we get

$$f(a,b) = a_{00}, \quad f_x(a,b) = a_{10}, \quad f_y(a,b) = a_{01}, \tag{2.2}$$
$$f_{xx}(a,b) = 2a_{20}, \quad f_{xy}(a,b) = a_{11}, \quad \text{etc.}$$

[Remember that $f_x(a,b)$ means that we are to find the partial derivative of f with respect to x and then put $x = a$, $y = b$, and similarly for the other derivatives.] Substituting the values for the coefficients into (2.1), we find

(2.3)
$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2!}[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2] \cdots .$$

This can be written in a simpler form if we put $x - a = h$ and $y - b = k$. Then the second-order terms (for example) become

$$\frac{1}{2!}[f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2]. \tag{2.4}$$

We can write this in the form

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) \tag{2.5}$$

if we understand that the parenthesis is to be squared and then a term of the form $h(\partial/\partial x)k(\partial/\partial y)f(a,b)$ is to mean $hkf_{xy}(a,b)$. It can be shown (Problem 7) that the third-order terms can be written in this notation as

$$\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(a,b) = \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b) + \cdots] \tag{2.6}$$

and so on for terms of any order. Thus we can write the series (2.3) in the form

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a,b). \tag{2.7}$$

The numbers appearing in the nth order terms are the familiar binomial coefficients [in the expansion of $(p+q)^n$] divided by $(n!)$. (See Chapter 1, Section 13C.)

▶ PROBLEMS, SECTION 2

Find the two-variable Maclaurin series for the following functions.

1. $\cos x \sinh y$ **2.** $\cos(x+y)$ **3.** $\dfrac{\ln(1+x)}{1+y}$

4. e^{xy} **5.** $\sqrt{1+xy}$ **6.** e^{x+y}

7. Verify the coefficients of the third-order terms [(2.6) or $n = 3$ in (2.7)] of the power series for $f(x,y)$ by finding the third-order partial derivatives in (2.1) and substituting $x = a$, $y = b$.

8. Find the two-variable Maclaurin series for $e^x \cos y$ and $e^x \sin y$ by finding the series for $e^z = e^{x+iy}$ and taking real and imaginary parts. (See Chapter 2.)

3. TOTAL DIFFERENTIALS

The graph (Figure 3.1) of the equation $y = f(x)$ is a curve in the (x, y) plane and

$$y' = \frac{dy}{dx} = \frac{d}{dx}f(x) \tag{3.1}$$

is the slope of the tangent to the curve at the point (x, y). In calculus, we use Δx to mean a change in x, and Δy means the corresponding change in y (see Figure 3.1). By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{3.2}$$

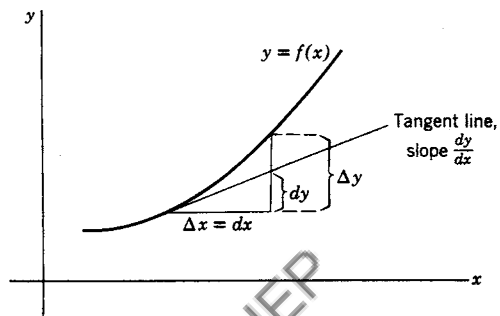


Figure 3.1

We shall now define the differential dx of the independent variable as

$$dx = \Delta x. \tag{3.3}$$

However, dy is not the same as Δy. From Figure 3.1 and equation (3.1), we can see that Δy is the change in y along the curve, but $dy = y'dx$ is the change in y along the tangent line. We say that dy is the tangent approximation (or linear approximation) to Δy.

Example. If $y = f(t)$ represents the distance a particle has gone as a function of t, then dy/dt is the speed. The actual distance the particle has gone between time t and time $t + dt$ is Δy. The tangent approximation $dy = (dy/dt)dt$ is the distance it would have gone if it had continued with the same speed dy/dt which it had at time t.

You can see from the graph (Figure 3.1) that dy is a good approximation to Δy if dx is small. We can say this more exactly using (3.2). Saying that dy/dx is the limit of $\Delta y/\Delta x$ as $\Delta x \to 0$ means that the difference $\Delta y/\Delta x - dy/dx \to 0$ as $\Delta x \to 0$. Let us call this difference ϵ; then we can say

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \epsilon, \qquad \text{where } \epsilon \to 0 \quad \text{as} \quad \Delta x \to 0, \tag{3.4}$$

or since $dx = \Delta x$

$$\Delta y = (y' + \epsilon)dx, \qquad \text{where } \epsilon \to 0 \quad \text{as} \quad \Delta x \to 0. \tag{3.5}$$

The differential $dy = y'dx$ is called the principal part of Δy; since ϵ is small for small dx, you can see from (3.5) that dy is then a good approximation to Δy.

In our example, suppose $y = t^2$, $t = 1$, $dt = 0.1$. Then

$$\begin{aligned}
\Delta y &= (1.1)^2 - 1^2 = 0.21,\\
dy &= \frac{dy}{dt}dt = 2 \cdot 1 \cdot (0.1) = 0.2,\\
\epsilon &= \frac{\Delta y}{\Delta t} - \frac{dy}{dt} = 2.1 - 2 = 0.1,\\
\Delta y &= (y' + \epsilon)dt = (2 + 0.1)(0.1) = dy + \epsilon dt = 0.2 + 0.01.
\end{aligned}$$

Thus dy is a good approximation to Δy.

For a function of two variables, $z = f(x, y)$, we want to do something similar to this. We have said that this equation represents a surface and that the derivatives $\partial f/\partial x$, $\partial f/\partial y$, at a point, are the slopes of the two tangent lines to the surface in the x and y directions at that point. The symbols $\Delta x = dx$ and $\Delta y = dy$ represent changes in the independent variables x and y. The quantity Δz means the corresponding change in z along the surface. We define dz by the equation

(3.6)
$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

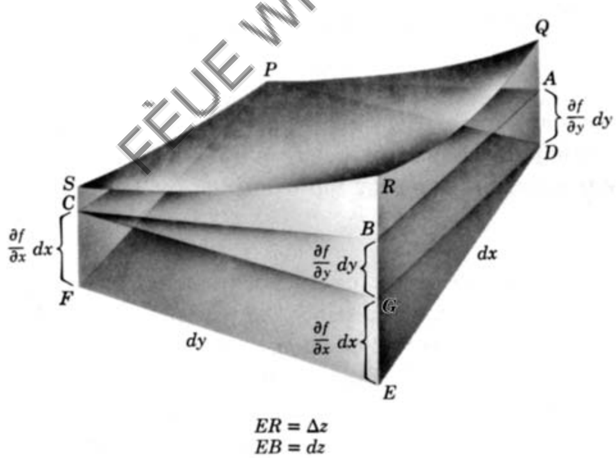


Figure 3.2

The differential dz is called the *total differential* of z. Let us consider the geometrical meaning of dz. Recall (Figure 3.1) that for $y = f(x)$, dy was the change in y along the tangent line; here we shall see that dz is the change in z along the tangent plane. In Figure 3.2, $PQRS$ is a surface, $PABC$ is the plane tangent to the surface at P, and $PDEF$ is a horizontal plane through P. Thus $PSCF$ is the plane $y =$ const. (through P), PS is the curve of intersection of this plane with the surface, and PC is the tangent line to this curve and so has the slope $\partial f/\partial x$; then (just as in Figure 3.1), if $PF = dx$, we have $CF = (\partial f/\partial x)\,dx$. Similarly, $PQAD$ is a plane $x =$ const., intersecting the surface in the curve PQ, whose tangent is PA; with

$PD = dy$, we have $DA = (\partial f/\partial y)\,dy$. From the figure, $GE = CF$, and $BG = AD$, so

$$EB = CF + DA = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = dz.$$

Thus, as we said, dz is the change in z along the tangent plane when x changes by dx and y by dy. In the figure, $ER = \Delta z$, the change in z along the surface.

From the geometry, we can reasonably expect dz to be a good approximation to Δz if dx and dy are small. However, we should like to say this more accurately in an equation corresponding to (3.5). We can do this if $\partial f/\partial x$ and $\partial f/\partial y$ are continuous functions. By definition

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{3.7}$$

By adding and subtracting a term, we get

$$\Delta z = f(x + \Delta x, y) - f(x, y) + f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y). \tag{3.8}$$

Recall from calculus that the mean value theorem (law of the mean) says that for a differentiable function $f(x)$,

$$f(x + \Delta x) - f(x) = (\Delta x) f'(x_1), \tag{3.9}$$

where x_1 is between x and $x + \Delta x$. Geometrically this says (Figure 3.3) that there is a tangent line somewhere between x and $x + \Delta x$ which has the same slope as the line AB. In the first two terms of the right side of (3.8), y is constant, and we can use (3.9) if we write $\partial f/\partial x$ for f'. In the last two terms of (3.8), x is constant and we can use an equation like (3.9) with y as the variable; y_1 will mean a value of y between y and $y + \Delta y$. Then (3.8) becomes

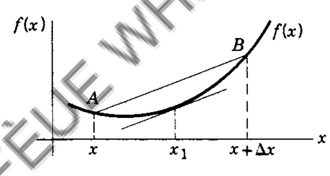


Figure 3.3

$$\Delta z = \frac{\partial f(x_1, y)}{\partial x}\,\Delta x + \frac{\partial f(x + \Delta x, y_1)}{\partial y}\,\Delta y. \tag{3.10}$$

If the partial derivatives of f are continuous, then their values in (3.10) at points *near* (x, y) differ from their values *at* (x, y) by quantities which approach zero as Δx and Δy approach zero. Let us call these quantities ϵ_1 and ϵ_2. Then we can write

(3.11)
$$\Delta z = \left(\frac{\partial f}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial f}{\partial y} + \epsilon_2\right)\Delta y = dz + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$
$$(\epsilon_1 \text{ and } \epsilon_2 \to 0 \text{ as } \Delta x \text{ and } \Delta y \to 0),$$

where $\partial f/\partial x$ and $\partial f/\partial y$ in (3.11) are evaluated at (x, y). Equation (3.11) [like (3.5) for the $y = f(x)$ case] tells us algebraically what we suspected from the geometry, that (if $\partial f/\partial x$ and $\partial f/\partial y$ are continuous) dz is a good approximation to Δz for small dx and dy. The differential dz is called the principal part of Δz.

Everything we have said about functions of two variables works just as well for functions of any number of variables. if $u = f(x, y, z, \cdots)$, then by definition

$$du = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz + \cdots \tag{3.12}$$

and du is a good approximation to Δu if the partial derivatives of f are continuous and dx, dy, dz, etc., are small.

PROBLEMS, SECTION 3

1. Consider a function $f(x, y)$ which can be expanded in a two-variable power series, (2.3) or (2.7). Let $x - a = h = \Delta x$, $y - b = k = \Delta y$; then $x = a + \Delta x$, $y = b + \Delta y$ so that $f(x, y)$ becomes $f(a + \Delta x, b + \Delta y)$. The change Δz in $z = f(x, y)$ when x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$ is then

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$

Use the series (2.7) to obtain (3.11) and to see explicitly what ϵ_1 and ϵ_2 are and that they approach zero as Δx and $\Delta y \to 0$.

4. APPROXIMATIONS USING DIFFERENTIALS

Let's consider some examples.

Example 1. Find approximately the value of

$$\frac{1}{\sqrt{0.25 - 10^{-20}}} - \frac{1}{\sqrt{0.25}}.$$

If $f(x) = 1/\sqrt{x}$, the desired difference is $\Delta f = f(0.25 - 10^{-20}) - f(0.25)$. But Δf is approximately $df = d(1/\sqrt{x})$ with $x = 0.25$ and $dx = -10^{-20}$.

$$d(1/\sqrt{x}) = (-1/2)x^{-3/2}dx = (-1/2)(0.25)^{-3/2}(-10^{-20}) = 4 \times 10^{-20}.$$

Now why not just use a computer or calculator for a problem like this? First note that we are subtracting two numbers which are almost equal to each other. If your calculator or computer isn't carrying enough digits, you may lose all accuracy in the subtraction (see Chapter 1, Section 15, Example 1). So it may take you more time to check on this and to type the problem into the computer than to find df which you can probably do in your head! However, there is another important point here which is shown in the next example. For theoretical purposes, we may want a *formula* rather than a numerical result.

Example 2. Show that when n is very large

$$\frac{1}{n^2} - \frac{1}{(n+1)^2} \cong \frac{2}{n^3}$$

($\cong$ means "approximately equal to"). If $f(x) = 1/x^2$, the desired difference is $\Delta f = f(n) - f(n+1)$. But Δf is approximately $df = d(1/x^2)$ with $x = n$ and $dx = -1$.

$$d\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}dx = -\frac{2}{n^3}(-1) = \frac{2}{n^3}.$$

(This result is used in obtaining the "correspondence principle" in quantum mechanics; see texts on quantum physics.) Also see Problem 17.

Example 3. The *reduced mass* μ of a system of two masses m_1 and m_2 is defined by $\mu^{-1} = m_1^{-1} + m_2^{-1}$. If m_1 is increased by 1%, what fractional change in m_2 leaves μ unchanged? Taking differentials of the equation and substituting $dm_1 = 0.01m_1$, we find

$$0 = -m_1^{-2}\,dm_1 - m_2^{-2}\,dm_2,$$

$$\frac{dm_2}{m_2^2} = -\frac{dm_1}{m_1^2} = -\frac{0.01m_1}{m_1^2} \quad \text{or} \quad \frac{dm_2}{m_2} = -0.01m_2/m_1.$$

For example, if $m_1 = m_2$, m_2 should be decreased by 1%; if $m_2 = 3m_1$, m_2 should be decreased by 3%; and so on.

Example 4. The electrical resistance R of a wire is proportional to its length and inversely proportional to the square of its radius, that is, $R = kl/r^2$. If the relative error in length measurement is 5% and the relative error in radius measurement is 10%, find the relative error in R in the worst possible case.

The relative error in l means the actual error in measuring l divided by the length measured. Since we might measure l either too large or too small, the relative error dl/l might be either $+0.05$ or -0.05 in the worst cases. Similarly $|dr/r|$ might be as large as 0.10. We want the largest value which $|dR/R|$ could have; we can find dR/R by differentiating $\ln R$. From $R = kl/r^2$ we find

$$\ln R = \ln k + \ln l - 2\ln r.$$

Then

$$\frac{dR}{R} = \frac{dl}{l} - 2\frac{dr}{r}.$$

In the worst case (that is, largest value of $|dR/R|$), dl/l and dr/r might have opposite signs so the two terms would add. Then we would have:

$$\text{Largest } \left|\frac{dR}{R}\right| = \left|\frac{dl}{l}\right| + 2\left|\frac{dr}{r}\right| = 0.05 + 2(0.10) = 0.25 \qquad \text{or} \qquad 25\%.$$

▶ **Example 5.** Estimate the change in

$$f(x) = \int_0^x \frac{\sin t}{t} dt$$

when x changes from $\pi/2$ to $(1+\epsilon)\pi/2$ where $\epsilon << 1/10$. Recall from calculus that $df/dx = (\sin x)/x$. Then we want $df = (df/dx)\,dx$ with $x = \pi/2$ and $dx = \epsilon\pi/2$. Thus

$$df = \frac{\sin \pi/2}{\pi/2}(\epsilon\pi/2) = \epsilon.$$

Note that the approximations we have been making correspond to using a Taylor series through the f' term. We can write Chapter 1 equation (12.8) with the replacements $x \to x + \Delta x$, $a \to x$, $x - a \to \Delta x$, to get

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + f''(x)(\Delta x)^2/2! + \cdots .$$

Dropping the $(\Delta x)^2$ and higher terms we have the approximation we have been using:

$$df \cong \Delta f = f(x + \Delta x) - f(x) \cong f'(x)\Delta x = f'(x)dx.$$

▶ PROBLEMS, SECTION 4

1. Use differentials to show that, for very large n, $\quad \dfrac{1}{(n+1)^3} - \dfrac{1}{n^3} \cong -\dfrac{3}{n^4}$.

2. Use differentials to show that, for large n and small a, $\quad \sqrt{n+a} - \sqrt{n} \cong \dfrac{a}{2\sqrt{n}}$.

Find the approximate value of $\sqrt{10^{26}+5} - \sqrt{10^{26}}$.

3. The thin lens formula is

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f},$$

where f is the focal length of the lens and o and i are the distances from the lens to the object and image. If $i = 15$ when $o = 10$, use differentials to find i when $o = 10.1$.

4. Do Problem 3 if $i = 12$ when $o = 18$, to find i if $o = 17.5$.

5. Let R be the resistance of $R_1 = 25$ ohms and $R_2 = 15$ ohms in parallel. (See Chapter 2, Problem 16.6.) If R_1 is changed to 25.1 ohms, find R_2 so that R is not changed.

6. The acceleration of gravity can be found from the length l and period T of a pendulum; the formula is $g = 4\pi^2 l/T^2$. Find the relative error in g in the worst case if the relative error in l is 5%, and the relative error in T is 2%.

7. Coulomb's law for the force between two charges q_1 and q_2 at distance r apart is $F = kq_1q_2/r^2$. Find the relative error in q_2 in the worst case if the relative error in q_1 is 3%; in r, 5%; and in F, 2%.

8. About how much (in percent) does an error of 1% in a and b affect a^2b^3?

9. Show that the approximate relative error $(df)/f$ of a product $f = gh$ is the sum of the approximate relative errors of the factors.

10. A force of 500 nt is measured with a possible error of 1 nt. Its component in a direction 60° away from its line of action is required, where the angle is subject to an error of 0.5°. What is (approximately) the largest possible error in the component?

11. Show how to make a quick estimate (to two decimal places) of $\sqrt{(4.98)^2-(3.03)^2}$ without using a computer or a calculator. *Hint:* Consider $f(x,y)=\sqrt{x^2-y^2}$.

12. As in Problem 11, estimate $\sqrt[3]{(2.05)^2+(1.98)^2}$.

13. Without using a computer or a calculator, estimate the change in length of a space diagonal of a box whose dimensions are changed from $200\times200\times100$ to $201\times202\times99$.

14. Estimate the change in

$$f(x)=\int_0^x \frac{e^{-t}}{t^2+0.51}\,dt$$

if x changes from 0.7 to 0.71.

15. For an ideal gas of N molecules, the number of molecules with speeds $\le v$ is given by the formula

$$n(v)=\frac{4a^3N}{\sqrt{\pi}}\int_0^v x^2e^{-a^2x^2}\,dx,$$

where a is a constant and N is the total number of molecules. If $N=10^{26}$, estimate the number of molecules with speeds between $v=1/a$ and $1.01/a$.

16. The operating equation for a synchrotron in the relativistic range is

$$qB=\omega m[1-(\omega R)^2/c^2]^{-1/2},$$

where q and m are the charge and rest mass of the particle being accelerated, B is the magnetic field strength, R is the orbit radius, ω is the angular frequency, and c is the speed of light. If ω and B are varied (all other quantities constant), show that the relation between $d\omega$ and dB can be written as

$$\frac{dB}{B^3}=\left(\frac{q}{m}\right)^2\frac{d\omega}{\omega^3},\qquad \text{or as}\qquad \frac{dB}{B}=\frac{d\omega}{\omega}[1-(\omega R/c)^2]^{-1}.$$

17. Here are some other ways of obtaining the formula in Example 2.
(a) Combine the two fractions to get $(2n+1)/[n^2(n+1)^2]$. Then note that for large n, $2n+1\cong 2n$ and $n+1\cong n$.
(b) Factor the expression as $\left(\frac{1}{n^2}\right)\left(1-\frac{1}{(1+\frac{1}{n})^2}\right)$, expand $\left(1+\frac{1}{n}\right)^{-2}$ by binomial series to two terms, and then simplify.

5. CHAIN RULE OR DIFFERENTIATING A FUNCTION OF A FUNCTION

You already know about the chain rule whether you have called it that or not. Look at this example.

Example 1. Find dy/dx if $y=\ln\sin 2x$.

You would say

$$\frac{dy}{dx}=\frac{1}{\sin 2x}\cdot\frac{d}{dx}(\sin 2x)=\frac{1}{\sin 2x}\cdot\cos 2x\cdot\frac{d}{dx}(2x)=2\cot 2x.$$

We *could* write this problem as

$$y=\ln u,\qquad \text{where}\qquad u=\sin v\qquad \text{and}\qquad v=2x.$$

Then we would say

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

This is an example of the chain rule. We shall want a similar equation for a function of several variables. Consider another example.

▶ **Example 2.** Find dz/dt if $z = 2t^2 \sin t$.

Differentiating the product, we get

$$\frac{dz}{dt} = 4t \sin t + 2t^2 \cos t.$$

We *could* have written this problem as

$$z = xy, \qquad \text{where } \ x = 2t^2 \qquad \text{and } \ y = \sin t,$$
$$\frac{dz}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}.$$

But since x is $\partial z/\partial y$ and y is $\partial z/\partial x$, we could also write

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{5.1}$$

We would like to be sure that (5.1) is a correct formula in general, when we are given any function $z(x, y)$ with continuous partial derivatives and x and y are differentiable functions of t. To see this, recall from our discussion of differentials that we had

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{5.2}$$

where ϵ_1 and $\epsilon_2 \to 0$ with Δx and Δy. Divide this equation by Δt and let $\Delta t \to 0$; since Δx and $\Delta y \to 0$, ϵ_1 and $\epsilon_2 \to 0$ also, and we get (5.1).

It is often convenient to use differentials rather than derivatives as in (5.1). We would like to be able to use (3.6), but in (3.6) x and y were independent variables and now they are functions of t. However, it is possible to show (Problem 8) that dz as defined in (3.6) is a good approximation to Δz even though x and y are related. We may then write

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \tag{5.3}$$

whether or not x and y are independent variables, and we may think of getting (5.1) by dividing (5.3) by dt. This is very convenient in doing problems. Thus we could do Example 2 in the following way:

$$dz = x\,dy + y\,dx = x\cos t\,dt + y \cdot 4t\,dt = (2t^2 \cos t + 4t \sin t)dt,$$
$$\frac{dz}{dt} = 2t^2 \cos t + 4t \sin t.$$

In doing problems, we may then use either differentials or derivatives. Here is another example.

Example 3. Find dz/dt given $z = x^y$, where $y = \tan^{-1} t$, $x = \sin t$.
Using differentials, we find

$$dz = yx^{y-1}\,dx + x^y \ln x\,dy = yx^{y-1}\cos t\,dt + x^y \ln x \cdot \frac{dt}{1+t^2},$$
$$\frac{dz}{dt} = yx^{y-1}\cos t + x^y \ln x \cdot \frac{1}{1+t^2}.$$

You may wonder in a problem like this why we don't just substitute x and y as functions of t into $z = x^y$ to get z as a function of t and then differentiate. Sometimes this may be the best thing to do but not always. For example, the resulting formula may be very complicated and it may save a lot of algebra to use (5.1) or (5.3). This is especially true if we want dz/dt for a numerical value of t. Then there are cases when we *cannot* substitute; for example, if x as a function of t is given by $x+e^x = t$, we cannot solve for x as a function of t in terms of elementary functions. But we *can* find dx/dt, and so we can find dz/dt by (5.1). Finding dx/dt from such an equation is called implicit differentiation; we shall discuss this process in the next section.

Computers can find derivatives, so why should we learn the methods shown here and in the following sections? Perhaps the most important reason is that the techniques are needed in theoretical derivations. However, there is also a practical reason: When a problem involves a number of variables, there may be many ways to express the answer. (You might like to verify that $dz/dt = z(y\cot t + \ln x \cos^2 y)$ is another form for the answer in Example 3 above). Your computer may not give you the form you want and it may be as easy to do the problem by hand as to convert the computer result. But in problems involving a lot of algebra, a computer can save time, so a good study method is to do problems both by hand and by computer and compare results.

PROBLEMS, SECTION 5

1. Given $z = xe^{-y}$, $x = \cosh t$, $y = \cos t$, find dz/dt.
2. Given $w = \sqrt{u^2+v^2}$, $u = \cos[\ln\tan(p+\frac{1}{4}\pi)]$, $v = \sin[\ln\tan(p+\frac{1}{4}\pi)]$, find dw/dp.
3. Given $r = e^{-p^2-q^2}$, $p = e^s$, $q = e^{-s}$, find dr/ds.
4. Given $x = \ln(u^2 - v^2)$, $u = t^2$, $v = \cos t$, find dx/dt.
5. If we are given $z = z(x, y)$ and $y = y(x)$, show that the chain rule (5.1) gives
$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$
6. Given $z = (x+y)^5$, $y = \sin 10x$, find dz/dx.
7. Given $c = \sin(a - b)$, $b = ae^{2a}$, find dc/da.
8. Prove the statement just after (5.2), that dz given by (3.6) is a good approximation to Δz even though dx and dy are not independent. *Hint:* let x and y be functions of t; then (5.2) is correct, but $\Delta x \neq dx$ and $\Delta y \neq dy$ (because x and y are not independent variables). However, $\Delta x/\Delta t$ is nearly dx/dt for small dt and $dt = \Delta t$ since t is the independent variable. You can then show that
$$\Delta x = \left(\frac{dx}{dt} + \epsilon_x\right) dt = dx + \epsilon_x dt,$$

and a similar formula for Δy , and get

$$\Delta z = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + (\text{terms containing } \epsilon\text{'s}) \cdot dt = dz + \epsilon dt,$$

where $\epsilon \to 0$ as $\Delta t \to 0$.

6. IMPLICIT DIFFERENTIATION

Some examples will show the use of implicit differentiation.

Example 1. Given $x + e^x = t$, find dx/dt and d^2x/dt^2.

If we give values to x, find the corresponding t values, and plot x against t, we have a graph whose slope is dx/dt. In other words, x *is* a function of t even though we cannot solve the equation for x in terms of elementary functions of t. To find dx/dt, we realize that x *is* a function of t and just differentiate each term of the equation with respect to t (this is called implicit differentiation). We get

$$\frac{dx}{dt} + e^x\frac{dx}{dt} = 1. \tag{6.1}$$

Solving for dx/dt, we get

$$\frac{dx}{dt} = \frac{1}{1+e^x}.$$

Alternatively, we could use differentials here, and write first $dx + e^x dx = dt$; dividing by dt then gives (6.1).

We can also find higher derivatives by implicit differentiation (but do *not* use differentials for this since we have not given any meaning to the derivative or differential of a differential). Let us differentiate each term of (6.1) with respect to t; we get

$$\frac{d^2x}{dt^2} + e^x\frac{d^2x}{dt^2} + e^x\left(\frac{dx}{dt}\right)^2 = 0. \tag{6.2}$$

Solving for d^2x/dt^2 and substituting the value already found for dx/dt, we get

$$\frac{d^2x}{dt^2} = \frac{-e^x\left(\frac{dx}{dt}\right)^2}{1+e^x} = \frac{-e^x}{(1+e^x)^3}. \tag{6.3}$$

This problem is even easier if we want only the numerical values of the derivatives at a point. For $x = 0$ and $t = 1$, (6.1) gives

$$\frac{dx}{dt} + 1\cdot\frac{dx}{dt} = 1 \qquad \text{or} \qquad \frac{dx}{dt} = \frac{1}{2},$$

and (6.2) gives

$$\frac{d^2x}{dt^2} + 1\cdot\frac{d^2x}{dt^2} + 1\cdot\left(\frac{1}{2}\right)^2 = 0 \qquad \text{or} \qquad \frac{d^2x}{dt^2} = -\frac{1}{8}.$$

Implicit differentiation is the best method to use in finding slopes of curves with complicated equations.

Example 2. Find the equation of the tangent line to the curve $x^3 - 3y^3 + xy + 21 = 0$ at the point $(1, 2)$.

We differentiate the given equation implicitly with respect to x to get

$$3x^2 - 9y^2\frac{dy}{dx} + x\frac{dy}{dx} + y = 0.$$

Substitute $x = 1, y = 2$:

$$3 - 36\frac{dy}{dx} + \frac{dy}{dx} + 2 = 0, \qquad \frac{dy}{dx} = \frac{5}{35} = \frac{1}{7}.$$

Then the equation of the tangent line is

$$\frac{y-2}{x-1} = \frac{1}{7} \qquad \text{or} \qquad x - 7y + 13 = 0.$$

By computer plotting the curve and the tangent line on the same axes, you can check to be sure that the line appears tangent to the curve.

PROBLEMS, SECTION 6

1. If $pv^a = C$ (where a and C are constants), find dv/dp and d^2v/dp^2.

2. If $ye^{xy} = \sin x$ find dy/dx and d^2y/dx^2 at $(0, 0)$.

3. If $x^y = y^x$, find dy/dx at $(2, 4)$.

4. If $xe^y = ye^x$, find dy/dx and d^2y/dx^2 for $y \neq 1$.

5. If $xy^3 - yx^3 = 6$ is the equation of a curve, find the slope and the equation of the tangent line at the point $(1, 2)$. Computer plot the curve and the tangent line on the same axes.

6. In Problem 5 find d^2y/dx^2 at $(1, 2)$.

7. If $y^3 - x^2y = 8$ is the equation of a curve, find the slope and the equation of the tangent line at the point $(3, -1)$. Computer plot the curve and the tangent line on the same axes.

8. In Problem 7 find d^2y/dx^2 at $(3, -1)$.

9. For the curve $x^{2/3} + y^{2/3} = 4$, find the equations of the tangent lines at $(2\sqrt{2}, -2\sqrt{2})$, at $(8, 0)$, and at $(0, 8)$. Computer plot the curve and the tangent lines on the same axes.

10. For the curve $xe^y + ye^x = 0$, find the equation of the tangent line at the origin. *Caution:* Substitute $x = y = 0$ as soon as you have differentiated. Computer plot the curve and the tangent line on the same axes.

11. In Problem 10, find y'' at the origin.

7. MORE CHAIN RULE

Above we have considered $z = f(x, y)$, where x and y are functions of t. Now suppose $z = f(x, y)$ as before, but x and y are each functions of two variables s and t. Then z is a function of s and t and we want to find $\partial z/\partial s$ and $\partial z/\partial t$. We show by some examples how to do problems like this.

▶ **Example 1.** Find $\partial z/\partial s$ and $\partial z/\partial t$ given

$$z = xy, \qquad x = \sin(s+t), \qquad y = s - t.$$

We take differentials of each of the three equations to get

$$dz = y\,dx + x\,dy, \qquad dx = \cos(s+t)(ds+dt), \qquad dy = ds - dt.$$

Substituting dx and dy into dz, we get

$$\begin{aligned} dz &= y\cos(s+t)(ds+dt) + x(ds-dt) \\ &= [y\cos(s+t)+x]\,ds + [y\cos(s+t)-x]\,dt. \end{aligned} \tag{7.1}$$

Now if s is constant, $ds = 0$, z is a function of one variable t, and we can divide (7.1) by dt [see (5.1) and the discussion following it]. For $dz \div dt$ on the left we write $\partial z/\partial t$ because that is the notation which properly describes what we are finding, namely, the rate of change of z with t when s is constant. Thus we have

$$\frac{\partial z}{\partial t} = y\cos(s+t) - x$$

and similarly

$$\frac{\partial z}{\partial s} = y\cos(s+t) + x.$$

Notice that in (7.1) the coefficient of ds is $\partial z/\partial s$ and the coefficient of dt is $\partial z/\partial t$ [also compare (5.3)]. If you realize this, you can simply read off $\partial z/\partial s$ and $\partial z/\partial t$ from (7.1).

We can do problems with more variables in the same way.

▶ **Example 2.** Find $\partial u/\partial s$, $\partial u/\partial t$, given $u = x^2 + 2xy - y\ln z$ and $x = s + t^2$, $y = s - t^2$, $z = 2t$.

We find

$$\begin{aligned} du &= 2x\,dx + 2x\,dy + 2y\,dx - \frac{y}{z}dz - \ln z\,dy \\ &= (2x+2y)(ds+2t\,dt) + (2x - \ln z)(ds - 2t\,dt) - \frac{y}{z}(2\,dt) \\ &= (4x + 2y - \ln z)\,ds + \left(4yt + 2t\ln z - \frac{2y}{z}\right)dt. \end{aligned}$$

Then

$$\frac{\partial u}{\partial s} = 4x + 2y - \ln z, \qquad \frac{\partial u}{\partial t} = 4yt + 2t\ln z - \frac{2y}{z}.$$

If we want just one derivative, say $\partial u/\partial t$, we can save some work by letting $ds = 0$ to start with. To make it clear that we have done this, we write

$$\begin{aligned} du_s &= (2x+2y)(2t\,dt) + (2x - \ln z)(-2t\,dt) - \frac{y}{z}(2\,dt) \\ &= \left(4yt + 2t\,\ln z - \frac{2y}{z}\right)dt. \end{aligned}$$

The subscript s indicates that s is being held constant. Then dividing by dt, we have $\partial u/\partial t$ as before. We could also use derivatives instead of differentials. By an equation like (5.1), we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}, \tag{7.2}$$

where we have written all the t derivatives as partials since u, x, y, and z depend on both s and t. Using (7.2), we get

$$\frac{\partial u}{\partial t} = (2x+2y)(2t) + (2x - \ln z)(-2t) + \left(-\frac{y}{z}\right)(2) = 4yt + 2t\ln z - \frac{2y}{z}.$$

It is sometimes useful to write chain rule formulas in matrix form (for matrix multiplication, see Chapter 3, Section 6). Given, as above, $u = f(x, y, z)$, $x(s, t)$, $y(s, t)$, $z(s, t)$, we can write equations like (7.2) in the following matrix form:

$$\begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix}. \tag{7.3}$$

[Sometimes (7.3) is written in the abbreviated form

$$\frac{\partial(u)}{\partial(s,t)} = \frac{\partial(u)}{\partial(x,y,z)}\frac{\partial(x,y,z)}{\partial(s,t)}$$

which is reminiscent of

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt};$$

but be careful of this for two reasons: (a) It may be helpful in remembering the formula but to use it you must understand that it means the matrix product (7.3). (b) The symbol $\partial(u, v)/\partial(x, y)$ usually means a determinant rather than a matrix of partial derivatives—see Chapter 5, Section 4].

Again in these problems, you may say, why not just substitute? Look at the following problem.

Example 3. Find dz/dt given $z = x - y$ and

$$x^2 + y^2 = t^2,$$
$$x \sin t = ye^y.$$

From the z equation, we have

$$dz = dx - dy.$$

We need dx and dy; here we cannot solve for x and y in terms of t. But we *can* find dx and dy in terms of dt from the other two equations and this is all we need. Take differentials of both equations to get

$$2x\,dx + 2y\,dy = 2t\,dt,$$
$$\sin t\,dx + x\cos t\,dt = (ye^y + e^y)dy.$$

Rearrange terms:

$$x\,dx + y\,dy = t\,dt,$$
$$\sin t\,dx - (y+1)e^y dy = -x\cos t\,dt.$$

Solve for dx and dy (in terms of dt) by determinants:

$$dx = \frac{\begin{vmatrix} t\,dt & y \\ -x\cos t\,dt & -(y+1)e^y \end{vmatrix}}{\begin{vmatrix} x & y \\ \sin t & -(y+1)e^y \end{vmatrix}} = \frac{-t(y+1)e^y + xy\cos t}{-x(y+1)e^y - y\sin t}\,dt,$$

and similarly for dy. Substituting dx and dy into the formula for dz and dividing by dt, we get dz/dt. A computer may save us some time with the algebra.

We can also do problems like this when x and y are given implicitly as functions of two variables s and t.

▶ **Example 4.** Find $\partial z/\partial s$ and $\partial z/\partial t$ given

$$z = x^2 + xy,$$
$$x^2 + y^3 = st + 5,$$
$$x^3 - y^2 = s^2 + t^2.$$

We have $dz = 2x\,dx + x\,dy + y\,dx$. To find dx and dy from the other two equations, we take differentials of each equation:

$$\begin{aligned} 2x\,dx + 3y^2\,dy &= s\,dt + t\,ds, \\ 3x^2\,dx - 2y\,dy &= 2s\,ds + 2t\,dt. \end{aligned} \tag{7.4}$$

We can solve these two equations for dx and dy in terms of ds and dt to get

$$dx = \frac{\begin{vmatrix} s\,dt + t\,ds & 3y^2 \\ 2s\,ds + 2t\,dt & -2y \end{vmatrix}}{\begin{vmatrix} 2x & 3y^2 \\ 3x^2 & -2y \end{vmatrix}} = \frac{(-2ys - 6ty^2)\,dt + (-2yt - 6sy^2)\,ds}{-4xy - 9x^2y^2}$$

and a similar expression for dy. We substitute these values of dx and dy into dz and find dz in terms of ds and dt just as in Example 1; we can then write $\partial z/\partial s$ and $\partial z/\partial t$ just as we did there (Problem 11). Notice that if we want only one derivative, say $\partial z/\partial t$, we could save some algebra by putting $ds = 0$ in (7.4). Also note that we can save some algebra if we want the derivatives only at one point. Suppose we were asked for $\partial z/\partial s$ and $\partial z/\partial t$ at $x = 3$, $y = 1$, $s = 1$, $t = 5$. We substitute these values into (7.4) to get

$$6\,dx + 3\,dy = dt + 5\,ds,$$
$$27\,dx - 2\,dy = 10\,dt + 2\,ds.$$

We solve these equations for dx and dy and substitute into dz just as before, but the algebra is easier with the numerical coefficients (Problem 11).

So far, we have been assuming that the independent variables were "natural" pairs like x and y, or s and t. For example, we wrote $\partial x/\partial s$ above, taking it for granted that the variable held constant was t. In some applications (particularly thermodynamics), it is not at all clear what the other independent variable is and we have to be more explicit. We write $(\partial x/\partial s)_t$; this means that s and t are the two independent variables, that x is thought of as a function of them, and then x is differentiated partially with respect to s. Suppose we try to find from the three equations of Example 4 a rather peculiar looking derivative.

Example 5. Given the equations of Example 4, find $(\partial s/\partial z)_x$.

First, let us see that the question makes sense. There are five variables in the three equations. If we give values to two of them, we can solve for the other three; that is, there are *two independent* variables, and the other three are functions of these two. If z and x are the independent ones, then s, t, and y are functions of z and x; we should be able to find their partial derivatives, for example $(\partial s/\partial z)_x$ which we wanted. To carry out the necessary work, we first rearrange equations (7.4) and the dz equation to get

$$\begin{aligned} -x\,dy &= (2x+y)\,dx - dz, \\ t\,ds + s\,dt - 3y^2\,dy &= 2x\,dx, \\ 2s\,ds + 2t\,dt + 2y\,dy &= 3x^2\,dx. \end{aligned}$$

From these three equations we could solve for ds, dt, and dy in terms of dx and dz (by determinants or by elimination—the same methods you use to solve any set of linear equations). Then we could find any partial derivative of $s(x,z)$, $t(x,z)$, or $y(x,z)$ with respect to x to z. For example, to find $(\partial y/\partial z)_x$, we get from the first equation

$$\begin{aligned} dy &= \frac{1}{x}\,dz - \frac{2x+y}{x}\,dx, \\ \left(\frac{\partial y}{\partial z}\right)_x &= \frac{1}{x}. \end{aligned}$$

Note that we would not need to differentiate all three equations if we wanted only this derivative; you should always look ahead to see how much differentiation is necessary! To find $(\partial s/\partial z)_x$, we must solve the three equations for ds in terms of dx and dz; we can save ourselves some work [if we want only $(\partial s/\partial z)_x$] by putting $dx = 0$ to start with. To make it clear that we have done this we write ds_x and dz_x. Then we get

$$ds_x = \frac{\begin{vmatrix} -dz_x & 0 & -x \\ 0 & s & -3y^2 \\ 0 & 2t & 2y \end{vmatrix}}{\begin{vmatrix} 0 & 0 & -x \\ t & s & -3y^2 \\ 2s & 2t & 2y \end{vmatrix}} = \frac{-(2sy+6ty^2)dz_x}{-x(2t^2-2s^2)},$$

$$\left(\frac{\partial s}{\partial z}\right)_x = \frac{sy+3ty^2}{x(t^2-s^2)}.$$

We could use a computer to save us some algebra in this problem.

▶ **Example 6.** Let x, y be rectangular coordinates and r, θ be polar coordinates in a plane. Then the equations relating them are

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned} \tag{7.5}$$

or

$$\begin{aligned} r &= \sqrt{x^2+y^2}, \\ \theta &= \tan^{-1}\frac{y}{x}. \end{aligned} \tag{7.6}$$

Suppose we want to find $\partial\theta/\partial x$. Remembering that if $y = f(x)$, dy/dx and dx/dy are reciprocals, you might be tempted to find $\partial\theta/\partial x$ by taking the reciprocal of $\partial x/\partial\theta$, which is easier to find than $\partial\theta/\partial x$. *This is wrong.* From (7.6) we get

$$\frac{\partial\theta}{\partial x} = \frac{-y/x^2}{1+(y^2/x^2)} = -\frac{y}{r^2}. \tag{7.7}$$

From $x = r\cos\theta$ we get

$$\frac{\partial x}{\partial\theta} = -r\sin\theta = -y. \tag{7.8}$$

These are not reciprocals. You should think carefully about the reason for this; $\partial\theta/\partial x$ means $(\partial\theta/\partial x)_y$, whereas $\partial x/\partial\theta$ means $(\partial x/\partial\theta)_r$. In one case y is held constant and in the other case r is held constant; this is why the two derivatives are not reciprocals. It *is* true that $(\partial\theta/\partial x)_y$ and $(\partial x/\partial\theta)_y$ are reciprocals. But to find $(\partial x/\partial\theta)_y$ directly, we have to express x as a function of θ and y. We find $x = y\cot\theta$, so we get

$$\left(\frac{\partial x}{\partial\theta}\right)_y = y(-\csc^2\theta) = \frac{-y}{\sin^2\theta} = \frac{-y}{y^2/r^2} = -\frac{r^2}{y}, \tag{7.9}$$

which *is* the reciprocal of $\partial\theta/\partial x$ in (7.7).

> This is a general rule: $\partial u/\partial v$ and $\partial v/\partial u$ are *not* usually reciprocals; they *are* reciprocals if the other independent variables (besides u or v) are the same in both cases.

You can see this clearly from the equations involving differentials. From the equation $\theta = \arctan(y/x)$, we can find

$$d\theta = \frac{x\,dy - y\,dx}{x^2}\bigg/\left(1+\frac{y^2}{x^2}\right) = \frac{x\,dy - y\,dx}{r^2}. \tag{7.10}$$

From $x = r\cos\theta$, we get

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta = \frac{x}{r}\,dr - y\,d\theta. \tag{7.11}$$

From (7.10), if y is constant, $dy = 0$, and we can write

$$d\theta_y = -\frac{y}{r^2}\,dx_y, \tag{7.12}$$

where the y subscript indicates that y is constant. From (7.12), we then find either

$$\left(\frac{\partial\theta}{\partial x}\right)_y = \frac{d\theta_y}{dx_y} \qquad \text{or} \qquad \left(\frac{\partial x}{\partial\theta}\right)_y = \frac{dx_y}{d\theta_y},$$

and these are reciprocals. From (7.11), however, we can find $(\partial x/\partial\theta)_r$ or $(\partial\theta/\partial x)_r$; these are again reciprocals of each other, but are different from the derivatives found from (7.12).

It is interesting to write equations like (7.11) in matrix notation:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta} \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = \mathrm{A}\begin{pmatrix} dr \\ d\theta \end{pmatrix}, \tag{7.13}$$

where A stands for the square matrix in (7.13). Similarly, we can write

$$\begin{pmatrix} dr \\ d\theta \end{pmatrix} = \begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\[2ex] \dfrac{\partial\theta}{\partial x} & \dfrac{\partial\theta}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \mathrm{A}^{-1}\begin{pmatrix} dx \\ dy \end{pmatrix}; \tag{7.14}$$

we have written the square matrix as A^{-1} since by (7.13),

$$\mathrm{A}^{-1}\begin{pmatrix} dx \\ dy \end{pmatrix} = \mathrm{A}^{-1}\mathrm{A}\begin{pmatrix} dr \\ d\theta \end{pmatrix} = \begin{pmatrix} dr \\ d\theta \end{pmatrix}.$$

Then, finding A^{-1} (Problem 9) and using (7.14), we have

$$\begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\[2ex] \dfrac{\partial\theta}{\partial x} & \dfrac{\partial\theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{1}{r}\sin\theta & \dfrac{1}{r}\cos\theta. \end{pmatrix} \tag{7.15}$$

We can simply read off the four partial derivatives of r, θ, with respect to x, y, from equation (7.15). (Also see Problem 9.) Also using (7.5), and specifically noting that x and y are independent variables, we have:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \left(\frac{\partial r}{\partial x}\right)_y = \cos\theta = \frac{x}{r}, & \frac{\partial r}{\partial y} &= \left(\frac{\partial r}{\partial y}\right)_x = \sin\theta = \frac{y}{r}, \\ \frac{\partial\theta}{\partial x} &= \left(\frac{\partial\theta}{\partial x}\right)_y = -\frac{1}{r}\sin\theta = -\frac{y}{r^2}, & \frac{\partial\theta}{\partial y} &= \left(\frac{\partial\theta}{\partial y}\right)_x = \frac{1}{r}\cos\theta = \frac{x}{r^2}. \end{aligned} \tag{7.16}$$

[In the notation mentioned just after (7.3), we could write

$$\mathrm{AA}^{-1} = \frac{\partial(x,y)}{\partial(r,\theta)}\frac{\partial(r,\theta)}{\partial(x,y)} = \text{unit matrix};$$

thus, although the individual pairs of partial derivatives discussed above are not reciprocals, the two matrices of partial derivatives are inverses.]

▶ PROBLEMS, SECTION 7

1. If $x = yz$ and $y = 2\sin(y+z)$, find dx/dy and d^2x/dy^2.

2. If $P = r\cos t$ and $r\sin t - 2te^r = 0$, find dP/dt.

3. If $z = xe^{-y}$ and $x = \cosh t$, $y = \cos s$, find $\partial z/\partial s$ and $\partial z/\partial t$.

4. If $w = e^{-r^2-s^2}$, $r = uv$, $s = u + 2v$, find $\partial w/\partial u$ and $\partial w/\partial v$.

5. If $u = x^2y^3z$ and $x = \sin(s+t)$, $y = \cos(s+t)$, $z = e^{st}$, find $\partial u/\partial s$ and $\partial u/\partial t$.

6. If $w = f(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$, find formulas for $\partial w/\partial r$, $\partial w/\partial\theta$, and $\partial^2 w/\partial r^2$.

7. If $x = r\cos\theta$ and $y = r\sin\theta$, find $(\partial y/\partial\theta)_r$ and $(\partial y/\partial\theta)_x$. Also find $(\partial\theta/\partial y)_x$ in two ways (by eliminating r from the given equations and then differentiating, or by taking differentials in both equations and then eliminating dr). When are $\partial y/\partial\theta$ and $\partial\theta/\partial y$ reciprocals?

8. If $xs^2 + yt^2 = 1$ and $x^2s + y^2t = xy - 4$, find $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, $\partial y/\partial t$, at $(x,y,s,t) = (1,-3,2,-1)$. *Hint:* To simplify the work, substitute the numerical values just after you have taken differentials.

9. Verify (7.16) in three ways:

(a) Differentiate equations (7.6).

(b) Take differentials of (7.5) and solve for dr and $d\theta$.

(c) Find A^{-1} in (7.15) from A in (7.13); note that this is (b) in matrix notation.

10. If $x^2 + y^2 = 2st - 10$ and $2xy = s^2 - t^2$, find $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, $\partial y/\partial t$ at $(x,y,s,t) = (4,2,5,3)$.

11. Finish Example 4 above, both for the general case and for the given numerical values. Substitute the numerical values into your general formulas to check your answers.

12. If $w = x + y$ with $x^3 + xy + y^3 = s$ and $x^2y + xy^2 = t$, find $\partial w/\partial s$, $\partial w/\partial t$.

13. If $m = pq$ with $a\sin p - p = q$ and $b\cos q + q = p$, find $(\partial p/\partial q)_m$, $(\partial p/\partial q)_a$, $(\partial p/\partial q)_b$, $(\partial b/\partial a)_p$, $(\partial a/\partial q)_m$.

14. If $u = x^2 + y^2 + xyz$ and $x^4 + y^4 + z^4 = 2x^2y^2z^2 + 10$, find $(\partial u/\partial x)_z$ at the point $(x,y,z) = (2,1,1)$.

15. Given $x^2u - y^2v = 1$, and $x + y = uv$. Find $(\partial x/\partial u)_v$, $(\partial x/\partial u)_y$.

16. Let $w = x^2 + xy + z^2$.

(a) If $x^3 + x = 3t$, $y^4 + y = 4t$, $z^5 + z = 5t$, find dw/dt.

(b) If $y^3 + xy = 1$ and $z^3 - xz = 2$, find dw/dx.

(c) If $x^3z + z^3y + y^3x = 0$, find $(\partial w/\partial x)_y$.

17. If $p^3 + sq = t$, and $q^3 + tp = s$, find $(\partial p/\partial s)_t$, $(\partial p/\partial s)_q$ at $(p,q,s,t) = (-1,2,3,5)$.

18. If $m = a + b$ and $n = a^2 + b^2$ find $(\partial b/\partial m)_n$ and $(\partial m/\partial b)_a$.

19. If $z = r + s^2$, $x + y = s^3 + r^3 - 3$, $xy = s^2 - r^2$, find $(\partial x/\partial z)_s$, $(\partial x/\partial z)_r$, $(\partial x/\partial z)_y$ at $(r,s,x,y,z) = (-1,2,3,1,3)$.

20. If $u^2 + v^2 = x^3 - y^3 + 4$, $u^2 - v^2 = x^2y^2 + 1$, find $(\partial u/\partial x)_y$, $(\partial u/\partial x)_v$, $(\partial x/\partial u)_y$, $(\partial x/\partial u)_v$ at $(x,y,u,v) = (2,-1,3,2)$.

21. Given $x^2+y^2+z^2=6$, and $w^3+z^3=5xy+12$, find the following partial derivatives at the point $(x,y,z,w)=(1,-2,1,1)$.

$$\left(\frac{\partial z}{\partial x}\right)_y,\quad \left(\frac{\partial z}{\partial x}\right)_w,\quad \left(\frac{\partial z}{\partial y}\right)_x,\quad \left(\frac{\partial z}{\partial y}\right)_w,\quad \left(\frac{\partial w}{\partial x}\right)_z,\quad \left(\frac{\partial x}{\partial w}\right)_z.$$

22. If $w=f(ax+by)$, show that $b\dfrac{\partial w}{\partial x}-a\dfrac{\partial w}{\partial y}=0$.
Hint: Let $ax+by=z$.

23. If $u=f(x-ct)+g(x+ct)$, show that $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$.

24. If $z=\cos(xy)$, show that $x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}=0$.

25. The formulas of this problem are useful in thermodynamics.

(a) Given $f(x,y,z)=0$, find formulas for

$$\left(\frac{\partial y}{\partial x}\right)_z,\quad \left(\frac{\partial x}{\partial y}\right)_z,\quad \left(\frac{\partial y}{\partial z}\right)_x,\quad \text{and}\quad \left(\frac{\partial z}{\partial x}\right)_y.$$

(b) Show that

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial x}\right)_z=1$$

and

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y=-1.$$

(c) If x,y,z are each functions of t, show that $\left(\dfrac{\partial y}{\partial z}\right)_x=\left(\dfrac{\partial y}{\partial t}\right)_x\Big/\left(\dfrac{\partial z}{\partial t}\right)_x$ and corresponding formulas for $\left(\dfrac{\partial z}{\partial x}\right)_y$ and $\left(\dfrac{\partial x}{\partial y}\right)_z$.

26. Given $f(x,y,z)=0$ and $g(x,y,z)=0$, find a formula for dy/dx.

27. Given $u(x,y)$ and $y(x,z)$, show that

$$\left(\frac{\partial u}{\partial x}\right)_z=\left(\frac{\partial u}{\partial x}\right)_y+\left(\frac{\partial u}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_z.$$

28. Given $s(v,T)$ and $v(p,T)$, we define $c_p=T(\partial s/\partial T)_p$, $c_v=T(\partial s/\partial T)_v$. (The c's are specific heats in thermodynamics.) Show that

$$c_p-c_v=T\left(\frac{\partial s}{\partial v}\right)_T\left(\frac{\partial v}{\partial T}\right)_p.$$

8. APPLICATION OF PARTIAL DIFFERENTIATION TO MAXIMUM AND MINIMUM PROBLEMS

You will recall that derivatives give slopes as well as rates and that you find maximum and minimum points of $y=f(x)$ by setting $dy/dx=0$. Often in applied problems we want to find maxima or minima of functions of more than one variable. Think of $z=f(x,y)$ which represents a surface. If there is a maximum point on it (like the top of a hill), then the curves for $x=$ const. and $y=$ const. which pass through the maximum point also have maxima at the same point. That is, $\partial z/\partial x$

and $\partial z/\partial y$ are zero at the maximum point. Recall that $dy/dx = 0$ was a necessary condition for a maximum point of $y = f(x)$, but not sufficient; the point might have been a minimum or perhaps a point of inflection with a horizontal tangent. Something similar can happen for $z = f(x, y)$. The point where $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$ may be a maximum point, a minimum point, or neither. (An interesting example of neither is a "saddle point"—a curve from front to back on a saddle has a minimum; one from side to side has a maximum. See Figure 8.1.) In finding maxima of $y = f(x)$, it is sometimes possible to tell from the geometry or physics that you have a maximum. If necessary you can find d^2y/dx^2; if it is negative, then you know you have a maximum point. There is a similar (rather complicated) second derivative test for functions of two variables (see Problems 1 to 7), but we use it only if we have to; usually we can tell from the problem whether we have a maximum, a minimum, or neither. Let us consider some examples of maximum or minimum problems.

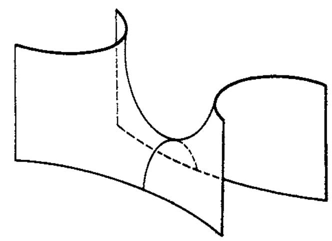


Figure 8.1

▶ **Example.** A pup tent (Figure 8.2) of given volume V, with ends but no floor, is to be made using the least possible material. Find the proportions.

Using the letters indicated in the figure, we find the volume V and the area A.

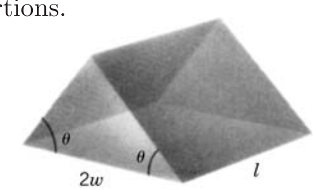


Figure 8.2

$$V = \frac{1}{2}\cdot 2w \cdot l \cdot w\tan\theta = w^2 l \tan\theta,$$
$$A = 2w^2\tan\theta + \frac{2lw}{\cos\theta}.$$

Since V is given, only two of the three variables w, l, and θ are independent, and we must eliminate one of them from A before we try to minimize A. Solving the V equation for l and substituting into A, we get

$$A = 2w^2\tan\theta + \frac{2w}{\cos\theta}\frac{V}{w^2\tan\theta} = 2w^2\tan\theta + \frac{2V}{w}\csc\theta.$$

We now have A as a function of two independent variables w and θ. To minimize A we find $\partial A/\partial w$ and $\partial A/\partial\theta$ and set them equal to zero.

$$\frac{\partial A}{\partial w} = 4w\tan\theta - \frac{2V\csc\theta}{w^2} = 0,$$
$$\frac{\partial A}{\partial \theta} = 2w^2\sec^2\theta - \frac{2V}{w}\csc\theta\cot\theta = 0.$$

Solving each of these equations for w^3 and setting the results equal, we get

$$w^3 = \frac{V\csc\theta}{2\tan\theta} = \frac{V\csc\theta\cot\theta}{\sec^2\theta} \quad\text{or}\quad \frac{\cos\theta}{2\sin^2\theta} = \frac{\cos\theta\cos^2\theta}{\sin^2\theta}.$$

You should convince yourself that neither $\sin\theta = 0$ nor $\cos\theta = 0$ is possible (the tent collapses to zero volume in both cases). Therefore we may assume $\sin\theta \neq 0$ and $\cos\theta \neq 0$ and cancel these factors, getting $\cos^2\theta = \frac{1}{2}$ or $\theta = 45°$. Then $\tan\theta = 1$, $V = w^2 l$, and from the $\partial A/\partial w$ equation we have $2w = l\sqrt{2}$. Then the height of the tent (at the peak) is $w\tan\theta = w = 1/\sqrt{2}$.

PROBLEMS, SECTION 8

1. Use the Taylor series about $x = a$ to verify the familiar "second derivative test" for a maximum or minimum point. That is, show that if $f'(a) = 0$, then $f''(a) > 0$ implies a minimum point at $x = a$ and $f''(a) < 0$ implies a maximum point at $x = a$. *Hint:* For a minimum point, say, you must show that $f(x) > f(a)$ for all x near enough to a.

2. Using the two-variable Taylor series [say (2.7)] prove the following "second derivative tests" for maximum or minimum points of functions of two variables. If $f_x = f_y = 0$ at (a, b), then

(a, b) is a minimum point if at (a, b), $f_{xx} > 0$, $f_{yy} > 0$, and $f_{xx}f_{yy} > f_{xy}^2$;

(a, b) is a maximum point if at (a, b), $f_{xx} < 0$, $f_{yy} < 0$, and $f_{xx}f_{yy} > f_{xy}^2$;

(a, b) is neither a maximum nor a minimum point if $f_{xx}f_{yy} < f_{xy}^2$. (Note that this includes $f_{xx}f_{yy} < 0$, that is, f_{xx} and f_{yy} of opposite sign.)

Hint: Let $f_{xx} = A$, $f_{xy} = B$, $f_{yy} = C$; then the second derivative terms in the Taylor series are $Ah^2 + 2Bhk + Ck^2$; this can be written $A(h + Bk/A)^2 + (C - B^2/A)k^2$. Find out when this expression is positive for *all* small h, k [that is, all (x, y) near (a, b)]; also find out when it is negative for all small h, k, and when it has both positive and negative values for small h, k.

Use the facts stated in Problem 2 to find the maximum and minimum points of the functions in Problems 3 to 6.

3. $x^2 + y^2 + 2x - 4y + 10$

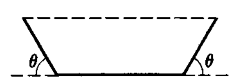
4. $x^2 - y^2 + 2x - 4y + 10$

5. $4 + x + y - x^2 - xy - \frac{1}{2}y^2$

6. $x^3 - y^3 - 2xy + 2$

7. Given $z = (y - x^2)(y - 2x^2)$, show that z has neither a maximum nor a minimum at $(0, 0)$, although z has a minimum on every straight line through $(0, 0)$.

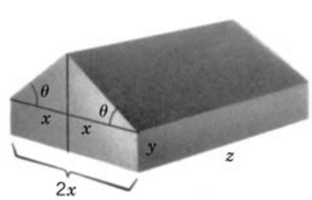
8. A roof gutter is to be made from a long strip of sheet metal, 24 cm wide, by bending up equal amounts at each side through equal angles. Find the angle and the dimensions that will make the carrying capacity of the gutter as large as possible.



9. An aquarium with rectangular sides and bottom (and no top) is to hold 5 gal. Find its proportions so that it will use the least amount of material.

10. Repeat Problem 9 if the bottom is to be three times as thick as the sides.

11. Find the most economical proportions for a tent as in the figure, with no floor.



12. Find the shortest distance from the origin to the surface $z = xy + 5$.

13. Given particles of masses m, $2m$, and $3m$ at the points $(0,1)$, $(1,0)$, and $(2,3)$, find the point P about which their total moment of inertia will be least. (Recall that to find the moment of inertia of m about P, you multiply m by the square of its distance from P.)

14. Repeat Problem 13 for masses m_1, m_2, m_3 at (x_1,y_1), (x_2,y_2), (x_3,y_3). Show that the point you find is the center of mass.

15. Find the point on the line through $(1,0,0)$ and $(0,1,0)$ that is closest to the line $x=y=z$. Also find the point on the line $x=y=z$ that is closest to the line through $(1,0,0)$ and $(0,1,0)$.

16. To find the best straight line fit to a set of data points (x_n,y_n) in the "least squares" sense means the following: Assume that the equation of the line is $y=mx+b$ and verify that the vertical deviation of the line from the point (x_n,y_n) is $y_n-(mx_n+b)$. Write $S=$ sum of the squares of the deviations, substitute the given values of x_n, y_n to give S as a function of m and b, and then find m and b to minimize S.

Carry through this routine for the set of points: $(-1,-2)$, $(0,0)$, $(1,3)$. Check your results by computer, and also computer plot (on the same axes) the given points and the approximating line.

17. Repeat Problem 16 for each of the following sets of data points.

(a) $(1,0)$, $(2,-1)$, $(3,-8)$

(b) $(-2,-6)$, $(-1,-3)$, $(0,0)$, $(1,9/2)$, $(2,7)$

(c) $(-2,4)$, $(-1,0)$, $(0,-1)$, $(1,-8)$, $(2,-10)$

▶ 9. MAXIMUM AND MINIMUM PROBLEMS WITH CONSTRAINTS; LAGRANGE MULTIPLIERS

Some examples will illustrate these methods.

▶ **Example 1.** A wire is bent to fit the curve $y=1-x^2$ (Figure 9.1). A string is stretched from the origin to a point (x,y) on the curve. Find (x,y) to minimize the length of the string.

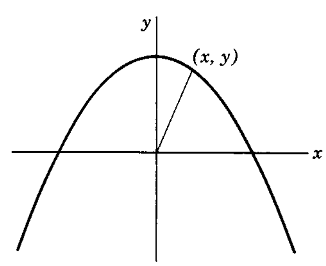


Figure 9.1

We want to minimize the distance $d=\sqrt{x^2+y^2}$ from the origin to the point (x,y); this is equivalent to minimizing $f=d^2=x^2+y^2$. But x and y are not independent; they are related by the equation of the curve. This extra relation between the variables is what we mean by a *constraint*. Problems involving constraints occur frequently in applications.

There are several ways to do a problem like this. We shall discuss the following methods: (a) elimination, (b) implicit differentiation, (c) Lagrange multipliers.

(a) Elimination The most obvious method is to eliminate y. Then we want to minimize

$$f = x^2 + (1 - x^2)^2 = x^2 + 1 - 2x^2 + x^4 = x^4 - x^2 + 1.$$

This is just an ordinary calculus problem:

$$\frac{df}{dx} = 4x^3 - 2x = 0, \quad x = 0, \quad \text{or} \quad x = \pm\sqrt{\frac{1}{2}}.$$

It is not immediately obvious which of these points is a maximum and which is a minimum, so in this simple problem it is worth while to find the second derivative:

$$\frac{d^2f}{dx^2} = 12x^2 - 2 = \begin{cases} -2 & \text{at} \quad x = 0 & \text{(relative maximum)}, \\ \ \ 4 & \text{at} \quad x = \pm\sqrt{1/2} & \text{(minimum)}. \end{cases}$$

The minimum we wanted then occurs at $x = \pm\sqrt{1/2}$, $y = 1/2$.

(b) Implicit Differentiation Suppose it had not been possible to solve for y and substitute; we could still do the problem. From $f = x^2 + y^2$, we find

$$df = 2x\,dx + 2y\,dy \quad \text{or} \quad \frac{df}{dx} = 2x + 2y\frac{dy}{dx}. \tag{9.1}$$

From an equation like $y = 1 - x^2$ relating x and y, we could find dy in terms of dx even if the equation were not solvable for y. Here we get

$$dy = -2x\,dx.$$

Eliminating dy from df, we have

$$df = (2x - 4xy)dx \quad \text{or} \quad \frac{df}{dx} = 2x - 4xy.$$

To minimize f, we set $df/dx = 0$ (or in the differential notation we set $df = 0$ for arbitrary dx). This gives

$$2x - 4xy = 0.$$

This equation must now be solved simultaneously with the equation of the curve $y = 1 - x^2$. We get $2x - 4x(1 - x^2) = 0$, $x = 0$ or $\pm\sqrt{1/2}$ as before.

To test for maxima or minima we need d^2f/dx^2. Differentiating df/dx in (9.1) with respect to x, we get

$$\frac{d^2f}{dx^2} = 2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}.$$

At $x = 0$, we find $y = 1$, $dy/dx = 0$, $d^2y/dx^2 = -2$, so

$$\frac{d^2f}{dx^2} = 2 - 4 = -2;$$

this is a maximum point. At $x = \pm\sqrt{1/2}$, we find

$$y = \frac{1}{2}, \quad \frac{dy}{dx} = \mp\sqrt{2}, \quad \frac{d^2y}{dx^2} = -2,$$

so

$$\frac{d^2 f}{dx^2} = 2 + 4 - 2 = 4;$$

this point is the required minimum. Notice particularly here that you could do every step of (b) even if the equation of the curve could not be solved for y.

We can do problems with several independent variables by methods similar to those we have just used in Example 1. Consider this problem.

▶ **Example 2.** Find the shortest distance from the origin to the plane $x - 2y - 2z = 3$.

We want to minimize the distance $d = \sqrt{x^2 + y^2 + z^2}$ from the origin to a point (x, y, z) on the plane. This is equivalent to minimizing $f = d^2 = x^2 + y^2 + z^2$ if $x - 2y - 2z = 3$. We can eliminate one variable, say x, from f using the equation of the plane. Then we have

$$f = (3 + 2y + 2z)^2 + y^2 + z^2.$$

Here f is a function of the two independent variables y and z, so to minimize f we set $\partial f/\partial y = 0, \partial f/\partial z = 0$.

$$\frac{\partial f}{\partial y} = 2(3 + 2y + 2z) \cdot 2 + 2y = 0,$$
$$\frac{\partial f}{\partial z} = 2(3 + 2y + 2z) \cdot 2 + 2z = 0.$$

Solving these equations for y and z, we get $y = z = -2/3$, so from the equation of the plane we get $x = 1/3$. Then

$$f_{\min} = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = 1, \quad d_{\min} = 1.$$

It is clear from the geometry that there *is* a minimum distance from the origin to a plane; therefore this is it without a second-derivative test. (Also see Chapter 3, Section 5 for another way to do this problem.)

Problems with any number of variables *can* be done this way, or by method (b) if the equations are implicit.

(c) Lagrange Multipliers However, methods (a) and (b) can involve an enormous amount of algebra. We can shortcut this algebra by a process known as the method of *Lagrange multipliers* or undetermined multipliers. We want to consider a problem like the one we discussed in (a) or (b). In general, we want to find the maximum or minimum of a function $f(x, y)$, where x and y are related by an equation $\phi(x, y) = \text{const}$. Then f is really a function of one variable (say x). To find the maximum or minimum points of f, we set $df/dx = 0$ or $df = 0$ as in (9.1). Since $\phi = \text{const.}$, we get $d\phi = 0$.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0,$$
$$(9.2) \qquad d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0.$$

In method (b) we solved the $d\phi$ equation for dy in terms of dx and substituted it into df; this often involves messy algebra. Instead, we shall multiply the $d\phi$ equation by λ (this is the undetermined multiplier—we shall find its value later) and add it to the df equation; then we have

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)dy=0. \tag{9.3}$$

We now pick λ so that

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0. \tag{9.4}$$

[That is, we pick $\lambda=-(\partial f/\partial y)/(\partial\phi/\partial y)$, but it isn't necessary to write it in this complicated form! In fact, this is exactly the point of the Lagrange multiplier λ; by using the abbreviation λ for a complicated expression, we avoid some algebra.] Then from (9.3) and (9.4) we have

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0. \tag{9.5}$$

Equations (9.4), (9.5), and $\phi(x,y)=$ const. can now be solved for the three unknowns x, y, λ. We don't actually want the value of λ, but often the algebra is simpler if we do find it and use it in finding x and y which we do want. Note that equations (9.4) and (9.5) are exactly the equations we would write if we had a function

$$F(x,y)=f(x,y)+\lambda\phi(x,y) \tag{9.6}$$

of two independent variables x and y and we wanted to find its maximum and minimum values. Actually, of course, x and y are not independent; they are related by the ϕ equation. However, (9.6) gives us a simple way of stating and remembering how to get equations (9.4) and (9.5). Thus we can state the method of Lagrange multipliers in the following way:

(9.7) To find the maximum or minimum values of $f(x,y)$ when x and y are related by the equation $\phi(x,y)=$ const., form the function $F(x,y)$ as in (9.6) and set the two partial derivatives of F equal to zero [equations (9.4) and (9.5)]. Then solve these two equations and the equation $\phi(x,y)=$ const. for the three unknowns x, y, and λ.

As a simple illustration of the method we shall do the problem of Example 1 by Lagrange multipliers. Here

$$f(x,y)=x^2+y^2, \qquad \phi(x,y)=y+x^2=1,$$

and we write the equations to minimize

$$F(x,y)=f+\lambda\phi=x^2+y^2+\lambda(y+x^2),$$

namely

$$\begin{aligned} \frac{\partial F}{\partial x} &= 2x + \lambda \cdot 2x = 0, \\ \frac{\partial F}{\partial y} &= 2y + \lambda = 0. \end{aligned} \tag{9.8}$$

We solve these simultaneously with the ϕ equation $y + x^2 = 1$. From the first equation in (9.8), either $x = 0$ or $\lambda = -1$. If $x = 0$, $y = 1$ from the ϕ equation (and $\lambda = -2$). If $\lambda = -1$, the second equation gives $y = \frac{1}{2}$, and then the ϕ equation gives $x^2 = \frac{1}{2}$. These are the same values we had before. The method offers nothing new in testing whether we have found a maximum or a minimum, so we shall not repeat that work; if it is possible to see from the geometry or the physics what we have found, we don't bother to test.

Lagrange multipliers simplify the work enormously in more complicated problems. Consider this problem.

▶ **Example 3.** Find the volume of the largest rectangular parallelepiped (that is, box), with edges parallel to the axes, inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let the point (x, y, z) be the corner in the first octant where the box touches the ellipsoid. Then (x, y, z) satisfies the ellipsoid equation and the volume of the box is $8xyz$ (since there are 8 octants). Our problem is to maximize $f(x, y, z) = 8xyz$, where x, y, z are related by the ellipsoid equation

$$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

By the method of Lagrange multipliers we write

$$F(x, y, z) = f + \lambda\phi = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

and set the three partial derivatives of F equal to 0:

$$\begin{aligned} \frac{\partial F}{\partial x} &= 8yz + \lambda \cdot \frac{2x}{a^2} = 0, \\ \frac{\partial F}{\partial y} &= 8xz + \lambda \cdot \frac{2y}{b^2} = 0, \\ \frac{\partial F}{\partial z} &= 8xy + \lambda \cdot \frac{2z}{c^2} = 0. \end{aligned}$$

We solve these three equations and the equation $\phi = 1$ simultaneously for x, y, z, and λ. (Although we don't *have* to find λ, it may be simpler to find it first.) Multiply the first equation by x, the second by y, and the third by z, and add to get

$$3 \cdot 8xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0.$$

Using the equation of the ellipsoid, we can simplify this to

$$24xyz + 2\lambda = 0 \qquad \text{or} \qquad \lambda = -12xyz.$$

Substituting λ into the $\partial F/\partial x$ equation, we find that

$$8yz - 12xyz \cdot \frac{2x}{a^2} = 0.$$

From the geometry it is clear that the corner of the box should not be where y or z is equal to zero, so we divide by yz and solve for x, getting

$$x^2 = \frac{1}{3}a^2.$$

The other two equations could be solved in the same way. However, it is pretty clear from symmetry that the solutions will be $y^2 = \frac{1}{3}b^2$ and $z^2 = \frac{1}{3}c^2$. Then the maximum volume is

$$8xyz = \frac{8abc}{3\sqrt{3}}.$$

You might contrast this fairly simple algebra with what would be involved in method (a). There you would have to solve the ellipsoid equation for, say, z, substitute this into the volume formula, and then differentiate the square root. Even by method (b) you would have to find $\partial z/\partial x$ or similar expressions from the ellipsoid equation.

We should show that the Lagrange multiplier method is justified for problems involving several independent variables. We want to find maximum or minimum values of $f(x, y, z)$ if $\phi(x, y, z) = \text{const.}$ (You might note at each step that the proof could easily be extended to more variables.) We take differentials of both the f and the ϕ equations. Since $\phi = \text{const.}$, we have $d\phi = 0$. We *put* $df = 0$ because we want maximum and minimum values of f. Thus we write

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0, \\ d\phi &= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0. \end{aligned} \tag{9.9}$$

We *could* find dz from the $d\phi$ equation and substitute it into the df equation; this corresponds to method (b) and may involve complicated algebra. Instead, we form the sum $F = f + \lambda\phi$ and find, using (9.9),

$$\begin{aligned} dF &= df + \lambda\, d\phi \\ &= \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right) dz. \end{aligned} \tag{9.10}$$

There are two independent variables in this problem (since x, y, and z are related by $\phi = \text{const.}$). Suppose x and y are the independent ones; then z is determined from the ϕ equation. Similarly, dx and dy may have any values we choose, and dz is determined. Let us select λ so that

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0. \tag{9.11}$$

Then from (9.10), for $dy = 0$, we get

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \tag{9.12}$$

and for $dx = 0$ we get

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0. \tag{9.13}$$

We can state a rule similar to (9.7) for obtaining equations (9.11), (9.12), and (9.13).

(9.14) To find the maximum and minimum values of $f(x, y, z)$ if $\phi(x, y, z) = \text{const.}$, we form the function $F = f + \lambda\phi$ and set the three partial derivatives of F equal to zero. We solve these equations and the equation $\phi = \text{const.}$ for x, y, z, and λ. (For a problem with still more variables there are more equations, but no change in method.)

It is interesting to consider the geometric meaning of equations (9.9) to (9.13). Recall that x, y, z are related by the equation $\phi(x, y, z) = \text{const.}$ We might, for example, think of solving the ϕ equation for $z = z(x, y)$. Then x and y are independent variables, and z is a function of them. Geometrically, $z = z(x, y)$ is a surface as in Figure 3.2. If we start at the point P of this surface (see Figure 3.2) and increase x by dx, y by dy, and z by dz as given in equation (3.6), we are at a point on the plane tangent to the surface at P. That is, the vector $d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$ ($\overrightarrow{PB}$ in Figure 3.2) lies in the tangent plane of the surface. Now the second equation in (9.9) is a dot product [see Chapter 3, equation (4.10)] of $d\mathbf{r}$ with the vector

$$\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}$$

(called the gradient of ϕ and written grad ϕ; see Chapter 6, Section 6ff.). We could write the second equation in (9.9) as $d\phi = (\text{grad } \phi) \cdot d\mathbf{r} = 0$. Recall [Chapter 3, equation (4.12)] that if the dot product of two vectors is zero, the vectors are perpendicular. Thus since $d\mathbf{r}$ lies anywhere in the plane tangent to the surface $\phi = \text{const.}$, (9.9) says that grad ϕ is perpendicular to this plane, or perpendicular to the surface $\phi = \text{const.}$ at P. The first of equations (9.9) says that grad f is also perpendicular to this plane. Thus grad ϕ and grad f are in the same direction, so their components are proportional; this is what equations (9.11), (9.12), and (9.13) say. We can also say that the surfaces $\phi = \text{const.}$ and $f = \text{const.}$ are tangent to each other at P; that is, they have the same tangent plane and their normals, grad ϕ and grad f, are in the same direction.

We can also use the method of Lagrange multipliers if there are several conditions (ϕ equations). Suppose we want to find the maximum or minimum of $f(x, y, z, w)$ if $\phi_1(x, y, z, w) = \text{const.}$ and $\phi_2(x, y, z, w) = \text{const.}$ There are two independent variables, say x and y. We write

$$\begin{aligned} df &= \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial w}\,dw = 0, \\ d\phi_1 &= \frac{\partial \phi_1}{\partial x}\,dx + \frac{\partial \phi_1}{\partial y}\,dy + \frac{\partial \phi_1}{\partial z}\,dz + \frac{\partial \phi_1}{\partial w}\,dw = 0, \\ d\phi_2 &= \frac{\partial \phi_2}{\partial x}\,dx + \frac{\partial \phi_2}{\partial y}\,dy + \frac{\partial \phi_2}{\partial z}\,dz + \frac{\partial \phi_2}{\partial w}\,dw = 0. \end{aligned} \tag{9.15}$$

Again we *could* use the $d\phi_1$ and $d\phi_2$ equations to eliminate dz and dw from df (method b), but the algebra is forbidding! Instead, by the Lagrange multiplier method we form the function $F = f + \lambda_1\phi_1 + \lambda_2\phi_2$ and write, using (9.15),

$$\begin{aligned} dF &= df + \lambda_1\,d\phi_1 + \lambda_2\,d\phi_2 \\ &= \left(\frac{\partial f}{\partial x} + \lambda_1\frac{\partial \phi_1}{\partial x} + \lambda_2\frac{\partial \phi_2}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda_1\frac{\partial \phi_1}{\partial y} + \lambda_2\frac{\partial \phi_2}{\partial y}\right)dy \\ &+ \left(\frac{\partial f}{\partial z} + \lambda_1\frac{\partial \phi_1}{\partial z} + \lambda_2\frac{\partial \phi_2}{\partial z}\right)dz + \left(\frac{\partial f}{\partial w} + \lambda_1\frac{\partial \phi_1}{\partial w} + \lambda_2\frac{\partial \phi_2}{\partial w}\right)dw. \end{aligned} \tag{9.16}$$

We determine λ_1 and λ_2 from the two equations

$$\begin{aligned} \frac{\partial f}{\partial z} + \lambda_1\frac{\partial \phi_1}{\partial z} + \lambda_2\frac{\partial \phi_2}{\partial z} &= 0, \\ \frac{\partial f}{\partial w} + \lambda_1\frac{\partial \phi_1}{\partial w} + \lambda_2\frac{\partial \phi_2}{\partial w} &= 0. \end{aligned} \tag{9.17}$$

Then for $dy = 0$, we have

$$\frac{\partial f}{\partial x} + \lambda_1\frac{\partial \phi_1}{\partial x} + \lambda_2\frac{\partial \phi_2}{\partial x} = 0 \tag{9.18}$$

and for $dx = 0$, we have

$$\frac{\partial f}{\partial y} + \lambda_1\frac{\partial \phi_1}{\partial y} + \lambda_2\frac{\partial \phi_2}{\partial y} = 0. \tag{9.19}$$

As before, we can remember the method of finding (9.17), (9.18), and (9.19) by thinking:

(9.20) To find the maximum or minimum of f subject to the conditions $\phi_1 = \text{const.}$ and $\phi_2 = \text{const.}$, define $F = f + \lambda_1\phi_1 + \lambda_2\phi_2$ and set each of the partial derivatives of F equal to zero. Solve these equations and the ϕ equations for the variables and the λ's.

Example 4. Find the minimum distance from the origin to the intersection of $xy = 6$ with $7x + 24z = 0$.

We are to minimize $x^2 + y^2 + z^2$ subject to the two conditions $xy = 6$ and $7x + 24z = 0$. By the Lagrange multiplier method, we find the three partial derivatives of

$$F = x^2 + y^2 + z^2 + \lambda_1(7x + 24z) + \lambda_2 xy$$

and set each of them equal to zero. We get

$$
\begin{aligned}
2x + 7\lambda_1 + \lambda_2 y &= 0,\\
2y + \lambda_2 x &= 0,\\
2z + 24\lambda_1 &= 0.
\end{aligned}
\tag{9.21}
$$

These equations can be solved with $xy = 6$ and $7x + 24z = 0$ to get (Problem 10)

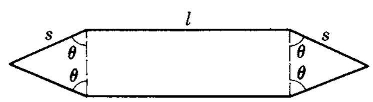
$$x = \pm 12/5, \qquad y = \pm 5/2, \qquad z = \mp 7/10.$$

Then the required minimum distance is (Problem 10)

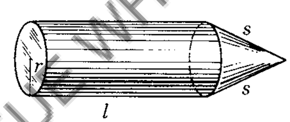
$$d = \sqrt{x^2 + y^2 + z^2} = 5/\sqrt{2} = 3.54.$$

▶ PROBLEMS, SECTION 9

1. What proportions will maximize the area shown in the figure (rectangle with isosceles triangles at its ends) if the perimeter is given?



2. What proportions will maximize the volume of a projectile in the form of a circular cylinder with one conical end and one flat end, if the surface area is given?



3. Find the largest rectangular parallelepiped (box) that can be shipped by parcel post (length plus girth = 108 in).

4. Find the largest box (with faces parallel to the coordinate axes) that can be inscribed in

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1.$$

5. Find the point on $2x + 3y + z - 11 = 0$ for which $4x^2 + y^2 + z^2$ is a minimum.

6. A box has three of its faces in the coordinate planes and one vertex on the plane $2x + 3y + 4z = 6$. Find the maximum volume for the box.

7. Repeat Problem 6 if the plane is $ax + by + cz = d$.

8. A point moves in the (x, y) plane on the line $2x + 3y - 4 = 0$. Where will it be when the sum of the squares of its distances from $(1, 0)$ and $(-1, 0)$ is smallest?

9. Find the largest triangle that can be inscribed in the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ (assume the triangle symmetric about one axis of the ellipse with one side perpendicular to this axis).

10. Complete Example 4 above.

11. Find the shortest distance from the origin to the line of intersection of the planes $2x + y - z = 1$ and $x - y + z = 2$.

12. Find the right triangular prism of given volume and least area if the base is required to be a right triangle.

10. ENDPOINT OR BOUNDARY POINT PROBLEMS

So far we have been assuming that if there is a maximum or minimum point, calculus will find it. Some simple examples (see Figures 10.1 to 10.4) show that this may not be true. Suppose, in a given problem, x can have values only between 0 and 1; this sort of restriction occurs frequently in applications. For example the *graph* of $f(x) = 2 - x^2$ exists for all real x, but if $x = |\cos\theta|$, θ real, the graph has no meaning except for $0 \le x \le 1$. As another example, suppose x is the length of a rectangle whose perimeter is 2; then $x < 0$ is meaningless in this problem since x is a length, and $x > 1$ is impossible because the perimeter is 2. Let us ask for the largest and smallest values of each of the functions in Figures 10.1 to 10.4 for $0 \le x \le 1$. In Figure 10.1, calculus will give us the minimum point, but the maximum of $f(x)$ *for* x *between* 0 *and* 1 occurs at $x = 1$ and cannot be obtained by calculus, since $f'(x) \neq 0$ there. In Figure 10.2, both the maximum and the minimum of $f(x)$ are at endpoints, the maximum at $x = 0$ and the minimum at $x = 1$. In Figure 10.3 a relative maximum at P and a relative minimum at Q are given by calculus, but the absolute minimum between 0 and 1 occurs at $x = 0$, and the absolute maximum at $x = 1$. Here is a practical example of this sort of function. It is said that geographers used to give as the highest point in Florida the top of the highest hill; then it was found that the highest point is on the Alabama border! [See H. A. Thurston, *American Mathematical Monthly*, vol. 68 (1961), pp. 650-652. A later paper, same journal, vol. 98, (1991), pp. 752-3, reports that the high point is actually just south of the Alabama border, but gives another example of a geographic boundary point maximum.] Figure 10.4 illustrates another way in which calculus may fail to give us a desired maximum or minimum point; here the derivative is discontinuous at the maximum point.

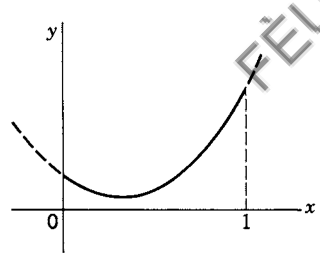


Figure 10.1

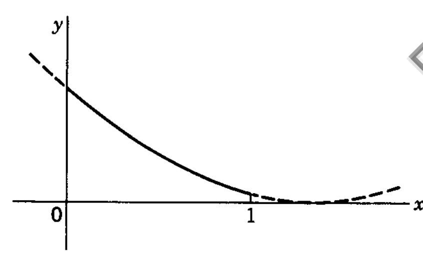


Figure 10.2

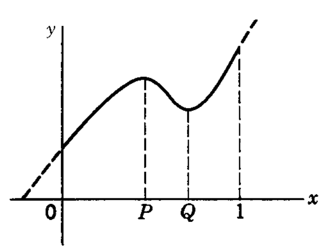


Figure 10.3

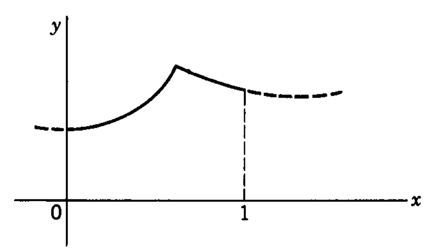


Figure 10.4

These are difficulties we must watch out for whenever there is any restriction on the values any of the variables may take (or any discontinuity in the functions or their derivatives). These restrictions are not usually stated in so many words;

you have to see them for yourself. For example, if $x^2 + y^2 = 25$, x and y are both between -5 and $+5$. If $y^2 = x^2 - 1$, then $|x|$ must be greater than or equal to 1. If $x = \csc\theta$, where θ is a first-quadrant angle, then $x \geq 1$. If $y = \sqrt{x}$, y' is discontinuous at the origin.

▶ **Example 1.** A piece of wire 40 cm long is to be used to form the perimeters of a square and a circle in such a way as to make the total area (of square and circle) a maximum. Call the radius of the circle r; then the circumference of the circle is $2\pi r$. A length $40 - 2\pi r$ is left for the four sides of the square, so one side is $10 - \frac{1}{2}\pi r$. The total area is

$$A = \pi r^2 + (10 - \tfrac{1}{2}\pi r)^2.$$

Then

$$\frac{dA}{dr} = 2\pi r + 2(10 - \tfrac{1}{2}\pi r)(-\tfrac{1}{2}\pi) = 2\pi r\left(1 + \frac{\pi}{4}\right) - 10\pi.$$

If $dA/dr = 0$, we get

$$r\left(1 + \frac{\pi}{4}\right) = 5, \qquad r = 2.8, \qquad A = 56+.$$

Now we might think that this is the maximum area. But let us apply the second derivative test to see whether we have a maximum. We find

$$\frac{d^2A}{dr^2} = 2\pi\left(1 + \frac{\pi}{4}\right) > 0;$$

we have found the *minimum* area! The problem asks for a maximum. One way to find it would be to sketch A as a function of r and look at the graph to see where A has its largest value. A simpler way is this. A is a continuous function of r with a continuous derivative. If there were an interior maximum (that is, one between $r = 0$ and $2\pi r = 40$), calculus would find it. Therefore the maximum must be at one end or the other.

$$\begin{aligned} &\text{At} \quad r = 0, \qquad A = 100. \\ &\text{At} \quad 2\pi r = 40, \qquad r = 20/\pi, \qquad A = 400/\pi = 127+. \end{aligned}$$

We see that A takes its largest value at $r = 20/\pi$; $A = 400/\pi = 127+$ is then the desired maximum. It corresponds to using all the wire to make a circle; the side of the square is zero.

A similar difficulty can arise in problems with more variables.

▶ **Example 2.** The temperature in a rectangular plate bounded by the lines $x = 0$, $y = 0$, $x = 3$, and $y = 5$ is

$$T = xy^2 - x^2y + 100.$$

Find the hottest and coldest points of the plate.

We first set the partial derivatives of T equal to zero to find any interior maxima and minima. We get

$$\begin{aligned} \frac{\partial T}{\partial x} &= y^2 - 2xy = 0, \\ \frac{\partial T}{\partial y} &= 2xy - x^2 = 0. \end{aligned}$$

The only solution of these equations is $x = y = 0$, for which $T = 100$.

We must next ask whether there are points around the boundary of the plate where T has a value larger or smaller than 100. To see that this might happen, think of a graph of T plotted as a function of x and y; this is a surface above the (x, y) plane. The mathematical surface does not have to stop at $x = 3$ and $y = 5$, but it has no meaning for our problem beyond these values. Just as for the curves in Figures 10.1 to 10.4, the graph of the temperature may be increasing or decreasing as we cross a boundary; calculus will not then give us a zero derivative even though the temperature at the boundary may be larger (or smaller) than at other points of the plate. Thus we must consider the complete boundary of the plate (*not* just the corners!). The lines $x = 0$, $y = 0$, $x = 3$, and $y = 5$ are the boundaries; we consider each of them in turn. On $x = 0$ and $y = 0$ the temperature is 100. On the line $x = 3$, we have

$$T = 3y^2 - 9y + 100.$$

We can use calculus to see whether T has maxima or minima as a function of y along this line. We have

$$\frac{dT}{dy} = 6y - 9 = 0,$$
$$y = \tfrac{3}{2}, \qquad T = 93\tfrac{1}{4}.$$

Similarly, along the line $y = 5$, we find

$$T = 25x - 5x^2 + 100,$$
$$\frac{dT}{dx} = 25 - 10x = 0,$$
$$x = \tfrac{5}{2}, \qquad T = 131\tfrac{1}{4}.$$

Finally, we must find T at the corners.

$$\text{At} \quad (0,0), (0,5), \text{ and } (3,0), \quad T = 100.$$
$$\text{At} \quad (3,5), \quad T = 130.$$

Putting all our results together, we see that the hottest point is $(\frac{5}{2}, 5)$ with $T = 131\frac{1}{4}$, and the coldest point is $(3, \frac{3}{2})$ with $T = 93\frac{1}{4}$.

Example 3. Find the point or points closest to the origin on the surfaces

$$x^2 - 4yz = 8, \tag{10.1a}$$
$$z^2 - x^2 = 1. \tag{10.1b}$$

We want to minimize $f = x^2 + y^2 + z^2$ subject to a condition [(a) or (b)]. If we eliminate x^2 in each case, we have

$$f = 8 + 4yz + y^2 + z^2, \tag{10.2a}$$
$$f = z^2 - 1 + y^2 + z^2 = 2z^2 + y^2 - 1. \tag{10.2b}$$

In both problems (a) and (b) the *mathematical function* $f(y, z)$ is defined for all y and z. *For our problems*, however, this is not true. In (a), since $x^2 \geq 0$, we have

$x^2 = 8 + 4yz \geq 0$ so we are interested in minimum values of $f(y,z)$ in (a) only in the region $yz \geq -2$. [Compare Example 2 where $T(x,y)$ was of interest only inside a rectangle.] Thus we look for "interior" minima in (a) satisfying $yz \geq -2$; then we substitute $z = -2/y$ into (10.2a) and find any minima on the boundary of the region of interest. In (b), since $x^2 = z^2 - 1 \geq 0$, we must have $z^2 \geq 1$. Again we try to find "interior" minima satisfying $z^2 \geq 1$; then we set $z^2 = 1$ and look for boundary minima. We now carry out these steps.

From (10.2a), we find

$$\left.\begin{aligned} \frac{\partial f}{\partial y} &= z + 2y = 0, \\ \frac{\partial f}{\partial z} &= y + 2z = 0, \end{aligned}\right\} \quad y = z = 0. \tag{10.3a}$$

These values satisfy the condition $yz > -2$ and so give points inside the region of interest. We find from (10.1a), $x^2 = 8$, $x = \pm 2\sqrt{2}$; the points are $(\pm 2\sqrt{2}, 0, 0)$ at distance $2\sqrt{2}$ from the origin. Next we consider the boundary $x = 0$, $z = -2/y$; from (10.2a),

$$f = 0 + y^2 + \frac{4}{y^2}, \qquad \frac{df}{dy} = 2y - \frac{8}{y^3} = 0,$$
$$y^4 = 4, \qquad y = \pm\sqrt{2}, \qquad z = -2/y = \mp\sqrt{2}.$$

Remembering that $x = 0$, we have the points $(0, \sqrt{2}, -\sqrt{2})$ and $(0, -\sqrt{2}, \sqrt{2})$ at distance 2 from the origin. Since $2 < 2\sqrt{2}$, these boundary points are closest to the origin.

$$\text{Answer to (a): } (0, \sqrt{2}, -\sqrt{2}), (0, -\sqrt{2}, \sqrt{2}). \tag{10.4a}$$

From (10.2b) we find

$$\left.\begin{aligned} \frac{\partial f}{\partial y} &= 2y = 0, \\ \frac{\partial f}{\partial z} &= 4z = 0. \end{aligned}\right\} \quad y = z = 0. \tag{10.3b}$$

Since $z = 0$ does not satisfy $z^2 \geq 1$, there is no minimum point *inside the region of interest*, so we look at the boundary $z^2 = 1$. From (10.1b), $x = 0$, and from (10.2b)

$$f = y^2 + 1, \qquad \frac{df}{dy} = 2y = 0, \qquad y = 0.$$

Thus we find the points $(0, 0, \pm 1)$ at distance 1 from the origin. Since the geometry tells us that there must be a point or points closest to the origin, and calculus tells us that these are the only possible minimum points, these must be the desired points.

$$\text{Answer to (b): } (0, 0, \pm 1). \tag{10.4b}$$

In both these problems, we could have avoided having to consider the boundary of the region of interest by eliminating z to obtain f as a function of x and y. Since

x and y are allowed by (10.1a) or (10.1b) to take *any* values, there are no boundaries to the region of interest. In (b) this is a satisfactory method; in (a) the algebra is complicated. In both problems, Lagrange multipliers offer a more routine method. For example, in (a) we write

$$
\begin{aligned}
&F = x^2 + y^2 + z^2 + \lambda(x^2 - 4yz); \\
&\frac{\partial F}{\partial x} = 2x(1+\lambda) = 0, && x = 0 \text{ or } \lambda = -1; \\
&\frac{\partial F}{\partial y} = 2y - 4\lambda z = 0; && \text{if } \lambda = -1,\ y = z = 0,\ x^2 = 8; \\
&\frac{\partial F}{\partial z} = 2z - 4\lambda y = 0; && \text{if } x = 0,\ \lambda = \frac{y}{2z} = \frac{z}{2y},\ y^2 = z^2 = 2.
\end{aligned}
$$

We obtain the same results as above, namely, the points $(\pm 2\sqrt{2}, 0, 0)$, $(0, \pm\sqrt{2}, \mp\sqrt{2})$; the points $(0, \sqrt{2}, -\sqrt{2})$, $(0, -\sqrt{2}, \sqrt{2})$ are closer to the origin by inspection. Part (b) can be done similarly (Problem 14).

We see that using Lagrange multipliers may simplify maximum and minimum problems. However, the Lagrange multiplier method still relies on calculus; consequently, it can work only if the maximum and minimum can be found by calculus using *some* set of variables (x and y, *not* y and z, in Example 3). For example, a problem in which the maximum or minimum occurs at endpoints in all variables cannot be done by *any* method that depends on setting derivatives equal to zero.

Example 4. Find the maximum value of $y - x$ for nonnegative x and y if $x^2 + y^2 = 1$.

Here we must have both x and y between 0 and 1. Then the values $y = 1$ and $x = 0$ give $y - x$ its largest value; these are both endpoint values which cannot be found by calculus.

PROBLEMS, SECTION 10

1. Find the shortest distance from the origin to $x^2 - y^2 = 1$.
2. Find the largest and smallest distances from the origin to the conic whose equation is $5x^2 - 6xy + 5y^2 - 32 = 0$ and hence determine the lengths of the semiaxes of this conic.
3. Repeat Problem 2 for the conic $6x^2 + 4xy + 3y^2 = 28$.

Find the shortest distance from the origin to each of the following quadric surfaces. *Hint:* See Example 3 above.

4. $3x^2 + y^2 - 4xz = 4$.
5. $2z^2 + 6xy = 3$.
6. $4y^2 + 2z^2 + 3xy = 18$.
7. Find the largest z for which $2x + 4y = 5$ and $x^2 + z^2 = 2y$.
8. If the temperature at the point (x, y, z) is $T = xyz$, find the hottest point (or points) on the surface of the sphere $x^2 + y^2 + z^2 = 12$, and find the temperature there.
9. The temperature T of the disk $x^2 + y^2 \le 1$ is given by $T = 2x^2 - 3y^2 - 2x$. Find the hottest and coldest points of the disk.

10. The temperature at a point (x, y, z) in the ball $x^2+y^2+z^2 \leq 1$ is given by $T = y^2+xz$. Find the largest and smallest values which T takes

(a) on the circle $y = 0$, $x^2 + z^2 = 1$,

(b) on the surface $x^2 + y^2 + z^2 = 1$,

(c) in the whole ball.

11. The temperature of a rectangular plate bounded by the lines $x = \pm 1$, $y = \pm 1$, is given by $T = 2x^2 - 3y^2 - 2x + 10$. Find the hottest and coldest points of the plate.

12. Find the largest and smallest values of the sum of the acute angles that a line through the origin makes with the three coordinate axes.

13. Find the largest and smallest values of the sum of the acute angles that a line through the origin makes with the three coordinate planes.

14. Do Example 3b using Lagrange multipliers.

11. CHANGE OF VARIABLES

One important use of partial differentiation is in making changes of variables (for example, from rectangular to polar coordinates). This may give a simpler expression or a simpler differential equation or one more suited to the physical problem one is doing. For example, if you are working with the vibration of a circular membrane, or the flow of heat in a circular cylinder, polar coordinates are better; for a problem about sound waves in a room, rectangular coordinates are better. Consider the following problems.

Example 1. Make the change of variables $r = x + vt$, $s = x - vt$ in the *wave equation*

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 0, \tag{11.1}$$

and solve the equation. (Also see Chapter 13, Sections 1, 4, and 6.)

We use the equations

$$\begin{aligned} r &= x + vt, \\ s &= x - vt, \end{aligned} \tag{11.2}$$

and equations like (7.2) to find

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial F}{\partial r} + \frac{\partial F}{\partial s} = \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)F, \\ \frac{\partial F}{\partial t} &= \frac{\partial F}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial t} = v\frac{\partial F}{\partial r} - v\frac{\partial F}{\partial s} = v\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)F. \end{aligned} \tag{11.3}$$

It is helpful to say in words what we have written in (11.3): To find the partial of a function with respect to x, we find its partial with respect to r plus its partial with respect to s; to find the partial with respect to t, we find the partial with respect to r minus the partial with respect to s and multiply by the constant v. It is useful to write this in *operator* notation (see Chapter 3, Section 7):

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r} + \frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial t} = v\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right). \tag{11.4}$$

Then from (11.3) and (11.4) we find

(11.5)

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\right) = \frac{\partial^2 F}{\partial r^2} + 2\frac{\partial^2 F}{\partial r \partial s} + \frac{\partial^2 F}{\partial s^2},$$

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial t}\right) = v\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)\left(v\frac{\partial F}{\partial r} - v\frac{\partial F}{\partial s}\right) = v^2\left(\frac{\partial^2 F}{\partial r^2} - 2\frac{\partial^2 F}{\partial r \partial s} + \frac{\partial^2 F}{\partial s^2}\right).$$

Substitute (11.5) into (11.1) to get

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 4\frac{\partial^2 F}{\partial r \partial s} = 0. \tag{11.6}$$

We can easily solve (11.6). We have

$$\frac{\partial^2 F}{\partial r \partial s} = \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial s}\right) = 0,$$

that is, the r derivative of $\partial F/\partial s$ is zero. Then $\partial F/\partial s$ must be independent of r, so $\partial F/\partial s$ = some function of s alone. We integrate with respect to s to find $F = f(s)+$"const."; the "constant" is a constant as far as s is concerned, but it may be any function of r, say $g(r)$, since $(\partial/\partial s)g(r) = 0$. Thus we find that the solution of (11.6) is

$$F = f(s) + g(r). \tag{11.7}$$

Then, using (11.2), we find the solution of (11.1):

$$F = f(x - vt) + g(x + vt), \tag{11.8}$$

where f and g are arbitrary functions. This is known as d'Alembert's solution of the wave equation. Also see Problem 7.23 and Chapter 13, Problem 1.2.

Example 2. Write the Laplace equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0 \tag{11.9}$$

in terms of polar coordinates r, θ, where

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned} \tag{11.10}$$

Note that equations (11.10) give the old variables x and y in terms of the new ones, r and θ, whereas (11.2) gave the new variables r and s in terms of the old ones. In this situation, there are several ways to get equations like (11.3). One way is to write

$$\begin{aligned} \frac{\partial F}{\partial r} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\,\frac{\partial F}{\partial x} + \sin\theta\,\frac{\partial F}{\partial y}, \\ \frac{\partial F}{\partial \theta} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\,\frac{\partial F}{\partial x} + r\cos\theta\,\frac{\partial F}{\partial y}, \end{aligned} \tag{11.11}$$

and then solve (11.11) for $\partial F/\partial x$ and $\partial F/\partial y$ (Problem 5). Another way is to find the needed partial derivatives of r and θ with respect to x and y [for methods and results, see Section 7, Example 6, equation (7.16) and Problem 7.9] and then write as in (11.3), using (7.16),

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial F}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F}{\partial \theta}, \\ \frac{\partial F}{\partial y} &= \frac{\partial F}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial F}{\partial r} + \frac{\cos\theta}{r}\frac{\partial F}{\partial \theta}. \end{aligned} \tag{11.12}$$

In finding the second derivatives, it will be convenient to use the abbreviations $G = \partial F/\partial x$ and $H = \partial F/\partial y$. Thus,

$$\begin{aligned} G &= \frac{\partial F}{\partial x} = \cos\theta\frac{\partial F}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F}{\partial \theta}, \\ H &= \frac{\partial F}{\partial y} = \sin\theta\frac{\partial F}{\partial r} + \frac{\cos\theta}{r}\frac{\partial F}{\partial \theta}. \end{aligned} \tag{11.13}$$

Then

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial G}{\partial x}, \qquad \frac{\partial^2 F}{\partial y^2} = \frac{\partial H}{\partial y}, \qquad \text{so} \qquad \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{\partial G}{\partial x} + \frac{\partial H}{\partial y}. \tag{11.14}$$

Now equations (11.12) are correct for *any* function F; in particular they are correct if we replace F by G or by H. Let us replace F by G in the first equation (11.12) and replace F by H in the second equation. Then we have

$$\begin{aligned} \frac{\partial G}{\partial x} &= \cos\theta\frac{\partial G}{\partial r} - \frac{\sin\theta}{r}\frac{\partial G}{\partial \theta}, \\ \frac{\partial H}{\partial y} &= \sin\theta\frac{\partial H}{\partial r} + \frac{\cos\theta}{r}\frac{\partial H}{\partial \theta}. \end{aligned} \tag{11.15}$$

Substituting (11.15) into (11.14), we get

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \cos\theta\frac{\partial G}{\partial r} + \sin\theta\frac{\partial H}{\partial r} + \frac{1}{r}\left(\cos\theta\frac{\partial H}{\partial \theta} - \sin\theta\frac{\partial G}{\partial \theta}\right). \tag{11.16}$$

We find the four partial derivatives of G and H which we need in (11.16), by differentiating the right-hand sides of equations (11.13).

$$\begin{aligned} \frac{\partial G}{\partial r} &= \cos\theta\frac{\partial^2 F}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 F}{\partial r\partial \theta} + \frac{\sin\theta}{r^2}\frac{\partial F}{\partial \theta}, \\ \frac{\partial H}{\partial r} &= \sin\theta\frac{\partial^2 F}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2 F}{\partial r\partial \theta} - \frac{\cos\theta}{r^2}\frac{\partial F}{\partial \theta}, \\ \frac{\partial H}{\partial \theta} &= \sin\theta\frac{\partial^2 F}{\partial \theta\partial r} + \cos\theta\frac{\partial F}{\partial r} + \frac{\cos\theta}{r}\frac{\partial^2 F}{\partial \theta^2} - \frac{\sin\theta}{r}\frac{\partial F}{\partial \theta}, \\ \frac{\partial G}{\partial \theta} &= \cos\theta\frac{\partial^2 F}{\partial \theta\partial r} - \sin\theta\frac{\partial F}{\partial r} - \frac{\sin\theta}{r}\frac{\partial^2 F}{\partial \theta^2} - \frac{\cos\theta}{r}\frac{\partial F}{\partial \theta}. \end{aligned} \tag{11.17}$$

We combine these to obtain the expressions needed in (11.16):

$$\begin{aligned} \cos\theta\frac{\partial G}{\partial r} + \sin\theta\frac{\partial H}{\partial r} &= \frac{\partial^2 F}{\partial r^2}, \\ \frac{1}{r}\left(\cos\theta\frac{\partial H}{\partial \theta} - \sin\theta\frac{\partial G}{\partial \theta}\right) &= \frac{1}{r}\left(\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial^2 F}{\partial \theta^2}\right). \end{aligned} \tag{11.18}$$

Finally, substituting (11.18) into (11.16) gives

$$\begin{aligned}\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}&=\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\\&=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}.\end{aligned}\tag{11.19}$$

We next discuss a simple kind of change of variables which is very useful in thermodynamics and mechanics. This process is sometimes known as a *Legendre transformation*. Suppose we are given a function $f(x,y)$; then we can write

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy.\tag{11.20}$$

Let us call $\partial f/\partial x=p$, and $\partial f/\partial y=q$; then we have

$$df=p\,dx+q\,dy.\tag{11.21}$$

If we now subtract from df the quantity $d(qy)$, we have

$$\begin{aligned}df-d(qy)&=p\,dx+q\,dy-q\,dy-y\,dq \qquad \text{or}\\ d(f-qy)&=p\,dx-y\,dq.\end{aligned}\tag{11.22}$$

If we define the function g by

$$g=f-qy,\tag{11.23}$$

then by (11.22)

$$dg=p\,dx-y\,dq.\tag{11.24}$$

Because dx and dq appear in (11.24), it is convenient to think of g as a function of x and q. The partial derivatives of g are then of simple form, namely,

$$\frac{\partial g}{\partial x}=p,\qquad \frac{\partial g}{\partial q}=-y.\tag{11.25}$$

Similarly, we could replace the $p\,dx$ term in df by $-x\,dp$ by considering the function $f-xp$. This sort of change of independent variables is called a Legendre transformation. (For applications, see Problems 10 to 13.) For a discussion of Legendre transformations, see Callen, Chapter 5.

From the equations above, we can find useful relations between partial derivatives. For example, from equations (11.24) and (11.25) we can write

$$\frac{\partial^2 g}{\partial q\partial x}=\left(\frac{\partial p}{\partial q}\right)_x \qquad \text{and} \qquad \frac{\partial^2 g}{\partial x\partial q}=-\left(\frac{\partial y}{\partial x}\right)_q.\tag{11.26}$$

Assuming $\dfrac{\partial^2 g}{\partial q\partial x}=\dfrac{\partial^2 g}{\partial x\partial q}$ (reciprocity relations, see end of Section 1), then we have

$$\left(\frac{\partial p}{\partial q}\right)_x=-\left(\frac{\partial y}{\partial x}\right)_q.\tag{11.27}$$

Many equations like these appear in thermodynamics (see Problems 12 and 13).

PROBLEMS, SECTION 11

1. In the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$$

put $s = y + 2x$, $t = y + 3x$ and show that the equation becomes $\partial^2 z/\partial s \partial t = 0$. Following the method of solving (11.6), solve the equation.

2. As in Problem 1, solve

$$2\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 10\frac{\partial^2 z}{\partial y^2} = 0$$

by making the change of variables $u = 5x - 2y$, $v = 2x + y$.

3. Suppose that $w = f(x, y)$ satisfies

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 1.$$

Put $x = u + v$, $y = u - v$, and show that w satisfies $\partial^2 w/\partial u \partial v = 1$. Hence solve the equation.

4. Verify the chain rule formulas

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial x},$$

and similar formulas for

$$\frac{\partial F}{\partial y}, \qquad \frac{\partial F}{\partial r}, \qquad \frac{\partial F}{\partial \theta},$$

using differentials. For example, write

$$dF = \frac{\partial F}{\partial r}dr + \frac{\partial F}{\partial \theta}d\theta$$

and substitute for dr and $d\theta$:

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy \qquad \text{(and similarly } d\theta\text{)}.$$

Collect coefficients of dx and dy; these are the values of $\partial F/\partial x$ and $\partial F/\partial y$.

5. Solve equations (11.11) to get equations (11.12).

6. Reduce the equation

$$x^2\left(\frac{d^2 y}{dx^2}\right) + 2x\left(\frac{dy}{dx}\right) - 5y = 0$$

to a differential equation with constant coefficients in d^2y/dz^2, dy/dz, and y by the change of variable $x = e^z$. (See Chapter 8, Section 7d.)

7. Change the independent variable from x to θ by $x = \cos\theta$ and show that the Legendre equation

$$(1 - x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

becomes

$$\frac{d^2 y}{d\theta^2} + \cot\theta\frac{dy}{d\theta} + 2y = 0.$$

8. Change the independent variable from x to $u = 2\sqrt{x}$ in the Bessel equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (1-x)y = 0$$

and show that the equation becomes

$$u^2\,\frac{d^2y}{du^2} + u\,\frac{dy}{du} + (u^2-4)y = 0.$$

9. If $x = e^s\cos t$, $y = e^s\sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right).$$

10. Given $du = T\,ds - p\,dv$, find a Legendre transformation giving

(a) a function $f(T, v)$;

(b) a function $h(s, p)$;

(c) a function $g(T, p)$.

Hint for (c): Perform a Legendre transformation on both terms in du.

11. Given $L(q, \dot q)$ such that $dL = \dot p\,dq + p\,d\dot q$, find $H(p, q)$ so that $dH = \dot q\,dp - \dot p\,dq$. *Comments:* L and H are functions used in mechanics called the Lagrangian and the Hamiltonian. The quantities $\dot q$ and $\dot p$ are actually time derivatives of p and q, but you make no use of the fact in this problem. Treat $\dot p$ and $\dot q$ as if they were two more variables having nothing to do with p and q. *Hint:* Use a Legendre transformation. On your first try you will probably get $-H$. Look at the text discussion of Legendre transformations and satisfy yourself that $g = qy - f$ would have been just as satisfactory as $g = f - qy$ in (11.23).

12. Using du in Problem 10, and the text method of obtaining (11.27), show that $\left(\frac{\partial T}{\partial v}\right)_s = -\left(\frac{\partial p}{\partial s}\right)_v$. (This is one of the Maxwell relations in thermodynamics.)

13. As in Problem 12, find three more Maxwell relations by using your results in Problem 10, parts (a), (b), (c).

12. DIFFERENTIATION OF INTEGRALS; LEIBNIZ' RULE

According to the definition of an integral as an antiderivative, if

$$f(x) = \frac{dF(x)}{dx}, \tag{12.1}$$

then

$$\int_a^x f(t)\,dt = F(t)\Big|_a^x = F(x) - F(a), \tag{12.2}$$

where a is a constant. If we differentiate (12.2) with respect to x, we have

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}[F(x) - F(a)] = \frac{dF(x)}{dx} = f(x) \tag{12.3}$$

by (12.1). Similarly,

$$\int_x^a f(t)\,dt = F(a) - F(x),$$

so

$$\frac{d}{dx}\int_x^a f(t)\,dt = -\frac{dF(x)}{dx} = -f(x). \tag{12.4}$$

▶ **Example 1.** Find $\dfrac{d}{dx}\displaystyle\int_{\pi/4}^{x} \sin t\,dt$.

By (12.3), we find immediately that the answer is $\sin x$. We can check this by finding the integral and then differentiating. We get

$$\int_{\pi/4}^{x} \sin t\,dt = -\cos t\Big|_{\pi/4}^{x} = -\cos x + \frac{1}{2}\sqrt{2}$$

and the derivative of this is $\sin x$ as before.

By replacing x in (12.3) by v, and replacing x in (12.4) by u, we can then write

$$\frac{d}{dv}\int_{a}^{v} f(t)\,dt = f(v) \tag{12.5}$$

and

$$\frac{d}{du}\int_{u}^{b} f(t)\,dt = -f(u). \tag{12.6}$$

Suppose u and v are functions of x and we want dI/dx where

$$I = \int_{u}^{v} f(t)\,dt.$$

When the integral is evaluated, the answer depends on the limits u and v. Finding dI/dx is then a partial differentiation problem; I is a function of u and v, which are functions of x. We can write

$$\frac{dI}{dx} = \frac{\partial I}{\partial u}\frac{du}{dx} + \frac{\partial I}{\partial v}\frac{dv}{dx}. \tag{12.7}$$

But $\partial I/\partial v$ means to differentiate I with respect to v when u is a constant; this is just (12.5), so $\partial I/\partial v = f(v)$. Similarly, $\partial I/\partial u$ means that v is constant and we can use (12.6) to get $\partial I/\partial u = -f(u)$. Then we have

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v)\,\frac{dv}{dx} - f(u)\,\frac{du}{dx}. \tag{12.8}$$

▶ **Example 2.** Find dI/dx if $I = \displaystyle\int_{0}^{x^{1/3}} t^2\,dt$.

By (12.8) we get

$$\frac{dI}{dx} = (x^{1/3})^2\frac{d}{dx}(x^{1/3}) = x^{2/3}\cdot\frac{1}{3}x^{-2/3} = \frac{1}{3}.$$

We *could* also integrate first and then differentiate with respect to x:

$$I = \int_{0}^{x^{1/3}} t^2 dt = \frac{t^3}{3}\Big|_{0}^{x^{1/3}} = \frac{x}{3}, \qquad \frac{dI}{dx} = \frac{1}{3}.$$

This last method seems so simple you may wonder why we need (12.8). Look at another example.

Example 3. Find dI/dx if

$$I = \int_{x^2}^{\sin^{-1} x} \frac{\sin t}{t}\, dt.$$

Here the indefinite integral cannot be evaluated in terms of elementary functions; however, we can find dI/dx by using (12.8). We get

$$\begin{aligned}\frac{dI}{dx} &= \frac{\sin(\sin^{-1} x)}{\sin^{-1} x} \frac{1}{\sqrt{1-x^2}} - \frac{\sin x^2}{x^2} \cdot 2x \\ &= \frac{x}{\sqrt{1-x^2}\,\sin^{-1} x} - \frac{2}{x} \sin x^2.\end{aligned}$$

Finally, we may want to find dI/dx when $I = \int_a^b f(x,t)\, dt$, where a and b are constants. Under not too restrictive conditions,

$$\frac{d}{dx}\int_a^b f(x,t)\, dt = \int_a^b \frac{\partial f(x,t)}{\partial x}\, dt; \tag{12.9}$$

that is, we can differentiate under the integral sign. [A set of sufficient conditions for this to be correct would be that $\int_a^b f(x,t)\, dt$ exists, $\partial f/\partial x$ is continuous and $|\partial f(x,t)/\partial x| \le g(t)$, where $\int_a^b g(t)\, dt$ exists. For most practical purposes this means that if both integrals in (12.9) exist, then (12.9) is correct.] Equation (12.9) is often useful in evaluating definite integrals.

Example 4. Find $\int_0^\infty t^n e^{-kt^2}\, dt$ for odd n, $k > 0$.

First we evaluate the integral

$$I = \int_0^\infty te^{-kt^2}\, dt = -\frac{1}{2k} e^{-kt^2}\Big|_0^\infty = \frac{1}{2k}.$$

Now we calculate successive derivatives of I with respect to k.

$$\frac{dI}{dk} = \int_0^\infty -t^2 te^{-kt^2}\, dt = -\frac{1}{2k^2} \qquad \text{or} \qquad \int_0^\infty t^3 e^{-kt^2}\, dt = \frac{1}{2k^2}.$$

Repeating the differentiation with respect to k, we get

$$\begin{aligned}\int_0^\infty -t^2 t^3 e^{-kt^2}\, dt &= -\frac{2}{2k^3} \qquad &\text{or} \qquad \int_0^\infty t^5 e^{-kt^2}\, dt &= \frac{1}{k^3}. \\ \int_0^\infty -t^2 t^5 e^{-kt^2}\, dt &= -\frac{3}{k^4} \qquad &\text{or} \qquad \int_0^\infty t^7 e^{-kt^2} &= \frac{3}{k^4}.\end{aligned}$$

Continuing in this way (Problem 17), we can find the integral of any odd power of t times e^{-kt^2}:

$$\int_0^\infty t^{2n+1} e^{-kt^2}\, dt = \frac{n!}{2k^{n+1}}. \tag{12.10}$$

Your computer may give you this result in terms of the gamma function (see Chapter 11, Sections 1 to 5). The relation is $n! = \Gamma(n+1)$.

▶ **Example 5.** Evaluate

$$I = \int_0^1 \frac{t^a - 1}{\ln t}\, dt, \qquad a > -1. \tag{12.11}$$

First we differentiate I with respect to a, and evaluate the resulting integral.

$$\frac{dI}{da} = \int_0^1 \frac{t^a \ln t}{\ln t}\, dt = \int_0^1 t^a\, dt = \left.\frac{t^{a+1}}{a+1}\right|_0^1 = \frac{1}{a+1}.$$

Now we integrate dI/da with respect to a to get I back again (plus an integration constant):

$$I = \int \frac{da}{a+1} = \ln(a+1) + C. \tag{12.12}$$

If $a = 0$, (12.11) gives $I = 0$ and (12.12) gives $I = C$, so $C = 0$ and we have from (12.12), $I = \ln(a+1)$.

It is convenient to collect formulas (12.8) and (12.9) into one formula known as *Leibniz' rule*:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\, dt = f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx} + \int_u^v \frac{\partial f}{\partial x}\, dt. \tag{12.13}$$

▶ **Example 6.** Find dI/dx if

$$I = \int_x^{2x} \frac{e^{xt}}{t}\, dt.$$

By (12.13) we get

$$\begin{aligned}
\frac{dI}{dx} &= \frac{e^{x\cdot 2x}}{2x}\cdot 2 - \frac{e^{x\cdot x}}{x}\cdot 1 + \int_x^{2x} \frac{te^{xt}}{t}\, dt \\
&= \frac{1}{x}(e^{2x^2} - e^{x^2}) + \left[\frac{e^{xt}}{x}\right]_x^{2x} \\
&= \frac{1}{x}(e^{2x^2} - e^{x^2} + e^{2x^2} - e^{x^2}) = \frac{2}{x}(e^{2x^2} - e^{x^2}).
\end{aligned}$$

Although you can do problems like this by computer, in many cases you can just write down the answer using (12.13) in less time than it takes to type the problem into the computer.

▶ PROBLEMS, SECTION 12

1. If $y = \int_0^{\sqrt{x}} \sin t^2 dt$, find dy/dx.

2. If $s = \int_u^v \frac{1 - e^t}{t} dt$, find $\partial s/\partial v$ and $\partial s/\partial u$ and also their limits as u and v tend to zero.

3. If $z = \int_{\sin x}^{\cos x} \frac{\sin t}{t}\,dt$, find $\frac{dz}{dx}$.

4. Use L'Hôpital's rule to evaluate $\lim_{x\to 2} \frac{1}{x-2}\int_2^x \frac{\sin t}{t}\,dt$.

5. If $u = \int_x^{y-x} \frac{\sin t}{t}\,dt$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial y}{\partial x}$ at $x = \pi/2$, $y = \pi$.
Hint: Use differentials.

6. If $w = \int_{xy}^{2x-3y} \frac{du}{\ln u}$, find $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, and $\frac{\partial y}{\partial x}$ at $x = 3$, $y = 1$.

7. If $\int_u^v e^{-t^2}\,dt = x$ and $u^v = y$, find $\left(\frac{\partial u}{\partial x}\right)_y$, $\left(\frac{\partial u}{\partial y}\right)_x$, and $\left(\frac{\partial y}{\partial x}\right)_u$ at $u = 2$, $v = 0$.

8. If $\int_0^x e^{-s^2}\,ds = u$, find $\frac{dx}{du}$.

9. If $y = \int_0^\pi \sin xt\,dt$, find dy/dx (a) by evaluating the integral and then differentiating, (b) by differentiating first and then evaluating the integral.

10. Find dy/dx explicitly if $y = \int_0^1 \frac{e^{xu}-1}{u}\,du$.

11. Find $\frac{d}{dx}\int_{3-x}^{x^2} (x-t)\,dt$ by evaluating the integral first, and by differentiating first.

12. Find $\frac{d}{dx}\int_x^{x^2} \frac{du}{\ln(x+u)}$.

13. Find $\frac{d}{dx}\int_{1/x}^{2/x} \frac{\sin xt}{t}\,dt$.

14. Given that $\int_0^\infty \frac{dx}{y^2+x^2} = \frac{\pi}{2y}$, differentiate with respect to y and so evaluate

$$\int_0^\infty \frac{dx}{(y^2+x^2)^2}.$$

15. Given that

$$\int_0^\infty e^{-ax}\sin kx\,dx = \frac{k}{a^2+k^2},$$

differentiate with respect to a to show that

$$\int_0^\infty xe^{-ax}\sin kx\,dx = \frac{2ka}{(a^2+k^2)^2}$$

and differentiate with respect to k to show that

$$\int_0^\infty xe^{-ax}\cos kx\,dx = \frac{a^2-k^2}{(a^2+k^2)^2}.$$

16. In kinetic theory we have to evaluate integrals of the form $I = \int_0^\infty t^n e^{-at^2}\,dt$. Given that $\int_0^\infty e^{-at^2}\,dt = \frac{1}{2}\sqrt{\pi/a}$, evaluate I for $n = 2, 4, 6, \cdots, 2m$.

17. Complete Example 4 to obtain (12.10).

18. Show that $u(x, y) = \dfrac{y}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{f(t)\,dt}{(x-t)^2+y^2}$ satisfies $u_{xx} + u_{yy} = 0$.

19. Show that $y = \displaystyle\int_0^x f(u)\sin(x-u)\,du$ satisfies $y'' + y = f(x)$.

20. (a) Show that $y = \displaystyle\int_0^x f(x-t)\,dt$ satisfies $(dy/dx) = f(x)$. (*Hint:* It is helpful to make the change of variable $x - t = u$ in the integral.)

(b) Show that $y = \displaystyle\int_0^x (x-u)f(u)\,du$ satisfies $y'' = f(x)$.

(c) Show that $y = \dfrac{1}{(n-1)!}\displaystyle\int_0^x (x-u)^{n-1} f(u)\,du$ satisfies $y^{(n)} = f(x)$.

▶ 13. MISCELLANEOUS PROBLEMS

1. A function $f(x, y, z)$ is called homogeneous of degree n if $f(tx, ty, tz) = t^n f(x, y, z)$. For example, $z^2 \ln(x/y)$ is homogeneous of degree 2 since

$$(tz)^2 \ln \frac{tx}{ty} = t^2\left(z^2 \ln \frac{x}{y}\right).$$

Euler's theorem on homogeneous functions says that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf.$$

Prove this theorem. *Hints:* Differentiate $f(tx, ty, tz) = t^n f(x, y, z)$ with respect to t, and then let $t = 1$. It is convenient to call $\partial f/\partial(tx) = f_1$ (that is, the partial derivative of f with respect to its first variable), $f_2 = \partial f/\partial(ty)$, and so on. Or, you can at first call $tx = u$, $ty = v$, $tz = w$. (Both the definition and the theorem can be extended to any number of variables.)

2. (a) Given the point $(2, 1)$ in the (x, y) plane and the line $3x + 2y = 4$, find the distance from the point to the line by using the method of Chapter 3, Section 5.

(b) Solve part (a) by writing a formula for the distance from $(2, 1)$ to (x, y) and minimizing the distance (use Lagrange multipliers).

(c) Derive the formula

$$D = \left|\frac{ax_0 + by_0 - c}{\sqrt{a^2 + b^2}}\right|$$

for the distance from (x_0, y_0) to $ax + by = c$ by the methods suggested in parts (a) and (b).

In Problems 3 to 6, assume that x, y and r, θ are rectangular and polar coordinates.

3. Find $\dfrac{\partial^2 y}{\partial x \partial \theta}$.

4. Find $\dfrac{\partial^2 r}{\partial \theta \partial y}$.

5. Given $z = y^2 - 2x^2$, find $\left(\dfrac{\partial z}{\partial x}\right)_r$, $\left(\dfrac{\partial z}{\partial \theta}\right)_x$, $\dfrac{\partial^2 z}{\partial x \partial \theta}$.

6. If $z = r^2 - x^2$, find $\left(\dfrac{\partial z}{\partial r}\right)_\theta$, $\left(\dfrac{\partial z}{\partial \theta}\right)_r$, $\dfrac{\partial^2 z}{\partial r \partial \theta}$, $\left(\dfrac{\partial z}{\partial x}\right)_y$.

7. About how much (in percent) does an error of 1% in x and y affect x^3y^2?

8. Assume that the earth is a perfect sphere. Suppose that a rope lies along the equator with its ends fastened so that it fits exactly. Now let the rope be made 2 ft longer, and let it be held up the same distance above the surface of the Earth at all points of the equator. About how high up is it? (For example, could you crawl under? Could a fly?) Answer the same questions for the moon.

9. If $z = xy$ and $\begin{cases} 2x^3 + 2y^3 = 3t^2, \\ 3x^2 + 3y^2 = 6t, \end{cases}$ find dz/dt.

10. If $w = (r\cos\theta)^{r\sin\theta}$, find $\partial w/\partial\theta$.

11. If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ by implicit differentiation.

12. Given $z = r^2 + s^2 + rst$, $r^4 + s^4 + t^4 = 2r^2s^2t^2 + 10$, find $(\partial z/\partial r)_t$ when $r = 2$, $s = t = 1$.

13. Given $\begin{cases} 2t + e^x = s - \cos y - 2, \\ 2s - t = \sin y + x - 1, \end{cases}$ find $\left(\dfrac{\partial s}{\partial t}\right)_y$ at $(x, y, s, t) = (0, \pi/2, -1, -2)$.

14. If $w = f(x, s, t)$, $s = 2x + y$, $t = 2x - y$, find $(\partial w/\partial x)_y$ in terms of f and its derivatives.

15. If $w = f(x, x^2 + y^2, 2xy)$, find $(\partial w/\partial x)_y$ (compare Problem 14).

16. If $z = \dfrac{1}{x} f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + z = 0$.

17. Find the shortest distance from the origin to the surface $x = yz + 10$.

18. Find the shortest distance from the origin to the line of intersection of the planes

$$\begin{aligned} 2x - 3y + z &= 5, \\ 3x - y - 2z &= 11, \end{aligned}$$

(a) using vector methods (see Chapter 3, Section 5);

(b) using Lagrange multipliers.

19. Find by the Lagrange multiplier method the largest value of the product of three positive numbers if their sum is 1.

20. Find the largest and smallest values of $y = 4x^3 + 9x^2 - 12x + 3$ if $x = \cos\theta$.

21. Find the hottest and coldest points on a bar of length 5 if $T = 4x - x^2$, where x is the distance measured from the left end.

22. Find the hottest and coldest points of the region $y^2 \le x < 5$ if $T = x^2 - y^2 - 3x$.

23. Find $\dfrac{d}{dt}\displaystyle\int_0^{\sin t} \frac{\sin^{-1} x}{x}\,dx$.

24. Find $\dfrac{d}{dx}\displaystyle\int_{t=1/x}^{t=2/x} \frac{\cosh xt}{t}\,dt$.

25. Find $\dfrac{d}{dx}\displaystyle\int_1^{1/x} \frac{e^{xt}}{t}\,dt$.

26. Find $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \frac{\sin xt}{t}\,dt$.

27. Show that $\dfrac{d}{dx}\displaystyle\int_{\cos x}^{\sin x} \sqrt{1 - t^2}\,dt = 1$.

28. In discussing the velocity distribution of molecules of an ideal gas, a function $F(x, y, z) = f(x)f(y)f(z)$ is needed such that $d(\ln F) = 0$ when $\phi = x^2 + y^2 + z^2 =$ const. Then by the Lagrange multiplier method $d(\ln F + \lambda\phi) = 0$. Use this to show that

$$F(x, y, z) = Ae^{-(\lambda/2)(x^2+y^2+z^2)}.$$

29. The time dependent temperature at a point of a long bar is given by

$$T(t) = 100^\circ \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{8/\sqrt{t}} e^{-\tau^2}\, d\tau\right).$$

When $t = 64$, $T = 15.73^\circ$. Use differentials to estimate how long it will be until $T = 17^\circ$.

30. Evaluate $\dfrac{d^2}{dx^2} \displaystyle\int_0^x \int_0^x f(s,t)\, ds\, dt$.

CHAPTER 5

Multiple Integrals; Applications of Integration

1. INTRODUCTION

In calculus and elementary physics, you have seen a number of uses for integration such as finding area, volume, mass, moment of inertia, and so on. In this chapter we want to consider these and other applications of both single and multiple integrals. We shall discuss both how to set up integrals to represent physical quantities and methods of evaluating them. In later chapters we will need to use both single and multiple integrals.

Computers and integral tables are very useful in evaluating integrals. But to use these tools efficiently, you need to understand the notation and meaning of integrals which we will discuss in this chapter. There is another important point here. A computer will give you an answer for a definite integral, but an indefinite integral has many possible answers (differing from each other by a constant of integration), and your computer or integral tables may not give you the form you need. (See problems below.) If this happens, here are some ideas you can try:

(a) Look in other integral tables, or try to induce your computer to change the form.

(b) See if some algebra will give the form you want (see Problem 1 below; also see Chapter 2, Section 15, Example 2).

(c) A simple substitution may give the desired result (see Problem 2 below).

(d) To check a claimed answer, differentiate it (by hand or computer) to see whether you get the integrand.

PROBLEMS, SECTION 1

Verify each of the following answers for an indefinite integral by one or more of the methods suggested above.

1. $\displaystyle\int 2\sin\theta\cos\theta\, d\theta = \sin^2\theta \quad \text{or} \quad -\cos^2\theta \quad \text{or} \quad -\tfrac{1}{2}\cos 2\theta.$ *Hint*: Use trig identities.

2. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} \quad \text{or} \quad \ln\left(x+\sqrt{x^2+a^2}\right).$ *Hint*: To find the $\sinh^{-1}$ form, make the substitution $x = a\sinh u$. Or see Chapter 2, Sections 15 and 17.

3. $\displaystyle\int \frac{dy}{\sqrt{y^2-a^2}} = \cosh^{-1}\frac{y}{a} \quad \text{or} \quad \ln\left(y+\sqrt{y^2-a^2}\right).$ *Hint*: See Problem 2 hints.

4. $$\int \sqrt{1+a^2x^2}\, dx = \frac{x}{2}\sqrt{1+a^2x^2} + \frac{1}{2a}\sinh^{-1} ax \quad \text{or}$$
$$\frac{x}{2}\sqrt{1+a^2x^2} + \frac{1}{2a}\ln\left(ax+\sqrt{1+a^2x^2}\right).$$

5. $\displaystyle\int \frac{K\, dr}{\sqrt{1-K^2r^2}} = \sin^{-1} Kr \quad \text{or} \quad -\cos^{-1} Kr \quad \text{or} \quad \tan^{-1}\frac{Kr}{\sqrt{1-K^2r^2}}.$
Hints: Sketch a right triangle with acute angles u and v and label the sides so that $\sin u = Kr$. Also note that $u+v=\pi/2$; then if u is an indefinite integral, so is $-v$ since they differ by a constant of integration.

6. $\displaystyle\int \frac{K\, dr}{r\sqrt{r^2-K^2}} = \cos^{-1}\frac{K}{r} \quad \text{or} \quad \sec^{-1}\frac{r}{K} \quad \text{or} \quad -\sin^{-1}\frac{K}{r} \quad \text{or} \quad -\tan^{-1}\frac{K}{\sqrt{r^2-K^2}}.$

▶ 2. DOUBLE AND TRIPLE INTEGRALS

Recall from calculus that $\int_a^b y\, dx = \int_a^b f(x)\, dx$ gives the area "under the curve" in Figure 2.1. Recall also the definition of the integral as the limit of a sum: We approximate the area by a sum of rectangles as in Figure 2.1; a representative rectangle (shaded) has width Δx. The geometry indicates that if we increase the number of rectangles and let all the widths $\Delta x \to 0$, the sum of the areas of the rectangles will tend to the area under the curve. We *define* $\int_a^b f(x)\, dx$ as the limit of the sum of the areas of the rectangles; then we *evaluate* the integral as an antiderivative, and *use* $\int_a^b f(x)\, dx$ to calculate the area under the curve.

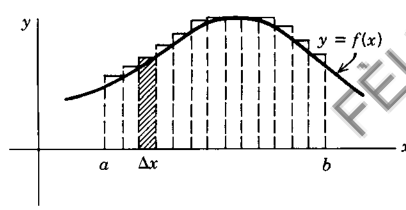


Figure 2.1

We are going to do something very similar in order to find the volume of the cylinder in Figure 2.2 under the surface $z = f(x,y)$. We cut the (x,y) plane into little rectangles of area $\Delta A = (\Delta x)(\Delta y)$ as shown in Figure 2.2; above each $\Delta x\, \Delta y$ is a tall slender box reaching up to the surface. We can approximate the desired volume by a sum of these boxes just as we approximated the area in Figure 2.1 by a sum of rectangles. As the number of boxes increases and all Δx and $\Delta y \to 0$, the geometry indicates that the sum of the volumes of the boxes will tend to the desired volume. We *define* the

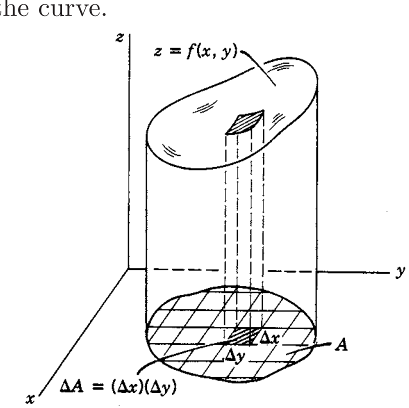


Figure 2.2

double integral of $f(x, y)$ over the area A in the (x, y) plane (Figure 2.2) as the limit of this sum, and we write it as $\iint_A f(x, y)\, dx\, dy$. Before we can use the double integral to compute volumes, however, we need to see how double integrals are evaluated. Even though we may use a computer to do the work, we need to understand the process in order to set up integrals correctly and find and correct errors. Doing some hand evaluation is a good way to learn this.

Iterated Integrals We now show by some examples the details of evaluating double integrals.

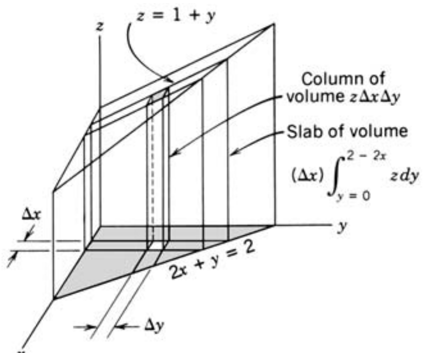


Figure 2.3

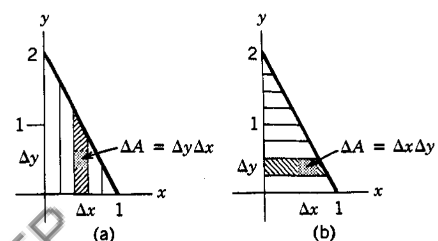


Figure 2.4

Example 1. Find the volume of the solid (Figure 2.3) below the plane $z = 1 + y$, bounded by the coordinate planes and the vertical plane $2x + y = 2$. From our discussion above, this is $\iint_A z\, dx\, dy = \iint_A (1 + y)\, dx\, dy$, where A is the shaded triangle in the (x, y) plane [A is shown also in Figure 2.4 (a and b)]. We are going to consider two ways of evaluating this double integral. We think of the triangle A cut up into little rectangles $\Delta A = \Delta x\, \Delta y$ (Figure 2.4) and the whole solid cut into vertical columns of height z and base ΔA (Figure 2.3). We want the (limit of the) sum of the volumes of these columns. First add up the columns (Figure 2.4a) for a fixed value of x producing the volume of a slab (Figure 2.3) of thickness Δx. This corresponds to integrating with respect to y (holding x constant, Figure 2.4a) from $y = 0$ to y on the line $2x + y = 2$, that is $y = 2 - 2x$; we find

$$\text{(2.1)} \qquad \begin{aligned} \int_{y=0}^{2-2x} z\, dy &= \int_{y=0}^{2-2x} (1 + y)\, dy = \left(y + \frac{y^2}{2}\right)\Big|_0^{2-2x} \\ &= (2 - 2x) + (2 - 2x)^2/2 = 4 - 6x + 2x^2. \end{aligned}$$

(What we have found is the area of the slab in Figure 2.3; its volume is the area times Δx.) Now we add up the volumes of the slabs; this corresponds to integrating (2.1) with respect to x from $x = 0$ to $x = 1$:

$$\text{(2.2)} \qquad \int_{x=0}^{1} (4 - 6x + 2x^2)\, dx = \frac{5}{3}.$$

We could summarize (2.1) and (2.2) by writing

$$(2.3)\qquad \int_{x=0}^{1}\left(\int_{y=0}^{2-2x}(1+y)\,dy\right)dx \quad\text{or}\quad \int_{x=0}^{1}\int_{y=0}^{2-2x}(1+y)\,dy\,dx$$
$$\text{or}\quad \int_{x=0}^{1}dx\int_{y=0}^{2-2x}dy\,(1+y).$$

We call (2.3) an *iterated* (repeated) integral. Multiple integrals are usually evaluated by using iterated integrals. Note that the large parentheses in (2.3) are not really necessary if we are always careful to state the variable in giving the limits on an integral; that is, always write $\int_{x=0}^{1}$, *not* just $\int_0^1$.

Now we could also add up the volume $z(\Delta A)$ by first integrating with respect to x (for fixed y, Figure 2.4b) from $x=0$ to $x=1-y/2$ giving the volume of a slab perpendicular to the y axis in Figure 2.3, and then add up the volumes of the slabs by integrating with respect to y from $y=0$ to $y=2$ (Figure 2.4b). We write

$$(2.4)\qquad \int_{y=0}^{2}\left(\int_{x=0}^{1-y/2}(1+y)\,dx\right)dy = \int_{y=0}^{2}(1+y)x\Big|_{x=0}^{1-y/2}dy$$
$$= \int_{y=0}^{2}(1+y)(1-y/2)\,dy$$
$$= \int_{y=0}^{2}(1+y/2-y^2/2)\,dy = \frac{5}{3}.$$

As the geometry would indicate, the results in (2.2) and (2.4) are the same; we have two methods of evaluating the double integral by using iterated integrals.

Often one of these two methods is more convenient than the other; we choose whichever method is easier. To see how to decide, study the following sketches of areas A over which we want to find $\iint_A f(x,y)\,dx\,dy$. In each case we think of combining little rectangles $dx\,dy$ to form strips (as shown) and then combining the strips to cover the whole area.

Areas shown in Figure 2.5: Integrate with respect to y first. Note that the top and bottom of area A are curves whose equations we know; the boundaries at $x=a$ and $x=b$ are either vertical straight lines or else points.

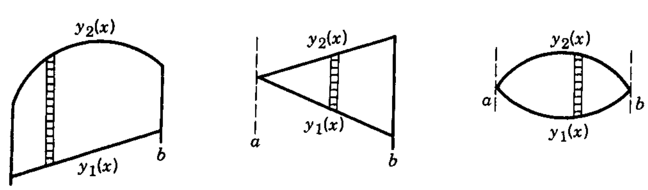


Figure 2.5

We find

$$(2.5)\qquad \iint\limits_A f(x,y)\,dx\,dy = \int_{x=a}^{b}\left(\int_{y=y_1(x)}^{y_2(x)}f(x,y)\,dy\right)dx.$$

Areas shown in Figure 2.6: Integrate with respect to x first. Note that the sides of area A are curves whose equations we know; the boundaries at $y = c$ and $y = d$ are either horizontal straight lines or else points.

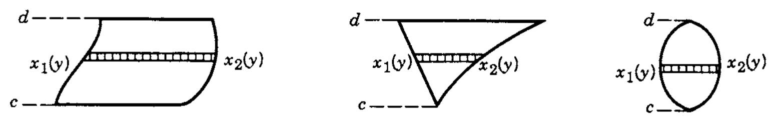


Figure 2.6

We find

$$\iint_A f(x,y)\,dx\,dy = \int_{y=c}^{d}\left(\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\right)dy. \tag{2.6}$$

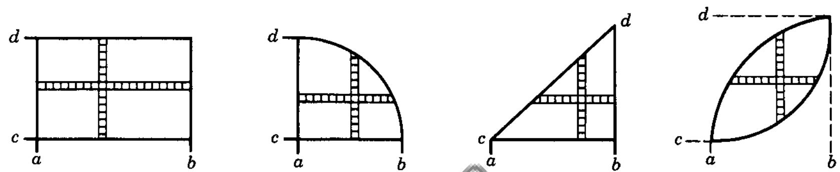


Figure 2.7

Areas shown in Figure 2.7: Integrate in either order. Note that these areas all satisfy the requirements for both (2.5) and (2.6).

We find

$$\begin{aligned}\iint_A f(x,y)\,dx\,dy &= \int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx \\ &= \int_{y=c}^{d}\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy.\end{aligned} \tag{2.7}$$

An important special case is a double integral over a rectangle (both x and y limits are constants) when $f(x,y)$ is a product, $f(x,y) = g(x)h(y)$. Then

$$\begin{aligned}\iint_A f(x,y)\,dx\,dy &= \int_{x=a}^{b}\int_{y=c}^{d} g(x)h(y)\,dy\,dx \\ &= \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).\end{aligned} \tag{2.8}$$

When areas are more complicated than those shown, we may break them into two or more simpler areas (Problems 9 and 10).

We have seen how to set up and evaluate double integrals to find areas and volumes. Recall, however, that we use single integrals for other purposes than finding areas. Similarly, now that we know how to evaluate a double integral, we can use it to find other quantities besides areas and volumes.

▶ **Example 2.** Find the mass of a rectangular plate bounded by $x = 0$, $x = 2$, $y = 0$, $y = 1$, if its density (mass per unit area) is $f(x, y) = xy$. The mass of a tiny rectangle $\Delta A = \Delta x\,\Delta y$ is approximately $f(x, y)\,\Delta x\,\Delta y$, where $f(x, y)$ is evaluated at some point in ΔA. We want to add up the masses of all the ΔA's; this is what we find by evaluating the double integral of $dM = xy\,dx\,dy$. We call dM an *element* of mass and think of adding up all the dM's to get M.

$$(2.9)\qquad M = \iint_A xy\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{1} xy\,dx\,dy$$
$$= \left(\int_0^2 x\,dx\right)\left(\int_0^1 y\,dy\right) = 2\cdot\frac{1}{2} = 1.$$

A triple integral of $f(x, y, z)$ over a volume V, written $\iiint_V f(x, y, z)\,dx\,dy\,dz$, is also defined as the limit of a sum and is evaluated by an iterated integral. If the integral is over a box, that is, all limits are constants, then we can do the x, y, z integrations in any order. If the volume is complicated, then we have to consider the geometry as we did for double integrals to decide on the best order and find the limits. This process can best be learned from examples (below and Section 3) and practice (see problems).

▶ **Example 3.** Find the volume of the solid in Figure 2.3 by using a triple integral. Here we imagine the whole solid cut into tiny boxes of volume $\Delta x\,\Delta y\,\Delta z$; an element of volume is $dx\,dy\,dz$. We first add up the volumes of the tiny boxes to get the volume of a column; this means integrating with respect to z from 0 to $1 + y$ with x and y constant. Then we add up the columns to get a slab and the slabs to get the whole volume just as we did in Example 1. Thus:

$$(2.10)\qquad V = \iiint_V dx\,dy\,dz$$
$$= \int_{x=0}^{1}\int_{y=0}^{2-2x}\left(\int_{z=0}^{1+y} dz\right)dy\,dx \quad\text{or}\quad \int_{x=0}^{1}\int_{y=0}^{2-2x}\int_{z=0}^{1+y} dz\,dy\,dx$$
$$= \int_{x=0}^{1}\int_{y=0}^{2-2x}(1+y)\,dy\,dx = \frac{5}{3},$$

as in (2.1) and (2.2). Or, we could have used (2.4).

▶ **Example 4.** Find the mass of the solid in Figure 2.3 if the density (mass per unit volume) is $x + z$. An element of mass is $dM = (x + z)\,dx\,dy\,dz$. We add up elements of mass just as we add up elements of volume; that is, the limits are the same as in Example 3.

$$(2.11)\qquad M = \int_{x=0}^{1}\int_{y=0}^{2-2x}\int_{z=0}^{1+y}(x+z)\,dz\,dy\,dx = 2$$

where we evaluate the integrals as we did (2.1) to (2.4). (Check the result by hand and by computer.)

PROBLEMS, SECTION 2

In the problems of this section, set up and evaluate the integrals by hand and check your results by computer.

1. $\displaystyle\int_{x=0}^{1}\int_{y=2}^{4} 3x\,dy\,dx$ **2.** $\displaystyle\int_{y=-2}^{1}\int_{x=1}^{2} 8xy\,dx\,dy$ **3.** $\displaystyle\int_{y=0}^{2}\int_{x=2y}^{4} dx\,dy$

4. $\displaystyle\int_{x=0}^{4}\int_{y=0}^{x/2} y\,dy\,dx$ **5.** $\displaystyle\int_{x=0}^{1}\int_{y=x}^{e^x} y\,dy\,dx$ **6.** $\displaystyle\int_{y=1}^{2}\int_{x=\sqrt{y}}^{y^2} x\,dx\,dy$

In Problems 7 to 18 evaluate the double integrals over the areas described. To find the limits, sketch the area and compare Figures 2.5 to 2.7.

7. $\iint_A (2x-3y)\,dx\,dy$, where A is the triangle with vertices $(0,0)$, $(2,1)$, $(2,0)$.

8. $\iint_A 6y^2\cos x\,dx\,dy$, where A is the area inclosed by the curves $y=\sin x$, the x axis, and the line $x=\pi/2$.

9. $\iint_A \sin x\,dx\,dy$ where A is the area shown in Figure 2.8.

10. $\iint_A y\,dx\,dy$ where A is the area in Figure 2.8.

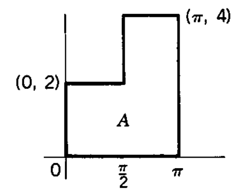


Figure 2.8

11. $\iint_A x\,dx\,dy$, where A is the area between the parabola $y=x^2$ and the straight line $2x-y+8=0$.

12. $\iint y\,dx\,dy$ over the triangle with vertices $(-1,0)$, $(0,2)$, and $(2,0)$.

13. $\iint 2xy\,dx\,dy$ over the triangle with vertices $(0,0)$, $(2,1)$, $(3,0)$.

14. $\iint x^2e^{x^2y}\,dx\,dy$ over the area bounded by $y=x^{-1}$, $y=x^{-2}$, and $x=\ln 4$.

15. $\iint dx\,dy$ over the area bounded by $y=\ln x$, $y=e+1-x$, and the x axis.

16. $\iint (9+2y^2)^{-1}\,dx\,dy$ over the quadrilateral with vertices $(1,3)$, $(3,3)$, $(2,6)$, $(6,6)$.

17. $\iint (x/y)\,dx\,dy$ over the triangle with vertices $(0,0)$, $(1,1)$, $(1,2)$.

18. $\iint y^{-1/2}\,dx\,dy$ over the area bounded by $y=x^2$, $x+y=2$, and the y axis.

In Problems 19 to 24, use double integrals to find the indicated volumes.

19. Above the square with vertices at $(0,0)$, $(2,0)$, $(0,2)$, and $(2,2)$, and under the plane $z=8-x+y$.

20. Above the rectangle with vertices $(0,0)$, $(0,1)$, $(2,0)$, and $(2,1)$, and below the surface $z^2=36x^2(4-x^2)$.

21. Above the triangle with vertices $(0,0)$, $(2,0)$, and $(2,1)$, and below the paraboloid $z=24-x^2-y^2$.

22. Above the triangle with vertices $(0,2)$, $(1,1)$, and $(2,2)$, and under the surface $z=xy$.

23. Under the surface $z=y(x+2)$, and over the area bounded by $x+y=0$, $y=1$, $y=\sqrt{x}$.

24. Under the surface $z=1/(y+2)$, and over the area bounded by $y=x$ and $y^2+x=2$.

In Problems 25 to 28, sketch the area of integration, observe that it is like the areas in Figure 2.7, and so write an equivalent integral with the integration in the opposite order. Check your work by evaluating the double integral both ways. Also check that your computer gives the same answer for both orders of integration.

25. $\displaystyle\int_{x=0}^{1}\int_{y=0}^{3-3x} dy\,dx$

26. $\displaystyle\int_{y=0}^{2}\int_{x=y/2}^{1} (x+y)\,dx\,dy$

27. $\displaystyle\int_{x=0}^{4}\int_{y=0}^{\sqrt{x}} y\sqrt{x}\,dy\,dx$

28. $\displaystyle\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y\,dx\,dy$

In Problems 29 to 32, observe that the inside integral cannot be expressed in terms of elementary functions. As in Problems 25 to 28, change the order of integration and so evaluate the double integral. Also try using your computer to evaluate these for both orders of integration.

29. $\displaystyle\int_{y=0}^{\pi}\int_{x=y}^{\pi} \frac{\sin x}{x}\,dx\,dy$

30. $\displaystyle\int_{x=0}^{2}\int_{y=x}^{2} e^{-y^2/2}\,dy\,dx$

31. $\displaystyle\int_{x=0}^{\ln 16}\int_{y=e^{x/2}}^{4} \frac{dy\,dx}{\ln y}$

32. $\displaystyle\int_{y=0}^{1}\int_{x=y^2}^{1} \frac{e^x}{\sqrt{x}}\,dx\,dy$

33. A lamina covering the quarter disk $x^2+y^2 \le 4$, $x>0$, $y>0$, has (area) density $x+y$. Find the mass of the lamina.

34. A dielectric lamina with charge density proportional to y covers the area between the parabola $y = 16 - x^2$ and the x axis. Find the total charge.

35. A triangular lamina is bounded by the coordinate axes and the line $x+y=6$. Find its mass if its density at each point P is proportional to the square of the distance from the origin to P.

36. A partially silvered mirror covers the square area with vertices at $(\pm1,\pm1)$. The fraction of incident light which it reflects at (x,y) is $(x-y)^2/4$. Assuming a uniform intensity of incident light, find the fraction reflected.

In Problems 37 to 40, evaluate the triple integrals.

37. $\displaystyle\int_{x=1}^{2}\int_{y=x}^{2x}\int_{z=0}^{y-x} dz\,dy\,dx$

38. $\displaystyle\int_{z=0}^{2}\int_{x=z}^{2}\int_{y=8x}^{z} dy\,dx\,dz$

39. $\displaystyle\int_{y=-2}^{3}\int_{z=1}^{2}\int_{x=y+z}^{2y+z} 6y\,dx\,dz\,dy$

40. $\displaystyle\int_{x=1}^{2}\int_{z=x}^{2x}\int_{y=0}^{1/z} z\,dy\,dz\,dx.$

41. Find the volume between the planes $z = 2x+3y+6$ and $z = 2x+7y+8$, and over the triangle with vertices $(0,0)$, $(3,0)$, and $(2,1)$.

42. Find the volume between the planes $z = 2x+3y+6$ and $z = 2x+7y+8$, and over the square in the (x,y) plane with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.

43. Find the volume between the surfaces $z = 2x^2+y^2+12$ and $z = x^2+y^2+8$, and over the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.

44. Find the mass of the solid in Problem 42 if the density is proportional to y.

45. Find the mass of the solid in Problem 43 if the density is proportional to x.

46. Find the mass of a cube of side 2 if the density is proportional to the square of the distance from the center of the cube.

47. Find the volume in the first octant bounded by the coordinate planes and the plane $x + 2y + z = 4$.

48. Find the volume in the first octant bounded by the cone $z^2 = x^2 - y^2$ and the plane $x = 4$.

49. Find the volume in the first octant bounded by the paraboloid $z = 1 - x^2 - y^2$, the plane $x + y = 1$, and all three coordinate planes.

50. Find the mass of the solid in Problem 48 if the density is z.

3. APPLICATIONS OF INTEGRATION; SINGLE AND MULTIPLE INTEGRALS

Many different physical quantities are given by integrals; let us do some problems to illustrate setting up and evaluating these integrals. The basic idea which we use in setting up the integrals in these problems is that an integral is the "limit of a sum." Thus we imagine the physical object (whose volume, moment of inertia, etc., we are trying to find) cut into a large number of small pieces called *elements*. We write an approximate formula for the volume, moment of inertia, etc., of an element and then sum over all elements of the object. The limit of this sum (as the number of elements tends to infinity and the size of each element tends to zero) is what we find by integration and is what we want in the physical problem.

Using a computer to evaluate the integrals saves time and we will concentrate mainly on setting up integrals. However, in order to do a skillful job of finding limits, deciding order of integration, detecting and correcting errors, making useful changes of variables, and understanding the meaning of the symbols used, it is important to learn to evaluate multiple integrals by hand. So a good study method is to do some integrals both by hand and by computer. A computer is also very useful to plot graphs of curves and surfaces to help you find the limits in a multiple integral.

Example 1. Given the curve $y = x^2$ from $x = 0$ to $x = 1$, find

(a) the area under the curve (that is, the area bounded by the curve, the x axis, and the line $x = 1$; see Figure 3.1);

(b) the mass of a plane sheet of material cut in the shape of this area if its density (mass per unit area) is xy;

(c) the arc length of the curve;

(d) the centroid of the area;

(e) the centroid of the arc;

(f) the moments of inertia about the x, y, and z axes of the lamina in (b).

(a) The area is

$$A = \int_{x=0}^{1} y\,dx = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

We could also find the area as a double integral of $dA = dy\,dx$ (see Figure 3.1). We have then

$$A = \int_{x=0}^{1}\int_{y=0}^{x^2} dy\,dx = \int_0^1 x^2\,dx$$

as before. Although the double integral is entirely unnecessary in finding the area in this problem, we shall need to use a double integral to find the mass in part (b).

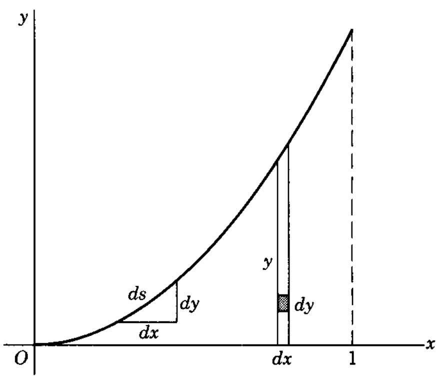


Figure 3.1

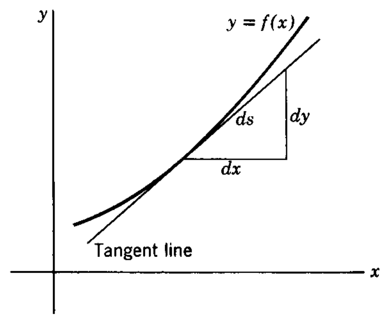


Figure 3.2

(b) The element of area, as in the double integral method in (a), is $dA = dy\,dx$. Since the density is $\rho = xy$, the element of mass is $dM = xy\,dy\,dx$, and the total mass is

$$M = \int_{x=0}^{1}\int_{y=0}^{x^2} xy\,dy\,dx = \int_0^1 x\,dx\left[\frac{y^2}{2}\right]_0^{x^2} = \int_0^1 \frac{x^5}{2}\,dx = \frac{1}{12}.$$

Observe that we could not do this problem as a single integral because the density depends on both x and y.

(c) The element of arc length ds is defined as indicated in Figures 3.1 and 3.2. Thus we have

$$(3.1)\qquad \begin{aligned} ds^2 &= dx^2 + dy^2,\\ ds &= \sqrt{dx^2+dy^2} = \sqrt{1+(dy/dx)^2}\,dx = \sqrt{(dx/dy)^2+1}\,dy. \end{aligned}$$

If $y = f(x)$ has a continuous first derivative $dy/\,dx$ (except possibly at a finite number of points), we can find the arc length of the curve $y = f(x)$ between a and b by calculating $\int_a^b ds$. For our example, we have

$$(3.2)\qquad \begin{aligned} &\frac{dy}{dx} = 2x, \qquad ds = \sqrt{1+4x^2}\,dx,\\ &s = \int_0^1 \sqrt{1+4x^2}\,dx = \frac{2\sqrt{5}+\ln(2+\sqrt{5})}{4} \end{aligned}$$

(see Problem 32).

(d) Recall from elementary physics that:

The *center of mass* of a body has coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ given by the equations

$$\int \bar{x}\,dM = \int x\,dM, \qquad \int \bar{y}\,dM = \int y\,dM, \qquad \int \bar{z}\,dM = \int z\,dM, \tag{3.3}$$

where dM is an element of mass and the integrals are over the whole body.

Although we have written single integrals in (3.3), they may be single, double, or triple integrals depending on the problem and the method of evaluation. Since $\bar{x}$, $\bar{y}$, and $\bar{z}$ are constants, we *can* take them outside the integrals in (3.3) and solve for them. However, you may find it easier to remember the definitions in the form (3.3). For the example we are doing, $\bar{z} = 0$ since the body is a sheet of material in the (x, y) plane. The element of mass is $dM = \rho\,dA = \rho\,dx\,dy$, where ρ is the density (mass per unit area in this problem). For a variable density as in (b), we would substitute the value of ρ into (3.3) and integrate both sides of each equation to find the coordinates of the center of mass. However, let us suppose the density is a constant. Then the first integral in (3.3) is

$$\int \bar{x}\rho\,dA = \int x\rho\,dA \quad \text{or} \quad \int \bar{x}\,dA = \int x\,dA. \tag{3.4}$$

Similarly, a *constant* density ρ can be canceled from all the equations in (3.3). The quantities $\bar{x}$, $\bar{y}$, $\bar{z}$, are then called the coordinates of the *centroid* of the area (or volume or arc).

The centroid of a body is the center of mass when we assume constant density.

In our example, we have

$$\begin{aligned} \int_{x=0}^{1}\int_{y=0}^{x^2} \bar{x}\,dy\,dx &= \int_{x=0}^{1}\int_{y=0}^{x^2} x\,dy\,dx \qquad \text{or} \qquad \bar{x}A = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}, \\ \int_{x=0}^{1}\int_{y=0}^{x^2} \bar{y}\,dy\,dx &= \int_{x=0}^{1}\int_{y=0}^{x^2} y\,dy\,dx \qquad \text{or} \qquad \bar{y}A = \left.\frac{x^5}{10}\right|_0^1 = \frac{1}{10}. \end{aligned} \tag{3.5}$$

(Double integrals are not really necessary for any of these but the last.) Using the value of A from part(a), we find $\bar{x} = \frac{3}{4}$, $\bar{y} = \frac{3}{10}$.

(e) The center of mass $(\bar{x}, \bar{y})$ of a wire bent in the shape of the curve $y = f(x)$ is given by

$$\int \bar{x}\rho\,ds = \int x\rho\,ds, \qquad \int \bar{y}\rho\,ds = \int y\rho\,ds, \tag{3.6}$$

where ρ is the density (mass per unit length), and the integrals are *single* integrals with ds given by (3.1). If ρ is constant, (3.6) defines the coordinates of the centroid.

In our example we have

$$(3.7)\qquad \begin{aligned} \int_0^1 \bar{x}\sqrt{1+4x^2}\,dx &= \int_0^1 x\sqrt{1+4x^2}\,dx, \\ \int_0^1 \bar{y}\sqrt{1+4x^2}\,dx &= \int_0^1 y\sqrt{1+4x^2}\,dx = \int_0^1 x^2\sqrt{1+4x^2}\,dx. \end{aligned}$$

Note carefully here that it is correct to put $y = x^2$ in the last integral of (3.7), but it would *not* have been correct to do this in the last integral of (3.5); the reason is that over the *area*, y could take values from zero to x^2, but *on* the arc, y takes only the value x^2. By calculating the integrals in (3.7) we can find $\bar{x}$ and $\bar{y}$.

(f) We need the following definition:

> The *moment of inertia* I of a point mass m about an axis is by definition the product ml^2 of m times the square of the distance l from m to the axis. For an extended object we must integrate $l^2 dM$ over the whole object, where l is the distance from dM to the axis.

In our example with variable density $\rho = xy$, we have $dM = xy\,dy\,dx$. The distance from dM to the x axis is y (Figure 3.3); similarly, the distance from dM to the y axis is x. The distance from dM to the z axis (the z axis is perpendicular to the paper in Figure 3.3) is $\sqrt{x^2+y^2}$. Then the three moments of inertia about the three coordinate axes are:

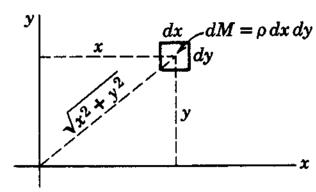


Figure 3.3

$$I_x = \int_{x=0}^1 \int_{y=0}^{x^2} y^2 xy\,dy\,dx = \int_0^1 \frac{x^9}{4}\,dx = \frac{1}{40},$$

$$I_y = \int_{x=0}^1 \int_{y=0}^{x^2} x^2 xy\,dy\,dx = \int_0^1 \frac{x^7}{2}\,dx = \frac{1}{16},$$

$$I_z = \int_{x=0}^1 \int_{y=0}^{x^2} (x^2+y^2)xy\,dy\,dx = I_x + I_y = \frac{7}{80}.$$

> The fact that $I_x + I_y = I_z$ for a plane lamina in the (x, y) plane is known as the perpendicular axis theorem.

It is customary to write moments of inertia as multiples of the mass; using $M = \frac{1}{12}$ from (b), we write

$$I_x = \frac{12}{40}M = \frac{3}{10}M, \qquad I_y = \frac{12}{16}M = \frac{3}{4}M, \qquad I_z = \frac{7\cdot 12}{80}M = \frac{21}{20}M.$$

▶ **Example 2.** Rotate the area of Example 1 about the x axis to form a volume and surface of revolution, and find

(a) the volume;

(b) the moment of inertia about the x axis of a solid of constant density occupying the given volume;

(c) the area of the curved surface;

(d) the centroid of the curved surface.

(a) We want to find the given volume.

> The easiest way to find a volume of revolution is to take as volume element a thin slab of the solid as shown in Figure 3.4. The slab has circular cross section of radius y and thickness dx; thus the volume element is $\pi y^2\,dx$.

Then the volume in our example is

$$V = \int_0^1 \pi y^2\,dx = \int_0^1 \pi x^4\,dx = \frac{\pi}{5}. \tag{3.8}$$

We have really avoided part of the integration here because we knew the formula for the area of a circle. In finding volumes of solids which are not solids of revolution, we may have to use double or triple integrals. Even for a solid of revolution we might need multiple integrals to find the mass if the density is variable.

To illustrate setting up such integrals, let us do the above problem using triple integrals. For this we need the equation of the surface which is (see Problem 16)

$$y^2 + z^2 = x^4, \qquad x > 0. \tag{3.9}$$

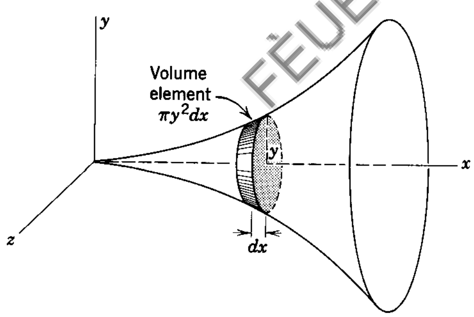


Figure 3.4

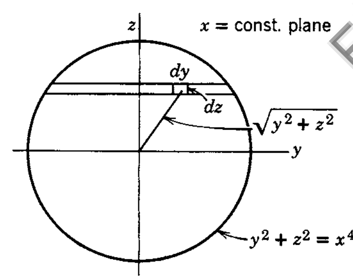


Figure 3.5

To set up a multiple integral for the volume of a solid, we cut the solid into slabs as in Figure 3.4 (not necessarily circular slabs, although they are in our example) and then as in Figure 3.5 we cut each slab into strips and each strip into tiny boxes of volume $dx\,dy\,dz$. The volume is

$$V = \iiint dx\,dy\,dz;$$

the only problem is to find the limits! To do this, we start by adding up tiny boxes to get a strip; as we have drawn Figure 3.5, this means to integrate with respect to y from one side of the circle $y^2 + z^2 = x^4$ to the other, that is, from

$$y = -\sqrt{x^4 - z^2} \qquad \text{to} \quad y = +\sqrt{x^4 - z^2}.$$

Next we add all the strips in a slab. This means that, in Figure 3.5, we integrate with respect to z from the bottom to the top of the circle $y^2 + z^2 = x^4$; thus the z limits are $z = \pm$ radius of circle $= \pm x^2$. And finally we add all the slabs to obtain the solid. This means to integrate in Figure 3.4 from $x = 0$ to $x = 1$; this is just what we did in our first simple method. The final integral is then

$$V = \int_{x=0}^{1} \int_{z=-x^2}^{x^2} \int_{y=-\sqrt{x^4-z^2}}^{\sqrt{x^4-z^2}} dy\,dz\,dx. \tag{3.10}$$

(See Problem 33).

Although the triple integral is an unnecessarily complicated way of finding a volume of revolution, this simple problem illustrates the general method of setting up an integral for any kind of volume. Once we have the volume as a triple integral, it is easy to write the integrals for the mass with a given variable density, for the coordinates of the centroid, for the moments of inertia, and so on. The limits of integration are the same as for the volume; we need only insert the proper expressions (density, etc.) in the integrand to get the mass, centroid, and so on.

(b) To find the moment of inertia of the solid about the x axis, we must integrate the quantity $l^2 dM$, where l is the distance from dM to the x axis; from Figure 3.5, since the x axis is perpendicular to the paper, $l^2 = y^2 + z^2$. The limits on the integrals are the same as in (3.10). We are assuming constant density, so the factor ρ can be written outside the integrals. Then we have

$$I_x = \rho \int_{x=0}^{1} \int_{z=-x^2}^{x^2} \int_{y=-\sqrt{x^4-z^2}}^{\sqrt{x^4-z^2}} (y^2 + z^2)\,dy\,dz\,dx = \frac{\pi}{18}\rho.$$

Since from (3.8) the mass of the solid is

$$M = \rho V = \frac{\pi}{5}\rho$$

we can write I_x (as is customary) as a multiple of M:

$$I_x = \frac{\pi}{18}\frac{5}{\pi}M = \frac{5}{18}M.$$

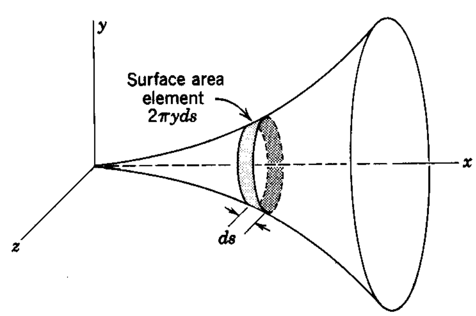


Figure 3.6

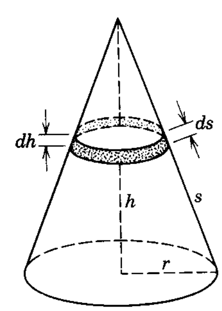


Figure 3.7

(c) We find the area of the surface of revolution by using as element the curved surface of a thin slab as in Figure 3.6. This is a strip of circumference $2\pi y$ and width ds. To see this clearly and to understand why we use ds here but dx in the volume element in (3.8), think of the slab as a thin section of a cone (Figure 3.7) between planes perpendicular to the axis of the cone. If you wanted to find the total volume $V = \frac{1}{3}\pi r^2 h$ of the cone, you would use the height h perpendicular to the base, but in finding the total curved surface area $S = \frac{1}{2} \cdot 2\pi r \cdot s$, you would use the *slant* height s. The same ideas hold in finding the volume and surface elements. The approximate volume of the thin slab is the area of a face of the slab times its thickness (dh in Figure 3.7, dx in Figure 3.4). But if you think of a narrow strip of paper just covering the curved surface of the thin slab, the width of the strip of paper is ds, and its length is the circumference of the thin slab.

The element of surface area (in Figure 3.6) is then

(3.11) $$dA = 2\pi y\, ds.$$

The total area is [using ds from (3.2)]

$$A = \int_{x=0}^{1} 2\pi y\, ds = \int_0^1 2\pi x^2 \sqrt{1+4x^2}\, dx.$$

(For more general surfaces, there is a way to calculate areas by double integration; we shall take this up in Section 5.)

(d) The y and z coordinates of the centroid of the surface area are zero by symmetry. For the x coordinate, we have by (3.4)

$$\int \bar{x}\, dA = \int x\, dA,$$

or, using $dA = 2\pi y\, ds$ and the total area A from (c), we have

$$\bar{x}A = \int_{x=0}^{1} x \cdot 2\pi y\, ds = \int_0^1 x \cdot 2\pi x^2 \sqrt{1+4x^2}\, dx.$$

PROBLEMS, SECTION 3

The following notation is used in the problems:

M = mass,

$\bar{x}, \bar{y}, \bar{z}$ = coordinates of center of mass (or centroid if the density is constant),

I = moment of inertia (about axis stated),

I_x, I_y, I_z = moments of inertia about x, y, z axes,

I_m = moment of inertia (about axis stated) through the center of mass.

Note: It is customary to give answers for I, I_m, I_x, etc., as multiples of M (for example, $I = \frac{1}{3}Ml^2$).

1. Prove the "parallel axis theorem": The moment of inertia I of a body about a given axis is $I = I_m + Md^2$, where M is the mass of the body, I_m is the moment of inertia of the body about an axis through the center of mass and parallel to the given axis, and d is the distance between the two axes.

2. For a thin rod of length l and uniform density ρ find

(a) M,

(b) I_m about an axis perpendicular to the rod,

(c) I about an axis perpendicular to the rod and passing through one end (see Problem 1).

3. A thin rod 10 ft long has a density which varies uniformly from 4 to 24 lb/ft. Find

(a) M,

(b) $\bar{x}$,

(c) I_m about an axis perpendicular to the rod,

(d) I about an axis perpendicular to the rod passing through the heavy end.

4. Repeat Problem 3 for a rod of length l with density varying uniformly from 2 to 1.

5. For a square lamina of uniform density, find I about

(a) a side,

(b) a diagonal,

(c) an axis through a corner and perpendicular to the plane of the lamina. *Hint:* See the perpendicular axis theorem, Example 1f.

6. A triangular lamina has vertices $(0,0)$, $(0,6)$ and $(6,0)$, and uniform density. Find:

(a) $\bar{x}$, $\bar{y}$,

(b) I_x,

(c) I_m about an axis parallel to the x axis. *Hint:* Use Problem 1 *carefully.*

7. A rectangular lamina has vertices $(0,0)$, $(0,2)$, $(3,0)$, $(3,2)$ and density xy. Find

(a) M,

(b) $\bar{x}$, $\bar{y}$,

(c) I_x, I_y,

(d) I_m about an axis parallel to the z axis. *Hint:* Use the parallel axis theorem and the perpendicular axis theorem.

8. For a uniform cube, find I about one edge.

9. For the pyramid inclosed by the coordinate planes and the plane $x+y+z=1$:

(a) Find its volume.

(b) Find the coordinates of its centroid.

(c) If the density is z, find M and $\bar{z}$.

10. A uniform chain hangs in the shape of the catenary $y = \cosh x$ between $x=-1$ and $x=1$. Find (a) its length, (b) $\bar{y}$.

11. A chain in the shape $y = x^2$ between $x=-1$ and $x=1$ has density $|x|$. Find (a) M, (b) $\bar{x}$, $\bar{y}$.

Prove the following two theorems of Pappus:

12. The area A inside a closed curve in the (x,y) plane, $y \geq 0$, is revolved about the x axis. The volume of the solid generated is equal to A times the circumference of the circle traced by the centroid of A. *Hint:* Write the integrals for the volume and for the centroid.

13. An arc in the (x, y) plane, $y \geq 0$, is revolved about the x axis. The surface area generated is equal to the length of the arc times the circumference of the circle traced by the centroid of the arc.

14. Use Problems 12 and 13 to find the volume and surface area of a torus (doughnut).

15. Use Problems 12 and 13 to find the centroids of a semicircular area and of a semicircular arc. *Hint:* Assume the formulas $A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$ for a sphere.

16. Let a curve $y = f(x)$ be revolved about the x axis, thus forming a surface of revolution. Show that the cross sections of this surface in any plane $x =$ const. [that is, parallel to the (y, z) plane] are circles of radius $f(x)$. Thus write the general equation of a surface of revolution and verify the special case $f(x) = x^2$ in (3.9).

In Problems 17 to 30, for the curve $y = \sqrt{x}$, between $x = 0$ and $x = 2$, find:

17. The area under the curve.

18. The arc length.

19. The volume of the solid generated when the area is revolved about the x axis.

20. The curved area of this solid.

21, 22, 23. The centroids of the arc, the volume, and the surface area.

24, 25, 26, 27. The moments of inertia about the x axis of a lamina in the shape of the plane area under the curve; of a wire bent along the arc of the curve; of the solid of revolution; and of a thin shell whose shape is the curved surface of the solid (assuming constant density for all these problems).

28. The mass of a wire bent in the shape of the arc if its density (mass per unit length) is $\sqrt{x}$.

29. The mass of the solid of revolution if the density (mass per unit volume) is $|xyz|$.

30. The moment of inertia about the y axis of the solid of revolution if the density is $|xyz|$.

31. (a) Revolve the curve $y = x^{-1}$, from $x = 1$ to $x = \infty$, about the x axis to create a surface and a volume. Write integrals for the surface area and the volume. Find the volume, and show that the surface area is infinite. *Hint:* The surface area integral is not easy to evaluate, but you can easily show that it is greater than $\int_1^\infty x^{-1}\, dx$ which you can evaluate.

(b) The following question is a challenge to your ability to fit together your mathematical calculations and physical facts: In (a) you found a finite volume and an infinite area. Suppose you fill the finite volume with a finite amount of paint and then pour off the excess leaving what sticks to the surface. Apparently you have painted an infinite area with a finite amount of paint! What is wrong? (Compare Problem 15.31c of Chapter 1.)

32. Use a computer or tables to evaluate the integral in (3.2) and verify that the answer is equivalent to the text answer. *Hint:* See Problem 1.4 and also Chapter 2, Sections 15 and 17.

33. Verify that (3.10) gives the same result as (3.8).

▶ 4. CHANGE OF VARIABLES IN INTEGRALS; JACOBIANS

In many applied problems, it is more convenient to use other coordinate systems instead of the rectangular coordinates we have been using. For example, in the plane we often use polar coordinates, and in three dimensions we often use cylindrical coordinates or spherical coordinates. It is important to know how to set up multiple integrals directly in these coordinate systems which occur so frequently in practice. That is, we need to know what the area, volume, and arc length elements are, what the variables r, θ, etc., mean geometrically, and how they are related to the rectangular coordinates. We are going to discuss finding elements of area, etc., geometrically for several important coordinate systems. However, if we are given equations like $x = r\cos\theta$, $y = r\sin\theta$, relating new variables to the rectangular ones, it is useful to know how to find the elements of area, etc., *algebraically*, without having to rely on the geometry. We are going to discuss this and illustrate it by verifying the results which we can get geometrically for several of the familiar coordinate systems.

In the plane, the polar coordinates r, θ are related to the rectangular coordinates x, y by the equations

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{4.1}$$

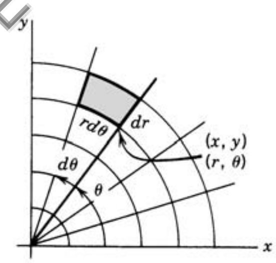


Figure 4.1

Recall that we found the area element $dy\,dx$ by drawing a grid of lines $x =$ const., $y =$ const., which cut the plane into little rectangles dx by dy; the area of one rectangle was then $dy\,dx$. We can make a similar construction for polar coordinates by drawing lines $\theta =$ const. and circles $r =$ const.; we then obtain the grid shown in Figure 4.1. Observe that the sides of the area element are not dr and $d\theta$, but dr and $r\,d\theta$, and its area is then

$$dA = dr \cdot r\,d\theta = r\,dr\,d\theta. \tag{4.2}$$

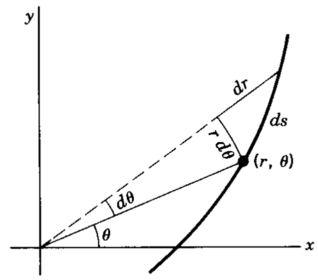


Figure 4.2

Similarly, we can see from Figure 4.2 that the arc length element ds is given by

(4.3)
$$ds^2 = dr^2 + r^2\,d\theta^2,$$
$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\;d\theta = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\;dr.$$

Example 1. Given a semicircular sheet of material of radius a and constant density ρ, find

(a) the centroid of the semicircular area;

(b) the moment of inertia of the sheet of material about the diameter forming the straight side of the semicircle.

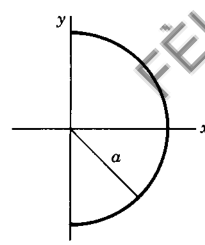


Figure 4.3

(a) In Figure 4.3, we see by symmetry that $\bar{y} = 0$. We want to find $\bar{x}$. By (3.4), we have

$$\int \bar{x} r\,dr\,d\theta = \int xr\,dr\,d\theta.$$

Changing the x to polar coordinates and putting in the limits, we get

$$\bar{x}\int_{r=0}^{a}\int_{\theta=-\pi/2}^{\pi/2} r\,dr d\theta = \int_{r=0}^{a}\int_{\theta=-\pi/2}^{\pi/2} r\cos\theta\,r\,dr\,d\theta.$$

We calculate the integrals and find $\bar{x}$:

$$\bar{x}\,\frac{a^2}{2}\,\pi = \frac{a^3}{3}\sin\theta\Big|_{-\pi/2}^{\pi/2} = \frac{a^3}{3}\cdot 2,$$
$$\bar{x} = \frac{4a}{3\pi}.$$

(b) We want the moment of inertia about the y axis in Figure 4.3; by definition this is $\int x^2\,dM$. In polar coordinates, $dM = \rho\,dA = \rho r\,dr\,d\theta$. We are given that the density ρ is constant. Then we have

$$I_y = \rho\int x^2 r\,dr\,d\theta = \rho\int_{r=0}^{a}\int_{\theta=-\pi/2}^{\pi/2} r^2\cos^2\theta\,r\,dr\,d\theta = \rho\frac{\pi a^4}{8}.$$

The mass of the semicircular object is

$$M = \rho \int r\,dr\,d\theta = \rho \int_{r=0}^{a} \int_{\theta=-\pi/2}^{\pi/2} r\,dr\,d\theta = \rho \frac{\pi a^2}{2}.$$

We write I_y in terms of M to get

$$I_y = \frac{2M}{\pi a^2} \frac{\pi a^4}{8} = \frac{Ma^2}{4}.$$

Spherical and Cylindrical Coordinates The two most important coordinate systems (besides rectangular) in three dimensions are the spherical and the cylindrical coordinate systems. Figures 4.4 and 4.5 and equations (4.4) and (4.5) show the geometrical meaning of the variables, their algebraic relation to x, y, z, the appearance of the volume elements, and the formulas for the volume, arc length, and surface area elements.

Cylindrical coordinates are just polar coordinates in the (x, y) plane with z for the third variable. Note that the spherical coordinates r and θ in Figure 4.5 are different from the cylindrical or polar coordinates r and θ in Figures 4.4 and 4.1. Since we seldom use both systems in the same problem, this should cause no confusion. (If necessary, use ρ or R for one of the r's and use ϕ instead of θ in cylindrical coordinates.) Watch out, however, for the discrepancy in notation for spherical coordinates in various texts. Most calculus books interchange θ and ϕ. This can be confusing later since the notation of Figure 4.5 is almost universal in applications to the physical sciences, and is often used in advanced mathematics (partial differential equations, special functions), in computer programs, and in reference books of formulas and tables. You will need to learn a number of useful formulas involving spherical coordinates [for example, (4.7), (4.19), and (4.20) below; also see Chapter 10, Section 9 and Chapter 13, Section 7]. It is best to learn these formulas in the notation that you will use in applications.

(4.4) Cylindrical coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ dV &= r\,dr\,d\theta\,dz \\ ds^2 &= dr^2 + r^2\,d\theta^2 + dz^2 \\ dA &= a\,d\theta\,dz \end{aligned}$$

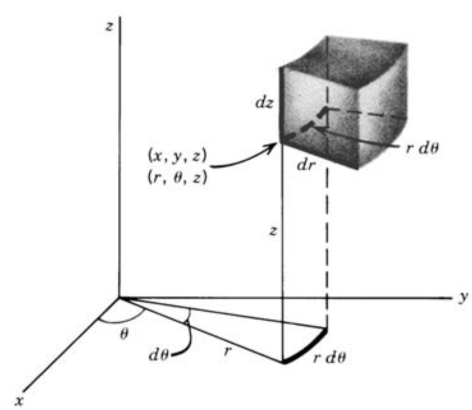


Figure 4.4

(4.5) Spherical coordinates:

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$
$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$
$$ds^2 = dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$
$$dA = a^2 \sin\theta \, d\theta \, d\phi$$

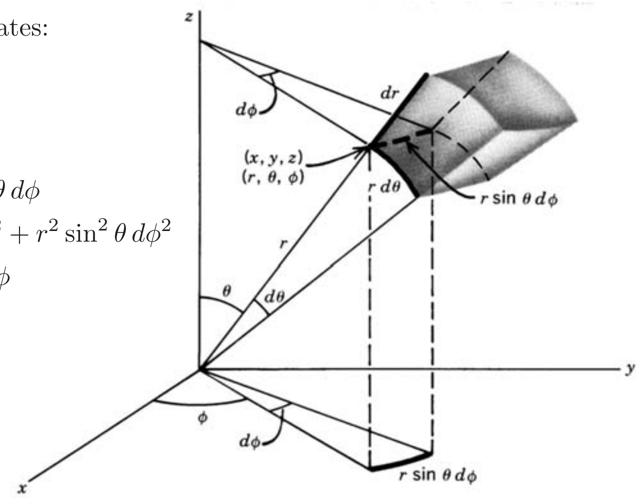


Figure 4.5

We will need the volume and surface area elements in these two systems [and also the arc length elements—see equations (4.18) and (4.19)]. To find the polar coordinate area element in Figure 4.1, we drew a grid of curves $r =$ const., $\theta =$ const. In three dimensions we want to draw a grid of surfaces. In cylindrical coordinates these surfaces are the cylinders $r =$ const., the half-planes $\theta =$ const. (through the z axis), and the planes $z =$ const. [parallel to the (x, y) plane]. One of the elements formed by this grid of surfaces is sketched in Figure 4.4. From the geometry we see that three edges of the element are dr, $r\,d\theta$, and dz, giving the volume element

(4.6) $$dV = r \, dr \, d\theta \, dz \qquad \text{(cylindrical coordinates).}$$

If r is constant, then the surface area element on the cylinder $r = a$ has edges $a\,d\theta$ and dz, so $dA = a\,d\theta\,dz$. Similarly for the spherical coordinate case, we draw the spheres $r =$ const., the cones $\theta =$ const., and the half-planes $\phi =$ const. The volume elements formed by this grid (Figure 4.5) have edges dr, $r\,d\theta$, and r sin $d\theta\,d\phi$; thus we get

(4.7) $$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi \qquad \text{(spherical coordinates).}$$

If r is constant, then the surface area element on the sphere $r = a$ has edges $a\,d\theta$ and a $\sin\theta\,d\phi$, so $dA = a^2 \sin\theta\,d\theta\,d\phi$.

Jacobians. For polar, cylindrical, and spherical coordinates, we have seen how to find area and volume elements from the geometry. However, it is convenient to know an algebraic way of finding them which we can use for unfamiliar coordinate systems

(Problems 16 and 17) or for any change of variables in a multiple integral (Problems 19 and 20). Here we state without proof (see Chapter 6, Section 3, Example 2) some theorems which tell us how to do this. First, in two dimensions, suppose x and y are given as functions of two new variables s and t. The *Jacobian* of x, y, with respect to s, t, is the determinant in (4.8) below; we also show abbreviations used for it.

$$J = J\left(\frac{x,y}{s,t}\right) = \frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{vmatrix}. \tag{4.8}$$

Then the area element $dy\,dx$ is replaced in the s, t system by the area element

$$dA = |J|\,ds\,dt \tag{4.9}$$

where $|J|$ is the absolute value of the Jacobian in (4.8).

Let us find the Jacobian of x, y with respect to the polar coordinates r, θ, and thus verify that (4.8) and our geometric method give the same result (4.2) for the polar coordinate area element. We have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r. \tag{4.10}$$

Thus by (4.9) the area element is $r\,dr\,d\theta$ as in (4.2).

The use of Jacobians extends to more variables. Also, it is not necessary to start with rectangular coordinates; let us state a more general theorem.

Suppose we have a triple integral

$$\iiint f(u,v,w)\,du\,dv\,dw \tag{4.11}$$

in some set of variables u, v, w. Let r, s, t be another set of variables, related to u, v, w by given equations

$$u = u(r,s,t), \qquad v = v(r,s,t), \qquad w = w(r,s,t).$$

Then if the determinant

$$(4.12)\qquad J = \frac{\partial(u,v,w)}{\partial(r,s,t)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{vmatrix}$$

is the Jacobian of u, v, w with respect to r, s, t, then the triple integral in the new variables is

$$(4.13)\qquad \iiint f \cdot |J| \cdot dr\, ds\, dt,$$

where, of course, f and J must both be expressed in terms of r, s, t, and the limits must be properly adjusted to correspond to the new variables.

We can use (4.12) to verify the volume element (4.6) for cylindrical coordinates (Problem 15) and the volume element (4.7) for spherical coordinates. Let us do the calculation for spherical coordinates. From (4.5), we have

$$(4.14)\qquad \begin{aligned} \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\,\cos\phi & r\,\cos\theta\,\cos\phi & -r\,\sin\theta\,\sin\phi \\ \sin\theta\,\sin\phi & r\,\cos\theta\,\sin\phi & r\,\sin\theta\,\cos\phi \\ \cos\theta & -r\,\sin\theta & 0 \end{vmatrix} \\ &= r^2\sin\theta[-\sin^2\phi(-\sin^2\theta-\cos^2\theta)-\cos^2\phi(-\sin^2\theta-\cos^2\theta)] \\ &= r^2\sin\theta. \end{aligned}$$

Thus the spherical coordinate volume element is $dV = r^2\,\sin\theta\, dr\, d\theta\, d\phi$ as in (4.7).

Example 2. Find the z coordinate of the centroid of a uniform solid cone (part of one nappe) of height h equal to the radius of the base r. Also find the moment of inertia of the solid about its axis.

If we take the cone as shown in Figure 4.6, its equation in cylindrical coordinates is $r = z$, since at any height z, the cross section is a circle of radius equal to the height. To find the mass we must integrate $dM = \rho r\,dr\,d\theta\,dz$, where ρ is the constant density. The limits of integration are

$$\theta:\ 0 \text{ to } 2\pi, \qquad r:\ 0 \text{ to } z, \qquad z:\ 0 \text{ to } h.$$

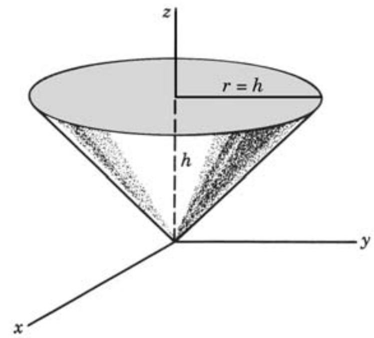


Figure 4.6

Then we have

$$M = \int \rho\,dV = \rho \int_{z=0}^{h} \int_{r=0}^{z} \int_{\theta=0}^{2\pi} r\,dr\,d\theta\,dz = \rho\cdot 2\pi \int_0^h \frac{z^2}{2}dz = \frac{\rho\pi h^3}{3},$$

$$\int \bar{z}\,dV = \int z\,dV = \int_{z=0}^{h}\int_{r=0}^{z}\int_{\theta=0}^{2\pi} zr\,dr\,d\theta\,dz$$

(4.15)
$$= 2\pi \int_0^h z\cdot\frac{1}{2}z^2\,dz = \frac{\pi h^4}{4},$$

$$\bar{z}\cdot\frac{\pi h^3}{3} = \frac{\pi h^4}{4},$$

$$\bar{z} = \frac{3}{4}h.$$

For the moment of inertia about the z axis we have

$$I = \rho \int_{z=0}^{h}\int_{r=0}^{z}\int_{\theta=0}^{2\pi} r^2 r\,dr\,d\theta\,dz = \rho\cdot 2\pi \int_0^h \frac{z^4}{4}\,dz = \rho\frac{\pi h^5}{10}.$$

Using the value of M from (4.15), we write I in the usual form as a multiple of M:

$$I = \frac{3M}{\pi h^3}\frac{\pi h^5}{10} = \frac{3}{10}Mh^2.$$

In the following examples and problems, note that we use *sphere* ($r = a$) to mean surface area, and *ball* ($r \le a$) to mean volume (just as we use *circle* to mean circumference and *disk* to mean area).

▶ **Example 3.** Find the moment of inertia of a solid ball of radius a about a diameter. In spherical coordinates, the equation of the ball is $r \le a$. Then the mass is

(4.16)
$$M = \rho\int dV = \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

$$= \rho\frac{a^3}{3}\cdot 2\cdot 2\pi = \frac{4}{3}\pi a^3\rho.$$

(to no one's surprise!). The moment of inertia about the z axis is

$$I = \int (x^2+y^2)\,dM = \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} (r^2\,\sin^2\,\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \rho\cdot\frac{a^5}{5}\cdot\frac{4}{3}\cdot 2\pi = \frac{8\pi a^5\rho}{15};$$

or, using the value of M, we get

$$I = \frac{2}{5} M a^2. \tag{4.17}$$

Example 4. Find the moment of inertia about the z axis of the solid ellipsoid inside

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

We want to evaluate

$$M = \rho \iiint dx\,dy\,dz \quad \text{and} \quad I = \rho \iiint (x^2 + y^2)\,dx\,dy\,dz$$

where the triple integrals are over the volume of the ellipsoid. Make the change of variables $x = ax'$, $y = by'$, $z = cz'$; then $x'^2 + y'^2 + z'^2 = 1$, so in the primed variables we integrate over the volume of a ball of radius 1. Then

$$M = \rho\,abc \iiint dx'\,dy'\,dz' = \rho\,abc \cdot \text{volume of ball of radius 1}.$$

Using (4.16), we have

$$M = \rho\,abc \cdot \frac{4}{3}\pi \cdot 1^3 = \frac{4}{3}\pi \rho\,abc.$$

Similarly, we find

$$I = \rho\,abc \iiint (a^2 x'^2 + b^2 y'^2)\,dV'$$

where the triple integral is over the volume of a ball of radius 1. Now, by symmetry,

$$\iiint x'^2 dV' = \iiint y'^2 dV' = \iiint z'^2 dV' = \frac{1}{3} \iiint r'^2 dV'$$

where $r'^2 = x'^2 + y'^2 + z'^2$, and we are integrating over the volume inside the sphere $r' = 1$. Let us use spherical coordinates in the primed system. Then

$$\iiint r'^2 dV' = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{1} r'^2 (r'^2 \sin\theta'\,dr'\,d\theta'\,d\phi')$$
$$= 4\pi \int_0^1 r'^4 dr' = \frac{4\pi}{5}.$$

Thus,

$$I = \rho\,abc \left[a^2 \iiint x'^2\,dV' + b^2 \iiint y'^2\,dV' \right] = \rho\,abc(a^2 + b^2)\frac{1}{3} \cdot \frac{4\pi}{5},$$

or, in terms of M,

$$I = \frac{1}{5} M (a^2 + b^2).$$

In order to find arc lengths using spherical or cylindrical coordinates, we need the arc length element ds. Recall that we found the polar coordinate arc length element ds (Figure 4.2) as the hypotenuse of the right triangle with sides dr and $r\,d\theta$. From Figure 4.1 you can see that ds can also be thought of as a diagonal of the area element. Similarly, in cylindrical and spherical coordinates (see Figures 4.4 and 4.5), the arc length element ds is a space diagonal of the volume element. In cylindrical coordinates (4.4), the sides of the volume element are dr, $r\,d\theta$, dz, so the arc length element is given by

$$(4.18) \qquad ds^2 = dr^2 + r^2\,d\theta^2 + dz^2 \qquad \text{(cylindrical coordinates).}$$

In spherical coordinates (4.5), the sides of the volume element are dr, $r\,d\theta$, $r\,\sin\theta\,d\phi$, so the arc length element is given by

$$(4.19) \qquad ds^2 = dr^2 + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\phi^2 \qquad \text{(spherical coordinates).}$$

It is also convenient to be able to find arc lengths algebraically. Let us do this for polar coordinates; the same method can be used in three dimensions. From (4.1) we have

$$\begin{aligned} dx &= \cos\theta\,dr - r\,\sin\theta\,d\theta, \\ dy &= \sin\theta\,dr + r\,\cos\theta\,d\theta. \end{aligned}$$

Squaring and adding these two equations, we get

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= (\cos^2\theta + \sin^2\theta)\,dr^2 + 0\cdot dr\,d\theta + r^2(\sin^2\theta + \cos^2\theta)\,d\theta^2 \\ &= dr^2 + r^2\,d\theta^2 \end{aligned}$$

as in (4.3). Using the same method for cylindrical and spherical coordinates (Problem 21) you can verify equations (4.18) and (4.19).

▶ **Example 5.** Express the velocity of a moving particle in spherical coordinates.

If s represents the distance the particle has moved along some path, then ds/dt is the velocity of the particle. Dividing (4.19) by dt^2, we find for the square of the velocity

$$(4.20) \qquad v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 \quad \text{(spherical coordinates).}$$

We have just seen how to find the arc length element ds in polar coordinates (or other systems) by calculating $\sqrt{dx^2 + dy^2}$. You might be tempted to try to find the area element by computing $dx\,dy$, but you would discover that this does not work—we must use the Jacobian [or geometry as in (4.2)] to get volume or area elements. You can see why by looking at Figure 4.1. The element of area $r\,dr\,d\theta$ at the point (x, y) is not the same as the element of area $dx\,dy$ at that point. Then consider Figure 4.2; the element of arc ds is the hypotenuse of the triangle with legs dr and $rd\theta$, and it is also the hypotenuse of the triangle with legs dx and dy. Thus ds is the *same element* for both x, y and r, θ and this is why we can compute ds in polar coordinates by calculating $\sqrt{dx^2 + dy^2}$. These comments hold for other coordinate systems, too. We can always find ds by computing $\sqrt{dx^2 + dy^2}$ or $\sqrt{dx^2 + dy^2 + dz^2}$, but we cannot compute area or volume elements directly from the rectangular ones—we must use the Jacobian or else geometrical methods.

PROBLEMS, SECTION 4

As needed, use a computer to plot graphs of figures and to check values of integrals.

1. For the disk $r \le a$, find by integration using polar coordinates:

(a) the area of the disk;

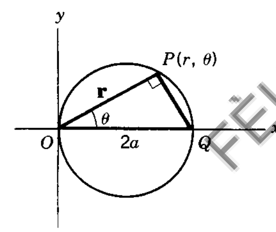
(b) the centroid of one quadrant of the disk;

(c) the moment of inertia of the disk about a diameter;

(d) the circumference of the circle $r = a$;

(e) the centroid of a quarter circle arc.

2. Using polar coordinates:



(a) Show that the equation of the circle sketched is $r = 2a\cos\theta$. *Hint:* Use the right triangle OPQ.

(b) By integration, find the area of the disk $r \le 2a\cos\theta$.

(c) Find the centroid of the area of the first quadrant half disk.

(d) Find the moments of inertia of the disk about each of the three coordinate axes, assuming constant area density.

(e) Find the length and the centroid of the semicircular arc in the first quadrant.

(f) Find the center of mass and the moments of inertia of the disk if the density is r.

(g) Find the area common to the disk sketched and the disk $r \le a$.

3. (a) Find the moment of inertia of a circular disk (uniform density) about an axis through its center and perpendicular to the plane of the disk.

(b) Find the moment of inertia of a solid right circular cylinder (uniform density) about its axis.

(c) Do (a) using Problem 1c and the perpendicular axis theorem (Section 3, Example 1f).

4. For the sphere $r = a$, find by integration:

(a) its surface area;

(b) the centroid of the curved surface area of a hemisphere;

(c) the moment of inertia of the whole spherical shell (that is, surface area) about a diameter (assuming constant area density);

(d) the volume of the ball $r \le a$;

(e) the centroid of a solid half ball.

5. (a) Write a triple integral in spherical coordinates for the volume inside the cone $z^2 = x^2 + y^2$ and between the planes $z = 1$ and $z = 2$. Evaluate the integral.

(b) Do (a) in cylindrical coordinates.

6. Find the mass of the solid in Problem 5 if the density is $(x^2 + y^2 + z^2)^{-1}$. Check your work by doing the problem in both spherical and cylindrical coordinates.

7. (a) Using spherical coordinates, find the volume cut from the ball $r \le a$ by the cone $\theta = \alpha < \pi/2$.

(b) Show that the z coordinate of the centroid of the volume in (a) is given by the formula $\bar{z} = 3a(1 + \cos\alpha)/8$.

8. For the solid in Problem 7, find I_z/M if $\alpha = \pi/3$ and the density is constant.

9. Let the solid in Problem 7 have density $= \cos\theta$. Show that then $I_z = \frac{3}{10} M a^2 \sin^2\alpha$.

10. (a) Find the volume inside the cone $3z^2 = x^2 + y^2$, above the plane $z = 2$ and inside the sphere $x^2 + y^2 + z^2 = 36$. *Hint:* Use spherical coordinates.

(b) Find the centroid of the volume in (a).

11. Write a triple integral in cylindrical coordinates for the volume inside the cylinder $x^2 + y^2 = 4$ and between $z = 2x^2 + y^2$ and the (x, y) plane. Evaluate the integral.

12. (a) Write a triple integral in cylindrical coordinates for the volume of the solid cut from a ball of radius 2 by a cylinder of radius 1, one of whose rulings is a diameter of the ball. *Hint:* Take the axis of the cylinder parallel to the z axis; a cross section of the cylinder then looks like the figure in Problem 2.

(b) Write a triple integral for the moment of inertia about the z axis of a uniform solid occupying this volume.

(c) Evaluate the integrals in (a) and (b), and find I as a multiple of the mass.

13. (a) Write a triple integral in cylindrical coordinates for the volume of the part of a ball between two parallel planes which intersect the ball.

(b) Evaluate the integral in (a). *Warning hint:* Do the r and θ integrals first.

(c) Find the centroid of this volume.

14. Express the integral

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy$$

as an integral in polar coordinates (r, θ) and so evaluate it.

15. Find the cylindrical coordinate volume element by Jacobians.

Find the Jacobians $\partial(x, y)/\partial(u, v)$ of the given transformations from variables x, y to variables u, v:

16. $x = \frac{1}{2}(u^2 - v^2)$,
$y = uv$, (u and v are called parabolic cylinder coordinates).

17. $x = a \cosh u \cos v,$
$y = a \sinh u \sin v,$ (u and v are called elliptic cylinder coordinates).

18. Prove the following theorems about Jacobians.

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1.$$

$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(s,t)}.$$

Hint: Multiply the determinants (as you would matrices) and show that each element in the product determinant can be written as a single partial derivative. Also see Chapter 4, Section 7.

19. In the integral

$$I = \int_0^\infty \int_0^\infty \frac{x^2+y^2}{1+(x^2-y^2)^2} e^{-2xy}\, dx\, dy$$

make the change of variables

$$u = x^2 - y^2$$
$$v = 2xy$$

and evaluate I. *Hint:* Use (4.8) and the accompanying discussion.

20. In the integral

$$I = \int_{x=0}^{1/2} \int_{y=x}^{1-x} \left(\frac{x-y}{x+y}\right)^2 dy\, dx,$$

make the change of variables

$$x = \frac{1}{2}(r-s),$$
$$y = \frac{1}{2}(r+s),$$

and evaluate I. *Hints:* See Problem 19. To find the r and s limits, sketch the area of integration in the (x,y) plane and sketch the r and s axes. Then show that to cover the same integration area, you may take the r and s limits to be: s from 0 to r, r from 0 to 1.

21. Verify equations (4.18) and (4.19).

22. Use equation (4.18) to set up an integral for the length of wire required to wind a coil spirally about a cylinder of radius 1 in., and length 1 ft, if there are three turns per inch.

23. A loxodrome or rhumb line is a curve on the earth's surface along which a ship sails without changing its course, that is, such that it crosses the meridians at a constant angle α. Show that then $\tan\alpha = \sin\theta\, d\phi/d\theta$ (θ and ϕ are spherical coordinates). Use (4.19) to set up an integral for the distance traveled by a ship along a rhumb line. Show that although a rhumb line winds infinitely many times around either the north or the south pole, its total length is finite.

24. Compute the gravitational attraction on a unit mass at the origin due to the mass (of constant density) occupying the volume inside the sphere $r = 2a$ and above the plane $z = a$. *Hint:* The magnitude of the gravitational force on the unit mass due to the element of mass dM at (r,θ,ϕ) is $(G/r^2)dM$. You want the z component of this since the other components of the total force are zero by symmetry. Use spherical coordinates.

25. The volume inside a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Then $dV = 4\pi r^2\,dr = A\,dr$, where A is the area of the sphere. What is the geometrical meaning of the fact that the derivative of the volume is the area? Could you use this fact to find the volume formula given the area formula?

26. Use the parallel axis theorem (Problem 3.1)

(a) and Example 3, to find the moment of inertia of a solid ball about a line tangent to it;

(b) and Problem 3b to find the moment of inertia of a solid cylinder about a ruling.

27. Use the spherical coordinates θ and ϕ to find the area of a zone of a sphere (that is, the spherical surface area between two parallel planes). *Hint:* See dA in (4.5).

28. Find the center of mass of a hemispherical shell of constant density (mass per unit area) by using double integrals and the area element dA in (4.5). [Compare your result in Problem 4(b).]

▶ 5. SURFACE INTEGRALS

In the preceding sections we found surface areas, moments of them, etc., for surfaces of revolution. We now want to consider a way of computing surface integrals in general whether the surface is a surface of revolution or not. Consider a part of a surface as in Figure 5.1 and its projection in the (x, y) plane. We assume that any line parallel to the z axis intersects the surface only once. If this is not true, we must work with part of the surface at a time, or project the surface into a different plane. For example, if the surface is closed, we could find the areas of the upper and lower parts separately. For a cylinder with axis parallel to the z axis we could project the front and back parts separately into the (y, z) plane.

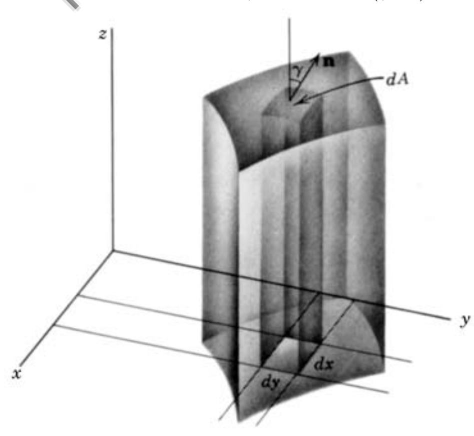


Figure 5.1

Let dA (Figure 5.1) be an element of surface area which projects onto $dx\,dy$ in the (x, y) plane and let γ be the acute angle between dA (that is, the tangent plane at dA) and the (x, y) plane. Then we have

$$dx\,dy = dA\cos\gamma \qquad \text{or} \qquad dA = \sec\gamma\,dx\,dy. \tag{5.1}$$

The surface area is then

$$\iint dA = \iint \sec\gamma \, dx\, dy \tag{5.2}$$

where the limits on x and y must be such that we integrate over the projected area in the (x, y) plane.

Now we must find $\sec\gamma$. The (acute) angle between two planes is the same as the (acute) angle between the normals to the planes. If $\mathbf{n}$ is a unit vector normal to the surface at dA (Figure 5.1), then γ is the (acute) angle between $\mathbf{n}$ and the z axis, that is, between the vectors $\mathbf{n}$ and $\mathbf{k}$, so $\cos\gamma = |\mathbf{n}\cdot\mathbf{k}|$. Let the equation of the surface be $\phi(x, y, z) = \text{const}$. Recall from Chapter 4 just after equation (9.14) that the vector

$$\text{grad}\,\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \tag{5.3}$$

is normal to the surface $\phi(x, y, z) = \text{const}$. (Also see Chapter 6, Section 6.) Then $\mathbf{n}$ is a unit vector in the direction of $\text{grad}\,\phi$, so

$$\mathbf{n} = (\text{grad}\,\phi)/|\text{grad}\,\phi|. \tag{5.4}$$

From (5.3) and (5.4) we find

$$\mathbf{n}\cdot\mathbf{k} = \frac{\mathbf{k}\cdot\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{\partial\phi/\partial z}{|\text{grad}\,\phi|},$$
$$\sec\gamma = \frac{1}{\cos\gamma} = \frac{1}{|\mathbf{n}\cdot\mathbf{k}|},$$

so

$$\sec\gamma = \frac{|\text{grad}\,\phi|}{|\partial\phi/\partial z|} = \frac{\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}}{|\partial\phi/\partial z|}. \tag{5.5}$$

Often the equation of a surface is given in the form $z = f(x, y)$. In this case $\phi(x, y, z) = z - f(x, y)$, so $\partial\phi/\partial z = 1$, and (5.5) simplifies to

$$\sec\gamma = \sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1}. \tag{5.6}$$

We then substitute (5.5) or (5.6) into (5.2) and integrate to find the area. To find centroids, moments of inertia, etc., we insert the proper factor into (5.2) as we have discussed in Section 3.

▶ **Example 1.** Find the area cut from the upper half of the sphere $x^2 + y^2 + z^2 = 1$ by the cylinder $x^2 + y^2 - y = 0$.

This is the same as the area on the sphere which projects onto the disk $x^2 + y^2 - y \le 0$ in the (x, y) plane. Thus we want to integrate (5.2) over the area of this disk. Figure 5.2 shows the disk of integration (shaded) and the equatorial circle of the sphere (large circle). We compute $\sec\gamma$ from the equation of the sphere; we could use (5.6), but it is easier in this problem to use (5.5):

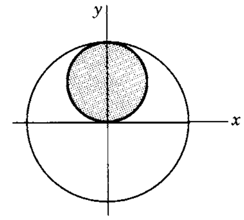


Figure 5.2

$$\phi = x^2 + y^2 + z^2,$$
$$\sec\gamma = \frac{|\operatorname{grad}\phi|}{|\partial\phi/\partial z|} = \frac{1}{2z}\sqrt{(2x)^2 + (2y)^2 + (2z)^2} = \frac{1}{z} = \frac{1}{\sqrt{1 - x^2 - y^2}}.$$

We find the limits of integration from the equation of the shaded disk, $x^2+y^2-y \le 0$. Because of the symmetry we can integrate over the first-quadrant part of the shaded area and double our result. Then the limits are:

$$x \text{ from } 0 \text{ to } \sqrt{y - y^2},$$
$$y \text{ from } 0 \text{ to } 1.$$

The desired area is

$$A = 2\int_{y=0}^{1}\int_{x=0}^{\sqrt{y-y^2}} \frac{dx\,dy}{\sqrt{1 - x^2 - y^2}}. \tag{5.7}$$

This integral is simpler in polar coordinates. The equation of the cylinder is then $r = \sin\theta$, so the limits are: r from 0 to $\sin\theta$, and θ from 0 to $\pi/2$. Thus (5.7) becomes

$$A = 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{1 - r^2}}. \tag{5.8}$$

This is still simpler if we make the change of variable $z = \sqrt{1 - r^2}$. Then $dz = -r\,dr/\sqrt{1 - r^2}$, and the limits $r = 0$ to $\sin\theta$ become $z = 1$ to $\cos\theta$. Thus (5.8) becomes

$$A = -2\int_{\theta=0}^{\pi/2}\int_{z=1}^{\cos\theta} dz\,d\theta = \pi - 2. \tag{5.9}$$

▶ PROBLEMS, SECTION 5

For these problems, the most important sketch is the projection in the plane of integration, which is easy to do by hand. However, you might like to use your computer to plot the corresponding 3 dimensional picture.

1. Find the area of the plane $x - 2y + 5z = 13$ cut out by the cylinder $x^2 + y^2 = 9$.

2. Find the surface area cut from the cone $2x^2 + 2y^2 = 5z^2$, $z > 0$, by the cylinder $x^2 + y^2 = 2y$.

3. Find the area of the paraboloid $x^2 + y^2 = z$ inside the cylinder $x^2 + y^2 = 9$.

4. Find the area of the part of the cone $2z^2 = x^2 + y^2$ in the first octant cut out by the planes $y = 0$, and $y = x/\sqrt{3}$, and the cylinder $x^2 + y^2 = 4$.

5. Find the area of the part of the cone $z^2 = 3(x^2 + y^2)$ which is inside the sphere $x^2 + y^2 + z^2 = 16$.

6. In Example 1, find the area of the cylinder inside the sphere.

7. Find the area of the part of the cylinder $y^2 + z^2 = 4$ in the first octant, cut out by the planes $x = 0$ and $y = x$.

8. Find the area of the part of the cylinder $z = x + y^2$ that lies below the second-quadrant area bounded by the x axis, $x = -1$, and $y^2 = -x$.

9. Find the area of the part of the cone $x^2+y^2 = z^2$ that is over the disk $(x-1)^2+y^2 \le 1$.

10. Find the area of the part of the sphere of radius a and center at the origin which is above the square in the (x, y) plane bounded by $x = \pm a/\sqrt{2}$ and $y = \pm a/\sqrt{2}$. *Hint for evaluating the integral*: Change to polar coordinates and evaluate the r integral first.

11. The part of the plane $x + y + z = 1$ which is in the first octant is a triangular area (sketch it). Find the area and its centroid by integration. You might like to check your work by geometry.

12. In Problem 11, let the triangle have a density (mass per unit area) equal to x. Find the total mass and the coordinates of the center of mass.

13. For the area of Example 1, find the z coordinate of the centroid.

14. For the area in Example 1, let the mass per unit area be equal to $|x|$. Find the total mass.

15. For a uniform mass distribution over the area of Example 1, find the moment of inertia about the z axis.

16. Find the centroid of the surface area in Problem 2.

6. MISCELLANEOUS PROBLEMS

As needed, use a computer to plot graphs and to check values of integrals.

1. Find the volume inside the cone $z^2 = x^2 + y^2$, above the (x, y) plane, and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. *Hint:* Use spherical coordinates.

2. Find the z coordinate of the centroid of the volume in Problem 1.

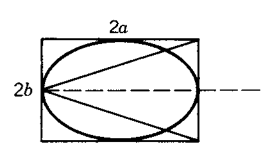
3. Find the mass of the solid in Problem 1 if the density is equal to z.

4. Find the moment of inertia of a hoop (wire bent to form a circle of radius R)

(a) about a diameter;

(b) about a tangent line.

5. The rectangle in the figure has sides $2a$ and $2b$; the curve is an ellipse. If the figure is rotated about the dotted line it generates three solids of revolution: a cone, an ellipsoid, and a cylinder. Show that the volumes are in the ratio 1 : 2 : 3. (See L.H. Lange, American Mathematical Monthly vol. 88 (1981), p. 339.)



6. (a) Find the area inside the circle $r = 2$, with $x > 0$ and $y > 1$.

(b) Find the centroid of the area in (a).

7. For a lamina of density 1 in the shape of the area of Problem 6, find its moment of inertia about the z axis.

8. For the solid bounded above by the sphere $x^2+y^2+z^2 = 4$ and below by a horizontal plane through $(0, 0, 1)$, find

(a) the volume (see Problem 6 and Problem 3.12);

(b) the z coordinate of the centroid (use cylindrical coordinates).

9. Find the centroid of the area above $y = x^2$ and below $y = c\,(c > 0)$.

10. (a) Find the centroid of the area between the x axis and one arch of $y = \sin x$.

(b) Find the volume formed if the area in (a) is rotated about the x axis.

(c) Find I_x of a mass of constant density occupying the volume in (b).

11. Show that the z coordinate of the centroid of the volume inside the elliptic cone

$$\frac{z^2}{h^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \quad 0 < z < h, \quad \text{is} \quad \bar{z} = \frac{3}{4}h.$$

(Note that the result is independent of a and b.) *Hint:* To evaluate the triple integrals, let $z = hz'$, $x = ax'$, $y = by'$, and then change to cylindrical coordinates in the primed system (see Example 4, Section 4). Compare Example 2, Section 4.

12. Find the mass of the solid inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

if the density is $|xyz|$. *Hint:* Evaluate the triple integral as in Example 4, Section 4.

13. Find the surface area of the part of the cylinder $x^2 + z^2 = a^2$ inside the cylinder $x^2 + y^2 = a^2$. Use your computer to graph the two cylinders on the same axes.

14. Find the volume that is inside both cylinders in Problem 13.

15. Find I_x and I_y for a mass distribution of constant density occupying the solid in Problem 14. *Hint:* Do the x integration last.

16. Find the centroid of the first quadrant part of the arc $x^{2/3} + y^{2/3} = a^{2/3}$. *Hint:* Let $x = a \cos^3\theta$, $y = a \sin^3\theta$.

17. Find the moment of inertia about a diagonal of a framework consisting of the four sides of a square of side a.

18. Find the center of mass of the solid right circular cone inside $r^2 = z^2$, $0 < z < h$, if the density is $r^2 = x^2 + y^2$. Use cylindrical coordinates.

19. For the cone in Problem 18, find I_x/M, I_y/M, I_z/M. Also find I/M about a line through the center of mass parallel to the x axis.

20. (a) Find the area of the surface $z = 1 + x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$.

(b) Find the volume inside the cylinder between the surface and the (x, y) plane. Use cylindrical coordinates.

21. Find the gravitational attraction on a unit mass at the origin due to a mass (of constant density) occupying the volume inside the cone $z^2 = x^2 + y^2$, $0 < z < h$. See Problem 4.24.

22. Find I_x/M, I_y/M, I_z/M, for a lamina in the shape of an ellipse $(x^2/a^2)+(y^2/b^2) = 1$. *Hint:* See Problem 11.

23. (a) Find the centroid of the solid paraboloid inside $z = x^2 + y^2$, $0 < z < c$.

(b) Repeat part (a) if the density is $\rho = r = \sqrt{x^2 + y^2}$.

24. Repeat Problem 23a for the paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \qquad 0 < z < c.$$

25. By changing to polar coordinates, evaluate

$$\int_0^\infty \int_0^\infty e^{-\sqrt{x^2+y^2}}\,dx\,dy.$$

26. Make the change of variables $u = x - y$, $v = x + y$, to evaluate the integral

$$\int_0^1 dy \int_0^{1-y} e^{(x-y)/(x+y)}\,dx.$$

27. Make the change of variables $u = y/x$, $v = x + y$, to evaluate the integral

$$\int_0^1 dx \int_0^x \frac{(x+y)e^{x+y}}{x^2}\,dy.$$

CHAPTER 6

Vector Analysis

1. INTRODUCTION

In Chapter 3, Sections 4 and 5, we have discussed the basic ideas of vector algebra. The principal topic of this chapter will be vector calculus. First (Sections 2 and 3) we shall consider some applications of vector products. Then (Section 4 ff.) we shall discuss differentiation and integration of vector functions. You have probably seen Newton's second law $\mathbf{F} = m\mathbf{a}$ written as $\mathbf{F} = m\,d^2\mathbf{r}/dt^2$. You may have met Gauss's law in electricity which uses a surface integral of the normal component of a vector (Section 10). Derivatives and integrals of vector functions are important in almost every area of applied mathematics. Such diverse fields as mechanics, quantum mechanics, electrodynamics, theory of heat, hydrodynamics, optics, etc., make use of the vector equations and theorems we shall discuss in this chapter.

2. APPLICATIONS OF VECTOR MULTIPLICATION

In Chapter 3, Section 4, we defined the scalar or dot product of vectors $\mathbf{A}$ and $\mathbf{B}$, and the vector or cross product of $\mathbf{A}$ and $\mathbf{B}$ as follows, where θ is the angle ($\le 180°$) between the vectors:

(2.1) $$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z.$$

(2.2) $\mathbf{A}\times\mathbf{B} = \mathbf{C}$, where $|\mathbf{C}| = AB\sin\theta$, and the direction of $\mathbf{C}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ and in the sense of the rotation of $\mathbf{A}$ to $\mathbf{B}$ through the angle θ (Figure 2.1).

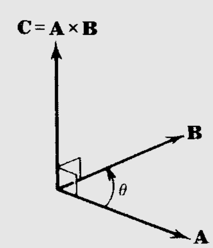


Figure 2.1

Let us consider some applications of these definitions.

Work In elementary physics you learned that work equals force times displacement. If the force and displacement are not parallel, then the component of the force perpendicular to the displacement does no work.

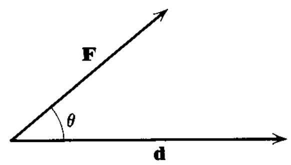


Figure 2.2

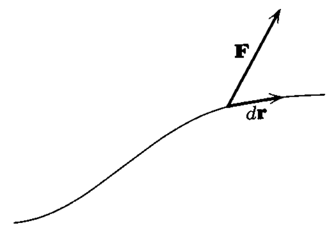


Figure 2.3

The work in this case is the component of the force parallel to the displacement, multiplied by the displacement; that is $W = (F\cos\theta)\cdot d = Fd\cos\theta$ (Figure 2.2). This can now conveniently be written as

$$W = Fd\cos\theta = \mathbf{F}\cdot\mathbf{d}. \tag{2.3}$$

If the force varies with distance, and perhaps also the direction of motion $\mathbf{d}$ changes with time, we can write, for an infinitesimal vector displacement $d\mathbf{r}$ (Figure 2.3)

$$dW = \mathbf{F}\cdot d\mathbf{r}. \tag{2.4}$$

We shall see later (Section 8) how to integrate dW in (2.4) to find the total work W done on a particle which is pushed along some path by a variable force $\mathbf{F}$.

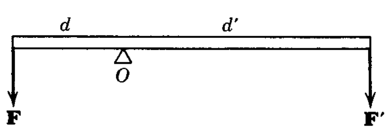


Figure 2.4

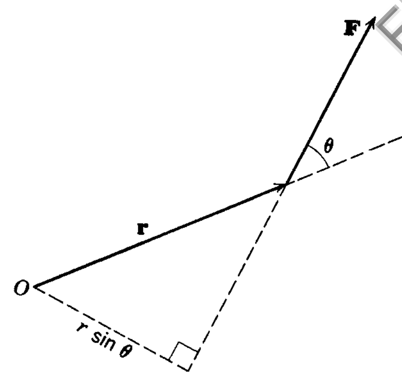


Figure 2.5

Torque In doing a seesaw or lever problem (Figure 2.4), you multiply force times distance; the quantity Fd is called the *torque* or *moment** of $\mathbf{F}$, and the distance d

*If the force $\mathbf{F}$ is due to a weight $w = mg$, then the torque about O in Figure 2.4 is $mg\cdot d = g\cdot(md)$; the moment of inertia (Chapter 5, Section 3) of m about O is md^2. The quantity md is called the moment (or *first* moment) of m about O, and the quantity md^2 is called the moment of inertia (or *second* moment) of m about O. By extension, we call mgd the moment of mg, or Fd the moment of $\mathbf{F}$. For an object which is not a point mass, the quantities md and md^2 become integrals (Chapter 5, Section 3).

from the fulcrum O to the line of action of $\mathbf{F}$ is the *lever arm* of $\mathbf{F}$. The lever arm is by definition the *perpendicular* distance from O to the line of action of $\mathbf{F}$. Then in general (Figure 2.5) the torque (or moment) of a force about O (really about an axis through O perpendicular to the paper) is defined as the magnitude of the force times its lever arm; in Figure 2.5 this is $Fr\sin\theta$. Now $\mathbf{r}\times\mathbf{F}$ has magnitude $rF\sin\theta$, so the magnitude of the torque is $|\mathbf{r}\times\mathbf{F}|$. We can also use the direction of $\mathbf{r}\times\mathbf{F}$ in describing the torque, in the following way. If you curve the fingers of your right hand in the direction of the rotation produced by applying the torque, then your thumb points in a direction parallel to the rotation axis. It is customary to call this the direction of the torque. By comparing Figures 2.5 and 2.1, we see that this is also the direction of $\mathbf{r}\times\mathbf{F}$. With this agreement, then, $\mathbf{r}\times\mathbf{F}$ is the torque or moment of $\mathbf{F}$ about an axis through O and perpendicular to the plane of the paper in Figure 2.5.

Angular Velocity In a similar way, a vector is used to represent the angular velocity of a rotating body. The direction of the vector is along the axis of rotation in the direction of progression of a right-handed screw turned the way the body is rotating. Suppose P in Figure 2.6 represents a point in a rigid body rotating with angular velocity $\boldsymbol{\omega}$. We can show that the linear velocity $\mathbf{v}$ of point P is $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$. First of all, $\mathbf{v}$ is in the right direction: It is perpendicular to the plane of $\mathbf{r}$ and $\boldsymbol{\omega}$ and in the right sense. Next we want to show that the magnitude of $\mathbf{v}$ is the same as $|\boldsymbol{\omega}\times\mathbf{r}|=\omega r\sin\theta$. But $r\sin\theta$ is the radius of the circle in which P is traveling, and ω is the angular velocity; thus $(r\sin\theta)\omega$ is $|\mathbf{v}|$, as we claimed.

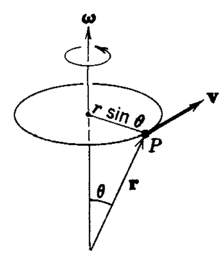


Figure 2.6

▶ 3. TRIPLE PRODUCTS

There are two products involving three vectors, one called the triple scalar product (because the answer is a scalar) and the other called the triple vector product (because the answer is a vector).

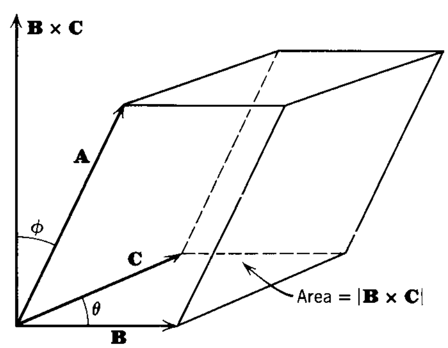


Figure 3.1

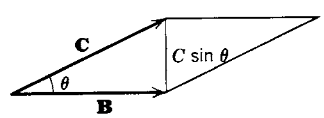


Figure 3.2

Triple Scalar Product This is written $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. There is a useful geometrical interpretation of the triple scalar product (see Figure 3.1). Construct a parallelepiped using $\mathbf{A},\mathbf{B},\mathbf{C}$ as three intersecting edges. Then $|\mathbf{B}\times\mathbf{C}|$ is the area of the base (Figure 3.2) because $|\mathbf{B}\times\mathbf{C}| = |\mathbf{B}|\,|\mathbf{C}|\sin\theta$, which is the area of a parallelogram with sides $|\mathbf{B}|$, $|\mathbf{C}|$, and angle θ. The height of the parallelepiped is $|\mathbf{A}|\cos\phi$ (Figure 3.1). Then the volume of the parallelepiped is

$$|\mathbf{B}|\,|\mathbf{C}|\sin\theta|\mathbf{A}|\cos\phi = |\mathbf{B}\times\mathbf{C}|\,|\mathbf{A}|\cos\phi = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

If $\phi > 90°$, this will come out negative, so in general we should say that the volume is $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$. Any side may be used as base, so, for example, $\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$ must also be either plus or minus the volume. There are six such triple scalar products, all equal except for sign [or twelve if you count both the type $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ and the type $(\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}$].

To write the triple scalar product in component form we first write $\mathbf{B}\times\mathbf{C}$ in determinant form [Chapter 3, equation (4.19)]:

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{3.1}$$

Now $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = A_x(\mathbf{B}\times\mathbf{C})_x + A_y(\mathbf{B}\times\mathbf{C})_y + A_z(\mathbf{B}\times\mathbf{C})_z$, and this is exactly what we get by expanding, by elements of the first row, the determinant in (3.2) below; this determinant is then equal to $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{3.2}$$

Recalling that an interchange of rows changes the sign of a determinant, we can now easily write out the six (or twelve) products mentioned above with their proper signs. You should convince yourself of, and then remember, the following facts: The *order* of the factors is all that counts; the dot and cross may be interchanged. If the order of factors is cyclic (one way around the circle in Figure 3.3), all such triple scalar products are equal. If you go the other way, you get another set all equal to each other and the negatives of the first set. For example,

$$\begin{aligned} (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} &= \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \\ &= \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \\ &= -(\mathbf{A}\times\mathbf{C})\cdot\mathbf{B}, \text{ etc.} \end{aligned} \tag{3.3}$$

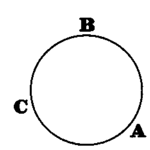


Figure 3.3

Since it doesn't matter where the dot and cross are, the triple scalar product is often written as $(\mathbf{ABC})$, meaning $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ or $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$.

Triple Vector Product This is written $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. Before we try to evaluate it, we can make the following observations. $\mathbf{B}\times\mathbf{C}$ is perpendicular to the plane of $\mathbf{B}$ and $\mathbf{C}$. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is perpendicular to the plane of $\mathbf{A}$ and $(\mathbf{B}\times\mathbf{C})$; we are particularly interested in the fact that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is perpendicular to $(\mathbf{B}\times\mathbf{C})$.

Now (see Figure 3.4) *any* vector perpendicular to $\mathbf{B}\times\mathbf{C}$ lies in the plane perpendicular to $\mathbf{B}\times\mathbf{C}$, that is, the plane of $\mathbf{B}$ and $\mathbf{C}$. Thus $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is *some* vector in the plane of $\mathbf{B}$ and $\mathbf{C}$, and can be written as some combination $a\mathbf{B}+b\mathbf{C}$, where a and b are scalars which we want to find. (See Chapter 3, Section 8, Problem 5.) One way to find a and b is to write out $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ in component form. We can simplify this work by choosing our coordinate system carefully; recall that a vector equation is true independently of the coordinate system. Given the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, we take the x axis along $\mathbf{B}$, and the y axis in the plane of $\mathbf{B}$ and $\mathbf{C}$; then $\mathbf{B}\times\mathbf{C}$ is in the z direction. The vectors in component form relative to these axes are:

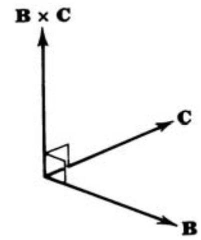


Figure 3.4

$$\begin{aligned}\mathbf{B}&=B_x\mathbf{i},\\ \mathbf{C}&=C_x\mathbf{i}+C_y\,\mathbf{j},\\ \mathbf{A}&=A_x\mathbf{i}+A_y\,\mathbf{j}+A_z\mathbf{k}.\end{aligned}\tag{3.4}$$

Using (3.4) we find

$$\begin{aligned}\mathbf{B}\times\mathbf{C}&=B_x\mathbf{i}\times(C_x\mathbf{i}+C_y\,\mathbf{j})=B_xC_y(\mathbf{i}\times\mathbf{j})=B_xC_y\mathbf{k},\\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C})&=A_xB_xC_y(\mathbf{i}\times\mathbf{k})+A_yB_xC_y(\mathbf{j}\times\mathbf{k})\\ &=A_xB_xC_y(-\mathbf{j})+A_yB_xC_y(\mathbf{i}).\end{aligned}\tag{3.5}$$

We would like to write $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ in (3.5) as a combination of $\mathbf{B}$ and $\mathbf{C}$; we can do this by adding and subtracting $A_xB_xC_x\mathbf{i}$:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=-A_xB_x(C_x\mathbf{i}+C_y\,\mathbf{j})+(A_yC_y+A_xC_x)B_x\mathbf{i}.\tag{3.6}$$

Each of these expressions is something simple in terms of the vectors in (3.4):

$$\begin{aligned}A_xB_x&=\mathbf{A}\cdot\mathbf{B}, &\qquad A_yC_y+A_xC_x&=\mathbf{A}\cdot\mathbf{C},\\ C_x\mathbf{i}+C_y\mathbf{j}&=\mathbf{C}, & B_x\mathbf{i}&=\mathbf{B}.\end{aligned}\tag{3.7}$$

Using (3.7) in (3.6), we get

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}.\tag{3.8}$$

This important formula should be learned, but not memorized in terms of letters, because that is confusing when you want some other combination of the same letters. Learn instead the following three facts:

> (3.9) The value of a triple vector product is a linear combination of the two vectors in the parenthesis [$\mathbf{B}$ and $\mathbf{C}$ in (3.8)]; the coefficient of each vector is the dot product of the other two; the middle vector in the triple product [$\mathbf{B}$ in (3.8)] always has the positive sign.

This method also covers triple vector products with the parenthesis first; by (3.9), the value of $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}$ is $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}-(\mathbf{A}\cdot\mathbf{C})\mathbf{B}$. This is correct since it is just the negative of what we had above for $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

Applications of the Triple Scalar Product We have shown that the torque of a force $\mathbf{F}$ about an axis may be written as $\mathbf{r} \times \mathbf{F}$ in one special case, namely when $\mathbf{r}$ and $\mathbf{F}$ are in a plane perpendicular to the axis. Now let us consider the general case of finding the torque produced by a force $\mathbf{F}$ about *any* given line (axis) L in Figure 3.5. Let $\mathbf{r}$ be a vector from some (that is, any) point on L to the line of action of $\mathbf{F}$; let O be the tail of $\mathbf{r}$. Then we define the torque about the *point* O to be $\mathbf{r} \times \mathbf{F}$. Note that this cannot contradict our previous discussion of torque because we were considering torque about a *line* before, and this definition is of torque about a *point*. However, we shall show how the two notions are connected. Also notice that $\mathbf{r} \times \mathbf{F}$ is not changed if the *head* of $\mathbf{r}$ is moved along $\mathbf{F}$; for this just adds a multiple of $\mathbf{F}$ to $\mathbf{r}$, and $\mathbf{F} \times \mathbf{F} = 0$ (see Problem 10).

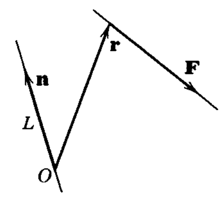


Figure 3.5

We shall now show that the torque of $\mathbf{F}$ about the line L through O is $\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F})$, where $\mathbf{n}$ is a unit vector along L. To simplify the calculation, choose the positive z axis in the direction $\mathbf{n}$; then $\mathbf{n} = \mathbf{k}$. Think of a door hinged to rotate about the z axis as in Figure 3.6. Let a force $\mathbf{F}$ be applied to it at the head of the vector $\mathbf{r}$. We first find the torque of $\mathbf{F}$ about the z axis by elementary methods and definition. Break $\mathbf{F}$ into its components; the z component is parallel to the rotation axis and produces no torque about it (pulling straight up or down on a door handle does not tend to open or close the door!). The x and y components can be seen better if we draw them in the (x, y) plane (Figure 3.7; note that the x and y axes are rotated $90°$ clockwise from their usual position in order to compare this figure more easily with Figure 3.6). The torque about the z axis produced by F_x and F_y is $xF_y - yF_x$ by the elementary definition of torque. We want to show that this is the same as $\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F})$ or here $\mathbf{k} \cdot (\mathbf{r} \times \mathbf{F})$. Using (3.2) we find

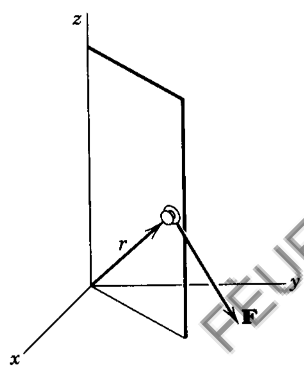


Figure 3.6

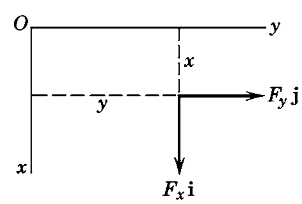


Figure 3.7

$$\mathbf{k} \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} 0 & 0 & 1 \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = xF_y - yF_x.$$

To summarize:

(3.10)	In Figure 3.5, the torque of $\mathbf{F}$ about point O is $\mathbf{r} \times \mathbf{F}$. The torque of $\mathbf{F}$ about the line L through O is $\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F})$ where $\mathbf{n}$ is a unit vector along L.

This proof can easily be given without reference to a coordinate system. Let the symbols $\parallel$ and $\perp$ stand for parallel and perpendicular to the given rotation axis $\mathbf{n}$. Then any vector ($\mathbf{F}$ or $\mathbf{r}$, say) can be written as the sum of a vector parallel to the axis and a vector perpendicular to the axis (that is, somewhere in the plane perpendicular to $\mathbf{n}$):

$$\mathbf{r} = \mathbf{r}_\perp + \mathbf{r}_\parallel, \qquad \mathbf{F} = \mathbf{F}_\perp + \mathbf{F}_\parallel.$$

Then the torque about O produced by $\mathbf{F}$ is

$$\begin{aligned}\mathbf{r} \times \mathbf{F} &= (\mathbf{r}_\perp + \mathbf{r}_\parallel)\times(\mathbf{F}_\perp + \mathbf{F}_\parallel)\\ &= \mathbf{r}_\perp\times\mathbf{F}_\perp + \mathbf{r}_\perp\times\mathbf{F}_\parallel + \mathbf{r}_\parallel\times\mathbf{F}_\perp + \mathbf{r}_\parallel\times\mathbf{F}_\parallel.\end{aligned}$$

The last term is zero (cross product of parallel vectors). Also $\mathbf{r}_\parallel$ and $\mathbf{F}_\parallel$ are parallel to $\mathbf{n}$; therefore their cross products with anything are in the plane perpendicular to $\mathbf{n}$, and the dot product of $\mathbf{n}$ with these is zero. Hence we have

$$\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F}) = \mathbf{n} \cdot (\mathbf{r}_\perp\times\mathbf{F}_\perp).$$

Now $\mathbf{r}_\perp$ and $\mathbf{F}_\perp$ are in a plane perpendicular to $\mathbf{n}$; thus the torque about $\mathbf{n}$ produced by $\mathbf{F}_\perp$ is (by Section 2) $\mathbf{r}_\perp\times\mathbf{F}_\perp$. But since only the component of $\mathbf{F}$ perpendicular to $\mathbf{n}$ produces a torque about $\mathbf{n}$, $\mathbf{r}_\perp\times\mathbf{F}_\perp$ is the total torque about $\mathbf{n}$ produced by $\mathbf{F}$. The vector torque $\mathbf{r}_\perp\times\mathbf{F}_\perp$ is in the $\pm\mathbf{n}$ direction since $\mathbf{r}_\perp$ and $\mathbf{F}_\perp$ are perpendicular to $\mathbf{n}$; the dot product of this vector torque with the unit vector $\mathbf{n}$ gives a scalar torque of the same magnitude; the $\pm$ sign indicates whether the torque is in the $+\mathbf{n}$ or the $-\mathbf{n}$ direction.

▶ **Example 1.** If $\mathbf{F} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ acts at the point $(1, 1, 1)$, find the torque of $\mathbf{F}$ about the line $\mathbf{r} = 3\mathbf{i} + 2\mathbf{k} + (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})t$.

We first find the vector torque about a point on the line, say the point $(3, 0, 2)$. By (3.10) and Figure 3.5, this is $\mathbf{r} \times \mathbf{F}$ where $\mathbf{r}$ is the vector *from* the point about which we want the torque, *to* the point at which $\mathbf{F}$ acts, that is, from $(3, 0, 2)$ to $(1, 1, 1)$; then $\mathbf{r} = (1, 1, 1) - (3, 0, 2) = (-2, 1, -1)$. The vector torque is

$$\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}.$$

The torque about the line is $\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F})$ where $\mathbf{n}$ is a unit vector along the line, namely $\mathbf{n} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. Then the torque about the line is

$$\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F}) = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}) = 1.$$

Example 2. As another application of the triple scalar product, let's find the Jacobian we used in Chapter 5, Section 4 for changing variables in a multiple integral. As you know, in rectangular coordinates the volume element is a rectangular box of volume $dx\,dy\,dz$. In other coordinate systems, the volume element may be approximately a parallelepiped as in Figure 3.1. We want a formula for the volume element in this case. (See, for example, the cylindrical and spherical coordinate volume elements in Chapter 5, Figures 4.4 and 4.5.)

Suppose we are given formulas for x, y, z as functions of new variables u, v, w. Then we want to find the vectors along the edges of the volume element in the u, v, w system. Suppose vector $\mathbf{A}$ in Figure 3.1 is along the direction in which u increases while v and w remain constant. Then if $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$ is a vector in this direction, we have

$$\mathbf{A} = \frac{\partial \mathbf{r}}{\partial u}\,du = \left(\mathbf{i}\,\frac{\partial x}{\partial u} + \mathbf{j}\,\frac{\partial y}{\partial u} + \mathbf{k}\,\frac{\partial z}{\partial u}\right)du.$$

Similarly if $\mathbf{B}$ is along the increasing v edge of the volume element and $\mathbf{C}$ is along the increasing w edge, we have

$$\mathbf{B} = \frac{\partial \mathbf{r}}{\partial v}\,dv = \left(\mathbf{i}\,\frac{\partial x}{\partial v} + \mathbf{j}\,\frac{\partial y}{\partial v} + \mathbf{k}\,\frac{\partial z}{\partial v}\right)dv,$$
$$\mathbf{C} = \frac{\partial \mathbf{r}}{\partial w}\,dw = \left(\mathbf{i}\,\frac{\partial x}{\partial w} + \mathbf{j}\,\frac{\partial y}{\partial w} + \mathbf{k}\,\frac{\partial z}{\partial w}\right)dw.$$

Then by (3.2)

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\[2ex] \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}\,du\,dv\,dw = J\,du\,dv\,dw$$

where J is the Jacobian of the transformation from x, y, z to u, v, w. Recall from the discussion of (3.2) that the triple scalar product may turn out to be positive or negative. Since we want a volume element to be positive, we use the absolute value of J. Thus the u, v, w volume element is $|J|\,du\,dv\,dw$ as stated in Chapter 5, Section 4.

Applications of the Triple Vector Product In Figure 3.8 (compare Figure 2.6), suppose the particle m is at rest on a rotating rigid body (for example, the earth). Then the *angular momentum* $\mathbf{L}$ of m about point O is defined by the equation $\mathbf{L} = \mathbf{r}\times(m\mathbf{v}) = m\mathbf{r}\times\mathbf{v}$. In the discussion of Figure 2.6, we showed that $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$. Thus, $\mathbf{L} = m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})$. See Problem 16 and also Chapter 10, Section 4.

As another example, it is shown in mechanics that the centripetal acceleration of m in Figure 3.8 is $\mathbf{a} = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$. See Problem 17.

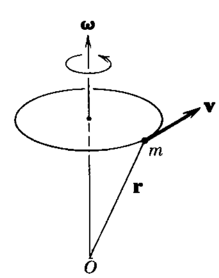


Figure 3.8

▶ PROBLEMS, SECTION 3

1. If $\mathbf{A} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{C} = \mathbf{j} + \mathbf{k}$, find $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, $\mathbf{A}(\mathbf{B}\cdot\mathbf{C})$, $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

For Problems 2 to 6, given $\mathbf{A} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{C} = \mathbf{j} - 5\mathbf{k}$:

2. Find the work done by the force $\mathbf{B}$ acting on an object which undergoes the displacement $\mathbf{C}$.

3. Find the total work done by forces $\mathbf{A}$ and $\mathbf{B}$ if the object undergoes the displacement $\mathbf{C}$. *Hint:* Can you add the two forces first?

4. Let O be the tail of $\mathbf{B}$ and let $\mathbf{A}$ be a force acting at the head of $\mathbf{B}$. Find the torque of $\mathbf{A}$ about O; about a line through O perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$; about a line through O parallel to $\mathbf{C}$.

5. Let $\mathbf{A}$ and $\mathbf{C}$ be drawn from a common origin and let $\mathbf{C}$ rotate about $\mathbf{A}$ with an angular velocity of 2 rad/sec. Find the velocity of the head of $\mathbf{C}$.

6. In Problem 5, draw $\mathbf{B}$ with its tail at the head of $\mathbf{A}$. If the figure is rotating as in Problem 5, find the velocity of the head of $\mathbf{B}$. With the same diagram, let $\mathbf{B}$ be a force; find the torque of $\mathbf{B}$ about the head of $\mathbf{C}$, and about the line $\mathbf{C}$.

7. A force $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ acts at the point $(1, 5, 2)$. Find the torque due to $\mathbf{F}$

(a) about the origin;

(b) about the y axis;

(c) about the line $x/2 = y/1 = z/(-2)$.

8. A vector force with components $(1, 2, 3)$ acts at the point $(3, 2, 1)$. Find the vector torque about the origin due to this force and find the torque about each of the coordinate axes.

9. The force $\mathbf{F} = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ acts at the point $(-5, 2, 1)$. Find the torque due to $\mathbf{F}$ about the origin and about the line $2x = -4y = -z$.

10. In Figure 3.5, let $\mathbf{r}'$ be another vector from O to the line of F. Show that $\mathbf{r}'\times\mathbf{F} = \mathbf{r}\times\mathbf{F}$. *Hint:* $\mathbf{r} - \mathbf{r}'$ is a vector along the line of $\mathbf{F}$ and so is a scalar multiple of $\mathbf{F}$. (The scalar has physical units of distance divided by force, but this fact is irrelevant for the vector proof.) Show also that moving the tail of $\mathbf{r}$ along $\mathbf{n}$ does not change $\mathbf{n}\cdot\mathbf{r}\times\mathbf{F}$. *Hint:* The triple scalar product is not changed by interchanging the dot and the cross.

11. Write out the twelve triple scalar products involving $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ and verify the facts stated just above (3.3).

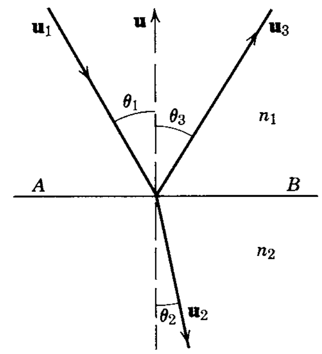
12. (a) Simplify $(\mathbf{A}\cdot\mathbf{B})^2 - [(\mathbf{A}\times\mathbf{B})\times\mathbf{B}]\cdot\mathbf{A}$ by using (3.9).

(b) Prove *Lagrange's identity*: $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$.

13. Prove that the triple scalar product of $(\mathbf{A}\times\mathbf{B})$, $(\mathbf{B}\times\mathbf{C})$, and $(\mathbf{C}\times\mathbf{A})$, is equal to the square of the triple scalar product of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. *Hint:* First let $(\mathbf{B}\times\mathbf{C}) = \mathbf{D}$, and evaluate $(\mathbf{A}\times\mathbf{B})\times\mathbf{D}$. [See Am. J. Phys. **66**, 739 (1998).]

14. Prove the *Jacobi identity*: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$. *Hint:* Expand each triple product as in equations (3.8) and (3.9).

15. In the figure $\mathbf{u}_1$ is a unit vector in the direction of an incident ray of light, and $\mathbf{u}_3$ and $\mathbf{u}_2$ are unit vectors in the directions of the reflected and refracted rays. If $\mathbf{u}$ is a unit vector normal to the surface AB, the laws of optics say that $\theta_1 = \theta_3$ and $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where n_1 and n_2 are constants (indices of refraction). Write these laws in vector form (using dot or cross products).



16. In the discussion of Figure 3.8, we found for the angular momentum, the formula $\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$. Use (3.9) to expand this triple product. If $\mathbf{r}$ is perpendicular to $\boldsymbol{\omega}$, show that you obtain the elementary formula, angular momentum $= mvr$.

17. Expand the triple product for $\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ given in the discussion of Figure 3.8. If $\mathbf{r}$ is perpendicular to $\boldsymbol{\omega}$ (Problem 16), show that $\mathbf{a} = -\omega^2\mathbf{r}$, and so find the elementary result that the acceleration is toward the center of the circle and of magnitude v^2/r.

18. Two moving charged particles exert forces on each other because each one creates a magnetic field in which the other moves (see Problem 4.6). These two forces are proportional to $\mathbf{v}_1 \times [\mathbf{v}_2 \times \mathbf{r}]$ and $\mathbf{v}_2 \times [\mathbf{v}_1 \times (-\mathbf{r})]$ where $\mathbf{r}$ is the vector joining the particles. By using (3.9), show that these forces are equal and opposite (Newton's third "law") if and only if $\mathbf{r} \times (\mathbf{v}_1 \times \mathbf{v}_2) = 0$. Compare Problem 14.

19. The force $\mathbf{F} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ acts at the point $(1, 1, 1)$.

(a) Find the torque of the force about the point $(2, -1, 5)$. *Careful!* The vector $\mathbf{r}$ goes from $(2, -1, 5)$ to $(1, 1, 1)$.

(b) Find the torque of the force about the line $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + (\mathbf{i} - \mathbf{j} + 2\mathbf{k})t$. Note that the line goes through the point $(2, -1, 5)$.

20. The force $\mathbf{F} = 2\mathbf{i} - 5\mathbf{k}$ acts at the point $(3, -1, 0)$. Find the torque of $\mathbf{F}$ about each of the following lines.

(a) $\mathbf{r} = (2\mathbf{i} - \mathbf{k}) + (3\mathbf{j} - 4\mathbf{k})t$.

(b) $\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})t$.

4. DIFFERENTIATION OF VECTORS

If $\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are fixed unit vectors and A_x, A_y, A_z are functions of t, then we define the derivative $d\mathbf{A}/dt$ by the equation

$$\frac{d\mathbf{A}}{dt} = \mathbf{i}\frac{dA_x}{dt} + \mathbf{j}\frac{dA_y}{dt} + \mathbf{k}\frac{dA_z}{dt}. \tag{4.1}$$

Thus the derivative of a vector $\mathbf{A}$ means a vector whose components are the derivatives of the components of $\mathbf{A}$.

▶ **Example 1.**

Let (x, y, z) be the coordinates of a moving particle at time t; then x, y, z are functions of t. The vector displacement of the particle from the origin at time t is

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z, \tag{4.2}$$

where $\mathbf{r}$ is a vector from the origin to the particle at time t. We say that $\mathbf{r}$ is the position vector or vector coordinate of the particle. The components of the velocity of the particle at time t are dx/dt, dy/dt, dz/dt so the velocity vector is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt}. \tag{4.3}$$

The acceleration vector is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{i}\frac{d^2x}{dt^2} + \mathbf{j}\frac{d^2y}{dt^2} + \mathbf{k}\frac{d^2z}{dt^2}. \tag{4.4}$$

The product of a scalar and a vector and the dot and cross products of vectors are differentiated by the ordinary calculus rules for differentiating a product, with one word of caution: The order of the factors must be kept in a cross product. You can easily prove equations (4.5) below by writing out components (Problem 1) and using (4.1).

$$\begin{aligned} \frac{d}{dt}(a\mathbf{A}) &= \frac{da}{dt}\mathbf{A} + a\frac{d\mathbf{A}}{dt}, \\ \frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\cdot\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\cdot\mathbf{B}, \\ \frac{d}{dt}(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}\times\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\times\mathbf{B}. \end{aligned} \tag{4.5}$$

The second term in $(d/dt)(\mathbf{A}\cdot\mathbf{B})$ *can* be written $\mathbf{B}\cdot d\mathbf{A}/dt$ if you like since $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$. But the corresponding term in $(d/dt)(\mathbf{A}\times\mathbf{B})$ must *not* be turned around unless you put a minus sign in front of it since $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$.

▶ **Example 2.** Consider the motion of a particle in a circle at constant speed. We can then write

$$\begin{aligned} r^2 &= \mathbf{r}\cdot\mathbf{r} = \text{ const.}, \\ v^2 &= \mathbf{v}\cdot\mathbf{v} = \text{ const.} \end{aligned} \tag{4.6}$$

If we differentiate these two equations using (4.5), we get

$$\begin{aligned} 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} &= 0 \quad \text{or} \quad \mathbf{r}\cdot\mathbf{v} = 0, \\ 2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} &= 0 \quad \text{or} \quad \mathbf{v}\cdot\mathbf{a} = 0. \end{aligned} \tag{4.7}$$

Also differentiating $\mathbf{r} \cdot \mathbf{v} = 0$, we get

$$\mathbf{r} \cdot \mathbf{a} + \mathbf{v} \cdot \mathbf{v} = 0 \quad \text{or} \quad \mathbf{r} \cdot \mathbf{a} = -v^2. \tag{4.8}$$

The first of equations (4.7) says that $\mathbf{r}$ is perpendicular to $\mathbf{v}$; the second says that $\mathbf{a}$ is perpendicular to $\mathbf{v}$. Therefore $\mathbf{a}$ and $\mathbf{r}$ are either parallel or antiparallel (since the motion is in a plane) and the angle θ between $\mathbf{a}$ and $\mathbf{r}$ is either 0° or 180°. From (4.8) and the definition of scalar product, we have

$$\mathbf{r} \cdot \mathbf{a} = |\mathbf{r}|\,|\mathbf{a}| \cos\theta = -v^2. \tag{4.9}$$

Thus we see that $\cos\theta$ is negative, so $\theta = 180°$. Then from (4.9) we get

$$|\mathbf{r}|\,|\mathbf{a}|(-1) = -v^2 \quad \text{or} \quad a = \frac{v^2}{r}. \tag{4.10}$$

We have just given a vector proof that for motion in a circle at constant speed the acceleration is toward the center of the circle and of magnitude v^2/r.

So far we have written vectors only in terms of their rectangular components using the unit basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. It is often convenient to use other coordinate systems, for example polar coordinates in two dimensions and spherical or cylindrical coordinates in three dimensions (see Chapter 5, Section 4, and Chapter 10, Sections 8 and 9). We shall consider using vectors in various coordinate systems in detail in Chapter 10, but it will be useful to discuss briefly here the use of plane polar coordinates. In Figure 4.1, think of starting at the point (x, y) or (r, θ) and moving along the line $\theta =$ const. in the direction of increasing r. We call this the "r direction"; we draw a unit vector (that is, a vector of length 1) in this direction

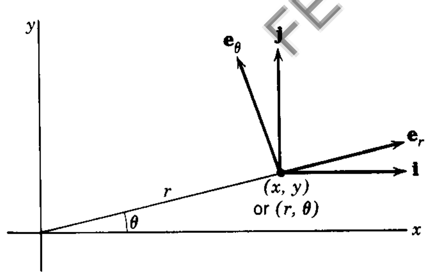


Figure 4.1

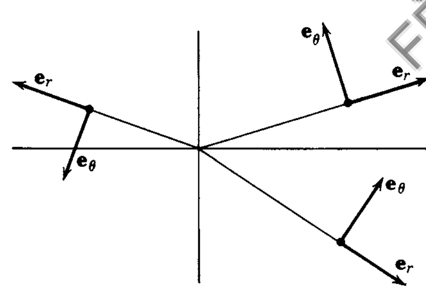


Figure 4.2

and label it $\mathbf{e}_r$. Similarly, think of moving along the circle $r =$ const. in the direction of increasing θ. We call this the "θ direction"; we draw a unit vector tangent to the circle and label it $\mathbf{e}_\theta$. These two vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the polar coordinate unit basis vectors just as $\mathbf{i}$ and $\mathbf{j}$ are the rectangular unit basis vectors. We can now write any given vector in terms of its components in the directions $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (by finding its projections in these directions). There is a complication here, however. In rectangular coordinates, the vectors $\mathbf{i}$ and $\mathbf{j}$ are constant in magnitude *and direction*. The polar coordinate unit basis vectors are constant in magnitude but their directions change from point to point (Figure 4.2). Thus in calculating the derivative of a vector written in polar coordinates, we must differentiate the basis vectors as well as the components [compare (4.1) where we differentiate the components only.] One straightforward way to do this is to express the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in terms of $\mathbf{i}$ and $\mathbf{j}$. From Figure 4.3, we see that the x and y components of $\mathbf{e}_r$ are $\cos\theta$ and $\sin\theta$. Thus we have

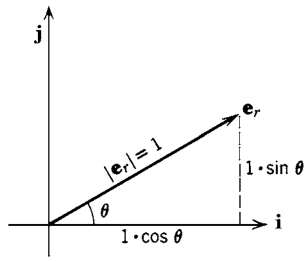


Figure 4.3

$$\mathbf{e}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta. \tag{4.11}$$

Similarly (Problem 7) we find

$$\mathbf{e}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta. \tag{4.12}$$

Differentiating $\mathbf{e}_r$ and $\mathbf{e}_\theta$ with respect to t, we get

$$\begin{aligned} \frac{d\mathbf{e}_r}{dt} &= -\mathbf{i}\sin\theta\,\frac{d\theta}{dt} + \mathbf{j}\cos\theta\,\frac{d\theta}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt}, \\ \frac{d\mathbf{e}_\theta}{dt} &= -\mathbf{i}\cos\theta\,\frac{d\theta}{dt} - \mathbf{j}\sin\theta\,\frac{d\theta}{dt} = -\mathbf{e}_r\frac{d\theta}{dt}. \end{aligned} \tag{4.13}$$

We can now use (4.13) in calculating the derivative of any vector which is written in terms of its polar components.

▶ **Example 3.** Given $\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta$, where A_r and A_θ are functions of t, find $d\mathbf{A}/dt$. We get

$$\frac{d\mathbf{A}}{dt} = \mathbf{e}_r\frac{dA_r}{dt} + A_r\frac{d\mathbf{e}_r}{dt} + \mathbf{e}_\theta\frac{dA_\theta}{dt} + A_\theta\frac{d\mathbf{e}_\theta}{dt}.$$

Using (4.13), we find

$$\frac{d\mathbf{A}}{dt} = \mathbf{e}_r\frac{dA_r}{dt} + \mathbf{e}_\theta A_r\frac{d\theta}{dt} + \mathbf{e}_\theta\frac{dA_\theta}{dt} - \mathbf{e}_r A_\theta\frac{d\theta}{dt}.$$

We can find higher-order derivatives if we like by differentiating again using (4.13) each time to evaluate the derivatives of $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

PROBLEMS, SECTION 4

1. Verify equations (4.5) by writing out the components.

2. Let the position vector (with its tail at the origin) of a moving particle be $\mathbf{r} = \mathbf{r}(t) = t^2\mathbf{i} - 2t\mathbf{j} + (t^2 + 2t)\mathbf{k}$, where t represents time.

(a) Show that the particle goes through the point $(4, -4, 8)$. At what time does it do this?

(b) Find the velocity vector and the speed of the particle at time t; at the time when it passes though the point $(4, -4, 8)$.

(c) Find the equations of the line tangent to the curve described by the particle and the plane normal to this curve, at the point $(4, -4, 8)$.

3. As in Problem 2, if the position vector of a particle is $\mathbf{r} = (4 + 3t)\mathbf{i} + t^3\mathbf{j} - 5t\mathbf{k}$, at what time does it pass through the point $(1, -1, 5)$? Find its velocity at this time. Find the equations of the line tangent to its path and the plane normal to the path, at $(1, -1, 5)$.

4. Let $\mathbf{r} = \mathbf{r}(t)$ be a vector whose *length* is always 1 (it may vary in direction). Prove that either $\mathbf{r}$ is a constant vector or $d\mathbf{r}/dt$ is perpendicular to $\mathbf{r}$. *Hint:* Differentiate $\mathbf{r} \cdot \mathbf{r}$.

5. The position of a particle at time t is given by $\mathbf{r} = \mathbf{i}\cos t + \mathbf{j}\sin t + \mathbf{k}t$. Show that both the speed and the magnitude of the acceleration are constant. Describe the motion.

6. The force acting on a moving charged particle in a magnetic field $\mathbf{B}$ is $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ where q is the electric charge of the particle, and $\mathbf{v}$ is its velocity. Suppose that a particle moves in the (x, y) plane with a uniform $\mathbf{B}$ in the z direction. Assuming Newton's second law, $m\,d\mathbf{v}/dt = \mathbf{F}$, show that the force and velocity are perpendicular and that both have constant magnitude. *Hint:* Find $(d/dt)(\mathbf{v} \cdot \mathbf{v})$.

7. Sketch a figure and verify equation (4.12).

8. In polar coordinates, the position vector of a particle is $\mathbf{r} = r\mathbf{e}_r$. Using (4.13), find the velocity and acceleration of the particle.

9. The angular momentum of a particle m is defined by $\mathbf{L} = m\mathbf{r} \times (d\mathbf{r}/dt)$ (see end of Section 3). Show that

$$\frac{d\mathbf{L}}{dt} = m\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}.$$

10. If $\mathbf{V}(t)$ is a vector function of t, find the indefinite integral

$$\int \left(\mathbf{V} \times \frac{d^2\mathbf{V}}{dt^2}\right) dt.$$

5. FIELDS

Many physical quantities have different values at different points in space. For example, the temperature in a room is different at different points: high near a register, low near an open window, and so on. The electric field around a point charge is large near the charge and decreases as we go away from the charge. Similarly, the gravitational force acting on a satellite depends on its distance from the earth. The velocity of flow of water in a stream is large in rapids and in narrow channels and small over flat areas and where the stream is wide. In all these examples there is a particular region of space which is of interest for the problem at hand; at every point of this region some physical quantity has a value. The term *field* is used to mean both the region and the value of the physical quantity in the region (for example, electric field, gravitational field). If the physical quantity is a scalar (for example, temperature), we speak of a *scalar field*. If the quantity is a vector (for example, electric field, force, or velocity), we speak of a *vector field*. Note again a point which we discussed in "endpoint problems" in Chapter 4, Section 10: Physical problems are often restricted to certain regions of space, and our mathematics must take account of this.

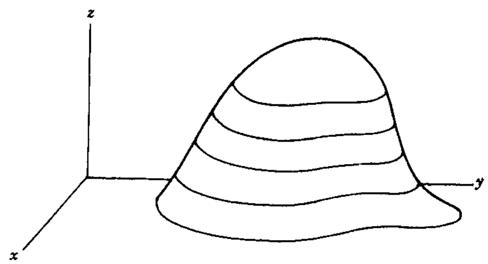


Figure 5.1

A simple example of a scalar field is the gravitational potential energy near the earth; its value is $V = mgz$ at every point of height z above some arbitrary reference level [which we take as the (x, y) plane]. Suppose that on a hill (Figure 5.1) we mark a series of curves each corresponding to some value of z (curves of constant elevation, often called *contour lines* or *level lines*). Any curve or surface on which a potential is constant is called an *equipotential*. Thus these level lines are equipotentials of the gravitational field since along any one curve the value of the gravitational potential energy mgz is constant. The horizontal planes which intersect the hill in these curves are equipotential surfaces (or level surfaces) of the gravitational field. (See Problems, Section 6 for more examples.)

As another example, let us ask for the equipotential surfaces in the field of an electric point charge q. The potential is $V = 9 \cdot 10^9 q/r$ (in SI units) at a point which is a distance r from the charge. The potential V is constant if r is constant; that is, the equipotentials of this electric field are spheres with centers at the charge. Similarly we could imagine drawing a set of surfaces (probably very irregular) in a room so that at every point of a single surface the temperature would be constant. These surfaces would be like equipotentials; they are called *isothermals* when the constant quantity is the temperature.

► 6. DIRECTIONAL DERIVATIVE; GRADIENT

Suppose that we know the temperature $T(x, y, z)$ at every point of a room, say, or of a metal bar. Starting at a given point we could ask for the rate of change of the temperature with distance (in degrees per centimeter) as we move away from the starting point. The chances are that the temperature increases in some directions and decreases in other directions, and that it increases more rapidly in some directions than others. Thus the rate of change of temperature with distance depends upon the *direction* in which we move; consequently it is called a *directional* derivative. In symbols, we want to find the limiting value of $\Delta T/\Delta s$ where Δs is an element of distance (arc length) in a given direction, and ΔT is the corresponding change in temperature; we write the directional derivative as dT/ds. We could also ask for the direction in which dT/ds has its largest value; this is physically the direction from which heat flows (that is, heat flows from hot to cold, in the opposite direction from the maximum rate of temperature increase).

Before we discuss how to calculate directional derivatives, consider another example. Suppose we are standing at a point on the side of the hill of Figure 5.1 (not at the top), and ask the question "In what direction does the hill slope downward most steeply from this point?" This is the direction in which you would start to slide if you lost your footing; it is the direction most people would probably call "straight" down. We want to make this vague idea more precise. Suppose we move a small distance Δs on the hill; the vertical distance Δz which we have gone may be positive (uphill) or negative (downhill) or zero (around the hill). Then $\Delta z/\Delta s$ and its limit dz/ds depend upon the *direction* in which we go; dz/ds is a directional derivative. The direction of steepest slope is the direction in which dz/ds has its largest absolute value. Notice that since the gravitational potential energy of a mass m is $V = mgz$, maximizing dz/ds is the same as maximizing dV/ds, where the equipotentials on the hill are $V(x, y) = mgz(x, y) = \text{const.}$

Let us now state and solve the general problem of finding a directional derivative. We are given a scalar field, that is, a function $\phi(x, y, z)$ [or $\phi(x, y)$ in a two-variable problem; the following discussion applies to two-variable problems if we simply drop terms and equations containing z]. We want to find $d\phi/ds$, the rate of change of ϕ with distance, at a given point (x_0, y_0, z_0) and in a given direction. Let $\mathbf{u} = \mathbf{i}a + \mathbf{j}b + \mathbf{k}c$ be a unit vector in the given direction. In Figure 6.1, we start at (x_0, y_0, z_0) and go a distance s $(s \geq 0)$ in the direction $\mathbf{u}$ to the point (x, y, z); the vector joining these points is $\mathbf{u}s$ since $\mathbf{u}$ is a unit vector. Then,

$$(x, y, z) - (x_0, y_0, z_0) = \mathbf{u}s = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})s$$

or

$$\begin{cases} x = x_0 + as, \\ y = y_0 + bs, \\ z = z_0 + cs. \end{cases} \tag{6.1}$$

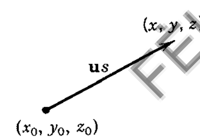


Figure 6.1

Equations (6.1) are the parametric equations of the line through (x_0, y_0, z_0) in the direction $\mathbf{u}$ [see Chapter 3, equation (5.8)] with the distance s (instead of t) as the parameter, and with $\mathbf{u}$ (instead of $\mathbf{A}$) as the vector along the line. From (6.1) we see that along the line, x, y, and z are each functions of a single variable, namely s [all the other letters in (6.1) are given constants]. If we substitute x, y, z in (6.1) into $\phi(x, y, z)$, then ϕ becomes a function of just the one variable s. That is, *along the straight line* (6.1), ϕ is a function of one variable, namely the distance along the line measured from (x_0, y_0, z_0). Since ϕ depends on s alone, we can find $d\phi/ds$:

$$\begin{aligned} \frac{d\phi}{ds} &= \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} \\ &= \frac{\partial\phi}{\partial x}a + \frac{\partial\phi}{\partial y}b + \frac{\partial\phi}{\partial z}c. \end{aligned} \tag{6.2}$$

This is the dot product of $\mathbf{u}$ with the vector $\mathbf{i}(\partial\phi/\partial x) + \mathbf{j}(\partial\phi/\partial y) + \mathbf{k}(\partial\phi/\partial z)$. This vector is called the *gradient* of ϕ and is written $\text{grad}\,\phi$ or $\boldsymbol{\nabla}\phi$ (read "del ϕ"). By definition

$$\nabla\phi = \operatorname{grad}\phi = \mathbf{i}\,\frac{\partial\phi}{\partial x} + \mathbf{j}\,\frac{\partial\phi}{\partial y} + \mathbf{k}\,\frac{\partial\phi}{\partial z}. \tag{6.3}$$

Then we can write (6.2) as

$$\frac{d\phi}{ds} = \nabla\phi\cdot\mathbf{u} \qquad \text{(directional derivative)}. \tag{6.4}$$

▶ **Example 1.** Find the directional derivative of $\phi = x^2y + xz$ at $(1, 2, -1)$ in the direction $\mathbf{A} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Here $\mathbf{u}$ is a unit vector obtained by dividing $\mathbf{A}$ by $|\mathbf{A}|$. Then we have

$$\mathbf{u} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

Using (6.3) we get

$$\nabla\phi = \mathbf{i}\,\frac{\partial\phi}{\partial x} + \mathbf{j}\,\frac{\partial\phi}{\partial y} + \mathbf{k}\,\frac{\partial\phi}{\partial z} = (2xy + z)\mathbf{i} + x^2\mathbf{j} + x\mathbf{k},$$
$$\nabla\phi \text{ at the point } (1, 2, -1) = 3\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Then from (6.4) we find

$$\frac{d\phi}{ds} \text{ at } (1, 2, -1) = \nabla\phi\cdot\mathbf{u} = 2 - \frac{2}{3} + \frac{1}{3} = \frac{5}{3}.$$

The gradient of a function has useful geometrical and physical meanings which we shall now investigate. From (6.4), using the definition of a dot product, and the fact that $|\mathbf{u}| = 1$, we have

$$\frac{d\phi}{ds} = |\nabla\phi|\cos\theta, \tag{6.5}$$

where θ is the angle between $\mathbf{u}$ and the vector $\nabla\phi$. Thus $d\phi/ds$ is the projection of $\nabla\phi$ on the direction $\mathbf{u}$ (Figure 6.2). We find the largest value of $d\phi/ds$ (namely $|\nabla\phi|$) if we go in the direction of $\nabla\phi$ (that is, $\theta = 0$ in Figure 6.2). If we go in the opposite direction (that is, $\theta = 180°$ in Figure 6.2) we find the largest rate of decrease of ϕ, namely $d\phi/ds = -|\nabla\phi|$.

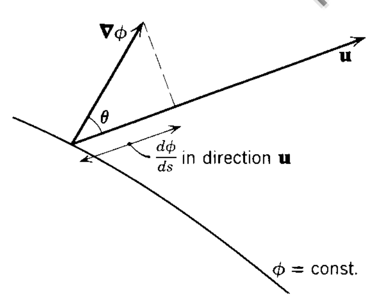


Figure 6.2

▶ **Example 2.** Suppose that the temperature T at the point (x, y, z) is given by the equation $T = x^2 - y^2 + xyz + 273$. In which direction is the temperature increasing most rapidly at $(-1, 2, 3)$, and at what rate? Here $\nabla T = (2x + yz)\mathbf{i} + (-2y + xz)\mathbf{j} + xy\mathbf{k} = 4\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$ at $(-1, 2, 3)$, and the increase in temperature is fastest in the direction of this vector. The rate of increase is $dT/ds = |\nabla T| = \sqrt{16 + 49 + 4} = \sqrt{69}$. We can also say that the temperature is decreasing most rapidly in the direction $-\nabla T$; in this direction, $dT/ds = -\sqrt{69}$. Heat flows in the direction $-\nabla T$ (that is, from hot to cold).

Next suppose $\mathbf{u}$ is tangent to the surface $\phi =$ const. at the point $P(x_0, y_0, z_0)$ (Figure 6.3). We want to show that $d\phi/ds$ in the direction $\mathbf{u}$ is then equal to zero. Consider $\Delta\phi/\Delta s$ for paths PA, PB, PC, etc., approaching the tangent $\mathbf{u}$. Since $\phi =$ const. on the surface, and P, A, B, C, etc. are all on the surface, $\Delta\phi = 0$, and $\Delta\phi/\Delta s = 0$ for such paths. But $d\phi/ds$ in the tangent direction is the limit of $\Delta\phi/\Delta s$ as $\Delta s \to 0$ (that is, as PA, PB, etc., approach $\mathbf{u}$), so $d\phi/ds$ in the direction $\mathbf{u}$ is zero also. Then for $\mathbf{u}$ along the tangent to $\phi =$ const., $\nabla\phi \cdot \mathbf{u} = 0$; this means that $\nabla\phi$ is perpendicular to $\mathbf{u}$. Since this is true for any $\mathbf{u}$ tangent to the surface at the point (x_0, y_0, z_0), then at that point:

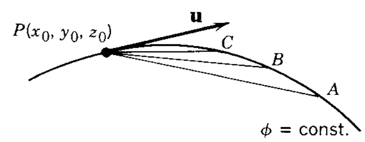


Figure 6.3

> The vector $\nabla\phi$ is perpendicular (normal) to the surface $\phi =$const.

Since $|\nabla\phi|$ is the value of the directional derivative in the direction normal (that is, perpendicular) to the surface, it is often called the *normal derivative* and written $|\nabla\phi| = d\phi/dn$.

We now see that the direction of largest rate of change of a given function ϕ with distance is perpendicular to the equipotentials (or level lines) $\phi =$ const. In the temperature problem, the direction of maximum dT/ds is then perpendicular to the isothermals. At any point this is the direction of ∇T and is called the direction of the temperature gradient. In the problem of the hill, the direction of steepest slope at any point is perpendicular to the level lines, that is, along ∇z or ∇V.

Example 3. Given the surface $x^3y^2z = 12$, find the equations of the tangent plane and normal line at $(1, -2, 3)$.

This is a level surface of the function $w = x^3y^2z$, so the normal direction is the direction of the gradient

$$\nabla w = 3x^2y^2z\mathbf{i} + 2x^3yz\mathbf{j} + x^3y^2\mathbf{k} = 36\mathbf{i} - 12\mathbf{j} + 4\mathbf{k} \quad \text{at} \quad (1, -2, 3).$$

A simpler vector in the same direction is $9\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. Then (see Chapter 3, Section 5) the equation of the tangent plane is

$$9(x-1) - 3(y+2) + (z-3) = 0,$$

and the equations of the normal line are

$$\frac{x-1}{9} = \frac{y+2}{-3} = \frac{z-3}{1}. \tag{6.6}$$

In (6.3) we have written the gradient in terms of its rectangular components. It is useful to write it in cylindrical and spherical coordinates also. (Note that this includes polar coordinates when $z = 0$). In cylindrical coordinates we want the components of $\nabla\phi$ in the directions $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z = \mathbf{k}$. According to (6.4), the component of ∇f in any direction $\mathbf{u}$ is the directional derivative df/ds in that direction. (We are changing the function from ϕ to f since ϕ is used as an angle in spherical, and sometimes in cylindrical and polar, coordinates.) The element of arc length ds in the r direction is dr so the directional derivative in the r direction is df/dr (θ and z constant) which we write as $\partial f/\partial r$. In the θ direction, the element of arc length is $r\,d\theta$ (Chapter 5, Section 4) so the directional derivative in the θ direction is $df/(r\,d\theta)$ (with r and z constant) which we write as $(1/r)\partial f/\partial\theta$. Thus we have in cylindrical coordinates (or polar without the z term)

$$(6.7) \qquad \nabla f = \mathbf{e}_r\frac{\partial f}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_z\frac{\partial f}{\partial z} \quad \text{in cylindrical coordinates.}$$

In a similar way we can show (Problem 21) that

$$(6.8) \qquad \nabla f = \mathbf{e}_r\frac{\partial f}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_\phi\frac{1}{r\sin\phi}\frac{\partial f}{\partial \phi} \quad \text{in spherical coordinates.}$$

PROBLEMS, SECTION 6

1. Find the gradient of $w = x^2y^3z$ at $(1, 2, -1)$.

2. Starting from the point $(1, 1)$, in what direction does the function $\phi = x^2 - y^2 + 2xy$ *decrease* most rapidly?

3. Find the derivative of $xy^2 + yz$ at $(1, 1, 2)$ in the direction of the vector $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

4. Find the derivative of $ze^x \cos y$ at $(1, 0, \pi/3)$ in the direction of the vector $\mathbf{i} + 2\mathbf{j}$.

5. Find the gradient of $\phi = z\sin y - xz$ at the point $(2, \pi/2, -1)$. Starting at this point, in what direction is ϕ *decreasing* most rapidly? Find the derivative of ϕ in the direction $2\mathbf{i} + 3\mathbf{j}$.

6. Find a vector normal to the surface $x^2 + y^2 - z = 0$ at the point $(3, 4, 25)$. Find the equations of the tangent plane and normal line to the surface at that point.

7. Find the direction of the line normal to the surface $x^2y + y^2z + z^2x + 1 = 0$ at the point $(1, 2, -1)$. Write the equations of the tangent plane and normal line at this point.

8. (a) Find the directional derivative of $\phi = x^2 + \sin y - xz$ in the direction $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ at the point $(1, \pi/2, -3)$.

(b) Find the equation of the tangent plane and the equations of the normal line to $\phi = 5$ at the point $(1, \pi/2, -3)$.

9. (a) Given $\phi = x^2 - y^2z$, find $\nabla\phi$ at $(1, 1, 1)$.

(b) Find the directional derivative of ϕ at $(1, 1, 1)$ in the direction $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(c) Find the equations of the normal line to the surface $x^2 - y^2z = 0$ at $(1, 1, 1)$.

For Problems 10 to 14, use a computer as needed to make plots of the given surfaces and the isothermal or equipotential curves. Try both 3D graphs and contour plots.

10. If the temperature in the (x, y) plane is given by $T = xy - x$, sketch a few isothermal curves, say for $T = 0, 1, 2, -1, -2$. Find the direction in which the temperature changes most rapidly with distance from the point $(1, 1)$, and the maximum rate of change. Find the directional derivative of T at $(1, 1)$ in the direction of the vector $3\mathbf{i} - 4\mathbf{j}$. Heat flows in the direction $-\boldsymbol{\nabla}T$ (perpendicular to the isothermals). Sketch a few curves along which heat would flow.

11. (a) Given $\phi = x^2 - y^2$, sketch on one graph the curves $\phi = 4$, $\phi = 1$, $\phi = 0$, $\phi = -1$, $\phi = -4$. If ϕ is the electrostatic potential, the curves $\phi =$ const. are equipotentials, and the electric field is given by $\mathbf{E} = -\boldsymbol{\nabla}\phi$. If ϕ is temperature, the curves $\phi =$const. are isothermals and $\boldsymbol{\nabla}\phi$ is the temperature gradient; heat flows in the direction $-\boldsymbol{\nabla}\phi$.

(b) Find and draw on your sketch the vectors $-\boldsymbol{\nabla}\phi$ at the points $(x, y) = (\pm1, \pm1)$, $(0, \pm2)$, $(\pm2, 0)$. Then, remembering that $\boldsymbol{\nabla}\phi$ is perpendicular to $\phi =$ const., sketch, without computation, several curves along which heat would flow [see(a)].

12. For Problem 11,

(a) Find the magnitude and direction of the electric field at $(2, 1)$.

(b) Find the direction in which the temperature is *decreasing* most rapidly at $(-3, 2)$.

(c) Find the rate of change of temperature with distance at $(1, 2)$ in the direction $3\mathbf{i} - \mathbf{j}$.

13. Let $\phi = e^x \cos y$. Let ϕ represent either temperature or electrostatic potential. Refer to Problem 11 for definitions and find:

(a) The direction in which the temperature is increasing most rapidly at $(1, -\pi/4)$ and the magnitude of the rate of increase.

(b) The rate of change of temperature with distance at $(0, \pi/3)$ in the direction $\mathbf{i} + \mathbf{j}\sqrt{3}$.

(c) The direction and magnitude of the electric field at $(0, \pi)$.

(d) The magnitude of the electric field at $x = -1$, any y.

14. (a) Suppose that a hill (as in Fig. 5.1) has the equation $z = 32 - x^2 - 4y^2$, where $z =$ height measured from some reference level (in hundreds of feet). Sketch a contour map (that is, draw on one graph a set of curves $z =$ const.); use the contours $z = 32, 19, 12, 7, 0$.

(b) If you start at the point $(3, 2)$ and in the direction $\mathbf{i} + \mathbf{j}$, are you going uphill or downhill, and how fast?

15. Repeat Problem 14b for the following points and directions.

(a) $(4, -2), \mathbf{i} + \mathbf{j}$ (b) $(-3, 1), 4\mathbf{i} + 3\mathbf{j}$

(c) $(2, 2), -3\mathbf{i} + \mathbf{j}$ (d) $(-4, -1), 4\mathbf{i} - 3\mathbf{j}$

16. Show by the Lagrange multiplier method that the maximum value of $d\phi/ds$ is $|\boldsymbol{\nabla}\phi|$. That is, maximize $d\phi/ds$ given by (6.3) subject to the condition $a^2+b^2+c^2 = 1$. You should get two values ($\pm$) for the Lagrange multiplier λ, and two values (maximum and minimum) for $d\phi/ds$. Which is the maximum and which is the minimum?

17. Find $\boldsymbol{\nabla}r$, where $r = \sqrt{x^2 + y^2}$, using (6.7) and also using (6.3). Show that your results are the same by using (4.11) and (4.12).

As in Problem 17, find the following gradients in two ways and show that your answers are equivalent.

18. $\boldsymbol{\nabla}x$ **19.** $\boldsymbol{\nabla}y$ **20.** $\boldsymbol{\nabla}(r^2)$

21. Verify equation (6.8); that is, find $\boldsymbol{\nabla}f$ in spherical coordinates as we did for cylindrical coordinates. *Hint*: What is ds in the ϕ direction? See Chapter 5, Figure 4.5.

▶ 7. SOME OTHER EXPRESSIONS INVOLVING ∇

If we write $\nabla\phi$ as $[\mathbf{i}(\partial/\partial x)+\mathbf{j}(\partial/\partial y)+\mathbf{k}(\partial/\partial z)]\phi$, we can then call the bracket ∇. By itself ∇ has no meaning (just as d/dx alone has no meaning; we must put some function after it to be differentiated). However, it is useful to use ∇ much as we use d/dx to indicate a certain operation.

We call ∇ a *vector operator* and write

(7.1)
$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

It is more complicated than d/dx (which is a *scalar operator*) because ∇ has vector properties too.

So far we have considered $\nabla\phi$ where ϕ is a scalar; we next want to consider whether ∇ can operate on a vector.

Suppose $\mathbf{V}(x,y,z)$ is a vector function, that is, the three components V_x, V_y, V_z of $\mathbf{V}$ are functions of x,y,z:

$$\mathbf{V}(x,y,z) = \mathbf{i}V_x(x,y,z) + \mathbf{j}V_y(x,y,z) + \mathbf{k}V_z(x,y,z).$$

(The subscripts mean components, *not* partial derivatives.) Physically, $\mathbf{V}$ represents a vector field (for example, the electric field about a point charge). At each point of space there is a vector $\mathbf{V}$, but the magnitude and direction of $\mathbf{V}$ may vary from point to point. We can form two useful combinations of ∇ and $\mathbf{V}$. We define the *divergence* of $\mathbf{V}$, abbreviated div $\mathbf{V}$ or $\nabla\cdot\mathbf{V}$, by (7.2):

(7.2)
$$\nabla\cdot\mathbf{V} = \operatorname{div}\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$

We define the *curl* of $\mathbf{V}$, written $\nabla\times\mathbf{V}$, by (7.3):

(7.3)
$$\begin{aligned}\nabla\times\mathbf{V} &= \operatorname{curl}\mathbf{V}\\ &= \mathbf{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}.\end{aligned}$$

You should study these expressions to see how we are using ∇ as "almost" a vector. The *definitions* of divergence and curl are the partial derivative expressions, of course. However, the similarity of the formulas (7.2) and (7.3) to those for $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ helps us to remember $\nabla\cdot\mathbf{V}$ and $\nabla\times\mathbf{V}$. But you must remember to put the partial derivative "components" of ∇ *before* the components of $\mathbf{V}$ in each term [for example, in evaluating the determinant in (7.3)]. Note that $\nabla \cdot \mathbf{V}$ is a scalar and $\nabla \times \mathbf{V}$ is a vector (compare $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$). We shall discuss later the meaning and some of the applications of the divergence and the curl of a vector function.

The quantity $\nabla\phi$ in (6.3) is a vector function; we can then let $\mathbf{V} = \nabla\phi$ in (7.2) and find $\nabla \cdot \nabla\phi = \operatorname{div}\operatorname{grad}\phi$. This is a very important expression called the *Laplacian* of ϕ; it is usually written as $\nabla^2\phi$. From (6.3) and (7.2), we have

$$\nabla^2\phi = \nabla \cdot \nabla\phi = \text{div grad } \phi = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial \phi}{\partial z} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \quad \text{(the Laplacian).} \tag{7.4}$$

The Laplacian is part of several important equations in mathematical physics:

$\nabla^2\phi = 0$	Laplace's equation.
$\nabla^2\phi = \dfrac{1}{a^2}\dfrac{\partial^2\phi}{\partial t^2}$	wave equation.
$\nabla^2\phi = \dfrac{1}{a^2}\dfrac{\partial\phi}{\partial t}$	diffusion, heat conduction, Schrödinger equation.

These equations arise in numerous problems in heat, hydrodynamics, electricity and magnetism, aerodynamics, elasticity, optics, etc.; we shall discuss solving such equations in Chapter 13.

There are many other more complicated expressions involving ∇ and one or more scalar or vector functions, which arise in various applications of vector analysis. For reference we list a table of such expressions at the end of the chapter (page 339). Notice that these are of two kinds: (1) expressions involving two applications of ∇ such as $\nabla \cdot \nabla\phi = \nabla^2\phi$; (2) combinations of ∇ with two functions (vectors or scalars) such as $\nabla \times (\phi\mathbf{V})$. We *can* verify these expressions simply by writing out components. However, it is usually simpler to use the same formulas we would use if ∇ were an ordinary vector, being careful to remember that ∇ is also a differential operator.

Example 1. Evaluate $\nabla \times (\nabla \times \mathbf{V})$. We use (3.8) for $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ being careful to write both ∇'s *before* the vector function $\mathbf{V}$ which they must differentiate. Then we get

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{V}) &= \nabla(\nabla \cdot \mathbf{V}) - (\nabla \cdot \nabla)\mathbf{V}\\ &= \nabla(\nabla \cdot \mathbf{V}) - \nabla^2\mathbf{V}.\end{aligned}$$

This is a vector as it should be; the Laplacian of a vector, $\nabla^2\mathbf{V}$, simply means a vector whose components are $\nabla^2 V_x$, $\nabla^2 V_y$, $\nabla^2 V_z$.

Example 2. Find $\nabla \cdot (\phi\mathbf{V})$, where ϕ is a scalar function and $\mathbf{V}$ is a vector function. Here we must differentiate a product, so our result will contain two terms. We could write these as

$$\nabla \cdot (\phi\mathbf{V}) = \nabla_\phi \cdot (\phi\mathbf{V}) + \nabla_\mathbf{V} \cdot (\phi\mathbf{V}), \tag{7.5}$$

where the subscripts on ∇ indicate which function is to be differentiated. Since ϕ is a scalar, it can be moved past the dot. Then

$$\nabla_\phi \cdot (\phi \mathbf{V}) = (\nabla_\phi \phi) \cdot \mathbf{V} = \mathbf{V} \cdot (\nabla \phi),$$

where we have removed the subscript in the last step since $\mathbf{V}$ no longer appears after ∇. Actually you may see in books $(\nabla\phi)\cdot\mathbf{V}$ meaning that only the ϕ is to be differentiated, but it is clearer to write it as $\mathbf{V}\cdot(\nabla\phi)$. [Be careful with $(\nabla\phi)\times\mathbf{V}$, however; assuming that this means that only ϕ is to be differentiated, the clear way to write it is $-\mathbf{V}\times(\nabla\phi)$; note the minus sign.] In the second term of (7.5), ϕ is a scalar and is not differentiated; thus it is just like a constant and we can write this term as $\phi(\nabla\cdot\mathbf{V})$. Collecting our results, we have

$$\nabla \cdot (\phi\mathbf{V}) = \mathbf{V}\cdot\nabla\phi + \phi(\nabla\cdot\mathbf{V}). \tag{7.6}$$

In Chapter 10, Section 9, we will derive the formulas for $\operatorname{div}\mathbf{V} = \nabla\cdot\mathbf{V}$ and $\nabla^2 f$ in cylindrical and spherical coordinates. However, it is useful to have the results for reference, so we state them here. Actually, these can be done as partial differentiation problems (see Chapter 4, Section 11), but the algebra is messy.

In cylindrical coordinates (or polar by omitting the z term):

$$\nabla\cdot\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial\theta}V_\theta + \frac{\partial}{\partial z}V_z \tag{7.7}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{7.8}$$

In spherical coordinates:

$$\nabla\cdot\mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{7.9}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}. \tag{7.10}$$

▶ PROBLEMS, SECTION 7

The purpose in doing the following simple problems is to become familiar with the formulas we have discussed. So a good study method is to do them by hand and then check your results by computer.
Compute the divergence and the curl of each of the following vector fields.

1. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

2. $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

3. $\mathbf{V} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$

4. $\mathbf{V} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

5. $\mathbf{V} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

6. $\mathbf{V} = x^2y\mathbf{i} + y^2x\mathbf{j} + xyz\mathbf{k}$

7. $\mathbf{V} = x\sin y\,\mathbf{i} + \cos y\,\mathbf{j} + xy\mathbf{k}$

8. $\mathbf{V} = \sinh z\,\mathbf{i} + 2y\,\mathbf{j} + x\cosh z\,\mathbf{k}$

Calculate the Laplacian ∇^2 of each of the following scalar fields.

9. $x^3 - 3xy^2 + y^3$

10. $\ln(x^2 + y^2)$

11. $\sqrt{x^2 - y^2}$

12. $(x + y)^{-1}$

13. $xy(x^2 + y^2 - 5z^2)$

14. $(x^2 + y^2 + z^2)^{-1/2}$

15. $xyz(x^2 - 2y^2 + z^2)$

16. $\ln(x^2 + y^2 + z^2)$

17. Verify formulas (b), (c), (d), (g), (h), (i), (j), (k) of the table of vector identities at the end of the chapter. *Hint* for (j): Start by expanding the two triple vector products on the right.

For $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, evaluate

18. $\nabla \times (\mathbf{k} \times \mathbf{r})$

19. $\nabla \cdot \left(\dfrac{\mathbf{r}}{|\mathbf{r}|}\right)$

20. $\nabla \times \left(\dfrac{\mathbf{r}}{|\mathbf{r}|}\right)$

8. LINE INTEGRALS

In Section 2, we discussed the fact that the work done by a force $\mathbf{F}$ on an object which undergoes an infinitesimal vector displacement $d\mathbf{r}$ can be written as

$$dW = \mathbf{F} \cdot d\mathbf{r}. \tag{8.1}$$

Suppose the object moves along some path (say A to B in Fig. 8.1), with the force $\mathbf{F}$ acting on it varying as it moves. For example, $\mathbf{F}$ might be the force on a charged particle in an electric field; then $\mathbf{F}$ would vary from point to point, that is $\mathbf{F}$ would be a function of x, y, z. However, *on a curve*, x, y, z are related by the equations of the curve. In three dimensions it takes two equations to determine a curve (as an intersection of two surfaces; for example, consider the equations of a straight line in Chapter 3, Section 5). Thus along a curve there is only *one* independent variable;

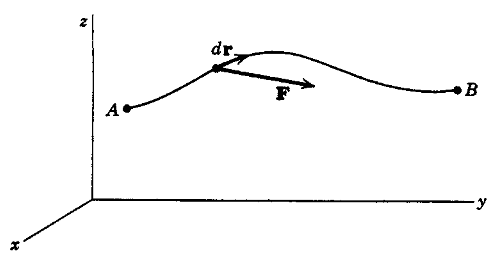


Figure 8.1

we can then write $\mathbf{F}$ and $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$ as functions of a single variable. The integral of $dW = \mathbf{F} \cdot d\mathbf{r}$ along the given curve then becomes an ordinary integral of a function of one variable and we can evaluate it to find the total work done by $\mathbf{F}$ in moving an object in Figure 8.1 from A to B. Such an integral is called a *line integral*. A line integral means an integral along a curve (or line), that is, a single integral as contrasted to a double integral over a surface or area, or a triple integral over a volume. The essential point to understand about a line integral is that there is *one* independent variable, because we are required to remain on a *curve*. In two dimensions, the equation of a curve might be written $y = f(x)$, where x is the independent variable. In three dimensions, the equations of a curve (for example, a straight line) can be written either like (6.6) (where we could take x as the independent variable and find y and z as functions of x), or (6.1) (where s is the independent variable and x, y, z are all functions of s). To evaluate a line integral, then, we must write it as a *single* integral using one independent variable.

▶ **Example 1.** Given the force $\mathbf{F} = xy\,\mathbf{i} - y^2\,\mathbf{j}$, find the work done by $\mathbf{F}$ along the paths indicated in Figure 8.2 from $(0,0)$ to $(2,1)$.

Since $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ on the (x,y) plane, we have

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy,$$
$$\mathbf{F}\cdot d\mathbf{r} = xy\,dx - y^2\,dy.$$

We want to evaluate

$$W = \int (xy\,dx - y^2\,dy). \tag{8.2}$$

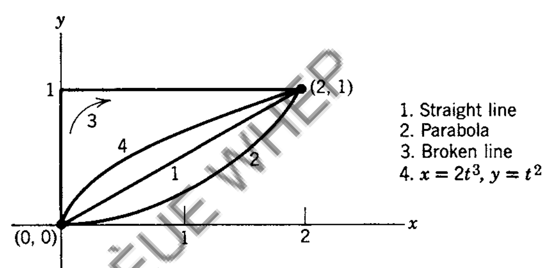


Figure 8.2

First we must write the integrand in terms of *one* variable. Along path 1 (a straight line), $y = \frac{1}{2}x$, $dy = \frac{1}{2}dx$. Substituting these values into (8.2), we obtain an integral in the one variable x. The limits for x (Figure 8.2) are 0 to 2. Thus, we get

$$W_1 = \int_0^2 \left[x\cdot\frac{1}{2}x\,dx - \left(\frac{1}{2}x\right)^2\cdot\frac{1}{2}\,dx\right] = \int_0^2 \frac{3}{8}x^2\,dx = \left.\frac{x^3}{8}\right|_0^2 = 1.$$

We could just as well use y as the independent variable and put $x = 2y$, $dx = 2dy$, and integrate from 0 to 1. (You should verify that the answer is the same.)

Along path 2 in Figure 8.2 (a parabola), $y = \frac{1}{4}x^2$, $dy = \frac{1}{2}x\,dx$. Then we get

$$W_2 = \int_0^2 \left(x\cdot\frac{1}{4}x^2\,dx - \frac{1}{16}x^4\cdot\frac{1}{2}x\,dx\right) = \int_0^2\left(\frac{1}{4}x^3 - \frac{1}{32}x^5\right)dx$$
$$= \left.\frac{x^4}{16} - \frac{x^6}{192}\right|_0^2 = \frac{2}{3}.$$

Along path 3 (the broken line), we have to use a different method. We integrate first from $(0,0)$ to $(0,1)$ and then from $(0,1)$ to $(2,1)$ and add the results. Along $(0,0)$ to $(0,1)$, $x=0$ and $dx=0$ so we must use y as the variable. Then we have

$$\int_{y=0}^{1}(0\cdot y\cdot 0-y^2dy)=-\frac{y^3}{3}\Big|_0^1=-\frac{1}{3}.$$

Along $(0,1)$ to $(2,1)$, $y=1$, $dy=0$, so we use x as the variable. We have

$$\int_{x=0}^{2}(x\cdot 1\cdot dx-1\cdot 0)=\frac{x^2}{2}\Big|_0^2=2.$$

Then the total $W_3=-\frac{1}{3}+2=\frac{5}{3}$.

Path 4 illustrates still another technique. Instead of using either x or y as the integration variable, we can use a parameter t. For $x=2t^3$, $y=t^2$, we have $dx=6t^2\,dt$, $dy=2t\,dt$. At the origin, $t=0$, and at $(2,1)$, $t=1$. Substituting these values into (8.2), we get

$$W_4=\int_0^1(2t^3\cdot t^2\cdot 6t^2\,dt-t^4\cdot 2t\,dt)=\int_0^1(12t^7-2t^5)\,dt=\frac{12}{8}-\frac{2}{6}=\frac{7}{6}.$$

Example 2. Find the value of

$$I=\int\frac{x\,dy-y\,dx}{x^2+y^2}$$

along each of the two paths indicated in Figure 8.3 from $(-1,0)$ to $(1,0)$. [Notice that we *could* have written $I=\int\mathbf{F}\cdot d\mathbf{r}$ with $\mathbf{F}=(-\mathbf{i}y+x\mathbf{j})/(x^2+y^2)$; however, there are also many other kinds of problems in which line integrals may arise.]

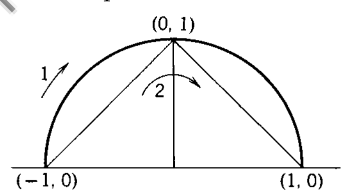


Figure 8.3

Along the circle it is simplest to use polar coordinates; then $r=1$ at all points of the circle and θ is the only variable. We then have

$$\begin{aligned} x&=\cos\theta, & dx&=-\sin\theta\,d\theta, \\ y&=\sin\theta, & dy&=\cos\theta\,d\theta, \qquad x^2+y^2=1, \end{aligned}$$

$$\frac{x\,dy-y\,dx}{x^2+y^2}=\frac{\cos^2\theta\,d\theta-\sin\theta(-\sin\theta)\,d\theta}{1}=d\theta.$$

At $(-1,0)$, $\theta=\pi$; at $(1,0)$, $\theta=0$. Then we get

$$I_1=\int_\pi^0 d\theta=-\pi.$$

Along path 2, we integrate from $(-1, 0)$ to $(0, 1)$ and from $(0, 1)$ to $(1, 0)$ and add the results. The first straight line has the equation $y = x + 1$; then $dy = dx$, and the integral is

$$\begin{aligned}\int_{-1}^{0} \frac{x\,dx - (x+1)\,dx}{x^2 + (x+1)^2} &= \int_{-1}^{0} \frac{-dx}{2x^2 + 2x + 1} = \int_{-1}^{0} \frac{-2\,dx}{(2x+1)^2 + 1} \\ &= -\arctan(2x+1)\Big|_{-1}^{0} \\ &= -\arctan 1 + \arctan(-1) \\ &= -\frac{\pi}{4} + \left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}.\end{aligned}$$

Along the second straight line $y = 1 - x$, $dy = -dx$, and the integral is

$$\begin{aligned}-\int_{0}^{1} \frac{x\,dx + (1-x)\,dx}{x^2 + (1-x)^2} &= \int_{0}^{1} \frac{-2\,dx}{(2x-1)^2 + 1} = -\arctan(2x-1)\Big|_{0}^{1} \\ &= -\frac{\pi}{2}.\end{aligned}$$

Adding the results for the integrals along the two parts of path 2, we get $I_2 = -\pi$.

Conservative Fields Notice that in Example 1 the answers were different for different paths, but in Example 2 they are the same. (See Section 11, however.) We can give a physical meaning to these facts if we interpret the integrals in all cases as the work done by a force on an object which moves along the path of integration. Suppose you want to get a heavy box across a sidewalk and up into a truck. Compare the work done in dragging the box across the sidewalk and then lifting it, with the work done in lifting it and then swinging it across in the air. In the first case work is done against friction in addition to the work required to lift the box; in the second case the only work done is that required to lift the box. Thus we see that the work done in moving an object from one point to another *may* depend on the path the object follows; in fact, it usually will when there is friction. Our example 1 was such a case. A force field for which $W = \int \mathbf{F} \cdot d\mathbf{r}$ depends upon the path as well as the endpoints is called *nonconservative*; physically this means that energy has been dissipated, say by friction. There are however, *conservative fields* for which $\int \mathbf{F} \cdot d\mathbf{r}$ is the same between two given points regardless of what path we calculate it along. For example, the work done in raising a mass m to the top of a mountain of height h is $W = mgh$ whether we lift the mass straight up a cliff or carry it up a slope, as long as no friction is involved. Thus the gravitational field is conservative.

It is useful to be able to recognize conservative and nonconservative fields before we do the integration. We shall see later (Section 11) that ordinarily $\operatorname{curl} \mathbf{F} = 0$ [see (7.3) for the definition of curl] is a necessary and sufficient condition for $\int \mathbf{F} \cdot d\mathbf{r}$ to be independent of the path, that is, $\operatorname{curl} \mathbf{F} = 0$ for conservative fields and $\operatorname{curl} \mathbf{F} \neq 0$ for nonconservative fields. (See Section 11 for a more careful discussion of this.) It is not hard to see why this is usually so. Suppose that for a given $\mathbf{F}$ there is a function $W(x, y, z)$ such that

(8.3)
$$\mathbf{F} = \nabla W = \mathbf{i}\frac{\partial W}{\partial x} + \mathbf{j}\frac{\partial W}{\partial y} + \mathbf{k}\frac{\partial W}{\partial z},$$
$$F_x = \frac{\partial W}{\partial x}, \quad F_y = \frac{\partial W}{\partial y}, \quad F_z = \frac{\partial W}{\partial z}.$$

Then assuming that $\partial^2 W/\partial x \partial y = \partial^2 W/\partial y \partial x$, etc. (see Chapter 4, end of Section 1), we get from (8.3)

(8.4)
$$\frac{\partial F_x}{\partial y} = \frac{\partial^2 W}{\partial y \partial x} = \frac{\partial F_y}{\partial x}, \quad \text{and similarly} \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}, \quad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}.$$

Using the definition (7.3) of $\operatorname{curl}\mathbf{F}$, we see that equations (8.4) say that the three components of $\operatorname{curl}\mathbf{F}$ are equal to zero. Thus if $\mathbf{F} = \nabla W$, then $\operatorname{curl}\mathbf{F} = 0$. Conversely (as we shall show later), if $\operatorname{curl}\mathbf{F} = 0$, then we can find a function $W(x, y, z)$ for which $\mathbf{F} = \nabla W$. Now if $\mathbf{F} = \nabla W$, we can write

(8.5)
$$\mathbf{F} \cdot d\mathbf{r} = \nabla W \cdot d\mathbf{r} = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz = dW,$$
$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B dW = W(B) - W(A),$$

where $W(B)$ and $W(A)$ mean the values of the function W at the endpoints A and B of the path of integration. Since the value of the integral depends only on the endpoints A and B, it is independent of the path along which we integrate from A to B, that is, $\mathbf{F}$ is conservative.

Potentials In mechanics, if $\mathbf{F} = \nabla W$ (that is, if $\mathbf{F}$ is conservative), then W is the work done by $\mathbf{F}$. For example, if a mass m falls a distance z under gravity, the work done on it is mgz. If, however, we lift the mass a distance z against gravity, the work done *by the force* $\mathbf{F}$ *of gravity* is $W = -mgz$ since the direction of motion is opposite to $\mathbf{F}$. The increase in potential energy of m in this case is $\phi = +mgz$, that is, $W = -\phi$, or $\mathbf{F} = -\nabla\phi$. The function ϕ is called the potential energy or the *scalar potential* of the force $\mathbf{F}$. (Of course, ϕ can be changed by adding any constant; this corresponds to a choice of the zero level of the potential energy and has no effect on $\mathbf{F}$.) More generally for any vector $\mathbf{V}$, if $\operatorname{curl}\mathbf{V} = 0$, there is a function ϕ, called the scalar potential of $\mathbf{V}$, such that $\mathbf{V} = -\nabla\phi$. (This is the customary definition of scalar potential in mechanics and electricity; in hydrodynamics many authors define the *velocity potential* so that $\mathbf{V} = +\nabla\phi$.)

Now suppose that we are given $\mathbf{F}$ or $dW = \mathbf{F} \cdot d\mathbf{r}$, and we find by calculation that $\operatorname{curl}\mathbf{F} = 0$. We then know that there *is* a function W and we want to know how to find it (up to an arbitrary additive constant of integration). To do this we can calculate the line integral in (8.5) from some reference point A to the variable point B along any convenient path; since the integral is independent of the path when $\operatorname{curl}\mathbf{F} = 0$, this process gives the value of W at the point B. (There is, of course, an additive constant in W whose value depends on our choice of the reference point A.)

▶ **Example 3.** Show that

$$\mathbf{F} = (2xy - z^3)\mathbf{i} + x^2\mathbf{j} - (3xz^2 + 1)\mathbf{k} \tag{8.6}$$

is conservative, and find a scalar potential ϕ such that $\mathbf{F} = -\boldsymbol{\nabla}\phi$.

We find

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy - z^3 & x^2 & -3xz^2 - 1 \end{vmatrix} = 0, \tag{8.7}$$

so $\mathbf{F}$ is conservative. Then

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B (2xy - z^3)\,dx + x^2\,dy - (3xz^2 + 1)\,dz \tag{8.8}$$

is independent of the path. Let us choose the origin as our reference point and integrate (8.8) from the origin to the point (x, y, z). As the path of integration, we choose the broken line from $(0, 0, 0)$ to $(x, 0, 0)$ to $(x, y, 0)$ to (x, y, z). From $(0, 0, 0)$ to $(x, 0, 0)$, we have $y = z = 0$, $dy = dz = 0$, so the integral is zero along this part of the path. From $(x, 0, 0)$ to $(x, y, 0)$, we have $x = \text{const.}$, $z = 0$, $dx = dz = 0$, so the integral is

$$\int_0^y x^2\,dy = x^2 \int_0^y dy = x^2 y.$$

From $(x, y, 0)$ to (x, y, z) we have $x = \text{const.}$, $y = \text{const.}$, $dx = dy = 0$, so the integral is

$$-\int_0^z (3xz^2 + 1)dz = -xz^3 - z.$$

Adding the three results, we get

$$W = x^2 y - xz^3 - z, \tag{8.9}$$

or

$$\phi = -W = -x^2 y + xz^3 + z. \tag{8.10}$$

▶ **Example 4.** Find the scalar potential for the electric field due to a point charge q at the origin.

Recall that the electric field at a point $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ means the force on a unit charge at $\mathbf{r}$ due to q and is (in Gaussian units)

$$\mathbf{E} = \frac{q}{r^2}\mathbf{e}_r = \frac{q}{r^2}\frac{\mathbf{r}}{r} = \frac{q}{r^3}\mathbf{r}. \tag{8.11}$$

(This is Coulomb's law in electricity.) If we take the zero level of the potential energy at infinity, then the scalar potential ϕ means the negative of the work done by the field on the unit charge as the charge moves from infinity to the point $\mathbf{r}$. This is

$$\phi = -\int_{\infty \text{ to } \mathbf{r}} \mathbf{E}\cdot d\mathbf{r} = q\int_{\mathbf{r} \text{ to } \infty} \frac{\mathbf{r}\cdot d\mathbf{r}}{r^3}. \tag{8.12}$$

It is simplest to evaluate the line integral using the spherical coordinate variable r along a radial line. This is justified by showing that $\operatorname{curl}\mathbf{E} = 0$, that is, that $\mathbf{E}$ is conservative (Problem 19). Since the differential of $(\mathbf{r}\cdot\mathbf{r})$ can be written as either $d(\mathbf{r}\cdot\mathbf{r}) = 2\mathbf{r}\cdot d\mathbf{r}$ or as $d(\mathbf{r}\cdot\mathbf{r}) = d(r^2) = 2r\,dr$, we have $\mathbf{r}\cdot d\mathbf{r} = r\,dr$ and (8.12) gives

$$\phi = q\int_r^\infty \frac{r\,dr}{r^3} = q\int_r^\infty \frac{dr}{r^2} = -\frac{q}{r}\bigg|_r^\infty = \frac{q}{r}. \tag{8.13}$$

It is interesting to obtain $\mathbf{r}\cdot d\mathbf{r} = r\,dr$ geometrically; in fact, for any vector $\mathbf{A}$, let us see that $\mathbf{A}\cdot d\mathbf{A} = A\,dA$. The vector $d\mathbf{A}$ means a change in the vector $\mathbf{A}$; a vector can change in both magnitude and direction (Figure 8.4). The scalar A means $|\mathbf{A}|$; the scalar dA means $d|\mathbf{A}|$. Thus dA is the increase in length of $\mathbf{A}$ and is *not* the same as $|d\mathbf{A}|$. In fact, from Figure 8.4, we see that

Figure 8.4

$$\mathbf{A}\cdot d\mathbf{A} = |\mathbf{A}|\,|d\mathbf{A}|\cos\alpha = A\,dA \tag{8.14}$$

since $dA = |d\mathbf{A}|\cos\alpha$. For the vector $\mathbf{r}$, we have

$$\begin{aligned} \mathbf{r} &= \mathbf{i}x + \mathbf{j}y + \mathbf{k}z,\\ d\mathbf{r} &= \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz,\\ |d\mathbf{r}| &= \sqrt{dx^2+dy^2+dz^2} = ds \qquad \text{(see Chapter 5)},\\ r &= |\mathbf{r}| = \sqrt{x^2+y^2+z^2},\\ dr &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x\,dx + 2y\,dy + 2z\,dz)\\ &= \frac{1}{r}(\mathbf{r}\cdot d\mathbf{r}), \end{aligned} \tag{8.15}$$

as above.

Exact Differentials The differential dW in (8.5) of a function $W(x, y, z)$ is called an *exact differential.* We could then say that $\operatorname{curl}\mathbf{F} = 0$ is a necessary and sufficient condition for $\mathbf{F}\cdot d\mathbf{r}$ to be an exact differential (but see Section 11). To make this clear, let us consider some examples in which $\mathbf{F}\cdot d\mathbf{r}$ is, or is not, an exact differential.

▶ **Example 5.** Consider the function W in (8.9).
Then

$$dW = (2xy - z^3)\,dx + x^2\,dy - (3xz^2+1)dz. \tag{8.16}$$

Here dW is an exact differential by definition since we got it by differentiating a function W. We can easily verify that if we write $dW = \mathbf{F}\cdot d\mathbf{r}$, then equations (8.4) are true:

$$\frac{\partial}{\partial x}(x^2) = 2x = \frac{\partial}{\partial y}(2xy - z^3),$$
$$\frac{\partial}{\partial x}(-3xz^2 - 1) = -3z^2 = \frac{\partial}{\partial z}(2xy - z^3), \tag{8.17}$$
$$\frac{\partial}{\partial y}(-3xz^2 - 1) = 0 = \frac{\partial}{\partial z}(x^2).$$

You should observe carefully how to get (8.17) from (8.16): the equations (8.17) say that, in (8.16), the partial derivative with respect to x of the coefficient of dy equals the partial derivative with respect to y of the coefficient of dx, and similarly for the other pairs of variables. The equations (8.17) are called the *reciprocity relations* in thermodynamics; in mechanics they are the components of $\operatorname{curl}\mathbf{F} = 0$ [see (8.7)]. In both cases they are true assuming that the mixed second partial derivatives are the same in either order, for example $\partial^2 W/\partial x\partial y = \partial^2 W/\partial y\partial x$ (see Chapter 4, end of Section 1).

We obtained dW in (8.16) by taking the differential of (8.9); now suppose we start with a given $dW = \mathbf{F}\cdot d\mathbf{r}$.

Example 6. Let us consider

$$dW = \mathbf{F}\cdot d\mathbf{r} = (2xy - z^3)\,dx + x^2\,dy + (3xz^2 + 1)\,dz. \tag{8.18}$$

This is almost the same as (8.16); just the sign of the dz term is changed. Then two of the equations corresponding to (8.17) do not hold, so $\operatorname{curl}\mathbf{F} \neq 0$, and dW is not an exact differential. We ask whether there is a function W of which (8.18) is the differential; the answer is "No" because if there were, the mixed second partial derivatives of W *would* be equal, and so $\operatorname{curl}\mathbf{F}$ would be zero. Equations like (8.18) often occur in applications. When dW is not exact, then $\mathbf{F}$ is a nonconservative force, and $\int \mathbf{F}\cdot d\mathbf{r}$, which is the work done by $\mathbf{F}$, depends not only on the points A and B but also upon the path along which the object moves. As we have said, this happens when there are friction forces.

PROBLEMS, SECTION 8

1. Evaluate the line integral $\int (x^2 - y^2)\,dx - 2xy\,dy$ along each of the following paths from $(0,0)$ to $(1,2)$.

(a) $y = 2x^2$.

(b) $x = t^2$, $y = 2t$.

(c) $y = 0$ from $x = 0$ to $x = 2$; then along the straight line joining $(2,0)$ to $(1,2)$.

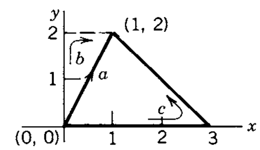
2. Evaluate the line integral $\oint (x + 2y)\,dx - 2x\,dy$ along each of the following closed paths, taken counterclockwise:

(a) the circle $x^2 + y^2 = 1$;

(b) the square with corners at $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$;

(c) the square with corners $(0,1)$, $(-1,0)$, $(0,-1)$, $(1,0)$.

3. Evaluate the line integral $\int xy\,dx + x\,dy$ from $(0,0)$ to $(1,2)$ along the paths shown in the sketch.

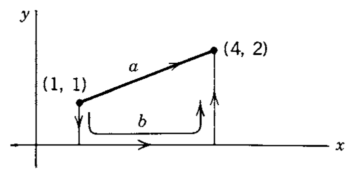


4. Evaluate the line integral $\int_C y^2\,dx + 2x\,dy + dz$, where C connects $(0,0,0)$ with $(1,1,1)$,

(a) along straight lines from $(0,0,0)$ to $(1,0,0)$ to $(1,0,1)$ to $(1,1,1)$;

(b) on the circle $x^2 + y^2 - 2y = 0$ to $(1,1,0)$ and then on a vertical line to $(1,1,1)$.

5. Find the work done by the force $\mathbf{F} = x^2y\mathbf{i} - xy^2\mathbf{j}$ along the paths shown from $(1,1)$ to $(4,2)$.

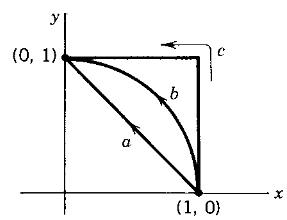


6. Find the work done by the force $\mathbf{F} = (2xy - 3)\mathbf{i} + x^2\mathbf{j}$ in moving an object from $(1,0)$ to $(0,1)$ along each of the three paths shown:

(a) straight line,

(b) circular arc,

(c) along lines parallel to the axes.



7. For the force field $\mathbf{F} = (y+z)\mathbf{i} - (x+z)\mathbf{j} + (x+y)\mathbf{k}$, find the work done in moving a particle around each of the following closed curves:

(a) the circle $x^2 + y^2 = 1$ in the (x,y) plane, taken counterclockwise;

(b) the circle $x^2 + z^2 = 1$ in the (z,x) plane, taken counterclockwise;

(c) the curve starting from the origin and going successively along the x axis to $(1,0,0)$, parallel to the z axis to $(1,0,1)$, parallel to the (y,z) plane to $(1,1,1)$, and back to the origin along $x = y = z$;

(d) from the origin to $(0,0,2\pi)$ on the curve $x = 1 - \cos t$, $y = \sin t$, $z = t$, and back to the origin along the z axis.

Verify that each of the following force fields is conservative. Then find, for each, a scalar potential ϕ such that $\mathbf{F} = -\nabla\phi$.

8. $\mathbf{F} = \mathbf{i} - z\mathbf{j} - y\mathbf{k}$.

9. $\mathbf{F} = (3x^2yz - 3y)\mathbf{i} + (x^3z - 3x)\mathbf{j} + (x^3y + 2z)\mathbf{k}$.

10. $\mathbf{F} = -k\mathbf{r}, \quad \mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z, \qquad k = \text{const.}$

11. $\mathbf{F} = y \sin 2x\,\mathbf{i} + \sin^2 x\,\mathbf{j}$.

12. $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + \mathbf{k}$.

13. $\mathbf{F} = z^2 \sinh y\,\mathbf{j} + 2z\cosh y\,\mathbf{k}$.

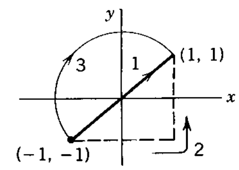
14. $\mathbf{F} = \dfrac{y}{\sqrt{1 - x^2y^2}}\,\mathbf{i} + \dfrac{x}{\sqrt{1 - x^2y^2}}\,\mathbf{j}$.

15. $\mathbf{F} = 2x\cos^2 y\,\mathbf{i} - (x^2 + 1)\sin 2y\,\mathbf{j}$.

16. Given $\mathbf{F}_1 = 2x\mathbf{i} - 2yz\mathbf{j} - y^2\mathbf{k}$ and $\mathbf{F}_2 = y\mathbf{i} - x\mathbf{j}$,

(a) Are these forces conservative? Find the potential corresponding to any conservative force.

(b) For any nonconservative force, find the work done if it acts on an object moving from $(-1,-1)$ to $(1,1)$ along each of the paths shown.



17. Which, if either, of the two force fields

$$\mathbf{F}_1 = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}, \qquad \mathbf{F}_2 = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$$

is conservative? Calculate for each field the work done in moving a particle around the circle $x = \cos t$, $y = \sin t$ in the (x, y) plane.

18. For the force field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$, calculate the work done in moving a particle from $(1, 0, 0)$ to $(-1, 0, \pi)$

(a) along the helix $x = \cos t$, $y = \sin t$, $z = t$;

(b) along the straight line joining the points.

Do you expect your answers to be the same? Why or why not?

19. Show that the electric field $\mathbf{E}$ of a point charge [equation (8.11)] is conservative. Write ϕ in (8.13) in rectangular coordinates, and find $\mathbf{E} = -\boldsymbol{\nabla}\phi$ using both rectangular coordinates (6.3) and cylindrical coordinates. Verify that your results are equivalent to (8.11).

20. For motion near the surface of the earth, we usually assume that the gravitational force on a mass m is

$$\mathbf{F} = -mg\mathbf{k},$$

but for motion involving an appreciable variation in distance r from the center of the earth, we must use

$$\mathbf{F} = -\frac{C}{r^2}\mathbf{e}_r = -\frac{C}{r^2}\frac{\mathbf{r}}{|\mathbf{r}|} = -\frac{C}{r^3}\mathbf{r},$$

where C is a constant. Show that both these $\mathbf{F}$'s are conservative, and find the potential for each.

21. Consider a uniform distribution of total mass m' over a spherical shell of radius r'. The potential energy ϕ of a mass m in the gravitational field of the spherical shell is

$$\phi = \begin{cases} \text{const.} & \text{if } m \text{ is inside the spherical shell,} \\ -\dfrac{Cm'}{r} & \text{if } m \text{ is outside the spherical shell, where } r \text{ is the distance} \\ & \text{from the center of the sphere to } m, \text{ and } C \text{ is a constant.} \end{cases}$$

Assuming that the earth is a spherical ball of radius R and constant density, find the potential and the force on a mass m outside and inside the earth. Evaluate the constants in terms of the acceleration of gravity g, to get

$$\begin{aligned} \mathbf{F} &= -\frac{mgR^2}{r^2}\mathbf{e}_r, &\text{and}\quad \phi &= -\frac{mgR^2}{r}, &&m \text{ outside the earth;} \\ \mathbf{F} &= -\frac{mgr}{R}\mathbf{e}_r, &\text{and}\quad \phi &= \frac{mg}{2R}(r^2 - 3R^2), &&m \text{ inside the earth.} \end{aligned}$$

Hint: To find the constants, recall that at the surface of the earth the magnitude of the force on m is mg.

9. GREEN'S THEOREM IN THE PLANE

The fundamental theorem of calculus says that the integral of the derivative of a function is the function, or more precisely:

$$\int_a^b \frac{d}{dt} f(t)\,dt = f(b) - f(a). \tag{9.1}$$

We are going to consider some useful generalizations of this theorem to two and three dimensions. The divergence theorem and Stokes' theorem (Sections 10 and 11) are very important in electrodynamics and other applications; in this section we will find two-dimensional forms of these theorems. First we develop an underlying useful theorem relating an area integral to the line integral around its boundary (see applications in examples and problems and also Chapter 14, Section 3).

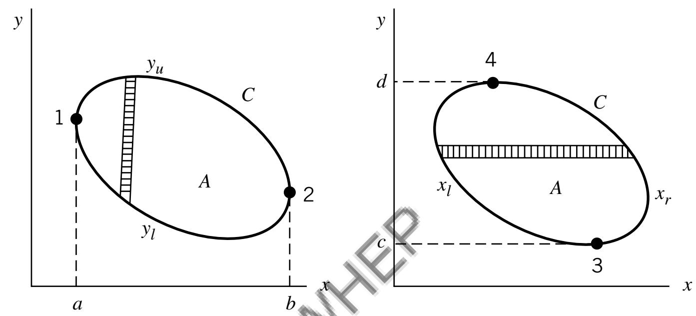


Figure 9.1

Recall that we know how to evaluate line integrals (Section 8), and that we learned in Chapter 5 to evaluate double integrals over areas in the (x, y) plane. We are going to consider areas (such as those in Figure 9.1 or in Chapter 5, Figure 2.7) for which we can evaluate the double integral over the area either with respect to x first or with respect to y first. Look at Figure 9.1. We want to find a relation between a double integral over the area A and a line integral around the curve C, for simple closed curves C. (A simple curve does not cross itself; for example, it is *not* a figure 8.) Now in Figure 9.1, the *upper* part of C between points 1 and 2 is given by an equation $y = y_u(x)$ and the *lower* part by an equation $y = y_l(x)$. (Think of solving the equation of a circle for $y_u(x) = \sqrt{1-x^2}$ and $y_l(x) = -\sqrt{1-x^2}$.) Similarly in Figure 9.1, we can find $x_l(y)$ and $x_r(y)$ for the left and right parts of C between points 3 and 4.

Let $P(x, y)$ and $Q(x, y)$ be continuous functions with continuous first derivatives. We are going to show that the double integral of $\partial P(x, y)/\partial y$ over the area A is equal to the line integral of P around C. We write the double integral using Figure 9.1 to integrate first with respect to y, and do the y integration by equation (9.1) with $t = y$ to get:

$$\iint_A \frac{\partial P(x,y)}{\partial y}\,dy\,dx = \int_a^b dx \int_{y_l}^{y_u} \frac{\partial P(x,y)}{\partial y}\,dy = \int_a^b [P(x,y_u) - P(x,y_l)]\,dx \tag{9.2}$$
$$= -\int_a^b P(x,y_l)\,dx - \int_b^a P(x,y_u)\,dx.$$

Now we have our answer—we just have to recognize it! Think how you would evaluate the line integral of $P(x,y)dx$ along the lower part of C in Figure 9.1 from point 1 to point 2. You would substitute $y = y_l(x)$ into $P(x,y)$ and integrate from $x = a$ to b (see Section 8).

$$\int_a^b P(x,y_l)\,dx = \text{line integral of } P\,dx \tag{9.3}$$
$$\text{along lower part of } C \text{ from point 1 to point 2.}$$

This is one of the terms in (9.2). Similarly, to find the line integral of $P(x,y)\,dx$ along the upper part of C from point 2 to point 1, we substitute $y = y_u(x)$ and integrate from b to a.

$$\int_b^a P(x,y_u)\,dx = \text{line integral of } P\,dx \tag{9.4}$$
$$\text{along upper part of } C \text{ from point 2 to point 1.}$$

Combining (9.3) and (9.4) gives us the line integral all the way around C in the counterclockwise direction, that is, so that A is always on our left as we go around C. (The symbol $\oint$ means an integral around a closed curve back to the starting point.) Then, from (9.2), we have

$$\oint_C P\,dx = -\iint_A \frac{\partial P(x,y)}{\partial y}dx\,dy. \tag{9.5}$$

Repeating the calculation but integrating first with respect to x, we find

$$\begin{aligned}\iint_A \frac{\partial Q}{\partial x}dx\,dy &= \int_c^d dy \int_{x_l}^{x_r} \frac{\partial Q}{\partial x}dx = \int_c^d [Q(x_r,y) - Q(x_l,y)]dy \\ &= \oint_C Q\,dy.\end{aligned} \tag{9.6}$$

Adding (9.5) and (9.6) and using the notation ∂A to mean the boundary of A (that is, C) we have

Green's theorem in the plane:

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_{\partial A} (P\,dx + Q\,dy) \tag{9.7}$$

The line integral is counterclockwise around the boundary of area A.

Using Green's theorem we can evaluate either a line integral around a closed path or a double integral over the area inclosed, whichever is easier to do. If the area is not of the simple type we have assumed, it may be possible to cut it into pieces (see Figure 9.2) so that our proof applies to each piece. Then the line integrals along the dotted cuts in Figure 9.2 are in opposite directions for adjacent pieces, and so cancel. Thus the theorem is valid for this more general area and its inclosing curve. In fact, we can even close up Figure 9.2 creating an area with a hole in the middle.

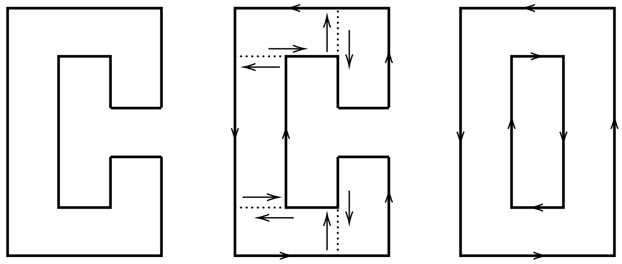


Figure 9.2

Green's theorem still holds, but now the line integral consists of a counterclockwise integral around the outside plus a clockwise integral around the hole as you can see in Figure 9.2. We say that this area is not "simply connected"—see further discussion of this in Section 11.

▶ **Example 1.** In Example 1, Section 8, we found the line integral (8.2) along several paths (Figure 8.2). Suppose we want the line integral in Figure 8.2 around the closed loop (Figure 9.3) from $(0,0)$ to $(2,1)$ and back as shown. From Section 8, Example 1, this is the work done along path 2 minus the work done along path 3 (since we are now going in the opposite direction); we find $W_2 - W_3 = \frac{2}{3} - \frac{5}{3} = -1$. Let us evaluate this using Green's theorem. From (8.2) and (9.7) we have

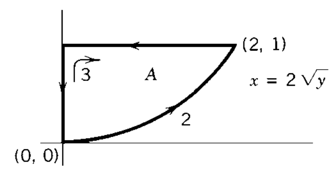


Figure 9.3

$$W = \oint_{\partial A} xy\,dx - y^2\,dy = \iint_A \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xy)\right] dx\,dy$$

$$= \iint_A -x\,dx\,dy = -\int_{y=0}^{1}\int_{x=0}^{2\sqrt{y}} x\,dx\,dy = -1$$

as before.

▶ **Example 2.** In Section 8, we discussed conservative forces for which work done is independent of the path. By Green's theorem (9.7), the work done by a force $\mathbf{F}$ around a closed path in the (x,y) plane is

$$W = \oint_{\partial A} (F_x\,dx + F_y\,dy) = \iint_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\,dy.$$

If $(\partial F_y/\partial x) - (\partial F_x/\partial y) = 0$ (note that this is the z component of $\operatorname{curl} \mathbf{F} = 0$) then W around any closed path is zero, which means that the work from one point to another is independent of the path (also see Section 11).

The functions $P(x,y)$ and $Q(x,y)$ in (9.7) are arbitrary; we may choose them to suit our purposes. Note that a two-dimensional vector function $\mathbf{i}V_x(x,y) + \mathbf{j}V_y(x,y)$

contains two functions, V_x and V_y. In the next two examples, we are going to define P and Q in terms of V_x and V_y in order to obtain two useful results.

▶ **Example 3.** We define:

$$Q = V_x, \quad P = -V_y, \quad \text{where} \quad \mathbf{V} = \mathbf{i}V_x + \mathbf{j}V_y. \tag{9.8}$$

Then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \operatorname{div} \mathbf{V} \tag{9.9}$$

by (7.2) with $V_z = 0$. Along the curve bounding an area A (Figure 9.4) the vector

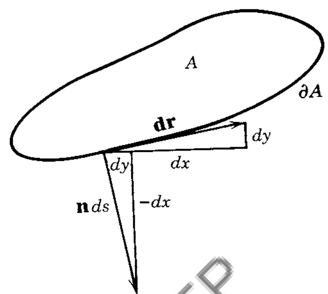


Figure 9.4

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy \quad \text{(tangent)} \tag{9.10}$$

is a tangent vector, and the vector

$$\mathbf{n}\,ds = \mathbf{i}\,dy - \mathbf{j}\,dx \quad \text{(outward normal)}, \tag{9.11}$$
where $\mathbf{n}$ is a unit vector and $ds = \sqrt{dx^2 + dy^2}$,

is a normal vector (perpendicular to the tangent) pointing out of area A. Using (9.11) and (9.8), we can write

$$\begin{aligned} P\,dx + Q\,dy &= -V_y\,dx + V_x\,dy = (\mathbf{i}V_x + \mathbf{j}V_y) \cdot (\mathbf{i}\,dy - \mathbf{j}\,x) \\ &= \mathbf{V} \cdot \mathbf{n}\,ds. \end{aligned} \tag{9.12}$$

Then substitute (9.9) and (9.12) into (9.7) to get

$$\iint_A \operatorname{div} \mathbf{V}\,dx\,dy = \int_{\partial A} \mathbf{V} \cdot \mathbf{n}\,ds. \tag{9.13}$$

This is the *divergence theorem* in two dimensions. It can be extended to three dimensions (also see Section 10). Let τ represent a volume; then $\partial\tau$ (read boundary of τ) means the closed surface area of τ. Let $d\tau$ mean a volume element and let $d\sigma$

mean an element of surface area. At each point of the surface, let $\mathbf{n}$ be a unit vector perpendicular to the surface and pointing outward. Then the divergence theorem in three dimensions says (also see Section 10)

$$\iiint_{\tau} \operatorname{div} \mathbf{V}\, d\tau = \iint_{\partial\tau} \mathbf{V}\cdot\mathbf{n}\, d\sigma. \quad \text{Divergence theorem} \tag{9.14}$$

Example 4. To see another application of (9.7) to vector functions, we let

$$Q = V_y, \quad P = V_x, \quad \text{where} \quad \mathbf{V} = \mathbf{i}V_x + \mathbf{j}V_y. \tag{9.15}$$

Then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = (\operatorname{curl} \mathbf{V})\cdot\mathbf{k} \tag{9.16}$$

by (7.3) with $V_z = 0$. Equations (9.10) and (9.15) give

$$P\,dx + Q\,dy = (\mathbf{i}V_x + \mathbf{j}V_y)\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy) = \mathbf{V}\cdot d\mathbf{r}. \tag{9.17}$$

Substituting (9.16) and (9.17) into (9.7), we get

$$\iint_A (\operatorname{curl} \mathbf{V})\cdot\mathbf{k}\, dx\, dy = \oint_{\partial A} \mathbf{V}\cdot d\mathbf{r}. \tag{9.18}$$

This is *Stokes' theorem* in two dimensions. It can be extended to three dimensions (Section 11). Let σ be an open surface (for example, a hemisphere); then $\partial\sigma$ means the curve bounding the surface (Figure 9.5). Let $\mathbf{n}$ be a unit vector normal to the surface. Then Stokes' theorem in three dimensions is (also see Section 11)

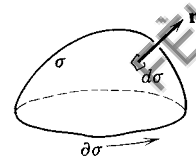


Figure 9.5

$$\iint_{\sigma} (\operatorname{curl} \mathbf{V})\cdot\mathbf{n}\, d\sigma = \int_{\partial\sigma} \mathbf{V}\cdot d\mathbf{r}. \quad \text{Stokes' theorem.} \tag{9.19}$$

The direction of integration for the line integral is as shown in Figure 9.5 (see also Section 11).

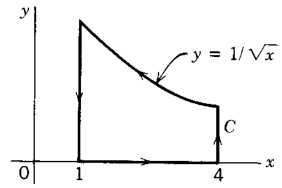
PROBLEMS, SECTION 9

1. Write out the equations corresponding to (9.3) and (9.4) for $\int Q\,dy$ between points 3 and 4 in Figure 9.2, and add them to get (9.6).

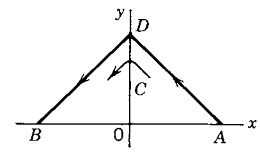
In Problems 2 to 5 use Green's theorem [formula (9.7)] to evaluate the given integrals.

2. $\oint 2x\,dy - 3y\,dx$ around the square with vertices $(0, 2)$, $(2, 0)$, $(-2, 0)$, and $(0, -2)$.

3. $\oint_C xy\,dx + x^2\,dy$, where C is as sketched.



4. $\int_C e^x \cos y\,dx - e^x \sin y\,dy$, where C is the broken line from $A = (\ln 2, 0)$ to $D = (0, 1)$ and then from D to $B = (-\ln 2, 0)$. *Hint:* Apply Green's theorem to the integral around the closed curve $ADBA$.



5. $\int_C (ye^x - 1)\,dx + e^x\,dy$, where C is the semicircle through $(0, -10)$, $(10, 0)$, and $(0, 10)$. (Compare Problem 4.)

6. For a simple closed curve C in the plane show by Green's theorem that the area inclosed is

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

7. Use Problem 6 to show that the area inside the ellipse $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$, is $A = \pi ab$.

8. Use Problem 6 to find the area inside the curve $x^{2/3} + y^{2/3} = 4$.

9. Apply Green's theorem with $P = 0$, $Q = \frac{1}{2}x^2$ to the triangle with vertices $(0, 0)$, $(0, 3)$, $(3, 0)$. You will then have $\iint x\,dx\,dy$ over the triangle expressed as a very simple line integral. Use this to locate the centroid of the triangle. (Compare Chapter 5, Section 3.)

Evaluate each of the following integrals in the easiest way you can.

10. $\oint (2y\,dx - 3x\,dy)$ around the square bounded by $x = 3$, $x = 5$, $y = 1$ and $y = 3$.

11. $\int_C (x\sin x - y)\,dx + (x - y^2)\,dy$, where C is the triangle in the (x, y) plane with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$.

12. $\int (y^2 - x^2)\,dx + (2xy + 3)\,dy$ along the x axis from $(0, 0)$ to $(\sqrt{5}, 0)$ and then along a circular arc from $(\sqrt{5}, 0)$ to $(1, 2)$.

▶ 10. THE DIVERGENCE AND THE DIVERGENCE THEOREM

We have defined (in Section 7) the *divergence* of a vector function $\mathbf{V}(x, y, z)$ as

$$\text{div}\,\mathbf{V} = \boldsymbol{\nabla}\cdot\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}. \tag{10.1}$$

We now want to investigate the meaning and use of the divergence in physical applications.

Consider a region in which water is flowing. We can imagine drawing at every point a vector $\mathbf{v}$ equal to the velocity of the water at that point. The vector function $\mathbf{v}$ then represents a vector field. The curves tangent to $\mathbf{v}$ are called stream lines. We could in the same way discuss the flow of a gas, of heat, of electricity, or of particles (say from a radioactive source). We are going to show that if $\mathbf{v}$ represents the velocity of flow of any of these things, then $\text{div}\,\mathbf{v}$ is related to the amount of the substance which flows out of a given volume. This could be different from zero either because of a change in density (more air flows out than in as a room is heated) or because there is a source or sink in the volume (alpha particles flow out of but not into a box containing an alpha-radioactive source). Exactly the same mathematics applies to the electric and magnetic fields where $\mathbf{v}$ is replaced by $\mathbf{E}$ or $\mathbf{B}$ and the quantity corresponding to outflow of a material substance is called flux.

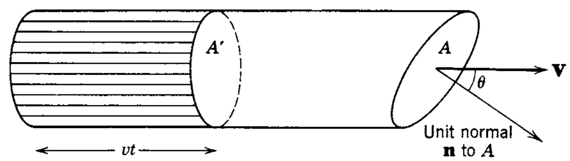


Figure 10.1

For our example of water flow, let $\mathbf{V} = \mathbf{v}\rho$, where ρ is the density of the water. Then the amount of water crossing in time t an area A' which is perpendicular to the direction of flow, is (see Figure 10.1) the amount of water in a cylinder of cross section A' and length vt. This amount of water is

$$(vt)(A')(\rho). \tag{10.2}$$

The same amount of water crosses area A (see Figure 10.1) whose normal is inclined at angle θ to $\mathbf{v}$. Since $A' = A\cos\theta$,

$$vtA'\rho = vt\rho A\cos\theta. \tag{10.3}$$

Then if water is flowing in the direction $\mathbf{v}$ making an angle θ with the normal $\mathbf{n}$ to a surface, the amount of water crossing *unit* area of the surface in *unit* time is

$$v\rho\cos\theta = V\cos\theta = \mathbf{V}\cdot\mathbf{n} \tag{10.4}$$

if $\mathbf{n}$ is a unit vector.

Now consider an element of volume $dx\,dy\,dz$ in the region through which the water is flowing (Figure 10.2). Water is flowing either in or out of the volume $dx\,dy\,dz$ through each of the six surfaces of the volume element; we shall calculate the net outward flow. In Figure 10.2, the rate at which water flows into $dx\,dy\,dz$

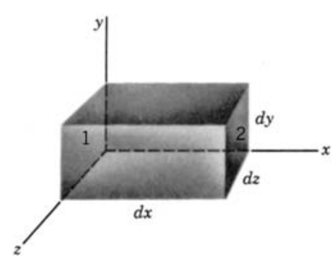


Figure 10.2

through surface 1 is [by (10.4)] $\mathbf{V}\cdot\mathbf{i}$ per unit area, or $(\mathbf{V}\cdot\mathbf{i})\,dy\,dz$ through the area $dy\,dz$ of surface 1. Since $\mathbf{V}\cdot\mathbf{i} = V_x$, we find that the rate at which water flows across surface 1 is $V_x\,dy\,dz$. A similar expression gives the rate at which water flows *out* through surface 2, except that V_x must be the x component of $\mathbf{V}$ at surface 2 instead of at surface 1. We want the difference of the two V_x values at two points, one on surface 1 and one on surface 2, directly opposite each other, that is, for the same y and z. These two values of V_x differ by ΔV_x which can be approximated (as in Chapter 4) by dV_x. For constant y and z, $dV_x = (\partial V_x/\partial x)\,dx$. Then the *net outflow* through these two surfaces is the outflow through surface 2 minus the inflow through surface 1, namely,

$$[(V_x \text{ at surface } 2) - (V_x \text{ at surface } 1)]dy\,dz = \left(\frac{\partial V_x}{\partial x}dx\right)dy\,dz. \tag{10.5}$$

We get similar expressions for the net outflow through the other two pairs of opposite surfaces:

$$\begin{aligned} &\frac{\partial V_y}{\partial y}\,dx\,dy\,dz \qquad \text{through top and bottom,} \quad \text{and} \\ &\frac{\partial V_z}{\partial z}\,dx\,dy\,dz \qquad \text{through the other two sides.} \end{aligned} \tag{10.6}$$

Then the total net rate of loss of water from $dx\,dy\,dz$ is

$$\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)dx\,dy\,dz = \operatorname{div}\mathbf{V}\,dx\,dy\,dz \quad \text{or} \quad \boldsymbol{\nabla}\cdot\mathbf{V}\,dx\,dy\,dz. \tag{10.7}$$

If we divide (10.7) by $dx\,dy\,dz$, we have the rate of loss of water per unit volume. This is the physical meaning of a divergence: It is the net rate of outflow *per unit volume* evaluated at a point (let $dx\,dy\,dz$ shrink to a point). This is outflow of actual substance for liquids, gases, or particles; it is called flux for electric and magnetic fields. You should note that this is somewhat like a density. Density is mass *per unit volume*, but it is evaluated *at a point* and may vary from point to point. Similarly, the divergence is evaluated at each point and may vary from point to point.

As we have said, div $\mathbf{V}$ may be different from zero either because of time variation of the density or because of sources and sinks. Let

ψ = *source density* minus *sink density*
= net mass of fluid being created (or added via something like a minute sprinkler system) per unit time per unit volume;
ρ = density of the fluid = mass per unit volume;
$\partial\rho/\partial t$ = time rate of increase of mass per unit volume.

Then:

Rate of increase of mass in $dx\,dy\,dz$ = rate of creation minus rate of outward flow,

or in symbols

$$\frac{\partial \rho}{\partial t}dx\,dy\,dz = \psi\,dx\,dy\,dz - \boldsymbol{\nabla}\cdot\mathbf{V}\,dx\,dy\,dz.$$

Canceling $dx\,dy\,dz$, we have

$$\frac{\partial\rho}{\partial t} = \psi - \boldsymbol{\nabla}\cdot\mathbf{V}$$

or

$$\boldsymbol{\nabla}\cdot\mathbf{V} = \psi - \frac{\partial\rho}{\partial t}. \tag{10.8}$$

If there are no sources or sinks, then $\psi = 0$; the resulting equation is often called the *equation of continuity.* (See Problem 15.)

$$\nabla \cdot \mathbf{V} + \frac{\partial \rho}{\partial t} = 0. \qquad \text{Equation of continuity} \tag{10.9}$$

If $\partial \rho / \partial t = 0$, then

$$\nabla \cdot \mathbf{V} = \psi. \tag{10.10}$$

In the case of the electric field, the "sources" and "sinks" are electric charges and the equation corresponding to (10.10) is $\operatorname{div} \mathbf{D} = \psi$, where ψ is the charge density and $\mathbf{D}$ is the electric displacement. For the magnetic field $\mathbf{B}$ you would expect the sources to be magnetic poles; however, there are no free magnetic poles, so $\operatorname{div} \mathbf{B} = 0$ always.

We have shown that the mass of fluid crossing a plane area A per unit time is $A\mathbf{V} \cdot \mathbf{n}$, where $\mathbf{n}$ is a unit vector normal to A, $\mathbf{v}$ and ρ are the velocity and density of the fluid, and $\mathbf{V} = \mathbf{v}\rho$. Consider any closed surface, and let $d\sigma$ represent an area element on the surface (Figure 10.3). For example: for a plane, $d\sigma = dx\,dy$; for a spherical surface,

$$d\sigma = r^2 \sin\theta \, d\theta \, d\phi.$$

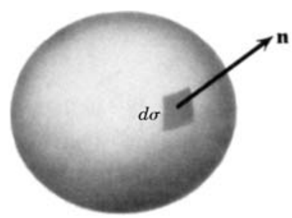


Figure 10.3

Let $\mathbf{n}$ be the unit vector normal to $d\sigma$ and pointing *out* of the surface ($\mathbf{n}$ varies in direction from point to point on the surface). Then the mass of fluid flowing out through $d\sigma$ is $\mathbf{V} \cdot \mathbf{n}\, d\sigma$ by (10.4) and the total outflow from the volume inclosed by the surface is

$$\iint \mathbf{V} \cdot \mathbf{n}\, d\sigma, \tag{10.11}$$

where the double integral is evaluated over the closed surface.

We showed previously [see (10.7)] that for the volume element $d\tau = dx\,dy\,dz$:

$$\text{The outflow from } d\tau \text{ is } \quad \nabla \cdot \mathbf{V}\, d\tau. \tag{10.12}$$

For simplicity, we proved this for a rectangular coordinate volume element $dx\,dy\,dz$. With extra effort we could prove it more generally, say for volume elements with slanted sides or for spherical coordinate volume elements. From now on we shall assume that $d\tau$ includes more general volume element shapes.

It is worth noticing here another way [besides (7.2)] of defining the divergence. If we write (10.11) for the surface of a volume element $d\tau$, we have two expressions for the total outflow from $d\tau$, and these must be equal. Thus

$$\nabla \cdot \mathbf{V}\, d\tau = \iint_{\substack{\text{surface} \\ \text{of } d\tau}} \mathbf{V} \cdot \mathbf{n}\, d\sigma. \tag{10.13}$$

The value of $\nabla \cdot \mathbf{V}$ on the left is, of course, an average value of $\nabla \cdot \mathbf{V}$ in $d\tau$, but if we divide (10.13) by $d\tau$ and let $d\tau$ shrink to a point, we have a definition of $\nabla \cdot \mathbf{V}$ at the point:

$$\nabla \cdot \mathbf{V} = \lim_{d\tau \to 0} \frac{1}{d\tau} \iint\limits_{\substack{\text{surface}\\ \text{of } d\tau}} \mathbf{V} \cdot \mathbf{n}\, d\sigma. \tag{10.14}$$

If we start with (10.14) as the definition of $\nabla \cdot \mathbf{V}$, then the discussion leading to (10.7) is a proof that $\nabla \cdot \mathbf{V}$ as defined in (10.14) is equal to $\nabla \cdot \mathbf{V}$ as defined in (7.2).

The Divergence Theorem See (10.17). The divergence theorem is also called Gauss's theorem, but be careful to distinguish this mathematical theorem from Gauss's law which is a law of physics; see (10.23).

Consider a large volume τ; imagine it cut up into volume elements $d\tau_i$ (a cross section of this is shown in Figure 10.4). The outflow from each $d\tau_i$ is $\nabla \cdot \mathbf{V}\, d\tau_i$; let us add together the outflow from all the $d\tau_i$ to get

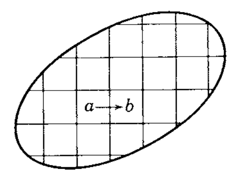


Figure 10.4

$$\sum_i \nabla \cdot \mathbf{V}\, d\tau_i. \tag{10.15}$$

We shall show that (10.15) is the outflow from the large volume τ. Consider the flow between the elements marked a and b in Figure 10.4 across their common face. An outflow from a to b is an inflow (negative outflow) from b to a, so that in the sum (10.15) such outflows across interior faces cancel. The total sum in (10.15) then equals just the total outflow from the large volume. As the size of the volume elements tends to zero, this sum approaches a triple integral over the volume,

$$\iiint \nabla \cdot \mathbf{V}\, d\tau. \tag{10.16}$$

We have shown that both (10.11) and (10.16) are equal to the total outflow from the large volume; hence they are equal to each other, and we have the divergence theorem as stated in (9.14):

$$\iiint\limits_{\text{volume } \tau} \nabla \cdot \mathbf{V}\, d\tau = \iint\limits_{\substack{\text{surface}\\ \text{inclosing } \tau}} \mathbf{V} \cdot \mathbf{n}\, d\sigma. \qquad \text{Divergence theorem} \tag{10.17}$$

($\mathbf{n}$ points out of the closed surface σ.)

Notice that the divergence theorem converts a volume integral into an integral over a closed surface or vice versa; we can then evaluate whichever one is the easier to do.

In (10.17) we have carefully written the volume integral with three integral signs and the surface integral with two integral signs. However, it is rather common to write only one integral sign for either case when the volume or area element is indicated by a single differential ($d\tau$, dV, etc., for volume; $d\sigma$, dA, dS, etc., for surface area). Thus we might write $\iiint d\tau$ or $\int d\tau$ or $\iiint dx\,dy\,dz$, all meaning the same thing. When the single integral sign is used to indicate a surface or volume integral, you must see from the notation (τ for volume, σ for area), or the words under the integral, what is really meant. To indicate a surface integral over a *closed* surface or a line integral around a *closed* curve, the symbol $\oint$ is often used. Thus we might write either $\iint d\sigma$ or $\oint d\sigma$ for a surface integral over a closed surface. A different notation for the integrand $\mathbf{V}\cdot\mathbf{n}\,d\sigma$ is often used. Instead of using a unit vector $\mathbf{n}$ and the scalar magnitude $d\sigma$, we may write the vector $d\boldsymbol{\sigma}$ meaning a vector of magnitude $d\sigma$ in the direction $\mathbf{n}$; thus $d\boldsymbol{\sigma}$ means exactly the same thing as $\mathbf{n}\,d\sigma$, and we may replace $\mathbf{V}\cdot\mathbf{n}\,d\sigma$ by $\mathbf{V}\cdot d\boldsymbol{\sigma}$ in (10.17).

Example of the Divergence Theorem Let $\mathbf{V} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ and evaluate $\oint \mathbf{V}\cdot\mathbf{n}\,d\sigma$ over the closed surface of the cylinder shown in Figure 10.5.

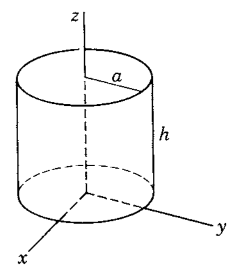


Figure 10.5

By the divergence theorem this is equal to $\int \boldsymbol{\nabla}\cdot\mathbf{V}\,d\tau$ over the volume of the cylinder. (Note that we are using single integral signs, but the notation and words make it clear which integral is a volume integral and which a surface integral.) We find from the definition of divergence

$$\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

Then by (10.17)

$$\oint\limits_{\substack{\text{surface of}\\ \text{cylinder}}} \mathbf{V}\cdot\mathbf{n}\,d\sigma = \int\limits_{\substack{\text{volume of}\\ \text{cylinder}}} \boldsymbol{\nabla}\cdot\mathbf{V}\,d\tau = \int 3\,d\tau = 3\int d\tau$$
$$= 3 \text{ times volume of cylinder} = 3\pi a^2 h.$$

It is harder to evaluate $\oint \mathbf{V}\cdot\mathbf{n}\,d\sigma$ directly, but we might do it to show an example of calculating a surface integral and to verify the divergence theorem in a special case. We need the surface normal $\mathbf{n}$. On the top surface (Figure 10.5) $\mathbf{n} = \mathbf{k}$, and there $\mathbf{V}\cdot\mathbf{n} = \mathbf{V}\cdot\mathbf{k} = z = h$. Then

$$\int\limits_{\substack{\text{top surface of}\\ \text{cylinder}}} \mathbf{V}\cdot\mathbf{n}\,d\sigma = h\int d\sigma = h\cdot\pi a^2.$$

On the bottom surface, $\mathbf{n} = -\mathbf{k}$, $\mathbf{V}\cdot\mathbf{n} = -z = 0$; hence the integral over the bottom surface is zero. On the curved surface we might see by inspection that the vector $\mathbf{i}x + \mathbf{j}y$ is normal to the surface, so for the curved surface we have

$$\mathbf{n} = \frac{\mathbf{i}x + \mathbf{j}y}{\sqrt{x^2+y^2}} = \frac{\mathbf{i}x + \mathbf{j}y}{a}.$$

If the vector $\mathbf{n}$ is not obvious by inspection, we can easily find it; recall (Section 6) that if the equation of a surface is $\phi(x,y,z) = \text{const.}$, then $\boldsymbol{\nabla}\phi$ is perpendicular to the surface. In this problem, the equation of the cylinder is $x^2 + y^2 = a^2$; then

$\phi = x^2 + y^2$, $\nabla\phi = 2x\mathbf{i} + 2y\mathbf{j}$, and we get the same unit vector $\mathbf{n}$ as above. Then for the curved surface we find

$$\mathbf{V}\cdot\mathbf{n} = \frac{x^2+y^2}{a} = \frac{a^2}{a} = a,$$

$$\int_{\substack{\text{curved}\\ \text{surface}}} \mathbf{V}\cdot\mathbf{n}\,d\sigma = a\int d\sigma = a\cdot(\text{area of curved surface}) = a\cdot 2\pi a h.$$

The value of $\oint \mathbf{V}\cdot\mathbf{n}\,d\sigma$ over the whole surface of the cylinder is then $\pi a^2 h + 2\pi a^2 h = 3\pi a^2 h$ as before.

Gauss's Law The divergence theorem is very important in electricity. In order to see how it is used, we need a law in electricity known as Gauss's law. Let us derive this law from the more familiar Coulomb's law (8.11). Coulomb's law (written this time in SI units) gives for the electric field at $\mathbf{r}$ due to a point charge q at the origin

(10.18) $$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\mathbf{e}_r. \qquad \text{Coulomb's law}$$

(ϵ_0 is a constant called the *permittivity of free space* and $\dfrac{1}{4\pi\epsilon_0} = 9\cdot 10^9$ in SI units.) The electric displacement $\mathbf{D}$ is defined (in free space) by $\mathbf{D} = \epsilon_0\mathbf{E}$; then

(10.19) $$\mathbf{D} = \frac{q}{4\pi r^2}\mathbf{e}_r.$$

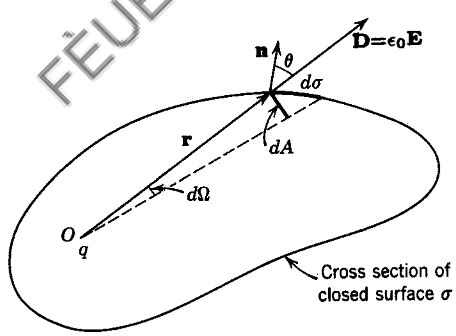


Figure 10.6

Let σ be a closed surface surrounding the point charge q at the origin; let $d\sigma$ be an element of area of the surface at the point $\mathbf{r}$, and let $\mathbf{n}$ be a unit normal to $d\sigma$ (Figures 10.3 and 10.6). Also (Figure 10.6) let dA be the projection of $d\sigma$ onto a sphere of radius r and center at O and let $d\Omega$ be the solid angle subtended by $d\sigma$ (and dA) at O. Then by definition of solid angle

(10.20) $$d\boldsymbol{\Omega} = \frac{1}{r^2}dA.$$

From Figure 10.6 and equations (10.19) and (10.20), we get

(10.21) $$\mathbf{D}\cdot\mathbf{n}\,d\sigma = D\cos\theta\,d\sigma = D\,dA = \frac{q}{4\pi r^2}\cdot r^2\,d\Omega = \frac{1}{4\pi}q\,d\Omega$$

We want to find the surface integral of $\mathbf{D} \cdot \mathbf{n}\, d\sigma$ over the closed surface σ; by (10.21) this is

$$\oint_{\substack{\text{closed}\\ \text{surface } \sigma}} \mathbf{D} \cdot \mathbf{n}\, d\sigma = \frac{q}{4\pi} \int_{\substack{\text{total}\\ \text{solid angle}}} d\Omega = \frac{q}{4\pi} \cdot 4\pi = q \qquad (q \text{ inside } \sigma). \tag{10.22}$$

This is a simple case of Gauss's law when we have only one point charge q; for most purposes we shall want Gauss's law in the forms (10.23) or (10.24) below. Before we derive these, we should note carefully that in (10.22) the charge q is *inside* the closed surface σ. If we repeat the derivation of (10.22) for a point charge q outside the surface (Problem 13), we find that in this case

$$\oint_{\text{closed } \sigma} \mathbf{D} \cdot \mathbf{n}\, d\sigma = 0.$$

Next suppose there are several charges q_i inside the closed surface. For each q_i and the D_i corresponding to it, we could write an equation like (10.22). But the total electric displacement vector $\mathbf{D}$ at a point due to all the q_i is the vector sum of the vectors $\mathbf{D}_i$. Thus we have

$$\oint_{\substack{\text{closed}\\ \text{surface } \sigma}} \mathbf{D} \cdot \mathbf{n}\, d\sigma = \sum_i \oint_{\substack{\text{closed}\\ \text{surface } \sigma}} \mathbf{D}_i \cdot \mathbf{n}\, d\sigma = \sum q_i.$$

Therefore for any charge distribution inside a closed surface

$$\oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D} \cdot \mathbf{n}\, d\sigma = \text{total charge inside the closed surface.} \qquad \text{Gauss's law} \tag{10.23}$$

If, instead of isolated charges, we have a charge distribution with charge density ρ (which may vary from point to point), then the total charge is $\int \rho\, d\tau$, so

$$\oint_{\substack{\text{closed}\\ \text{surface } \sigma}} \mathbf{D} \cdot \mathbf{n}\, d\sigma = \int_{\substack{\text{volume}\\ \text{bounded by } \sigma}} \rho\, d\tau. \qquad \text{Gauss's law} \tag{10.24}$$

Since (by Problem 13) charges outside the closed surface σ do not contribute to the integral, (10.23) and (10.24) are correct if $\mathbf{D}$ is the total electric displacement due to all charges inside and outside the surface. The total charge on the right-hand side of these equations is, however, just the charge inside the surface σ. Either (10.23) or (10.24) is called Gauss's law.

We now want to see the use of the divergence theorem in connection with Gauss's law. By the divergence theorem, the surface integral on the left-hand side of (10.23) or (10.24) is equal to

$$\int_{\substack{\text{volume}\\ \text{bounded by } \sigma}} \boldsymbol{\nabla} \cdot \mathbf{D}\, d\tau.$$

Then (10.24) can be written as

$$\int \nabla \cdot \mathbf{D}\, d\tau = \int \rho\, d\tau.$$

Since this is true for *every* volume, we must have $\nabla \cdot \mathbf{D} = \rho$; this is one of the Maxwell equations in electricity. What we have done is to start by assuming Coulomb's law; we have derived Gauss's law from it, and then by use of the divergence theorem, we have derived the Maxwell equation $\nabla \cdot \mathbf{D} = \rho$. From a more sophisticated viewpoint, we might take the Maxwell equation as one of our basic assumptions in electricity. We could then use the divergence theorem to obtain Gauss's law:

$$\text{(10.25)} \qquad \oint_{\substack{\text{closed} \\ \text{surface } \sigma}} \mathbf{D} \cdot \mathbf{n}\, d\sigma = \int_{\substack{\text{volume } \tau \\ \text{inside } \sigma}} \nabla \cdot \mathbf{D}\, d\tau = \int_{\text{volume } \tau} \rho\, d\tau$$
$$= \text{total charge inclosed by } \sigma.$$

From Gauss's law we could then derive Coulomb's law (Problem 14); more generally we can often use Gauss's law to obtain the electric field produced by a given charge distribution as in the following example.

▶ **Example.** Find $\mathbf{E}$ just above a very large conducting plate carrying a surface charge of C coulombs per square meter on each surface.

The electric field inside a conductor is zero when we are considering an electrostatics problem (otherwise current would flow). From the symmetry of the problem (all horizontal directions are equivalent), we can say that $\mathbf{E}$ (and $\mathbf{D}$) must be vertical as shown in Figure 10.7. We now find $\oint \mathbf{D} \cdot \mathbf{n}\, d\sigma$ over the box whose cross section is shown by the dotted lines. The integral over the bottom surface is zero since $\mathbf{D} = 0$ inside the conductor. The integral over the vertical sides is zero because $\mathbf{D}$ is perpendicular to $\mathbf{n}$ there. On the top surface $\mathbf{D} \cdot \mathbf{n} = |\mathbf{D}|$ and $\int \mathbf{D} \cdot \mathbf{n}\, d\sigma = |\mathbf{D}| \cdot$ (surface area). By (10.25) this is equal to the charge inclosed by the box, which is $C \cdot$ (surface area). Thus we have $|\mathbf{D}| \cdot$ (surface area) $= C \cdot$ (surface area) , or $|\mathbf{D}| = C$ and $|\mathbf{E}| = C/\epsilon_0$.

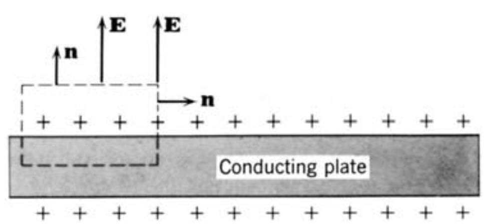


Figure 10.7

▶ PROBLEMS, SECTION 10

1. Evaluate both sides of (10.17) if $\mathbf{V} = \mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, and τ is the volume $x^2 + y^2 + z^2 \le 1$, and so verify the divergence theorem in this case.

2. Given $\mathbf{V} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, integrate $\mathbf{V} \cdot \mathbf{n}\, d\sigma$ over the whole surface of the cube of side 1 with four of its vertices at $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$. Evaluate the same integral by means of the divergence theorem.

Evaluate each of the integrals in Problems 3 to 8 as either a volume integral or a surface integral, whichever is easier.

3. $\iint \mathbf{r}\cdot\mathbf{n}\,d\sigma$ over the whole surface of the cylinder bounded by $x^2+y^2=1$, $z=0$, and $z=3$; $\mathbf{r}$ means $\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$.

4. $\iint \mathbf{V}\cdot\mathbf{n}\,d\sigma$ if $\mathbf{V}=x\cos^2 y\,\mathbf{i}+xz\,\mathbf{j}+z\sin^2 y\,\mathbf{k}$ over the surface of a sphere with center at the origin and radius 3.

5. $\iiint(\nabla\cdot\mathbf{F})\,d\tau$ over the region $x^2+y^2+z^2\le 25$, where

$$\mathbf{F}=(x^2+y^2+z^2)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

6. $\iiint \nabla\cdot\mathbf{V}\,d\tau$ over the unit cube in the first octant, where

$$\mathbf{V}=(x^3-x^2)y\mathbf{i}+(y^3-2y^2+y)x\mathbf{j}+(z^2-1)\mathbf{k}.$$

7. $\iint \mathbf{r}\cdot\mathbf{n}\,d\sigma$ over the entire surface of the cone with base $x^2+y^2\le 16$, $z=0$, and vertex at $(0,0,3)$, where $\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$.

8. $\iiint \nabla\cdot\mathbf{V}\,d\tau$ over the volume $x^2+y^2\le 4$, $0\le z\le 5$, $\mathbf{V}=(\sqrt{x^2+y^2}\,)(\mathbf{i}x+\mathbf{j}y)$.

9. If $\mathbf{F}=x\mathbf{i}+y\mathbf{j}$, calculate $\iint \mathbf{F}\cdot\mathbf{n}\,d\sigma$ over the part of the surface $z=4-x^2-y^2$ that is above the (x,y) plane, by applying the divergence theorem to the volume bounded by the surface and the piece that it cuts out of the (x,y) plane. *Hint:* What is $\mathbf{F}\cdot\mathbf{n}$ *on* the (x,y) plane?

10. Evaluate $\iint \mathbf{V}\cdot\mathbf{n}\,d\sigma$ over the *curved* surface of the hemisphere $x^2+y^2+z^2=9$, $z\ge 0$, if $\mathbf{V}=y\mathbf{i}+xz\mathbf{j}+(2z-1)\mathbf{k}$. *Careful:* See Problem 9.

11. Given that $\mathbf{B}=\operatorname{curl}\mathbf{A}$, use the divergence theorem to show that $\oint \mathbf{B}\cdot\mathbf{n}\,d\sigma$ over any closed surface is zero.

12. A cylindrical capacitor consists of two long concentric metal cylinders. If there is a charge of k coulombs per meter on the inside cylinder of radius R_1, and $-k$ coulombs per meter on the outside cylinder of radius R_2, find the electric field $\mathbf{E}$ between the cylinders. *Hint:* Use Gauss's law and the method indicated in Figure 10.7. What is $\mathbf{E}$ inside the inner cylinder? Outside the outer cylinder? (Again use Gauss's law.) Find, either by inspection or by direct integration, the potential ϕ such that $\mathbf{E}=-\nabla\phi$ for each of the three regions above. In each case $\mathbf{E}$ is not affected by adding an arbitrary constant to ϕ. Adjust the additive constant to make ϕ a continuous function for all space.

13. Draw a figure similar to Figure 10.6 but with q outside the surface. A vector (like $\mathbf{r}$ in the figure) from q to the surface now intersects it twice, and for each solid angle $d\Omega$ there are two $d\sigma$'s, one where $\mathbf{r}$ enters and one where it leaves the surface. Show that $\mathbf{D}\cdot\mathbf{n}\,d\sigma$ is given by (10.21) for the $d\sigma$ where $\mathbf{r}$ leaves the surface and the negative of (10.21) for the $d\sigma$ where $\mathbf{r}$ enters the surface. Hence show that the total $\oint \mathbf{D}\cdot\mathbf{n}\,d\sigma$ over the closed surface is zero.

14. Obtain Coulomb's law from Gauss's law by considering a spherical surface σ with center at q.

15. Suppose the density ρ of a fluid varies from point to point as well as with time, that is, $\rho=\rho(x,y,z,t)$. If we follow the fluid along a streamline, then x, y, z are functions of t such that the fluid velocity is

$$\mathbf{v}=\mathbf{i}\frac{dx}{dt}+\mathbf{j}\frac{dy}{dt}+\mathbf{k}\frac{dz}{dt}.$$

Show that then $d\rho/dt = \partial\rho/\partial t + \mathbf{v}\cdot\nabla\rho$. Combine this equation with (10.9) to get

$$\rho\nabla\cdot\mathbf{v} + \frac{d\rho}{dt} = 0.$$

(Physically, $d\rho/dt$ is the rate of change of density with time as we follow the fluid along a streamline; $\partial\rho/\partial t$ is the corresponding rate at a fixed point.) For a steady state (that is, time-independent), $\partial\rho/\partial t = 0$, but $d\rho/dt$ is not necessarily zero. For an incompressible fluid, $d\rho/dt = 0$; show that then $\nabla\cdot\mathbf{v} = 0$. (Note that incompressible does not necessarily mean constant density since $d\rho/dt = 0$ does not imply either time or space independence of ρ; consider, for example, a flow of water mixed with blobs of oil.)

16. The following equations are variously known as Green's first and second identities or formulas or theorems. Derive them, as indicated, from the divergence theorem.

(1) $$\int_{\substack{\text{volume } \tau \\ \text{inside } \sigma}} (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,d\tau = \oint_{\substack{\text{closed} \\ \text{surface } \sigma}} (\phi\nabla\psi)\cdot\mathbf{n}\,d\sigma.$$

To prove this, let $\mathbf{V} = \phi\nabla\psi$ in the divergence theorem.

(2) $$\int_{\substack{\text{volume } \tau \\ \text{inside } \sigma}} (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d\tau = \oint_{\substack{\text{closed} \\ \text{surface } \sigma}} (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\,d\sigma.$$

To prove this, copy Theorem 1 above as is and also with ϕ and ψ interchanged; then subtract the two equations.

▶ 11. THE CURL AND STOKES' THEOREM

We have already defined curl $\mathbf{V} = \nabla\times\mathbf{V}$ [see (7.3)] and have considered one application of the curl, namely, to determine whether or not a line integral between two points is independent of the path of integration (Section 8). Here is another application of the curl. Suppose a rigid body is rotating with constant angular velocity $\boldsymbol{\omega}$; this means that $|\boldsymbol{\omega}|$ is the magnitude of the angular velocity and $\boldsymbol{\omega}$ is a vector along the axis of rotation (see Figure 2.6). Then we showed in Section 2 that the velocity $\mathbf{v}$ of a particle in the rigid body is $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, where $\mathbf{r}$ is a radius vector from a point on the rotation axis to the particle. Let us calculate $\nabla\times\mathbf{v} = \nabla\times(\boldsymbol{\omega}\times\mathbf{r})$; we can evaluate this by the method described in Section 7. We use the formula for the triple vector product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, being careful to remember that ∇ is not an ordinary vector—it has both vector and differential-operator properties, and so must be written before variables that it differentiates. Then

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{r}) = (\nabla\cdot\mathbf{r})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{r}. \tag{11.1}$$

Since $\boldsymbol{\omega}$ is constant, the first term of (11.1) means

$$\boldsymbol{\omega}(\nabla\cdot\mathbf{r}) = \boldsymbol{\omega}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3\boldsymbol{\omega}. \tag{11.2}$$

In the second term of (11.1) we intentionally wrote $\boldsymbol{\omega}\cdot\nabla$ instead of $\nabla\cdot\boldsymbol{\omega}$ since $\boldsymbol{\omega}$ is constant, and ∇ operates only on $\mathbf{r}$; this term means

$$\left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z = \boldsymbol{\omega}$$

since $\partial y/\partial x = \partial z/\partial x = 0$, etc. Then

$$\text{(11.3)} \qquad \boldsymbol{\nabla} \times \mathbf{v} = \boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \mathbf{r}) = 2\boldsymbol{\omega} \qquad \text{or} \qquad \boldsymbol{\omega} = \frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{v}).$$

This result gives a clue as to the name curl $\mathbf{v}$ (or rotation $\mathbf{v}$ or rot $\mathbf{v}$ as it is sometimes called). For this simple case curl $\mathbf{v}$ gave the angular velocity of rotation. In a more complicated case such as flow of fluid, the value of curl $\mathbf{v}$ at a point is a measure of the angular velocity of the fluid in the neighborhood of the point. When $\boldsymbol{\nabla} \times \mathbf{v} = 0$ everywhere in some region, the velocity field $\mathbf{v}$ is called *irrotational* in that region. Notice that this is the same mathematical condition as for a force $\mathbf{F}$ to be *conservative.*

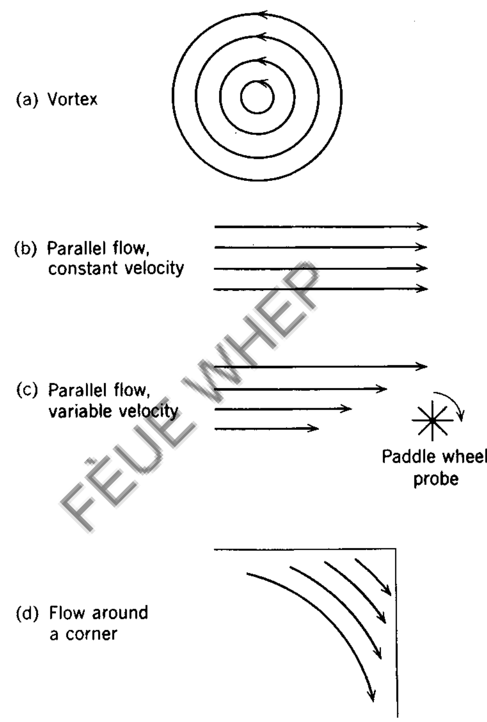


Figure 11.1

Consider a vector field $\mathbf{V}$ (for example, $\mathbf{V} = \mathbf{v}\rho$ for flow of water, or $\mathbf{V} =$ force $\mathbf{F}$). We define the *circulation* as the line integral $\oint \mathbf{V} \cdot d\mathbf{r}$ around a closed plane curve. If $\mathbf{V}$ is a force $\mathbf{F}$, then this integral is equal to the work done by the force. For flow of water, we can get a physical picture of the meaning of the circulation in the following way. Think of placing a tiny paddle-wheel probe (Figure 11.1c) in any of the flow patterns pictured in Figure 11.1. If the velocity of the fluid is greater on one side of the wheel than on the other, for example, as in (c), then the wheel will turn. Suppose we calculate the circulation $\oint \mathbf{V} \cdot d\mathbf{r}$ around the axis of the paddle wheel along a closed curve in a plane perpendicular to the axis (plane of the paper in Figure 11.1). If $\mathbf{V} = \mathbf{v}\rho$ is larger on one side of the wheel than the other, then the circulation is different from zero, but if [as in (b)] $\mathbf{V}$ is the same on both sides, then the circulation is zero. We shall show that the component of curl $\mathbf{V}$ along the axis of the paddle wheel equals

$$\lim_{d\sigma\to 0}\frac{1}{d\sigma}\oint \mathbf{V}\cdot d\mathbf{r} \tag{11.4}$$

where $d\sigma$ is the area inclosed by the curve along which we calculate the circulation. The paddle wheel then acts as a "curl meter" to measure $\operatorname{curl}\mathbf{V}$; if it does not rotate, $\operatorname{curl}\mathbf{V}=0$; if it does, then $\operatorname{curl}\mathbf{V}\neq 0$. In (a), $\operatorname{curl}\mathbf{V}\neq 0$ at the center of the vortex. In (b), $\operatorname{curl}\mathbf{V}=0$. In (c), $\operatorname{curl}\mathbf{V}\neq 0$ in spite of the fact that the flow lines are parallel. In (d), it is possible to have $\operatorname{curl}\mathbf{V}=0$ even though the stream lines go around a corner; in fact, for the flow of water around a corner, $\operatorname{curl}\mathbf{V}=0$. What you should realize is that the value of $\operatorname{curl}\mathbf{V}$ at a point depends upon the circulation in the neighborhood of the point and not on the overall flow pattern.

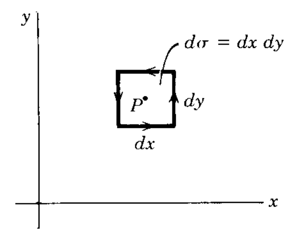


Figure 11.2

We want to show the relation between the circulation $\oint\mathbf{V}\cdot d\mathbf{r}$ and $\operatorname{curl}\mathbf{V}$ for a given vector field $\mathbf{V}$. Given a point P and a direction $\mathbf{n}$, let us find the component of $\operatorname{curl}\mathbf{V}$ in the direction $\mathbf{n}$ at P. Draw a plane through P perpendicular to $\mathbf{n}$ and choose axes so that it is the (x,y) plane with $\mathbf{n}$ parallel to $\mathbf{k}$. Find the circulation around an element of area $d\sigma$ centered on P. (See Figures 9.5 and 11.2.) By (9.18) with area A replaced by the *element of area* $d\sigma$, and with $\mathbf{n}=\mathbf{k}$

$$\oint_{\text{around } d\sigma}\mathbf{V}\cdot d\mathbf{r}=\iint_{d\sigma}(\operatorname{curl}\mathbf{V})\cdot\mathbf{k}\,dx\,dy=\iint_{d\sigma}(\operatorname{curl}\mathbf{V})\cdot\mathbf{n}\,d\sigma \tag{11.5}$$

Note that, since we proved (9.7) and so (9.18) for non-rectangular areas A (see Section 9), $d\sigma$ here may be more general than $dx\,dy$, say with curved or slanted sides.

We assume that the components of $\mathbf{V}$ have continuous first derivatives; then $\operatorname{curl}\mathbf{V}$ is continuous. Thus the value of $(\operatorname{curl}\mathbf{V})\cdot\mathbf{n}$ over $d\sigma$ is nearly the same as $(\operatorname{curl}\mathbf{V})\cdot\mathbf{n}$ at P, so the double integral in (11.5) is approximately the value of $(\operatorname{curl}\mathbf{V})\cdot\mathbf{n}$ at P multiplied by $d\sigma$. If we divide (11.5) by $d\sigma$ and take the limit as $d\sigma\to 0$, we have an exact equation

$$(\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n}=\lim_{d\sigma\to 0}\frac{1}{d\sigma}\oint_{\text{around } d\sigma}\mathbf{V}\cdot d\mathbf{r}. \tag{11.6}$$

This equation can be used as a definition of $\operatorname{curl}\mathbf{V}$; then the discussion above shows that [see equation (9.16)] the components of $\operatorname{curl}\mathbf{V}$ are those given in our previous definition (7.3).

In evaluating the line integral we must go around the area element $d\sigma$ as in Figure 11.2 keeping the area to our left. Another way of saying this is that we go around $d\sigma$ in the direction indicated by $\mathbf{n}$ and the right-hand rule; that is, if the thumb of your right hand points in the direction $\mathbf{n}$, your fingers curve in the direction you must go around the boundary of $d\sigma$ in evaluating the line integral. (See Figure 11.2 with $\mathbf{n}=\mathbf{k}$.)

Stokes' Theorem This theorem relates an integral over an open surface to the line integral around the curve bounding the surface (Figure 11.3). A butterfly net is a good example of what we are talking about; the net is the surface and the supporting rim is the curve bounding the surface. The surfaces we consider here (and which arise in applications) will be surfaces which could be obtained by deforming a hemisphere (or the butterfly net of Figure 11.3). In particular, the surfaces we consider must be *two*-sided. You can easily construct a *one*-sided surface by taking a long strip of paper, giving it a half twist, and joining the ends (Figure 11.4). A belt of this shape is sometimes used for driving machinery. This surface is called a Moebius strip, and you can verify that it has only one side by tracing your finger around it or imagining trying to paint one side. Stokes' theorem does not apply to such surfaces because we cannot define the sense of the normal vector $\mathbf{n}$ to such a surface. We require the bounding curve to be simple (that is, it must not cross itself) and closed.

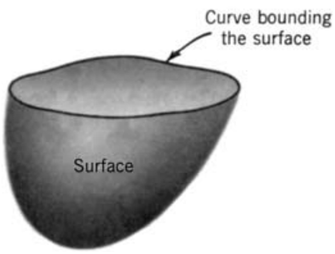


Figure 11.3

Figure 11.4

Consider the kind of surface we have described and imagine it divided into area elements $d\sigma$ by a network of curves as in Figure 11.5. Draw a unit vector $\mathbf{n}$ perpendicular to each area element; $\mathbf{n}$, of course, varies from element to element, but all $\mathbf{n}$'s must be on the same side of the two-sided surface. Each area element

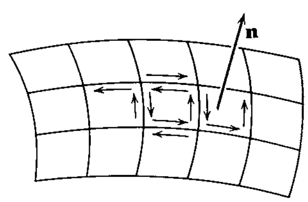


Figure 11.5

is approximately an element of the tangent plane to the surface at a point in $d\sigma$. Then, as in (11.5), we have

$$\oint_{\text{around } d\sigma} \mathbf{V}\cdot d\mathbf{r} = \iint_{d\sigma} (\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n}\,d\sigma \tag{11.7}$$

for each element. Recall from Section 9 and the comment just after equation (11.5), that $d\sigma$ includes area elements such as those along the edges in Figure 11.5. Then if we sum the equations in (11.7) for all the area elements of the whole surface area, we get

$$\sum_{\text{all } d\sigma} \oint \mathbf{V}\cdot d\mathbf{r} = \iint_{\text{surface } \sigma} (\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n}\,d\sigma. \tag{11.8}$$

From Figure 11.5 we see that all the interior line integrals cancel because along a border between two $d\sigma$'s the two integrals are in opposite directions. Then the left side of (11.8) becomes simply the line integral around the outside curve bounding the surface. Thus we have Stokes' theorem as stated in (9.19):

$$\oint_{\substack{\text{curve}\\ \text{bounding } \sigma}} \mathbf{V}\cdot d\mathbf{r} = \iint_{\text{surface } \sigma} (\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n}\,d\sigma. \qquad \text{Stokes' theorem} \tag{11.9}$$

You should have it clearly in mind that this is for an open surface bounded by a simple closed curve. Recall the example of a butterfly net. Notice that Stokes' theorem says that the line integral $\oint \mathbf{V}\cdot d\mathbf{r}$ is equal to the surface integral of $(\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n}$ over *any* surface of which the curve is a boundary; in other words, you don't change the value of the integral by deforming the butterfly net! An easy way to determine the direction of integration for the line integral is to imagine collapsing the surface and its bounding curve into a plane; then the "surface" is just the plane area inside the curve and $\mathbf{n}$ is normal to the plane. The direction of integration is then given by the right-hand rule as discussed just after equation (11.6).

▶ **Example 1.** Given $\mathbf{V} = 4y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, find $\int(\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n}\,d\sigma$ over the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$.

Using (7.3), we find that $\boldsymbol{\nabla}\times\mathbf{V} = -3\mathbf{k}$. There are several ways we could do the problem: (a) integrate the expression as it stands; (b) use Stokes' theorem and evaluate $\oint \mathbf{V}\cdot d\mathbf{r}$ around the circle $x^2 + y^2 = a^2$ in the (x, y) plane; (c) use Stokes' theorem to say that the integral is the same over *any* surface bounded by this circle, for example, the plane area inside the circle! Since this plane area is in the (x, y) plane, we have

$$\mathbf{n} = \mathbf{k}, \qquad (\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n} = -3\mathbf{k}\cdot\mathbf{k} = -3,$$

so the integral is

$$-3\int d\sigma = -3\cdot\pi a^2 = -3\pi a^2.$$

This is the easiest way to do the problem; however, for this simple case it is not too hard by the other methods. We shall leave (b) for you to do and do (a). Since the surface is a sphere with center at the origin, $\mathbf{r}$ is normal to it (but for any surface we could get the normal from the gradient). Then on the surface

$$\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{a} = \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{a},$$

$$(\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n} = -3\mathbf{k}\cdot\frac{\mathbf{r}}{a} = -3\frac{z}{a}.$$

We want to evaluate $\int -3(z/a)\,d\sigma$ over the hemisphere. In spherical coordinates (see Chapter 5, Section 4) we have

$$z = r\cos\theta,$$
$$d\sigma = r^2 \sin\theta\, d\theta\, d\phi.$$

For our surface $r = a$. Then the integral is

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} -3\frac{a\cos\theta}{a} a^2 \sin\theta\, d\theta\, d\phi = -3a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$
$$= -3a^2 \cdot 2\pi \cdot \frac{1}{2} = -3\pi a^2$$

(as before).

Ampère's Law Stokes' theorem is of interest in electromagnetic theory. (Compare the use of the divergence theorem in connection with Gauss's law in Section 10.) Ampère's circuital law (in SI units) says that

$$\oint_C \mathbf{H}\cdot d\mathbf{r} = I,$$

where $\mathbf{H} = \mathbf{B}/\mu_0$, $\mathbf{B}$ is the magnetic field, μ_0 is a constant (called the *permeability of free space*), C is a closed curve, and I is the current "linking" C, that is crossing any surface area bounded by C. The surface area and the curve C are related just as in Stokes' theorem (butterfly net and its rim). If we think of a bundle of wires linking a closed curve C (Figure 11.6) and then spreading out, we can see that the same current crosses any surface whose bounding curve is C.

Just as Gauss's law (10.23) is useful in computing electric fields, so Ampère's law is useful in computing magnetic fields. Consider, for example, a long straight wire carrying a current I (Figure 11.7). At a distance r from the wire, $\mathbf{H}$ is tangent

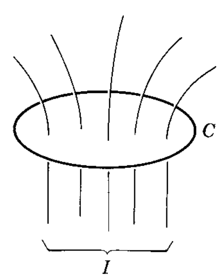


Figure 11.6

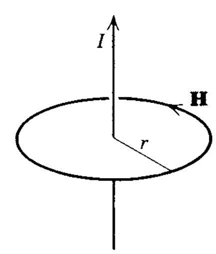


Figure 11.7

to a circle of radius r in a plane perpendicular to the wire. By symmetry, $|\mathbf{H}|$ same at all points of the circle. We can then find $|\mathbf{H}|$ by Ampère's law. Taking C to be the circle of radius r, we have

$$\oint_C \mathbf{H}\cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{H}| r\, d\theta = |\mathbf{H}| r \cdot 2\pi = I$$

or

$$|\mathbf{H}| = \frac{I}{2\pi r}.$$

If, in Figure 11.6, $\mathbf{J}$ is the current density (current crossing unit area perpendicular to $\mathbf{J}$), then $\mathbf{J}\cdot\mathbf{n}\,d\sigma$ is the current across a surface element $d\sigma$ [compare (10.4)] and $\iint_\sigma \mathbf{J}\cdot\mathbf{n}\,d\sigma$, over any surface σ bounded by C, is the total current I linking C. Then by Ampère's law

$$\oint_C \mathbf{H}\cdot d\mathbf{r} = \iint_\sigma \mathbf{J}\cdot\mathbf{n}\,d\sigma.$$

By Stokes' theorem

$$\oint_C \mathbf{H}\cdot d\mathbf{r} = \iint_\sigma (\nabla\times\mathbf{H})\cdot\mathbf{n}\,d\sigma,$$

so we have

$$\iint_\sigma (\nabla\times\mathbf{H})\cdot\mathbf{n}\,d\sigma = \iint_\sigma \mathbf{J}\cdot\mathbf{n}\,d\sigma.$$

Since this is true for any σ, we have $\nabla\times\mathbf{H} = \mathbf{J}$, which is one of the Maxwell equations. Alternatively, we could start with the Maxwell equation and apply Stokes' theorem to get Ampère's law.

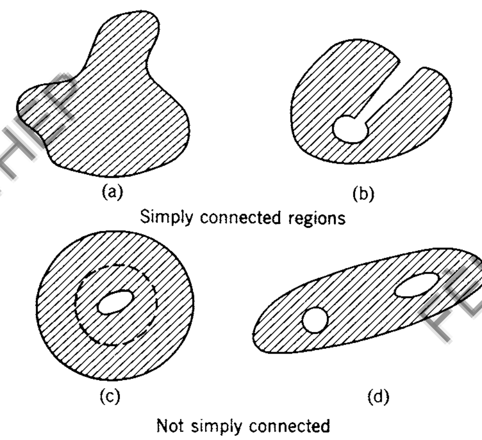


Figure 11.8

Conservative Fields We next want to state carefully, and use Stokes' theorem to prove, under what conditions a given field $\mathbf{F}$ is conservative (see Section 8). First, recall that in physical problems we are often interested only in a particular region of space, and our formulas (say for $\mathbf{F}$) may very well be correct *only* in that region. For example, the gravitational pull of the earth on an object is proportional to $1/r^2$ for $r \geq$ earths' radius R, but this is not a correct formula for $r < R$ (see Problem 8.21). The electric field in the region between the plates of a cylindrical capacitor is proportional to $1/r$ (problem 10.12), but only *in this region* is this formula correct. We must, then, consider the *kind of region* in which a given field $\mathbf{F}$ is defined. Consider the shaded regions in Figure 11.8. We say that a region is *simply connected* if any simple† closed curve in the region can be shrunk to a point without encountering any points not in the region. You can see in Figure 11.8c that the dotted curve surrounds the "hole" and so cannot be shrunk to a point in the region; this region is then not simply connected. The "hole" is sometimes only a single point, but this is enough to make the region *not* simply connected. In three dimensions the region between cylindrical capacitor plates (infinitely long) is not simply connected since a loop of string around the inner cylinder (see cross section, Figure 11.8c) cannot be drawn up to a knot. Similarly, the interior of an inner tube is not simply connected. The region between two concentric spheres *is* simply connected, however. You should see this by realizing that you could pull up into a

†A simple curve does not cross itself; for example, a figure eight is not a simple curve.

knot, a loop of string placed anywhere in this region. We shall now state and prove our theorem.

(11.10)

If the components of $\mathbf{F}$ and their first partial derivatives are continuous in a simply connected region, then any one of the following five conditions implies all the others.

(a) $\operatorname{curl}\mathbf{F} = 0$ at every point of the region.

(b) $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ around every simple closed curve in the region.

(c) $\mathbf{F}$ is conservative, that is $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration from A to B. (The path must, of course, lie entirely in the region.)

(d) $\mathbf{F} \cdot d\mathbf{r}$ is an exact differential of a single valued function.

(e) $\mathbf{F} = \operatorname{grad} W$, W single-valued.

We shall show that each of these conditions implies the one following it. We can use Stokes' theorem to prove (b) assuming (a). First select any simple closed curve and let it be the bounding curve for the surface in Stokes' theorem. Since the region is simply connected we can think of shrinking the curve to a point in the region; as it shrinks it traces out a surface which we use as the Stokes' theorem surface. Assuming (a), we have $\operatorname{curl}\mathbf{F} = 0$ at every point of the region and so also at every point of the surface. Thus the surface integral in Stokes' theorem is zero and therefore the line integral around the closed curve equals zero. This gives (b).

To show that (b) implies (c), consider any two paths I and II from A to B (Figure 11.9). From (b) we have

$$\underset{\text{path I}}{\int_A^B} \mathbf{F} \cdot d\mathbf{r} + \underset{\text{path II}}{\int_B^A} \mathbf{F} \cdot d\mathbf{r} = 0.$$

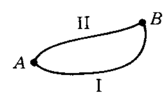


Figure 11.9

Since an integral from A to B is the negative of an integral from B to A, we have

$$\underset{\text{path I}}{\int_A^B} \mathbf{F} \cdot d\mathbf{r} - \underset{\text{path II}}{\int_A^B} \mathbf{F} \cdot d\mathbf{r} = 0.$$

which is (c).

To show that (c) implies (d), select some reference point O in the region and calculate $\int \mathbf{F} \cdot d\mathbf{r}$ from the reference point to every other point of the region. For each point P we find a single value of the integral no matter what path of integration we choose from O to P. Let this value be the value of the function W at the point P. We then have a single-valued function W such that

$$\underset{0 \text{ to } P}{\int} \mathbf{F} \cdot d\mathbf{r} = W(P).$$

Then (since $\mathbf{F}$ is continuous), $dW = \mathbf{F} \cdot d\mathbf{r}$, that is, $\mathbf{F} \cdot d\mathbf{r}$ is the differential of a single-valued function W. Since $dW = \boldsymbol{\nabla} W \cdot d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}$ for arbitrary $d\mathbf{r}$, we have $\mathbf{F} = \boldsymbol{\nabla} W$ which is (e).

Finally, (e) implies (a) as we proved in Section 8. (The continuity of the components of $\mathbf{F}$ and their partial derivatives makes the second-order mixed partial derivatives of W equal.) Thus we have shown that any one of the five conditions (a) to (e) implies the others under the conditions of the theorem. It is worth observing carefully the requirement that $\mathbf{F}$ and its partial derivatives must be continuous in a simply connected region. A simple example makes this clear. Look at Example 2 in Section 8; you can easily compute $\operatorname{curl} \mathbf{F}$ and find that it is zero everywhere except at the origin (where it is undefined). You might then be tempted to assume that $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ around any closed path. But we found that $\mathbf{F} \cdot d\mathbf{r} = d\theta$, and the integral of $d\theta$ along a circle with center at the origin is 2π. What is wrong? The trouble is that $\mathbf{F}$ does not have continuous partial derivatives at the origin, and any simply connected region containing the circle of integration must contain the origin. Then $\operatorname{curl} \mathbf{F}$ is not zero at *every point* inside the integration curve. Notice also that $\mathbf{F} \cdot d\mathbf{r} = d\theta$ is an exact differential, but not of a single-valued function; θ increases by 2π every time we go around the origin.

A vector field $\mathbf{V}$ is called *irrotational* (or *conservative* or *lamellar*) if $\operatorname{curl} \mathbf{V} = 0$; in this case $\mathbf{V} = \operatorname{grad} W$, where W (or its negative) is called the *scalar potential*. If $\operatorname{div} \mathbf{V} = 0$, the vector field is called *solenoidal*; in this case $\mathbf{V} = \operatorname{curl} \mathbf{A}$, where A is a vector function called the *vector potential*. It is easy to prove (Problem 7.17d) that if $\mathbf{V} = \boldsymbol{\nabla} \times \mathbf{A}$, then $\operatorname{div} \mathbf{V} = 0$. It is also possible to construct an $\mathbf{A}$ (actually an infinite number of $\mathbf{A}$'s) so that $\mathbf{V} = \operatorname{curl} \mathbf{A}$ if we know that $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$.

▶ **Example 2.** Given $\mathbf{V} = \mathbf{i}(x^2 - yz) - \mathbf{j}2yz + \mathbf{k}(z^2 - 2zx)$, find $\mathbf{A}$ such that $\mathbf{V} = \boldsymbol{\nabla} \times \mathbf{A}$. We find

$$\begin{aligned} \operatorname{div} \mathbf{V} &= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(-2yz) + \frac{\partial}{\partial z}(z^2 - 2zx) \\ &= 2x - 2z + 2z - 2x = 0. \end{aligned}$$

Thus $\mathbf{V}$ is solenoidal and we proceed to find $\mathbf{A}$. We are looking for an $\mathbf{A}$ such that

$$\mathbf{V} = \operatorname{curl} \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{i}(x^2 - yz) - \mathbf{j}2yz + \mathbf{k}(z^2 - 2zx). \tag{11.11}$$

There are many $\mathbf{A}$'s satisfying this equation; we shall show first how to find one of them and then a general formula for all. It is possible to find an $\mathbf{A}$ with one zero component; let us take $A_x = 0$. Then the y and z components of $\operatorname{curl} \mathbf{A}$ each involve just one component of $\mathbf{A}$. From (11.11), the y and z components of $\operatorname{curl} \mathbf{A}$ are

$$-2yz = -\frac{\partial A_z}{\partial x}, \qquad z^2 - 2zx = \frac{\partial A_y}{\partial x}. \tag{11.12}$$

If we integrate (11.12) partially with respect to x (that is, with y and z constant), we find A_y and A_z except for possible functions of y and z which could be added without changing (11.12):

$$\begin{aligned} A_y &= z^2x - zx^2 + f_1(y,z), \\ A_z &= 2xyz + f_2(y,z). \end{aligned} \tag{11.13}$$

Substituting (11.13) into the x component of (11.11), we get

$$x^2 - yz = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 2xz + \frac{\partial f_2}{\partial y} - 2zx + x^2 - \frac{\partial f_1}{\partial z}. \tag{11.14}$$

We now select f_1 and f_2 to satisfy (11.14). There is much leeway here and this can easily be done by inspection. We could take $f_2 = 0$, $f_1 = \frac{1}{2}yz^2$, or $f_1 = 0$, $f_2 = -\frac{1}{2}y^2z$, and so forth. Using the second choice, we have

$$\mathbf{A} = \mathbf{j}(z^2x - zx^2) + \mathbf{k}(2xyz - \frac{1}{2}y^2z). \tag{11.15}$$

You may wonder why this process works and what $\operatorname{div}\mathbf{V} = 0$ has to do with it. We can answer both these questions by following the above process with a general $\mathbf{V}$ rather than a special example. Given that $\operatorname{div}\mathbf{V} = 0$, we want an $\mathbf{A}$ such that $\mathbf{V} = \operatorname{curl}\mathbf{A}$. We try to find one with $A_x = 0$. Then the y and z components of $\mathbf{V} = \operatorname{curl}\mathbf{A}$ are

$$V_y = -\frac{\partial A_z}{\partial x}, \qquad V_z = \frac{\partial A_y}{\partial x}. \tag{11.16}$$

Then we have

$$A_y = \int V_z\,dx + f(y,z), \qquad A_z = -\int V_y\,dx + g(y,z). \tag{11.17}$$

The x component of $\mathbf{V} = \operatorname{curl}\mathbf{A}$ is

$$V_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\int\left(\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)dx + h(y,z). \tag{11.18}$$

Since $\operatorname{div}\mathbf{V} = 0$, we can put

$$-\left(\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) = \frac{\partial V_x}{\partial x} \tag{11.19}$$

into (11.18), getting

$$V_x = \int \frac{\partial V_x}{\partial x}dx + h(y,z).$$

This is correct with proper choice of $h(y,z)$.

When we know one $\mathbf{A}$, for which a given $\mathbf{V}$ is equal to $\operatorname{curl}\mathbf{A}$, all others are of the form

$$\mathbf{A} + \boldsymbol{\nabla}u, \tag{11.20}$$

where u is any scalar function. For (see Problem 7.17b), $\boldsymbol{\nabla}\times\boldsymbol{\nabla}u = 0$, so the addition of $\boldsymbol{\nabla}u$ to $\mathbf{A}$ does not affect $\mathbf{V}$. Also we can show that all possible $\mathbf{A}$'s are of the form (11.20). For if $\mathbf{V} = \operatorname{curl} \mathbf{A}_1$ and $\mathbf{V} = \operatorname{curl} \mathbf{A}_2$, then $\operatorname{curl}(\mathbf{A}_1 - \mathbf{A}_2) = 0$, so $\mathbf{A}_1 - \mathbf{A}_2$ is the gradient of some scalar function.

A careful statement and proof that $\operatorname{div} \mathbf{V} = 0$ is a necessary and sufficient condition for $\mathbf{V} = \operatorname{curl} \mathbf{A}$ requires that $\mathbf{V}$ have continuous partial derivatives at every point of a region which is simply connected in the sense that every closed surface (rather than closed curve) can be shrunk to a point in the region (for example, the region between two concentric spheres is not simply connected in this sense).

▶ PROBLEMS, SECTION 11

1. Do case (b) of Example 1 above.

2. Given the vector $\mathbf{A} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$.

(a) Find $\boldsymbol{\nabla} \times \mathbf{A}$.

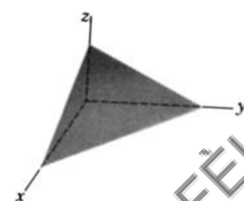
(b) Evaluate $\iint (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\boldsymbol{\sigma}$ over a rectangle in the (x, y) plane bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$.

(c) Evaluate $\oint \mathbf{A} \cdot d\mathbf{r}$ around the boundary of the rectangle and thus verify Stokes' theorem for this case.

Use either Stokes' theorem or the divergence theorem to evaluate each of the following integrals in the easiest possible way.

3. $\iint_{\text{surface } \sigma} \operatorname{curl}(x^2\mathbf{i} + z^2\mathbf{j} - y^2\mathbf{k}) \cdot \mathbf{n}\, d\sigma$, where σ is the part of the surface $z = 4 - x^2 - y^2$ above the (x, y) plane.

4. $\iint \operatorname{curl}(y\mathbf{i} + 2\mathbf{j}) \cdot \mathbf{n}\, d\sigma$, where σ is the surface in the first octant made up of part of the plane $2x + 3y + 4z = 12$, and triangles in the (x, z) and (y, z) planes, as indicated in the figure.



5. $\iint \mathbf{r} \cdot \mathbf{n}\, d\sigma$ over the surface in Problem 4, where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. *Hint:* See Problem 10.9.

6. $\iint \mathbf{V} \cdot \mathbf{n}\, d\sigma$ over the closed surface of the tin can bounded by $x^2 + y^2 = 9$, $z = 0$, $z = 5$, if

$$\mathbf{V} = 2xy\mathbf{i} - y^2\mathbf{j} + (z + xy)\mathbf{k}.$$

7. $\iint (\operatorname{curl} \mathbf{V}) \cdot \mathbf{n}\, d\sigma$ over any surface whose bounding curve is in the (x, y) plane, where

$$\mathbf{V} = (x - x^2 z)\mathbf{i} + (yz^3 - y^2)\mathbf{j} + (x^2 y - xz)\mathbf{k}.$$

8. $\iint \operatorname{curl}(x^2 y\mathbf{i} - xz\mathbf{k}) \cdot \mathbf{n}\, d\sigma$ over the *closed* surface of the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

Warning: Stokes' theorem applies only to an open surface. *Hints:* Could you cut the given surface into two halves? Also see (d) in the table of vector identities (page 339).

9. $\iint \mathbf{V} \cdot \mathbf{n}\, d\sigma$ over the entire surface of the volume in the first octant bounded by $x^2 + y^2 + z^2 = 16$ and the coordinate planes, where

$$\mathbf{V} = (x + x^2 - y^2)\mathbf{i} + (2xyz - 2xy)\mathbf{j} - xz^2\mathbf{k}.$$

10. $\iint (\operatorname{curl} \mathbf{V}) \cdot \mathbf{n}\, d\sigma$ over the part of the surface $z = 9 - x^2 - 9y^2$ above the (x, y) plane, if $\mathbf{V} = 2xy\mathbf{i} + (x^2 - 2x)\mathbf{j} - x^2 z^2\mathbf{k}$.

11. $\iint \mathbf{V}\cdot\mathbf{n}\,d\sigma$ over the entire surface of a cube in the first octant with edges of length 2 along the coordinate axes, where

$$\mathbf{V} = (x^2 - y^2)\mathbf{i} + 3y\mathbf{j} - 2xz\mathbf{k}.$$

12. $\oint \mathbf{V}\cdot d\mathbf{r}$ around the circle $(x-2)^2 + (y-3)^2 = 9$, $z = 0$, where

$$\mathbf{V} = (x^2 + yz^2)\mathbf{i} + (2x - y^3)\mathbf{j}.$$

13. $\iint (2x\mathbf{i} - 2y\mathbf{j} + 5\mathbf{k})\cdot\mathbf{n}\,d\sigma$ over the surface of a sphere of radius 2 and center at the origin.

14. $\oint (y\mathbf{i} - x\mathbf{j} + z\mathbf{k})\cdot d\mathbf{r}$ around the circumference of the circle of radius 2, center at the origin, in the (x, y) plane.

15. $\oint_c y\,dx + z\,dy + x\,dz$, where C is the curve of intersection of the surfaces whose equations are $x + y = 2$ and $x^2 + y^2 + z^2 = 2(x + y)$.

16. What is wrong with the following "proof" that there are no magnetic fields? By electromagnetic theory, $\nabla\cdot\mathbf{B} = 0$, and $\mathbf{B} = \nabla\times\mathbf{A}$. (The error is *not* in these equations.) Using them, we find

$$\iiint \nabla\cdot\mathbf{B}\,d\tau = 0 = \iint \mathbf{B}\cdot\mathbf{n}\,d\sigma \qquad \text{(by the divergence theorem)}$$

$$= \iint (\nabla\times\mathbf{A})\cdot\mathbf{n}\,d\sigma = \int \mathbf{A}\cdot d\mathbf{r} \qquad \text{(by Stokes' theorem).}$$

Since $\int \mathbf{A}\cdot d\mathbf{r} = 0$, $\mathbf{A}$ is conservative, or $\mathbf{A} = \nabla\psi$. Then $\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\nabla\psi = 0$, so $\mathbf{B} = 0$.

17. Derive the following vector integral theorems.

(a) $$\int\limits_{\text{volume } \tau} \nabla\phi\,d\tau = \oint\limits_{\substack{\text{surface}\\ \text{inclosing } \tau}} \phi\mathbf{n}\,d\sigma.$$

Hint: In the divergence theorem (10.17), substitute $\mathbf{V} = \phi\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector, to obtain $\mathbf{C}\cdot\int\nabla\phi\,d\tau = \mathbf{C}\cdot\oint\phi\mathbf{n}\,d\sigma$. Since $\mathbf{C}$ is arbitrary, let $\mathbf{C} = \mathbf{i}$ to show that the x components of the two integrals are equal; similarly, let $\mathbf{C} = \mathbf{j}$ and $\mathbf{C} = \mathbf{k}$ to show that the y components are equal and the z components are equal.

(b) $$\int\limits_{\text{volume } \tau} \nabla\times\mathbf{V}\,d\tau = \oint\limits_{\substack{\text{surface}\\ \text{inclosing } \tau}} \mathbf{n}\times\mathbf{V}\,d\sigma.$$

Hint: Replace $\mathbf{V}$ in the divergence theorem by $\mathbf{V}\times\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector. Follow the last part of the hint in (a).

(c) $$\int\limits_{\substack{\text{curve}\\ \text{bounding } \sigma}} \phi\,d\mathbf{r} = \oint\limits_{\text{surface } \sigma} (\mathbf{n}\times\nabla\phi)d\sigma.$$

(d) $$\oint\limits_{\substack{\text{curve}\\ \text{bounding } \sigma}} d\mathbf{r}\times\mathbf{V} = \int\limits_{\text{surface } \sigma} (\mathbf{n}\times\nabla)\times\mathbf{V}\,d\sigma.$$

Hints for (c) and (d): Use the substitutions suggested in (a) and (b) but in Stokes' theorem (11.9) instead of the divergence theorem.

(e) $$\int\limits_{\text{volume } \tau} \phi\nabla\cdot\mathbf{V}\,d\tau = \oint\limits_{\substack{\text{surface}\\ \text{inclosing } \tau}} \phi\mathbf{V}\cdot\mathbf{n}\,d\sigma - \int\limits_{\text{volume } \tau} \mathbf{V}\cdot\nabla\phi\,d\tau.$$

Hint: Integrate (7.6) over volume τ and use the divergence theorem.

(f) $$\int_{\text{volume } \tau} \mathbf{V}\cdot(\nabla\times\mathbf{U})\,d\tau = \int_{\text{volume } \tau} \mathbf{U}\cdot(\nabla\times\mathbf{V})\,d\tau + \oint_{\substack{\text{surface}\\ \text{inclosing } \tau}} (\mathbf{U}\times\mathbf{V})\cdot\mathbf{n}\,d\sigma.$$

Hint: Integrate (h) in the Table of Vector Identities (page 339) and use the divergence theorem.

(g) $$\int_{\text{surface of } \sigma} \phi(\nabla\times\mathbf{V})\cdot\mathbf{n}\,d\sigma = \int_{\text{surface of } \sigma} (\nabla\times\nabla\phi)\cdot\mathbf{n}\,d\sigma + \oint_{\substack{\text{curve}\\ \text{bounding } \sigma}} \phi\mathbf{V}\cdot d\mathbf{r}.$$

Hint: Integrate (g) in the Table of Vector Identities (page 339) and use Stokes' Theorem.

Find vector fields $\mathbf{A}$ such that $\mathbf{V} = \operatorname{curl}\mathbf{A}$ for each given $\mathbf{V}$.

18. $\mathbf{V} = (x^2 - yz + y)\mathbf{i} + (x - 2yz)\mathbf{j} + (z^2 - 2zx + x + y)\mathbf{k}$

19. $\mathbf{V} = \mathbf{i}(x^2 - 2xz) + \mathbf{j}(y^2 - 2xy) + \mathbf{k}(z^2 - 2yz + xy)$

20. $\mathbf{V} = \mathbf{i}(ze^{zy} + x\sin zx) + \mathbf{j}x\cos xz - \mathbf{k}z\sin zx$

21. $\mathbf{V} = -\mathbf{k}$

22. $\mathbf{V} = (y + z)\mathbf{i} + (x - z)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

12. MISCELLANEOUS PROBLEMS

1. If $\mathbf{A}$ and $\mathbf{B}$ are unit vectors with an angle θ between them, and $\mathbf{C}$ is a unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, evaluate $[(\mathbf{A}\times\mathbf{B})\times(\mathbf{B}\times\mathbf{C})]\times(\mathbf{C}\times\mathbf{A})$.

2. If $\mathbf{A}$ and $\mathbf{B}$ are the diagonals of a parallelogram, find a vector formula for the area of the parallelogram.

3. The force on a charge q moving with velocity $\mathbf{v} = d\mathbf{r}/dt$ in a magnetic field $\mathbf{B}$ is $\mathbf{F} = q(\mathbf{v}\times\mathbf{B})$. We can write $\mathbf{B}$ as $\mathbf{B} = \nabla\times\mathbf{A}$ where $\mathbf{A}$ (called the vector potential) is a vector function of x, y, z, t. If the position vector $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ of the charge q is a function of time t, show that

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{A}.$$

Thus show that

$$\mathbf{F} = q\mathbf{v}\times(\nabla\times\mathbf{A}) = q\left[\nabla(\mathbf{v}\cdot\mathbf{A}) - \frac{d\mathbf{A}}{dt} + \frac{\partial\mathbf{A}}{\partial t}\right].$$

4. Show that $\nabla\cdot(\mathbf{U}\times\mathbf{r}) = \mathbf{r}\cdot(\nabla\times\mathbf{U})$ where $\mathbf{U}$ is a vector function of x, y, z, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

5. Use Green's theorem (Section 9) to do Problem 8.2.

6. Find the torque about the point $(1, -2, 1)$ due to the force $\mathbf{F} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ acting at the point $(1, 1, -3)$.

7. Let $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ act at the point $(5, 1, 3)$.

(a) Find the torque of $\mathbf{F}$ about the point $(4, 1, 0)$.

(b) Find the torque of $\mathbf{F}$ about the line $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})t$.

8. The force $\mathbf{F} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ acts at the point $(0, 1, 2)$. Find the torque of $\mathbf{F}$ about the line $\mathbf{r} = (2\mathbf{i} - \mathbf{j})t$.

9. Let $\mathbf{F} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ act at the point $(2, 1, 0)$. Find the torque of $\mathbf{F}$ about the line $\mathbf{r} = (3\mathbf{j} + 4\mathbf{k}) - 2\mathbf{i}t$.

10. Given $u = xy + \sin z$, find

(a) the gradient of u at $(1, 2, \pi/2)$;

(b) how fast u is increasing, in the direction $4\mathbf{i} + 3\mathbf{j}$, at $(1, 2, \pi/2)$;

(c) the equation of the tangent plane to the surface $u = 3$ at $(1, 2, \pi/2)$.

11. Given $\phi = z^2 - 3xy$, find

(a) grad ϕ;

(b) the directional derivative of ϕ at the point $(1, 2, 3)$ in the direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$;

(c) the equations of the tangent plane and of the normal line to $\phi = 3$ at the point $(1, 2, 3)$.

12. Given $u = xy + yz + z \sin x$, find

(a) $\boldsymbol{\nabla} u$ at $(0, 1, 2)$;

(b) the directional derivative of u at $(0, 1, 2)$ in the direction $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$;

(c) the equations of the tangent plane and of the normal line to the level surface $u = 2$ at $(0, 1, 2)$;

(d) a unit vector in the direction of most rapid increase of u at $(0, 1, 2)$.

13. Given $\phi = x^2 - yz$ and the point $P(3, 4, 1)$, find

(a) $\boldsymbol{\nabla}\phi$ at P;

(b) a unit vector normal to the surface $\phi = 5$ at P;

(c) a vector in the direction of most rapid increase of ϕ at P;

(d) the magnitude of the vector in (c);

(e) the derivative of ϕ at P in a direction parallel to the line $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + (6\mathbf{i} - \mathbf{j} - 4\mathbf{k})t$.

14. If the temperature is $T = x^2 - xy + z^2$, find

(a) the direction of heat flow at $(2, 1, -1)$;

(b) the rate of change of temperature in the direction $\mathbf{j} - \mathbf{k}$ at $(2, 1, -1)$.

15. Show that

$$\mathbf{F} = y^2 z \sinh(2xz)\mathbf{i} + 2y \cosh^2(xz)\mathbf{j} + y^2 x \sinh(2xz)\mathbf{k}$$

is conservative, and find a scalar potential ϕ such that $\mathbf{F} = -\boldsymbol{\nabla}\phi$.

16. Given $\mathbf{F}_1 = 2xz\mathbf{i} + y\mathbf{j} + x^2\mathbf{k}$ and $\mathbf{F}_2 = y\mathbf{i} - x\mathbf{j}$:

(a) Which $\mathbf{F}$, if either, is conservative?

(b) If one of the given $\mathbf{F}$'s is conservative, find a function W so that $\mathbf{F} = \boldsymbol{\nabla}\mathbf{W}$.

(c) If one of the $\mathbf{F}$'s is nonconservative, use it to evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ along the straight line from $(0, 1)$ to $(1, 0)$.

(d) Do part (c) by applying Green's theorem to the triangle with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$.

17. Find the value of $\int \mathbf{F} \cdot d\mathbf{r}$ along the circle $x^2 + y^2 = 2$ from $(1, 1)$ to $(1, -1)$ if

$$\mathbf{F} = (2x - 3y)\mathbf{i} - (3x - 2y)\mathbf{j}.$$

18. Is $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + z\mathbf{k}$ conservative? Evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the paths

(a) broken line $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$,

(b) straight line connecting the points.

19. Given $\mathbf{F}_1 = -2y\mathbf{i} + (z-2x)\mathbf{j} + (y+z)\mathbf{k}$, $\mathbf{F}_2 = y\mathbf{i} + 2x\mathbf{j}$:

(a) Is $\mathbf{F}_1$ conservative? Is $\mathbf{F}_2$ conservative?

(b) Find the work done by $\mathbf{F}_2$ on a particle that moves around the ellipse $x = \cos\theta$, $y = 2\sin\theta$ from $\theta = 0$ to $\theta = 2\pi$.

(c) For any conservative force in this problem find a potential function V such that $\mathbf{F} = -\boldsymbol{\nabla} V$.

(d) Find the work done by $\mathbf{F}_1$ on a particle that moves along the straight line from $(0,1,0)$ to $(0,2,5)$.

(e) Use Green's theorem and the result of Problem 9.7 to do Part (b) above.

In Problems 20 to 31, evaluate each integral in the simplest way possible.

20. $\iint \mathbf{P}\cdot\mathbf{n}\,d\sigma$ over the upper half of the sphere $r = 1$ if $\mathbf{P} = \operatorname{curl}(\mathbf{j}x - \mathbf{k}z)$.

21. $\iint (\boldsymbol{\nabla}\times\mathbf{V})\cdot\mathbf{n}\,d\sigma$ over the surface consisting of the four slanting faces of a pyramid whose base is the square in the (x,y) plane with corners at $(0,0)$, $(0,2)$, $(2,0)$, $(2,2)$ and whose top vertex is at $(1,1,2)$, where

$$\mathbf{V} = (x^2 z - 2)\mathbf{i} + (x + y - z)\mathbf{j} - xyz\mathbf{k}.$$

22. $\iint \mathbf{V}\cdot\mathbf{n}\,d\sigma$ over the entire surface of the sphere $(x-2)^2 + (y+3)^2 + z^2 = 9$, if

$$\mathbf{V} = (3x - yz)\mathbf{i} + (z^2 - y^2)\mathbf{j} + (2yz + x^2)\mathbf{k}.$$

23. $\iint \mathbf{F}\cdot\mathbf{n}\,d\sigma$ where $\mathbf{F} = (y^2 - x^2)\mathbf{i} + (2xy - y)\mathbf{j} + 3z\mathbf{k}$ and σ is the entire surface of the tin can bounded by the cylinder $x^2 + y^2 = 16$, $z = 3$, $z = -3$.

24. $\iint \mathbf{r}\cdot\mathbf{n}\,d\sigma$ over the entire surface of the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

25. $\iint \mathbf{V}\cdot\mathbf{n}\,d\sigma$ over the curved part of the hemisphere in Problem 24, if $\mathbf{V} = \operatorname{curl}(y\mathbf{i} - x\mathbf{j})$.

26. $\iint (\operatorname{curl}\mathbf{V})\cdot\mathbf{n}\,d\sigma$ over the entire surface of the cube in the first octant with three faces in the three coordinate planes and the other three faces intersecting at $(2,2,2)$, where

$$\mathbf{V} = (2 - y)\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k}.$$

27. Problem 26, but integrate over the open surface obtained by leaving out the face of the cube in the (x,y) plane.

28. $\oint \mathbf{F}\cdot d\mathbf{r}$ around the circle $x^2 + y^2 + 2x = 0$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

29. $\oint \mathbf{V}\cdot d\mathbf{r}$ around the boundary of the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$, if $\mathbf{V} = x^2\mathbf{i} + 5x\mathbf{j}$.

30. $\int_C (x^2 - y)dx + (x + y^3)dy$, where C is the parallelogram with vertices at $(0,0)$, $(2,0)$, $(1,1)$, $(3,1)$.

31. $\int (y^2 - x^2)\,dx + (2xy + 3)dy$ along the x axis from $(0,0)$ to $(\sqrt{5},0)$ and then along a circular arc from $(\sqrt{5},0)$ to $(1,2)$. *Hint:* Use Green's theorem.

Table of Vector Identities Involving ∇

Note carefully that ϕ and ψ are scalar functions; $\mathbf{U}$ and $\mathbf{V}$ are vector functions. Formulas are given in rectangular coordinates; for other coordinate systems, see Chapter 10, Section 9.

(a) $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \operatorname{div}\operatorname{grad}\phi = \nabla^2\phi = \text{Laplacian } \phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

(b) $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \operatorname{curl}\operatorname{grad}\phi = 0$

(c) $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V}) = \operatorname{grad}\operatorname{div}\mathbf{V}$

$$= \mathbf{i}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z}\right) + \mathbf{j}\left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y \partial z}\right) + \mathbf{k}\left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

(d) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{V}) = \operatorname{div}\operatorname{curl}\mathbf{V} = 0$

(e) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{V}) = \operatorname{curl}\operatorname{curl}\mathbf{V} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V}) - \nabla^2\mathbf{V} = \operatorname{grad}\operatorname{div}\mathbf{V} - \text{Laplacian } \mathbf{V}$

(f) $\boldsymbol{\nabla} \cdot (\phi\mathbf{V}) = \phi(\boldsymbol{\nabla} \cdot \mathbf{V}) + \mathbf{V} \cdot (\boldsymbol{\nabla}\phi)$

(g) $\boldsymbol{\nabla} \times (\phi\mathbf{V}) = \phi(\boldsymbol{\nabla} \times \mathbf{V}) - \mathbf{V} \times (\boldsymbol{\nabla}\phi)$

(h) $\boldsymbol{\nabla} \cdot (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot (\boldsymbol{\nabla} \times \mathbf{U}) - \mathbf{U} \cdot (\boldsymbol{\nabla} \times \mathbf{V})$

(i) $\boldsymbol{\nabla} \times (\mathbf{U} \times \mathbf{V}) = (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{U} - (\mathbf{U} \cdot \boldsymbol{\nabla})\mathbf{V} - \mathbf{V}(\boldsymbol{\nabla} \cdot \mathbf{U}) + \mathbf{U}(\boldsymbol{\nabla} \cdot \mathbf{V})$

(j) $\boldsymbol{\nabla}(\mathbf{U} \cdot \mathbf{V}) = \mathbf{U} \times (\boldsymbol{\nabla} \times \mathbf{V}) + (\mathbf{U} \cdot \boldsymbol{\nabla})\mathbf{V} + \mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{U}) + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{U}$

(k) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi) = 0$

CHAPTER 7

Fourier Series and Transforms

1. INTRODUCTION

Problems involving vibrations or oscillations occur frequently in physics and engineering. You can think of examples you have already met: a vibrating tuning fork, a pendulum, a weight attached to a spring, water waves, sound waves, alternating electric currents, etc. In addition, there are many more examples which you will meet as you continue to study physics. Some of them—for example, heat conduction, electric and magnetic fields, light—do not appear in elementary work to have anything oscillatory about them, but will turn out in your more advanced work to involve the sines and cosines which are used in describing simple harmonic motion and wave motion.

In Chapter 1 we discussed the use of power series to approximate complicated functions. In many problems, series called Fourier series, whose terms are sines and cosines, are more useful than power series. In this chapter we shall see how to find and use Fourier series. Then, in Chapter 13 (Sections 2 to 4), we shall consider several of the physics problems which Fourier was trying to solve when he invented Fourier series.

Since sines and cosines are periodic functions, Fourier series can represent only periodic functions. We will see in Section 12 how to represent a non-periodic function by a Fourier integral (Fourier transform).

2. SIMPLE HARMONIC MOTION AND WAVE MOTION; PERIODIC FUNCTIONS

We shall need much of the notation and terminology used in discussing simple harmonic motion and wave motion. Let's discuss these two topics briefly.

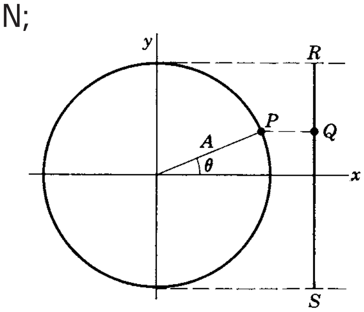


Figure 2.1

Let particle P (Figure 2.1) move at constant speed around a circle of radius A. At the same time, let particle Q move up and down along the straight line segment RS in such a way that the y coordinates of P and Q are always equal. If ω is the angular velocity of P in radians per second, and

(Figure 2.1) $\theta = 0$ when $t = 0$, then at a later time t

$$\theta = \omega t. \tag{2.1}$$

The y coordinate of Q (which is equal to the y coordinate of P) is

$$y = A \sin\theta = A \sin\omega t. \tag{2.2}$$

The back and forth motion of Q is called *simple harmonic motion*. By definition, an object is executing simple harmonic motion if its displacement from equilibrium can be written as $A \sin\omega t$ [or $A\cos\omega t$ or $A\sin(\omega t+\phi)$, but these two functions differ from $A\sin\omega t$ only in choice of origin; such functions are called *sinusoidal functions*]. You can think of many physical examples of this sort of simple vibration: a pendulum, a tuning fork, a weight bobbing up and down at the end of a spring.

The x and y coordinates of particle P in Figure 2.1 are

$$x = A\cos\omega t, \qquad y = A\sin\omega t. \tag{2.3}$$

If we think of P as the point $z = x+iy$ in the complex plane, we could replace (2.3) by a single equation to describe the motion of P:

$$\begin{aligned} z = x + iy &= A(\cos\omega t + i\sin\omega t) \\ &= Ae^{i\omega t}. \end{aligned} \tag{2.4}$$

It is often worth while to use this complex notation even to describe the motion of Q; we then understand that the actual position of Q is equal to the imaginary part of z (or with different starting conditions the real part of z). For example, the velocity of Q is the imaginary part of

$$\frac{dz}{dt} = \frac{d}{dt}(Ae^{i\omega t}) = Ai\omega e^{i\omega t} = Ai\omega(\cos\omega t + i\sin\omega t). \tag{2.5}$$

[The imaginary part of (2.5) is $A\omega\cos\omega t$, which is dy/dt from (2.2).]

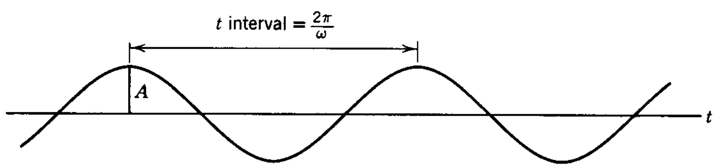


Figure 2.2

It is useful to draw a graph of x and y in (2.2) and (2.3) as a function of t. Figure 2.2 represents any of the functions $\sin\omega t$, $\cos\omega t$, $\sin(\omega t + \phi)$ if we choose the origin correctly. The number A is called the *amplitude of the vibration* or the *amplitude of the function*. Physically it is the maximum displacement of Q from its equilibrium position. The *period of the simple harmonic motion* or the *period of the function* is the time for one complete oscillation, that is, $2\pi/\omega$ (See Figure 2.2).

We could write the velocity of Q from (2.5) as

$$\frac{dy}{dt} = A\omega\cos\omega t = B\cos\omega t. \tag{2.6}$$

Here B is the maximum value of the velocity and is called the *velocity amplitude*. Note that the velocity has the same period as the displacement. If the mass of the particle Q is m, its kinetic energy is:

$$\text{Kinetic energy} = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}mB^2\cos^2\omega t. \tag{2.7}$$

We are considering an idealized harmonic oscillator which does not lose energy. Then the total energy (kinetic plus potential) must be equal to the largest value of the kinetic energy, that is, $\frac{1}{2}mB^2$. Thus we have:

$$\text{Total energy} = \frac{1}{2}mB^2. \tag{2.8}$$

Notice that the energy is proportional to the square of the (velocity) amplitude; we shall be interested in this result later when we discuss sound.

Waves are another important example of an oscillatory phenomenon. The mathematical ideas of wave motion are useful in many fields; for example, we talk about water waves, sound waves, and radio waves.

▶ **Example 1.** Consider water waves in which the shape of the water surface is (unrealistically!) a sine curve. If we take a photograph (at the instant $t = 0$) of the water surface, the equation of this picture could be written (relative to appropriate axes)

$$y = A\sin\frac{2\pi x}{\lambda}, \tag{2.9}$$

where x represents horizontal distance and λ is the distance between wave crests. Usually λ is called the *wavelength*, but mathematically it is the same as the period of this function of x. Now suppose we take another photograph when the waves have moved forward a distance vt (v is the velocity of the waves and t is the time between photographs). Figure 2.3 shows the two photographs superimposed. Observe that the value of y at the point x on the graph labeled t, is just the same as the value of y at the point $(x - vt)$ on the graph labeled $t = 0$. If (2.9) is the equation representing the waves at $t = 0$, then

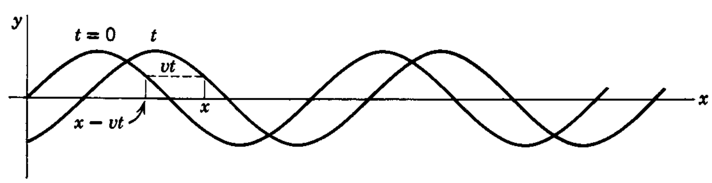


Figure 2.3

$$y = A\sin\frac{2\pi}{\lambda}(x - vt) \tag{2.10}$$

represents the waves at time t. We can interpret (2.10) in another way. Suppose you stand at one point in the water [fixed x in (2.10)] and observe the up and down motion of the water, that is, y in (2.10) as a function of t (for fixed x). This is a simple harmonic motion of amplitude A and period λ/v. You are doing something analogous to this when you stand still and listen to a sound (sound waves pass your ear and you observe their frequency) or when you listen to the radio (radio waves pass the receiver and it reacts to their frequency).

We see that y in (2.10) is a periodic function either of x (t fixed) or of t (x fixed); both interpretations are useful. It makes no difference in the basic mathematics, however, what letter we use for the independent variable. To simplify our notation we shall ordinarily use x as the variable, but if the physical problem calls for it, you can replace x by t.

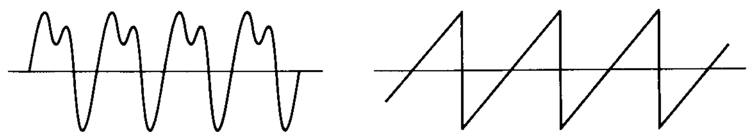


Figure 2.4

Sines and cosines are periodic functions; once you have drawn $\sin x$ from $x = 0$ to $x = 2\pi$, the rest of the graph from $x = -\infty$ to $x = +\infty$ is just a repetition over and over of the 0 to 2π graph. The number 2π is the period of $\sin x$. A periodic function need not be a simple sine or cosine, but may be any sort of complicated graph that repeats itself (Figure 2.4). The interval of repetition is the period.

Example 2. If we are describing the vibration of a seconds pendulum, the period is 2 sec (time for one complete back-and-forth oscillation). The reciprocal of the period is the *frequency*, the number of oscillations per second; for the seconds pendulum, the frequency is $\frac{1}{2}$ sec^{-1}. When radio announcers say, "operating on a frequency of 780 kilohertz," they mean that 780,000 radio waves reach you per second, or that the period of one wave is (1/780,000) sec.

By definition, the function $f(x)$ is periodic if $f(x+p) = f(x)$ for every x; the number p is the period. The period of $\sin x$ is 2π since $\sin(x+2\pi) = \sin x$; similarly, the period of $\sin 2\pi x$ is 1 since $\sin 2\pi(x+1) = \sin(2\pi x + 2\pi) = \sin 2\pi x$, and the period of $\sin(\pi x/l)$ is $2l$ since $\sin(\pi/l)(x+2l) = \sin(\pi x/l)$. In general, the period of $\sin 2\pi x/T$ is T.

PROBLEMS, SECTION 2

In Problems 1 to 6 find the amplitude, period, frequency, and velocity amplitude for the motion of a particle whose distance s from the origin is the given function.

1. $s = 3\cos 5t$

2. $s = 2\sin(4t-1)$

3. $s = \frac{1}{2}\cos(\pi t - 8)$

4. $s = 5\sin(t-\pi)$

5. $s = 2\sin 3t \cos 3t$

6. $s = 3\sin(2t+\pi/8) + 3\sin(2t-\pi/8)$

In Problems 7 to 10 you are given a complex function $z = f(t)$. In each case, show that a particle whose coordinate is (a) $x = \operatorname{Re} z$, (b) $y = \operatorname{Im} z$ is undergoing simple harmonic motion, and find the amplitude, period, frequency, and velocity amplitude of the motion.

7. $z = 5e^{it}$

8. $z = 2e^{-it/2}$

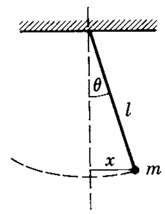
9. $z = 2e^{i\pi t}$

10. $z = -4e^{i(2t+3\pi)}$

11. The charge q on a capacitor in a simple a-c circuit varies with time according to the equation $q = 3\sin(120\pi t + \pi/4)$. Find the amplitude, period, and frequency of this oscillation. By definition, the current flowing in the circuit at time t is $I = dq/dt$. Show that I is also a sinusoidal function of t, and find its amplitude, period, and frequency.

12. Repeat Problem 11: (a) if $q = \operatorname{Re} 4e^{30i\pi t}$; (b) if $q = \operatorname{Im} 4e^{30i\pi t}$.

13. A simple pendulum consists of a point mass m suspended by a (weightless) cord or rod of length l, as shown, and swinging in a vertical plane under the action of gravity. Show that for small oscillations (small θ), both θ and x are sinusoidal functions of time, that is, the motion is simple harmonic. *Hint:* Write the differential equation $\mathbf{F} = m\mathbf{a}$ for the particle m. Use the approximation $\sin\theta = \theta$ for small θ, and show that $\theta = A\sin\omega t$ is a solution of your equation. What are A and ω?



14. The displacements x of two simple pendulums (see Problem 13) are $4\sin(\pi t/3)$ and $3\sin(\pi t/4)$. They start together at $x = 0$. How long will it be before they are together again at $x = 0$? *Hint:* Sketch or computer plot the graphs.

15. As in Problem 14, the displacements x of two simple pendulums are $x = -2\cos(t/2)$ and $3\sin(t/3)$. They are *not* together at $t = 0$; plot graphs to see when they are first together.

16. As in Problem 14, let the displacements be $y_1 = 3\sin(t/\sqrt{2})$ and $y_2 = \sin t$. The pendulums start together at $t = 0$. Make computer plots to estimate when they will be together again and then, by computer, solve the equation $y_1 = y_2$ for the root near your estimate.

17. Show that equation (2.10) for a wave can be written in all these forms:

$$y = A\sin\frac{2\pi}{\lambda}(x - vt) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$
$$= A\sin\omega\left(\frac{x}{v} - t\right) = A\sin\left(\frac{2\pi x}{\lambda} - 2\pi f t\right) = A\sin\frac{2\pi}{T}\left(\frac{x}{v} - t\right).$$

Here λ is the wavelength, f is the frequency, v is the wave velocity, T is the period, and $\omega = 2\pi f$ is called the *angular frequency. Hint:* Show that $v = \lambda f$.

In Problems 18 to 20, find the amplitude, period, frequency, wave velocity, and wavelength of the given wave. By computer, plot on the same axes, y as a function of x for the given values of t, and label each graph with its value of t. Similarly, plot on the same axes, y as a function of t for the given values of x, and label each curve with its value of x.

18. $y = 2\sin\frac{2}{3}\pi(x - 3t)$; $\quad t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$; $\quad x = 0, 1, 2, 3.$

19. $y = \cos 2\pi(x - \frac{1}{4}t)$; $\quad t = 0, 1, 2, 3$; $\quad x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}.$

20. $y = 3\sin\pi(x - \frac{1}{2}t)$; $\quad t = 0, 1, 2, 3$; $\quad x = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.$

21. Write the equation for a sinusoidal wave of wavelength 4, amplitude 20, and velocity 6. (See Problem 17.) Make computer plots of y as a function of t for $x = 0, 1, 2, 3$, and of y as a function of x for $t = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. If this wave represents the shape of a long rope which is being shaken back and forth at one end, find the velocity $\partial y/\partial t$ of particles of the rope as a function of x and t. (Note that this velocity has nothing to do with the wave velocity v, which is the rate at which crests of the wave move forward.)

22. Do Problem 21 for a wave of amplitude 4, period 6, and wavelength 3. Make computer plots of y as a function of x when $t = 0, 1, 2, 3$, and of y as a function of t when $x = \frac{1}{2}, 1, \frac{3}{2}, 2$.

23. Write an equation for a sinusoidal sound wave of amplitude 1 and frequency 440 hertz (1 hertz means 1 cycle per second). (Take the velocity of sound to be 350 m/sec.)

24. The velocity of sound in sea water is about 1530 m/sec. Write an equation for a sinusoidal sound wave in the ocean, of amplitude 1 and frequency 1000 hertz.

25. Write an equation for a sinusoidal radio wave of amplitude 10 and frequency 600 kilohertz. *Hint:* The velocity of a radio wave is the velocity of light, $c = 3{\cdot}10^8$ m/sec.

3. APPLICATIONS OF FOURIER SERIES

We have said that the vibration of a tuning fork is an example of simple harmonic motion. When we hear the musical note produced, we say that a sound wave has passed through the air from the tuning fork to our ears. As the tuning fork vibrates it pushes against the air molecules, creating alternately regions of high and low

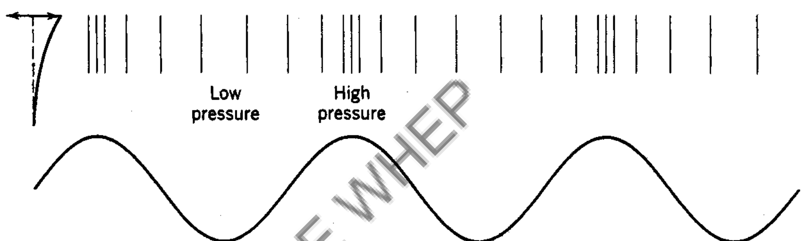


Figure 3.1

pressure (Figure 3.1). If we measure the pressure as a function of x and t from the tuning fork to us, we find that the pressure is of the form of (2.10); if we measure the pressure where we are as a function of t as the wave passes, we find that the pressure is a periodic function of t. The sound wave is a pure sine wave of a definite frequency (in the language of music, a pure tone). Now suppose that several pure tones are heard simultaneously. In the resultant sound wave, the pressure will not be a single sine function but a sum of several sine functions. If you strike a piano key you do not get a sound wave of just one frequency. Instead, you get a fundamental accompanied by a number of overtones (harmonics) of frequencies $2, 3, 4, \cdots,$ times the frequency of the fundamental. Higher frequencies mean shorter periods. If $\sin\omega t$ and $\cos\omega t$ correspond to the fundamental frequency, then $\sin n\omega t$ and $\cos n\omega t$ correspond to the higher harmonics. The combination of the fundamental and the harmonics is a complicated periodic function with the period of the fundamental (Problem 5). Given the complicated function, we could ask how to write it as a sum of terms corresponding to the various harmonics. In general it might require all the harmonics, that is, an infinite series of terms. This is called a Fourier series. Expanding a function in a Fourier series then amounts to breaking it down into its various harmonics. In fact, this process is sometimes called harmonic analysis.

There are applications to other fields besides sound. Radio waves, visible light, and x rays are all examples of a kind of wave motion in which the "waves" correspond to varying electric and magnetic fields. Exactly the same mathematical equations apply as for water waves and sound waves. We could then ask what light frequencies (these correspond to the color) are in a given light beam and in what proportions. To find the answer, we would expand the given function describing the wave in a Fourier series.

You have probably seen a sine curve used to represent an alternating current (a-c) or voltage in electricity. This is a periodic function, but so are the functions shown in Figure 3.2. Any of these and many others might represent signals (voltages or currents) which are to be applied to an electric circuit. Then we could ask

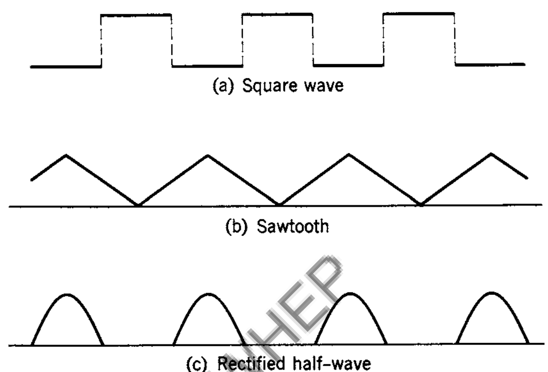


Figure 3.2

what a-c frequencies (harmonics) make up a given signal and in what proportions. When an electric signal is passed through a network (say a radio), some of the harmonics may be lost. If most of the important ones get through with their relative intensities preserved, we say that the radio possesses "high fidelity." To find out which harmonics are the important ones in a given signal, we expand it in a Fourier series. The terms of the series with large coefficients then represent the important harmonics (frequencies).

Since sines and cosines are themselves periodic, it seems rather natural to use series of them, rather than power series, to represent periodic functions. There is another important reason. The coefficients of a power series are obtained, you will recall (Chapter 1, Section 12), by finding successive derivatives of the function being expanded; consequently, only continuous functions with derivatives of all orders can be expanded in power series. Many periodic functions in practice are not continuous or not differentiable (Figure 3.2). Fortunately, Fourier series (unlike power series) can represent discontinuous functions or functions whose graphs have corners. On the other hand, Fourier series do not usually converge as rapidly as power series and much more care is needed in manipulating them. For example, a power series can be differentiated term by term (Chapter 1, Section 11), but differentiating a Fourier series term by term sometimes produces a series which doesn't converge. (See end of Section 9.)

Our problem then is to expand a given periodic function in a series of sines and cosines. We shall take this up in Section 5 after doing some preliminary work.

PROBLEMS, SECTION 3

For each of the following combinations of a fundamental musical tone and some of its overtones, make a computer plot of individual harmonics (all on the same axes) and then a plot of the sum. Note that the sum has the period of the fundamental (Problem 5).

1. $\sin t - \frac{1}{9}\sin 3t$

2. $2\cos t + \cos 2t$

3. $\sin \pi t + \sin 2\pi t + \frac{1}{3}\sin 3\pi t$

4. $\cos 2\pi t + \cos 4\pi t + \frac{1}{2}\cos 6\pi t$

5. Using the definition (end of Section 2) of a periodic function, show that a sum of terms corresponding to a fundamental musical tone and its overtones has the period of the fundamental.

In Problems 6 and 7, use a trigonometry formula to write the two terms as a single harmonic. Find the period and amplitude. Compare computer plots of your result and the given problem.

6. $\sin 2x + \sin 2(x + \pi/3)$

7. $\cos \pi x - \cos \pi(x - 1/2)$

8. A periodic modulated (AM) radio signal has the form

$$y = (A + B\sin 2\pi f t)\sin 2\pi f_c\left(t - \frac{x}{v}\right).$$

The factor $\sin 2\pi f_c(t - x/v)$ is called the carrier wave; it has a very high frequency (called radio frequency; f_c is of the order of 10^6 cycles per second). The amplitude of the carrier wave is $(A + B\sin 2\pi f t)$. This amplitude varies with time—hence the term "amplitude modulation"—with the much smaller frequency of the sound being transmitted (called audio frequency; f is of the order of 10^2 cycles per second). In order to see the general appearance of such a wave, use the following simple but unrealistic data to sketch a graph of y as a function of t for $x = 0$ over two periods of the *amplitude* function: $A = 3$, $B = 1$, $f = 1$, $f_c = 20$. Using trigonometric formulas, show that y can be written as a sum of three waves of frequencies f_c, $f_c + f$, and $f_c - f$; the first of these is the carrier wave and the other two are called side bands.

4. AVERAGE VALUE OF A FUNCTION

The concept of the average value of a function is often useful. You know how to find the average of a set of numbers: you add them and divide by the number of numbers.

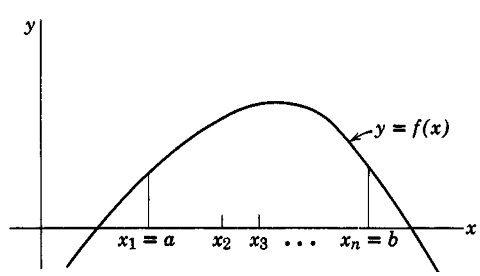


Figure 4.1

This process suggests that we ought to get an approximation to the average value of a function $f(x)$ on the interval (a, b) by averaging a number of values of $f(x)$ (Figure 4.1):

(4.1) Average of $f(x)$ on (a, b) is approximately equal to
$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

This should become a better approximation as n increases. Let the points $x_1, x_2, \cdots$ be Δx apart. Multiply the numerator and the denominator of the approximate average by Δx. Then (4.1) becomes:

(4.2) Average of $f(x)$ on (a, b) is approximately equal to
$$\frac{[f(x_1) + \cdots + f(x_n)]\Delta x}{n\,\Delta x}.$$

Now $n\,\Delta x = b - a$, the length of the interval over which we are averaging, no matter what n and Δx are. If we let $n \to \infty$ and $\Delta x \to 0$, the numerator approaches $\int_a^b f(x)\,dx$, and we have

(4.3) $$\text{Average of } f(x) \text{ on } (a, b) = \frac{\int_a^b f(x)\,dx}{b - a}.$$

In applications, it may happen that the average value of a given function is zero.

▶ **Example 1.** The average of $\sin x$ over any number of periods is zero. The average value of the velocity of a simple harmonic oscillator over any number of vibrations is zero. In such cases the average of the square of the function may be of interest.

▶ **Example 2.** If the alternating electric current flowing through a wire is described by a sine function, the square root of the average of the sine squared is known as the root-mean-square or effective value of the current, and is what you would measure with an a-c ammeter. In the example of the simple harmonic oscillator, the average kinetic energy (average of $\frac{1}{2}mv^2$) is $\frac{1}{2}m$ times the average of v^2.

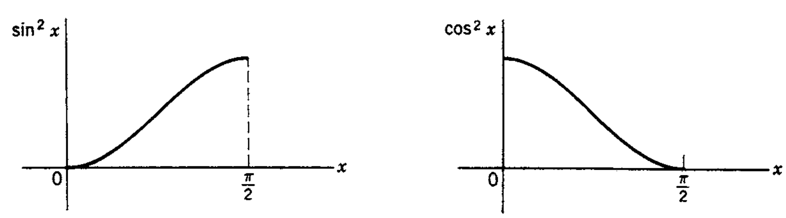


Figure 4.2

Now you can, of course, find the average value of $\sin^2 x$ over a period (say $-\pi$ to π) by evaluating the integral in (4.3). There is an easier way. Look at the graphs of $\cos^2 x$ and $\sin^2 x$ (Figure 4.2). You can probably convince yourself that the area under them is the same for any quarter-period from 0 to $\pi/2$, $\pi/2$ to π, etc. (Also see Problems 2 and 13.) Then

$$\int_{-\pi}^{\pi} \sin^2 x\,dx = \int_{-\pi}^{\pi} \cos^2 x\,dx. \tag{4.4}$$

Similarly (for integral $n \neq 0$),

$$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \cos^2 nx\,dx. \tag{4.5}$$

But since $\sin^2 nx + \cos^2 nx = 1$,

$$\int_{-\pi}^{\pi} (\sin^2 nx + \cos^2 nx)\,dx = \int_{-\pi}^{\pi} dx = 2\pi. \tag{4.6}$$

Using (4.5), we get

$$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \cos^2 nx\,dx = \pi. \tag{4.7}$$

Then using (4.3) we see that:

$$\begin{aligned} &\text{The average value (over a period) of } \sin^2 nx \\ &= \text{the average value (over a period) of } \cos^2 nx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 nx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 nx\,dx = \frac{\pi}{2\pi} = \frac{1}{2}. \end{aligned} \tag{4.8}$$

We can say all this more simply in words. By (4.5), the average value of $\sin^2 nx$ equals the average value of $\cos^2 nx$. The average value of $\sin^2 nx + \cos^2 nx = 1$ is 1. Therefore the average value of $\sin^2 nx$ or of $\cos^2 nx$ is $\frac{1}{2}$. (In each case the average value is taken over one or more periods.)

PROBLEMS, SECTION 4

1. Show that if $f(x)$ has period p, the average value of f is the same over any interval of length p. *Hint:* Write $\int_a^{a+p} f(x)\,dx$ as the sum of two integrals (a to p, and p to $a+p$) and make the change of variable $x = t + p$ in the second integral.

2. (a) Prove that $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx$ by making the change of variable $x = \frac{1}{2}\pi - t$ in one of the integrals.

(b) Use the same method to prove that the averages of $\sin^2(n\pi x/l)$ and $\cos^2(n\pi x/l)$ are the same over a period.

In Problems 3 to 12, find the average value of the function on the given interval. Use equation (4.8) if it applies. If an average value is zero, you may be able to decide this from a quick sketch which shows you that the areas above and below the x axis are the same.

3. $\sin x + 2\sin 2x + 3\sin 3x$ on $(0, 2\pi)$

4. $1 - e^{-x}$ on $(0, 1)$

5. $\cos^2 \dfrac{x}{2}$ on $\left(0, \dfrac{\pi}{2}\right)$

6. $\sin x$ on $(0, \pi)$

7. $x - \cos^2 6x$ on $\left(0, \dfrac{\pi}{6}\right)$

8. $\sin 2x$ on $\left(\dfrac{\pi}{6}, \dfrac{7\pi}{6}\right)$

9. $\sin^2 3x$ on $(0, 4\pi)$

10. $\cos x$ on $(0, 3\pi)$

11. $\sin x + \sin^2 x$ on $(0, 2\pi)$

12. $\cos^2 \dfrac{7\pi x}{2}$ on $\left(0, \dfrac{8}{7}\right)$

13. Using (4.3) and equations similar to (4.5) to (4.7), show that

$$\int_a^b \sin^2 kx\,dx = \int_a^b \cos^2 kx\,dx = \frac{1}{2}(b-a)$$

if $k(b-a)$ is an integral multiple of π, or if kb and ka are both integral multiples of $\pi/2$.

Use the results of Problem 13 to evaluate the following integrals without calculation.

14. (a) $\displaystyle\int_0^{4\pi/3} \sin^2\left(\frac{3x}{2}\right) dx$ (b) $\displaystyle\int_{-\pi/2}^{3\pi/2} \cos^2\left(\frac{x}{2}\right) dx$

15. (a) $\displaystyle\int_{-1/4}^{11/4} \cos^2 \pi x\,dx$ (b) $\displaystyle\int_{-1}^{2} \sin^2\left(\frac{\pi x}{3}\right) dx$

16. (a) $\displaystyle\int_0^{2\pi/\omega} \sin^2 \omega t\,dt$ (b) $\displaystyle\int_0^{2} \cos^2 2\pi t\,dt$

▶ 5. FOURIER COEFFICIENTS

We want to expand a given periodic function in a series of sines and cosines. To simplify our formulas at first, we start with functions of period 2π; that is, we shall expand periodic functions of period 2π in terms of the functions $\sin nx$ and $\cos nx$. (Later we shall see how we can change the formulas to fit a different period—see Section 8.) The functions $\sin x$ and $\cos x$ have period 2π; so do $\sin nx$ and $\cos nx$ for any integral n since $\sin n(x + 2\pi) = \sin(nx + 2n\pi) = \sin nx$. (It is true that $\sin nx$ and $\cos nx$ also have shorter periods, namely $2\pi/n$, but the fact that they repeat every 2π is what we are interested in here, for this makes them reasonable functions to use in an expansion of a function of period 2π.) Then, given a function $f(x)$ of period 2π, we write

$$\begin{aligned} f(x) = \frac{1}{2}a_0 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \\ &+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots, \end{aligned} \tag{5.1}$$

and derive formulas for the coefficients a_n and b_n. (The reason for writing $\frac{1}{2}a_0$ as the constant term will be clear later—it makes the formulas for the coefficients simpler to remember—but you must not forget the $\frac{1}{2}$ in the series!)

In finding formulas for a_n and b_n in (5.1) we need the following integrals:

(5.2) The average value of $\sin mx \cos nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

The average value of $\sin mx \sin nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n \neq 0, \\ 0, & m = n = 0. \end{cases}$$

The average value of $\cos mx \cos nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n \neq 0, \\ 1, & m = n = 0. \end{cases}$$

We have already shown that the average values of $\sin^2 nx$ and $\cos^2 nx$ are $\frac{1}{2}$. The last integral in (5.2) is the average value of 1 which is 1. To show that the other average values in (5.2) are zero (unless $m = n \neq 0$), we could use the trigonometry formulas for products like $\sin\theta\cos\phi$ and then integrate. An easier way is to use the formulas for the sines and cosines in terms of complex exponentials. [See (7.1) or Chapter 2, Section 11.] We shall show this method for one integral

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = \int_{-\pi}^{\pi} \frac{e^{imx} - e^{-imx}}{2i} \cdot \frac{e^{inx} + e^{-inx}}{2} dx. \tag{5.3}$$

We can see the result without actually multiplying these out. All terms in the product are of the form e^{ikx}, where k is an integer $\neq 0$ (except for the cross-product terms when $n = m$, and these cancel). We can show that the integral of each such term is zero:

$$\int_{-\pi}^{\pi} e^{ikx}\, dx = \left.\frac{e^{ikx}}{ik}\right|_{-\pi}^{\pi} = \frac{e^{ik\pi} - e^{-ik\pi}}{ik} = 0 \tag{5.4}$$

because $e^{ik\pi} = e^{-ik\pi} = \cos k\pi$ (since $\sin k\pi = 0$). The other integrals in (5.2) may be evaluated similarly (Problem 12).

We now show how to find a_n and b_n in (5.1). To find a_0, we find the average value on $(-\pi, \pi)$ of each term of (5.1).

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx &= \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} dx + a_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos x\, dx \\ &\quad + a_2 \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos 2x\, dx + \cdots + b_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x \, dx + \cdots . \end{aligned} \tag{5.5}$$

By (5.2), all the integrals on the right-hand side of (5.5) are zero except the first, because they are integrals of $\sin mx \cos nx$ or of $\cos mx \cos nx$ with $n = 0$ and $m \neq 0$ (that is, $m \neq n$). Then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} dx = \frac{a_0}{2},$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx. \tag{5.6}$$

Given $f(x)$ to be expanded in a Fourier series, we can now evaluate a_0 by calculating the integral in (5.6).

To find a_1, multiply both sides of (5.1) by $\cos x$ and again find the average value of each term:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\cos x\,dx = \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,dx + a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx \\ + a_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2x\cos x\,dx + \cdots \\ + b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x\,\cos x\,dx + \cdots .\end{aligned} \tag{5.7}$$

This time, by (5.2), all terms on the right are zero except the a_1 term and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\cos x\,dx = a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{1}{2}a_1.$$

Solving for a_1, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx.$$

The method should be clear by now, so we shall next find a general formula for a_n. Multiply both sides of (5.1) by $\cos nx$ and find the average value of each term:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos nx\,dx + a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,\cos nx\,dx \\ + a_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2x\,\cos nx\,dx + \cdots \\ + b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x\,\cos nx\,dx + \cdots .\end{aligned} \tag{5.8}$$

By (5.2), all terms on the right are zero except the a_n term and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 nx\,dx = \frac{1}{2}a_n.$$

Solving for a_n, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx. \tag{5.9}$$

Notice that this includes the $n = 0$ formula, but only because we called the constant term $\frac{1}{2}a_0$.

To obtain a formula for b_n, we multiply both sides of (5.1) by $\sin nx$ and take average values just as we did in deriving (5.9). We find (Problem 13)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx. \tag{5.10}$$

The formulas (5.9) and (5.10) will be used repeatedly in problems and should be memorized.

Example 1. Expand in a Fourier series the function $f(x)$ sketched in Figure 5.1. This function might represent, for example, a periodic voltage pulse. The terms of our Fourier series would then correspond to the different a-c frequencies which are combined in this "square wave" voltage, and the magnitude of the Fourier coefficients would indicate the relative importance of the various frequencies.

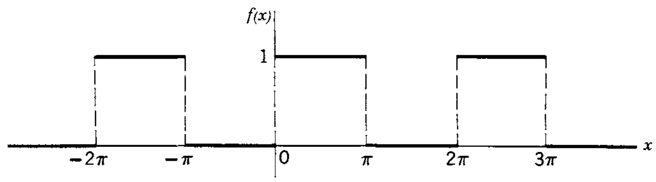


Figure 5.1

Note that $f(x)$ is a function of period 2π. Often in problems you will be given $f(x)$ for only one period; you should always sketch several periods so that you see clearly the periodic function you are expanding. For example, in this problem, instead of a sketch, you might have been given

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases} \tag{5.11}$$

It is then understood that $f(x)$ is to be continued periodically with period 2π outside the interval $(-\pi, \pi)$.

We use equations (5.9) and (5.10) to find a_n and b_n:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,dx + \int_0^{\pi} 1\cdot\cos nx\,dx\right] \\ &= \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = \begin{cases} \dfrac{1}{\pi}\cdot\dfrac{1}{n}\sin nx\Big|_0^{\pi} = 0 & \text{for } n \neq 0, \\ \dfrac{1}{\pi}\cdot\pi = 1 & \text{for } n = 0. \end{cases} \end{aligned}$$

Thus $a_0 = 1$, and all other $a_n = 0$.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin nx\,dx + \int_0^{\pi} 1\cdot\sin nx\,dx\right] \\ &= \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = -\frac{1}{n\pi}[(-1)^n - 1] \\ &= \begin{cases} 0 & \text{for even } n, \\ \dfrac{2}{n\pi} & \text{for odd } n. \end{cases} \end{aligned}$$

Putting these values for the coefficients into (5.1), we have

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \tag{5.12}$$

▶ **Example 2.** We can now find the Fourier series for some other functions without more evaluation of coefficients. For example, consider

$$(5.13)\qquad g(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$$

Sketch this and verify that $g(x) = 2f(x) - 1$, where $f(x)$ is the function in Example 1. Then from (5.12), the Fourier series for $g(x)$ is

$$(5.14)\qquad g(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Similarly, verify that $h(x) = f(x + \pi/2)$ is Fig. 5.1 shifted $\pi/2$ to the left (sketch it), and its Fourier series is (replace x in (5.12) by $x + \pi/2$)

$$h(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \cdots\right)$$

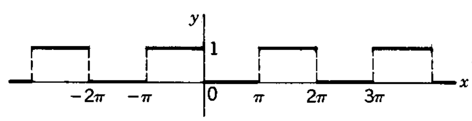
since $\sin(x + \pi/2) = \cos x$, $\sin(x + 3\pi/2) = -\cos 3x$, etc.

▶ PROBLEMS, SECTION 5

In each of the following problems you are given a function on the interval $-\pi < x < \pi$. Sketch several periods of the corresponding periodic function of period 2π. Expand the periodic function in a sine-cosine Fourier series.

1. $f(x) = \begin{cases} 1, & -\pi < x < 0, \\ 0, & 0 < x < \pi. \end{cases}$

In this case the sketch is:



Your answer for the series is: $f(x) = \dfrac{1}{2} - \dfrac{2}{\pi}\left(\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5}\cdots\right)$.

Can you use the ideas of Example 2 to find this result without computation?

2. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \dfrac{\pi}{2}, \\ 0, & \dfrac{\pi}{2} < x < \pi. \end{cases}$

Answer: $$f(x) = \frac{1}{4} + \frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5}\cdots\right) + \frac{1}{\pi}\left(\frac{\sin x}{1} + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\cdots\right).$$

3. $f(x) = \begin{cases} 0, & -\pi < x < \dfrac{\pi}{2}, \\ 1, & \dfrac{\pi}{2} < x < \pi. \end{cases}$

Answer: $$f(x) = \frac{1}{4} - \frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5}\cdots\right) + \frac{1}{\pi}\left(\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6}\cdots\right).$$

4. $f(x) = \begin{cases} -1, & -\pi < x < \dfrac{\pi}{2}, \\ 1, & \dfrac{\pi}{2} < x < \pi. \end{cases}$

Could you use Problem 3 to solve Problem 4 without computation?

5. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ -1, & 0 < x < \dfrac{\pi}{2}, \\ 1, & \dfrac{\pi}{2} < x < \pi. \end{cases}$

6. $f(x) = \begin{cases} 1, & -\pi < x < -\dfrac{\pi}{2}, \quad \text{and} \quad 0 < x < \dfrac{\pi}{2}; \\ 0, & -\dfrac{\pi}{2} < x < 0, \quad \text{and} \quad \dfrac{\pi}{2} < x < \pi. \end{cases}$

7. $f(x) = \begin{cases} 0, & -\pi < x < 0; \\ x, & 0 < x < \pi. \end{cases}$

Answer: $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left(\cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \cdots\right) + \left(\sin x - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \cdots\right).$

8. $f(x) = 1 + x, \quad -\pi < x < \pi.$

Answer: $f(x) = 1 + 2\left(\sin x - \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x - \dfrac{1}{4}\sin 4x + \cdots\right).$

9. $f(x) = \begin{cases} -x, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$

Answer: $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots\right).$

10. $f(x) = \begin{cases} \pi + x, & -\pi < x < 0, \\ \pi - x, & 0 < x < \pi. \end{cases}$

11. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases}$

Answer: $f(x) = \dfrac{1}{\pi} + \dfrac{1}{2}\sin x - \dfrac{2}{\pi}\left(\dfrac{\cos 2x}{2^2 - 1} + \dfrac{\cos 4x}{4^2 - 1} + \dfrac{\cos 6x}{6^2 - 1} + \cdots\right).$

12. Show that in (5.2) the average values of $\sin mx \sin nx$ and of $\cos mx \cos nx$, $m \neq n$, are zero (over a period), by using the complex exponential forms for the sines and cosines as in (5.3).

13. Write out the details of the derivation of equation (5.10).

6. DIRICHLET CONDITIONS

Now we have a series, but there are still some questions that we ought to get answered. Does it converge, and if so, does it converge to the values of $f(x)$? You will find, if you try, that for most values of x the series in (5.12) does not respond to any of the tests for convergence that we discussed in Chapter 1. What is the sum of the series at $x = 0$ where $f(x)$ jumps from 0 to 1? You can see from the series (5.12) that the sum at $x = 0$ is $\frac{1}{2}$, but what does this have to do with $f(x)$?

These questions would not be easy for us to answer for ourselves, but they are answered for us for most practical purposes by the *theorem of Dirichlet*:

> If $f(x)$ is periodic of period 2π, and if between $-\pi$ and π it is single-valued, has a finite number of maximum and minimum values, and a finite number of discontinuities, and if $\int_{-\pi}^{\pi} |f(x)|\, dx$ is finite, then the Fourier series (5.1) [with coefficients given by (5.9) and (5.10)] converges to $f(x)$ at all the points where $f(x)$ is continuous; at jumps the Fourier series converges to the midpoint of the jump. (This includes jumps that occur at $\pm\pi$ for the periodic function.)

To see what all this means, we shall consider some special functions. We have already discussed what a periodic function means. A function $f(x)$ is single-valued if there is just one value of $f(x)$ for each x. For example, if $x^2 + y^2 = 1$, y is not a single-valued function of x, unless we select just $y = +\sqrt{1-x^2}$ or just $y = -\sqrt{1-x^2}$. An example of a function with an infinite number of maxima and minima is $\sin(1/x)$, which oscillates infinitely many times as $x \to 0$. If we imagine a function constructed from $\sin(1/x)$ by making $f(x) = 1$ for every x for which $\sin(1/x) > 0$, and $f(x) = -1$ for every x for which $\sin(1/x) < 0$, this function would have an infinite number of discontinuities. Now most functions in applied work do not behave like these, but will satisfy the Dirichlet conditions.

Finally, if $y = 1/x$, we find

$$\int_{-\pi}^{\pi} \left|\frac{1}{x}\right| dx = 2\int_{0}^{\pi} \frac{1}{x}\, dx = 2\ln x\Big|_0^1 = \infty,$$

so the function $1/x$ is ruled out by the Dirichlet conditions. On the other hand, if $f(x) = 1/\sqrt{|x|}$, then

$$\int_{-\pi}^{\pi} \frac{1}{\sqrt{|x|}}\, dx = 2\int_{0}^{\pi} \frac{dx}{\sqrt{x}} = 4\sqrt{x}\Big|_0^{\pi} = 4\sqrt{\pi},$$

so the periodic function which is $1/\sqrt{|x|}$ between $-\pi$ and π can be expanded in a Fourier series. In most problems it is not necessary to find the value of $\int_{-\pi}^{\pi} |f(x)|\, dx$; let us see why. If $f(x)$ is bounded (that is, all its values lie between $\pm M$ for some positive constant M), then

$$\int_{-\pi}^{\pi} |f(x)| dx \le \int_{-\pi}^{\pi} M\, dx = M \cdot 2\pi$$

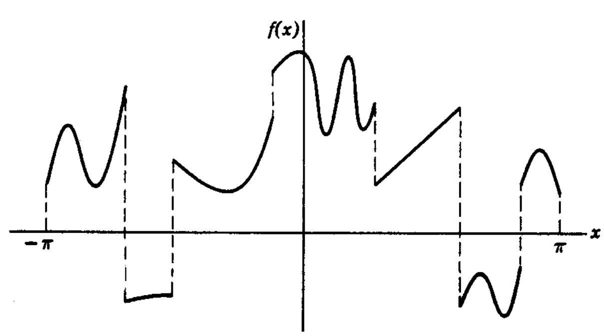


Figure 6.1

and so is finite. Thus you can simply verify that the function you are considering is bounded (if it is) instead of evaluating the integral. Figure 6.1 is an (exaggerated!) example of a function which satisfies the Dirichlet conditions on $(-\pi, \pi)$.

We see, then, that rather than testing Fourier series for convergence as we did power series, we instead check the given function; if it satisfies the Dirichlet conditions we are then sure that the Fourier series, when we get it, will converge to the function at points of continuity and to the midpoint of a jump. For example, consider the function $f(x)$ in Figure 5.1. Between $-\pi$ and π the given $f(x)$ is single-valued (one value for each x), bounded (between $+1$ and 0), has a finite number of maximum and minimum values (one of each), and a finite number of discontinuities (at $-\pi$, 0, and π), and therefore satisfies the Dirichlet conditions. Dirichlet's theorem then assures us that the series (5.12) actually converges to the function $f(x)$ in Figure 5.1 at all points except $x = n\pi$ where it converges to $1/2$.

In Chapter 3, Sections 10 and 14, we defined a *basis* for ordinary 3-dimensional space as a set of linearly independent vectors (like $\mathbf{i}, \mathbf{j}, \mathbf{k}$) in terms of which we could write every vector in the space. We then extended this idea to an n-dimensional space and to a space in which the basis vectors were functions. By analogy, we say here that the functions $\sin nx$, $\cos nx$ are a set of basis functions for the (infinite dimensional) space of all functions (satisfying Dirichlet conditions) defined on $(-\pi, \pi)$ or any 2π interval. (Also see "completeness relation" in Section 11. And for more examples of such sets of basis functions, see Chapters 12 and 13.)

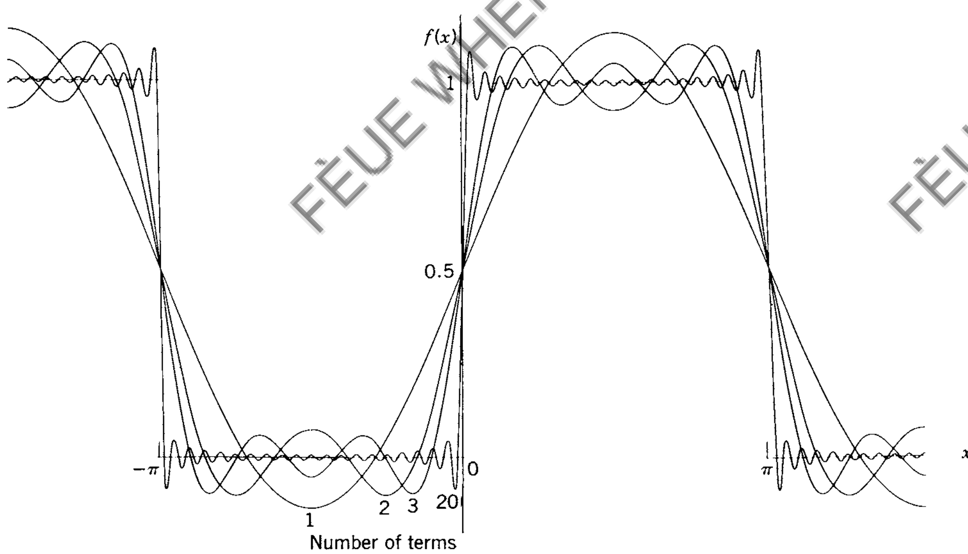


Figure 6.2

It is interesting to see a graph of the sum of a large number of terms of a Fourier series. Figure 6.2 shows several different partial sums of the series in (5.12) for the function in Figure 5.1. We can see that the sum of many terms of the series closely approximates the function away from the jumps and goes through the midpoint of the jump. The "overshoot" on either side of a jump bears comment. It does not disappear as we add more and more terms of the series. It simply becomes a narrower and narrower spike of height equal to about 9% of the jump. This fact is called the *Gibbs phenomenon*.

We ought to say here that the converse of Dirichlet's theorem is not true—if a function fails to satisfy the Dirichlet conditions, it still *may* be expandable in a Fourier series. The periodic function which is $\sin(1/x)$ on $(-\pi, \pi)$ is an example of such a function. However, such functions are rarely met with in practice.

▶ **Example.** Fourier series can be useful in summing numerical series. Look at Problem 5.2 (sketch it). From Dirichlet's theorem, we see that the Fourier series converges to 1/2 at $x = 0$. Let $x = 0$ in the Fourier series to get

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

since $\sin 0 = 0$ and $\cos 0 = 1$. Thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

▶ PROBLEMS, SECTION 6

1 to **11**. For each of the periodic functions in Problems 5.1 to 5.11, use Dirichlet's theorem to find the value to which the Fourier series converges at $x = 0, \pm\pi/2, \pm\pi, \pm 2\pi$.

12. Use a computer to produce graphs like Fig. 6.2 showing Fourier series approximations to the functions in Problems 5.1 to 5.3, and 5.7 to 5.11. You might like to set up a computer animation showing the Gibbs phenomenon as the number of terms increases.

13. Repeat the example using the same Fourier series but at $x = \pi/2$.

14. Use Problem 5.7 to show that $\sum_{\text{odd } n} 1/n^2 = \pi^2/8$. Try $x = 0$, and $x = \pi$. What do you find at $x = \pi/2$?

15. Use Problem 5.11 to show that $\dfrac{1}{2^2 - 1} + \dfrac{1}{4^2 - 1} + \dfrac{1}{6^2 - 1} + \cdots = \dfrac{1}{2}$.

▶ 7. COMPLEX FORM OF FOURIER SERIES

Recall that real sines and cosines can be expressed in terms of complex exponentials by the formulas [Chapter 2, (11.3)]

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}, \qquad \cos nx = \frac{e^{inx} + e^{-inx}}{2}. \tag{7.1}$$

If we substitute equations (7.1) into a Fourier series like (5.12), we get a series of terms of the forms e^{inx} and e^{-inx}. This is the complex form of a Fourier series. We can also find the complex form directly; this is often easier than finding the sine-cosine form. We can then, if we like, work back the other way and [using Euler's formula, Chapter 2, (9.3)] get the sine-cosine form from the exponential form.

We want to see how to find the coefficients in the complex form directly. We assume a series

$$\boxed{\begin{aligned} f(x) &= c_0 + c_1 e^{ix} + c_{-1}e^{-ix} + c_2 e^{2ix} + c_{-2}e^{-2ix} + \cdots \\ &= \sum_{n=-\infty}^{n=+\infty} c_n e^{inx} \end{aligned}} \tag{7.2}$$

and try to find the c_n's. From (5.4) we know that the average value of e^{ikx} on $(-\pi, \pi)$ is zero when k is an integer not equal to zero. To find c_0, we find the average values of the terms in (7.2):

$$\text{(7.3)} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = c_0 \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} dx + \begin{cases}\text{average values of terms of the}\\ \text{form } e^{ikx} \text{ with } k \text{ an integer} \neq 0\end{cases}$$

$$= c_0 + 0,$$

$$\text{(7.4)} \qquad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

To find c_n, we multiply (7.2) by e^{-inx} and again find the average value of each term. Note the minus sign in the exponent. In finding a_n, the coefficient of $\cos nx$ in equation (5.1), we multiplied by $\cos nx$; but here in finding the coefficient c_n of e^{inx}, we multiply by the complex conjugate e^{-inx}.

$$\text{(7.5)} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = c_0\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\,dx + c_1\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{ix}\,dx$$
$$+ c_{-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{-ix}\,dx + \cdots.$$

The terms on the right are the average values of exponentials e^{ikx}, where the k values are integers. Therefore all these terms are zero except the one where $k = 0$; this is the term containing c_n. We then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = c_n \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{inx}\,dx = c_n \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} dx = c_n,$$

$$\boxed{\text{(7.6)} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.}$$

Note that this formula contains the one for c_0 (no $\frac{1}{2}$ to worry about here!). Also, since (7.6) is valid for negative as well as positive n, you have only one formula to memorize here! You can easily show that for *real* $f(x)$, $c_{-n} = \bar{c}_n$ (Problem 12).

Example. Let us expand the same $f(x)$ we did before, namely (5.11). We have from (7.6)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}\cdot 0\cdot dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\cdot 1\cdot dx$$
$$\text{(7.7)} \qquad = \frac{1}{2\pi}\frac{e^{-inx}}{-in}\bigg|_0^{\pi} = \frac{1}{-2\pi in}(e^{-in\pi} - 1) = \begin{cases}\frac{1}{\pi in}, & n \text{ odd},\\ 0, & n \text{ even} \neq 0,\end{cases}$$
$$c_0 = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2}.$$

Then

$$(7.8)\qquad f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{1}{2} + \frac{1}{i\pi}\left(\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \cdots\right) + \frac{1}{i\pi}\left(\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \frac{e^{-5ix}}{-5} + \cdots\right).$$

It is interesting to verify that this is the same as the sine-cosine series we had before. We *could* use Euler's formula for each exponential, but it is easier to collect terms like this:

$$(7.9)\qquad \begin{aligned} f(x) &= \frac{1}{2} + \frac{2}{\pi}\left(\frac{e^{ix} - e^{-ix}}{2i} + \frac{1}{3}\frac{e^{3ix} - e^{-3ix}}{2i} + \cdots\right) \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \cdots\right) \end{aligned}$$

which is the same as (5.12).

▶ PROBLEMS, SECTION 7

1 to **11.** Expand the same functions as in Problems 5.1 to 5.11 in Fourier series of complex exponentials e^{inx} on the interval $(-\pi, \pi)$ and verify in each case that the answer is equivalent to the one found in Section 5.

12. Show that if a real $f(x)$ is expanded in a complex exponential Fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$, then $c_{-n} = \bar{c}_n$, where $\bar{c}_n$ means the complex conjugate of c_n.

13. If $f(x) = \frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx + \sum_1^{\infty} b_n \sin nx = \sum_{-\infty}^{\infty} c_n e^{inx}$, use Euler's formula to find a_n and b_n in terms of c_n and c_{-n}, and to find c_n and c_{-n} in terms of a_n and b_n.

▶ 8. OTHER INTERVALS

The functions $\sin nx$ and $\cos nx$ and e^{inx} have period 2π. We have been considering $(-\pi, \pi)$ as the basic interval of length 2π. Given $f(x)$ on $(-\pi, \pi)$, we have first sketched it for this interval, and then repeated our sketch for the intervals $(\pi, 3\pi)$, $(3\pi, 5\pi)$, $(-3\pi, -\pi)$, etc. There are (infinitely) many other intervals of length 2π, any one of which could serve as the basic interval. If we are given $f(x)$ on *any* interval of length 2π, we can sketch $f(x)$ for that given basic interval and then repeat it periodically with period 2π. We then want to expand the periodic function so obtained, in a Fourier series. Recall that in evaluating the Fourier coefficients, we used average values *over a period*. The formulas for the coefficients are then unchanged (except for the limits of integration) if we use other basic intervals of length 2π. In practice, the intervals $(-\pi, \pi)$ and $(0, 2\pi)$ are the ones most frequently used. For $f(x)$ defined on $(0, 2\pi)$ and then repeated periodically, (5.9), (5.10), and (7.6) would read

$$(8.1)\qquad \begin{gathered} a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \\ c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx, \end{gathered}$$

and (5.1) and (7.2) are unchanged.

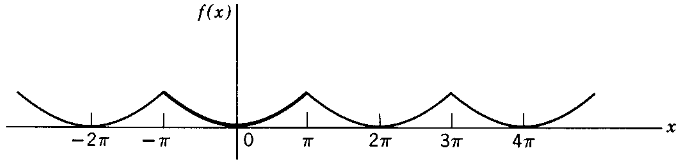


Figure 8.1

Notice how important it is to sketch a graph to see clearly what function you are talking about. For example, given $f(x) = x^2$ on $(-\pi, \pi)$, the extended function of period 2π is shown in Figure 8.1. But given $f(x) = x^2$ on $(0, 2\pi)$, the extended periodic function is different (see Figure 8.2). On the other hand, given $f(x)$ as in our example (5.11), or given $f(x) = 1$ on $(0, \pi)$, $f(x) = 0$ on $(\pi, 2\pi)$, you can easily verify by sketching that the graphs of the extended functions are identical. In this case you would get the same answer from either formulas (5.9), (5.10), and (7.6) or formulas (8.1).

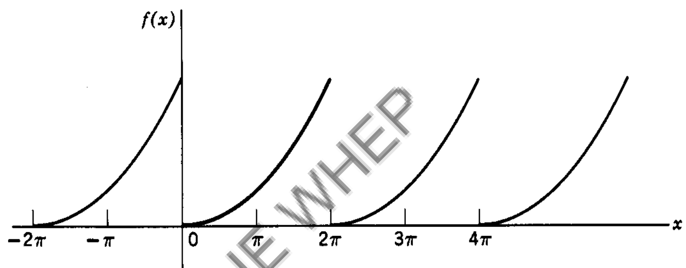


Figure 8.2

Physics problems do not always come to us with intervals of length 2π. Fortunately, it is easy now to change to other intervals. Consider intervals of length $2l$, say $(-l, l)$ or $(0, 2l)$. The function $\sin(n\pi x/l)$ has period $2l$, since

$$\sin \frac{n\pi}{l}(x + 2l) = \sin\left(\frac{n\pi x}{l} + 2n\pi\right) = \sin \frac{n\pi x}{l}.$$

Similarly, $\cos(n\pi x/l)$ and $e^{in\pi x/l}$ have period $2l$. Equations (5.1) and (7.2) are now replaced by

$$\begin{aligned} f(x) &= \frac{a_0}{2} + a_1 \cos \frac{\pi x}{l} + a_2 \cos \frac{2\pi x}{l} + \cdots \\ &\qquad + b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + \cdots \\ &= \frac{a_0}{2} + \sum_1^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right), \\ f(x) &= \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}. \end{aligned} \tag{8.2}$$

We have already found the average values *over a period* of all the functions we need to use to find a_n, b_n, and c_n here. The period is now of length $2l$, say $-l$ to l, so in finding average values of the terms we replace

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \quad \text{by} \quad \frac{1}{2l}\int_{-l}^{l}.$$

Recall that the average of the square of either the sine or the cosine over a period is $\frac{1}{2}$ and the average of $e^{in\pi x/l}\cdot e^{-in\pi x/l} = 1$ is 1. Then the formulas (5.9), (5.10), and (7.6) for the coefficients become

$$\text{(8.3)}\qquad \begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx,\\ c_n &= \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx. \end{aligned}$$

For the basic interval $(0, 2l)$ we need only change the integration limits to 0 to $2l$. The Dirichlet theorem just needs π replaced by l in order to apply here.

▶ **Example.** Given $f(x) = \begin{cases} 0, & 0 < x < l, \\ 1, & l < x < 2l. \end{cases}$

Expand $f(x)$ in an exponential Fourier series of period $2l$. [The function is given by the same formulas as (5.11) but on a different interval.]

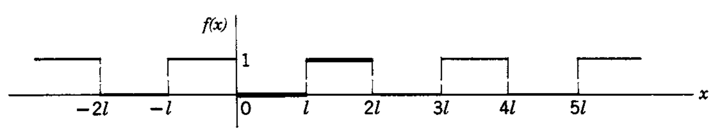


Figure 8.3

First we sketch a graph of $f(x)$ repeated with period $2l$ (Figure 8.3). By equations (8.3), we find

$$\text{(8.4)}\qquad \begin{aligned} c_n &= \frac{1}{2l}\int_0^l 0\cdot dx + \frac{1}{2l}\int_l^{2l} 1\cdot e^{-in\pi x/l}\,dx \\ &= \frac{1}{2l}\left.\frac{e^{-in\pi x/l}}{-in\pi/l}\right|_l^{2l} = \frac{1}{-2in\pi}(e^{-2in\pi} - e^{-in\pi}) \\ &= \frac{1}{-2in\pi}(1 - e^{in\pi}) = \begin{cases} 0, & \text{even } n \neq 0, \\ -\dfrac{1}{in\pi}, & \text{odd } n, \end{cases} \\ c_0 &= \frac{1}{2l}\int_l^{2l} dx = \frac{1}{2}. \end{aligned}$$

Then,

$$(8.5) \qquad f(x) = \frac{1}{2} - \frac{1}{i\pi}\left(e^{i\pi x/l} - e^{-i\pi x/l} + \frac{1}{3}e^{3i\pi x/l} - \frac{1}{3}e^{-3i\pi x/l} + \cdots\right)$$
$$= \frac{1}{2} - \frac{2}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \cdots\right).$$

PROBLEMS, SECTION 8

1 to 9. In Problems 5.1 to 5.9, define each function by the formulas given but on the interval $(-l, l)$. [That is, replace $\pm\pi$ by $\pm l$ and $\pm\pi/2$ by $\pm l/2$.] Expand each function in a sine-cosine Fourier series and in a complex exponential Fourier series.

10. (a) Sketch several periods of the function $f(x)$ of period 2π which is equal to x on $-\pi < x < \pi$. Expand $f(x)$ in a sine-cosine Fourier series and in a complex exponential Fourier series.

Answer: $f(x) = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots)$.

(b) Sketch several periods of the function $f(x)$ of period 2π which is equal to x on $0 < x < 2\pi$. Expand $f(x)$ in a sine-cosine Fourier series and in a complex exponential Fourier series. Note that this is not the same function or the same series as (a).

Answer: $f(x) = \pi - 2\sum_1^\infty \frac{\sin nx}{n}$.

In Problems 11 to 14, parts (a) and (b), you are given in each case one period of a function. Sketch several periods of the function and expand it in a sine-cosine Fourier series, and in a complex exponential Fourier series.

11. (a) $f(x) = x^2$, $-\pi < x < \pi$; (b) $f(x) = x^2$, $0 < x < 2\pi$.

12. (a) $f(x) = e^x$, $-\pi < x < \pi$; (b) $f(x) = e^x$, $0 < x < 2\pi$.

13. (a) $f(x) = 2 - x$, $-2 < x < 2$; (b) $f(x) = 2 - x$, $0 < x < 4$.

14. (a) $f(x) = \sin \pi x$, $-\frac{1}{2} < x < \frac{1}{2}$; (b) $f(x) = \sin \pi x$, $0 < x < 1$.

15. Sketch (or computer plot) each of the following functions on the interval $(-1, 1)$ and expand it in a complex exponential series and in a sine-cosine series.

(a) $f(x) = x, \quad -1 < x < 1.$

Answer: $f(x) = \frac{2}{\pi}\sum_1^\infty (-1)^{n+1}\frac{\sin n\pi x}{n}$.

(b) $f(x) = \begin{cases} 1 + 2x, & -1 < x < 0, \\ 1 - 2x, & 0 < x < 1. \end{cases}$

Answer: $f(x) = \frac{8}{\pi^2}\sum_{\text{odd } n=1}^\infty \frac{\cos n\pi x}{n^2}$.

(c) $f(x) = \begin{cases} x + x^2, & -1 < x < 0, \\ x - x^2, & 0 < x < 1. \end{cases}$

Answer: $f(x) = \frac{8}{\pi^3}\sum_{\text{odd } n=1}^\infty \frac{\sin n\pi x}{n^3}$.

Each of the following functions is given over one period. Sketch several periods of the corresponding periodic function and expand it in an appropriate Fourier series.

16. $f(x) = x, \quad 0 < x < 2.$ *Answer:* $f(x) = 1 - \dfrac{2}{\pi}\displaystyle\sum_{1}^{\infty} \dfrac{\sin n\pi x}{n}.$

17. $f(x) = \begin{cases} 0, & -1 < x < 0, \\ 1, & 0 < x < 3. \end{cases}$

18. $f(x) = x^2, \quad 0 < x < 10.$

19. $f(x) = \begin{cases} 0, & -\frac{1}{2} < x < 0, \\ x, & 0 < x < \frac{1}{2}. \end{cases}$

20. $f(x) = \begin{cases} x/2, & 0 < x < 2, \\ 1, & 2 < x < 3. \end{cases}$

21. Write out the details of the derivation of the formulas (8.3).

9. EVEN AND ODD FUNCTIONS

An *even* function is one like x^2 or $\cos x$ (Figure 9.1) whose graph for negative x is just a reflection in the y axis of its graph for positive x. In formulas, the value of $f(x)$ is the same for a given x and its negative; that is

$$(9.1) \qquad f(x) \quad \text{is even if} \quad f(-x) = f(x).$$

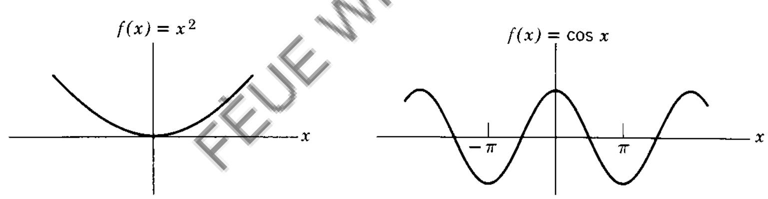


Figure 9.1

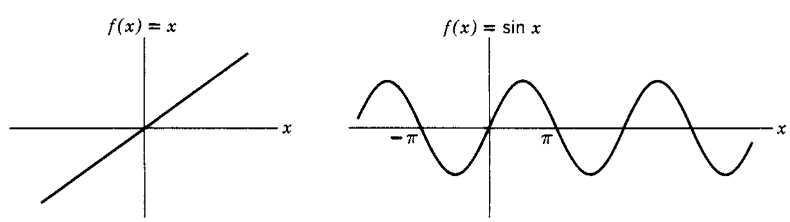


Figure 9.2

An odd function is one like x or $\sin x$ (Figure 9.2) for which the values of $f(x)$ and $f(-x)$ are negatives of each other. By definition

$$(9.2) \qquad f(x) \quad \text{is odd if} \quad f(-x) = -f(x).$$

Notice that even powers of x are even, and odd powers of x are odd; in fact, this is the reason for the names. You should verify (Problem 14) the following rules for the product of two functions: An even function times an even function, or an odd function times an odd function, gives an even function; an odd function times an even function gives an odd function. Some functions are even, some are odd, and some (for example, e^x) are neither. However, any function can be written as the sum of an even function and an odd function, like this:

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)];$$

the first part is even and the second part is odd. For example,

$$e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \cosh x + \sinh x;$$

$\cosh x$ is even and $\sinh x$ is odd (look at the graphs).

Integrals of even functions or of odd functions, over symmetric intervals like $(-\pi, \pi)$ or $(-l, l)$, can be simplified. Look at the graph of $\sin x$ and think about $\int_{-\pi}^{\pi} \sin x\, dx$. The negative area from $-\pi$ to 0 cancels the positive area from 0 to π, so the integral is zero. This integral is still zero for any interval $(-l, l)$ which is symmetric about the origin, as you can see from the graph. The same is true for *any* odd $f(x)$; the areas to the left and to the right cancel. Next look at the cosine graph and the integral $\int_{-\pi/2}^{\pi/2} \cos x\, dx$. You see that the area from $-\pi/2$ to 0 is the same as the area from 0 to $\pi/2$. We could then just as well find the integral from 0 to $\pi/2$ and multiply it by 2. In general, if $f(x)$ is even, the integral of $f(x)$ from $-l$ to l is twice the integral from 0 to l. Then we have

$$\int_{-l}^{l} f(x)\, dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd,} \\ 2\displaystyle\int_0^l f(x)\, dx & \text{if } f(x) \text{ is even.} \end{cases} \tag{9.3}$$

Suppose now that we are given a function on the interval $(0, l)$. If we want to represent it by a Fourier series of period $2l$, we must have $f(x)$ defined on $(-l, 0)$ too. There are several things we could do. We *could* define it to be zero (or, indeed, anything else) on $(-l, 0)$ and go ahead as we have done previously to find either an exponential or a sine-cosine series of period $2l$. However, it often happens in practice that we need (for physical reasons—see Chapter 13) to have an even function (or, in a different problem, an odd function). We first sketch the given function on $(0, l)$ (heavy lines in Figures 9.3 and 9.4). Then we extend the function on $(-l, 0)$ to be even or to be odd as required. To sketch more periods, just repeat the $(-l, l)$ sketch. (If the graph is complicated, it is helpful to trace it with a finger of one

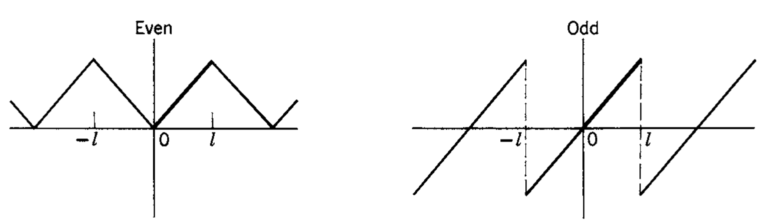


Figure 9.3

hand while you use the other hand to copy exactly what you are tracing. Turn the paper upside down to avoid crossing hands.)

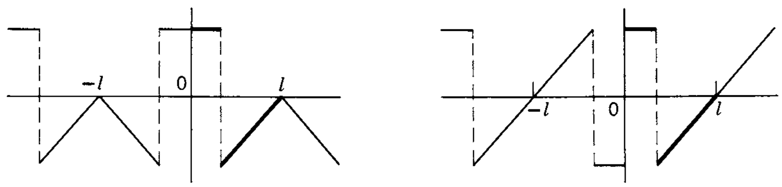


Figure 9.4

For even or odd functions, the coefficient formulas for a_n and b_n simplify. First suppose $f(x)$ is odd. Since sines are odd and cosines are even, $f(x)\sin(n\pi x/l)$ is even and $f(x)\cos(n\pi x/l)$ is odd. Then a_n is the integral, over a symmetric interval $(-l, l)$, of an odd function, namely $f(x)\cos(n\pi x/l)$; a_n is therefore zero. But b_n is the integral of an even function over a symmetric interval and is therefore twice the 0 to l integral. We have:

(9.4) $$\text{If } f(x) \text{ is odd,} \quad \begin{cases} b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \\ a_n = 0. \end{cases}$$

We say that we have expanded $f(x)$ in a sine series ($a_n = 0$ so there are no cosine terms). Similarly, if $f(x)$ is even, all the b_n's are zero, and the a_n's are integrals of even functions. We have:

(9.5) $$\text{If } f(x) \text{ is even,} \quad \begin{cases} a_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \\ b_n = 0. \end{cases}$$

We say that $f(x)$ is expanded in a cosine series. (Remember that the constant term is $a_0/2$.)

You have now learned to find several different kinds of Fourier series that represent a given function $f(x)$ on, let us say, the interval $(0, 1)$. How do you know which to use in a given problem? You have to decide this from the physical problem when you are using Fourier series. There are two things to check: (1) the basic period involved in the physical problem; the functions in your series should have this period; and (2) the physical problem may require either an even function or an odd function for its solution; in these cases you must find the appropriate series. Now consider $f(x)$ defined on $(0, 1)$. We could find for it a sine-cosine or an exponential series of period 1 (that is, $l = \frac{1}{2}$):

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{2in\pi x} \qquad \text{where} \qquad c_n = \int_0^1 f(x)e^{-2in\pi x}\,dx.$$

(The choice between sine-cosine and exponential series is just one of convenience in evaluating the coefficients—the series are really identical.) But we could also find two other Fourier series representing the same $f(x)$ on $(0,1)$. These series would have period 2 (that is, $l=1$). One would be a cosine series

$$f(x)=\sum_{n=0}^{\infty} a_n \cos n\pi x, \qquad a_n = 2\int_0^1 f(x)\cos n\pi x\,dx, \qquad b_n=0,$$

and represent an even function; the other would be a sine series and represent an odd function. In the problems, you may just be told to expand a function in a cosine series, say. You must then see for yourself what the period is when you have sketched an even function, and so choose the proper l in $\cos(n\pi x/l)$ and in the formula for a_n.

▶ **Example.** Represent $f(x)=\begin{cases}1, & 0<x<\frac{1}{2},\\ 0, & \frac{1}{2}<x<1,\end{cases}$
by (a) a Fourier sine series, (b) a Fourier cosine series, (c) a Fourier series (the last ordinarily means a sine-cosine or exponential series whose period is the interval over which the function is given; in this case the period is 1).

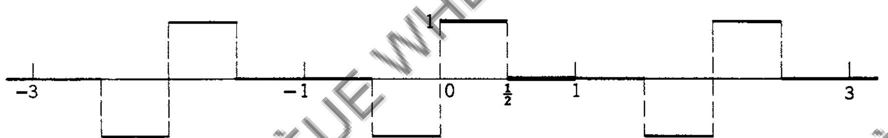


Figure 9.5

(a) Sketch the given function between 0 and 1. Extend it to the interval $(-1,0)$ making it odd. The period is now 2, that is, $l=1$. Continue the function with period 2 (Figure 9.5). Since we now have an odd function, $a_n=0$ and

$$\begin{aligned}
b_n &= \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^{1/2}\sin n\pi x\,dx\\
&= -\frac{2}{n\pi}\cos n\pi x\Big|_0^{1/2} = -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2}-1\right),\\
b_1 &= \frac{2}{\pi}, \quad b_2=\frac{4}{2\pi}, \quad b_3=\frac{2}{3\pi}, \quad b_4=0, \quad \cdots .
\end{aligned}$$

Thus we obtain the *Fourier sine series* for $f(x)$:

$$f(x)=\frac{2}{\pi}\left(\sin\pi x+\frac{2\sin 2\pi x}{2}+\frac{\sin 3\pi x}{3}+\frac{\sin 5\pi x}{5}+\frac{2\sin 6\pi x}{6}+\cdots\right).$$

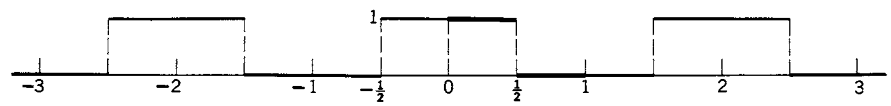


Figure 9.6

(b) Sketch an even function of period 2 (Figure 9.6).
Here $l = 1$, $b_n = 0$, and

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1,$$
$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx = \frac{2}{n\pi}\sin n\pi x\Big|_0^{1/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Then the *Fourier cosine series* for $f(x)$ is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\cos\pi x}{1} - \frac{\cos 3\pi x}{3} + \frac{\cos 5\pi x}{5} - \cdots\right).$$

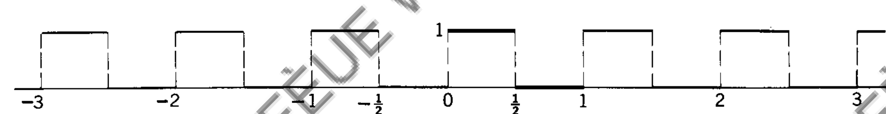


Figure 9.7

(c) Sketch the given function on $(0,1)$ and continue it with period 1 (Figure 9.7). Here $2l = 1$, and we find c_n as we did in the example of Section 8. As in that example, the exponential series here can then be put in sine-cosine form.

$$c_n = \int_0^1 f(x)e^{-2in\pi x}\,dx = \int_0^{1/2} e^{-2in\pi x}\,dx$$
$$= \frac{1-e^{-in\pi}}{2in\pi} = \frac{1-(-1)^n}{2in\pi} = \begin{cases} \dfrac{1}{in\pi}, & n \text{ odd},\\ 0, & n \text{ even} \neq 0.\end{cases}$$
$$c_0 = \int_0^{1/2} dx = \frac{1}{2}.$$
$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left(e^{2i\pi x} - e^{-2i\pi x} + \frac{1}{3}e^{6\pi ix} - \frac{1}{3}e^{-6\pi ix} + \cdots\right)$$
$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin 2\pi x + \frac{\sin 6\pi x}{3} + \cdots\right).$$

Alternatively we can find both a_n and b_n directly.

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1.$$

$$a_n = 2\int_0^{1/2} \cos 2n\pi x\,dx = 0.$$

$$b_n = 2\int_0^{1/2} \sin 2n\pi x\,dx = \frac{1}{n\pi}(1-\cos n\pi) = \frac{1}{n\pi}[1-(-1)^n].$$

$$b_1 = \frac{2}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{2}{3\pi}, \quad b_4 = 0, \quad \cdots.$$

There is one other very useful point to notice about even and odd functions. If you are given a function on $(-l, l)$ to expand in a sine-cosine series (of period $2l$) and happen to notice that it is an even function, you should realize that the b_n's are all going to be zero and you do not have to work them out. Also the a_n's can be written as twice an integral from 0 to l just as in (9.5). Similarly, if the given function is odd, you can use (9.4). Recognizing this may save you a good deal of algebra.

Differentiating Fourier Series Now that we have a supply of Fourier series for reference, let's discuss the question of differentiating a Fourier series term by term. First consider a Fourier series in which a_n and b_n are proportional to $1/n$. Since the derivative of $\frac{1}{n}\sin nx$ is $\cos nx$ (and a similar result for the cosine terms), we see that the differentiated series has no $1/n$ factors to make it converge. Now you might suspect (correctly) that if you can't differentiate the Fourier series, then the function $f(x)$ which it represents can't be differentiated either, at least not at all points. Turn back to examples and problems for which the Fourier series have coefficients proportional to $1/n$ and look at the graphs (or sketch them). Note in every case that $f(x)$ is discontinuous (that is, has jumps) at some points, and so can't be differentiated there. Next consider Fourier series with a_n and b_n proportional to $1/n^2$. If we differentiate such a series once, there are still $1/n$ factors left but we can't differentiate it twice. In that case we would (correctly) expect the function to be continuous with a discontinuous first derivative. (Look for examples.) Continuing, if a_n and b_n are proportional to $1/n^3$, we can find two derivatives, but the second derivative is discontinuous, and so on for Fourier coefficients proportional to higher powers of $1/n$. (See Problems 26 and 27.)

It is interesting to plot (by computer) a given function together with enough terms of its Fourier series to give a reasonable fit. In Section 5 we did this for discontinuous functions and it took many terms of the series. You will find (see Problems 26 and 27) that the more continuous derivatives a function has, the fewer terms of its Fourier series are required to approximate it. We can understand this: The higher order terms oscillate more rapidly (compare $\sin x$, $\sin 2x$, $\sin 10x$), and this rapid oscillation is what is needed to fit a curve which is changing rapidly (for example, a jump). But if $f(x)$ has several continuous derivatives, then it is quite "smooth" and doesn't require so much of the rapid oscillation of the higher order terms. This is reflected in the dependence of the Fourier coefficients on a power of $1/n$.

▶ PROBLEMS, SECTION 9

The functions in Problems 1 to 3 are neither even nor odd. Write each of them as the sum of an even function and an odd function.

1. (a) e^{inx} (b) xe^x

2. (a) $\ln|1-x|$ (b) $(1+x)(\sin x + \cos x)$

3. (a) $x^5 - x^4 + x^3 - 1$ (b) $1 + e^x$

4. Using what you know about even and odd functions, prove the first part of (5.2).

Each of the functions in Problems 5 to 12 is given over one period. For each function, sketch several periods and decide whether it is even or odd. Then use (9.4) or (9.5) to expand it in an appropriate Fourier series.

5. $f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

6. $f(x) = \begin{cases} -1, & -l < x < 0, \\ 1, & 0 < x < l. \end{cases}$

Answer: $f(x) = \dfrac{4}{\pi}\left(\sin\dfrac{\pi x}{l} + \dfrac{1}{3}\sin\dfrac{3\pi x}{l} + \dfrac{1}{5}\sin\dfrac{5\pi x}{l} + \cdots\right).$

7. $f(x) = \begin{cases} 1, & -1 < x < 1, \\ 0, & -2 < x < -1 \text{ and } 1 < x < 2. \end{cases}$

8. $f(x) = x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

9. $f(x) = x^2, \quad -\frac{1}{2} < x < \frac{1}{2}.$

Answer: $f(x) = \dfrac{1}{12} - \dfrac{1}{\pi^2}\left(\cos 2\pi x - \dfrac{1}{2^2}\cos 4\pi x + \dfrac{1}{3^2}\cos 6\pi x - \cdots\right).$

10. $f(x) = |x|, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

11. $f(x) = \cosh x, \quad -\pi < x < \pi.$

Answer: $f(x) = \dfrac{2\sinh\pi}{\pi}\left(\dfrac{1}{2} - \dfrac{1}{2}\cos x + \dfrac{1}{5}\cos 2x - \dfrac{1}{10}\cos 3x + \dfrac{1}{17}\cos 4x - \cdots\right).$

12. $f(x) = \begin{cases} x+1, & -1 < x < 0, \\ x-1, & 0 < x < 1. \end{cases}$

13. Give algebraic proofs of (9.3). *Hint:* Write $\int_{-l}^{l} = \int_{-l}^{0} + \int_{0}^{l}$, make the change of variable $x = -t$ in $\int_{-l}^{0}$, and use the definition of even or odd function.

14. Give algebraic proofs that for even and odd functions:

(a) even times even = even; odd times odd = even; even times odd = odd;

(b) the derivative of an even function is odd; the derivative of an odd function is even.

15. Given $f(x) = x$ for $0 < x < 1$, sketch the even function f_c of period 2 and the odd function f_s of period 2, each of which equals $f(x)$ on $0 < x < 1$. Expand f_c in a cosine series and f_s in a sine series.

Answer: $f_c(x) = \dfrac{1}{2} - \dfrac{4}{\pi^2}\left(\cos\pi x + \dfrac{1}{3^2}\cos 3\pi x + \cdots\right),$

$f_s(x) = \dfrac{2}{\pi}\left(\sin\pi x - \dfrac{1}{2}\sin 2\pi x + \dfrac{1}{3}\sin 3\pi x - \cdots\right).$

16. Let $f(x) = \sin^2 x$, $0 < x < \pi$. Sketch (or computer plot) the even function f_c of period 2π, the odd function f_s of period 2π, and the function f_p of period π, each of which is equal to $f(x)$ on $(0, \pi)$. Expand each of these functions in an appropriate Fourier series.

In Problems 17 to 22 you are given $f(x)$ on an interval, say $0 < x < b$. Sketch several periods of the even function f_c of period $2b$, the odd function f_s of period $2b$, and the function f_p of period b, each of which equals $f(x)$ on $0 < x < b$. Expand each of the three functions in an appropriate Fourier series.

17. $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x < 1. \end{cases}$

18. $f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 3. \end{cases}$

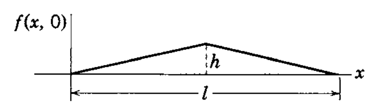
19. $f(x) = |\cos x|, \quad 0 < x < \pi.$

20. $f(x) = x^2, \quad 0 < x < 1.$

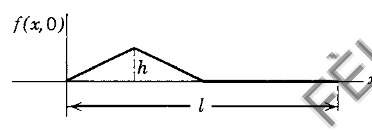
21. $f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 < x < 2. \end{cases}$

22. $f(x) = \begin{cases} 10, & 0 < x < 10, \\ 20, & 10 < x < 20. \end{cases}$

23. If a violin string is plucked (pulled aside and let go), it is possible to find a formula $f(x, t)$ for the displacement at time t of any point x of the vibrating string from its equilibrium position. It turns out that in solving this problem we need to expand the function $f(x, 0)$, whose graph is the initial shape of the string, in a Fourier sine series. Find this series if a string of length l is pulled aside a small distance h at its center, as shown.



24. If, in Problem 23, the string is stopped at the center and half of it is plucked, then the function to be expanded in a sine series is shown here. Find the series. *Caution:* Note that $f(x, 0) = 0$ for $l/2 < x < l$.



25. Suppose that $f(x)$ and its derivative $f'(x)$ are both expanded in Fourier series on $(-\pi, \pi)$. Call the coefficients in the $f(x)$ series a_n and b_n and the coefficients in the $f'(x)$ series a'_n and b'_n. Write the integral for a_n [equation (5.9)] and integrate it by parts to get an integral of $f'(x) \sin nx$. Recognize this integral in terms of b'_n [equation (5.10) for $f'(x)$] and so show that $b'_n = -na_n$. (In the integration by parts, the integrated term is zero because $f(\pi) = f(-\pi)$ since f is continuous—sketch several periods.). Find a similar relation for a'_n and b_n. Now show that this is the result you get by differentiating the $f(x)$ series term by term. Thus you have shown that the Fourier series for $f'(x)$ is correctly given by differentiating the $f(x)$ series term by term (assuming that $f'(x)$ is expandable in a Fourier series).

In Problems 26 and 27, find the indicated Fourier series. Then differentiate your result repeatedly (both the function and the series) until you get a discontinuous function. Use a computer to plot $f(x)$ and the derivative functions. For each graph, plot on the same axes one or more terms of the corresponding Fourier series. Note the number of terms needed for a good fit (see comment at the end of the section).

26. $f(x) = \begin{cases} 3x^2 + 2x^3, & -1 < x < 0, \\ 3x^2 - 2x^3, & 0 < x < 1. \end{cases}$

27. $f(x) = (x^2 - \pi^2)^2, \quad -\pi < x < \pi.$

10. AN APPLICATION TO SOUND

We have said that when a sound wave passes through the air and we hear it, the air pressure where we are varies with time. Suppose the excess pressure above (and below) atmospheric pressure in a sound wave is given by the graph in Figure 10.1. (We shall not be concerned here with the units of p; however, reasonable units in Figure 10.1 would be p in 10^{-6} atmospheres.) Let us ask what frequencies we hear when we listen to this sound. To find out, we expand $p(t)$ in a Fourier series. The period of $p(t)$ is $\frac{1}{262}$; that is, the sound wave repeats itself 262 times per second. We have called the period $2l$ in our formulas, so here $l = \frac{1}{524}$. The functions we have called $\sin(n\pi x/l)$ here become $\sin 524n\pi t$. We can save some work by observing

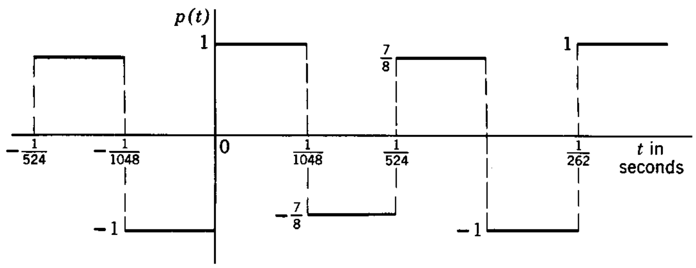


Figure 10.1

that $p(t)$ is an odd function; there are then only sine terms in its Fourier series and we need to compute only b_n. Using (9.4), we have

$$\begin{aligned} b_n &= 2(524)\int_0^{1/524} p(t)\sin 524n\pi t\,dt \\ &= 1048\int_0^{1/1048} \sin 524n\pi t\,dt - \frac{7}{8}(1048)\int_{1/1048}^{1/524}\sin 524n\pi t\,dt \\ &= 1048\left(-\frac{\cos\frac{n\pi}{2}-1}{524n\pi} + \frac{7}{8}\,\frac{\cos n\pi - \cos\frac{n\pi}{2}}{524n\pi}\right) \\ &= \frac{2}{n\pi}\left(-\frac{15}{8}\cos\frac{n\pi}{2} + 1 + \frac{7}{8}\cos n\pi\right). \end{aligned} \tag{10.1}$$

From this we can compute the values of b_n for the first few values of n:

$$\begin{aligned} b_1 &= \frac{2}{\pi}\left(1-\frac{7}{8}\right) = \frac{2}{\pi}\left(\frac{1}{8}\right) = \frac{1}{\pi}\cdot\frac{1}{4} & \qquad b_5 &= \frac{1}{5\pi}\cdot\frac{1}{4} \\ b_2 &= \frac{2}{2\pi}\left(\frac{15}{8}+1+\frac{7}{8}\right) = \frac{1}{2\pi}\left(\frac{15}{2}\right) & b_6 &= \frac{1}{6\pi}\left(\frac{15}{2}\right) \\ b_3 &= \frac{2}{3\pi}\left(1-\frac{7}{8}\right) = \frac{1}{3\pi}\cdot\frac{1}{4} & b_7 &= \frac{1}{7\pi}\cdot\frac{1}{4} \\ b_4 &= \frac{2}{4\pi}\left(-\frac{15}{8}+1+\frac{7}{8}\right) = 0 & b_8 &= 0, \quad \text{etc.} \end{aligned} \tag{10.2}$$

Then we have

$$(10.3)\qquad p(t)=\frac{1}{4\pi}\left(\frac{\sin 524\pi t}{1}+\frac{30\sin(524\cdot 2\pi t)}{2}+\frac{\sin(524\cdot 3\pi t)}{3}\right.$$
$$\left.+\frac{\sin(524\cdot 5\pi t)}{5}+\frac{30\sin(524\cdot 6\pi t)}{6}+\frac{\sin(524\cdot 7\pi t)}{7}+\cdots\right).$$

We can see just by looking at the coefficients that the most important term is the second one. The first term corresponds to the fundamental with frequency 262 vibrations per second (this is approximately middle C on a piano). But it is much weaker in this case than the first overtone (second harmonic) corresponding to the second term; this tone has frequency 524 vibrations per second (approximately high C). (You might like to use a computer to play one or several terms of the series.) The sixth harmonic (corresponding to $n = 6$) and also the harmonics for $n = 10$, 14, 18, 22, and 26 are all more prominent (that is, have larger coefficients) than the fundamental. We can be even more specific about the relative importance of the various frequencies. Recall that in discussing a simple harmonic oscillator, we showed that its average energy was proportional to the square of its velocity amplitude. It can be proved that the intensity of a sound wave (average energy striking unit area of your ear per second) is proportional to the average of the square of the excess pressure. Thus for a sinusoidal pressure variation $A\sin 2\pi ft$, the intensity is proportional to A^2. In the Fourier series for $p(t)$, the intensities of the various harmonics are then proportional to the squares of the corresponding Fourier coefficients. (The intensity corresponds roughly to the loudness of the tone—not exactly because the ear is not uniformly sensitive to all frequencies.) The relative intensities of the harmonics in our example are then:

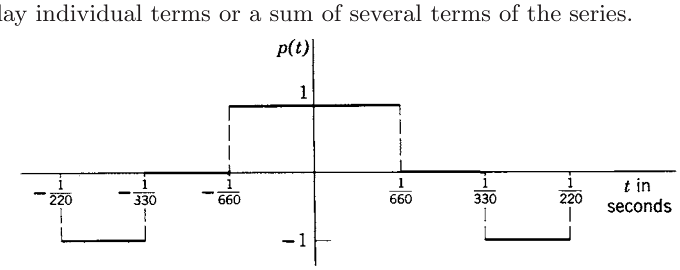
n	$=$ 1	2	3	4	5	6	7	8	9	10	$\cdots$
Relative intensity	$=$ 1	225	$\frac{1}{9}$	0	$\frac{1}{25}$	25	$\frac{1}{49}$	0	$\frac{1}{81}$	9	$\cdots$

From this we see even more clearly that we would hear principally the second harmonic with frequency 524 (high C).

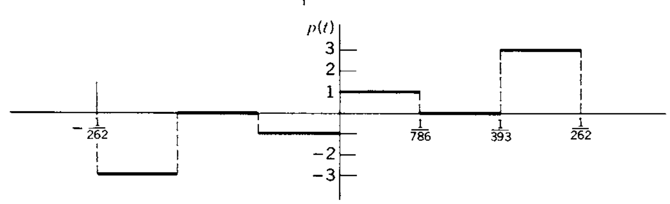
PROBLEMS, SECTION 10

In Problems 1 to 3, the graphs sketched represent one period of the excess pressure $p(t)$ in a sound wave. Find the important harmonics and their relative intensities. Use a computer to play individual terms or a sum of several terms of the series.

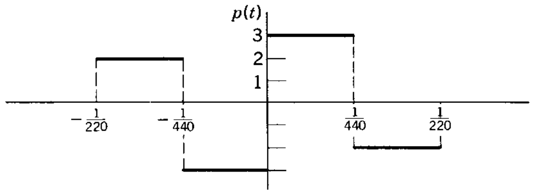
1.



2.

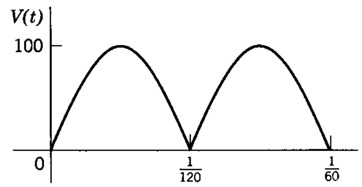


3.

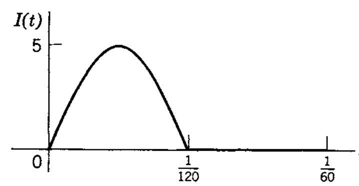


In Problems 4 to 10, the sketches show several practical examples of electrical signals (voltages or currents). In each case we want to know the harmonic content of the signal, that is, what frequencies it contains and in what proportions. To find this, expand each function in an appropriate Fourier series. Assume in each case that the part of the graph shown is repeated sixty times per second.

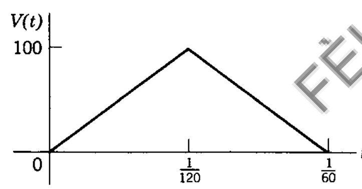
4. Output of a simple d-c generator; the shape of the curve is the absolute value of a sine function. Let the maximum voltage be 100 v.



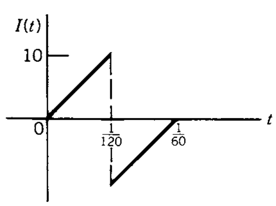
5. Rectified half-wave; the curve is a sine function for half the cycle and zero for the other half. Let the maximum current be 5 amp. *Hint:* Be careful! The value of l here is $1/60$, but $I(t) = \sin t$ only from $t = 0$ to $t = 1/120$.



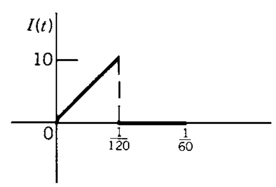
6. Triangular wave; the graph consists of two straight lines whose equations you must write! The maximum voltage of 100 v occurs at the middle of the cycle.



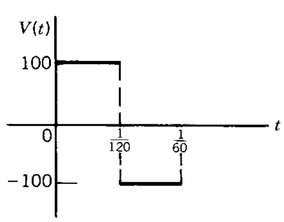
7. Sawtooth



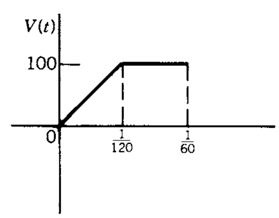
8. Rectified sawtooth



9. Square wave



10. Periodic ramp function



11. PARSEVAL'S THEOREM

We shall now find a relation between the average of the square (or absolute square) of $f(x)$ and the coefficients in the Fourier series for $f(x)$, assuming that $\int_{-\pi}^{\pi} |f(x)|^2\,dx$ is finite. The result is known as *Parseval's theorem* or the *completeness relation.* You should understand that the point of the theorem is *not* to get the average of the square of a given $f(x)$ by using its Fourier series. [Given $f(x)$, it is easy to get its average square just by doing the integration in (11.2) below.] The point of the theorem is to show the *relation* between the average of the square of $f(x)$ and the Fourier coefficients. We can derive a form of Parseval's theorem from any of the various Fourier expansions we have made; let us use (5.1).

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx. \tag{11.1}$$

We square $f(x)$ and then average the square over $(-\pi, \pi)$:

$$\text{The average of } [f(x)]^2 \text{ is } \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx. \tag{11.2}$$

When we square the Fourier series in (11.1) we get many terms. To avoid writing out a large number of them, consider instead what types of terms there are and what the averages of the different kinds of terms are. First, there are the squares of the individual terms. Using the fact that the average of the square of a sine or cosine over a period is $\frac{1}{2}$, we have:

$$\begin{array}{lll} \text{The average of } (\frac{1}{2}a_0)^2 & \text{is} & (\frac{1}{2}a_0)^2. \\ \text{The average of } (a_n \cos nx)^2 & \text{is} & a_n^2 \cdot \frac{1}{2}. \\ \text{The average of } (b_n \sin nx)^2 & \text{is} & b_n^2 \cdot \frac{1}{2}. \end{array} \tag{11.3}$$

Then there are cross-product terms of the forms $2 \cdot \frac{1}{2}a_0 a_n \cos nx$, $2 \cdot \frac{1}{2}a_0 b_n \sin nx$, and $2a_n b_m \cos nx \sin mx$ with $m \neq n$ (we write n in the cosine factor and m in the sine factor since every sine term must be multiplied times every cosine term). By (5.2), the average values of terms of all these types are zero. Then we have

$$\text{The average of } [f(x)]^2 \text{ (over a period)} = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + \frac{1}{2}\sum_1^\infty b_n^2. \tag{11.4}$$

This is one form of Parseval's theorem. You can easily verify (Problem 1) that the theorem is unchanged if $f(x)$ has period $2l$ instead of 2π and its square is averaged over any period of length $2l$. You can also verify (Problem 3) that if $f(x)$ is written as a complex exponential Fourier series, and if in addition we include the possibility that $f(x)$ itself may be complex, then we find:

$$\text{The average of } |f(x)|^2 \text{ (over a period)} = \sum_{-\infty}^{\infty} |c_n|^2. \tag{11.5}$$

Parseval's theorem is also called the *completeness relation.* In the problem of representing a given sound wave as a sum of harmonics, suppose we had left one of the harmonics out of the series. It seems plausible physically, and it can be proved mathematically, that with one or more harmonics left out, we would not be able to represent sound waves containing the omitted harmonics. We say that the set of functions $\sin nx$, $\cos nx$ is a *complete set* of functions on any interval of length 2π; that is, any function (satisfying Dirichlet conditions) can be expanded in a Fourier series whose terms are constants times $\sin nx$ and $\cos nx$. If we left out some values of n, we would have an incomplete set of basis functions (see basis, page 357) and could not use it to expand some given functions. For example, suppose that you made a mistake in finding the period (that is, the value of l) of your given function and tried to use the set of functions $\sin 2nx$, $\cos 2nx$ in expanding a given function of period 2π. You would get a wrong answer because you used an incomplete set of basis functions (with the $\sin x$, $\cos x$, $\sin 3x$, $\cos 3x$, $\cdots$, terms missing). If your Fourier series is wrong because the set of basis functions you use is incomplete, then the results you get from Parseval's theorem (11.4) or (11.5) will be wrong too. In fact, if we use an incomplete basis set in, say, (11.5), then there are missing (non-negative) terms on the right-hand side, so the equation becomes the inequality: Average of $|f(x)|^2 \geq \sum_{-\infty}^{\infty} |c_n|^2$. This is known as Bessel's inequality. Conversely, if (11.4) and (11.5) are correct for *all* $f(x)$, then the set of basis functions used is a complete set. This is why Parseval's theorem is often called the completeness relation. (Also see page 377 and Chapter 12, Section 6.)

Let us look at some examples of the physical meaning and the use of Parseval's theorem.

▶ **Example 1.** In Section 10 we said that the intensity (energy per square centimeter per second) of a sound wave is proportional to the average value of the square of the excess pressure. If for simplicity we write (10.3) with letters instead of numerical values, we have

$$p(t) = \sum_{1}^{\infty} b_n \sin 2\pi n f t. \tag{11.6}$$

For this case, Parseval's theorem (11.4) says that:

$$\text{The average of } [p(t)]^2 = \sum_{1}^{\infty} b_n^2 \cdot \frac{1}{2} = \sum_{1}^{\infty} \text{ the average of } b_n^2 \sin^2 2n\pi f t\ . \tag{11.7}$$

Now the intensity or energy (per square centimeter per second) of the sound wave is proportional to the average of $[p(t)]^2$, and the energy associated with the nth harmonic is proportional to the average of $b_n^2 \sin^2 2n\pi f t$. Thus Parseval's theorem says that the total energy of the sound wave is equal to the sum of the energies associated with the various harmonics.

▶ **Example 2.** Let us use Parseval's theorem to find the sum of an infinite series. From Problem 8.15(a) written in complex exponential form we get:

The function $f(x)$ of period 2 which is equal to x on $(-1, 1)$

$$= -\frac{i}{\pi}\left(e^{i\pi x} - e^{-i\pi x} - \frac{1}{2}e^{2i\pi x} + \frac{1}{2}e^{-2i\pi x} + \frac{1}{3}e^{3i\pi x} - \frac{1}{3}e^{-3i\pi x} + \cdots\right).$$

Let us find the average of $[f(x)]^2$ on $(-1, 1)$.

$$\text{The average of } [f(x)]^2 = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\bigg|_{-1}^{1}\right] = \frac{1}{3}.$$

By Parseval's theorem (11.5), this is equal to $\sum_{-\infty}^{\infty}|c_n|^2$, so we have

$$\frac{1}{3} = \sum_{-\infty}^{\infty}|c_n|^2 = \frac{1}{\pi^2}(1 + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{9} + \cdots = \frac{2}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}.$$

Then we get the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2}\cdot\frac{1}{3} = \frac{\pi^2}{6}.$$

We have seen that a function given on $(0, l)$ can be expanded in a sine series by defining it on $(-l, 0)$ to make it odd, or in a cosine series by defining it on $(-l, 0)$ to make it even. Here is another useful example of defining a function to suit our purposes. (We will need this in Chapter 13.) Suppose we want to expand a function defined on $(0, l)$ in terms of the basis functions $\sin(n + \frac{1}{2})\frac{\pi x}{l} = \sin\frac{(2n+1)\pi x}{2l}$. Can we do it, that is, do these functions make up a complete set for this problem? Note that our proposed basis functions have period $4l$, say $(-2l, 2l)$ (observe the $2l$ in the denominator where you are used to l). So given $f(x)$ on $(0, l)$, we can define it as we like on $(l, 2l)$ and on $(-2l, 0)$. We know (by the Dirichlet theorem) that the functions $\sin\frac{n\pi x}{2l}$ and $\cos\frac{n\pi x}{2l}$, *all* n, make up a complete set on $(-2l, 2l)$. We need to see how, on $(0, l)$ we can use just the sines (that's easy—make the function odd) and only the odd values of n. It turns out (see Problem 11) that if we define $f(x)$ on $(l, 2l)$ to make it symmetric around $x = l$, then all the b_n's for even n are equal to zero. So our desired basis set is indeed a complete set on $(0, l)$. Similarly we can show (Problem 11) that the functions $\cos\frac{(2n+1)\pi x}{2l}$ make up a complete set on $(0, l)$.

PROBLEMS, SECTION 11

1. Prove (11.4) for a function of period $2l$ expanded in a sine-cosine series.

2. Prove that if $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, then the average value of $[f(x)]^2$ is $\sum_{-\infty}^{\infty} c_n c_{-n}$. Show by Problem 7.12 that for real $f(x)$ this becomes (11.5).

3. If $f(x)$ is complex, we usually want the average of the square of the absolute value of $f(x)$. Recall that $|f(x)|^2 = f(x)\cdot\bar{f}(x)$, where $\bar{f}(x)$ means the complex conjugate of $f(x)$. Show that if a complex $f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}$, then (11.5) holds.

4. When a current I flows through a resistance R, the heat energy dissipated per second is the average value of RI^2. Let a periodic (not sinusoidal) current $I(t)$ be expanded in a Fourier series $I(t) = \sum_{-\infty}^{\infty} c_n e^{120in\pi t}$. Give a physical meaning to Parseval's theorem for this problem.

Use Parseval's theorem and the results of the indicated problems to find the sum of the series in Problems 5 to 9.

5. The series $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$, using Problem 9.6.

6. The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$, using Problem 9.9.

7. The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, using Problem 5.8.

8. The series $\displaystyle\sum_{\text{odd } n}\frac{1}{n^4}$, using Problem 9.10.

9. The series $\displaystyle\frac{1}{3^2}+\frac{1}{15^2}+\frac{1}{35^2}+\cdots$, using Problem 5.11.

10. A general form of Parseval's theorem says that if two functions are expanded in Fourier series

$$f(x)=\frac{1}{2}a_0+\sum_1^{\infty}a_n\cos nx+\sum_1^{\infty}b_n\sin nx,$$

$$g(x)=\frac{1}{2}a_0'+\sum_1^{\infty}a_n'\cos nx+\sum_1^{\infty}b_n'\sin nx,$$

then the average value of $f(x)g(x)$ is $\frac{1}{4}a_0a_0'+\frac{1}{2}\sum_1^{\infty}a_na_n'+\frac{1}{2}\sum_1^{\infty}b_nb_n'$. Prove this.

11. (a) Let $f(x)$ on $(0,2l)$ satisfy $f(2l-x)=f(x)$, that is, $f(x)$ is symmetric about $x=l$. If you expand $f(x)$ on $(0,2l)$ in a sine series $\sum b_n\sin\frac{n\pi x}{2l}$, show that for even n, $b_n=0$. *Hint:* Note that the period of the sines is $4l$. Sketch an $f(x)$ which is symmetric about $x=l$, and on the same axes sketch a few sines to see that the even ones are antisymmetric about $x=l$. Alternatively, write the integral for b_n as an integral from 0 to l plus an integral from l to $2l$, and replace x by $2l-x$ in the second integral.

(b) Similarly, show that if we define $f(2l-x)=-f(x)$, the cosine series has $a_n=0$ for even n.

▶ 12. FOURIER TRANSFORMS

We have been expanding *periodic* functions in series of sines, cosines, and complex exponentials. Physically, we could think of the terms of these Fourier series as representing a set of harmonics. In music these would be an infinite set of frequencies nf, $n=1, 2, 3, \cdots$; notice that this set, although infinite, does not by any means include all possible frequencies. In electricity, a Fourier series could represent a periodic voltage; again we could think of this as made up of an infinite but discrete (that is, not continuous) set of a-c voltages of frequencies $n\omega$. Similarly, in discussing light, a Fourier series could represent light consisting of a discrete set of wavelengths λ/n, $n=1, 2, \cdots$, that is, a discrete set of colors. Two related questions might occur to us here. First, is it possible to represent a function which is *not* periodic by something analogous to a Fourier series? Second, can we somehow extend or modify Fourier series to cover the case of a continuous spectrum of wavelengths of light, or a sound wave containing a continuous set of frequencies?

If you recall that an integral is a limit of a sum, it may not surprise you very much to learn that the Fourier *series* (that is, a *sum* of terms) is replaced by a Fourier *integral* in the above cases. The Fourier integral can be used to represent nonperiodic functions, for example a single voltage pulse not repeated, or a flash of light, or a sound which is not repeated. The Fourier integral also represents a continuous set (spectrum) of frequencies, for example a whole range of musical tones or colors of light rather than a discrete set.

Recall from equations (8.2) and (8.3), these complex Fourier series formulas:

$$(12.1)\qquad \begin{aligned} f(x) &= \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}, \\ c_n &= \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx. \end{aligned}$$

The period of $f(x)$ is $2l$ and the frequencies of the terms in the series are $n/(2l)$. We now want to consider the case of continuous frequencies.

Definition of Fourier Transforms We state without proof (see plausibility arguments below) the formulas corresponding to (12.1) for a continuous range of frequencies.

$$(12.2)\qquad \boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha, \\ g(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx. \end{aligned}}$$

Compare (12.2) and (12.1); $g(\alpha)$ corresponds to c_n, α corresponds to n, and $\int_{-\infty}^{\infty}$ corresponds to $\sum_{-\infty}^{\infty}$. This agrees with our discussion of the physical meaning and use of Fourier integrals. The quantity α is a continuous analog of the integral-valued variable n, and so the set of coefficients c_n has become a function $g(\alpha)$; the sum over n has become an integral over α. The two functions $f(x)$ and $g(\alpha)$ are called a pair of *Fourier transforms*. Usually, $g(\alpha)$ is called the Fourier transform of $f(x)$, and $f(x)$ is called the inverse Fourier transform of $g(\alpha)$, but since the two integrals differ in form only in the sign in the exponent, it is rather common simply to call either a Fourier transform of the other. You should check the notation of any book or computer program you are using. Another point on which various references differ is the position of the factor $1/(2\pi)$ in (12.2); it is possible to have it multiply the $f(x)$ integral instead of the $g(\alpha)$ integral, or to have the factor $1/\sqrt{2\pi}$ multiply each of the integrals.

The *Fourier integral theorem* says that, if a function $f(x)$ satisfies the Dirichlet conditions (Section 6) on every finite interval, and if $\int_{-\infty}^{\infty}|f(x)|\,dx$ is finite, then (12.2) is correct. That is, if $g(\alpha)$ is computed and substituted into the integral for $f(x)$ [compare the procedure of computing the c_n's for a Fourier series and substituting them into the series for $f(x)$], then the integral gives the value of $f(x)$ anywhere that $f(x)$ is continuous; at jumps of $f(x)$, the integral gives the midpoint of the jump (again compare Fourier series, Section 6). The following discussion is not a mathematical proof of this theorem but is intended to help you see more clearly how Fourier integrals are related to Fourier series.

It might seem reasonable to think of trying to represent a function which is not periodic by letting the period $(-l, l)$ increase to $(-\infty, \infty)$. Let us try to do this, starting with (12.1). If we call $n\pi/l = \alpha_n$ and $\alpha_{n+1} - \alpha_n = \pi/l = \Delta\alpha$, then $1/(2l) = \Delta\alpha/(2\pi)$ and (12.1) can be rewritten as

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\alpha_n x}, \tag{12.3}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-i\alpha_n x}\,dx = \frac{\Delta\alpha}{2\pi}\int_{-l}^{l} f(u)e^{-i\alpha_n u}\,du. \tag{12.4}$$

(We have changed the dummy integration variable in c_n from x to u to avoid later confusion.) Substituting (12.4) into (12.3), we have

$$\begin{aligned} f(x) &= \sum_{-\infty}^{\infty}\left[\frac{\Delta\alpha}{2\pi}\int_{-l}^{l} f(u)e^{-i\alpha_n u}\,du\right] e^{i\alpha_n x} \\ &= \sum_{-\infty}^{\infty}\frac{\Delta\alpha}{2\pi}\int_{-l}^{l} f(u)e^{i\alpha_n(x-u)}\,du = \frac{1}{2\pi}\sum_{-\infty}^{\infty} F(\alpha_n)\,\Delta\alpha, \end{aligned} \tag{12.5}$$

where

$$F(\alpha_n) = \int_{-l}^{l} f(u)e^{i\alpha_n(x-u)}\,du. \tag{12.6}$$

Now $\sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha$ looks rather like the formula in calculus for the sum whose limit, as $\Delta\alpha$ tends to zero, is an integral. If we let l tend to infinity [that is, let the period of $f(x)$ tend to infinity], then $\Delta\alpha = \pi/l \to 0$, and the sum $\sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha$ goes over formally to $\int_{-\infty}^{\infty} F(\alpha)d\alpha$; we have dropped the subscript n on α now that it is a continuous variable. We also let l tend to infinity and $\alpha_n = \alpha$ in (12.6) to get

$$F(\alpha) = \int_{-\infty}^{\infty} f(u)e^{i\alpha(x-u)}\,du. \tag{12.7}$$

Replacing $\sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha$ in (12.5) by $\int_{-\infty}^{\infty} F(\alpha)\,d\alpha$ and substituting from (12.7) for $F(\alpha)$ gives

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\alpha(x-u)}\,du\,d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha\int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\,du. \end{aligned} \tag{12.8}$$

If we define $g(\alpha)$ by

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\,du, \tag{12.9}$$

then (12.8) gives

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha. \tag{12.10}$$

These equations are the same as (12.2). Notice that the actual requirement for the factor $1/(2\pi)$ is that the *product* of the constants multiplying the two integrals for $g(\alpha)$ and $f(x)$ should be $1/(2\pi)$; this accounts for the various notations we have discussed before.

Just as we have sine series representing odd functions and cosine series representing even functions (Section 9), so we have sine and cosine Fourier integrals which represent odd or even functions respectively. Let us prove that if $f(x)$ is odd, then $g(\alpha)$ is odd too, and show that in this case (12.2) reduces to a pair of sine transforms. The corresponding proof for even $f(x)$ is similar (Problem 1). We substitute

$$e^{-i\alpha x} = \cos\alpha x - i\sin\alpha x$$

into (12.9) to get

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(\cos\alpha x - i\sin\alpha x)\,dx. \tag{12.11}$$

Since $\cos\alpha x$ is even and we are assuming that $f(x)$ is odd, the product $f(x)\cos\alpha x$ is odd. Recall that the integral of an odd function over a symmetric interval about the origin (here, $-\infty$ to $+\infty$) is zero, so the term $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ in (12.11) is zero. The product $f(x)\sin\alpha x$ is even (product of two odd functions); recall that the integral of an even function over a symmetric interval is twice the integral over positive x. Substituting these results into (12.11), we have

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(-i\sin\alpha x)\,dx = -\frac{i}{\pi}\int_{0}^{\infty} f(x)\sin\alpha x\,dx. \tag{12.12}$$

From (12.12), we can see that replacing α by $-\alpha$ changes the sign of $\sin\alpha x$ and so changes the sign of $g(\alpha)$. That is, $g(-\alpha) = -g(\alpha)$, so $g(\alpha)$ is an odd function as we claimed. Then expanding the exponential in (12.10) and arguing as we did to obtain (12.12), we find

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,dx = 2i\int_{0}^{\infty} g(\alpha)\sin\alpha x\,d\alpha. \tag{12.13}$$

If we substitute $g(\alpha)$ from (12.12) into (12.13) to obtain an equation like (12.8), the numerical factor is $(-i/\pi)(2i) = 2/\pi$; thus the imaginary factors are not needed. The factor $2/\pi$ may multiply either of the two integrals or each integral may be multiplied by $\sqrt{2/\pi}$. Let us make the latter choice in giving the following definition.

Fourier Sine Transforms We define $f_s(x)$ and $g_s(\alpha)$, a pair of *Fourier sine transforms* representing *odd functions*, by the equations

$$\begin{aligned} f_s(x) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} g_s(\alpha)\sin\alpha x\,d\alpha,\\ g_s(x) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_s(x)\sin\alpha x\,dx. \end{aligned} \tag{12.14}$$

We discuss even functions in a similar way (Problem 1).

Fourier Cosine Transforms We define $f_c(x)$ and $g_c(\alpha)$, a pair of *Fourier cosine transforms* representing *even functions*, by the equations

$$(12.15)\qquad \begin{aligned} f_c(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty g_c(x)\cos\alpha x\,d\alpha,\\ g_c(\alpha) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f_c(x)\cos\alpha x\,dx. \end{aligned}$$

▶ **Example 1.** Let us represent a nonperiodic function as a Fourier integral. The function

$$f(x) = \begin{cases} 1, & -1 < x < 1, \\ 0, & |x| > 1, \end{cases}$$

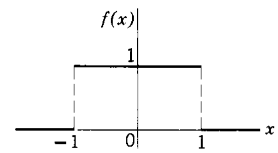


Figure 12.1

shown in Figure 12.1 might represent an impulse in mechanics (that is, a force applied only over a short time such as a bat hitting a baseball), or a sudden short surge of current in electricity, or a short pulse of sound or light which is not repeated. Since the given function is not periodic, it cannot be expanded in a Fourier *series*, since a Fourier series always represents a *periodic* function. Instead, we write $f(x)$ as a Fourier integral as follows. Using (12.9), we calculate $g(\alpha)$; this process is like finding the c_n's for a Fourier series. We find

$$(12.16)\qquad \begin{aligned} g(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx = \frac{1}{2\pi}\int_{-1}^{1} e^{-i\alpha x}\,dx \\ &= \frac{1}{2\pi}\frac{e^{-i\alpha x}}{-i\alpha}\bigg|_{-1}^{1} = \frac{1}{\pi\alpha}\,\frac{e^{-i\alpha}-e^{i\alpha}}{-2i} = \frac{\sin\alpha}{\pi\alpha}. \end{aligned}$$

We substitute $g(\alpha)$ from (12.16) into the formula (12.10) for $f(x)$ (this is like substituting the evaluated coefficients into a Fourier series). We get

$$(12.17)\qquad \begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{\sin\alpha}{\pi\alpha} e^{i\alpha x}\,dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha(\cos\alpha x + i\sin\alpha x)}{\alpha}\,d\alpha = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha \end{aligned}$$

since $(\sin\alpha)/\alpha$ is an even function. We thus have an integral representing the function $f(x)$ shown in Figure 12.1.

▶ **Example 2.** We can use (12.17) to evaluate a definite integral. Using $f(x)$ in Figure 12.1, we find

$$(12.18)\qquad \int_0^\infty \frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1, \quad \frac{\pi}{4} \text{ for } |x| = 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

Notice that we have used the fact that the Fourier integral represents the midpoint of the jump in $f(x)$ at $|x| = 1$. If we let $x = 0$, we get

$$(12.19)\qquad \int_0^\infty \frac{\sin\alpha}{\alpha}\,d\alpha = \frac{\pi}{2}.$$

We could have done this problem by observing that $f(x)$ is an even function and so can be represented by a cosine transform. The final results (12.17) to (12.19) would be just the same (Problem 2).

In Section 9, we sometimes started with a function defined only for $x > 0$ and extended it to be even or odd so that we could represent it by a cosine series or by a sine series. Similarly, for Fourier transforms, we can represent a function defined for $x > 0$ by either a Fourier cosine integral (by defining it for $x < 0$ so that it is even), or by a Fourier sine integral (by defining it for $x < 0$ so that it is odd). (See Problem 2 and Problems 27 to 30.)

Parseval's Theorem for Fourier Integrals Recall (Section 11) that Parseval's theorem for a Fourier series $f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}$ relates $\int_{-l}^{l} |f|^2\,dx$ and $\sum_{-\infty}^{\infty} |c_n|^2$. In physical applications (see Section 11), Parseval's theorem says that the total energy (say in a sound wave, or in an electrical signal) is equal to the sum of the energies associated with the various harmonics. Remember that a Fourier integral represents a continuous spectrum of frequencies and that $g(\alpha)$ corresponds to c_n. Then we might expect that $\sum_{-\infty}^{\infty} |c_n|^2$ would be replaced by $\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha$ (that is, a "sum" over a continuous rather than a discrete spectrum) and that Parseval's theorem would relate $\int_{-\infty}^{\infty} |f|^2 dx$ and $\int_{-\infty}^{\infty} |g|^2 d\alpha$. Let us try to find the relation.

We will first find a generalized form of Parseval's theorem involving two functions $f_1(x)$, $f_2(x)$ and their Fourier transforms $g_1(\alpha)$, $g_2(\alpha)$. Let $\bar{g}_1(\alpha)$ be the complex conjugate of $g_1(\alpha)$; from (12.1), we have

$$\bar{g}_1(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(x)e^{i\alpha x}\,dx. \tag{12.20}$$

We now multiply (12.20) by $g_2(\alpha)$ and integrate with respect to α:

$$\int_{-\infty}^{\infty} \bar{g}_1(\alpha)g_2(\alpha)\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \bar{f}_1(x)e^{i\alpha x}\,dx\right] g_2(\alpha)d\alpha. \tag{12.21}$$

Let us rearrange (12.21) so that we integrate first with respect to α. [This is justified assuming that the absolute values of the functions f_1 and f_2 are integrable on $(-\infty, \infty)$.]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(x)\,dx\left[\int_{-\infty}^{\infty} g_2(\alpha)e^{i\alpha x}\,d\alpha\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(x)f_2(x)\,dx \tag{12.22}$$

by (12.2). Thus

$$\int_{-\infty}^{\infty} \bar{g}_1(\alpha)g_2(\alpha)\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_1(x)f_2(x)\,dx. \tag{12.23}$$

(Compare this with the corresponding Fourier series theorem in Problem 11.10.) If we set $f_1 = f_2 = f$ and $g_1 = g_2 = g$, we get Parseval's theorem:

$$\int_{-\infty}^{\infty} |g(\alpha)|^2\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2\,dx. \tag{12.24}$$

▶ PROBLEMS, SECTION 12

1. Following a method similar to that used in obtaining equations (12.11) to (12.14), show that if $f(x)$ is even, then $g(\alpha)$ is even too. Show that in this case $f(x)$ and $g(\alpha)$ can be written as Fourier cosine transforms and obtain (12.15).

2. Do Example 1 above by using a cosine transform (12.15). Obtain (12.17); for $x > 0$, the 0 to ∞ integral represents the function

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

Represent this function also by a Fourier sine integral (see the paragraph just before Parseval's theorem).

In Problems 3 to 12, find the exponential Fourier transform of the given $f(x)$ and write $f(x)$ as a Fourier integral [that is, find $g(\alpha)$ in equation (12.2) and substitute your result into the first integral in equation (12.2)].

3. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$

4. $f(x) = \begin{cases} 1, & \pi/2 < |x| < \pi \\ 0, & \text{otherwise} \end{cases}$

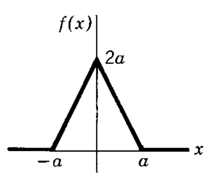
5. $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

7. $f(x) = \begin{cases} |x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

8. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

9.



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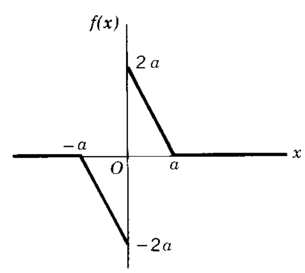


11. $f(x) = \begin{cases} \cos x, & -\pi/2 < x < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$

12. $f(x) = \begin{cases} \sin x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$

Hint: In Problems 11 and 12, use complex exponentials.

In Problems 13 to 16, find the Fourier cosine transform of the function in the indicated problem, and write $f(x)$ as a Fourier integral [use equation (12.15)]. Verify that the cosine integral for $f(x)$ is the same as the exponential integral found previously.

13. Problem 4.

14. Problem 7.

15. Problem 9.

16. Problem 11.

In Problems 17 to 20, find the Fourier sine transform of the function in the indicated problem, and write $f(x)$ as a Fourier integral [use equation (12.14)]. Verify that the sine integral for $f(x)$ is the same as the exponential integral found previously.

17. Problem 3.

18. Problem 6.

19. Problem 10.

20. Problem 12.

21. Find the Fourier transform of $f(x) = e^{-x^2/(2\sigma^2)}$. *Hint:* Complete the square in the x terms in the exponent and make the change of variable $y = x + \sigma^2 i\alpha$. Use tables or computer to evaluate the definite integral.

22. The function $j_1(\alpha) = (\alpha\cos\alpha - \sin\alpha)/\alpha$ is of interest in quantum mechanics. [It is called a spherical Bessel function; see Chapter 12, equation (17.4).] Using Problem 18, show that

$$\int_0^\infty j_1(\alpha)\sin\alpha x\,d\alpha = \begin{cases} \pi x/2, & -1 < x < 1, \\ 0, & |x| > 1. \end{cases}$$

23. Using Problem 17, show that

$$\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\alpha\,d\alpha = \frac{\pi}{2},$$
$$\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\pi\alpha\,d\alpha = \frac{\pi}{4}.$$

24. (a) Find the exponential Fourier transform of $f(x) = e^{-|x|}$ and write the inverse transform. You should find

$$\int_0^\infty \frac{\cos\alpha x}{\alpha^2+1}\,d\alpha = \frac{\pi}{2}e^{-|x|}.$$

(b) Obtain the result in (a) by using the Fourier cosine transform equations (12.15).

(c) Find the Fourier cosine transform of $f(x) = 1/(1+x^2)$. *Hint:* Write your result in (b) with x and α interchanged.

25. (a) Represent as an exponential Fourier transform the function

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi, \\ 0, & \text{otherwise}. \end{cases}$$

Hint: Write $\sin x$ in complex exponential form.

(b) Show that your result can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \cos\alpha(x-\pi)}{1-\alpha^2}\,d\alpha.$$

26. Using Problem 15, show that

$$\int_0^\infty \frac{1-\cos\alpha}{\alpha^2}\,d\alpha = \frac{\pi}{2}.$$

Represent each of the following functions (a) by a Fourier cosine integral; (b) by a Fourier sine integral. *Hint:* See the discussion just before Parseval's theorem.

27. $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ 0, & x > \pi/2 \end{cases}$

28. $f(x) = \begin{cases} 1, & 2 < x < 4 \\ 0, & 0 < x < 2,\ x > 4 \end{cases}$

29. $f(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 3 \\ 0, & x > 3 \end{cases}$

30. $f(x) = \begin{cases} 1 - x/2, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$

Verify Parseval's theorem (12.24) for the special cases in Problems 31 to 33.

31. $f(x)$ as in Figure 12.1. *Hint:* Integrate by parts and use (12.18) to evaluate

$$\int_{-\infty}^{\infty} |g(\alpha)|^2 \, d\alpha.$$

32. $f(x)$ and $g(\alpha)$ as in Problem 21.

33. $f(x)$ and $g(\alpha)$ as in Problem 24a.

34. Show that if (12.2) is written with the factor $1/\sqrt{2\pi}$ multiplying each integral, then the corresponding form of Parseval's theorem (12.24) is

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |g(\alpha)|^2 \, d\alpha.$$

35. Starting with the symmetrized integrals as in Problem 34, make the substitutions $\alpha = 2\pi p/h$ (where p is the new variable, h is a constant), $f(x) = \psi(x)$, $g(\alpha) = \sqrt{h/2\pi}\,\phi(p)$; show that then

$$\psi(x) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \phi(p) e^{2\pi i p x/h} \, dp,$$
$$\phi(p) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i p x/h} \, dx,$$
$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = \int_{-\infty}^{\infty} |\phi(p)|^2 \, dp.$$

This notation is often used in quantum mechanics.

36. Normalize $f(x)$ in Problem 21; that is find the factor N so that $\int_{-\infty}^{\infty} |Nf(x)|^2 = 1$. Let $\psi(x) = Nf(x)$, and find $\phi(p)$ as given in Problem 35. Verify Parseval's theorem, that is, show that $\int_{-\infty}^{\infty} |\phi(p)|^2 \, dp = 1$.

▶ 13. MISCELLANEOUS PROBLEMS

1. The displacement (from equilibrium) of a particle executing simple harmonic motion may be either $y = A \sin \omega t$ or $y = A \sin(\omega t + \phi)$ depending on our choice of time origin. Show that the average of the kinetic energy of a particle of mass m (over a period of the motion) is the same for the two formulas (as it must be since both describe the same physical motion). Find the average value of the kinetic energy for the $\sin(\omega t + \phi)$ case in two ways:

(a) By selecting the integration limits (as you may by Problem 4.1) so that a change of variable reduces the integral to the $\sin \omega t$ case.

(b) By expanding $\sin(\omega t + \phi)$ by the trigonometric addition formulas and using (5.2) to write the average values.

2. The symbol $[x]$ means the greatest integer less than or equal to x (for example, $[3] = 3$, $[2.1] = 2$, $[-4.5] = -5$). Expand $x - [x] - \frac{1}{2}$ in an exponential Fourier series of period 1. *Hint:* Sketch the function.

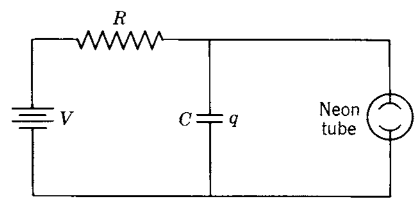
Answer: $\dfrac{i}{2\pi}\left(\cdots - \dfrac{e^{-4\pi i x}}{2} - \dfrac{e^{-2\pi i x}}{1} + \dfrac{e^{2\pi i x}}{1} + \dfrac{e^{4\pi i x}}{2} + \cdots\right).$

3. We have said that Fourier series can represent discontinuous functions although power series cannot. It might occur to you to wonder why we could not substitute the power series for $\sin nx$ and $\cos nx$ (which converge for all x) into a Fourier series and collect terms to obtain a power series for a discontinuous function. As an example of what happens if we try this, consider the series in Problem 9.5. Show that the coefficients of x, if collected, form a divergent series; similarly, the coefficients of x^3 form a divergent series, and so on.

4. The diagram shows a "relaxation" oscillator. The charge q on the capacitor builds up until the neon tube fires and discharges the capacitor (we assume instantaneously). Then the cycle repeats itself over and over.

(a) The charge q on the capacitor satisfies the differential equation

$$R\frac{dq}{dt} + \frac{q}{c} = V,$$



where R is the resistance, C is the capacitance, and V is the constant d-c voltage, as shown in the diagram. Show that if $q = 0$ when $t = 0$, then at any later time t (during one cycle, that is, before the neon tube fires)

$$q = CV(1 - e^{-t/RC}).$$

(b) Suppose the neon tube fires at $t = \frac{1}{2}RC$. Sketch q as a function of t for several cycles.

(c) Expand the periodic q in part (b) in an appropriate Fourier series.

5. Consider one arch of $f(x) = \sin x$. Show that the average value of $f(x)$ over the middle third of the arch is twice the average value over the end thirds.

6. Let $f(t) = e^{i\omega t}$ on $(-\pi, \pi)$. Expand $f(t)$ in a complex exponential Fourier series of period 2π. (Assume $\omega \neq$ integer.)

7. Given $f(x) = |x|$ on $(-\pi, \pi)$, expand $f(x)$ in an appropriate Fourier series of period 2π.

8. From facts you know, find in your head the average value of

(a) $x^3 - 3\sinh 2x + \sin^2 \pi x + \cos 3\pi x$ on $(-5, 5)$.

(b) $2\sin^2 3x - 4\cos x + 5x\cosh 2x - x\cos^2 x$ on $(-\pi, \pi)$.

9. Given $f(x) = \begin{cases} x, & 0 < x < 1, \\ -2, & 1 < x < 2. \end{cases}$

(a) Sketch at least three periods of the graph of the function represented by the sine series for $f(x)$. Without finding any series, answer the following questions:

(b) To what value does the sine series in (a) converge at $x = 1$? At $x = 2$? At $x = 0$? At $x = -1$?

(c) If the given function is continued with period 2 and then is represented by a complex exponential series $\sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$, what is the value of $\sum_{n=-\infty}^{\infty} |c_n|^2$?

10. (a) Sketch at least three periods of the graph of the function represented by the cosine series for $f(x)$ in Problem 9.

(b) Sketch at least three periods of the graph of the exponential Fourier series of period 2 for $f(x)$ in Problem 9.

(c) To what value does the cosine series in (a) converge at $x = 0$? At $x = 1$? At $x = 2$? At $x = -2$?

(d) To what value does the exponential series in (b) converge at $x = 0$? At $x = 1$? At $x = \frac{3}{2}$? At $x = -2$.

11. Find the three Fourier series in Problems 9 and 10.

12. What would be the apparent frequency of a sound wave represented by

$$p(t) = \sum_{n=1}^{\infty} \frac{\cos 60n\pi t}{100(n-3)^2 + 1}?$$

13. (a) Given $f(x) = (\pi - x)/2$ on $(0, \pi)$, find the sine series of period 2π for $f(x)$.

(b) Use your result in (a) to evaluate $\sum 1/n^2$.

14. (a) Find the Fourier series of period 2 for $f(x) = (x-1)^2$ on $(0, 2)$.

(b) Use your result in (a) to evaluate $\sum 1/n^4$.

15. Given

$$f(x) = \begin{cases} 1, & -2 < x < 0, \\ -1, & 0 < x < 2, \end{cases}$$

find the exponential Fourier transform $g(\alpha)$ and the sine transform $g_s(\alpha)$. Write $f(x)$ as an integral and use your result to evaluate

$$\int_0^\infty \frac{(\cos 2\alpha - 1)\sin 2\alpha}{\alpha}\, d\alpha.$$

16. Given

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2, \\ 0, & x \ge 2, \end{cases}$$

find the cosine transform of $f(x)$ and use it to write $f(x)$ as an integral. Use your result to evaluate

$$\int_0^\infty \frac{\cos^2 \alpha \sin^2 \alpha/2}{\alpha^2}\, d\alpha.$$

17. Show that the Fourier sine transform of $x^{-1/2}$ is $\alpha^{-1/2}$. *Hint:* Make the change of variable $z = \alpha x$. The integral $\int_0^\infty z^{-1/2} \sin z\, dz$ can be found by computer or in tables.

18. Let $f(x)$ and $g(\alpha)$ be a pair of Fourier transforms. Show that df/dx and $i\alpha g(\alpha)$ are a pair of Fourier transforms. *Hint:* Differentiate the first integral in (12.2) under the integral sign with respect to x. Use (12.23) to show that

$$\int_{-\infty}^{\infty} \alpha |g(\alpha)|^2\, d\alpha = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \bar{f}(x) \frac{d}{dx} f(x)\, dx.$$

Comment: This result is of interest in quantum mechanics where it would read, in the notation of Problem 12.35:

$$\int_{-\infty}^{\infty} p|\phi(p)|^2\, dp = \int_{-\infty}^{\infty} \psi^*(x) \left(\frac{-ih}{2\pi} \frac{d}{dx} \right) \psi(x)\, dx.$$

19. Find the form of Parseval's theorem (12.24) for sine transforms (12.14) and for cosine transforms (12.15).

20. Find the exponential Fourier transform of

$$f(x) = \begin{cases} 2a - |x|, & |x| < 2a, \\ 0, & |x| > 2a, \end{cases}$$

and use your result with Parseval's theorem to evaluate

$$\int_0^\infty \frac{\sin^4 a\alpha}{\alpha^4}\, d\alpha.$$

21. Define a function $h(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$, assuming that the series converges to a function satisfying Dirichlet conditions (Section 6). Verify that $h(x)$ does have period 2π.

(a) Expand $h(x)$ in an exponential Fourier series $h(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$; show that $c_n = g(n)$ where $g(\alpha)$ is the Fourier transform of $f(x)$. *Hint:* Write c_n as an integral from 0 to 2π and make the change of variable $u = x + 2k\pi$. Note that $e^{-2ink\pi} = 1$, and the sum on k gives a single integral from $-\infty$ to ∞.

(b) Let $x = 0$ in (a) to get *Poisson's summation formula* $\sum_{-\infty}^{\infty} f(2k\pi) = \sum_{-\infty}^{\infty} g(n)$. This result has many applications; for example: statistical mechanics, communication theory, theory of optical instruments, scattering of light in a liquid, and so on. (See Problem 22.)

22. Use Poisson's formula (Problem 21b) and Problem 20 to show that

$$\sum_{-\infty}^{\infty} \frac{\sin^2 n\theta}{n^2} = \pi\theta, \quad 0 < \theta < \pi.$$

(This sum is needed in the theory of scattering of light in a liquid.) *Hint:* Consider $f(x)$ and $g(\alpha)$ as in Problem 20. Note that $f(2k\pi) = 0$ except for $k = 0$ if $a < \pi$. Put $\alpha = n$, $a = \theta$.

23. Use Parseval's theorem and Problem 12.11 to evaluate

$$\int_0^\infty \frac{\cos^2(\alpha\pi/2)}{(1-\alpha^2)^2}\, d\alpha.$$

CHAPTER 8

Ordinary Differential Equations

1. INTRODUCTION

A great many applied problems involve rates, that is, derivatives. An equation containing derivatives is called a *differential equation*. If it contains partial derivatives, it is called a *partial differential equation*; otherwise it is called an *ordinary differential equation*. In this chapter we shall consider methods of solving ordinary differential equations which occur frequently in applications. Let us look at a few examples.

Newton's second law in vector form is $\mathbf{F} = m\mathbf{a}$. If we write the acceleration as $d\mathbf{v}/dt$, where $\mathbf{v}$ is the velocity, or as $d^2\mathbf{r}/dt^2$, where $\mathbf{r}$ is the displacement, we have a differential equation (or a set of differential equations, one for each component). Thus any mechanics problem in which we want to describe the motion of a body (automobile, electron, or satellite) under the action of a given force, involves the solution of a differential equation or a set of differential equations.

The rate at which heat Q escapes through a window or from a hot water pipe is proportional to the area A and to the rate of change of temperature with distance in the direction of flow of heat. Thus we have

$$\frac{dQ}{dt} = kA\frac{dT}{dx} \tag{1.1}$$

(k is called the thermal conductivity and depends on the material through which the heat is flowing). Here we have two different derivatives in the differential equation. In such a problem we might know either dT/dx or dQ/dt and solve the differential equation to find either T as a function of x, or Q as a function of t. (See Problems 2.23 to 2.25.)

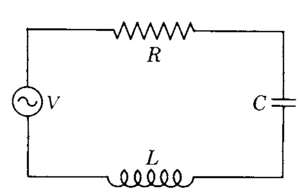


Figure 1.1

Consider a simple series circuit (Figure 1.1) containing a resistance R, a capacitance C, an inductance L, and a source of emf V. If the current flowing around the circuit at time t is $I(t)$ and the charge on the capacitor is $q(t)$, then $I = dq/dt$. The voltage across R is RI, the voltage across C is q/C, and the voltage across L is

$L(dI/dt)$. Then at any time we must have

$$L\frac{dI}{dt} + RI + \frac{q}{C} = V. \tag{1.2}$$

If we differentiate this equation with respect to t and substitute $dq/dt = I$, we have

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt} \tag{1.3}$$

as the differential equation satisfied by the current I in a simple series circuit with given L, R, and C, and a given $V(t)$.

There are many more examples of physical problems leading to differential equations; we shall consider some of them later in the text and problems. You might find it interesting at this point to browse through the problems to see the wide range of topics giving rise to differential equations.

The *order* of a differential equation is the order of the highest derivative in the equation. Thus the equations

$$\begin{aligned} y' + xy^2 &= 1, \\ xy' + y &= e^x, \\ \frac{dv}{dt} &= -g, \\ L\frac{dI}{dt} + RI &= V, \end{aligned} \tag{1.4}$$

are first-order equations, while (1.3) and

$$m\frac{d^2r}{dt^2} = -kr$$

are second-order equations. A *linear* differential equation (with x as independent and y as dependent variable) is one of the form

$$a_0y + a_1y' + a_2y'' + a_3y''' + \cdots = b,$$

where the a's and b are either constants or functions of x. The first equation in (1.4) is not linear because of the y^2 term; all the other equations we have mentioned so far are linear. Some other examples of nonlinear equations are:

$$\begin{aligned} y' &= \cot y && \text{(not linear because of the term } \cot y\text{);} \\ yy' &= 1 && \text{(not linear because of the product } yy'\text{);} \\ y'^2 &= xy && \text{(not linear because of the term } y'^2\text{).} \end{aligned}$$

Many of the differential equations which occur in applied problems are linear and of the first or second order; we shall be particularly interested in these.

> A *solution* of a differential equation (in the variables x and y) is a relation between x and y which, if substituted into the differential equation, gives an identity.

▶ **Example 1.** The relation

$$y = \sin x + C \tag{1.5}$$

is a solution of the differential equation

$$y' = \cos x \tag{1.6}$$

because if we substitute (1.5) into (1.6) we get the identity $\cos x = \cos x$.

▶ **Example 2.** The equation $y'' = y$ has solutions $y = e^x$ or $y = e^{-x}$ or $y = Ae^x + Be^{-x}$ as you can verify by substitution.

If we integrate $y' = f(x)$, the expression for y, namely $y = \int f(x)\,dx + C$, contains one arbitrary constant of integration. If we integrate $y'' = g(x)$ twice to get $y(x)$, then y contains two independent integration constants. We might expect that in general the solution of a differential equation of the nth order would contain n independent arbitrary constants. Note that in Example 1 above, the solution of the first-order equation $y' = \cos x$ contained one arbitrary constant C, and in Example 2 the solution $y = Ae^x + Be^{-x}$ of the second-order equation $y'' = y$ contained two arbitrary constants A and B.

> Any *linear* differential equation of order n has a solution containing n independent arbitrary constants, from which *all* solutions of the differential equation can be obtained by letting the constants have particular values. This solution is called the *general* solution of the linear differential equation.

(This may not be true for nonlinear equations; see Section 2.)

In applications, we usually want a *particular* solution, that is, one which satisfies the differential equation and some other requirements as well. Here are some examples of this.

▶ **Example 3.** Find the distance which an object falls under gravity in t seconds if it starts from rest.

Let x be the distance the object has fallen in time t. The acceleration of the object is g, the acceleration of gravity. Then we have

$$\frac{d^2x}{dt^2} = g. \tag{1.7}$$

Integrating, we get

$$\frac{dx}{dt} = gt + \text{const.} = gt + v_0, \tag{1.8}$$

$$x = \tfrac{1}{2}gt^2 + v_0 t + x_0, \tag{1.9}$$

where v_0 and x_0 are the values of v and x at $t = 0$. Now (1.9) is the *general* solution of (1.7) (because it is a solution of a second-order linear differential equation and contains two independent arbitrary constants). We want the *particular* solution for which $v_0 = 0$ (since the object starts from rest), and $x_0 = 0$ (since the distance the object has fallen is zero at $t = 0$). Then the desired particular solution is

$$x = \tfrac{1}{2}gt^2. \tag{1.10}$$

Example 4. Find the solution of $y'' = y$ which passes through the origin and through the point $(\ln 2, \frac{3}{4})$.

The general solution of the differential equation is

$$y = Ae^x + Be^{-x}$$

(see Example 2). If the given points satisfy the equation of the curve, we must have

$$0 = A + B \qquad \text{or } A = -B,$$
$$\tfrac{3}{4} = Ae^{\ln 2} + Be^{-\ln 2} = A \cdot 2 + B \cdot \tfrac{1}{2} = 2A - \tfrac{1}{2}A = \tfrac{3}{2}A.$$

Thus we get

$$A = -B = \tfrac{1}{2},$$

and the desired particular solution is

$$y = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x.$$

The given conditions which are to be satisfied by the particular solution are called *boundary conditions*, or when they are conditions at $t = 0$ they may be called *initial conditions*. For linear equations, the desired particular solution can be found from the general solution by determining the values of the constants as we did in Example 4. (For nonlinear equations, see Section 2.)

As you study methods of solving various types of differential equations in the following sections, you may wonder whether you can use computer solutions and not bother to learn these techniques. Just as for indefinite integrals (see Chapter 5, Section 1), there may be various forms for the solution of a differential equation, and your computer may not give the one you need. In order to make intelligent use of computer solutions, you need to know something about what to expect, and an effective way of gaining this knowledge is to solve some equations by hand. (See Example 1, Section 3.) By comparing your solutions with computer solutions, you will learn what you can (and cannot) expect from your computer.

The graphing capabilities of your computer are very useful in differential equations. Consider a first-order equation, say $y' = f(x, y)$. If the solution of this differential equation is $y = y(x)$, the differential equation gives the slope y' of the solution curve at each point (x, y). Suppose, for a large number of points, we draw a short line (or vector) centered on each point and with the slope y' at that point. (This would be a big job by hand, but your computer does it easily.) This plot is called a *slope field*, or a *direction field*, or a *vector field*. From such a diagram we can see the general trend of the solution curves even without solving the equation.

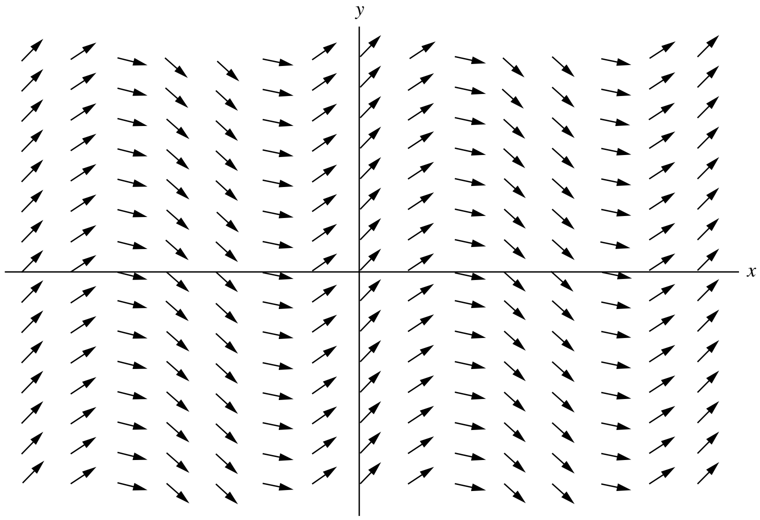


Figure 1.2

▶ **Example 5.** In Figure 1.2, we have plotted a "slope field" for the differential equation $y' = \cos x$. Note how you can trace the general shape of the solution curves, even without knowing from Example 1 that their equations are $y = \sin x + C$.

▶ PROBLEMS, SECTION 1

1. Verify the statement of Example 2. Also verify that $y = \cosh x$ and $y = \sinh x$ are solutions of $y'' = y$.

2. Solve Example 4 using the general solution $y = a \sinh x + b \cosh x$.

3. Verify that $y = \sin x$, $y = \cos x$, $y = e^{ix}$, and $y = e^{-ix}$ are all solutions of $y'' = -y$.

4. Find the distance which an object moves in time t if it starts from rest and has an acceleration $d^2x/dt^2 = ge^{-kt}$. Show that for small t the result is approximately (1.10), and for very large t, the speed dx/dt is approximately constant. The constant is called the terminal speed. (This problem corresponds roughly to the motion of a parachutist.)

5. Find the position x of a particle at time t if its acceleration is $d^2x/dt^2 = A \sin \omega t$.

6. A substance evaporates at a rate proportional to the exposed surface. If a spherical mothball of radius $\frac{1}{2}$ cm has radius 0.4 cm after 6 months, how long will it take:

(a) For the radius to be $\frac{1}{4}$ cm?

(b) For the volume of the mothball to be half of what it was originally?

7. The momentum p of an electron at speed v near the speed c of light increases according to the formula $p = mv/\sqrt{1 - v^2/c^2}$, where m is a constant (mass of the electron). If an electron is subject to a constant force F, Newton's second law describing its motion is

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mv}{\sqrt{1 - v^2/c^2}} = F.$$

Find $v(t)$ and show that $v \to c$ as $t \to \infty$. Find the distance traveled by the electron in time t if it starts from rest.

2. SEPARABLE EQUATIONS

Every time you evaluate an integral

$$y = \int f(x)\,dx, \tag{2.1}$$

you are solving a differential equation, namely

$$y' = \frac{dy}{dx} = f(x). \tag{2.2}$$

This is a simple example of an equation which can be written with only y terms on one side of the equation and only x terms on the other:

$$dy = f(x)\,dx. \tag{2.3}$$

Whenever we can separate the variables this way, we call the equation *separable*, and we get the solution by just integrating each side of the equation.

Example 1. The rate at which a radioactive substance decays is proportional to the remaining number of atoms. If there are N_0 atoms at $t = 0$, find the number at time t.

The differential equation for this problem is

$$\frac{dN}{dt} = -\lambda N. \tag{2.4}$$

(The proportionality constant λ is called the decay constant.) This is a separable equation; we write it as $dN/N = -\lambda\,dt$. Then integrating both sides, we get $\ln N = -\lambda t + \text{const.}$ Since we are given $N = N_0$ at $t = 0$, we see that the constant is $\ln N_0$. Solving for N, we have

$$N = N_0 e^{-\lambda t}. \tag{2.5}$$

(For further discussion of radioactive decay problems, see Section 3, Example 2, and Problems 2.19b and 3.19 to 3.21.)

Example 2. Solve the differential equation

$$xy' = y + 1. \tag{2.6}$$

To separate variables, we divide both sides of (2.6) by $x(y+1)$; this gives

$$\frac{y'}{y+1} = \frac{1}{x} \qquad \text{or} \qquad \frac{dy}{y+1} = \frac{dx}{x}. \tag{2.7}$$

Integrating each side of (2.7), we have

$$\ln(y+1) = \ln x + \text{const.} = \ln x + \ln a = \ln(ax). \tag{2.8}$$

(We have called the constant of integration $\ln a$ for simplicity.) Then (2.8) gives the solution of (2.6), namely

$$y + 1 = ax. \tag{2.9}$$

This general solution represents a *family* of curves in the (x, y) plane, one curve for each value of the constant a. Or we may call the general solution (2.9) a *family of solutions* of the differential equation (2.6). Finding a particular solution means selecting one particular curve from the family.

Orthogonal Trajectories In Figure 2.1, the straight lines through $(0, -1)$ are the family of curves given by the solutions (2.9) of the differential equation (2.6). They might represent, for example, the lines of electric force due to an electric charge at $(0, -1)$. The circles in Figure 2.1 are then curves of constant electrostatic potential (called equipotentials—see Chapter 6, Sections 5 and 6). Note that the lines of force intersect the equipotential curves at right angles; each family of curves is called a set of *orthogonal trajectories* of the other family. It is often of interest to find the orthogonal trajectories of a given family of curves. Let us do this for the family (2.9). (In this case we know in advance that our answer will be the set of circles in Figure 2.1.)

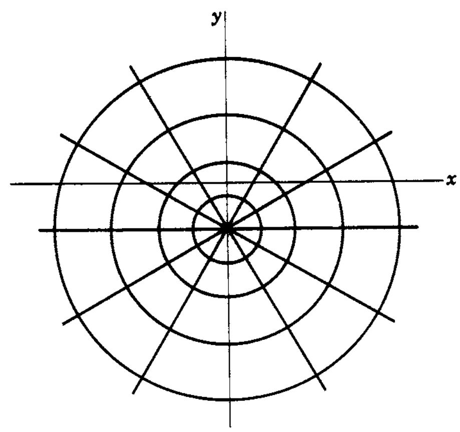


Figure 2.1

First we find the slope of a line of the family (2.9), namely,

$$y' = a. \tag{2.10}$$

For each a this gives the slope of *one* line. We want a formula (as a function of x and y) which gives the slope, at any point of the plane, of the line through that point. To obtain this, we eliminate a between (2.9) and (2.10) to get

$$y' = \frac{y+1}{x}. \tag{2.11}$$

[Or, given (2.6) rather than (2.9), we could simply solve for y'.] Now recall from analytic geometry that the slopes of two perpendicular lines are negative reciprocals. Then at each point we want the slope of the orthogonal trajectory curve to be the negative reciprocal of the slope of the line given by (2.11). Thus,

$$y' = -\frac{x}{y+1} \tag{2.12}$$

gives the slope of the orthogonal trajectories, and we solve (2.12) to obtain the equation of the orthogonal trajectory curves. Now (2.12) is separable; we obtain

$$\begin{aligned}
(y+1)\,dy &= -x\,dx,\\
\tfrac{1}{2}y^2 + y &= -\tfrac{1}{2}x^2 + C,\\
x^2 + y^2 + 2y &= 2C,\\
x^2 + (y+1)^2 &= 2C + 1.
\end{aligned}$$

This is, as we expected, a family of circles with centers at the point $(0, -1)$.

Nonlinear Differential Equations We have said that for linear differential equations of order n there is always a general solution containing n independent constants, and *all* solutions can be obtained by specializing the constants. You should be aware that this may not be true for some nonlinear equations, and routine methods of solution (including computer) may sometimes give partially incorrect or incomplete solutions. It is beyond our scope to discuss this in detail (see differential equations books), but here are some examples.

Example 3. Solve the differential equation $y' = \sqrt{1-y^2}$ and computer plot the slope field and a set of solution curves. Find particular solutions satisfying (a) $y = 0$ when $x = 0$, and (b) $y = 1$ when $x = 0$.

We separate variables and integrate to get

$$\frac{dy}{\sqrt{1-y^2}} = dx, \quad \arcsin y = x + \alpha, \quad y = \sin(x+\alpha).$$

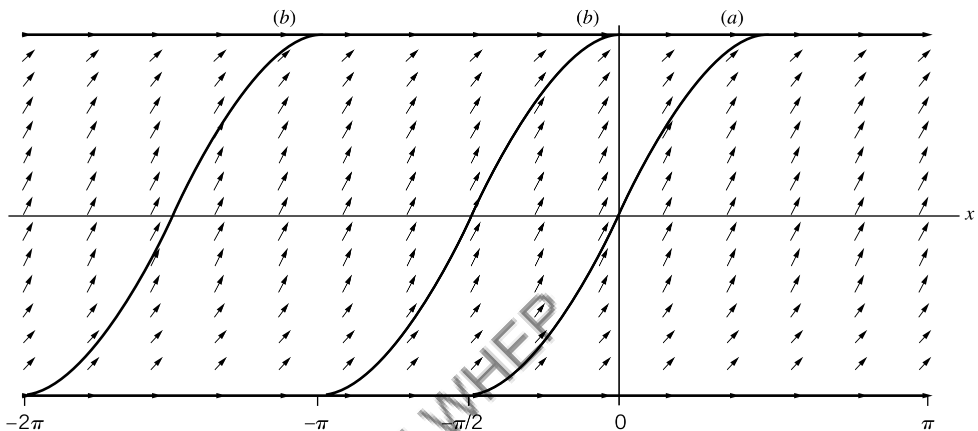


Figure 2.2

A computer gives the same answer. However if we look at either a computer plot of the slope field (Figure 2.2), or the differential equation itself, we see that the slope y' is always non-negative. Thus the solution of the given differential equation includes only the parts of the sine curves with non-negative slopes (Figure 2.2). A second difficulty is that part of the solution is missing. From either the slope field (Figure 2.2) or directly from the differential equation we can see that $y \equiv 1$ and $y \equiv -1$ are solutions not obtainable from the sine solution by any choice of α. (These are sometimes called *singular* solutions.) The fact that we did not find these solutions by separation of variables should not surprise us when we note that in separating variables we divided by $\sqrt{1-y^2}$ and this step is not valid if $y^2 = 1$.

Now for the particular solution (a) passing through $(0,0)$, the sine solution gives either $y = \sin x$ or $y = \sin(x+\pi) = -\sin x$. But since we know that y' is non-negative, only the $y = \sin x$ solution is correct in the vicinity of $x = 0$. In fact (Figure 2.2), $y = \sin x$ is a correct particular solution from $x = -\pi/2$ to $x = \pi/2$. We could construct a continuous solution from $-\infty$ to ∞ by letting $y = -1$ from $-\infty$ to $-\pi/2$, $y = \sin x$ from $-\pi/2$ to $\pi/2$, and $y = 1$ from $\pi/2$ to ∞. Thus for case (a) [solution passing through the origin] we find just *one* particular solution.

For particular solution (b) [passing through $(0,1)$], we find either $y = \sin(x+\frac{\pi}{2}) = \cos x$, or $y \equiv 1$; the $\cos x$ solution is valid from $x = -\pi$ to $x = 0$. As in (a), we can extend it by using parts of $y = -1$ and $y = 1$; this is one particular solution. But there are an infinite number of other particular solutions passing through $(0,1)$ obtained by moving this one solution any distance to the left (Figure 2.2).

▶ PROBLEMS, SECTION 2

For each of the following differential equations, separate variables and find a solution containing one arbitrary constant. Then find the value of the constant to give a particular solution satisfying the given boundary condition. Computer plot a slope field and some of the solution curves.

1. $xy' = y,$ $\qquad y = 3$ when $x = 2.$

2. $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0,$ $\qquad y = \frac{1}{2}$ when $x = \frac{1}{2}.$

3. $y' \sin x = y \ln y,$ $\qquad y = e$ when $x = \pi/3.$

4. $(1+y^2)\,dx + xy\,dy = 0,$ $\qquad y = 0$ when $x = 5.$

5. $xy' - xy = y,$ $\qquad y = 1$ when $x = 1.$

6. $y' = \dfrac{2xy^2 + x}{x^2 y - y},$ $\qquad y = 0$ when $x = \sqrt{2}.$

7. $y\,dy + (xy^2 - 8x)\,dx = 0,$ $\qquad y = 3$ when $x = 1.$

8. $y' + 2xy^2 = 0,$ $\qquad y = 1$ when $x = 2.$

9. $(1+y)y' = y,$ $\qquad y = 1$ when $x = 1.$

10. $y' - xy = x,$ $\qquad y = 1$ when $x = 0.$

11. $2y' = 3(y-2)^{1/3},$ $\qquad y = 3$ when $x = 1.$

12. $(x + xy)y' + y = 0,$ $\qquad y = 1$ when $x = 1.$

In Problems 13 to 15, find a solution (or solutions) of the differential equation not obtainable by specializing the constant in your solution of the original problem. *Hint:* See Example 3.

13. Problem 2. **14.** Problem 8. **15.** Problem 11.

16. By separation of variables, find a solution of the equation $y' = \sqrt{y}$ containing one arbitrary constant. Find a particular solution satisfying $y = 0$ when $x = 0$. Show that $y \equiv 0$ is a solution of the differential equation which cannot be obtained by specializing the arbitrary constant in your solution above. Computer plot a slope field and some of the solution curves. Show that there are an infinite number of solution curves passing through any point on the x axis, but just one through any point for which $y > 0$. *Hint:* See Example 3. Problems 17 and 18 are physical problems leading to this differential equation.

17. The speed of a particle on the x axis, $x \ge 0$, is always numerically equal to the square root of its displacement x. If $x = 0$ when $t = 0$, find x as a function of t. Show that the given conditions are satisfied if the particle remains at the origin for any arbitrary length of time t_0 and then moves away; find x for $t > t_0$ for this case.

18. Let the rate of growth dN/dt of a colony of bacteria be proportional to the square root of the number present at any time. If there are no bacteria present at $t = 0$, how many are there at a later time? Observe here that the routine separation of variables solution gives an unreasonable answer, and the correct answer, $N \equiv 0$, is not obtainable from the routine solution. (You have to think, not just follow rules!)

19. (a) Consider a light beam traveling downward into the ocean. As the beam progresses, it is partially absorbed and its intensity decreases. The rate at which the intensity is decreasing with depth at any point is proportional to the intensity at that depth. The proportionality constant μ is called the *linear absorption coefficient*. Show that if the intensity at the surface is I_0, the intensity at a distance s below the surface is $I = I_0 e^{-\mu s}$. The linear absorption coefficient for water is of the order of 10^{-2} ft^{-1} (the exact value depending on the wavelength of the light and the impurities in the water). For this value of μ, find the intensity as a fraction of the surface intensity at a depth of 1 ft, 50 ft, 500 ft, 1 mile. When the intensity of a light beam has been reduced to half its surface intensity ($I = \frac{1}{2}I_0$), the distance the light has penetrated into the absorbing substance is called the *half-value thickness* of the substance. Find the half-value thickness in terms of μ. Find the half-value thickness for water for the value of μ given above.

(b) Note that the differential equation and its solution in this problem are mathematically the same as those in Example 1, although the physical problem and the terminology are different. In discussing radioactive decay, we call λ the *decay constant*, and we define the *half-life* T of a radioactive substance as the time when $N = \frac{1}{2}N_0$ (compare half-value thickness). Find the relation between λ and T.

20. Consider the following special cases of the simple series circuit [Figure 1.1 and equation (1.2)].

(a) RC circuit (that is, $L = 0$) with $V = 0$; find q as a function of t if q_0 is the charge on the capacitor at $t = 0$.

(b) RL circuit (that is, no capacitor; this means $1/C = 0$) with $V = 0$; find $I(t)$ given $I = I_0$ at $t = 0$.

(c) Again note that these are the same differential equations as in Problem 19 and Example 1. The terminology is again different; we define the time constant τ for a circuit as the time required for the charge (or current) to fall to $1/e$ times its initial value. Find the time constant for the circuits (a) and (b). If the same equation, say $y = y_0 e^{-at}$, represented either radioactive decay or light absorption or an RC or RL circuit, what would be the relations among the half-life, the half-value thickness, and the time constant?

21. Suppose the rate at which bacteria in a culture grow is proportional to the number present at any time. Write and solve the differential equation for the number N of bacteria as a function of time t if there are N_0 bacteria when $t = 0$. Again note that (except for a change of sign) this is the same differential equation and solution as in the preceding problems.

22. Solve the equation for the rate of growth of bacteria if the rate of increase is proportional to the number present but the population is being reduced at a constant rate by the removal of bacteria for experimental purposes.

23. Heat is escaping at a constant rate [dQ/dt in (1.1) is constant] through the walls of a long cylindrical pipe. Find the temperature T at a distance r from the axis of the cylinder if the inside wall has radius $r = 1$ and temperature $T = 100$ and the outside wall has $r = 2$ and $T = 0$.

24. Do Problem 23 for a spherical cavity containing a constant source of heat. Use the same radii and temperatures as in Problem 23.

25. Show that the thickness of the ice on a lake increases with the square root of the time in cold weather, making the following simplifying assumptions. Let the water temperature be a constant $10°$C, the air temperature a constant $-10°$, and assume that at any given time the ice forms a slab of uniform thickness x. The rate of formation of ice is proportional to the rate at which heat is transferred from the water to the air. Let $t = 0$ when $x = 0$.

26. An object of mass m falls from rest under gravity subject to an air resistance proportional to its speed. Taking the y axis as positive down, show that the differential equation of motion is $m(dv/dt) = mg - kv$, where k is a positive constant. Find v as a function of t, and find the limiting value of v as t tends to infinity; this limit is called the *terminal speed*. Can you find the terminal speed directly from the differential equation without solving it? *Hint:* What is dv/dt after v has reached an essentially constant value?

Consider the following specific examples of this problem.

(a) A person drops from an airplane with a parachute. Find a reasonable value of k.

(b) In the Millikan oil drop experiment to measure the charge of an electron, tiny electrically charged drops of oil fall through air under gravity or rise under the combination of gravity and an electric field. Measurements can be made only after they have reached terminal speed. Find a formula for the time required for a drop starting at rest to reach 99% of its terminal speed.

27. According to Newton's law of cooling, the rate at which the temperature of an object changes is proportional to the difference between its temperature and that of its surroundings. A cup of coffee at $200°$ in a room of temperature $70°$ is stirred continually and reaches $100°$ after 10 min. At what time was it at $120°$?

28. A glass of milk at $38°$ is removed from the refrigerator and left in a room at temperature $70°$. If the temperature of the milk is $54°$ after 10 min, what will its temperature be in half an hour? (See Problem 27.)

29. A solution containing 90% by volume of alcohol (in water) runs at 1 gal/min into a 100-gal tank of pure water where it is continually mixed. The mixture is withdrawn at the rate of 1 gal/min. When will it start coming out 50% alcohol?

30. If P dollars are left in the bank at interest I percent per year compounded continuously, find the amount A at time t. *Hint:* Find dA, the interest on A dollars for time dt.

Find the orthogonal trajectories of each of the following families of curves. In each case, sketch or computer plot several of the given curves and several of their orthogonal trajectories. Be careful to eliminate the constant from y' for the original curves; this constant takes different values for different curves of the original family, and you want an expression for y' which is valid for all curves of the family crossed by the orthogonal trajectory you are trying to find. See equations (2.10) to (2.12).

31. $x^2 + y^2 = \text{const.}$

32. $y = kx^2$.

33. $y = kx^n$. (Assume that n is a given number; the different curves of the family have different values of k.)

34. $xy = k$.

35. $(y - 1)^2 = x^2 + k$.

3. LINEAR FIRST-ORDER EQUATIONS

A first-order equation contains y' but no higher derivatives. A *linear* first-order equation means one which can be written in the form

$$y' + Py = Q, \tag{3.1}$$

where P and Q are functions of x. To see how to solve (3.1), let us first consider the simpler equation when $Q = 0$. The equation

$$y' + Py = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -Py \tag{3.2}$$

is separable. As in Section 2, we obtain the solution as follows:

$$\begin{aligned} \frac{dy}{y} &= -P\,dx, \\ \ln y &= -\int P\,dx + C, \\ y &= e^{-\int P\,dx + C} = Ae^{-\int P\,dx} \end{aligned} \tag{3.3}$$

where $A = e^C$. Let us simplify the notation for future use; we write

$$I = \int P\,dx. \tag{3.4}$$

Then

$$\frac{dI}{dx} = P \tag{3.5}$$

and we can write (3.3) as $y = Ae^{-I}$ or

$$ye^I = A. \tag{3.6}$$

We can now see how to solve (3.1). If we differentiate (3.6) with respect to x and use (3.5), we get

$$\frac{d}{dx}(ye^I) = y'e^I + ye^I\frac{dI}{dx} = y'e^I + ye^I P = e^I(y' + Py), \tag{3.7}$$

which is the left-hand side of (3.1) multiplied by e^I. (We call e^I an integrating factor—see Section 4.) Thus, we can write (3.1) (times e^I) as

$$\frac{d}{dx}(ye^I) = e^I(y' + Py) = Qe^I. \tag{3.8}$$

Since Q and e^I are functions of x only, we can now integrate both sides of (3.8) with respect to x to get

$$\left.\begin{aligned} ye^I &= \int Qe^I\,dx + c, \qquad \text{or} \\ y &= e^{-I}\int Qe^I\,dx + ce^{-I}, \end{aligned}\right\} \qquad \text{where} \quad I = \int P\,dx. \tag{3.9}$$

This is the general solution of (3.1). Note that it contains one arbitrary constant as expected for a first-order linear equation. The term ce^{-I} is a solution of equation (3.2); the first term in y is one particular solution of (3.1). Borrowing notation which we shall use in Section 6, let's call the term $ce^{-I} = y_c$ and the particular solution $= y_p$. Then $y_p + y_c$ is a solution of (3.1) for any value of c. Also note that $y_p e^I = \int Qe^I\,dx$ is an indefinite integral which, as we know (see Chapter 5, Section 1), has infinitely many answers differing from each other by constants of integration. Thus the particular solution obtained by you and by your computer may not be the same (see Example 1 and Problems).

▶ **Example 1.** Solve $(1+x^2)y' + 6xy = 2x$. In the form of (3.1), this is

$$y' + \frac{6x}{1+x^2}\,y = \frac{2x}{1+x^2}.$$

From (3.9), we get

$$\begin{aligned}
I &= \int \frac{6x}{1+x^2}\,dx = 3\ln(1+x^2)\\
e^I &= e^{3\ln(1+x^2)} = (1+x^2)^3\\
ye^I &= \int \frac{2x}{1+x^2}(1+x^2)^3\,dx = \int 2x(1+x^2)^2\,dx = \tfrac{1}{3}(1+x^2)^3 + c\\
y &= \tfrac{1}{3} + \frac{c}{(1+x^2)^3}.
\end{aligned}$$

A computer gives the answer

$$y = \frac{3x^2+3x^4+x^6}{3(1+x^2)^3} + \frac{A}{(1+x^2)^3}.$$

Let us show that the answers agree (see comments just after (3.9)). If we put $A = c + 1/3$ in the computer solution above and combine terms, we get

$$y = \frac{3x^2+3x^4+x^6+1}{3(1+x^2)^3} + \frac{c}{(1+x^2)^3} = \frac{(1+x^2)^3}{3(1+x^2)^3} + \frac{c}{(1+x^2)^3},$$

which, after cancelling, is our solution above. We see that the computer program chose a more complicated particular solution y_p which differed from our y_p by a multiple of $y_c = 1/(1+x^2)^3$. Always be aware of the possibility of simplifying a particular solution by adding a multiple of y_c.

▶ **Example 2.** Radium decays to radon which decays to polonium. If at $t=0$, a sample is pure radium, how much radon does it contain at time t?

$$\begin{aligned}
\text{Let}\qquad N_0 &= \text{number of radium atoms at } t = 0,\\
N_1 &= \text{number of radium atoms at time } t,\\
N_2 &= \text{number of radon atoms at time } t,\\
\lambda_1 \text{ and } \lambda_2 &= \text{decay constants for Ra and Rn.}
\end{aligned}$$

As in Section 2, we have for radium

$$\frac{dN_1}{dt} = -\lambda_1 N_1, \qquad N_1 = N_0 e^{-\lambda_1 t}.$$

The rate at which radon is being created is the rate at which radium is decaying, namely $\lambda_1 N_1$ or $\lambda_1 N_0 e^{-\lambda_1 t}$. But the radon is also decaying at the rate $\lambda_2 N_2$. Hence, we have

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2, \qquad \text{or}$$
$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_1 = \lambda_1 N_0 e^{-\lambda_1 t}.$$

This equation is of the form (3.1), and we solve it as follows:

$$I = \int \lambda_2\,dt = \lambda_2 t,$$
$$(3.10) \qquad N_2 e^{\lambda_2 t} = \int \lambda_1 N_0 e^{-\lambda_1 t} e^{\lambda_2 t}\,dt + c$$
$$= \lambda_1 N_0 \int e^{(\lambda_2-\lambda_1)t}\,dt + c = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{(\lambda_2-\lambda_1)t} + c,$$

if $\lambda_1 \neq \lambda_2$. (For the case $\lambda_1 = \lambda_2$, see Problem 19.) Since $N_2 = 0$ at $t = 0$ (we assumed pure Ra at $t = 0$), we must have

$$0 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} + c \quad \text{or} \quad c = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}.$$

Substituting this value of c into (3.10) and solving for N_2, we get

$$N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}).$$

PROBLEMS, SECTION 3

Using (3.9), find the general solution of each of the following differential equations. Compare a computer solution and, if necessary, reconcile it with yours. *Hint*: See comments just after (3.9), and Example 1.

1. $y' + y = e^x$

2. $x^2 y' + 3xy = 1$

3. $dy + (2xy - xe^{-x^2})\,dx = 0$

4. $2xy' + y = 2x^{5/2}$

5. $y' \cos x + y = \cos^2 x$

6. $y' + y/\sqrt{x^2+1} = 1/(x + \sqrt{x^2+1}\,)$

7. $(1 + e^x)y' + 2e^x y = (1 + e^x)e^x$

8. $(x \ln x)y' + y = \ln x$

9. $(1 - x^2)y' = xy + 2x\sqrt{1 - x^2}$

10. $y' + y \tanh x = 2e^x$

11. $y' + y \cos x = \sin 2x$

12. $\dfrac{dx}{dy} = \cos y - x \tan y$

13. $dx + (x - e^y)\,dy = 0$

14. $\dfrac{dy}{dx} = \dfrac{3y}{3y^{2/3} - x}$

Hint: For Problems 12 to 14, solve for x in terms of y.

15. Water with a small salt content (5 lb in 1000 gal) is flowing into a very salty lake at the rate of $4 \cdot 10^5$ gal per hr. The salty water is flowing out at the rate of 10^5 gal per hr. If at some time (say $t = 0$) the volume of the lake is 10^9 gal, and its salt content is 10^7 lb, find the salt content at time t. Assume that the salt is mixed uniformly with the water in the lake at all times.

16. Find the general solution of (1.2) for an RL circuit $(1/C = 0)$ with $V = V_0 \cos \omega t$ $(\omega = \text{const.})$.

17. Find the general solution of (1.3) for an RC circuit $(L = 0)$, with $V = V_0 \cos \omega t$.

18. Do Problems 16 and 17 using $V = V_0 e^{i\omega t}$, and find the solutions for 16 and 17 by taking real parts of the complex solutions.

19. If $\lambda_1 = \lambda_2 = \lambda$ in (3.10), then $\int e^{(\lambda_2 - \lambda_1)t}\, dt = \int dt$. Find N_2 for this case.

20. Extend the radioactive decay problem (Example 2) one more stage, that is, let λ_3 be the decay constant of polonium and find how much polonium there is at time t.

21. Generalize Problem 20 to any number of stages.

22. Find the orthogonal trajectories of the family of curves $x = y + 1 + ce^y$. (See the instructions above Problem 2.31.)

23. Find the orthogonal trajectories of the family of curves $y = -e^{x^2} \operatorname{erf} x + Ce^{x^2}$. *Hint:* See Chapter 11, equation (9.1) for definition of erf x, and Chapter 4, Section 12, for differentiation of an integral. Solve for x in terms of y.

▶ 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS

Separable equations and linear equations are the two types of first-order equations you are most apt to meet in elementary applications. However, we shall also mention briefly a few other methods of solving special first-order equations. You will find more details in the problems and in most differential equations books.

The Bernoulli Equation The differential equation

$$y' + Py = Qy^n, \tag{4.1}$$

where P and Q are functions of x, is known as the Bernoulli equation. It is not linear but is easily reduced to a linear equation. We make the change of variable

$$z = y^{1-n}. \tag{4.2}$$

Then

$$z' = (1-n)y^{-n}y'. \tag{4.3}$$

Next multiply (4.1) by $(1-n)y^{-n}$ and make the substitutions (4.2) and (4.3) to get

$$\begin{aligned}(1-n)y^{-n}y' + (1-n)Py^{1-n} &= (1-n)Q,\\ z' + (1-n)Pz &= (1-n)Q.\end{aligned}$$

This is now a first-order linear equation which we can solve as we did the linear equations above. (See Section 7 for an example of a physical problem in which we need to solve a Bernoulli equation.)

Exact Equations; Integrating Factors Recall from Chapter 6, Section 8, that the expression $P(x, y)\,dx + Q(x, y)\,dy$ is an *exact differential* [that is, the differential of a function $F(x, y)$] if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{4.4}$$

If (4.4) holds, then there is a function $F(x, y)$ such that

$$P = \frac{\partial F}{\partial x}, \quad Q = \frac{\partial F}{\partial y}, \quad P\,dx + Q\,dy = dF. \tag{4.5}$$

In Chapter 6 we considered ways of finding F when (4.4) holds. The differential equation

$$P\,dx + Q\,dy = 0 \qquad \text{or} \quad y' = -\frac{P}{Q} \tag{4.6}$$

is called *exact* if (4.4) holds. In this case

$$P\,dx + Q\,dy = dF = 0,$$

and the solution of (4.6) is then

$$F(x, y) = \text{const.} \tag{4.7}$$

We find F as in Chapter 6, Section 8.

An equation which is not exact may often be made exact by multiplying it by an appropriate factor.

Example 1. The equation

$$x\,dy - y\,dx = 0 \tag{4.8}$$

is not exact [by (4.4)]. But the equation

$$\frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}\,dy - \frac{y}{x^2}\,dx = d\left(\frac{y}{x}\right) = 0, \tag{4.9}$$

obtained by dividing (4.8) by x^2, *is* exact [use (4.4)], and its solution is

$$\frac{y}{x} = \text{const.} \tag{4.10}$$

We multiplied (4.8) by $1/x^2$ to make the equation exact; the factor $1/x^2$ is called an *integrating factor*. To see another example of an integrating factor, look back at Section 3. The expression e^I is an integrating factor for equations (3.1) and (3.2); as you can see in (3.8), multiplying (3.1) by e^I makes it an exact equation.

The method of finding an integrating factor and solving the resulting exact equation is useful mainly in simple cases when we can see the result by inspection. It is not usually worth while to spend much time searching for integrating factors.

Homogeneous Equations A *homogeneous function* of x and y of degree n means a function which can be written as $x^n f(y/x)$. For example, $x^3 - xy^2 = x^3[1-(y/x)^2]$ is a homogeneous function of degree 3. (Also see Problem 21.) An equation of the form

$$P(x,y)\,dx + Q(x,y)\,dy = 0, \tag{4.11}$$

where P and Q are homogeneous functions of *the same degree* is called *homogeneous*. (The term homogeneous is also used in another sense; see Section 5.) If we divide two homogeneous functions of the same degree, the x^n factors cancel and we have a function of y/x. Thus, from (4.11) we can write

$$y' = \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} = f\left(\frac{y}{x}\right), \tag{4.12}$$

and we can say that a differential equation is homogeneous if it can be written as y' = a function of y/x. This suggests that we solve homogeneous equations by making the change of variables $v = y/x$, or

$$y = xv. \tag{4.13}$$

This substitution does, in fact, give us a separable equation in x and v (see Problem 22). We solve it to find a relation between v and x and then put back $v = y/x$ to find the solution of (4.11).

Also see Problem 23 for another way to solve homogeneous equations.

Change of Variables We have solved both Bernoulli equations and homogeneous equations by making changes of variables. Other equations may yield to this method also. If a differential equation contains some combination of the variables x, y (especially if this combination appears more than once), we try replacing this combination by a new variable. See Problems 11, 15, and 16 for examples.

▶ PROBLEMS, SECTION 4

Use the methods of this section to solve the following differential equations. Compare computer solutions and reconcile differences.

1. $y' + y = xy^{2/3}$

2. $y' + \frac{1}{x}y = 2x^{3/2}y^{1/2}$

3. $3xy^2y' + 3y^3 = 1$

4. $(2xe^{3y} + e^x)\,dx + (3x^2e^{3y} - y^2)\,dy = 0$

5. $(x-y)\,dy + (y+x+1)\,dx = 0$

6. $(\cos x\cos y + \sin^2 x)\,dx - (\sin x \sin y + \cos^2 y)\,dy = 0$

7. $x^2\,dy + (y^2 - xy)\,dx = 0$

8. $y\,dy = (-x + \sqrt{x^2+y^2}\,)\,dx$

9. $xy\,dx + (y^2 - x^2)\,dy = 0$

10. $(y^2 - xy)\,dx + (x^2 + xy)\,dy = 0$

11. $y' = \cos(x+y)$ *Hint:* Let $u = x + y$; then $u' = 1 + y'$.

12. $y' = \frac{y}{x} - \tan\frac{y}{x}$

13. $yy' - 2y^2\cot x = \sin x\cos x$

14. $(x-1)y' + y - x^{-2} + 2x^{-3} = 0$ **15.** $xy' + y = e^{xy}$ *Hint:* Let $u = xy$

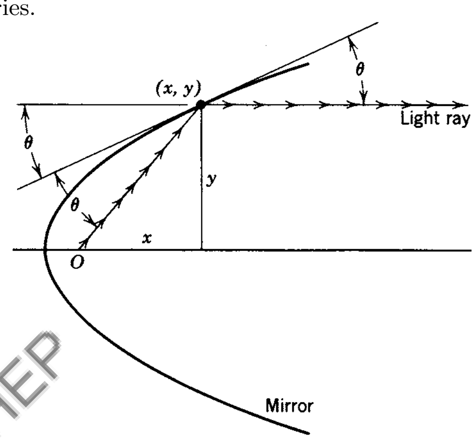
16. Solve the differential equation $yy'^2 + 2xy' - y = 0$ by changing from variables y, x, to r, x, where $y^2 = r^2 - x^2$; then $yy' = rr' - x$.

17. If an incompressible fluid flows in a corner bounded by walls meeting at the origin at an angle of $60°$, the streamlines of the flow satisfy the equation $2xy\,dx + (x^2 - y^2)\,dy = 0$. Find the streamlines.

18. Find the family of orthogonal trajectories of the circles $(x-h)^2 + y^2 = h^2$. (See the instructions above Problem 2.31.)

19. Find the family of curves satisfying the differential equation $(x+y)\,dy + (x-y)\,dx = 0$ and also find their orthogonal trajectories.

20. Find the shape of a mirror which has the property that rays from a point O on the axis are reflected into a parallel beam. *Hint:* Take the point O at the origin. Show from the figure that $\tan 2\theta = y/x$. Use the formula for $\tan 2\theta$ to express this in terms of $\tan\theta = dy/dx$ and solve the resulting differential equation. (*Hint:* See Problem 16.)



21. As in text just before (4.11), show that

(a) $x^2 - 5xy + y^3/x$ is a homogeneous function of degree 2;

(b) $x^{-1}(y^4 - x^3y) - xy^2\sin(x/y)$ is homogeneous of degree 3;

(c) $x^2y^3 + x^5\ln(y/x) - y^6/\sqrt{x^2+y^2}$ is homogeneous of degree 5;

(d) $x^2 + y$, $x + \cos y$, and $y + 1$ are not homogeneous.

See Chapter 4, Section 13, Problem 1 for a more general definition of a homogeneous function of any number of variables.

22. Show that the change of variables (4.13) in (4.11) or (4.12) gives a separable equation. *Hints:* Substitute $y = xv$ and $dy = x\,dv + v\,dx$ from (4.13) into (4.12) and rearrange terms to get the equation

(a) $$[f(v) - v]\,dx = x\,dv.$$

Alternatively, suppose P and Q are homogeneous of degree n; that is $P(x,y) = x^nP(1, y/x) = x^nP(1,v)$ and a similar equation for Q. Substitute these results and $dy = x\,dv + v\,dx$ into (4.11), divide by x^n, and rearrange terms to get

(b) $$[P(1,v) + Q(1,v)\,v]\,dx + Q(1,v)x\,dv = 0.$$

Write both (a) and (b) with variables separated.

23. Show that $(xP + yQ)^{-1}$ is an integrating factor for (4.11). *Hint:* You want to show that $(P\,dx + Q\,dy)/(xP + yQ)$ is an exact differential (see Chapter 6, Section 8). Remember that P and Q are homogeneous of the same degree. Divide numerator and denominator by Q and use $P/Q = -f(y/x)$ from (4.12). Now find the needed partial derivatives. *Comment:* If $(x\,P + y\,Q)$ turns out to be very simple, this may be an easier way to solve a homogeneous equation than the $v = y/x$ substitution (see Problem 24).

24. Solve Problems 9 and 10 by using an integrating factor as discussed in Problem 23.

25. An equation of the form $y' = f(x)y^2 + g(x)y + h(x)$ is called a *Riccati* equation. If we know one particular solution y_p, then the substitution $y = y_p + \frac{1}{z}$ gives a linear first-order equation for z. We can solve this for z and substitute back to find a solution of the y equation containing one arbitrary constant (see Problem 26). Following this method, check the given y_p, and then solve

(a) $y' = xy^2 - \dfrac{2}{x}y - \dfrac{1}{x^3}, \quad y_p = \dfrac{1}{x^2};$

(b) $y' = \dfrac{2}{x}y^2 + \dfrac{1}{x}y - 2x, \quad y_p = x;$

(c) $y' = e^{-x}y^2 + y - e^x, \quad y_p = e^x.$

26. Show that the substitution given in Problem 25 does in general give a solution of the Riccati equation. *Hints:* First show that the substitution $y = y_p + u$ yields the following equation for u: $u' - (g + 2fy_p)u = fu^2$. Note by text equation (4.1) that this is a Bernoulli equation with $n = 2$, so by equation (4.2) we let $z = u^{-1}$. Show that the z equation is the linear first-order equation $z' + (g + 2fy_p)z = -f$. Note that we could have obtained the z equation in one step by substituting $y = y_p + z^{-1}$ in the original equation as claimed in Problem 25.

▶ 5. SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE

Because of their importance in applications, we are going to consider carefully the solution of differential equations of the form

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0, \tag{5.1}$$

where a_2, a_1, a_0 are constants; also we shall consider (Section 6) the corresponding equation when the right-hand side of (5.1) is a function of x. Equations of the form (5.1) are called *homogeneous* because every term contains y or a derivative of y. Equations of the form (6.1) are called *inhomogeneous* because they contain a term which does not depend on y. (Note, however, that this use of the term homogeneous is completely unrelated to its use in Section 4.) Although we shall concentrate on second-order equations, which are the ones that occur most frequently in applications, most of our discussion can be extended immediately to linear equations of higher order with constant coefficients (see Problems 21 to 30).

These problems are pretty simple by hand; you may be able to write down answers faster than you can type the problem into a computer! Remember that a computer may not give an answer in the form you need. To use computer solutions effectively, you need to know what to expect, and you can learn this by studying the following methods and doing some problems by hand. Let us consider an equation of the form (5.1).

▶ **Example 1.** Solve the equation

$$y'' + 5y' + 4y = 0. \tag{5.2}$$

It is convenient to let D stand for d/dx; then

$$Dy = \frac{dy}{dx} = y', \qquad D^2y = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''. \tag{5.3}$$

Expressions involving D, such as $D+1$ or D^2+5D+4, are called *differential operators*. (See Problem 31.) In this notation (5.2) becomes

$$D^2y+5Dy+4y=0 \quad \text{or} \quad (D^2+5D+4)y=0. \tag{5.4}$$

The *algebraic* expression D^2+5D+4 can be factored as $(D+1)(D+4)$ or $(D+4)(D+1)$. You should satisfy yourself that

$$(D+1)(D+4)y=(D+4)(D+1)y=(D^2+5D+4)y \tag{5.5}$$

when $D=d/dx$, and, in fact, that a similar statement is true for $(D-a)(D-b)$ where a and b are any *constants*. (This is not necessarily true if a and b are functions of x; see Problem 31.) Then we can write (5.2) or (5.4) as

$$(D+1)(D+4)y=0 \quad \text{or} \quad (D+4)(D+1)y=0. \tag{5.6}$$

To solve (5.4) [or (5.6) which is the same equation rewritten], we shall first solve the simpler equations

$$(D+4)y=0 \quad \text{and} \quad (D+1)y=0. \tag{5.7}$$

These are separable equations (Section 2) with solutions

$$y=c_1e^{-4x}, \quad y=c_2e^{-x}. \tag{5.8}$$

Now if $(D+4)y=0$, then

$$(D+1)(D+4)y=(D+1)\cdot 0=0,$$

so any solution of $(D+4)y=0$ is a solution of the differential equation (5.6) or (5.4). Similarly, any solution of $(D+1)y=0$ is a solution of (5.6) or (5.4). Since the two solutions (5.8) are linearly independent [Problem 13; also see Chapter 3, equation (8.5)], a linear combination of them contains two arbitrary constants and so is the general solution. Thus

$$y=c_1e^{-4x}+c_2e^{-x} \tag{5.9}$$

is the general solution of (5.4). Note that we can think of the two solutions e^{-4x} and e^{-x} as basis vectors of a 2-dimensional linear vector space (see Chapter 3, Section 14). Then the general solution (5.9) gives all the vectors of that space. (See Problem 21.)

Now we must investigate whether we can solve all second-order linear equations with constant coefficients (and zero right-hand side) by this method. We first wrote the differential equation using D for d/dx, and then factored the D expression to get (5.5). In this last step, we treated D as if it were an algebraic letter instead of d/dx; this is justified by checking the result (5.5) when $D=d/dx$. Recall from algebra that saying that the algebraic expression D^2+5D+4 has the factors $(D+4)$ and $(D+1)$ is equivalent to saying that the quadratic equation

$$D^2+5D+4=0 \tag{5.10}$$

has roots -4 and -1. The equation (5.10) is called the *auxiliary* (or characteristic) equation for the given differential equation (5.2). From equations (5.6) to (5.9), we see that to solve a linear second-order equation with constant coefficients, we should first solve the auxiliary equation; if the roots of the auxiliary equation are a and b ($a \neq b$), the general solution of the differential equation is a linear combination of e^{ax} and e^{bx}.

(5.11) $y = c_1e^{ax} + c_2e^{bx}$ is the general solution of $(D-a)(D-b)y = 0, \quad a \neq b.$

(If $a = b$, we get only one solution this way; we shall consider this case shortly.) Recall from algebra that the roots of a quadratic equation (with real coefficients; see Problem 19) can be real and unequal, real and equal, or a complex conjugate pair. The equation (5.2) which we have solved is an example in which the roots are real and unequal. Let us consider the other two cases.

Equal Roots of the Auxiliary Equation If the two roots of the auxiliary equation are equal, then the differential equation can be written

$$(D-a)(D-a)y = 0, \tag{5.12}$$

where a is the value of the two equal roots. From our previous discussion (5.5) to (5.11), we know that one solution of (5.12) is $y = c_1e^{ax}$. But our previous second solution $y = c_2e^{bx}$ in (5.11) is not a second solution here since $b = a$. To find the second solution for this case, we let

$$u = (D-a)y. \tag{5.13}$$

Then (5.12) becomes

$$(D-a)u = 0,$$

from which we get

$$u = Ae^{ax}. \tag{5.14}$$

We substitute (5.14) into (5.13) to get

$$(D-a)y = Ae^{ax} \quad \text{or} \quad y' - ay = Ae^{ax}.$$

This is a first-order linear equation which we solve as in Section 3:

$$ye^{-ax} = \int e^{-ax}Ae^{ax}\,dx = \int A\,dx = Ax + B.$$

Thus

(5.15) $y = (Ax + B)e^{ax}$ is the general solution of (5.12).

This is the general solution of (5.1) for the case of equal roots of the auxiliary equation. The solution e^{ax} we already know; what is new here is the fact that xe^{ax} is a second (linearly independent; see Problem 14) solution of the differential equation when a is a double root of the auxiliary equation. Equations (5.11) and (5.15) then give the general solution of (5.1) for both unequal and equal roots of the auxiliary equation.

Complex Conjugate Roots of the Auxiliary Equation Suppose the roots of the auxiliary equation are $\alpha \pm i\beta$. These are unequal roots, so by (5.11) the general solution of the differential equation is

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} = e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}). \tag{5.16}$$

There are two other very useful forms of (5.16). If we substitute $e^{\pm i\beta x} = \cos\beta x \pm i\sin\beta x$ [see Chapter 2, equation (9.3)] into (5.16), then the parenthesis becomes a linear combination of $\sin\beta x$ and $\cos\beta x$ and we can write (5.16) as

$$y = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x), \tag{5.17}$$

where c_1 and c_2 are new arbitrary constants. We can also write (5.17) in the form

$$y = ce^{\alpha x}\sin(\beta x + \gamma), \tag{5.18}$$

where c and γ are now the arbitrary constants. An easy way to see that this is correct is to expand $\sin(\beta x + \gamma)$ by the trigonometric addition formula; this gives a linear combination of $\sin\beta x$ and $\cos\beta x$ as in (5.17). Although it is not hard to express any one of the sets of arbitrary constants [A, B in (5.16); c_1, c_2 in (5.17); and c, γ in (5.18)] in terms of either of the other sets, there is seldom any need to do this. In solving actual problems we simply write whichever one of the three forms seems best for the problem at hand and then determine the arbitrary constants in that form from the given data.

Example 2. Solve the differential equation

$$y'' - 6y' + 9y = 0. \tag{5.19}$$

We can write the equation as

$$(D^2 - 6D + 9)y = 0 \quad \text{or} \quad (D-3)(D-3)y = 0. \tag{5.20}$$

Since the roots of the auxiliary equation are equal, we know that the solution is of the form (5.15) and we simply write the result

$$y = (Ax + B)e^{3x}. \tag{5.21}$$

▶ **Example 3.** In Section 16, Chapter 2, we discussed the differential equation for the motion of a mass m oscillating at the end of a spring, and we solved it by guessing the solution. Now let's solve it by the methods of this chapter. The differential equation is [see Chapter 2, equation (16.21)]

$$m\frac{d^2y}{dt^2} = -ky \quad \text{or} \quad \frac{d^2y}{dt^2} = -\frac{k}{m}y = -\omega^2 y \quad \text{if} \quad \omega^2 = \frac{k}{m}. \tag{5.22}$$

We can write this differential equation as

$$D^2y + \omega^2 y = 0 \quad \text{or} \quad (D^2 + \omega^2)y = 0 \tag{5.23}$$

where $D = d/dt$. The roots of the auxiliary equation are $D = \pm i\omega$; the solution may be written in any of the three forms, (5.16), (5.17), or (5.18):

$$\begin{aligned} y &= Ae^{i\omega t} + Be^{-i\omega t} \\ &= c_1 \sin\omega t + c_2 \cos\omega t \\ &= c\sin(\omega t + \gamma). \end{aligned} \tag{5.24}$$

An object whose displacement from equilibrium satisfies (5.22) or (5.24) is said to be executing *simple harmonic motion*. (Recall Chapter 7, Section 2.)

Equations (5.24) are general solutions of (5.22), each containing two arbitrary constants. Let us find a particular solution corresponding to given initial conditions.

▶ **Example 4.** Suppose the mass is held at rest at a distance 10 cm below equilibrium and then suddenly let go. If we agree to call y positive when m is above the equilibrium position, then at $t = 0$, we have $y = -10$, and $dy/dt = 0$. Using the second solution in (5.24), we get

$$\frac{dy}{dt} = c_1\omega\cos\omega t - c_2\omega\sin\omega t,$$

so the initial conditions give

$$\begin{aligned} -10 &= c_1 \cdot 0 + c_2 \cdot 1, \\ 0 &= c_1\omega \cdot 1 - c_2\omega \cdot 0. \end{aligned}$$

Thus we find

$$c_1 = 0, \qquad c_2 = -10,$$

and the particular solution we wanted is

$$y = -10\cos\omega t. \tag{5.25}$$

You can verify that either of the other solutions in (5.24) gives the same particular solution (5.25) for the same initial conditions (Problem 32).

This solution is pretty unrealistic from the practical viewpoint. Equations (5.24) and (5.25) imply that the mass m, once started, will simply oscillate up and down forever! This is certainly not true; what *will* happen is that the oscillations will gradually die down. The reason for the discrepancy between the physical facts and our mathematical answer is that we have neglected "friction" forces.

Example 5. A fairly reasonable assumption for this problem and many other similar ones is that there is a retarding force proportional to the velocity; let us call this force $-l(dy/dt)$ $(l > 0)$. Then (5.22), revised to include this force, becomes

$$m\frac{d^2y}{dt^2} = -ky - l\frac{dy}{dt} \qquad (l > 0) \tag{5.26}$$

or with the abbreviations

$$\omega^2 = \frac{k}{m}, \qquad 2b = \frac{l}{m} \qquad (b > 0)$$

it is

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0. \tag{5.27}$$

To solve (5.27), we find the roots of the auxiliary equation

$$D^2 + 2bD + \omega^2 = 0, \tag{5.28}$$

which are

$$D = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2} = -b \pm \sqrt{b^2 - \omega^2}. \tag{5.29}$$

There are three possible types of answer here depending on the relative size of b^2 and ω^2, and there are three special names given to the corresponding types of motion. We say that the motion is

$$\begin{array}{ll} \text{overdamped if} & b^2 > \omega^2, \\ \text{critically damped if} & b^2 = \omega^2, \\ \text{underdamped or oscillatory if} & b^2 < \omega^2. \end{array}$$

Let us discuss the corresponding general solutions of the differential equation for the three cases.

Overdamped Motion Since $\sqrt{b^2 - \omega^2}$ is real and less than b, both roots of the auxiliary equation are negative, and the general solution is a linear combination of two negative exponentials:

$$y = Ae^{-\lambda t} + Be^{-\mu t}, \qquad \text{where} \quad \begin{cases} \lambda = b + \sqrt{b^2 - \omega^2}, \\ \mu = b - \sqrt{b^2 - \omega^2}. \end{cases} \tag{5.30}$$

Critically Damped Motion Since $b = \omega$, the auxiliary equation has equal roots and the general solution is

$$y = (A + Bt)e^{-bt}. \tag{5.31}$$

In both overdamped and critically damped motion, the mass m is subject to such a large retarding force that it slows down and returns to equilibrium rather than oscillating repeatedly.