

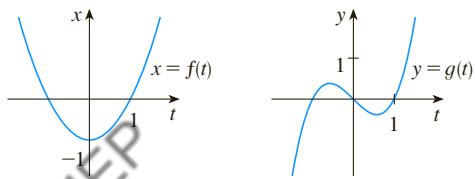
EXERCISES

- 1–4** Sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

1. $x = t^2 + 4t, \quad y = 2 - t, \quad -4 \leq t \leq 1$
2. $x = 1 + e^{2t}, \quad y = e^t$
3. $x = \cos \theta, \quad y = \sec \theta, \quad 0 \leq \theta < \pi/2$
4. $x = 2 \cos \theta, \quad y = 1 + \sin \theta$

5. Write three different sets of parametric equations for the curve $y = \sqrt{x}$.

6. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t), y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



7. (a) Plot the point with polar coordinates $(4, 2\pi/3)$. Then find its Cartesian coordinates.
 (b) The Cartesian coordinates of a point are $(-3, 3)$. Find two sets of polar coordinates for the point.
 8. Sketch the region consisting of points whose polar coordinates satisfy $1 \leq r < 2$ and $\pi/6 \leq \theta \leq 5\pi/6$.

9–16 Sketch the polar curve.

9. $r = 1 + \sin \theta$	10. $r = \sin 4\theta$
11. $r = \cos 3\theta$	12. $r = 3 + \cos 3\theta$
13. $r = 1 + \cos 2\theta$	14. $r = 2 \cos(\theta/2)$
15. $r = \frac{3}{1 + 2 \sin \theta}$	16. $r = \frac{3}{2 - 2 \cos \theta}$

- 17–18** Find a polar equation for the curve represented by the given Cartesian equation.

17. $x + y = 2$

18. $x^2 + y^2 = 2$

- 19.** The curve with polar equation $r = (\sin \theta)/\theta$ is called a **cochleoid**. Use a graph of r as a function of θ in Cartesian coordinates to sketch the cochleoid by hand. Then graph it with a machine to check your sketch.
20. Graph the ellipse $r = 2/(4 - 3 \cos \theta)$ and its directrix. Also graph the ellipse obtained by rotation about the origin through an angle $2\pi/3$.

- 21–24** Find the slope of the tangent line to the given curve at the point corresponding to the specified value of the parameter.

21. $x = \ln t, \quad y = 1 + t^2; \quad t = 1$
22. $x = t^3 + 6t + 1, \quad y = 2t - t^2; \quad t = -1$
23. $r = e^{-\theta}; \quad \theta = \pi$
24. $r = 3 + \cos 3\theta; \quad \theta = \pi/2$

- 25–26** Find dy/dx and d^2y/dx^2 .

25. $x = t + \sin t, \quad y = t - \cos t$
26. $x = 1 + t^2, \quad y = t - t^3$

- 27.** Use a graph to estimate the coordinates of the lowest point on the curve $x = t^3 - 3t, y = t^2 + t + 1$. Then use calculus to find the exact coordinates.

- 28.** Find the area enclosed by the loop of the curve in Exercise 27.

- 29.** At what points does the curve

$$x = 2a \cos t - a \cos 2t \quad y = 2a \sin t - a \sin 2t$$

have vertical or horizontal tangents? Use this information to help sketch the curve.

- 30.** Find the area enclosed by the curve in Exercise 29.

- 31.** Find the area enclosed by the curve $r^2 = 9 \cos 5\theta$.

- 32.** Find the area enclosed by the inner loop of the curve $r = 1 - 3 \sin \theta$.

- 33.** Find the points of intersection of the curves $r = 2$ and $r = 4 \cos \theta$.

- 34.** Find the points of intersection of the curves $r = \cot \theta$ and $r = 2 \cos \theta$.

- 35.** Find the area of the region that lies inside both of the circles $r = 2 \sin \theta$ and $r = \sin \theta + \cos \theta$.

- 36.** Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin \theta$.

- 37–40** Find the length of the curve.

37. $x = 3t^2, \quad y = 2t^3, \quad 0 \leq t \leq 2$
38. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leq t \leq 1$
39. $r = 1/\theta, \quad \pi \leq \theta \leq 2\pi$
40. $r = \sin^3(\theta/3), \quad 0 \leq \theta \leq \pi$

- 41–42** Find the area of the surface obtained by rotating the given curve about the x -axis.

41. $x = 4\sqrt{t}$, $y = \frac{t^3}{3} + \frac{1}{2t^2}$, $1 \leq t \leq 4$

42. $x = 2 + 3t$, $y = \cosh 3t$, $0 \leq t \leq 1$

- 43.** The curves defined by the parametric equations

$$x = \frac{t^2 - c}{t^2 + 1} \quad y = \frac{t(t^2 - c)}{t^2 + 1}$$

are called **strophoids** (from a Greek word meaning “to turn or twist”). Investigate how these curves vary as c varies.

- 44.** A family of curves has polar equations $r^a = |\sin 2\theta|$ where a is a positive number. Investigate how the curves change as a changes.

- 45–48** Find the foci and vertices and sketch the graph.

45. $\frac{x^2}{9} + \frac{y^2}{8} = 1$

46. $4x^2 - y^2 = 16$

47. $6y^2 + x - 36y + 55 = 0$

48. $25x^2 + 4y^2 + 50x - 16y = 59$

- 49.** Find an equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$.

- 50.** Find an equation of the parabola with focus $(2, 1)$ and directrix $x = -4$.

- 51.** Find an equation of the hyperbola with foci $(0, \pm 4)$ and asymptotes $y = \pm 3x$.

- 52.** Find an equation of the ellipse with foci $(3, \pm 2)$ and major axis with length 8.

- 53.** Find an equation for the ellipse that shares a vertex and a focus with the parabola $x^2 + y = 100$ and that has its other focus at the origin.

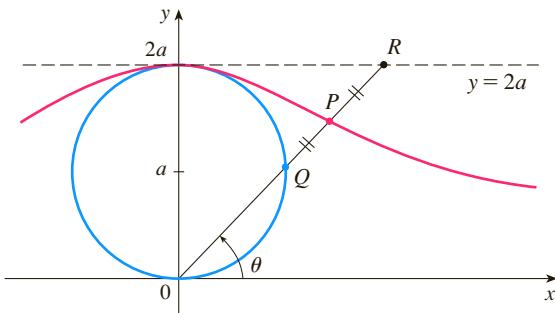
- 54.** Show that if m is any real number, then there are exactly two lines of slope m that are tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and their equations are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

- 55.** Find a polar equation for the ellipse with focus at the origin, eccentricity $\frac{1}{3}$, and directrix with equation $r = 4 \sec \theta$.

- 56.** Show that the angles between the polar axis and the asymptotes of the hyperbola $r = ed/(1 - e \cos \theta)$, $e > 1$, are given by $\cos^{-1}(\pm 1/e)$.

- 57.** In the figure the circle of radius a is stationary, and for every θ , the point P is the midpoint of the segment QR . The curve traced out by P for $0 < \theta < \pi$ is called the **longbow curve**. Find parametric equations for this curve.



- 58.** A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t}{1+t^3} \quad y = \frac{3t^2}{1+t^3}$$

- (a) Show that if (a, b) lies on the curve, then so does (b, a) ; that is, the curve is symmetric with respect to the line $y = x$. Where does the curve intersect this line?
 (b) Find the points on the curve where the tangent lines are horizontal or vertical.
 (c) Show that the line $y = -x - 1$ is a slant asymptote.
 (d) Sketch the curve.
 (e) Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.
 (f) Show that the polar equation can be written in the form

$$r = \frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta}$$

- (g) Find the area enclosed by the loop of this curve.
 (h) Show that the area of the loop is the same as the area that lies between the asymptote and the infinite branches of the curve. (Use a computer algebra system to evaluate the integral.)

Problems Plus

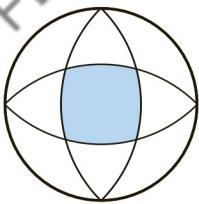


FIGURE FOR PROBLEM 1

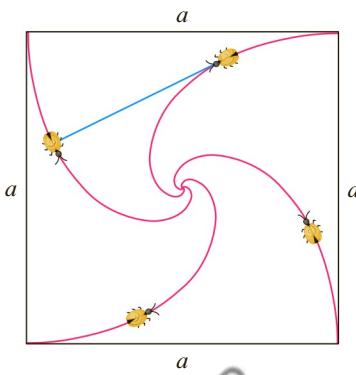


FIGURE FOR PROBLEM 4

1. The outer circle in the figure has radius 1 and the centers of the interior circular arcs lie on the outer circle. Find the area of the shaded region.

2. (a) Find the highest and lowest points on the curve $x^4 + y^4 = x^2 + y^2$.
 (b) Sketch the curve. (Notice that it is symmetric with respect to both axes and both of the lines $y = \pm x$, so it suffices to consider $y \geq x \geq 0$ initially.)
 (c) Use polar coordinates and a computer algebra system to find the area enclosed by the curve.

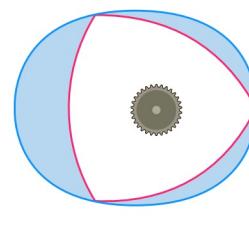
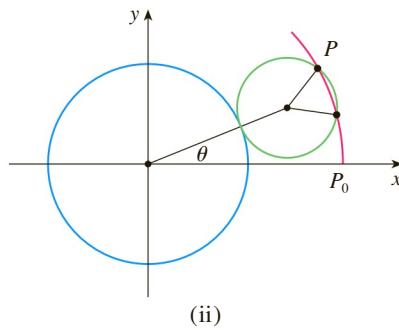
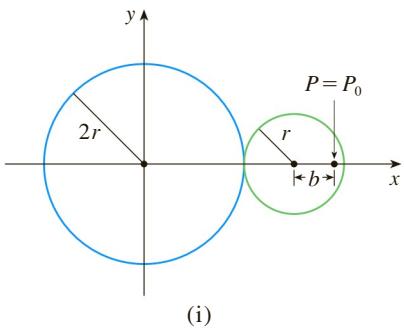
3. What is the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c \sin \theta$, where $0 \leq c \leq 1$? Illustrate your answer by graphing several members of the family in this viewing rectangle.

4. Four bugs are placed at the four corners of a square with side length a . The bugs crawl counterclockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the center of the square along spiral paths.
 (a) Find the polar equation of a bug's path assuming the pole is at the center of the square. (Use the fact that the line joining one bug to the next is tangent to the bug's path.)
 (b) Find the distance traveled by a bug by the time it meets the other bugs at the center.
 5. Show that any tangent line to a hyperbola touches the hyperbola halfway between the points of intersection of the tangent and the asymptotes.
 6. A circle C of radius $2r$ has its center at the origin. A circle of radius r rolls without slipping in the counterclockwise direction around C . A point P is located on a fixed radius of the rolling circle at a distance b from its center, $0 < b < r$. [See parts (i) and (ii) of the figure.] Let L be the line from the center of C to the center of the rolling circle and let θ be the angle that L makes with the positive x -axis.
 (a) Using θ as a parameter, show that parametric equations of the path traced out by P are

$$x = b \cos 3\theta + 3r \cos \theta \quad y = b \sin 3\theta + 3r \sin \theta$$

Note: If $b = 0$, the path is a circle of radius $3r$; if $b = r$, the path is an *epicycloid*. The path traced out by P for $0 < b < r$ is called an *epitrochoid*.

- (b) Graph the curve for various values of b between 0 and r .
 (c) Show that an equilateral triangle can be inscribed in the epitrochoid and that its centroid is on the circle of radius b centered at the origin.
Note: This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the curve.
 (d) In most rotary engines the sides of the equilateral triangles are replaced by arcs of circles centered at the opposite vertices as in part (iii) of the figure. (Then the diameter of the rotor is constant.) Show that the rotor will fit in the epitrochoid if $b \leq \frac{3}{2}(2 - \sqrt{3})r$.



11

Infinite Sequences and Series

Betelgeuse is a red supergiant star, one of the largest and brightest of the observable stars. In the project on page 823 you are asked to compare the radiation emitted by Betelgeuse with that of other stars.



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INFINITE SEQUENCES AND SERIES WERE introduced briefly in *A Preview of Calculus* in connection with Zeno's paradoxes and the decimal representation of numbers. Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series. For instance, in finding areas he often integrated a function by first expressing it as a series and then integrating each term of the series. We will pursue his idea in Section 11.10 in order to integrate such functions as e^{-x^2} . (Recall that we have previously been unable to do this.) Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way, as we will see in Section 11.11. In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it.

11.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *n th term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1} .

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n .

NOTATION The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

EXAMPLE 1 Some sequences can be defined by giving a formula for the n th term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

- | | | |
|---|---------------------------------------|--|
| (a) $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$ | $a_n = \frac{n}{n+1}$ | $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots \right\}$ |
| (b) $\left\{ \frac{(-1)^n(n+1)}{3^n} \right\}$ | $a_n = \frac{(-1)^n(n+1)}{3^n}$ | $\left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots \right\}$ |
| (c) $\left\{ \sqrt{n-3} \right\}_{n=3}^{\infty}$ | $a_n = \sqrt{n-3}, n \geq 3$ | $\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$ |
| (d) $\left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty}$ | $a_n = \cos \frac{n\pi}{6}, n \geq 0$ | $\left\{ 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos \frac{n\pi}{6}, \dots \right\}$ |

EXAMPLE 2 Find a formula for the general term a_n of the sequence

$$\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots \right\}$$

assuming that the pattern of the first few terms continues.

SOLUTION We are given that

$$a_1 = \frac{3}{5} \quad a_2 = -\frac{4}{25} \quad a_3 = \frac{5}{125} \quad a_4 = -\frac{6}{625} \quad a_5 = \frac{7}{3125}$$

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the n th term will have numerator $n + 2$. The denominators are the

powers of 5, so a_n has denominator 5^n . The signs of the terms are alternately positive and negative, so we need to multiply by a power of -1 . In Example 1(b) the factor $(-1)^n$ meant we started with a negative term. Here we want to start with a positive term and so we use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore

$$a_n = (-1)^{n-1} \frac{n+2}{5^n}$$

EXAMPLE 3 Here are some sequences that don't have a simple defining equation.

- (a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n .
- (b) If we let a_n be the digit in the n th decimal place of the number e , then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \dots\}$$

- (c) **The Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Exercise 83).

A sequence such as the one in Example 1(a), $a_n = n/(n + 1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2. Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad \dots \quad (n, a_n) \quad \dots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n + 1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n \rightarrow \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 3.4.

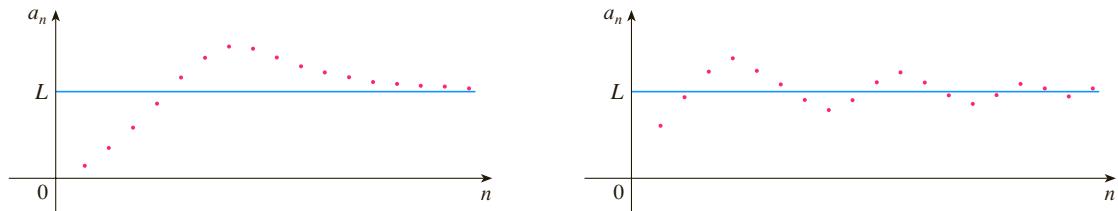
1 Definition A sequence $\{a_n\}$ has the **limit L** and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \rightarrow \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit L .

FIGURE 3
Graphs of two sequences with
 $\lim_{n \rightarrow \infty} a_n = L$



A more precise version of Definition 1 is as follows.

2 Definition A sequence $\{a_n\}$ has the **limit L** and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

Compare this definition with
Definition 3.4.5.

Definition 2 is illustrated by Figure 4, in which the terms a_1, a_2, a_3, \dots are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

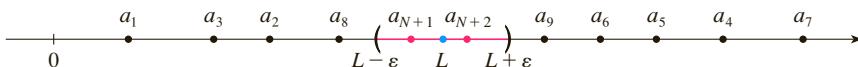


FIGURE 4

Another illustration of Definition 2 is given in Figure 5. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N .

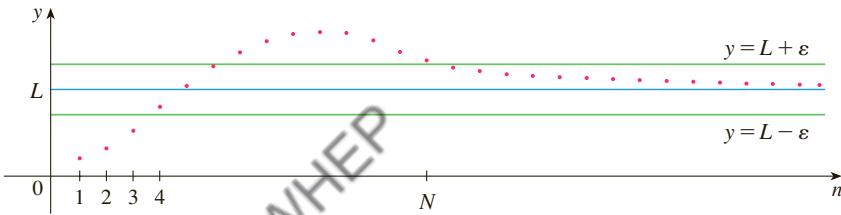


FIGURE 5

If you compare Definition 2 with Definition 3.4.5, you will see that the only difference between $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{x \rightarrow \infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 6.

3 Theorem If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.

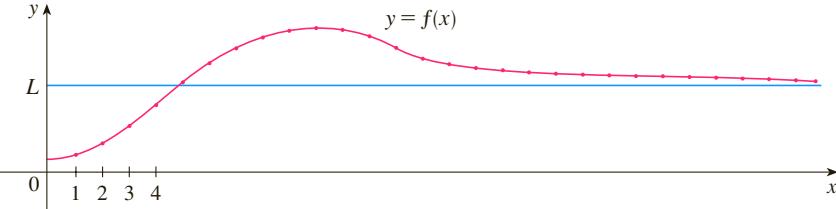


FIGURE 6

In particular, since we know that $\lim_{x \rightarrow \infty} (1/x^r) = 0$ when $r > 0$ (Theorem 3.4.4), we have

$$\boxed{4} \quad \lim_{n \rightarrow \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n \rightarrow \infty} a_n = \infty$. The following precise definition is similar to Definition 3.4.7.

5 Definition $\lim_{n \rightarrow \infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if } n > N \quad \text{then } a_n > M$$

If $\lim_{n \rightarrow \infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞ .

The Limit Laws given in Section 1.6 also hold for the limits of sequences and their proofs are similar.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n \quad \lim_{n \rightarrow \infty} c = c$$

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{if } \lim_{n \rightarrow \infty} b_n \neq 0$$

$$\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

Squeeze Theorem for Sequences

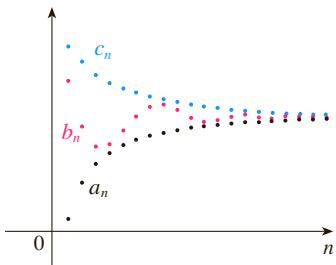


FIGURE 7

The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

This shows that the guess we made earlier from Figures 1 and 2 was correct.

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

Another useful fact about limits of sequences is given by the following theorem, whose proof is left as Exercise 87.

6 Theorem

If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

EXAMPLE 4 Find $\lim_{n \rightarrow \infty} \frac{n}{n+1}$.

SOLUTION The method is similar to the one we used in Section 3.4: Divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n}{n+1} &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} \\ &= \frac{1}{1 + 0} = 1\end{aligned}$$

Here we used Equation 4 with $r = 1$.

EXAMPLE 5 Is the sequence $a_n = \frac{n}{\sqrt{10+n}}$ convergent or divergent?

SOLUTION As in Example 4, we divide numerator and denominator by n :

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{10+n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent. ■

EXAMPLE 6 Calculate $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$.

SOLUTION Notice that both numerator and denominator approach infinity as $n \rightarrow \infty$. We can't apply l'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l'Hospital's Rule to the related function $f(x) = (\ln x)/x$ and obtain

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$$

Therefore, by Theorem 3, we have

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

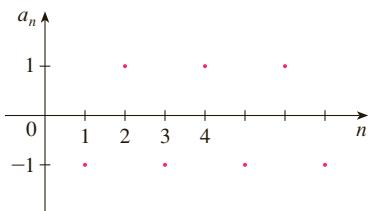


FIGURE 8

The graph of the sequence in Example 8 is shown in Figure 9 and supports our answer.

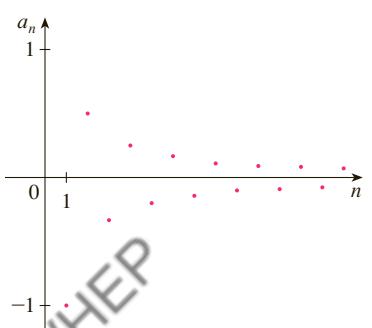


FIGURE 9

Creating Graphs of Sequences

Some computer algebra systems have special commands that enable us to create sequences and graph them directly. With most graphing calculators, however, sequences can be graphed by using parametric equations. For instance, the sequence in Example 10 can be graphed by entering the parametric equations

$$x = t \quad y = t!/t^t$$

and graphing in dot mode, starting with $t = 1$ and setting the t -step equal to 1.

The result is shown in Figure 10.



FIGURE 10

EXAMPLE 7 Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

SOLUTION If we write out the terms of the sequence, we obtain

$$\{-1, 1, -1, 1, -1, 1, -1, \dots\}$$

The graph of this sequence is shown in Figure 8. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim_{n \rightarrow \infty} (-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent. ■

EXAMPLE 8 Evaluate $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$ if it exists.

SOLUTION We first calculate the limit of the absolute value:

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Therefore, by Theorem 6,

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent. The proof is left as Exercise 88.

7 Theorem If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then

$$\lim_{n \rightarrow \infty} f(a_n) = f(L)$$

EXAMPLE 9 Find $\lim_{n \rightarrow \infty} \sin(\pi/n)$.

SOLUTION Because the sine function is continuous at 0, Theorem 7 enables us to write

$$\lim_{n \rightarrow \infty} \sin(\pi/n) = \sin\left(\lim_{n \rightarrow \infty} (\pi/n)\right) = \sin 0 = 0$$

EXAMPLE 10 Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdots \cdot n$.

SOLUTION Both numerator and denominator approach infinity as $n \rightarrow \infty$ but here we have no corresponding function for use with l'Hospital's Rule ($x!$ is not defined when x is not an integer). Let's write out a few terms to get a feeling for what happens to a_n as n gets large:

$$a_1 = 1 \quad a_2 = \frac{1 \cdot 2}{2 \cdot 2} \quad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdots \cdot n}{n \cdot n \cdot n \cdots \cdot n}$$

It appears from these expressions and the graph in Figure 10 that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation 8 that

$$a_n = \frac{1}{n} \left(\frac{2 \cdot 3 \cdots \cdot n}{n \cdot n \cdots \cdot n} \right)$$

Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator. So

$$0 < a_n \leq \frac{1}{n}$$

We know that $1/n \rightarrow 0$ as $n \rightarrow \infty$. Therefore $a_n \rightarrow 0$ as $n \rightarrow \infty$ by the Squeeze Theorem. ■

EXAMPLE 11 For what values of r is the sequence $\{r^n\}$ convergent?

SOLUTION We know from Section 3.4 and the graphs of the exponential functions in Section 6.2 (or Section 6.4*) that $\lim_{x \rightarrow \infty} a^x = \infty$ for $a > 1$ and $\lim_{x \rightarrow \infty} a^x = 0$ for $0 < a < 1$. Therefore, putting $a = r$ and using Theorem 3, we have

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

It is obvious that

$$\lim_{n \rightarrow \infty} 1^n = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} 0^n = 0$$

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n \rightarrow \infty} |r^n| = \lim_{n \rightarrow \infty} |r|^n = 0$$

and therefore $\lim_{n \rightarrow \infty} r^n = 0$ by Theorem 6. If $r \leq -1$, then $\{r^n\}$ diverges as in Example 7. Figure 11 shows the graphs for various values of r . (The case $r = -1$ is shown in Figure 8.)

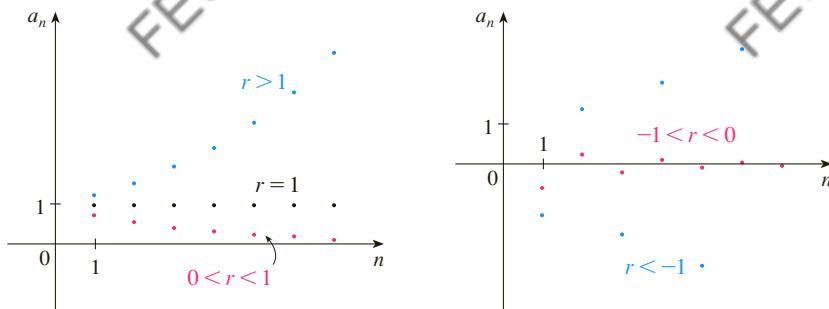


FIGURE 11
The sequence $a_n = r^n$

The results of Example 11 are summarized for future use as follows.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \dots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

EXAMPLE 12 The sequence $\left\{ \frac{3}{n+5} \right\}$ is decreasing because

The right side is smaller because it has a larger denominator.

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

and so $a_n > a_{n+1}$ for all $n \geq 1$. ■

EXAMPLE 13 Show that the sequence $a_n = \frac{n}{n^2 + 1}$ is decreasing.

SOLUTION 1 We must show that $a_{n+1} < a_n$, that is,

$$\frac{n+1}{(n+1)^2 + 1} < \frac{n}{n^2 + 1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\begin{aligned} \frac{n+1}{(n+1)^2 + 1} < \frac{n}{n^2 + 1} &\iff (n+1)(n^2 + 1) < n[(n+1)^2 + 1] \\ &\iff n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\ &\iff 1 < n^2 + n \end{aligned}$$

Since $n \geq 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore $a_{n+1} < a_n$ and so $\{a_n\}$ is decreasing.

SOLUTION 2 Consider the function $f(x) = \frac{x}{x^2 + 1}$:

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} < 0 \quad \text{whenever } x^2 > 1$$

Thus f is decreasing on $(1, \infty)$ and so $f(n) > f(n+1)$. Therefore $\{a_n\}$ is decreasing. ■

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

It is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = n/(n+1)$ is bounded because $0 < a_n < 1$ for all n .

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq 1$ but is divergent from Example 7] and not every

monotonic sequence is convergent ($a_n = n \rightarrow \infty$). But if a sequence is both bounded *and* monotonic, then it must be convergent. This fact is proved as Theorem 12, but intuitively you can understand why it is true by looking at Figure 12. If $\{a_n\}$ is increasing and $a_n \leq M$ for all n , then the terms are forced to crowd together and approach some number L .

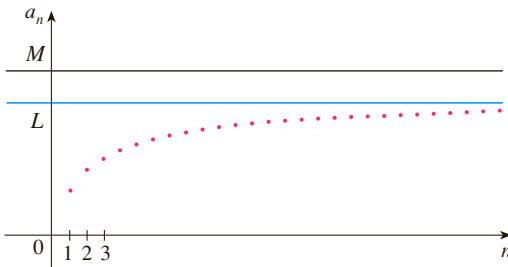


FIGURE 12

The proof of Theorem 12 is based on the **Completeness Axiom** for the set \mathbb{R} of real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leq M$ for all x in S), then S has a **least upper bound** b . (This means that b is an upper bound for S , but if M is any other upper bound, then $b \leq M$.) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

PROOF Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n \mid n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L . Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon \quad \text{for some integer } N$$

But the sequence is increasing so $a_n \geq a_N$ for every $n > N$. Thus if $n > N$, we have

$$a_n > L - \varepsilon$$

so

$$0 \leq L - a_n < \varepsilon$$

since $a_n \leq L$. Thus

$$|L - a_n| < \varepsilon \quad \text{whenever } n > N$$

so $\lim_{n \rightarrow \infty} a_n = L$.

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing. ■

The proof of Theorem 12 shows that a sequence that is increasing and bounded above is convergent. (Likewise, a decreasing sequence that is bounded below is convergent.) This fact is used many times in dealing with infinite series.

EXAMPLE 14 Investigate the sequence $\{a_n\}$ defined by the *recurrence relation*

$$a_1 = 2 \quad a_{n+1} = \frac{1}{2}(a_n + 6) \quad \text{for } n = 1, 2, 3, \dots$$

SOLUTION We begin by computing the first several terms:

$$\begin{array}{lll} a_1 = 2 & a_2 = \frac{1}{2}(2 + 6) = 4 & a_3 = \frac{1}{2}(4 + 6) = 5 \\ a_4 = \frac{1}{2}(5 + 6) = 5.5 & a_5 = 5.75 & a_6 = 5.875 \\ a_7 = 5.9375 & a_8 = 5.96875 & a_9 = 5.984375 \end{array}$$

Mathematical induction is often used in dealing with recursive sequences. See page 99 for a discussion of the Principle of Mathematical Induction.

These initial terms suggest that the sequence is increasing and the terms are approaching 6. To confirm that the sequence is increasing, we use mathematical induction to show that $a_{n+1} > a_n$ for all $n \geq 1$. This is true for $n = 1$ because $a_2 = 4 > a_1$. If we assume that it is true for $n = k$, then we have

$$a_{k+1} > a_k$$

$$\text{so} \quad a_{k+1} + 6 > a_k + 6$$

$$\text{and} \quad \frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$$

Thus

$$a_{k+2} > a_{k+1}$$

We have deduced that $a_{n+1} > a_n$ is true for $n = k + 1$. Therefore the inequality is true for all n by induction.

Next we verify that $\{a_n\}$ is bounded by showing that $a_n < 6$ for all n . (Since the sequence is increasing, we already know that it has a lower bound: $a_n \geq a_1 = 2$ for all n .) We know that $a_1 < 6$, so the assertion is true for $n = 1$. Suppose it is true for $n = k$. Then

$$a_k < 6$$

$$\text{so} \quad a_k + 6 < 12$$

$$\text{and} \quad \frac{1}{2}(a_k + 6) < \frac{1}{2}(12) = 6$$

Thus

$$a_{k+1} < 6$$

This shows, by mathematical induction, that $a_n < 6$ for all n .

Since the sequence $\{a_n\}$ is increasing and bounded, Theorem 12 guarantees that it has a limit. The theorem doesn't tell us what the value of the limit is. But now that we know $L = \lim_{n \rightarrow \infty} a_n$ exists, we can use the given recurrence relation to write

$$\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2}(a_n + 6) = \frac{1}{2} \left(\lim_{n \rightarrow \infty} a_n + 6 \right) = \frac{1}{2}(L + 6)$$

A proof of this fact is requested in Exercise 70.

Since $a_n \rightarrow L$, it follows that $a_{n+1} \rightarrow L$ too (as $n \rightarrow \infty$, $n + 1 \rightarrow \infty$ also). So we have

$$L = \frac{1}{2}(L + 6)$$

Solving this equation for L , we get $L = 6$, as we predicted. ■

11.1 EXERCISES

1. (a) What is a sequence?
 (b) What does it mean to say that $\lim_{n \rightarrow \infty} a_n = 8$?
 (c) What does it mean to say that $\lim_{n \rightarrow \infty} a_n = \infty$?
 2. (a) What is a convergent sequence? Give two examples.
 (b) What is a divergent sequence? Give two examples.

3–12 List the first five terms of the sequence.

3. $a_n = \frac{2^n}{2n + 1}$

4. $a_n = \frac{n^2 - 1}{n^2 + 1}$

5. $a_n = \frac{(-1)^{n-1}}{5^n}$

6. $a_n = \cos \frac{n\pi}{2}$

7. $a_n = \frac{1}{(n + 1)!}$

8. $a_n = \frac{(-1)^n n}{n! + 1}$

9. $a_1 = 1, a_{n+1} = 5a_n - 3$

10. $a_1 = 6, a_{n+1} = \frac{a_n}{n}$

11. $a_1 = 2, a_{n+1} = \frac{a_n}{1 + a_n}$

12. $a_1 = 2, a_2 = 1, a_{n+1} = a_n - a_{n-1}$

13–18 Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

13. $\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots \right\}$

14. $\left\{ 4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, \dots \right\}$

15. $\left\{ -3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \dots \right\}$

16. $\{5, 8, 11, 14, 17, \dots\}$

17. $\left\{ \frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \dots \right\}$

18. $\{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$

19–22 Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

19. $a_n = \frac{3n}{1 + 6n}$

20. $a_n = 2 + \frac{(-1)^n}{n}$

21. $a_n = 1 + \left(-\frac{1}{2}\right)^n$

22. $a_n = 1 + \frac{10^n}{9^n}$

23–56 Determine whether the sequence converges or diverges. If it converges, find the limit.

23. $a_n = \frac{3 + 5n^2}{n + n^2}$

24. $a_n = \frac{3 + 5n^2}{1 + n}$

25. $a_n = \frac{n^4}{n^3 - 2n}$

26. $a_n = 2 + (0.86)^n$

27. $a_n = 3^n 7^{-n}$

28. $a_n = \frac{3\sqrt{n}}{\sqrt{n} + 2}$

29. $a_n = e^{-1/\sqrt{n}}$

30. $a_n = \frac{4^n}{1 + 9^n}$

31. $a_n = \sqrt{\frac{1 + 4n^2}{1 + n^2}}$

32. $a_n = \cos\left(\frac{n\pi}{n + 1}\right)$

33. $a_n = \frac{n^2}{\sqrt{n^3 + 4n}}$

34. $a_n = e^{2n/(n+2)}$

35. $a_n = \frac{(-1)^n}{2\sqrt{n}}$

36. $a_n = \frac{(-1)^{n+1} n}{n + \sqrt{n}}$

37. $\left\{ \frac{(2n - 1)!}{(2n + 1)!} \right\}$

38. $\left\{ \frac{\ln n}{\ln 2n} \right\}$

39. $\{\sin n\}$

40. $a_n = \frac{\tan^{-1} n}{n}$

41. $\{n^2 e^{-n}\}$

42. $a_n = \ln(n + 1) - \ln n$

43. $a_n = \frac{\cos^2 n}{2^n}$

44. $a_n = \sqrt[3]{2^{1+3n}}$

45. $a_n = n \sin(1/n)$

46. $a_n = 2^{-n} \cos n\pi$

47. $a_n = \left(1 + \frac{2}{n}\right)^n$

48. $a_n = \sqrt[n]{n}$

49. $a_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$

50. $a_n = \frac{(\ln n)^2}{n}$

51. $a_n = \arctan(\ln n)$

52. $a_n = n - \sqrt{n + 1} \sqrt{n + 3}$

53. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$

54. $\left\{ \frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots \right\}$

55. $a_n = \frac{n!}{2^n}$

56. $a_n = \frac{(-3)^n}{n!}$

 57–63 Use a graph of the sequence to decide whether the sequence is convergent or divergent. If the sequence is convergent, guess the value of the limit from the graph and then prove your guess. (See the margin note on page 739 for advice on graphing sequences.)

57. $a_n = (-1)^n \frac{n}{n+1}$

58. $a_n = \frac{\sin n}{n}$

59. $a_n = \arctan\left(\frac{n^2}{n^2 + 4}\right)$

60. $a_n = \sqrt[n]{3^n + 5^n}$

61. $a_n = \frac{n^2 \cos n}{1 + n^2}$

62. $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$

63. $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n}$

64. (a) Determine whether the sequence defined as follows is convergent or divergent:

$$a_1 = 1 \quad a_{n+1} = 4 - a_n \quad \text{for } n \geq 1$$

(b) What happens if the first term is $a_1 = 2$?

65. If \$1000 is invested at 6% interest, compounded annually, then after n years the investment is worth $a_n = 1000(1.06)^n$ dollars.

- (a) Find the first five terms of the sequence $\{a_n\}$.
 (b) Is the sequence convergent or divergent? Explain.

66. If you deposit \$100 at the end of every month into an account that pays 3% interest per year compounded monthly, the amount of interest accumulated after n months is given by the sequence

$$I_n = 100 \left(\frac{1.0025^n - 1}{0.0025} - n \right)$$

- (a) Find the first six terms of the sequence.
 (b) How much interest will you have earned after two years?

67. A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month and the farmer harvests 300 catfish per month.

- (a) Show that the catfish population P_n after n months is given recursively by

$$P_n = 1.08P_{n-1} - 300 \quad P_0 = 5000$$

- (b) How many catfish are in the pond after six months?

68. Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence.

69. For what values of r is the sequence $\{nr^n\}$ convergent?

70. (a) If $\{a_n\}$ is convergent, show that

$$\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n$$

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \geq 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

71. Suppose you know that $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

72–78 Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

72. $a_n = \cos n$

73. $a_n = \frac{1}{2n+3}$

74. $a_n = \frac{1-n}{2+n}$

75. $a_n = n(-1)^n$

76. $a_n = 2 + \frac{(-1)^n}{n}$

77. $a_n = 3 - 2ne^{-n}$

78. $a_n = n^3 - 3n + 3$

79. Find the limit of the sequence

$$\left\{ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots \right\}$$

80. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

- (a) By induction or otherwise, show that $\{a_n\}$ is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that $\lim_{n \rightarrow \infty} a_n$ exists.

- (b) Find $\lim_{n \rightarrow \infty} a_n$.

81. Show that the sequence defined by

$$a_1 = 1 \quad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n . Deduce that $\{a_n\}$ is convergent and find its limit.

82. Show that the sequence defined by

$$a_1 = 2 \quad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

- 83.** (a) Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the n th month? Show that the answer is f_n , where $\{f_n\}$ is the Fibonacci sequence defined in Example 3(c).

- (b) Let $a_n = f_{n+1}/f_n$ and show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit.

- 84.** (a) Let $a_1 = a$, $a_2 = f(a)$, $a_3 = f(a_2) = f(f(a))$, \dots , $a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n \rightarrow \infty} a_n = L$, show that $f(L) = L$.
- (b) Illustrate part (a) by taking $f(x) = \cos x$, $a = 1$, and estimating the value of L to five decimal places.

- 85.** (a) Use a graph to guess the value of the limit

$$\lim_{n \rightarrow \infty} \frac{n^5}{n!}$$

- (b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.001$ in Definition 2.

- 86.** Use Definition 2 directly to prove that $\lim_{n \rightarrow \infty} r^n = 0$ when $|r| < 1$.

- 87.** Prove Theorem 6.

[Hint: Use either Definition 2 or the Squeeze Theorem.]

- 88.** Prove Theorem 7.

- 89.** Prove that if $\lim_{n \rightarrow \infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n \rightarrow \infty} (a_n b_n) = 0$.

- 90.** Let $a_n = \left(1 + \frac{1}{n}\right)^n$.

- (a) Show that if $0 \leq a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n$$

- (b) Deduce that $b^n[(n + 1)a - nb] < a^{n+1}$.
- (c) Use $a = 1 + 1/(n + 1)$ and $b = 1 + 1/n$ in part (b) to show that $\{a_n\}$ is increasing.
- (d) Use $a = 1$ and $b = 1 + 1/(2n)$ in part (b) to show that $a_{2n} < 4$.
- (e) Use parts (c) and (d) to show that $a_n < 4$ for all n .
- (f) Use Theorem 12 to show that $\lim_{n \rightarrow \infty} (1 + 1/n)^n$ exists. (The limit is e . See Equation 6.4.9 or 6.4*.9.)

- 91.** Let a and b be positive numbers with $a > b$. Let a_1 be their arithmetic mean and b_1 their geometric mean:

$$a_1 = \frac{a + b}{2} \quad b_1 = \sqrt{ab}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \quad b_{n+1} = \sqrt{a_n b_n}$$

- (a) Use mathematical induction to show that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

- (b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.
- (c) Show that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean** of the numbers a and b .

- 92.** (a) Show that if $\lim_{n \rightarrow \infty} a_{2n} = L$ and $\lim_{n \rightarrow \infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n \rightarrow \infty} a_n = L$.
- (b) If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1 + a_n}$$

find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n \rightarrow \infty} a_n = \sqrt{2}$. This gives the **continued fraction expansion**

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}$$

- 93.** The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.

- (a) Show that if $\{p_n\}$ is convergent, then the only possible values for its limit are 0 and $b - a$.
- (b) Show that $p_{n+1} < (b/a)p_n$.
- (c) Use part (b) to show that if $a > b$, then $\lim_{n \rightarrow \infty} p_n = 0$; in other words, the population dies out.
- (d) Now assume that $a < b$. Show that if $p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a$. Show also that if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a$. Deduce that if $a < b$, then $\lim_{n \rightarrow \infty} p_n = b - a$.

LABORATORY PROJECT**CAS LOGISTIC SEQUENCES**

A sequence that arises in ecology as a model for population growth is defined by the **logistic difference equation**

$$p_{n+1} = kp_n(1 - p_n)$$

where p_n measures the size of the population of the n th generation of a single species. To keep the numbers manageable, p_n is a fraction of the maximal size of the population, so $0 \leq p_n \leq 1$. Notice that the form of this equation is similar to the logistic differential equation in Section 9.4. The discrete model—with sequences instead of continuous functions—is preferable for modeling insect populations, where mating and death occur in a periodic fashion.

An ecologist is interested in predicting the size of the population as time goes on, and asks these questions: Will it stabilize at a limiting value? Will it change in a cyclical fashion? Or will it exhibit random behavior?

Write a program to compute the first n terms of this sequence starting with an initial population p_0 , where $0 < p_0 < 1$. Use this program to do the following.

1. Calculate 20 or 30 terms of the sequence for $p_0 = \frac{1}{2}$ and for two values of k such that $1 < k < 3$. Graph each sequence. Do the sequences appear to converge? Repeat for a different value of p_0 between 0 and 1. Does the limit depend on the choice of p_0 ? Does it depend on the choice of k ?
2. Calculate terms of the sequence for a value of k between 3 and 3.4 and plot them. What do you notice about the behavior of the terms?
3. Experiment with values of k between 3.4 and 3.5. What happens to the terms?
4. For values of k between 3.6 and 4, compute and plot at least 100 terms and comment on the behavior of the sequence. What happens if you change p_0 by 0.001? This type of behavior is called *chaotic* and is exhibited by insect populations under certain conditions.

11.2 Series

The current record for computing a decimal approximation for π was obtained by Shigeru Kondo and Alexander Yee in 2011 and contains more than 10 trillion decimal places.

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ \dots$$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \dots$$

where the three dots (\dots) indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π .

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

1

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

Does it make sense to talk about the sum of infinitely many terms?

It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms we get the cumulative sums $1, 3, 6, 10, 15, 21, \dots$ and, after the n th term, we get $n(n + 1)/2$, which becomes very large as n increases.

However, if we start to add the terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$$

we get $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \dots, 1 - 1/2^n, \dots$. The table shows that as we add more and more terms, these *partial sums* become closer and closer to 1. (See also Figure 11 in *A Preview of Calculus*, page 6.) In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1. So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series (1) has a sum. We consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n \rightarrow \infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\sum a_n$.

2 Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s . Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

Compare with the improper integral

$$\int_1^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_1^t f(x) dx$$

To find this integral we integrate from 1 to t and then let $t \rightarrow \infty$. For a series, we sum from 1 to n and then let $n \rightarrow \infty$.

EXAMPLE 1 Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{2n}{3n+5} = \lim_{n \rightarrow \infty} \frac{2}{3 + \frac{5}{n}} = \frac{2}{3}$$

In Example 1 we were *given* an expression for the sum of the first n terms, but it's usually not easy to *find* such an expression. In Example 2, however, we look at a famous series for which we *can* find an explicit formula for s_n .

EXAMPLE 2 An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio r** . (We have already considered the special case where $a = \frac{1}{2}$ and $r = \frac{1}{2}$ on page 748.)

If $r = 1$, then $s_n = a + a + \cdots + a = na \rightarrow \pm\infty$. Since $\lim_{n \rightarrow \infty} s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

3

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

If $-1 < r < 1$, we know from (11.1.9) that $r^n \rightarrow 0$ as $n \rightarrow \infty$, so

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n \rightarrow \infty} r^n = \frac{a}{1 - r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1 - r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by (11.1.9) and so, by Equation 3, $\lim_{n \rightarrow \infty} s_n$ does not exist. Therefore the geometric series diverges in those cases.

Figure 1 provides a geometric demonstration of the result in Example 2. If the triangles are constructed as shown and s is the sum of the series, then, by similar triangles,

$$\frac{s}{a} = \frac{a}{a - ar} \quad \text{so} \quad s = \frac{a}{1 - r}$$

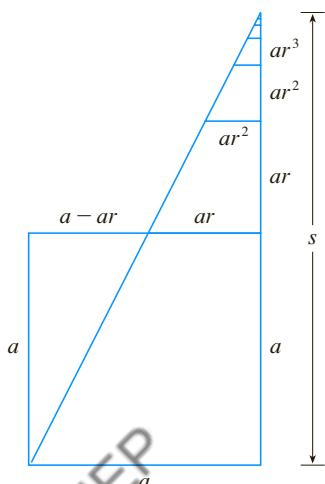


FIGURE 1

We summarize the results of Example 2 as follows.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

In words: The sum of a convergent geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

EXAMPLE 3 Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$$

SOLUTION The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent by (4) and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots = \frac{5}{1 - (-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3$$

What do we really mean when we say that the sum of the series in Example 3 is 3? Of course, we can't literally add an infinite number of terms, one by one. But, according to Definition 2, the total sum is the limit of the sequence of partial sums. So, by taking the sum of sufficiently many terms, we can get as close as we like to the number 3. The table shows the first ten partial sums s_n and the graph in Figure 2 shows how the sequence of partial sums approaches 3.

n	s_n
1	5.000000
2	1.666667
3	3.888889
4	2.407407
5	3.395062
6	2.736626
7	3.175583
8	2.882945
9	3.078037
10	2.947975

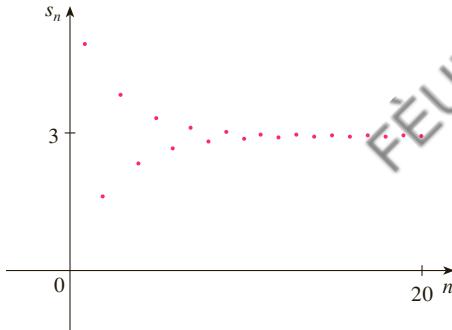


FIGURE 2

EXAMPLE 4 Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

SOLUTION Let's rewrite the n th term of the series in the form ar^{n-1} :

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

We recognize this series as a geometric series with $a = 4$ and $r = \frac{4}{3}$. Since $r > 1$, the series diverges by (4).

Another way to identify a and r is to write out the first few terms:

$$4 + \frac{16}{3} + \frac{64}{9} + \dots$$

EXAMPLE 5 A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the n th day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL.

(a) Find the concentration after three days.

- (b) What is the concentration after the n th dose?
 (c) What is the limiting concentration?

SOLUTION

(a) Just before the daily dose of medication is administered, the concentration is reduced to 30% of the preceding day's concentration, that is, $0.3C_n$. With the new dose, the concentration is increased by 0.2 mg/mL and so

$$C_{n+1} = 0.2 + 0.3C_n$$

Starting with $C_0 = 0$ and putting $n = 0, 1, 2$ into this equation, we get

$$C_1 = 0.2 + 0.3C_0 = 0.2$$

$$C_2 = 0.2 + 0.3C_1 = 0.2 + 0.2(0.3) = 0.26$$

$$C_3 = 0.2 + 0.3C_2 = 0.2 + 0.2(0.3) + 0.2(0.3)^2 = 0.278$$

The concentration after three days is 0.278 mg/mL.

- (b) After the n th dose the concentration is

$$C_n = 0.2 + 0.2(0.3) + 0.2(0.3)^2 + \cdots + 0.2(0.3)^{n-1}$$

This is a finite geometric series with $a = 0.2$ and $r = 0.3$, so by Formula 3 we have

$$C_n = \frac{0.2[1 - (0.3)^n]}{1 - 0.3} = \frac{2}{7}[1 - (0.3)^n] \text{ mg/mL}$$

- (c) Because $0.3 < 1$, we know that $\lim_{n \rightarrow \infty} (0.3)^n = 0$. So the limiting concentration is

$$\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} \frac{2}{7}[1 - (0.3)^n] = \frac{2}{7}(1 - 0) = \frac{2}{7} \text{ mg/mL}$$

EXAMPLE 6 Write the number $2.\overline{317} = 2.3171717\dots$ as a ratio of integers.

SOLUTION

$$2.3171717\dots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term we have a geometric series with $a = 17/10^3$ and $r = 1/10^2$. Therefore

$$\begin{aligned} 2.\overline{317} &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

EXAMPLE 7 Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

SOLUTION Notice that this series starts with $n = 0$ and so the first term is $x^0 = 1$. (With series, we adopt the convention that $x^0 = 1$ even when $x = 0$.)

TEC Module 11.2 explores a series that depends on an angle θ in a triangle and enables you to see how rapidly the series converges when θ varies.

Thus

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

This is a geometric series with $a = 1$ and $r = x$. Since $|r| = |x| < 1$, it converges and (4) gives

$$5 \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

EXAMPLE 8 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

(see Section 7.4). Thus we have

$$\begin{aligned} s_n &= \sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1 - 0 = 1$$

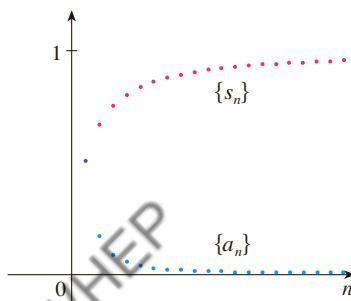
Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Notice that the terms cancel in pairs. This is an example of a **telescoping sum**: Because of all the cancellations, the sum collapses (like a pirate's collapsing telescope) into just two terms.

Figure 3 illustrates Example 8 by showing the graphs of the sequence of terms $a_n = 1/[n(n+1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \rightarrow 0$ and $s_n \rightarrow 1$. See Exercises 78 and 79 for two geometric interpretations of Example 8.

FIGURE 3



EXAMPLE 9 Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is divergent.

SOLUTION For this particular series it's convenient to consider the partial sums $s_2, s_4, s_8, s_{16}, s_{32}, \dots$ and show that they become large.

$$\begin{aligned} s_2 &= 1 + \frac{1}{2} \\ s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} \\ s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \\ s_{16} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2} \end{aligned}$$

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that $s_{2^n} \rightarrow \infty$ as $n \rightarrow \infty$ and so $\{s_n\}$ is divergent. Therefore the harmonic series diverges. ■

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.

PROOF Let $s_n = a_1 + a_2 + \dots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n \rightarrow \infty} s_n = s$. Since $n - 1 \rightarrow \infty$ as $n \rightarrow \infty$, we also have $\lim_{n \rightarrow \infty} s_{n-1} = s$. Therefore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = s - s = 0$$

NOTE 1 With any series $\sum a_n$ we associate two sequences: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and, as Theorem 6 asserts, the limit of the sequence $\{a_n\}$ is 0.



NOTE 2 The converse of Theorem 6 is not true in general. If $\lim_{n \rightarrow \infty} a_n = 0$, we cannot conclude that $\sum a_n$ is convergent. Observe that for the harmonic series $\sum 1/n$ we have $a_n = 1/n \rightarrow 0$ as $n \rightarrow \infty$, but we showed in Example 9 that $\sum 1/n$ is divergent.

7 Test for Divergence If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the

series $\sum_{n=1}^{\infty} a_n$ is divergent.

The Test for Divergence follows from Theorem 6 because, if the series is not divergent, then it is convergent, and so $\lim_{n \rightarrow \infty} a_n = 0$.

EXAMPLE 10 Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$ diverges.

SOLUTION

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{5n^2 + 4} = \lim_{n \rightarrow \infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence. ■

NOTE 3 If we find that $\lim_{n \rightarrow \infty} a_n \neq 0$, we know that Σa_n is divergent. If we find that $\lim_{n \rightarrow \infty} a_n = 0$, we know *nothing* about the convergence or divergence of Σa_n . Remember the warning in Note 2: if $\lim_{n \rightarrow \infty} a_n = 0$, the series Σa_n might converge or it might diverge.

8 Theorem If Σa_n and Σb_n are convergent series, then so are the series Σca_n (where c is a constant), $\Sigma(a_n + b_n)$, and $\Sigma(a_n - b_n)$, and

$$(i) \quad \sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$$

$$(ii) \quad \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$(iii) \quad \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

These properties of convergent series follow from the corresponding Limit Laws for Sequences in Section 11.1. For instance, here is how part (ii) of Theorem 8 is proved:

Let

$$s_n = \sum_{i=1}^n a_i \quad s = \sum_{n=1}^{\infty} a_n \quad t_n = \sum_{i=1}^n b_i \quad t = \sum_{n=1}^{\infty} b_n$$

The n th partial sum for the series $\Sigma(a_n + b_n)$ is

$$u_n = \sum_{i=1}^n (a_i + b_i)$$

and, using Equation 4.2.10, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (a_i + b_i) = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n a_i + \sum_{i=1}^n b_i \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i + \lim_{n \rightarrow \infty} \sum_{i=1}^n b_i \\ &= \lim_{n \rightarrow \infty} s_n + \lim_{n \rightarrow \infty} t_n = s + t \end{aligned}$$

Therefore $\Sigma (a_n + b_n)$ is convergent and its sum is

$$\sum_{n=1}^{\infty} (a_n + b_n) = s + t = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

EXAMPLE 11 Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$.

SOLUTION The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

In Example 8 we found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So, by Theorem 8, the given series is convergent and

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) &= 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 3 \cdot 1 + 1 = 4 \end{aligned}$$

NOTE 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

it follows that the entire series $\sum_{n=1}^{\infty} n/(n^3 + 1)$ is convergent. Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^N a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

11.2 EXERCISES

1. (a) What is the difference between a sequence and a series?
(b) What is a convergent series? What is a divergent series?
2. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.

3–4 Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given.

3. $s_n = 2 - 3(0.8)^n$

4. $s_n = \frac{n^2 - 1}{4n^2 + 1}$

5–8 Calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

5. $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^2}$

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

7. $\sum_{n=1}^{\infty} \sin n$

8. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

9–14 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

9. $\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$

10. $\sum_{n=1}^{\infty} \cos n$

11. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 4}}$

12. $\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$

13. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

14. $\sum_{n=1}^{\infty} \left(\sin \frac{1}{n} - \sin \frac{1}{n+1} \right)$

15. Let $a_n = \frac{2n}{3n+1}$.

- (a) Determine whether $\{a_n\}$ is convergent.
 (b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

16. (a) Explain the difference between

$$\sum_{i=1}^n a_i \quad \text{and} \quad \sum_{j=1}^n a_j$$

(b) Explain the difference between

$$\sum_{i=1}^n a_i \quad \text{and} \quad \sum_{i=1}^n a_j$$

17–26 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

17. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \dots$

18. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \dots$

19. $10 - 2 + 0.4 - 0.08 + \dots$

20. $2 + 0.5 + 0.125 + 0.03125 + \dots$

21. $\sum_{n=1}^{\infty} 12(0.73)^{n-1}$

22. $\sum_{n=1}^{\infty} \frac{5}{\pi^n}$

23. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

24. $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$

25. $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

26. $\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$

27–42 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

27. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \dots$

28. $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \dots$

29. $\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$

30. $\sum_{k=1}^{\infty} \frac{k^2}{k^2 - 2k + 5}$

31. $\sum_{n=1}^{\infty} 3^{n+1} 4^{-n}$

32. $\sum_{n=1}^{\infty} [(-0.2)^n + (0.6)^{n-1}]$

33. $\sum_{n=1}^{\infty} \frac{1}{4 + e^{-n}}$

34. $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n}$

35. $\sum_{k=1}^{\infty} (\sin 100)^k$

36. $\sum_{n=1}^{\infty} \frac{1}{1 + \left(\frac{2}{3}\right)^n}$

37. $\sum_{n=1}^{\infty} \ln \left(\frac{n^2 + 1}{2n^2 + 1} \right)$

38. $\sum_{k=0}^{\infty} (\sqrt{2})^{-k}$

39. $\sum_{n=1}^{\infty} \arctan n$

40. $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$

41. $\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)} \right)$

42. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

43–48 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 8). If it is convergent, find its sum.

43. $\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$

44. $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

45. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

46. $\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$

47. $\sum_{n=1}^{\infty} (e^{1/n} - e^{1/(n+1)})$

48. $\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$

49. Let $x = 0.99999 \dots$

- (a) Do you think that $x < 1$ or $x = 1$?
 (b) Sum a geometric series to find the value of x .
 (c) How many decimal representations does the number 1 have?
 (d) Which numbers have more than one decimal representation?

50. A sequence of terms is defined by

$$a_1 = 1 \quad a_n = (5 - n)a_{n-1}$$

Calculate $\sum_{n=1}^{\infty} a_n$.

51–56 Express the number as a ratio of integers.

51. $0.\overline{8} = 0.8888 \dots$

52. $0.\overline{46} = 0.46464646 \dots$

53. $2.\overline{516} = 2.516516516 \dots$

54. $10.\overline{135} = 10.13535353 \dots$

55. $1.234\overline{567}$

56. $5.\overline{71358}$

57–63 Find the values of x for which the series converges. Find the sum of the series for those values of x .

57. $\sum_{n=1}^{\infty} (-5)^n x^n$

58. $\sum_{n=1}^{\infty} (x+2)^n$

59. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

60. $\sum_{n=0}^{\infty} (-4)^n (x-5)^n$

61. $\sum_{n=0}^{\infty} \frac{2^n}{x^n}$

62. $\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$

63. $\sum_{n=0}^{\infty} e^{nx}$

- 64.** We have seen that the harmonic series is a divergent series whose terms approach 0. Show that

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$$

is another series with this property.

- CAS 65–66** Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

65. $\sum_{n=1}^{\infty} \frac{3n^2 + 3n + 1}{(n^2 + n)^3}$

66. $\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$

- 67.** If the n th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

- 68.** If the n th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

- 69.** A doctor prescribes a 100-mg antibiotic tablet to be taken every eight hours. Just before each tablet is taken, 20% of the drug remains in the body.

- (a) How much of the drug is in the body just after the second tablet is taken? After the third tablet?
 (b) If Q_n is the quantity of the antibiotic in the body just after the n th tablet is taken, find an equation that expresses Q_{n+1} in terms of Q_n .
 (c) What quantity of the antibiotic remains in the body in the long run?

- 70.** A patient is injected with a drug every 12 hours. Immediately before each injection the concentration of the drug has been reduced by 90% and the new dose increases the concentration by 1.5 mg/L.

- (a) What is the concentration after three doses?
 (b) If C_n is the concentration after the n th dose, find a formula for C_n as a function of n .
 (c) What is the limiting value of the concentration?

- 71.** A patient takes 150 mg of a drug at the same time every day. Just before each tablet is taken, 5% of the drug remains in the body.

- (a) What quantity of the drug is in the body after the third tablet? After the n th tablet?
 (b) What quantity of the drug remains in the body in the long run?

- 72.** After injection of a dose D of insulin, the concentration of insulin in a patient's system decays exponentially and so it can be written as De^{-at} , where t represents time in hours and a is a positive constant.

- (a) If a dose D is injected every T hours, write an expression for the sum of the residual concentrations just before the $(n + 1)$ st injection.

- (b) Determine the limiting pre-injection concentration.
 (c) If the concentration of insulin must always remain at or above a critical value C , determine a minimal dosage D in terms of C , a , and T .

- 73.** When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money that he or she receives. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save* and, of course, $c + s = 1$.

- (a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n .
 (b) Show that $\lim_{n \rightarrow \infty} S_n = kD$, where $k = 1/s$. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

Note: The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

- 74.** A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh , where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters.

- (a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.
 (b) Calculate the total time that the ball travels. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)
 (c) Suppose that each time the ball strikes the surface with velocity v it rebounds with velocity $-kv$, where $0 < k < 1$. How long will it take for the ball to come to rest?

- 75.** Find the value of c if

$$\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$$

- 76.** Find the value of c such that

$$\sum_{n=0}^{\infty} e^{nc} = 10$$

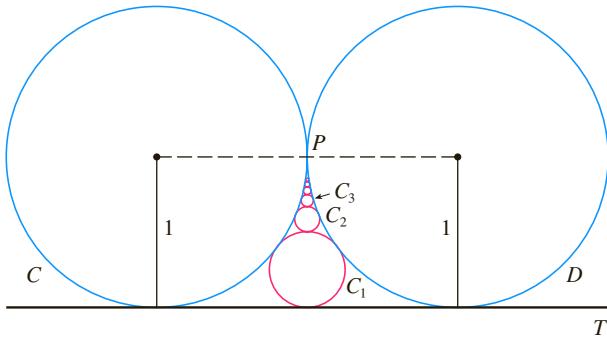
- 77.** In Example 9 we showed that the harmonic series is divergent. Here we outline another method, making use of the fact that $e^x > 1 + x$ for any $x > 0$. (See Exercise 6.2.109.)

If s_n is the n th partial sum of the harmonic series, show that $e^{s_n} > n + 1$. Why does this imply that the harmonic series is divergent?

- 78.** Graph the curves $y = x^n$, $0 \leq x \leq 1$, for $n = 0, 1, 2, 3, 4, \dots$ on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact, shown in Example 8, that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

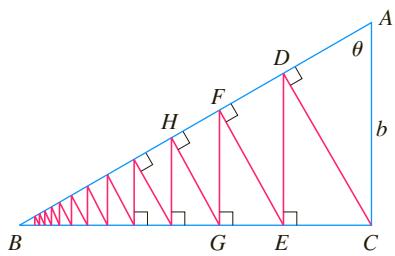
79. The figure shows two circles C and D of radius 1 that touch at P . The line T is a common tangent line; C_1 is the circle that touches C , D , and T ; C_2 is the circle that touches C , D , and C_1 ; C_3 is the circle that touches C , D , and C_2 . This procedure can be continued indefinitely and produces an infinite sequence of circles $\{C_n\}$. Find an expression for the diameter of C_n and thus provide another geometric demonstration of Example 8.



80. A right triangle ABC is given with $\angle A = \theta$ and $|AC| = b$. CD is drawn perpendicular to AB , DE is drawn perpendicular to BC , $EF \perp AB$, and this process is continued indefinitely, as shown in the figure. Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \dots$$

in terms of b and θ .



81. What is wrong with the following calculation?

$$0 = 0 + 0 + 0 + \dots$$

$$= (1 - 1) + (1 - 1) + (1 - 1) + \dots$$

$$= 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

$$= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots$$

$$= 1 + 0 + 0 + 0 + \dots = 1$$

(Guido Ubaldus thought that this proved the existence of God because “something has been created out of nothing.”)

82. Suppose that $\sum_{n=1}^{\infty} a_n$ ($a_n \neq 0$) is known to be a convergent series. Prove that $\sum_{n=1}^{\infty} 1/a_n$ is a divergent series.

83. Prove part (i) of Theorem 8.

84. If $\sum a_n$ is divergent and $c \neq 0$, show that $\sum ca_n$ is divergent.
85. If $\sum a_n$ is convergent and $\sum b_n$ is divergent, show that the series $\sum (a_n + b_n)$ is divergent. [Hint: Argue by contradiction.]
86. If $\sum a_n$ and $\sum b_n$ are both divergent, is $\sum (a_n + b_n)$ necessarily divergent?
87. Suppose that a series $\sum a_n$ has positive terms and its partial sums s_n satisfy the inequality $s_n \leq 1000$ for all n . Explain why $\sum a_n$ must be convergent.
88. The Fibonacci sequence was defined in Section 11.1 by the equations

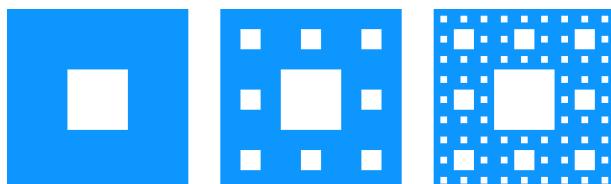
$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3$$

Show that each of the following statements is true.

$$(a) \frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}$$

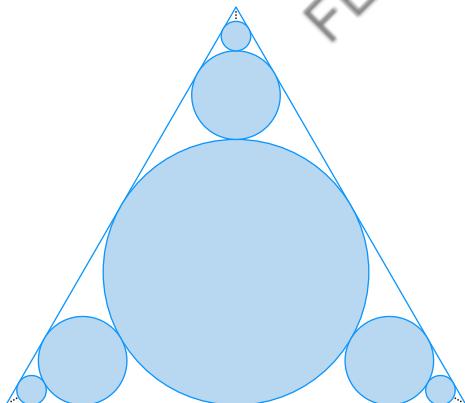
$$(b) \sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = 1 \quad (c) \sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = 2$$

89. The **Cantor set**, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. We start with the closed interval $[0, 1]$ and remove the open interval $(\frac{1}{3}, \frac{2}{3})$. That leaves the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in $[0, 1]$ after all those intervals have been removed.
- (a) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.
- (b) The **Sierpinski carpet** is a two-dimensional counterpart of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.



90. (a) A sequence $\{a_n\}$ is defined recursively by the equation $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \geq 3$, where a_1 and a_2 can be any real numbers. Experiment with various values of a_1 and a_2 and use your calculator to guess the limit of the sequence.

- (b) Find $\lim_{n \rightarrow \infty} a_n$ in terms of a_1 and a_2 by expressing $a_{n+1} - a_n$ in terms of $a_2 - a_1$ and summing a series.
- 91.** Consider the series $\sum_{n=1}^{\infty} n/(n + 1)!$.
- Find the partial sums s_1, s_2, s_3 , and s_4 . Do you recognize the denominators? Use the pattern to guess a formula for s_n .
 - Use mathematical induction to prove your guess.
 - Show that the given infinite series is convergent, and find its sum.
- 92.** In the figure at the right there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.



11.3 The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\sum 1/[n(n + 1)]$ because in each of those cases we could find a simple formula for the n th partial sum s_n . But usually it isn't easy to discover such a formula. Therefore, in the next few sections, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. (In some cases, however, our methods will enable us to find good estimates of the sum.) Our first test involves improper integrals.

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

n	$s_n = \sum_{i=1}^n \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

There's no simple formula for the sum s_n of the first n terms, but the computer-generated table of approximate values given in the margin suggests that the partial sums are approaching a number near 1.64 as $n \rightarrow \infty$ and so it looks as if the series is convergent.

We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = 1/x^2$ and rectangles that lie below the curve. The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.

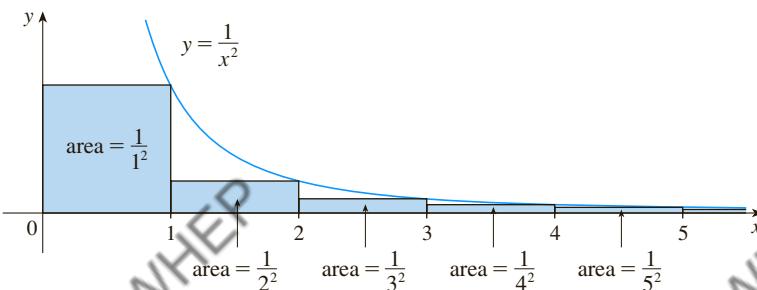


FIGURE 1

So the sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = 1/x^2$ for $x \geq 1$, which is the value of the integral $\int_1^{\infty} (1/x^2) dx$. In Section 7.8 we discovered that this improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^{\infty} \frac{1}{x^2} dx = 2$$

Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Monotonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 2$$

[The exact sum of this series was found by the Swiss mathematician Leonhard Euler (1707–1783) to be $\pi^2/6$, but the proof of this fact is quite difficult. (See Problem 6 in the Problems Plus following Chapter 15.)]

Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots$$

The table of values of s_n suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent. Again we use a picture for confirmation. Figure 2 shows the curve $y = 1/\sqrt{x}$, but this time we use rectangles whose tops lie *above* the curve.

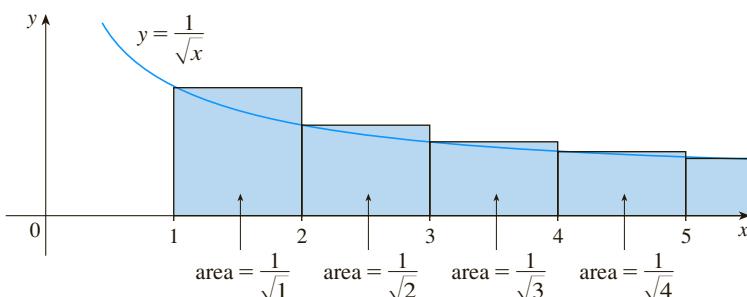


FIGURE 2

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = 1/\sqrt{x}$ at the *left* endpoint of the interval. So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve $y = 1/\sqrt{x}$ for $x \geq 1$, which is

equal to the integral $\int_1^\infty (1/\sqrt{x}) dx$. But we know from Section 7.8 that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite; that is, the series is divergent.

The same sort of geometric reasoning that we used for these two series can be used to prove the following test. (The proof is given at the end of this section.)

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^\infty a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x) dx$ is convergent. In other words:

- (i) If $\int_1^\infty f(x) dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.
- (ii) If $\int_1^\infty f(x) dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

NOTE When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^\infty \frac{1}{(n-3)^2} \quad \text{we use} \quad \int_4^\infty \frac{1}{(x-3)^2} dx$$

Also, it is not necessary that f be *always* decreasing. What is important is that f be *ultimately* decreasing, that is, decreasing for x larger than some number N . Then $\sum_{n=N}^\infty a_n$ is convergent, so $\sum_{n=1}^\infty a_n$ is convergent by Note 4 of Section 11.2.

EXAMPLE 1 Test the series $\sum_{n=1}^\infty \frac{1}{n^2 + 1}$ for convergence or divergence.

SOLUTION The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\begin{aligned} \int_1^\infty \frac{1}{x^2 + 1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \tan^{-1} x \Big|_1^t \\ &= \lim_{t \rightarrow \infty} \left(\tan^{-1} t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Thus $\int_1^\infty 1/(x^2 + 1) dx$ is a convergent integral and so, by the Integral Test, the series $\sum_{n=1}^\infty 1/(n^2 + 1)$ is convergent. ■

EXAMPLE 2 For what values of p is the series $\sum_{n=1}^\infty \frac{1}{n^p}$ convergent?

SOLUTION If $p < 0$, then $\lim_{n \rightarrow \infty} (1/n^p) = \infty$. If $p = 0$, then $\lim_{n \rightarrow \infty} (1/n^p) = 1$. In either case $\lim_{n \rightarrow \infty} (1/n^p) \neq 0$, so the given series diverges by the Test for Divergence (11.2.7).

If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We found in Chapter 7 [see (7.8.2)] that

$$\int_1^\infty \frac{1}{x^p} dx \text{ converges if } p > 1 \text{ and diverges if } p \leq 1$$

In order to use the Integral Test we need to be able to evaluate $\int_1^\infty f(x) dx$ and therefore we have to be able to find an antiderivative of f . Frequently this is difficult or impossible, so we need other tests for convergence too.

It follows from the Integral Test that the series $\sum 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leq 1$. (For $p = 1$, this series is the harmonic series discussed in Example 11.2.9.) ■

The series in Example 2 is called the ***p*-series**. It is important in the rest of this chapter, so we summarize the results of Example 2 for future reference as follows.

1 The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

EXAMPLE 3

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$$

is convergent because it is a *p*-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \dots$$

is divergent because it is a *p*-series with $p = \frac{1}{3} < 1$. ■

NOTE We should *not* infer from the Integral Test that the sum of the series is equal to the value of the integral. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_1^{\infty} \frac{1}{x^2} dx = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x) dx$$

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

SOLUTION The function $f(x) = (\ln x)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus $f'(x) < 0$ when $\ln x > 1$, that is, when $x > e$. It follows that f is decreasing when $x > e$ and so we can apply the Integral Test:

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \frac{(\ln t)^2}{2} = \infty \end{aligned}$$

Since this improper integral is divergent, the series $\sum (\ln n)/n$ is also divergent by the Integral Test.

■ Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\sum a_n$ is convergent and we now want to find an approximation to the sum s of the series. Of course, any partial sum s_n is an approximation to s because $\lim_{n \rightarrow \infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the **remainder**

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$$

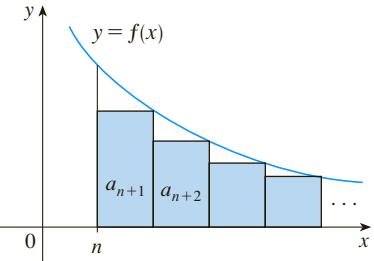


FIGURE 3

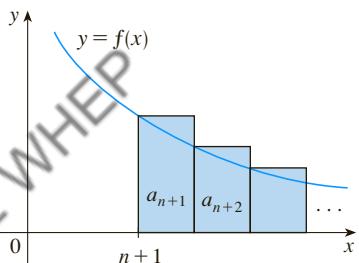


FIGURE 4

The remainder R_n is the error made when s_n , the sum of the first n terms, is used as an approximation to the total sum.

We use the same notation and ideas as in the Integral Test, assuming that f is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under $y = f(x)$ for $x > n$ in Figure 3, we see that

$$R_n = a_{n+1} + a_{n+2} + \dots \leq \int_n^\infty f(x) dx$$

Similarly, we see from Figure 4 that

$$R_n = a_{n+1} + a_{n+2} + \dots \geq \int_{n+1}^\infty f(x) dx$$

So we have proved the following error estimate.

2 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^\infty f(x) dx \leq R_n \leq \int_n^\infty f(x) dx$$

EXAMPLE 5

- (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
 (b) How many terms are required to ensure that the sum is accurate to within 0.0005?

SOLUTION In both parts (a) and (b) we need to know $\int_n^\infty f(x) dx$. With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have

$$\int_n^\infty \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{2x^2} \right]_n^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2} \right) = \frac{1}{2n^2}$$

- (a) Approximating the sum of the series by the 10th partial sum, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate in (2), we have

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3} dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \leq 0.0005$. Since

$$R_n \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}$$

we want

$$\frac{1}{2n^2} < 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \quad \text{or} \quad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005. ■

If we add s_n to each side of the inequalities in (2), we get

3

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx$$

because $s_n + R_n = s$. The inequalities in (3) give a lower bound and an upper bound for s . They provide a more accurate approximation to the sum of the series than the partial sum s_n does.

EXAMPLE 6 Use (3) with $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

SOLUTION The inequalities in (3) become

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3} dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3} dx$$

From Example 5 we know that

$$\int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}$$

so

$$s_{10} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leq s \leq 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval. So

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

If we compare Example 6 with Example 5, we see that the improved estimate in (3) can be much better than the estimate $s \approx s_n$. To make the error smaller than 0.0005 we had to use 32 terms in Example 5 but only 10 terms in Example 6.

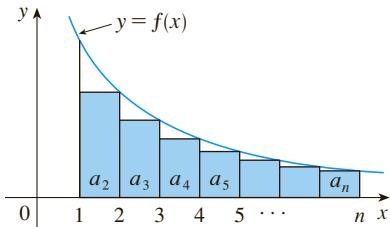


FIGURE 5

Proof of the Integral Test

We have already seen the basic idea behind the proof of the Integral Test in Figures 1 and 2 for the series $\sum 1/n^2$ and $\sum 1/\sqrt{n}$. For the general series $\sum a_n$, look at Figures 5 and 6. The area of the first shaded rectangle in Figure 5 is the value of f at the right endpoint of $[1, 2]$, that is, $f(2) = a_2$. So, comparing the areas of the shaded rectangles with the area under $y = f(x)$ from 1 to n , we see that

4

$$a_2 + a_3 + \dots + a_n \leq \int_1^n f(x) dx$$

(Notice that this inequality depends on the fact that f is decreasing.) Likewise, Figure 6 shows that

5

$$\int_1^n f(x) dx \leq a_1 + a_2 + \dots + a_{n-1}$$

(i) If $\int_1^\infty f(x) dx$ is convergent, then (4) gives

$$\sum_{i=2}^n a_i \leq \int_1^n f(x) dx \leq \int_1^\infty f(x) dx$$

since $f(x) \geq 0$. Therefore

$$s_n = a_1 + \sum_{i=2}^n a_i \leq a_1 + \int_1^\infty f(x) dx = M, \text{ say}$$

Since $s_n \leq M$ for all n , the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since $a_{n+1} = f(n+1) \geq 0$. Thus $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem (11.1.12). This means that $\sum a_n$ is convergent.

(ii) If $\int_1^\infty f(x) dx$ is divergent, then $\int_1^n f(x) dx \rightarrow \infty$ as $n \rightarrow \infty$ because $f(x) \geq 0$. But (5) gives

$$\int_1^n f(x) dx \leq \sum_{i=1}^{n-1} a_i = s_{n-1}$$

and so $s_{n-1} \rightarrow \infty$. This implies that $s_n \rightarrow \infty$ and so $\sum a_n$ diverges. ■

11.3 EXERCISES

1. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}} dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \geq 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x) dx$$

$$\sum_{i=1}^5 a_i$$

$$\sum_{i=2}^6 a_i$$

- 3–8 Use the Integral Test to determine whether the series is convergent or divergent.

$$3. \sum_{n=1}^{\infty} n^{-3}$$

$$4. \sum_{n=1}^{\infty} n^{-0.3}$$

$$5. \sum_{n=1}^{\infty} \frac{2}{5n-1}$$

$$6. \sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$$

$$7. \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$$8. \sum_{n=1}^{\infty} n^2 e^{-n^3}$$

9–26 Determine whether the series is convergent or divergent.

9. $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

10. $\sum_{n=3}^{\infty} n^{-0.9999}$

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \dots$

12. $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \dots$

13. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \dots$

14. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \dots$

15. $\sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2}$

16. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1 + n^{3/2}}$

17. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

18. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 2}$

19. $\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4}$

20. $\sum_{n=3}^{\infty} \frac{3n - 4}{n^2 - 2n}$

21. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

22. $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

23. $\sum_{k=1}^{\infty} k e^{-k}$

24. $\sum_{k=1}^{\infty} k e^{-k^2}$

25. $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3}$

26. $\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$

27–28 Explain why the Integral Test can't be used to determine whether the series is convergent.

27. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

28. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{1 + n^2}$

29–32 Find the values of p for which the series is convergent.

29. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

30. $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}$

31. $\sum_{n=1}^{\infty} n(1 + n^2)^p$

32. $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

33. The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ ?

34. Leonhard Euler was able to calculate the exact sum of the p -series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(See page 760.) Use this fact to find the sum of each series.

(a) $\sum_{n=2}^{\infty} \frac{1}{n^2}$

(b) $\sum_{n=3}^{\infty} \frac{1}{(n + 1)^2}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

35. Euler also found the sum of the p -series with $p = 4$:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's result to find the sum of the series.

(a) $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$

(b) $\sum_{k=5}^{\infty} \frac{1}{(k - 2)^4}$

- 36.** (a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.
 (b) Use (3) with $n = 10$ to give an improved estimate of the sum.
 (c) Compare your estimate in part (b) with the exact value given in Exercise 35.
 (d) Find a value of n so that s_n is within 0.00001 of the sum.

- 37.** (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. How good is this estimate?
 (b) Improve this estimate using (3) with $n = 10$.
 (c) Compare your estimate in part (b) with the exact value given in Exercise 34.
 (d) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

- 38.** Find the sum of the series $\sum_{n=1}^{\infty} ne^{-2n}$ correct to four decimal places.

- 39.** Estimate $\sum_{n=1}^{\infty} (2n + 1)^{-6}$ correct to five decimal places.

- 40.** How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$ would you need to add to find its sum to within 0.01?

- 41.** Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty} n^{-1.001}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

- 42.** (a) Show that the series $\sum_{n=1}^{\infty} (\ln n)^2/n^2$ is convergent.
 (b) Find an upper bound for the error in the approximation $s \approx s_n$.
 (c) What is the smallest value of n such that this upper bound is less than 0.05?
 (d) Find s_n for this value of n .

- 43.** (a) Use (4) to show that if s_n is the n th partial sum of the harmonic series, then

$$s_n \leq 1 + \ln n$$

- (b) The harmonic series diverges, but very slowly. Use part (a) to show that the sum of the first million terms is less than 15 and the sum of the first billion terms is less than 22.

- 44.** Use the following steps to show that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$$

has a limit. (The value of the limit is denoted by γ and is called Euler's constant.)

- (a) Draw a picture like Figure 6 with $f(x) = 1/x$ and interpret t_n as an area [or use (5)] to show that $t_n > 0$ for all n .

- (b) Interpret

$$t_n - t_{n+1} = [\ln(n+1) - \ln n] - \frac{1}{n+1}$$

as a difference of areas to show that $t_n - t_{n+1} > 0$. Therefore $\{t_n\}$ is a decreasing sequence.

- (c) Use the Monotonic Sequence Theorem to show that $\{t_n\}$ is convergent.

- 45.** Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ converges.

- 46.** Find all values of c for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n+1} \right)$$

11.4 The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

1

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\sum_{n=1}^{\infty} 1/2^n$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series (1) is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is. The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series (1) has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series). This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent. The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.
- (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

PROOF

It is important to keep in mind the distinction between a sequence and a series. A sequence is a list of numbers, whereas a series is a sum. With every series $\sum a_n$ there are associated two sequences: the sequence $\{a_n\}$ of terms and the sequence $\{s_n\}$ of partial sums.

Standard Series for Use with the Comparison Test

$$(i) \text{ Let } s_n = \sum_{i=1}^n a_i \quad t_n = \sum_{i=1}^n b_i \quad t = \sum_{n=1}^{\infty} b_n$$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geq s_n$). Also $t_n \rightarrow t$, so $t_n \leq t$ for all n . Since $a_i \leq b_i$, we have $s_n \leq t_n$. Thus $s_n \leq t$ for all n . This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus $\sum a_n$ converges.

(ii) If $\sum b_n$ is divergent, then $t_n \rightarrow \infty$ (since $\{t_n\}$ is increasing). But $a_i \geq b_i$ so $s_n \geq t_n$. Thus $s_n \rightarrow \infty$. Therefore $\sum a_n$ diverges. ■

In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p -series [$\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$; see (11.3.1)]
- A geometric series [$\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$; see (11.2.4)]

EXAMPLE 1 Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

SOLUTION For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test, a_n is the left side and b_n is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p -series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test. ■

NOTE 1 Although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all n , we need verify only that it holds for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

EXAMPLE 2 Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

SOLUTION We used the Integral Test to test this series in Example 11.3.4, but we can also test it by comparing it with the harmonic series. Observe that $\ln k > 1$ for $k \geq 3$ and so

$$\frac{\ln k}{k} > \frac{1}{k} \quad k \geq 3$$

We know that $\sum 1/k$ is divergent (p -series with $p = 1$). Thus the given series is divergent by the Comparison Test. ■

NOTE 2 The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply. Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because $\sum b_n = \sum (\frac{1}{2})^n$ is convergent and $a_n > b_n$. Nonetheless, we have the feeling that $\sum 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\sum (\frac{1}{2})^n$. In such cases the following test can be used.

Exercises 40 and 41 deal with the cases $c = 0$ and $c = \infty$.

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

PROOF Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n , there is an integer N such that

$$m < \frac{a_n}{b_n} < M \quad \text{when } n > N$$

and so

$$mb_n < a_n < Mb_n \quad \text{when } n > N$$

If $\sum b_n$ converges, so does $\sum Mb_n$. Thus $\sum a_n$ converges by part (i) of the Comparison Test. If $\sum b_n$ diverges, so does $\sum mb_n$ and part (ii) of the Comparison Test shows that $\sum a_n$ diverges. ■

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \quad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test. ■

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$\begin{aligned} a_n &= \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \quad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n \rightarrow \infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}} \\ &= \lim_{n \rightarrow \infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{2 + 0}{2\sqrt{0 + 1}} = 1 \end{aligned}$$

Since $\sum b_n = 2 \sum 1/n^{1/2}$ is divergent (p -series with $p = \frac{1}{2} < 1$), the given series diverges by the Limit Comparison Test. ■

Notice that in testing many series we find a suitable comparison series $\sum b_n$ by keeping only the highest powers in the numerator and denominator.

■ Estimating Sums

If we have used the Comparison Test to show that a series $\sum a_n$ converges by comparison with a series $\sum b_n$, then we may be able to estimate the sum $\sum a_n$ by comparing remainders. As in Section 11.3, we consider the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \dots$$

For the comparison series $\sum b_n$ we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \dots$$

Since $a_n \leq b_n$ for all n , we have $R_n \leq T_n$. If $\sum b_n$ is a p -series, we can estimate its remainder T_n as in Section 11.3. If $\sum b_n$ is a geometric series, then T_n is the sum of a geometric series and we can sum it exactly (see Exercises 35 and 36). In either case we know that R_n is smaller than T_n .

EXAMPLE 5 Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3 + 1)$. Estimate the error involved in this approximation.

SOLUTION Since

$$\frac{1}{n^3 + 1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. The remainder T_n for the comparison series $\sum 1/n^3$ was estimated in Example 11.3.5 using the Remainder Estimate for the Integral Test. There we found that

$$T_n \leq \int_n^\infty \frac{1}{x^3} dx = \frac{1}{2n^2}$$

Therefore the remainder R_n for the given series satisfies

$$R_n \leq T_n \leq \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \leq \frac{1}{2(100)^2} = 0.00005$$

Using a programmable calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \approx \sum_{n=1}^{100} \frac{1}{n^3 + 1} \approx 0.6864538$$

with error less than 0.00005. ■

11.4 EXERCISES

1. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be convergent.
 - If $a_n > b_n$ for all n , what can you say about $\sum a_n$? Why?
 - If $a_n < b_n$ for all n , what can you say about $\sum a_n$? Why?
2. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be divergent.
 - If $a_n > b_n$ for all n , what can you say about $\sum a_n$? Why?
 - If $a_n < b_n$ for all n , what can you say about $\sum a_n$? Why?

3–32 Determine whether the series converges or diverges.

$$3. \sum_{n=1}^{\infty} \frac{1}{n^3 + 8}$$

$$4. \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$$

$$5. \sum_{n=1}^{\infty} \frac{n + 1}{n\sqrt{n}}$$

$$6. \sum_{n=1}^{\infty} \frac{n - 1}{n^3 + 1}$$

$$7. \sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n}$$

$$8. \sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$$

$$9. \sum_{k=1}^{\infty} \frac{\ln k}{k}$$

$$10. \sum_{k=1}^{\infty} \frac{k \sin^2 k}{1 + k^3}$$

$$11. \sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3 + 4k + 3}}$$

$$12. \sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$$

$$13. \sum_{n=1}^{\infty} \frac{1 + \cos n}{e^n}$$

$$14. \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4 + 1}}$$

$$15. \sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$$

$$16. \sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$17. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$$

$$18. \sum_{n=1}^{\infty} \frac{2}{\sqrt{n} + 2}$$

$$19. \sum_{n=1}^{\infty} \frac{n + 1}{n^3 + n}$$

$$20. \sum_{n=1}^{\infty} \frac{n^2 + n + 1}{n^4 + n^2}$$

$$21. \sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n}$$

$$22. \sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$$

$$23. \sum_{n=1}^{\infty} \frac{5 + 2n}{(1 + n^2)^2}$$

$$24. \sum_{n=1}^{\infty} \frac{n + 3^n}{n + 2^n}$$

$$25. \sum_{n=1}^{\infty} \frac{e^n + 1}{ne^n + 1}$$

$$26. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$$

$$27. \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$$

$$28. \sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$$

$$29. \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$30. \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$31. \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

$$32. \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

33–36 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

$$33. \sum_{n=1}^{\infty} \frac{1}{5 + n^5}$$

$$34. \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^4}$$

$$35. \sum_{n=1}^{\infty} 5^{-n} \cos^2 n$$

$$36. \sum_{n=1}^{\infty} \frac{1}{3^n + 4^n}$$

37. The meaning of the decimal representation of a number $0.d_1d_2d_3\dots$ (where the digit d_i is one of the numbers 0, 1, 2, ..., 9) is that

$$0.d_1d_2d_3d_4\dots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \dots$$

Show that this series always converges.

38. For what values of p does the series $\sum_{n=2}^{\infty} 1/(n^p \ln n)$ converge?
39. Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ also converges.
40. (a) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is convergent. Prove that if

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$$

then $\sum a_n$ is also convergent.

- (b) Use part (a) to show that the series converges.

$$(i) \sum_{n=1}^{\infty} \frac{\ln n}{n^3} \quad (ii) \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n} e^n}$$

41. (a) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent. Prove that if

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$$

then $\sum a_n$ is also divergent.

- (b) Use part (a) to show that the series diverges.
- (i) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
42. Give an example of a pair of series $\sum a_n$ and $\sum b_n$ with positive terms where $\lim_{n \rightarrow \infty} (a_n/b_n) = 0$ and $\sum b_n$ diverges, but $\sum a_n$ converges. (Compare with Exercise 40.)
43. Show that if $a_n > 0$ and $\lim_{n \rightarrow \infty} na_n \neq 0$, then $\sum a_n$ is divergent.
44. Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1 + a_n)$ is convergent.
45. If $\sum a_n$ is a convergent series with positive terms, is it true that $\sum \sin(a_n)$ is also convergent?
46. If $\sum a_n$ and $\sum b_n$ are both convergent series with positive terms, is it true that $\sum a_n b_n$ is also convergent?

11.5 Alternating Series

The convergence tests that we have looked at so far apply only to series with positive terms. In this section and the next we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series*, whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \dots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the n th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots \quad b_n > 0$$

satisfies

$$(i) \quad b_{n+1} \leq b_n \quad \text{for all } n$$

$$(ii) \quad \lim_{n \rightarrow \infty} b_n = 0$$

then the series is convergent.

Before giving the proof let's look at Figure 1, which gives a picture of the idea behind the proof. We first plot $s_1 = b_1$ on a number line. To find s_2 we subtract b_2 , so s_2 is to the left of s_1 . Then to find s_3 we add b_3 , so s_3 is to the right of s_2 . But, since $b_3 < b_2$, s_3 is to the left of s_1 . Continuing in this manner, we see that the partial sums oscillate back and forth. Since $b_n \rightarrow 0$, the successive steps are becoming smaller and smaller. The even partial sums s_2, s_4, s_6, \dots are increasing and the odd partial sums s_1, s_3, s_5, \dots are decreasing. Thus it seems plausible that both are converging to some number s , which is the sum of the series. Therefore we consider the even and odd partial sums separately in the following proof.

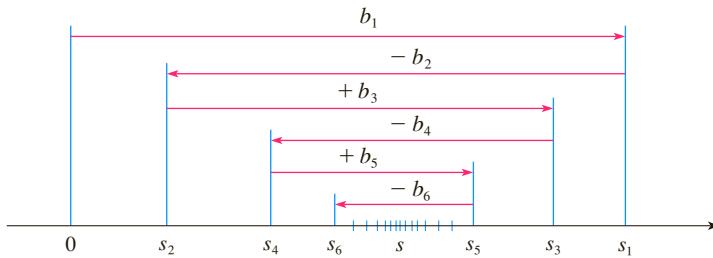


FIGURE 1

PROOF OF THE ALTERNATING SERIES TEST We first consider the even partial sums:

$$s_2 = b_1 - b_2 \geq 0 \quad \text{since } b_2 \leq b_1$$

$$s_4 = s_2 + (b_3 - b_4) \geq s_2 \quad \text{since } b_4 \leq b_3$$

$$\text{In general } s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2} \quad \text{since } b_{2n} \leq b_{2n-1}$$

$$\text{Thus } 0 \leq s_2 \leq s_4 \leq s_6 \leq \dots \leq s_{2n} \leq \dots$$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in parentheses is positive, so $s_{2n} \leq b_1$ for all n . Therefore the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call its limit s , that is,

$$\lim_{n \rightarrow \infty} s_{2n} = s$$

Now we compute the limit of the odd partial sums:

$$\begin{aligned} \lim_{n \rightarrow \infty} s_{2n+1} &= \lim_{n \rightarrow \infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n \rightarrow \infty} s_{2n} + \lim_{n \rightarrow \infty} b_{2n+1} \\ &= s + 0 && \text{[by condition (ii)]} \\ &= s \end{aligned}$$

Since both the even and odd partial sums converge to s , we have $\lim_{n \rightarrow \infty} s_n = s$ [see Exercise 11.1.92(a)] and so the series is convergent.

Figure 2 illustrates Example 1 by showing the graphs of the terms $a_n = (-1)^{n-1}/n$ and the partial sums s_n . Notice how the values of s_n zigzag across the limiting value, which appears to be about 0.7. In fact, it can be proved that the exact sum of the series is $\ln 2 \approx 0.693$ (see Exercise 36).

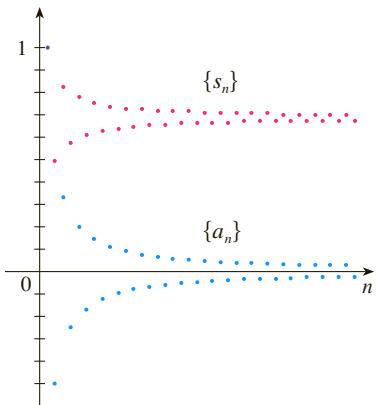


FIGURE 2

EXAMPLE 1 The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

$$(i) \quad b_{n+1} < b_n \quad \text{because} \quad \frac{1}{n+1} < \frac{1}{n}$$

$$(ii) \quad \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

so the series is convergent by the Alternating Series Test. ■

EXAMPLE 2 The series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating, but

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{3n}{4n-1} = \lim_{n \rightarrow \infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4}$$

so condition (ii) is not satisfied. Instead, we look at the limit of the n th term of the series:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist, so the series diverges by the Test for Divergence. ■

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1}$ for convergence or divergence.

SOLUTION The given series is alternating so we try to verify conditions (i) and (ii) of the Alternating Series Test.

Unlike the situation in Example 1, it is not obvious that the sequence given by $b_n = n^2/(n^3 + 1)$ is decreasing. However, if we consider the related function $f(x) = x^2/(x^3 + 1)$, we find that

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$$

Since we are considering only positive x , we see that $f'(x) < 0$ if $2 - x^3 < 0$, that is, $x > \sqrt[3]{2}$. Thus f is decreasing on the interval $(\sqrt[3]{2}, \infty)$. This means that $f(n+1) < f(n)$ and therefore $b_{n+1} < b_n$ when $n \geq 2$. (The inequality $b_2 < b_1$ can be verified directly but all that really matters is that the sequence $\{b_n\}$ is eventually decreasing.)

Condition (ii) is readily verified:

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n^2}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0$$

Thus the given series is convergent by the Alternating Series Test. ■

Instead of verifying condition (i) of the Alternating Series Test by computing a derivative, we could verify that $b_{n+1} < b_n$ directly by using the technique of Solution 1 of Example 11.1.13.

■ Estimating Sums

A partial sum s_n of any convergent series can be used as an approximation to the total sum s , but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$. The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1} , which is the absolute value of the first neglected term.

You can see geometrically why the Alternating Series Estimation Theorem is true by looking at Figure 1 (on page 773). Notice that $s - s_4 < b_5$, $|s - s_5| < b_6$, and so on. Notice also that s lies between any two consecutive partial sums.

Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

$$(i) \quad b_{n+1} \leq b_n \quad \text{and} \quad (ii) \quad \lim_{n \rightarrow \infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

PROOF We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1} . (There we showed that s is larger than all the even partial sums. A similar argument shows that s is smaller than all the odd sums.) It follows that

$$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$$

EXAMPLE 4 Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

SOLUTION We first observe that the series is convergent by the Alternating Series Test because

$$(i) \quad \frac{1}{(n+1)!} = \frac{1}{n!(n+1)} < \frac{1}{n!}$$

$$(ii) \quad 0 < \frac{1}{n!} < \frac{1}{n} \rightarrow 0 \quad \text{so} \quad \frac{1}{n!} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$\begin{aligned} s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \dots \\ &= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \dots \end{aligned}$$

Notice that

$$b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$$

and

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$$

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \leq b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places.

In Section 11.10 we will prove that $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all x , so what we have obtained in Example 4 is actually an approximation to the number e^{-1} .

 **NOTE** The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. **The rule does not apply to other types of series.**

11.5 EXERCISES

1. (a) What is an alternating series?
 (b) Under what conditions does an alternating series converge?
 (c) If these conditions are satisfied, what can you say about the remainder after n terms?

2–20 Test the series for convergence or divergence.

2. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \dots$

3. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \dots$

4. $\frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \frac{1}{\ln 6} + \frac{1}{\ln 7} - \dots$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$

7. $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

9. $\sum_{n=1}^{\infty} (-1)^n e^{-n}$

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

13. $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

15. $\sum_{n=0}^{\infty} \frac{\sin(n+\frac{1}{2})\pi}{1+\sqrt{n}}$

17. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

6. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$

10. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$

12. $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

16. $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$

18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

20. $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$

Series Estimation Theorem to estimate the sum correct to four decimal places.

21. $\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$

22. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

23–26 Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ ($|\text{error}| < 0.00005$)

24. $\sum_{n=1}^{\infty} \frac{(-\frac{1}{3})^n}{n}$ ($|\text{error}| < 0.0005$)

25. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 2^n}$ ($|\text{error}| < 0.0005$)

26. $\sum_{n=1}^{\infty} \left(-\frac{1}{n}\right)^n$ ($|\text{error}| < 0.00005$)

27–30 Approximate the sum of the series correct to four decimal places.

27. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$

29. $\sum_{n=1}^{\infty} (-1)^n n e^{-2n}$

30. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 4^n}$

31. Is the 50th partial sum s_{50} of the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ an overestimate or an underestimate of the total sum? Explain.

32–34 For what values of p is each series convergent?

32. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

33. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$

34. $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$

 **21–22** Graph both the sequence of terms and the sequence of partial sums on the same screen. Use the graph to make a rough estimate of the sum of the series. Then use the Alternating

35. Show that the series $\sum (-1)^{n-1} b_n$, where $b_n = 1/n$ if n is odd and $b_n = 1/n^2$ if n is even, is divergent. Why does the Alternating Series Test not apply?

36. Use the following steps to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

Let h_n and s_n be the partial sums of the harmonic and alternating harmonic series.

- (a) Show that $s_{2n} = h_{2n} - h_n$.

- (b) From Exercise 11.3.44 we have

$$h_n - \ln n \rightarrow \gamma \quad \text{as } n \rightarrow \infty$$

and therefore

$$h_{2n} - \ln(2n) \rightarrow \gamma \quad \text{as } n \rightarrow \infty$$

Use these facts together with part (a) to show that $s_{2n} \rightarrow \ln 2$ as $n \rightarrow \infty$.

11.6 Absolute Convergence and the Ratio and Root Tests

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \dots$$

whose terms are the absolute values of the terms of the original series.

We have convergence tests for series with positive terms and for alternating series. But what if the signs of the terms switch back and forth irregularly? We will see in Example 3 that the idea of absolute convergence sometimes helps in such cases.

1 Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

EXAMPLE 1 The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

is a convergent p -series ($p = 2$). ■

EXAMPLE 2 We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent (see Example 11.5.1), but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

which is the harmonic series (p -series with $p = 1$) and is therefore divergent.

2 Definition A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent. Thus it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

3 Theorem If a series $\sum a_n$ is absolutely convergent, then it is convergent.

PROOF Observe that the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, so $\sum 2|a_n|$ is convergent. Therefore, by the Comparison Test, $\sum (a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is therefore convergent.

EXAMPLE 3 Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \dots$$

is convergent or divergent.

SOLUTION This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: the signs change irregularly.) We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \leq 1$ for all n , we have

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

We know that $\sum 1/n^2$ is convergent (p -series with $p = 2$) and therefore $\sum |\cos n|/n^2$ is convergent by the Comparison Test. Thus the given series $\sum (\cos n)/n^2$ is absolutely convergent and therefore convergent by Theorem 3. ■

The following test is very useful in determining whether a given series is absolutely convergent.

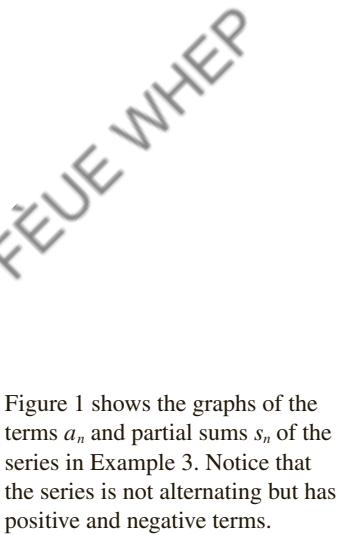


FIGURE 1

The Ratio Test

(i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

PROOF

(i) The idea is to compare the given series with a convergent geometric series. Since $L < 1$, we can choose a number r such that $L < r < 1$. Since

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \text{and} \quad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than r ; that is, there exists an integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| < r \quad \text{whenever } n \geq N$$

or, equivalently,

$$4 \quad |a_{n+1}| < |a_n|r \quad \text{whenever } n \geq N$$

Putting n successively equal to $N, N + 1, N + 2, \dots$ in (4), we obtain

$$|a_{N+1}| < |a_N|r$$

$$|a_{N+2}| < |a_{N+1}|r < |a_N|r^2$$

$$|a_{N+3}| < |a_{N+2}|r < |a_N|r^3$$

and, in general,

$$5 \quad |a_{N+k}| < |a_N|r^k \quad \text{for all } k \geq 1$$

Now the series

$$\sum_{k=1}^{\infty} |a_N|r^k = |a_N|r + |a_N|r^2 + |a_N|r^3 + \dots$$

is convergent because it is a geometric series with $0 < r < 1$. So the inequality (5), together with the Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \dots$$

is also convergent. It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. (Recall that a finite number of terms doesn't affect convergence.) Therefore $\sum a_n$ is absolutely convergent.

(ii) If $|a_{n+1}/a_n| \rightarrow L > 1$ or $|a_{n+1}/a_n| \rightarrow \infty$, then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1; that is, there exists an integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| > 1 \quad \text{whenever } n \geq N$$

This means that $|a_{n+1}| > |a_n|$ whenever $n \geq N$ and so

$$\lim_{n \rightarrow \infty} a_n \neq 0$$

Therefore $\sum a_n$ diverges by the Test for Divergence. ■

NOTE Part (iii) of the Ratio Test says that if $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$, the test gives no information. For instance, for the convergent series $\sum 1/n^2$ we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

The Ratio Test is usually conclusive if the n th term of the series contains an exponential or a factorial, as we will see in Examples 4 and 5.

Therefore, if $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$, the series $\sum a_n$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

EXAMPLE 4 Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3 / 3^n$:

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \\ &= \frac{1}{3} \left(\frac{n+1}{n} \right)^3 = \frac{1}{3} \left(1 + \frac{1}{n} \right)^3 \rightarrow \frac{1}{3} < 1 \end{aligned}$$

Thus, by the Ratio Test, the given series is absolutely convergent.

EXAMPLE 5 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \\ &= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \rightarrow e \quad \text{as } n \rightarrow \infty\end{aligned}$$

(see Equation 6.4.9 or 6.4*.9). Since $e > 1$, the given series is divergent by the Ratio Test. ■

NOTE Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \dots \cdot n}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} \geq n$$

it follows that a_n does not approach 0 as $n \rightarrow \infty$. Therefore the given series is divergent by the Test for Divergence.

The following test is convenient to apply when n th powers occur. Its proof is similar to the proof of the Ratio Test and is left as Exercise 49.

The Root Test

- (i) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- (ii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\sum a_n$ could converge or diverge. (If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

EXAMPLE 6 Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{2n+3}{3n+2}\right|} = \frac{\sqrt[n]{2n+3}}{\sqrt[n]{3n+2}} = \frac{\sqrt[n]{2+\frac{3}{n}}}{\sqrt[n]{3+\frac{2}{n}}} \rightarrow \frac{2}{3} < 1$$

Thus the given series is absolutely convergent (and therefore convergent) by the Root Test. ■

Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series. By a **rearrangement** of an infinite series $\sum a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\sum a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \dots$$

It turns out that

if $\sum a_n$ is an absolutely convergent series with sum s ,
then any rearrangement of $\sum a_n$ has the same sum s .

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$\text{6} \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots = \ln 2$$

(See Exercise 11.5.36.) If we multiply this series by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = \frac{1}{2} \ln 2$$

Inserting zeros between the terms of this series, we have

$$\text{7} \quad 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \dots = \frac{1}{2} \ln 2$$

Now we add the series in Equations 6 and 7 using Theorem 11.2.8:

$$\text{8} \quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots = \frac{3}{2} \ln 2$$

Notice that the series in (8) contains the same terms as in (6) but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

if $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r .

A proof of this fact is outlined in Exercise 52.

11.6 EXERCISES

1. What can you say about the series $\sum a_n$ in each of the following cases?

$$(a) \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 8$$

$$(b) \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8$$

$$(c) \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

- 2–6 Determine whether the series is absolutely convergent or conditionally convergent.

$$2. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

$$3. \sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$$

$$4. \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 1}$$

5. $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$

6. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 4}$

7-24 Use the Ratio Test to determine whether the series is convergent or divergent.

7. $\sum_{n=1}^{\infty} \frac{n}{5^n}$

9. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$

11. $\sum_{k=1}^{\infty} \frac{1}{k!}$

13. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

15. $\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$

17. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

19. $\sum_{n=1}^{\infty} \frac{n^{100} 100^n}{n!}$

21. $1 - \frac{2!}{1 \cdot 3} + \frac{3!}{1 \cdot 3 \cdot 5} - \frac{4!}{1 \cdot 3 \cdot 5 \cdot 7} + \dots + (-1)^{n-1} \frac{n!}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} + \dots$

22. $\frac{2}{3} + \frac{2 \cdot 5}{3 \cdot 5} + \frac{2 \cdot 5 \cdot 8}{3 \cdot 5 \cdot 7} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 5 \cdot 7 \cdot 9} + \dots$

23. $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)}{n!}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \dots \cdot (3n+2)}$

25-30 Use the Root Test to determine whether the series is convergent or divergent.

25. $\sum_{n=1}^{\infty} \left(\frac{n^2 + 1}{2n^2 + 1} \right)^n$

26. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

27. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$

28. $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1} \right)^{5n}$

29. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)^{n^2}$

30. $\sum_{n=0}^{\infty} (\arctan n)^n$

31-38 Use any test to determine whether the series is absolutely convergent, conditionally convergent, or divergent.

31. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

32. $\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n} \right)^n$

33. $\sum_{n=1}^{\infty} \frac{(-9)^n}{n 10^{n+1}}$

34. $\sum_{n=1}^{\infty} \frac{n 5^{2n}}{10^{n+1}}$

35. $\sum_{n=2}^{\infty} \left(\frac{n}{\ln n} \right)^n$

36. $\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{1+n\sqrt{n}}$

37. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$

38. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

39. The terms of a series are defined recursively by the equations

$$a_1 = 2 \quad a_{n+1} = \frac{5n+1}{4n+3} a_n$$

Determine whether $\sum a_n$ converges or diverges.

40. A series $\sum a_n$ is defined by the equations

$$a_1 = 1 \quad a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n$$

Determine whether $\sum a_n$ converges or diverges.

41-42 Let $\{b_n\}$ be a sequence of positive numbers that converges to $\frac{1}{2}$. Determine whether the given series is absolutely convergent.

41. $\sum_{n=1}^{\infty} \frac{b_n^n \cos n\pi}{n}$

42. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 b_3 \dots b_n}$

43. For which of the following series is the Ratio Test inconclusive (that is, it fails to give a definite answer)?

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(c) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

44. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

45. (a) Show that $\sum_{n=0}^{\infty} x^n/n!$ converges for all x .
 (b) Deduce that $\lim_{n \rightarrow \infty} x^n/n! = 0$ for all x .

46. Let $\sum a_n$ be a series with positive terms and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n \rightarrow \infty} r_n = L < 1$, so $\sum a_n$ converges by the Ratio Test. As usual, we let R_n be the remainder after n terms, that is,

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$$

- (a) If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that

$$R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}$$

- (b) If $\{r_n\}$ is an increasing sequence, show that

$$R_n \leq \frac{a_{n+1}}{1 - L}$$

- 47.** (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/(n2^n)$. Use Exercise 46 to estimate the error in using s_5 as an approximation to the sum of the series.
 (b) Find a value of n so that s_n is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.

- 48.** Use the sum of the first 10 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Use Exercise 46 to estimate the error.

- 49.** Prove the Root Test. [Hint for part (i): Take any number r such that $L < r < 1$ and use the fact that there is an integer N such that $\sqrt[n]{|a_n|} < r$ whenever $n \geq N$.]

- 50.** Around 1910, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

William Gosper used this series in 1985 to compute the first 17 million digits of π .

- (a) Verify that the series is convergent.
 (b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use two terms?

- 51.** Given any series $\sum a_n$, we define a series $\sum a_n^+$ whose terms are all the positive terms of $\sum a_n$ and a series $\sum a_n^-$ whose terms are all the negative terms of $\sum a_n$. To be specific, we let

$$a_n^+ = \frac{a_n + |a_n|}{2} \quad a_n^- = \frac{a_n - |a_n|}{2}$$

Notice that if $a_n > 0$, then $a_n^+ = a_n$ and $a_n^- = 0$, whereas if $a_n < 0$, then $a_n^- = a_n$ and $a_n^+ = 0$.

- (a) If $\sum a_n$ is absolutely convergent, show that both of the series $\sum a_n^+$ and $\sum a_n^-$ are convergent.
 (b) If $\sum a_n$ is conditionally convergent, show that both of the series $\sum a_n^+$ and $\sum a_n^-$ are divergent.

- 52.** Prove that if $\sum a_n$ is a conditionally convergent series and r is any real number, then there is a rearrangement of $\sum a_n$ whose sum is r . [Hints: Use the notation of Exercise 51. Take just enough positive terms a_n^+ so that their sum is greater than r . Then add just enough negative terms a_n^- so that the cumulative sum is less than r . Continue in this manner and use Theorem 11.2.6.]

- 53.** Suppose the series $\sum a_n$ is conditionally convergent.
 (a) Prove that the series $\sum n^2 a_n$ is divergent.
 (b) Conditional convergence of $\sum a_n$ is not enough to determine whether $\sum na_n$ is convergent. Show this by giving an example of a conditionally convergent series such that $\sum na_n$ converges and an example where $\sum na_n$ diverges.

11.7 Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

- If the series is of the form $\sum 1/n^p$, it is a *p*-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
- If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
- If the series has a form that is similar to a *p*-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a *p*-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n \rightarrow \infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the n th power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \rightarrow 1$ as $n \rightarrow \infty$ for all p -series and therefore all rational or algebraic functions of n . Thus the Ratio Test should not be used for such series.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x) dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

EXAMPLE 1
$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

Since $a_n \rightarrow \frac{1}{2} \neq 0$ as $n \rightarrow \infty$, we should use the Test for Divergence. ■

EXAMPLE 2
$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3 + 1}}{3n^3 + 4n^2 + 2}$$

Since a_n is an algebraic function of n , we compare the given series with a p -series. The comparison series for the Limit Comparison Test is $\sum b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

EXAMPLE 3
$$\sum_{n=1}^{\infty} n e^{-n^2}$$

Since the integral $\int_1^\infty x e^{-x^2} dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works. ■

EXAMPLE 4
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4 + 1}$$

Since the series is alternating, we use the Alternating Series Test. ■

EXAMPLE 5
$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

Since the series involves $k!$, we use the Ratio Test. ■

EXAMPLE 6
$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$$

Since the series is closely related to the geometric series $\sum 1/3^n$, we use the Comparison Test. ■

11.7 EXERCISES

1–38 Test the series for convergence or divergence.

$$1. \sum_{n=1}^{\infty} \frac{n^2 - 1}{n^3 + 1}$$

$$3. \sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 1}{n^3 + 1}$$

$$5. \sum_{n=1}^{\infty} \frac{e^n}{n^2}$$

$$7. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

$$9. \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$$

$$11. \sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n} \right)$$

$$13. \sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

$$15. \sum_{k=1}^{\infty} \frac{2^{k-1} 3^{k+1}}{k^k}$$

$$17. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

$$18. \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$$

$$2. \sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$$

$$4. \sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$$

$$6. \sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$$

$$8. \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$$

$$10. \sum_{n=1}^{\infty} n^2 e^{-n^3}$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$$

$$14. \sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$$

$$16. \sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$$

$$19. \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$$

$$21. \sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$$

$$23. \sum_{n=1}^{\infty} \tan(1/n)$$

$$25. \sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

$$27. \sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$$

$$29. \sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$$

$$31. \sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$$

$$33. \sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

$$35. \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

$$37. \sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$$

$$20. \sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$$

$$22. \sum_{k=1}^{\infty} \frac{1}{2+\sin k}$$

$$24. \sum_{n=1}^{\infty} n \sin(1/n)$$

$$26. \sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$$

$$28. \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

$$30. \sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$$

$$32. \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$$

$$34. \sum_{n=1}^{\infty} \frac{1}{n+n \cos^2 n}$$

$$36. \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

$$38. \sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$$

11.8 Power Series

A power series is a series of the form

$$1. \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is a variable and the c_n 's are constants called the **coefficients** of the series. For each fixed x , the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x . The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n , the power series becomes the geometric series

$$2. \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$. (See Equation 11.2.5.)

In fact if we put $x = \frac{1}{2}$ in the geometric series (2) we get the convergent series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

but if we put $x = 2$ in (2) we get the divergent series

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + \dots$$

More generally, a series of the form

$$\boxed{3} \quad \sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots$$

is called a **power series in $(x - a)$** or a **power series centered at a** or a **power series about a** . Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 3 we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$. Notice also that when $x = a$, all of the terms are 0 for $n \geq 1$ and so the power series (3) always converges when $x = a$.

EXAMPLE 1 For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

SOLUTION We use the Ratio Test. If we let a_n , as usual, denote the n th term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

Notice that

$$\begin{aligned} (n+1)! &= (n+1)n(n-1) \cdots 3 \cdot 2 \cdot 1 \\ &= (n+1)n! \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n \rightarrow \infty} (n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus the given series converges only when $x = 0$. ■

EXAMPLE 2 For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

SOLUTION Let $a_n = (x-3)^n/n$. Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\ &= \frac{1}{1 + \frac{1}{n}} |x-3| \rightarrow |x-3| \quad \text{as } n \rightarrow \infty \end{aligned}$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. Now

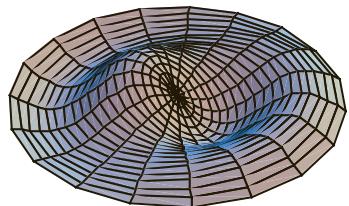
$$|x-3| < 1 \iff -1 < x-3 < 1 \iff 2 < x < 4$$

so the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x-3| = 1$ so we must consider $x = 2$ and $x = 4$ separately. If we put $x = 4$ in the series, it becomes $\sum 1/n$, the harmonic series, which is divergent. If $x = 2$, the series is $\sum (-1)^n/n$, which converges by the Alternating Series Test. Thus the given power series converges for $2 \leq x < 4$. ■



Membrane courtesy of National Film Board of Canada



Notice how closely the computer-generated model (which involves Bessel functions and cosine functions) matches the photograph of a vibrating rubber membrane.

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry. In particular, the sum of the power series in the next example is called a **Bessel function**, after the German astronomer Friedrich Bessel (1784–1846), and the function given in Exercise 35 is another example of a Bessel function. In fact, these functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, these functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead.

EXAMPLE 3 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

SOLUTION Let $a_n = (-1)^n x^{2n}/[2^{2n}(n!)^2]$. Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\ &= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \\ &= \frac{x^2}{4(n+1)^2} \rightarrow 0 < 1 \quad \text{for all } x \end{aligned}$$

Thus, by the Ratio Test, the given series converges for all values of x . In other words, the domain of the Bessel function J_0 is $(-\infty, \infty) = \mathbb{R}$. ■

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function in Example 3 as the sum of a series we mean that, for every real number x ,

$$J_0(x) = \lim_{n \rightarrow \infty} s_n(x) \quad \text{where} \quad s_n(x) = \sum_{i=0}^n \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$$

The first few partial sums are

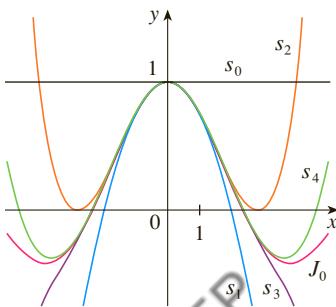
$$s_0(x) = 1$$

$$s_1(x) = 1 - \frac{x^2}{4}$$

$$s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

$$s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147,456}$$

**FIGURE 1**

Partial sums of the Bessel function J_0

Figure 1 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function J_0 , but notice that the approximations become better when more terms are included. Figure 2 shows a more complete graph of the Bessel function.

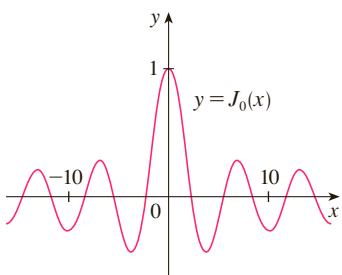


FIGURE 2

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval [a finite interval for the geometric series and the series in Example 2, the infinite interval $(-\infty, \infty)$ in Example 3, and a collapsed interval $[0, 0] = \{0\}$ in Example 1]. The following theorem, proved in Appendix F, says that this is true in general.

4 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x - a)^n$, there are only three possibilities:

- The series converges only when $x = a$.
- The series converges for all x .
- There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a . In case (ii) the interval is $(-\infty, \infty)$. In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$. When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints. Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a - R, a + R) \quad (a - R, a + R] \quad [a - R, a + R) \quad [a - R, a + R]$$

The situation is illustrated in Figure 3.

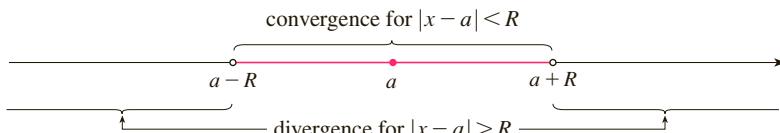


FIGURE 3

We summarize here the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n! x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x - 3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R . The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

EXAMPLE 4 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

SOLUTION Let $a_n = (-3)^n x^n / \sqrt{n+1}$. Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| = \left| -3x \sqrt{\frac{n+1}{n+2}} \right| \\ &= 3 \sqrt{\frac{1 + (1/n)}{1 + (2/n)}} |x| \rightarrow 3|x| \quad \text{as } n \rightarrow \infty \end{aligned}$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$. This means that the radius of convergence is $R = \frac{1}{3}$.

We know the series converges in the interval $(-\frac{1}{3}, \frac{1}{3})$, but we must now test for convergence at the endpoints of this interval. If $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$$

which diverges. (Use the Integral Test or simply observe that it is a p -series with $p = \frac{1}{2} < 1$.) If $x = \frac{1}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test. Therefore the given power series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$, so the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$. ■

EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

SOLUTION If $a_n = n(x+2)^n / 3^{n+1}$, then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right| \\ &= \left(1 + \frac{1}{n}\right) \frac{|x+2|}{3} \rightarrow \frac{|x+2|}{3} \quad \text{as } n \rightarrow \infty \end{aligned}$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x + 2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1 . When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$. ■

11.8 EXERCISES

1. What is a power series?

2. (a) What is the radius of convergence of a power series?

How do you find it?

(b) What is the interval of convergence of a power series?

How do you find it?

3–28 Find the radius of convergence and interval of convergence of the series.

3. $\sum_{n=1}^{\infty} (-1)^n nx^n$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[n]{n}}$

5. $\sum_{n=1}^{\infty} \frac{x^n}{2n - 1}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

7. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

8. $\sum_{n=1}^{\infty} n^n x^n$

9. $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$

10. $\sum_{n=1}^{\infty} 2^n n^2 x^n$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} x^n$

13. $\sum_{n=1}^{\infty} \frac{n}{2^n (n^2 + 1)} x^n$

14. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

15. $\sum_{n=0}^{\infty} \frac{(x - 2)^n}{n^2 + 1}$

16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n - 1)2^n} (x - 1)^n$

17. $\sum_{n=2}^{\infty} \frac{(x + 2)^n}{2^n \ln n}$

18. $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{8^n} (x + 6)^n$

19. $\sum_{n=1}^{\infty} \frac{(x - 2)^n}{n^n}$

20. $\sum_{n=1}^{\infty} \frac{(2x - 1)^n}{5^n \sqrt[n]{n}}$

21. $\sum_{n=1}^{\infty} \frac{n}{b^n} (x - a)^n, \quad b > 0$

22. $\sum_{n=2}^{\infty} \frac{b^n}{\ln n} (x - a)^n, \quad b > 0$

23. $\sum_{n=1}^{\infty} n!(2x - 1)^n$

24. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)}$

25. $\sum_{n=1}^{\infty} \frac{(5x - 4)^n}{n^3}$

26. $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

27. $\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdots (2n - 1)}$

28. $\sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdots (2n - 1)}$

29. If $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, can we conclude that each of the following series is convergent?

(a) $\sum_{n=0}^{\infty} c_n (-2)^n$

(b) $\sum_{n=0}^{\infty} c_n (-4)^n$

30. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?

(a) $\sum_{n=0}^{\infty} c_n$

(b) $\sum_{n=0}^{\infty} c_n 8^n$

(c) $\sum_{n=0}^{\infty} c_n (-3)^n$

(d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

31. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

- 32.** Let p and q be real numbers with $p < q$. Find a power series whose interval of convergence is
 (a) (p, q) (b) $[p, q]$ (c) $[p, q)$ (d) $[p, q]$

- 33.** Is it possible to find a power series whose interval of convergence is $[0, \infty)$? Explain.

- 34.** Graph the first several partial sums $s_n(x)$ of the series $\sum_{n=0}^{\infty} x^n$, together with the sum function $f(x) = 1/(1-x)$, on a common screen. On what interval do these partial sums appear to be converging to $f(x)$?

- 35.** The function J_1 defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

is called the *Bessel function of order 1*.

- (a) Find its domain.
 (b) Graph the first several partial sums on a common screen.
 (c) If your CAS has built-in Bessel functions, graph J_1 on the same screen as the partial sums in part (b) and observe how the partial sums approximate J_1 .

- 36.** The function A defined by

$$A(x) = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \dots$$

is called an *Airy function* after the English mathematician and astronomer Sir George Airy (1801–1892).

- (a) Find the domain of the Airy function.
 (b) Graph the first several partial sums on a common screen.

- (c) If your CAS has built-in Airy functions, graph A on the same screen as the partial sums in part (b) and observe how the partial sums approximate A .

- 37.** A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \dots$$

that is, its coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geq 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

- 38.** If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \geq 0$, find the interval of convergence of the series and a formula for $f(x)$.

- 39.** Show that if $\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} = c$, where $c \neq 0$, then the radius of convergence of the power series $\sum c_n x^n$ is $R = 1/c$.

- 40.** Suppose that the power series $\sum c_n (x - a)^n$ satisfies $c_n \neq 0$ for all n . Show that if $\lim_{n \rightarrow \infty} |c_n/c_{n+1}|$ exists, then it is equal to the radius of convergence of the power series.

- 41.** Suppose the series $\sum c_n x^n$ has radius of convergence 2 and the series $\sum d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (c_n + d_n)x^n$?

- 42.** Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R . What is the radius of convergence of the power series $\sum c_n x^{2n}$?

11.9 Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials. (Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.)

We start with an equation that we have seen before:

1

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

We first encountered this equation in Example 11.2.7, where we obtained it by observing that the series is a geometric series with $a = 1$ and $r = x$. But here our point of view is different. We now regard Equation 1 as expressing the function $f(x) = 1/(1-x)$ as a sum of a power series.

A geometric illustration of Equation 1 is shown in Figure 1. Because the sum of a series is the limit of the sequence of partial sums, we have

$$\frac{1}{1-x} = \lim_{n \rightarrow \infty} s_n(x)$$

where

$$s_n(x) = 1 + x + x^2 + \cdots + x^n$$

is the n th partial sum. Notice that as n increases, $s_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.

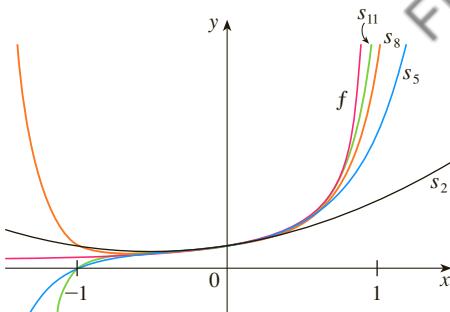


FIGURE 1 $f(x) = \frac{1}{1-x}$ and some partial sums

EXAMPLE 1 Express $1/(1+x^2)$ as the sum of a power series and find the interval of convergence.

SOLUTION Replacing x by $-x^2$ in Equation 1, we have

$$\begin{aligned} \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \dots \end{aligned}$$

Because this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$. Therefore the interval of convergence is $(-1, 1)$. (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.) ■

EXAMPLE 2 Find a power series representation for $1/(x+2)$.

SOLUTION In order to put this function in the form of the left side of Equation 1, we first factor a 2 from the denominator:

$$\begin{aligned} \frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \end{aligned}$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$. So the interval of convergence is $(-2, 2)$. ■

EXAMPLE 3 Find a power series representation of $x^3/(x+2)$.

SOLUTION Since this function is just x^3 times the function in Example 2, all we have to do is to multiply that series by x^3 :

$$\begin{aligned} \frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\ &= \frac{1}{2} x^3 - \frac{1}{4} x^4 + \frac{1}{8} x^5 - \frac{1}{16} x^6 + \dots \end{aligned}$$

It's legitimate to move x^3 across the sigma sign because it doesn't depend on n . [Use Theorem 11.2.8(i) with $c = x^3$.]

Another way of writing this series is as follows:

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

As in Example 2, the interval of convergence is $(-2, 2)$. ■

■ Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$ whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem (which we won't prove) says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

2 Theorem If the power series $\sum c_n(x - a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

$$\begin{aligned} \text{(i)} \quad f'(x) &= c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1} \\ \text{(ii)} \quad \int f(x) dx &= C + c_0(x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} + \dots \\ &= C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} \end{aligned}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R .

In part (ii), $\int c_0 dx = c_0 x + C_1$ is written as $c_0(x - a) + C$, where $C = C_1 + ac_0$, so all the terms of the series have the same form.

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

$$\text{(iii)} \quad \frac{d}{dx} \left[\sum_{n=0}^{\infty} c_n(x - a)^n \right] = \sum_{n=0}^{\infty} \frac{d}{dx} [c_n(x - a)^n]$$

$$\text{(iv)} \quad \int \left[\sum_{n=0}^{\infty} c_n(x - a)^n \right] dx = \sum_{n=0}^{\infty} \int c_n(x - a)^n dx$$

We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. Equations (iii) and (iv) assert that the same is true for infinite sums, provided we are dealing with *power series*. (For other types of series of functions the situation is not as simple; see Exercise 38.)

NOTE 2 Although Theorem 2 says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there. (See Exercise 39.)

NOTE 3 The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations. We will discuss this method in Chapter 17.

EXAMPLE 4 In Example 11.8.3 we saw that the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is defined for all x . Thus, by Theorem 2, J_0 is differentiable for all x and its derivative is found by term-by-term differentiation as follows:

$$J'_0(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n-1}}{2^{2n}(n!)^2}$$

EXAMPLE 5 Express $1/(1 - x)^2$ as a power series by differentiating Equation 1. What is the radius of convergence?

SOLUTION Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} nx^{n-1}$$

If we wish, we can replace n by $n + 1$ and write the answer as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

According to Theorem 2, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely, $R = 1$.

EXAMPLE 6 Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

SOLUTION We notice that the derivative of this function is $1/(1 + x)$. From Equation 1 we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots \quad |x| < 1$$

Integrating both sides of this equation, we get

$$\begin{aligned} \ln(1+x) &= \int \frac{1}{1+x} dx = \int (1 - x + x^2 - x^3 + \dots) dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + C \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C \quad |x| < 1 \end{aligned}$$

To determine the value of C we put $x = 0$ in this equation and obtain $\ln(1+0) = C$.

Thus $C = 0$ and

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad |x| < 1$$

The radius of convergence is the same as for the original series: $R = 1$. ■

EXAMPLE 7 Find a power series representation for $f(x) = \tan^{-1}x$.

SOLUTION We observe that $f'(x) = 1/(1 + x^2)$ and find the required series by integrating the power series for $1/(1 + x^2)$ found in Example 1.

$$\begin{aligned} \tan^{-1}x &= \int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + \dots) dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \end{aligned}$$

To find C we put $x = 0$ and obtain $C = \tan^{-1}0 = 0$. Therefore

$$\begin{aligned} \tan^{-1}x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

Since the radius of convergence of the series for $1/(1 + x^2)$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1. ■

EXAMPLE 8

- Evaluate $\int [1/(1 + x^7)]dx$ as a power series.
- Use part (a) to approximate $\int_0^{0.5} [1/(1 + x^7)]dx$ correct to within 10^{-7} .

SOLUTION

- The first step is to express the integrand, $1/(1 + x^7)$, as the sum of a power series. As in Example 1, we start with Equation 1 and replace x by $-x^7$:

$$\begin{aligned} \frac{1}{1 + x^7} &= \frac{1}{1 - (-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} - \dots \end{aligned}$$

Now we integrate term by term:

$$\begin{aligned} \int \frac{1}{1 + x^7} dx &= \int \sum_{n=0}^{\infty} (-1)^n x^{7n} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \\ &= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots \end{aligned}$$

This series converges for $|-x^7| < 1$, that is, for $|x| < 1$.

- (b) In applying the Fundamental Theorem of Calculus, it doesn't matter which antiderivative we use, so let's use the antiderivative from part (a) with $C = 0$:

$$\begin{aligned}\int_0^{0.5} \frac{1}{1+x^7} dx &= \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots \right]_0^{0.5} \\ &= \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \dots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \dots\end{aligned}$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem. If we stop adding after the term with $n = 3$, the error is smaller than the term with $n = 4$:

$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So we have

$$\int_0^{0.5} \frac{1}{1+x^7} dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374$$

11.9 EXERCISES

1. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$? Why?
2. Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

3–10 Find a power series representation for the function and determine the interval of convergence.

$$3. f(x) = \frac{1}{1+x}$$

$$4. f(x) = \frac{5}{1-4x^2}$$

$$5. f(x) = \frac{2}{3-x}$$

$$6. f(x) = \frac{4}{2x+3}$$

$$7. f(x) = \frac{x^2}{x^4+16}$$

$$8. f(x) = \frac{x}{2x^2+1}$$

$$9. f(x) = \frac{x-1}{x+2}$$

$$10. f(x) = \frac{x+a}{x^2+a^2}, \quad a > 0$$

11–12 Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

$$11. f(x) = \frac{2x-4}{x^2-4x+3}$$

$$12. f(x) = \frac{2x+3}{x^2+3x+2}$$

13. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

What is the radius of convergence?

- (b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

- (c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

14. (a) Use Equation 1 to find a power series representation for

$f(x) = \ln(1-x)$. What is the radius of convergence?

- (b) Use part (a) to find a power series for $f(x) = x \ln(1-x)$.

- (c) By putting $x = \frac{1}{2}$ in your result from part (a), express $\ln 2$ as the sum of an infinite series.

15–20 Find a power series representation for the function and determine the radius of convergence.

$$15. f(x) = \ln(5-x)$$

$$16. f(x) = x^2 \tan^{-1}(x^3)$$

$$17. f(x) = \frac{x}{(1+4x)^2}$$

$$18. f(x) = \left(\frac{x}{2-x} \right)^3$$

$$19. f(x) = \frac{1+x}{(1-x)^2}$$

$$20. f(x) = \frac{x^2+x}{(1-x)^3}$$

- 21–24** Find a power series representation for f , and graph f and several partial sums $s_n(x)$ on the same screen. What happens as n increases?

21. $f(x) = \frac{x^2}{x^2 + 1}$

22. $f(x) = \ln(1 + x^4)$

23. $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

24. $f(x) = \tan^{-1}(2x)$

- 25–28** Evaluate the indefinite integral as a power series. What is the radius of convergence?

25. $\int \frac{t}{1-t^8} dt$

26. $\int \frac{t}{1+t^3} dt$

27. $\int x^2 \ln(1+x) dx$

28. $\int \frac{\tan^{-1} x}{x} dx$

- 29–32** Use a power series to approximate the definite integral to six decimal places.

29. $\int_0^{0.3} \frac{x}{1+x^3} dx$

30. $\int_0^{1/2} \arctan(x/2) dx$

31. $\int_0^{0.2} x \ln(1+x^2) dx$

32. $\int_0^{0.3} \frac{x^2}{1+x^4} dx$

- 33.** Use the result of Example 7 to compute $\arctan 0.2$ correct to five decimal places.

- 34.** Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

- 35.** (a) Show that J_0 (the Bessel function of order 0 given in Example 4) satisfies the differential equation

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$$

- (b) Evaluate $\int_0^1 J_0(x) dx$ correct to three decimal places.

- 36.** The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}}$$

- (a) Show that J_1 satisfies the differential equation

$$x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x) = 0$$

- (b) Show that $J_0'(x) = -J_1(x)$.

- 37.** (a) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution of the differential equation

$$f'(x) = f(x)$$

- (b) Show that $f(x) = e^x$.

- 38.** Let $f_n(x) = (\sin nx)/n^2$. Show that the series $\sum f_n(x)$ converges for all values of x but the series of derivatives $\sum f'_n(x)$ diverges when $x = 2n\pi$, n an integer. For what values of x does the series $\sum f''_n(x)$ converge?

- 39.** Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for f , f' , and f'' .

- 40.** (a) Starting with the geometric series $\sum_{n=0}^{\infty} x^n$, find the sum of the series

$$\sum_{n=1}^{\infty} n x^{n-1} \quad |x| < 1$$

- (b) Find the sum of each of the following series.

$$(i) \sum_{n=1}^{\infty} n x^n, \quad |x| < 1 \quad (ii) \sum_{n=1}^{\infty} \frac{n}{2^n}$$

- (c) Find the sum of each of the following series.

$$(i) \sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1$$

$$(ii) \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} \quad (iii) \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

- 41.** Use the power series for $\tan^{-1} x$ to prove the following expression for π as the sum of an infinite series:

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

- 42.** (a) By completing the square, show that

$$\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \frac{\pi}{3\sqrt{3}}$$

- (b) By factoring $x^3 + 1$ as a sum of cubes, rewrite the integral in part (a). Then express $1/(x^3 + 1)$ as the sum of a power series and use it to prove the following formula for π :

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2} \right)$$

11.10 Taylor and Maclaurin Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

We start by supposing that f is any function that can be represented by a power series

$$\boxed{1} \quad f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \dots \quad |x - a| < R$$

Let's try to determine what the coefficients c_n must be in terms of f . To begin, notice that if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

By Theorem 11.9.2, we can differentiate the series in Equation 1 term by term:

$$\boxed{2} \quad f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \dots \quad |x - a| < R$$

and substitution of $x = a$ in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$\boxed{3} \quad f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \dots \quad |x - a| < R$$

Again we put $x = a$ in Equation 3. The result is

$$f''(a) = 2c_2$$

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

$$\boxed{4} \quad f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \dots \quad |x - a| < R$$

and substitution of $x = a$ in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdots n c_n = n! c_n$$

Solving this equation for the n th coefficient c_n , we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus we have proved the following theorem.

5 Theorem If f has a power series representation (expansion) at a , that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \quad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see that if f has a power series expansion at a , then it must be of the following form.

$$\begin{aligned} 6 \quad f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots \end{aligned}$$

Taylor and Maclaurin

The Taylor series is named after the English mathematician Brook Taylor (1685–1731) and the Maclaurin series is named in honor of the Scottish mathematician Colin Maclaurin (1698–1746) despite the fact that the Maclaurin series is really just a special case of the Taylor series. But the idea of representing particular functions as sums of power series goes back to Newton, and the general Taylor series was known to the Scottish mathematician James Gregory in 1668 and to the Swiss mathematician John Bernoulli in the 1690s. Taylor was apparently unaware of the work of Gregory and Bernoulli when he published his discoveries on series in 1715 in his book *Methodus incrementorum directa et inversa*. Maclaurin series are named after Colin Maclaurin because he popularized them in his calculus textbook *Treatise of Fluxions* published in 1742.

The series in Equation 6 is called the **Taylor series of the function f at a** (or **about a** or **centered at a**). For the special case $a = 0$ the Taylor series becomes

$$7 \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

NOTE We have shown that if f can be represented as a power series about a , then f is equal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series. An example of such a function is given in Exercise 84.

EXAMPLE 1 Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

SOLUTION If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n . Therefore the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

To find the radius of convergence we let $a_n = x^n/n!$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0 < 1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$.

The conclusion we can draw from Theorem 5 and Example 1 is that if e^x has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So how can we determine whether e^x does have a power series representation?

Let's investigate the more general question: under what circumstances is a function equal to the sum of its Taylor series? In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n \end{aligned}$$

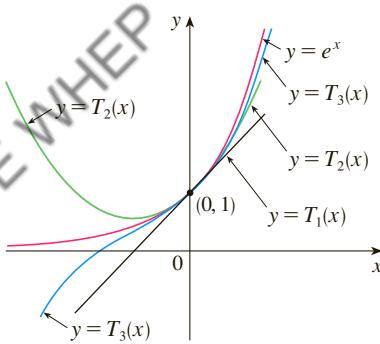


FIGURE 1

As n increases, $T_n(x)$ appears to approach e^x in Figure 1. This suggests that e^x is equal to the sum of its Taylor series.

Notice that T_n is a polynomial of degree n called the **n th-degree Taylor polynomial of f at a** . For instance, for the exponential function $f(x) = e^x$, the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1, 2$, and 3 are

$$T_1(x) = 1 + x \quad T_2(x) = 1 + x + \frac{x^2}{2!} \quad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The graphs of the exponential function and these three Taylor polynomials are drawn in Figure 1.

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n \rightarrow \infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \quad \text{so that} \quad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n \rightarrow \infty} R_n(x) = 0$, then it follows that

$$\lim_{n \rightarrow \infty} T_n(x) = \lim_{n \rightarrow \infty} [f(x) - R_n(x)] = f(x) - \lim_{n \rightarrow \infty} R_n(x) = f(x)$$

We have therefore proved the following theorem.

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the n th-degree Taylor polynomial of f at a and

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In trying to show that $\lim_{n \rightarrow \infty} R_n(x) = 0$ for a specific function f , we usually use the following theorem.

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } |x - a| \leq d$$

To see why this is true for $n = 1$, we assume that $|f''(x)| \leq M$. In particular, we have $f''(x) \leq M$, so for $a \leq x \leq a + d$ we have

$$\int_a^x f''(t) dt \leq \int_a^x M dt$$

Formulas for the Taylor Remainder Term

As alternatives to Taylor's Inequality, we have the following formulas for the remainder term. If $f^{(n+1)}$ is continuous on an interval I and $x \in I$, then

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt$$

This is called the *integral form of the remainder term*. Another formula, called *Lagrange's form of the remainder term*, states that there is a number z between x and a such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-a)^{n+1}$$

This version is an extension of the Mean Value Theorem (which is the case $n = 0$).

Proofs of these formulas, together with discussions of how to use them to solve the examples of Sections 11.10 and 11.11, are given on the website

www.stewartcalculus.com

Click on *Additional Topics* and then on *Formulas for the Remainder Term in Taylor series*.

An antiderivative of f'' is f' , so by Part 2 of the Fundamental Theorem of Calculus, we have

$$f'(x) - f'(a) \leq M(x - a) \quad \text{or} \quad f'(x) \leq f'(a) + M(x - a)$$

Thus

$$\int_a^x f'(t) dt \leq \int_a^x [f'(a) + M(t - a)] dt$$

$$f(x) - f(a) \leq f'(a)(x - a) + M \frac{(x - a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x - a) \leq \frac{M}{2}(x - a)^2$$

But $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$. So

$$R_1(x) \leq \frac{M}{2}(x - a)^2$$

A similar argument, using $f''(x) \geq -M$, shows that

$$R_1(x) \geq -\frac{M}{2}(x - a)^2$$

So

$$|R_1(x)| \leq \frac{M}{2} |x - a|^2$$

Although we have assumed that $x > a$, similar calculations show that this inequality is also true for $x < a$.

This proves Taylor's Inequality for the case where $n = 1$. The result for any n is proved in a similar way by integrating $n + 1$ times. (See Exercise 83 for the case $n = 2$.)

NOTE In Section 11.11 we will explore the use of Taylor's Inequality in approximating functions. Our immediate use of it is in conjunction with Theorem 8.

In applying Theorems 8 and 9 it is often helpful to make use of the following fact.

10

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x$$

This is true because we know from Example 1 that the series $\sum x^n/n!$ converges for all x and so its n th term approaches 0.

EXAMPLE 2 Prove that e^x is equal to the sum of its Maclaurin series.

SOLUTION If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n . If d is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$. So Taylor's Inequality, with $a = 0$ and $M = e^d$, says that

$$|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1} \quad \text{for } |x| \leq d$$

Notice that the same constant $M = e^d$ works for every value of n . But, from Equation 10, we have

$$\lim_{n \rightarrow \infty} \frac{e^d}{(n+1)!} |x|^{n+1} = e^d \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that $\lim_{n \rightarrow \infty} |R_n(x)| = 0$ and therefore $\lim_{n \rightarrow \infty} R_n(x) = 0$ for all values of x . By Theorem 8, e^x is equal to the sum of its Maclaurin series, that is,

$$11 \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

In particular, if we put $x = 1$ in Equation 11, we obtain the following expression for the number e as a sum of an infinite series:

$$12 \quad e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

EXAMPLE 3 Find the Taylor series for $f(x) = e^x$ at $a = 2$.

SOLUTION We have $f^{(n)}(2) = e^2$ and so, putting $a = 2$ in the definition of a Taylor series (6), we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x - 2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x - 2)^n$$

Again it can be verified, as in Example 1, that the radius of convergence is $R = \infty$. As in Example 2 we can verify that $\lim_{n \rightarrow \infty} R_n(x) = 0$, so

$$13 \quad e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x - 2)^n \quad \text{for all } x$$

We have two power series expansions for e^x , the Maclaurin series in Equation 11 and the Taylor series in Equation 13. The first is better if we are interested in values of x near 0 and the second is better if x is near 2.

EXAMPLE 4 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x .

SOLUTION We arrange our computation in two columns as follows:

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \\ f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \end{array}$$

Figure 2 shows the graph of $\sin x$ together with its Taylor (or Maclaurin) polynomials

$$\begin{aligned} T_1(x) &= x \\ T_3(x) &= x - \frac{x^3}{3!} \\ T_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \end{aligned}$$

Notice that, as n increases, $T_n(x)$ becomes a better approximation to $\sin x$.

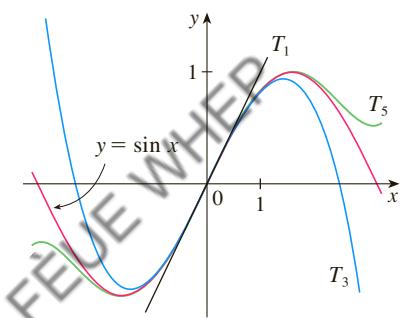


FIGURE 2

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\begin{aligned} f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Since $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, we know that $|f^{(n+1)}(x)| \leq 1$ for all x . So we can take $M = 1$ in Taylor's Inequality:

14 $|R_n(x)| \leq \frac{M}{(n+1)!} |x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$

By Equation 10 the right side of this inequality approaches 0 as $n \rightarrow \infty$, so $|R_n(x)| \rightarrow 0$ by the Squeeze Theorem. It follows that $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$, so $\sin x$ is equal to the sum of its Maclaurin series by Theorem 8. ■

We state the result of Example 4 for future reference.

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$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \end{aligned}$$

EXAMPLE 5 Find the Maclaurin series for $\cos x$.

SOLUTION We could proceed directly as in Example 4, but it's easier to differentiate the Maclaurin series for $\sin x$ given by Equation 15:

$$\begin{aligned} \cos x &= \frac{d}{dx} (\sin x) = \frac{d}{dx} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \\ &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

The Maclaurin series for e^x , $\sin x$, and $\cos x$ that we found in Examples 2, 4, and 5 were discovered, using different methods, by Newton. These equations are remarkable because they say we know everything about each of these functions if we know all its derivatives at the single number 0.

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$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x\end{aligned}$$

EXAMPLE 6 Find the Maclaurin series for the function $f(x) = x \cos x$.

SOLUTION Instead of computing derivatives and substituting in Equation 7, it's easier to multiply the series for $\cos x$ (Equation 16) by x :

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

EXAMPLE 7 Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

SOLUTION Arranging our work in columns, we have

$$\begin{array}{ll} f(x) = \sin x & f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \\ f'(x) = \cos x & f'\left(\frac{\pi}{3}\right) = \frac{1}{2} \\ f''(x) = -\sin x & f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\ f'''(x) = -\cos x & f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2} \end{array}$$

We have obtained two different series representations for $\sin x$, the Maclaurin series in Example 4 and the Taylor series in Example 7. It is best to use the Maclaurin series for values of x near 0 and the Taylor series for x near $\pi/3$. Notice that the third Taylor polynomial T_3 in Figure 3 is a good approximation to $\sin x$ near $\pi/3$ but not as good near 0. Compare it with the third Maclaurin polynomial T_3 in Figure 2, where the opposite is true.

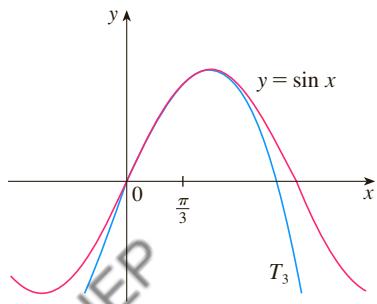


FIGURE 3

and this pattern repeats indefinitely. Therefore the Taylor series at $\pi/3$ is

$$\begin{aligned}f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!} \left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!} \left(x - \frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!} \left(x - \frac{\pi}{3}\right)^3 + \dots \\ = \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!} \left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!} \left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!} \left(x - \frac{\pi}{3}\right)^3 + \dots\end{aligned}$$

The proof that this series represents $\sin x$ for all x is very similar to that in Example 4. [Just replace x by $x - \pi/3$ in (14).] We can write the series in sigma notation if we separate the terms that contain $\sqrt{3}$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!} \left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!} \left(x - \frac{\pi}{3}\right)^{2n+1}$$

The power series that we obtained by indirect methods in Examples 5 and 6 and in Section 11.9 are indeed the Taylor or Maclaurin series of the given functions because Theorem 5 asserts that, no matter how a power series representation $f(x) = \sum c_n(x - a)^n$ is obtained, it is always true that $c_n = f^{(n)}(a)/n!$. In other words, the coefficients are uniquely determined.

EXAMPLE 8 Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

SOLUTION Arranging our work in columns, we have

$$\begin{array}{ll} f(x) = (1 + x)^k & f(0) = 1 \\ f'(x) = k(1 + x)^{k-1} & f'(0) = k \\ f''(x) = k(k - 1)(1 + x)^{k-2} & f''(0) = k(k - 1) \\ f'''(x) = k(k - 1)(k - 2)(1 + x)^{k-3} & f'''(0) = k(k - 1)(k - 2) \\ \vdots & \vdots \\ f^{(n)}(x) = k(k - 1) \cdots (k - n + 1)(1 + x)^{k-n} & f^{(n)}(0) = k(k - 1) \cdots (k - n + 1) \end{array}$$

Therefore the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k - 1) \cdots (k - n + 1)}{n!} x^n$$

This series is called the **binomial series**. Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of k none of the terms is 0 and so we can try the Ratio Test. If the n th term is a_n , then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{k(k - 1) \cdots (k - n + 1)(k - n)x^{n+1}}{(n + 1)!} \cdot \frac{n!}{k(k - 1) \cdots (k - n + 1)x^n} \right| \\ &= \frac{|k - n|}{n + 1} |x| = \frac{\left| 1 - \frac{k}{n} \right|}{1 + \frac{1}{n}} |x| \rightarrow |x| \quad \text{as } n \rightarrow \infty \end{aligned}$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$. ■

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k - 1)(k - 2) \cdots (k - n + 1)}{n!}$$

and these numbers are called the **binomial coefficients**.

The following theorem states that $(1 + x)^k$ is equal to the sum of its Maclaurin series. It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult. The proof outlined in Exercise 85 is much easier.

17 The Binomial Series If k is any real number and $|x| < 1$, then

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

Although the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, ± 1 , depends on the value of k . It turns out that the series converges at 1 if $-1 < k \leq 0$ and at both endpoints if $k \geq 0$. Notice that if k is a positive integer and $n > k$, then the expression for $\binom{k}{n}$ contains a factor $(k - k)$, so $\binom{k}{n} = 0$ for $n > k$. This means that the series terminates and reduces to the ordinary Binomial Theorem when k is a positive integer. (See Reference Page 1.)

EXAMPLE 9 Find the Maclaurin series for the function $f(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

SOLUTION We rewrite $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1 - \frac{x}{4}\right)}} = \frac{1}{2\sqrt{1 - \frac{x}{4}}} = \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-1/2}$$

Using the binomial series with $k = -\frac{1}{2}$ and with x replaced by $-x/4$, we have

$$\begin{aligned} \frac{1}{\sqrt{4-x}} &= \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-1/2} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n \\ &= \frac{1}{2} \left[1 + \left(-\frac{1}{2}\right) \left(-\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} \left(-\frac{x}{4}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} \left(-\frac{x}{4}\right)^3 \right. \\ &\quad \left. + \dots + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2}) \cdots (-\frac{1}{2} - n + 1)}{n!} \left(-\frac{x}{4}\right)^n + \dots \right] \\ &= \frac{1}{2} \left[1 + \frac{1}{8}x + \frac{1 \cdot 3}{2!8^2}x^2 + \frac{1 \cdot 3 \cdot 5}{3!8^3}x^3 + \dots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!8^n}x^n + \dots \right] \end{aligned}$$

We know from (17) that this series converges when $|-x/4| < 1$, that is, $|x| < 4$, so the radius of convergence is $R = 4$.

We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

Table 1

Important Maclaurin
Series and Their Radii
of Convergence

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$	$R = 1$

EXAMPLE 10 Find the sum of the series $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \dots$.

SOLUTION With sigma notation we can write the given series as

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}$$

Then from Table 1 we see that this series matches the entry for $\ln(1+x)$ with $x = \frac{1}{2}$. So

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} = \ln\left(1 + \frac{1}{2}\right) = \ln \frac{3}{2}$$

TEC Module 11.10/11.11 enables you to see how successive Taylor polynomials approach the original function.

One reason that Taylor series are important is that they enable us to integrate functions that we couldn't previously handle. In fact, in the introduction to this chapter we mentioned that Newton often integrated functions by first expressing them as power series and then integrating the series term by term. The function $f(x) = e^{-x^2}$ can't be integrated by techniques discussed so far because its antiderivative is not an elementary function (see Section 7.5). In the following example we use Newton's idea to integrate this function.

EXAMPLE 11

- (a) Evaluate $\int e^{-x^2} dx$ as an infinite series.
- (b) Evaluate $\int_0^1 e^{-x^2} dx$ correct to within an error of 0.001.

SOLUTION

- (a) First we find the Maclaurin series for $f(x) = e^{-x^2}$. Although it's possible to use the direct method, let's find it simply by replacing x with $-x^2$ in the series for e^x given in

Table 1. Thus, for all values of x ,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

Now we integrate term by term:

$$\begin{aligned} \int e^{-x^2} dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + (-1)^n \frac{x^{2n}}{n!} + \dots \right) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \dots \end{aligned}$$

This series converges for all x because the original series for e^{-x^2} converges for all x .

(b) The Fundamental Theorem of Calculus gives

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \dots \right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \dots \\ &\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475 \end{aligned}$$

We can take $C = 0$ in the antiderivative in part (a).

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$$

Another use of Taylor series is illustrated in the next example. The limit could be found with l'Hospital's Rule, but instead we use a series.

EXAMPLE 12 Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$.

SOLUTION Using the Maclaurin series for e^x , we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x \rightarrow 0} \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) - 1 - x}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}{x^2} \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \dots \right) = \frac{1}{2} \end{aligned}$$

Some computer algebra systems compute limits in this way.

because power series are continuous functions.

Multiplication and Division of Power Series

If power series are added or subtracted, they behave like polynomials (Theorem 11.2.8 shows this). In fact, as the following example illustrates, they can also be multiplied and divided like polynomials. We find only the first few terms because the calculations for the later terms become tedious and the initial terms are the most important ones.

EXAMPLE 13 Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

SOLUTION

(a) Using the Maclaurin series for e^x and $\sin x$ in Table 1, we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(x - \frac{x^3}{3!} + \dots \right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$\begin{array}{r} 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\ \times \quad x \quad - \frac{1}{6}x^3 + \dots \\ \hline x + \quad x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \dots \\ + \quad \quad - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \dots \\ \hline x + \quad x^2 + \frac{1}{3}x^3 + \dots \end{array}$$

Thus

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \dots$$

(b) Using the Maclaurin series in Table 1, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots}$$

We use a procedure like long division:

$$\begin{array}{r} x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \\ 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots \overline{)x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots} \\ x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \dots \\ \hline \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots \\ \frac{1}{3}x^3 - \frac{1}{6}x^5 + \dots \\ \hline \frac{2}{15}x^5 + \dots \end{array}$$

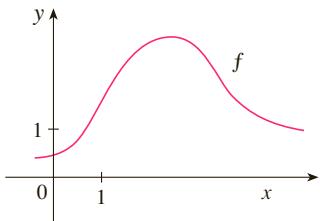
Thus

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

Although we have not attempted to justify the formal manipulations used in Example 13, they are legitimate. There is a theorem which states that if both $f(x) = \sum c_n x^n$ and $g(x) = \sum b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials, then the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$. For division we require $b_0 \neq 0$; the resulting series converges for sufficiently small $|x|$.

11.10 EXERCISES

1. If $f(x) = \sum_{n=0}^{\infty} b_n(x - 5)^n$ for all x , write a formula for b_8 .
2. The graph of f is shown.



- (a) Explain why the series

$$1.6 - 0.8(x - 1) + 0.4(x - 1)^2 - 0.1(x - 1)^3 + \dots$$

is *not* the Taylor series of f centered at 1.

- (b) Explain why the series

$$2.8 + 0.5(x - 2) + 1.5(x - 2)^2 - 0.1(x - 2)^3 + \dots$$

is *not* the Taylor series of f centered at 2.

3. If $f^{(n)}(0) = (n + 1)!$ for $n = 0, 1, 2, \dots$, find the Maclaurin series for f and its radius of convergence.

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n + 1)}$$

What is the radius of convergence of the Taylor series?

- 5–10** Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a .

5. $f(x) = xe^x, a = 0$

6. $f(x) = \frac{1}{1+x}, a = 2$

7. $f(x) = \sqrt[3]{x}, a = 8$

8. $f(x) = \ln x, a = 1$

9. $f(x) = \sin x, a = \pi/6$

10. $f(x) = \cos^2 x, a = 0$

- 11–18** Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \rightarrow 0$.] Also find the associated radius of convergence.

11. $f(x) = (1 - x)^{-2}$

12. $f(x) = \ln(1 + x)$

13. $f(x) = \cos x$

14. $f(x) = e^{-2x}$

15. $f(x) = 2^x$

16. $f(x) = x \cos x$

17. $f(x) = \sinh x$

18. $f(x) = \cosh x$

- 19–26** Find the Taylor series for $f(x)$ centered at the given value of a . [Assume that f has a power series expansion. Do not show that $R_n(x) \rightarrow 0$.] Also find the associated radius of convergence.

19. $f(x) = x^5 + 2x^3 + x, a = 2$

20. $f(x) = x^6 - x^4 + 2, a = -2$

21. $f(x) = \ln x, a = 2$

22. $f(x) = 1/x, a = -3$

23. $f(x) = e^{2x}, a = 3$

24. $f(x) = \cos x, a = \pi/2$

25. $f(x) = \sin x, a = \pi$

26. $f(x) = \sqrt{x}, a = 16$

27. Prove that the series obtained in Exercise 13 represents $\cos x$ for all x .

28. Prove that the series obtained in Exercise 25 represents $\sin x$ for all x .

29. Prove that the series obtained in Exercise 17 represents $\sinh x$ for all x .

30. Prove that the series obtained in Exercise 18 represents $\cosh x$ for all x .

- 31–34** Use the binomial series to expand the function as a power series. State the radius of convergence.

31. $\sqrt[4]{1-x}$

32. $\sqrt[3]{8+x}$

33. $\frac{1}{(2+x)^3}$

34. $(1-x)^{3/4}$

- 35–44** Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

35. $f(x) = \arctan(x^2)$

36. $f(x) = \sin(\pi x/4)$

37. $f(x) = x \cos 2x$

38. $f(x) = e^{3x} - e^{2x}$

39. $f(x) = x \cos(\frac{1}{2}x^2)$

40. $f(x) = x^2 \ln(1 + x^3)$

41. $f(x) = \frac{x}{\sqrt{4+x^2}}$

42. $f(x) = \frac{x^2}{\sqrt{2+x}}$

43. $f(x) = \sin^2 x$ [Hint: Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.]

44. $f(x) = \begin{cases} \frac{x - \sin x}{x^3} & \text{if } x \neq 0 \\ \frac{1}{6} & \text{if } x = 0 \end{cases}$

- 45–48** Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials on the same screen. What do you notice about the relationship between these polynomials and f ?

45. $f(x) = \cos(x^2)$

46. $f(x) = \ln(1 + x^2)$

47. $f(x) = xe^{-x}$

48. $f(x) = \tan^{-1}(x^3)$

- 49.** Use the Maclaurin series for $\cos x$ to compute $\cos 5^\circ$ correct to five decimal places.

50. Use the Maclaurin series for e^x to calculate $1/\sqrt[10]{e}$ correct to five decimal places.

- 51.** (a) Use the binomial series to expand $1/\sqrt{1-x^2}$.
 (b) Use part (a) to find the Maclaurin series for $\sin^{-1}x$.

- 52.** (a) Expand $1/\sqrt[4]{1+x}$ as a power series.
 (b) Use part (a) to estimate $1/\sqrt[4]{1.1}$ correct to three decimal places.

53–56 Evaluate the indefinite integral as an infinite series.

53. $\int \sqrt{1+x^3} dx$

54. $\int x^2 \sin(x^2) dx$

55. $\int \frac{\cos x - 1}{x} dx$

56. $\int \arctan(x^2) dx$

57–60 Use series to approximate the definite integral to within the indicated accuracy.

57. $\int_0^{1/2} x^3 \arctan x dx$ (four decimal places)

58. $\int_0^1 \sin(x^4) dx$ (four decimal places)

59. $\int_0^{0.4} \sqrt{1+x^4} dx$ ($|\text{error}| < 5 \times 10^{-6}$)

60. $\int_0^{0.5} x^2 e^{-x^2} dx$ ($|\text{error}| < 0.001$)

61–65 Use series to evaluate the limit.

61. $\lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{x^2}$

62. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x}$

63. $\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

64. $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}$

65. $\lim_{x \rightarrow 0} \frac{x^3 - 3x + 3 \tan^{-1}x}{x^5}$

66. Use the series in Example 13(b) to evaluate

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$$

We found this limit in Example 6.8.4 using l'Hospital's Rule three times. Which method do you prefer?

67–72 Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

67. $y = e^{-x^2} \cos x$

68. $y = \sec x$

69. $y = \frac{x}{\sin x}$

70. $y = e^x \ln(1+x)$

71. $y = (\arctan x)^2$

72. $y = e^x \sin^2 x$

73–80 Find the sum of the series.

73. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$

74. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

75. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n 5^n}$

76. $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

77. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

78. $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \dots$

79. $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \dots$

80. $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \dots$

81. Show that if p is an n th-degree polynomial, then

$$p(x+1) = \sum_{i=0}^n \frac{p^{(i)}(x)}{i!}$$

82. If $f(x) = (1+x^3)^{30}$, what is $f^{(58)}(0)$?

83. Prove Taylor's Inequality for $n = 2$, that is, prove that if $|f'''(x)| \leq M$ for $|x-a| \leq d$, then

$$|R_2(x)| \leq \frac{M}{6} |x-a|^3 \quad \text{for } |x-a| \leq d$$

84. (a) Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not equal to its Maclaurin series.

(b) Graph the function in part (a) and comment on its behavior near the origin.

85. Use the following steps to prove (17).

(a) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. Differentiate this series to show that

$$g'(x) = \frac{kg(x)}{1+x} \quad -1 < x < 1$$

(b) Let $h(x) = (1+x)^{-k} g(x)$ and show that $h'(x) = 0$.

(c) Deduce that $g(x) = (1+x)^k$.

86. In Exercise 10.2.53 it was shown that the length of the ellipse $x = a \sin \theta$, $y = b \cos \theta$, where $a > b > 0$, is

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta$$

where $e = \sqrt{a^2 - b^2}/a$ is the eccentricity of the ellipse.

Expand the integrand as a binomial series and use the result of Exercise 7.1.50 to express L as a series in powers of the eccentricity up to the term in e^6 .

LABORATORY PROJECT CAS AN ELUSIVE LIMIT

This project deals with the function

$$f(x) = \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$$

1. Use your computer algebra system to evaluate $f(x)$ for $x = 1, 0.1, 0.01, 0.001$, and 0.0001 . Does it appear that f has a limit as $x \rightarrow 0$?
2. Use the CAS to graph f near $x = 0$. Does it appear that f has a limit as $x \rightarrow 0$?
3. Try to evaluate $\lim_{x \rightarrow 0} f(x)$ with l'Hospital's Rule, using the CAS to find derivatives of the numerator and denominator. What do you discover? How many applications of l'Hospital's Rule are required?
4. Evaluate $\lim_{x \rightarrow 0} f(x)$ by using the CAS to find sufficiently many terms in the Taylor series of the numerator and denominator. (Use the command `taylor` in Maple or `Series` in Mathematica.)
5. Use the limit command on your CAS to find $\lim_{x \rightarrow 0} f(x)$ directly. (Most computer algebra systems use the method of Problem 4 to compute limits.)
6. In view of the answers to Problems 4 and 5, how do you explain the results of Problems 1 and 2?

WRITING PROJECT

HOW NEWTON DISCOVERED THE BINOMIAL SERIES

The Binomial Theorem, which gives the expansion of $(a + b)^k$, was known to Chinese mathematicians many centuries before the time of Newton for the case where the exponent k is a positive integer. In 1665, when he was 22, Newton was the first to discover the infinite series expansion of $(a + b)^k$ when k is a fractional exponent (positive or negative). He didn't publish his discovery, but he stated it and gave examples of how to use it in a letter (now called the *epistola prior*) dated June 13, 1676, that he sent to Henry Oldenburg, secretary of the Royal Society of London, to transmit to Leibniz. When Leibniz replied, he asked how Newton had discovered the binomial series. Newton wrote a second letter, the *epistola posterior* of October 24, 1676, in which he explained in great detail how he arrived at his discovery by a very indirect route. He was investigating the areas under the curves $y = (1 - x^2)^{n/2}$ from 0 to x for $n = 0, 1, 2, 3, 4, \dots$. These are easy to calculate if n is even. By observing patterns and interpolating, Newton was able to guess the answers for odd values of n . Then he realized he could get the same answers by expressing $(1 - x^2)^{n/2}$ as an infinite series.

Write a report on Newton's discovery of the binomial series. Start by giving the statement of the binomial series in Newton's notation (see the *epistola prior* on page 285 of [4] or page 402 of [2]). Explain why Newton's version is equivalent to Theorem 17 on page 807. Then read Newton's *epistola posterior* (page 287 in [4] or page 404 in [2]) and explain the patterns that Newton discovered in the areas under the curves $y = (1 - x^2)^{n/2}$. Show how he was able to guess the areas under the remaining curves and how he verified his answers. Finally, explain how these discoveries led to the binomial series. The books by Edwards [1] and Katz [3] contain commentaries on Newton's letters.

1. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 178–187.
2. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987).
3. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 463–466.
4. D. J. Struik, ed., *A Sourcebook in Mathematics, 1200–1800* (Princeton, NJ: Princeton University Press, 1969).

11.11 Applications of Taylor Polynomials

In this section we explore two types of applications of Taylor polynomials. First we look at how they are used to approximate functions—computer scientists like them because polynomials are the simplest of functions. Then we investigate how physicists and engineers use them in such fields as relativity, optics, blackbody radiation, electric dipoles, the velocity of water waves, and building highways across a desert.

■ Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

In Section 11.10 we introduced the notation $T_n(x)$ for the n th partial sum of this series and called it the n th-degree Taylor polynomial of f at a . Thus

$$\begin{aligned} T_n(x) &= \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n \end{aligned}$$

Since f is the sum of its Taylor series, we know that $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ and so T_n can be used as an approximation to f : $f(x) \approx T_n(x)$.

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

is the same as the linearization of f at a that we discussed in Section 2.9. Notice also that T_1 and its derivative have the same values at a that f and f' have. In general, it can be shown that the derivatives of T_n at a agree with those of f up to and including derivatives of order n .

To illustrate these ideas let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials, as shown in Figure 1. The graph of T_1 is the tangent line to $y = e^x$ at $(0, 1)$; this tangent line is the best linear approximation to e^x near $(0, 1)$. The graph of T_2 is the parabola $y = 1 + x + x^2/2$, and the graph of T_3 is the cubic curve $y = 1 + x + x^2/2 + x^3/6$, which is a closer fit to the exponential curve $y = e^x$ than T_2 . The next Taylor polynomial T_4 would be an even better approximation, and so on.

The values in the table give a numerical demonstration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$. We see that when $x = 0.2$ the convergence is very rapid, but when $x = 3$ it is somewhat slower. In fact, the farther x is from 0, the more slowly $T_n(x)$ converges to e^x .

When using a Taylor polynomial T_n to approximate a function f , we have to ask the questions: How good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

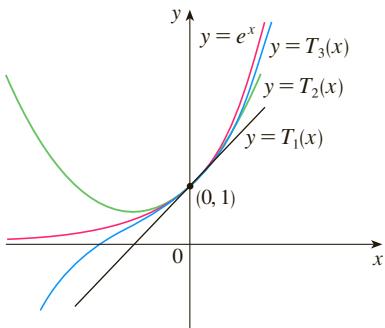


FIGURE 1

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

There are three possible methods for estimating the size of the error:

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.
2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
3. In all cases we can use Taylor's Inequality (Theorem 11.10.9), which says that if $|f^{(n+1)}(x)| \leq M$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

EXAMPLE 1

- (a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
 (b) How accurate is this approximation when $7 \leq x \leq 9$?

SOLUTION

$$(a) \quad f(x) = \sqrt[3]{x} = x^{1/3} \quad f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \quad f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3}$$

Thus the second-degree Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 \end{aligned}$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

- (b) The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem in this example. But we can use Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \leq \frac{M}{3!} |x-8|^3$$

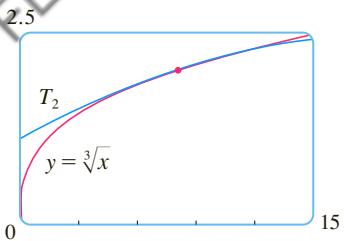
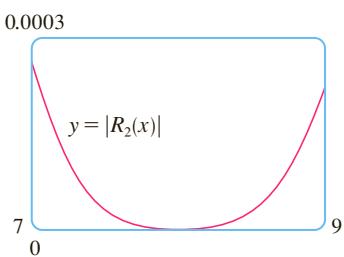
where $|f'''(x)| \leq M$. Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leq \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

Therefore we can take $M = 0.0021$. Also $7 \leq x \leq 9$, so $-1 \leq x-8 \leq 1$ and $|x-8| \leq 1$. Then Taylor's Inequality gives

$$|R_2(x)| \leq \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \leq x \leq 9$, the approximation in part (a) is accurate to within 0.0004. ■

**FIGURE 2****FIGURE 3**

Let's use a graphing device to check the calculation in Example 1. Figure 2 shows that the graphs of $y = \sqrt[3]{x}$ and $y = T_2(x)$ are very close to each other when x is near 8. Figure 3 shows the graph of $|R_2(x)|$ computed from the expression

$$|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$$

We see from the graph that

$$|R_2(x)| < 0.0003$$

when $7 \leq x \leq 9$. Thus the error estimate from graphical methods is slightly better than the error estimate from Taylor's Inequality in this case.

EXAMPLE 2

- (a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$? Use this approximation to find $\sin 12^\circ$ correct to six decimal places.

- (b) For what values of x is this approximation accurate to within 0.00005?

SOLUTION

- (a) Notice that the Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

is alternating for all nonzero values of x , and the successive terms decrease in size because $|x| < 1$, so we can use the Alternating Series Estimation Theorem. The error in approximating $\sin x$ by the first three terms of its Maclaurin series is at most

$$\left| \frac{x^7}{7!} \right| = \frac{|x|^7}{5040}$$

If $-0.3 \leq x \leq 0.3$, then $|x| \leq 0.3$, so the error is smaller than

$$\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$$

To find $\sin 12^\circ$ we first convert to radian measure:

$$\begin{aligned} \sin 12^\circ &= \sin\left(\frac{12\pi}{180}\right) = \sin\left(\frac{\pi}{15}\right) \\ &\approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \frac{1}{5!} \approx 0.20791169 \end{aligned}$$

Thus, correct to six decimal places, $\sin 12^\circ \approx 0.207912$.

- (b) The error will be smaller than 0.00005 if

$$\frac{|x|^7}{5040} < 0.00005$$

Solving this inequality for x , we get

$$|x|^7 < 0.252 \quad \text{or} \quad |x| < (0.252)^{1/7} \approx 0.821$$

So the given approximation is accurate to within 0.00005 when $|x| < 0.82$. ■

TEC Module 11.10/11.11 graphically shows the remainders in Taylor polynomial approximations.

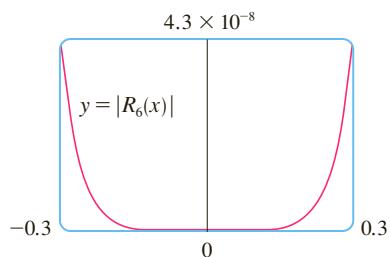


FIGURE 4

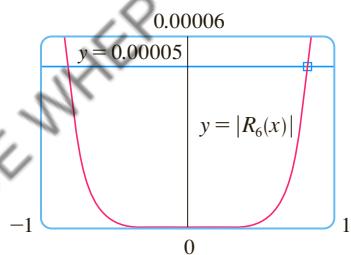


FIGURE 5

What if we use Taylor's Inequality to solve Example 2? Since $f^{(7)}(x) = -\cos x$, we have $|f^{(7)}(x)| \leq 1$ and so

$$|R_6(x)| \leq \frac{1}{7!} |x|^7$$

So we get the same estimates as with the Alternating Series Estimation Theorem.

What about graphical methods? Figure 4 shows the graph of

$$|R_6(x)| = \left| \sin x - \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \right) \right|$$

and we see from it that $|R_6(x)| < 4.3 \times 10^{-8}$ when $|x| \leq 0.3$. This is the same estimate that we obtained in Example 2. For part (b) we want $|R_6(x)| < 0.00005$, so we graph both $y = |R_6(x)|$ and $y = 0.00005$ in Figure 5. By placing the cursor on the right intersection point we find that the inequality is satisfied when $|x| < 0.82$. Again this is the same estimate that we obtained in the solution to Example 2.

If we had been asked to approximate $\sin 72^\circ$ instead of $\sin 12^\circ$ in Example 2, it would have been wise to use the Taylor polynomials at $a = \pi/3$ (instead of $a = 0$) because they are better approximations to $\sin x$ for values of x close to $\pi/3$. Notice that 72° is close to 60° (or $\pi/3$ radians) and the derivatives of $\sin x$ are easy to compute at $\pi/3$.

Figure 6 shows the graphs of the Maclaurin polynomial approximations

$$T_1(x) = x$$

$$T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

to the sine curve. You can see that as n increases, $T_n(x)$ is a good approximation to $\sin x$ on a larger and larger interval.

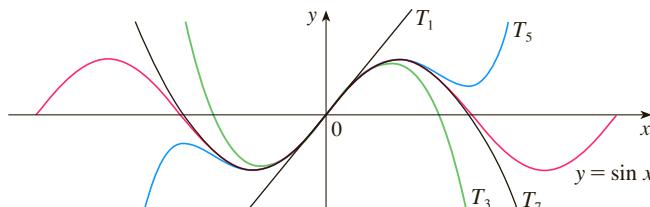


FIGURE 6

One use of the type of calculation done in Examples 1 and 2 occurs in calculators and computers. For instance, when you press the \sin or e^x key on your calculator, or when a computer programmer uses a subroutine for a trigonometric or exponential or Bessel function, in many machines a polynomial approximation is calculated. The polynomial is often a Taylor polynomial that has been modified so that the error is spread more evenly throughout an interval.

■ Applications to Physics

Taylor polynomials are also used frequently in physics. In order to gain insight into an equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series. In other words, the physicist uses a Taylor polynomial as an approximation to the function. Taylor's Inequality can then be used to gauge the accuracy of the approximation. The following example shows one way in which this idea is used in special relativity.

EXAMPLE 3 In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

- (a) Show that when v is very small compared with c , this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.
- (b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leq 100$ m/s.

SOLUTION

- (a) Using the expressions given for K and m , we get

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2 \left[\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right]$$

With $x = -v^2/c^2$, the Maclaurin series for $(1 + x)^{-1/2}$ is most easily computed as a binomial series with $k = -\frac{1}{2}$. (Notice that $|x| < 1$ because $v < c$.) Therefore we have

$$\begin{aligned} (1 + x)^{-1/2} &= 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \dots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots \end{aligned}$$

The upper curve in Figure 7 is the graph of the expression for the kinetic energy K of an object with velocity v in special relativity. The lower curve shows the function used for K in classical Newtonian physics. When v is much smaller than the speed of light, the curves are practically identical.

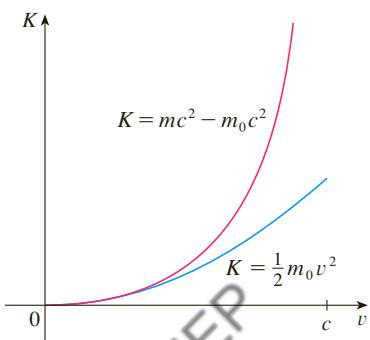


FIGURE 7

$$\begin{aligned} \text{and} \quad K &= m_0c^2 \left[\left(1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \dots \right)^{-1/2} - 1 \right] \\ &= m_0c^2 \left(\frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \dots \right) \end{aligned}$$

If v is much smaller than c , then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K \approx m_0c^2 \left(\frac{1}{2} \frac{v^2}{c^2} \right) = \frac{1}{2}m_0v^2$$

(b) If $x = -v^2/c^2$, $f(x) = m_0 c^2 [(1 + x)^{-1/2} - 1]$, and M is a number such that $|f''(x)| \leq M$, then we can use Taylor's Inequality to write

$$|R_1(x)| \leq \frac{M}{2!} x^2$$

We have $f''(x) = \frac{3}{4} m_0 c^2 (1 + x)^{-5/2}$ and we are given that $|v| \leq 100$ m/s, so

$$|f''(x)| = \frac{3m_0 c^2}{4(1 - v^2/c^2)^{5/2}} \leq \frac{3m_0 c^2}{4(1 - 100^2/c^2)^{5/2}} \quad (=M)$$

Thus, with $c = 3 \times 10^8$ m/s,

$$|R_1(x)| \leq \frac{1}{2} \cdot \frac{3m_0 c^2}{4(1 - 100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

So when $|v| \leq 100$ m/s, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.2 \times 10^{-10})m_0$. ■

Another application to physics occurs in optics. Figure 8 is adapted from *Optics*, 4th ed., by Eugene Hecht (San Francisco, 2002), page 153. It depicts a wave from the point source S meeting a spherical interface of radius R centered at C . The ray SA is refracted toward P .

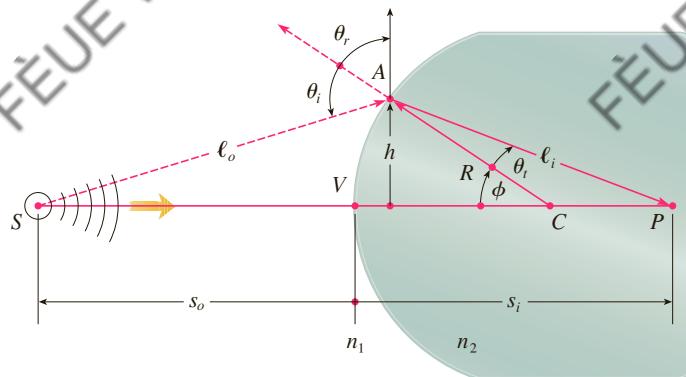


FIGURE 8

Refraction at a spherical interface

Source: Adapted from E. Hecht, *Optics*, 4e (Upper Saddle River, NJ: Pearson Education, 2002).

Using Fermat's principle that light travels so as to minimize the time taken, Hecht derives the equation

$$\boxed{1} \quad \frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R} \left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o} \right)$$

where n_1 and n_2 are indexes of refraction and ℓ_o , ℓ_i , s_o , and s_i are the distances indicated in Figure 8. By the Law of Cosines, applied to triangles ACS and ACP , we have

$$\boxed{2} \quad \ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R) \cos \phi}$$

$$\ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R) \cos \phi}$$

Here we use the identity

$$\cos(\pi - \phi) = -\cos \phi$$

Because Equation 1 is cumbersome to work with, Gauss, in 1841, simplified it by using the linear approximation $\cos \phi \approx 1$ for small values of ϕ . (This amounts to using the Taylor polynomial of degree 1.) Then Equation 1 becomes the following simpler equation [as you are asked to show in Exercise 34(a)]:

$$\boxed{3} \quad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

The resulting optical theory is known as *Gaussian optics*, or *first-order optics*, and has become the basic theoretical tool used to design lenses.

A more accurate theory is obtained by approximating $\cos \phi$ by its Taylor polynomial of degree 3 (which is the same as the Taylor polynomial of degree 2). This takes into account rays for which ϕ is not so small, that is, rays that strike the surface at greater distances h above the axis. In Exercise 34(b) you are asked to use this approximation to derive the more accurate equation

$$\boxed{4} \quad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2 \left[\frac{n_1}{2s_o} \left(\frac{1}{s_o} + \frac{1}{R} \right)^2 + \frac{n_2}{2s_i} \left(\frac{1}{R} - \frac{1}{s_i} \right)^2 \right]$$

The resulting optical theory is known as *third-order optics*.

Other applications of Taylor polynomials to physics and engineering are explored in Exercises 32, 33, 35, 36, 37, and 38, and in the Applied Project on page 823.

11.11 EXERCISES

1. (a) Find the Taylor polynomials up to degree 5 for $f(x) = \sin x$ centered at $a = 0$. Graph f and these polynomials on a common screen.
 (b) Evaluate f and these polynomials at $x = \pi/4, \pi/2$, and π .
 (c) Comment on how the Taylor polynomials converge to $f(x)$.
2. (a) Find the Taylor polynomials up to degree 3 for $f(x) = \tan x$ centered at $a = 0$. Graph f and these polynomials on a common screen.
 (b) Evaluate f and these polynomials at $x = \pi/6, \pi/4$, and $\pi/3$.
 (c) Comment on how the Taylor polynomials converge to $f(x)$.

- 3–10 Find the Taylor polynomial $T_3(x)$ for the function f centered at the number a . Graph f and T_3 on the same screen.

3. $f(x) = e^x, a = 1$
4. $f(x) = \sin x, a = \pi/6$
5. $f(x) = \cos x, a = \pi/2$
6. $f(x) = e^{-x} \sin x, a = 0$
7. $f(x) = \ln x, a = 1$

8. $f(x) = x \cos x, a = 0$

9. $f(x) = xe^{-2x}, a = 0$

10. $f(x) = \tan^{-1} x, a = 1$

- 11–12 Use a computer algebra system to find the Taylor polynomials T_n centered at a for $n = 2, 3, 4, 5$. Then graph these polynomials and f on the same screen.

11. $f(x) = \cot x, a = \pi/4$

12. $f(x) = \sqrt[3]{1 + x^2}, a = 0$

13–22

- (a) Approximate f by a Taylor polynomial with degree n at the number a .
- (b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.
- (c) Check your result in part (b) by graphing $|R_n(x)|$.

13. $f(x) = 1/x, a = 1, n = 2, 0.7 \leq x \leq 1.3$

14. $f(x) = x^{-1/2}, a = 4, n = 2, 3.5 \leq x \leq 4.5$

15. $f(x) = x^{2/3}$, $a = 1$, $n = 3$, $0.8 \leq x \leq 1.2$
16. $f(x) = \sin x$, $a = \pi/6$, $n = 4$, $0 \leq x \leq \pi/3$
17. $f(x) = \sec x$, $a = 0$, $n = 2$, $-0.2 \leq x \leq 0.2$
18. $f(x) = \ln(1 + 2x)$, $a = 1$, $n = 3$, $0.5 \leq x \leq 1.5$
19. $f(x) = e^{x^2}$, $a = 0$, $n = 3$, $0 \leq x \leq 0.1$
20. $f(x) = x \ln x$, $a = 1$, $n = 3$, $0.5 \leq x \leq 1.5$
21. $f(x) = x \sin x$, $a = 0$, $n = 4$, $-1 \leq x \leq 1$
22. $f(x) = \sinh 2x$, $a = 0$, $n = 5$, $-1 \leq x \leq 1$

23. Use the information from Exercise 5 to estimate $\cos 80^\circ$ correct to five decimal places.
24. Use the information from Exercise 16 to estimate $\sin 38^\circ$ correct to five decimal places.
25. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.
26. How many terms of the Maclaurin series for $\ln(1 + x)$ do you need to use to estimate $\ln 1.4$ to within 0.001?

27–29 Use the Alternating Series Estimation Theorem or Taylor's Inequality to estimate the range of values of x for which the given approximation is accurate to within the stated error. Check your answer graphically.

27. $\sin x \approx x - \frac{x^3}{6}$ ($|\text{error}| < 0.01$)
28. $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$ ($|\text{error}| < 0.005$)
29. $\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5}$ ($|\text{error}| < 0.05$)

30. Suppose you know that

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

and the Taylor series of f centered at 4 converges to $f(x)$ for all x in the interval of convergence. Show that the fifth-degree Taylor polynomial approximates $f(5)$ with error less than 0.0002.

31. A car is moving with speed 20 m/s and acceleration 2 m/s² at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate the distance traveled during the next minute?

32. The resistivity ρ of a conducting wire is the reciprocal of the conductivity and is measured in units of ohm-meters ($\Omega\text{-m}$). The resistivity of a given metal depends on the temperature according to the equation

$$\rho(t) = \rho_{20} e^{\alpha(t-20)}$$

where t is the temperature in $^\circ\text{C}$. There are tables that list the values of α (called the temperature coefficient) and ρ_{20} (the resistivity at 20°C) for various metals. Except at very low temperatures, the resistivity varies almost linearly with temperature and so it is common to approximate the expression for $\rho(t)$ by its first- or second-degree Taylor polynomial at $t = 20$.

- (a) Find expressions for these linear and quadratic approximations.
- (b) For copper, the tables give $\alpha = 0.0039/\text{ }^\circ\text{C}$ and $\rho_{20} = 1.7 \times 10^{-8} \Omega\text{-m}$. Graph the resistivity of copper and the linear and quadratic approximations for $-250^\circ\text{C} \leq t \leq 1000^\circ\text{C}$.
- (c) For what values of t does the linear approximation agree with the exponential expression to within one percent?

33. An electric dipole consists of two electric charges of equal magnitude and opposite sign. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expanding this expression for E as a series in powers of d/D , show that E is approximately proportional to $1/D^3$ when P is far away from the dipole.



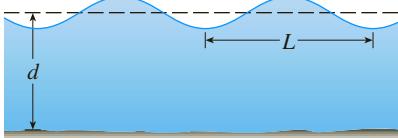
34. (a) Derive Equation 3 for Gaussian optics from Equation 1 by approximating $\cos \phi$ in Equation 2 by its first-degree Taylor polynomial.
- (b) Show that if $\cos \phi$ is replaced by its third-degree Taylor polynomial in Equation 2, then Equation 1 becomes Equation 4 for third-order optics. [Hint: Use the first two terms in the binomial series for ℓ_o^{-1} and ℓ_i^{-1} . Also, use $\phi \approx \sin \phi$.]
35. If a water wave with length L moves with velocity v across a body of water with depth d , as in the figure on page 822, then

$$v^2 = \frac{gL}{2\pi} \tanh \frac{2\pi d}{L}$$

- (a) If the water is deep, show that $v \approx \sqrt{gL/(2\pi)}$.
- (b) If the water is shallow, use the Maclaurin series for \tanh to show that $v \approx \sqrt{gd}$. (Thus in shallow water the

velocity of a wave tends to be independent of the length of the wave.)

- (c) Use the Alternating Series Estimation Theorem to show that if $L > 10d$, then the estimate $v^2 \approx gd$ is accurate to within $0.014gL$.

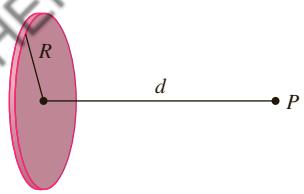


36. A uniformly charged disk has radius R and surface charge density σ as in the figure. The electric potential V at a point P at a distance d along the perpendicular central axis of the disk is

$$V = 2\pi k_e \sigma (\sqrt{d^2 + R^2} - d)$$

where k_e is a constant (called Coulomb's constant). Show that

$$V \approx \frac{\pi k_e R^2 \sigma}{d} \quad \text{for large } d$$



37. If a surveyor measures differences in elevation when making plans for a highway across a desert, corrections must be made for the curvature of the earth.

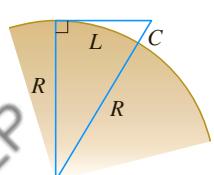
- (a) If R is the radius of the earth and L is the length of the highway, show that the correction is

$$C = R \sec(L/R) - R$$

- (b) Use a Taylor polynomial to show that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3}$$

- (c) Compare the corrections given by the formulas in parts (a) and (b) for a highway that is 100 km long. (Take the radius of the earth to be 6370 km.)



38. The period of a pendulum with length L that makes a maximum angle θ_0 with the vertical is

$$T = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin(\frac{1}{2}\theta_0)$ and g is the acceleration due to gravity. (In Exercise 7.7.42 we approximated this integral using Simpson's Rule.)

- (a) Expand the integrand as a binomial series and use the result of Exercise 7.1.50 to show that

$$T = 2\pi \sqrt{\frac{L}{g}} \left[1 + \frac{1^2}{2^2} k^2 + \frac{1^2 3^2}{2^2 4^2} k^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2} k^6 + \dots \right]$$

If θ_0 is not too large, the approximation $T \approx 2\pi\sqrt{L/g}$, obtained by using only the first term in the series, is often used. A better approximation is obtained by using two terms:

$$T \approx 2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{1}{4} k^2 \right)$$

- (b) Notice that all the terms in the series after the first one have coefficients that are at most $\frac{1}{4}$. Use this fact to compare this series with a geometric series and show that

$$2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{1}{4} k^2 \right) \leq T \leq 2\pi \sqrt{\frac{L}{g}} \frac{4 - 3k^2}{4 - 4k^2}$$

- (c) Use the inequalities in part (b) to estimate the period of a pendulum with $L = 1$ meter and $\theta_0 = 10^\circ$. How does it compare with the estimate $T \approx 2\pi\sqrt{L/g}$? What if $\theta_0 = 42^\circ$?

39. In Section 3.8 we considered Newton's method for approximating a root r of the equation $f(x) = 0$, and from an initial approximation x_1 we obtained successive approximations x_2, x_3, \dots , where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use Taylor's Inequality with $n = 1$, $a = x_n$, and $x = r$ to show that if $f''(x)$ exists on an interval I containing r, x_n , and x_{n+1} , and $|f''(x)| \leq M$, $|f'(x)| \geq K$ for all $x \in I$, then

$$|x_{n+1} - r| \leq \frac{M}{2K} |x_n - r|^2$$

[This means that if x_n is accurate to d decimal places, then x_{n+1} is accurate to about $2d$ decimal places. More precisely, if the error at stage n is at most 10^{-m} , then the error at stage $n + 1$ is at most $(M/2K)10^{-2m}$.]

APPLIED PROJECT**RADIATION FROM THE STARS**

Luke Dodd / Science Source



Any object emits radiation when heated. A *blackbody* is a system that absorbs all the radiation that falls on it. For instance, a matte black surface or a large cavity with a small hole in its wall (like a blast furnace) is a blackbody and emits blackbody radiation. Even the radiation from the sun is close to being blackbody radiation.

Proposed in the late 19th century, the Rayleigh-Jeans Law expresses the energy density of blackbody radiation of wavelength λ as

$$f(\lambda) = \frac{8\pi kT}{\lambda^4}$$

where λ is measured in meters, T is the temperature in kelvins (K), and k is Boltzmann's constant. The Rayleigh-Jeans Law agrees with experimental measurements for long wavelengths but disagrees drastically for short wavelengths. [The law predicts that $f(\lambda) \rightarrow \infty$ as $\lambda \rightarrow 0^+$ but experiments have shown that $f(\lambda) \rightarrow 0$.] This fact is known as the *ultraviolet catastrophe*.

In 1900 Max Planck found a better model (known now as Planck's Law) for blackbody radiation:

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1}$$

where λ is measured in meters, T is the temperature (in kelvins), and

$$h = \text{Planck's constant} = 6.6262 \times 10^{-34} \text{ J}\cdot\text{s}$$

$$c = \text{speed of light} = 2.997925 \times 10^8 \text{ m/s}$$

$$k = \text{Boltzmann's constant} = 1.3807 \times 10^{-23} \text{ J/K}$$

1. Use l'Hospital's Rule to show that

$$\lim_{\lambda \rightarrow 0^+} f(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} f(\lambda) = 0$$

for Planck's Law. So this law models blackbody radiation better than the Rayleigh-Jeans Law for short wavelengths.

2. Use a Taylor polynomial to show that, for large wavelengths, Planck's Law gives approximately the same values as the Rayleigh-Jeans Law.
3. Graph f as given by both laws on the same screen and comment on the similarities and differences. Use $T = 5700$ K (the temperature of the sun). (You may want to change from meters to the more convenient unit of micrometers: 1 mm = 10^{-6} m.)
4. Use your graph in Problem 3 to estimate the value of λ for which $f(\lambda)$ is a maximum under Planck's Law.
5. Investigate how the graph of f changes as T varies. (Use Planck's Law.) In particular, graph f for the stars Betelgeuse ($T = 3400$ K), Procyon ($T = 6400$ K), and Sirius ($T = 9200$ K), as well as the sun. How does the total radiation emitted (the area under the curve) vary with T ? Use the graph to comment on why Sirius is known as a blue star and Betelgeuse as a red star.

11 REVIEW

CONCEPT CHECK

1. (a) What is a convergent sequence?
 (b) What is a convergent series?
 (c) What does $\lim_{n \rightarrow \infty} a_n = 3$ mean?
 (d) What does $\sum_{n=1}^{\infty} a_n = 3$ mean?
2. (a) What is a bounded sequence?
 (b) What is a monotonic sequence?
 (c) What can you say about a bounded monotonic sequence?
3. (a) What is a geometric series? Under what circumstances is it convergent? What is its sum?
 (b) What is a p -series? Under what circumstances is it convergent?
4. Suppose $\sum a_n = 3$ and s_n is the n th partial sum of the series. What is $\lim_{n \rightarrow \infty} a_n$? What is $\lim_{n \rightarrow \infty} s_n$?
5. State the following.
 - (a) The Test for Divergence
 - (b) The Integral Test
 - (c) The Comparison Test
 - (d) The Limit Comparison Test
 - (e) The Alternating Series Test
 - (f) The Ratio Test
 - (g) The Root Test
6. (a) What is an absolutely convergent series?
 (b) What can you say about such a series?
 (c) What is a conditionally convergent series?
7. (a) If a series is convergent by the Integral Test, how do you estimate its sum?

Answers to the Concept Check can be found on the back endpapers.

- (b) If a series is convergent by the Comparison Test, how do you estimate its sum?
- (c) If a series is convergent by the Alternating Series Test, how do you estimate its sum?
8. (a) Write the general form of a power series.
 (b) What is the radius of convergence of a power series?
 (c) What is the interval of convergence of a power series?
9. Suppose $f(x)$ is the sum of a power series with radius of convergence R .
 - (a) How do you differentiate f ? What is the radius of convergence of the series for f' ?
 - (b) How do you integrate f ? What is the radius of convergence of the series for $\int f(x) dx$?
10. (a) Write an expression for the n th-degree Taylor polynomial of f centered at a .
 (b) Write an expression for the Taylor series of f centered at a .
 (c) Write an expression for the Maclaurin series of f .
 (d) How do you show that $f(x)$ is equal to the sum of its Taylor series?
 (e) State Taylor's Inequality.
11. Write the Maclaurin series and the interval of convergence for each of the following functions.

(a) $1/(1 - x)$	(b) e^x	(c) $\sin x$
(d) $\cos x$	(e) $\tan^{-1} x$	(f) $\ln(1 + x)$
12. Write the binomial series expansion of $(1 + x)^k$. What is the radius of convergence of this series?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum a_n$ is convergent.
2. The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ is convergent.
3. If $\lim_{n \rightarrow \infty} a_n = L$, then $\lim_{n \rightarrow \infty} a_{2n+1} = L$.
4. If $\sum c_n 6^n$ is convergent, then $\sum c_n (-2)^n$ is convergent.
5. If $\sum c_n 6^n$ is convergent, then $\sum c_n (-6)^n$ is convergent.
6. If $\sum c_n x^n$ diverges when $x = 6$, then it diverges when $x = 10$.
7. The Ratio Test can be used to determine whether $\sum 1/n^3$ converges.
8. The Ratio Test can be used to determine whether $\sum 1/n!$ converges.
9. If $0 \leq a_n \leq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

10. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$
11. If $-1 < \alpha < 1$, then $\lim_{n \rightarrow \infty} \alpha^n = 0$.
12. If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent.
13. If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \dots$ converges for all x , then $f'''(0) = 2$.
14. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n + b_n\}$ is divergent.
15. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n b_n\}$ is divergent.
16. If $\{a_n\}$ is decreasing and $a_n > 0$ for all n , then $\{a_n\}$ is convergent.
17. If $a_n > 0$ and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.

18. If $a_n > 0$ and $\lim_{n \rightarrow \infty} (a_{n+1}/a_n) < 1$, then $\lim_{n \rightarrow \infty} a_n = 0$.
19. $0.99999\dots = 1$
20. If $\lim_{n \rightarrow \infty} a_n = 2$, then $\lim_{n \rightarrow \infty} (a_{n+3} - a_n) = 0$.

EXERCISES

1–8 Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

1. $a_n = \frac{2 + n^3}{1 + 2n^3}$

2. $a_n = \frac{9^{n+1}}{10^n}$

3. $a_n = \frac{n^3}{1 + n^2}$

4. $a_n = \cos(n\pi/2)$

5. $a_n = \frac{n \sin n}{n^2 + 1}$

6. $a_n = \frac{\ln n}{\sqrt{n}}$

7. $\{(1 + 3/n)^{4n}\}$

8. $\{(-1)^n/n!\}$

9. A sequence is defined recursively by the equations $a_1 = 1$, $a_{n+1} = \frac{1}{3}(a_n + 4)$. Show that $\{a_n\}$ is increasing and $a_n < 2$ for all n . Deduce that $\{a_n\}$ is convergent and find its limit.

10. Show that $\lim_{n \rightarrow \infty} n^4 e^{-n} = 0$ and use a graph to find the smallest value of N that corresponds to $\varepsilon = 0.1$ in the precise definition of a limit.

11–22 Determine whether the series is convergent or divergent.

11. $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$

12. $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$

13. $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

14. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

15. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

16. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

17. $\sum_{n=1}^{\infty} \frac{\cos 3n}{1 + (1.2)^n}$

18. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1 + 2n^2)^n}$

19. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!}$

20. $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

21. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$

22. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$

21. If a finite number of terms are added to a convergent series, then the new series is still convergent.

22. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then $\sum_{n=1}^{\infty} a_n b_n = AB$.

23–26 Determine whether the series is conditionally convergent, absolutely convergent, or divergent.

23. $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$

24. $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$

25. $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1) 3^n}{2^{2n+1}}$

26. $\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n}$

27–31 Find the sum of the series.

27. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}}$

28. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

29. $\sum_{n=1}^{\infty} [\tan^{-1}(n+1) - \tan^{-1}n]$

30. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n} (2n)!}$

31. $1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + \frac{e^4}{4!} - \dots$

32. Express the repeating decimal $4.17326326326\dots$ as a fraction.

33. Show that $\cosh x \geq 1 + \frac{1}{2}x^2$ for all x .

34. For what values of x does the series $\sum_{n=1}^{\infty} (\ln x)^n$ converge?

35. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ correct to four decimal places.

36. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/n^6$ and estimate the error in using it as an approximation to the sum of the series.
 (b) Find the sum of this series correct to five decimal places.

37. Use the sum of the first eight terms to approximate the sum of the series $\sum_{n=1}^{\infty} (2 + 5^n)^{-1}$. Estimate the error involved in this approximation.

38. (a) Show that the series $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$ is convergent.

(b) Deduce that $\lim_{n \rightarrow \infty} \frac{n^n}{(2n)!} = 0$.

39. Prove that if the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the series

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n} \right) a_n$$

is also absolutely convergent.

40–43 Find the radius of convergence and interval of convergence of the series.

40. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$

41. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n 4^n}$

42. $\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$

43. $\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$

44. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

45. Find the Taylor series of $f(x) = \sin x$ at $a = \pi/6$.

46. Find the Taylor series of $f(x) = \cos x$ at $a = \pi/3$.

47–54 Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (definition of a Maclaurin series) or known series such as geometric series, binomial series, or the Maclaurin series for e^x , $\sin x$, $\tan^{-1} x$, and $\ln(1+x)$.

47. $f(x) = \frac{x^2}{1+x}$

48. $f(x) = \tan^{-1}(x^2)$

49. $f(x) = \ln(4-x)$

50. $f(x) = xe^{2x}$

51. $f(x) = \sin(x^4)$

52. $f(x) = 10^x$

53. $f(x) = 1/\sqrt[4]{16-x}$

54. $f(x) = (1-3x)^{-5}$

55. Evaluate $\int \frac{e^x}{x} dx$ as an infinite series.

56. Use series to approximate $\int_0^1 \sqrt{1+x^4} dx$ correct to two decimal places.

57–58

(a) Approximate f by a Taylor polynomial with degree n at the number a .

(b) Graph f and T_n on a common screen.

(c) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

(d) Check your result in part (c) by graphing $|R_n(x)|$.

57. $f(x) = \sqrt{x}, \quad a = 1, \quad n = 3, \quad 0.9 \leq x \leq 1.1$

58. $f(x) = \sec x, \quad a = 0, \quad n = 2, \quad 0 \leq x \leq \pi/6$

59. Use series to evaluate the following limit.

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$$

60. The force due to gravity on an object with mass m at a height h above the surface of the earth is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the earth and g is the acceleration due to gravity for an object on the surface of the earth.

(a) Express F as a series in powers of h/R .

(b) Observe that if we approximate F by the first term in the series, we get the expression $F \approx mg$ that is usually used when h is much smaller than R . Use the Alternating Series Estimation Theorem to estimate the range of values of h for which the approximation $F \approx mg$ is accurate to within one percent. (Use $R = 6400$ km.)

61. Suppose that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all x .

(a) If f is an odd function, show that

$$c_0 = c_2 = c_4 = \dots = 0$$

(b) If f is an even function, show that

$$c_1 = c_3 = c_5 = \dots = 0$$

62. If $f(x) = e^{x^2}$, show that $f^{(2n)}(0) = \frac{(2n)!}{n!}$.

Problems Plus

Before you look at the solution of the example, cover it up and first try to solve the problem yourself.

EXAMPLE Find the sum of the series $\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$.

SOLUTION The problem-solving principle that is relevant here is *recognizing something familiar*. Does the given series look anything like a series that we already know? Well, it does have some ingredients in common with the Maclaurin series for the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

We can make this series look more like our given series by replacing x by $x+2$:

$$e^{x+2} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} = 1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \dots$$

But here the exponent in the numerator matches the number in the denominator whose factorial is taken. To make that happen in the given series, let's multiply and divide by $(x+2)^3$:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} &= \frac{1}{(x+2)^3} \sum_{n=0}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!} \\ &= (x+2)^{-3} \left[\frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \dots \right] \end{aligned}$$

We see that the series between brackets is just the series for e^{x+2} with the first three terms missing. So

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = (x+2)^{-3} \left[e^{x+2} - 1 - (x+2) - \frac{(x+2)^2}{2!} \right]$$

Problems

1. If $f(x) = \sin(x^3)$, find $f^{(15)}(0)$.

2. A function f is defined by

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n} - 1}{x^{2n} + 1}$$

Where is f continuous?

3. (a) Show that $\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2 \cot x$.
 (b) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

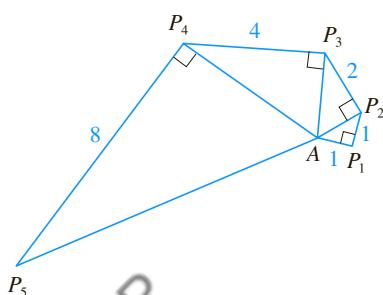


FIGURE FOR PROBLEM 4

4. Let $\{P_n\}$ be a sequence of points determined as in the figure. Thus $|AP_1| = 1$, $|P_nP_{n+1}| = 2^{n-1}$, and angle AP_nP_{n+1} is a right angle. Find $\lim_{n \rightarrow \infty} \angle P_nAP_{n+1}$.

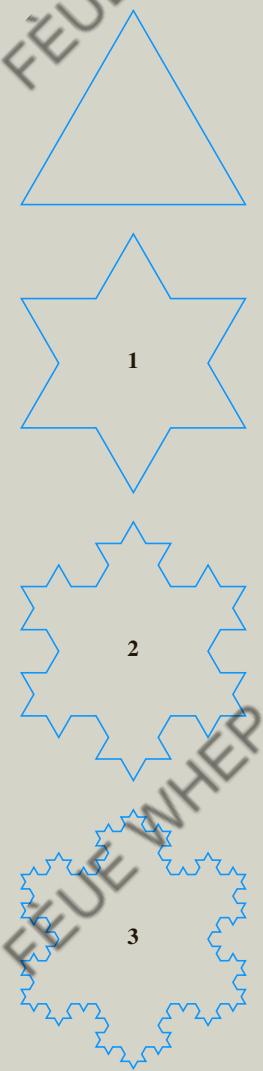


FIGURE FOR PROBLEM 5

5. To construct the **snowflake curve**, start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part (see the figure). Step 2 is to repeat step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely.
- Let s_n , l_n , and p_n represent the number of sides, the length of a side, and the total length of the n th approximating curve (the curve obtained after step n of the construction), respectively. Find formulas for s_n , l_n , and p_n .
 - Show that $p_n \rightarrow \infty$ as $n \rightarrow \infty$.
 - Sum an infinite series to find the area enclosed by the snowflake curve.
- Note:* Parts (b) and (c) show that the snowflake curve is infinitely long but encloses only a finite area.

6. Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \dots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

7. (a) Show that for $xy \neq -1$,

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

- (b) Show that $\arctan \frac{120}{119} - \arctan \frac{1}{239} = \pi/4$.

- (c) Deduce the following formula of John Machin (1680–1751):

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

- (d) Use the Maclaurin series for arctan to show that

$$0.1973955597 < \arctan \frac{1}{5} < 0.1973955616$$

- (e) Show that

$$0.004184075 < \arctan \frac{1}{239} < 0.004184077$$

- (f) Deduce that, correct to seven decimal places, $\pi \approx 3.1415927$.

Machin used this method in 1706 to find π correct to 100 decimal places. Recently, with the aid of computers, the value of π has been computed to increasingly greater accuracy. In 2013 Shigeru Kondo and Alexander Yee computed the value of π to more than 12 trillion decimal places!

8. (a) Prove a formula similar to the one in Problem 7(a) but involving arccot instead of arctan.

- (b) Find the sum of the series $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1)$.

9. Use the result of Problem 7(a) to find the sum of the series $\sum_{n=1}^{\infty} \arctan(2/n^2)$.

10. If $a_0 + a_1 + a_2 + \dots + a_k = 0$, show that

$$\lim_{n \rightarrow \infty} (a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \dots + a_k\sqrt{n+k}) = 0$$

If you don't see how to prove this, try the problem-solving strategy of *using analogy* (see page 98). Try the special cases $k = 1$ and $k = 2$ first. If you can see how to prove the assertion for these cases, then you will probably see how to prove it in general.

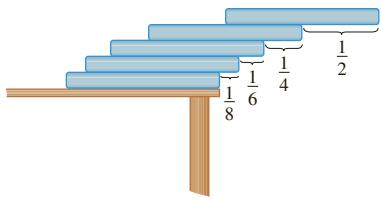


FIGURE FOR PROBLEM 12

11. Find the interval of convergence of $\sum_{n=1}^{\infty} n^3 x^n$ and find its sum.

12. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough. Use the following method of stacking: The top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. (Try it yourself with a deck of cards.) Consider centers of mass.

13. Find the sum of the series $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

14. If $p > 1$, evaluate the expression

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots}$$

15. Suppose that circles of equal diameter are packed tightly in n rows inside an equilateral triangle. (The figure illustrates the case $n = 4$.) If A is the area of the triangle and A_n is the total area occupied by the n rows of circles, show that

$$\lim_{n \rightarrow \infty} \frac{A_n}{A} = \frac{\pi}{2\sqrt{3}}$$

16. A sequence $\{a_n\}$ is defined recursively by the equations

$$a_0 = a_1 = 1 \quad n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$$

Find the sum of the series $\sum_{n=0}^{\infty} a_n$.

17. If the curve $y = e^{-x/10} \sin x$, $x \geq 0$, is rotated about the x -axis, the resulting solid looks like an infinite decreasing string of beads.

- (a) Find the exact volume of the n th bead. (Use either a table of integrals or a computer algebra system.)
 (b) Find the total volume of the beads.

18. Starting with the vertices $P_1(0, 1)$, $P_2(1, 1)$, $P_3(1, 0)$, $P_4(0, 0)$ of a square, we construct further points as shown in the figure: P_5 is the midpoint of P_1P_2 , P_6 is the midpoint of P_2P_3 , P_7 is the midpoint of P_3P_4 , and so on. The polygonal spiral path $P_1P_2P_3P_4P_5P_6P_7\dots$ approaches a point P inside the square.

- (a) If the coordinates of P_n are (x_n, y_n) , show that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$ and find a similar equation for the y -coordinates.
 (b) Find the coordinates of P .

19. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

20. Carry out the following steps to show that

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \dots = \ln 2$$

- (a) Use the formula for the sum of a finite geometric series (11.2.3) to get an expression for

$$1 - x + x^2 - x^3 + \dots + x^{2n-2} - x^{2n-1}$$

- (b) Integrate the result of part (a) from 0 to 1 to get an expression for

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

as an integral.

- (c) Deduce from part (b) that

$$\left| \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(2n-1)(2n)} - \int_0^1 \frac{dx}{1+x} \right| < \int_0^1 x^{2n} dx$$

- (d) Use part (c) to show that the sum of the given series is $\ln 2$.

- 21.** Find all the solutions of the equation

$$1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$$

[Hint: Consider the cases $x \geq 0$ and $x < 0$ separately.]

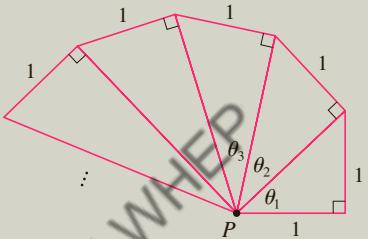


FIGURE FOR PROBLEM 22

- 22.** Right-angled triangles are constructed as in the figure. Each triangle has height 1 and its base is the hypotenuse of the preceding triangle. Show that this sequence of triangles makes indefinitely many turns around P by showing that $\sum \theta_n$ is a divergent series.

- 23.** Consider the series whose terms are the reciprocals of the positive integers that can be written in base 10 notation without using the digit 0. Show that this series is convergent and the sum is less than 90.

- 24.** (a) Show that the Maclaurin series of the function

$$f(x) = \frac{x}{1-x-x^2} \quad \text{is} \quad \sum_{n=1}^{\infty} f_n x^n$$

where f_n is the n th Fibonacci number, that is, $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. [Hint: Write $x/(1-x-x^2) = c_0 + c_1x + c_2x^2 + \cdots$ and multiply both sides of this equation by $1-x-x^2$.]

- (b) By writing $f(x)$ as a sum of partial fractions and thereby obtaining the Maclaurin series in a different way, find an explicit formula for the n th Fibonacci number.

- 25.** Let

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$$

$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots$$

$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

Show that $u^3 + v^3 + w^3 - 3uvw = 1$.

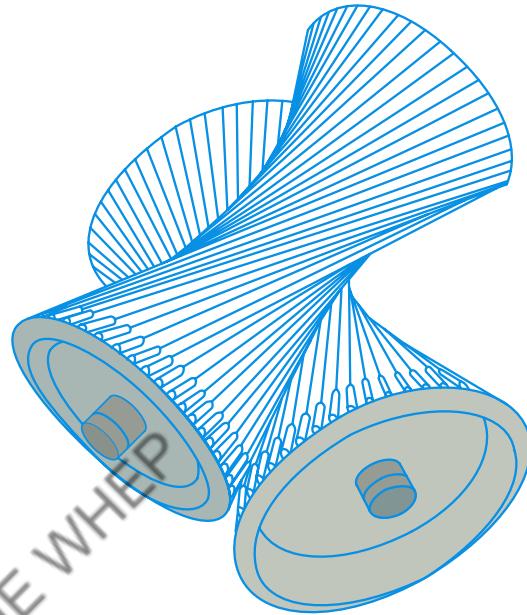
- 26.** Prove that if $n > 1$, the n th partial sum of the harmonic series is not an integer.

Hint: Let 2^k be the largest power of 2 that is less than or equal to n and let M be the product of all odd integers that are less than or equal to n . Suppose that $s_n = m$, an integer. Then $M2^k s_n = M2^k m$. The right side of this equation is even. Prove that the left side is odd by showing that each of its terms is an even integer, except for the last one.

12

Vectors and the Geometry of Space

Each of these gears has the shape of a hyperboloid, a type of surface we will study in Section 12.6. The shape allows the gears to transmit motion between skew (neither parallel nor intersecting) axes.



IN THIS CHAPTER WE INTRODUCE vectors and coordinate systems for three-dimensional space. This will be the setting for our study of the calculus of functions of two variables in Chapter 14 because the graph of such a function is a surface in space. In this chapter we will see that vectors provide particularly simple descriptions of lines and planes in space.

12.1 Three-Dimensional Coordinate Systems

■ 3D Space

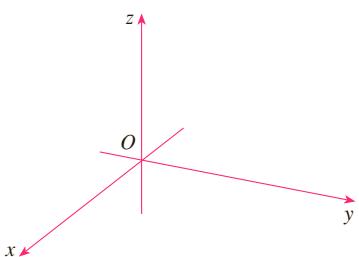


FIGURE 1
Coordinate axes

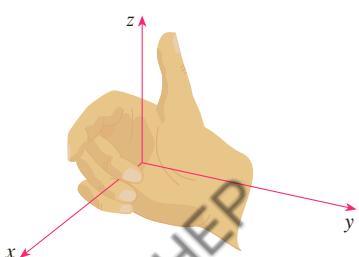


FIGURE 2
Right-hand rule

To locate a point in a plane, we need two numbers. We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x -coordinate and b is the y -coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a, b, c) of real numbers.

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x -axis, y -axis, and z -axis. Usually we think of the x - and y -axes as being horizontal and the z -axis as being vertical, and we draw the orientation of the axes as in Figure 1. The direction of the z -axis is determined by the **right-hand rule** as illustrated in Figure 2: If you curl the fingers of your right hand around the z -axis in the direction of a 90° counterclockwise rotation from the positive x -axis to the positive y -axis, then your thumb points in the positive direction of the z -axis.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a). The xy -plane is the plane that contains the x - and y -axes; the yz -plane contains the y - and z -axes; the xz -plane contains the x - and z -axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.

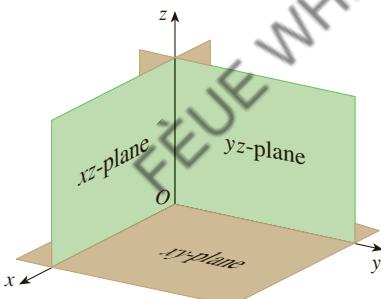
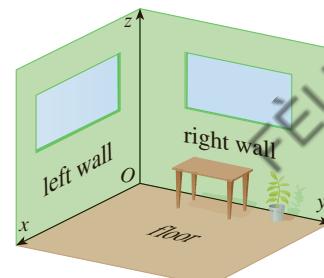


FIGURE 3

(a) Coordinate planes



(b)

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the xz -plane, the wall on your right is in the yz -plane, and the floor is in the xy -plane. The x -axis runs along the intersection of the floor and the left wall. The y -axis runs along the intersection of the floor and the right wall. The z -axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O .

Now if P is any point in space, let a be the (directed) distance from the yz -plane to P , let b be the distance from the xz -plane to P , and let c be the distance from the xy -plane to P . We represent the point P by the ordered triple (a, b, c) of real numbers and we call a , b , and c the **coordinates** of P ; a is the x -coordinate, b is the y -coordinate, and c is the z -coordinate. Thus, to locate the point (a, b, c) , we can start at the origin O and move a units along the x -axis, then b units parallel to the y -axis, and then c units parallel to the z -axis as in Figure 4.

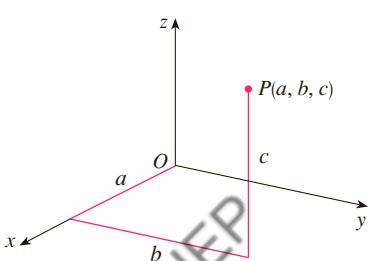


FIGURE 4

The point $P(a, b, c)$ determines a rectangular box as in Figure 5. If we drop a perpendicular from P to the xy -plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P onto the xy -plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz -plane and xz -plane, respectively.

As numerical illustrations, the points $(-4, 3, -5)$ and $(3, -2, -6)$ are plotted in Figure 6.

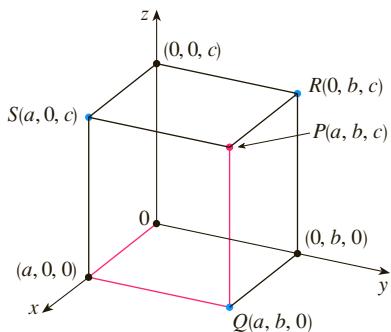


FIGURE 5

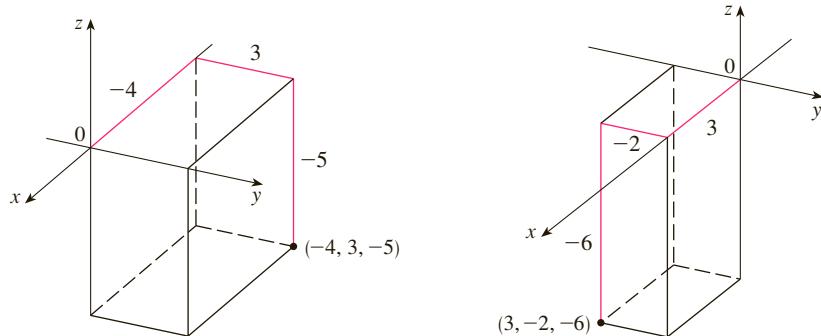


FIGURE 6

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by \mathbb{R}^3 . We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in \mathbb{R}^3 . It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

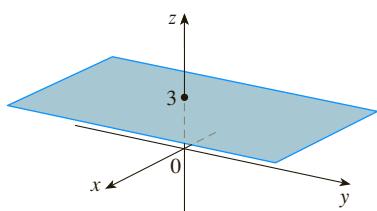
Surfaces

In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in \mathbb{R}^2 . In three-dimensional analytic geometry, an equation in x , y , and z represents a *surface* in \mathbb{R}^3 .

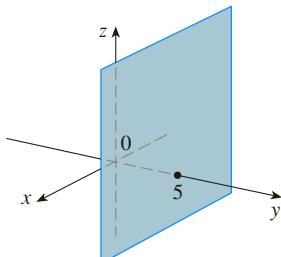
EXAMPLE 1 What surfaces in \mathbb{R}^3 are represented by the following equations?

SOLUTION

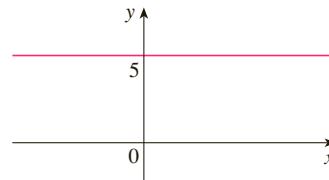
(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$, which is the set of all points in \mathbb{R}^3 whose z -coordinate is 3 (x and y can each be any value). This is the horizontal plane that is parallel to the xy -plane and three units above it as in Figure 7(a).



(a) $z = 3$, a plane in \mathbb{R}^3



(b) $y = 5$, a plane in \mathbb{R}^3



(c) $y = 5$, a line in \mathbb{R}^2

FIGURE 7

(b) The equation $y = 5$ represents the set of all points in \mathbb{R}^3 whose y -coordinate is 5. This is the vertical plane that is parallel to the xz -plane and five units to the right of it as in Figure 7(b). ■

NOTE When an equation is given, we must understand from the context whether it represents a curve in \mathbb{R}^2 or a surface in \mathbb{R}^3 . In Example 1, $y = 5$ represents a plane in \mathbb{R}^3 , but of course $y = 5$ can also represent a line in \mathbb{R}^2 if we are dealing with two-dimensional analytic geometry. See Figure 7(b) and (c).

In general, if k is a constant, then $x = k$ represents a plane parallel to the yz -plane, $y = k$ is a plane parallel to the xz -plane, and $z = k$ is a plane parallel to the xy -plane. In Figure 5, the faces of the rectangular box are formed by the three coordinate planes $x = 0$ (the yz -plane), $y = 0$ (the xz -plane), and $z = 0$ (the xy -plane), and the planes $x = a$, $y = b$, and $z = c$.

EXAMPLE 2

- (a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 3$$

- (b) What does the equation $x^2 + y^2 = 1$ represent as a surface in \mathbb{R}^3 ?

SOLUTION

(a) Because $z = 3$, the points lie in the horizontal plane $z = 3$ from Example 1(a). Because $x^2 + y^2 = 1$, the points lie on the circle with radius 1 and center on the z -axis. See Figure 8.

(b) Given that $x^2 + y^2 = 1$, with no restrictions on z , we see that the point (x, y, z) could lie on a circle in any horizontal plane $z = k$. So the surface $x^2 + y^2 = 1$ in \mathbb{R}^3 consists of all possible horizontal circles $x^2 + y^2 = 1$, $z = k$, and is therefore the circular cylinder with radius 1 whose axis is the z -axis. See Figure 9.

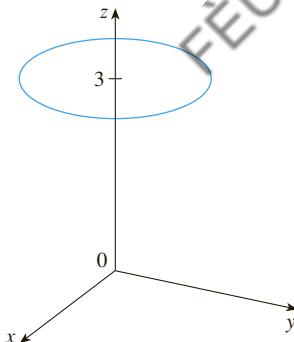


FIGURE 8
The circle $x^2 + y^2 = 1$, $z = 3$

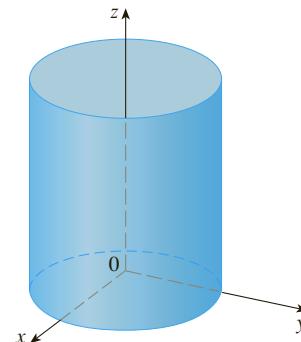


FIGURE 9
The cylinder $x^2 + y^2 = 1$

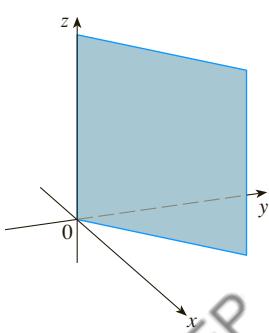


FIGURE 10
The plane $y = x$

EXAMPLE 3 Describe and sketch the surface in \mathbb{R}^3 represented by the equation $y = x$.

SOLUTION The equation represents the set of all points in \mathbb{R}^3 whose x - and y -coordinates are equal, that is, $\{(x, x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$. This is a vertical plane that intersects the xy -plane in the line $y = x$, $z = 0$. The portion of this plane that lies in the first octant is sketched in Figure 10.

Distance and Spheres

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

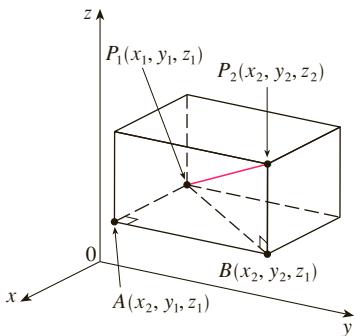


FIGURE 11

To see why this formula is true, we construct a rectangular box as in Figure 11, where P_1 and P_2 are opposite vertices and the faces of the box are parallel to the coordinate planes. If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \quad |AB| = |y_2 - y_1| \quad |BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of the Pythagorean Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

and

$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these equations, we get

$$\begin{aligned} |P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

$$\text{Therefore } |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EXAMPLE 4 The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = 3$$

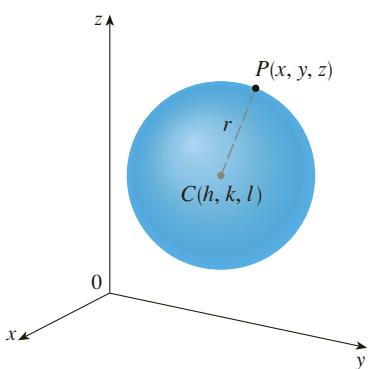


FIGURE 12

EXAMPLE 5 Find an equation of a sphere with radius r and center $C(h, k, l)$.

SOLUTION By definition, a sphere is the set of all points $P(x, y, z)$ whose distance from C is r . (See Figure 12.) Thus P is on the sphere if and only if $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$ or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

The result of Example 5 is worth remembering.

Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O , then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

EXAMPLE 6 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

SOLUTION We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$

$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$. ■

EXAMPLE 7 What region in \mathbb{R}^3 is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4 \quad z \leq 0$$

SOLUTION The inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4$$

can be rewritten as

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2$$

so they represent the points (x, y, z) whose distance from the origin is at least 1 and at most 2. But we are also given that $z \leq 0$, so the points lie on or below the xy -plane. Thus the given inequalities represent the region that lies between (or on) the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and beneath (or on) the xy -plane. It is sketched in Figure 13.

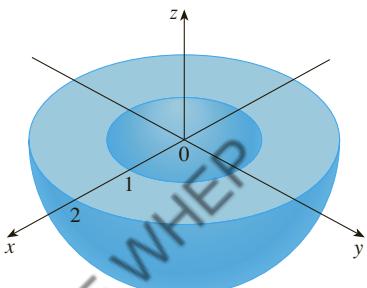


FIGURE 13

12.1 EXERCISES

- Suppose you start at the origin, move along the x -axis a distance of 4 units in the positive direction, and then move downward a distance of 3 units. What are the coordinates of your position?
- Sketch the points $(1, 5, 3)$, $(0, 2, -3)$, $(-3, 0, 2)$, and $(2, -2, -1)$ on a single set of coordinate axes.
- Which of the points $A(-4, 0, -1)$, $B(3, 1, -5)$, and $C(2, 4, 6)$ is closest to the yz -plane? Which point lies in the xz -plane?
- What are the projections of the point $(2, 3, 5)$ on the xy -, yz -, and xz -planes? Draw a rectangular box with the origin and $(2, 3, 5)$ as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.
- What does the equation $x = 4$ represent in \mathbb{R}^2 ? What does it represent in \mathbb{R}^3 ? Illustrate with sketches.
- What does the equation $y = 3$ represent in \mathbb{R}^3 ? What does $z = 5$ represent? What does the pair of equations $y = 3$, $z = 5$ represent? In other words, describe the set of points (x, y, z) such that $y = 3$ and $z = 5$. Illustrate with a sketch.

- Describe and sketch the surface in \mathbb{R}^3 represented by the equation $x + y = 2$.

- Describe and sketch the surface in \mathbb{R}^3 represented by the equation $x^2 + z^2 = 9$.

- 9–10** Find the lengths of the sides of the triangle PQR . Is it a right triangle? Is it an isosceles triangle?

- 9.** $P(3, -2, -3)$, $Q(7, 0, 1)$, $R(1, 2, 1)$

- 10.** $P(2, -1, 0)$, $Q(4, 1, 1)$, $R(4, -5, 4)$

- 11.** Determine whether the points lie on a straight line.

- (a) $A(2, 4, 2)$, $B(3, 7, -2)$, $C(1, 3, 3)$
(b) $D(0, -5, 5)$, $E(1, -2, 4)$, $F(3, 4, 2)$

- 12.** Find the distance from $(4, -2, 6)$ to each of the following.

- | | |
|---------------------|---------------------|
| (a) The xy -plane | (b) The yz -plane |
| (c) The xz -plane | (d) The x -axis |
| (e) The y -axis | (f) The z -axis |

13. Find an equation of the sphere with center $(-3, 2, 5)$ and radius 4. What is the intersection of this sphere with the yz -plane?
14. Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.
15. Find an equation of the sphere that passes through the point $(4, 3, -1)$ and has center $(3, 8, 1)$.
16. Find an equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

17–20 Show that the equation represents a sphere, and find its center and radius.

17. $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$

18. $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$

19. $2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$

20. $3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$

21. (a) Prove that the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

- (b) Find the lengths of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$. (A *median* of a triangle is a line segment that joins a vertex to the midpoint of the opposite side.)

22. Find an equation of a sphere if one of its diameters has endpoints $(5, 4, 3)$ and $(1, 6, -9)$.
23. Find equations of the spheres with center $(2, -3, 6)$ that touch (a) the xy -plane, (b) the yz -plane, (c) the xz -plane.
24. Find an equation of the largest sphere with center $(5, 4, 9)$ that is contained in the first octant.

25–38 Describe in words the region of \mathbb{R}^3 represented by the equation(s) or inequality.

25. $x = 5$

26. $y = -2$

27. $y < 8$

28. $z \geq -1$

29. $0 \leq z \leq 6$

30. $y^2 = 4$

31. $x^2 + y^2 = 4$, $z = -1$

32. $x^2 + y^2 = 4$

33. $x^2 + y^2 + z^2 = 4$

34. $x^2 + y^2 + z^2 \leq 4$

35. $1 \leq x^2 + y^2 + z^2 \leq 5$

36. $x = z$

37. $x^2 + z^2 \leq 9$

38. $x^2 + y^2 + z^2 > 2z$

39–42 Write inequalities to describe the region.

39. The region between the yz -plane and the vertical plane $x = 5$

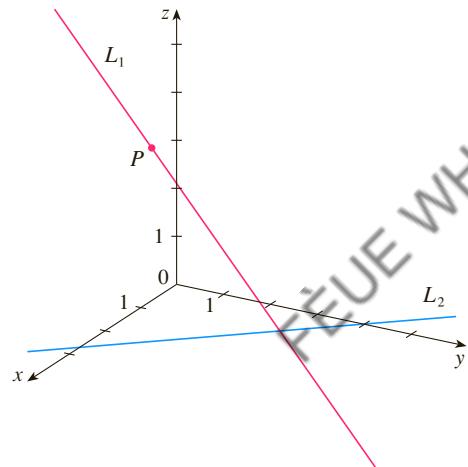
40. The solid cylinder that lies on or below the plane $z = 8$ and on or above the disk in the xy -plane with center the origin and radius 2

41. The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$

42. The solid upper hemisphere of the sphere of radius 2 centered at the origin

43. The figure shows a line L_1 in space and a second line L_2 , which is the projection of L_1 onto the xy -plane. (In other words, the points on L_2 are directly beneath, or above, the points on L_1 .)

- (a) Find the coordinates of the point P on the line L_1 .
 (b) Locate on the diagram the points A , B , and C , where the line L_1 intersects the xy -plane, the yz -plane, and the xz -plane, respectively.



44. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

45. Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

46. Find the volume of the solid that lies inside both of the spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$$

and $x^2 + y^2 + z^2 = 4$

47. Find the distance between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$.

48. Describe and sketch a solid with the following properties. When illuminated by rays parallel to the z -axis, its shadow is a circular disk. If the rays are parallel to the y -axis, its shadow is a square. If the rays are parallel to the x -axis, its shadow is an isosceles triangle.

12.2 Vectors

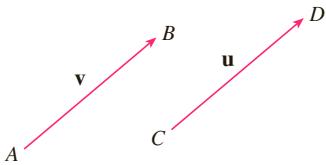


FIGURE 1
Equivalent vectors

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface (\mathbf{v}) or by putting an arrow above the letter (\vec{v}).

For instance, suppose a particle moves along a line segment from point A to point B . The corresponding **displacement vector** \mathbf{v} , shown in Figure 1, has **initial point** A (the tail) and **terminal point** B (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$. Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as \mathbf{v} even though it is in a different position. We say that \mathbf{u} and \mathbf{v} are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$. The **zero vector**, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

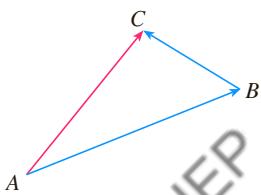


FIGURE 2

Combining Vectors

Suppose a particle moves from A to B , so its displacement vector is \overrightarrow{AB} . Then the particle changes direction and moves from B to C , with displacement vector \overrightarrow{BC} as in Figure 2. The combined effect of these displacements is that the particle has moved from A to C . The resulting displacement vector \overrightarrow{AC} is called the *sum* of \overrightarrow{AB} and \overrightarrow{BC} and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

In general, if we start with vectors \mathbf{u} and \mathbf{v} , we first move \mathbf{v} so that its tail coincides with the tip of \mathbf{u} and define the sum of \mathbf{u} and \mathbf{v} as follows.

Definition of Vector Addition If \mathbf{u} and \mathbf{v} are vectors positioned so the initial point of \mathbf{v} is at the terminal point of \mathbf{u} , then the **sum** $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of \mathbf{u} to the terminal point of \mathbf{v} .

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.

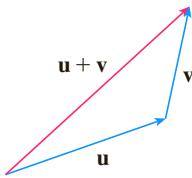


FIGURE 3
The Triangle Law

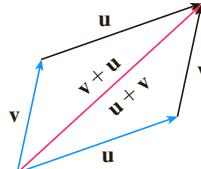


FIGURE 4
The Parallelogram Law

In Figure 4 we start with the same vectors \mathbf{u} and \mathbf{v} as in Figure 3 and draw another copy of \mathbf{v} with the same initial point as \mathbf{u} . Completing the parallelogram, we see that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. This also gives another way to construct the sum: if we place \mathbf{u} and \mathbf{v} so they start at the same point, then $\mathbf{u} + \mathbf{v}$ lies along the diagonal of the parallelogram with \mathbf{u} and \mathbf{v} as sides. (This is called the **Parallelogram Law**.)

EXAMPLE 1 Draw the sum of the vectors \mathbf{a} and \mathbf{b} shown in Figure 5.

SOLUTION First we move \mathbf{b} and place its tail at the tip of \mathbf{a} , being careful to draw a copy of \mathbf{b} that has the same length and direction. Then we draw the vector $\mathbf{a} + \mathbf{b}$ [see

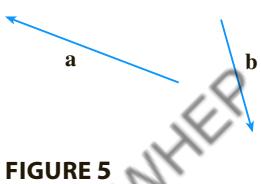


FIGURE 5

Figure 6(a)] starting at the initial point of \mathbf{a} and ending at the terminal point of the copy of \mathbf{b} .

Alternatively, we could place \mathbf{b} so it starts where \mathbf{a} starts and construct $\mathbf{a} + \mathbf{b}$ by the Parallelogram Law as in Figure 6(b).

TEC Visual 12.2 shows how the Triangle and Parallelogram Laws work for various vectors \mathbf{a} and \mathbf{b} .

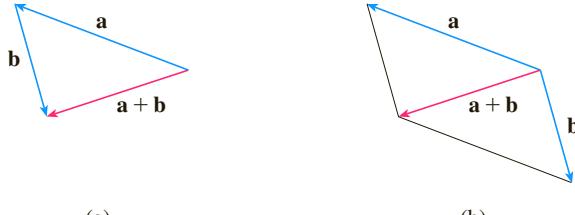


FIGURE 6

(a)

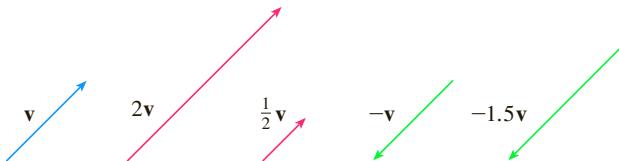
(b)

It is possible to multiply a vector by a real number c . (In this context we call the real number c a **scalar** to distinguish it from a vector.) For instance, we want $2\mathbf{v}$ to be the same vector as $\mathbf{v} + \mathbf{v}$, which has the same direction as \mathbf{v} but is twice as long. In general, we multiply a vector by a scalar as follows.

Definition of Scalar Multiplication If c is a scalar and \mathbf{v} is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of \mathbf{v} and whose direction is the same as \mathbf{v} if $c > 0$ and is opposite to \mathbf{v} if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

This definition is illustrated in Figure 7. We see that real numbers work like scaling factors here; that's why we call them scalars. Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as \mathbf{v} but points in the opposite direction. We call it the **negative** of \mathbf{v} .

FIGURE 7
Scalar multiples of \mathbf{v}



By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

So we can construct $\mathbf{u} - \mathbf{v}$ by first drawing the negative of \mathbf{v} , $-\mathbf{v}$, and then adding it to \mathbf{u} by the Parallelogram Law as in Figure 8(a). Alternatively, since $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$, the vector $\mathbf{u} - \mathbf{v}$, when added to \mathbf{v} , gives \mathbf{u} . So we could construct $\mathbf{u} - \mathbf{v}$ as in Figure 8(b) by means of the Triangle Law. Notice that if \mathbf{u} and \mathbf{v} both start from the same initial point, then $\mathbf{u} - \mathbf{v}$ connects the tip of \mathbf{v} to the tip of \mathbf{u} .

FIGURE 8
Drawing $\mathbf{u} - \mathbf{v}$

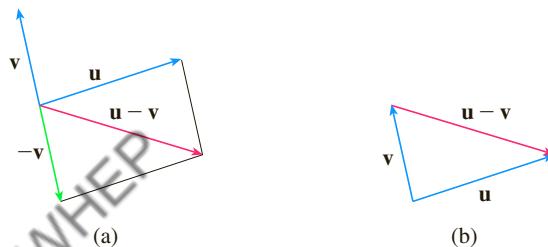




FIGURE 9

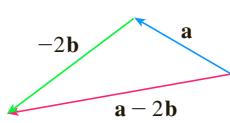


FIGURE 10

EXAMPLE 2 If \mathbf{a} and \mathbf{b} are the vectors shown in Figure 9, draw $\mathbf{a} - 2\mathbf{b}$.

SOLUTION We first draw the vector $-2\mathbf{b}$ pointing in the direction opposite to \mathbf{b} and twice as long. We place it with its tail at the tip of \mathbf{a} and then use the Triangle Law to draw $\mathbf{a} + (-2\mathbf{b})$ as in Figure 10. ■

Components

For some purposes it's best to introduce a coordinate system and treat vectors algebraically. If we place the initial point of a vector \mathbf{a} at the origin of a rectangular coordinate system, then the terminal point of \mathbf{a} has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3) , depending on whether our coordinate system is two- or three-dimensional (see Figure 11). These coordinates are called the **components** of \mathbf{a} and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

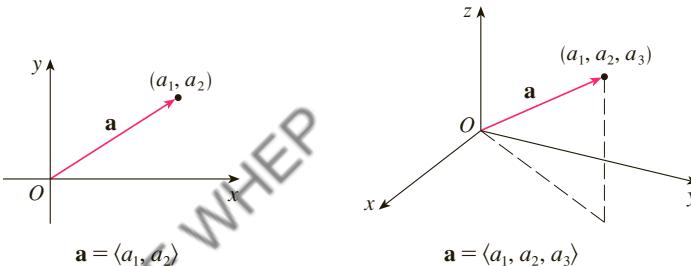


FIGURE 11

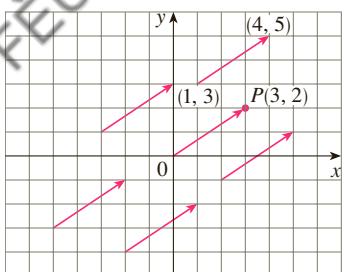


FIGURE 12

Representations of $\mathbf{a} = \langle 3, 2 \rangle$

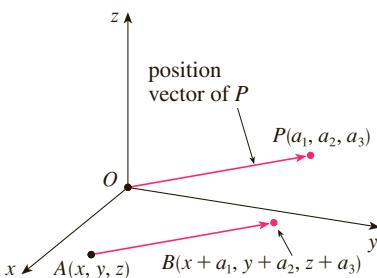


FIGURE 13

Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

For instance, the vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$ whose terminal point is $P(3, 2)$. What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We can think of all these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3, 2 \rangle$. The particular representation \overrightarrow{OP} from the origin to the point $P(3, 2)$ is called the **position vector** of the point P .

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.) Let's consider any other representation \overrightarrow{AB} of \mathbf{a} , where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$. Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$. Thus we have the following result.

- 1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector \mathbf{a} with representation \overrightarrow{AB} is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

EXAMPLE 3 Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

SOLUTION By (1), the vector corresponding to \overrightarrow{AB} is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle = \langle -4, 4, -3 \rangle$$

The **magnitude** or **length** of the vector \mathbf{v} is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment OP , we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

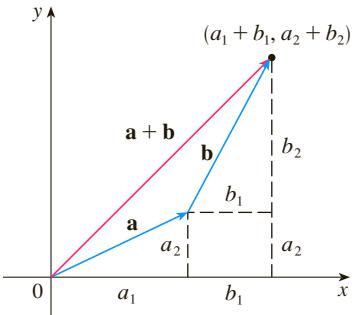


FIGURE 14

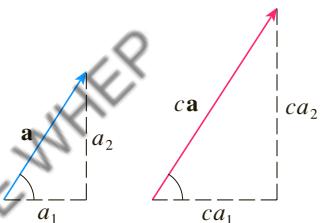


FIGURE 15

How do we add vectors algebraically? Figure 14 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive. In other words, *to add algebraic vectors we add corresponding components*. Similarly, *to subtract vectors we subtract corresponding components*. From the similar triangles in Figure 15 we see that the components of $c\mathbf{a}$ are ca_1 and ca_2 . So *to multiply a vector by a scalar we multiply each component by that scalar*.

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \quad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle \\ c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

EXAMPLE 4 If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

SOLUTION $|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle \\ &= \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle \end{aligned}$$

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle \\ &= \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle \end{aligned}$$

$$3\mathbf{b} = 3\langle -2, 1, 5 \rangle = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$$

$$\begin{aligned} 2\mathbf{a} + 5\mathbf{b} &= 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle \\ &= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle \end{aligned}$$

Vectors in n dimensions are used to list various quantities in an organized way. For instance, the components of a six-dimensional vector

$$\mathbf{p} = \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle$$

might represent the prices of six different ingredients required to make a particular product. Four-dimensional vectors $\langle x, y, z, t \rangle$ are used in relativity theory, where the first three components specify a position in space and the fourth represents time.

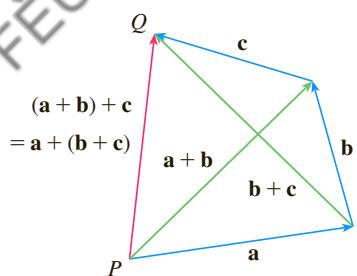


FIGURE 16

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors. More generally, we will later need to consider the set V_n of all n -dimensional vectors. An n -dimensional vector is an ordered n -tuple:

$$\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$$

where a_1, a_2, \dots, a_n are real numbers that are called the components of \mathbf{a} . Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Properties of Vectors If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors in V_n and c and d are scalars, then

- | | |
|--|---|
| 1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$ | 2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
8. $1\mathbf{a} = \mathbf{a}$ |
|--|---|

These eight properties of vectors can be readily verified either geometrically or algebraically. For instance, Property 1 can be seen from Figure 16 (it's equivalent to the Parallelogram Law) or as follows for the case $n = 2$:

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \\ &= \langle b_1 + a_1, b_2 + a_2 \rangle = \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle \\ &= \mathbf{b} + \mathbf{a} \end{aligned}$$

We can see why Property 2 (the associative law) is true by looking at Figure 16 and applying the Triangle Law several times: the vector \overrightarrow{PQ} is obtained either by first constructing $\mathbf{a} + \mathbf{b}$ and then adding \mathbf{c} or by adding \mathbf{a} to the vector $\mathbf{b} + \mathbf{c}$.

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

These vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x -, y -, and z -axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)

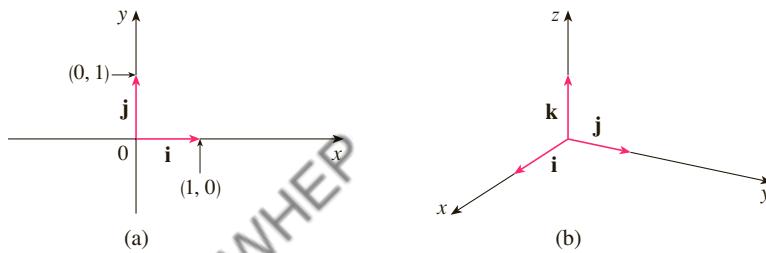
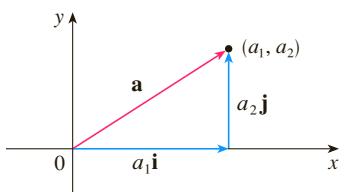
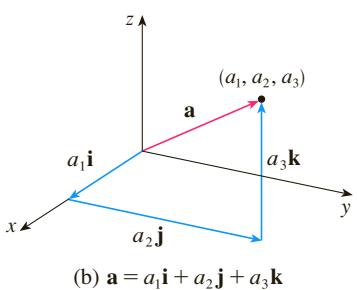


FIGURE 17

Standard basis vectors in V_2 and V_3



$$(a) \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$$



$$(b) \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

FIGURE 18

Gibbs

Josiah Willard Gibbs (1839–1903), a professor of mathematical physics at Yale College, published the first book on vectors, *Vector Analysis*, in 1881. More complicated objects, called quaternions, had earlier been invented by Hamilton as mathematical tools for describing space, but they weren't easy for scientists to use. Quaternions have a scalar part and a vector part. Gibb's idea was to use the vector part separately. Maxwell and Heaviside had similar ideas, but Gibb's approach has proved to be the most convenient way to study space.

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle \\ \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\end{aligned}$$

Thus any vector in V_3 can be expressed in terms of \mathbf{i} , \mathbf{j} , and \mathbf{k} . For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.

EXAMPLE 5 If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of \mathbf{i} , \mathbf{j} , and \mathbf{k} .

SOLUTION Using Properties 1, 2, 5, 6, and 7 of vectors, we have

$$\begin{aligned}2\mathbf{a} + 3\mathbf{b} &= 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k}) \\ &= 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} = 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}\end{aligned}$$

A **unit vector** is a vector whose length is 1. For instance, \mathbf{i} , \mathbf{j} , and \mathbf{k} are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as \mathbf{a} is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|} \mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so \mathbf{u} has the same direction as \mathbf{a} . Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|} |\mathbf{a}| = 1$$

EXAMPLE 6 Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

SOLUTION The given vector has length

$$|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

so, by Equation 4, the unit vector with the same direction is

$$\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Applications

Vectors are useful in many aspects of physics and engineering. In Chapter 13 we will see how they describe the velocity and acceleration of objects moving in space. Here we look at forces.

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

EXAMPLE 7 A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) \mathbf{T}_1 and \mathbf{T}_2 in both wires and the magnitudes of the tensions.

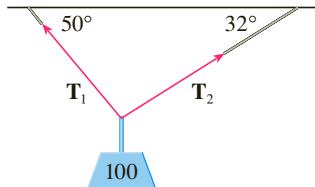


FIGURE 19

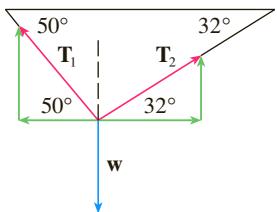


FIGURE 20

SOLUTION We first express \mathbf{T}_1 and \mathbf{T}_2 in terms of their horizontal and vertical components. From Figure 20 we see that

$$5 \quad \mathbf{T}_1 = -|\mathbf{T}_1| \cos 50^\circ \mathbf{i} + |\mathbf{T}_1| \sin 50^\circ \mathbf{j}$$

$$6 \quad \mathbf{T}_2 = |\mathbf{T}_2| \cos 32^\circ \mathbf{i} + |\mathbf{T}_2| \sin 32^\circ \mathbf{j}$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w} = -100 \mathbf{j}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100 \mathbf{j}$$

Thus

$$(-|\mathbf{T}_1| \cos 50^\circ + |\mathbf{T}_2| \cos 32^\circ) \mathbf{i} + (|\mathbf{T}_1| \sin 50^\circ + |\mathbf{T}_2| \sin 32^\circ) \mathbf{j} = 100 \mathbf{j}$$

Equating components, we get

$$-|\mathbf{T}_1| \cos 50^\circ + |\mathbf{T}_2| \cos 32^\circ = 0$$

$$|\mathbf{T}_1| \sin 50^\circ + |\mathbf{T}_2| \sin 32^\circ = 100$$

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1| \sin 50^\circ + \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ} \sin 32^\circ = 100$$

$$|\mathbf{T}_1| \left(\sin 50^\circ + \cos 50^\circ \frac{\sin 32^\circ}{\cos 32^\circ} \right) = 100$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb}$$

and $|\mathbf{T}_2| = \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb}$

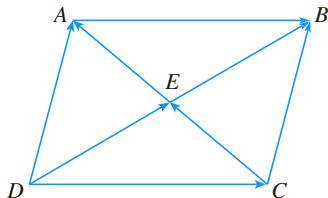
Substituting these values in (5) and (6), we obtain the tension vectors

$$\mathbf{T}_1 \approx -55.05 \mathbf{i} + 65.60 \mathbf{j}$$

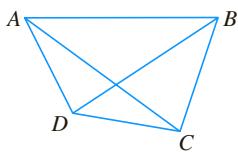
$$\mathbf{T}_2 \approx 55.05 \mathbf{i} + 34.40 \mathbf{j}$$

12.2 EXERCISES

- Are the following quantities vectors or scalars? Explain.
 - The cost of a theater ticket
 - The current in a river
 - The initial flight path from Houston to Dallas
 - The population of the world
- What is the relationship between the point $(4, 7)$ and the vector $\langle 4, 7 \rangle$? Illustrate with a sketch.
- Name all the equal vectors in the parallelogram shown.



- Write each combination of vectors as a single vector.
 - $\vec{AB} + \vec{BC}$
 - $\vec{CD} + \vec{DB}$
 - $\vec{DB} - \vec{AB}$
 - $\vec{DC} + \vec{CA} + \vec{AB}$



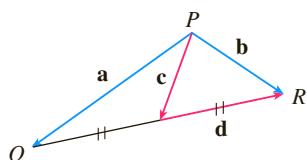
- Copy the vectors in the figure and use them to draw the following vectors.
 - $\mathbf{u} + \mathbf{v}$
 - $\mathbf{u} + \mathbf{w}$
 - $\mathbf{v} + \mathbf{w}$
 - $\mathbf{u} - \mathbf{v}$
 - $\mathbf{v} + \mathbf{u} + \mathbf{w}$
 - $\mathbf{u} - \mathbf{w} - \mathbf{v}$



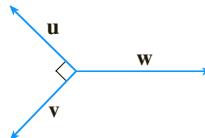
- Copy the vectors in the figure and use them to draw the following vectors.
 - $\mathbf{a} + \mathbf{b}$
 - $\mathbf{a} - \mathbf{b}$
 - $\frac{1}{2}\mathbf{a}$
 - $-3\mathbf{b}$
 - $\mathbf{a} + 2\mathbf{b}$
 - $2\mathbf{b} - \mathbf{a}$



- In the figure, the tip of \mathbf{c} and the tail of \mathbf{d} are both the midpoint of QR . Express \mathbf{c} and \mathbf{d} in terms of \mathbf{a} and \mathbf{b} .



- If the vectors in the figure satisfy $|\mathbf{u}| = |\mathbf{v}| = 1$ and $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, what is $|\mathbf{w}|$?



- 9–14** Find a vector \mathbf{a} with representation given by the directed line segment \overrightarrow{AB} . Draw \overrightarrow{AB} and the equivalent representation starting at the origin.

- $A(-2, 1), B(1, 2)$
- $A(3, -1), B(2, 3)$
- $A(0, 3, 1), B(2, 3, -1)$
- $A(0, 6, -1), B(3, 4, 4)$

- 15–18** Find the sum of the given vectors and illustrate geometrically.

- $\langle -1, 4 \rangle, \langle 6, -2 \rangle$
- $\langle 3, -1 \rangle, \langle -1, 5 \rangle$
- $\langle 3, 0, 1 \rangle, \langle 0, 8, 0 \rangle$
- $\langle 1, 3, -2 \rangle, \langle 0, 0, 6 \rangle$

- 19–22** Find $\mathbf{a} + \mathbf{b}$, $4\mathbf{a} + 2\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

- $\mathbf{a} = \langle -3, 4 \rangle, \mathbf{b} = \langle 9, -1 \rangle$
- $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}, \mathbf{b} = -\mathbf{i} - 2\mathbf{j}$
- $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$
- $\mathbf{a} = \langle 8, 1, -4 \rangle, \mathbf{b} = \langle 5, -2, 1 \rangle$

- 23–25** Find a unit vector that has the same direction as the given vector.

- $\langle 6, -2 \rangle$
- $-5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
- $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

- 26**. Find the vector that has the same direction as $\langle 6, 2, -3 \rangle$ but has length 4.

- 27–28** What is the angle between the given vector and the positive direction of the x -axis?

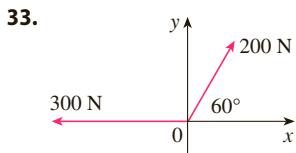
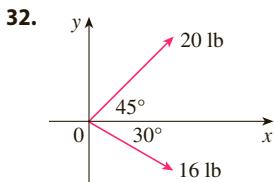
- $\mathbf{i} + \sqrt{3}\mathbf{j}$
- $8\mathbf{i} + 6\mathbf{j}$

- 29**. If \mathbf{v} lies in the first quadrant and makes an angle $\pi/3$ with the positive x -axis and $|\mathbf{v}| = 4$, find \mathbf{v} in component form.

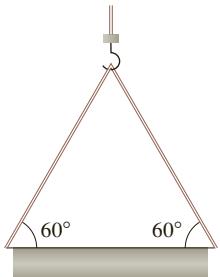
- 30**. If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of 38° above the horizontal, find the horizontal and vertical components of the force.

- 31**. A quarterback throws a football with angle of elevation 40° and speed 60 ft/s. Find the horizontal and vertical components of the velocity vector.

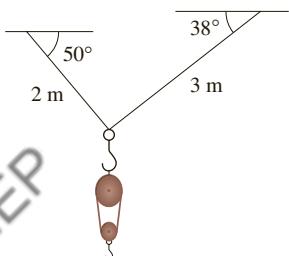
- 32–33** Find the magnitude of the resultant force and the angle it makes with the positive x -axis.



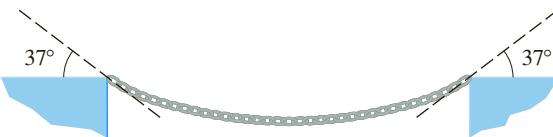
- 34.** The magnitude of a velocity vector is called *speed*. Suppose that a wind is blowing *from* the direction N 45° W at a speed of 50 km/h. (This means that the direction from which the wind blows is 45° west of the northerly direction.) A pilot is steering a plane in the direction N 60° E at an airspeed (speed in still air) of 250 km/h. The *true course*, or *track*, of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane.
- 35.** A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the woman relative to the surface of the water.
- 36.** A crane suspends a 500-lb steel beam horizontally by support cables (with negligible weight) attached from a hook to each end of the beam. The support cables each make an angle of 60° with the beam. Find the tension vector in each support cable and the magnitude of each tension.



- 37.** A block-and-tackle pulley hoist is suspended in a warehouse by ropes of lengths 2 m and 3 m. The hoist weighs 350 N. The ropes, fastened at different heights, make angles of 50° and 38° with the horizontal. Find the tension in each rope and the magnitude of each tension.



- 38.** The tension \mathbf{T} at each end of a chain has magnitude 25 N (see the figure). What is the weight of the chain?



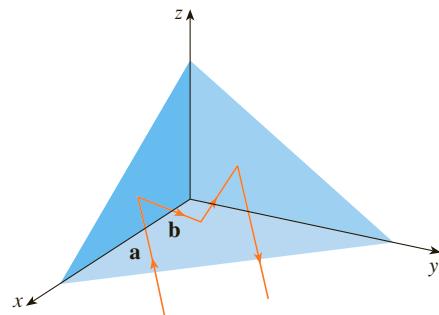
- 39.** A boatman wants to cross a canal that is 3 km wide and wants to land at a point 2 km upstream from his starting point. The current in the canal flows at 3.5 km/h and the speed of his boat is 13 km/h.
- In what direction should he steer?
 - How long will the trip take?
- 40.** Three forces act on an object. Two of the forces are at an angle of 100° to each other and have magnitudes 25 N and 12 N. The third is perpendicular to the plane of these two forces and has magnitude 4 N. Calculate the magnitude of the force that would exactly counterbalance these three forces.
- 41.** Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.
- 42.**
 - Find the unit vectors that are parallel to the tangent line to the curve $y = 2 \sin x$ at the point $(\pi/6, 1)$.
 - Find the unit vectors that are perpendicular to the tangent line.
 - Sketch the curve $y = 2 \sin x$ and the vectors in parts (a) and (b), all starting at $(\pi/6, 1)$.
- 43.** If A , B , and C are the vertices of a triangle, find

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$$

- 44.** Let C be the point on the line segment AB that is twice as far from B as it is from A . If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.
- 45.**
 - Draw the vectors $\mathbf{a} = \langle 3, 2 \rangle$, $\mathbf{b} = \langle 2, -1 \rangle$, and $\mathbf{c} = \langle 7, 1 \rangle$.
 - Show, by means of a sketch, that there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.
 - Use the sketch to estimate the values of s and t .
 - Find the exact values of s and t .
- 46.** Suppose that \mathbf{a} and \mathbf{b} are nonzero vectors that are not parallel and \mathbf{c} is any vector in the plane determined by \mathbf{a} and \mathbf{b} . Give a geometric argument to show that \mathbf{c} can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for suitable scalars s and t . Then give an argument using components.
- 47.** If $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, describe the set of all points (x, y, z) such that $|\mathbf{r} - \mathbf{r}_0| = 1$.
- 48.** If $\mathbf{r} = \langle x, y \rangle$, $\mathbf{r}_1 = \langle x_1, y_1 \rangle$, and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$, describe the set of all points (x, y) such that $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$, where $k > |\mathbf{r}_1 - \mathbf{r}_2|$.
- 49.** Figure 16 gives a geometric demonstration of Property 2 of vectors. Use components to give an algebraic proof of this fact for the case $n = 2$.

50. Prove Property 5 of vectors algebraically for the case $n = 3$. Then use similar triangles to give a geometric proof.
51. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.
52. Suppose the three coordinate planes are all mirrored and a light ray given by the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ first strikes the xz -plane, as shown in the figure. Use the fact that the angle of incidence equals the angle of reflection to show that the direction of the reflected ray is given by $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. Deduce that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray. (American space scientists used this principle, together with laser beams

and an array of corner mirrors on the moon, to calculate very precisely the distance from the earth to the moon.)



12.3 The Dot Product

So far we have added two vectors and multiplied a vector by a scalar. The question arises: is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows. Another is the cross product, which is discussed in the next section.

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of \mathbf{a} and \mathbf{b} is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Thus, to find the dot product of \mathbf{a} and \mathbf{b} , we multiply corresponding components and add. The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**). Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

EXAMPLE 1

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = (-1)(6) + 7(2) + 4\left(-\frac{1}{2}\right) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

2 Properties of the Dot Product If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(ca) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (cb)$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$
3.
$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}\end{aligned}$$

The proofs of the remaining properties are left as exercises. ■

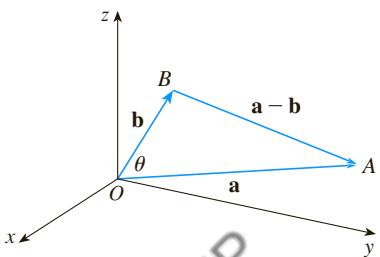


FIGURE 1

The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle θ between \mathbf{a} and \mathbf{b}** , which is defined to be the angle between the representations of \mathbf{a} and \mathbf{b} that start at the origin, where $0 \leq \theta \leq \pi$. In other words, θ is the angle between the line segments \overrightarrow{OA} and \overrightarrow{OB} in Figure 1. Note that if \mathbf{a} and \mathbf{b} are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

The formula in the following theorem is used by physicists as the *definition* of the dot product.

3 Theorem If θ is the angle between the vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

PROOF If we apply the Law of Cosines to triangle OAB in Figure 1, we get

$$4 \quad |AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos \theta$$

(Observe that the Law of Cosines still applies in the limiting cases when $\theta = 0$ or π , or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.) But $|OA| = |\mathbf{a}|$, $|OB| = |\mathbf{b}|$, and $|AB| = |\mathbf{a} - \mathbf{b}|$, so Equation 4 becomes

$$5 \quad |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos \theta$$

Using Properties 1, 2, and 3 of the dot product, we can rewrite the left side of this equation as follows:

$$\begin{aligned}|\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2\end{aligned}$$

Therefore Equation 5 gives

$$|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos \theta$$

Thus

$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos \theta$$

or

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos \theta$$

EXAMPLE 2 If the vectors \mathbf{a} and \mathbf{b} have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

SOLUTION Using Theorem 3, we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\pi/3) = 4 \cdot 6 \cdot \frac{1}{2} = 12$$

The formula in Theorem 3 also enables us to find the angle between two vectors.

6 Corollary If θ is the angle between the nonzero vectors \mathbf{a} and \mathbf{b} , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

EXAMPLE 3 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

SOLUTION Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \quad \text{and} \quad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

we have, from Corollary 6,

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between \mathbf{a} and \mathbf{b} is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84^\circ)$$

Two nonzero vectors \mathbf{a} and \mathbf{b} are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos \theta = 0$, so $\theta = \pi/2$. The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal.

7

Two vectors \mathbf{a} and \mathbf{b} are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

EXAMPLE 4 Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

SOLUTION Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by (7).

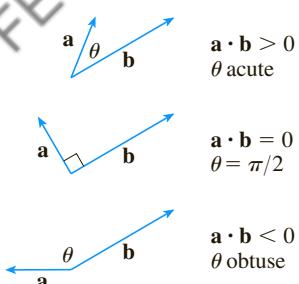


FIGURE 2

TEC Visual 12.3A shows an animation of Figure 2.

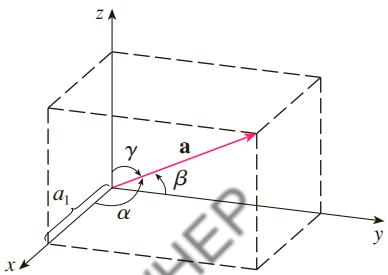


FIGURE 3

Because $\cos \theta > 0$ if $0 \leq \theta < \pi/2$ and $\cos \theta < 0$ if $\pi/2 < \theta \leq \pi$, we see that $\mathbf{a} \cdot \mathbf{b}$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of $\mathbf{a} \cdot \mathbf{b}$ as measuring the extent to which \mathbf{a} and \mathbf{b} point in the same direction. The dot product $\mathbf{a} \cdot \mathbf{b}$ is positive if \mathbf{a} and \mathbf{b} point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2). In the extreme case where \mathbf{a} and \mathbf{b} point in exactly the same direction, we have $\theta = 0$, so $\cos \theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}|$$

If \mathbf{a} and \mathbf{b} point in exactly opposite directions, then we have $\theta = \pi$ and so $\cos \theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}| |\mathbf{b}|$.

■ Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector \mathbf{a} are the angles α , β , and γ (in the interval $[0, \pi]$) that \mathbf{a} makes with the positive x -, y -, and z -axes, respectively. (See Figure 3.)

The cosines of these direction angles, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, are called the **direction cosines** of the vector \mathbf{a} . Using Corollary 6 with \mathbf{b} replaced by \mathbf{i} , we obtain

$$8 \quad \cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}| |\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.)

Similarly, we also have

$$9 \quad \cos \beta = \frac{a_2}{|\mathbf{a}|} \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that

$$10 \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

We can also use Equations 8 and 9 to write

$$\begin{aligned} \mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle \\ &= |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \end{aligned}$$

Therefore

$$11 \quad \frac{1}{|\mathbf{a}|} \mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

which says that the direction cosines of \mathbf{a} are the components of the unit vector in the direction of \mathbf{a} .

EXAMPLE 5 Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

$$\cos \alpha = \frac{1}{\sqrt{14}} \quad \cos \beta = \frac{2}{\sqrt{14}} \quad \cos \gamma = \frac{3}{\sqrt{14}}$$

and so

$$\alpha = \cos^{-1} \left(\frac{1}{\sqrt{14}} \right) \approx 74^\circ \quad \beta = \cos^{-1} \left(\frac{2}{\sqrt{14}} \right) \approx 58^\circ \quad \gamma = \cos^{-1} \left(\frac{3}{\sqrt{14}} \right) \approx 37^\circ$$

TEC Visual 12.3B shows how Figure 4 changes when we vary \mathbf{a} and \mathbf{b} .

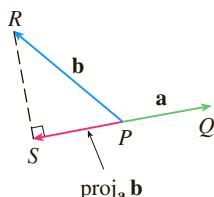
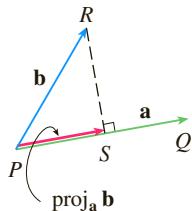


FIGURE 4
Vector projections

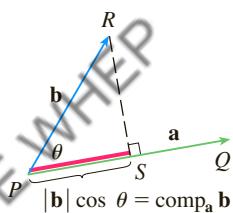


FIGURE 5
Scalar projection

■ Projections

Figure 4 shows representations \vec{PQ} and \vec{PR} of two vectors \mathbf{a} and \mathbf{b} with the same initial point P . If S is the foot of the perpendicular from R to the line containing \vec{PQ} , then the vector with representation \vec{PS} is called the **vector projection** of \mathbf{b} onto \mathbf{a} and is denoted by $\text{proj}_{\mathbf{a}} \mathbf{b}$. (You can think of it as a shadow of \mathbf{b} .)

The **scalar projection** of \mathbf{b} onto \mathbf{a} (also called the **component of \mathbf{b} along \mathbf{a}**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}| \cos \theta$, where θ is the angle between \mathbf{a} and \mathbf{b} . (See Figure 5.) This is denoted by $\text{comp}_{\mathbf{a}} \mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leq \pi$. The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = |\mathbf{a}| (|\mathbf{b}| \cos \theta)$$

shows that the dot product of \mathbf{a} and \mathbf{b} can be interpreted as the length of \mathbf{a} times the scalar projection of \mathbf{b} onto \mathbf{a} . Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of \mathbf{b} along \mathbf{a} can be computed by taking the dot product of \mathbf{b} with the unit vector in the direction of \mathbf{a} . We summarize these ideas as follows.

Scalar projection of \mathbf{b} onto \mathbf{a} : $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of \mathbf{b} onto \mathbf{a} : $\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of \mathbf{a} .

EXAMPLE 6 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of \mathbf{b} onto \mathbf{a} is

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of \mathbf{a} :

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14} \mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

One use of projections occurs in physics in calculating work. In Section 5.4 we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \vec{PR}$ pointing in some other direction, as in Figure 6. If the force moves the object from P to Q , then the **displacement vector** is $\mathbf{D} = \vec{PQ}$. The **work** done by this force is defined to be the product of the component of the force along \mathbf{D} and the distance moved:

$$W = (|\mathbf{F}| \cos \theta) |\mathbf{D}|$$

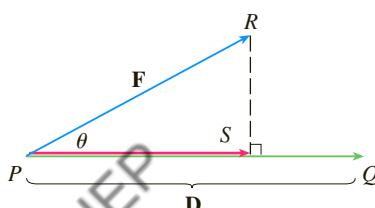


FIGURE 6

But then, from Theorem 3, we have

12

$$W = |\mathbf{F}| |\mathbf{D}| \cos \theta = \mathbf{F} \cdot \mathbf{D}$$

Thus the work done by a constant force \mathbf{F} is the dot product $\mathbf{F} \cdot \mathbf{D}$, where \mathbf{D} is the displacement vector.

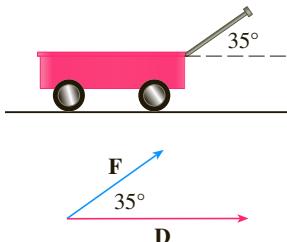


FIGURE 7

EXAMPLE 7 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

SOLUTION If \mathbf{F} and \mathbf{D} are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}| |\mathbf{D}| \cos 35^\circ \\ &= (70)(100) \cos 35^\circ \approx 5734 \text{ N}\cdot\text{m} = 5734 \text{ J} \end{aligned}$$

EXAMPLE 8 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \vec{PQ} = \langle 2, 5, 2 \rangle$, so by Equation 12, the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} = \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \\ &= 6 + 20 + 10 = 36 \end{aligned}$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J.

12.3 EXERCISES

1. Which of the following expressions are meaningful? Which are meaningless? Explain.
- (a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ (b) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
 (c) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ (d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
 (e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$ (f) $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$
- 2–10 Find $\mathbf{a} \cdot \mathbf{b}$.
2. $\mathbf{a} = \langle 5, -2 \rangle$, $\mathbf{b} = \langle 3, 4 \rangle$
3. $\mathbf{a} = \langle 1.5, 0.4 \rangle$, $\mathbf{b} = \langle -4, 6 \rangle$
4. $\mathbf{a} = \langle 6, -2, 3 \rangle$, $\mathbf{b} = \langle 2, 5, -1 \rangle$
5. $\mathbf{a} = \langle 4, 1, \frac{1}{4} \rangle$, $\mathbf{b} = \langle 6, -3, -8 \rangle$
6. $\mathbf{a} = \langle p, -p, 2p \rangle$, $\mathbf{b} = \langle 2q, q, -q \rangle$
7. $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$
8. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$
9. $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$, the angle between \mathbf{a} and \mathbf{b} is 30°
10. $|\mathbf{a}| = 80$, $|\mathbf{b}| = 50$, the angle between \mathbf{a} and \mathbf{b} is $3\pi/4$
- 11–12 If \mathbf{u} is a unit vector, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$.
- 11.
-
- 12.
-
13. (a) Show that $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.
 (b) Show that $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.
14. A street vendor sells a hamburgers, b hot dogs, and c soft drinks on a given day. He charges \$4 for a hamburger, \$2.50 for a hot dog, and \$1 for a soft drink. If $\mathbf{A} = \langle a, b, c \rangle$ and $\mathbf{P} = \langle 4, 2.5, 1 \rangle$, what is the meaning of the dot product $\mathbf{A} \cdot \mathbf{P}$?
- 15–20 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)
15. $\mathbf{a} = \langle 4, 3 \rangle$, $\mathbf{b} = \langle 2, -1 \rangle$
16. $\mathbf{a} = \langle -2, 5 \rangle$, $\mathbf{b} = \langle 5, 12 \rangle$

17. $\mathbf{a} = \langle 1, -4, 1 \rangle$, $\mathbf{b} = \langle 0, 2, -2 \rangle$
 18. $\mathbf{a} = \langle -1, 3, 4 \rangle$, $\mathbf{b} = \langle 5, 2, 1 \rangle$
 19. $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$
 20. $\mathbf{a} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 4\mathbf{j} + 2\mathbf{k}$

21–22 Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

21. $P(2, 0)$, $Q(0, 3)$, $R(3, 4)$
 22. $A(1, 0, -1)$, $B(3, -2, 0)$, $C(1, 3, 3)$

23–24 Determine whether the given vectors are orthogonal, parallel, or neither.

23. (a) $\mathbf{a} = \langle 9, 3 \rangle$, $\mathbf{b} = \langle -2, 6 \rangle$
 (b) $\mathbf{a} = \langle 4, 5, -2 \rangle$, $\mathbf{b} = \langle 3, -1, 5 \rangle$
 (c) $\mathbf{a} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$
 (d) $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 9\mathbf{j} - 2\mathbf{k}$
24. (a) $\mathbf{u} = \langle -5, 4, -2 \rangle$, $\mathbf{v} = \langle 3, 4, -1 \rangle$
 (b) $\mathbf{u} = 9\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = -6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$
 (c) $\mathbf{u} = \langle c, c, c \rangle$, $\mathbf{v} = \langle c, 0, -c \rangle$

25. Use vectors to decide whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is right-angled.

26. Find the values of x such that the angle between the vectors $\langle 2, 1, -1 \rangle$, and $\langle 1, x, 0 \rangle$ is 45° .

27. Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

28. Find two unit vectors that make an angle of 60° with $\mathbf{v} = \langle 3, 4 \rangle$.

29–30 Find the acute angle between the lines.

29. $2x - y = 3$, $3x + y = 7$
 30. $x + 2y = 7$, $5x - y = 2$

31–32 Find the acute angles between the curves at their points of intersection. (The angle between two curves is the angle between their tangent lines at the point of intersection.)

31. $y = x^2$, $y = x^3$
 32. $y = \sin x$, $y = \cos x$, $0 \leq x \leq \pi/2$

33–37 Find the direction cosines and direction angles of the vector. (Give the direction angles correct to the nearest degree.)

33. $\langle 2, 1, 2 \rangle$ 34. $\langle 6, 3, -2 \rangle$
 35. $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ 36. $\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}$
 37. $\langle c, c, c \rangle$, where $c > 0$

38. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ .

39–44 Find the scalar and vector projections of \mathbf{b} onto \mathbf{a} .

39. $\mathbf{a} = \langle -5, 12 \rangle$, $\mathbf{b} = \langle 4, 6 \rangle$
 40. $\mathbf{a} = \langle 1, 4 \rangle$, $\mathbf{b} = \langle 2, 3 \rangle$
 41. $\mathbf{a} = \langle 4, 7, -4 \rangle$, $\mathbf{b} = \langle 3, -1, 1 \rangle$
 42. $\mathbf{a} = \langle -1, 4, 8 \rangle$, $\mathbf{b} = \langle 12, 1, 2 \rangle$
 43. $\mathbf{a} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
 44. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - \mathbf{k}$

45. Show that the vector $\text{orth}_{\mathbf{a}} \mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}} \mathbf{b}$ is orthogonal to \mathbf{a} . (It is called an **orthogonal projection** of \mathbf{b} .)

46. For the vectors in Exercise 40, find $\text{orth}_{\mathbf{a}} \mathbf{b}$ and illustrate by drawing the vectors \mathbf{a} , \mathbf{b} , $\text{proj}_{\mathbf{a}} \mathbf{b}$, and $\text{orth}_{\mathbf{a}} \mathbf{b}$.

47. If $\mathbf{a} = \langle 3, 0, -1 \rangle$, find a vector \mathbf{b} such that $\text{comp}_{\mathbf{a}} \mathbf{b} = 2$.

48. Suppose that \mathbf{a} and \mathbf{b} are nonzero vectors.

- (a) Under what circumstances is $\text{comp}_{\mathbf{a}} \mathbf{b} = \text{comp}_{\mathbf{b}} \mathbf{a}$?
 (b) Under what circumstances is $\text{proj}_{\mathbf{a}} \mathbf{b} = \text{proj}_{\mathbf{b}} \mathbf{a}$?

49. Find the work done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.

50. A tow truck drags a stalled car along a road. The chain makes an angle of 30° with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

51. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of 40° above the horizontal moves the sled 80 ft. Find the work done by the force.

52. A boat sails south with the help of a wind blowing in the direction S36°E with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.

53. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

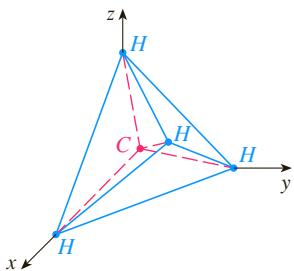
$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

54. If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

55. Find the angle between a diagonal of a cube and one of its edges.

56. Find the angle between a diagonal of a cube and a diagonal of one of its faces.
57. A molecule of methane, CH_4 , is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5° . [Hint: Take the vertices of the tetrahedron to be the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$, as shown in the figure. Then the centroid is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.]



58. If $\mathbf{c} = |\mathbf{a}| \mathbf{b} + |\mathbf{b}| \mathbf{a}$, where \mathbf{a} , \mathbf{b} , and \mathbf{c} are all nonzero vectors, show that \mathbf{c} bisects the angle between \mathbf{a} and \mathbf{b} .
59. Prove Properties 2, 4, and 5 of the dot product (Theorem 2).

60. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

61. Use Theorem 3 to prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}|$$

62. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

- (a) Give a geometric interpretation of the Triangle Inequality.
 (b) Use the Cauchy-Schwarz Inequality from Exercise 61 to prove the Triangle Inequality. [Hint: Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

63. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

- (a) Give a geometric interpretation of the Parallelogram Law.
 (b) Prove the Parallelogram Law. (See the hint in Exercise 62.)

64. Show that if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then the vectors \mathbf{u} and \mathbf{v} must have the same length.

65. If θ is the angle between vectors \mathbf{a} and \mathbf{b} , show that

$$\text{proj}_{\mathbf{a}} \mathbf{b} \cdot \text{proj}_{\mathbf{b}} \mathbf{a} = (\mathbf{a} \cdot \mathbf{b}) \cos^2 \theta$$

12.4 The Cross Product

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector \mathbf{c} that is perpendicular to both \mathbf{a} and \mathbf{b} , as we will see in the next section and in Chapters 13 and 14. If $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

1

$$a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$$

2

$$b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$$

To eliminate c_3 we multiply (1) by b_3 and (2) by a_3 and subtract:

3

$$(a_1 b_3 - a_3 b_1) c_1 + (a_2 b_3 - a_3 b_2) c_2 = 0$$

Equation 3 has the form $p c_1 + q c_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of (3) is

$$c_1 = a_2 b_3 - a_3 b_2 \quad c_2 = a_3 b_1 - a_1 b_3$$

Substituting these values into (1) and (2), we then get

$$c_3 = a_1 b_2 - a_2 b_1$$

This means that a vector perpendicular to both \mathbf{a} and \mathbf{b} is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

The resulting vector is called the *cross product* of \mathbf{a} and \mathbf{b} and is denoted by $\mathbf{a} \times \mathbf{b}$.

Hamilton

The cross product was invented by the Irish mathematician Sir William Rowan Hamilton (1805–1865), who had created a precursor of vectors, called quaternions. When he was five years old Hamilton could read Latin, Greek, and Hebrew. At age eight he added French and Italian and when ten he could read Arabic and Sanskrit. At the age of 21, while still an undergraduate at Trinity College in Dublin, Hamilton was appointed Professor of Astronomy at the university and Royal Astronomer of Ireland!

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Notice that the **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors \mathbf{a} and \mathbf{b} , unlike the dot product, is a vector. For this reason it is also called the **vector product**. Note that $\mathbf{a} \times \mathbf{b}$ is defined only when \mathbf{a} and \mathbf{b} are *three-dimensional* vectors.

In order to make Definition 4 easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(Multiply across the diagonals and subtract.) For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\text{5} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38$$

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} , we see that the cross product of the vectors $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ and $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ is

$$\text{6} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

In view of the similarity between Equations 5 and 6, we often write

$$\text{7} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain

Equation 6. The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\ &= (-15 - 28) \mathbf{i} - (-5 - 8) \mathbf{j} + (7 - 6) \mathbf{k} = -43 \mathbf{i} + 13 \mathbf{j} + \mathbf{k} \quad \blacksquare\end{aligned}$$

EXAMPLE 2 Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector \mathbf{a} in V_3 .

SOLUTION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\begin{aligned}\mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= (a_2 a_3 - a_3 a_2) \mathbf{i} - (a_1 a_3 - a_3 a_1) \mathbf{j} + (a_1 a_2 - a_2 a_1) \mathbf{k} \\ &= 0 \mathbf{i} - 0 \mathbf{j} + 0 \mathbf{k} = \mathbf{0}\end{aligned}$$

We constructed the cross product $\mathbf{a} \times \mathbf{b}$ so that it would be perpendicular to both \mathbf{a} and \mathbf{b} . This is one of the most important properties of a cross product, so let's emphasize and verify it in the following theorem and give a formal proof.

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

PROOF In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{a} , we compute their dot product as follows:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1) \\ &= a_1 a_2 b_3 - a_1 b_2 a_3 - a_1 a_2 b_3 + b_1 a_2 a_3 + a_1 b_2 a_3 - b_1 a_2 a_3 \\ &= 0\end{aligned}$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} . ■

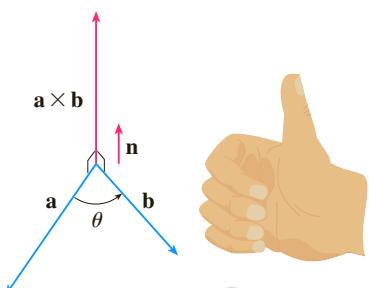


FIGURE 1

The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

If \mathbf{a} and \mathbf{b} are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through \mathbf{a} and \mathbf{b} . It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule*: if the fingers of your right hand curl in the direction of

a rotation (through an angle less than 180°) from \mathbf{a} to \mathbf{b} , then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by the following theorem.

9 Theorem If θ is the angle between \mathbf{a} and \mathbf{b} (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

TEC Visual 12.4 shows how $\mathbf{a} \times \mathbf{b}$ changes as \mathbf{b} changes.

PROOF From the definitions of the cross product and length of a vector, we have

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\ &= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_3^2 \\ &\quad + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2 \theta \quad (\text{by Theorem 12.3.3}) \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2 \theta) \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \theta \end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2 \theta} = \sin \theta$ because $\sin \theta \geq 0$ when $0 \leq \theta \leq \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta \quad \blacksquare$$

Geometric characterization of $\mathbf{a} \times \mathbf{b}$

Since a vector is completely determined by its magnitude and direction, we can now say that $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both \mathbf{a} and \mathbf{b} , whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}| |\mathbf{b}| \sin \theta$. In fact, that is exactly how physicists *define* $\mathbf{a} \times \mathbf{b}$.

10 Corollary Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

PROOF Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel if and only if $\theta = 0$ or π . In either case $\sin \theta = 0$, so $|\mathbf{a} \times \mathbf{b}| = 0$ and therefore $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. \blacksquare

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2. If \mathbf{a} and \mathbf{b} are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}| \sin \theta$, and area

$$A = |\mathbf{a}| (|\mathbf{b}| \sin \theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

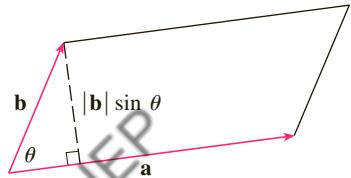


FIGURE 2

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b} .

EXAMPLE 3 Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to both \vec{PQ} and \vec{PR} and is therefore perpendicular to the plane through P , Q , and R . We know from (12.2.1) that

$$\vec{PQ} = (-2 - 1)\mathbf{i} + (5 - 4)\mathbf{j} + (-1 - 6)\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\vec{PR} = (1 - 1)\mathbf{i} + (-1 - 4)\mathbf{j} + (1 - 6)\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\begin{aligned}\vec{PQ} \times \vec{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} \\ &= (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}\end{aligned}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane. ■

EXAMPLE 4 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION In Example 3 we computed that $\vec{PQ} \times \vec{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$|\vec{PQ} \times \vec{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$. ■

If we apply Theorems 8 and 9 to the standard basis vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} using $\theta = \pi/2$, we obtain

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{i} = \mathbf{j}$
$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

□ Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

-  So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products. The following theorem summarizes the properties of vector products.

11 Properties of the Cross Product If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product. We give the proof of Property 5 and leave the remaining proofs as exercises.

PROOF OF PROPERTY 5 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$$\begin{aligned} \mathbf{12} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

■ Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$\mathbf{13} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . (See Figure 3.) The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between \mathbf{a} and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}| |\cos \theta|$. (We must use $|\cos \theta|$ instead of $\cos \theta$ in case $\theta > \pi/2$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}| |\mathbf{a}| |\cos \theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus we have proved the following formula.

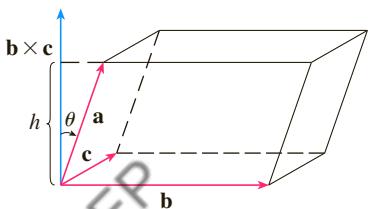


FIGURE 3

14 The volume of the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

If we use the formula in (14) and discover that the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

EXAMPLE 5 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

SOLUTION We use Equation 13 to compute their scalar triple product:

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} \\ &= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} \\ &= 1(18) - 4(36) - 7(-18) = 0\end{aligned}$$

Therefore, by (14), the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is 0. This means that \mathbf{a} , \mathbf{b} , and \mathbf{c} are coplanar. ■

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 is called the **vector triple product** of \mathbf{a} , \mathbf{b} , and \mathbf{c} . Property 6 will be used to derive Kepler's First Law of planetary motion in Chapter 13. Its proof is left as Exercise 50.

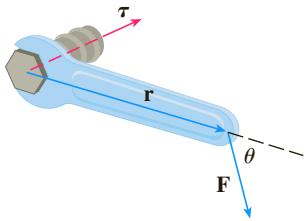


FIGURE 4

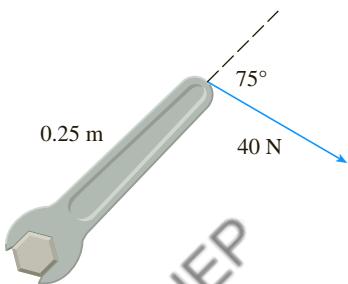


FIGURE 5

Torque

The idea of a cross product occurs often in physics. In particular, we consider a force \mathbf{F} acting on a rigid body at a point given by a position vector \mathbf{r} . (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.) The **torque** τ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\tau = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 9, the magnitude of the torque vector is

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta$$

where θ is the angle between the position and force vectors. Observe that the only component of \mathbf{F} that can cause a rotation is the one perpendicular to \mathbf{r} , that is, $|\mathbf{F}| \sin \theta$. The magnitude of the torque is equal to the area of the parallelogram determined by \mathbf{r} and \mathbf{F} .

EXAMPLE 6 A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

SOLUTION The magnitude of the torque vector is

$$\begin{aligned} |\tau| &= |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin 75^\circ = (0.25)(40) \sin 75^\circ \\ &= 10 \sin 75^\circ \approx 9.66 \text{ N}\cdot\text{m} \end{aligned}$$

If the bolt is right-threaded, then the torque vector itself is

$$\tau = |\tau| \mathbf{n} \approx 9.66 \mathbf{n}$$

where \mathbf{n} is a unit vector directed down into the page (by the right-hand rule). ■

12.4 EXERCISES

- 1–7** Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both \mathbf{a} and \mathbf{b} .

1. $\mathbf{a} = \langle 2, 3, 0 \rangle$, $\mathbf{b} = \langle 1, 0, 5 \rangle$
2. $\mathbf{a} = \langle 4, 3, -2 \rangle$, $\mathbf{b} = \langle 2, -1, 1 \rangle$
3. $\mathbf{a} = 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
4. $\mathbf{a} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$
5. $\mathbf{a} = \frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
6. $\mathbf{a} = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}$
7. $\mathbf{a} = \langle t, 1, 1/t \rangle$, $\mathbf{b} = \langle t^2, t^2, 1 \rangle$

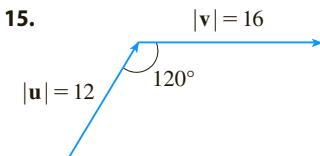
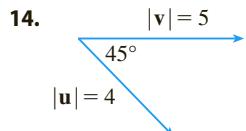
- 8.** If $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. Sketch \mathbf{a} , \mathbf{b} , and $\mathbf{a} \times \mathbf{b}$ as vectors starting at the origin.

- 9–12** Find the vector, not with determinants, but by using properties of cross products.

9. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$
10. $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$
11. $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$
12. $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

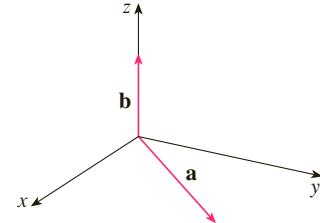
- 13.** State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.
- (a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
 - (b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$
 - (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
 - (d) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$
 - (e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$
 - (f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

- 14–15** Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.



- 16.** The figure shows a vector \mathbf{a} in the xy -plane and a vector \mathbf{b} in the direction of \mathbf{k} . Their lengths are $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.
- (a) Find $|\mathbf{a} \times \mathbf{b}|$.

- (b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.



17. If $\mathbf{a} = \langle 2, -1, 3 \rangle$ and $\mathbf{b} = \langle 4, 2, 1 \rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.
18. If $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 2, 1, -1 \rangle$, and $\mathbf{c} = \langle 0, 1, 3 \rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.
19. Find two unit vectors orthogonal to both $\langle 3, 2, 1 \rangle$ and $\langle -1, 1, 0 \rangle$.
20. Find two unit vectors orthogonal to both $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.
21. Show that $\mathbf{0} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$ for any vector \mathbf{a} in V_3 .
22. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all vectors \mathbf{a} and \mathbf{b} in V_3 .
- 23–26 Prove the property of cross products (Theorem 11).
23. Property 1: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
24. Property 2: $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
25. Property 3: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
26. Property 4: $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
27. Find the area of the parallelogram with vertices $A(-3, 0)$, $B(-1, 3)$, $C(5, 2)$, and $D(3, -1)$.
28. Find the area of the parallelogram with vertices $P(1, 0, 2)$, $Q(3, 3, 3)$, $R(7, 5, 8)$, and $S(5, 2, 7)$.
- 29–32 (a) Find a nonzero vector orthogonal to the plane through the points P , Q , and R , and (b) find the area of triangle PQR .
29. $P(1, 0, 1)$, $Q(-2, 1, 3)$, $R(4, 2, 5)$
30. $P(0, 0, -3)$, $Q(4, 2, 0)$, $R(3, 3, 1)$
31. $P(0, -2, 0)$, $Q(4, 1, -2)$, $R(5, 3, 1)$

32. $P(2, -3, 4)$, $Q(-1, -2, 2)$, $R(3, 1, -3)$

33–34 Find the volume of the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

33. $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle -1, 1, 2 \rangle$, $\mathbf{c} = \langle 2, 1, 4 \rangle$

34. $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

35–36 Find the volume of the parallelepiped with adjacent edges PQ , PR , and PS .

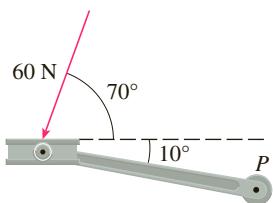
35. $P(-2, 1, 0)$, $Q(2, 3, 2)$, $R(1, 4, -1)$, $S(3, 6, 1)$

36. $P(3, 0, 1)$, $Q(-1, 2, 5)$, $R(5, 1, -1)$, $S(0, 4, 2)$

37. Use the scalar triple product to verify that the vectors $\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$ are coplanar.

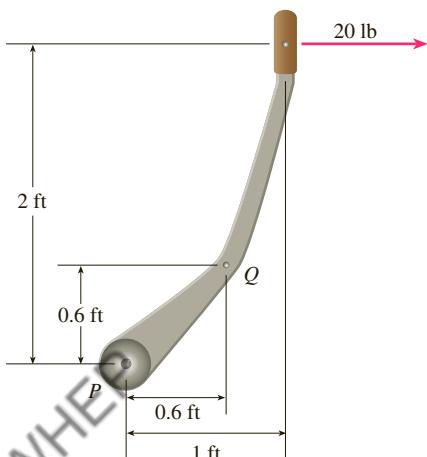
38. Use the scalar triple product to determine whether the points $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$, and $D(3, 6, -4)$ lie in the same plane.

39. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P .



40. (a) A horizontal force of 20 lb is applied to the handle of a gearshift lever as shown. Find the magnitude of the torque about the pivot point P .

(b) Find the magnitude of the torque about P if the same force is applied at the elbow Q of the lever.



41. A wrench 30 cm long lies along the positive y -axis and grips a bolt at the origin. A force is applied in the direction $\langle 0, 3, -4 \rangle$ at the end of the wrench. Find the magnitude of the force needed to supply 100 N·m of torque to the bolt.

42. Let $\mathbf{v} = 5\mathbf{j}$ and let \mathbf{u} be a vector with length 3 that starts at the origin and rotates in the xy -plane. Find the maximum and minimum values of the length of the vector $\mathbf{u} \times \mathbf{v}$. In what direction does $\mathbf{u} \times \mathbf{v}$ point?

43. If $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle$, find the angle between \mathbf{a} and \mathbf{b} .

44. (a) Find all vectors \mathbf{v} such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, -5 \rangle$$

(b) Explain why there is no vector \mathbf{v} such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$$

45. (a) Let P be a point not on the line L that passes through the points Q and R . Show that the distance d from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \vec{QR}$ and $\mathbf{b} = \vec{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

46. (a) Let P be a point not on the plane that passes through the points Q , R , and S . Show that the distance d from P to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \vec{QR}$, $\mathbf{b} = \vec{QS}$, and $\mathbf{c} = \vec{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

47. Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

48. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, show that

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

49. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

50. Prove Property 6 of cross products, that is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

51. Use Exercise 50 to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

52. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

53. Suppose that $\mathbf{a} \neq \mathbf{0}$.

- If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
- If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
- If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

54. If $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \quad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

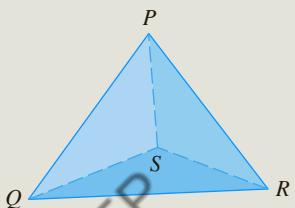
(These vectors occur in the study of crystallography. Vectors of the form $n_1 \mathbf{v}_1 + n_2 \mathbf{v}_2 + n_3 \mathbf{v}_3$, where each n_i is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of $\mathbf{k}_1, \mathbf{k}_2$, and \mathbf{k}_3 form the *reciprocal lattice*.)

- Show that \mathbf{k}_i is perpendicular to \mathbf{v}_j if $i \neq j$.
- Show that $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.

- Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.

DISCOVERY PROJECT

THE GEOMETRY OF A TETRAHEDRON



A tetrahedron is a solid with four vertices, P, Q, R , and S , and four triangular faces, as shown in the figure.

- Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and \mathbf{v}_4 be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R , and S , respectively, and directions perpendicular to the respective faces and pointing outward. Show that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

- The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.
 - Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices P, Q, R , and S .
 - Find the volume of the tetrahedron whose vertices are $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(1, 1, 2)$, and $S(3, -1, 2)$.
- Suppose the tetrahedron in the figure has a trirectangular vertex S . (This means that the three angles at S are all right angles.) Let A, B , and C be the areas of the three faces that meet at S , and let D be the area of the opposite face PQR . Using the result of Problem 1, or otherwise, show that

$$D^2 = A^2 + B^2 + C^2$$

(This is a three-dimensional version of the Pythagorean Theorem.)

12.5 Equations of Lines and Planes

■ Lines

A line in the xy -plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L . In three dimensions the direction of a line is conveniently described by a vector, so we let \mathbf{v} be a vector parallel to L . Let $P(x, y, z)$ be an arbitrary point on L and let \mathbf{r}_0 and \mathbf{r} be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and \overrightarrow{OP}). If \mathbf{a} is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$. But, since \mathbf{a} and \mathbf{v} are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$. Thus

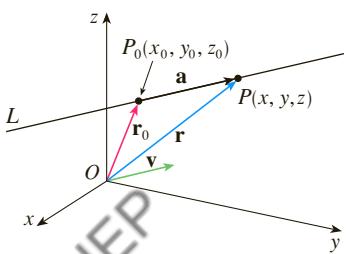


FIGURE 1

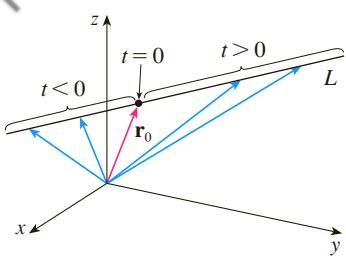


FIGURE 2

1

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a **vector equation** of L . Each value of the **parameter** t gives the position vector \mathbf{r} of a point on L . In other words, as t varies, the line is traced out by the tip of the vector \mathbf{r} . As Figure 2 indicates, positive values of t correspond to points on L that lie on one side of P_0 , whereas negative values of t correspond to points that lie on the other side of P_0 .

If the vector \mathbf{v} that gives the direction of the line L is written in component form as $\mathbf{v} = \langle a, b, c \rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Each value of the parameter t gives a point (x, y, z) on L .

2 Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c \rangle$ are

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

Figure 3 shows the line L in Example 1 and its relation to the given point and to the vector that gives its direction.

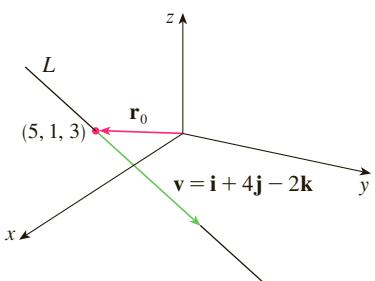


FIGURE 3

EXAMPLE 1

- (a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
 (b) Find two other points on the line.

SOLUTION

- (a) Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

- (b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$. ■

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change. For instance, if, instead of $(5, 1, 3)$, we choose the point $(6, 5, 1)$ in Example 1, then the parametric equations of the line become

$$x = 6 + t \quad y = 5 + 4t \quad z = 1 - 2t$$

Or, if we stay with the point $(5, 1, 3)$ but choose the parallel vector $2\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$, we arrive at the equations

$$x = 5 + 2t \quad y = 1 + 8t \quad z = 3 - 4t$$

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L , then the numbers a , b , and c are called **direction numbers** of L . Since any vector parallel to \mathbf{v} could also be used, we see that any three numbers proportional to a , b , and c could also be used as a set of direction numbers for L .

Another way of describing a line L is to eliminate the parameter t from Equations 2. If none of a , b , or c is 0, we can solve each of these equations for t :

$$t = \frac{x - x_0}{a} \quad t = \frac{y - y_0}{b} \quad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

3

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called **symmetric equations** of L . Notice that the numbers a , b , and c that appear in the denominators of Equations 3 are direction numbers of L , that is, components of a vector parallel to L . If one of a , b , or c is 0, we can still eliminate t . For instance, if $a = 0$, we could write the equations of L as

$$x = x_0 \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.

EXAMPLE 2

- (a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
(b) At what point does this line intersect the xy -plane?

SOLUTION

- (a) We are not explicitly given a vector parallel to the line, but observe that the vector \mathbf{v} with representation \overrightarrow{AB} is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are $a = 1$, $b = -5$, and $c = 4$. Taking the point $(2, 4, -3)$ as P_0 , we see that parametric equations (2) are

$$x = 2 + t \quad y = 4 - 5t \quad z = -3 + 4t$$

and symmetric equations (3) are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

- (b) The line intersects the xy -plane when $z = 0$, so we put $z = 0$ in the symmetric equations and obtain

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{3}{4}$$

This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$, so the line intersects the xy -plane at the point $(\frac{11}{4}, \frac{1}{4}, 0)$. ■

Figure 4 shows the line L in Example 2 and the point P where it intersects the xy -plane.

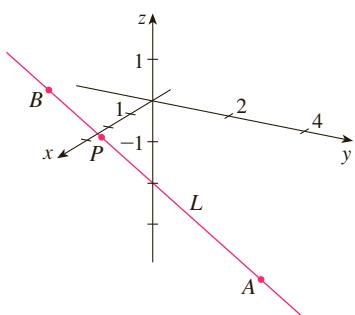


FIGURE 4

In general, the procedure of Example 2 shows that direction numbers of the line L through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are $x_1 - x_0$, $y_1 - y_0$, and $z_1 - z_0$ and so symmetric equations of L are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Often, we need a description, not of an entire line, but of just a line segment. How, for instance, could we describe the line segment AB in Example 2? If we put $t = 0$ in the parametric equations in Example 2(a), we get the point $(2, 4, -3)$ and if we put $t = 1$ we get $(3, -1, 1)$. So the line segment AB is described by the parametric equations

$$x = 2 + t \quad y = 4 - 5t \quad z = -3 + 4t \quad 0 \leq t \leq 1$$

or by the corresponding vector equation

$$\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t \rangle \quad 0 \leq t \leq 1$$

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector \mathbf{r}_0 in the direction of a vector \mathbf{v} is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. If the line also passes through (the tip of) \mathbf{r}_1 , then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and so its vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

The line segment from \mathbf{r}_0 to \mathbf{r}_1 is given by the parameter interval $0 \leq t \leq 1$.

4 The line segment from \mathbf{r}_0 to \mathbf{r}_1 is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

EXAMPLE 3 Show that the lines L_1 and L_2 with parametric equations

$$L_1: \quad x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$

$$L_2: \quad x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

SOLUTION The lines are not parallel because the corresponding direction vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. (Their components are not proportional.) If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$1 + t = 2s$$

$$-2 + 3t = 3 + s$$

$$4 - t = -3 + 4s$$

But if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$, and these values don't satisfy the third equation. Therefore there are no values of t and s that satisfy the three equations, so L_1 and L_2 do not intersect. Thus L_1 and L_2 are skew lines. ■

The lines L_1 and L_2 in Example 3, shown in Figure 5, are skew lines.

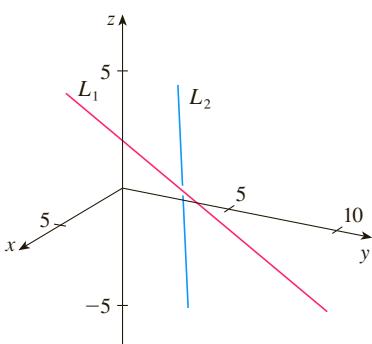


FIGURE 5

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the

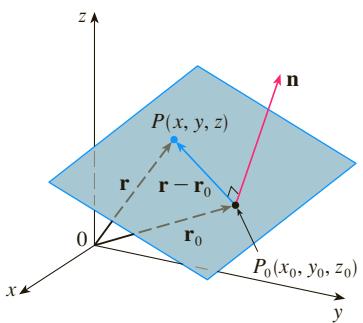


FIGURE 6

“direction” of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector \mathbf{n} that is orthogonal to the plane. This orthogonal vector \mathbf{n} is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let \mathbf{r}_0 and \mathbf{r} be the position vectors of P_0 and P . Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. (See Figure 6.) The normal vector \mathbf{n} is orthogonal to every vector in the given plane. In particular, \mathbf{n} is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

5

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be rewritten as

6

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Then the vector equation (5) becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

7 A scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

EXAMPLE 4 Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

SOLUTION Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

To find the x -intercept we set $y = z = 0$ in this equation and obtain $x = 6$. Similarly, the y -intercept is 4 and the z -intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7). ■

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

8

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$. Equation 8 is called a **linear equation** in x , y , and z . Conversely, it can be shown that if a , b , and c are not all 0, then the linear equation (8) represents a plane with normal vector $\langle a, b, c \rangle$. (See Exercise 83.)

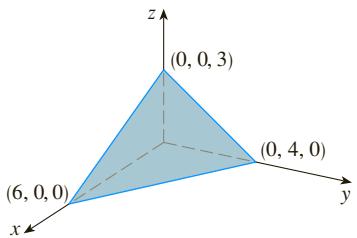


FIGURE 7

Figure 8 shows the portion of the plane in Example 5 that is enclosed by triangle PQR .

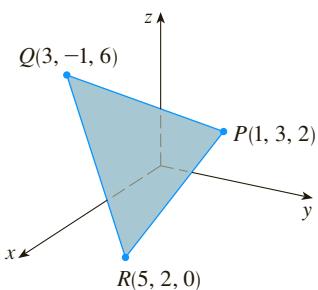


FIGURE 8

EXAMPLE 5 Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

SOLUTION The vectors \mathbf{a} and \mathbf{b} corresponding to \vec{PQ} and \vec{PR} are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \quad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both \mathbf{a} and \mathbf{b} lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector \mathbf{n} , an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

EXAMPLE 6 Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

SOLUTION We substitute the expressions for x , y , and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

This simplifies to $-10t = 20$, so $t = -2$. Therefore the point of intersection occurs when the parameter value is $t = -2$. Then $x = 2 + 3(-2) = -4$, $y = -4(-2) = 8$, $z = 5 - 2 = 3$ and so the point of intersection is $(-4, 8, 3)$.

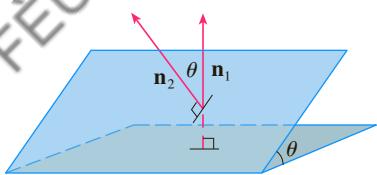


FIGURE 9

Figure 10 shows the planes in Example 7 and their line of intersection L .

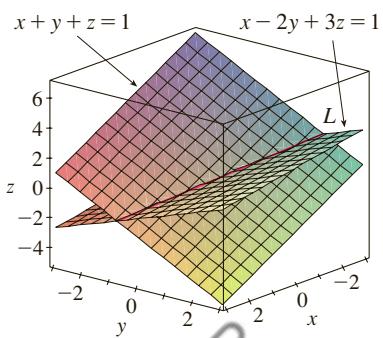


FIGURE 10

EXAMPLE 7

- Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
- Find symmetric equations for the line of intersection L of these two planes.

SOLUTION

- The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \quad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

and so, if θ is the angle between the planes, Corollary 12.3.6 gives

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1} \sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

- We first need to find a point on L . For instance, we can find the point where the line intersects the xy -plane by setting $z = 0$ in the equations of both planes. This gives the

equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1, y = 0$. So the point $(1, 0, 0)$ lies on L .

Now we observe that, since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector \mathbf{v} parallel to L is given by the cross product

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and so the symmetric equations of L can be written as

$$\frac{x - 1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

NOTE Since a linear equation in x, y , and z represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points (x, y, z) that satisfy both $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ lie on both of these planes, and so the pair of linear equations represents the line of intersection of the planes (if they are not parallel). For instance, in Example 7 the line L was given as the line of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. The symmetric equations that we found for L could be written as

$$\frac{x - 1}{5} = \frac{y}{-2} \quad \text{and} \quad \frac{y}{-2} = \frac{z}{-3}$$

which is again a pair of linear equations. They exhibit L as the line of intersection of the planes $(x - 1)/5 = y/(-2)$ and $y/(-2) = z/(-3)$. (See Figure 11.)

In general, when we write the equations of a line in the symmetric form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

we can regard the line as the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{and} \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Distances

EXAMPLE 8 Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

SOLUTION Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let \mathbf{b} be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of \mathbf{b} onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. (See Section 12.3.) Thus

$$\begin{aligned} D &= |\operatorname{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

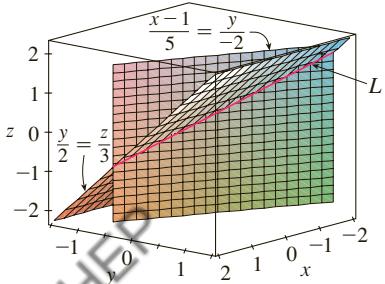


FIGURE 11

Figure 11 shows how the line L in Example 7 can also be regarded as the line of intersection of planes derived from its symmetric equations.

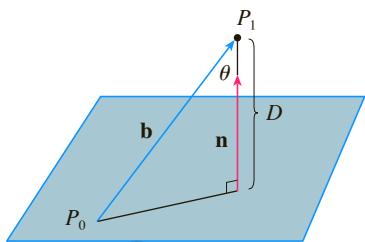


FIGURE 12

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane and so we have $ax_0 + by_0 + cz_0 + d = 0$. Thus the formula for D can be written as

9

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

EXAMPLE 9 Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

SOLUTION First we note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so $(\frac{1}{2}, 0, 0)$ is a point in this plane. By Formula 9, the distance between $(\frac{1}{2}, 0, 0)$ and the plane $5x + y - z - 1 = 0$ is

$$D = \frac{|5(\frac{1}{2}) + 1(0) - 1(0) - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is $\sqrt{3}/6$.

EXAMPLE 10 In Example 3 we showed that the lines

$$L_1: \quad x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$

$$L_2: \quad x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

are skew. Find the distance between them.

SOLUTION Since the two lines L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2 . The distance between L_1 and L_2 is the same as the distance between P_1 and P_2 , which can be computed as in Example 9. The common normal vector to both planes must be orthogonal to both $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$ (the direction of L_1) and $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$ (the direction of L_2). So a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put $s = 0$ in the equations of L_2 , we get the point $(0, 3, -3)$ on L_2 and so an equation for P_2 is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \quad \text{or} \quad 13x - 6y - 5z + 3 = 0$$

If we now set $t = 0$ in the equations for L_1 , we get the point $(1, -2, 4)$ on P_1 . So the distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$. By Formula 9, this distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$

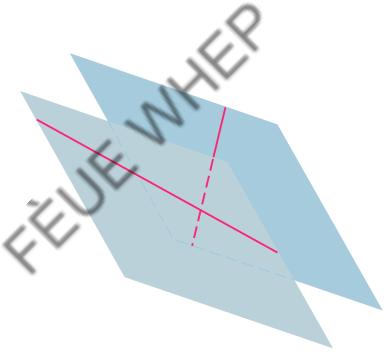


FIGURE 13

Skew lines, like those in Example 10, always lie on (nonidentical) parallel planes.

12.5 EXERCISES

1. Determine whether each statement is true or false in \mathbb{R}^3 .
- Two lines parallel to a third line are parallel.
 - Two lines perpendicular to a third line are parallel.
 - Two planes parallel to a third plane are parallel.
 - Two planes perpendicular to a third plane are parallel.
 - Two lines parallel to a plane are parallel.
 - Two lines perpendicular to a plane are parallel.
 - Two planes parallel to a line are parallel.
 - Two planes perpendicular to a line are parallel.
 - Two planes either intersect or are parallel.
 - Two lines either intersect or are parallel.
 - A plane and a line either intersect or are parallel.

2-5 Find a vector equation and parametric equations for the line.

- The line through the point $(6, -5, 2)$ and parallel to the vector $\langle 1, 3, -\frac{2}{3} \rangle$
- The line through the point $(2, 2.4, 3.5)$ and parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- The line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t, y = 6 - 3t, z = 3 + 9t$
- The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$

6-12 Find parametric equations and symmetric equations for the line.

- The line through the origin and the point $(4, 3, -1)$
- The line through the points $(0, \frac{1}{2}, 1)$ and $(2, 1, -3)$
- The line through the points $(1, 2.4, 4.6)$ and $(2.6, 1.2, 0.3)$
- The line through the points $(-8, 1, 4)$ and $(3, -2, 4)$
- The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$
- The line through $(-6, 2, 3)$ and parallel to the line $\frac{1}{2}x = \frac{1}{3}y = z + 1$
- The line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$

- Is the line through $(-4, -6, 1)$ and $(-2, 0, -3)$ parallel to the line through $(10, 18, 4)$ and $(5, 3, 14)$?
- Is the line through $(-2, 4, 0)$ and $(1, 1, 1)$ perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$?
- (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3 \rangle$.
(b) Find the points in which the required line in part (a) intersects the coordinate planes.
- (a) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.

- (b) In what points does this line intersect the coordinate planes?

17. Find a vector equation for the line segment from $(6, -1, 9)$ to $(7, 6, 0)$.

18. Find parametric equations for the line segment from $(-2, 18, 31)$ to $(11, -4, 48)$.

19-22 Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19. $L_1: x = 3 + 2t, y = 4 - t, z = 1 + 3t$

$L_2: x = 1 + 4s, y = 3 - 2s, z = 4 + 5s$

20. $L_1: x = 5 - 12t, y = 3 + 9t, z = 1 - 3t$

$L_2: x = 3 + 8s, y = -6s, z = 7 + 2s$

21. $L_1: \frac{x - 2}{1} = \frac{y - 3}{-2} = \frac{z - 1}{-3}$

$L_2: \frac{x - 3}{1} = \frac{y + 4}{3} = \frac{z - 2}{-7}$

22. $L_1: \frac{x}{1} = \frac{y - 1}{-1} = \frac{z - 2}{3}$

$L_2: \frac{x - 2}{2} = \frac{y - 3}{-2} = \frac{z}{7}$

23-40 Find an equation of the plane.

23. The plane through the origin and perpendicular to the vector $\langle 1, -2, 5 \rangle$

24. The plane through the point $(5, 3, 5)$ and with normal vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

25. The plane through the point $(-1, \frac{1}{2}, 3)$ and with normal vector $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

26. The plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t, y = 2 - t, z = 3 + 4t$

27. The plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$

28. The plane through the point $(3, -2, 8)$ and parallel to the plane $z = x + y$

29. The plane through the point $(1, \frac{1}{2}, \frac{1}{3})$ and parallel to the plane $x + y + z = 0$

30. The plane that contains the line $x = 1 + t, y = 2 - t, z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

31. The plane through the points $(0, 1, 1), (1, 0, 1)$, and $(1, 1, 0)$

32. The plane through the origin and the points $(3, -2, 1)$ and $(1, 1, 1)$

33. The plane through the points $(2, 1, 2), (3, -8, 6)$, and $(-2, -3, 1)$

- 34.** The plane through the points $(3, 0, -1)$, $(-2, -2, 3)$, and $(7, 1, -4)$
- 35.** The plane that passes through the point $(3, 5, -1)$ and contains the line $x = 4 - t$, $y = 2t - 1$, $z = -3t$
- 36.** The plane that passes through the point $(6, -1, 3)$ and contains the line with symmetric equations
 $x/3 = y + 4 = z/2$
- 37.** The plane that passes through the point $(3, 1, 4)$ and contains the line of intersection of the planes $x + 2y + 3z = 1$ and $2x - y + z = -3$
- 38.** The plane that passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $2z = 5x + 4y$
- 39.** The plane that passes through the point $(1, 5, 1)$ and is perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$
- 40.** The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$

41–44 Use intercepts to help sketch the plane.

41. $2x + 5y + z = 10$ **42.** $3x + y + 2z = 6$
43. $6x - 3y + 4z = 6$ **44.** $6x + 5y - 3z = 15$

45–47 Find the point at which the line intersects the given plane.

45. $x = 2 - 2t$, $y = 3t$, $z = 1 + t$; $x + 2y - z = 7$
46. $x = t - 1$, $y = 1 + 2t$, $z = 3 - t$; $3x - y + 2z = 5$
47. $5x = y/2 = z + 2$; $10x - 7y + 3z + 24 = 0$

- 48.** Where does the line through $(-3, 1, 0)$ and $(-1, 5, 6)$ intersect the plane $2x + y - z = -2$?
- 49.** Find direction numbers for the line of intersection of the planes $x + y + z = 1$ and $x + z = 0$.
- 50.** Find the cosine of the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

51–56 Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them. (Round to one decimal place.)

51. $x + 4y - 3z = 1$, $-3x + 6y + 7z = 0$
52. $9x - 3y + 6z = 2$, $2y = 6x + 4z$
53. $x + 2y - z = 2$, $2x - 2y + z = 1$
54. $x - y + 3z = 1$, $3x + y - z = 2$
55. $2x - 3y = z$, $4x = 3 + 6y + 2z$
56. $5x + 2y + 3z = 2$, $y = 4x - 6z$

57–58 (a) Find parametric equations for the line of intersection of the planes and (b) find the angle between the planes.

57. $x + y + z = 1$, $x + 2y + 2z = 1$
58. $3x - 2y + z = 1$, $2x + y - 3z = 3$

59–60 Find symmetric equations for the line of intersection of the planes.

59. $5x - 2y - 2z = 1$, $4x + y + z = 6$
60. $z = 2x - y - 5$, $z = 4x + 3y - 5$

61. Find an equation for the plane consisting of all points that are equidistant from the points $(1, 0, -2)$ and $(3, 4, 0)$.

62. Find an equation for the plane consisting of all points that are equidistant from the points $(2, 5, 5)$ and $(-6, 3, 1)$.

63. Find an equation of the plane with x -intercept a , y -intercept b , and z -intercept c .

64. (a) Find the point at which the given lines intersect:

$$\mathbf{r} = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$$

$$\mathbf{r} = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle$$

(b) Find an equation of the plane that contains these lines.

65. Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

66. Find parametric equations for the line through the point $(0, 1, 2)$ that is perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$ and intersects this line.

67. Which of the following four planes are parallel? Are any of them identical?

$$P_1: 3x + 6y - 3z = 6 \quad P_2: 4x - 12y + 8z = 5$$

$$P_3: 9y = 1 + 3x + 6z \quad P_4: z = x + 2y - 2$$

68. Which of the following four lines are parallel? Are any of them identical?

$$L_1: x = 1 + 6t, \quad y = 1 - 3t, \quad z = 12t + 5$$

$$L_2: x = 1 + 2t, \quad y = t, \quad z = 1 + 4t$$

$$L_3: 2x - 2 = 4 - 4y = z + 1$$

$$L_4: \mathbf{r} = \langle 3, 1, 5 \rangle + t\langle 4, 2, 8 \rangle$$

69–70 Use the formula in Exercise 12.4.45 to find the distance from the point to the given line.

69. $(4, 1, -2)$; $x = 1 + t$, $y = 3 - 2t$, $z = 4 - 3t$
70. $(0, 1, 3)$; $x = 2t$, $y = 6 - 2t$, $z = 3 + t$

71–72 Find the distance from the point to the given plane.

71. $(1, -2, 4)$, $3x + 2y + 6z = 5$

72. $(-6, 3, 5)$, $x - 2y - 4z = 8$

73–74 Find the distance between the given parallel planes.

73. $2x - 3y + z = 4$, $4x - 6y + 2z = 3$

74. $6z = 4y - 2x$, $9z = 1 - 3x + 6y$

75. Show that the distance between the parallel planes

$$ax + by + cz + d_1 = 0 \text{ and } ax + by + cz + d_2 = 0 \text{ is}$$

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

76. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

77. Show that the lines with symmetric equations $x = y = z$ and $x + 1 = y/2 = z/3$ are skew, and find the distance between these lines.

78. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

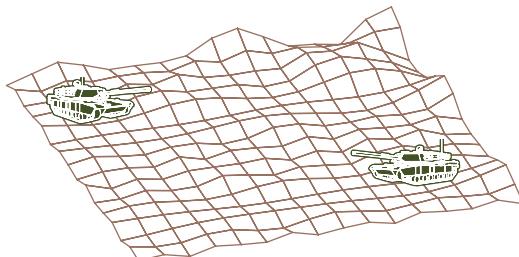
79. Let L_1 be the line through the origin and the point $(2, 0, -1)$. Let L_2 be the line through the points $(1, -1, 1)$ and $(4, 1, 3)$. Find the distance between L_1 and L_2 .

80. Let L_1 be the line through the points $(1, 2, 6)$ and $(2, 4, 8)$. Let L_2 be the line of intersection of the planes P_1 and P_2 , where P_1 is the plane $x - y + 2z + 1 = 0$ and P_2 is the plane

through the points $(3, 2, -1)$, $(0, 0, 1)$, and $(1, 2, 1)$. Calculate the distance between L_1 and L_2 .

81. Two tanks are participating in a battle simulation. Tank A is at point $(325, 810, 561)$ and tank B is positioned at point $(765, 675, 599)$.

- (a) Find parametric equations for the line of sight between the tanks.
- (b) If we divide the line of sight into 5 equal segments, the elevations of the terrain at the four intermediate points from tank A to tank B are 549, 566, 586, and 589. Can the tanks see each other?



82. Give a geometric description of each family of planes.

- | | |
|---|----------------------|
| (a) $x + y + z = c$ | (b) $x + y + cz = 1$ |
| (c) $y \cos \theta + z \sin \theta = 1$ | |

83. If a , b , and c are not all 0, show that the equation $ax + by + cz + d = 0$ represents a plane and $\langle a, b, c \rangle$ is a normal vector to the plane.

Hint: Suppose $a \neq 0$ and rewrite the equation in the form

$$a\left(x + \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0$$

LABORATORY PROJECT PUTTING 3D IN PERSPECTIVE



Computer graphics programmers face the same challenge as the great painters of the past: how to represent a three-dimensional scene as a flat image on a two-dimensional plane (a screen or a canvas). To create the illusion of perspective, in which closer objects appear larger than those farther away, three-dimensional objects in the computer's memory are projected onto a rectangular screen window from a viewpoint where the eye, or camera, is located. The viewing volume—the portion of space that will be visible—is the region contained by the four planes that pass through the viewpoint and an edge of the screen window. If objects in the scene extend beyond these four planes, they must be truncated before pixel data are sent to the screen. These planes are therefore called *clipping planes*.

1. Suppose the screen is represented by a rectangle in the yz -plane with vertices $(0, \pm 400, 0)$ and $(0, \pm 400, 600)$, and the camera is placed at $(1000, 0, 0)$. A line L in the scene passes through the points $(230, -285, 102)$ and $(860, 105, 264)$. At what points should L be clipped by the clipping planes?
2. If the clipped line segment is projected onto the screen window, identify the resulting line segment.

3. Use parametric equations to plot the edges of the screen window, the clipped line segment, and its projection onto the screen window. Then add eight lines connecting the viewpoint to each end of the clipped segments to verify that the projection is correct.
4. A rectangle with vertices $(621, -147, 206)$, $(563, 31, 242)$, $(657, -111, 86)$, and $(599, 67, 122)$ is added to the scene. The line L intersects this rectangle. To make the rectangle appear opaque, a programmer can use *hidden line rendering*, which removes portions of objects that are behind other objects. Identify the portion of L that should be removed.

12.6 Cylinders and Quadric Surfaces

We have already looked at two special types of surfaces: planes (in Section 12.5) and spheres (in Section 12.1). Here we investigate two other types of surfaces: cylinders and quadric surfaces.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

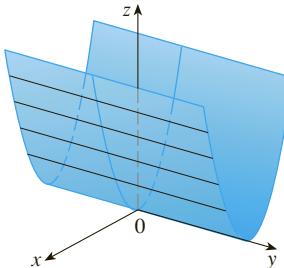
Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

EXAMPLE 1 Sketch the graph of the surface $z = x^2$.

SOLUTION Notice that the equation of the graph, $z = x^2$, doesn't involve y . This means that any vertical plane with equation $y = k$ (parallel to the xz -plane) intersects the graph in a curve with equation $z = x^2$. So these vertical traces are parabolas. Figure 1 shows how the graph is formed by taking the parabola $z = x^2$ in the xz -plane and moving it in the direction of the y -axis. The graph is a surface, called a **parabolic cylinder**, made up of infinitely many shifted copies of the same parabola. Here the rulings of the cylinder are parallel to the y -axis.

FIGURE 1
The surface $z = x^2$ is a parabolic cylinder.



We noticed that the variable y is missing from the equation of the cylinder in Example 1. This is typical of a surface whose rulings are parallel to one of the coordinate axes. If one of the variables x , y , or z is missing from the equation of a surface, then the surface is a cylinder.

EXAMPLE 2 Identify and sketch the surfaces.

(a) $x^2 + y^2 = 1$

(b) $y^2 + z^2 = 1$

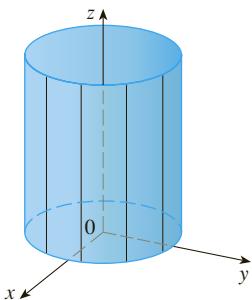


FIGURE 2
 $x^2 + y^2 = 1$

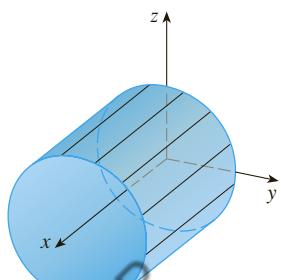


FIGURE 3
 $y^2 + z^2 = 1$

SOLUTION

(a) Since z is missing and the equations $x^2 + y^2 = 1, z = k$ represent a circle with radius 1 in the plane $z = k$, the surface $x^2 + y^2 = 1$ is a circular cylinder whose axis is the z -axis. (See Figure 2.) Here the rulings are vertical lines.

(b) In this case x is missing and the surface is a circular cylinder whose axis is the x -axis. (See Figure 3.) It is obtained by taking the circle $y^2 + z^2 = 1, x = 0$ in the yz -plane and moving it parallel to the x -axis. ■

NOTE When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the xy -plane is the circle with equations $x^2 + y^2 = 1, z = 0$.

■ Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables x, y , and z . The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where A, B, C, \dots, J are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane. (See Section 10.5 for a review of conic sections.)

EXAMPLE 3 Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

SOLUTION By substituting $z = 0$, we find that the trace in the xy -plane is $x^2 + y^2/9 = 1$, which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane $z = k$ is

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \quad z = k$$

which is an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$.

Similarly, vertical traces parallel to the yz - and xz -planes are also ellipses:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \quad x = k \quad (\text{if } -1 < k < 1)$$

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \quad y = k \quad (\text{if } -3 < k < 3)$$

Figure 4 shows how drawing some traces indicates the shape of the surface. It's called an **ellipsoid** because all of its traces are ellipses. Notice that it is symmetric with respect to each coordinate plane; this is a reflection of the fact that its equation involves only even powers of x, y , and z . ■

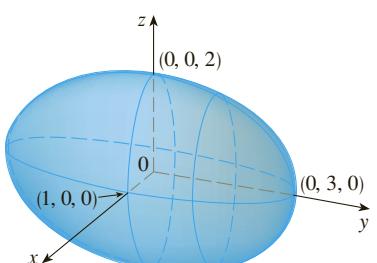
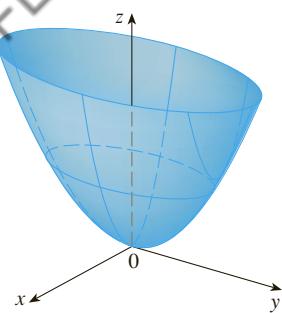


FIGURE 4

The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

EXAMPLE 4 Use traces to sketch the surface $z = 4x^2 + y^2$.

SOLUTION If we put $x = 0$, we get $z = y^2$, so the yz -plane intersects the surface in a parabola. If we put $x = k$ (a constant), we get $z = y^2 + 4k^2$. This means that if we

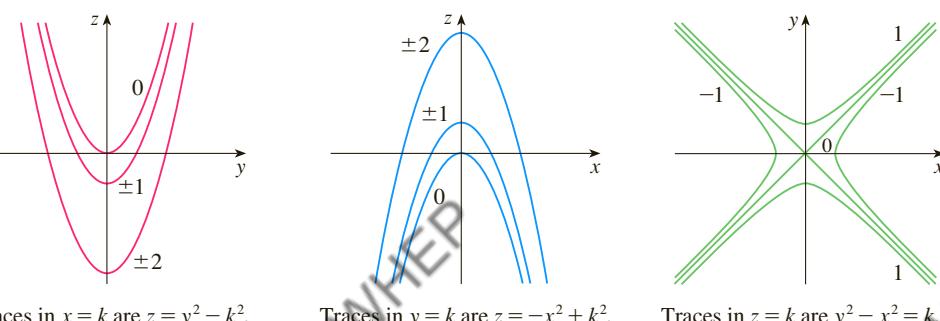
**FIGURE 5**

The surface $z = 4x^2 + y^2$ is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

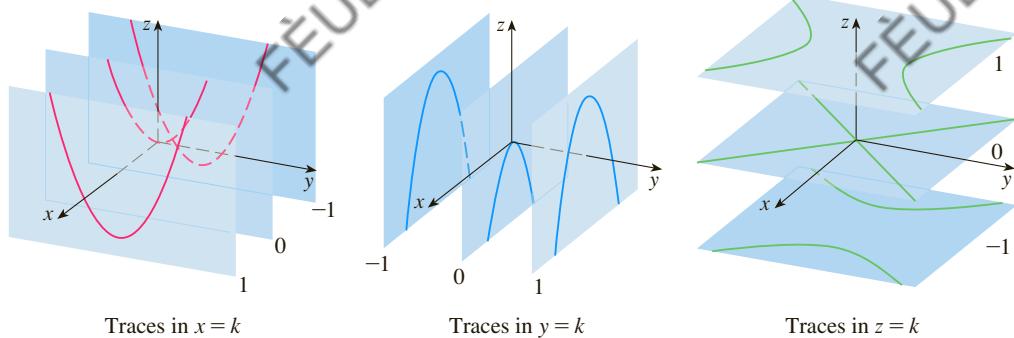
slice the graph with any plane parallel to the yz -plane, we obtain a parabola that opens upward. Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a parabola that opens upward. If we put $z = k$, we get the horizontal traces $4x^2 + y^2 = k$, which we recognize as a family of ellipses. Knowing the shapes of the traces, we can sketch the graph in Figure 5. Because of the elliptical and parabolic traces, the quadric surface $z = 4x^2 + y^2$ is called an **elliptic paraboloid**. ■

EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

SOLUTION The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.

**FIGURE 6**

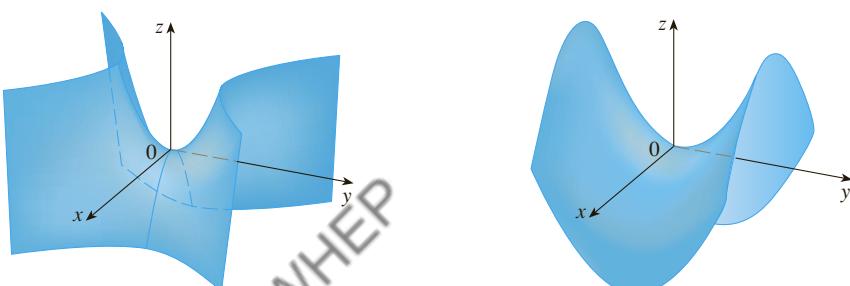
Vertical traces are parabolas;
horizontal traces are hyperbolas.
All traces are labeled with the
value of k .

**FIGURE 7**

Traces moved to their
correct planes

TEC In Module 12.6A you can investigate how traces determine the shape of a surface.

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**. Notice that the shape of the surface near the origin resembles that of a saddle. This surface will be investigated further in Section 14.7 when we discuss saddle points.

**FIGURE 8**

Two views of the surface $z = y^2 - x^2$,
a hyperbolic paraboloid

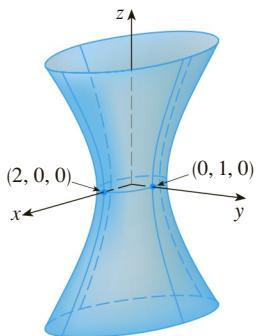


FIGURE 9

EXAMPLE 6 Sketch the surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$.

SOLUTION The trace in any horizontal plane $z = k$ is the ellipse

$$\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \quad z = k$$

but the traces in the xz - and yz -planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \quad \text{and} \quad y^2 - \frac{z^2}{4} = 1 \quad x = 0$$

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9. ■

The idea of using traces to draw a surface is employed in three-dimensional graphing software. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k , and parts of the graph are eliminated using hidden line removal. Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the z -axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

TEC In Module 12.6B you can see how changing a , b , and c in Table 1 affects the shape of the quadric surface.

EXAMPLE 7 Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

SOLUTION Dividing by -4 , we first put the equation in standard form:

$$-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$$

Comparing this equation with Table 1, we see that it represents a hyperboloid of two sheets, the only difference being that in this case the axis of the hyperboloid is the y -axis. The traces in the xy - and yz -planes are the hyperbolas

$$-x^2 + \frac{y^2}{4} = 1 \quad z = 0 \quad \text{and} \quad \frac{y^2}{4} - \frac{z^2}{2} = 1 \quad x = 0$$

The surface has no trace in the xz -plane, but traces in the vertical planes $y = k$ for $|k| > 2$ are the ellipses

$$x^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1 \quad y = k$$

which can be written as

$$\frac{x^2}{\frac{k^2}{4} - 1} + \frac{z^2}{2\left(\frac{k^2}{4} - 1\right)} = 1 \quad y = k$$

These traces are used to make the sketch in Figure 10.

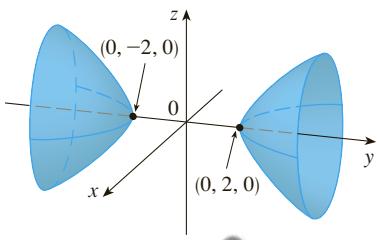


FIGURE 10
 $4x^2 - y^2 + 2z^2 + 4 = 0$

EXAMPLE 8 Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

SOLUTION By completing the square we rewrite the equation as

$$y - 1 = (x - 3)^2 + 2z^2$$

Comparing this equation with Table 1, we see that it represents an elliptic paraboloid. Here, however, the axis of the paraboloid is parallel to the y -axis, and it has been shifted so that its vertex is the point $(3, 1, 0)$. The traces in the plane $y = k$ ($k > 1$) are the ellipses

$$(x - 3)^2 + 2z^2 = k - 1 \quad y = k$$

The trace in the xy -plane is the parabola with equation $y = 1 + (x - 3)^2$, $z = 0$. The paraboloid is sketched in Figure 11.

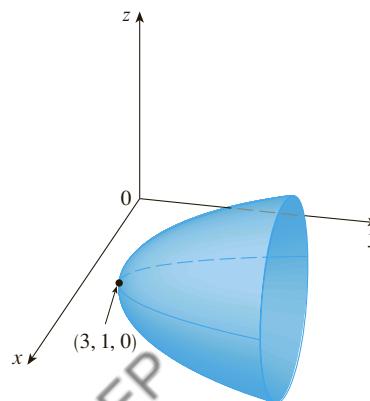


FIGURE 11
 $x^2 + 2z^2 - 6x - y + 10 = 0$

■ Applications of Quadric Surfaces

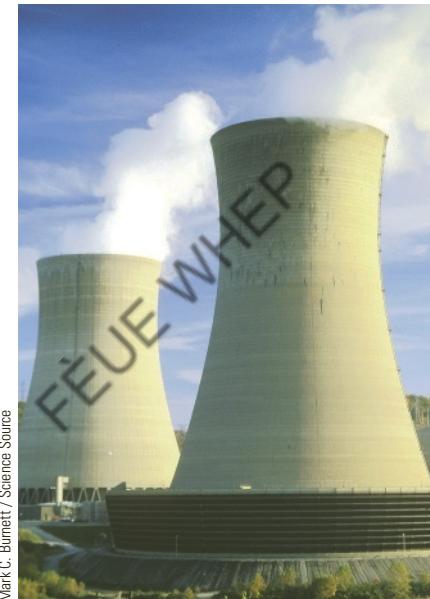
Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. (See Exercise 49.)

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals. In a radio telescope, for instance, signals from distant stars that strike the bowl are all reflected to the receiver at the focus and are therefore amplified. (The idea is explained in Problem 18 on page 202.) The same principle applies to microphones and satellite dishes in the shape of paraboloids.

Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes. (The cogs of the gears are the generating lines of the hyperboloids. See Exercise 51.)



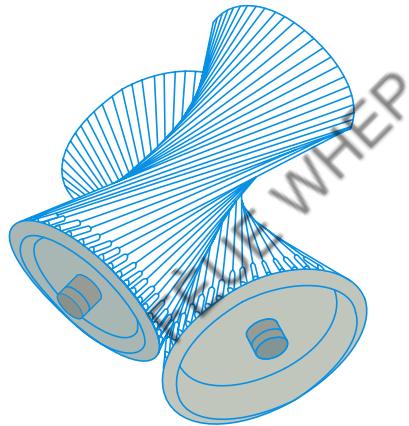
David Frazier / Spirit / Corbis



Mark C. Burnett / Science Source

A satellite dish reflects signals to the focus of a paraboloid.

Nuclear reactors have cooling towers in the shape of hyperboloids.



Hyperboloids produce gear transmission.

12.6 EXERCISES

- (a) What does the equation $y = x^2$ represent as a curve in \mathbb{R}^2 ?
 (b) What does it represent as a surface in \mathbb{R}^3 ?
 (c) What does the equation $z = y^2$ represent?
- (a) Sketch the graph of $y = e^x$ as a curve in \mathbb{R}^2 .
 (b) Sketch the graph of $y = e^x$ as a surface in \mathbb{R}^3 .
 (c) Describe and sketch the surface $z = e^y$.

3–8 Describe and sketch the surface.

3. $x^2 + z^2 = 1$

4. $4x^2 + y^2 = 4$

- $z = 1 - y^2$
- $xy = 1$
- $y = z^2$
- $z = \sin y$

- (a) Find and identify the traces of the quadric surface $x^2 + y^2 - z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of one sheet in Table 1.
 (b) If we change the equation in part (a) to $x^2 - y^2 + z^2 = 1$, how is the graph affected?
 (c) What if we change the equation in part (a) to $x^2 + y^2 + 2y - z^2 = 0$?

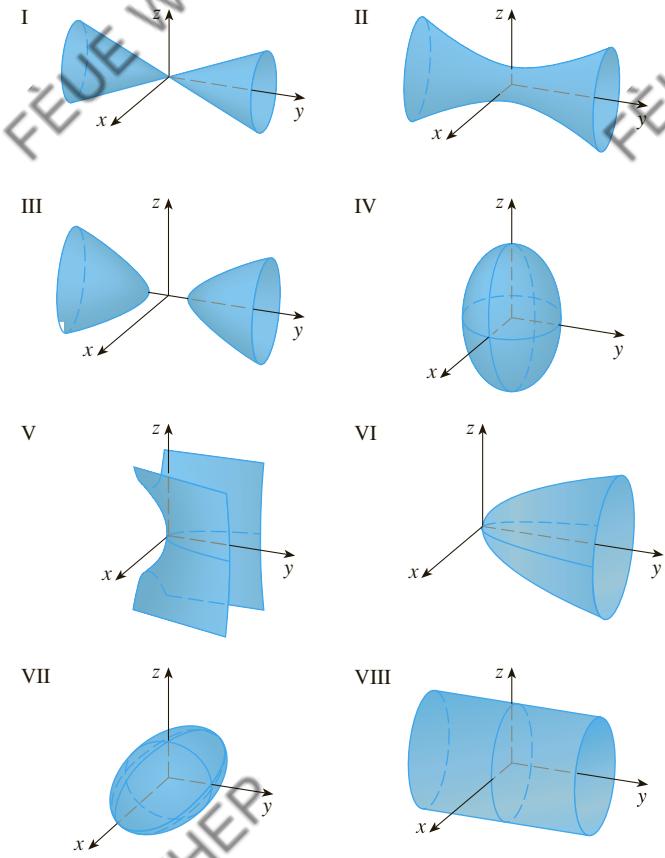
- 10.** (a) Find and identify the traces of the quadric surface $-x^2 - y^2 + z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of two sheets in Table 1.
 (b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, what happens to the graph? Sketch the new graph.

11–20 Use traces to sketch and identify the surface.

- 11.** $x = y^2 + 4z^2$ **12.** $4x^2 + 9y^2 + 9z^2 = 36$
13. $x^2 = 4y^2 + z^2$ **14.** $z^2 - 4x^2 - y^2 = 4$
15. $9y^2 + 4z^2 = x^2 + 36$ **16.** $3x^2 + y + 3z^2 = 0$
17. $\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} = 1$ **18.** $3x^2 - y^2 + 3z^2 = 0$
19. $y = z^2 - x^2$ **20.** $x = y^2 - z^2$

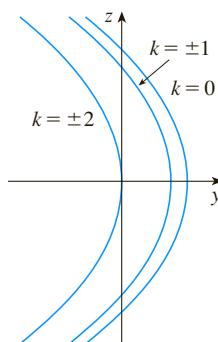
21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

- 21.** $x^2 + 4y^2 + 9z^2 = 1$ **22.** $9x^2 + 4y^2 + z^2 = 1$
23. $x^2 - y^2 + z^2 = 1$ **24.** $-x^2 + y^2 - z^2 = 1$
25. $y = 2x^2 + z^2$ **26.** $y^2 = x^2 + 2z^2$
27. $x^2 + 2z^2 = 1$ **28.** $y = x^2 - z^2$

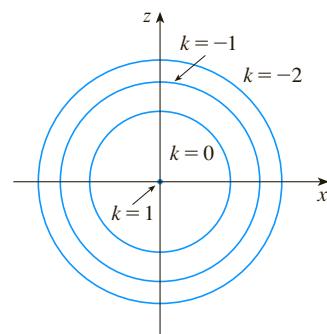


29–30 Sketch and identify a quadric surface that could have the traces shown.

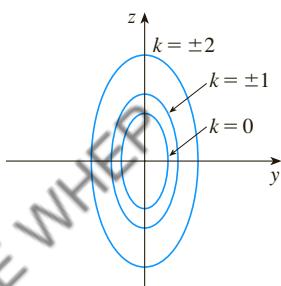
29. Traces in $x = k$



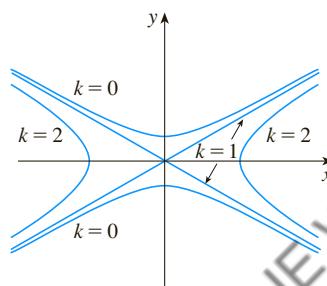
Traces in $y = k$



30. Traces in $x = k$



Traces in $z = k$



31–38 Reduce the equation to one of the standard forms, classify the surface, and sketch it.

- 31.** $y^2 = x^2 + \frac{1}{9}z^2$ **32.** $4x^2 - y + 2z^2 = 0$
33. $x^2 + 2y - 2z^2 = 0$ **34.** $y^2 = x^2 + 4z^2 + 4$
35. $x^2 + y^2 - 2x - 6y - z + 10 = 0$ **36.** $x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$
37. $x^2 - y^2 + z^2 - 4x - 2z = 0$ **38.** $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$

39–42 Use a computer with three-dimensional graphing software to graph the surface. Experiment with viewpoints and with domains for the variables until you get a good view of the surface.

- 39.** $-4x^2 - y^2 + z^2 = 1$ **40.** $x^2 - y^2 - z = 0$
41. $-4x^2 - y^2 + z^2 = 0$ **42.** $x^2 - 6x + 4y^2 - z = 0$

43. Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \leq z \leq 2$.

44. Sketch the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

- 45.** Find an equation for the surface obtained by rotating the curve $y = \sqrt{x}$ about the x -axis.
- 46.** Find an equation for the surface obtained by rotating the line $z = 2y$ about the z -axis.
- 47.** Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.
- 48.** Find an equation for the surface consisting of all points P for which the distance from P to the x -axis is twice the distance from P to the yz -plane. Identify the surface.
- 49.** Traditionally, the earth's surface has been modeled as a sphere, but the World Geodetic System of 1984 (WGS-84) uses an ellipsoid as a more accurate model. It places the center of the earth at the origin and the north pole on the positive z -axis. The distance from the center to the poles is 6356.523 km and the distance to a point on the equator is 6378.137 km.
- Find an equation of the earth's surface as used by WGS-84.
 - Curves of equal latitude are traces in the planes $z = k$. What is the shape of these curves?
 - Meridians (curves of equal longitude) are traces in planes of the form $y = mx$. What is the shape of these meridians?
- 50.** A cooling tower for a nuclear reactor is to be constructed in the shape of a hyperboloid of one sheet (see the photo on page 879). The diameter at the base is 280 m and the minimum diameter, 500 m above the base, is 200 m. Find an equation for the tower.
- 51.** Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines with parametric equations $x = a + t, y = b + t, z = c + 2(b - a)t$ and $x = a + t, y = b - t, z = c - 2(b + a)t$ both lie entirely on this paraboloid. (This shows that the hyperbolic paraboloid is what is called a **ruled surface**; that is, it can be generated by the motion of a straight line. In fact, this exercise shows that through each point on the hyperbolic paraboloid there are two generating lines. The only other quadric surfaces that are ruled surfaces are cylinders, cones, and hyperboloids of one sheet.)
- 52.** Show that the curve of intersection of the surfaces $x^2 + 2y^2 - z^2 + 3x = 1$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$ lies in a plane.
-  **53.** Graph the surfaces $z = x^2 + y^2$ and $z = 1 - y^2$ on a common screen using the domain $|x| \leq 1.2, |y| \leq 1.2$ and observe the curve of intersection of these surfaces. Show that the projection of this curve onto the xy -plane is an ellipse.

12 REVIEW

CONCEPT CHECK

- What is the difference between a vector and a scalar?
- How do you add two vectors geometrically? How do you add them algebraically?
- If \mathbf{a} is a vector and c is a scalar, how is $c\mathbf{a}$ related to \mathbf{a} geometrically? How do you find $c\mathbf{a}$ algebraically?
- How do you find the vector from one point to another?
- How do you find the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?
- How are dot products useful?
- Write expressions for the scalar and vector projections of \mathbf{b} onto \mathbf{a} . Illustrate with diagrams.
- How do you find the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?
- How are cross products useful?
- (a) How do you find the area of the parallelogram determined by \mathbf{a} and \mathbf{b} ?
(b) How do you find the volume of the parallelepiped determined by \mathbf{a}, \mathbf{b} , and \mathbf{c} ?

Answers to the Concept Check can be found on the back endpapers.

- How do you find a vector perpendicular to a plane?
- How do you find the angle between two intersecting planes?
- Write a vector equation, parametric equations, and symmetric equations for a line.
- Write a vector equation and a scalar equation for a plane.
- (a) How do you tell if two vectors are parallel?
(b) How do you tell if two vectors are perpendicular?
(c) How do you tell if two planes are parallel?
- (a) Describe a method for determining whether three points P, Q , and R lie on the same line.
(b) Describe a method for determining whether four points P, Q, R , and S lie in the same plane.
- (a) How do you find the distance from a point to a line?
(b) How do you find the distance from a point to a plane?
(c) How do you find the distance between two lines?
- What are the traces of a surface? How do you find them?
- Write equations in standard form of the six types of quadric surfaces.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then $\mathbf{u} \cdot \mathbf{v} = \langle u_1 v_1, u_2 v_2 \rangle$.

2. For any vectors \mathbf{u} and \mathbf{v} in V_3 , $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

3. For any vectors \mathbf{u} and \mathbf{v} in V_3 , $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}| |\mathbf{v}|$.

4. For any vectors \mathbf{u} and \mathbf{v} in V_3 , $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}|$.

5. For any vectors \mathbf{u} and \mathbf{v} in V_3 , $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

6. For any vectors \mathbf{u} and \mathbf{v} in V_3 , $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

7. For any vectors \mathbf{u} and \mathbf{v} in V_3 , $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.

8. For any vectors \mathbf{u} and \mathbf{v} in V_3 and any scalar k ,

$$k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$$

9. For any vectors \mathbf{u} and \mathbf{v} in V_3 and any scalar k ,

$$k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v}$$

10. For any vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V_3 ,

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$$

11. For any vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V_3 ,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

12. For any vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V_3 ,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

13. For any vectors \mathbf{u} and \mathbf{v} in V_3 , $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.

14. For any vectors \mathbf{u} and \mathbf{v} in V_3 , $(\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$.

15. The vector $\langle 3, -1, 2 \rangle$ is parallel to the plane

$$6x - 2y + 4z = 1$$

16. A linear equation $Ax + By + Cz + D = 0$ represents a line in space.

17. The set of points $\{(x, y, z) \mid x^2 + y^2 = 1\}$ is a circle.

18. In \mathbb{R}^3 the graph of $y = x^2$ is a paraboloid.

19. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

20. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

21. If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

22. If \mathbf{u} and \mathbf{v} are in V_3 , then $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| |\mathbf{v}|$.

EXERCISES

1. (a) Find an equation of the sphere that passes through the point $(6, -2, 3)$ and has center $(-1, 2, 1)$.

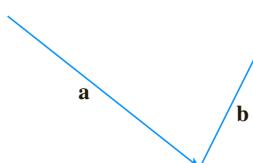
- (b) Find the curve in which this sphere intersects the yz -plane.

- (c) Find the center and radius of the sphere

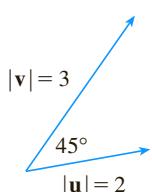
$$x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0$$

2. Copy the vectors in the figure and use them to draw each of the following vectors.

- (a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$ (c) $-\frac{1}{2}\mathbf{a}$ (d) $2\mathbf{a} + \mathbf{b}$



3. If \mathbf{u} and \mathbf{v} are the vectors shown in the figure, find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$. Is $\mathbf{u} \times \mathbf{v}$ directed into the page or out of it?



4. Calculate the given quantity if

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

$$\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{c} = \mathbf{j} - 5\mathbf{k}$$

- | | |
|---|--|
| (a) $2\mathbf{a} + 3\mathbf{b}$ | (b) $ \mathbf{b} $ |
| (c) $\mathbf{a} \cdot \mathbf{b}$ | (d) $\mathbf{a} \times \mathbf{b}$ |
| (e) $ \mathbf{b} \times \mathbf{c} $ | (f) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ |
| (g) $\mathbf{c} \times \mathbf{c}$ | (h) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ |
| (i) $\text{comp}_{\mathbf{a}} \mathbf{b}$ | (j) $\text{proj}_{\mathbf{a}} \mathbf{b}$ |
| (k) The angle between \mathbf{a} and \mathbf{b} (correct to the nearest degree) | |

5. Find the values of x such that the vectors $\langle 3, 2, x \rangle$ and $\langle 2x, 4, x \rangle$ are orthogonal.

6. Find two unit vectors that are orthogonal to both $\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

7. Suppose that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2$. Find

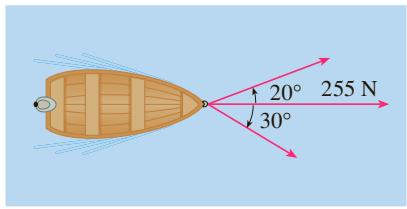
- | | |
|---|---|
| (a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ | (b) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$ |
| (c) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$ | (d) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$ |

8. Show that if \mathbf{a} , \mathbf{b} , and \mathbf{c} are in V_3 , then

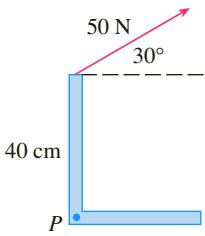
$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$$

9. Find the acute angle between two diagonals of a cube.

10. Given the points $A(1, 0, 1)$, $B(2, 3, 0)$, $C(-1, 1, 4)$, and $D(0, 3, 2)$, find the volume of the parallelepiped with adjacent edges AB , AC , and AD .
11. (a) Find a vector perpendicular to the plane through the points $A(1, 0, 0)$, $B(2, 0, -1)$, and $C(1, 4, 3)$.
 (b) Find the area of triangle ABC .
12. A constant force $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$ moves an object along the line segment from $(1, 0, 2)$ to $(5, 3, 8)$. Find the work done if the distance is measured in meters and the force in newtons.
13. A boat is pulled onto shore using two ropes, as shown in the diagram. If a force of 255 N is needed, find the magnitude of the force in each rope.



14. Find the magnitude of the torque about P if a 50-N force is applied as shown.



15–17 Find parametric equations for the line.

15. The line through $(4, -1, 2)$ and $(1, 1, 5)$
16. The line through $(1, 0, -1)$ and parallel to the line $\frac{1}{3}(x - 4) = \frac{1}{2}y = z + 2$
17. The line through $(-2, 2, 4)$ and perpendicular to the plane $2x - y + 5z = 12$

18–20 Find an equation of the plane.

18. The plane through $(2, 1, 0)$ and parallel to $x + 4y - 3z = 1$
19. The plane through $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$
20. The plane through $(1, 2, -2)$ that contains the line $x = 2t$, $y = 3 - t$, $z = 1 + 3t$

21. Find the point in which the line with parametric equations $x = 2 - t$, $y = 1 + 3t$, $z = 4t$ intersects the plane $2x - y + z = 2$.

22. Find the distance from the origin to the line $x = 1 + t$, $y = 2 - t$, $z = -1 + 2t$.

23. Determine whether the lines given by the symmetric equations

$$\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z - 3}{4}$$

$$\text{and } \frac{x + 1}{6} = \frac{y - 3}{-1} = \frac{z + 5}{2}$$

are parallel, skew, or intersecting.

24. (a) Show that the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$ are neither parallel nor perpendicular.
 (b) Find, correct to the nearest degree, the angle between these planes.

25. Find an equation of the plane through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and perpendicular to the plane $x + y - 2z = 1$.

26. (a) Find an equation of the plane that passes through the points $A(2, 1, 1)$, $B(-1, -1, 10)$, and $C(1, 3, -4)$.
 (b) Find symmetric equations for the line through B that is perpendicular to the plane in part (a).
 (c) A second plane passes through $(2, 0, 4)$ and has normal vector $\langle 2, -4, -3 \rangle$. Show that the acute angle between the planes is approximately 43° .
 (d) Find parametric equations for the line of intersection of the two planes.

27. Find the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.

28–36 Identify and sketch the graph of each surface.

28. $x = 3$

29. $x = z$

30. $y = z^2$

31. $x^2 = y^2 + 4z^2$

32. $4x - y + 2z = 4$

33. $-4x^2 + y^2 - 4z^2 = 4$

34. $y^2 + z^2 = 1 + x^2$

35. $4x^2 + 4y^2 - 8y + z^2 = 0$

36. $x = y^2 + z^2 - 2y - 4z + 5$

37. An ellipsoid is created by rotating the ellipse $4x^2 + y^2 = 16$ about the x -axis. Find an equation of the ellipsoid.

38. A surface consists of all points P such that the distance from P to the plane $y = 1$ is twice the distance from P to the point $(0, -1, 0)$. Find an equation for this surface and identify it.

Problems Plus

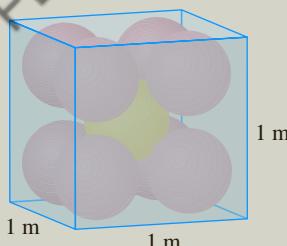


FIGURE FOR PROBLEM 1

- Each edge of a cubical box has length 1 m. The box contains nine spherical balls with the same radius r . The center of one ball is at the center of the cube and it touches the other eight balls. Each of the other eight balls touches three sides of the box. Thus the balls are tightly packed in the box (see the figure). Find r . (If you have trouble with this problem, read about the problem-solving strategy entitled *Use Analogy* on page 98.)
- Let B be a solid box with length L , width W , and height H . Let S be the set of all points that are a distance at most 1 from some point of B . Express the volume of S in terms of L , W , and H .
- Let L be the line of intersection of the planes $cx + y + z = c$ and $x - cy + cz = -1$, where c is a real number.
 - Find symmetric equations for L .
 - As the number c varies, the line L sweeps out a surface S . Find an equation for the curve of intersection of S with the horizontal plane $z = t$ (the trace of S in the plane $z = t$).
 - Find the volume of the solid bounded by S and the planes $z = 0$ and $z = 1$.
- A plane is capable of flying at a speed of 180 km/h in still air. The pilot takes off from an airfield and heads due north according to the plane's compass. After 30 minutes of flight time, the pilot notices that, due to the wind, the plane has actually traveled 80 km at an angle 5° east of north.
 - What is the wind velocity?
 - In what direction should the pilot have headed to reach the intended destination?
- Suppose \mathbf{v}_1 and \mathbf{v}_2 are vectors with $|\mathbf{v}_1| = 2$, $|\mathbf{v}_2| = 3$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 5$. Let $\mathbf{v}_3 = \text{proj}_{\mathbf{v}_1}\mathbf{v}_2$, $\mathbf{v}_4 = \text{proj}_{\mathbf{v}_2}\mathbf{v}_3$, $\mathbf{v}_5 = \text{proj}_{\mathbf{v}_3}\mathbf{v}_4$, and so on. Compute $\sum_{n=1}^{\infty} |\mathbf{v}_n|$.
- Find an equation of the largest sphere that passes through the point $(-1, 1, 4)$ and is such that each of the points (x, y, z) inside the sphere satisfies the condition

$$x^2 + y^2 + z^2 < 136 + 2(x + 2y + 3z)$$

- Suppose a block of mass m is placed on an inclined plane, as shown in the figure. The block's descent down the plane is slowed by friction; if θ is not too large, friction will prevent the block from moving at all. The forces acting on the block are the weight \mathbf{W} , where $|\mathbf{W}| = mg$ (g is the acceleration due to gravity); the normal force \mathbf{N} (the normal component of the reactionary force of the plane on the block), where $|\mathbf{N}| = n$; and the force \mathbf{F} due to friction, which acts parallel to the inclined plane, opposing the direction of motion. If the block is at rest and θ is increased, $|\mathbf{F}|$ must also increase until ultimately $|\mathbf{F}|$ reaches its maximum, beyond which the block begins to slide. At this angle θ_s , it has been observed that $|\mathbf{F}|$ is proportional to n . Thus, when $|\mathbf{F}|$ is maximal, we can say that $|\mathbf{F}| = \mu_s n$, where μ_s is called the *coefficient of static friction* and depends on the materials that are in contact.
 - Observe that $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$ and deduce that $\mu_s = \tan(\theta_s)$.
 - Suppose that, for $\theta > \theta_s$, an additional outside force \mathbf{H} is applied to the block, horizontally from the left, and let $|\mathbf{H}| = h$. If h is small, the block may still slide down the plane; if h is large enough, the block will move up the plane. Let h_{\min} be the smallest value of h that allows the block to remain motionless (so that $|\mathbf{F}|$ is maximal).

By choosing the coordinate axes so that \mathbf{F} lies along the x -axis, resolve each force into components parallel and perpendicular to the inclined plane and show that

$$h_{\min} \sin \theta + mg \cos \theta = n \quad \text{and} \quad h_{\min} \cos \theta + \mu_s n = mg \sin \theta$$

- Show that $h_{\min} = mg \tan(\theta - \theta_s)$

Does this equation seem reasonable? Does it make sense for $\theta = \theta_s$? Does it make sense as $\theta \rightarrow 90^\circ$? Explain.

- (d) Let h_{\max} be the largest value of h that allows the block to remain motionless. (In which direction is \mathbf{F} heading?) Show that

$$h_{\max} = mg \tan(\theta + \theta_s)$$

Does this equation seem reasonable? Explain.

- 8.** A solid has the following properties. When illuminated by rays parallel to the z -axis, its shadow is a circular disk. If the rays are parallel to the y -axis, its shadow is a square. If the rays are parallel to the x -axis, its shadow is an isosceles triangle. (In Exercise 12.1.48 you were asked to describe and sketch an example of such a solid, but there are many such solids.) Assume that the projection onto the xz -plane is a square whose sides have length 1.
- (a) What is the volume of the largest such solid?
(b) Is there a smallest volume?

13

Vector Functions

The paths of objects moving through space like the jet planes pictured here can be described by vector functions. In Section 13.1 we will see how to use these vector functions to determine whether or not two such objects will collide.



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THE FUNCTIONS THAT WE HAVE been using so far have been real-valued functions. We now study functions whose values are vectors because such functions are needed to describe curves and surfaces in space. We will also use vector-valued functions to describe the motion of objects through space. In particular, we will use them to derive Kepler's laws of planetary motion.

13.1 Vector Functions and Space Curves

In general, a function is a rule that assigns to each element in the domain an element in the range. A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions \mathbf{r} whose values are three-dimensional vectors. This means that for every number t in the domain of \mathbf{r} there is a unique vector in V_3 denoted by $\mathbf{r}(t)$. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f , g , and h are real-valued functions called the **component functions** of \mathbf{r} and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

EXAMPLE 1 If

$$\mathbf{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t} \rangle$$

then the component functions are

$$f(t) = t^3 \quad g(t) = \ln(3 - t) \quad h(t) = \sqrt{t}$$

By our usual convention, the domain of \mathbf{r} consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined. The expressions t^3 , $\ln(3 - t)$, and \sqrt{t} are all defined when $3 - t > 0$ and $t \geq 0$. Therefore the domain of \mathbf{r} is the interval $[0, 3]$.

Limits and Continuity

The **limit** of a vector function \mathbf{r} is defined by taking the limits of its component functions as follows.

If $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L}$, this definition is equivalent to saying that the length and direction of the vector $\mathbf{r}(t)$ approach the length and direction of the vector \mathbf{L} .

1 If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Equivalently, we could have used an ϵ - δ definition (see Exercise 54). Limits of vector functions obey the same rules as limits of real-valued functions (see Exercise 53).

EXAMPLE 2 Find $\lim_{t \rightarrow 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$.

SOLUTION According to Definition 1, the limit of \mathbf{r} is the vector whose components are the limits of the component functions of \mathbf{r} :

$$\begin{aligned} \lim_{t \rightarrow 0} \mathbf{r}(t) &= \left[\lim_{t \rightarrow 0} (1 + t^3) \right] \mathbf{i} + \left[\lim_{t \rightarrow 0} te^{-t} \right] \mathbf{j} + \left[\lim_{t \rightarrow 0} \frac{\sin t}{t} \right] \mathbf{k} \\ &= \mathbf{i} + \mathbf{k} \quad (\text{by Equation 2.4.2}) \end{aligned}$$

A vector function \mathbf{r} is **continuous at a** if

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of Definition 1, we see that \mathbf{r} is continuous at a if and only if its component functions f , g , and h are continuous at a .

Space Curves

There is a close connection between continuous vector functions and space curves. Suppose that f , g , and h are continuous real-valued functions on an interval I . Then the set C of all points (x, y, z) in space, where

2

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

and t varies throughout the interval I , is called a **space curve**. The equations in (2) are called **parametric equations of C** and t is called a **parameter**. We can think of C as being traced out by a moving particle whose position at time t is $(f(t), g(t), h(t))$. If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C . Thus any continuous vector function \mathbf{r} defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 1.

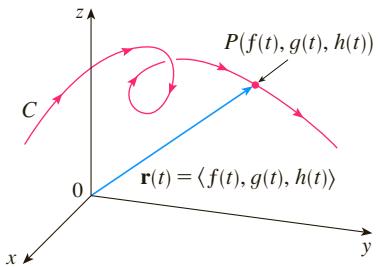


FIGURE 1

C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

TEC Visual 13.1A shows several curves being traced out by position vectors, including those in Figures 1 and 2.

EXAMPLE 3 Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$$

SOLUTION The corresponding parametric equations are

$$x = 1 + t \quad y = 2 + 5t \quad z = -1 + 6t$$

which we recognize from Equations 12.5.2 as parametric equations of a line passing through the point $(1, 2, -1)$ and parallel to the vector $\langle 1, 5, 6 \rangle$. Alternatively, we could observe that the function can be written as $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle 1, 5, 6 \rangle$, and this is the vector equation of a line as given by Equation 12.5.1. ■

Plane curves can also be represented in vector notation. For instance, the curve given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$ (see Example 10.1.1) could also be described by the vector equation

$$\mathbf{r}(t) = \langle t^2 - 2t, t + 1 \rangle = (t^2 - 2t)\mathbf{i} + (t + 1)\mathbf{j}$$

where $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

EXAMPLE 4 Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$$

SOLUTION The parametric equations for this curve are

$$x = \cos t \quad y = \sin t \quad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ for all values of t , the curve must lie on the circular cylinder $x^2 + y^2 = 1$. The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy -plane. (The projection of the curve onto the xy -plane has vector equation $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. See Example 10.1.2.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 2, is called a **helix**. ■

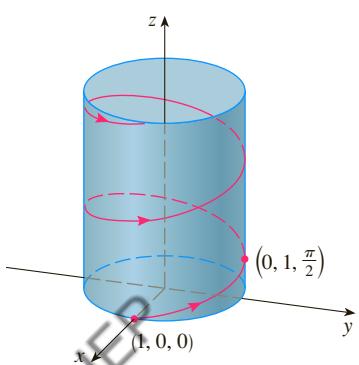


FIGURE 2

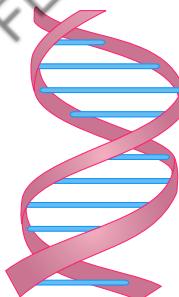


FIGURE 3
A double helix

Figure 4 shows the line segment PQ in Example 5.

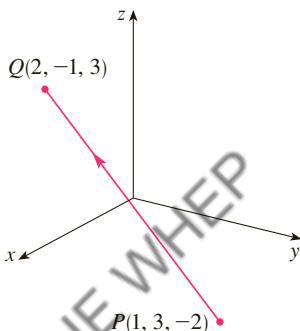


FIGURE 4

The corkscrew shape of the helix in Example 4 is familiar from its occurrence in coiled springs. It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells). In 1953 James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helices that are intertwined as in Figure 3.

In Examples 3 and 4 we were given vector equations of curves and asked for a geometric description or sketch. In the next two examples we are given a geometric description of a curve and are asked to find parametric equations for the curve.

EXAMPLE 5 Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

SOLUTION In Section 12.5 we found a vector equation for the line segment that joins the tip of the vector \mathbf{r}_0 to the tip of the vector \mathbf{r}_1 :

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

(See Equation 12.5.4.) Here we take $\mathbf{r}_0 = \langle 1, 3, -2 \rangle$ and $\mathbf{r}_1 = \langle 2, -1, 3 \rangle$ to obtain a vector equation of the line segment from P to Q :

$$\mathbf{r}(t) = (1 - t)\langle 1, 3, -2 \rangle + t\langle 2, -1, 3 \rangle \quad 0 \leq t \leq 1$$

$$\text{or} \quad \mathbf{r}(t) = \langle 1 + t, 3 - 4t, -2 + 5t \rangle \quad 0 \leq t \leq 1$$

The corresponding parametric equations are

$$x = 1 + t \quad y = 3 - 4t \quad z = -2 + 5t \quad 0 \leq t \leq 1$$

EXAMPLE 6 Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

SOLUTION Figure 5 shows how the plane and the cylinder intersect, and Figure 6 shows the curve of intersection C , which is an ellipse.

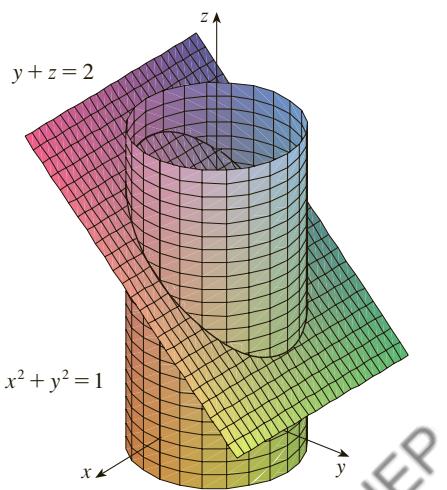


FIGURE 5

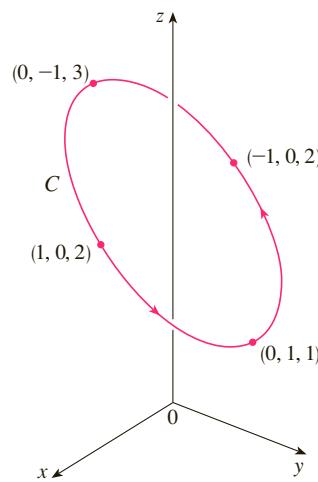


FIGURE 6

The projection of C onto the xy -plane is the circle $x^2 + y^2 = 1, z = 0$. So we know from Example 10.1.2 that we can write

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

From the equation of the plane, we have

$$z = 2 - y = 2 - \sin t$$

So we can write parametric equations for C as

$$x = \cos t \quad y = \sin t \quad z = 2 - \sin t \quad 0 \leq t \leq 2\pi$$

The corresponding vector equation is

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + (2 - \sin t) \mathbf{k} \quad 0 \leq t \leq 2\pi$$

This equation is called a *parametrization* of the curve C . The arrows in Figure 6 indicate the direction in which C is traced as the parameter t increases. ■

■ Using Computers to Draw Space Curves

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation we need to use technology. For instance, Figure 7 shows a computer-generated graph of the curve with parametric equations

$$x = (4 + \sin 20t) \cos t \quad y = (4 + \sin 20t) \sin t \quad z = \cos 20t$$

It's called a **toroidal spiral** because it lies on a torus. Another interesting curve, the **trefoil knot**, with equations

$$x = (2 + \cos 1.5t) \cos t \quad y = (2 + \cos 1.5t) \sin t \quad z = \sin 1.5t$$

is graphed in Figure 8. It wouldn't be easy to plot either of these curves by hand.

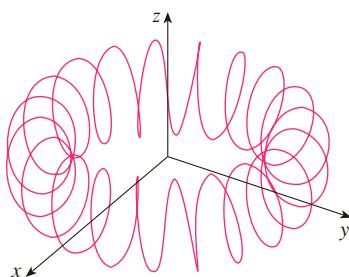


FIGURE 7
A toroidal spiral

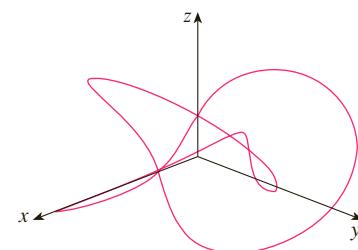


FIGURE 8
A trefoil knot

Even when a computer is used to draw a space curve, optical illusions make it difficult to get a good impression of what the curve really looks like. (This is especially true in Figure 8. See Exercise 52.) The next example shows how to cope with this problem.

EXAMPLE 7 Use a computer to draw the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. This curve is called a **twisted cubic**.

SOLUTION We start by using the computer to plot the curve with parametric equations $x = t, y = t^2, z = t^3$ for $-2 \leq t \leq 2$. The result is shown in Figure 9(a), but it's hard to see the true nature of the curve from that graph alone. Most three-dimensional

computer graphing programs allow the user to enclose a curve or surface in a box instead of displaying the coordinate axes. When we look at the same curve in a box in Figure 9(b), we have a much clearer picture of the curve. We can see that it climbs from a lower corner of the box to the upper corner nearest us, and it twists as it climbs.

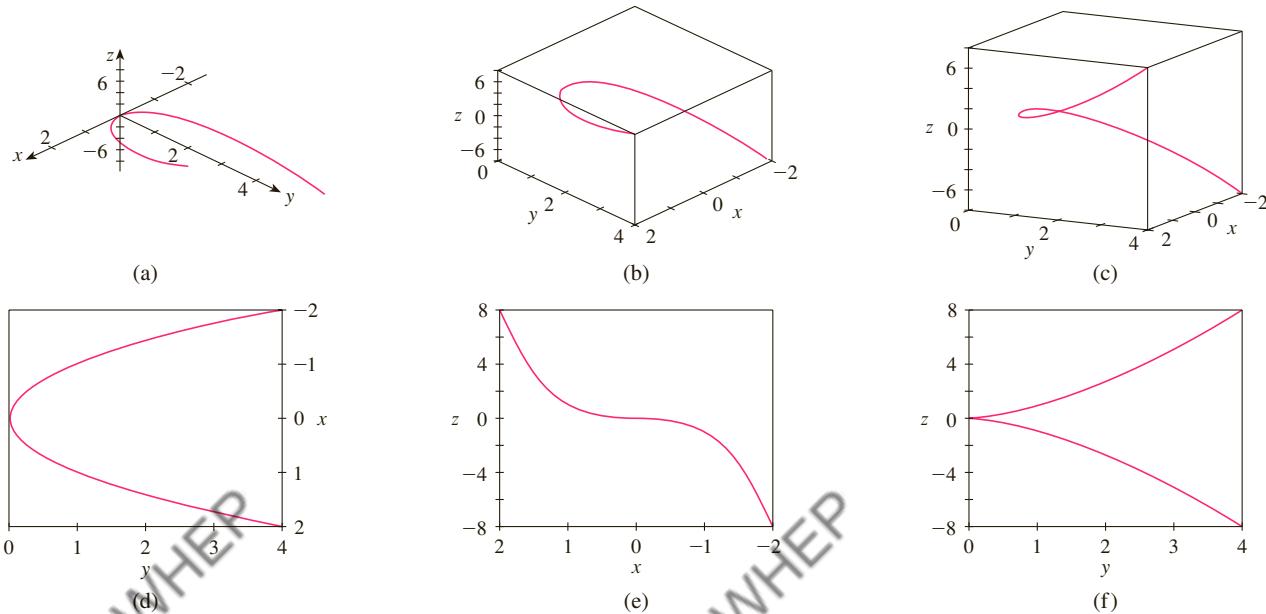


FIGURE 9 Views of the twisted cubic

TEC In Visual 13.1B you can rotate the box in Figure 9 to see the curve from any viewpoint.

We get an even better idea of the curve when we view it from different vantage points. Part (c) shows the result of rotating the box to give another viewpoint. Parts (d), (e), and (f) show the views we get when we look directly at a face of the box. In particular, part (d) shows the view from directly above the box. It is the projection of the curve onto the xy -plane, namely, the parabola $y = x^2$. Part (e) shows the projection onto the xz -plane, the cubic curve $z = x^3$. It's now obvious why the given curve is called a twisted cubic. ■

Another method of visualizing a space curve is to draw it on a surface. For instance, the twisted cubic in Example 7 lies on the parabolic cylinder $y = x^2$. (Eliminate the parameter from the first two parametric equations, $x = t$ and $y = t^2$.) Figure 10 shows both the cylinder and the twisted cubic, and we see that the curve moves upward from the origin along the surface of the cylinder. We also used this method in Example 4 to visualize the helix lying on the circular cylinder (see Figure 2).

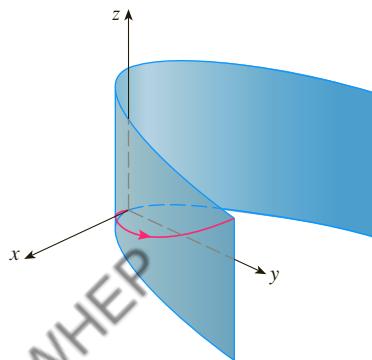


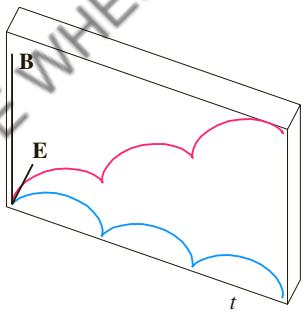
FIGURE 10

A third method for visualizing the twisted cubic is to realize that it also lies on the cylinder $z = x^3$. So it can be viewed as the curve of intersection of the cylinders $y = x^2$ and $z = x^3$. (See Figure 11.)

TEC Visual 13.1C shows how curves arise as intersections of surfaces.

FIGURE 11

Some computer algebra systems provide us with a clearer picture of a space curve by enclosing it in a tube. Such a plot enables us to see whether one part of a curve passes in front of or behind another part of the curve. For example, Figure 13 shows the curve of Figure 12(b) as rendered by the `tubeplot` command in Maple.

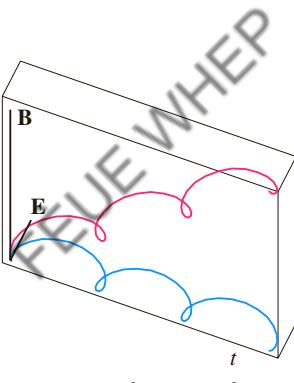


$$(a) \mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, t \rangle$$

FIGURE 12

Motion of a charged particle in orthogonally oriented electric and magnetic fields

We have seen that an interesting space curve, the helix, occurs in the model of DNA. Another notable example of a space curve in science is the trajectory of a positively charged particle in orthogonally oriented electric and magnetic fields \mathbf{E} and \mathbf{B} . Depending on the initial velocity given the particle at the origin, the path of the particle is either a space curve whose projection onto the horizontal plane is the cycloid we studied in Section 10.1 [Figure 12(a)] or a curve whose projection is the trochoid investigated in Exercise 10.1.40 [Figure 12(b)].



$$(b) \mathbf{r}(t) = \left\langle t - \frac{3}{2} \sin t, 1 - \frac{3}{2} \cos t, t \right\rangle$$

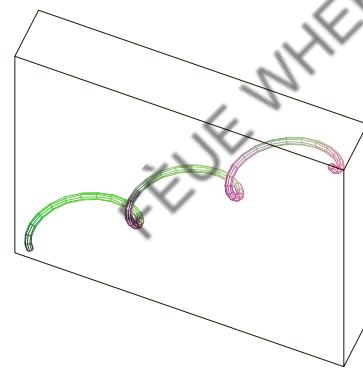


FIGURE 13

For further details concerning the physics involved and animations of the trajectories of the particles, see the following websites:

- www.physics.ucla.edu/plasma-exp/Beam/
- www.phy.ntnu.edu.tw/ntnужava/index.php?topic=36

13.1 EXERCISES

1–2 Find the domain of the vector function.

$$1. \mathbf{r}(t) = \left\langle \ln(t+1), \frac{t}{\sqrt{9-t^2}}, 2^t \right\rangle$$

$$2. \mathbf{r}(t) = \cos t \mathbf{i} + \ln t \mathbf{j} + \frac{1}{t-2} \mathbf{k}$$

3–6 Find the limit.

$$3. \lim_{t \rightarrow 0} \left(e^{-3t} \mathbf{i} + \frac{t^2}{\sin^2 t} \mathbf{j} + \cos 2t \mathbf{k} \right)$$

$$4. \lim_{t \rightarrow 1} \left(\frac{t^2 - t}{t - 1} \mathbf{i} + \sqrt{t + 8} \mathbf{j} + \frac{\sin \pi t}{\ln t} \mathbf{k} \right)$$

5. $\lim_{t \rightarrow \infty} \left\langle \frac{1+t^2}{1-t^2}, \tan^{-1} t, \frac{1-e^{-2t}}{t} \right\rangle$

6. $\lim_{t \rightarrow \infty} \left\langle te^{-t}, \frac{t^3+t}{2t^3-1}, t \sin \frac{1}{t} \right\rangle$

7–14 Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

7. $\mathbf{r}(t) = \langle \sin t, t \rangle$

8. $\mathbf{r}(t) = \langle t^2 - 1, t \rangle$

9. $\mathbf{r}(t) = \langle t, 2-t, 2t \rangle$

10. $\mathbf{r}(t) = \langle \sin \pi t, t, \cos \pi t \rangle$

11. $\mathbf{r}(t) = \langle 3, t, 2-t^2 \rangle$

12. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \mathbf{k}$

13. $\mathbf{r}(t) = t^2 \mathbf{i} + t^4 \mathbf{j} + t^6 \mathbf{k}$

14. $\mathbf{r}(t) = \cos t \mathbf{i} - \cos t \mathbf{j} + \sin t \mathbf{k}$

15–16 Draw the projections of the curve on the three coordinate planes. Use these projections to help sketch the curve.

15. $\mathbf{r}(t) = \langle t, \sin t, 2 \cos t \rangle$

16. $\mathbf{r}(t) = \langle t, t, t^2 \rangle$

17–20 Find a vector equation and parametric equations for the line segment that joins P to Q .

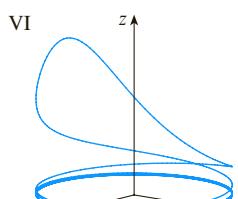
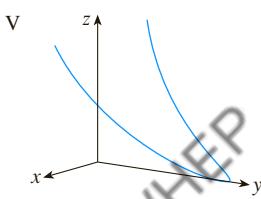
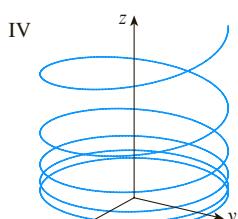
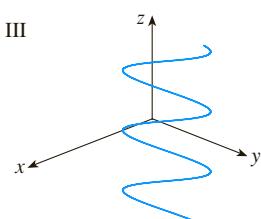
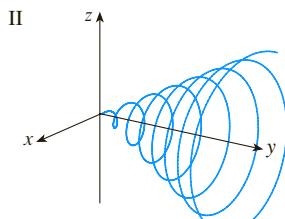
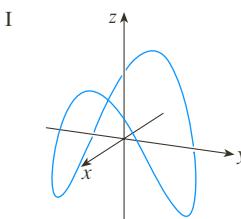
17. $P(2, 0, 0), Q(6, 2, -2)$

18. $P(-1, 2, -2), Q(-3, 5, 1)$

19. $P(0, -1, 1), Q(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$

20. $P(a, b, c), Q(u, v, w)$

21–26 Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.



21. $x = t \cos t, y = t, z = t \sin t, t \geq 0$

22. $x = \cos t, y = \sin t, z = 1/(1+t^2)$

23. $x = t, y = 1/(1+t^2), z = t^2$

24. $x = \cos t, y = \sin t, z = \cos 2t$

25. $x = \cos 8t, y = \sin 8t, z = e^{0.8t}, t \geq 0$

26. $x = \cos^2 t, y = \sin^2 t, z = t$

27. Show that the curve with parametric equations $x = t \cos t, y = t \sin t, z = t$ lies on the cone $z^2 = x^2 + y^2$, and use this fact to help sketch the curve.

28. Show that the curve with parametric equations $x = \sin t, y = \cos t, z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

29. Find three different surfaces that contain the curve $\mathbf{r}(t) = 2t \mathbf{i} + e^t \mathbf{j} + e^{2t} \mathbf{k}$.

30. Find three different surfaces that contain the curve $\mathbf{r}(t) = t^2 \mathbf{i} + \ln t \mathbf{j} + (1/t) \mathbf{k}$.

31. At what points does the curve $\mathbf{r}(t) = t \mathbf{i} + (2t - t^2) \mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

32. At what points does the helix $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

33–37 Use a computer to graph the curve with the given vector equation. Make sure you choose a parameter domain and viewpoints that reveal the true nature of the curve.

33. $\mathbf{r}(t) = \langle \cos t \sin 2t, \sin t \sin 2t, \cos 2t \rangle$

34. $\mathbf{r}(t) = \langle t^e, e^{-t}, t \rangle$

35. $\mathbf{r}(t) = \langle \sin 3t \cos t, \frac{1}{4}t, \sin 3t \sin t \rangle$

36. $\mathbf{r}(t) = \langle \cos(8 \cos t) \sin t, \sin(8 \cos t) \sin t, \cos t \rangle$

37. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$

38. Graph the curve with parametric equations $x = \sin t, y = \sin 2t, z = \cos 4t$. Explain its shape by graphing its projections onto the three coordinate planes.

39. Graph the curve with parametric equations

$$x = (1 + \cos 16t) \cos t$$

$$y = (1 + \cos 16t) \sin t$$

$$z = 1 + \cos 16t$$

Explain the appearance of the graph by showing that it lies on a cone.

40. Graph the curve with parametric equations

$$x = \sqrt{1 - 0.25 \cos^2 10t} \cos t$$

$$y = \sqrt{1 - 0.25 \cos^2 10t} \sin t$$

$$z = 0.5 \cos 10t$$

Explain the appearance of the graph by showing that it lies on a sphere.

- 41.** Show that the curve with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$ passes through the points $(1, 4, 0)$ and $(9, -8, 28)$ but not through the point $(4, 7, -6)$.
- 42–46** Find a vector function that represents the curve of intersection of the two surfaces.
- 42.** The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$
- 43.** The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$
- 44.** The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$
- 45.** The hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$
- 46.** The semiellipsoid $x^2 + y^2 + 4z^2 = 4$, $y \geq 0$, and the cylinder $x^2 + z^2 = 1$

47. Try to sketch by hand the curve of intersection of the circular cylinder $x^2 + y^2 = 4$ and the parabolic cylinder $z = x^2$. Then find parametric equations for this curve and use these equations and a computer to graph the curve.

48. Try to sketch by hand the curve of intersection of the parabolic cylinder $y = x^2$ and the top half of the ellipsoid $x^2 + 4y^2 + 4z^2 = 16$. Then find parametric equations for this curve and use these equations and a computer to graph the curve.

49. If two objects travel through space along two different curves, it's often important to know whether they will collide. (Will a missile hit its moving target? Will two aircraft collide?) The curves might intersect, but we need to know whether the objects are in the same position *at the same time*. Suppose the trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \quad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geq 0$. Do the particles collide?

50. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \quad \mathbf{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Do the particles collide? Do their paths intersect?

51. (a) Graph the curve with parametric equations

$$x = \frac{27}{26} \sin 8t - \frac{8}{39} \sin 18t$$

$$y = -\frac{27}{26} \cos 8t + \frac{8}{39} \cos 18t$$

$$z = \frac{144}{65} \sin 5t$$

(b) Show that the curve lies on the hyperboloid of one sheet $144x^2 + 144y^2 - 25z^2 = 100$.

52. The view of the trefoil knot shown in Figure 8 is accurate, but it doesn't reveal the whole story. Use the parametric equations

$$x = (2 + \cos 1.5t) \cos t$$

$$y = (2 + \cos 1.5t) \sin t$$

$$z = \sin 1.5t$$

to sketch the curve by hand as viewed from above, with gaps indicating where the curve passes over itself. Start by showing that the projection of the curve onto the xy -plane has polar coordinates $r = 2 + \cos 1.5t$ and $\theta = t$, so r varies between 1 and 3. Then show that z has maximum and minimum values when the projection is halfway between $r = 1$ and $r = 3$.

When you have finished your sketch, use a computer to draw the curve with viewpoint directly above and compare with your sketch. Then use the computer to draw the curve from several other viewpoints. You can get a better impression of the curve if you plot a tube with radius 0.2 around the curve. (Use the `tubeplot` command in Maple or the `tubecurve` or `Tube` command in Mathematica.)

53. Suppose \mathbf{u} and \mathbf{v} are vector functions that possess limits as $t \rightarrow a$ and let c be a constant. Prove the following properties of limits.

$$(a) \lim_{t \rightarrow a} [\mathbf{u}(t) + \mathbf{v}(t)] = \lim_{t \rightarrow a} \mathbf{u}(t) + \lim_{t \rightarrow a} \mathbf{v}(t)$$

$$(b) \lim_{t \rightarrow a} c\mathbf{u}(t) = c \lim_{t \rightarrow a} \mathbf{u}(t)$$

$$(c) \lim_{t \rightarrow a} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \lim_{t \rightarrow a} \mathbf{u}(t) \cdot \lim_{t \rightarrow a} \mathbf{v}(t)$$

$$(d) \lim_{t \rightarrow a} [\mathbf{u}(t) \times \mathbf{v}(t)] = \lim_{t \rightarrow a} \mathbf{u}(t) \times \lim_{t \rightarrow a} \mathbf{v}(t)$$

54. Show that $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{b}$ if and only if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |t - a| < \delta \text{ then } |\mathbf{r}(t) - \mathbf{b}| < \varepsilon$$

13.2 Derivatives and Integrals of Vector Functions

Later in this chapter we are going to use vector functions to describe the motion of planets and other objects through space. Here we prepare the way by developing the calculus of vector functions.

Derivatives

The derivative \mathbf{r}' of a vector function \mathbf{r} is defined in much the same way as for real-valued functions:

1

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Notice that when $0 < h < 1$, multiplying the secant vector by $1/h$ stretches the vector, as shown in Figure 1(b).

if this limit exists. The geometric significance of this definition is shown in Figure 1. If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$, then \overrightarrow{PQ} represents the vector $\mathbf{r}(t+h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector. If $h > 0$, the scalar multiple $(1/h)(\mathbf{r}(t+h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$. As $h \rightarrow 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by \mathbf{r} at the point P , provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq 0$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. We will also have occasion to consider the **unit tangent vector**, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

TEC Visual 13.2 shows an animation of Figure 1.

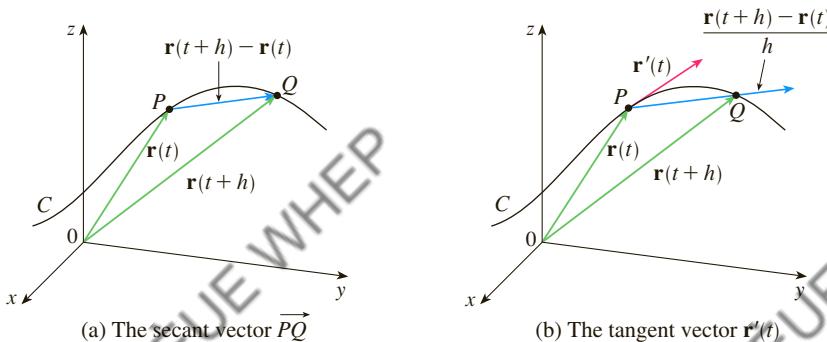


FIGURE 1

The following theorem gives us a convenient method for computing the derivative of a vector function \mathbf{r} : just differentiate each component of \mathbf{r} .

2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

PROOF

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\mathbf{r}(t + \Delta t) - \mathbf{r}(t)] \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t) \rangle - \langle f(t), g(t), h(t) \rangle] \\ &= \lim_{\Delta t \rightarrow 0} \left\langle \frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\ &= \left\langle \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\ &= \langle f'(t), g'(t), h'(t) \rangle\end{aligned}$$

EXAMPLE 1

- (a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$.
 (b) Find the unit tangent vector at the point where $t = 0$.

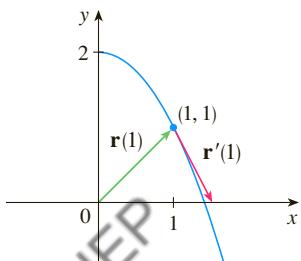
SOLUTION

- (a) According to Theorem 2, we differentiate each component of \mathbf{r} :

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + (1 - t)e^{-t}\mathbf{j} + 2\cos 2t\mathbf{k}$$

- (b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point $(1, 0, 0)$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

**FIGURE 2**

Notice from Figure 2 that the tangent vector points in the direction of increasing t . (See Exercise 58.)

EXAMPLE 2 For the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2 - t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.**SOLUTION** We have

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}$$

The curve is a plane curve and elimination of the parameter from the equations $x = \sqrt{t}$, $y = 2 - t$ gives $y = 2 - x^2$, $x \geq 0$. In Figure 2 we draw the position vector $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$ starting at the origin and the tangent vector $\mathbf{r}'(1)$ starting at the corresponding point $(1, 1)$.

EXAMPLE 3 Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2 \cos t \quad y = \sin t \quad z = t$$

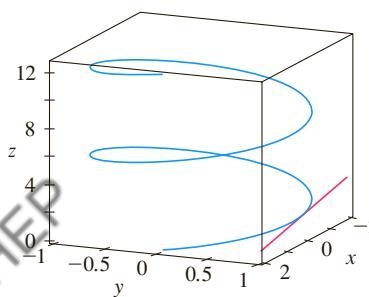
at the point $(0, 1, \pi/2)$.

SOLUTION The vector equation of the helix is $\mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle$, so

$$\mathbf{r}'(t) = \langle -2 \sin t, \cos t, 1 \rangle$$

The parameter value corresponding to the point $(0, 1, \pi/2)$ is $t = \pi/2$, so the tangent vector there is $\mathbf{r}'(\pi/2) = \langle -2, 0, 1 \rangle$. The tangent line is the line through $(0, 1, \pi/2)$ parallel to the vector $\langle -2, 0, 1 \rangle$, so by Equations 12.5.2 its parametric equations are

$$x = -2t \quad y = 1 \quad z = \frac{\pi}{2} + t$$



The helix and the tangent line in Example 3 are shown in Figure 3.

FIGURE 3

In Section 13.4 we will see how $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ can be interpreted as the velocity and acceleration vectors of a particle moving through space with position vector $\mathbf{r}(t)$ at time t .

Just as for real-valued functions, the **second derivative** of a vector function \mathbf{r} is the derivative of \mathbf{r}' , that is, $\mathbf{r}'' = (\mathbf{r}')'$. For instance, the second derivative of the function in Example 3 is

$$\mathbf{r}''(t) = \langle -2 \cos t, -\sin t, 0 \rangle$$

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose \mathbf{u} and \mathbf{v} are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt} [\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt} [c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt} [f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt} [\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

This theorem can be proved either directly from Definition 1 or by using Theorem 2 and the corresponding differentiation formulas for real-valued functions. The proof of Formula 4 follows; the remaining formulas are left as exercises.

PROOF OF FORMULA 4 Let

$$\mathbf{u}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle \quad \mathbf{v}(t) = \langle g_1(t), g_2(t), g_3(t) \rangle$$

$$\text{Then } \mathbf{u}(t) \cdot \mathbf{v}(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t) = \sum_{i=1}^3 f_i(t)g_i(t)$$

so the ordinary Product Rule gives

$$\begin{aligned} \frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \frac{d}{dt} \sum_{i=1}^3 f_i(t)g_i(t) = \sum_{i=1}^3 \frac{d}{dt} [f_i(t)g_i(t)] \\ &= \sum_{i=1}^3 [f'_i(t)g_i(t) + f_i(t)g'_i(t)] \\ &= \sum_{i=1}^3 f'_i(t)g_i(t) + \sum_{i=1}^3 f_i(t)g'_i(t) \\ &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \end{aligned}$$

■

EXAMPLE 4 Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t .

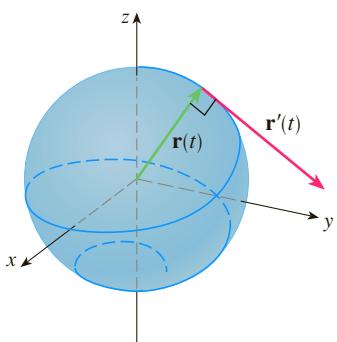


FIGURE 4

SOLUTION Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$. (See Figure 4.) ■

■ Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of \mathbf{r} in terms of the integrals of its component functions f , g , and h as follows. (We use the notation of Chapter 4.)

$$\begin{aligned}\int_a^b \mathbf{r}(t) dt &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbf{r}(t_i^*) \Delta t \\ &= \lim_{n \rightarrow \infty} \left[\left(\sum_{i=1}^n f(t_i^*) \Delta t \right) \mathbf{i} + \left(\sum_{i=1}^n g(t_i^*) \Delta t \right) \mathbf{j} + \left(\sum_{i=1}^n h(t_i^*) \Delta t \right) \mathbf{k} \right]\end{aligned}$$

and so

$$\boxed{\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where \mathbf{R} is an antiderivative of \mathbf{r} , that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t) dt$ for indefinite integrals (antiderivatives).

EXAMPLE 5 If $\mathbf{r}(t) = 2 \cos t \mathbf{i} + \sin t \mathbf{j} + 2t \mathbf{k}$, then

$$\begin{aligned}\int \mathbf{r}(t) dt &= \left(\int 2 \cos t dt \right) \mathbf{i} + \left(\int \sin t dt \right) \mathbf{j} + \left(\int 2t dt \right) \mathbf{k} \\ &= 2 \sin t \mathbf{i} - \cos t \mathbf{j} + t^2 \mathbf{k} + \mathbf{C}\end{aligned}$$

where \mathbf{C} is a vector constant of integration, and

$$\int_0^{\pi/2} \mathbf{r}(t) dt = \left[2 \sin t \mathbf{i} - \cos t \mathbf{j} + t^2 \mathbf{k} \right]_0^{\pi/2} = 2 \mathbf{i} + \mathbf{j} + \frac{\pi^2}{4} \mathbf{k}$$

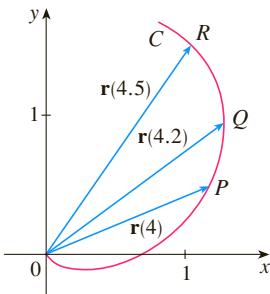
13.2 EXERCISES

1. The figure shows a curve C given by a vector function $\mathbf{r}(t)$.

- (a) Draw the vectors $\mathbf{r}(4.5) - \mathbf{r}(4)$ and $\mathbf{r}(4.2) - \mathbf{r}(4)$.
 (b) Draw the vectors

$$\frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} \quad \text{and} \quad \frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$$

- (c) Write expressions for $\mathbf{r}'(4)$ and the unit tangent vector $\mathbf{T}(4)$.
 (d) Draw the vector $\mathbf{T}(4)$.



2. (a) Make a large sketch of the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t \rangle$, $0 \leq t \leq 2$, and draw the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.
 (b) Draw the vector $\mathbf{r}'(1)$ starting at $(1, 1)$, and compare it with the vector

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$

Explain why these vectors are so close to each other in length and direction.

3-8

- (a) Sketch the plane curve with the given vector equation.
 (b) Find $\mathbf{r}'(t)$.
 (c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t .

3. $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle$, $t = -1$

4. $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $t = 1$

5. $\mathbf{r}(t) = e^{2t} \mathbf{i} + e^t \mathbf{j}$, $t = 0$

6. $\mathbf{r}(t) = e^t \mathbf{i} + 2t \mathbf{j}$, $t = 0$

7. $\mathbf{r}(t) = 4 \sin t \mathbf{i} - 2 \cos t \mathbf{j}$, $t = 3\pi/4$

8. $\mathbf{r}(t) = (\cos t + 1) \mathbf{i} + (\sin t - 1) \mathbf{j}$, $t = -\pi/3$

- 9-16 Find the derivative of the vector function.

9. $\mathbf{r}(t) = \langle \sqrt{t-2}, 3, 1/t^2 \rangle$

10. $\mathbf{r}(t) = \langle e^{-t}, t - t^3, \ln t \rangle$

11. $\mathbf{r}(t) = t^2 \mathbf{i} + \cos(t^2) \mathbf{j} + \sin^2 t \mathbf{k}$

12. $\mathbf{r}(t) = \frac{1}{1+t} \mathbf{i} + \frac{t}{1+t} \mathbf{j} + \frac{t^2}{1+t} \mathbf{k}$

13. $\mathbf{r}(t) = t \sin t \mathbf{i} + e^t \cos t \mathbf{j} + \sin t \cos t \mathbf{k}$

14. $\mathbf{r}(t) = \sin^2 at \mathbf{i} + te^{bt} \mathbf{j} + \cos^2 ct \mathbf{k}$

15. $\mathbf{r}(t) = \mathbf{a} + t \mathbf{b} + t^2 \mathbf{c}$

16. $\mathbf{r}(t) = t \mathbf{a} \times (\mathbf{b} + t \mathbf{c})$

- 17-20 Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter t .

17. $\mathbf{r}(t) = \langle t^2 - 2t, 1 + 3t, \frac{1}{3}t^3 + \frac{1}{2}t^2 \rangle$, $t = 2$

18. $\mathbf{r}(t) = \langle \tan^{-1} t, 2e^{2t}, 8te^t \rangle$, $t = 0$

19. $\mathbf{r}(t) = \cos t \mathbf{i} + 3t \mathbf{j} + 2 \sin 2t \mathbf{k}$, $t = 0$

20. $\mathbf{r}(t) = \sin^2 t \mathbf{i} + \cos^2 t \mathbf{j} + \tan^2 t \mathbf{k}$, $t = \pi/4$

21. If $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

22. If $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$, find $\mathbf{T}(0)$, $\mathbf{r}''(0)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

- 23-26 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

23. $x = t^2 + 1$, $y = 4\sqrt{t}$, $z = e^{t^2-t}$; $(2, 4, 1)$

24. $x = \ln(t+1)$, $y = t \cos 2t$, $z = 2^t$; $(0, 0, 1)$

25. $x = e^{-t} \cos t$, $y = e^{-t} \sin t$, $z = e^{-t}$; $(1, 0, 1)$

26. $x = \sqrt{t^2 + 3}$, $y = \ln(t^2 + 3)$, $z = t$; $(2, \ln 4, 1)$

27. Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point $(3, 4, 2)$.

28. Find the point on the curve $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, e^t \rangle$, $0 \leq t \leq \pi$, where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

- CAS** 29-31 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

29. $x = t$, $y = e^{-t}$, $z = 2t - t^2$; $(0, 1, 0)$

30. $x = 2 \cos t$, $y = 2 \sin t$, $z = 4 \cos 2t$; $(\sqrt{3}, 1, 2)$

31. $x = t \cos t$, $y = t$, $z = t \sin t$; $(-\pi, \pi, 0)$

32. (a) Find the point of intersection of the tangent lines to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2 \sin \pi t, \cos \pi t \rangle$ at the points where $t = 0$ and $t = 0.5$.

- (b) Illustrate by graphing the curve and both tangent lines.

33. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.

34. At what point do the curves $\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$ and $\mathbf{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$ intersect? Find their angle of intersection correct to the nearest degree.

35–40 Evaluate the integral.

35. $\int_0^2 (t\mathbf{i} - t^3\mathbf{j} + 3t^5\mathbf{k}) dt$

36. $\int_1^4 (2t^{3/2}\mathbf{i} + (t+1)\sqrt{t}\mathbf{k}) dt$

37. $\int_0^1 \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k} \right) dt$

38. $\int_0^{\pi/4} (\sec t \tan t \mathbf{i} + t \cos 2t \mathbf{j} + \sin^2 2t \cos 2t \mathbf{k}) dt$

39. $\int (\sec^2 t \mathbf{i} + t(t^2+1)^3 \mathbf{j} + t^2 \ln t \mathbf{k}) dt$

40. $\int \left(te^{2t} \mathbf{i} + \frac{t}{1-t} \mathbf{j} + \frac{1}{\sqrt{1-t^2}} \mathbf{k} \right) dt$

41. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

42. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\mathbf{i} + e^t\mathbf{j} + te^t\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

43. Prove Formula 1 of Theorem 3.

44. Prove Formula 3 of Theorem 3.

45. Prove Formula 5 of Theorem 3.

46. Prove Formula 6 of Theorem 3.

47. If $\mathbf{u}(t) = \langle \sin t, \cos t, t \rangle$ and $\mathbf{v}(t) = \langle t, \cos t, \sin t \rangle$, use Formula 4 of Theorem 3 to find

$$\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)]$$

48. If \mathbf{u} and \mathbf{v} are the vector functions in Exercise 47, use Formula 5 of Theorem 3 to find

$$\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)]$$

49. Find $f'(2)$, where $f(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$, $\mathbf{u}(2) = \langle 1, 2, -1 \rangle$, $\mathbf{u}'(2) = \langle 3, 0, 4 \rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$.

50. If $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$, where \mathbf{u} and \mathbf{v} are the vector functions in Exercise 49, find $\mathbf{r}'(2)$.

51. If $\mathbf{r}(t) = \mathbf{a} \cos \omega t + \mathbf{b} \sin \omega t$, where \mathbf{a} and \mathbf{b} are constant vectors, show that $\mathbf{r}(t) \times \mathbf{r}'(t) = \omega \mathbf{a} \times \mathbf{b}$.

52. If \mathbf{r} is the vector function in Exercise 51, show that $\mathbf{r}''(t) + \omega^2 \mathbf{r}(t) = \mathbf{0}$.

53. Show that if \mathbf{r} is a vector function such that \mathbf{r}'' exists, then

$$\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

54. Find an expression for $\frac{d}{dt} [\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))]$.

55. If $\mathbf{r}(t) \neq \mathbf{0}$, show that $\frac{d}{dt} |\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t)$.

[Hint: $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$]

56. If a curve has the property that the position vector $\mathbf{r}(t)$ is always perpendicular to the tangent vector $\mathbf{r}'(t)$, show that the curve lies on a sphere with center the origin.

57. If $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$, show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

58. Show that the tangent vector to a curve defined by a vector function $\mathbf{r}(t)$ points in the direction of increasing t .

[Hint: Refer to Figure 1 and consider the cases $h > 0$ and $h < 0$ separately.]

13.3 Arc Length and Curvature

Length of a Curve

In Section 10.2 we defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula

$$1 \quad L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

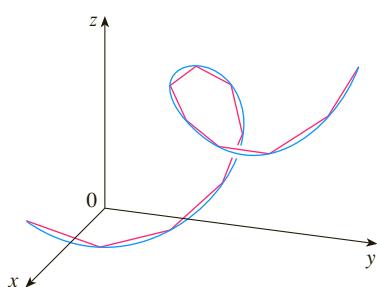


FIGURE 1

The length of a space curve is the limit of lengths of inscribed polygons.

The length of a space curve is defined in exactly the same way (see Figure 1). Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f' , g' , and h' are continuous. If the curve is traversed exactly once as t increases from a to b , then it can be shown

that its length is

2

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \end{aligned}$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

3

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

because, for plane curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

In the next section we will see that if $\mathbf{r}(t)$ is the position vector of a moving object at time t , then $\mathbf{r}'(t)$ is the velocity vector and $|\mathbf{r}'(t)|$ is the speed. Thus Equation 3 says that to compute distance traveled, we integrate speed.

Figure 2 shows the arc of the helix whose length is computed in Example 1.

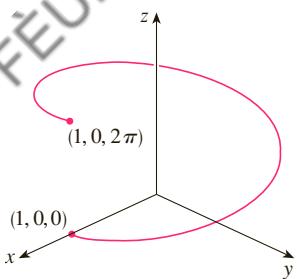


FIGURE 2

EXAMPLE 1 Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

SOLUTION Since $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$, we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from $(1, 0, 0)$ to $(1, 0, 2\pi)$ is described by the parameter interval $0 \leq t \leq 2\pi$ and so, from Formula 3, we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$$

A single curve C can be represented by more than one vector function. For instance, the twisted cubic

4

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \quad 1 \leq t \leq 2$$

could also be represented by the function

5

$$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \quad 0 \leq u \leq \ln 2$$

where the connection between the parameters t and u is given by $t = e^u$. We say that Equations 4 and 5 are **parametrizations** of the curve C . If we were to use Equation 3 to compute the length of C using Equations 4 and 5, we would get the same answer. In general, it can be shown that when Equation 3 is used to compute arc length, the answer is independent of the parametrization that is used.

■ The Arc Length Function

Now we suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \leq t \leq b$$

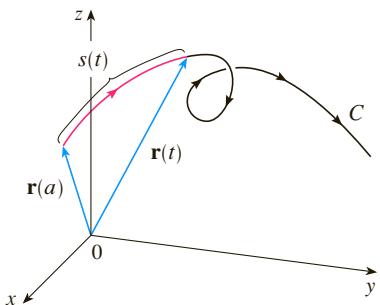


FIGURE 3

where \mathbf{r}' is continuous and C is traversed exactly once as t increases from a to b . We define its **arc length function** s by

$$\boxed{6} \quad s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. (See Figure 3.) If we differentiate both sides of Equation 6 using Part 1 of the Fundamental Theorem of Calculus, we obtain

$$\boxed{7} \quad \frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by Equation 6, then we may be able to solve for t as a function of s : $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t : $\mathbf{r} = \mathbf{r}(t(s))$. Thus, if $s = 3$ for instance, $\mathbf{r}(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

EXAMPLE 2 Reparametrize the helix $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t .

SOLUTION The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$. From Example 1 we have

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2}$$

and so $s = s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t \sqrt{2} du = \sqrt{2}t$

Therefore $t = s/\sqrt{2}$ and the required reparametrization is obtained by substituting for t :

$$\mathbf{r}(t(s)) = \cos(s/\sqrt{2}) \mathbf{i} + \sin(s/\sqrt{2}) \mathbf{j} + (s/\sqrt{2}) \mathbf{k}$$

■

Curvature

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I . A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function \mathbf{r} , recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve. From Figure 4 you can see that $\mathbf{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.) Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of \mathbf{T} .

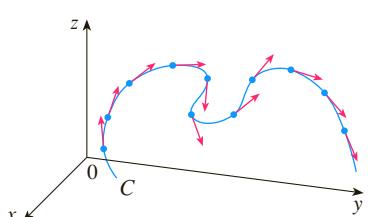


FIGURE 4
Unit tangent vectors at equally spaced points on C

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where \mathbf{T} is the unit tangent vector.

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s , so we use the Chain Rule (Theorem 13.2.3, Formula 6) to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \quad \text{and} \quad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right|$$

But $ds/dt = |\mathbf{r}'(t)|$ from Equation 7, so

9

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

EXAMPLE 3 Show that the curvature of a circle of radius a is $1/a$.

SOLUTION We can take the circle to have center the origin, and then a parametrization is

$$\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$$

Therefore $\mathbf{r}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j}$ and $|\mathbf{r}'(t)| = a$

so $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t \mathbf{i} + \cos t \mathbf{j}$

and $\mathbf{T}'(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}$

This gives $|\mathbf{T}'(t)| = 1$, so using Formula 9, we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}$$

■

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition. We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 Theorem The curvature of the curve given by the vector function \mathbf{r} is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

PROOF Since $\mathbf{T} = \mathbf{r}' / |\mathbf{r}'|$ and $|\mathbf{r}'| = ds/dt$, we have

$$\mathbf{r}' = |\mathbf{r}'| \mathbf{T} = \frac{ds}{dt} \mathbf{T}$$

so the Product Rule (Theorem 13.2.3, Formula 3) gives

$$\mathbf{r}'' = \frac{d^2 s}{dt^2} \mathbf{T} + \frac{ds}{dt} \mathbf{T}'$$

Using the fact that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$ (see Example 12.4.2), we have

$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt} \right)^2 (\mathbf{T} \times \mathbf{T}')$$

Now $|\mathbf{T}(t)| = 1$ for all t , so \mathbf{T} and \mathbf{T}' are orthogonal by Example 13.2.4. Therefore, by Theorem 12.4.9,

$$|\mathbf{r}' \times \mathbf{r}''| = \left(\frac{ds}{dt} \right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\frac{ds}{dt} \right)^2 |\mathbf{T}| |\mathbf{T}'| = \left(\frac{ds}{dt} \right)^2 |\mathbf{T}'|$$

Thus

$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{(ds/dt)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$

and

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

EXAMPLE 4 Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0, 0, 0)$.

SOLUTION We first compute the required ingredients:

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \quad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2 \mathbf{i} - 6t \mathbf{j} + 2 \mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

Theorem 10 then gives

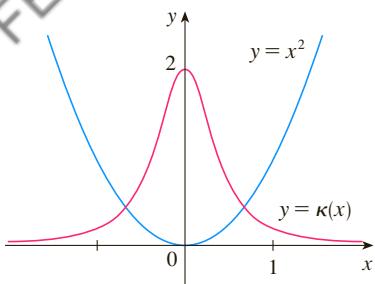
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

At the origin, where $t = 0$, the curvature is $\kappa(0) = 2$. ■

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x \mathbf{i} + f(x) \mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x) \mathbf{j}$ and $\mathbf{r}''(x) = f''(x) \mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x) \mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

11

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

**FIGURE 5**

The parabola $y = x^2$ and its curvature function

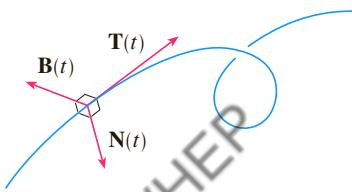
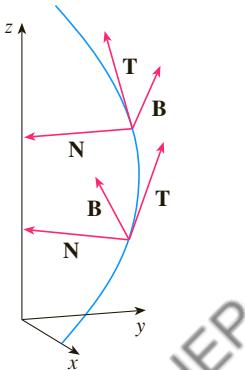
**FIGURE 6**

Figure 7 illustrates Example 6 by showing the vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} at two locations on the helix. In general, the vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} , starting at the various points on a curve, form a set of orthogonal vectors, called the **TNB frame**, that moves along the curve as t varies. This **TNB frame** plays an important role in the branch of mathematics known as differential geometry and in its applications to the motion of spacecraft.

**FIGURE 7**

EXAMPLE 5 Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

SOLUTION Since $y' = 2x$ and $y'' = 2$, Formula 11 gives

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

The curvature at $(0, 0)$ is $\kappa(0) = 2$. At $(1, 1)$ it is $\kappa(1) = 2/5^{3/2} \approx 0.18$. At $(2, 4)$ it is $\kappa(2) = 2/17^{3/2} \approx 0.03$. Observe from the expression for $\kappa(x)$ or the graph of κ in Figure 5 that $\kappa(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. This corresponds to the fact that the parabola appears to become flatter as $x \rightarrow \pm\infty$. ■

■ The Normal and Binormal Vectors

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t , we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$ by Example 13.2.4, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that, typically, $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where $\kappa \neq 0$ we can define the **principal unit normal vector** $\mathbf{N}(t)$ (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**. It is perpendicular to both \mathbf{T} and \mathbf{N} and is also a unit vector. (See Figure 6.)

EXAMPLE 6 Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$$

SOLUTION We first compute the ingredients needed for the unit normal vector:

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \quad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t \mathbf{i} - \sin t \mathbf{j}) \quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t \mathbf{i} - \sin t \mathbf{j} = \langle -\cos t, -\sin t, 0 \rangle$$

This shows that the normal vector at any point on the helix is horizontal and points toward the z -axis. The binormal vector is

$$\begin{aligned} \mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle \end{aligned}$$

TEC Visual 13.3B shows how the TNB frame moves along several curves.

The plane determined by the normal and binormal vectors \mathbf{N} and \mathbf{B} at a point P on a curve C is called the **normal plane** of C at P . It consists of all lines that are orthogonal to the tangent vector \mathbf{T} . The plane determined by the vectors \mathbf{T} and \mathbf{N} is called the **osculating plane** of C at P . The name comes from the Latin *osculum*, meaning “kiss.” It is the plane that comes closest to containing the part of the curve near P . (For a plane curve, the osculating plane is simply the plane that contains the curve.)

The circle that lies in the osculating plane of C at P , has the same tangent as C at P , lies on the concave side of C (toward which \mathbf{N} points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the **osculating circle** (or the **circle of curvature**) of C at P . It is the circle that best describes how C behaves near P ; it shares the same tangent, normal, and curvature at P .

EXAMPLE 7 Find equations of the normal plane and osculating plane of the helix in Example 6 at the point $P(0, 1, \pi/2)$.

SOLUTION The point P corresponds to $t = \pi/2$ and the normal plane there has normal vector $\mathbf{r}'(\pi/2) = \langle -1, 0, 1 \rangle$, so an equation is

$$-1(x - 0) + 0(y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \quad \text{or} \quad z = x + \frac{\pi}{2}$$

The osculating plane at P contains the vectors \mathbf{T} and \mathbf{N} , so its normal vector is $\mathbf{T} \times \mathbf{N} = \mathbf{B}$. From Example 6 we have

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle \quad \mathbf{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

A simpler normal vector is $\langle 1, 0, 1 \rangle$, so an equation of the osculating plane is

$$1(x - 0) + 0(y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \quad \text{or} \quad z = -x + \frac{\pi}{2}$$

EXAMPLE 8 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

SOLUTION From Example 5, the curvature of the parabola at the origin is $\kappa(0) = 2$. So the radius of the osculating circle at the origin is $1/\kappa = \frac{1}{2}$ and its center is $(0, \frac{1}{2})$. Its equation is therefore

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

For the graph in Figure 9 we use parametric equations of this circle:

$$x = \frac{1}{2} \cos t \quad y = \frac{1}{2} + \frac{1}{2} \sin t$$

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

Figure 8 shows the helix and the osculating plane in Example 7.

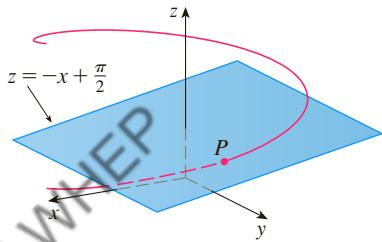


FIGURE 8

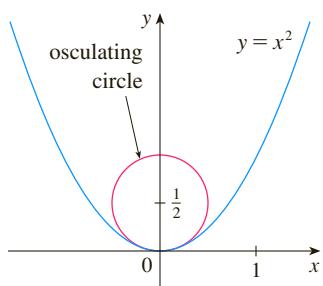


FIGURE 9

Notice that the circle and the parabola appear to bend similarly at the origin.

TEC Visual 13.3C shows how the osculating circle changes as a point moves along a curve.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

13.3 EXERCISES

1–6 Find the length of the curve.

1. $\mathbf{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle, -5 \leq t \leq 5$

2. $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle, 0 \leq t \leq 1$

3. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, 0 \leq t \leq 1$

4. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \ln \cos t\mathbf{k}, 0 \leq t \leq \pi/4$

5. $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \leq t \leq 1$

6. $\mathbf{r}(t) = t^2\mathbf{i} + 9t\mathbf{j} + 4t^{3/2}\mathbf{k}, 1 \leq t \leq 4$

7–9 Find the length of the curve correct to four decimal places. (Use a calculator to approximate the integral.)

7. $\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle, 0 \leq t \leq 2$

8. $\mathbf{r}(t) = \langle t, e^{-t}, te^{-t} \rangle, 1 \leq t \leq 3$

9. $\mathbf{r}(t) = \langle \cos \pi t, 2t, \sin 2\pi t \rangle$, from $(1, 0, 0)$ to $(1, 4, 0)$

10. Graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$. Find the total length of this curve correct to four decimal places.

11. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$.

12. Find, correct to four decimal places, the length of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

13–14 (a) Find the arc length function for the curve measured from the point P in the direction of increasing t and then reparametrize the curve with respect to arc length starting from P , and (b) find the point 4 units along the curve (in the direction of increasing t) from P .

13. $\mathbf{r}(t) = (5-t)\mathbf{i} + (4t-3)\mathbf{j} + 3t\mathbf{k}$, $P(4, 1, 3)$

14. $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} + \sqrt{2}e^t\mathbf{k}$, $P(0, 1, \sqrt{2})$

15. Suppose you start at the point $(0, 0, 3)$ and move 5 units along the curve $x = 3 \sin t$, $y = 4t$, $z = 3 \cos t$ in the positive direction. Where are you now?

16. Reparametrize the curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2+1} - 1 \right) \mathbf{i} + \frac{2t}{t^2+1} \mathbf{j}$$

with respect to arc length measured from the point $(1, 0)$ in the direction of increasing t . Express the reparametrization in its simplest form. What can you conclude about the curve?

17–20

- (a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.
 (b) Use Formula 9 to find the curvature.

17. $\mathbf{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle$

18. $\mathbf{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle$, $t > 0$

19. $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$

20. $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$

21–23 Use Theorem 10 to find the curvature.

21. $\mathbf{r}(t) = t^3\mathbf{j} + t^2\mathbf{k}$

22. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k}$

23. $\mathbf{r}(t) = \sqrt{6}t^2\mathbf{i} + 2t\mathbf{j} + 2t^3\mathbf{k}$

24. Find the curvature of $\mathbf{r}(t) = \langle t^2, \ln t, t \ln t \rangle$ at the point $(1, 0, 0)$.

25. Find the curvature of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$.

26. Graph the curve with parametric equations $x = \cos t$, $y = \sin t$, $z = \sin 5t$ and find the curvature at the point $(1, 0, 0)$.

27–29 Use Formula 11 to find the curvature.

27. $y = x^4$

28. $y = \tan x$

29. $y = xe^x$

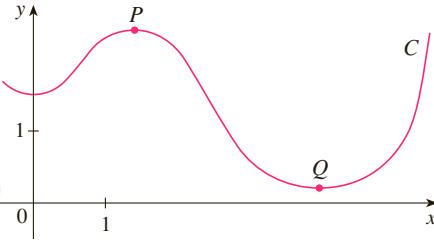
30–31 At what point does the curve have maximum curvature? What happens to the curvature as $x \rightarrow \infty$?

30. $y = \ln x$

31. $y = e^x$

32. Find an equation of a parabola that has curvature 4 at the origin.

- 33.** (a) Is the curvature of the curve C shown in the figure greater at P or at Q ? Explain.
 (b) Estimate the curvature at P and at Q by sketching the osculating circles at those points.



- 34–35 Use a graphing calculator or computer to graph both the curve and its curvature function $\kappa(x)$ on the same screen. Is the graph of κ what you would expect?

34. $y = x^4 - 2x^2$

35. $y = x^{-2}$

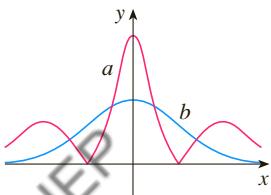
- 36–37 Plot the space curve and its curvature function $\kappa(t)$. Comment on how the curvature reflects the shape of the curve.

36. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, 4 \cos(t/2) \rangle, 0 \leq t \leq 8\pi$

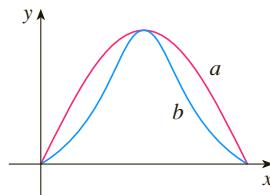
37. $\mathbf{r}(t) = \langle te^t, e^{-t}, \sqrt{2}t \rangle, -5 \leq t \leq 5$

- 38–39 Two graphs, a and b , are shown. One is a curve $y = f(x)$ and the other is the graph of its curvature function $y = \kappa(x)$. Identify each curve and explain your choices.

38.



39.



40. (a) Graph the curve $\mathbf{r}(t) = \langle \sin 3t, \sin 2t, \sin 3t \rangle$. At how many points on the curve does it appear that the curvature has a local or absolute maximum?
 (b) Use a CAS to find and graph the curvature function. Does this graph confirm your conclusion from part (a)?

41. The graph of $\mathbf{r}(t) = \langle t - \frac{3}{2} \sin t, 1 - \frac{3}{2} \cos t, t \rangle$ is shown in Figure 13.1.12(b). Where do you think the curvature is largest? Use a CAS to find and graph the curvature function. For which values of t is the curvature largest?

42. Use Theorem 10 to show that the curvature of a plane parametric curve $x = f(t), y = g(t)$ is

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t .

- 43–45 Use the formula in Exercise 42 to find the curvature.

43. $x = t^2, y = t^3$

44. $x = a \cos \omega t, y = b \sin \omega t$

45. $x = e^t \cos t, y = e^t \sin t$

46. Consider the curvature at $x = 0$ for each member of the family of functions $f(x) = e^{cx}$. For which members is $\kappa(0)$ largest?

- 47–48 Find the vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} at the given point.

47. $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle, (1, \frac{2}{3}, 1)$

48. $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle, (1, 0, 0)$

- 49–50 Find equations of the normal plane and osculating plane of the curve at the given point.

49. $x = \sin 2t, y = -\cos 2t, z = 4t; (0, 1, 2\pi)$

50. $x = \ln t, y = 2t, z = t^2; (0, 2, 1)$

51. Find equations of the osculating circles of the ellipse $9x^2 + 4y^2 = 36$ at the points $(2, 0)$ and $(0, 3)$. Use a graphing calculator or computer to graph the ellipse and both osculating circles on the same screen.

52. Find equations of the osculating circles of the parabola $y = \frac{1}{2}x^2$ at the points $(0, 0)$ and $(1, \frac{1}{2})$. Graph both osculating circles and the parabola on the same screen.

53. At what point on the curve $x = t^3, y = 3t, z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

54. Is there a point on the curve in Exercise 53 where the osculating plane is parallel to the plane $x + y + z = 1$? [Note: You will need a CAS for differentiating, for simplifying, and for computing a cross product.]

55. Find equations of the normal and osculating planes of the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ at the point $(1, 1, 1)$.

56. Show that the osculating plane at every point on the curve $\mathbf{r}(t) = \langle t + 2, 1 - t, \frac{1}{2}t^2 \rangle$ is the same plane. What can you conclude about the curve?

57. Show that at every point on the curve

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$$

the angle between the unit tangent vector and the z -axis is the same. Then show that the same result holds true for the unit normal and binormal vectors.

58. The *rectifying plane* of a curve at a point is the plane that contains the vectors \mathbf{T} and \mathbf{B} at that point. Find the rectifying plane of the curve $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + \tan t \mathbf{k}$ at the point $(\sqrt{2}/2, \sqrt{2}/2, 1)$.

59. Show that the curvature κ is related to the tangent and normal vectors by the equation

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

60. Show that the curvature of a plane curve is $\kappa = |d\phi/ds|$, where ϕ is the angle between \mathbf{T} and \mathbf{i} ; that is, ϕ is the angle of inclination of the tangent line. (This shows that the definition of curvature is consistent with the definition for plane curves given in Exercise 10.2.69.)

- 61.** (a) Show that $d\mathbf{B}/ds$ is perpendicular to \mathbf{B} .
 (b) Show that $d\mathbf{B}/ds$ is perpendicular to \mathbf{T} .
 (c) Deduce from parts (a) and (b) that $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$ for some number $\tau(s)$ called the **torsion** of the curve.
 (The torsion measures the degree of twisting of a curve.)
 (d) Show that for a plane curve the torsion is $\tau(s) = 0$.
- 62.** The following formulas, called the **Frenet-Serret formulas**, are of fundamental importance in differential geometry:

1. $d\mathbf{T}/ds = \kappa\mathbf{N}$
2. $d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B}$
3. $d\mathbf{B}/ds = -\tau\mathbf{N}$

(Formula 1 comes from Exercise 59 and Formula 3 comes from Exercise 61.) Use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ to deduce Formula 2 from Formulas 1 and 3.

- 63.** Use the Frenet-Serret formulas to prove each of the following. (Primes denote derivatives with respect to t . Start as in the proof of Theorem 10.)
- (a) $\mathbf{r}'' = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$
 - (b) $\mathbf{r}' \times \mathbf{r}'' = \kappa(s')^3\mathbf{B}$
 - (c) $\mathbf{r}''' = [s''' - \kappa^2(s')^3]\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}$
 - (d) $\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$

- 64.** Show that the circular helix $\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$, where a and b are positive constants, has constant curvature and constant torsion. [Use the result of Exercise 63(d).]

- 65.** Use the formula in Exercise 63(d) to find the torsion of the curve $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$.

- 66.** Find the curvature and torsion of the curve $x = \sinh t$, $y = \cosh t$, $z = t$ at the point $(0, 1, 0)$.

- 67.** The DNA molecule has the shape of a double helix (see Figure 3 on page 890). The radius of each helix is about 10 angstroms ($1 \text{ \AA} = 10^{-8} \text{ cm}$). Each helix rises about 34 \AA during each complete turn, and there are about 2.9×10^8 complete turns. Estimate the length of each helix.

- 68.** Let's consider the problem of designing a railroad track to make a smooth transition between sections of straight track. Existing track along the negative x -axis is to be joined smoothly to a track along the line $y = 1$ for $x \geq 1$.

- (a) Find a polynomial $P = P(x)$ of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

is continuous and has continuous slope and continuous curvature.

- (b) Graph F .

13.4 Motion in Space: Velocity and Acceleration

In this section we show how the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve. In particular, we follow in the footsteps of Newton by using these methods to derive Kepler's First Law of planetary motion.

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Notice from Figure 1 that, for small values of h , the vector

1

$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$. Its magnitude measures the size of the displacement vector per unit time. The vector (1) gives the average velocity over a time interval of length h and its limit is the **velocity vector** $\mathbf{v}(t)$ at time t :

2

$$\mathbf{v}(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

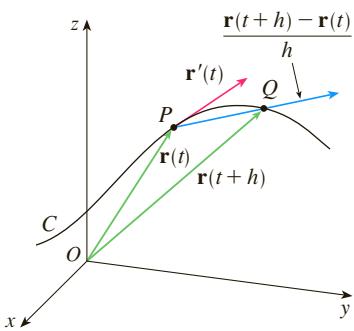


FIGURE 1

Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

The **speed** of the particle at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. This is appropriate because, from (2) and from Equation 13.3.7, we have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt} = \text{rate of change of distance with respect to time}$$

As in the case of one-dimensional motion, the **acceleration** of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

EXAMPLE 1 The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3 \mathbf{i} + t^2 \mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and illustrate geometrically.

SOLUTION The velocity and acceleration at time t are

$$\mathbf{v}(t) = \mathbf{r}'(t) = 3t^2 \mathbf{i} + 2t \mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = 6t \mathbf{i} + 2 \mathbf{j}$$

and the speed is

$$|\mathbf{v}(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

When $t = 1$, we have

$$\mathbf{v}(1) = 3\mathbf{i} + 2\mathbf{j} \quad \mathbf{a}(1) = 6\mathbf{i} + 2\mathbf{j} \quad |\mathbf{v}(1)| = \sqrt{13}$$

These velocity and acceleration vectors are shown in Figure 2.

EXAMPLE 2 Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

SOLUTION

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, e^t, (1+t)e^t \rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, e^t, (2+t)e^t \rangle$$

$$|\mathbf{v}(t)| = \sqrt{4t^2 + e^{2t} + (1+t)^2 e^{2t}}$$

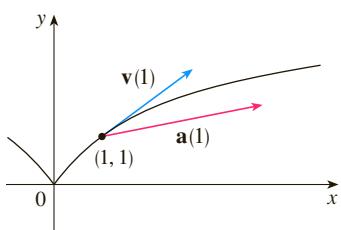


FIGURE 2

TEC Visual 13.4 shows animated velocity and acceleration vectors for objects moving along various curves.

Figure 3 shows the path of the particle in Example 2 with the velocity and acceleration vectors when $t = 1$.

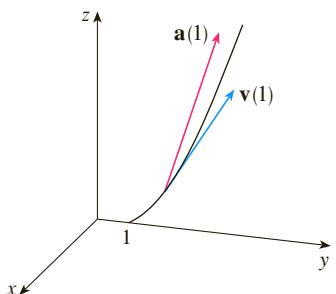


FIGURE 3

The vector integrals that were introduced in Section 13.2 can be used to find position vectors when velocity or acceleration vectors are known, as in the next example.

EXAMPLE 3 A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t \mathbf{i} + 6t \mathbf{j} + \mathbf{k}$. Find its velocity and position at time t .

SOLUTION Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we have

$$\begin{aligned} \mathbf{v}(t) &= \int \mathbf{a}(t) dt = \int (4t \mathbf{i} + 6t \mathbf{j} + \mathbf{k}) dt \\ &= 2t^2 \mathbf{i} + 3t^2 \mathbf{j} + t \mathbf{k} + \mathbf{C} \end{aligned}$$

To determine the value of the constant vector \mathbf{C} , we use the fact that $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. The preceding equation gives $\mathbf{v}(0) = \mathbf{C}$, so $\mathbf{C} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and

$$\begin{aligned} \mathbf{v}(t) &= 2t^2 \mathbf{i} + 3t^2 \mathbf{j} + t \mathbf{k} + \mathbf{i} - \mathbf{j} + \mathbf{k} \\ &= (2t^2 + 1) \mathbf{i} + (3t^2 - 1) \mathbf{j} + (t + 1) \mathbf{k} \end{aligned}$$

The expression for $\mathbf{r}(t)$ that we obtained in Example 3 was used to plot the path of the particle in Figure 4 for $0 \leq t \leq 3$.

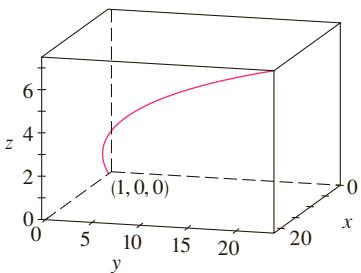


FIGURE 4

Since $\mathbf{v}(t) = \mathbf{r}'(t)$, we have

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t) dt \\ &= \int [(2t^2 + 1)\mathbf{i} + (3t^2 - 1)\mathbf{j} + (t + 1)\mathbf{k}] dt \\ &= \left(\frac{2}{3}t^3 + t\right)\mathbf{i} + (t^3 - t)\mathbf{j} + \left(\frac{1}{2}t^2 + t\right)\mathbf{k} + \mathbf{D}\end{aligned}$$

Putting $t = 0$, we find that $\mathbf{D} = \mathbf{r}(0) = \mathbf{i}$, so the position at time t is given by

$$\mathbf{r}(t) = \left(\frac{2}{3}t^3 + t + 1\right)\mathbf{i} + (t^3 - t)\mathbf{j} + \left(\frac{1}{2}t^2 + t\right)\mathbf{k}$$

In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^t \mathbf{a}(u) du \quad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^t \mathbf{v}(u) du$$

If the force that acts on a particle is known, then the acceleration can be found from **Newton's Second Law of Motion**. The vector version of this law states that if, at any time t , a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

EXAMPLE 4 An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.

SOLUTION To find the force, we first need to know the acceleration:

$$\mathbf{v}(t) = \mathbf{r}'(t) = -a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = -a\omega^2 \cos \omega t \mathbf{i} - a\omega^2 \sin \omega t \mathbf{j}$$

Therefore Newton's Second Law gives the force as

$$\mathbf{F}(t) = m\mathbf{a}(t) = -m\omega^2(a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j})$$

Notice that $\mathbf{F}(t) = -m\omega^2 \mathbf{r}(t)$. This shows that the force acts in the direction opposite to the radius vector $\mathbf{r}(t)$ and therefore points toward the origin (see Figure 5). Such a force is called a *centripetal* (center-seeking) force.

The object moving with position P has angular speed $\omega = d\theta/dt$, where θ is the angle shown in Figure 5.

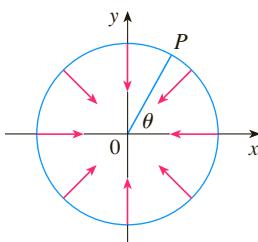


FIGURE 5

■ Projectile Motion

EXAMPLE 5 A projectile is fired with angle of elevation α and initial velocity \mathbf{v}_0 . (See Figure 6.) Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?

SOLUTION We set up the axes so that the projectile starts at the origin. Since the force due to gravity acts downward, we have

$$\mathbf{F} = m\mathbf{a} = -mg\mathbf{j}$$

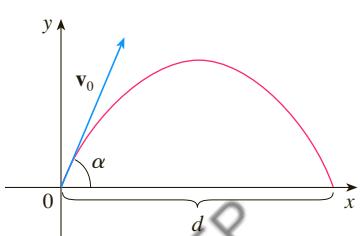


FIGURE 6

where $g = |\mathbf{a}| \approx 9.8 \text{ m/s}^2$. Thus

$$\mathbf{a} = -g\mathbf{j}$$

Since $\mathbf{v}'(t) = \mathbf{a}$, we have

$$\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{C}$$

where $\mathbf{C} = \mathbf{v}(0) = \mathbf{v}_0$. Therefore

$$\mathbf{r}'(t) = \mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$$

Integrating again, we obtain

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{D}$$

But $\mathbf{D} = \mathbf{r}(0) = \mathbf{0}$, so the position vector of the projectile is given by

$$3 \quad \mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0$$

If we write $|\mathbf{v}_0| = v_0$ (the initial speed of the projectile), then

$$\mathbf{v}_0 = v_0 \cos \alpha \mathbf{i} + v_0 \sin \alpha \mathbf{j}$$

and Equation 3 becomes

$$\mathbf{r}(t) = (v_0 \cos \alpha)t \mathbf{i} + [(v_0 \sin \alpha)t - \frac{1}{2}gt^2]\mathbf{j}$$

The parametric equations of the trajectory are therefore

$$4 \quad x = (v_0 \cos \alpha)t \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$

The horizontal distance d is the value of x when $y = 0$. Setting $y = 0$, we obtain $t = 0$ or $t = (2v_0 \sin \alpha)/g$. This second value of t then gives

$$d = x = (v_0 \cos \alpha) \frac{2v_0 \sin \alpha}{g} = \frac{v_0^2(2 \sin \alpha \cos \alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{g}$$

Clearly, d has its maximum value when $\sin 2\alpha = 1$, that is, $\alpha = 45^\circ$.

EXAMPLE 6 A projectile is fired with muzzle speed 150 m/s and angle of elevation 45° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

SOLUTION If we place the origin at ground level, then the initial position of the projectile is $(0, 10)$ and so we need to adjust Equations 4 by adding 10 to the expression for y . With $v_0 = 150 \text{ m/s}$, $\alpha = 45^\circ$, and $g = 9.8 \text{ m/s}^2$, we have

$$x = 150 \cos(45^\circ)t = 75\sqrt{2}t$$

$$y = 10 + 150 \sin(45^\circ)t - \frac{1}{2}(9.8)t^2 = 10 + 75\sqrt{2}t - 4.9t^2$$

Impact occurs when $y = 0$, that is, $4.9t^2 - 75\sqrt{2}t - 10 = 0$. Using the quadratic formula to solve this equation (and taking only the positive value of t), we get

$$t = \frac{75\sqrt{2} + \sqrt{11,250 + 196}}{9.8} \approx 21.74$$

Then $x \approx 75\sqrt{2}(21.74) \approx 2306$, so the projectile hits the ground about 2306 m away.

The velocity of the projectile is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 75\sqrt{2} \mathbf{i} + (75\sqrt{2} - 9.8t) \mathbf{j}$$

So its speed at impact is

$$|\mathbf{v}(21.74)| = \sqrt{(75\sqrt{2})^2 + (75\sqrt{2} - 9.8 \cdot 21.74)^2} \approx 151 \text{ m/s} \quad \blacksquare$$

■ Tangential and Normal Components of Acceleration

When we study the motion of a particle, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal. If we write $v = |\mathbf{v}|$ for the speed of the particle, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

and so

$$\mathbf{v} = v\mathbf{T}$$

If we differentiate both sides of this equation with respect to t , we get

$$\boxed{5} \quad \mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'$$

If we use the expression for the curvature given by Equation 13.3.9, then we have

$$\boxed{6} \quad \kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \quad \text{so} \quad |\mathbf{T}'| = \kappa v$$

The unit normal vector was defined in the preceding section as $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, so (6) gives

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v\mathbf{N}$$

and Equation 5 becomes

$$\boxed{7} \quad \mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$$

Writing a_T and a_N for the tangential and normal components of acceleration, we have

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

$$\boxed{8} \quad a_T = v' \quad \text{and} \quad a_N = \kappa v^2$$

This resolution is illustrated in Figure 7.

Let's look at what Formula 7 says. The first thing to notice is that the binormal vector \mathbf{B} is absent. No matter how an object moves through space, its acceleration always lies in the plane of \mathbf{T} and \mathbf{N} (the osculating plane). (Recall that \mathbf{T} gives the direction of motion and \mathbf{N} points in the direction the curve is turning.) Next we notice that the tangential component of acceleration is v' , the rate of change of speed, and the normal component of acceleration is κv^2 , the curvature times the square of the speed. This makes sense if we think of a passenger in a car—a sharp turn in a road means a large value of the curvature κ , so the component of the acceleration perpendicular to the motion is large and the passenger is thrown against a car door. High speed around the turn has the same effect; in fact, if you double your speed, a_N is increased by a factor of 4.

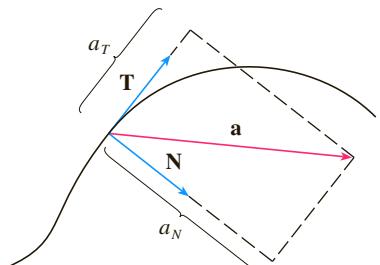


FIGURE 7

Although we have expressions for the tangential and normal components of acceleration in Equations 8, it's desirable to have expressions that depend only on \mathbf{r} , \mathbf{r}' , and \mathbf{r}'' . To this end we take the dot product of $\mathbf{v} = v\mathbf{T}$ with \mathbf{a} as given by Equation 7:

$$\begin{aligned}\mathbf{v} \cdot \mathbf{a} &= v\mathbf{T} \cdot (v'\mathbf{T} + \kappa v^2 \mathbf{N}) \\ &= vv'\mathbf{T} \cdot \mathbf{T} + \kappa v^3 \mathbf{T} \cdot \mathbf{N} \\ &= vv' \quad (\text{since } \mathbf{T} \cdot \mathbf{T} = 1 \text{ and } \mathbf{T} \cdot \mathbf{N} = 0)\end{aligned}$$

Therefore

$$9 \quad a_T = v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

Using the formula for curvature given by Theorem 13.3.10, we have

$$10 \quad a_N = \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} |\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

EXAMPLE 7 A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

SOLUTION

$$\mathbf{r}(t) = t^2 \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$$

$$\mathbf{r}'(t) = 2t \mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}$$

$$\mathbf{r}''(t) = 2 \mathbf{i} + 2 \mathbf{j} + 6t \mathbf{k}$$

$$|\mathbf{r}'(t)| = \sqrt{8t^2 + 9t^4}$$

Therefore Equation 9 gives the tangential component as

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

Since

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = 6t^2 \mathbf{i} - 6t^2 \mathbf{j}$$

Equation 10 gives the normal component as

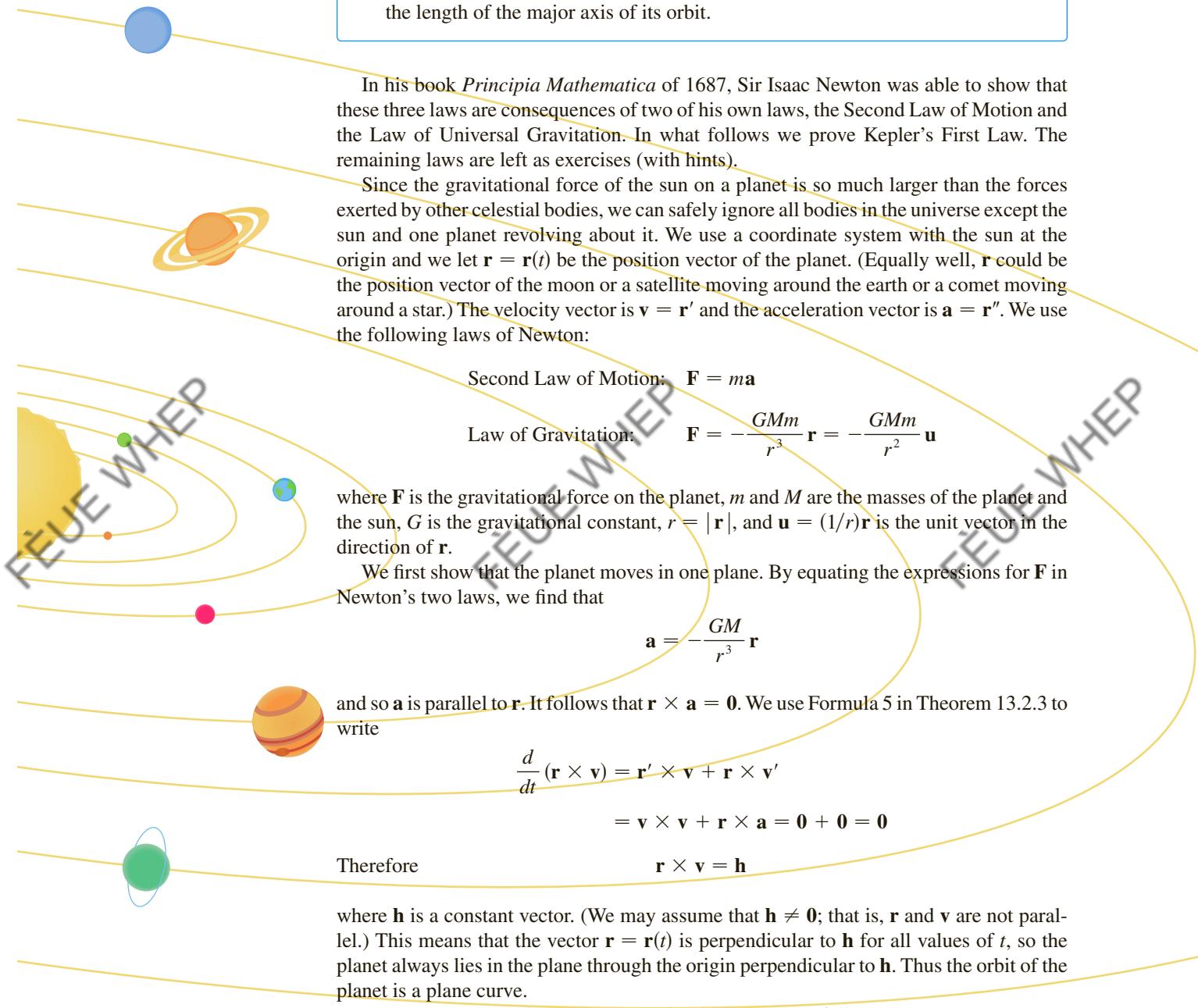
$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{6\sqrt{2}t^2}{\sqrt{8t^2 + 9t^4}}$$

■ Kepler's Laws of Planetary Motion

We now describe one of the great accomplishments of calculus by showing how the material of this chapter can be used to prove Kepler's laws of planetary motion. After 20 years of studying the astronomical observations of the Danish astronomer Tycho Brahe, the German mathematician and astronomer Johannes Kepler (1571–1630) formulated the following three laws.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.



To prove Kepler's First Law we rewrite the vector \mathbf{h} as follows:

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}' = r\mathbf{u} \times (r\mathbf{u}') \\ &= r\mathbf{u} \times (r\mathbf{u}' + r'\mathbf{u}) = r^2(\mathbf{u} \times \mathbf{u}') + rr'(\mathbf{u} \times \mathbf{u}) \\ &= r^2(\mathbf{u} \times \mathbf{u}') \end{aligned}$$

Then

$$\begin{aligned}\mathbf{a} \times \mathbf{h} &= \frac{-GM}{r^2} \mathbf{u} \times (r^2 \mathbf{u} \times \mathbf{u}') = -GM \mathbf{u} \times (\mathbf{u} \times \mathbf{u}') \\ &= -GM[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}'] \quad (\text{by Theorem 12.4.11, Property 6})\end{aligned}$$

But $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$ and, since $|\mathbf{u}(t)| = 1$, it follows from Example 13.2.4 that

$$\mathbf{u} \cdot \mathbf{u}' = 0$$

Therefore

$$\mathbf{a} \times \mathbf{h} = GM \mathbf{u}'$$

and so

$$(\mathbf{v} \times \mathbf{h})' = \mathbf{v}' \times \mathbf{h} = \mathbf{a} \times \mathbf{h} = GM \mathbf{u}'$$

Integrating both sides of this equation, we get

$$11 \quad \mathbf{v} \times \mathbf{h} = GM \mathbf{u} + \mathbf{c}$$

where \mathbf{c} is a constant vector.

At this point it is convenient to choose the coordinate axes so that the standard basis vector \mathbf{k} points in the direction of the vector \mathbf{h} . Then the planet moves in the xy -plane. Since both $\mathbf{v} \times \mathbf{h}$ and \mathbf{u} are perpendicular to \mathbf{h} , Equation 11 shows that \mathbf{c} lies in the xy -plane. This means that we can choose the x - and y -axes so that the vector \mathbf{i} lies in the direction of \mathbf{c} , as shown in Figure 8.

If θ is the angle between \mathbf{c} and \mathbf{r} , then (r, θ) are polar coordinates of the planet. From Equation 11 we have

$$\begin{aligned}\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= \mathbf{r} \cdot (GM \mathbf{u} + \mathbf{c}) = GM \mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{c} \\ &= GM r \mathbf{u} \cdot \mathbf{u} + |\mathbf{r}| |\mathbf{c}| \cos \theta = GM r + rc \cos \theta\end{aligned}$$

where $c = |\mathbf{c}|$. Then

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c \cos \theta} = \frac{1}{GM} \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e \cos \theta}$$

where $e = c/(GM)$. But

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = |\mathbf{h}|^2 = h^2$$

where $h = |\mathbf{h}|$. So

$$r = \frac{h^2/(GM)}{1 + e \cos \theta} = \frac{eh^2/c}{1 + e \cos \theta}$$

Writing $d = h^2/c$, we obtain the equation

$$12 \quad r = \frac{ed}{1 + e \cos \theta}$$

Comparing with Theorem 10.6.6, we see that Equation 12 is the polar equation of a conic section with focus at the origin and eccentricity e . We know that the orbit of a planet is a closed curve and so the conic must be an ellipse.

This completes the derivation of Kepler's First Law. We will guide you through the derivation of the Second and Third Laws in the Applied Project on page 920. The proofs of these three laws show that the methods of this chapter provide a powerful tool for describing some of the laws of nature.

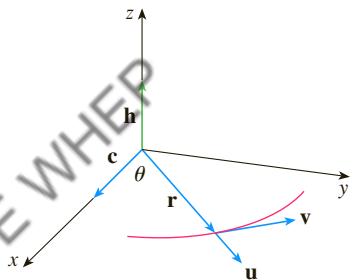


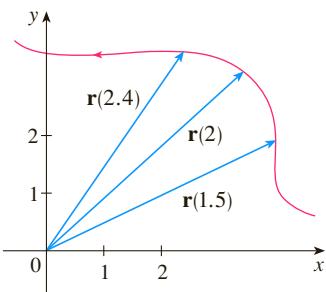
FIGURE 8

13.4 EXERCISES

1. The table gives coordinates of a particle moving through space along a smooth curve.
- Find the average velocities over the time intervals $[0, 1]$, $[0.5, 1]$, $[1, 2]$, and $[1, 1.5]$.
 - Estimate the velocity and speed of the particle at $t = 1$.

t	x	y	z
0	2.7	9.8	3.7
0.5	3.5	7.2	3.3
1.0	4.5	6.0	3.0
1.5	5.9	6.4	2.8
2.0	7.3	7.8	2.7

2. The figure shows the path of a particle that moves with position vector $\mathbf{r}(t)$ at time t .
- Draw a vector that represents the average velocity of the particle over the time interval $2 \leq t \leq 2.4$.
 - Draw a vector that represents the average velocity over the time interval $1.5 \leq t \leq 2$.
 - Write an expression for the velocity vector $\mathbf{v}(2)$.
 - Draw an approximation to the vector $\mathbf{v}(2)$ and estimate the speed of the particle at $t = 2$.



- 3–8 Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t .

- $\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle$, $t = 2$
- $\mathbf{r}(t) = \left\langle t^2, 1/t^2 \right\rangle$, $t = 1$
- $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$, $t = \pi/3$
- $\mathbf{r}(t) = e^t \mathbf{i} + e^{2t} \mathbf{j}$, $t = 0$
- $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + 2\mathbf{k}$, $t = 1$
- $\mathbf{r}(t) = t \mathbf{i} + 2 \cos t \mathbf{j} + \sin t \mathbf{k}$, $t = 0$

- 9–14 Find the velocity, acceleration, and speed of a particle with the given position function.

- $\mathbf{r}(t) = \left\langle t^2 + t, t^2 - t, t^3 \right\rangle$
- $\mathbf{r}(t) = \left\langle 2 \cos t, 3t, 2 \sin t \right\rangle$

11. $\mathbf{r}(t) = \sqrt{2} t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}$

12. $\mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{j} + \ln t \mathbf{k}$

13. $\mathbf{r}(t) = e^t(\cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k})$

14. $\mathbf{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle$, $t \geq 0$

- 15–16 Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

15. $\mathbf{a}(t) = 2 \mathbf{i} + 2t \mathbf{k}$, $\mathbf{v}(0) = 3 \mathbf{i} - \mathbf{j}$, $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

16. $\mathbf{a}(t) = \sin t \mathbf{i} + 2 \cos t \mathbf{j} + 6t \mathbf{k}$,
 $\mathbf{v}(0) = -\mathbf{k}$, $\mathbf{r}(0) = \mathbf{j} - 4 \mathbf{k}$

17–18

- (a) Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.
(b) Use a computer to graph the path of the particle.

17. $\mathbf{a}(t) = 2t \mathbf{i} + \sin t \mathbf{j} + \cos 2t \mathbf{k}$, $\mathbf{v}(0) = \mathbf{i}$, $\mathbf{r}(0) = \mathbf{j}$

18. $\mathbf{a}(t) = t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}$, $\mathbf{v}(0) = \mathbf{k}$, $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

19. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. When is the speed a minimum?

20. What force is required so that a particle of mass m has the position function $\mathbf{r}(t) = t^3 \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$?

21. A force with magnitude 20 N acts directly upward from the xy -plane on an object with mass 4 kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Find its position function and its speed at time t .

22. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

23. A projectile is fired with an initial speed of 200 m/s and angle of elevation 60° . Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

24. Rework Exercise 23 if the projectile is fired from a position 100 m above the ground.

25. A ball is thrown at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

26. A projectile is fired from a tank with initial speed 400 m/s. Find two angles of elevation that can be used to hit a target 3000 m away.

27. A rifle is fired with angle of elevation 36° . What is the muzzle speed if the maximum height of the bullet is 1600 ft?

28. A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/s at an angle 50° above the horizontal. Is it a home run? (In other words, does the ball clear the fence?)

- 29.** A medieval city has the shape of a square and is protected by walls with length 500 m and height 15 m. You are the commander of an attacking army and the closest you can get to the wall is 100 m. Your plan is to set fire to the city by catapulting heated rocks over the wall (with an initial speed of 80 m/s). At what range of angles should you tell your men to set the catapult? (Assume the path of the rocks is perpendicular to the wall.)
- 30.** Show that a projectile reaches three-quarters of its maximum height in half the time needed to reach its maximum height.
- 31.** A ball is thrown eastward into the air from the origin (in the direction of the positive x -axis). The initial velocity is $50 \mathbf{i} + 80 \mathbf{k}$, with speed measured in feet per second. The spin of the ball results in a southward acceleration of 4 ft/s^2 , so the acceleration vector is $\mathbf{a} = -4\mathbf{j} - 32\mathbf{k}$. Where does the ball land and with what speed?
- 32.** A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at an angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?
- 33.** Water traveling along a straight portion of a river normally flows fastest in the middle, and the speed slows to almost zero at the banks. Consider a long straight stretch of river flowing north, with parallel banks 40 m apart. If the maximum water speed is 3 m/s, we can use a quadratic function as a basic model for the rate of water flow x units from the west bank: $f(x) = \frac{3}{400}x(40 - x)$.
- A boat proceeds at a constant speed of 5 m/s from a point A on the west bank while maintaining a heading perpendicular to the bank. How far down the river on the opposite bank will the boat touch shore? Graph the path of the boat.
 - Suppose we would like to pilot the boat to land at the point B on the east bank directly opposite A . If we maintain a constant speed of 5 m/s and a constant heading, find the angle at which the boat should head. Then graph the actual path the boat follows. Does the path seem realistic?
- 34.** Another reasonable model for the water speed of the river in Exercise 33 is a sine function: $f(x) = 3 \sin(\pi x/40)$. If a boater would like to cross the river from A to B with constant heading and a constant speed of 5 m/s, determine the angle at which the boat should head.
- 35.** A particle has position function $\mathbf{r}(t)$. If $\mathbf{r}'(t) = \mathbf{c} \times \mathbf{r}(t)$, where \mathbf{c} is a constant vector, describe the path of the particle.
- 36.** (a) If a particle moves along a straight line, what can you say about its acceleration vector?
 (b) If a particle moves with constant speed along a curve, what can you say about its acceleration vector?

37–40 Find the tangential and normal components of the acceleration vector.

37. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + t^3\mathbf{j}, \quad t \geq 0$

38. $\mathbf{r}(t) = 2t^2\mathbf{i} + (\frac{2}{3}t^3 - 2t)\mathbf{j}$

39. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

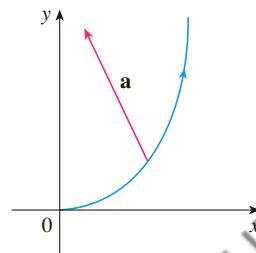
40. $\mathbf{r}(t) = t\mathbf{i} + 2e^t\mathbf{j} + e^{2t}\mathbf{k}$

41–42 Find the tangential and normal components of the acceleration vector at the given point.

41. $\mathbf{r}(t) = \ln t\mathbf{i} + (t^2 + 3t)\mathbf{j} + 4\sqrt{t}\mathbf{k}, \quad (0, 4, 4)$

42. $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t^3}\mathbf{k}, \quad (1, 1, 1)$

43. The magnitude of the acceleration vector \mathbf{a} is 10 cm/s^2 . Use the figure to estimate the tangential and normal components of \mathbf{a} .



- 44.** If a particle with mass m moves with position vector $\mathbf{r}(t)$, then its **angular momentum** is defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t)$ and its **torque** as $\boldsymbol{\tau}(t) = m\mathbf{r}(t) \times \mathbf{a}(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$. Deduce that if $\boldsymbol{\tau}(t) = \mathbf{0}$ for all t , then $\mathbf{L}(t)$ is constant. (This is the *law of conservation of angular momentum*.)

- 45.** The position function of a spaceship is

$$\mathbf{r}(t) = (3 + t)\mathbf{i} + (2 + \ln t)\mathbf{j} + \left(7 - \frac{4}{t^2 + 1}\right)\mathbf{k}$$

and the coordinates of a space station are $(6, 4, 9)$. The captain wants the spaceship to coast into the space station. When should the engines be turned off?

- 46.** A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t . If the exhaust gases escape with velocity \mathbf{v}_e relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$m \frac{d\mathbf{v}}{dt} = \frac{dm}{dt} \mathbf{v}_e$$

- Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln \frac{m(0)}{m(t)} \mathbf{v}_e$.
- For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

APPLIED PROJECT**KEPLER'S LAWS**

Johannes Kepler stated the following three laws of planetary motion on the basis of massive amounts of data on the positions of the planets at various times.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Kepler formulated these laws because they fitted the astronomical data. He wasn't able to see why they were true or how they related to each other. But Sir Isaac Newton, in his *Principia Mathematica* of 1687, showed how to deduce Kepler's three laws from two of Newton's own laws, the Second Law of Motion and the Law of Universal Gravitation. In Section 13.4 we proved Kepler's First Law using the calculus of vector functions. In this project we guide you through the proofs of Kepler's Second and Third Laws and explore some of their consequences.

1. Use the following steps to prove Kepler's Second Law. The notation is the same as in the proof of the First Law in Section 13.4. In particular, use polar coordinates so that $\mathbf{r} = (r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j}$.

(a) Show that $\mathbf{h} = r^2 \frac{d\theta}{dt} \mathbf{k}$.

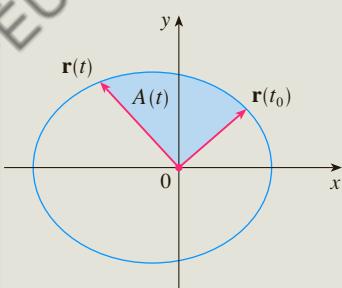
(b) Deduce that $r^2 \frac{d\theta}{dt} = h$.

- (c) If $A = A(t)$ is the area swept out by the radius vector $\mathbf{r} = \mathbf{r}(t)$ in the time interval $[t_0, t]$ as in the figure, show that

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

- (d) Deduce that

$$\frac{dA}{dt} = \frac{1}{2} h = \text{constant}$$



This says that the rate at which A is swept out is constant and proves Kepler's Second Law.

2. Let T be the period of a planet about the sun; that is, T is the time required for it to travel once around its elliptical orbit. Suppose that the lengths of the major and minor axes of the ellipse are $2a$ and $2b$.

- (a) Use part (d) of Problem 1 to show that $T = 2\pi ab/h$.

(b) Show that $\frac{h^2}{GM} = ed = \frac{b^2}{a}$.

(c) Use parts (a) and (b) to show that $T^2 = \frac{4\pi^2}{GM} a^3$.

This proves Kepler's Third Law. [Notice that the proportionality constant $4\pi^2/(GM)$ is independent of the planet.]

3. The period of the earth's orbit is approximately 365.25 days. Use this fact and Kepler's Third Law to find the length of the major axis of the earth's orbit. You will need the mass of the sun, $M = 1.99 \times 10^{30}$ kg, and the gravitational constant, $G = 6.67 \times 10^{-11}$ N·m²/kg².
4. It's possible to place a satellite into orbit about the earth so that it remains fixed above a given location on the equator. Compute the altitude that is needed for such a satellite. The earth's mass is 5.98×10^{24} kg; its radius is 6.37×10^6 m. (This orbit is called the Clarke Geosynchronous Orbit after Arthur C. Clarke, who first proposed the idea in 1945. The first such satellite, *Syncom II*, was launched in July 1963.)

13 REVIEW

CONCEPT CHECK

- What is a vector function? How do you find its derivative and its integral?
- What is the connection between vector functions and space curves?
- How do you find the tangent vector to a smooth curve at a point? How do you find the tangent line? The unit tangent vector?
- If \mathbf{u} and \mathbf{v} are differentiable vector functions, c is a scalar, and f is a real-valued function, write the rules for differentiating the following vector functions.

(a) $\mathbf{u}(t) + \mathbf{v}(t)$	(b) $c\mathbf{u}(t)$	(c) $f(t)\mathbf{u}(t)$
(d) $\mathbf{u}(t) \cdot \mathbf{v}(t)$	(e) $\mathbf{u}(t) \times \mathbf{v}(t)$	(f) $\mathbf{u}(f(t))$
- How do you find the length of a space curve given by a vector function $\mathbf{r}(t)$?

Answers to the Concept Check can be found on the back endpapers.

- (a) What is the definition of curvature?
 (b) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{T}'(t)$.
 (c) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.
 (d) Write a formula for the curvature of a plane curve with equation $y = f(x)$.
- (a) Write formulas for the unit normal and binormal vectors of a smooth space curve $\mathbf{r}(t)$.
 (b) What is the normal plane of a curve at a point? What is the osculating plane? What is the osculating circle?
- (a) How do you find the velocity, speed, and acceleration of a particle that moves along a space curve?
 (b) Write the acceleration in terms of its tangential and normal components.
- State Kepler's Laws.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

- The curve with vector equation $\mathbf{r}(t) = t^3 \mathbf{i} + 2t^3 \mathbf{j} + 3t^3 \mathbf{k}$ is a line.
- The curve $\mathbf{r}(t) = \langle 0, t^2, 4t \rangle$ is a parabola.
- The curve $\mathbf{r}(t) = \langle 2t, 3 - t, 0 \rangle$ is a line that passes through the origin.
- The derivative of a vector function is obtained by differentiating each component function.
- If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions, then

$$\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}'(t)$$

- If $\mathbf{r}(t)$ is a differentiable vector function, then

$$\frac{d}{dt} |\mathbf{r}(t)| = |\mathbf{r}'(t)|$$

- If $\mathbf{T}(t)$ is the unit tangent vector of a smooth curve, then the curvature is $\kappa = |d\mathbf{T}/dt|$.
- The binormal vector is $\mathbf{B}(t) = \mathbf{N}(t) \times \mathbf{T}(t)$.
- Suppose f is twice continuously differentiable. At an inflection point of the curve $y = f(x)$, the curvature is 0.
- If $\kappa(t) = 0$ for all t , the curve is a straight line.
- If $|\mathbf{r}(t)| = 1$ for all t , then $|\mathbf{r}'(t)|$ is a constant.
- If $|\mathbf{r}(t)| = 1$ for all t , then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t .
- The osculating circle of a curve C at a point has the same tangent vector, normal vector, and curvature as C at that point.
- Different parametrizations of the same curve result in identical tangent vectors at a given point on the curve.

EXERCISES

1. (a) Sketch the curve with vector function

$$\mathbf{r}(t) = t\mathbf{i} + \cos \pi t \mathbf{j} + \sin \pi t \mathbf{k} \quad t \geq 0$$

- (b) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

2. Let $\mathbf{r}(t) = \langle \sqrt{2-t}, (e^t - 1)/t, \ln(t+1) \rangle$.

- (a) Find the domain of \mathbf{r} .
 (b) Find $\lim_{t \rightarrow 0} \mathbf{r}(t)$.
 (c) Find $\mathbf{r}'(t)$.

3. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$.

4. Find parametric equations for the tangent line to the curve $x = 2 \sin t, y = 2 \sin 2t, z = 2 \sin 3t$ at the point $(1, \sqrt{3}, 2)$. Graph the curve and the tangent line on a common screen.

5. If $\mathbf{r}(t) = t^2 \mathbf{i} + t \cos \pi t \mathbf{j} + \sin \pi t \mathbf{k}$, evaluate $\int_0^1 \mathbf{r}(t) dt$.

6. Let C be the curve with equations $x = 2 - t^3, y = 2t - 1, z = \ln t$. Find (a) the point where C intersects the xz -plane, (b) parametric equations of the tangent line at $(1, 1, 0)$, and (c) an equation of the normal plane to C at $(1, 1, 0)$.

7. Use Simpson's Rule with $n = 6$ to estimate the length of the arc of the curve with equations $x = t^2, y = t^3, z = t^4, 0 \leq t \leq 3$.

8. Find the length of the curve $\mathbf{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle, 0 \leq t \leq 1$.

9. The helix $\mathbf{r}_1(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$ at the point $(1, 0, 0)$. Find the angle of intersection of these curves.

10. Reparametrize the curve $\mathbf{r}(t) = e^t \mathbf{i} + e^t \sin t \mathbf{j} + e^t \cos t \mathbf{k}$ with respect to arc length measured from the point $(1, 0, 1)$ in the direction of increasing t .

11. For the curve given by $\mathbf{r}(t) = \langle \sin^3 t, \cos^3 t, \sin^2 t \rangle, 0 \leq t \leq \pi/2$, find

- (a) the unit tangent vector,
 (b) the unit normal vector,
 (c) the unit binormal vector, and
 (d) the curvature.

12. Find the curvature of the ellipse $x = 3 \cos t, y = 4 \sin t$ at the points $(3, 0)$ and $(0, 4)$.

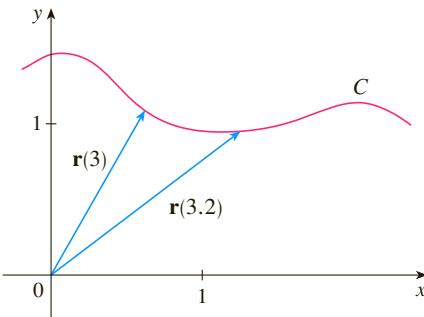
13. Find the curvature of the curve $y = x^4$ at the point $(1, 1)$.

14. Find an equation of the osculating circle of the curve $y = x^4 - x^2$ at the origin. Graph both the curve and its osculating circle.

15. Find an equation of the osculating plane of the curve $x = \sin 2t, y = t, z = \cos 2t$ at the point $(0, \pi, 1)$.

16. The figure shows the curve C traced by a particle with position vector $\mathbf{r}(t)$ at time t .

- (a) Draw a vector that represents the average velocity of the particle over the time interval $3 \leq t \leq 3.2$.
 (b) Write an expression for the velocity $\mathbf{v}(3)$.
 (c) Write an expression for the unit tangent vector $\mathbf{T}(3)$ and draw it.



17. A particle moves with position function

$$\mathbf{r}(t) = t \ln t \mathbf{i} + t \mathbf{j} + e^{-t} \mathbf{k}$$

Find the velocity, speed, and acceleration of the particle.

18. Find the velocity, speed, and acceleration of a particle moving with position function $\mathbf{r}(t) = (2t^2 - 3)\mathbf{i} + 2t\mathbf{j}$. Sketch the path of the particle and draw the position, velocity, and acceleration vectors for $t = 1$.

19. A particle starts at the origin with initial velocity $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 6t\mathbf{i} + 12t^2\mathbf{j} - 6t\mathbf{k}$. Find its position function.

20. An athlete throws a shot at an angle of 45° to the horizontal at an initial speed of 43 ft/s. It leaves his hand 7 ft above the ground.

- (a) Where is the shot 2 seconds later?
 (b) How high does the shot go?
 (c) Where does the shot land?

21. A projectile is launched with an initial speed of 40 m/s from the floor of a tunnel whose height is 30 m. What angle of elevation should be used to achieve the maximum possible horizontal range of the projectile? What is the maximum range?

22. Find the tangential and normal components of the acceleration vector of a particle with position function

$$\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$$

23. A disk of radius 1 is rotating in the counterclockwise direction at a constant angular speed ω . A particle starts at the center of the disk and moves toward the edge along a fixed radius so that its position at time t , $t \geq 0$, is given by

$\mathbf{r}(t) = t\mathbf{R}(t)$, where

$$\mathbf{R}(t) = \cos \omega t \mathbf{i} + \sin \omega t \mathbf{j}$$

- (a) Show that the velocity \mathbf{v} of the particle is

$$\mathbf{v} = \cos \omega t \mathbf{i} + \sin \omega t \mathbf{j} + t\mathbf{v}_d$$

where $\mathbf{v}_d = \mathbf{R}'(t)$ is the velocity of a point on the edge of the disk.

- (b) Show that the acceleration \mathbf{a} of the particle is

$$\mathbf{a} = 2\mathbf{v}_d + t\mathbf{a}_d$$

where $\mathbf{a}_d = \mathbf{R}''(t)$ is the acceleration of a point on the edge of the disk. The extra term $2\mathbf{v}_d$ is called the *Coriolis acceleration*; it is the result of the interaction of the rotation of the disk and the motion of the particle. One can obtain a physical demonstration of this acceleration by walking toward the edge of a moving merry-go-round.

- (c) Determine the Coriolis acceleration of a particle that moves on a rotating disk according to the equation

$$\mathbf{r}(t) = e^{-t} \cos \omega t \mathbf{i} + e^{-t} \sin \omega t \mathbf{j}$$

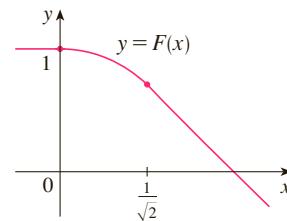
24. In designing *transfer curves* to connect sections of straight railroad tracks, it's important to realize that the acceleration of the train should be continuous so that the reactive force exerted by the train on the track is also continuous. Because of the formulas for the components of acceleration in Section 13.4, this will be the case if the curvature varies continuously.

- (a) A logical candidate for a transfer curve to join existing tracks given by $y = 1$ for $x \leq 0$ and $y = \sqrt{2} - x$ for $x \geq 1/\sqrt{2}$ might be the function $f(x) = \sqrt{1 - x^2}$, $0 < x < 1/\sqrt{2}$, whose graph is the arc of the circle

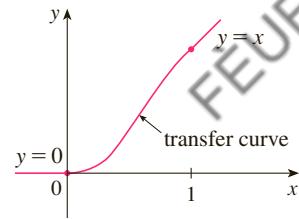
shown in the figure. It looks reasonable at first glance. Show that the function

$$F(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ \sqrt{1 - x^2} & \text{if } 0 < x < 1/\sqrt{2} \\ \sqrt{2} - x & \text{if } x \geq 1/\sqrt{2} \end{cases}$$

is continuous and has continuous slope, but does not have continuous curvature. Therefore f is not an appropriate transfer curve.



- (b) Find a fifth-degree polynomial to serve as a transfer curve between the following straight line segments: $y = 0$ for $x \leq 0$ and $y = x$ for $x \geq 1$. Could this be done with a fourth-degree polynomial? Use a graphing calculator or computer to sketch the graph of the "connected" function and check to see that it looks like the one in the figure.



Problems Plus

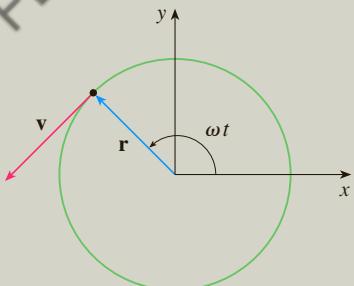


FIGURE FOR PROBLEM 1

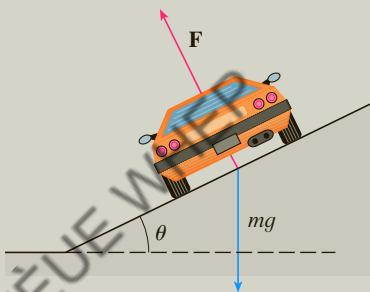


FIGURE FOR PROBLEM 2

1. A particle P moves with constant angular speed ω around a circle whose center is at the origin and whose radius is R . The particle is said to be in *uniform circular motion*. Assume that the motion is counterclockwise and that the particle is at the point $(R, 0)$ when $t = 0$. The position vector at time $t \geq 0$ is $\mathbf{r}(t) = R \cos \omega t \mathbf{i} + R \sin \omega t \mathbf{j}$.

- Find the velocity vector \mathbf{v} and show that $\mathbf{v} \cdot \mathbf{r} = 0$. Conclude that \mathbf{v} is tangent to the circle and points in the direction of the motion.
- Show that the speed $|\mathbf{v}|$ of the particle is the constant ωR . The *period* T of the particle is the time required for one complete revolution. Conclude that

$$T = \frac{2\pi R}{|\mathbf{v}|} = \frac{2\pi}{\omega}$$

- Find the acceleration vector \mathbf{a} . Show that it is proportional to \mathbf{r} and that it points toward the origin. An acceleration with this property is called a *centripetal acceleration*. Show that the magnitude of the acceleration vector is $|\mathbf{a}| = R\omega^2$.
- Suppose that the particle has mass m . Show that the magnitude of the force \mathbf{F} that is required to produce this motion, called a *centripetal force*, is

$$|\mathbf{F}| = \frac{m|\mathbf{v}|^2}{R}$$

2. A circular curve of radius R on a highway is banked at an angle θ so that a car can safely traverse the curve without skidding when there is no friction between the road and the tires. The loss of friction could occur, for example, if the road is covered with a film of water or ice. The rated speed v_r of the curve is the maximum speed that a car can attain without skidding. Suppose a car of mass m is traversing the curve at the rated speed v_r . Two forces are acting on the car: the vertical force, mg , due to the weight of the car, and a force \mathbf{F} exerted by, and normal to, the road (see the figure).

The vertical component of \mathbf{F} balances the weight of the car, so that $|\mathbf{F}| \cos \theta = mg$. The horizontal component of \mathbf{F} produces a centripetal force on the car so that, by Newton's Second Law and part (d) of Problem 1,

$$|\mathbf{F}| \sin \theta = \frac{mv_r^2}{R}$$

- Show that $v_r^2 = Rg \tan \theta$.
 - Find the rated speed of a circular curve with radius 400 ft that is banked at an angle of 12° .
 - Suppose the design engineers want to keep the banking at 12° , but wish to increase the rated speed by 50%. What should the radius of the curve be?
3. A projectile is fired from the origin with angle of elevation α and initial speed v_0 . Assuming that air resistance is negligible and that the only force acting on the projectile is gravity, g , we showed in Example 13.4.5 that the position vector of the projectile is

$$\mathbf{r}(t) = (v_0 \cos \alpha)t \mathbf{i} + [(v_0 \sin \alpha)t - \frac{1}{2}gt^2] \mathbf{j}$$

We also showed that the maximum horizontal distance of the projectile is achieved when $\alpha = 45^\circ$ and in this case the range is $R = v_0^2/g$.

- At what angle should the projectile be fired to achieve maximum height and what is the maximum height?
- Fix the initial speed v_0 and consider the parabola $x^2 + 2Ry - R^2 = 0$, whose graph is shown in the figure at the left. Show that the projectile can hit any target inside or on the boundary of the region bounded by the parabola and the x -axis, and that it can't hit any target outside this region.

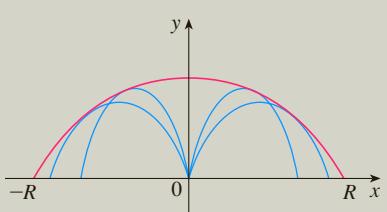


FIGURE FOR PROBLEM 3

- (c) Suppose that the gun is elevated to an angle of inclination α in order to aim at a target that is suspended at a height h directly over a point D units downrange (see the figure below). The target is released at the instant the gun is fired. Show that the projectile always hits the target, regardless of the value v_0 , provided the projectile does not hit the ground “before” D .

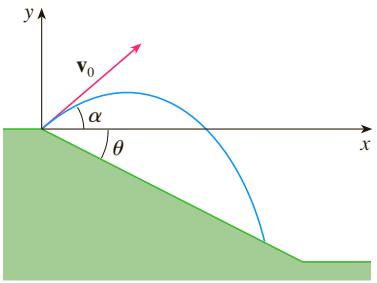
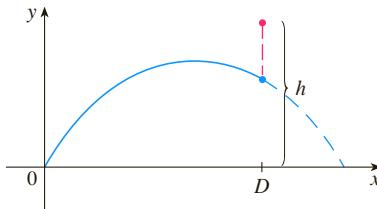


FIGURE FOR PROBLEM 4

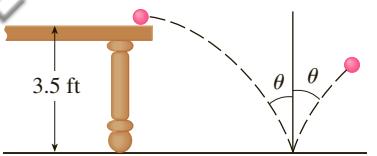


FIGURE FOR PROBLEM 5

4. (a) A projectile is fired from the origin down an inclined plane that makes an angle θ with the horizontal. The angle of elevation of the gun and the initial speed of the projectile are α and v_0 , respectively. Find the position vector of the projectile and the parametric equations of the path of the projectile as functions of the time t . (Ignore air resistance.)
 (b) Show that the angle of elevation α that will maximize the downhill range is the angle halfway between the plane and the vertical.
 (c) Suppose the projectile is fired up an inclined plane whose angle of inclination is θ . Show that, in order to maximize the (uphill) range, the projectile should be fired in the direction halfway between the plane and the vertical.
 (d) In a paper presented in 1686, Edmond Halley summarized the laws of gravity and projectile motion and applied them to gunnery. One problem he posed involved firing a projectile to hit a target a distance R up an inclined plane. Show that the angle at which the projectile should be fired to hit the target but use the least amount of energy is the same as the angle in part (c). (Use the fact that the energy needed to fire the projectile is proportional to the square of the initial speed, so minimizing the energy is equivalent to minimizing the initial speed.)
5. A ball rolls off a table with a speed of 2 ft/s. The table is 3.5 ft high.
- Determine the point at which the ball hits the floor and find its speed at the instant of impact.
 - Find the angle θ between the path of the ball and the vertical line drawn through the point of impact (see the figure).
 - Suppose the ball rebounds from the floor at the same angle with which it hits the floor, but loses 20% of its speed due to energy absorbed by the ball on impact. Where does the ball strike the floor on the second bounce?
6. Find the curvature of the curve with parametric equations

$$x = \int_0^t \sin\left(\frac{1}{2}\pi\theta^2\right) d\theta \quad y = \int_0^t \cos\left(\frac{1}{2}\pi\theta^2\right) d\theta$$

7. If a projectile is fired with angle of elevation α and initial speed v , then parametric equations for its trajectory are

$$x = (v \cos \alpha)t \quad y = (v \sin \alpha)t - \frac{1}{2}gt^2$$

(See Example 13.4.5.) We know that the range (horizontal distance traveled) is maximized when $\alpha = 45^\circ$. What value of α maximizes the total distance traveled by the projectile? (State your answer correct to the nearest degree.)

8. A cable has radius r and length L and is wound around a spool with radius R without overlapping. What is the shortest length along the spool that is covered by the cable?
 9. Show that the curve with vector equation

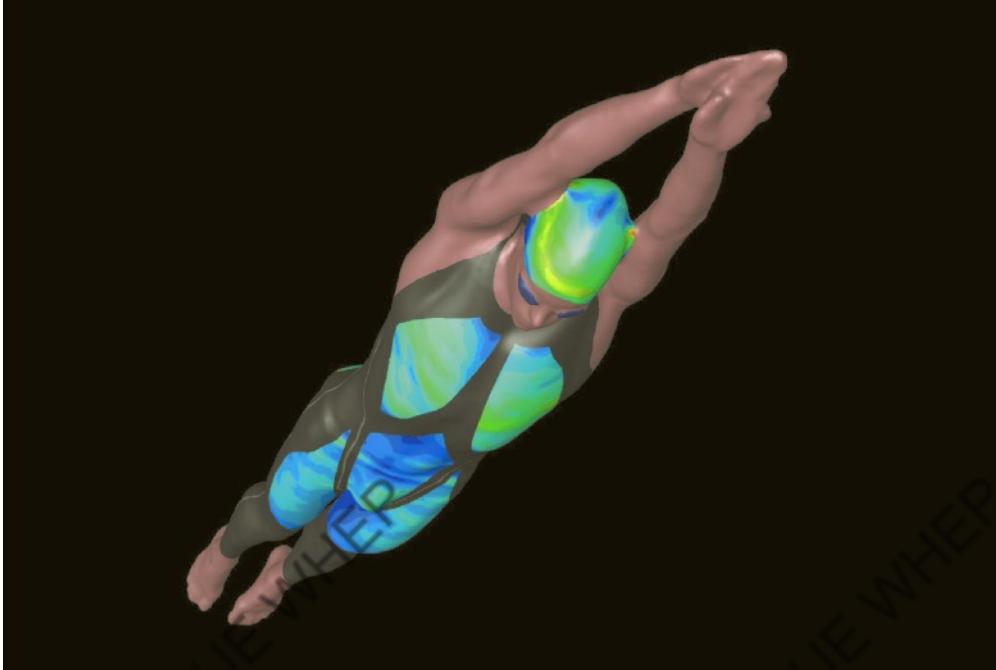
$$\mathbf{r}(t) = \langle a_1 t^2 + b_1 t + c_1, a_2 t^2 + b_2 t + c_2, a_3 t^2 + b_3 t + c_3 \rangle$$

lies in a plane and find an equation of the plane.

14

Partial Derivatives

In 2008 Speedo introduced the LZR Racer and, because it reduced drag in the water, many swimming records were broken. In the project on page 976 you are asked to use partial derivatives to explain why a small decrease in drag can have a big effect on performance.



Courtesy Speedo and ANSYS, Inc.

SO FAR WE HAVE DEALT with the calculus of functions of a single variable. But, in the real world, physical quantities often depend on two or more variables, so in this chapter we turn our attention to functions of several variables and extend the basic ideas of differential calculus to such functions.

14.1 Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)

■ Functions of Two Variables

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y , or as a function of the pair (x, y) . We indicate this functional dependence by writing $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h . In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h , and we write $V(r, h) = \pi r^2 h$.

Definition A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.

The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

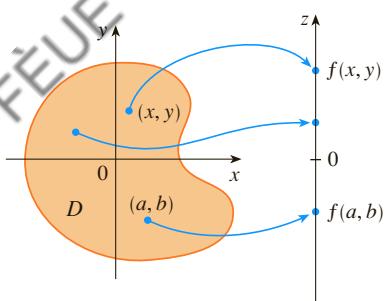


FIGURE 1

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y) . The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

A function of two variables is just a function whose domain is a subset of \mathbb{R}^2 and whose range is a subset of \mathbb{R} . One way of visualizing such a function is by means of an arrow diagram (see Figure 1), where the domain D is represented as a subset of the xy -plane and the range is a set of numbers on a real line, shown as a z -axis. For instance, if $f(x, y)$ represents the temperature at a point (x, y) in a flat metal plate with the shape of D , we can think of the z -axis as a thermometer displaying the recorded temperatures.

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is a well-defined real number.

EXAMPLE 1 For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

$$(a) f(x, y) = \frac{\sqrt{x+y+1}}{x-1} \qquad (b) f(x, y) = x \ln(y^2 - x)$$

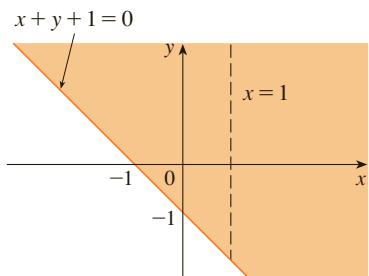
SOLUTION

$$(a) f(3, 2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$$

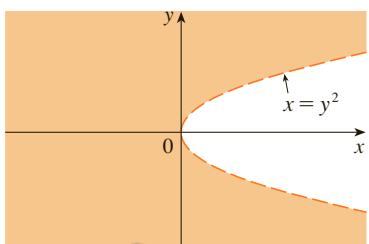
The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y) \mid x + y + 1 \geq 0, x \neq 1\}$$

The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or

**FIGURE 2**

Domain of $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$

**FIGURE 3**

Domain of $f(x, y) = x \ln(y^2 - x)$

The Wind-Chill Index

The wind-chill index measures how cold it feels when it's windy. It is based on a model of how fast a human face loses heat. It was developed through clinical trials in which volunteers were exposed to a variety of temperatures and wind speeds in a refrigerated wind tunnel.

above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain. (See Figure 2.)

$$(b) \quad f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$$

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is $D = \{(x, y) \mid x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$. (See Figure 3.) ■

Not all functions can be represented by explicit formulas. The function in the next example is described verbally and by numerical estimates of its values.

EXAMPLE 2 In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v . So W is a function of T and v , and we can write $W = f(T, v)$. Table 1 records values of W compiled by the US National Weather Service and the Meteorological Service of Canada.

Table 1 Wind-chill index as a function of air temperature and wind speed

		Wind speed (km/h)											
		5	10	15	20	25	30	40	50	60	70	80	
		5	4	3	2	1	1	0	-1	-1	-2	-2	-3
		0	-2	-3	-4	-5	-6	-6	-7	-8	-9	-9	-10
		-5	-7	-9	-11	-12	-12	-13	-14	-15	-16	-16	-17
		-10	-13	-15	-17	-18	-19	-20	-21	-22	-23	-23	-24
		-15	-19	-21	-23	-24	-25	-26	-27	-29	-30	-30	-31
		-20	-24	-27	-29	-30	-32	-33	-34	-35	-36	-37	-38
		-25	-30	-33	-35	-37	-38	-39	-41	-42	-43	-44	-45
		-30	-36	-39	-41	-43	-44	-46	-48	-49	-50	-51	-52
		-35	-41	-45	-48	-49	-51	-52	-54	-56	-57	-58	-60
		-40	-47	-51	-54	-56	-57	-59	-61	-63	-64	-65	-67

For instance, the table shows that if the temperature is -5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about -15°C with no wind. So

$$f(-5, 50) = -15$$

EXAMPLE 3 In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate. The function they used to model production was of the form

1

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is

Table 2

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

the amount of capital invested (the monetary worth of all machinery, equipment, and buildings). In Section 14.3 we will show how the form of Equation 1 follows from certain economic assumptions.

Cobb and Douglas used economic data published by the government to obtain Table 2. They took the year 1899 as a baseline and P , L , and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures.

Cobb and Douglas used the method of least squares to fit the data of Table 2 to the function

2

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

(See Exercise 81 for the details.)

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.

The production function (1) has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**. Its domain is $\{(L, K) \mid L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative. ■

EXAMPLE 4 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\} = \{(x, y) \mid x^2 + y^2 \leq 9\}$$

which is the disk with center $(0, 0)$ and radius 3. (See Figure 4.) The range of g is

$$\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

Since z is a positive square root, $z \geq 0$. Also, because $9 - x^2 - y^2 \leq 9$, we have

$$\sqrt{9 - x^2 - y^2} \leq 3$$

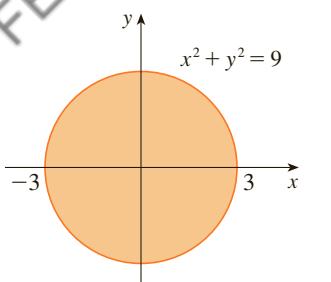
So the range is

$$\{z \mid 0 \leq z \leq 3\} = [0, 3]$$

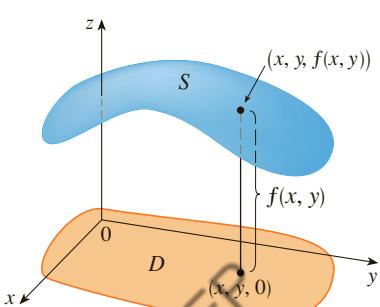
Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D , then the **graph** of f is the set of all points (x, y, z) in \mathbb{R}^3 such that $z = f(x, y)$ and (x, y) is in D .

**FIGURE 4**

Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

**FIGURE 5**

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy -plane (see Figure 5).

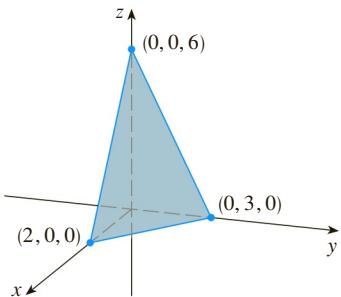


FIGURE 6

EXAMPLE 5 Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

SOLUTION The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x -intercept. Similarly, the y -intercept is 3 and the z -intercept is 6. This helps us sketch the portion of the graph that lies in the first octant in Figure 6. ■

The function in Example 5 is a special case of the function

$$f(x, y) = ax + by + c$$

which is called a **linear function**. The graph of such a function has the equation

$$z = ax + by + c \quad \text{or} \quad ax + by - z + c = 0$$

so it is a plane. In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

EXAMPLE 6 Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere (see Figure 7). ■

NOTE An entire sphere can't be represented by a single function of x and y . As we saw in Example 6, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.

EXAMPLE 7 Use a computer to draw the graph of the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

SOLUTION Figure 8 shows the graph of P for values of the labor L and capital K that lie between 0 and 300. The computer has drawn the surface by plotting vertical traces. We see from these traces that the value of the production P increases as either L or K increases, as is to be expected.

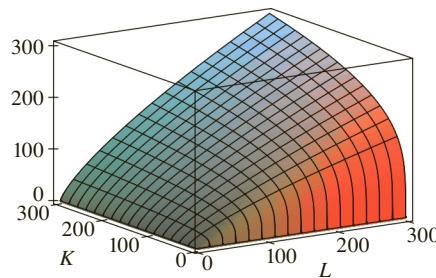


FIGURE 8

EXAMPLE 8 Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

SOLUTION Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y) , so the domain is \mathbb{R}^2 , the entire xy -plane. The range of h is the set $[0, \infty)$ of all nonnegative real numbers. [Notice that $x^2 \geq 0$ and $y^2 \geq 0$, so $h(x, y) \geq 0$ for all x and y .]

and y .] The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid that we sketched in Example 12.6.4. Horizontal traces are ellipses and vertical traces are parabolas (see Figure 9).

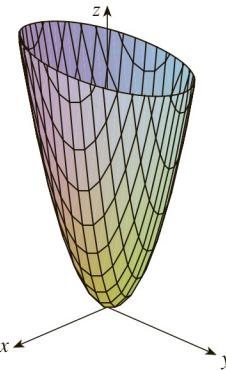
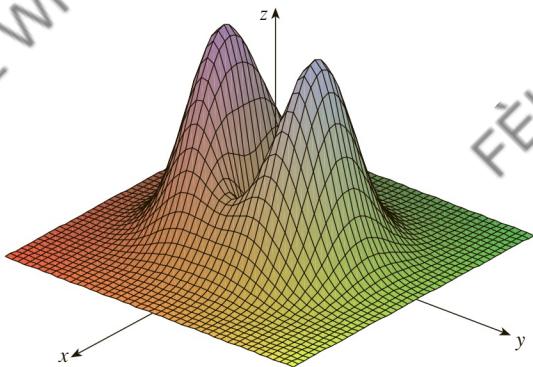


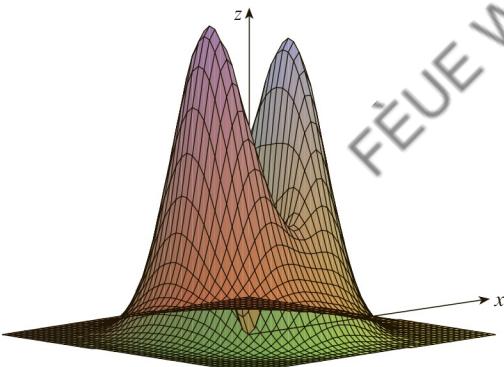
FIGURE 9
Graph of $h(x, y) = 4x^2 + y^2$

Computer programs are readily available for graphing functions of two variables. In most such programs, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k and parts of the graph are eliminated using hidden line removal.

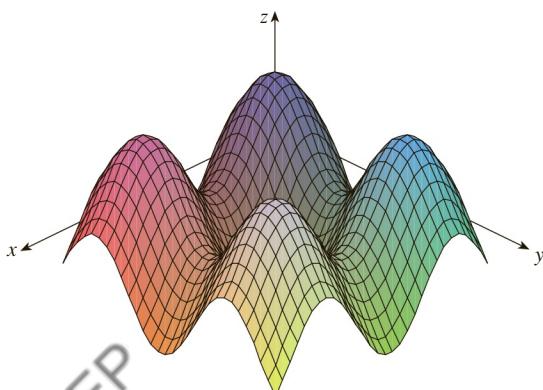
Figure 10 shows computer-generated graphs of several functions. Notice that we get an especially good picture of a function when rotation is used to give views from dif-



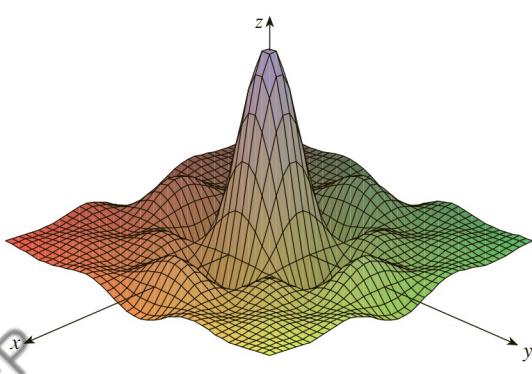
$$(a) f(x, y) = (x^2 + 3y^2)e^{-(x^2-y^2)}$$



$$(b) f(x, y) = (x^2 + 3y^2)e^{-(x^2-y^2)}$$



$$(c) f(x, y) = \sin x + \sin y$$



$$(d) f(x, y) = \frac{\sin x \sin y}{xy}$$

FIGURE 10

ferent vantage points. In parts (a) and (b) the graph of f is very flat and close to the xy -plane except near the origin; this is because $e^{-x^2-y^2}$ is very small when x or y is large.

■ Level Curves

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k . In other words, it shows where the graph of f has height k .

You can see from Figure 11 the relation between level curves and horizontal traces. The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy -plane. So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph. The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

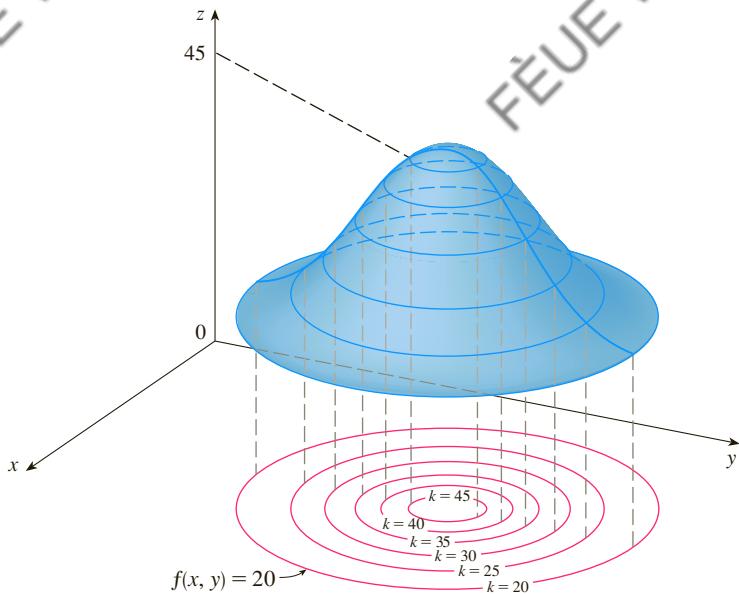


FIGURE 11

TEC Visual 14.1A animates Figure 11 by showing level curves being lifted up to graphs of functions.

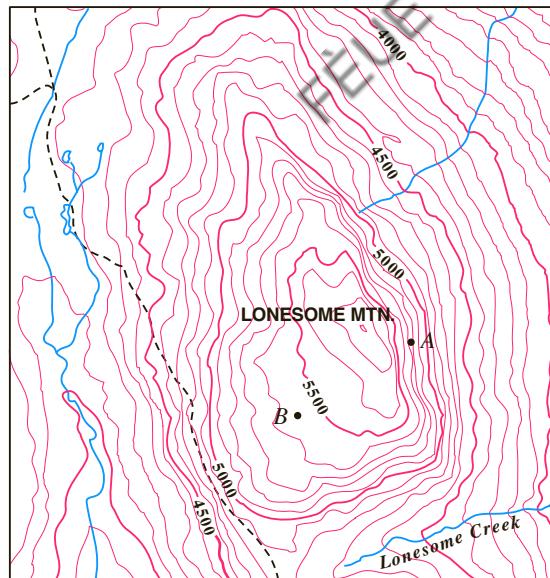


FIGURE 12

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12. The level curves are curves of constant elevation above sea level. If you walk along one of these contour lines, you neither ascend nor descend. Another common example is the temperature function introduced in the opening paragraph of this section. Here the level curves are called **isothermals** and join loca-

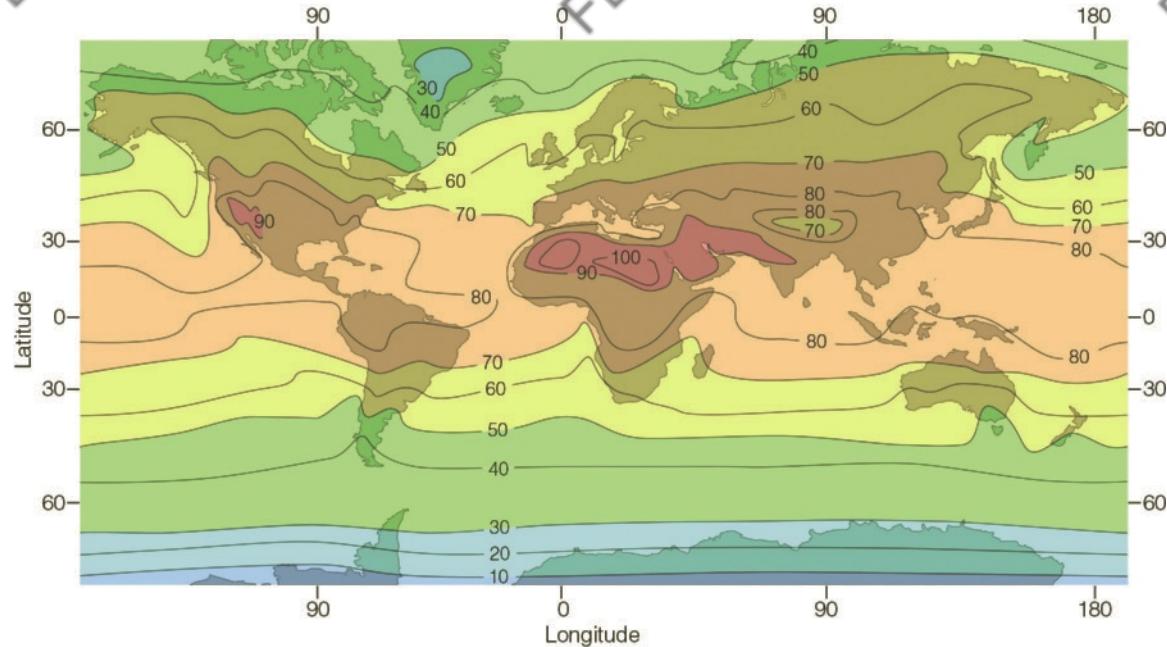


FIGURE 13 Average air temperature near sea level in July (°F)

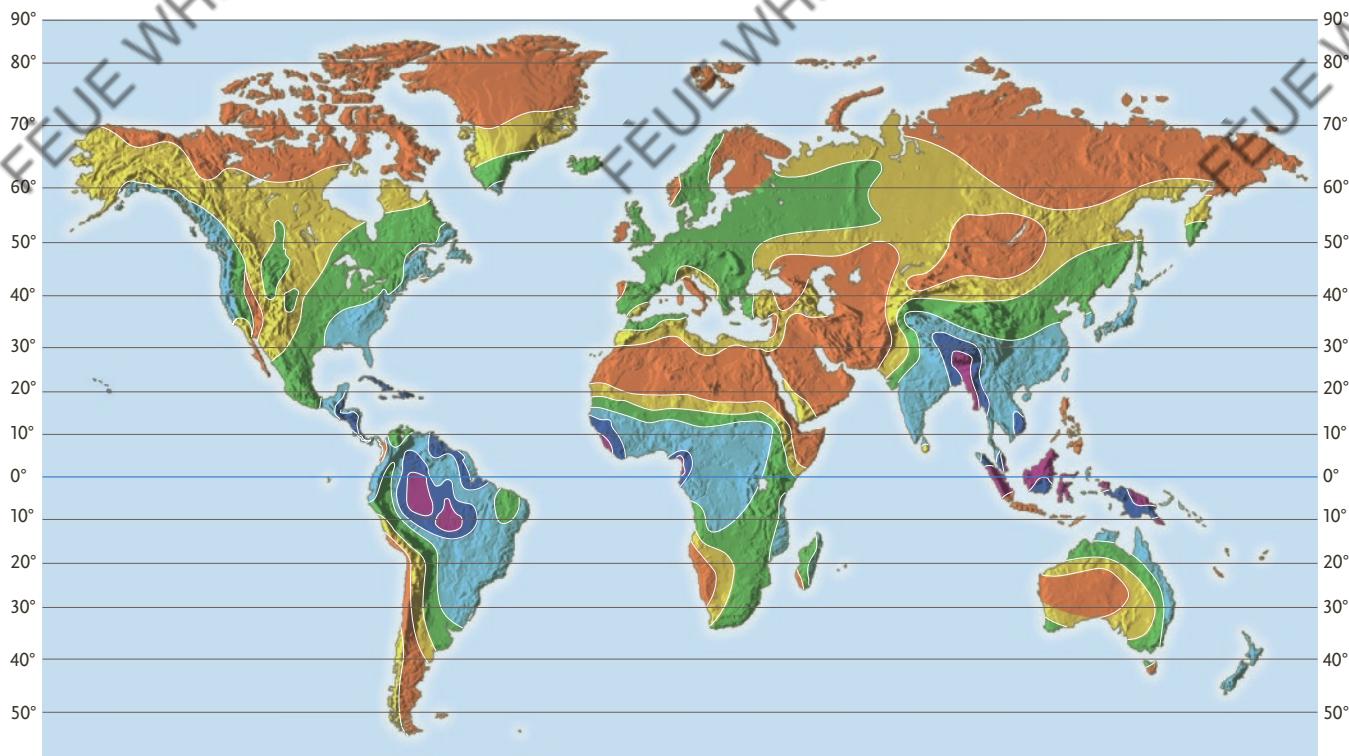


FIGURE 14
Precipitation

KEY					
Precipitation (cm/yr)					
Under 25	50 to 100	200 to 250			
25 to 50	100 to 200	Over 250			

tions with the same temperature. Figure 13 shows a weather map of the world indicating the average July temperatures. The isotherms are the curves that separate the colored bands.

In weather maps of atmospheric pressure at a given time as a function of longitude and latitude, the level curves are called **isobars** and join locations with the same pressure. (See Exercise 34.) Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure, and are strongest where the isobars are tightly packed.

A contour map of world-wide precipitation is shown in Figure 14. Here the level curves are not labeled but they separate the colored regions and the amount of precipitation in each region is indicated in the color key.

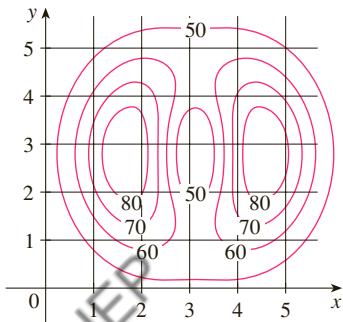


FIGURE 15

EXAMPLE 9 A contour map for a function f is shown in Figure 15. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

SOLUTION The point $(1, 3)$ lies partway between the level curves with z -values 70 and 80. We estimate that

$$f(1, 3) \approx 73$$

Similarly, we estimate that

$$f(4, 5) \approx 56$$

EXAMPLE 10 Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

SOLUTION The level curves are

$$6 - 3x - 2y = k \quad \text{or} \quad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6$, and 12 are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They are sketched in Figure 16. The level curves are equally spaced parallel lines because the graph of f is a plane (see Figure 6).

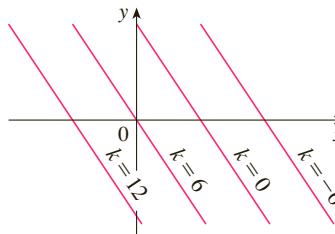


FIGURE 16
Contour map of
 $f(x, y) = 6 - 3x - 2y$

EXAMPLE 11 Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \quad \text{for } k = 0, 1, 2, 3$$

SOLUTION The level curves are

$$\sqrt{9 - x^2 - y^2} = k \quad \text{or} \quad x^2 + y^2 = 9 - k^2$$

This is a family of concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases

$k = 0, 1, 2, 3$ are shown in Figure 17. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7. (See TEC Visual 14.1A.)

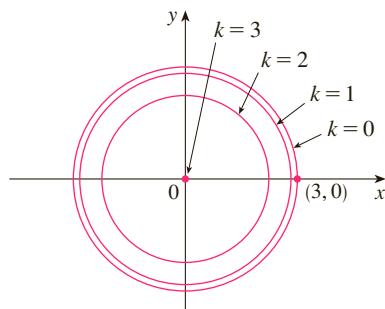


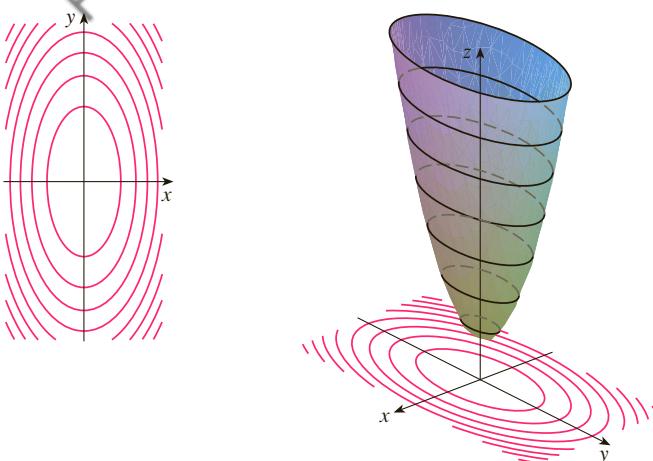
FIGURE 17
Contour map of
 $g(x, y) = \sqrt{9 - x^2 - y^2}$

EXAMPLE 12 Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

SOLUTION The level curves are

$$4x^2 + y^2 + 1 = k \quad \text{or} \quad \frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{k-1} = 1$$

which, for $k > 1$, describes a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k-1}$ and $\sqrt{k-1}$. Figure 18(a) shows a contour map of h drawn by a computer. Figure 18(b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure 18 how the graph of h is put together from the level curves.



TEC Visual 14.1B demonstrates the connection between surfaces and their contour maps.

FIGURE 18
The graph of $h(x, y) = 4x^2 + y^2 + 1$
is formed by lifting the level curves.

(a) Contour map

(b) Horizontal traces are raised level curves

EXAMPLE 13 Plot level curves for the Cobb-Douglas production function of Example 3.

SOLUTION In Figure 19 we use a computer to draw a contour plot for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

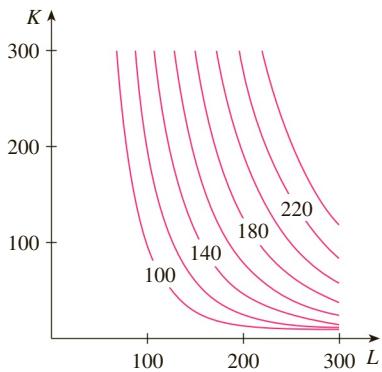


FIGURE 19

Level curves are labeled with the value of the production P . For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of $P = 140$. We see that, for a fixed value of P , as L increases K decreases, and vice versa. ■

For some purposes, a contour map is more useful than a graph. That is certainly true in Example 13. (Compare Figure 19 with Figure 8.) It is also true in estimating function values, as in Example 9.

Figure 20 shows some computer-generated level curves together with the corresponding computer-generated graphs. Notice that the level curves in part (c) crowd together near the origin. That corresponds to the fact that the graph in part (d) is very steep near the origin.

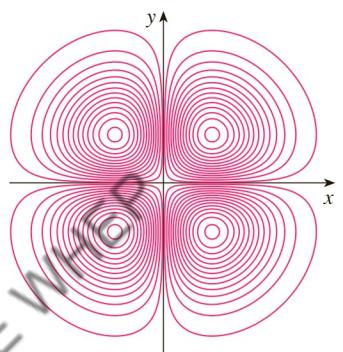
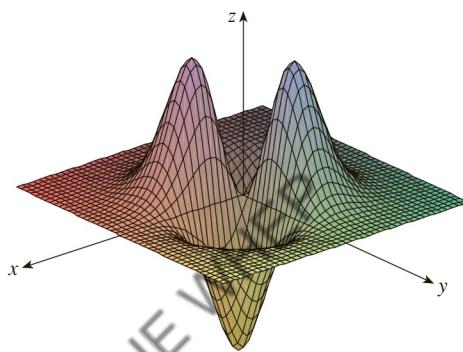
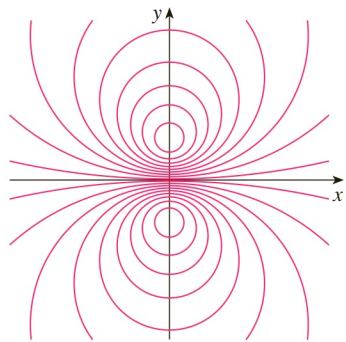
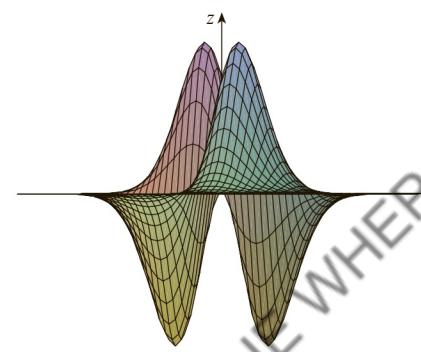
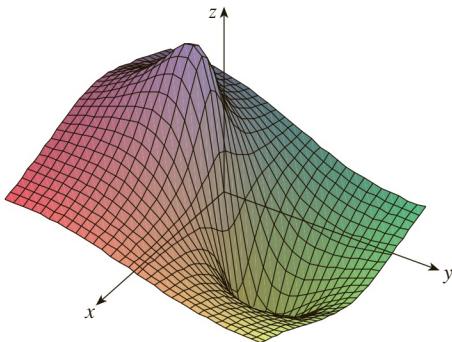
(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$ (b) Two views of $f(x, y) = -xye^{-x^2-y^2}$ (c) Level curves of $f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$ (d) $f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$

FIGURE 20

■ Functions of Three or More Variables

A **function of three variables**, f , is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t , so we could write $T = f(x, y, t)$.

EXAMPLE 14 Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

SOLUTION The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$. ■

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

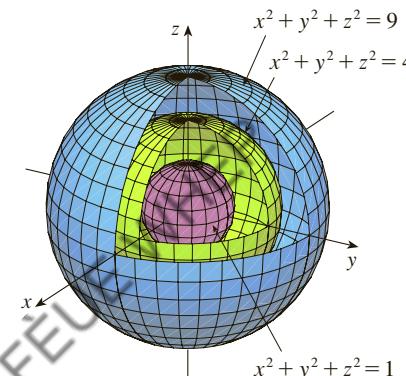


FIGURE 21

EXAMPLE 15 Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius \sqrt{k} . (See Figure 21.) Thus, as (x, y, z) varies over any sphere with center O , the value of $f(x, y, z)$ remains fixed. ■

Functions of any number of variables can be considered. A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \dots, x_n)$ to an n -tuple (x_1, x_2, \dots, x_n) of real numbers. We denote by \mathbb{R}^n the set of all such n -tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the i th ingredient, and x_i units of the i th ingredient are used, then the total cost C of the ingredients is a function of the n variables x_1, x_2, \dots, x_n :

3

$$C = f(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

The function f is a real-valued function whose domain is a subset of \mathbb{R}^n . Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \dots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \dots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors \mathbf{c} and \mathbf{x} in V_n .

In view of the one-to-one correspondence between points (x_1, x_2, \dots, x_n) in \mathbb{R}^n and their position vectors $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ in V_n , we have three ways of looking at a function f defined on a subset of \mathbb{R}^n :

1. As a function of n real variables x_1, x_2, \dots, x_n
2. As a function of a single point variable (x_1, x_2, \dots, x_n)
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$

We will see that all three points of view are useful.

14.1 EXERCISES

- In Example 2 we considered the function $W = f(T, v)$, where W is the wind-chill index, T is the actual temperature, and v is the wind speed. A numerical representation is given in Table 1 on page 929.
 - What is the value of $f(-15, 40)$? What is its meaning?
 - Describe in words the meaning of the question “For what value of v is $f(-20, v) = -30$?” Then answer the question.
 - Describe in words the meaning of the question “For what value of T is $f(T, 20) = -49$?” Then answer the question.
 - What is the meaning of the function $W = f(-5, v)$? Describe the behavior of this function.
 - What is the meaning of the function $W = f(T, 50)$? Describe the behavior of this function.
- The *temperature-humidity index* I (or humidex, for short) is the perceived air temperature when the actual temperature is T and the relative humidity is h , so we can write $I = f(T, h)$. The following table of values of I is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

Table 3 Apparent temperature as a function of temperature and humidity

		Relative humidity (%)					
		20	30	40	50	60	70
Actual temperature (°F)	80	77	78	79	81	82	83
	85	82	84	86	88	90	93
	90	87	90	93	96	100	106
	95	93	96	101	107	114	124
	100	99	104	110	120	132	144

- What is the value of $f(95, 70)$? What is its meaning?
- For what value of h is $f(90, h) = 100$?
- For what value of T is $f(T, 50) = 88$?
- What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of h .
- A manufacturer has modeled its yearly production function P (the monetary value of its entire production in millions of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Find $P(120, 20)$ and interpret it.

- Verify for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

discussed in Example 3 that the production will be doubled if both the amount of labor and the amount of capital are doubled. Determine whether this is also true for the general production function

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

- A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet.

- Find $f(160, 70)$ and interpret it.
- What is your own surface area?

- The wind-chill index W discussed in Example 2 has been modeled by the following function:

$$W(T, v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

Check to see how closely this model agrees with the values in Table 1 for a few values of T and v .

- The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in Table 4.
 - What is the value of $f(40, 15)$? What is its meaning?
 - What is the meaning of the function $h = f(30, t)$? Describe the behavior of this function.
 - What is the meaning of the function $h = f(v, 30)$? Describe the behavior of this function.

Table 4

		Duration (hours)						
		5	10	15	20	30	40	50
Wind speed (knots)	10	2	2	2	2	2	2	2
	15	4	4	5	5	5	5	5
	20	5	7	8	8	9	9	9
	30	9	13	16	17	18	19	19
	40	14	21	25	28	31	33	33
	50	19	29	36	40	45	48	50
	60	24	37	47	54	62	67	69

- A company makes three sizes of cardboard boxes: small, medium, and large. It costs \$2.50 to make a small box,

\$4.00 for a medium box, and \$4.50 for a large box. Fixed costs are \$8000.

- (a) Express the cost of making x small boxes, y medium boxes, and z large boxes as a function of three variables: $C = f(x, y, z)$.

- (b) Find $f(3000, 5000, 4000)$ and interpret it.
(c) What is the domain of f ?

9. Let $g(x, y) = \cos(x + 2y)$.

- (a) Evaluate $g(2, -1)$.
(b) Find the domain of g .
(c) Find the range of g .

10. Let $F(x, y) = 1 + \sqrt{4 - y^2}$.

- (a) Evaluate $F(3, 1)$.
(b) Find and sketch the domain of F .
(c) Find the range of F .

11. Let $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} + \ln(4 - x^2 - y^2 - z^2)$.

- (a) Evaluate $f(1, 1, 1)$.
(b) Find and describe the domain of f .

12. Let $g(x, y, z) = x^3 y^2 z \sqrt{10 - x - y - z}$.

- (a) Evaluate $g(1, 2, 3)$.
(b) Find and describe the domain of g .

13–22 Find and sketch the domain of the function.

13. $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$

14. $f(x, y) = \sqrt[4]{x - 3y}$

15. $f(x, y) = \ln(9 - x^2 - 9y^2)$

16. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

17. $g(x, y) = \frac{x - y}{x + y}$

18. $g(x, y) = \frac{\ln(2 - x)}{1 - x^2 - y^2}$

19. $f(x, y) = \frac{\sqrt{y - x^2}}{1 - x^2}$

20. $f(x, y) = \sin^{-1}(x + y)$

21. $f(x, y, z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$

22. $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$

23–31 Sketch the graph of the function.

23. $f(x, y) = y$

24. $f(x, y) = x^2$

25. $f(x, y) = 10 - 4x - 5y$

26. $f(x, y) = \cos y$

27. $f(x, y) = \sin x$

28. $f(x, y) = 2 - x^2 - y^2$

29. $f(x, y) = x^2 + 4y^2 + 1$

30. $f(x, y) = \sqrt{4x^2 + y^2}$

31. $f(x, y) = \sqrt{4 - 4x^2 - y^2}$

32. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = \frac{1}{1 + x^2 + y^2}$

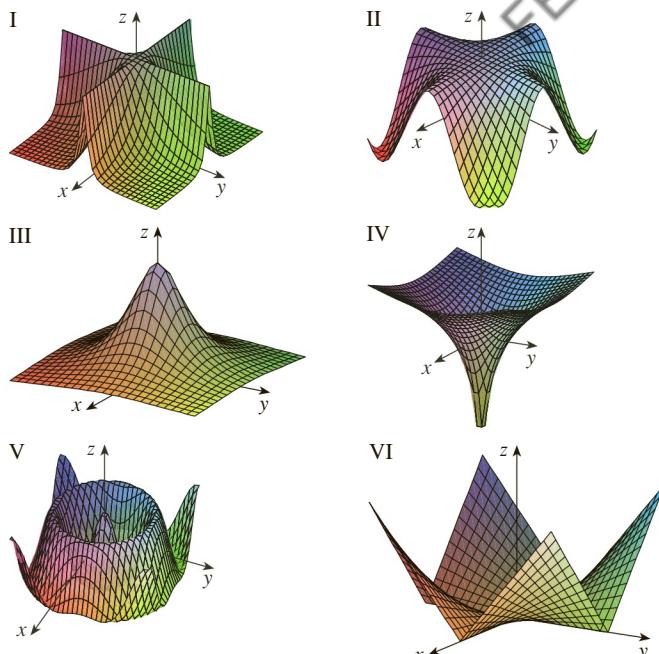
(b) $f(x, y) = \frac{1}{1 + x^2 y^2}$

(c) $f(x, y) = \ln(x^2 + y^2)$

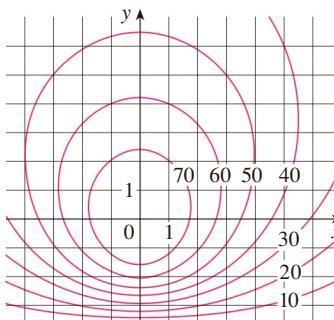
(d) $f(x, y) = \cos \sqrt{x^2 + y^2}$

(e) $f(x, y) = |xy|$

(f) $f(x, y) = \cos(xy)$

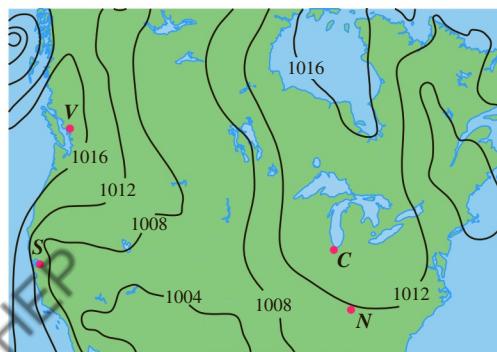


33. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?

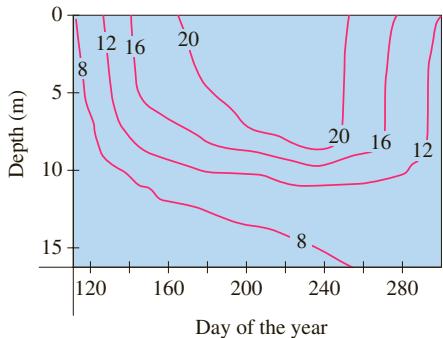


34. Shown is a contour map of atmospheric pressure in North America on August 12, 2008. On the level curves (called isobars) the pressure is indicated in millibars (mb).

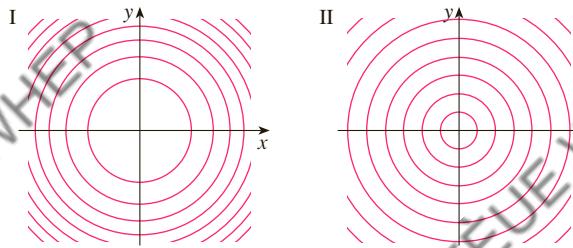
- (a) Estimate the pressure at C (Chicago), N (Nashville), S (San Francisco), and V (Vancouver).
(b) At which of these locations were the winds strongest?



- 35.** Level curves (isothermals) are shown for the typical water temperature (in °C) in Long Lake (Minnesota) as a function of depth and time of year. Estimate the temperature in the lake on June 9 (day 160) at a depth of 10 m and on June 29 (day 180) at a depth of 5 m.

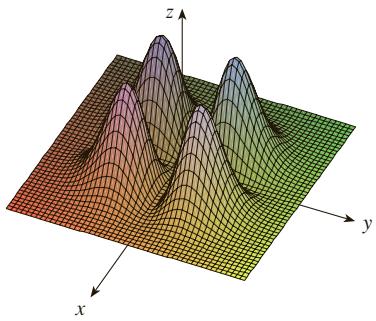


- 36.** Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?



- 37.** Locate the points A and B on the map of Lonesome Mountain (Figure 12). How would you describe the terrain near A ? Near B ?

- 38.** Make a rough sketch of a contour map for the function whose graph is shown.



- 39.** The *body mass index* (BMI) of a person is defined by

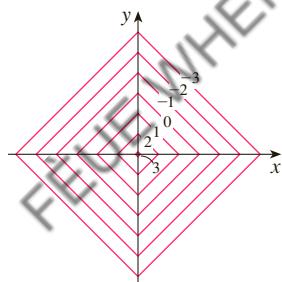
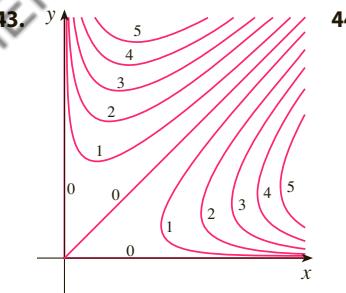
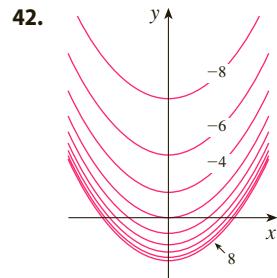
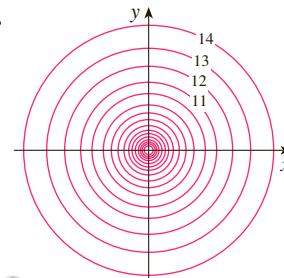
$$B(m, h) = \frac{m}{h^2}$$

where m is the person's mass (in kilograms) and h is the height (in meters). Draw the level curves $B(m, h) = 18.5$, $B(m, h) = 25$, $B(m, h) = 30$, and $B(m, h) = 40$. A rough guideline is that a person is underweight if the BMI is less than

18.5; optimal if the BMI lies between 18.5 and 25; overweight if the BMI lies between 25 and 30; and obese if the BMI exceeds 30. Shade the region corresponding to optimal BMI. Does someone who weighs 62 kg and is 152 cm tall fall into this category?

- 40.** The body mass index is defined in Exercise 39. Draw the level curve of this function corresponding to someone who is 200 cm tall and weighs 80 kg. Find the weights and heights of two other people with that same level curve.

- 41–44** A contour map of a function is shown. Use it to make a rough sketch of the graph of f .



- 45–52** Draw a contour map of the function showing several level curves.

45. $f(x, y) = x^2 - y^2$

46. $f(x, y) = xy$

47. $f(x, y) = \sqrt{x} + y$

48. $f(x, y) = \ln(x^2 + 4y^2)$

49. $f(x, y) = ye^x$

50. $f(x, y) = y - \arctan x$

51. $f(x, y) = \sqrt[3]{x^2 + y^2}$

52. $f(x, y) = y/(x^2 + y^2)$

- 53–54** Sketch both a contour map and a graph of the function and compare them.

53. $f(x, y) = x^2 + 9y^2$

54. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

- 55.** A thin metal plate, located in the xy -plane, has temperature $T(x, y)$ at the point (x, y) . Sketch some level curves (isothermals) if the temperature function is given by

$$T(x, y) = \frac{100}{1 + x^2 + 2y^2}$$

- 56.** If $V(x, y)$ is the electric potential at a point (x, y) in the xy -plane, then the level curves of V are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where c is a positive constant.

57–60 Use a computer to graph the function using various domains and viewpoints. Get a printout of one that, in your opinion, gives a good view. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

57. $f(x, y) = xy^2 - x^3$ (monkey saddle)

58. $f(x, y) = xy^3 - yx^3$ (dog saddle)

59. $f(x, y) = e^{-(x^2+y^2)/3}(\sin(x^2) + \cos(y^2))$

60. $f(x, y) = \cos x \cos y$

61–66 Match the function (a) with its graph (labeled A–F below) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

61. $z = \sin(xy)$

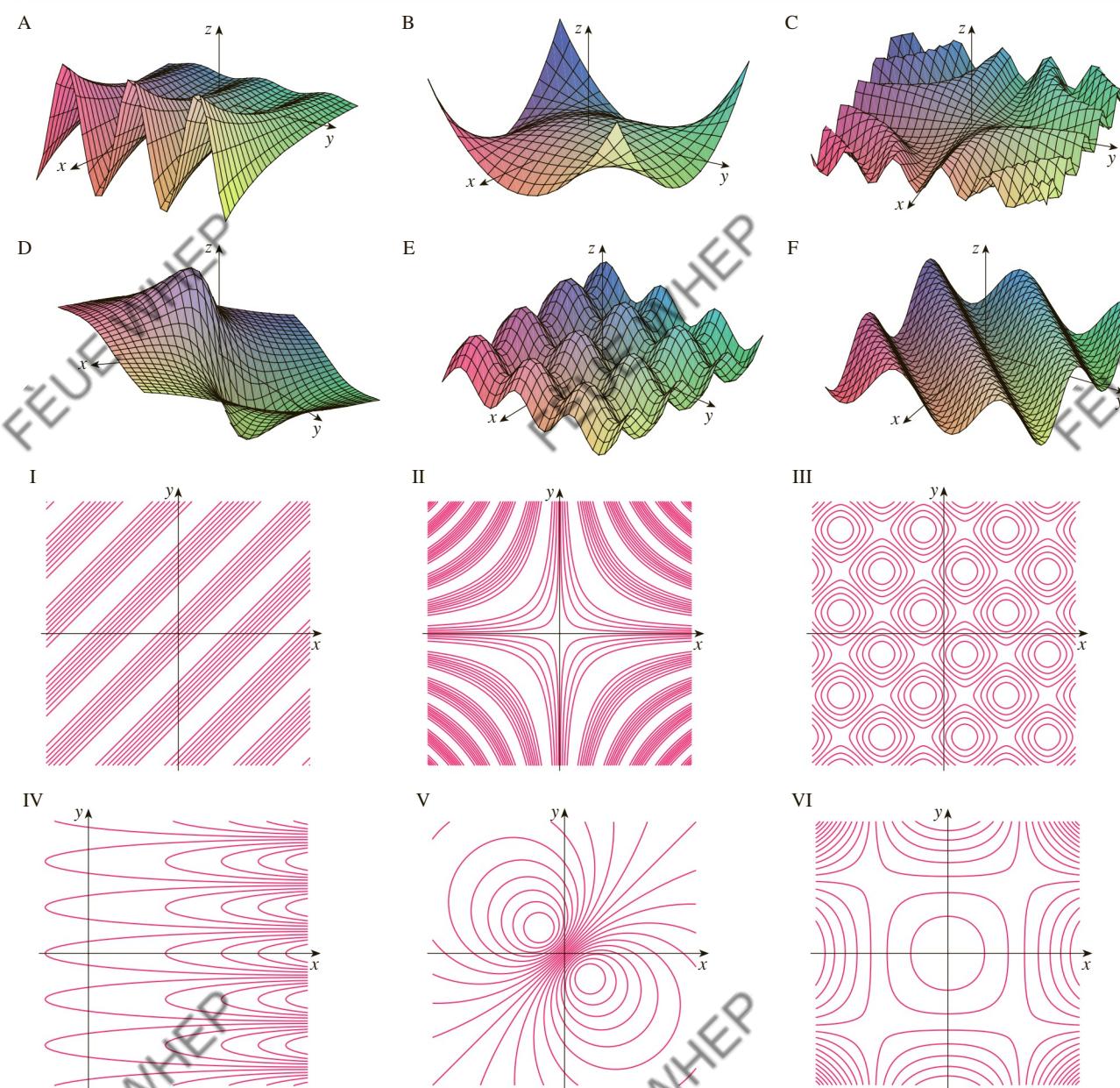
62. $z = e^x \cos y$

63. $z = \sin(x - y)$

64. $z = \sin x - \sin y$

65. $z = (1 - x^2)(1 - y^2)$

66. $z = \frac{x - y}{1 + x^2 + y^2}$



67–70 Describe the level surfaces of the function.

67. $f(x, y, z) = x + 3y + 5z$
68. $f(x, y, z) = x^2 + 3y^2 + 5z^2$
69. $f(x, y, z) = y^2 + z^2$
70. $f(x, y, z) = x^2 - y^2 - z^2$

71–72 Describe how the graph of g is obtained from the graph of f .

71. (a) $g(x, y) = f(x, y) + 2$
 (b) $g(x, y) = 2f(x, y)$
 (c) $g(x, y) = -f(x, y)$
 (d) $g(x, y) = 2 - f(x, y)$
72. (a) $g(x, y) = f(x - 2, y)$
 (b) $g(x, y) = f(x, y + 2)$
 (c) $g(x, y) = f(x + 3, y - 4)$

73–74 Use a computer to graph the function using various domains and viewpoints. Get a printout that gives a good view of the “peaks and valleys.” Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be “local maximum points”? What about “local minimum points”?

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$
74. $f(x, y) = xye^{-x^2-y^2}$

75–76 Graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both x and y become large? What happens as (x, y) approaches the origin?

$$75. f(x, y) = \frac{x + y}{x^2 + y^2} \quad 76. f(x, y) = \frac{xy}{x^2 + y^2}$$

77. Investigate the family of functions $f(x, y) = e^{cx^2+y^2}$. How does the shape of the graph depend on c ?

78. Use a computer to investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2-y^2}$$

How does the shape of the graph depend on the numbers a and b ?

79. Use a computer to investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadric surface to another.

80. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f(x, y) = e^{\sqrt{x^2+y^2}}$$

$$f(x, y) = \ln\sqrt{x^2 + y^2}$$

$$f(x, y) = \sin(\sqrt{x^2 + y^2})$$

$$\text{and} \quad f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if g is a function of one variable, how is the graph of

$$f(x, y) = g(\sqrt{x^2 + y^2})$$

obtained from the graph of g ?

81. (a) Show that, by taking logarithms, the general Cobb-Douglas function $P = bL^\alpha K^{1-\alpha}$ can be expressed as

$$\ln \frac{P}{K} = \ln b + \alpha \ln \frac{L}{K}$$

- (b) If we let $x = \ln(L/K)$ and $y = \ln(P/K)$, the equation in part (a) becomes the linear equation $y = \alpha x + \ln b$. Use Table 2 (in Example 3) to make a table of values of $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. Then use a graphing calculator or computer to find the least squares regression line through the points $(\ln(L/K), \ln(P/K))$.
- (c) Deduce that the Cobb-Douglas production function is $P = 1.01L^{0.75}K^{0.25}$.

14.2 Limits and Continuity

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

Table 1 Values of $f(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2 Values of $g(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \quad \text{and} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

In general, we use the notation

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f . In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b) , but not equal to (a, b) . A more precise definition follows.

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b) . Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b) is L** and we write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = L \quad \text{and} \quad f(x, y) \rightarrow L \quad \text{as} \quad (x, y) \rightarrow (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L , and $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between the point (x, y) and the point (a, b) . Thus Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by

making the distance from (x, y) to (a, b) sufficiently small (but not 0). Figure 1 illustrates Definition 1 by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L , then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.

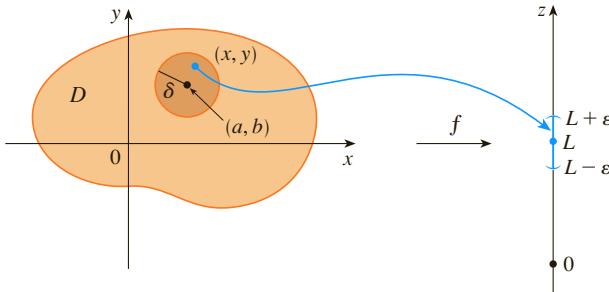


FIGURE 1

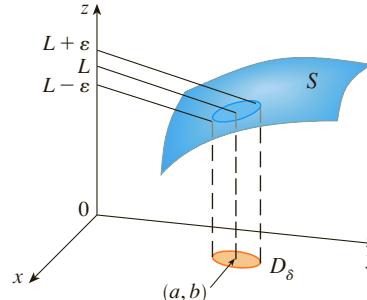


FIGURE 2

Another illustration of Definition 1 is given in Figure 2 where the surface S is the graph of f . If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

For functions of a single variable, when we let x approach a , there are only two possible directions of approach, from the left or from the right. We recall from Chapter 1 that if $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f .

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b) . It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b) . Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a path C_2 , where $L_1 \neq L_2$, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

EXAMPLE 1 Show that $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x -axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \rightarrow 1 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y -axis by putting $x = 0$. Then $f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \rightarrow -1 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } y\text{-axis}$$

(See Figure 4.) Since f has two different limits along two different lines, the given limit

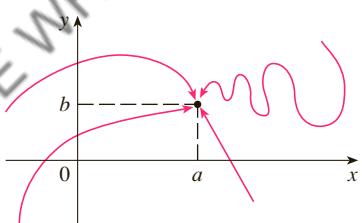


FIGURE 3

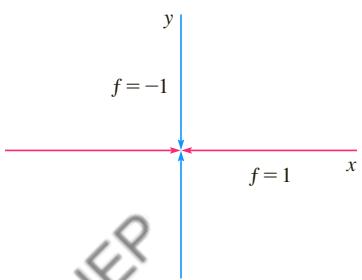


FIGURE 4

does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.) ■

EXAMPLE 2 If $f(x, y) = xy/(x^2 + y^2)$, does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach $(0, 0)$ along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$ along $y = x$

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist. ■

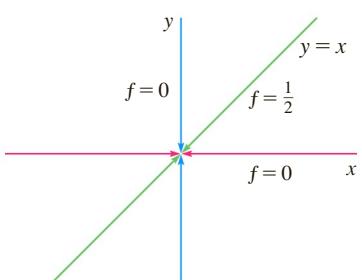


FIGURE 5

TEC In Visual 14.2 a rotating line on the surface in Figure 6 shows different limits at the origin from different directions.

Figure 6 sheds some light on Example 2. The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line except the origin.

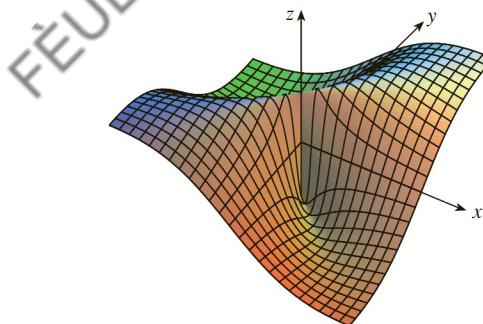


FIGURE 6

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

EXAMPLE 3 If $f(x, y) = \frac{xy^2}{x^2 + y^4}$, does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \rightarrow (0, 0)$ along any line through the origin. If the line is not the y -axis, then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

So $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along $y = mx$

We get the same result as $(x, y) \rightarrow (0, 0)$ along the line $x = 0$. Thus f has the same limiting value along every line through the origin. But that does not show that the given

Figure 7 shows the graph of the function in Example 3. Notice the ridge above the parabola $x = y^2$.

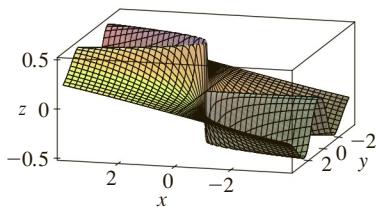


FIGURE 7

limit is 0, for if we now let $(x, y) \rightarrow (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

$$\text{so } f(x, y) \rightarrow \frac{1}{2} \quad \text{as } (x, y) \rightarrow (0, 0) \text{ along } x = y^2$$

Since different paths lead to different limiting values, the given limit does not exist. ■

Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 1.6 can be extended to functions of two variables: the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

$$\boxed{2} \quad \lim_{(x, y) \rightarrow (a, b)} x = a \quad \lim_{(x, y) \rightarrow (a, b)} y = b \quad \lim_{(x, y) \rightarrow (a, b)} c = c$$

The Squeeze Theorem also holds.

EXAMPLE 4 Find $\lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2}$ if it exists.

SOLUTION As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\text{that is, if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \frac{3x^2|y|}{x^2 + y^2} < \varepsilon$$

But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$, so $x^2/(x^2 + y^2) \leq 1$ and therefore

$$\boxed{3} \quad \frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition 1,

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0$$

■ Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous

function is $\lim_{x \rightarrow a} f(x) = f(a)$. Continuous functions of two variables are also defined by the direct substitution property.

4 Definition A function f of two variables is called **continuous at (a, b)** if

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on D** if f is continuous at every point (a, b) in D .

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains. Let's use this fact to give examples of continuous functions.

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The limits in (2) show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f , g , and h by multiplication and addition, it follows that *all polynomials are continuous on \mathbb{R}^2* . Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

EXAMPLE 5 Evaluate $\lim_{(x, y) \rightarrow (1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

SOLUTION Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x, y) \rightarrow (1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11 \quad \blacksquare$$

EXAMPLE 6 Where is the function $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at $(0, 0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. ■

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x,y) \rightarrow (0,0)} g(x, y)$ does not exist (see Example 1). ■

Figure 8 shows the graph of the continuous function in Example 8.

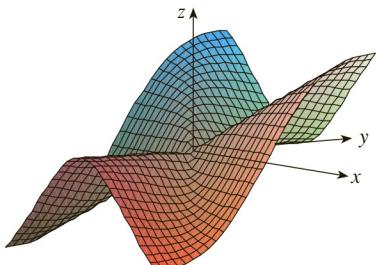


FIGURE 8

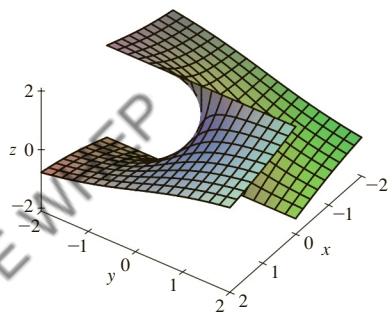


FIGURE 9

The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} \frac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 4, we have

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on \mathbb{R}^2 . ■

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f , then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

EXAMPLE 9 Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where $x = 0$. The graph in Figure 9 shows the break in the graph of h above the y -axis. ■

■ Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z) \rightarrow (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f . Because the distance between two points (x, y, z) and (a, b, c) in \mathbb{R}^3 is given by $\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$, we can write the precise definition as follows: for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if (x, y, z) is in the domain of f and $0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta$

then $|f(x, y, z) - L| < \varepsilon$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \rightarrow (a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in \mathbb{R}^3 except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

If we use the vector notation introduced at the end of Section 14.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

5 If f is defined on a subset D of \mathbb{R}^n , then $\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } \mathbf{x} \in D \text{ and } 0 < |\mathbf{x} - \mathbf{a}| < \delta \text{ then } |f(\mathbf{x}) - L| < \varepsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and (5) is just the definition of a limit for functions of a single variable. For the case $n = 2$, we have $\mathbf{x} = \langle x, y \rangle$, $\mathbf{a} = \langle a, b \rangle$, and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x - a)^2 + (y - b)^2}$, so (5) becomes Definition 1. If $n = 3$, then $\mathbf{x} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a, b, c \rangle$, and (5) becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

14.2 EXERCISES

- Suppose that $\lim_{(x, y) \rightarrow (3, 1)} f(x, y) = 6$. What can you say about the value of $f(3, 1)$? What if f is continuous?
- Explain why each function is continuous or discontinuous.
 - The outdoor temperature as a function of longitude, latitude, and time
 - Elevation (height above sea level) as a function of longitude, latitude, and time
 - The cost of a taxi ride as a function of distance traveled and time

3–4 Use a table of numerical values of $f(x, y)$ for (x, y) near the origin to make a conjecture about the value of the limit of $f(x, y)$ as $(x, y) \rightarrow (0, 0)$. Then explain why your guess is correct.

$$3. f(x, y) = \frac{x^2 y^3 + x^3 y^2 - 5}{2 - xy}$$

$$4. f(x, y) = \frac{2xy}{x^2 + 2y^2}$$

5–22 Find the limit, if it exists, or show that the limit does not exist.

$$5. \lim_{(x, y) \rightarrow (3, 2)} (x^2 y^3 - 4y^2)$$

$$6. \lim_{(x, y) \rightarrow (2, -1)} \frac{x^2 y + xy^2}{x^2 - y^2}$$

$$7. \lim_{(x, y) \rightarrow (\pi, \pi/2)} y \sin(x - y)$$

$$8. \lim_{(x, y) \rightarrow (3, 2)} e^{\sqrt{2x-y}}$$

$$9. \lim_{(x, y) \rightarrow (0, 0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$$

$$10. \lim_{(x, y) \rightarrow (0, 0)} \frac{5y^4 \cos^2 x}{x^4 + y^4}$$

$$11. \lim_{(x, y) \rightarrow (0, 0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$$

$$12. \lim_{(x, y) \rightarrow (1, 0)} \frac{xy - y}{(x - 1)^2 + y^2}$$

$$13. \lim_{(x, y) \rightarrow (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

$$14. \lim_{(x, y) \rightarrow (0, 0)} \frac{x^3 - y^3}{x^2 + xy + y^2}$$

$$15. \lim_{(x, y) \rightarrow (0, 0)} \frac{xy^2 \cos y}{x^2 + y^4}$$

$$16. \lim_{(x, y) \rightarrow (0, 0)} \frac{xy^4}{x^4 + y^4}$$

$$17. \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

$$18. \lim_{(x, y) \rightarrow (0, 0)} \frac{xy^4}{x^2 + y^8}$$

$$19. \lim_{(x, y, z) \rightarrow (\pi, 0, 1/3)} e^{y^2} \tan(xz)$$

$$20. \lim_{(x, y, z) \rightarrow (0, 0, 0)} \frac{xy + yz}{x^2 + y^2 + z^2}$$

21. $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$

22. $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$

23–24 Use a computer graph of the function to explain why the limit does not exist.

23. $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$

24. $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^6}$

25–26 Find $h(x, y) = g(f(x, y))$ and the set of points at which h is continuous.

25. $g(t) = t^2 + \sqrt{t}, \quad f(x, y) = 2x + 3y - 6$

26. $g(t) = t + \ln t, \quad f(x, y) = \frac{1 - xy}{1 + x^2y^2}$

27–28 Graph the function and observe where it is discontinuous. Then use the formula to explain what you have observed.

27. $f(x, y) = e^{1/(x-y)}$

28. $f(x, y) = \frac{1}{1 - x^2 - y^2}$

29–38 Determine the set of points at which the function is continuous.

29. $F(x, y) = \frac{xy}{1 + e^{x-y}}$

30. $F(x, y) = \cos \sqrt{1 + x - y}$

31. $F(x, y) = \frac{1 + x^2 + y^2}{1 - x^2 - y^2}$

32. $H(x, y) = \frac{e^x + e^y}{e^{xy} - 1}$

33. $G(x, y) = \sqrt{x} + \sqrt{1 - x^2 - y^2}$

34. $G(x, y) = \ln(1 + x - y)$

35. $f(x, y, z) = \arcsin(x^2 + y^2 + z^2)$

36. $f(x, y, z) = \sqrt{y - x^2} \ln z$

37. $f(x, y) = \begin{cases} \frac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

38. $f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

39–41 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geq 0$, note that $r \rightarrow 0^+$ as $(x, y) \rightarrow (0, 0)$.]

39. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$

40. $\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \ln(x^2 + y^2)$

41. $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$

42. At the beginning of this section we considered the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and guessed on the basis of numerical evidence that $f(x, y) \rightarrow 1$ as $(x, y) \rightarrow (0, 0)$. Use polar coordinates to confirm the value of the limit. Then graph the function.

43. Graph and discuss the continuity of the function

$$f(x, y) = \begin{cases} \frac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

44. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 0 \quad \text{or} \quad y \geq x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

- (a) Show that $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^a$ with $0 < a < 4$.
- (b) Despite part (a), show that f is discontinuous at $(0, 0)$.
- (c) Show that f is discontinuous on two entire curves.

45. Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on \mathbb{R}^n . [Hint: Consider $|\mathbf{x} - \mathbf{a}|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.]

46. If $\mathbf{c} \in V_n$, show that the function f given by $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is continuous on \mathbb{R}^n .

14.3 Partial Derivatives

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined