then the complex roots of z may be found.

►*Solve the equation* $z^6 - z^5 + 4z^4 - 6z^3 + 2z^2 - 8z + 8 = 0$.

We first factorise to give

$$(z^3 - 2)(z^2 + 4)(z - 1) = 0.$$

Hence $z^3 = 2$ or $z^2 = -4$ or $z = 1$. The solutions to the quadratic equation are $z = \pm 2i$; to find the complex cube roots, we first write the equation in the form

$$z^3 = 2 = 2e^{2ik\pi},$$

where k is any integer. If we now take the cube root, we get

$$z = 2^{1/3} e^{2ik\pi/3}.$$

To avoid the duplication of solutions, we use the fact that $-\pi < \arg z \leq \pi$ and find

$$z_1 = 2^{1/3},$$
$$z_2 = 2^{1/3} e^{2\pi i/3} = 2^{1/3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right),$$
$$z_3 = 2^{1/3} e^{-2\pi i/3} = 2^{1/3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right).$$

The complex numbers z_1, z_2 and z_3, together with $z_4 = 2i$, $z_5 = -2i$ and $z_6 = 1$ are the solutions to the original polynomial equation.

As expected from the fundamental theorem of algebra, we find that the total number of complex roots (six, in this case) is equal to the largest power of z in the polynomial. ◀

A useful result is that the roots of a polynomial with real coefficients occur in conjugate pairs (i.e. if z_1 is a root, then z_1^* is a second distinct root, unless z_1 is real). This may be proved as follows. Let the polynomial equation of which z is a root be

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Taking the complex conjugate of this equation,

$$a_n^*(z^*)^n + a_{n-1}^*(z^*)^{n-1} + \cdots + a_1^* z^* + a_0^* = 0.$$

But the a_n are real, and so z^* satisfies

$$a_n(z^*)^n + a_{n-1}(z^*)^{n-1} + \cdots + a_1 z^* + a_0 = 0,$$

and is also a root of the original equation.

3.5 Complex logarithms and complex powers

The concept of a complex exponential has already been introduced in section 3.3, where it was assumed that the definition of an exponential as a series was valid for complex numbers as well as for real numbers. Similarly we can define the logarithm of a complex number and we can use complex numbers as exponents.

Let us denote the natural logarithm of a complex number z by $w = \operatorname{Ln} z$, where the notation Ln will be explained shortly. Thus, w must satisfy

$$z = e^w.$$

Using (3.20), we see that

$$z_1 z_2 = e^{w_1} e^{w_2} = e^{w_1 + w_2},$$

and taking logarithms of both sides we find

$$\operatorname{Ln}(z_1 z_2) = w_1 + w_2 = \operatorname{Ln} z_1 + \operatorname{Ln} z_2, \tag{3.34}$$

which shows that the familiar rule for the logarithm of the product of two real numbers also holds for complex numbers.

We may use (3.34) to investigate further the properties of $\mathrm{Ln}\, z$. We have already noted that the argument of a complex number is multivalued, i.e. $\arg z = \theta + 2n\pi$, where n is any integer. Thus, in polar form, the complex number z should strictly be written as

$$z = re^{i(\theta+2n\pi)}.$$

Taking the logarithm of both sides, and using (3.34), we find

$$\mathrm{Ln}\, z = \ln r + i(\theta + 2n\pi), \qquad (3.35)$$

where $\ln r$ is the natural logarithm of the real positive quantity r and so is written normally. Thus from (3.35) we see that $\mathrm{Ln}\, z$ is itself multivalued. To avoid this multivalued behaviour it is conventional to define another function $\ln z$, the *principal value* of $\mathrm{Ln}\, z$, which is obtained from $\mathrm{Ln}\, z$ by restricting the argument of z to lie in the range $-\pi < \theta \leq \pi$.

►*Evaluate* $\mathrm{Ln}\,(-i)$.

By rewriting $-i$ as a complex exponential, we find

$$\mathrm{Ln}\,(-i) = \mathrm{Ln}\, \left[e^{i(-\pi/2+2n\pi)}\right] = i(-\pi/2 + 2n\pi),$$

where n is any integer. Hence $\mathrm{Ln}\,(-i) = -i\pi/2,\ 3i\pi/2,\ \ldots$. We note that $\ln(-i)$, the principal value of $\mathrm{Ln}\,(-i)$, is given by $\ln(-i) = -i\pi/2$. ◄

If z and t are both complex numbers then the zth power of t is defined by

$$t^z = e^{z\,\mathrm{Ln}\, t}.$$

Since $\mathrm{Ln}\, t$ is multivalued, so too is this definition.

►*Simplify the expression* $z = i^{-2i}$.

Firstly we take the logarithm of both sides of the equation to give

$$\mathrm{Ln}\, z = -2i\, \mathrm{Ln}\, i.$$

Now inverting the process we find

$$e^{\mathrm{Ln}\, z} = z = e^{-2i\mathrm{Ln}\, i}.$$

We can write $i = e^{i(\pi/2+2n\pi)}$, where n is any integer, and hence

$$\begin{aligned}\mathrm{Ln}\, i &= \mathrm{Ln}\, \left[e^{i\left(\pi/2+2n\pi\right)}\right] \\ &= i\left(\pi/2 + 2n\pi\right).\end{aligned}$$

We can now simplify z to give

$$\begin{aligned}i^{-2i} &= e^{-2i\times i\left(\pi/2+2n\pi\right)} \\ &= e^{(\pi+4n\pi)},\end{aligned}$$

which, perhaps surprisingly, is a real quantity rather than a complex one. ◄

Complex powers and the logarithms of complex numbers are discussed further in chapter 24.

3.6 Applications to differentiation and integration

We can use the exponential form of a complex number together with de Moivre's theorem (see section 3.4) to simplify the differentiation of trigonometric functions.

►*Find the derivative with respect to x of* $e^{3x}\cos 4x$.

We could differentiate this function straightforwardly using the product rule (see subsection 2.1.2). However, an alternative method in this case is to use a complex exponential. Let us consider the complex number

$$z = e^{3x}(\cos 4x + i\sin 4x) = e^{3x}e^{4ix} = e^{(3+4i)x},$$

where we have used de Moivre's theorem to rewrite the trigonometric functions as a complex exponential. This complex number has $e^{3x}\cos 4x$ as its real part. Now, differentiating z with respect to x we obtain

$$\frac{dz}{dx} = (3+4i)e^{(3+4i)x} = (3+4i)e^{3x}(\cos 4x + i\sin 4x), \tag{3.36}$$

where we have again used de Moivre's theorem. Equating real parts we then find

$$\frac{d}{dx}\left(e^{3x}\cos 4x\right) = e^{3x}(3\cos 4x - 4\sin 4x).$$

By equating the imaginary parts of (3.36), we also obtain, as a bonus,

$$\frac{d}{dx}\left(e^{3x}\sin 4x\right) = e^{3x}(4\cos 4x + 3\sin 4x). \blacktriangleleft$$

In a similar way the complex exponential can be used to evaluate integrals containing trigonometric and exponential functions.

►*Evaluate the integral* $I = \int e^{ax}\cos bx\, dx$.

Let us consider the integrand as the real part of the complex number

$$e^{ax}(\cos bx + i\sin bx) = e^{ax}e^{ibx} = e^{(a+ib)x},$$

where we use de Moivre's theorem to rewrite the trigonometric functions as a complex exponential. Integrating we find

$$\begin{aligned}\int e^{(a+ib)x}\,dx &= \frac{e^{(a+ib)x}}{a+ib} + c \\ &= \frac{(a-ib)e^{(a+ib)x}}{(a-ib)(a+ib)} + c \\ &= \frac{e^{ax}}{a^2+b^2}\left(ae^{ibx} - ibe^{ibx}\right) + c,\end{aligned} \tag{3.37}$$

where the constant of integration c is in general complex. Denoting this constant by $c = c_1 + ic_2$ and equating real parts in (3.37) we obtain

$$I = \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c_1,$$

which agrees with result (2.37) found using integration by parts. Equating imaginary parts in (3.37) we obtain, as a bonus,

$$J = \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c_2. \blacktriangleleft$$

3.7 Hyperbolic functions

The *hyperbolic functions* are the complex analogues of the trigonometric functions. The analogy may not be immediately apparent and their definitions may appear at first to be somewhat arbitrary. However, careful examination of their properties reveals the purpose of the definitions. For instance, their close relationship with the trigonometric functions, both in their identities and in their calculus, means that many of the familiar properties of trigonometric functions can also be applied to the hyperbolic functions. Further, hyperbolic functions occur regularly, and so giving them special names is a notational convenience.

3.7.1 Definitions

The two fundamental hyperbolic functions are $\cosh x$ and $\sinh x$, which, as their names suggest, are the hyperbolic equivalents of $\cos x$ and $\sin x$. They are defined by the following relations:

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \tag{3.38}$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}). \tag{3.39}$$

Note that $\cosh x$ is an even function and $\sinh x$ is an odd function. By analogy with the trigonometric functions, the remaining hyperbolic functions are

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{3.40}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{3.41}$$
$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \tag{3.42}$$
$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \tag{3.43}$$

All the hyperbolic functions above have been defined in terms of the real variable x. However, this was simply so that they may be plotted (see figures 3.11–3.13); the definitions are equally valid for any complex number z.

3.7.2 Hyperbolic–trigonometric analogies

In the previous subsections we have alluded to the analogy between trigonometric and hyperbolic functions. Here, we discuss the close relationship between the two groups of functions.

Recalling (3.32) and (3.33) we find

$$\cos ix = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sin ix = \tfrac{1}{2}i(e^x - e^{-x}).$$

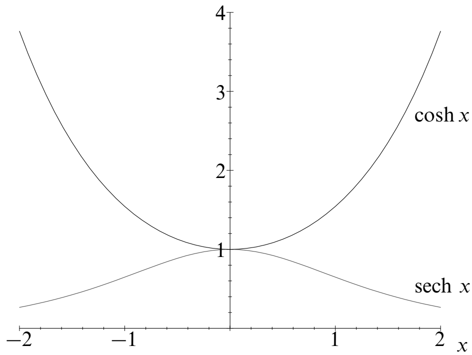

Figure 3.11 Graphs of $\cosh x$ and $\text{sech}x$.

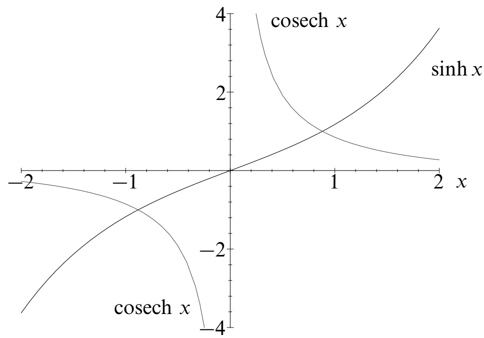

Figure 3.12 Graphs of $\sinh x$ and $\text{cosech}x$.

Hence, by the definitions given in the previous subsection,

$$\cosh x = \cos ix, \tag{3.44}$$

$$i \sinh x = \sin ix, \tag{3.45}$$

$$\cos x = \cosh ix, \tag{3.46}$$

$$i \sin x = \sinh ix. \tag{3.47}$$

These useful equations make the relationship between hyperbolic and trigono-

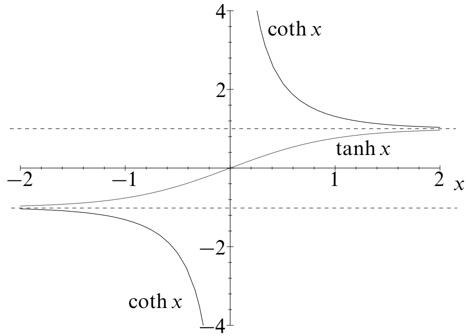

Figure 3.13 Graphs of $\tanh x$ and $\coth x$.

metric functions transparent. The similarity in their calculus is discussed further in subsection 3.7.6.

3.7.3 Identities of hyperbolic functions

The analogies between trigonometric functions and hyperbolic functions having been established, we should not be surprised that all the trigonometric identities also hold for hyperbolic functions, with the following modification. Wherever $\sin^2 x$ occurs it must be replaced by $-\sinh^2 x$, and vice versa. Note that this replacement is necessary even if the $\sin^2 x$ is hidden, e.g. $\tan^2 x = \sin^2 x/\cos^2 x$ and so must be replaced by $(-\sinh^2 x/\cosh^2 x) = -\tanh^2 x$.

►*Find the hyperbolic identity analogous to* $\cos^2 x + \sin^2 x = 1$.

Using the rules stated above $\cos^2 x$ is replaced by $\cosh^2 x$, and $\sin^2 x$ by $-\sinh^2 x$, and so the identity becomes

$$\cosh^2 x - \sinh^2 x = 1.$$

This can be verified by direct substitution, using the definitions of $\cosh x$ and $\sinh x$; see (3.38) and (3.39). ◄

Some other identities that can be proved in a similar way are

$$\operatorname{sech}^2 x = 1 - \tanh^2 x, \tag{3.48}$$

$$\operatorname{cosech}^2 x = \coth^2 x - 1, \tag{3.49}$$

$$\sinh 2x = 2 \sinh x \cosh x, \tag{3.50}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x. \tag{3.51}$$

3.7.4 Solving hyperbolic equations

When we are presented with a hyperbolic equation to solve, we may proceed by analogy with the solution of trigonometric equations. However, it is almost always easier to express the equation directly in terms of exponentials.

► *Solve the hyperbolic equation* $\cosh x - 5\sinh x - 5 = 0$.

Substituting the definitions of the hyperbolic functions we obtain

$$\tfrac{1}{2}(e^x + e^{-x}) - \tfrac{5}{2}(e^x - e^{-x}) - 5 = 0.$$

Rearranging, and then multiplying through by $-e^x$, gives in turn

$$-2e^x + 3e^{-x} - 5 = 0$$

and

$$2e^{2x} + 5e^x - 3 = 0.$$

Now we can factorise and solve:

$$(2e^x - 1)(e^x + 3) = 0.$$

Thus $e^x = 1/2$ or $e^x = -3$. Hence $x = -\ln 2$ or $x = \ln(-3)$. The interpretation of the logarithm of a negative number has been discussed in section 3.5. ◄

3.7.5 Inverses of hyperbolic functions

Just like trigonometric functions, hyperbolic functions have inverses. If $y = \cosh x$ then $x = \cosh^{-1} y$, which serves as a definition of the inverse. By using the fundamental definitions of hyperbolic functions, we can find closed-form expressions for their inverses. This is best illustrated by example.

► *Find a closed-form expression for the inverse hyperbolic function* $y = \sinh^{-1} x$.

First we write x as a function of y, i.e.

$$y = \sinh^{-1} x \quad \Rightarrow \quad x = \sinh y.$$

Now, since $\cosh y = \frac{1}{2}(e^y + e^{-y})$ and $\sinh y = \frac{1}{2}(e^y - e^{-y})$,

$$\begin{aligned} e^y &= \cosh y + \sinh y \\ &= \sqrt{1 + \sinh^2 y} + \sinh y \\ e^y &= \sqrt{1 + x^2} + x, \end{aligned}$$

and hence

$$y = \ln(\sqrt{1 + x^2} + x). \ ◄$$

In a similar fashion it can be shown that

$$\cosh^{-1} x = \ln(\sqrt{x^2 - 1} + x).$$

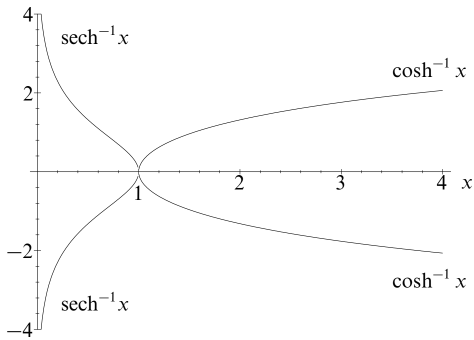

Figure 3.14 Graphs of $\cosh^{-1} x$ and $\mathrm{sech}^{-1}x$.

►*Find a closed-form expression for the inverse hyperbolic function* $y = \tanh^{-1} x$.

First we write x as a function of y, i.e.

$$y = \tanh^{-1} x \qquad \Rightarrow \qquad x = \tanh y.$$

Now, using the definition of $\tanh y$ and rearranging, we find

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \qquad \Rightarrow \qquad (x+1)e^{-y} = (1-x)e^y.$$

Thus, it follows that

$$e^{2y} = \frac{1+x}{1-x} \qquad \Rightarrow \qquad e^y = \sqrt{\frac{1+x}{1-x}},$$
$$y = \ln\sqrt{\frac{1+x}{1-x}},$$
$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \blacktriangleleft$$

Graphs of the inverse hyperbolic functions are given in figures 3.14–3.16.

3.7.6 Calculus of hyperbolic functions

Just as the identities of hyperbolic functions closely follow those of their trigonometric counterparts, so their calculus is similar. The derivatives of the two basic

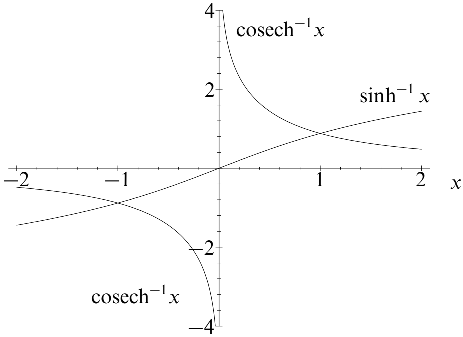

Figure 3.15 Graphs of $\sinh^{-1} x$ and $\operatorname{cosech}^{-1} x$.

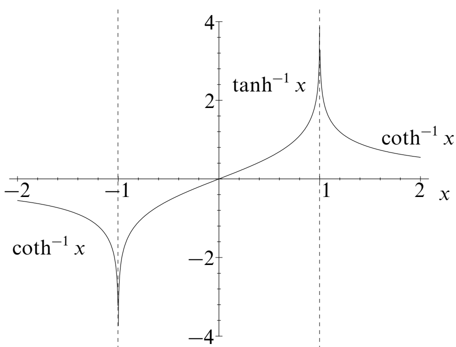

Figure 3.16 Graphs of $\tanh^{-1} x$ and $\coth^{-1} x$.

hyperbolic functions are given by

$$\frac{d}{dx}(\cosh x) = \sinh x, \tag{3.52}$$

$$\frac{d}{dx}(\sinh x) = \cosh x. \tag{3.53}$$

They may be deduced by considering the definitions (3.38), (3.39) as follows.

►*Verify the relation* $(d/dx)\cosh x = \sinh x$.

Using the definition of $\cosh x$,

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

and differentiating directly, we find

$$\begin{aligned}\frac{d}{dx}(\cosh x) &= \tfrac{1}{2}(e^x - e^{-x})\\ &= \sinh x. ◄\end{aligned}$$

Clearly the integrals of the fundamental hyperbolic functions are also defined by these relations. The derivatives of the remaining hyperbolic functions can be derived by product differentiation and are presented below only for completeness.

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x, \qquad (3.54)$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x, \qquad (3.55)$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x, \qquad (3.56)$$

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x. \qquad (3.57)$$

The inverse hyperbolic functions also have derivatives, which are given by the following:

$$\frac{d}{dx}\left(\cosh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{x^2 - a^2}}, \qquad (3.58)$$

$$\frac{d}{dx}\left(\sinh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{x^2 + a^2}}, \qquad (3.59)$$

$$\frac{d}{dx}\left(\tanh^{-1}\frac{x}{a}\right) = \frac{a}{a^2 - x^2}, \quad \text{for } x^2 < a^2, \qquad (3.60)$$

$$\frac{d}{dx}\left(\coth^{-1}\frac{x}{a}\right) = \frac{-a}{x^2 - a^2}, \quad \text{for } x^2 > a^2. \qquad (3.61)$$

These may be derived from the logarithmic form of the inverse (see subsection 3.7.5).

► *Evaluate $(d/dx)\sinh^{-1} x$ using the logarithmic form of the inverse.*

From the results of section 3.7.5,

$$\begin{aligned}
\frac{d}{dx}\left(\sinh^{-1} x\right) &= \frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right] \\
&= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) \\
&= \frac{1}{\sqrt{x^2+1}}. \blacktriangleleft
\end{aligned}$$

3.8 Exercises

3.1 Two complex numbers z and w are given by $z = 3 + 4i$ and $w = 2 - i$. On an Argand diagram, plot

(a) $z + w$, (b) $w - z$, (c) wz, (d) z/w,
(e) $z^* w + w^* z$, (f) w^2, (g) $\ln z$, (h) $(1 + z + w)^{1/2}$.

3.2 By considering the real and imaginary parts of the product $e^{i\theta}e^{i\phi}$ prove the standard formulae for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$.

3.3 By writing $\pi/12 = (\pi/3) - (\pi/4)$ and considering $e^{i\pi/12}$, evaluate $\cot(\pi/12)$.

3.4 Find the locus in the complex z-plane of points that satisfy the following equations.

(a) $z - c = \rho\left(\dfrac{1+it}{1-it}\right)$, where c is complex, ρ is real and t is a real parameter that varies in the range $-\infty < t < \infty$.
(b) $z = a + bt + ct^2$, in which t is a real parameter and a, b, and c are complex numbers with b/c real.

3.5 Evaluate

(a) $\mathrm{Re}(\exp 2iz)$, (b) $\mathrm{Im}(\cosh^2 z)$, (c) $(-1 + \sqrt{3}i)^{1/2}$,
(d) $|\exp(i^{1/2})|$, (e) $\exp(i^3)$, (f) $\mathrm{Im}(2^{i+3})$, (g) i^i, (h) $\ln[(\sqrt{3} + i)^3]$.

3.6 Find the equations in terms of x and y of the sets of points in the Argand diagram that satisfy the following:

(a) $\mathrm{Re}\, z^2 = \mathrm{Im}\, z^2$;
(b) $(\mathrm{Im}\, z^2)/z^2 = -i$;
(c) $\arg[z/(z-1)] = \pi/2$.

3.7 Show that the locus of all points $z = x + iy$ in the complex plane that satisfy

$$|z - ia| = \lambda|z + ia|, \quad \lambda > 0,$$

is a circle of radius $|2\lambda a/(1 - \lambda^2)|$ centred on the point $z = ia[(1 + \lambda^2)/(1 - \lambda^2)]$. Sketch the circles for a few typical values of λ, including $\lambda < 1$, $\lambda > 1$ and $\lambda = 1$.

3.8 The two sets of points $z = a$, $z = b$, $z = c$, and $z = A$, $z = B$, $z = C$ are the corners of two similar triangles in the Argand diagram. Express in terms of $a, b, \ldots, C$

(a) the equalities of corresponding angles, and
(b) the constant ratio of corresponding sides,

in the two triangles.

By noting that any complex quantity can be expressed as

$$z = |z| \exp(i \arg z),$$

deduce that

$$a(B - C) + b(C - A) + c(A - B) = 0.$$

3.9 For the real constant a find the loci of all points $z = x + iy$ in the complex plane that satisfy

(a) $\operatorname{Re}\left\{\ln\left(\dfrac{z - ia}{z + ia}\right)\right\} = c, \qquad c > 0,$

(b) $\operatorname{Im}\left\{\ln\left(\dfrac{z - ia}{z + ia}\right)\right\} = k, \qquad 0 \le k \le \pi/2.$

Identify the two families of curves and verify that in case (b) all curves pass through the two points $\pm ia$.

3.10 The most general type of transformation between one Argand diagram, in the z-plane, and another, in the Z-plane, that gives one and only one value of Z for each value of z (and conversely) is known as the *general bilinear transformation* and takes the form

$$z = \frac{aZ + b}{cZ + d}.$$

(a) Confirm that the transformation from the Z-plane to the z-plane is also a general bilinear transformation.
(b) Recalling that the equation of a circle can be written in the form

$$\left|\frac{z - z_1}{z - z_2}\right| = \lambda, \qquad \lambda \ne 1,$$

show that the general bilinear transformation transforms circles into circles (or straight lines). What is the condition that z_1, z_2 and λ must satisfy if the transformed circle is to be a straight line?

3.11 Sketch the parts of the Argand diagram in which

(a) $\operatorname{Re} z^2 < 0$, $|z^{1/2}| \le 2$;
(b) $0 \le \arg z^* \le \pi/2$;
(c) $|\exp z^3| \to 0$ as $|z| \to \infty$.

What is the area of the region in which all three sets of conditions are satisfied?

3.12 Denote the nth roots of unity by $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$.

(a) Prove that

$$\text{(i)} \sum_{r=0}^{n-1} \omega_n^r = 0, \qquad \text{(ii)} \prod_{r=0}^{n-1} \omega_n^r = (-1)^{n+1}.$$

(b) Express $x^2 + y^2 + z^2 - yz - zx - xy$ as the product of two factors, each linear in x, y and z, with coefficients dependent on the third roots of unity (and those of the x terms arbitrarily taken as real).

3.13 Prove that $x^{2m+1} - a^{2m+1}$, where m is an integer ≥ 1, can be written as

$$x^{2m+1} - a^{2m+1} = (x-a)\prod_{r=1}^{m}\left[x^2 - 2ax\cos\left(\frac{2\pi r}{2m+1}\right) + a^2\right].$$

3.14 The complex position vectors of two parallel interacting equal fluid vortices moving with their axes of rotation always perpendicular to the z-plane are z_1 and z_2. The equations governing their motions are

$$\frac{dz_1^*}{dt} = -\frac{i}{z_1 - z_2}, \qquad \frac{dz_2^*}{dt} = -\frac{i}{z_2 - z_1}.$$

Deduce that (a) $z_1 + z_2$, (b) $|z_1 - z_2|$ and (c) $|z_1|^2 + |z_2|^2$ are all constant in time, and hence describe the motion geometrically.

3.15 Solve the equation

$$z^7 - 4z^6 + 6z^5 - 6z^4 + 6z^3 - 12z^2 + 8z + 4 = 0,$$

(a) by examining the effect of setting z^3 equal to 2, and then
(b) by factorising and using the binomial expansion of $(z+a)^4$.

Plot the seven roots of the equation on an Argand plot, exemplifying that complex roots of a polynomial equation always occur in conjugate pairs if the polynomial has real coefficients.

3.16 The polynomial $f(z)$ is defined by

$$f(z) = z^5 - 6z^4 + 15z^3 - 34z^2 + 36z - 48.$$

(a) Show that the equation $f(z) = 0$ has roots of the form $z = \lambda i$, where λ is real, and hence factorize $f(z)$.
(b) Show further that the cubic factor of $f(z)$ can be written in the form $(z+a)^3 + b$, where a and b are real, and hence solve the equation $f(z) = 0$ completely.

3.17 The binomial expansion of $(1+x)^n$, discussed in chapter 1, can be written for a positive integer n as

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r,$$

where ${}^nC_r = n!/[r!(n-r)!]$.

(a) Use de Moivre's theorem to show that the sum

$$S_1(n) = {}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots + (-1)^m\,{}^nC_{2m}, \qquad n-1 \leq 2m \leq n,$$

has the value $2^{n/2}\cos(n\pi/4)$.

(b) Derive a similar result for the sum

$$S_2(n) = {}^nC_1 - {}^nC_3 + {}^nC_5 - \cdots + (-1)^m\,{}^nC_{2m+1}, \qquad n-1 \leq 2m+1 \leq n,$$

and verify it for the cases $n = 6$, 7 and 8.

3.18 By considering $(1 + \exp i\theta)^n$, prove that

$$\sum_{r=0}^{n} {}^nC_r \cos r\theta = 2^n \cos^n(\theta/2)\cos(n\theta/2),$$

$$\sum_{r=0}^{n} {}^nC_r \sin r\theta = 2^n \cos^n(\theta/2)\sin(n\theta/2),$$

where ${}^nC_r = n!/[r!(n-r)!]$.

3.19 Use de Moivre's theorem with $n = 4$ to prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1,$$

and deduce that

$$\cos\frac{\pi}{8} = \left(\frac{2+\sqrt{2}}{4}\right)^{1/2}.$$

3.20 Express $\sin^4\theta$ entirely in terms of the trigonometric functions of multiple angles and deduce that its average value over a complete cycle is $\frac{3}{8}$.

3.21 Use de Moivre's theorem to prove that

$$\tan 5\theta = \frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1},$$

where $t = \tan\theta$. Deduce the values of $\tan(n\pi/10)$ for $n = 1, 2, 3, 4$.

3.22 Prove the following results involving hyperbolic functions.

(a) That

$$\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right).$$

(b) That, if $y = \sinh^{-1} x$,

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

3.23 Determine the conditions under which the equation

$$a\cosh x + b\sinh x = c, \qquad c > 0,$$

has zero, one, or two real solutions for x. What is the solution if $a^2 = c^2 + b^2$?

3.24 Use the definitions and properties of hyperbolic functions to do the following:

(a) Solve $\cosh x = \sinh x + 2\,\mathrm{sech}\,x$.
(b) Show that the real solution x of $\tanh x = \mathrm{cosech}\,x$ can be written in the form $x = \ln(u + \sqrt{u})$. Find an explicit value for u.
(c) Evaluate $\tanh x$ when x is the real solution of $\cosh 2x = 2\cosh x$.

3.25 Express $\sinh^4 x$ in terms of hyperbolic cosines of multiples of x, and hence find the real solutions of

$$2\cosh 4x - 8\cosh 2x + 5 = 0.$$

3.26 In the theory of special relativity, the relationship between the position and time coordinates of an event, as measured in two frames of reference that have parallel x-axes, can be expressed in terms of hyperbolic functions. If the coordinates are x and t in one frame and x' and t' in the other, then the relationship take the form

$$x' = x\cosh\phi - ct\sinh\phi,$$
$$ct' = -x\sinh\phi + ct\cosh\phi.$$

Express x and ct in terms of x', ct' and ϕ and show that

$$x^2 - (ct)^2 = (x')^2 - (ct')^2.$$

3.27 A closed barrel has as its curved surface the surface obtained by rotating about the x-axis the part of the curve

$$y = a[2 - \cosh(x/a)]$$

lying in the range $-b \leq x \leq b$, where $b < a\cosh^{-1} 2$. Show that the total surface area, A, of the barrel is given by

$$A = \pi a[9a - 8a\exp(-b/a) + a\exp(-2b/a) - 2b].$$

3.28 The principal value of the logarithmic function of a complex variable is defined to have its argument in the range $-\pi < \arg z \leq \pi$. By writing $z = \tan w$ in terms of exponentials show that

$$\tan^{-1} z = \frac{1}{2i} \ln\left(\frac{1+iz}{1-iz}\right).$$

Use this result to evaluate

$$\tan^{-1}\left(\frac{2\sqrt{3}-3i}{7}\right).$$

3.9 Hints and answers

3.1 (a) $5+3i$; (b) $-1-5i$; (c) $10+5i$; (d) $2/5+11i/5$; (e) 4; (f) $3-4i$; (g) $\ln 5 + i[\tan^{-1}(4/3) + 2n\pi]$; (h) $\pm(2.521 + 0.595i)$.

3.3 Use $\sin\pi/4 = \cos\pi/4 = 1/\sqrt{2}$, $\sin\pi/3 = 1/2$ and $\cos\pi/3 = \sqrt{3}/2$. $\cot\pi/12 = 2+\sqrt{3}$.

3.5 (a) $\exp(-2y)\cos 2x$; (b) $(\sin 2y \sinh 2x)/2$; (c) $\sqrt{2}\exp(\pi i/3)$ or $\sqrt{2}\exp(4\pi i/3)$; (d) $\exp(1/\sqrt{2})$ or $\exp(-1/\sqrt{2})$; (e) $0.540 - 0.841i$; (f) $8\sin(\ln 2) = 5.11$; (g) $\exp(-\pi/2 - 2\pi n)$; (h) $\ln 8 + i(6n+1/2)\pi$.

3.7 Starting from $|x+iy-ia| = \lambda|x+iy+ia|$, show that the coefficients of x and y are equal, and write the equation in the form $x^2 + (y-\alpha)^2 = r^2$.

3.9 (a) Circles enclosing $z = -ia$, with $\lambda = \exp c > 1$.
(b) The condition is that $\arg[(z-ia)/(z+ia)] = k$. This can be rearranged to give $a(z+z^*) = (a^2 - |z|^2)\tan k$, which becomes in x, y coordinates the equation of a circle with centre $(-a\cot k, 0)$ and radius $a\,\mathrm{cosec}\,k$.

3.11 All three conditions are satisfied in $3\pi/2 \leq \theta \leq 7\pi/4$, $|z| \leq 4$; area $= 2\pi$.

3.13 Denoting $\exp[2\pi i/(2m+1)]$ by Ω, express $x^{2m+1} - a^{2m+1}$ as a product of factors like $(x - a\Omega^r)$ and then combine those containing Ω^r and Ω^{2m+1-r}. Use the fact that $\Omega^{2m+1} = 1$.

3.15 The roots are $2^{1/3}\exp(2\pi ni/3)$ for $n = 0, 1, 2$; $1 \pm 3^{1/4}$; $1 \pm 3^{1/4}i$.

3.17 Consider $(1+i)^n$. (b) $S_2(n) = 2^{n/2}\sin(n\pi/4)$. $S_2(6) = -8$, $S_2(7) = -8$, $S_2(8) = 0$.

3.19 Use the binomial expansion of $(\cos\theta + i\sin\theta)^4$.

3.21 Show that $\cos 5\theta = 16c^5 - 20c^3 + 5c$, where $c = \cos\theta$, and correspondingly for $\sin 5\theta$. Use $\cos^{-2}\theta = 1 + \tan^2\theta$. The four required values are $[(5-\sqrt{20})/5]^{1/2}$, $(5-\sqrt{20})^{1/2}$, $[(5+\sqrt{20})/5]^{1/2}$, $(5+\sqrt{20})^{1/2}$.

3.23 Reality of the root(s) requires $c^2 + b^2 \geq a^2$ and $a + b > 0$. With these conditions, there are two roots if $a^2 > b^2$, but only one if $b^2 > a^2$.
For $a^2 = c^2 + b^2$, $x = \frac{1}{2}\ln[(a-b)/(a+b)]$.

3.25 Reduce the equation to $16\sinh^4 x = 1$, yielding $x = \pm 0.481$.

3.27 Show that $ds = (\cosh x/a)\,dx$;
curved surface area $= \pi a^2[8\sinh(b/a) - \sinh(2b/a)] - 2\pi ab$.
flat ends area $= 2\pi a^2[4 - 4\cosh(b/a) + \cosh^2(b/a)]$.

4

Series and limits

4.1 Series

Many examples exist in the physical sciences of situations where we are presented with a *sum of terms* to evaluate. For example, we may wish to add the contributions from successive slits in a diffraction grating to find the total light intensity at a particular point behind the grating.

A series may have either a finite or infinite number of terms. In either case, the sum of the first N terms of a series (often called a partial sum) is written

$$S_N = u_1 + u_2 + u_3 + \cdots + u_N,$$

where the terms of the series u_n, $n = 1, 2, 3, \ldots, N$ are numbers, that may in general be complex. If the terms are complex then S_N will in general be complex also, and we can write $S_N = X_N + iY_N$, where X_N and Y_N are the partial sums of the real and imaginary parts of each term separately and are therefore real. If a series has only N terms then the partial sum S_N is of course the sum of the series. Sometimes we may encounter series where each term depends on some variable, x, say. In this case the partial sum of the series will depend on the value assumed by x. For example, consider the infinite series

$$S(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

This is an example of a power series; these are discussed in more detail in section 4.5. It is in fact the Maclaurin expansion of $\exp x$ (see subsection 4.6.3). Therefore $S(x) = \exp x$ and, of course, varies according to the value of the variable x. A series might just as easily depend on a complex variable z.

A general, random sequence of numbers can be described as a series and a sum of the terms found. However, for cases of practical interest, there will usually be