Solution

The differential equation may be rewritten

$$-y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + x.y^{-2} = e^{-x^2}.$$

Substituting $z = y^{-2}$, we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$ and, hence,

$$\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}x} + xz = e^{-x^2}$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 2e^{-x^2}.$$

An integrating factor for this equation is

$$e^{\int\ 2x\ \mathrm{d}x} = e^{x^2}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{x^2}\right) = 2,$$

giving

$$ze^{x^2} = 2x + C,$$

where C is an arbitrary constant.

Finally, replacing z by y^{-2},

$$y^2 = \frac{e^{x^2}}{2x + C}.$$

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = xy^2.$$

Solution

The differential equation may be rewritten

$$y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}.y^{-1} = x.$$

On substituting $z = y^{-1}$ we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$ so that

$$-\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{1}{x}.z = x$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{x}.z = -x.$$

An integrating factor for this equation is

$$e^{\int \left(-\frac{1}{x}\right)\,\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(z \times \frac{1}{x}\right) = -1,$$

giving

$$\frac{z}{x} = -x + C,$$

where C is an arbitrary constant.

The general solution of the given differential equation is therefore

$$\frac{1}{xy} = -x + C \quad \text{or} \quad y = \frac{1}{Cx - x^2}.$$

15.3.3 EXERCISES

Use an integrating factor to solve the following differential equations subject to the given boundary condition:

1.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$

where $y = 10$ when $x = 0$.

2.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 10,$$

where $y = 4$ when $x = 0$.

3.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 3x,$$

where $y = 2$ when $x = -1$.

4.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{1-x} = 1 - x^2,$$

where $y = 0$ when $x = -1$.

5.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \cos x,$$

where $y = \frac{5}{2}$ when $x = \frac{\pi}{2}$.

6.

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = x,$$

where $y = 0$ when $x = 1$.

7.

$$3y - 2\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{4x},$$

where $y = 1$ when $x = 0$.

8.

$$2y - x\frac{\mathrm{d}y}{\mathrm{d}x} = x(x-1)y^4,$$

where $y^3 = 14$ when $x = 1$.

15.3.4 ANSWERS TO EXERCISES

1.

$$y = 10e^{-\frac{2}{3}x}.$$

2.

$$y = 6e^{\frac{5}{3}x} - 2.$$

3.

$$yx = x^3 - 1.$$

4.

$$y = \frac{1}{2}(1-x)(1+x)^2.$$

5.

$$y = \frac{\sin x}{2} + \frac{2}{\sin x}.$$

6.

$$y = 1 + x^2 - \sqrt{2(1+x^2)}.$$

7.

$$y^2 = \frac{7e^{3x}}{e^{7x}+6}.$$

8.

$$y^3 = \frac{56x^6}{21x^6 - 24x^7 + 7}.$$

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"JUST THE MATHS"

UNIT NUMBER

15.4

ORDINARY DIFFERENTIAL EQUATIONS 4 (Second order equations (A))

by

A.J.Hobson

UNIT 15.4 - ORDINARY DIFFERENTIAL EQUATIONS 4

SECOND ORDER EQUATIONS (A)

15.4.1 INTRODUCTION

In the discussion which follows, we shall consider a particular kind of second order ordinary differential equation which is called **"linear, with constant coefficients"**; it has the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

where a, b and c are the constant coefficients.

The various cases of solution which arise depend on the values of the coefficients, together with the type of function, $f(x)$, on the right hand side. These cases will now be dealt with in turn.

15.4.2 SECOND ORDER HOMOGENEOUS EQUATIONS

The term **"homogeneous"**, in the context of <u>second</u> order differential equations, is used to mean that the function, $f(x)$, on the right hand side is zero. It should not be confused with the previous use of this term in the context of <u>first</u> order differential equations.

We therefore consider equations of the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

Note:
A very simple case of this equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0,$$

which, on integration twice, gives the general solution

$$y = Ax + B,$$

where A and B are arbitrary constants. We should therefore expect two arbitrary constants in the solution of any second order linear differential equation with constant coefficients.

The Standard General Solution

The equivalent of

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

in the discussion of first order differential equations would have been

$$b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0; \text{ that is, } \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{b}y = 0$$

and this could have been solved using an integrating factor of

$$e^{\int \frac{c}{b} \, \mathrm{d}x} = e^{\frac{c}{b}x},$$

giving the general solution

$$y = Ae^{-\frac{c}{b}x},$$

where A is an arbitrary constant.

It seems reasonable, therefore, to make a trial solution of the form $y = Ae^{mx}$, where $A \neq 0$, in the second order case.

We shall need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ame^{mx} \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Am^2e^{mx}.$$

Hence, on substituting the trial solution, we require that

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0;$$

and, by cancelling Ae^{mx}, this condition reduces to

$$am^2 + bm + c = 0,$$

a quadratic equation, called the **"auxiliary equation"**, having the same (constant) coefficients as the orginal differential equation.

In general, it will have two solutions, say $m = m_1$ and $m = m_2$, giving corresponding solutions $y = Ae^{m_1 x}$ and $y = Be^{m_2 x}$ of the differential equation.

However, the linearity of the differential equation implies that the sum of any two solutions is also a solution, so that

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

is another solution; and, since this contains two arbitrary constants, we shall take it to be the general solution.

Notes:

(i) It may be shown that there are no solutions other than those of the above form though special cases are considered later.

(ii) It will be possible to determine particular values of A and B if an appropriate number of boundary conditions for the differential equation are specified. These will usually be a set of given values for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a certain value of x.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

and also the particular solution for which $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -5$ when $x = 0$.

Solution

The auxiliary equation is $m^2 + 5m + 6 = 0$,

which can be factorised as

$$(m+2)(m+3)=0.$$

Its solutions are therefore $m=-2$ and $m=-3$.

Hence, the differential equation has general solution

$$y=Ae^{-2x}+Be^{-3x},$$

where A and B are arbitrary constants.

Applying the boundary conditions, we shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-2Ae^{-2x}-3Be^{-3x}.$$

Hence,

$$\begin{array}{rcl} 2 & = & A+B, \\ -5 & = & -2A-3B \end{array}$$

giving $A=1$, $B=1$ and a particular solution

$$y=e^{-2x}+e^{-3x}.$$

15.4.3 SPECIAL CASES OF THE AUXILIARY EQUATION

(a) The auxilary equation has coincident solutions

Suppose that both solutions of the auxiliary equation are the same number, m_1.

In other words, the quadratic expression am^2+bm+c is a **"perfect square"**, which means that it is actually $a(m-m_1)^2$.

Apparently, the general solution of the differential equation is

$$y=Ae^{m_1x}+Be^{m_1x},$$

which does not genuinely contain <u>two</u> arbitrary constants since it can be rewritten as

$$y = Ce^{m_1 x} \quad \text{where} \quad C = A + B.$$

It will not, therefore, count as the <u>general</u> solution, though the fault seems to lie with the constants A and B rather than with m_1.

Consequently, let us now examine a new trial solution of the form

$$y = ze^{m_1 x},$$

where z denotes a function of x rather than a constant.

We shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = zm_1 e^{m_1 x} + e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = zm_1^2 e^{m_1 x} + 2m_1 e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x} + e^{m_1 x}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}.$$

On substituting these into the differential equation, we obtain the condition that

$$e^{m_1 x}\left[a\left(zm_1^2 + 2m_1\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right) + b\left(zm_1 + \frac{\mathrm{d}z}{\mathrm{d}x}\right) + cz\right] = 0$$

or

$$z(am_1^2 + bm_1 + c) + \frac{\mathrm{d}z}{\mathrm{d}x}(2am_1 + b) + a\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0.$$

The first term on the left hand side of this condition is zero since m_1 is already a solution of the auxiliary equation; and the second term is also zero since the auxiliary equation, $am^2 + bm + c = 0$, is equivalent to $a(m - m_1)^2 = 0$; that is, $am^2 - 2am_1 m + am_1^2 = 0$. Thus $b = -2am_1$.

We conclude that $\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = 0$ with the result that $z = Ax + B$, where A and B are arbitrary constants.

The general solution of the differential equation in the case of coincident solutions to the auxiliary equation is therefore

$$y = (Ax + B)e^{m_1 x}.$$

EXAMPLE

Determine the general solution of the differential equation

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Solution

The auxilary equation is

$$4m^2 + 4m + 1 = 0 \quad \text{or} \quad (2m + 1)^2 = 0$$

and it has coincident solutions at $m = -\frac{1}{2}$.

The general solution is therefore

$$y = (Ax + B)e^{-\frac{1}{2}x}.$$

(b) The auxiliary equation has complex solutions

If the auxiliary equation has complex solutions, they will automatically appear as a pair of **"complex conjugates"**, say $m = \alpha \pm j\beta$.

Using these two solutions instead of the previous m_1 and m_2, the general solution of the differential equation will be

$$y = Pe^{(\alpha + j\beta)x} + Qe^{(\alpha - j\beta)x},$$

where P and Q are arbitrary constants.

But, by properties of complex numbers, a neater form of this result is obtainable as follows:

$$y = e^{\alpha x}[P(\cos\beta x + j\sin\beta x) + Q(\cos\beta x - j\sin\beta x)]$$

or

$$y = e^{\alpha x}[(P+Q)\cos\beta x + j(P-Q)\sin\beta x].$$

Replacing $P+Q$ and $j(P-Q)$ (which are just arbitrary quantities) by A and B, we obtain the standard general solution for the case in which the auxiliary equation has complex solutions. It is

$$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x].$$

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = 0.$$

Solution

The auxiliary equation is

$$m^2 - 6m + 13 = 0,$$

which has solutions given by

$$m = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\times 13\times 1}}{2\times 1} = \frac{6 \pm j4}{2} = 3 \pm j2.$$

The general solution is therefore

$$y = e^{3x}[A\cos 2x + B\sin 2x],$$

where A and B are arbitrary constants.

15.4.4 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)
   $$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0;$$

   (b)
   $$\frac{d^2r}{d\theta^2} + 6\frac{dr}{d\theta} + 9r = 0;$$

   (c)
   $$\frac{d^2\theta}{dt^2} + 4\frac{d\theta}{dt} + 5\theta = 0.$$

2. Solve the following differential equations, subject to the given boundary conditions:

   (a)
   $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0,$$
   where $y = 2$ and $\frac{dy}{dx} = 1$ when $x = 0$;

   (b)
   $$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = 0,$$
   where $x = 3$ and $\frac{dx}{dt} = 5$ when $t = 0$;

   (c)
   $$4\frac{d^2z}{ds^2} - 12\frac{dz}{ds} + 9z = 0,$$
   where $z = 1$ and $\frac{dz}{ds} = \frac{5}{2}$ when $s = 0$;

(d)

$$\frac{d^2r}{d\theta^2} - 2\frac{dr}{d\theta} + 2r = 0,$$

where $r = 5$ and $\frac{dr}{d\theta} = 7$ when $\theta = 0$.

15.4.5 ANSWERS TO EXERCISES

1. (a)

$$y = Ae^{-3x} + Be^{-4x};$$

(b)

$$r = (A\theta + B)e^{-3\theta};$$

(c)

$$\theta = e^{-2t}[A\cos 2t + B\sin 2t].$$

2. (a)

$$y = 3e^x - e^{2x};$$

(b)

$$x = 2e^t + e^{3t};$$

(c)

$$z = (s+1)e^{\frac{3}{2}s};$$

(d)

$$r = e^{\theta}[5\cos\theta + 2\sin\theta].$$

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"JUST THE MATHS"

UNIT NUMBER

15.5

ORDINARY DIFFERENTIAL EQUATIONS 5 (Second order equations (B))

by

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UNIT 15.5 - ORDINARY DIFFERENTIAL EQUATIONS 5

SECOND ORDER EQUATIONS (B)

15.5.1 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

The following discussion will examine the solution of the second order linear differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

in which a, b and c are constants, but $f(x)$ is not identically equal to zero.

The Particular Integral and Complementary Function

(i) Suppose that $y = u(x)$ is any particular solution of the differential equation; that is, it contains no arbitrary constants. In the present context, we shall refer to such particular solutions as **"particular integrals"** and systematic methods of finding them will be discussed later.

It follows that

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu = f(x).$$

(ii) Suppose also that we make the substitution $y = u(x) + v(x)$ in the original differential equation to give

$$a\frac{\mathrm{d}^2(u+v)}{\mathrm{d}x^2} + b\frac{\mathrm{d}(u+v)}{\mathrm{d}x} + c(u+v) = f(x).$$

That is,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu + a\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + b\frac{\mathrm{d}v}{\mathrm{d}x} + cv = f(x);$$

and, hence,

$$a\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + b\frac{\mathrm{d}v}{\mathrm{d}x} + cv = 0.$$

This means that the function $v(x)$ is the general solution of the homogeneous differential equation whose auxiliary equation is

$$am^2 + bm + c = 0.$$

In future, $v(x)$ will be called the **"complementary function"** in the general solution of the original (non-homogeneous) differential equation. It complements the particular integral to provide the general solution.

Summary

$$\boxed{\textbf{General solution} = \textbf{particular integral} + \textbf{complementary function.}}$$

15.5.2 DETERMINATION OF SIMPLE PARTICULAR INTEGRALS

(a) Particular integrals, when $f(x)$ is a constant, k.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = k,$$

it is easy to see that a particular integral will be $y = \frac{k}{c}$, since its first and second derivatives are both zero, while $cy = k$.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 20.$$

Solution

(i) By inspection, we may observe that a particular integral is $y = 2$.

(ii) The auxiliary equation is

$$m^2 + 7m + 10 = 0 \quad \text{or} \quad (m+2)(m+5) = 0,$$

having solutions $m = -2$ and $m = -5$.

(iii) The complementary function is

$$Ae^{-2x} + Be^{-5x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 2 + Ae^{-2x} + Be^{-5x}.$$

(b) Particular integrals, when $f(x)$ is of the form $px + q$.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = px + q,$$

it is possible to determine a particular integral by assuming one which has the same form as the right hand side; that is, in this case, another expression consisting of a multiple of x and constant term. The method is, again, illustrated by an example.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 11\frac{\mathrm{d}y}{\mathrm{d}x} + 28y = 84x - 5.$$

Solution

(i) First, we assume a particular integral of the form

$$y = \alpha x + \beta,$$

which implies that $\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

Substituting into the differential equation, we require that

$$-11\alpha + 28(\alpha x + \beta) \equiv 84x - 5.$$

Hence, $28\alpha = 84$ and $-11\alpha + 28\beta = -5$, giving $\alpha = 3$ and $\beta = 1$.

Thus, the particular integral is

$$y = 3x + 1.$$

(ii) The auxiliary equation is

$$m^2 - 11m + 28 = 0 \quad \text{or} \quad (m-4)(m-7) = 0,$$

having solutions $m = 4$ and $m = 7$.

(iii) The complementary function is

$$Ae^{4x} + Be^{7x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 3x + 1 + Ae^{4x} + Be^{7x}.$$

Note:
In examples of the above types, the complementary function must not be prefixed by "$y =$", since the given differential equation, as a whole, is not normally satisfied by the complementary function alone.

15.5.3 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6;$$

   (b)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 16y = 7;$$

   (c)
   $$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - y = x + 1;$$

   (d)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 18x + 28.$$

2. Solve, completely, the following differential equations, subject to the given boundary condidtions:

   (a)
   $$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 100,$$
   where $y = -26$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 5$ when $x = 0$;

   (b)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 12x + 16,$$
   where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$;

   (c)
   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 10x + 14,$$
   where $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.5.4 ANSWERS TO EXERCISES

1. (a)

$$y = -3 + Ae^{x} + Be^{2x};$$

(b)

$$y = \frac{7}{16} + A\cos 4x + B\sin 4x;$$

(c)

$$y = 1 - x + Ae^{x} + Be^{-\frac{1}{3}x};$$

(d)

$$y = 2x + 5 + (Ax + B)e^{3x}.$$

2. (a)

$$y = -25 + e^{4x} - 2e^{\frac{1}{2}x};$$

(b)

$$y = 3x + 1 - (x + 1)e^{-2x};$$

(c)

$$y = x + 2 + e^{3x}(\cos x - 2\sin x).$$

"JUST THE MATHS"

UNIT NUMBER

15.6

ORDINARY DIFFERENTIAL EQUATIONS 6 (Second order equations (C))

by

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UNIT 15.6 - ORDINARY DIFFERENTIAL EQUATIONS 6

SECOND ORDER EQUATIONS (C)

15.6.1 RECAP

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

it was seen, in Unit 15.5, that

(a) when $f(x) \equiv k$, a given **constant**, a particular integral is $y = \frac{k}{c}$;

(b) when $f(x) \equiv px+q$, a **linear** function in which p and q are given constants, it is possible to obtain a particular integral by assuming that y also has the form of a linear function; that is, we make a **"trial solution"**, $y = \alpha x + \beta$.

15.6.2 FURTHER TYPES OF PARTICULAR INTEGRAL

We now examine particular integrals for other cases of $f(x)$, the method being illustrated by examples. Also, for reasons relating to certain problematic cases discussed in Unit 15.7, we shall determine the complementary function **before** determining the particular integral.

1. $f(x) \equiv px^2 + qx + r$, a **quadratic** function in which p, q and r are given constants; $p \neq 0$.

$$\text{Trial solution :} \quad y = \alpha x^2 + \beta x + \gamma.$$

Note:
This is the trial solution even if q or r (or both) are zero.

EXAMPLE

Determine the general solution of the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 4x^2 + 10x - 23.$$

Solution

The auxilary equation is

$$2m^2 - 7m - 4 = 0 \ \text{ or } \ (2m+1)(m-4) = 0,$$

having solutions $m = 4$ and $m = -\frac{1}{2}$.
Thus, the complementary function is

$$Ae^{4x} + Be^{-\frac{1}{2}x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form $y = \alpha x^2 + \beta x + \gamma$, giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x + \beta$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha$.
We thus require that

$$4\alpha - 14\alpha x - 7\beta - 4\alpha x^2 - 4\beta x - 4\gamma \equiv 4x^2 + 10x - 23.$$

That is,

$$-4\alpha x^2 - (14\alpha + 4\beta)x + 4\alpha - 7\beta - 4\gamma \equiv 4x^2 + 10x - 23.$$

Comparing corresponding coefficents on both sides, this means that

$$-4\alpha = 4, \quad -(14\alpha + 4\beta) = 10 \ \text{ and } \ 4\alpha - 7\beta - 4\gamma = -23,$$

which give $\alpha = -1$, $\beta = 1$ and $\gamma = 3$.
Hence, the particular integral is

$$y = 3 + x - x^2.$$

Finally, the general solution is

$$y = 3 + x - x^2 + Ae^{4x} + Be^{-\frac{1}{2}x}.$$

2. $f(x) \equiv p\sin kx + q\cos kx$, a **trigonometric** function in which p, q and k are given constants.

$$\text{Trial solution :} \quad y = \alpha\sin kx + \beta\cos kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 8\cos 3x - 19\sin 3x.$$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0,$$

which has complex number solutions given by

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm j.$$

Hence, the complementary function is

$$e^x(A\cos x + B\sin x),$$

where A and B are arbitrary constants.

To determine a particular integral, we may make a trial solution of the form

$$y = \alpha\sin 3x + \beta\cos 3x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha\cos 3x - 3\beta\sin 3x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\alpha\sin 3x - 9\beta\cos 3x$.

We thus require that

$$-9\alpha\sin 3x - 9\beta\cos 3x - 6\alpha\cos 3x + 6\beta\sin 3x + 2\alpha\sin 3x + 2\beta\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

That is,

$$(-9\alpha + 6\beta + 2\alpha)\sin 3x + (-9\beta - 6\alpha + 2\beta)\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

Comparing corresponding coefficients on both sides, we have

$$\begin{aligned} -7\alpha + 6\beta &= -19, \\ -6\alpha - 7\beta &= 8. \end{aligned}$$

These equations are satisfied by $\alpha = 1$ and $\beta = -2$, so that the particular integral is

$$y = \sin 3x - 2\cos 3x.$$

Finally, the general solution is

$$y = \sin 3x - 2\cos 3x + e^x(A\cos x + B\sin x).$$

3. $f(x) \equiv pe^{kx}$, an **exponential** function in which p and k are given constants.

$$\text{Trial solution :} \quad y = \alpha e^{kx}.$$

EXAMPLE

Determine the general solution of the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y = 50e^{3x}.$$

Solution

The auxiliary equation is

$$9m^2 + 6m + 1 = 0 \quad \text{or} \quad (3m+1)^2 = 0,$$

which has coincident solutions at $m = -\frac{1}{3}$.
The complementary function is therefore

$$(Ax + B)e^{-\frac{1}{3}x}.$$

To find a particular integral, we may make a trial solution of the form

$$y = \alpha e^{3x},$$

which gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha e^{3x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 9\alpha e^{3x}$.
Hence, on substituting into the differential equation, it is necessary that

$$81\alpha e^{3x} + 18\alpha e^{3x} + \alpha e^{3x} = 50e^{3x}.$$

That is, $100\alpha = 50$, from which we deduce that $\alpha = \frac{1}{2}$ and a particular integral is

$$y = \frac{1}{2}e^{3x}.$$

Finally, the general solution is

$$y = \frac{1}{2}e^{3x} + (Ax + B)e^{-\frac{1}{3}x}.$$

4. $f(x) \equiv p \sinh kx + q \cosh kx$, a **hyperbolic** function in which p, q and k are given constants.

$$\text{Trial solution : } \quad y = \alpha \sinh kx + \beta \cosh kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 93 \cosh 5x - 75 \sinh 5x.$$

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \quad \text{or} \quad (m-2)(m-3) = 0,$$

which has solutions $m = 2$ and $m = 3$ so that the complementary function is

$$Ae^{2x} + Be^{3x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form

$$y = \alpha \sinh 5x + \beta \cosh 5x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 5\alpha \cosh 5x + 5\beta \sinh 5x$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 25\alpha \sinh 5x + 25\beta \cosh 5x$.
Substituting into the differential equation, the left-hand-side becomes

$$25\alpha \sinh 5x + 25\beta \cosh 5x - 25\alpha \cosh 5x - 25\beta \sinh 5x + 6\alpha \sinh 5x + 6\beta \cosh 5x.$$

This simplifies to

$$(31\alpha - 25\beta) \sinh 5x + (31\beta - 25\alpha) \cosh 5x,$$

so that we require

$$\begin{aligned} 31\alpha - 25\beta &= -75, \\ -25\alpha + 31\beta &= 93, \end{aligned}$$

and these are satisfied by $\alpha = 0$ and $\beta = 3$.
The particular integral is thus

$$y = 3\cosh 5x$$

and, hence, the general solution is

$$y = 3\cosh 5x + Ae^{2x} + Be^{3x}.$$

5. Combinations of Different Types of Function

In cases where $f(x)$ is the sum of two or more of the various types of function discussed previously, then the particular integrals for each type (determined separately) may be added together to give an overall particular integral.

15.6.3 EXERCISES

1. Determine the general solution for each of the following differential equations:

(a)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 4x^2 + 2x - 4;$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 8\cos 2x - \sin 2x;$$

(c)

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 12\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 27e^{-x};$$

(d)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = \cosh 3x - \sinh 3x.$$

2. Solve completely the following differential equations subject to the given boundary conditions:

   (a)
$$\frac{d^2y}{dx^2} - y = 10 - 5x^2 - x + 16e^{-3x},$$
   where $y = 13$ and $\frac{dy}{dx} = -2$ when $x = 0$;

   (b)
$$4\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 3y = 9x + 6\cos x - 19\sin x,$$
   where $y = -2$ and $\frac{dy}{dx} = 0$ when $x = 0$.

15.6.4 ANSWERS TO EXERCISES

1. (a)
$$y = x^2 - 2x + 1 + Ae^{-x} + Be^{-4x};$$
   (b)
$$y = \sin 2x + e^{-2x}(A\cos x + B\sin x);$$
   (c)
$$y = 3e^{-x} + (Ax + B)e^{-\frac{3}{2}x};$$
   (d)
$$y = \frac{1}{8}(\cosh 3x - \sinh 3x) + Ae{-}2x + Be^{5x}.$$

2. (a)
$$y = 5x^2 + x + 2e^{-3x} + 3e^{x} - 2e^{-x};$$
   (b)
$$y = 3x - 8 + 2\cos x + \sin x + 2e^{-\frac{1}{2}x} + 2e^{-\frac{3}{2}x}.$$

"JUST THE MATHS"

UNIT NUMBER

15.7

ORDINARY DIFFERENTIAL EQUATIONS 7 (Second order equations (D))

by

A.J.Hobson

UNIT 15.7 - ORDINARY DIFFERENTIAL EQUATIONS 7

SECOND ORDER EQUATIONS (D)

15.7.1 PROBLEMATIC CASES OF PARTICULAR INTEGRALS

Difficulties can arise if all or part of any trial solution would already be included in the complementary function. We illustrate with some examples:

EXAMPLES

1. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{2x}.$$

Solution

The auxiliary equation is $m^2 - 3m + 2 = 0$, with solutions $m = 1$ and $m = 2$ and hence the complementary function is $Ae^x + Be^{2x}$, where A and B are arbitrary constants.

A trial solution of $y = \alpha e^{2x}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha e^{2x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\alpha e^{2x}$$

and, on substituting these into the differential equation, it is necessary that

$$4\alpha e^{2x} - 6\alpha e^{2x} + 2\alpha e^{2x} \equiv e^{2x}.$$

That is, $0 \equiv e^{2x}$ which is impossible.

However, if $y = \alpha e^{2x}$ has proved to be unsatisfactory, let us investigate, as an alternative, $y = F(x)e^{2x}$ (where $F(x)$ is a function of x instead of a constant).

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2F(x)e^{2x} + F'(x)e^{2x}$$

and, hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4F(x)e^{2x} + 2F'(x)e^{2x} + F''(x)e^{2x} + 2F'(x)e^{2x}.$$

On substituting these into the differential equation, it is necessary that

$$(4F(x) + 2F'(x) + F''(x) + 2F'(x) - 6F(x) - 3F'(x) + 2F(x))\, e^{2x} \equiv e^{2x}.$$

That is,

$$F''(x) + F'(x) = 1,$$

which is satisfied by the function $F(x) \equiv x$ and thus a suitable particular integral is

$$y = xe^{2x}.$$

Note:
It may be shown, in other cases too that, if the standard trial solution is already contained in the complementary function, then it is is necessary to multiply it by x in order to obtain a suitable particular integral.

2. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin x.$$

Solution
The auxilary equation is $m^2 + 1 = 0$, with solutions $m = \pm j$ and, hence, the complementary function is $A \sin x + B \cos x$, where A and B are arbitrary constants.
A trial solution of $y = \alpha \sin x + \beta \cos x$ gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\alpha \sin x - \beta \cos x;$$

and, on substituting into the differential equation, it is necessary that $0 \equiv \sin x$, which is impossible.
Here, we may try $y = x(\alpha \sin x + \beta \cos x)$, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha \sin x + \beta \cos x + x(\alpha \cos x - \beta \sin x) = (\alpha - \beta x) \sin x + (\beta + \alpha x) \cos x$$

and, therefore,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\alpha-\beta x) \cos x - \beta \sin x - (\beta+\alpha x) \sin x + \alpha \cos x = (2\alpha-\beta x) \cos x - (2\beta+\alpha x) \sin x.$$

Substituting into the differential equation, we thus require that

$$(2\alpha - \beta x)\cos x - (2\beta + \alpha x)\sin x + x(\alpha \sin x + \beta \cos x) \equiv \sin x,$$

which simplifies to

$$2\alpha \cos x - 2\beta \sin x \equiv \sin x.$$

Thus $2\alpha = 0$ and $-2\beta = 1$.

An appropriate particular integral is now

$$y = -\frac{1}{2}x\cos x.$$

3. Determine the complementary function and a particular integral for the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y == 50e^{-\frac{1}{3}x}.$$

Solution

The auxiliary equation is $9m^2 + 6m + 1 = 0$, or $(3m+1)^2 = 0$, which has coincident solutions $m = -\frac{1}{3}$ and so the complementary function is

$$(Ax + B)e^{-\frac{1}{3}x}.$$

In this example, both $e^{-\frac{1}{3}x}$ **and** $xe^{-\frac{1}{3}x}$ are contained in the complementary function.

Thus, in the trial solution, it is necessary to multiply by a **further** x, giving

$$y = \alpha x^2 e^{-\frac{1}{3}x}.$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x e^{-\frac{1}{3}x} - \frac{1}{3}x^2 e^{\frac{1}{3}x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} + \frac{1}{9}\alpha x^2 e^{-\frac{1}{3}x}.$$

Substituting these into the differential equation, it is necessary that

$$\left(18\alpha - 12\alpha x + \alpha x^2 + 12\alpha x - 2\alpha x^2 + \alpha x^2\right) e^{-\frac{1}{3}x} = 50e^{-\frac{1}{3}x}$$

and, hence, $18\alpha = 50$ or $\alpha = \frac{25}{9}$.
An appropriate particular integral is

$$y = \frac{25}{9}x^2 e^{-\frac{1}{3}x}.$$

4. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \sinh 2x.$$

Solution

The auxiliary equation is $m^2 - 5m + 6 = 0$ or $(m-2)(m-3) = 0$ which has solutions $m = 2$ and $m = 3$ and, hence, the complementary function is

$$Ae^{2x} + Be^{3x}.$$

However, since $\sinh 2x \equiv \frac{1}{2}(e^{2x} - e^{-2x})$, **part** of it is contained in the complementary function and we must find a particular integral for each part separately.
(a) For $\frac{1}{2}e^{2x}$, we may try

$$y = x\alpha e^{2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha e^{2x} + 2x\alpha e^{2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{2x} + 2\alpha e^{2x} + 4x\alpha e^{2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\alpha + 4x\alpha - 5\alpha - 10x\alpha + 6x\alpha)\, e^{2x} \equiv \frac{1}{2}e^{2x},$$

which gives $\alpha = -\frac{1}{2}$.

(b) For $-\frac{1}{2}e^{-2x}$, we may try

$$y = \beta e^{-2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\beta e^{-2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\beta e^{-2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\beta + 10\beta + 6\beta)e^{-2x} \equiv -\frac{1}{2}e^{-2x},$$

which gives $\beta = -\frac{1}{40}$.
The overall particular integral is thus

$$y = -\frac{1}{2}xe^{2x} - \frac{1}{40}e^{-2x}.$$

15.7.2 EXERCISES

Solve completely the following differential equations subject to the given boundary conditions:

1.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-x},$$

where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{2}$ when $x = 0$.

2.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y = 2\sin 3x,$$

where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8}{3}$ when $x = 0$.

3.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 10\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 8e^{3x} + 25x^2 - 20x + 27,$$

where $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 13$ when $x = 0$.

4.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cosh x,$$

where $y = \frac{7}{12}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$.

5.

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 24e^{-\frac{1}{2}x}$$

where $y = 6$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.7.3 ANSWERS TO EXERCISES

1.

$$y = \frac{1}{2}xe^{-x} + Ae^{-x} + Be^{-3x}.$$

2.

$$y = -\frac{1}{3}x\cos 3x + 2\cos 3x + \sin 3x.$$

3.

$$y = 2e^{3x} + x^2 + 1 + (2 - 3x)e^{5x}.$$

4.

$$y = \frac{1}{12}\left(e^{-x} - 6xe^{x} - e^{x} + 7e^{2x}\right).$$

5.

$$y = 3x^2e^{-\frac{1}{2}x} + (5x + 6)e^{-\frac{1}{2}x}.$$

"JUST THE MATHS"

UNIT NUMBER

15.8

ORDINARY DIFFERENTIAL EQUATIONS 8 (Simultaneous equations (A))

by

A.J.Hobson

UNIT 15.8 - ORDINARY DIFFERENTIAL EQUATIONS 8

SIMULTANEOUS EQUATIONS (A)

15.8.1 THE SUBSTITUTION METHOD

The methods discussed in previous Units for the solution of <u>second</u> order ordinary linear differential equations with constant coefficients may now be used for cases of two <u>first</u> order differential equations which must be satisfied simultaneously. The technique will be illustrated by the following examples:

EXAMPLES

1. Determine the general solutions for y and z in the case when

$$\begin{aligned} 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + 4y - z &= e^{-x}, - - - - - - - - - - - - (1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 3z &= 5e^{-x}. - - - - - - - - - - - (2) \end{aligned}$$

Solution

First, we eliminate one of the dependent variables from the two equations; in this case, we eliminate z.

From equation (2),

$$z = \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right)$$

and, on substituting this into equation (1), we obtain

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{3}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5e^{-x}\right) + 4y - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right) = e^{-x}.$$

That is, $$-\frac{2}{3}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{2}{3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{3}y = \frac{8}{3}e^{-x}$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = -4e^{-x}.$$

The auxiliary equation is

$$m^2 + m - 2 = 0 \quad \text{or} \quad (m-1)(m+2) = 0,$$

giving a complementary function of $Ae^x + Be^{-2x}$, where A and B are arbitrary constants. A particular integral will be of the form ke^{-x}, where $k - k - 2k = -4$ and hence $k = 2$. Thus,

$$y = 2e^{-x} + Ae^x + Be^{-2x}.$$

Finally, from the formula for z in terms of y,

$$z = \frac{1}{3}\left(-2e^{-x} + Ae^x - 2Be^{-2x} + 16e^{-x} + 8Ae^x + 8Be^{-2x} - 5e^{-x}\right).$$

That is,

$$z = 3e^{-x} + 3Ae^{-x} + 2Be^{-2x}.$$

Note:
The above example would have been a little more difficult if the second differential equation had contained a term in $\frac{\mathrm{d}z}{\mathrm{d}x}$. But, if this were the case, we could eliminate $\frac{\mathrm{d}z}{\mathrm{d}x}$ between the two equations in order to obtain a statement with the same form as Equation (2).

2. Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} + 2y &= e^x, - - - - - - - - - (1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} - 2z &= 1 + x, - - - - - - - -(2) \end{aligned}$$

given that $y = 1$ and $z = 2$ when $x = 0$.

Solution:

From equation (2), we have

$$z = \frac{1}{2}\left[\frac{\mathrm{d}y}{\mathrm{d}x} - 1 - x\right].$$

Substituting into the first differential equation gives

$$\frac{1}{2}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 1\right] + 2y = e^x$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 2e^x + 1.$$

The auxiliary equation is therefore $m^2 + 4 = 0$, having solutions $m = \pm j2$, which means that the complementary function is

$$A\cos 2x + B\sin 2x,$$

where A and B are arbitrary constants.
The particular integral will be of the form $y = pe^x + q$,
where

$$pe^x + 4pe^x + 4q = 2e^x + 1.$$

We require, then, that $5p = 2$ and $4q = 1$; and so the general solution for y is

$$y = A\cos 2x + B\sin 2x + \frac{2}{5}e^x + \frac{1}{4}.$$

Using the earlier formula for z, we obtain

$$z = \frac{1}{2}\left[-2A\sin 2x + 2B\cos 2x + \frac{2}{5}e^x - 1 - x\right] = B\cos 2x - A\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

Applying the boundary conditions,

$$1 = A + \frac{2}{5} + \frac{1}{4} \quad \text{giving} \quad A = \frac{7}{20}$$

and

$$2 = B + \frac{1}{5} - \frac{1}{2} \quad \text{giving} \quad B = \frac{23}{10}.$$

The required solutions are therefore

$$y = \frac{7}{20}\cos 2x + \frac{23}{10}\sin 2x + \frac{2}{5}e^x + \frac{1}{4}$$

and

$$z = \frac{23}{10}\cos 2x - \frac{7}{20}\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

15.8.2 EXERCISES

Solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}+2z &= e^{-x},\\ \frac{\mathrm{d}z}{\mathrm{d}x}+3z &= y,\end{aligned}$$

given that $y=1$ and $z=0$ when $x=0$.

2.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}-z &= \sin x,\\ \frac{\mathrm{d}z}{\mathrm{d}x}+y &= \cos x,\end{aligned}$$

given that $y=3$ and $z=4$ when $x=0$.

3.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}+2y-3z &= 1,\\ \frac{\mathrm{d}z}{\mathrm{d}x}-y &= e^{-2x},\end{aligned}$$

given that $y=0$ and $z=0$ when $x=0$.

4.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2z,\\ \frac{\mathrm{d}z}{\mathrm{d}x} &= 8y,\end{aligned}$$

given that $y=1$ and $z=0$ when $x=0$.

5.

$$\begin{aligned}\frac{dy}{dx} + 4\frac{dz}{dx} + 6z &= 0,\\ 5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y &= 0,\end{aligned}$$

given that $y = 3$ and $z = 0$ when $x = 0$.

Hint: First eliminate the $\frac{dz}{dx}$ terms to obtain a formula for z in terms of y and $\frac{dy}{dx}$.

6.

$$\begin{aligned}10\frac{dy}{dx} - 3\frac{dz}{dx} + 6y + 5z &= 0,\\ 2\frac{dy}{dx} - \frac{dz}{dx} + 2y + z &= 2e^{-x},\end{aligned}$$

given that $y = 2$ and $z = -1$ when $x = 0$.

Hint: First, eliminate the $\frac{dy}{dx}$ and z terms in one step, to obtain a formula for y in terms of $\frac{dz}{dx}$ and x.

15.8.3 ANSWERS TO EXERCISES

1.

$$y = (2x+1)e^{-x} \quad \text{and} \quad z = xe^{-x}.$$

2.

$$y = (x+4)\sin x + 3\cos x \quad \text{and} \quad z = (x+4)\cos x - 3\sin x.$$

3.

$$y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-3x} - e^{-2x} \quad \text{and} \quad z = \frac{1}{2}e^{x} - \frac{1}{6}e^{-3x} - \frac{1}{3}.$$

4.

$$y = \frac{1}{2}e^{4x} - \frac{1}{2}e^{-4x} \equiv \sinh 4x \quad \text{and} \quad z = e^{4x} + e^{-4x} \equiv 2\cosh 4x.$$

5.

$$y = 2e^{-x} + e^{-2x} \quad \text{and} \quad z = e^{-x} - e^{-2x}.$$

6.

$$y = \sin x + 2e^{-x} \quad \text{and} \quad z = e^{-x} - 2\cos x.$$

"JUST THE MATHS"

UNIT NUMBER

15.9

ORDINARY DIFFERENTIAL EQUATIONS 9 (Simultaneous equations (B))

by

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UNIT 15.9 - ORDINARY DIFFERENTIAL EQUATIONS 9

SIMULTANEOUS EQUATIONS (B)

15.9.1 INTRODUCTION

For students who have studied the principles of eigenvalues and eigenvectors (see Unit 9.6), a second method of solving two simultaneous linear differential equations is to interpret them as a single equation using matrix notation. The discussion will be limited to the simpler kinds of example, and we shall find it convenient to use t, x_1 and x_2 rather than x, y and z.

15.9.2 MATRIX METHODS FOR HOMOGENEOUS SYSTEMS

To introduce the technique, we begin by considering two simultaneous differential equations of the form

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2. \end{aligned}$$

which are of the "homogeneous" type, since no functions of t, other than x_1 and x_2, appear on the right hand sides.

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathrm{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(ii) Secondly, in a similar way to the method appropriate to a single differential equation, we make a trial solution of the form

$$\mathrm{X} = \mathrm{K}e^{\lambda t},$$

where

$$\mathrm{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

is a constant matrix of order 2×1.

This requires that

$$\lambda \mathrm{K} e^{\lambda t} = \mathrm{MK} e^{\lambda t} \quad \text{or} \quad \lambda \mathrm{K} = \mathrm{MK},$$

which we may recognise as the condition that λ is an eigenvalue of the matrix M, and K is an eigenvector of M.

The solutions for λ are obtained from the "characteristic equation"

$$|\mathrm{M} - \lambda \mathrm{I}| = 0.$$

In other words,

$$\begin{vmatrix} a - \lambda & b \\ c & b - \lambda \end{vmatrix} = 0,$$

leading to a quadratic equation having real and distinct solutions ($\lambda = \lambda_1$ and $\lambda = \lambda_2$), real and coincident solutions (λ only) or conjugate complex solutions ($\lambda = l \pm jm$).

(iii) The possibilities for the matrix K are obtained by solving the homogeneous linear equations

$$\begin{aligned} (a - \lambda_1 k_1 + b k_2 &= 0, \\ c k_1 + (d - \lambda_1) k_2 &= 0, \end{aligned}$$

giving $k_1 : k_2 = 1 : \alpha$ (say)

and

$$
\begin{aligned}
(a-\lambda_2)k_1 + bk_2 &= 0, \\
ck_1 + (d-\lambda_2)k_2 &= 0,
\end{aligned}
$$

giving $k_1 : k_2 = 1 : \beta$ (say).

Finally, it may be shown that, according to the types of solution to the auxiliary equation, the solution of the differential equation will take one of the following three forms, in which A and B are arbitrary constants:

(a)

$$
A\begin{bmatrix} 1 \\ \alpha \end{bmatrix} e^{\lambda_1 t} + B\begin{bmatrix} 1 \\ \beta \end{bmatrix} e^{\lambda_2 t};
$$

(b)

$$
\left\{(At+B)\begin{bmatrix} 1 \\ \alpha \end{bmatrix} + \frac{A}{b}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} e^{\lambda t};
$$

or

(c)

$$
e^{lt}\left\{\begin{bmatrix} A \\ pA+qB \end{bmatrix}\cos mt + \begin{bmatrix} B \\ pB-qA \end{bmatrix}\sin mt\right\},
$$

where, in (c), $1 : \alpha = 1 : p + jq$ and $1 : \beta = 1 : p - jq$.

EXAMPLES

1. Determine the general solution of the simultaneous differential equations

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -4x_1 + 5x_2, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= -x_1 + 2x_2.
\end{aligned}
$$

Solution

The characteristic equation is

$$\begin{vmatrix} -4-\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 + 2\lambda - 3 = 0 \quad \text{or} \quad (\lambda - 1)(\lambda + 3) = 0.$$

When $\lambda = 1$, we need to solve the homogeneous equations

$$\begin{aligned} -5k_1 + 5k_2 &= 0, \\ -k_1 + k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : 1$.
When $\lambda = -3$, we need to solve the homogeneous equations

$$\begin{aligned} -k_1 + 5k_2 &= 0, \\ -k_1 + 5k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{1}{5}$.
The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} e^{-3t}$$

or, alternatively,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{-3t},$$

where A and B are arbitrary constants.

2. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 - x_2,\\ \frac{dx_2}{dt} &= x_1 + 3x_2.\end{aligned}$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 4 = 0 \quad \text{or} \quad (\lambda - 2)^2 = 0.$$

When $\lambda = 2$, we need to solve the homogeneous equations

$$\begin{aligned}-k_1 - k_2 &= 0,\\ k_1 + k_2 &= 0,\end{aligned}$$

both of which give $k_1 : k_2 = 1 : -1$.

The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{(At+B)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - A\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} e^{2t},$$

where A and B are arbitrary constants.

3. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 - 5x_2,\\ \frac{dx_2}{dt} &= 2x_1 + 3x_2.\end{aligned}$$

Solution
The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 13 = 0,$$

which gives $\lambda = 2 \pm j3$.
When $\lambda = 2 + j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1-j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1-j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1-j3}{5}$.
When $\lambda = 2 - j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1+j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1+j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1+j3}{5}$.
The general solution is therefore

$$\frac{e^{2t}}{5}\left\{\begin{bmatrix} A \\ -A+3B \end{bmatrix}\cos 3t + \begin{bmatrix} B \\ -B-3A \end{bmatrix}\sin 3t\right\},$$

where A and B are arbitrary constants.
Note:
From any set of simultaneous differential equations of the form

$$\begin{aligned} a\frac{\mathrm{d}x_1}{\mathrm{d}t} + b\frac{\mathrm{d}x_2}{\mathrm{d}t} + cx_1 + dx_2 = 0, \\ a'\frac{\mathrm{d}x_1}{\mathrm{d}t} + b'\frac{\mathrm{d}x_2}{\mathrm{d}t} + b'x_1 + c'x_2 = 0, \end{aligned}$$

it is possible to eliminate $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ in turn, in order to obtain two equivalent equations of the form discussed in the above examples.

15.9.3 EXERCISES

1. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 + 3x_2,\\ \frac{dx_2}{dt} &= 3x_1 + x_2.\end{aligned}$$

2. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 + 2x_2,\\ \frac{dx_2}{dt} &= 4x_1 + x_2,\end{aligned}$$

given that $x_1 = 3$ and $x_2 = -3$ when $t = 0$.

3. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + 9x_2,\\ \frac{dx_2}{dt} &= 11x_1 + x_2,\end{aligned}$$

given that $x_1 = 20$ and $x_2 = 20$ when $t = 0$.

4. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} - \frac{dx_2}{dt} + 2x_1 - 2x_2 &= 0,\\ \frac{dx_1}{dt} + 2\frac{dx_2}{dt} - 7x_1 - 5x_2 &= 0,\end{aligned}$$

given that $x_1 = 2$ and $x_2 = 0$ when $t = 0$.

5. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2.\end{aligned}$$

6. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2.\end{aligned}$$

15.9.4 ANSWERS TO EXERCISES

1.
$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

2.
$$\begin{bmatrix}1\\1\end{bmatrix}e^{5t} + 2\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}.$$

3.
$$2\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t} + \begin{bmatrix}18\\22\end{bmatrix}e^{10t}.$$

4.
$$\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + \begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

5.
$$\left\{(At+B)\begin{bmatrix}1\\-1\end{bmatrix} + \frac{A}{2}\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}.$$

6.
$$e^{7t}\left\{\begin{bmatrix}A\\-A+2B\end{bmatrix}\cos 2t + \begin{bmatrix}B\\-B-2A\end{bmatrix}\sin 2t\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

15.10

ORDINARY DIFFERENTIAL EQUATIONS 10 (Simultaneous equations (C))

by

A.J.Hobson

UNIT 15.10 - ORDINARY DIFFERENTIAL EQUATIONS 10

SIMULTANEOUS EQUATIONS (C)

15.10.1 MATRIX METHODS FOR NON-HOMOGENEOUS SYSTEMS

In Units 15.5, 15.6 and 15.7 , it was seen that, for a single linear differential equation with constant coefficients, the general solution is made up of a particular integral and a complementary function (the latter being the general solution of the corresponding homogeneous differential equation).

In the work which follows, a similar principle is applied to a pair of simultaneous non-homogeneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2 + f(t),\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2 + g(t).\end{aligned}$$

The method will be illustrated by the following example, in which $f(t) \equiv 0$:

EXAMPLE

Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, - - - - - - - - - - - - -(1)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -4x_1 - 5x_2 + g(t), - - - - -(2)\end{aligned}$$

where $g(t)$ is (a) t, (b) e^{2t} (c) $\sin t$, (d) e^{-t}.

Solutions

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t),$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t) \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \quad \text{and} \quad \mathrm{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Secondly, we consider the corresponding "homogeneous" system

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX},$$

for which the characteristic equation is

$$\begin{vmatrix} 0-\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0,$$

and gives

$$\lambda(5+\lambda)+4=0 \quad \text{or} \quad \lambda^2+5\lambda+4=0 \quad \text{or} \quad (\lambda+1)(\lambda+4)=0.$$

(iii) The eigenvectors of M are obtained from the homogeneous equations

$$\begin{aligned} -\lambda k_1 + k_2 &= 0, \\ -4k_1 - (5+\lambda)k_2 &= 0. \end{aligned}$$

Hence, in the case when $\lambda = -1$, we solve

$$\begin{aligned} k_1 + k_2 &= 0, \\ -4k_1 - 4k_2 &= 0, \end{aligned}$$

and these are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -1$.

Also, when $\lambda = -4$, we solve

$$\begin{aligned} 4k_1 + k_2 &= 0, \\ -4k_1 - k_2 &= 0 \end{aligned}$$

which are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -4$.

The complementary function may now be written in the form

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t},$$

where A and B are arbitrary constants.

(iv) In order to obtain a particular integral for the equation

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t),$$

we note the second term on the right hand side and investigate a trial solution of a similar form. The three cases in this example are as follows:

(a) $g(t) \equiv t$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P} + \mathrm{Q}t,$$

where P and Q are constant matrices of order 2×1.

We require that

$$\mathrm{Q} = \mathrm{M}(\mathrm{P} + \mathrm{Q}t) + \mathrm{N}t,$$

whereupon, equating the matrix coefficients of t and the constant matrices,

$$\mathrm{MQ} + \mathrm{N} = \mathbf{0} \quad \text{and} \quad \mathrm{Q} = \mathrm{MP},$$

giving

$$\mathrm{Q} = -\mathrm{M}^{-1}\mathrm{N} \quad \text{and} \quad \mathrm{P} = \mathrm{M}^{-1}\mathrm{Q}.$$

Thus, using

$$\mathrm{M}^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix},$$

we obtain

$$\mathrm{Q} = -\frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

and

$$\mathrm{P} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} t.$$

(b) $g(t) \equiv e^{2t}$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P}e^{2t}$$

We require that

$$2\mathrm{P}e^{2t} = \mathrm{MP}e^{2t} + \mathrm{N}e^{2t}.$$

That is,

$$2\mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The matrix, P, may now be determined from the formula

$$(2\mathrm{I} - \mathrm{M})\mathrm{P} = \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix}.\mathrm{P} = \mathrm{N}.$$

Hence,

$$\mathrm{P} = \frac{1}{18}\begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix}.\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-4t} + \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}e^{2t}.$$

(c) $g(t) \equiv \sin t$

$$\text{Trial solution } \mathrm{X} = \mathrm{P}\sin t + \mathrm{Q}\cos t.$$

We require that

$$\mathrm{P}\cos t - \mathrm{Q}\sin t = \mathrm{M}(\mathrm{P}\sin t + \mathrm{Q}\cos t) + \mathrm{N}\sin t.$$

Equating the matrix coefficients of $\cos t$ and $\sin t$,

$$\mathrm{P} = \mathrm{MQ} \text{ and } -\mathrm{Q} = \mathrm{MP} + \mathrm{N},$$

which means that

$$-\mathrm{Q} = \mathrm{M}^2\mathrm{Q} + \mathrm{N} \text{ or } (\mathrm{M}^2 + \mathrm{I})\mathrm{Q} = -\mathrm{N}.$$

Thus,

$$Q = -(M^2 + I)^{-1}N,$$

where

$$M^2 + I = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 20 & 22 \end{bmatrix}$$

and, hence,

$$Q = -\frac{1}{34}\begin{bmatrix} 22 & 5 \\ -20 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

Also,

$$P = MQ = \frac{1}{34}\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{34}\begin{bmatrix} 3 \\ 5 \end{bmatrix}\sin t + \frac{1}{34}\begin{bmatrix} -5 \\ 3 \end{bmatrix}\cos t.$$

(d) $g(t) \equiv e^{-t}$

In this case, the function, $g(t)$, is already included in the complementary function and it becomes necessary to assume a particular integral of the form

$$X = (P + Qt)e^{-t},$$

where P and Q are constant matrices of order 2×1.

We require that

$$Qe^{-t} - (P + Qt)e^{-t} = M(P + Qt)e^{-t} + Ne^{-t},$$

whereupon, equating the matrix coefficients of te^{-t} and e^{-t}, we obtain

$$-\mathrm{Q} = \mathrm{MQ} \quad \text{and} \quad \mathrm{Q} - \mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The first of these conditions shows that Q is an eigenvector of the matrix M corresponding the eigenvalue -1 and so, from earlier work,

$$\mathrm{Q} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any constant k.

Also,

$$(\mathrm{M} + \mathrm{I})\mathrm{P} = \mathrm{Q} - \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} p_1 + p_2 &= k, \\ -4p_1 - 4p_2 &= -k - 1. \end{aligned}$$

Using $p_1 + p_2 = k$ and $p_1 + p_2 = \frac{k+1}{4}$, we deduce that $k = \frac{1}{3}$ and that the matrix P is given by

$$\mathrm{P} = \begin{bmatrix} l \\ \frac{1}{3} - l \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + l \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any number, l.

Taking $l = 0$ for simplicity, a particular integral is therefore

$$\mathrm{X} = \frac{1}{3}\left\{\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}t\right\}e^{-t}.$$

and the general solution is

$$\mathrm{X} = A\begin{bmatrix}1\\-1\end{bmatrix}e^{-t} + B\begin{bmatrix}1\\-4\end{bmatrix}e^{-4t} + \frac{1}{3}\left\{\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}t\right\}e^{-t}.$$

Note:
In examples for which neither $f(t)$ nor $g(t)$ is identically equal to zero, the particular integral may be found by adding together the separate forms of particular integral for $f(t)$ and $g(t)$ and writing the system of differential equations in the form

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}_1 f(t) + \mathrm{N}_2 g(t),$$

where

$$\mathrm{N}_1 = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } \mathrm{N}_2 = \begin{bmatrix}0\\1\end{bmatrix}.$$

For instance, if $f(t) \equiv t$ and $g(t) \equiv e^{2t}$, the particular integral would take the form

$$\mathrm{X} = \mathrm{P} + \mathrm{Q}t + \mathrm{R}e^{2t},$$

where P, Q and R are matrices of order 2×1.

15.10.2 EXERCISES

1. Determine the general solutions of the following systems of simultaneous differential equations:

   (a)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + 3x_2 + 5t,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 3x_1 + x_2 + e^{3t}.\end{aligned}$$

(b)

$$\frac{dx_1}{dt} = 3x_1 + 2x_2 + t^2,$$
$$\frac{dx_2}{dt} = 4x_1 + x_2 + e^{-2t}.$$

2. Determine the complete solutions of the following systems of differential equations, subject to the conditions given:

(a)

$$\frac{dx_1}{dt} = -x_1 + 9x_2 + 3,$$
$$\frac{dx_2}{dt} = 11x_1 + x_2 + e^{10t},$$

given that $x_1 = \frac{1}{225}$ and $x_2 = -\frac{1}{100}$ when $t = 0$.

(b)

$$\frac{dx_1}{dt} = 5x_1 + 2x_2 + 2t^2 + t,$$
$$\frac{dx_2}{dt} = -2x_1 + x_2,$$

given that $x_1 = \frac{32}{27}$ and $x_2 = -\frac{12}{27}$ when $t = 0$.

(c)

$$\frac{dx_1}{dt} = 8x_1 + x_2 + \sin t,$$
$$\frac{dx_2}{dt} = -5x_1 + 6x_2 + \cos t,$$

given that $x_1 = 0$ and $x_2 = 0$ when $t = 0$.

15.10.3 ANSWERS TO EXERCISES

1. (a)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t}+B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}+\frac{1}{32}\begin{bmatrix}-25\\15\end{bmatrix}+\frac{1}{8}\begin{bmatrix}5\\-15\end{bmatrix}t-\frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}e^{3t};$$

   (b)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{5t}+B\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}+\frac{2}{125}\begin{bmatrix}41\\-84\end{bmatrix}+\frac{2}{25}\begin{bmatrix}-9\\16\end{bmatrix}t+\frac{1}{5}\begin{bmatrix}1\\-4\end{bmatrix}t^2+\frac{1}{7}\begin{bmatrix}2\\-5\end{bmatrix}e^{-2t}.$$

2. (a)

$$-\frac{7}{45}\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t}+\frac{13}{900}\begin{bmatrix}9\\11\end{bmatrix}e^{10t}+\frac{3}{100}\begin{bmatrix}1\\-11\end{bmatrix}+\frac{1}{180}\begin{bmatrix}0\\1\end{bmatrix}e^{10t}+\frac{1}{20}\begin{bmatrix}9\\11\end{bmatrix}te^{10t};$$

   (b)

$$\left\{(2t+1)\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}+\frac{1}{27}\begin{bmatrix}5\\-12\end{bmatrix}+\frac{1}{27}\begin{bmatrix}1\\-22\end{bmatrix}t-\frac{2}{9}\begin{bmatrix}1\\2\end{bmatrix}t^2;$$

   (c)

$$\frac{1}{145}\left\{e^{7t}\left(\begin{bmatrix}-1\\25\end{bmatrix}\cos 2t+\begin{bmatrix}-12\\-10\end{bmatrix}\sin 2t\right)+\begin{bmatrix}-17\\-10\end{bmatrix}\sin t+\begin{bmatrix}1\\-25\end{bmatrix}\cos t\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

16.1

LAPLACE TRANSFORMS 1
(Definitions and rules)

by

A.J.Hobson

UNIT 16.1 - LAPLACE TRANSFORMS 1 - DEFINITIONS AND RULES

16.1.1 INTRODUCTION

The theory of **"Laplace Transforms"** to be discussed in the following notes will be for the purpose of solving certain kinds of **"differential equation"**; that is, an equation which involves a derivative or derivatives.

The particular differential equation problems to be encountered will be limited to the two types listed below:

(a) Given the **"first order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

together with the value of x when $t = 0$ (that is, $x(0)$), determine a formula for x in terms of t, which does not include any derivatives.

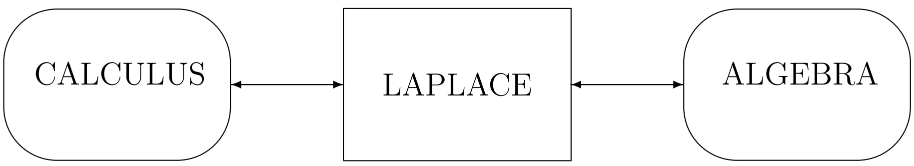
(b) Given the **"second order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

together with the values of x and $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $t = 0$ (that is, $x(0)$ and $x'(0)$), determine a formula for x in terms of t which does not include any derivatives.

Roughly speaking, the method of Laplace Transforms is used to convert a calculus problem (the differential equation) into an algebra problem (frequently an exercise on partial fractions and/or completing the square).

The solution of the algebra problem is then fed backwards through what is called an **"Inverse Laplace Transform"** and the solution of the differential equation is obtained.



The background to the development of Laplace Transforms would be best explained using certain other techniques of solving differential equations which may not have been part of earlier work. This background will therefore be omitted here.

DEFINITION
The Laplace Transform of a given function $f(t)$, defined for $t > 0$, is defined by the definite integral

$$\int_0^\infty e^{-st} f(t) \ \mathrm{d}t,$$

where s is an **arbitrary positive number**.

Notes
(i) The Laplace Transform is usually denoted by $L[f(t)]$ or $F(s)$, since the result of the definite integral in the definition will be an expression involving s.

(ii) Although s is an arbitrary positive number, it is occasionally necessary to assume that it is large enough to avoid difficulties in the calculations; (see the note to the second standard result below).

16.1.2 LAPLACE TRANSFORMS OF SIMPLE FUNCTIONS

The following is a list of standard results on which other Laplace Transforms will be based:

1. $f(t) \equiv t^n$.

$$F(s) = \int_0^\infty e^{-st} t^n \ \mathrm{d}t = I_n \ \text{ say.}$$

Hence,

$$I_n = \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} \ \mathrm{d}t = \frac{n}{s}.I_{n-1},$$

using the fact that e^{-st} tends to zero much faster than any other function of t can tend to infinity. That is, a decaying exponential will always have the dominating effect.

We conclude that

$$I_n = \frac{n}{s}.\frac{(n-1)}{s}.\frac{(n-2)}{s}....\frac{2}{s}.\frac{1}{s}.I_0 = \frac{n!}{s^n}.I_0.$$

But,

$$I_0 = \int_0^\infty e^{-st} \ \mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}.$$

Thus,

$$L[t^n] = \frac{n!}{s^{n+1}}.$$

Note:
This result also shows that

$$L[1] = \frac{1}{s},$$

since $1 = t^0$.

2. $f(t) \equiv e^{-at}$.

$$F(s) = \int_0^\infty e^{-st}e^{-at}\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}\ \mathrm{d}t$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}.$$

Hence,

$$L[e^{-at}] = \frac{1}{s+a}.$$

Note:
A slightly different form of this result, less commonly used in applications to science and engineering, is

$$L[e^{bt}] = \frac{1}{s-b};$$

but, to obtain this result by integration, we would need to assume that $s > b$ to ensure that $e^{-(s-b)t}$ is genuinely a **decaying** exponential.

3. $f(t) \equiv \cos at$.

$$F(s) = \int_0^\infty e^{-st}\cos at\ \mathrm{d}t = \left[\frac{e^{-st}\sin at}{a}\right]_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at\ \mathrm{d}t$$

using Integration by Parts, once.
Using Integration by Parts a second time,

$$F(s) = 0 + \frac{s}{a}\left\{\left[-\frac{e^{-st}\cos at}{a}\right]_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos at\ \mathrm{d}t\right\},$$

which gives

$$F(s) = \frac{s}{a^2} - \frac{s^2}{a^2}.F(s).$$

That is,

$$F(s) = \frac{s}{s^2+a^2}.$$

In other words,

$$L[\cos at] = \frac{s}{s^2+a^2}.$$

4. $f(t) \equiv \sin at$.
   The method is similar to that for $\cos at$, and we obtain
$$L[\sin at] = \frac{a}{s^2 + a^2}.$$

16.1.3 ELEMENTARY LAPLACE TRANSFORM RULES

The following list of results is of use in finding the Laplace Transform of a function which is made up of **basic** functions, such as those encountered in the previous section.

1. LINEARITY

If A and B are constants, then
$$L[Af(t) + Bg(t)] = AL[f(t)] + BL[g(t)].$$

Proof:

This follows easily from the linearity of an integral.

EXAMPLE

Determine the Laplace Transform of the function,
$$2t^5 + 7\cos 4t - 1.$$

Solution
$$L[2t^5 + 7\cos 4t - 1] = 2.\frac{5!}{s^6} + 7.\frac{s}{s^2 + 4^2} - \frac{1}{s} = \frac{240}{s^6} + \frac{7s}{s^2 + 16} - \frac{1}{s}.$$

2. THE TRANSFORM OF A DERIVATIVE

The two results which follow are of special use when solving first and second order differential equations. We shall begin by discussing them in relation to an arbitrary function, $f(t)$; then we shall restate them in the form which will be needed for solving differential equations.

(a)
$$L[f'(t)] = sL[f(t)] - f(0).$$

Proof:
$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\ \mathrm{d}t = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\ \mathrm{d}t$$

using integration by parts.

Thus,
$$L[f'(t)] = -f(0) + sL[f(t)],$$

as required.

(b)
$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0).$$

Proof:
Treating $f''(t)$ as the first derivative of $f'(t)$, we have
$$L[f''(t)] = sL[f'(t)] - f'(0),$$
which gives the required result on substituting from (a) the expression for $L[f'(t)]$.
Alternative Forms (Using $L[x(t)] = X(s)$)**:**
(i)
$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$
(ii)
$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \text{ or } s[sX(s) - x(0)] - x'(0).$$

3. THE (First) SHIFTING THEOREM

$$L[e^{-at}f(t)] = F(s+a).$$

Proof:
$$L[e^{-at}f(t)] = \int_0^\infty e^{-st}e^{-at}f(t)\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}f(t)\ \mathrm{d}t,$$
which can be regarded as the effect of replacing s by $s+a$ in $L[f(t)]$. In other words, $F(s+a)$.
Notes:
(i) Sometimes, this result is stated in the form
$$L[e^{bt}f(t)] = F(s-b)$$
but, in science and engineering, the exponential is more likely to be a **decaying** exponential.
(ii) There is, in fact, a Second Shifting Theorem, encountered in more advanced courses; but we do not include it in this Unit (see Unit 16.5).
EXAMPLE
Determine the Laplace Transform of the function, $e^{-2t}\sin 3t$.
Solution
First of all, we note that
$$L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.$$
Replacing s by $(s+2)$ in this result, the First Shifting Theorem gives
$$L[e^{-2t}\sin 3t] = \frac{3}{(s+2)^2+9}.$$

4. MULTIPLICATION BY t

$$L[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}[F(s)].$$

Proof:

It may be shown that

$$\frac{\mathrm{d}}{\mathrm{d}s}[F(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\mathrm{d}t = \int_0^\infty -te^{-st}f(t)\ \mathrm{d}t = -\ L[tf(t)].$$

EXAMPLE

Determine the Laplace Transform of the function,

$$t\cos 7t.$$

Solution

$$L[t\cos 7t] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2+7^2}\right] = -\ \frac{(s^2+7^2).1 - s.2s}{(s^2+7^2)^2} = \frac{s^2-49}{(s^2+49)^2}.$$

THE USE OF A TABLE OF LAPLACE TRANSFORMS AND RULES

For the purposes of these Units, the following **brief** table may be used to determine the Laplace Transforms of functions of t without having to use integration:

$f(t)$	$L[f(t)] = F(s)$
K (a constant)	$\frac{K}{s}$
e^{-at}	$\frac{1}{s+a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
$\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$

16.1.4 FURTHER LAPLACE TRANSFORM RULES

1.
$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

2.
$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \quad \text{or} \quad s[sX(s) - x(0)] - x'(0).$$

3. **The Initial Value Theorem**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s),$$

provided that the indicated limits exist.

4. **The Final Value Theorem**

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s),$$

provided that the indicated limits exist.

5. **The Convolution Theorem**

$$L\left[\int_0^t f(T)g(t-T)\ \mathrm{d}T\right] = F(s)G(s).$$

16.1.5 EXERCISES

1. Use a table the table of Laplace Transforms to find $L[f(t)]$ in the following cases:

(a)

$$3t^2 + 4t - 1;$$

(b)

$$t^3 + 3t^2 + 3t + 1 \quad (\equiv (t+1)^3);$$

(c)

$$2e^{5t} - 3e^t + e^{-7t};$$

(d)

$$2\sin 3t - 3\cos 2t;$$

(e)

$$t\sin 6t;$$

(f)

$$t(e^t + e^{-2t});$$

(g)

$$\frac{1}{2}(1-\cos 2t) \quad (\equiv \sin^2 t).$$

2. Using the First Shifting Theorem, obtain the Laplace Transforms of the following functions of t:

(a)

$$e^{-3t}\cos 5t;$$

(b)

$$t^2e^{2t};$$

(c)

$$e^{-2t}\left(2t^3+3t-2\right);$$

(d)

$$\cosh 2t.\sin t;$$

(e)

$$e^{-at}f'(t),$$

where $L[f(t)] = F(s)$.

3. (a) If

$$x = t^3e^{-t},$$

determine the Laplace Transform of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ without differentiating x more than once with respect to t.

(b) If

$$\frac{\mathrm{d}x}{\mathrm{d}t}+x=e^t,$$

where $x(0) = 0$, show that

$$X(s)=\frac{1}{s^2-1}.$$

4. Verify the Initial and Final Value Theorems for the function

$$f(t) = te^{-3t}.$$

16.1.6 ANSWERS TO EXERCISES

1. (a)

$$\frac{6}{s^3}+\frac{4}{s^2}-\frac{1}{s};$$

(b)

$$\frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s};$$

(c)

$$\frac{2}{s-5}-\frac{3}{s-1}+\frac{1}{s+7};$$

(d)

$$\frac{6}{s^2+9}-\frac{3s}{s^2+4};$$

(e)

$$\frac{12s}{(s^2+36)^2};$$

(f)

$$\frac{1}{(s-1)^2}+\frac{1}{(s+2)^2};$$

(g)

$$\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right].$$

2. (a)

$$\frac{s+3}{(s+3)^2+25};$$

(b)

$$\frac{2}{(s-2)^3};$$

(c)

$$\frac{12}{(s+2)^4}+\frac{3}{(s+2)^2}-\frac{2}{s+2};$$

(d)

$$\frac{1}{2}\left[\frac{1}{(s-2)^2+1}+\frac{1}{(s+2)^2+1}\right];$$

(e)

$$(s+a)F(s+a)-f(0).$$

3. (a)

$$\frac{6s^2}{(s+1)^4};$$

(b) On the left hand side, use the formula for $L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]$.

4.

$$\lim_{t\to 0} f(t)=0 \quad \text{and} \quad \lim_{t\to\infty} f(t)=0.$$

"JUST THE MATHS"

UNIT NUMBER

16.2

LAPLACE TRANSFORMS 2
(Inverse Laplace Transforms)

by

A.J.Hobson

UNIT 16.2 - LAPLACE TRANSFORMS 2
INVERSE LAPLACE TRANSFORMS

In order to solve differential equations, we now examine how to determine a function of the variable, t, whose Laplace Transform is already known.

16.2.1 THE DEFINITION OF AN INVERSE LAPLACE TRANSFORMS

A function of t, whose Laplace Transform is the given expression, $F(s)$, is called the **"Inverse Laplace Transform"** of $f(t)$ and may be denoted by the symbol

$$L^{-1}[F(s)].$$

Notes:
(i) Since two functions which coincide for $t > 0$ will have the same Laplace Transform, we can determine the Inverse Laplace Transform of $F(s)$ only for **positive** values of t.

(ii) Inverse Laplace Transforms are **linear** since

$$L^{-1}\left[AF(s) + BG(s)\right]$$

is a function of t whose Laplace Transform is

$$AF(s) + BG(s);$$

and, by the linearity of Laplace Transforms, discussed in Unit 16.1, such a function is

$$AL^{-1}[F(s)] + BL^{-1}[G(s)].$$

16.2.2 METHODS OF DETERMINING AN INVERSE LAPLACE TRANSFORM

The type of differential equation to be encountered in simple practical problems usually lead to Laplace Transforms which are **"rational functions of** s**"**. We shall restrict the discussion to such cases, as illustrated in the following examples, where the table of standard Laplace Transforms is used whenever possible. The partial fractions are discussed in detail, but other, shorter, methods may be used if known (for example, the "Cover-up Rule" and "Keily's Method"; see Unit 1.9)

EXAMPLES

1. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{s^3} + \frac{4}{s-2}.$$

Solution

$$f(t) = \frac{3}{2}t^2 + 4e^{2t} \quad t > 0$$

2. Determine the Inverse Laplace Transform of

$$F(s) = \frac{2s+3}{s^2+3s} = \frac{2s+3}{s(s+3)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{2s+3}{s(s+3)} \equiv \frac{A}{s} + \frac{B}{s+3},$$

giving

$$2s+3 \equiv A(s+3) + Bs$$

Note:

Although the s of a Laplace Transform is an arbitrary **positive** number, we may temporarily ignore that in order to complete the partial fractions. Otherwise, entire partial fractions exercises would have to be carried out by equating coefficients of appropriate powers of s on both sides.

Putting $s = 0$ and $s = -3$ gives

$$3 = 3A \text{ and } -3 = -3B;$$

so that

$$A = 1 \text{ and } B = 1.$$

Hence,

$$F(s) = \frac{1}{s} + \frac{1}{s+3}$$

Finally,

$$f(t) = 1 + e^{-3t} \quad t > 0.$$

3. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{s^2+9}.$$

Solution

$$f(t) = \frac{1}{3}\sin 3t \quad t > 0.$$

4. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+2}{s^2+5}.$$

Solution

$$f(t) = \cos t\sqrt{5} + \frac{2}{\sqrt{5}}\sin t\sqrt{5} \quad t > 0.$$

5. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3s^2 + 2s + 4}{(s+1)(s^2+4)}.$$

Solution
Applying the principles of partial fractions,

$$\frac{3s^2 + 2s + 4}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs + C}{s^2+4}.$$

That is,

$$3s^2 + 2s + 4 \equiv A(s^2+4) + (Bs+C)(s+1).$$

Substituting $s = -1$, we obtain

$$5 = 5A \text{ which implies that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$3 = A + B \text{ so that } B = 2.$$

Equating constant terms on both sides,

$$4 = 4A + C \text{ so that } C = 0.$$

We conclude that

$$F(s) = \frac{1}{s+1} + \frac{2s}{s^2+4}.$$

Hence,

$$f(t) = e^{-t} + 2\cos 2t \quad t > 0.$$

6. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s+2)^5}.$$

Solution
Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{n!}{s^{n+1}}$, we obtain

$$f(t) = \frac{1}{24}t^4 e^{-2t} \quad t > 0.$$

7. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{(s-7)^2 + 9}.$$

Solution
Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{a}{s^2+a^2}$, we obtain

$$f(t) = e^{7t}\sin 3t \quad t > 0.$$

8. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s}{s^2 + 4s + 13}.$$

Solution

The denominator will not factorise conveniently, so we **complete the square**, giving

$$F(s) = \frac{s}{(s+2)^2 + 9}.$$

In order to use the First Shifting Theorem, we must try to include $s+2$ in the numerator; so we write

$$F(s) = \frac{(s+2) - 2}{(s+2)^2 + 9} = \frac{s+2}{(s+2)^2 + 3^2} - \frac{2}{3}.\frac{3}{(s+2)^2 + 3^2}.$$

Hence,

$$f(t) = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t = \frac{1}{3}e^{-2t}\left[3\cos 3t - 2\sin 3t\right] \quad t > 0.$$

9. Determine the Inverse Laplace Transform of

$$F(s) = \frac{8(s+1)}{s(s^2 + 4s + 8)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{8(s+1)}{s(s^2 + 4s + 8)} \equiv \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}.$$

Mutiplying up, we obtain

$$8(s+1) \equiv A(s^2 + 4s + 8) + (Bs + C)s.$$

Substituting $s = 0$ gives

$$8 = 8A \text{ so that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B \text{ which gives } B = -1.$$

Equating coefficients of s on both sides,

$$8 = 4A + C \text{ which gives } C = 4.$$

Thus,

$$F(s) = \frac{1}{s} + \frac{-s + 4}{s^2 + 4s + 8}.$$

The quadratic denominator will not factorise conveniently, so we complete the square to give

$$F(s) = \frac{1}{s} + \frac{-s+4}{(s+2)^2+4},$$

which, on rearrangement, becomes

$$F(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+2^2} + \frac{6}{(s+2)^2+2^2}.$$

Thus, from the First Shifting Theorem,

$$f(t) = 1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t \quad t > 0.$$

10. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+10}{s^2-4s-12}.$$

Solution

This time, the denominator **will** factorise, into $(s+2)(s-6)$, and partial fractions give

$$\frac{s+10}{(s+2)(s-6)} \equiv \frac{A}{s+2} + \frac{B}{s-6}.$$

Hence,

$$s + 10 \equiv \mathrm{A}(s-6) + \mathrm{B}(s+2).$$

Putting $s = -2$,

$$8 = -8A \text{ giving } A = -1.$$

Putting $s = 6$,

$$16 = 8B \text{ giving } B = 2.$$

We conclude that

$$F(s) = \frac{-1}{s+2} + \frac{2}{s-6}.$$

Finally,

$$f(t) = -e^{-2t} + 2e^{6t} \quad t > 0.$$

However, if we did not factorise the denominator, a different form of solution could be obtained as follows:

$$F(s) = \frac{(s-2)+12}{(s-2)^2-4^2} = \frac{s-2}{(s-2)^2-4^2} + 3.\frac{4}{(s-2)^2+4^2}.$$

Hence,

$$f(t) = e^{2t}[\cosh 4t + 3\sinh 4t] \quad t > 0.$$

11. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s-1)(s+2)}.$$

Solution

The Inverse Laplace Transform of this function could certainly be obtained by using partial fractions, but we note here how it could be obtained from the Convolution Theorem.

Writing

$$F(s) = \frac{1}{(s-1)}.\frac{1}{(s+2)},$$

we obtain

$$f(t) = \int_0^t e^T.e^{-2(t-T)}\ \mathrm{d}T = \int_0^t e^{(3T-2t)}\ \mathrm{d}T = \left[\frac{e^{3T-2t}}{3}\right]_0^t.$$

That is,

$$f(t) = \frac{e^t}{3} - \frac{e^{-2t}}{3} \quad t > 0.$$

16.2.3 EXERCISES

Determine the Inverse Laplace Transforms of the following rational functions of s:

1. (a)

$$\frac{1}{(s-1)^2};$$

(b)

$$\frac{1}{(s+1)^2+4};$$

(c)

$$\frac{s+2}{(s+2)^2+9};$$

(d)

$$\frac{s-2}{(s-3)^3};$$

(e)

$$\frac{1}{(s^2+4)^2};$$

(f)

$$\frac{s+1}{s^2+2s+5};$$

(g)

$$\frac{s-3}{s^2-4s+5};$$

(h)

$$\frac{s-3}{(s-1)^2(s-2)};$$

(i)

$$\frac{5}{(s+1)(s^2-2s+2)};$$

(j)

$$\frac{2s-9}{(s-3)(s+2)};$$

(k)

$$\frac{3}{s(s^2+9)};$$

(l)

$$\frac{2s-1}{(s-1)(s^2+2s+2)}.$$

2. Use the Convolution Theorem to obtain the Inverse Laplace Transform of

$$\frac{s}{(s^2+1)^2}.$$

16.2.4 ANSWERS TO EXERCISES

1. (a)

$$te^t \quad t > 0;$$

(b)

$$\frac{1}{2}e^{-t}\sin 2t \quad t > 0;$$

(c)

$$e^{-2t}\cos 3t \quad t > 0;$$

(d)

$$e^{3t}\left[t + \frac{1}{2}t^2\right] \quad t > 0;$$

(e)

$$\frac{1}{16}[\sin 2t - 2t\cos 2t] \quad t > 0;$$

(f)

$$e^{-t}\cos 2t \quad t > 0;$$

(g)

$$e^{2t}[\cos t - \sin t] \quad t > 0;$$

(h)

$$2te^{t} + e^{t} - e^{2t} \quad t > 0;$$

(i)

$$e^{-t} + e^{t}[2\sin t - \cos t] \quad t > 0;$$

(j)

$$\frac{1}{5}[13e^{-2t} - 3e^{3t}] \quad t > 0;$$

(k)

$$\frac{1}{3}[1 - \cos 3t] \quad t > 0;$$

(l)

$$\frac{1}{5}[e^{t} - e^{-t}\cos t + 8e^{-t}\sin t] \quad t > 0.$$

2.

$$\frac{1}{2}t\sin t \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.3

LAPLACE TRANSFORMS 3
(Differential equations)

by

A.J.Hobson

UNIT 16.3 - LAPLACE TRANSFORMS 3 - DIFFERENTIAL EQUATIONS

16.3.1 EXAMPLES OF SOLVING DIFFERENTIAL EQUATIONS

In the work which follows, the problems considered will usually take the form of a linear differential equation of the second order with constant coefficients.

That is,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

However, the method will apply equally well to the corresponding first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t).$$

The technique will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 0,$$

given that $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 3] + 4[sX(s) - 3] + 13X(s) = 0.$$

Hence,

$$(s^2 + 4s + 13)X(s) = 3s + 12,$$

giving

$$X(s) \equiv \frac{3s + 12}{s^2 + 4s + 13}.$$

The denominator does not factorise, therefore we complete the square to obtain

$$X(s) \equiv \frac{3s + 12}{(s + 2)^2 + 9} \equiv \frac{3(s + 2) + 6}{(s + 2)^2 + 9} \equiv 3.\frac{s + 2}{(s + 2)^2 + 9} + 2.\frac{3}{(s + 2)^2 + 9}.$$

Thus,

$$x(t) = 3e^{-2t}\cos 3t + 2e^{-2t}\sin 3t \quad t > 0$$

or

$$x(t) = e^{-2t}[3\cos 3t + 2\sin 3t] \quad t > 0.$$

2. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 50\sin t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] - 4 + 6[sX(s) - 1] + 9X(s) = \frac{50}{s^2+1},$$

giving

$$(s^2 + 6s + 9)X(s) = \frac{50}{s^2+1} + s + 10.$$

Hint: Do <u>not</u> combine the terms on the right into a single fraction - it won't help !

Thus,

$$X(s) \equiv \frac{50}{(s^2+6s+9)(s^2+1)} + \frac{s+10}{s^2+6s+9}$$

or

$$X(s) \equiv \frac{50}{(s+3)^2(s^2+1)} + \frac{s+10}{(s+3)^2}.$$

Using the principles of partial fractions in the first term on the right,

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}.$$

Hence,

$$50 \equiv A(s^2+1) + B(s+3)(s^2+1) + (Cs+D)(s+3)^2.$$

Substituting $s = -3$,

$$50 = 10A \text{ giving } A = 5.$$

Equating coefficients of s^3 on both sides,

$$0 = B + C. \quad (1)$$

Equating the coefficients of s on both sides (we shall not need the s^2 coefficients in this example),

$$0 = B + 9C + 6D. \quad (2)$$

Equating the constant terms on both sides,

$$50 = A + 3B + 9D = 5 + 3B + 9D. \quad (3)$$

Putting $C = -B$ into (2), we obtain

$$-8B + 6D = 0, \quad (4)$$

and we already have

$$3B + 9D = 45. \quad (3)$$

These last two solve easily to give $B = 3$ and $D = 4$ so that $C = -3$.

We conclude that

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{5}{(s+3)^2} + \frac{3}{s+3} + \frac{-3s+4}{s^2+1}.$$

In addition to this, we also have

$$\frac{s+10}{(s+3)^2} \equiv \frac{s+3}{(s+3)^2} + \frac{7}{(s+3)^2} \equiv \frac{1}{s+3} + \frac{7}{(s+3)^2}.$$

The total for $X(s)$ is therefore given by

$$X(s) \equiv \frac{12}{(s+3)^2} + \frac{4}{s+3} - 3.\frac{s}{s^2+1} + 4.\frac{1}{s^2+1}.$$

Finally,

$$x(t) = 12te^{-3t} + 4e^{-3t} - 3\cos t + 4\sin t \quad t > 0.$$

3. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} - 3x = 4e^t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -2$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] + 2 + 4[sX(s) - 1] - 3X(s) = \frac{4}{s-1}.$$

This gives

$$(s^2 + 4s - 3)X(s) = \frac{4}{s-1} + s + 2.$$

Therefore,

$$X(s) \equiv \frac{4}{(s-1)(s^2+4s-3)} + \frac{s+2}{s^2+4s-3}.$$

Applying the principles of partial fractions,

$$\frac{4}{(s-1)(s^2+4s-3)} \equiv \frac{A}{s-1} + \frac{Bs+C}{s^2+4s-3}.$$

Hence,

$$4 \equiv A(s^2 + 4s - 3) + (Bs + C)(s - 1).$$

Substituting $s = 1$, we obtain

$$4 = 2A; \quad \text{that is,} \quad A = 2.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B, \quad \text{so that} \quad B = -2.$$

Equating constant terms on both sides,

$$4 = -3A - C, \quad \text{so that} \quad C = -10.$$

Thus, in total,

$$X(s) \equiv \frac{2}{s-1} + \frac{-s-8}{s^2+4s-3} \equiv \frac{2}{s-1} + \frac{-s-8}{(s+2)^2-7}$$

or

$$X(s) \equiv \frac{2}{s-1} - \frac{s+2}{(s+2)^2-7} - \frac{6}{(s+2)^2-7}.$$

Finally,

$$x(t) = 2e^t - e^{-2t}\text{cosh}t\sqrt{7} - \frac{6}{\sqrt{7}}e^{-2t}\text{sinh}t\sqrt{7} \quad t > 0.$$

16.3.2 THE GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

On some occasions, we may either be given no boundary conditions at all; or else the boundary conditions given do not tell us the values of $x(0)$ and $x'(0)$.

In such cases, we simply let $x(0) = A$ and $x'(0) = B$ to obtain a solution in terms of A and B called the **"general solution"**.

If any non-standard boundary conditions are provided, we then substitute them into the general solution to obtain particular values of A and B.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 0$$

and, hence, determine the particular solution in the case when $x(\frac{\pi}{2}) = -3$ and $x'(\frac{\pi}{2}) = 10$.

Solution

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - A) - B + 4X(s) = 0.$$

That is,

$$(s^2 + 4)X(s) = As + B.$$

Hence,

$$X(s) \equiv \frac{As + B}{s^2 + 4} \equiv A.\frac{s}{s^2 + 4} + B.\frac{1}{s^2 + 4}.$$

This gives

$$x(t) = A\cos 2t + \frac{B}{2}\sin 2t \quad t > 0;$$

but, since A and B are **arbitrary** constants, this may be written in the simpler form

$$x(t) = A\cos 2t + B\sin 2t \quad t > 0,$$

in which $\frac{B}{2}$ has been rewritten as B.

To apply the boundary conditions, we require also the formula for $x'(t)$, namely

$$x'(t) = -\ 2A\sin 2t + 2B\cos 2t.$$

Hence, $-\ 3 = -\ A$ and $10 = -\ 2B$ giving $A = 3$ and $B =$ - 5.

Therefore, the particular solution is

$$x(t) = 3\cos 2t - 5\sin 2t \quad t > 0.$$

16.3.3 EXERCISES

1. Solve the following differential equations subject to the conditions given:

   (a)
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 0,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (b)
$$4\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0,$$
   given that $x(0) = 4$ and $x'(0) = 1$;

   (c)
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} - 8x = 2t,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (d)
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4x = 2e^{2t},$$
   given that $x(0) = 1$ and $x'(0) = 10.5$;

   (e)
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 3\cos^2 t,$$
   given that $x(0) = 1$ and $x'(0) = 2$.
   Hint: $\cos 2t \equiv 2\cos^2 t - 1$.

2. Determine the particular solution of the differential equation
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} = e^t(t-3)$$
   in the case when $x(0) = 2$ and $x(3) = -1$.

   Hint:
   Since $x(0)$ is given, just let $x'(0) = B$ to obtain a solution in terms of B; then substitute the second boundary condition at the end.

16.3.4 ANSWERS TO EXERCISES

1. (a)

$$X(s) = \frac{3s-5}{s^2-2s+5},$$

giving

$$x(t) = e^t(3\cos 2t - \sin 2t) \quad t > 0;$$

(b)

$$X(s) = \frac{4}{s+\frac{1}{2}} + \frac{3}{(s+\frac{1}{2})^2},$$

giving

$$x(t) = 4e^{-\frac{1}{2}t} + 3te^{-\frac{1}{2}t} = e^{-\frac{1}{2}t}[4+3t] \quad t > 0;$$

(c)

$$X(s) = \frac{27}{12}.\frac{1}{s-2} + \frac{39}{48}.\frac{1}{s+4} - \frac{1}{4}.\frac{1}{s^2} - \frac{1}{16}.\frac{1}{s},$$

giving

$$x(t) = \frac{27}{12}e^{2t} + \frac{39}{48}e^{-4t} - \frac{1}{4}t - \frac{1}{16} \quad t > 0;$$

(d)

$$X(s) = \frac{\frac{1}{2}}{(s-2)^2} + \frac{3}{s-2} - \frac{2}{s+2},$$

giving

$$x(t) = \frac{1}{2}te^{2t} + 3e^{2t} - 2e^{-2t} \quad t > 0;$$

(e)

$$X(s) = \frac{3}{2}.\frac{s}{(s^2+4)^2} + \frac{3}{8}.\frac{1}{s} + \frac{5}{8}.\frac{s}{s^2+4} + \frac{2}{s^2+4},$$

giving

$$x(t) = \frac{3}{8}t\sin 2t + \frac{3}{8} + \frac{5}{8}\cos 2t + \sin 2t \quad t > 0.$$

2.

$$x(t) = 3e^t - te^t - 1 \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.4

LAPLACE TRANSFORMS 4
(Simultaneous differential equations)

by

A.J.Hobson

UNIT 16.4 - LAPLACE TRANSFORMS 4
SIMULTANEOUS DIFFERENTIAL EQUATIONS

16.4.1 AN EXAMPLE OF SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In this Unit, we shall consider a pair of differential equations involving an independent variable, t, such as a time variable, and two dependent variables, x and y, such as electric currents or linear displacements.

The general format is as follows:

$$\begin{aligned} a_1\frac{\mathrm{d}x}{\mathrm{d}t} + b_1\frac{\mathrm{d}y}{\mathrm{d}t} + c_1x + d_1y &= f_1(t), \\ a_2\frac{\mathrm{d}x}{\mathrm{d}t} + b_2\frac{\mathrm{d}y}{\mathrm{d}t} + c_2x + d_2y &= f_2(t). \end{aligned}$$

To solve these equations simultaneously, we take the Laplace Transform of each equation obtaining two simultaneous algebraic equations from which we may determine $X(s)$ and $Y(s)$, the Laplace Transforms of $x(t)$ and $y(t)$ respectively.

EXAMPLE

Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= e^t, \\ \frac{\mathrm{d}x}{\mathrm{d}t} - 2y &= 1 + t, \end{aligned}$$

given that $x(0) = 1$ and $y(0) = 2$.

Solution

Taking the Laplace Transforms of the differential equations,

$$sY(s) - 2 + 2X(s) = \frac{1}{s-1},$$

$$sX(s) - 1 - 2Y(s) \quad = \quad \frac{1}{s} + \frac{1}{s^2}.$$

That is,

$$\begin{aligned} 2X(s) + sY(s) &= \frac{1}{s-1} + 2, \qquad (1) \\ sX(s) - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2} + 1. \qquad (2) \end{aligned}$$

Using $(1) \times 2 + (2) \times s$, we obtain

$$(4 + s^2)X(s) = \frac{2}{s-1} + 4 + 1 + \frac{1}{s} + s.$$

Hence,

$$X(s) = \frac{2}{(s-1)(s^2+4)} + \frac{5}{s^2+4} + \frac{1}{s(s^2+4)} + \frac{s}{s^2+4}.$$

Applying the methods of partial fractions, this gives

$$X(s) = \frac{2}{5}.\frac{1}{s-1} + \frac{7}{20}.\frac{s}{s^2+4} + \frac{23}{5}.\frac{1}{s^2+4} + \frac{1}{4}.\frac{1}{s}.$$

Thus,

$$x(t) = \frac{2}{5}e^t + \frac{1}{4} + \frac{7}{20}\cos 2t + \frac{23}{10}\sin 2t \quad t > 0.$$

We could now start again by eliminating x from equations (1) and (2) in order to calculate y, and this is often necessary; but, since

$$2y = \frac{\mathrm{d}x}{\mathrm{d}t} - 1 - t$$

in the current example,

$$y(t) = \frac{1}{5}e^t - \frac{1}{2} - \frac{7}{20}\sin 2t + \frac{23}{10}\cos 2t - \frac{t}{2} \quad t > 0.$$

16.4.2 EXERCISES

Use Laplace Transforms to solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2y &= e^{-t}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} + 3y &= x,\end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

2.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} - y &= \sin t, \\ \frac{\mathrm{d}y}{\mathrm{d}t} + x &= \cos t,\end{aligned}$$

given that $x = 3$ and $y = 4$ when $t = 0$.

3.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 3y &= 1, \\ \frac{\mathrm{d}y}{\mathrm{d}t} - x + 2y &= e^{-2t},\end{aligned}$$

given that $x = 0$ and $y = 0$ when $t = 0$.

4.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 8x,\end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

5.

$$\begin{aligned} 10\frac{\mathrm{d}x}{\mathrm{d}t} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 6x + 5y &= 0, \\ 2\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} + 2x + y &= 2e^{-t}, \end{aligned}$$

given that $x = 2$ and $y = -1$ when $t = 0$.

6.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 6y &= 0, \\ 5\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 6x &= 0, \end{aligned}$$

given that $x = 3$ and $y = 0$ when $t = 0$.

7.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 2y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2z, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 2x, \end{aligned}$$

given that $x = 1$, $y = 0$ and $z = -1$ when $t = 0$.

16.4.3 ANSWERS TO EXERCISES

1.

$$x = (2t+1)e^{-t} \quad \text{and} \quad y = te^{-t}.$$

2.

$$x = (t+4)\sin t + 3\cos t \quad \text{and} \quad y = (t+4)\cos t - 3\sin t.$$

3.

$$x = 2 - e^{-2t}[1 + \sqrt{3}\sinh t\sqrt{3} + \cosh t\sqrt{3}]$$

and

$$y = 1 - e^{-2t}\left[\cosh t\sqrt{3} + \frac{1}{\sqrt{3}}\sinh t\sqrt{3}\right].$$

4.

$$x = \sinh 4t \quad \text{and} \quad y = 2\cosh 4t.$$

5.

$$x = 4\cos t - 2e^{-t} \quad \text{and} \quad y = e^{-t} - 2\cos t.$$

6.

$$x = 2e^{-t} + e^{-2t} \quad \text{amd} \quad y = e^{-t} - e^{-2t}.$$

7.

$$x = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} + \cos t\sqrt{3}\right],$$

$$y = \frac{-2}{\sqrt{3}}e^{-t}\sin t\sqrt{3}$$

and

$$z = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} - \cos t\sqrt{3}\right].$$

"JUST THE MATHS"

UNIT NUMBER

16.5

LAPLACE TRANSFORMS 5
(The Heaviside step function)

by

A.J.Hobson

UNIT 16.5 - LAPLACE TRANSFORMS 5

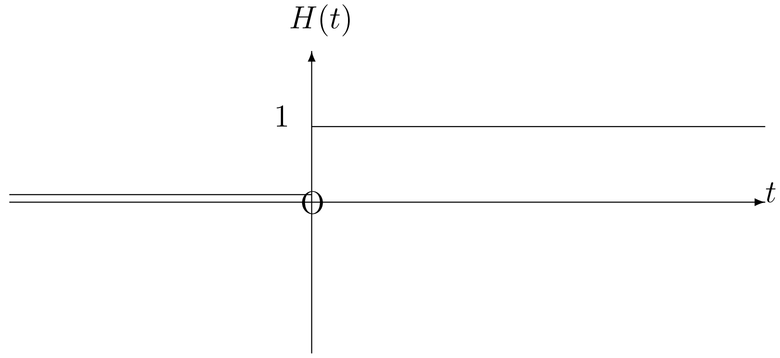
THE HEAVISIDE STEP FUNCTION

16.5.1 THE DEFINITION OF THE HEAVISIDE STEP FUNCTION

The Heaviside Step Function, $H(t)$, is defined by the statements

$$H(t) = \begin{cases} 0 & \text{for } t < 0; \\ 1 & \text{for } t > 0. \end{cases}$$

Note:
$H(t)$ is undefined when $t = 0$.

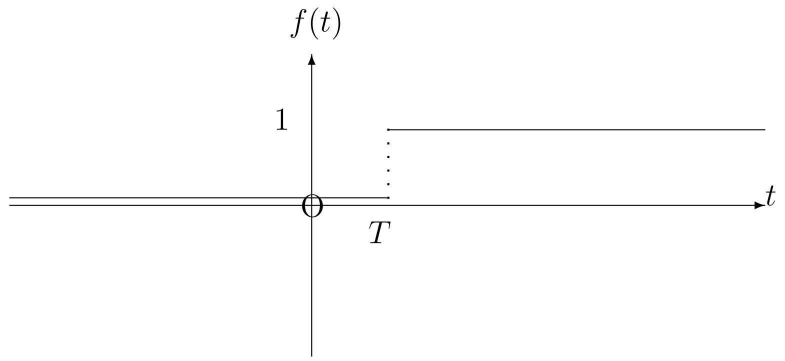


EXAMPLE

Express, in terms of $H(t)$, the function, $f(t)$, given by the statements

$$f(t) = \begin{cases} 0 & \text{for } t < T; \\ 1 & \text{for } t > T. \end{cases}$$

Solution



Clearly, $f(t)$ is the same type of function as $H(t)$, but we have effectively moved the origin to the point $(T, 0)$. Hence,

$$f(t) \equiv H(t - T).$$

Note:
The function $H(t-T)$ is of importance in constructing what are known as **"pulse functions"** (see later).

16.5.2 THE LAPLACE TRANSFORM OF H(t - T)

From the definition of a Laplace Transform,

$$L[H(t-T)] = \int_0^\infty \ e^{-st} H(t-T) \ \mathrm{d}t$$

$$= \int_0^T \ e^{-st}.0 \ \mathrm{d}t + \int_T^\infty \ e^{-st}.1 \ \mathrm{d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_T^\infty = \frac{e^{-sT}}{s}.$$

Note:
In the special case when $T = 0$, we have

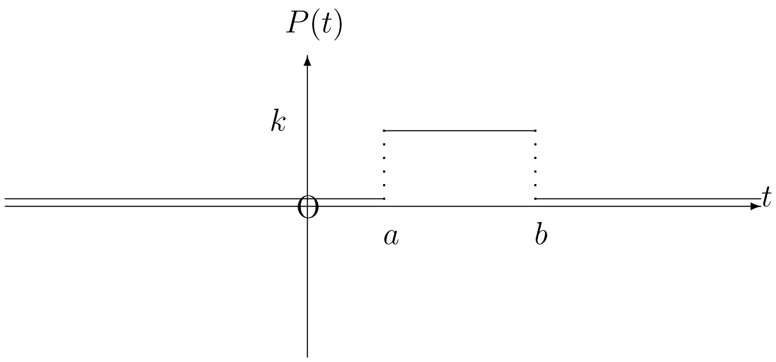
$$L[H(t)] = \frac{1}{s},$$

which can be expected since $H(t)$ and 1 are identical over the range of integration.

16.5.3 PULSE FUNCTIONS

If $a < b$, a **"rectangular pulse"**, $P(t)$, of duration, $b - a$, and magnitude, k, is defined by the statements,

$$P(t) = \begin{cases} k & \text{for } a < t < b; \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$



We can show that, in terms of Heaviside functions, the above pulse may be represented by

$$P(t) \equiv k[H(t-a) - H(t-b)].$$

Proof:

(i) If $t < a$, then $H(t-a) = 0$ and $H(t-b) = 0$. Hence, the above right-hand side $= 0$.

(ii) If $t > b$, then $H(t-a) = 1$ and $H(t-b) = 1$. Hence, the above right-hand side $= 0$.

(iii) If $a < t < b$, then $H(t-a) = 1$ and $H(t-b) = 0$. Hence, the above right-hand side $= k$.

EXAMPLE

Determine the Laplace Transform of a pulse, $P(t)$, of duration, $b - a$, having magnitude, k.

Solution

$$L[P(t)] = k\left[\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right] = k.\frac{e^{-sa} - e^{-sb}}{s}.$$

Notes:

(i) The **"strength"** of the pulse, described above, is defined as the area of the rectangle with base, $b - a$, and height, k. That is,

$$\text{strength} = k(b - a).$$

(ii) In general, the expression,

$$[H(t-a) - H(t-b)]f(t),$$

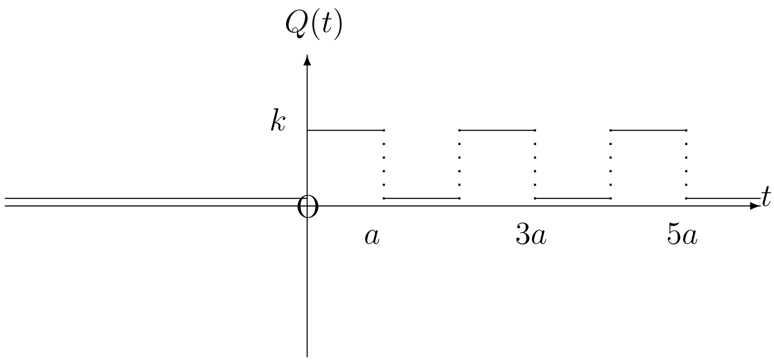
may be considered to **"switch on"** the function, $f(t)$, between $t = a$ and $t = b$ but **"switch off"** the function, $f(t)$, when $t < a$ or $t > b$.

(iii) Similarly, the expression,

$$H(t-a)f(t),$$

may be considered to **"switch on"** the function, $f(t)$, when $t > a$ but **"switch off"** the function, $f(t)$, when $t < a$.

For example, the train of rectangular pulses, $Q(t)$, in the following diagram:



may be represented by the function

$$Q(t) \equiv k\left\{[H(t) - H(t-a)] + [H(t-2a) - H(t-3a)] + [H(t-4a) - H(t-5a)] + \ldots\ldots\ldots\ldots\right\}.$$

16.5.4 THE SECOND SHIFTING THEOREM

THEOREM

$$L[H(t-T)f(t-T)] = e^{-sT}L[f(t)].$$

Proof:

Left-hand side =

$$\int_0^\infty e^{-st}H(t-T)f(t-T)\ \mathrm{d}t$$
$$= \int_0^T 0\ \mathrm{d}t + \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t$$
$$= \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t.$$

Making the substitution $u = t - T$, we obtain

$$\int_0^\infty e^{-s(u+T)}f(u)\ \mathrm{d}u$$
$$= e^{-sT}\int_0^\infty e^{-su}f(u)\ \mathrm{d}u = e^{-sT}L[f(t)].$$

EXAMPLES

1. Express, in terms of Heaviside functions, the function

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t > 1; \\ 0 & \text{for } 0 < t < 1. \end{cases}$$

and, hence, determine its Laplace Transform.

Solution

For values of $t > 0$, we may write

$$f(t) = (t-1)^2H(t-1).$$

Therefore, using $T = 1$ in the second shifting theorem,

$$L[f(t)] = e^{-s}L[t^2] = e^{-s}.\frac{2}{s^3}.$$

2. Determine the inverse Laplace Transform of the expression

$$\frac{e^{-7s}}{s^2 + 4s + 5}.$$

Solution

First, we find the inverse Laplace Transform of the expression,

$$\frac{1}{s^2 + 4s + 5} \equiv \frac{1}{(s+2)^2 + 1}.$$

From the first shifting theorem, this will be the function

$$e^{-2t} \sin t, \quad t > 0.$$

From the second shifting theorem, the required function will be

$$H(t-7)e^{-2(t-7)} \sin(t-7), \quad t > 0.$$

16.5.5 EXERCISES

1. (a) For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} e^{-t} & \text{for } 0 < t < 3; \\ 0 & \text{for } t > 3. \end{cases}$$

   (b) Determine the Laplace Transform of the function, $f(t)$, in part (a).

2. For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} f_1(t) & \text{for } 0 < t < a; \\ f_2(t) & \text{for } t > a. \end{cases}$$

3. For values of $t > 0$, express the following functions in terms of Heaviside functions:

   (a)

$$f(t) = \begin{cases} t^2 & \text{for } 0 < t < 2; \\ 4t & \text{for } t > 2. \end{cases}$$

   (b)

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi; \\ \sin 2t & \text{for } \pi < t < 2\pi; \\ \sin 3t & \text{for } t > 2\pi. \end{cases}$$

4. Use the second shifting theorem to determine the Laplace Transform of the function,

$$f(t) \equiv t^3 H(t-1).$$

   Hint:
   Write $t^3 \equiv [(t-1)+1]^3$.

5. Determine the inverse Laplace Transforms of the following:

   (a)
$$\frac{e^{-2s}}{s^2};$$

   (b)
$$\frac{8e^{-3s}}{s^2+4};$$

   (c)
$$\frac{se^{-2s}}{s^2+3s+2};$$

   (d)
$$\frac{e^{-3s}}{s^2-2s+5}.$$

6. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = H(t-2),$$

   given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

16.5.6 ANSWERS TO EXERCISES

1. (a)
$$e^{-t}[H(t) - H(t-3)];$$

   (b)
$$L[f(t)] = \frac{1 - e^{-3(s+1)}}{s+1}.$$

2.

$$f(t) \equiv f_1(t)[H(t) - H(t-a)] + f_2(t)H(t-a)].$$

3. (a)

$$f(t) \equiv t^2[H(t) - H(t-2)] + 4tH(t-2);$$

(b)

$$f(t) \equiv \sin t[H(t) - H(t-\pi)] + \sin 2t[H(t-\pi) - H(t-2\pi)] + \sin 3t[H(t-2\pi).$$

4.

$$L[f(t)] = \left[\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right] e^{-s}.$$

5. (a)

$$H(t-2)(t-2);$$

(b)

$$4H(t-3)\sin 2(t-3);$$

(c)

$$H(t-2)[2e^{-2(t-2)} - e^{-(t-2)};$$

(d)

$$\frac{1}{2}H(t-3)e^{(t-3)}\sin 2(t-3).$$

6.

$$x = \frac{1}{2}\sin 2t + \frac{1}{4}H(t-2)[1 - \cos 2(t-2)].$$

"JUST THE MATHS"

UNIT NUMBER

16.6

LAPLACE TRANSFORMS 6
(The Dirac unit impulse function)

by

A.J.Hobson

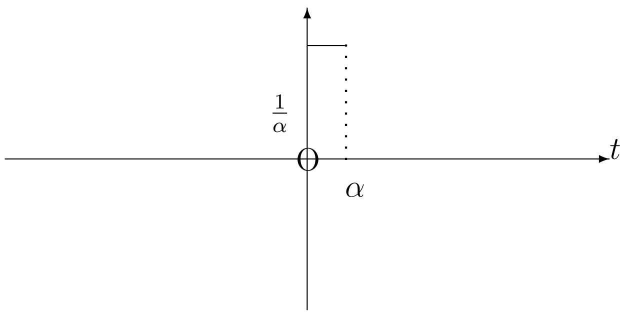
UNIT 16.6 - LAPLACE TRANSFORMS 6
THE DIRAC UNIT IMPULSE FUNCTION

16.6.1 THE DEFINITION OF THE DIRAC UNIT IMPULSE FUNCTION

A pulse of large magnitude, short duration and finite strength is called an **"impulse"**. In particular, a **"unit impulse"** is an impulse of strength 1.

ILLUSTRATION

Consider a pulse, of duration α, between $t = 0$ and $t = \alpha$, having magnitude, $\frac{1}{\alpha}$. The strength of the pulse is then 1.



From Unit 16.5, this pulse is given by

$$\frac{H(t) - H(t-\alpha)}{\alpha}.$$

If we now allow α to tend to zero, we obtain a unit impulse located at $t = 0$. This leads to the following definition:

DEFINITION 2

The **"Dirac unit impulse function"** , $\delta(t)$ is defined to be an impulse of unit strength located at $t = 0$. It is given by

$$\delta(t) = \lim_{\alpha \to 0} \frac{H(t) - H(t-\alpha)}{\alpha}.$$

Notes:

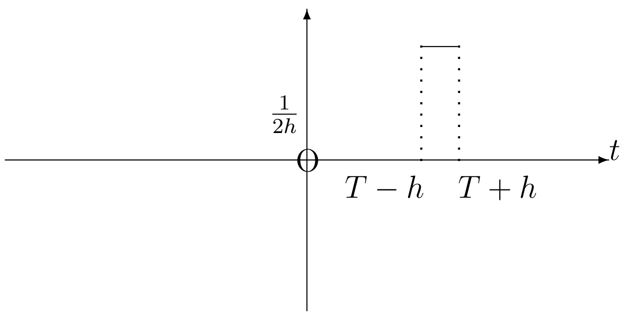
(i) An impulse of unit strength located at $t = T$ is represented by $\delta(t - T)$.

(ii) An alternative definition of the function $\delta(t - T)$ is as follows:

$$\delta(t - T) = \begin{cases} 0 & \text{for } t \neq T; \\ \infty & \text{for } t = T. \end{cases}$$

and

$$\lim_{h \to 0} \int_{T-h}^{T+h} \delta(t - T) \; \mathrm{d}t = 1.$$



THEOREM

$$\int_a^b f(t)\delta(t - T) \; \mathrm{d}t = f(T) \;\; \text{if} \;\; a < T < b.$$

Proof:

Since $\delta(t - T)$ is equal to zero everywhere except at $t = T$, the left-hand side of the above formula reduces to

$$\lim_{h \to 0} \int_{T-h}^{T+h} f(t)\delta(t - T) \; \mathrm{d}t.$$

But, in the small interval from $T - h$ to $T + h$, $f(t)$ is approximately constant and equal to $f(T)$. Hence, the left-hand side may be written

$$f(T) \left[\lim_{h \to 0} \int_{T-h}^{T+h} \delta(T - T) \; \mathrm{d}t \right],$$

which reduces to $f(T)$, using note (ii) in the definition of the Dirac unit impulse function.

16.6.2 THE LAPLACE TRANSFORM OF THE DIRAC UNIT IMPULSE FUNCTION

RESULT

$$L[\delta(t-T)] = e^{-sT};$$

and, in particular,

$$L[\delta(t)] = 1.$$

Proof:

From the definition of a Laplace Transform,

$$L[\delta(t-T) = \int_0^\infty e^{-st}\delta(t-T)\ \mathrm{d}t.$$

But, from the Theorem discussed above, with $f(t) = e^{-st}$, we have

$$L[\delta(t-T)] = e^{-sT}.$$

EXAMPLES

1. Solve the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = \delta(t),$$

given that $x = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$3sX(s) + 4X(s) = 1.$$

That is,

$$X(s) = \frac{1}{3s+4} = \frac{1}{3}.\frac{1}{s+\frac{4}{3}}.$$

Hence,

$$x(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

2. Show that, for any function, $f(t)$,

$$\int_0^\infty f(t)\delta'(t-a)\ \mathrm{d}t = -f'(a).$$

Solution

Using Integration by Parts, the left-hand side of the formula may be written

$$[f(t)\delta(t-a)]_0^\infty - \int_0^\infty f'(t)\delta(t-a)\ \mathrm{d}t.$$

The first term of this reduces to zero, since $\delta(t-a)$ is equal to zero except when $t = a$. The required result follows from the Theorem discussed earlier, with $T = a$.

16.6.3 TRANSFER FUNCTIONS

In scientific applications, the solution of an ordinary differential equation having the form

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

is sometimes called the **"response of a system to the function** $f(t)$**"**.

The term **"system"** may, for example, refer to an oscillatory electrical circuit or a mechanical vibration.

It is also customary to refer to $f(t)$ as the **"input"** and $x(t)$ as the **"output"** of a system.

In the work which follows, we shall consider the special case in which $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$; that is, we shall assume zero initial conditions.

Impulse response function and transfer function

Consider, for the moment, the differential equation having the form,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + cu = \delta(t).$$

Here, we refer to the function, $u(t)$, as the **"impulse response function"** of the original system.

The Laplace Transform of its differential equation is given by

$$(as^2 + bs + c)U(s) = 1.$$

Hence,

$$U(s) = \frac{1}{as^2 + bs + c},$$

which is called the **"transfer function"** of the original system.

EXAMPLE

Determine the transfer function and impulse response function for the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = f(t),$$

assuming zero initial conditions.

Solution

To find $U(s)$ and $u(t)$, we have

$$3\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = \delta t,$$

so that

$$(3s + 4)U(s) = 1$$

and, hence, the transfer function is

$$U(s) = \frac{1}{3s + 4} = \frac{1}{3}.\frac{1}{s + \frac{4}{3}}.$$

Taking the inverse Laplace Transform of $U(s)$ gives the impulse response function,

$$u(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

System response for any input

Assuming zero initial conditions, the Laplace Transform of the differential equation

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + bx + cx = f(t)$$

is given by

$$(as^2 + bs + c)X(s) = F(s),$$

which means that

$$X(s) = \frac{F(s)}{as^2 + bs + c} = F(s).U(s).$$

In order to find the response of the system to the function $f(t)$, we need the inverse Laplace Transform of $F(s).U(s)$ which may possibly be found using partial fractions but may, if necessary, be found by using the Convolution Theorem referred to in Unit 16.1

The Convolution Theorem shows, in this case, that

$$L\left[\int_0^t f(T).u(t-T)\ \mathrm{d}T\right] = F(s).U(s);$$

in other words,

$$L^{-1}[F(s).U(s)] = \int_0^t f(T).u(t-T)\ \mathrm{d}T.$$

EXAMPLE

The impulse response of a system is known to be $u(t) = \frac{10e^{-t}}{3}$.

Determine the response, $x(t)$, of the system to an input of $f(t) \equiv \sin 3t$.

Solution

First, we note that

$$U(s) = \frac{10}{3(s+1)} \quad \text{and} \quad F(s) = \frac{3}{s^2+9}.$$

Hence,

$$X(s) = \frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} + \frac{-s+1}{s^2+9},$$

using partial fractions.

Thus

$$x(t) = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0.$$

Alternatively, using the Convolution Theorem,

$$x(t) = \int_0^t \sin 3T.\frac{10e^{-(t-T)}}{3}\ \mathrm{d}T;$$

but the integration here can be made simpler if we replace $\sin 3T$ by e^{j3T} and use the imaginary part, only, of the result.

Hence,

$$x(t) = I_m\left(\int_0^t \frac{10}{3}e^{-t}.e^{(1+j3)T}\ \mathrm{d}T\right)$$

$$= I_m\left(\frac{10}{3}\left[e^{-t}\frac{e^{(1+j3)T}}{1+j3}\right]_0^t\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{e^{-t}.e^{(1+j3)t} - e^{-t}}{1+j3}\right]\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{[(\cos 3t - e^{-t}) + j\sin 3t](1-j3)}{10}\right]\right)$$

$$= \frac{10}{3}\left[\frac{\sin 3t - 3\cos 3t + 3e^{-t}}{10}\right]$$

$$= e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0,$$

as before.

Note:
Clearly, in this example, the method using partial fractions is simpler.

16.6.4 STEADY-STATE RESPONSE TO A SINGLE FREQUENCY INPUT

In the differential equation,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

suppose that the quadratic denominator of the transfer function, $U(s)$, has negative real roots; that is, it gives rise to an impulse response, $u(t)$, involving negative powers of e and, hence, tending to zero as t tends to infinity.

Suppose also that $f(t)$ takes one of the forms, $\cos\omega t$ or $\sin\omega t$, which may be regarded, respectively, as the real and imaginary parts of the function, $e^{j\omega t}$.

It turns out that the response, $x(t)$, will consist of a **"transient"** part which tends to zero as t tends to infinity, together with a non-transient part forming the **"steady-state response"**.

We illustrate with an example:

EXAMPLE

Consider that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = e^{j7t},$$

where $x = 2$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - 2) - 1 + 3(sX(s) - 2) + 2X(s) = \frac{1}{s - j7}.$$

That is,

$$(s^2 + 3s + 2)X(s) = 2s + 7 + \frac{1}{s - j7},$$

giving

$$X(s) = \frac{2s + 7}{s^2 + 3s + 2} + \frac{1}{(s - j7)(s^2 + 3s + 2)} = \frac{2s + 7}{(s + 2)(s + 1)} + \frac{1}{(s - j7)(s + 2)(s + 1)}.$$

Using the "cover-up" rule for partial fractions, we obtain

$$X(s) = \frac{5}{s + 1} - \frac{3}{s + 2} + \frac{1}{(-1 - j7)(s + 1)} + \frac{1}{(2 + j7)(s + 2)} + \frac{U(j7)}{(s - j7)},$$

where

$$U(s) \equiv \frac{1}{s^2 + 3s + 2}$$

is the transfer function.

Taking inverse Laplace Transforms,

$$x(t) = 5e^{-t} - 3e^{-2t} + \frac{1}{-1-j7}e^{-t} + \frac{1}{2+j7}e^{-2t} + U(j7)e^{j7t}.$$

The first four terms on the right-hand side tend to zero as t tends to infinity, so that the final term represents the steady state response; we need its real part if $f(t) \equiv \cos 7t$ and its imaginary part if $f(t) \equiv \sin 7t$.

Summary

The above example illustrates the result that the steady-state response, $s(t)$, of a system to an input of $e^{j\omega t}$ is given by

$$s(t) = U(j\omega)e^{j\omega t}.$$

16.6.5 EXERCISES

1. Evaluate
$$\int_0^\infty e^{-4t}\delta'(t-2)\ \mathrm{d}t.$$

2. In the following cases, solve the differential equation
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = f(t),$$
where $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$:

   (a)
$$f(t) \equiv \delta(t);$$

   (b)
$$f(t) \equiv \delta(t-2).$$

3. Determine the transfer function and impulse response function for the differential equation,
$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t),$$
assuming zero initial conditions.

4. The impulse response function of a system is known to be $u(t) = e^{3t}$. Determine the response, $x(t)$, of the system to an input of $f(t) \equiv 6\cos 3t$.

5. Determine the steady-state response to the system

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t)$$

in the cases when

(a)

$$f(t) \equiv e^{j2t};$$

(b)

$$f(t) \equiv 3\cos 2t.$$

16.6.6 ANSWERS TO EXERCISES

1.

$$4e^{-8}.$$

2. (a)

$$x = \sin 2t \quad t > 0;$$

(b)

$$x = \sin t + H(t-2)\sin(t-2) \quad t \neq 2.$$

3.

$$U(s) = \frac{1}{2s^2 - 3s + 1} \quad \text{and} \quad u(t) = [e^t - e^{\frac{1}{2}t}].$$

4.

$$\frac{1}{13}\left[18e^{3t} - 18\cos 2t + 12\sin 2t\right] \quad t > 0.$$

5. (a)

$$\frac{(1-j6)e^{j2t}}{37} \quad t > 0;$$

(b)

$$\frac{1}{37}(\cos 2t + 6\sin 2t) \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.7

LAPLACE TRANSFORMS 7
(An appendix)

by

A.J.Hobson

One view of how Laplace Transforms might have arisen

UNIT 16.7 - LAPLACE TRANSFORMS 7 (AN APPENDIX)

ONE VIEW OF HOW LAPLACE TRANSFORMS MIGHT HAVE ARISEN.

(i) Let us consider that our main problem is to solve a second order linear differential equation with constant coefficients, the general form of which is

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

(ii) Assuming that the solution of an equivalent first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

has already been included in previous knowledge, we examine a typical worked example as follows:

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = e^{2t},$$

given that $x = 0$ when $t = 0$.

Solution

A method called the **"integrating factor method"** uses the coefficient of x to find a function of t which multiplies both sides of the given differential equation to convert it to an **"exact"** differential equation.

The integrating factor in the current example is e^{3t} since the coefficient of x is 3.

We obtain, therefore,

$$e^{3t}\left[\frac{\mathrm{d}x}{\mathrm{d}t} + 3x\right] = e^{5t}.$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] = e^{5t}.$$

On integrating both sides with respect to t,

$$xe^{3t} = \frac{e^{5t}}{5} + C$$

or

$$x = \frac{e^{2t}}{5} + Ce^{-3t}.$$

Putting $x = 0$ and $t = 0$, we have

$$0 = \frac{1}{5} + C.$$

Hence, $C = -\frac{1}{5}$ and the complete solution becomes

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5}.$$

(iii) As a lead up to what follows, we shall now examine a different way of setting out the above working in which we do not leave the substitution of the boundary condition until the very end.

We multiply both sides of the differential equation by e^{3t} as before, but we then integrate both sides of the new "exact" equation from 0 to t.

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] \mathrm{d}t = \int_0^t e^{5t} \mathrm{d}t.$$

That is,

$$\left[xe^{3t}\right]_0^t = \left[\frac{e^{5t}}{5}\right]_0^t,$$

giving

$$xe^{3t} - 0 = \frac{e^{5t}}{5} - \frac{1}{5}.$$

since $x = 0$ when $t = 0$.

In other words,

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5},$$

as before.

(iv) Now let us consider whether an example of a second order linear differential equation could be solved by a similar method.

EXAMPLE

Solve the differential equation,

$$\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 21x = e^{9t},$$

given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

Solution

Supposing that there might be an integrating factor for this equation, we shall take it to be e^{st} where s, at present, is unknown, but assumed to be positive.

Multiplying throughout by e^{st} and integrating from 0 to t, as in the previous example,

$$\int_0^t e^{st}\left[\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 21x\right] dt = \int_0^t e^{(s+9)t} dt = \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

Now, using integration by parts, with the boundary condition,

$$\int_0^t e^{st}\frac{dx}{dt} dt = e^{st}x - s\int_0^t e^{st}x dt$$

and

$$\int_0^t e^{st}\frac{d^2x}{dt^2} dt = e^{st}\frac{dx}{dt} - s\int_0^t e^{st}\frac{dx}{dt} dt = e^{st}\frac{dx}{dt} - se^{st}x + s^2\int_0^t e^{st}x dt.$$

On substituting these results into the differential equation, we may collect together (on the left hand side) terms which involve $\int_0^t e^{st}x dt$ and e^{st} as follows:

$$(s^2 + 10s + 21)\int_0^t e^{st}x dt + e^{st}\left[\frac{dx}{dt} - (s+10)x\right] = \frac{e^{(s+9)t}}{s+9} - \frac{1}{s+9}.$$

(v) OBSERVATIONS

(a) If we had used e^{-st} instead of e^{st}, the quadratic expression in s, above, would have had the same coefficients as the original differential equation; that is, $(s^2 - 10s + 21)$.

(b) Using e^{-st} with $s > 0$, if we had integrated from 0 to ∞ instead of 0 to t, the second term on the left hand side above would have been absent, since $e^{-\infty} = 0$.

(vi) Having made our observations, we start again, multiplying both sides of the differential equation by e^{-st} and integrating from 0 to ∞ to obtain

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x \text{ d}t = \left[\frac{e^{(-s+9)t}}{-s+9}\right]_0^\infty = \frac{-1}{-s+9} = \frac{1}{s-9}.$$

Of course, this works only if $s > 9$, but we can easily assume that it is so. Hence,

$$\int_0^\infty e^{-st}x \text{ d}t = \frac{1}{(s-9)(s^2-10s+21)} = \frac{1}{(s-9)(s-3)(s-7)}.$$

Applying the principles of partial fractions, we obtain

$$\int_0^\infty e^{-st}x \text{ d}t = \frac{1}{12}.\frac{1}{s-9} + \frac{1}{24}.\frac{1}{s-3} - \frac{1}{8}.\frac{1}{s-7}.$$

(vii) But, finally, it can be shown by an independent method of solution that

$$x = \frac{e^{9t}}{12} + \frac{e^{3t}}{24} - \frac{e^{7t}}{8}.$$

and we may conclude that the solution of the differential equation is closely linked to the integral

$$\int_0^\infty e^{-st}x \text{ d}t,$$

which is called the **"Laplace Transform"** of $x(t)$.

"JUST THE MATHS"

UNIT NUMBER

16.8

Z-TRANSFORMS 1
(Definition and rules)

by

A.J.Hobson

UNIT 16.8 - Z TRANSFORMS 1 - DEFINITION AND RULES

16.8.1 INTRODUCTION - Linear Difference Equations

Closely linked with the concept of a linear differential equation with constant coefficients is that of a **"linear difference equation with constant coefficients"**.

Two particular types of difference equation to be discussed in the present section may be defined as follows:

DEFINITION 1

A first-order linear difference equation with constant coefficients has the general form,

$$a_1u_{n+1} + a_0u_n = f(n),$$

where a_0, a_1 are constants, n is a positive integer, $f(n)$ is a given function of n (possibly zero) and u_n is the general term of an infinite sequence of numbers, $\{u_n\} \equiv u_0,\ u_1,\ u_2,\ u_3,\ \ldots.$

DEFINITION 2

A second-order linear difference equation with constant coefficients has the general form,

$$a_2u_{n+2} + a_1u_{n+1} + a_0u_n = f(n),$$

where a_0, a_1, a_2 are constants, n is an integer, $f(n)$ is a given function of n (possibly zero) and u_n is the general term of an infinite sequence of numbers, $\{u_n\} \equiv u_0,\ u_1,\ u_2\ , u_3,\ \ldots.$

Notes:

(i) We shall assume that the sequences under discussion are such that $u_n = 0$ whenever $n < 0$.

(ii) Difference equations are usually associated with given "boundary conditions", such as the value of u_0 for a first-order equation or the values of u_0 and u_1 for a second-order equation.

ILLUSTRATION

Certain **simple** difference equations may be solved by very elementary methods.

For example, suppose that we wish to solve the difference equation,

$$u_{n+1} - (n+1)u_n = 0,$$

subject to the boundary condition that $u_0 = 1$.

We may rewrite the difference equation as

$$u_{n+1} = (n+1)u_n$$

and, by using this formula repeatedly, we obtain

$$u_1 = u_0 = 1, \quad u_2 = 2u_1 = 2, \quad u_3 = 3u_2 = 3 \times 2, \quad u_4 = 4u_3 = 4 \times 3 \times 2, \quad . \ . \ . \ .$$

In general, for this illustration, $u_n = n!$.

However, not all difference equations can be solved as easily as this and we shall now discuss the Z-Transform method of solving more advanced types.

16.8.2 STANDARD DEFINITION AND RESULTS

THE DEFINITION OF A Z-TRANSFORM (WITH EXAMPLES)

The Z-Transform of the sequence of numbers, $\{u_n\} \equiv u_0, \ u_1, \ u_2, \ u_3, \ . \ . \ . \ .$, is defined by the formula,

$$Z\{u_n\} = \sum_{r=0}^{\infty} u_r z^{-r},$$

provided that the series converges (allowing for z to be a complex number if necessary).

EXAMPLES

1. Determine the Z-Transform of the sequence,

$$\{u_n\} \equiv \{a^n\},$$

   where a is a non-zero constant.

Solution

$$Z\{a^n\} = \sum_{r=0}^{\infty} a^r z^{-r}.$$

That is,

$$Z\{a^n\} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ \ldots\ldots = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a},$$

by properties of infinite geometric series.
Thus,

$$Z\{a^n\} = \frac{z}{z-a}.$$

2. Determine the Z-Transform of the sequence,

$$\{u_n\} = \{n\}.$$

Solution

$$Z\{n\} = \sum_{r=0}^{\infty} rz^{-r}.$$

That is,

$$Z\{n\} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ \ldots\ldots,$$

which may be rearranged as

$$Z\{n\} = \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ \ldots\ldots\right) + \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ \ldots\ldots\right) + \left(\frac{1}{z^3} + \frac{1}{z^4} + \ \ldots\ldots\right),$$

giving

$$Z\{n\} = \frac{\frac{1}{z}}{1-\frac{1}{z}} + \frac{\frac{1}{z^2}}{1-\frac{1}{z}} + \frac{\frac{1}{z^3}}{1-\frac{1}{z}} + \ \ldots\ldots = \frac{1}{1-\frac{1}{z}}\left[\frac{\frac{1}{z}}{1-\frac{1}{z}}\right],$$

by properties of infinite geometric series.
Thus,

$$Z\{n\} = \frac{z}{(1-z)^2} = \frac{z}{(z-1)^2}.$$

Note:
Other Z-Transforms may be obtained, in the same way as in the above examples, from the definition.

We list, here, for reference, a short table of standard Z-Transforms, including those already proven:

A SHORT TABLE OF Z-TRANSFORMS

$\{u_n\}$	$Z\{u_n\}$	Region of Existence
$\{1\}$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$\{a^n\}$ (a constant)	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$\{n\}$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\left\{e^{-nT}\right\}$ (T constant)	$\frac{z}{z-e^{-T}}$	$\lvert z\rvert > e^{-T}$
$\sin nT$ (T constant)	$\frac{z\sin T}{z^2-2z\cos T+1}$	$\lvert z\rvert > 1$
$\cos nT$ (T constant)	$\frac{z(z-\cos T)}{z^2-2z\cos T+1}$	$\lvert z\rvert > 1$
1 for $n = 0$ 0 for $n > 0$ (Unit pulse sequence)	1	All z
0 for $n = 0$ $\{a^{n-1}\}$ for $n > 0$	$\frac{1}{z-a}$	$\lvert z\rvert > \lvert a\rvert$

16.8.3 PROPERTIES OF Z-TRANSFORMS

(a) Linearity

If $\{u_n\}$ and $\{v_n\}$ are sequences of numbers, while A and B are constants, then

$$Z\{Au_n + Bv_n\} \equiv A.Z\{u_n\} + B.Z\{v_n\}.$$

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty}(Au_r + Bv_r)z^{-r} \equiv A\sum_{r=0}^{\infty} u_r z^{-r} + B\sum_{r=0}^{\infty} v_r z^{-r},$$

which, in turn, is equivalent to the right-hand side.

EXAMPLE

$$Z\{5.2^n - 3n\} = \frac{5z}{z-2} - \frac{3z}{(z-1)^2}.$$

(b) The First Shifting Theorem

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.Z\{u_n\},$$

where $\{u_{n-1}\}$ denotes the sequence whose first term, corresponding to $n = 0$, is taken as zero and whose subsequent terms, corresponding to $n = 1, 2, 3, 4, \ldots.$, are the terms $u_0, u_1, u_3, u_4, \ldots$ of the original sequence.

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} u_{r-1}z^{-r} \equiv \frac{u_0}{z} + \frac{u_1}{z^2} + \frac{u_2}{z^3} + \frac{u_3}{z^4} + \ldots.,$$

since it is assumed that $u_n = 0$ whenever $n < 0$.

Thus,

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \ \ldots.\right],$$

which is equivalent to the right-hand side.

Note:
A more general form of the first shifting theorem states that

$$Z\{u_{n-k}\} \equiv \frac{1}{z^k}.Z\{u_n\},$$

where $\{u_{n-k}\}$ denotes the sequence whose first k terms, corresponding to $n = 0,\ 1,\ 2,\ \ldots.,k-1$, are taken as zero and whose subsequent terms, corresponding to $n = k,\ k+1,\ k+2,\ \ldots.$ are the terms $u_0,\ u_1,\ u_2,\ \ldots.$ of the original sequence.

ILLUSTRATION

Given that $\{u_n\} \equiv \{4^n\}$, we may say that

$$Z\{u_{n-2}\} \equiv \frac{1}{z^2}.Z\{u_n\} \equiv \frac{1}{z^2}.\frac{z}{z-4} \equiv \frac{1}{z(z-4)}.$$

Note:
In this illustration, the sequence, $\{u_{n-2}\}$ has terms $0,\ 0,\ 1,\ 4,\ 4^2,\ 4^3,\ \ldots$ and, by applying the definition of a Z-Transform directly, we would obtain

$$Z\{u_{n-2}\} = \frac{1}{z^2} + \frac{4}{z^3} + \frac{4^2}{z^4} + \frac{4^3}{z^5}\ \ldots.,$$

which gives

$$Z\{u_n\} \equiv \frac{1}{z^2}.\frac{1}{1-\frac{4}{z}} \equiv \frac{1}{z(z-4)},$$

by properties of infinite geometric series.

(c) The Second Shifting Theorem

$$Z\{u_{n+1}\} \equiv z.Z\{u_n\} - z.u_0$$

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} u_{r+1}z^{-r} \equiv u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \frac{u_4}{z^4} + \ \ldots .$$

This may be rearranged as

$$z.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \ \ldots .\right] - z.u_0$$

which, in turn, is equivalent to the right-hand side.

Note:
This **"recursive relationship"** may be applied repeatedly. For example, we may deduce that

$$Z\{u_{n+2}\} \equiv z.Z\{u_{n+1}\} - z.u_1 \equiv z^2.Z\{u_n\} - z^2.u_0 - z.u_1$$

16.8.4 EXERCISES

1. Determine, from first principles, the Z-Transforms of the following sequences, $\{u_n\}$:

   (a)

   $$\{u_n\} \equiv \{e^{-n}\};$$

   (b)

   $$\{u_n\} \equiv \{\cos \pi n\}.$$

2. Determine the Z-Transform of the following sequences:

   (a)

$$\{u_n\} \equiv \{7.(3)^n - 4.(-1)^n\};$$

   (b)

$$\{u_n\} \equiv \left\{6n + 2e^{-5n}\right\};$$

   (c)

$$\{u_n\} \equiv \{13 + \sin 2n - \cos 2n\}.$$

3. Determine the Z-Transform of $\{u_{n-1}\}$ and $\{u_{n-2}\}$ for the sequences in question 1.

4. Determine the Z-Transform of $\{u_{n+1}\}$ and $\{u_{n+2}\}$ for the sequences in question 1.

16.8.5 ANSWERS TO EXERCISES

1. (a)

$$\frac{ez}{ez-1};$$

   (b)

$$\frac{z}{z+1}.$$

2. (a)

$$\frac{7z}{z-3} - \frac{4z}{z+1};$$

   (b)

$$\frac{6z}{(z-1)^2} + \frac{2z}{z-e^{-5}};$$

   (c)

$$\frac{13z}{z-1} + \frac{z(\sin 2 + \cos 2 - z)}{z^2 - 2z\cos 2 + 1}.$$

3. (a)

$$Z\{u_{n-1}\} \equiv \frac{e}{ez-1}\ (n>0), \quad Z\{u_{n-2}\} \equiv \frac{e}{z(ez-1)}\ (n>1);$$

(b)

$$Z\{u_{n-1}\} \equiv \frac{1}{z+1}\ (n>0), \quad Z\{u_{n-2}\} \equiv \frac{1}{z(z+1)}\ (n>1).$$

Note:
$u_{-2}=0$ and $u_{-1}=0$.

4. (a)

$$Z\{u_{n+1}\} \equiv \frac{z}{ez-1}, \quad Z\{u_{n+2}\} \equiv \frac{z}{e(ez-1)};$$

(b)

$$Z\{u_{n+1}\} \equiv -\frac{z}{z+1}, \quad Z\{u_{n+2}\} \equiv \frac{z}{z+1}.$$

"JUST THE MATHS"

UNIT NUMBER

16.9

Z-TRANSFORMS 2
(Inverse Z-Transforms)

by

A.J.Hobson

UNIT 16.9 - Z TRANSFORMS 2

INVERSE Z - TRANSFORMS

16.9.1 THE USE OF PARTIAL FRACTIONS

When solving linear difference equations by means of Z-Transforms, it is necessary to be able to determine a sequence, $\{u_n\}$, of numbers, whose Z-Transform is a known function, $F(z)$, of z. Such a sequence is called the **"inverse Z-Transform of $F(z)$"** and may be denoted by $\mathrm{Z}^{-1}[F(z)]$.

For simple difference equations, the function $F(z)$ turns out to be a rational function of z, and the method of partial fractions may be used to determine the corresponding inverse Z-Transform.

EXAMPLES

1. Determine the inverse Z-Transform of the function

$$F(z) \equiv \frac{10z(z+5)}{(z-1)(z-2)(z+3)}.$$

Solution
Bearing in mind that

$$Z\{a^n\} = \frac{z}{z-a},$$

for any non-zero constant, a, we shall write

$$F(z) \equiv z.\left[\frac{10(z+5)}{(z-1)(z-2)(z+3)}\right],$$

which gives

$$F(z) \equiv z.\left[\frac{-15}{z-1} + \frac{14}{z-2} + \frac{1}{z+3}\right]$$

or

$$F(z) \equiv \frac{z}{z+3} + 14\frac{z}{z-2} - 15\frac{z}{z-1}.$$

Hence,

$$\mathrm{Z}^{-1}[F(z)] = \{(-3)^n + 14(2)^n - 15\}.$$

2. Determine the Inverse Z-Transform of the function

$$F(z) \equiv \frac{1}{z-a}.$$

Solution

In this example, there is no factor, z, in the function $F(z)$ and we shall see that it is necessary to make use of the first shifting theorem.

First, we may write

$$F(z) \equiv \frac{1}{z}\left[\frac{z}{z-a}\right]$$

and, since the inverse Z-Transform of the expression inside the brackets is a^n, the first shifting theorem tells us that

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ a^{n-1} & \text{when } n > 0. \end{cases}$$

Note:

This may now be taken as a standard result.

3. Determine the inverse Z-Transform of the function

$$F(z) \equiv \frac{4(2z+1)}{(z+1)(z-3)}.$$

Solution

Expressing $F(z)$ in partial fractions, we obtain

$$F(z) \equiv \frac{1}{z+1} + \frac{7}{(z-3)}.$$

Hence,

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ (-1)^{n-1} + 7.(3)^{n-1} & \text{when } n > 0. \end{cases}$$

16.9.2 EXERCISES

1. Determine the inverse Z-Transforms of each of the following functions, $F(z)$:

   (a)
$$F(z) \equiv \frac{z}{z-1};$$

   (b)
$$F(z) \equiv \frac{z}{z+1};$$

   (c)
$$F(z) \equiv \frac{2z}{2z-1};$$

   (d)
$$F(z) \equiv \frac{z}{3z+1};$$

   (e)
$$F(z) \equiv \frac{z}{(z-1)(z+2)};$$

   (f)
$$F(z) \equiv \frac{z}{(2z+1)(z-3)};$$

   (g)
$$F(z) \equiv \frac{z^2}{(2z+1)(z-1)}.$$

2. Determine the inverse Z-Transform of each of the following functions, $F(z)$, and list the first five terms of the sequence obtained:

   (a)
$$F(z) \equiv \frac{1}{z-1};$$

   (b)
$$F(z) \equiv \frac{z+2}{z+1};$$

(c)

$$F(z) \equiv \frac{z-3}{(z-1)(z-2)};$$

(d)

$$F(z) \equiv \frac{2z^2-7z+7}{(z-1)^2(z-2)}.$$

16.9.3 ANSWERS TO EXERCISES

1. (a)

$$\mathrm{Z}^{-1}[F(z)] = \{1\}$$

(b)

$$\mathrm{Z}^{-1}[F(z)] = \{(-1)^n\}$$

(c)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\left(\frac{1}{2}\right)^n\right\};$$

(d)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}\left(-\frac{1}{3}\right)^n\right\};$$

(e)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}[1-(-2)^n]\right\};$$

(f)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{7}\left[(3)^n - \left(-\frac{1}{2}\right)^n\right]\right\};$$

(g)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}+\frac{1}{6}\left(-\frac{1}{2}\right)^n\right\}.$$

2. (a)

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ 1 & \text{when } n > 0; \end{cases}$$

The first five terms are 0,1,1,1,1

(b)

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 1 & \text{when } n = 0; \\ (-1)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 1,1,−1,1,−1

(c)

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ 2 - (2)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 0,1,0,−2,−6

(d)

$$\mathrm{Z}^{-1}[F(z] = \begin{cases} 0 & \text{when } n = 0; \\ 3 - 2n + (2)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 0,2,1,1,3

"JUST THE MATHS"

UNIT NUMBER

16.10

Z-TRANSFORMS 3
(Solution of linear difference equations)

by

A.J.Hobson

UNIT 16.10 - Z TRANSFORMS 3

THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS

Linear difference equations may be solved by constructing the Z-Transform of both sides of the equation. The method will be illustrated with linear difference equations of the first and second orders (with constant coefficients).

16.10.1 FIRST ORDER LINEAR DIFFERENCE QUATIONS

EXAMPLES

1. Solve the linear difference equation,

$$u_{n+1} - 2u_n = (3)^{-n},$$

given that $u_0 = 2/5$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.\frac{2}{5}.$$

Taking the Z-Transform of the difference equation, we obtain

$$z.Z\{u_n\} - \frac{2}{5}.z - 2Z\{u_n\} = \frac{z}{z - \frac{1}{3}},$$

so that, on rearrangement,

$$Z\{u_n\} = \frac{2}{5}.\frac{z}{z-2} + \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}$$

$$\equiv \frac{2}{5}.\frac{z}{z-2} + z.\left[\frac{-\frac{3}{5}}{z - \frac{1}{3}} + \frac{\frac{3}{5}}{z-2}\right]$$

$$\equiv \frac{z}{z-2} - \frac{3}{5}.\frac{z}{z - \frac{1}{3}}.$$

Taking the inverse Z-Transform of this function of z gives the solution

$$\{u_n\} \equiv \left\{(2)^n - \frac{3}{5}(3)^{-n}\right\}.$$

2. Solve the linear difference equation,

$$u_{n+1} + u_n = f(n),$$

given that

$$f(n) \equiv \begin{cases} 1 & \text{when } n = 0; \\ 0 & \text{when } n > 0. \end{cases}$$

and $u_0 = 5$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.5$$

Taking the Z-Transform of the difference equation, we obtain

$$z.Z\{u_n\} - 5z + Z\{u_n\} = 1,$$

which, on rearrangement, gives

$$Z\{u_n\} = \frac{1}{z+1} + \frac{5z}{z+1}.$$

Hence,

$$\{u_n\} = \begin{cases} 5 & \text{when } n = 0; \\ (-1)^{n-1} + 5(-1)^n \equiv 4(-1)^n & \text{when } n > 0. \end{cases}$$

16.10.2 SECOND ORDER LINEAR DIFFERENCE EQUATIONS

EXAMPLES

1. Solve the linear difference equation

$$u_{n+2} = u_{n+1} + u_n,$$

given that $u_0 = 0$ and $u_1 = 1$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n - z.0 \equiv z.Z\{u_n\}$$

and

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z.1 \equiv z^2 Z\{u_n\} - z.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2.Z\{u_n\} - z = z.Z\{u_n\} + Z\{u_n\},$$

so that, on rearrangement,

$$Z\{u_n\} = \frac{z}{z^2 - z - 1},$$

which may be written

$$Z\{u_n\} = \frac{z}{(z-\alpha)(z-\beta)},$$

where, from the quadratic formula,

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

Using partial fractions,

$$Z\{u_n\} = \frac{1}{\alpha - \beta}\left[\frac{z}{z-\alpha} - \frac{z}{z-\beta}\right].$$

Taking the inverse Z-Transform of this function of z gives the solution

$$\{u_n\} \equiv \left\{\frac{1}{\alpha - \beta}\left[(\alpha)^n - (\beta)^n\right]\right\}.$$

2. Solve the linear difference equation

$$u_{n+2} - 7u_{n+1} + 10u_n = 16n,$$

given that $u_0 = 6$ and $u_1 = 2$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - 6z$$

and

$$Z\{u_{n+2}\} = z^2.Z\{u_n\} - 6z^2 - 2z.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2.Z\{u_n\} - 6z^2 - 2z - 7\left[z.Z\{u_n\} - 6z\right] + 10Z\{u_n\} = \frac{16z}{(z-1)^2},$$

which, on rearrangement, gives

$$Z\{u_n\}[z^2 - 7z + 10] - 6z^2 + 40z = \frac{16z}{(z-1)^2};$$

and, hence,

$$Z\{u_n\} = \frac{16z}{(z-1)^2(z-5)(z-2)} + \frac{6z^2 - 40z}{(z-5)(z-2)}.$$

Using partial fractions, we obtain

$$Z\{u_n\} = z.\left[\frac{4}{z-2} - \frac{3}{z-5} + \frac{4}{(z-1)^2} + \frac{5}{z-1}\right].$$

The solution of the difference equation is therefore

$$\{u_n\} \equiv \{4(2)^n - 3(5)^n + 4n + 5\}.$$

3. Solve the linear difference equation

$$u_{n+2} + 2u_n = 0$$

given that $u_0 = 1$ and $u_1 = \sqrt{2}$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z^2 - z\sqrt{2}.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2 Z\{u_n\} - z^2 - z\sqrt{2} + 2Z\{u_n\} = 0,$$

which, on rearrangement, gives

$$Z\{u_n\} = \frac{z^2 + z\sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{(z + j\sqrt{2})(z - j\sqrt{2})}.$$

Using partial fractions,

$$Z\{u_n\} = z\left[\frac{\sqrt{2}(1+j)}{j2\sqrt{2}(z - j\sqrt{2})} + \frac{\sqrt{2}(1-j)}{-j2\sqrt{2}(z + j\sqrt{2})}\right] \equiv z.\left[\frac{(1-j)}{2(z - j\sqrt{2})} + \frac{(1+j)}{2(z + j\sqrt{2})}\right],$$

so that

$$\{u_n\} \equiv \left\{\frac{1}{2}(1-j)(j\sqrt{2})^n + \frac{1}{2}(1+j)(-j\sqrt{2})^n\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n \left[(1-j)(j)^n + (1+j)(-j)^n\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n \left[\sqrt{2}e^{-j\frac{\pi}{4}}.e^{j\frac{n\pi}{2}} + \sqrt{2}e^{j\frac{\pi}{4}}.e^{-j\frac{n\pi}{2}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1} \left[e^{j\frac{(2n-1)\pi}{4}} + e^{-j\frac{(2n-1)\pi}{4}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}.2\cos\frac{(2n-1)\pi}{4}\right\}$$

$$\equiv \left\{(\sqrt{2})^{n+1}\cos\frac{(2n-1)\pi}{4}\right\}.$$

16.10.3 EXERCISES

1. Solve the following first-order linear difference equations:

   (a)
   $$3u_{n+1} + 2u_n = (-1)^n,$$
   given that $u_0 = 0$;

   (b)
   $$u_{n+1} - 5u_n = 3(2)^n,$$
   given that $u_0 = 1$;

   (c)
   $$u_{n+1} + u_n = n,$$
   given that $u_0 = 1$;

   (d)
   $$u_{n+1} + 2u_n = f(n),$$
   where
   $$f(n) \equiv \begin{cases} 3 & \text{when } n = 0; \\ 0 & \text{when } n > 0; \end{cases}$$
   and $u_0 = 2$;

   (e)
   $$u_{n+1} - 3u_n = \sin\frac{n\pi}{2} + \frac{1}{2}\cos\frac{n\pi}{2},$$
   given that $u_0 = 0$.

2. Solve the following second-order linear difference equations:

   (a)
   $$u_{n+2} - 2u_{n+1} + u_n = 0,$$
   given that $u_0 = 0$ and $u_1 = 1$;

   (b)
   $$u_{n+2} - 4u_n = n,$$
   given that $u_0 = 0$ and $u_1 = 1$;

(c)

$$u_{n+2} - 8u_{n+1} - 9u_n = 24,$$

given that $u_0 = 2$ and $u_1 = 0$;

(d)

$$6u_{n+2} + 5u_{n+1} - u_n = 20,$$

given that $u_0 = 3$ and $u_1 = 8$;

(e)

$$u_{n+2} + 2u_{n+1} - 15u_n = 32\cos n\pi,$$

given that $u_0 = 0$ and $u_1 = 0$;

(f)

$$u_{n+2} - 3u_{n+1} + 3u_n = 5,$$

given that $u_0 = 5$ and $u_1 = 8$.

16.10.4 ANSWERS TO EXERCISES

1. (a)

$$\{u_n\} \equiv \left\{\left(-\frac{2}{3}\right)^n - (-1)^n\right\};$$

(b)

$$\{u_n\} \equiv \{2(5)^n - (2)^n\};$$

(c)

$$\{u_n\} \equiv \left\{\frac{1}{2}n - \frac{1}{4} + \frac{5}{4}(-1)^n\right\};$$

(d)

$$\{u_n\} \equiv 2(-2)^n + 3(-2)^{n-1} \quad \text{when} \quad n > 0;$$

(e)

$$\{u_n\} \equiv \left\{\frac{1}{4}\left[(3^n - \sqrt{2}\cos\frac{(2n-1)\pi}{4}\right]\right\}.$$

2. (a)

$$\{u_n\} \equiv \{n\};$$

(b)

$$\{u_n\} \equiv \left\{\frac{1}{2}(2)^n - \frac{1}{3}n - \frac{5}{18}(-2)^n - \frac{2}{9}\right\};$$

(c)

$$\{u_n\} \equiv \left\{\frac{1}{2}(9)^n + 3(-1)^n - \frac{3}{2}\right\};$$

(d)

$$\{u_n\} \equiv \left\{2 + (6)^{1-n} - 5(-1)^n\right\};$$

(e)

$$\{u_n\} \equiv \left\{2(-1)^{n+1} + (3)^n + (-5)^n\right\};$$

(f)

$$\{u_n\} \equiv \left\{5 + (2\sqrt{3})^{n+1}\cos\frac{(n-3)\pi}{6}\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

17.1

NUMERICAL MATHEMATICS 1
(Approximate solution of equations)

by

A.J.Hobson

UNIT 17.1 - NUMERICAL MATHEMATICS 1

THE APPROXIMATE SOLUTION OF ALGEBRAIC EQUATIONS

17.1.1 INTRODUCTION

In the work which follows, we shall consider the solution of the equation

$$f(x) = 0,$$

where $f(x)$ is a given function of x.

It is assumed that examples of such equations will have been encountered earlier at an elementary level; as, for instance, with quadratic equations where there is simple formula for obtaining solutions.

However, the equation

$$f(x) = 0$$

cannot, in general, be solved algebraically to give **exact** solutions and we have to be satisfied, at most, with **approximate** solutions. Nevertheless, it is often possible to find approximate solutions which are correct to any specified degree of accuracy; and this is satisfactory for the applications of mathematics to science and engineering.

It is certainly possible to consider **graphical** methods of solving the equation

$$f(x) = 0,$$

where we try to plot a graph of the equation

$$y = f(x),$$

then determine where the graph crosses the x-axis. But this method can be laborious and inaccurate and will not be discussed, here, as a viable method.

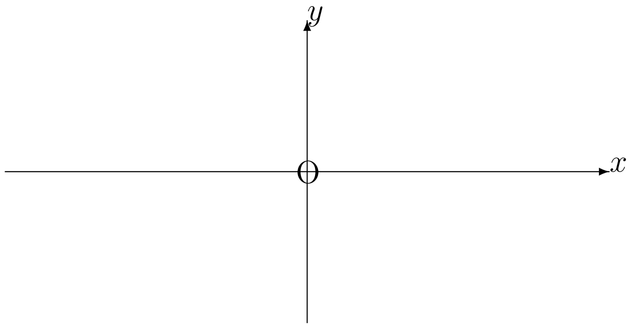
Three so called **"iterative"** methods will be included, below, where repeated use of the method is able to improve the accuracy of an approximate solution, already obtained.

17.1.2 THE BISECTION METHOD

Suppose a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. We may obtain these by trial and error or by sketching, roughly, the graph of the equation

$$y = f(x),$$

in order to estimate convenient values a and b between which the graph crosses the x-axis; whole numbers will usually suffice.



If we let $c = (a + b)/2$, there are three possibilities;

(i) $f(c) = 0$, in which case we have solved the equation;

(ii) $f(c) < 0$, in which case there is a solution between c and b enabling us repeat the procedure with these two numbers;

(iii) $f(c) > 0$, in which case there is a solution between c and a enabling us to repeat the procedure with these two numbers.

Each time we apply the method, we bisect the interval between the two numbers being used so that, eventually, the two numbers used will be very close together. The method stops when two consecutive values of the mid-point agree with each other to the required number of decimal places or significant figures.

Convenient labels for the numbers used at each stage (or iteration) are

$$a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, ..., a_n, b_n, c_n, ...$$

EXAMPLE

Determine, correct to three decimal places, the positive solution of the equation

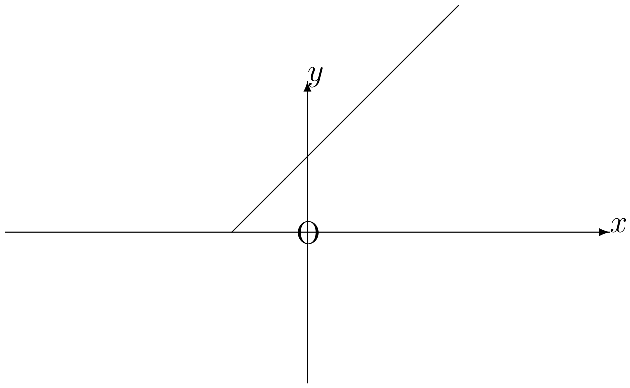
$$e^x = x + 2.$$

Solution

We could first observe, from a rough sketch of the graphs of

$$y = e^x \quad \text{and} \quad y = x + 2,$$

that the graphs intersect each other at a positive value of x. This confirms that there is indeed a positive solution to our equation.



But now let

$$f(x) = e^x - x - 2$$

and look for two numbers between which $f(x)$ changes sign from positive to negative. By trial and error, suitable numbers are 1 and 2, since

$$f(1) = e - 3 < 0 \text{ and } f(2) = e^2 - 5 > 0.$$

The rest of the solution may be set out in the form of a table as follows:

n	a_n	b_n	c_n	$f(c_n)$
0	1.00000	2.00000	1.50000	0.98169
1	1.00000	1.50000	1.25000	0.24034
2	1.00000	1.25000	1.12500	- 0.04478
3	1.12500	1.25000	1.18750	0.09137
4	1.12500	1.18750	1.15625	0.02174
5	1.12500	1.15625	1.14062	- 0.01191
6	1.14063	1.15625	1.14844	0.00483
7	1.14063	1.14844	1.14454	- 0.00354
8	1.14454	1.14844	1.14649	0.00064
9	1.14454	1.14649	1.14552	- 0.00144

As a general rule, it is appropriate to work to two more places of decimals than that of the required accuracy; and so, in this case, we work to five.

We can stop at stage 9, since c_8 and c_9 are the same value when rounded to three places of decimals. The required solution is therefore $\boxed{x = 1.146}$

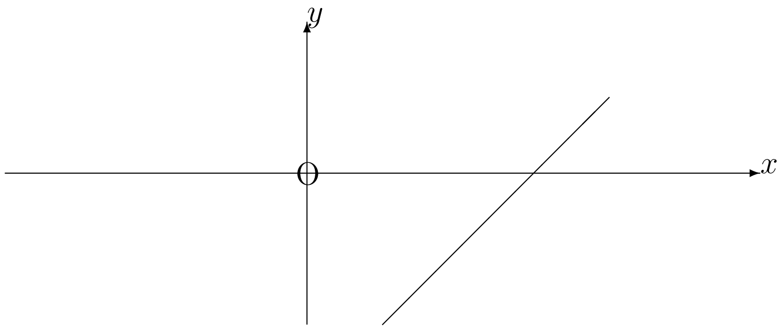
17.1.3 THE RULE OF FALSE POSITION

This method is commonly known by its Latin name, **"Regula Falsi"**, and tries to compensate a little for the shortcomings of the Bisection Method.

Instead of taking c as the average of a and b, we consider that the two points, $(a, f(a))$ and $(b, f(b))$, on the graph of the equation,

$$y = f(x),$$

are joined by a straight line; and the point at which this straight line crosses the x-axis is taken as c.



From elementary co-ordinate geometry, the equation of the straight line is given by

$$\frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}.$$

Hence, when $y = 0$, we obtain

$$x = a - \frac{(b - a)f(a)}{f(b) - f(a)}.$$

That is,

$$x = \frac{a[f(b) - f(a)] - (b - a)f(a)}{f(b) - f(a)}.$$

Hence,

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

In setting out the tabular form of a Regula Falsi solution, the c_n column uses the general formula

$$c_n = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}.$$

EXAMPLE

For the equation

$$f(x) \equiv x^3 + 2x - 1 = 0$$

use the Regula Falsi method with $a_0 = 0$ and $b_0 = 1$ to determine the first approximation, c_0, to the solution between $x = 0$ and $x = 1$.

Solution

We have $f(0) = -1$ and $f(1) = 2$, so that there is certainly a solution between $x = 0$ and $x = 1$.

From the general formula,

$$c_0 = \frac{0 \times 2 - 1 \times (-1)}{2 - (-1)} = \frac{1}{3}$$

and, if we were to continue with the method, we would observe that $f(1/3) < 0$ so that $a_1 = 1/3$ and $b_1 = 1$.

Note:
The Bisection Method would have given $c_0 = \frac{1}{2}$.

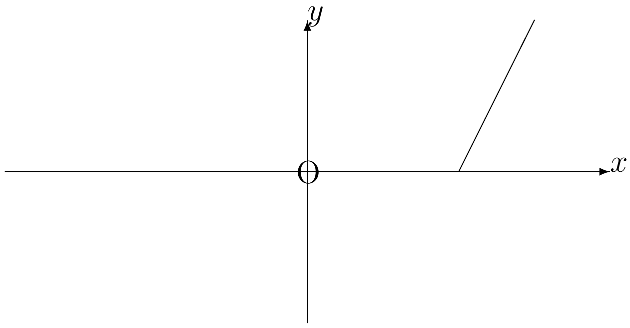
17.1.4 THE NEWTON-RAPHSON METHOD

This method is based on the guessing of an approximate solution, $x = x_0$, to the equation $f(x) = 0$.

We then draw the tangent to the curve whose equation is

$$y = f(x)$$

at the point $x_0, f(x_0)$ to find out where this tangent crosses the x-axis. The point obtained is normally a better approximation x_1 to the solution.



In the diagram,

$$f'(x_0) = \frac{\text{AB}}{\text{AC}} = \frac{f(x_0)}{h}.$$

Hence,

$$h = \frac{f(x_0)}{f'(x_0)},$$

so that a better approximation to the exact solution at point D is given by

$$x_1 = x_0 - h.$$

Repeating the process, gives rise to the following iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Notes:

(i) To guess the starting approximation, x_0, it is normally sufficient to use a similar technique to that in the Bisection Method; that is, we find a pair of whole numbers, a and b, such that $f(a) < 0$ and $f(b) > 0$; then we take $x_0 = (a + b)/2$. In some exercises, however, an alternative starting approximation may be suggested in order to speed up the rate of convergence to the final solution.

(ii) There are situations where the Newton-Raphson Method fails to give a better approximation; as, for example, when the tangent to the curve has a very small gradient, and consequently meets the x-axis at a relatively great distance from the previous approximation. In this Unit, we shall consider only examples in which the successive approximations converge rapidly to the required solution.

EXAMPLE

Use the Newton-Raphson method to calculate $\sqrt{5}$, correct to three places of decimals.

Solution

We are required to solve the equation

$$f(x) \equiv x^2 - 5 = 0.$$

By trial and error, we find that a solution exists between $x = 2$ and $x = 3$ since $f(2) = -1 < 0$ and $f(3) = 4 > 0$. Hence, we use $x_0 = 2.5$

Furthermore,

$$f'(x) = 2x,$$

so that

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}.$$

Thus,

$$\begin{aligned} x_1 &= 2.5 - \frac{1.25}{5} = 2.250, \\ x_2 &= 2.250 - \frac{0.0625}{4.5} \simeq 2.236, \\ x_3 &= 2.236 - \frac{-0.000304}{4.472} \simeq 2.236 \end{aligned}$$

At each stage, we round off the result to the required number of decimal places and use the rounded figure in the next iteration.

The last two iterations give the same result to three places of decimals and this is therefore the required result.

17.1.5 EXERCISES

1. Determine the smallest positive solution to the following equations (i) by the Bisection Method and (ii) by the Regula Falsi Method, giving your answers correct to four significant figures:

   (a)
   $$x - 2\sin^2 x = 0;$$
   (b)
   $$e^x - \cos(x^2) - 1 = 0.$$

2. Use the Newton-Raphson Method to determine the smallest positive solution to each of the following equations, correct to five decimal places:

   (a)
   $$x^4 = 5;$$
   (b)
   $$x^3 + x^2 - 4x + 1 = 0;$$
   (c)
   $$x - 2 = \ln x.$$

17.1.6 ANSWERS TO EXERCISES

1. (a)
   $$x \simeq 1.849;$$
   (b)
   $$x \simeq 0.6486$$

2. (a)
   $$x \simeq 1.49535;$$
   (b)
   $$x \simeq 0.27389;$$
   (c)
   $$x \simeq 3.14619$$

"JUST THE MATHS"

UNIT NUMBER

17.2

NUMERICAL MATHEMATICS 2
(Approximate integration (A))

by

A.J.Hobson

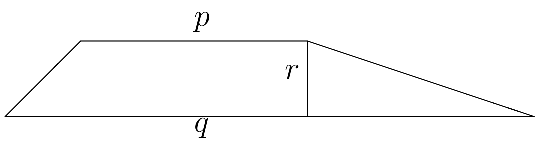
UNIT 17.2 - NUMERICAL MATHEMATICS 2

APPROXIMATE INTEGRATION (A)

17.2.1 THE TRAPEZOIDAL RULE

The rule which is explained below is based on the formula for the area of a trapezium. If the parallel sides of a trapezium are of length p and q while the perpendicular distance between them is r, then the area A is given by

$$A = \frac{r(p+q)}{2}.$$



Let us assume first that the curve whose equation is

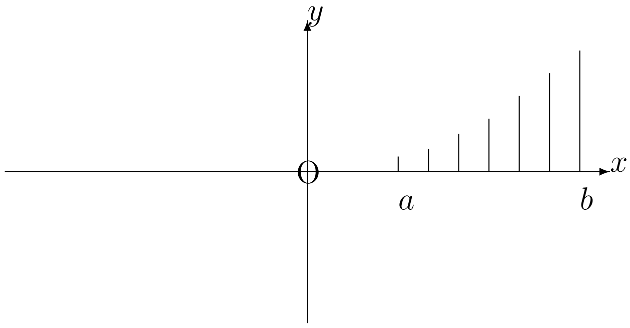
$$y = f(x)$$

lies wholly above the x-axis between $x = a$ and $x = b$. It has already been established, in Unit 13.1, that the definite integral

$$\int_a^b f(x)\ \mathrm{d}x$$

can be regarded as the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

However, suppose we divided this area into several narrow strips of equal width, h, by marking the values $x_1, x_2, x_3, ..., x_n$ along the x-axis (where $x_1 = a$ and $x_n = b$) and drawing in the corresponding lines of length $y_1, y_2, y_3, ..., y_n$ parallel to the y-axis.



Each narrow strip of width h may be considered approximately as a trapezium whose parallel sides are of lengths y_i and y_{i+1}, where $i = 1, 2, 3, ..., n-1$.

Thus, the area under the curve, and hence the value of the definite integral, approximates to

$$\frac{h}{2}[(y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + ... + (y_{n-1} + y_n)].$$

That is,

$$\int_a^b f(x)\ \mathrm{d}x \simeq \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + y_4 + ... + y_{n-1})];$$

or, what amounts to the same thing,

$$\int_a^b f(x)\ \mathrm{d}x = \frac{h}{2}[\text{First} + \text{Last} + 2 \times\ \text{The Rest}].$$

Note:
Care must be taken at the beginning to ascertain whether or not the curve $y = f(x)$ crosses the x-axis between $x = a$ and $x = b$. If it does, then allowance must be made for the fact that areas below the x-axis are negative and should be calculated separately from those above the x-axis.

EXAMPLE

Use the trapezoidal rule with five divisions of the x-axis in order to evaluate, approximately, the definite integral:

$$\int_0^1 e^{x^2}\ \mathrm{d}x.$$

Solution

First we make up a table of values as follows:

x	0	0.2	0.4	0.6	0.8	1.0
e^{x^2}	1	1.041	1.174	1.433	1.896	2.718

Then, using $h = 0.2$, we have

$$\int_0^1 e^{x^2}\ \mathrm{d}x \simeq \frac{0.2}{2}[1 + 2.718 + 2(1.041 + 1.174 + 1.433 + 1.896)] \simeq 1.481$$

17.2.2 EXERCISES

Use the trapezoidal rule with six divisions of the x-axis to determine an approximation for each of the following, working to three decimal places throughout:

1.
$$\int_1^7 x \ln x\ \mathrm{d}x.$$

2.
$$\int_{-2}^1 \frac{1}{5 - x^2}\ \mathrm{d}x.$$

3.
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\ \mathrm{d}x.$$

4.
$$\int_0^{\frac{\pi}{2}} \sin\sqrt{x^2 + 1}\ \mathrm{d}x.$$

5.
$$\int_0^{\frac{\pi}{2}} \ln(1 + \sin x)\ \mathrm{d}x.$$

17.2.3 ANSWERS TO EXERCISES

1. 35.836 2. 0.931 3. 0.348 4. 1.468 5. 0.737

"JUST THE MATHS"

UNIT NUMBER

17.3

NUMERICAL MATHEMATICS 3
(Approximate integration (B))

by

A.J.Hobson

UNIT 17.3 - NUMERICAL MATHEMATICS 3

APPROXIMATE INTEGRATION (B)

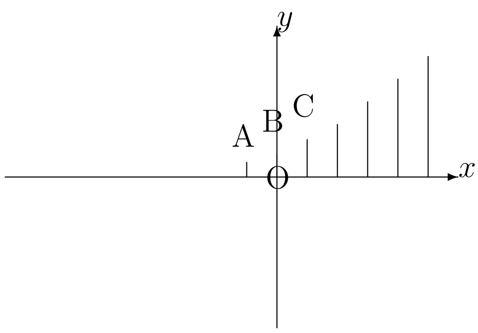
17.3.1 SIMPSON'S RULE

A better approximation to

$$\int_a^b f(x)\mathrm{d}x$$

than that provided by the Trapezoidal rule (Unit 17.2) may be obtained by usingan **even** number of narrow strips of width, h, and considering them in pairs.

To begin with, we examine a **special** case in which the first strip lies to the left of the y-axis as in the following diagram:



The arc of the curve passing through the points A$(-h, y_1)$, B$(0, y_2)$ and C(h, y_3) may be regarded as an arc of a parabola whose equation is

$$y = Lx^2 + Mx + N,$$

provided that the coefficients L, M and N satisfy the equations

$$\begin{aligned} y_1 &= Lh^2 - Mh + N, \\ y_2 &= N, \\ y_3 &= Lh^2 + Mh + N. \end{aligned}$$

Also, the area of the first pair of strips is given by

$$\text{Area} = \int_{-h}^{h} (Lx^2 + Mx + N)\ \mathrm{d}x$$

$$= \left[L\frac{x^3}{3} + M\frac{x^2}{2} + Nx\right]_{-h}^{h}$$

$$= \frac{2Lh^3}{3} + 2Nh$$

$$= \frac{h}{3}[2Lh^2 + 6N],$$

which, from the simultaneous equations earlier, gives

$$\text{Area} = \frac{h}{3}[y_1 + y_3 + 4y_2].$$

But the area of **every** pair of strips will be dependent only on the three corresponding y co-ordinates, together with the value of h.

Hence, the area of the next pair of strips will be

$$\frac{h}{3}[y_3 + y_5 + 4y_4],$$

and the area of the pair after that will be

$$\frac{h}{3}[y_5 + y_7 + 4y_6].$$

Thus, the total area is given by

$$\text{Area} = \frac{h}{3}[y_1 + y_n + 4(y_2 + y_4 + y_6 + \ldots) + 2(y_3 + y_5 + y_7 + \ldots),$$

usually interpreted as

$$\text{Area} = \frac{h}{3}[\text{First}+\text{Last}+4\times\text{The even numbered } y \text{ co} - \text{ords.}+2\times\text{The remaining } y \text{ co} - \text{ords.}]$$

or

$$\text{Area} = \frac{h}{3}[\text{F} + \text{L} + 4\text{E} + 2\text{R}]$$

This result is known as SIMPSON'S RULE.

Notes:

(i) Since the area of the pairs of strips depends only on the three corresponding y co-ordinates, together with the value of h, the Simpson's rule formula provides an approximate value of the definite integral

$$\int_a^b f(x) \, \mathrm{d}x$$

whatever the values of a and b are, as long as the curve does not cross the x-axis between $x = a$ and $x = b$.

(ii) If the curve **does** cross the x-axis between $x = a$ and $x = b$, it is necessary to consider separately the positive parts of the area above the x-axis and the negative parts below the x-axis.

(iii) The approximate evaluation, by Simpson's rule, of a definite integral should be set out in **tabular form**, as illustrated in the examples overleaf.

EXAMPLES

1. Working to a maximum of three places of decimals throughout, use Simpson's rule with ten divisions to evaluate, approximately, the definite integral

$$\int_0^1 e^{x^2}\,\mathrm{d}x.$$

Solution

x_i	$y_i = e^{x_i^2}$	F & L	E	R
0	1	1		
0.1	1.010		1.010	
0.2	1.041			1.041
0.3	1.094		1.094	
0.4	1.174			1.174
0.5	1.284		1.284	
0.6	1.433			1.433
0.7	1.632		1.632	
0.8	1.896			1.896
0.9	2.248		2.248	
1.0	2.718	2.718		
	F + L $\rightarrow$	3.718	7.268	5.544
	4E $\rightarrow$	29.072	$\times 4$	$\times 2$
	2R $\rightarrow$	11.088	29.072	11.088
	(F + L) + 4E + 2R $\rightarrow$	43.878	//////	//////

Hence,

$$\int_0^1 e^{x^2}\,\mathrm{d}x \simeq \frac{0.1}{3} \times 43.878 \simeq 1.463$$

2. Working to a maximum of three places of decimals throughout, use Simpson's rule with eight divisions between $x = -1$ and $x = 1$ and four divisions between $x = 1$ and $x = 2$ in order to evaluate, approximately, the area between the curve whose equation is

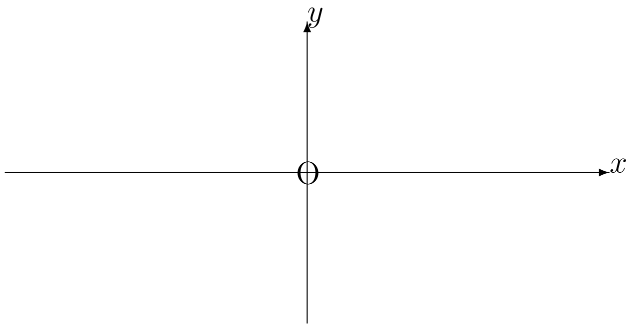
$$y = (x^2 - 1)e^{-x}$$

and the x-axis from $x = -1$ to $x = 2$.

Solution

We note that the curve crosses the x-axis when $x = -1$ and when $x = 1$, the y co-ordinates being negative in the interval between these two values of x and positive outside this interval.

Hence, we need to evaluate the negative area between $x = -1$ and $x = 1$ and the positive area between $x = 1$ and $x = 2$; then we add their numerical values together to find the total area.



(a) The Negative Area

x_i	$y_i = (x^2 - 1)e^{-x}$	F & L	E	R
-1	0	0		
-0.75	-0.926		-0.926	
-0.5	-1.237			-1.237
-0.25	-1.204		-1.204	
0	-1			-1
0.25	-0.730		-0.730	
0.50	-0.455			-0.455
0.75	-0.207		-0.207	
1	0	0		
	F + L $\rightarrow$	0	-2.860	-2.692
	4E $\rightarrow$	-11.440	$\times 4$	$\times 2$
	2R $\rightarrow$	-5.384	-11.440	-5.384
	(F + L) + 4E + 2R $\rightarrow$	-16.824	//////	//////

(b) The Positive Area

x_i	$y_i = (x^2 - 1)e^{-x}$	F & L	E	R
1	0	0		
1.25	0.161		0.161	
1.5	0.279			0.279
1.75	0.358		0.358	
2	0.406	0.406		
	F + L $\rightarrow$	0.406	0.519	0.279
	4E $\rightarrow$	2.076	$\times 4$	$\times 2$
	2R $\rightarrow$	0.558	2.076	0.558
	(F + L) + 4E + 2R $\rightarrow$	3.040	//////	//////

The total area is thus

$$\frac{0.25}{3} \times (16.824 + 3.040) \simeq 1.655$$

17.3.2 EXERCISES

Use Simpson's rule with six divisions of the x-axis to find an approximation for each of the following, working to a maximum of three decimal places throughout:

1.
$$\int_1^7 x \ln x \ \mathrm{d}x.$$

2.
$$\int_{-2}^1 \frac{1}{5 - x^2} \ \mathrm{d}x.$$

3.
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \ \mathrm{d}x.$$

4.
$$\int_0^{\frac{\pi}{2}} \sin\sqrt{x^2 + 1} \ \mathrm{d}x.$$

5.
$$\int_0^{\frac{\pi}{2}} \ln(1 + \sin x) \ \mathrm{d}x.$$

17.3.3 ANSWERS TO EXERCISES

1. 35.678 2. 0.882 3. 0.347 4. 1.469 5. 0.743

"JUST THE MATHS"

UNIT NUMBER

17.4

NUMERICAL MATHEMATICS 4
(Further Gaussian elimination)

by

A.J.Hobson

UNIT 17.4 - NUMERICAL MATHEMATICS 4
FURTHER GAUSSIAN ELIMINATION

The **elementary** method of Gaussian Elimination, for simultaneous linear equations, was discussed in Unit 9.4. We introduce, here, a more **general** method, suitable for use with sets of equations having **decimal** coefficients.

17.4.1 GAUSSIAN ELIMINATION BY "PARTIAL PIVOTING" WITH A CHECK COLUMN

Let us first consider an example in which the coefficients are **integers**.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 2x + y + z &= 3, \\ x - 2y - z &= 2, \\ 3x - y + z &= 8. \end{aligned}$$

Solution

We may set out the solution, in the form of a **table** (rather than a **matrix**) indicating each of the **"pivot elements"** in a box as follows:

	x	y	z	constant	Σ
	$\boxed{2}$	1	1	3	7
$\frac{1}{2}$	1	-2	-1	2	0
$\frac{3}{2}$	3	-1	1	8	11
		$\boxed{\frac{-5}{2}}$	$\frac{-3}{2}$	$\frac{1}{2}$	$\frac{-7}{2}$
1		$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
			1	3	4

INSTRUCTIONS

(i) Divide the coefficients of x in lines 2 and 3 by the coefficient of x in line 1 and write the respective results at the side of lines 2 and 3; (that is, $\frac{1}{2}$ and $\frac{3}{2}$ in this case).

(ii) Eliminate x by subtracting $\frac{1}{2}$ times line 1 from line 2 and $\frac{3}{2}$ times line 1 from line 3.

(iii) Repeat the process starting with lines 4 and 5.

(iv) line 6 implies that $z = 3$ and by substitution back into earlier lines, we obtain the values $y = -2$ and $x = 1$.

OBSERVATIONS

Difficulties could arise if the pivot element were very small compared with the other quantities in the same column, since the errors involved in dividing by small numbers are likely to be large.

A better choice of pivot element would be the one with the **largest** numerical value in its column.

We shall consider an example, now, in which this choice of pivot is made. The working will be carried out using fractional quantities; though, in practice, decimals would normally be used instead.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} x - y + 2z &= 5, \\ 2x + y - z &= 1, \\ x + 3y - z &= 4. \end{aligned}$$

Solution

	x	y	z	constant	Σ
$\frac{1}{2}$	1	-1	2	5	7
	$\boxed{2}$	1	-1	1	3
$\frac{1}{2}$	1	3	-1	4	7

On eliminating x, we obtain the new table:

	y	z	constant	Σ
$\frac{-3}{5}$	$\frac{-3}{2}$	$\frac{5}{2}$	$\frac{9}{2}$	$\frac{11}{2}$
	$\boxed{\frac{5}{2}}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Eliminating y takes us to the final table as follows:

z	constant	Σ
$\frac{11}{5}$	$\frac{33}{5}$	$\frac{44}{5}$

We conclude that
$11z = 33$ and, hence, $\boxed{z = 3}$.

Substituting into the second table (either line will do), we have
$5y - 3 = 7$ and, hence, $\boxed{y = 2}$.

Substituting into the original table (any line will do), we have
$x - 2 + 6 = 5$, so that $\boxed{x = 1}$.

Notes:

(i) In questions which involve decimal quantities stated to n decimal places, the calculations should be carried out to $n + 2$ decimal places to allow for rounding up.

(ii) A final check on accuracy in the above example is obtained by adding the original three equations together and verifying that the solution obtained also satisfies the further equation

$$4x + 3y = 10.$$

(iii) It is not essential to set out the solution in the form of separate tables (at each step) with their own headings. A continuation of the first table is acceptable.

17.4.2 EXERCISES

1. Use Gaussian Elimination by Partial Pivoting with a check column to solve the following sets of simultaneous linear equations:

   (a)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5, \\ 3x_1 - x_2 + 2x_3 &= 8, \\ 4x_1 - 6x_2 - 4x_3 &= -2. \end{aligned}$$

   (b)

$$\begin{aligned} 5i_1 - i_2 + 2i_3 &= 3, \\ 2i_1 + 4i_2 + i_3 &= 8, \\ i_1 + 3i_2 - 3i_3 &= 2; \end{aligned}$$

(c)

$$\begin{aligned} i_1 + 2i_2 + 3i_3 &= -4, \\ 2i_1 + 6i_2 - 3i_3 &= 33, \\ 4i_1 - 2i_2 + i_3 &= 3; \end{aligned}$$

(d)

$$\begin{aligned} 7i_1 - 4i_2 \qquad &= 12, \\ -4i_1 + 12i_2 - 6i_3 &= 0, \\ -6i_2 + 14i_3 &= 0; \end{aligned}$$

2. Use Gaussian Elimination with Partial Pivoting and a check column to solve the following sets of simultaneous linear equations:

   (a)

$$\begin{aligned} 1.202x_1 - 4.371x_2 + 0.651x_3 &= 19.447, \\ -3.141x_1 + 2.243x_2 - 1.626x_3 &= -13.702, \\ 0.268x_1 - 0.876x_2 + 1.341x_3 &= 6.849; \end{aligned}$$

   (b)

$$\begin{aligned} -2.381x_1 + 1.652x_2 - 1.243x_3 &= 12.337, \\ 2.151x_1 - 3.427x_2 + 3.519x_3 &= 9.212, \\ 1.882x_1 + 2.734x_2 - 1.114x_3 &= 5.735; \end{aligned}$$

17.4.3 ANSWERS TO EXERCISES

1. (a) $x_1 = -1, \quad x_2 = -3, \quad x_3 = 4;$
   (b) $i_1 = 0.5, \quad i_2 = 1.5, \quad i_3 = 1.0;$
   (c) $i_1 = 3.0, \quad i_2 = 2.5, \quad i_3 = -4.0;$
   (d) $i_1 = 2.26, \quad i_2 = 0.96, \quad i_3 = 0.41$
2. (a) $x_1 = 0.229, \quad x_2 = -4.024, \quad x_3 = 2.433;$
   (b) $x_1 = -5.753, \quad x_2 = 14.187, \quad x_3 = 19.951$

"JUST THE MATHS"

UNIT NUMBER

17.5

NUMERICAL MATHEMATICS 5
(Iterative methods)
for solving
(simultaneous linear equations)

by

A.J.Hobson

UNIT 17.5 - NUMERICAL MATHEMATICS 5

ITERATIVE METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATIONS

17.5.1 INTRODUCTION

An iterative method is one which is used repeatedly until the results obtained acquire a pre-assigned degree of accuracy. For example, if results are required to be accurate to five places of decimals, the number of **"iterations"** (that is, stages of the method) is continued until two consecutive iterations give the same result when rounded off to that number of decimal places. It is usually enough for the calculations themselves to be carried out to **two extra** places of decimals.

A similar interpretation holds for accuracy which requires a certain number of **significant figures**.

In the work which follows, we shall discuss two standard methods of solving a set of simultaneous linear equations of the form

$$
\begin{aligned}
a_1x + b_1y + c_1z &= k_1, \\
a_2x + b_2y + c_2z &= k_2, \\
a_3x + b_3y + c_3z &= k_3,
\end{aligned}
$$

when the system is **"diagonally dominant"**, which, in this case, means that

$$
\begin{aligned}
|a_1| &> |b_1| + |c_1|, \\
|b_2| &> |a_2| + |c_2|, \\
|c_3| &> |a_3| + |b_3|.
\end{aligned}
$$

The methods would be adaptable to a different number of simultaneous equations.

17.5.2 THE GAUSS-JACOBI ITERATION

This method begins by making x the subject of the first equation, y the subject of the second equation and z the subject of the third equation.
An initial approximation such as $x_0 = 1, y_0 = 1, z_0 = 1$ is substituted on the new right-hand sides to give values $x = x_1, y = y_1$ and $z = z_1$ on the new left-hand sides.

A continuation of the process leads to the following general scheme for the results of the $(n + 1)$-th iteration:

$$\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad - b_1 y_n - c_1 z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_n \qquad - c_2 z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_n - b_3 y_n \qquad\right).
\end{aligned}$$

This scheme will now be illustrated by numerical examples:

EXAMPLES

1. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$\begin{aligned}
5x + y - z &= 4, \\
x + 4y + 2z &= 15, \\
x - 2y + 5z &= 12,
\end{aligned}$$

obtaining x, y and z correct to the nearest whole number.

Solution

We have

$$\begin{aligned}
x_{n+1} &= 0.8 \qquad - 0.2y_n + 0.2z_n, \\
y_{n+1} &= 3.75 - 0.25x_n \qquad - 0.5z_n, \\
z_{n+1} &= 2.4 - 0.2x_n + 0.4y_n.
\end{aligned}$$

Using

$$x_0 = 1, y_0 = 1, z_0 = 1,$$

we obtain

$$x_1 = 0.8, y_1 = 3.0, z_1 = 2.6,$$
$$x_2 = 0.72, y_2 = 2.25, z_2 = 3.44,$$
$$x_3 = 1.038, y_3 = 1.85, z_3 = 3.156$$

The results of the last two iterations both give

$$x = 1, y = 2, z = 3,$$

when rounded to the nearest whole number.

In fact, these whole numbers are clearly seen to be the **exact** solutions.

2. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$\begin{aligned} x + 7y - z &= 3, \\ 5x + y + z &= 9, \\ -3x + 2y + 7z &= 17, \end{aligned}$$

obtaining x, y and z correct to the nearest whole number.

Solution

This set of equations is not diagonally dominant; but they can be rewritten as

$$\begin{aligned} 7y + x - z &= 3, \\ y + 5x + z &= 9, \\ 2y - 3x + 7z &= 17, \end{aligned}$$

which **is** a diagonally dominant set. We could also interchange the first two of the original equations.

We have now

$$\begin{aligned} y_{n+1} &= 0.43 - 0.14x_n + 0.14z_n, \\ x_{n+1} &= 1.8 - 0.2y_n - 0.2z_n, \\ z_{n+1} &= 2.43 + 0.43x_n - 0.29y_n. \end{aligned}$$

Using

$$y_0 = 1, x_0 = 1, z_0 = 1,$$

we obtain

$$y_1 = 0.43, x_1 = 1.4, z_1 = 2.57,$$

$$y_2 = 0.59, x_2 = 1.2, z_2 = 2.91,$$

$$y_3 = 0.67, x_3 = 1.1, z_3 = 2.78$$

This is now enough to conclude that $x = 1, y = 1, z = 3$ to the nearest whole number though, this time, they are not the exact solutions.

17.5.3 THE GAUSS-SEIDEL ITERATION

This method differs from the Gauss-Jacobi Iteration in that successive approximations are used within each step **as soon as they become available**.

It turns out that the rate of convergence of this method is usually faster than that of the Gauss-Jacobi method.

The scheme of the calculations is according to the following pattern:

$$\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \phantom{- a_2x_{n+1}} - b_1y_n - c_1z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2x_{n+1} - c_2z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3x_{n+1} - b_3y_{n+1} \right).
\end{aligned}$$

EXAMPLES

1. Use the Gauss-Seidel method to solve the simultaneous linear equations

$$\begin{aligned}
5x + y - z &= 4, \\
x + 4y + 2z &= 15, \\
x - 2y + 5z &= 12.
\end{aligned}$$

Solution

This time, we write:

$$\begin{aligned} x_{n+1} &= 0.8 \qquad\qquad - 0.2y_n + 0.2z_n, \\ y_{n+1} &= 3.75 - 0.25x_{n+1} \qquad\quad - 0.5z_n, \\ z_{n+1} &= 2.4 - 0.2x_{n+1} + 0.4y_{n+1}, \end{aligned}$$

and the sequence of successive results is as follows:

$$x_0 = 1, y_0 = 1, z_0 = 1,$$

$$x_1 = 0.8, y_1 = 3.05, z_1 = 3.46,$$

$$x_2 = 0.88, y_2 = 1.80, z_2 = 2.94,$$

$$x_3 = 1.03, y_3 = 2.02, z_3 = 3.00$$

In this particular example, the rate of convergence is about the same as for the Gauss-Jacobi method, giving $x = 1, y = 2, z = 3$ to the nearest whole number; but we would normally expect the Gauss-Seidel method to converge at a faster rate.

2. Use the Gauss Seidel method to solve the simultaneous linear equations:

$$\begin{aligned} 7y + x - z &= 3, \\ y + 5x + z &= 9, \\ 2y - 3x + 7z &= 17. \end{aligned}$$

Solution

These equations give rise to the following iterative scheme:

$$\begin{aligned} y_{n+1} &= 0.43 - 0.14x_n \qquad\quad + 0.14z_n, \\ x_{n+1} &= 1.8 \qquad\qquad - 0.2y_{n+1} - 0.2z_n, \\ z_{n+1} &= 2.43 + 0.43x_{n+1} - 0.29y_{n+1}, \end{aligned}$$

The sequence of successive results is:

$$y_0 = 1, x_0 = 1, z_0 = 1,$$

$$y_1 = 0.43, x_1 = 1.51, z_1 = 2.96,$$
$$y_2 = 0.63, x_2 = 1.08, z_2 = 2.71,$$
$$y_3 = 0.66, x_3 = 1.13, z_3 = 2.73$$

Once more, to the nearest whole number, the solutions are $x = 1, y = 1, z = 3$.

17.5.4 EXERCISES

1. Setting $x_0 = y_0 = z_0 = 1$ and working to three places of decimals, complete four iterations of

   (a) The Gauss-Jacobi method

   and

   (b) The Gauss-Seidel method

   for the system of simultaneous linear equations

$$\begin{aligned} 7x - y + z &= 7.3, \\ 2x - 8y - z &= -6.4, \\ x + 2y + 9z &= 13.6 \end{aligned}$$

   To how many decimal places are your results accurate ?

2. Rearrange the following equations to form a diagonally dominant system and perform the first four iterations of the Gauss-Seidel method, setting $x_0 = 1$, $y_0 = 1$ and $z_0 = 1$ and working to two places of decimals:

$$\begin{aligned} x + 5y - z &= 8, \\ -9x + 3y + 2z &= 3, \\ x + 2y + 7z &= 26. \end{aligned}$$

   Estimate the accuracy of your results and suggest the exact solutions (checking that they are valid).

3. Use an appropriate number of iterations of the Gauss-Seidel method to solve accurately, to three places of decimals, the simultaneous linear equations

$$\begin{aligned} 7x + y + z &= 5, \\ -2x + 9y + 3z &= 4, \\ x + 4y + 8z &= 3. \end{aligned}$$

17.5.5 ANSWERS TO EXERCISES

1. (a) $x_4 \simeq 1.014$, $y_4 \simeq 0.850$, $z_4 \simeq 1.199$, which are accurate to one decimal place.
   (b) $x_4 \simeq 0.999$, $y_4 \simeq 0.900$, $z_4 \simeq 1.200$, which are accurate to two decimal places.
2. On interchanging the first two equations, $x_4 \simeq 0.98$, $y_4 \simeq 2.00$, $z_4 \simeq 2.99$, which are accurate to the nearest whole number. The exact solutions are $x = 1$, $y = 2$, $z = 3$.
3. $x \simeq 0.630$, $y \simeq 0.582$, $z \simeq 0.004$.

"JUST THE MATHS"

UNIT NUMBER

17.6

NUMERICAL MATHEMATICS 6
(Numerical solution)
of
(ordinary differential equations (A))

by

A.J.Hobson

UNIT 17.6 - NUMERICAL MATHEMATICS 6

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (A)

17.6.1 EULER'S UNMODIFIED METHOD

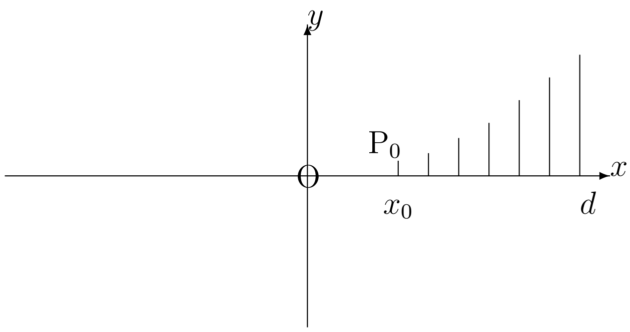
Every first order ordinary differential equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y);$$

and, if it is given that $y = y_0$ when $x = x_0$, then the solution for y in terms of x represents some curve through the point $\mathrm{P}_0(x_0, y_0)$.

Suppose that we wish to find the solution for y at $x = d$, where $d > x_0$.

We sub-divide the interval from $x = x_0$ to $x = d$ into n equal parts of width, δx.



Letting $x_1, x_2, x_3, ...$ be the points of subdivision, we have

$$x_1 = x_0 + \delta x,$$

$$x_2 = x_0 + 2\delta x,$$

$$x_3 = x_0 + 3\delta x,$$

$$...,$$

$$...,$$

$$d = x_n = x_0 + n\delta x.$$

If $y_1, y_2, y_3, ...$ are the y co-ordinates of $x_1, x_2, x_3, ...$, we are required to find y_n.

From elementary calculus, the increase in y, when x increases by δx, is given approximately by $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; and since, in our case, $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, we have

$$y_1 = y_0 + f(x_0, y_0)\delta x,$$
$$y_2 = y_1 + f(x_1, y_1)\delta x,$$
$$y_3 = y_2 + f(x_2, y_2)\delta x,$$
$$\dots,$$
$$\dots,$$
$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\delta x,$$

each stage using the previously calculated y value.

Note:
The method will be the same if $d < x_0$, except that δx will be negative.

In general, each intermediate value of y is given by the formula

$$y_{i+1} = y_i + f(x_i, y_i)\delta x.$$

EXAMPLE

Use Euler's method with 5 sub-intervals to continue, to $x = 0.5$, the solution of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y = 1$ when $x = 0$; (that is, $y(0) = 1$).

Solution

i	x_i	y_i	$f(x_i, y_i)$	$y_{i+1} = y_i + f(x_i, y_i)\delta x$
0	0	1	0	1
1	0.1	1	0.1	1.01
2	0.2	1.01	0.202	1.0302
3	0.3	1.0302	0.30906	1.061106
4	0.4	1.061106	0.4244424	1.1035524
5	0.5	1.1035524	-	-

Accuracy
The differential equation in the above example is simple to solve by an elementary method,

such as separation of the variables. It is therefore useful to compare the exact result so obtained with the approximation which comes from Eulers' method.

$$\int \frac{\mathrm{d}y}{y} = \int x\mathrm{d}x.$$

Therefore

$$\ln y = \frac{x^2}{2} + C;$$

that is,

$$y = Ae^{\frac{x^2}{2}}.$$

At $x = 0$, we are told that $y = 1$ and, hence, $A = 1$, giving

$$y = e^{\frac{x^2}{2}}.$$

But a table of values of x against y in the previous interval reveals the following:

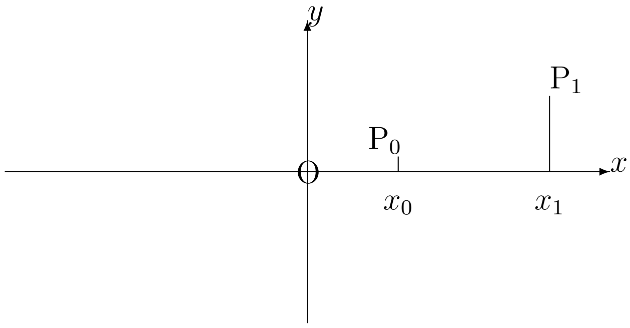
x	$e^{\frac{x^2}{2}}$
0	1
0.1	1.00501
0.2	1.0202
0.3	1.04603
0.4	1.08329
0.5	1.13315

There is thus an error in our approximate value of 0.0296, which is about 2.6%. Attempts to determine y for values of x which are greater than 0.5 would result in a very rapid growth of error.

17.6.2 EULER'S MODIFIED METHOD

In the previous method, we used the gradient to the solution curve at the point P_0 in order to find an approximate position for the point P_1, and so on up to P_n.

But the approximation turns out to be much better if, instead, we use the **average** of the two gradients at P_0 and P_1 for which we need use only x_0, y_0 and δx in order to calculate approximately.



The gradient, m_0, at P_0, is given by

$$m_0 = f(x_0, y_0).$$

The gradient, m_1, at P_1, is given approximately by

$$m_1 = f(x_0 + \delta x, y_0 + \delta y_0),$$

where $\delta y_0 = f(x_0, y_0)\delta x$.

Note:
We cannot call $y_0 + \delta y_0$ by the name y_1, as we did with the unmodified method, because this label is now reserved for the new and **better** approximation at $x = x_0 + \delta x$.

The average gradient, between P_0 and P_1, is given by

$$m_0^* = \frac{1}{2}(m_0 + m_1).$$

Hence, our approximation to y at the point P_1 is given by

$$y_1 = y_0 + m_0^* \delta x.$$

Similarly, we proceed from y_1 to y_2, and so on until we reach y_n.

In general, the intermediate values of y are given by

$$y_{i+1} = y_i + m_i^* \delta x.$$

EXAMPLE

Solve the example in the previous section using Euler's Modified method.

Solution

i	x_i	y_i	$m_i = f(x_i, y_i)$	$\delta y_i = f(x_i, y_i)\delta x$	$m_{i+1} = f(x_i + \delta x, y_i + \delta y_i)$	$m_i^* = \frac{1}{2}(m_i + m_{i+1})$	$y_{i+1} = y_i + m_i^* \delta x$
0	0	1	0	0	0.1	0.05	1.005
1	0.1	1.005	0.1005	0.0101	0.2030	0.1518	1.0202
2	0.2	1.0202	0.2040	0.0204	0.3122	0.2581	1.0460
3	0.3	1.0460	0.3138	0.0314	0.4310	0.3724	1.0832
4	0.4	1.0832	0.4333	0.0433	0.5633	0.4983	1.1330
5	0.5	1.1330	——	——	——	——	——

17.6.3 EXERCISES

1. (a) Taking intervals $\delta x = 0.2$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0,$$
and that $y(0) = 1$
Compare your solution with the exact solution given by
$$y = e^{-x}.$$

(b) Taking intervals $\delta x = 0.1$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y}{x}$$
and that $y(0.5) = 0.5$.
Compare your solution with the exact solution given by
$$y = x^2 + \frac{x}{2}.$$

(c) Taking intervals $\delta x = 0.2$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + e^{-x},$$
and that $y(0) = 0$.
Compare your solution with the exact solution given by
$$y = \sinh x.$$

(d) Given that $y(1) = 2$, use Euler's unmodified method to continue the solution of the differential equation,

$$\frac{dy}{dx} = x^2 + \frac{y}{2},$$

to obtain values of y for values of x from $x = 1$ to $x = 1.5$, in steps of 0.1.

2. Repeat all parts of question 1 using Euler's modified method.

17.6.4 ANSWERS TO EXERCISES

1. (a) 0.33, 0.37, 11% low;
   (b) 1.45 1.50. 3% low;
   (c) 1.113, 1.175, 5% low;
   (d) 2.0000, 2.200, 2.431 2.697 3.001, 3.347.
2. (a) 0.371, 0.368, 0.8% high;
   (b) 1.495, 1.50, 0.33% low;
   (c) 1.175 , accurate to three decimal places;
   (d) 2.000, 2.216, 2.465, 2.751, 3.079, 3.452.

Note:
In questions 1(d) and 2(d), the actual values are 2.000, 2.245, 2.496, 2.784, 3.113 and 3.489, from the exact solution of the differential equation.

"JUST THE MATHS"

UNIT NUMBER

17.7

NUMERICAL MATHEMATICS 7
(Numerical solution)
of
(ordinary differential equations (B))

by

A.J.Hobson

UNIT 17.7 - NUMERICAL MATHEMATICS 7

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (B)

17.7.1 PICARD'S METHOD

This method of solving a differential equation approximately is one of successive approximation; that is, it is an **iterative** method in which the numerical results become more and more accurate, the more times it is used.

An approximate value of y (taken, at first, to be a constant) is substituted into the right hand side of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

The equation is then integrated with respect to x giving y in terms of x as a second approximation, into which given numerical values are substituted and the result rounded off to an assigned number of decimal places or significant figures.

The iterative process is continued until two consecutive numerical solutions are the same when rounded off to the required number of decimal places.

A hint on notation
Imagine, for example, that we wished to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

given that $y = y_0 = 7$ when $x = x_0 = 2$.

This ofcourse can be solved exactly to give

$$y = x^3 + C,$$

which requires that

$$7 = 2^3 + C.$$

Hence,

$$y - 7 = x^3 - 2^3;$$

or, in more general terms

$$y - y_0 = x^3 - x_0^3.$$

Thus,

$$\int_{y_0}^{y} \mathrm{d}y = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

In other words,

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

The rule, in future, therefore, will be to integrate both sides of the given differential equation with respect to x, from x_0 to x.

EXAMPLES

1. Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2,$$

and that $y = 0$ when $x = 0$, determine the value of y when $x = 0.3$, correct to four places of decimals.

Solution

To begin the solution, we proceed as follows:

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x} (x + y^2) \mathrm{d}x,$$

where $x_0 = 0$.

Hence,

$$y - y_0 = \int_{x_0}^{x} (x + y^2) \mathrm{d}x,$$

where $y_0 = 0$.

That is,

$$y = \int_{0}^{x} (x + y^2) \mathrm{d}x.$$

(a) First Iteration

We do not know y in terms of x yet, so we replace y by the constant value y_0 in the function to be integrated.

The result of the first iteration is thus given, at $x = 0.3$, by

$$y_1 = \int_{0}^{x} x \mathrm{d}x = \frac{x^2}{2} \simeq 0.0450$$

(b) Second Iteration
Now we use
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_1^2 = x + \frac{x^4}{4}.$$
Therefore,
$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4}\right)\mathrm{d}x,$$
which gives
$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20}.$$
The result of the second iteration is thus given by
$$y_2 = \frac{x^2}{2} + \frac{x^5}{20} \simeq 0.0451$$
at $x = 0.3$.

(c) Third Iteration
Now we use
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_2^2$$
$$= x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}.$$
Therefore,
$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\right)\mathrm{d}x,$$
which gives
$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.$$
The result of the third iteration is thus given by
$$y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \simeq 0.0451 \quad at \quad x = 0.3$$
Hence, $y = 0.0451$, correct to four decimal places, at $x = 0.3$.

2. If
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y}{x}$$
and $y = 2$ when $x = 1$, perform three iterations of Picard's method to estimate a value for y when $x = 1.2$. Work to four places of decimals throughout and state how accurate is the result of the third iteration.

Solution

(a) First Iteration

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $x_0 = 1$.

That is,

$$y - y_0 = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $y_0 = 2$.

Hence,

$$y - 2 = \int_{1}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x.$$

Replacing y by $y_0 = 2$ in the function being integrated, we have

$$y - 2 = \int_{1}^{x} \left(2 - \frac{2}{x}\right) \mathrm{d}x.$$

Therefore,

$$\begin{aligned} y &= 2 + [2x - 2\ln x]_1^x \\ &= 2 + 2x - 2\ln x - 2 + 2\ln 1 = 2(x - \ln x). \end{aligned}$$

The result of the first iteration is thus given by

$$y_1 = 2(x - \ln x) \simeq 2.0354,$$

when $x = 1.2$.

(b) Second Iteration

In this case we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_1}{x} = 2 - \frac{2(x - \ln x)}{x} = \frac{2\ln x}{x}.$$

Hence,

$$\int_{1}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{1}^{x} \frac{2lnx}{x}\mathrm{d}x.$$

That is,

$$y - 2 = \left[(\ln x)^2\right]_1^x = (\ln x)^2.$$

The result of the second iteration is thus given by

$$y_2 = 2 + (\ln x)^2 \simeq 2.0332,$$

when $x = 1.2$.

(c) Third Iteration

Finally, we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_2}{x} = 2 - \frac{2}{x} - \frac{(\ln x)^2}{x}.$$

Hence,

$$\int_1^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_1^x \left[2 - \frac{2}{x} - \frac{(lnx)^2}{x}\right] \mathrm{d}x.$$

That is,

$$y - 2 = \left[2x - 2\ln x - \frac{(\ln x)^3}{3}\right]_1^x$$

$$= 2x - 2\ln x - \frac{(\ln x)^3}{3} - 2.$$

The result of the third iteration is thus given by

$$y_3 = 2x - 2\ln x - \frac{(\ln x)^3}{3} \simeq 2.0293,$$

when $x = 1.2$.

The results of the last two iterations are identical when rounded off to two places of decimals, namely 2.03. Hence, the accuracy of the third iteration is two decimal place accuracy.

17.7.2 EXERCISES

1. Use Picard's method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + e^x$$

at $x = 1$, correct to two significant figures, given that $y = 0$ when $x = 0$.

2. Use Picard's method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + \frac{y}{2}$$

at $x = 0.5$, correct to two decimal places, given that $y = 1$ when $x = 0$.

3. Given the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - xy,$$
where $y(0) = 0$, use Picard's method to obtain y as a series of powers of x which will give two decimal place accuracy in the interval $0 \leq x \leq 1$.

What is the solution when $x = 1$?

17.7.3 ANSWERS TO EXERCISES

1.
$$y(1) \simeq 2.7$$

2.
$$y(0.5) \simeq 1.33$$

3.
$$y = x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \frac{x^9}{945} - \ldots\ldots$$

$$y(1) \simeq 0.72$$

"JUST THE MATHS"

UNIT NUMBER

17.8

NUMERICAL MATHEMATICS 8 (Numerical solution) of (ordinary differential equations (C))

by

A.J.Hobson

UNIT 17.8 - NUMERICAL MATHEMATICS 8

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (C)

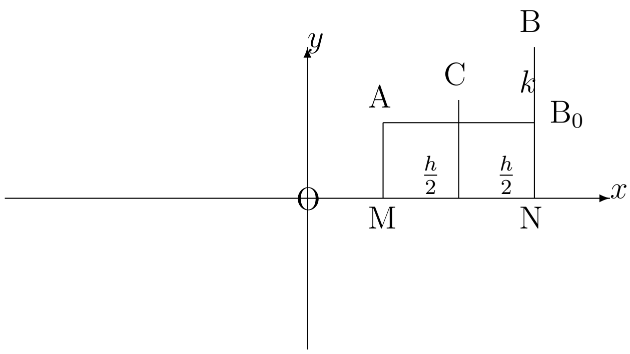
17.8.1 RUNGE'S METHOD

We solve, approximately, the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

subject to the condition that $y = y_0$ when $x = x_0$.

Consider the **graph** of the solution, passing through the two points, $\mathrm{A}(x_0, y_0)$ and $\mathrm{B}(x_0 + h, y_0 + k)$.



We can say that

$$\int_{x_0}^{x_0+h} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x_0+h} f(x, y)\mathrm{d}x.$$

That is,

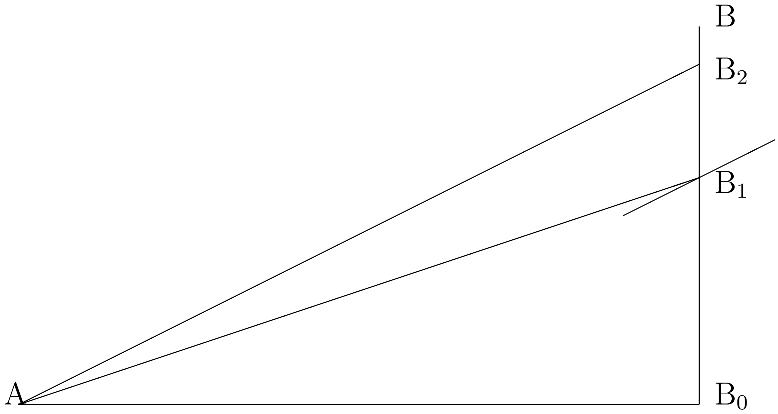
$$y_{\mathrm{B}} - y_{\mathrm{A}} = \int_{x_0}^{x_0+h} f(x, y)\mathrm{d}x.$$

Suppose that C is the intersection with the curve of the perpendicular bisector of MN. Then, by Simpson's Rule (See Unit 17.3),

$$\int_{x_0}^{x_0+h} f(x, y)\mathrm{d}x = \frac{h/2}{3}\left[f(\mathrm{A}) + \mathrm{f(B)} + 4\mathrm{f(C)}\right].$$

(i) The value of $f(\mathrm{A})$
This is already given, namely, $f(x_0, y_0)$.

(ii) The Value of $f(\mathrm{B})$



In the diagram, if the tangent at A meets $\mathrm{B_0B}$ in B_1, then the gradient at A is given by

$$\frac{\mathrm{B_1B_0}}{\mathrm{AB_0}} = f(x_0, y_0).$$

Therefore,

$$\mathrm{B_1B_0} = \mathrm{AB_0}f(x_0, y_0) = hf(x_0, y_0).$$

Calling this value k_1, as an initial approximation to k, we have

$$k_1 = hf(x_0, y_0).$$

As a rough approximation to the gradient of the solution curve passing through B, we now take the gradient of the solution curve passing through B_1. Its value is

$$f(x_0 + h, y_0 + k_1).$$

To find a better approximation, we assume that a straight line of gradient $f(x_0+h, y_0+k_1)$, drawn at A, meets $\mathrm{B_0B}$ in B_2, a point nearer to B than B_1.

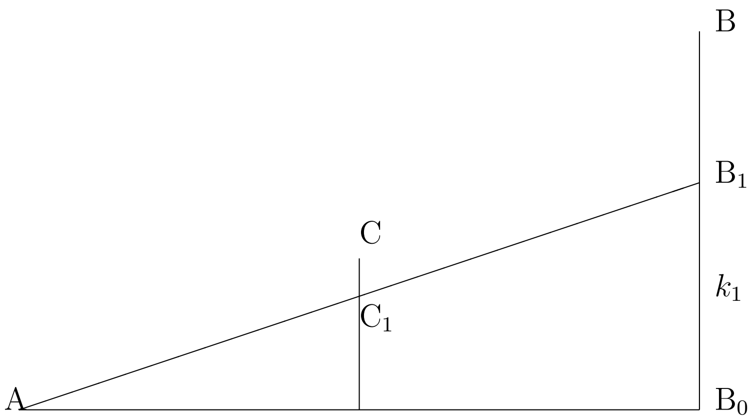
Letting $\mathrm{B_0B_2} = k_2$, we have

$$k_2 = hf(x_0 + h, y_0 + k_1).$$

The co-ordinates of B_2 are (x_0+h, y_0+k_2) and the gradient of the solution curve through B_2 is taken as a closer approximation than before to the gradient of the solution curve through B. Its value is

$$f(x_0+h, y_0+k_2).$$

(iii) The Value of f(C)



Let C_1 be the intersection of the ordinate through C and the tangent at A. Then C_1 is the point,

$$\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right),$$

and the gradient at C_1 of the solution curve through C_1 is

$$f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right).$$

We take this to be an approximation to the gradient at C for the arc, AB.

We saw earlier that

$$y_{\mathrm{B}}-y_{\mathrm{A}}=\int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x.$$

Therefore,

$$y_{\mathrm{B}}-y_{\mathrm{A}}=\frac{h}{6}[f(\mathrm{A})+\mathrm{f(B)}+4\mathrm{f(C)}].$$

That is,

$$y=y_0+\frac{h}{6}\left[f(x_0,y_0)+f(x_0+h,y_0+k_2)+4f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)\right].$$

PRACTICAL LAYOUT

If

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

and $y = y_0$ when $x = x_0$, then the value of y when $x = x_0 + h$ is determined by the following sequence of calculations:

1. $k_1 = hf(x_0, y_0)$.
2. $k_2 = hf(x_0 + h, y_0 + k_1)$.
3. $k_3 = hf(x_0 + h, y_0 + k_2)$.
4. $k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$.
5. $k = \frac{1}{6}(k_1 + k_3 + 4k_4)$.
6. $y \simeq y_0 + k$.

EXAMPLE

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 3y$$

at $x = 0.1$, given that $y = 1$ when $x = 0$.

Solution

We use $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

1. $k_1 = 0.1(5 - 3) = 0.2$
2. $k_2 = 0.1(5 - 3[1.2]) = 0.14$
3. $k_3 = 0.1(5 - 3[1.14]) = 0.158$
4. $k_4 = 0.1(5 - 3[1.1]) = 0.17$
5. $k = \frac{1}{6}(0.2 + 0.158 + 4[0.17]) = 0.173$
6. $y \simeq 1.173$ at $x = 0.1$

Note:

It can be shown that the error in the result is of the order h^5; that is, the error is equivalent to some constant multiplied by h^5.

17.8.2 EXERCISES

1. Use Runge's Method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2$$

at $x = 0.3$, given that $y = 0$ when $x = 0$.

Work to four places of decimals throughout.

2. Use Runge's Method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{y+x}$$

at $x = 1.1$, given that $y(1) = 1$.

Work to three places of decimals throughout.

3. Use Runge's Method with **successive** increments of $h = 0.1$ to find the solution at $x = 0.5$ of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y(0) = 1$. Work to four decimal places throughout.

Compare your results with those given by the exact solution

$$y = e^{\frac{1}{2}x^2}.$$

4. Use Runge's Method with $h = 0.2$ to determine the solution at $x = 1$ of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + e^{-x},$$

given that $y(0) = 0$. Work to four decimal places throughout.

17.8.3 ANSWERS TO EXERCISES

1.

$$y(0.3) \simeq 0.0454$$

2.

$$y(1.1) \simeq 1.049$$

3.

$$y(0.5) \simeq 1.3318$$

4.

$$y(1) \simeq 1.1752$$

"JUST THE MATHS"

UNIT NUMBER

18.1

STATISTICS 1
(The presentation of data)

by

A.J.Hobson

UNIT 18.1 - STATISTICS 1 - THE PRESENTATION OF DATA

18.1.1 INTRODUCTION

(i) The collection of numerical information often leads to large masses of data which, if they are to be understood, or presented effectively, must be summarised and analysed in some way. This is the purpose of the subject of **"Statistics"**.

(ii) The source from which a set of data is collected is called a **"population"**. For example, a population of 1000 ball-bearings could provide data relating to their diameters.

(iii) Statistical problems may be either **"descriptive problems"** (in which all the data is known and can be analysed) or **"inference problems"** (in which data collected from a **"sample"** population is used to infer properties of a larger population). For example, the annual pattern of rainfall over several years in a particular place could be used to estimate the rainfall pattern in other years.

(iv) The variables measured in a statistical problem may be either **"discrete"** (in which case they may take only certain values) or **"continuous"** (in which case they make take any values within the limits of the problem itself. For example, the number of students passing an examination from a particular class of students is a discrete variable; but the diameter of ball-bearings from a stock of 1000 is a continuous variable.

(v) Various methods are seen in the commercial presentation of data but, in this series of Units, we shall be concerned with just two methods, one of which is tabular and the other graphical.

18.1.2 THE TABULATION OF DATA

(a) Ungrouped Data
Suppose we have a collection of measurements given by numbers. Some may occur only once, while others may be repeated several times.

If we write down the numbers as they appear, the processing of them is likely to be cumbersome. This is known as **"ungrouped (or raw) data"**, as, for example, in the following table which shows rainfall figures (in inches), for a certain location, in specified months over a 90 year period:

TABLE 1 - Ungrouped (or Raw) Data

18.6	13.8	10.4	15.0	16.0	22.1	16.2	36.1	11.6	7.8
22.6	17.9	25.3	32.8	16.6	13.6	8.5	23.7	14.2	22.9
17.7	26.3	9.2	24.9	17.9	26.5	26.6	16.5	18.1	24.8
16.6	32.3	14.0	11.6	20.0	33.8	15.8	15.2	24.0	16.4
24.1	23.2	17.3	10.5	15.0	20.2	20.2	17.3	16.6	16.9
22.0	23.9	24.0	12.2	21.8	12.2	22.0	9.6	8.0	20.4
17.2	18.3	13.0	10.6	17.2	8.9	16.8	14.2	15.7	8.0
17.7	16.1	17.8	11.6	10.4	13.6	8.4	12.6	8.1	11.6
21.1	20.5	19.8	24.8	9.7	25.1	31.8	24.9	20.0	17.6

(b) Ranked Data
A slightly more convenient method of tabulating a collection of data would be to arrange them in rank order, so making it easier to see how many times each number appears. This is known as **"ranked data"**.

The next table shows the previous rainfall figures in this form

TABLE 2 - Ranked Data

7.8	8.0	8.0	8.1	8.4	8.5	8.9	9.2	9.6	9.7
10.4	10.4	10.5	10.6	11.6	11.6	11.6	11.6	12.2	12.2
12.6	13.0	13.6	13.6	13.8	14.0	14.2	14.2	15.0	15.0
15.2	15.7	15.8	16.0	16.1	16.2	16.4	16.5	16.6	16.6
16.6	16.8	16.9	17.2	17.2	17.3	17.3	17.6	17.7	17.7
17.8	17.9	17.9	18.1	18.3	18.6	19.8	20.0	20.0	20.2
20.2	20.4	20.5	21.1	21.8	22.0	22.0	22.1	22.6	22.9
23.2	23.7	23.9	24.0	24.0	24.1	24.8	24.8	24.9	24.9
25.1	25.3	26.3	26.5	26.6	31.8	32.3	32.8	33.8	36.1

(c) Frequency Distribution Tables
Thirdly, it is possible to save a little space by making a table in which each individual item of the ranked data is written down once only, but paired with the number of times it occurs. The data is then presented in the form of a **"frequency distribution table"**.

TABLE 3 - Frequency Distribution Table

Value	Frequency	Value	Frequency	Value	Frequency
7.8	1	15.8	1	21.1	1
8.0	2	16.0	1	21.8	1
8.1	1	16.1	1	22.0	2
8.4	1	16.2	1	22.1	1
8.5	1	16.4	1	22.6	1
8.9	1	16.5	1	22.9	1
9.2	1	16.6	3	23.2	1
9.6	1	16.8	1	23.7	1
9.7	1	16.9	1	23.9	1
10.4	2	17.2	2	24.0	2
10.5	1	17.3	2	24.1	1
10.6	1	17.6	1	24.8	2
11.6	4	17.7	2	24.9	2
12.2	2	17.8	1	25.1	1
12.6	1	17.9	2	25.3	1
13.0	1	18.1	1	26.3	1
13.6	2	18.3	1	26.5	1
13.8	1	18.6	1	26.6	1
14.0	1	19.8	1	31.8	1
14.2	2	20.0	2	32.3	1
15.0	2	20.2	2	32.8	1
15.2	1	20.4	1	33.8	1
15.7	1	20.5	1	36.1	1

(d) Grouped Frequency Distribution Tables
For about forty or more items in a set of numerical data, it usually most convenient to group them together into between 10 and 25 **"classes"** of values, each covering a specified range or **"class interval"** (for example, $7.5 - 10.5$, $10.5 - 13.5$, $13.5 - 16.5$,......).

Each item is counted every time it appears in order to obtain the **"class frequency"** and each class interval has the same **"class width"**.

Too few classes means that the data is over-summarised, while too many classes means that there is little advantage in summarising at all.

Here, we use the convention that the lower boundary of the class is included while the upper boundary is excluded.

Each item in a particular class is considered to be approximately equal to the "class mid-point"; that is, the average of the two **"class boundaries"**.

A **"grouped frequency distribution table"** normally has columns which show the class intervals, class mid-points, class frequencies, and **"cumulative frequencies"**, the last of these being a running total of the frequencies themselves. There may also be a column of **"tallied frequencies"**, if the table is being constructed from the raw data without having first arranged the values in rank order.

TABLE 4 - Grouped Frequency Distribution

<table>
<tr><th>Class Interval</th><th>Class Mid-point</th><th>Tallied Frequency</th><th>Frequency</th><th>Cumulative Frequency</th></tr>
<tr><td>7.5 – 10.5</td><td>9</td><td>////| ////| //</td><td>12</td><td>12</td></tr>
<tr><td>10.5 – 13.5</td><td>12</td><td>////| ////|</td><td>10</td><td>22</td></tr>
<tr><td>13.5 – 16.5</td><td>15</td><td>////| ////| ////|</td><td>15</td><td>37</td></tr>
<tr><td>16.5 – 19.5</td><td>18</td><td>////| ////| ////| ////</td><td>19</td><td>56</td></tr>
<tr><td>19.5 – 22.5</td><td>21</td><td>////| ////| //</td><td>12</td><td>68</td></tr>
<tr><td>22.5 – 25.5</td><td>24</td><td>////| ////| ////</td><td>14</td><td>82</td></tr>
<tr><td>25.5 – 28.5</td><td>27</td><td>///</td><td>3</td><td>85</td></tr>
<tr><td>28.5 – 31.5</td><td>30</td><td></td><td>0</td><td>85</td></tr>
<tr><td>31.5 – 34.5</td><td>33</td><td>////</td><td>4</td><td>89</td></tr>
<tr><td>34.5 – 37.5</td><td>36</td><td>/</td><td>1</td><td>90</td></tr>
</table>

Notes:

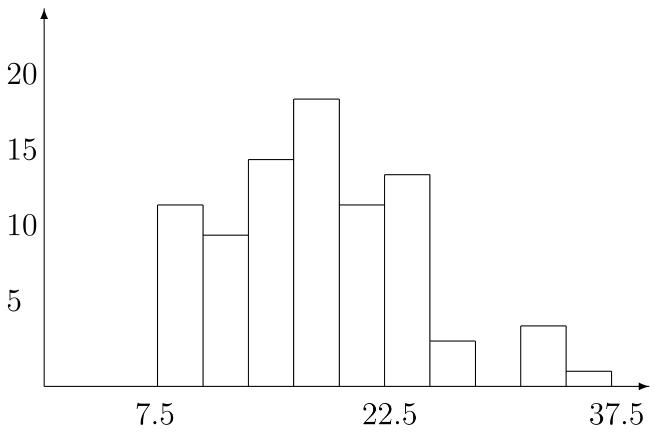
(i) The cumulative frequency shows, at a glance, how many items in the data are less than a specified value. In the above table, for example, 82 items are less than 25.5

(ii) It is sometimes more useful to use the ratio of the cumulative frequency to the total number of observations. This ratio is called the **"relative cumulative frequency"** and, in the above table, for example, the percentage of items in the data which are less than 25.5 is

$$\frac{82}{90} \times 100 \simeq 91\%$$

18.1.3 THE GRAPHICAL REPRESENTATION OF DATA

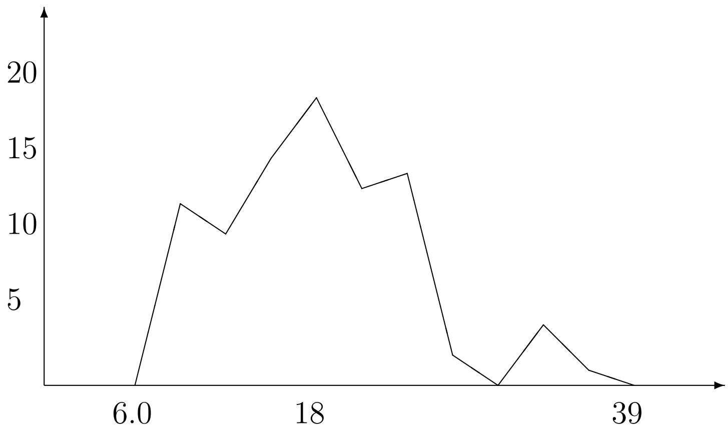
(a) The Histogram
A **"histogram"** is a diagram which is directly related to a grouped frequency distribution table and consists of a collection of rectangles whose height represents the class frequency (to some suitable scale) and whose breadth represents the class width.



The histogram shows, at a glance, not just the class intervals with the highest and lowest frequencies, but also how the frequencies are distributed.

In the case of examination results, for example, there is usually a group of high frequencies around the central class intervals and lower ones at the ends. Such an ideal situation would be called a **"Normal Distribution"**.

(b) The Frequency Polygon
Using the fact that each class interval may be represented, on average, by its class mid-point, we may plot the class mid-points against the class frequencies to obtain a display of single points. By joining up these points with straight line segments and including two extra class mid-points, we obtain a **"frequency polygon"**.



Notes:

(i) Although the frequency polygon officially plots only the class mid-points against their frequencies, it is sometimes convenient to read-off intermediate points in order to estimate additional data. For example, we might estimate that the value 11.0 occurred 11 times when, in fact, it did not occur at all.

We may use this technique only for continuous variables.

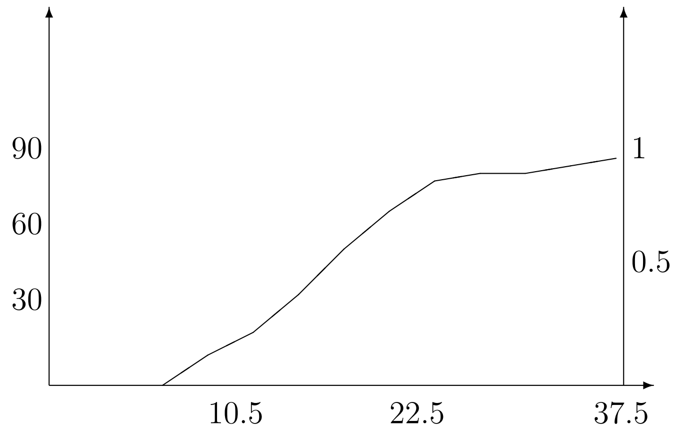
(ii) Frequency polygons are more useful than histograms if we wish to compare two or more frequency distributions. A clearer picture is obtained if we plot them on the same diagram.

(iii) If the class intervals are made smaller and smaller while, at the same time, the total number of items in the data is increased more and more, the points of the frequency polygon will be very close together. The smooth curve joining them is called the **"frequency curve"** and is of greater use for estimating intermediate values.

(c) The Cumulative Frequency Polygon (or Ogive)

The earlier use of the cumulative frequency to estimate the number (or proportion) of values less than a certain amount may be applied graphically by plotting the upper class-boundary against the cumulative frequency, then joining up the points plotted with straight line segments. The graph obtained is called the **"cumulative frequency polygon"** or **"ogive"**.

We may also use a second vertical axis at the right-hand end of the diagram showing the relative cumulative frequency. The range of this axis will always be 0 to 1.



18.1.4 EXERCISES

1. State whether the following variables are discrete or continuous:
   (a) The denominators of a set of 10,000 rational numbers;
   (b) The difference between a fixed integer and each one of any set of real numbers;
   (c) The volume of fruit squash in a sample of 200 bottles taken from different firms;
   (d) The minimum plug-gap setting specified by the makers of 30 different cars;
   (e) The points gained by 80 competitors in a skating championship.

2. The following marks were obtained in an examination taken by 100 students:

Marks	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50
No. of Students	2	3	7	7	8

Marks	50 – 55	55 – 60	60 – 65	65 – 70	70 – 75
No. of Students	25	18	12	10	8

   (a) Draw a histogram for this data;
   (b) Draw a cumulative frequency polygon;
   (c) Estimate the mark exceeded by the top 25% of the students;
   (d) Suggest a pass-mark if 15% of the students are to fail.

3. In a test of 35 glue-laminated beams, the following values of the "spring constant" in kilopounds per inch were found:

   SPRING CONSTANT $\times$ 100

6.72	6.77	6.82	6.70	6.78	6.70	6.62
6.75	6.66	6.66	6.64	6.76	6.73	6.80
6.72	6.76	6.76	6.68	6.66	6.62	6.72
6.76	6.70	6.78	6.76	6.67	6.70	6.72
6.74	6.81	6.79	6.78	6.66	6.76	6.72

   (a) Arrange this data in rank order and hence construct a frequency distribution table;

   (b) Construct a grouped frequency distribution table with class intervals of length 0.02 starting with $6.61 - 6.63$ and include columns to show the class mid-points and the cumulative frequency.

4. The following data shows the diameters of 25 ball-bearings in cms.

0.386	0.391	0.396	0.380	0.397
0.376	0.384	0.401	0.382	0.404
0.390	0.404	0.380	0.390	0.388
0.381	0.393	0.387	0.377	0.399
0.400	0.383	0.384	0.390	0.379

   Construct a grouped frequency distribution table with class intervals $0.375 - 0.380$, $0.380 - 0.385$ etc. and construct

   (a) the histogram;

   (b) the frequency polygon;

   (c) the cumulative frequency polygon (ogive).

18.1.5 SELECTED ANSWERS TO EXERCISES

1. (a) discrete

   (b) continuous

   (c) continuous

   (d) discrete

   (e) discrete

2. (c) 63% is the mark exceeded by the top 25% of students

   (d) 43% needs to be the pass-mark if 15% of the students are to fail.

"JUST THE MATHS"

UNIT NUMBER

18.2

STATISTICS 2
(Measures of central tendency)

by

A.J.Hobson

UNIT 18.2 - STATISTICS 2

MEASURES OF CENTRAL TENDENCY

18.2.1 INTRODUCTION

Having shown, in Unit 18.1, how statistical data may be presented in a clear and concise form, we shall now be concerned with the methods of analysing the data in order to obtain the maximum amount of information from it.

In the previous Unit, it was stated that statistical problems may be either "descriptive problems" (in which all the data is known and can be analysed) or "inference problems" (in which data collected from a "sample" population is used to infer properties of a larger population).

In both types of problem, it is useful to be able to measure some value around which all items in the data may be considered to cluster. This is called **"a measure of central tendency"**; and we find it by using several types of average value as follows:

18.2.2 THE ARITHMETIC MEAN (BY CODING)

To obtain the **"arithmetic mean"** of a finite collection of n numbers, we may simply add all the numbers together and then divide by n. This elementary rule applies even if some of the numbers occur more than once and even if some of the numbers are negative.

However, the purpose of this section is to introduce some short-cuts (called **"coding"**) in the calculation of the arithmetic mean of large collections of data. The methods will be illustrated by the following example, in which the number of items of data is not over-large:

EXAMPLE

The solid contents, x, of water (in parts per million) was measured in eleven samples and the following data was obtained:

4520	4490	4500	4500
4570	4540	4520	4590
4520	4570	4520	

Determine the arithmetic mean, $\overline{x}$, of the data.

Solution

(i) Direct Calculation

By adding together the eleven numbers, then dividing by 11, we obtain

$$\overline{x} = 49840 \div 11 \simeq 4530.91$$

(ii) Using Frequencies

We could first make a frequency table having a column of distinct values x_i, $(i = 1, 2, 3, ..., 11)$, a column of frequencies f_i, $(i = 1, 2, 3......, 11)$ and a column of corresponding values $f_i x_i$.

The arithmetic mean is then calculated from the formula

$$\overline{x} = \frac{1}{11} \sum_{i=1}^{11} f_i x_i.$$

In the present example, the table would be

x_i	f_i	$f_i x_i$
4490	1	4490
4500	2	9000
4520	4	18080
4540	1	4540
4570	2	9140
4590	1	4590
	Total	49840

The arithmetic mean is then $\overline{x} = 49840 \div 11 \simeq 4530.91$ as before.

(iii) Reduction by a constant

With such large data-values, as in the present example, it can be convenient to reduce all of the values by a constant, k, before calculating the arithmetic mean.

It is easy to show that, by adding the constant, k, to the arithmetic mean of the reduced data, we obtain the arithmetic mean of the original data.

Proof:

For n values, $x_1, x_2, x_3, ..., x_n$, the arithmetic mean is given by

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{n}.$$

If each value is reduced by a constant, k, the arithmetic mean of the reduced data is

$$\frac{(x_1 - k) + (x_2 - k) + (x_3 - k) + \ldots\ldots + (x_n - k)}{n} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{n} - \frac{nk}{n} = \overline{x} - k.$$

(iv) Division by a constant

In a similar way to the previous paragraph, each value in a collection of data could by divided by a constant, k, before calculating the arithmetic mean.

This time, we may show that the arithmetic mean of the original data is obtained on multiplying the arithmetic mean of the reduced data by k.

Proof:

$$\frac{\frac{x_1}{k} + \frac{x_2}{k} + \frac{x_3}{k} + \ldots\ldots + \frac{x_n}{k}}{n} = \frac{\overline{x}}{k}.$$

In order to summarise the shortcuts used in the present example, the following table shows a combination of the use of frequencies and of the two types of reduction made to the data:

x_i	$x_i - 4490$	$x'_i = (x_i - 4490) \div 10$	f_i	$f_i x'_i$
4490	0	0	1	0
4500	10	1	2	2
4520	30	3	4	12
4540	50	5	1	5
4570	80	8	2	16
4590	100	10	1	10
			Total	45

$$\text{The fictitious arithmetic mean, } \overline{x}' = \frac{45}{11} \simeq 4.0909$$

$$\text{The actual arithmetic mean, } \overline{x} \simeq (4.0909 \times 10) + 4490 \simeq 4530.91$$

(v) The approximate arithmetic mean for a grouped distribution

For a large number of items of data, we may (without losing too much accuracy) take all items within a class interval to be equal to the class mid-point.

A calculation similar to that in the previous paragraph may then be performed if we reduce each mid-point by the first mid-point and divide by the class width (or other convenient number).

EXAMPLE

Calculate, approximately, the arithmetic mean of the data in TABLE 4 on page 4 of Unit 18.1

Solution

Class Interval	Class Mid-point x_i	$x_i - 9$	$(x_i - 9) \div 3 = x_i'$	Frequency f_i	$f_i x_i'$
7.5 – 10.5	9	0	0	12	0
10.5 – 13.5	12	3	1	10	10
13.5 – 16.5	15	6	2	15	30
16.5 – 19.5	18	9	3	19	57
19.5 – 22.5	21	12	4	12	48
22.5 – 25.5	24	15	5	14	70
25.5 – 28.5	27	18	6	3	18
28.5 – 31.5	30	21	7	0	0
31.5 – 34.5	33	24	8	4	32
34.5 – 37.5	36	27	9	1	9
			Totals	90	274

$$\text{Fictitious arithmetic mean} \quad \overline{x}' = \frac{274}{90} \simeq 3.0444$$

$$\text{Actual arithmetic mean} = 3.044 \times 3 + 9 \simeq 18.13.$$

Notes:

(i) By direct calculation from TABLE 1 in Unit 18.1, it may be shown that the arithmetic mean is 17.86 correct to two places of decimals; and this indicates an error of about 1.5%.

(ii) The arithmetic mean is widely used where samples are taken of a larger population. It usually turns out that two samples of the same population have arithmetic means which are close in value.

18.2.3 THE MEDIAN

Collections of data often include one or more values which are widely out of character with the rest; and the arithmetic mean can be significantly affected by such extreme values.

For example, the values 8,12,13,15,21,23 have an arithmetic mean of $\frac{92}{6} \simeq 15.33$; but the values 5,12,13,15,21,36 have an arithmetic mean of $\frac{102}{6} \simeq 17.00$.

A second type of average, not so much affected, is defined as follows:

DEFINITION

The **"median"** of a collection of data is the middle value when the data is arranged in rank order. For an even number of values in the collection of data, the median is the arithmetic mean of the centre two values.

EXAMPLES

1. For both 8,12,13,15,21,23 and 5,12,13,15,21,36, the median is given by

$$\frac{13+15}{2} = 14.$$

2. For a grouped distribution, the problem is more complex since we no longer have access to the individual values from the data.

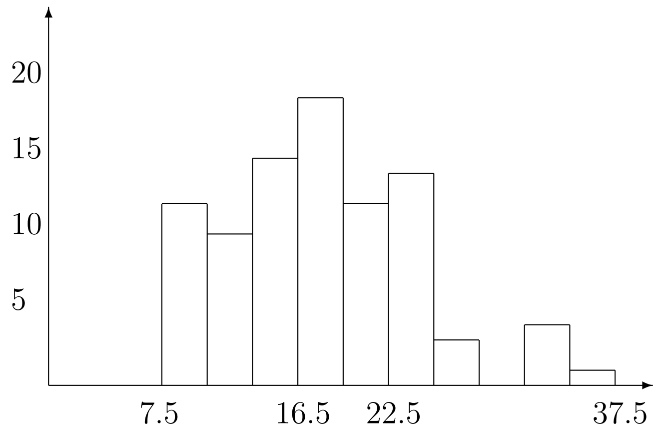
   However, the area of a histogram is directly proportional to the total number of values which it represents, since the base of all the rectangles are the same width and each height represents a frequency.

   We may thus take the median to be the value for which the vertical line through it divides the histogram into two equal areas.

   For non-symmetrical histograms, the median is often a better measure of central tendency than the arithmetic mean.

   Illustration

   Consider the histogram from Unit 18.1, representing rainfall figures over a 90 year period.



The total area of the histogram $= 90 \times 3 = 270$.

Half the area of the histogram $= 135$.

The area up as far as $16.5 = 3 \times 37 = 111$ while the area up as far as $19.5 = 3 \times 56 = 168$; hence the Median must lie between 16.5 and 19.5

The Median $= 16.5 + x$ where $18x = 135 - 111 = 24$ since 18 is the frequency of the class interval $16.5 - 19.5$

That is,

$$x = \frac{24}{18} = \frac{4}{3} \simeq 1.33,$$

giving a Median of 17.83

Notes:

(i) The median, in this case, is close to the arithmetic mean since the distribution is fairly symmetrical.

(ii) If a sequence of zero frequencies occurs, it may be necessary to take the arithmetic mean of two class mid-points, which are not consecutive to each other.

(iii) Another example of the advantage of median over arithmetic mean would be the average life of 100 electric lamps. To find the arithmetic mean, all 100 must be tested; but to find the median, the testing may stop after the 51st.

18.2.4 THE MODE

DEFINITIONS

1. For a collection of individual items of data, the **"mode"** is the value having the highest frequency.
2. In a grouped frequency distribution, the mid-point of the class interval with the highest frequency is called the **"crude mode"** and the class interval itself is called the **"modal class"**.

Note:
Like the median, the mode is not much affected by changes in the extreme values of the data. However, some distributions may have several different modes, which is a disadvantage of this measure of central tendency.

EXAMPLE

For the histogram discussed earlier, the mode is 18.0; but if the class interval, 22.5 – 25.5, had 5 more members, then 24.0 would be a mode as well.

18.2.5 QUANTILES

To conclude this Unit, we shall define three more standard measurements which, in fact, extend the idea of a median; and we may recall that a median divides a collection of values in such a way that half of them fall on either side of it.

Collectively, these three new measurements are called **"quantiles"** but may be considered separately by their own names as follows:

(a) Quartiles

These are the three numbers dividing a ranked collection of values (or the area of a histogram) into 4 equal parts.

(b) Deciles

These are the nine numbers dividing a ranked collection of values (or the area of a histogram) into 10 equal parts.

(c) Percentiles

These are the ninety nine numbers dividing a ranked collection of values (or the area of a histogram) into 100 equal parts.

Note:
For collections of individual values, quartiles may need to be calculated as the arithmetic mean of two consecutive values.

EXAMPLES

1. (a) The 25th percentile = The 1st quartile.
   (b) The 5th Decile = The median.
   (c) The 85th Percentile = the point at which 85% of the values fall below it and 15% above it.
2. For the collection of values

$$5, 12, 13, 19, 25, 26, 30, 33,$$

   the quartiles are 12.5, 22 and 28.

3. For the collection of values

$$5, 12, 13, 19, 25, 26, 30,$$

   the quartiles are 12.5, 19 and 25.5

18.2.6 EXERCISES

1. The arithmetic mean of 75 observations is 52.6 and the arithmetic mean of 25 similar observations is 48.4; determine the Arithmetic Mean of all 100 observations.
2. Of 500 students, whose mean height is 67.8 inches, 150 are women. If the mean height of 150 women is 62.0 inches, what is the mean height of the men ?
3. By coding the following collection of data, determine the arithmetic mean correct to three places of decimals:

$$1.847, 1.843, 1.842, 1.847, 1.848, 1.841, 1.845$$

4. Using a histogram of the frequency distribution shown, determine
   (a) the arithmetic mean;
   (b) the median class;
   (c) the median;

(d) the modal class;
(e) the crude mode.

class interval	$15-25$	$25-35$	$35-45$	$45-55$	$55-65$	$65-75$
Frequency	4	11	19	14	0	2

5. The number of a certain component issued, per day, from stock over a 40 day period is given as follows:

83	80	91	81	88	82	87	97	83	99
75	85	72	92	84	90	87	78	93	98
86	80	93	86	88	83	82	101	89	82
85	95	80	89	84	92	76	81	103	94

Using class intervals $70-75$, $75-80$, $80-85$ etc., draw up a frequency distribution table and construct a histogram.

From the histogram, determine the median and the 7th Decile.

18.2.7 ANSWERS TO EXERCISES

1.
$$51.55$$
2.
$$70.3 \text{ inches.}$$
3.
$$1.845$$
4. (a)
$$40.20$$
(b)
$$35-45.$$
(c)
$$40.26$$
(d)
$$35-45.$$
(e)
$$40.$$
5.
$$\text{Median} = 86.5, \quad 7\text{th} \;\; \text{Decile} = 91.25$$

1997

"JUST THE MATHS"

UNIT NUMBER

18.3

STATISTICS 3
(Measures of dispersion (or scatter))

by

A.J.Hobson

UNIT 18.3 - STATISTICS 3

MEASURES OF DISPERSION (OR SCATTER)

18.3.1 INTRODUCTION

Averages typify a whole collection of values, but they give little information about how the values are distributed within the whole collection.

For example, 99.9, 100.0, 100.1 is a collection which has an arithmetic mean of 100.0 and so is 99.0,100.0,101.0; but the second collection is more widely dispersed than the first.

It is the purpose of this Unit to examine two types of quantity which typify the distance of all the values in a collection from their arithmetic mean. They are known as measures of dispersion (or scatter) and the smaller these quantities are, the more clustered are the values around the arithmetic mean.

18.3.2 THE MEAN DEVIATION

If the n values x_1, x_2, x_3, ...,x_n have an arithmetic mean of $\overline{x}$, then $x_1 - \overline{x}$, $x_2 - \overline{x}$, $x_3 - \overline{x}$,......,$x_n - \overline{x}$ are called the **"deviations"** of x_1, x_2, x_3,......, x_n from the arithmetic mean.

Note:
The deviations add up to zero since

$$\sum_{i=1}^{n} (x_i - \overline{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \overline{x} = n\overline{x} - n\overline{x} = 0$$

DEFINITI0N
The **"mean deviation"** (*or, more accurately, the "mean absolute deviation*) is defined by the formula

$$\text{M.D.} = \frac{1}{n}\sum_{i=1}^{n} |x_i - \overline{x}|$$