38. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

39. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, if it has constant density.

40. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leqslant z \leqslant 4$, if its density function is $\rho(x, y, z) = 10 - z$.

41. (a) Give an integral expression for the moment of inertia I_z about the z-axis of a thin sheet in the shape of a surface S if the density function is ρ.
(b) Find the moment of inertia about the z-axis of the funnel in Exercise 40.

42. Let S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If S has constant density k, find (a) the center of mass and (b) the moment of inertia about the z-axis.

43. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\,\mathbf{j} + x^2\,\mathbf{k}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leqslant z \leqslant 1$.

44. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geqslant 0$.

45. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leqslant a^2$, $z \geqslant 0$, if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$$

46. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

47. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leqslant x \leqslant 4$.

48. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

49. Let $\mathbf{F}$ be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\,\mathbf{r}\,|^3$ for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show that the flux of $\mathbf{F}$ across a sphere S with center the origin is independent of the radius of S.

16.8 Stokes' Theorem

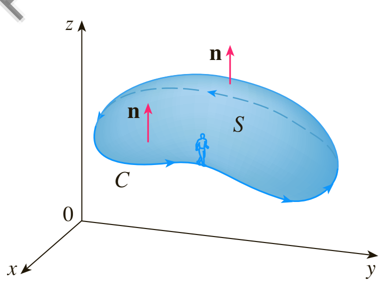


FIGURE 1

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 1 shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the **positive orientation of the boundary curve C** shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \quad \text{and} \quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

George Stokes

Stokes' Theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903). Stokes was a professor at Cambridge University (in fact he held the same position as Newton, Lucasian Professor of Mathematics) and was especially noted for his studies of fluid flow and light. What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin). Stokes learned of this theorem in a letter from Thomson in 1850 and asked students to prove it on an examination at Cambridge University in 1854. We don't know if any of those students was able to do so.

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral over S of the normal component of the curl of $\mathbf{F}$.

The positively oriented boundary curve of the oriented surface S is often written as ∂S, so Stokes' Theorem can be expressed as

$$\boxed{1} \qquad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. As before, there is an integral involving derivatives on the left side of Equation 1 (recall that curl $\mathbf{F}$ is a sort of derivative of $\mathbf{F}$) and the right side involves the values of $\mathbf{F}$ only on the *boundary* of S.

In fact, in the special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is $\mathbf{k}$, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$$

This is precisely the vector form of Green's Theorem given in Equation 16.5.12. Thus we see that Green's Theorem is really a special case of Stokes' Theorem.

Although Stokes' Theorem is too difficult for us to prove in its full generality, we can give a proof when S is a graph and $\mathbf{F}$, S, and C are well behaved.

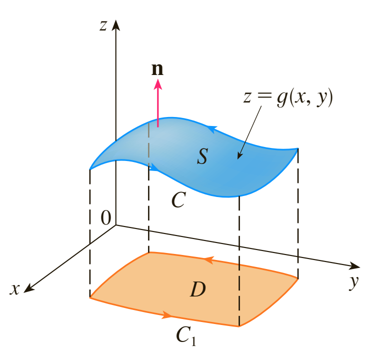


FIGURE 2

PROOF OF A SPECIAL CASE OF STOKES' THEOREM We assume that the equation of S is $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. If the orientation of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1. (See Figure 2.) We are also given that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, where the partial derivatives of P, Q, and R are continuous.

Since S is a graph of a function, we can apply Formula 16.7.10 with $\mathbf{F}$ replaced by curl $\mathbf{F}$. The result is

$$\boxed{2} \qquad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dA$$

where the partial derivatives of P, Q, and R are evaluated at $(x, y, g(x, y))$. If

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

is a parametric representation of C_1, then a parametric representation of C is

$$x = x(t) \qquad y = y(t) \qquad z = g(x(t), y(t)) \qquad a \leqslant t \leqslant b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt \\
&= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right] dt \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x} \right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y} \right)\frac{dy}{dt} \right] dt \\
&= \int_{C_1} \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x} \right) \right] dA
\end{aligned}$$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that P, Q, and R are functions of x, y, and z and that z is itself a function of x and y, we get

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \Bigg[&\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x\,\partial y} \right) \\
&- \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y\,\partial x} \right) \Bigg] dA
\end{aligned}$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$$ ■

EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

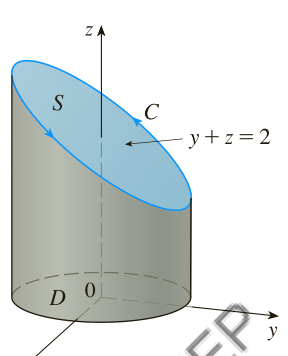


FIGURE 3

SOLUTION The curve C (an ellipse) is shown in Figure 3. Although $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

$$\text{curl}\, \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\,\mathbf{k}$$

Although there are many surfaces with boundary C, the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, then C has the induced positive orientation. The projection D of S onto the xy-plane is the disk $x^2 + y^2 \leqslant 1$ and so using Equation 16.7.10 with $z = g(x, y) = 2 - y$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin\theta \right]_0^1 d\theta = \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3} \sin\theta \right) d\theta$$

$$= \tfrac{1}{2}(2\pi) + 0 = \pi$$

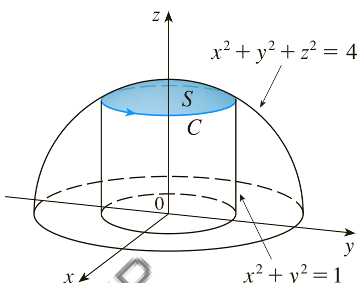


FIGURE 4

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See Figure 4.)

SOLUTION To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$ (since $z > 0$). Thus C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$. A vector equation of C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \qquad 0 \leqslant t \leqslant 2\pi$$

so

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Also, we have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{3}\cos t\,\mathbf{i} + \sqrt{3}\sin t\,\mathbf{j} + \cos t \sin t\,\mathbf{k}$$

Therefore, by Stokes' Theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t \right) dt$$

$$= \sqrt{3} \int_0^{2\pi} 0\, dt = 0$$

Note that in Example 2 we computed a surface integral simply by knowing the values of $\mathbf{F}$ on the boundary curve C. This means that if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral!

In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

3
$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

We now use Stokes' Theorem to throw some light on the meaning of the curl vector. Suppose that C is an oriented closed curve and $\mathbf{v}$ represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T}\, ds$$

and recall that $\mathbf{v} \cdot \mathbf{T}$ is the component of $\mathbf{v}$ in the direction of the unit tangent vector $\mathbf{T}$. This means that the closer the direction of $\mathbf{v}$ is to the direction of $\mathbf{T}$, the larger the value of $\mathbf{v} \cdot \mathbf{T}$. Thus $\int_C \mathbf{v} \cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around C and is called the **circulation** of $\mathbf{v}$ around C. (See Figure 5.)

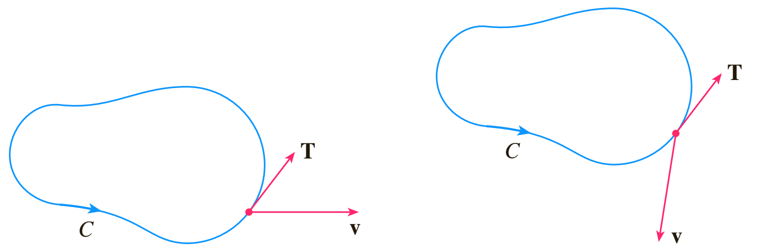


FIGURE 5

(a) $\int_C \mathbf{v} \cdot d\mathbf{r} > 0$, positive circulation

(b) $\int_C \mathbf{v} \cdot d\mathbf{r} < 0$, negative circulation

Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a and center P_0. Then $(\text{curl } \mathbf{F})(P) \approx (\text{curl } \mathbf{F})(P_0)$ for all points P on S_a because curl $\mathbf{F}$ is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a:

$$\int_{C_a} \mathbf{v} \cdot d\mathbf{r} = \iint_{S_a} \text{curl } \mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \text{curl } \mathbf{v} \cdot \mathbf{n}\, dS$$

$$\approx \iint_{S_a} \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0)\, dS = \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \pi a^2$$

Imagine a tiny paddle wheel placed in the fluid at a point P, as in Figure 6; the paddle wheel rotates fastest when its axis is parallel to curl $\mathbf{v}$.

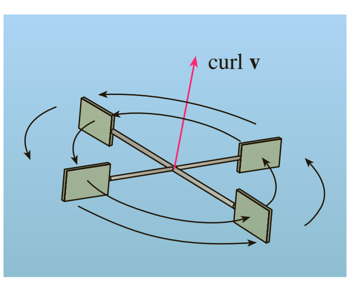


FIGURE 6

This approximation becomes better as $a \to 0$ and we have

4
$$\text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}$$

Equation 4 gives the relationship between the curl and the circulation. It shows that curl $\mathbf{v} \cdot \mathbf{n}$ is a measure of the rotating effect of the fluid about the axis $\mathbf{n}$. The curling effect is greatest about the axis parallel to curl $\mathbf{v}$.

Finally, we mention that Stokes' Theorem can be used to prove Theorem 16.5.4 (which states that if curl $\mathbf{F} = \mathbf{0}$ on all of $\mathbb{R}^3$, then $\mathbf{F}$ is conservative). From our previous work (Theorems 16.3.3 and 16.3.4), we know that $\mathbf{F}$ is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C. Given C, suppose we can find an orientable surface S whose boundary is C. (This can be done, but the proof requires advanced techniques.) Then Stokes' Theorem gives

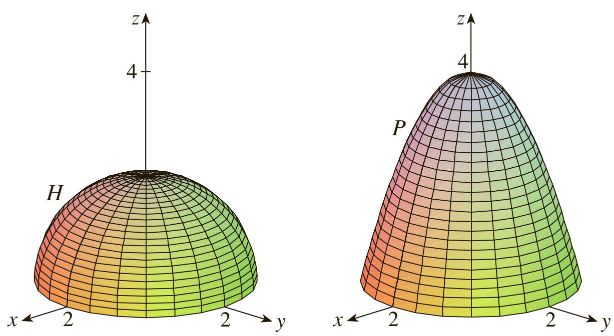
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

16.8 EXERCISES

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



2–6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = x^2 \sin z\, \mathbf{i} + y^2\, \mathbf{j} + xy\, \mathbf{k}$,
S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward

3. $\mathbf{F}(x, y, z) = ze^y\, \mathbf{i} + x \cos y\, \mathbf{j} + xz \sin y\, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geqslant 0$, oriented in the direction of the positive y-axis

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2)\, \mathbf{i} + x^2y\, \mathbf{j} + x^2z^2\, \mathbf{k}$,
S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leqslant x \leqslant 2$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz\, \mathbf{i} + xy\, \mathbf{j} + x^2yz\, \mathbf{k}$,
S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward

6. $\mathbf{F}(x, y, z) = e^{xy}\, \mathbf{i} + e^{xz}\, \mathbf{j} + x^2z\, \mathbf{k}$,
S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis

7–10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

7. $\mathbf{F}(x, y, z) = (x + y^2)\, \mathbf{i} + (y + z^2)\, \mathbf{j} + (z + x^2)\, \mathbf{k}$,
C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

8. $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\, \mathbf{j} + \left(xy - \sqrt{z}\right) \mathbf{k}$,
C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant

9. $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$, C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

10. $\mathbf{F}(x, y, z) = 2y\, \mathbf{i} + xz\, \mathbf{j} + (x + y)\, \mathbf{k}$, C is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$

11. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2z\, \mathbf{i} + xy^2\, \mathbf{j} + z^2\, \mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

(b) Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

12. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2y\, \mathbf{i} + \frac{1}{3}x^3\, \mathbf{j} + xy\, \mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

13–15 Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

13. $\mathbf{F}(x, y, z) = -y\, \mathbf{i} + x\, \mathbf{j} - 2\, \mathbf{k}$,
S is the cone $z^2 = x^2 + y^2$, $0 \leqslant z \leqslant 4$, oriented downward

14. $\mathbf{F}(x, y, z) = -2yz\, \mathbf{i} + y\, \mathbf{j} + 3x\, \mathbf{k}$,
S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward

15. $\mathbf{F}(x, y, z) = y\, \mathbf{i} + z\, \mathbf{j} + x\, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geqslant 0$, oriented in the direction of the positive y-axis

16. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\, dx - 2x\, dy + 3y\, dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

17. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\, \mathbf{i} + 2xy\, \mathbf{j} + 4y^2\, \mathbf{k}$$

Find the work done.

18. Evaluate

$$\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leqslant t \leqslant 2\pi$. [*Hint:* Observe that C lies on the surface $z = 2xy$.]

19. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

20. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 24 and 26 in Section 16.5 to show the following.

(a) $\int_C (f\,\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$

(b) $\int_C (f\,\nabla f) \cdot d\mathbf{r} = 0$

(c) $\int_C (f\,\nabla g + g\nabla f) \cdot d\mathbf{r} = 0$

WRITING PROJECT THREE MEN AND TWO THEOREMS

The photograph shows a stained-glass window at Cambridge University in honor of George Green.

Courtesy of the Masters and Fellows of Gonville and Caius College, Cambridge University, England

Although two of the most important theorems in vector calculus are named after George Green and George Stokes, a third man, William Thomson (also known as Lord Kelvin), played a large role in the formulation, dissemination, and application of both of these results. All three men were interested in how the two theorems could help to explain and predict physical phenomena in electricity and magnetism and fluid flow. The basic facts of the story are given in the margin notes on pages 1137 and 1175.

Write a report on the historical origins of Green's Theorem and Stokes' Theorem. Explain the similarities and relationship between the theorems. Discuss the roles that Green, Thomson, and Stokes played in discovering these theorems and making them widely known. Show how both theorems arose from the investigation of electricity and magnetism and were later used to study a variety of physical problems.

The dictionary edited by Gillispie [2] is a good source for both biographical and scientific information. The book by Hutchinson [5] gives an account of Stokes' life and the book by Thompson [8] is a biography of Lord Kelvin. The articles by Grattan-Guinness [3] and Gray [4] and the book by Cannell [1] give background on the extraordinary life and works of Green. Additional historical and mathematical information is found in the books by Katz [6] and Kline [7].

1. D. M. Cannell, *George Green, Mathematician and Physicist 1793–1841: The Background to His Life and Work* (Philadelphia: Society for Industrial and Applied Mathematics, 2001).

2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Green by P. J. Wallis in Volume XV and the articles on Thomson by Jed Buchwald and on Stokes by E. M. Parkinson in Volume XIII.

3. I. Grattan-Guinness, "Why Did George Green Write his Essay of 1828 on Electricity and Magnetism?" *Amer. Math. Monthly,* Vol. 102 (1995), pp. 387–96.

4. J. Gray, "There Was a Jolly Miller." *The New Scientist,* Vol. 139 (1993), pp. 24–27.

5. G. E. Hutchinson, *The Enchanted Voyage and Other Studies* (Westport, CT: Greenwood Press, 1978).

6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 678–80.

7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 683–85.

8. Sylvanus P. Thompson, *The Life of Lord Kelvin* (New York: Chelsea, 1976).

16.9 The Divergence Theorem

In Section 16.5 we rewrote Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

where C is the positively oriented boundary curve of the plane region D. If we were seeking to extend this theorem to vector fields on $\mathbb{R}^3$, we might make the guess that

1
$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\, dV$$

where S is the boundary surface of the solid region E. It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem. Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function (div $\mathbf{F}$ in this case) over a region to the integral of the original function $\mathbf{F}$ over the boundary of the region.

At this stage you may wish to review the various types of regions over which we were able to evaluate triple integrals in Section 15.6. We state and prove the Divergence Theorem for regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.) The boundary of E is a closed surface, and we use the convention, introduced in Section 16.7, that the positive orientation is outward; that is, the unit normal vector $\mathbf{n}$ is directed outward from E.

The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801–1862), who published this result in 1826.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Thus the Divergence Theorem states that, under the given conditions, the flux of $\mathbf{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E.

PROOF Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV$$

If $\mathbf{n}$ is the unit outward normal of S, then the surface integral on the left side of the Divergence Theorem is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three equations:

$$\boxed{2} \qquad \iint_S P\,\mathbf{i}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial P}{\partial x}\,dV$$

$$\boxed{3} \qquad \iint_S Q\,\mathbf{j}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial Q}{\partial y}\,dV$$

$$\boxed{4} \qquad \iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

To prove Equation 4 we use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. By Equation 15.6.6, we have

$$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x, y, z)\,dz\right] dA$$

and therefore, by the Fundamental Theorem of Calculus,

$$\boxed{5} \qquad \iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \big[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\big]\,dA$$

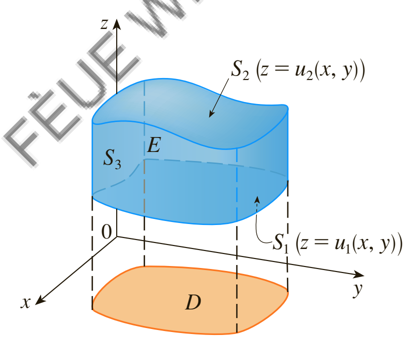


FIGURE 1

The boundary surface S consists of three pieces: the bottom surface S_1, the top surface S_2, and possibly a vertical surface S_3, which lies above the boundary curve of D. (See Figure 1. It might happen that S_3 doesn't appear, as in the case of a sphere.) Notice that on S_3 we have $\mathbf{k}\cdot\mathbf{n} = 0$, because $\mathbf{k}$ is vertical and $\mathbf{n}$ is horizontal, and so

$$\iint_{S_3} R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_{S_3} 0\,dS = 0$$

Thus, regardless of whether there is a vertical surface, we can write

$$\boxed{6} \qquad \iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS + \iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\mathbf{n}$ points upward, so from Equation 16.7.10 (with $\mathbf{F}$ replaced by $R\,\mathbf{k}$) we have

$$\iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D R(x, y, u_2(x, y))\,dA$$

On S_1 we have $z = u_1(x, y)$, but here the outward normal $\mathbf{n}$ points downward, so we multiply by -1:

$$\iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS = -\iint_D R(x, y, u_1(x, y))\,dA$$

Therefore Equation 6 gives

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D \big[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\big]\,dA$$

Comparison with Equation 5 shows that

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

Notice that the method of proof of the Divergence Theorem is very similar to that of Green's Theorem.

Equations 2 and 3 are proved in a similar manner using the expressions for E as a type 2 or type 3 region, respectively. ■

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION First we compute the divergence of $\mathbf{F}$:

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leqslant 1$. Thus the Divergence Theorem gives the flux as

The solution in Example 1 should be compared with the solution in Example 16.7.4.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_B \operatorname{div}\mathbf{F}\,dV = \iiint_B 1\,dV = V(B) = \tfrac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$ ■

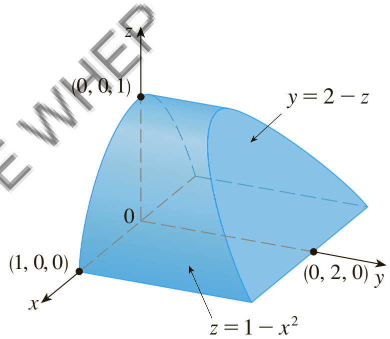


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)

SOLUTION It would be extremely difficult to evaluate the given surface integral directly. (We would have to evaluate four surface integrals corresponding to the four pieces of S.) Furthermore, the divergence of $\mathbf{F}$ is much less complicated than $\mathbf{F}$ itself:

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y$$

Therefore we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as a type 3 region:

$$E = \left\{(x, y, z) \mid -1 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2,\ 0 \leqslant y \leqslant 2 - z\right\}$$

Then we have

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 3y\,dV \\
&= 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y\,dy\,dz\,dx = 3\int_{-1}^{1}\int_0^{1-x^2} \frac{(2-z)^2}{2}\,dz\,dx \\
&= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2-z)^3}{3}\right]_0^{1-x^2} dx = -\tfrac{1}{2}\int_{-1}^{1}\left[(x^2+1)^3 - 8\right]dx \\
&= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7)\,dx = \frac{184}{35}
\end{aligned}$$ ■

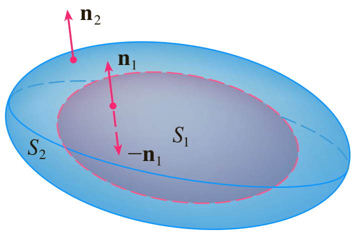


FIGURE 3

Although we have proved the Divergence Theorem only for simple solid regions, it can be proved for regions that are finite unions of simple solid regions. (The procedure is similar to the one we used in Section 16.4 to extend Green's Theorem.)

For example, let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2. Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2. (See Figure 3.) Applying the Divergence Theorem to S, we get

$$\boxed{7} \qquad \iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS$$

$$= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

EXAMPLE 3 In Example 16.1.5 we considered the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z\rangle$ is a position vector. Use the Divergence Theorem to show that the electric flux of $\mathbf{E}$ through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q$$

SOLUTION The difficulty is that we don't have an explicit equation for S_2 because it is *any* closed surface enclosing the origin. The simplest such surface would be a sphere, so we let S_1 be a small sphere with radius a and center the origin. You can verify that $\operatorname{div} \mathbf{E} = 0$. (See Exercise 23.) Therefore Equation 7 gives

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iiint_E \operatorname{div} \mathbf{E}\, dV = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS$$

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at $\mathbf{x}$ is $\mathbf{x}/|\mathbf{x}|$. Therefore

$$\mathbf{E} \cdot \mathbf{n} = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \frac{\varepsilon Q}{|\mathbf{x}|^4}\mathbf{x} \cdot \mathbf{x} = \frac{\varepsilon Q}{|\mathbf{x}|^2} = \frac{\varepsilon Q}{a^2}$$

since the equation of S_1 is $|\mathbf{x}| = a$. Thus we have

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{\varepsilon Q}{a^2}\iint_{S_1} dS = \frac{\varepsilon Q}{a^2}A(S_1) = \frac{\varepsilon Q}{a^2}4\pi a^2 = 4\pi\varepsilon Q$$

This shows that the electric flux of $\mathbf{E}$ is $4\pi\varepsilon Q$ through *any* closed surface S_2 that contains the origin. [This is a special case of Gauss's Law (Equation 16.7.11) for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/(4\pi\varepsilon_0)$.] ■

Another application of the Divergence Theorem occurs in fluid flow. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ. Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area. If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then $\operatorname{div} \mathbf{F}(P) \approx \operatorname{div} \mathbf{F}(P_0)$ for all points P in B_a since $\operatorname{div} \mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F}\, dV \approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0)\, dV = \operatorname{div} \mathbf{F}(P_0) V(B_a)$$

This approximation becomes better as $a \to 0$ and suggests that

$$\boxed{8} \qquad \operatorname{div} \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

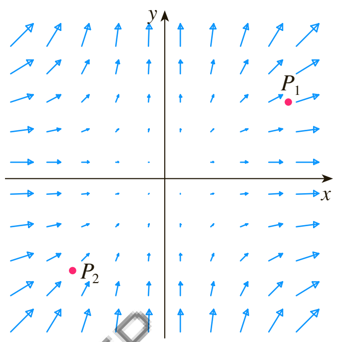


FIGURE 4
The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

Equation 8 says that $\operatorname{div} \mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence*.) If $\operatorname{div} \mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**. If $\operatorname{div} \mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1. Thus the net flow is outward near P_1, so $\operatorname{div} \mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so $\operatorname{div} \mathbf{F}(P_2) < 0$ and P_2 is a sink. We can use the formula for $\mathbf{F}$ to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have $\operatorname{div} \mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

16.9 EXERCISES

1–4 Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

1. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

2. $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2yz\,\mathbf{j} + 4z^2\,\mathbf{k}$,
E is the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

3. $\mathbf{F}(x, y, z) = \langle z, y, x \rangle$,
E is the solid ball $x^2 + y^2 + z^2 \leqslant 16$

4. $\mathbf{F}(x, y, z) = \langle x^2, -y, z \rangle$,
E is the solid cylinder $y^2 + z^2 \leqslant 9$, $0 \leqslant x \leqslant 2$

5–15 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

5. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$,
S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

6. $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$,
S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where a, b, and c are positive numbers

7. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

8. $\mathbf{F}(x, y, z) = (x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + (z^3 + x^3)\,\mathbf{k}$,
S is the sphere with center the origin and radius 2

9. $\mathbf{F}(x, y, z) = xe^y\,\mathbf{i} + (z - e^y)\,\mathbf{j} - xy\,\mathbf{k}$,
S is the ellipsoid $x^2 + 2y^2 + 3z^2 = 4$

10. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k}$,
S is the surface of the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, and c are positive numbers

11. $\mathbf{F}(x, y, z) = (2x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + 3y^2z\,\mathbf{k}$,
S is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane

12. $\mathbf{F}(x, y, z) = (xy + 2xz)\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j} + (xy - z^2)\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = y - 2$ and $z = 0$

13. $\mathbf{F} = |\mathbf{r}|\,\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leqslant 1$ in the xy-plane

14. $\mathbf{F} = |\mathbf{r}|^2\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S is the sphere with radius R and center the origin

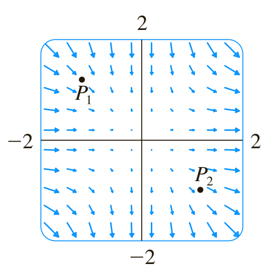
CAS **15.** $\mathbf{F}(x, y, z) = e^y \tan z\,\mathbf{i} + y\sqrt{3 - x^2}\,\mathbf{j} + x \sin y\,\mathbf{k}$,
S is the surface of the solid that lies above the xy-plane and below the surface $z = 2 - x^4 - y^4$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$

CAS **16.** Use a computer algebra system to plot the vector field $\mathbf{F}(x, y, z) = \sin x \cos^2 y\,\mathbf{i} + \sin^3 y \cos^4 z\,\mathbf{j} + \sin^5 z \cos^6 x\,\mathbf{k}$ in the cube cut from the first octant by the planes $x = \pi/2$, $y = \pi/2$, and $z = \pi/2$. Then compute the flux across the surface of the cube.

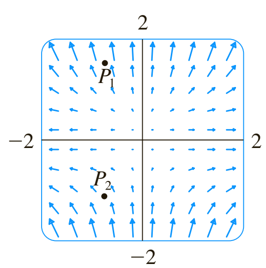
17. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = z^2 x\,\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2 z + y^2)\,\mathbf{k}$ and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. [*Hint:* Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leqslant 1$, oriented downward, and $S_2 = S \cup S_1$.]

18. Let $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2)\,\mathbf{i} + z^3 \ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

19. A vector field $\mathbf{F}$ is shown. Use the interpretation of divergence derived in this section to determine whether $\operatorname{div} \mathbf{F}$ is positive or negative at P_1 and at P_2.



20. (a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the figure? Give an explanation based solely on the picture.
(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).



CAS **21–22** Plot the vector field and guess where $\operatorname{div} \mathbf{F} > 0$ and where $\operatorname{div} \mathbf{F} < 0$. Then calculate $\operatorname{div} \mathbf{F}$ to check your guess.

21. $\mathbf{F}(x, y) = \langle xy, x + y^2 \rangle$ **22.** $\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$

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23. Verify that $\operatorname{div} \mathbf{E} = 0$ for the electric field $\mathbf{E}(\mathbf{x}) = \dfrac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$.

24. Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\, dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

25–30 Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

25. $\displaystyle\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = 0$, where $\mathbf{a}$ is a constant vector

26. $\displaystyle V(E) = \frac{1}{3}\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

27. $\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ **28.** $\displaystyle\iint_S D_{\mathbf{n}} f\, dS = \iiint_E \nabla^2 f\, dV$

29. $\displaystyle\iint_S (f\nabla g) \cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$

30. $\displaystyle\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\, dV$

---

31. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\, dS = \iiint_E \nabla f\, dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [*Hint:* Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.]

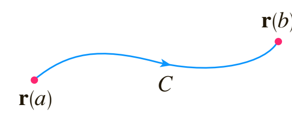
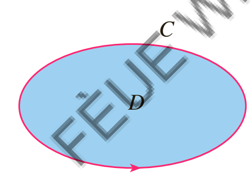
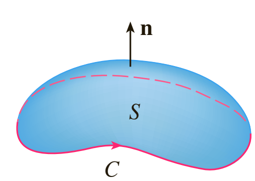
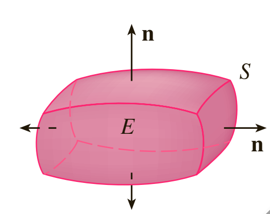
32. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid, and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity (see Section 8.3). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n}\, dS$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 31 to show that $\mathbf{F} = -W\mathbf{k}$, where W is the weight of the liquid displaced by the solid. (Note that $\mathbf{F}$ is directed upward because z is directed downward.) The result is *Archimedes' Principle:* The buoyant force on an object equals the weight of the displaced liquid.

16.10 Summary

The main results of this chapter are all higher-dimensional versions of the Fundamental Theorem of Calculus. To help you remember them, we collect them together here (without hypotheses) so that you can see more easily their essential similarity. Notice that in each case we have an integral of a "derivative" over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Green's Theorem	$\iint\limits_D \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint\limits_E \operatorname{div} \mathbf{F}\,dV = \iint\limits_S \mathbf{F} \cdot d\mathbf{S}$	

16 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. What is a vector field? Give three examples that have physical meaning.

2. (a) What is a conservative vector field?
(b) What is a potential function?

3. (a) Write the definition of the line integral of a scalar function f along a smooth curve C with respect to arc length.
(b) How do you evaluate such a line integral?
(c) Write expressions for the mass and center of mass of a thin wire shaped like a curve C if the wire has linear density function $\rho(x, y)$.
(d) Write the definitions of the line integrals along C of a scalar function f with respect to x, y, and z.
(e) How do you evaluate these line integrals?

4. (a) Define the line integral of a vector field $\mathbf{F}$ along a smooth curve C given by a vector function $\mathbf{r}(t)$.
(b) If $\mathbf{F}$ is a force field, what does this line integral represent?
(c) If $\mathbf{F} = \langle P, Q, R\rangle$, what is the connection between the line integral of $\mathbf{F}$ and the line integrals of the component functions P, Q, and R?

5. State the Fundamental Theorem for Line Integrals.

6. (a) What does it mean to say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path?
(b) If you know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, what can you say about $\mathbf{F}$?

7. State Green's Theorem.

8. Write expressions for the area enclosed by a curve C in terms of line integrals around C.

9. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.
(a) Define curl $\mathbf{F}$. (b) Define div $\mathbf{F}$.
(c) If $\mathbf{F}$ is a velocity field in fluid flow, what are the physical interpretations of curl $\mathbf{F}$ and div $\mathbf{F}$?

10. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, how do you determine whether $\mathbf{F}$ is conservative? What if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$?

11. (a) What is a parametric surface? What are its grid curves?
(b) Write an expression for the area of a parametric surface.
(c) What is the area of a surface given by an equation $z = g(x, y)$?

12. (a) Write the definition of the surface integral of a scalar function f over a surface S.
(b) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
(c) What if S is given by an equation $z = g(x, y)$?
(d) If a thin sheet has the shape of a surface S, and the density at (x, y, z) is $\rho(x, y, z)$, write expressions for the mass and center of mass of the sheet.

13. (a) What is an oriented surface? Give an example of a non-orientable surface.
(b) Define the surface integral (or flux) of a vector field $\mathbf{F}$ over an oriented surface S with unit normal vector $\mathbf{n}$.
(c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
(d) What if S is given by an equation $z = g(x, y)$?

14. State Stokes' Theorem.

15. State the Divergence Theorem.

16. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{F}$ is a vector field, then div $\mathbf{F}$ is a vector field.

2. If $\mathbf{F}$ is a vector field, then curl $\mathbf{F}$ is a vector field.

3. If f has continuous partial derivatives of all orders on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl} \nabla f) = 0$.

4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.

5. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ and $P_y = Q_x$ in an open region D, then $\mathbf{F}$ is conservative.

6. $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$

7. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields and $\operatorname{div}\mathbf{F} = \operatorname{div}\mathbf{G}$, then $\mathbf{F} = \mathbf{G}$.

8. The work done by a conservative force field in moving a particle around a closed path is zero.

9. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$$

10. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\operatorname{curl}(\mathbf{F} \cdot \mathbf{G}) = \operatorname{curl} \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$$

11. If S is a sphere and $\mathbf{F}$ is a constant vector field, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$.

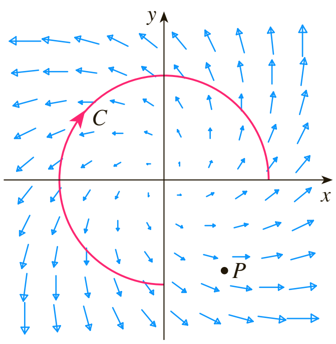
12. There is a vector field $\mathbf{F}$ such that

$$\operatorname{curl} \mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

13. The area of the region bounded by the positively oriented, piecewise smooth, simple closed curve C is $A = \oint_C y\, dx$.

EXERCISES

1. A vector field **F**, a curve C, and a point P are shown.
(a) Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero? Explain.
(b) Is div $\mathbf{F}(P)$ positive, negative, or zero? Explain.



2–9 Evaluate the line integral.

2. $\int_C x\, ds$,
C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$

3. $\int_C yz \cos x\, ds$,
C: $x = t$, $y = 3\cos t$, $z = 3\sin t$, $0 \leqslant t \leqslant \pi$

4. $\int_C y\, dx + (x + y^2)\, dy$, C is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation

5. $\int_C y^3\, dx + x^2\, dy$, C is the arc of the parabola $x = 1 - y^2$ from $(0, -1)$ to $(0, 1)$

6. $\int_C \sqrt{xy}\, dx + e^y\, dy + xz\, dz$,
C is given by $\mathbf{r}(t) = t^4\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

7. $\int_C xy\, dx + y^2\, dy + yz\, dz$,
C is the line segment from $(1, 0, -1)$, to $(3, 4, 2)$

8. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = \sin t\,\mathbf{i} + (1 + t)\,\mathbf{j}$, $0 \leqslant t \leqslant \pi$

9. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = e^z\,\mathbf{i} + xz\,\mathbf{j} + (x + y)\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - t\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

10. Find the work done by the force field

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$$

in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$ along
(a) a straight line
(b) the helix $x = 3\cos t$, $y = t$, $z = 3\sin t$

11–12 Show that **F** is a conservative vector field. Then find a function f such that $\mathbf{F} = \nabla f$.

11. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + (e^y + x^2 e^{xy})\,\mathbf{j}$

12. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j} - \sin z\,\mathbf{k}$

13–14 Show that **F** is conservative and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.

13. $\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3)\,\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\,\mathbf{j}$,
C: $\mathbf{r}(t) = (t + \sin \pi t)\,\mathbf{i} + (2t + \cos \pi t)\,\mathbf{j}$, $0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$,
C is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$

15. Verify that Green's Theorem is true for the line integral $\int_C xy^2\, dx - x^2y\, dy$, where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.

16. Use Green's Theorem to evaluate

$$\int_C \sqrt{1 + x^3}\, dx + 2xy\, dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$.

17. Use Green's Theorem to evaluate $\int_C x^2y\, dx - xy^2\, dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

18. Find curl **F** and div **F** if

$$\mathbf{F}(x, y, z) = e^{-x}\sin y\,\mathbf{i} + e^{-y}\sin z\,\mathbf{j} + e^{-z}\sin x\,\mathbf{k}$$

19. Show that there is no vector field **G** such that

$$\text{curl}\,\mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}$$

20. If **F** and **G** are vector fields whose component functions have continuous first partial derivatives, show that

$$\text{curl}(\mathbf{F} \times \mathbf{G}) = \mathbf{F}\,\text{div}\,\mathbf{G} - \mathbf{G}\,\text{div}\,\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

21. If C is any piecewise-smooth simple closed plane curve and f and g are differentiable functions, show that $\int_C f(x)\, dx + g(y)\, dy = 0$.

22. If f and g are twice differentiable functions, show that

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

23. If f is a harmonic function, that is, $\nabla^2 f = 0$, show that the line integral $\int f_y\, dx - f_x\, dy$ is independent of path in any simple region D.

24. (a) Sketch the curve C with parametric equations

$$x = \cos t \qquad y = \sin t \qquad z = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(b) Find $\int_C 2xe^{2y}\, dx + (2x^2e^{2y} + 2y\cot z)\, dy - y^2\csc^2 z\, dz$.

25. Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

26. (a) Find an equation of the tangent plane at the point $(4, -2, 1)$ to the parametric surface S given by

$$\mathbf{r}(u, v) = v^2\,\mathbf{i} - uv\,\mathbf{j} + u^2\,\mathbf{k} \qquad 0 \leqslant u \leqslant 3,\ -3 \leqslant v \leqslant 3$$

(b) Use a computer to graph the surface S and the tangent plane found in part (a).

(c) Set up, but do not evaluate, an integral for the surface area of S.

CAS (d) If

$$\mathbf{F}(x, y, z) = \frac{z^2}{1 + x^2}\,\mathbf{i} + \frac{x^2}{1 + y^2}\,\mathbf{j} + \frac{y^2}{1 + z^2}\,\mathbf{k}$$

find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ correct to four decimal places.

27–30 Evaluate the surface integral.

27. $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$

28. $\iint_S (x^2z + y^2z)\,dS$, where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$

29. $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} - 2y\,\mathbf{j} + 3x\,\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation

30. $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation

31. Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane and S has upward orientation.

32. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented upward.

33. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counterclockwise as viewed from above.

34. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

35. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, where E is the unit ball $x^2 + y^2 + z^2 \leqslant 1$.

36. Compute the outward flux of

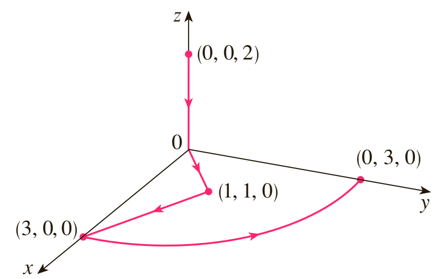
$$\mathbf{F}(x, y, z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

37. Let

$$\mathbf{F}(x, y, z) = (3x^2yz - 3y)\,\mathbf{i} + (x^3z - 3x)\,\mathbf{j} + (x^3y + 2z)\,\mathbf{k}$$

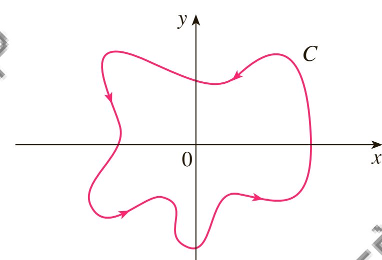
Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve with initial point $(0, 0, 2)$ and terminal point $(0, 3, 0)$ shown in the figure.



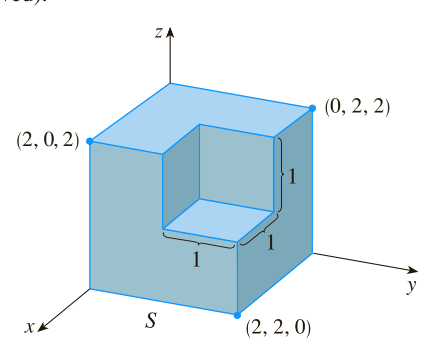
38. Let

$$\mathbf{F}(x, y) = \frac{(2x^3 + 2xy^2 - 2y)\,\mathbf{i} + (2y^3 + 2x^2y + 2x)\,\mathbf{j}}{x^2 + y^2}$$

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is shown in the figure.



39. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



40. If the components of $\mathbf{F}$ have continuous second partial derivatives and S is the boundary surface of a simple solid region, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

41. If $\mathbf{a}$ is a constant vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve C, show that

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

Problems Plus

1. Let S be a smooth parametric surface and let P be a point such that each line that starts at P intersects S at most once. The **solid angle** $\Omega(S)$ subtended by S at P is the set of lines starting at P and passing through S. Let $S(a)$ be the intersection of $\Omega(S)$ with the surface of the sphere with center P and radius a. Then the measure of the solid angle (in *steradians*) is defined to be

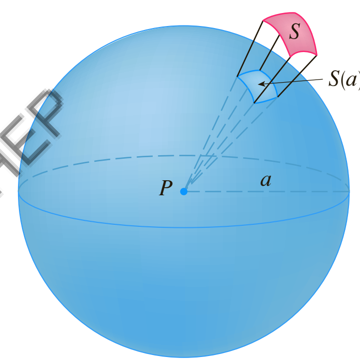
$$|\Omega(S)| = \frac{\text{area of } S(a)}{a^2}$$

Apply the Divergence Theorem to the part of $\Omega(S)$ between $S(a)$ and S to show that

$$|\Omega(S)| = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}\, dS$$

where $\mathbf{r}$ is the radius vector from P to any point on S, $r = |\mathbf{r}|$, and the unit normal vector $\mathbf{n}$ is directed away from P.

This shows that the definition of the measure of a solid angle is independent of the radius a of the sphere. Thus the measure of the solid angle is equal to the area subtended on a *unit* sphere. (Note the analogy with the definition of radian measure.) The total solid angle subtended by a sphere at its center is thus 4π steradians.



2. Find the positively oriented simple closed curve C for which the value of the line integral

$$\int_C (y^3 - y)\, dx - 2x^3\, dy$$

is a maximum.

3. Let C be a simple closed piecewise-smooth space curve that lies in a plane with unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and has positive orientation with respect to $\mathbf{n}$. Show that the plane area enclosed by C is

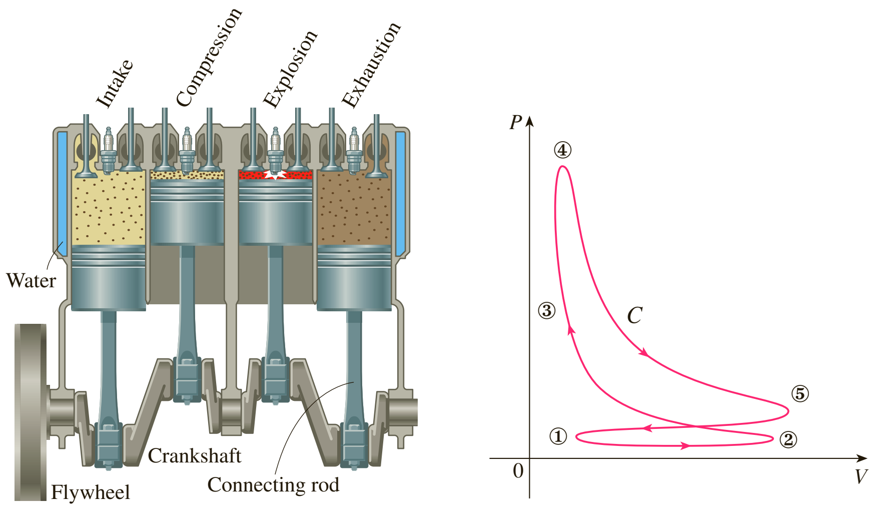
$$\tfrac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$$

4. Investigate the shape of the surface with parametric equations $x = \sin u$, $y = \sin v$, $z = \sin(u + v)$. Start by graphing the surface from several points of view. Explain the appearance of the graphs by determining the traces in the horizontal planes $z = 0$, $z = \pm 1$, and $z = \pm\frac{1}{2}$.

5. Prove the following identity:

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \operatorname{curl} \mathbf{G} + \mathbf{G} \times \operatorname{curl} \mathbf{F}$$

6. The figure depicts the sequence of events in each cylinder of a four-cylinder internal combustion engine. Each piston moves up and down and is connected by a pivoted arm to a rotating crankshaft. Let $P(t)$ and $V(t)$ be the pressure and volume within a cylinder at time t, where $a \leqslant t \leqslant b$ gives the time required for a complete cycle. The graph shows how P and V vary through one cycle of a four-stroke engine.



During the intake stroke (from ① to ②) a mixture of air and gasoline at atmospheric pressure is drawn into a cylinder through the intake valve as the piston moves downward. Then the piston rapidly compresses the mix with the valves closed in the compression stroke (from ② to ③) during which the pressure rises and the volume decreases. At ③ the sparkplug ignites the fuel, raising the temperature and pressure at almost constant volume to ④. Then, with valves closed, the rapid expansion forces the piston downward during the power stroke (from ④ to ⑤). The exhaust valve opens, temperature and pressure drop, and mechanical energy stored in a rotating flywheel pushes the piston upward, forcing the waste products out of the exhaust valve in the exhaust stroke. The exhaust valve closes and the intake valve opens. We're now back at ① and the cycle starts again.

(a) Show that the work done on the piston during one cycle of a four-stroke engine is $W = \int_C P\,dV$, where C is the curve in the PV-plane shown in the figure.

[*Hint:* Let $x(t)$ be the distance from the piston to the top of the cylinder and note that the force on the piston is $\mathbf{F} = AP(t)\,\mathbf{i}$, where A is the area of the top of the piston. Then $W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i}$, $a \leqslant t \leqslant b$. An alternative approach is to work directly with Riemann sums.]

(b) Use Formula 16.4.5 to show that the work is the difference of the areas enclosed by the two loops of C.

17

Second-Order Differential Equations

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The motion of a shock absorber in a motorcycle is described by the differential equations that we solve in Section 17.3.

THE BASIC IDEAS OF DIFFERENTIAL equations were explained in Chapter 9; there we concentrated on first-order equations. In this chapter we study second-order linear differential equations and learn how they can be applied to solve problems concerning the vibrations of springs and the analysis of electric circuits. We will also see how infinite series can be used to solve differential equations.

17.1 Second-Order Linear Equations

A **second-order linear differential equation** has the form

$$\boxed{1} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where P, Q, R, and G are continuous functions. We saw in Section 9.1 that equations of this type arise in the study of the motion of a spring. In Section 17.3 we will further pursue this application as well as the application to electric circuits.

In this section we study the case where $G(x) = 0$, for all x, in Equation 1. Such equations are called **homogeneous** linear equations. Thus the form of a second-order linear homogeneous differential equation is

$$\boxed{2} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

If $G(x) \neq 0$ for some x, Equation 1 is **nonhomogeneous** and is discussed in Section 17.2.

Two basic facts enable us to solve homogeneous linear equations. The first of these says that if we know two solutions y_1 and y_2 of such an equation, then the **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution.

3 Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and c_1 and c_2 are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation 2.

PROOF Since y_1 and y_2 are solutions of Equation 2, we have

$$P(x)y_1'' + Q(x)y_1' + R(x)y_1 = 0$$

and

$$P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0$$

Therefore, using the basic rules for differentiation, we have

$$\begin{aligned}
&P(x)y'' + Q(x)y' + R(x)y \\
&\qquad = P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\
&\qquad = P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\
&\qquad = c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] \\
&\qquad = c_1(0) + c_2(0) = 0
\end{aligned}$$

Thus $y = c_1y_1 + c_2y_2$ is a solution of Equation 2. ■

The other fact we need is given by the following theorem, which is proved in more advanced courses. It says that the general solution is a linear combination of two **linearly independent** solutions y_1 and y_2. This means that neither y_1 nor y_2 is a constant multiple of the other. For instance, the functions $f(x) = x^2$ and $g(x) = 5x^2$ are linearly dependent, but $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

4 Theorem If y_1 and y_2 are linearly independent solutions of Equation 2 on an interval, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Theorem 4 is very useful because it says that if we know *two* particular linearly independent solutions, then we know *every* solution.

In general, it's not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions P, Q, and R are constant functions, that is, if the differential equation has the form

5 $$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

It's not hard to think of some likely candidates for particular solutions of Equation 5 if we state the equation verbally. We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0. We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2 e^{rx}$. If we substitute these expressions into Equation 5, we see that $y = e^{rx}$ is a solution if

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

or

$$(ar^2 + br + c)e^{rx} = 0$$

But e^{rx} is never 0. Thus $y = e^{rx}$ is a solution of Equation 5 if r is a root of the equation

6 $$ar^2 + br + c = 0$$

Equation 6 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. Notice that it is an algebraic equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Sometimes the roots r_1 and r_2 of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

7 $$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$.

CASE I $b^2 - 4ac > 0$

In this case the roots r_1 and r_2 of the auxiliary equation are real and distinct, so $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solutions of Equation 5. (Note that $e^{r_2 x}$ is not a constant multiple of $e^{r_1 x}$.) Therefore, by Theorem 4, we have the following fact.

8 If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

In Figure 1 the graphs of the basic solutions $f(x) = e^{2x}$ and $g(x) = e^{-3x}$ of the differential equation in Example 1 are shown in blue and red, respectively. Some of the other solutions, linear combinations of f and g, are shown in black.

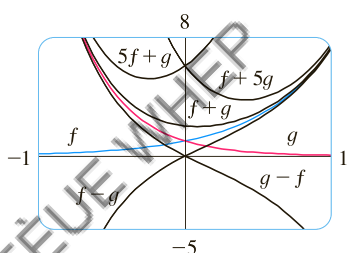


FIGURE 1

EXAMPLE 1 Solve the equation $y'' + y' - 6y = 0$.

SOLUTION The auxiliary equation is

$$r^2 + r - 6 = (r - 2)(r + 3) = 0$$

whose roots are $r = 2, -3$. Therefore, by (8), the general solution of the given differential equation is

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

We could verify that this is indeed a solution by differentiating and substituting into the differential equation. ■

EXAMPLE 2 Solve $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

SOLUTION To solve the auxiliary equation $3r^2 + r - 1 = 0$, we use the quadratic formula:

$$r = \frac{-1 \pm \sqrt{13}}{6}$$

Since the roots are real and distinct, the general solution is

$$y = c_1 e^{(-1+\sqrt{13})x/6} + c_2 e^{(-1-\sqrt{13})x/6}$$ ■

CASE II $b^2 - 4ac = 0$

In this case $r_1 = r_2$; that is, the roots of the auxiliary equation are real and equal. Let's denote by r the common value of r_1 and r_2. Then, from Equations 7, we have

9 $$r = -\frac{b}{2a} \qquad \text{so} \qquad 2ar + b = 0$$

We know that $y_1 = e^{rx}$ is one solution of Equation 5. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + 0(xe^{rx}) = 0 \end{aligned}$$

In the first term, $2ar + b = 0$ by Equations 9; in the second term, $ar^2 + br + c = 0$ because r is a root of the auxiliary equation. Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 4 provides us with the general solution.

> **10** If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is
>
> $$y = c_1 e^{rx} + c_2 x e^{rx}$$

Figure 2 shows the basic solutions $f(x) = e^{-3x/2}$ and $g(x) = xe^{-3x/2}$ in Example 3 and some other members of the family of solutions. Notice that all of them approach 0 as $x \to \infty$.

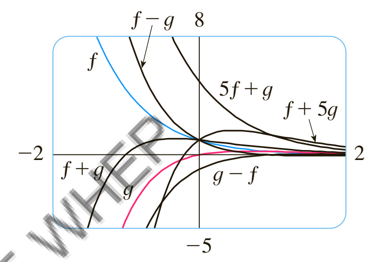


FIGURE 2

EXAMPLE 3 Solve the equation $4y'' + 12y' + 9y = 0$.

SOLUTION The auxiliary equation $4r^2 + 12r + 9 = 0$ can be factored as

$$(2r + 3)^2 = 0$$

so the only root is $r = -\frac{3}{2}$. By (10) the general solution is

$$y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$$

■

CASE III $b^2 - 4ac < 0$

In this case the roots r_1 and r_2 of the auxiliary equation are complex numbers. (See Appendix H for information about complex numbers.) We can write

$$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta$$

where α and β are real numbers. [In fact, $\alpha = -b/(2a)$, $\beta = \sqrt{4ac - b^2}/(2a)$.] Then, using Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

from Appendix H, we write the solution of the differential equation as

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ &= C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x] \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \end{aligned}$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$. This gives all solutions (real or complex) of the differential equation. The solutions are real when the constants c_1 and c_2 are real. We summarize the discussion as follows.

> **11** If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is
>
> $$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

EXAMPLE 4 Solve the equation $y'' - 6y' + 13y = 0$.

SOLUTION The auxiliary equation is $r^2 - 6r + 13 = 0$. By the quadratic formula, the roots are

$$r = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

By (11), the general solution of the differential equation is

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

Figure 3 shows the graphs of the solutions in Example 4, $f(x) = e^{3x} \cos 2x$ and $g(x) = e^{3x} \sin 2x$, together with some linear combinations. All solutions approach 0 as $x \to -\infty$.

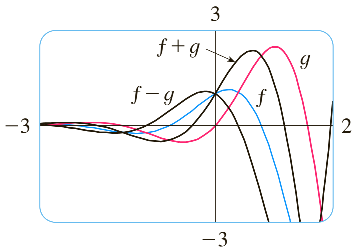


FIGURE 3

Initial-Value and Boundary-Value Problems

An **initial-value problem** for the second-order Equation 1 or 2 consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

where y_0 and y_1 are given constants. If P, Q, R, and G are continuous on an interval and $P(x) \neq 0$ there, then a theorem found in more advanced books guarantees the existence and uniqueness of a solution to this initial-value problem. Examples 5 and 6 illustrate the technique for solving such a problem.

EXAMPLE 5 Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

SOLUTION From Example 1 we know that the general solution of the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}$$

Differentiating this solution, we get

$$y'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

To satisfy the initial conditions we require that

$$y(0) = c_1 + c_2 = 1 \tag{12}$$

$$y'(0) = 2c_1 - 3c_2 = 0 \tag{13}$$

From (13), we have $c_2 = \frac{2}{3}c_1$ and so (12) gives

$$c_1 + \tfrac{2}{3}c_1 = 1 \qquad c_1 = \tfrac{3}{5} \qquad c_2 = \tfrac{2}{5}$$

Thus the required solution of the initial-value problem is

$$y = \tfrac{3}{5}e^{2x} + \tfrac{2}{5}e^{-3x}$$

Figure 4 shows the graph of the solution of the initial-value problem in Example 5. Compare with Figure 1.

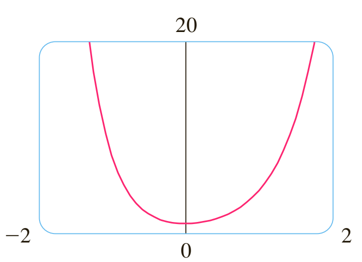


FIGURE 4

EXAMPLE 6 Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \qquad y'(0) = 3$$

SOLUTION The auxiliary equation is $r^2 + 1 = 0$, or $r^2 = -1$, whose roots are $\pm i$. Thus $\alpha = 0$, $\beta = 1$, and since $e^{0x} = 1$, the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x$$

Since

$$y'(x) = -c_1 \sin x + c_2 \cos x$$

The solution to Example 6 is graphed in Figure 5. It appears to be a shifted sine curve and, indeed, you can verify that another way of writing the solution is

$$y = \sqrt{13}\sin(x + \phi) \quad \text{where } \tan\phi = \tfrac{2}{3}$$

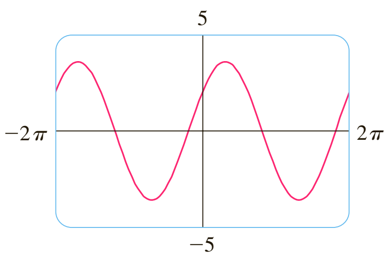


FIGURE 5

the initial conditions become

$$y(0) = c_1 = 2 \qquad y'(0) = c_2 = 3$$

Therefore the solution of the initial-value problem is

$$y(x) = 2\cos x + 3\sin x$$

■

A **boundary-value problem** for Equation 1 or 2 consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad y(x_1) = y_1$$

In contrast with the situation for initial-value problems, a boundary-value problem does not always have a solution. The method is illustrated in Example 7.

EXAMPLE 7 Solve the boundary-value problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3$$

SOLUTION The auxiliary equation is

$$r^2 + 2r + 1 = 0 \qquad \text{or} \qquad (r + 1)^2 = 0$$

whose only root is $r = -1$. Therefore the general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

Figure 6 shows the graph of the solution of the boundary-value problem in Example 7.

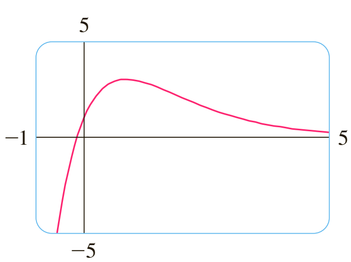


FIGURE 6

The boundary conditions are satisfied if

$$y(0) = c_1 = 1$$

$$y(1) = c_1 e^{-1} + c_2 e^{-1} = 3$$

The first condition gives $c_1 = 1$, so the second condition becomes

$$e^{-1} + c_2 e^{-1} = 3$$

Solving this equation for c_2 by first multiplying through by e, we get

$$1 + c_2 = 3e \qquad \text{so} \qquad c_2 = 3e - 1$$

Thus the solution of the boundary-value problem is

$$y = e^{-x} + (3e - 1)xe^{-x}$$

■

Summary: Solutions of $ay'' + by' + cy = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

17.1 EXERCISES

1–13 Solve the differential equation.

1. $y'' - y' - 6y = 0$
2. $y'' - 6y' + 9y = 0$
3. $y'' + 2y = 0$
4. $y'' + y' - 12y = 0$
5. $4y'' + 4y' + y = 0$
6. $9y'' + 4y = 0$
7. $3y'' = 4y'$
8. $y = y''$
9. $y'' - 4y' + 13y = 0$
10. $3y'' + 4y' - 3y = 0$
11. $2\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - y = 0$
12. $\dfrac{d^2R}{dt^2} + 6\dfrac{dR}{dt} + 34R = 0$
13. $3\dfrac{d^2V}{dt^2} + 4\dfrac{dV}{dt} + 3V = 0$

14–16 Graph the two basic solutions along with several other solutions of the differential equation. What features do the solutions have in common?

14. $4\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$
15. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$
16. $2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$

17–24 Solve the initial-value problem.

17. $y'' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3$
18. $y'' - 2y' - 3y = 0, \quad y(0) = 2, \quad y'(0) = 2$
19. $9y'' + 12y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$
20. $3y'' - 2y' - y = 0, \quad y(0) = 0, \quad y'(0) = -4$
21. $y'' - 6y' + 10y = 0, \quad y(0) = 2, \quad y'(0) = 3$
22. $4y'' - 20y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -3$
23. $y'' - y' - 12y = 0, \quad y(1) = 0, \quad y'(1) = 1$
24. $4y'' + 4y' + 3y = 0, \quad y(0) = 0, \quad y'(0) = 1$

25–32 Solve the boundary-value problem, if possible.

25. $y'' + 16y = 0, \quad y(0) = -3, \quad y(\pi/8) = 2$
26. $y'' + 6y' = 0, \quad y(0) = 1, \quad y(1) = 0$
27. $y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y(1) = 0$
28. $y'' - 8y' + 17y = 0, \quad y(0) = 3, \quad y(\pi) = 2$
29. $y'' = y', \quad y(0) = 1, \quad y(1) = 2$
30. $4y'' - 4y' + y = 0, \quad y(0) = 4, \quad y(2) = 0$
31. $y'' + 4y' + 20y = 0, \quad y(0) = 1, \quad y(\pi) = 2$
32. $y'' + 4y' + 20y = 0, \quad y(0) = 1, \quad y(\pi) = e^{-2\pi}$

33. Let L be a nonzero real number.
    (a) Show that the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.
    (b) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.
34. If a, b, and c are all positive constants and $y(x)$ is a solution of the differential equation $ay'' + by' + cy = 0$, show that $\lim_{x\to\infty} y(x) = 0$.
35. Consider the boundary-value problem $y'' - 2y' + 2y = 0$, $y(a) = c$, $y(b) = d$.
    (a) If this problem has a unique solution, how are a and b related?
    (b) If this problem has no solution, how are a, b, c, and d related?
    (c) If this problem has infinitely many solutions, how are a, b, c, and d related?

17.2 Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$ay'' + by' + cy = G(x) \tag{1}$$

where a, b, and c are constants and G is a continuous function. The related homogeneous equation

$$ay'' + by' + cy = 0 \tag{2}$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation (1).

3 Theorem The general solution of the nonhomogeneous differential equation (1) can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of Equation 1 and y_c is the general solution of the complementary Equation 2.

PROOF We verify that if y is any solution of Equation 1, then $y - y_p$ is a solution of the complementary Equation 2. Indeed

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) &= ay'' - ay_p'' + by' - by_p' + cy - cy_p \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= G(x) - G(x) = 0 \end{aligned}$$

This shows that every solution is of the form $y(x) = y_p(x) + y_c(x)$. It is easy to check that every function of this form is a solution. ■

We know from Section 17.1 how to solve the complementary equation. (Recall that the solution is $y_c = c_1y_1 + c_2y_2$, where y_1 and y_2 are linearly independent solutions of Equation 2.) Therefore Theorem 3 says that we know the general solution of the nonhomogeneous equation as soon as we know a particular solution y_p. There are two methods for finding a particular solution: The method of undetermined coefficients is straightforward but works only for a restricted class of functions G. The method of variation of parameters works for every function G but is usually more difficult to apply in practice.

The Method of Undetermined Coefficients

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial. It is reasonable to guess that there is a particular solution y_p that is a polynomial of the same degree as G because if y is a polynomial, then $ay'' + by' + cy$ is also a polynomial. We therefore substitute $y_p(x)$ = a polynomial (of the same degree as G) into the differential equation and determine the coefficients.

EXAMPLE 1 Solve the equation $y'' + y' - 2y = x^2$.

SOLUTION The auxiliary equation of $y'' + y' - 2y = 0$ is

$$r^2 + r - 2 = (r - 1)(r + 2) = 0$$

with roots $r = 1, -2$. So the solution of the complementary equation is

$$y_c = c_1e^x + c_2e^{-2x}$$

Since $G(x) = x^2$ is a polynomial of degree 2, we seek a particular solution of the form

$$y_p(x) = Ax^2 + Bx + C$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A$ so, substituting into the given differential equation, we have

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

or
$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$$

Figure 1 shows four solutions of the differential equation in Example 1 in terms of the particular solution y_p and the functions $f(x) = e^x$ and $g(x) = e^{-2x}$.

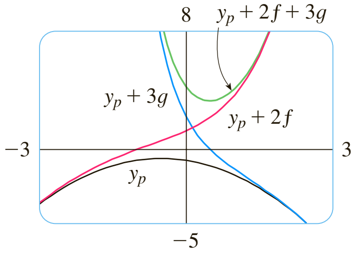


FIGURE 1

Polynomials are equal when their coefficients are equal. Thus

$$-2A = 1 \qquad 2A - 2B = 0 \qquad 2A + B - 2C = 0$$

The solution of this system of equations is

$$A = -\tfrac{1}{2} \qquad B = -\tfrac{1}{2} \qquad C = -\tfrac{3}{4}$$

A particular solution is therefore

$$y_p(x) = -\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

and, by Theorem 3, the general solution is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-2x} - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

If $G(x)$ (the right side of Equation 1) is of the form Ce^{kx}, where C and k are constants, then we take as a trial solution a function of the same form, $y_p(x) = Ae^{kx}$, because the derivatives of e^{kx} are constant multiples of e^{kx}.

EXAMPLE 2 Solve $y'' + 4y = e^{3x}$.

Figure 2 shows solutions of the differential equation in Example 2 in terms of y_p and the functions $f(x) = \cos 2x$ and $g(x) = \sin 2x$. Notice that all solutions approach ∞ as $x \to \infty$ and all solutions (except y_p) resemble sine functions when x is negative.

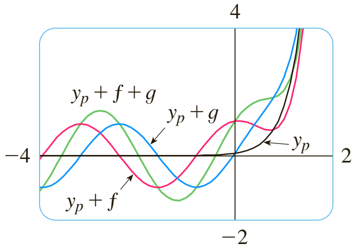


FIGURE 2

SOLUTION The auxiliary equation is $r^2 + 4 = 0$ with roots $\pm 2i$, so the solution of the complementary equation is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

For a particular solution we try $y_p(x) = Ae^{3x}$. Then $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting into the differential equation, we have

$$9Ae^{3x} + 4(Ae^{3x}) = e^{3x}$$

so $13Ae^{3x} = e^{3x}$ and $A = \frac{1}{13}$. Thus a particular solution is

$$y_p(x) = \tfrac{1}{13}e^{3x}$$

and the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \tfrac{1}{13}e^{3x}$$

If $G(x)$ is either $C \cos kx$ or $C \sin kx$, then, because of the rules for differentiating the sine and cosine functions, we take as a trial particular solution a function of the form

$$y_p(x) = A \cos kx + B \sin kx$$

EXAMPLE 3 Solve $y'' + y' - 2y = \sin x$.

SOLUTION We try a particular solution

$$y_p(x) = A \cos x + B \sin x$$

Then $\qquad y_p' = -A \sin x + B \cos x \qquad y_p'' = -A \cos x - B \sin x$

so substitution in the differential equation gives

$$(-A \cos x - B \sin x) + (-A \sin x + B \cos x) - 2(A \cos x + B \sin x) = \sin x$$

or
$$(-3A + B) \cos x + (-A - 3B) \sin x = \sin x$$

This is true if

$$-3A + B = 0 \qquad \text{and} \qquad -A - 3B = 1$$

The solution of this system is

$$A = -\tfrac{1}{10} \qquad B = -\tfrac{3}{10}$$

so a particular solution is

$$y_p(x) = -\tfrac{1}{10} \cos x - \tfrac{3}{10} \sin x$$

In Example 1 we determined that the solution of the complementary equation is $y_c = c_1 e^x + c_2 e^{-2x}$. Thus the general solution of the given equation is

$$y(x) = c_1 e^x + c_2 e^{-2x} - \tfrac{1}{10}(\cos x + 3 \sin x)$$

If $G(x)$ is a product of functions of the preceding types, then we take the trial solution to be a product of functions of the same type. For instance, in solving the differential equation

$$y'' + 2y' + 4y = x \cos 3x$$

we would try

$$y_p(x) = (Ax + B) \cos 3x + (Cx + D) \sin 3x$$

If $G(x)$ is a sum of functions of these types, we use the easily verified *principle of superposition*, which says that if y_{p_1} and y_{p_2} are solutions of

$$ay'' + by' + cy = G_1(x) \qquad ay'' + by' + cy = G_2(x)$$

respectively, then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

EXAMPLE 4 Solve $y'' - 4y = xe^x + \cos 2x$.

SOLUTION The auxiliary equation is $r^2 - 4 = 0$ with roots ± 2, so the solution of the complementary equation is $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. For the equation $y'' - 4y = xe^x$ we try

$$y_{p_1}(x) = (Ax + B)e^x$$

Then $y_{p_1}' = (Ax + A + B)e^x$, $y_{p_1}'' = (Ax + 2A + B)e^x$, so substitution in the equation gives

$$(Ax + 2A + B)e^x - 4(Ax + B)e^x = xe^x$$

or
$$(-3Ax + 2A - 3B)e^x = xe^x$$

Thus $-3A = 1$ and $2A - 3B = 0$, so $A = -\frac{1}{3}$, $B = -\frac{2}{9}$, and

$$y_{p_1}(x) = \left(-\tfrac{1}{3}x - \tfrac{2}{9}\right)e^x$$

For the equation $y'' - 4y = \cos 2x$, we try

$$y_{p_2}(x) = C\cos 2x + D\sin 2x$$

Substitution gives

$$-4C\cos 2x - 4D\sin 2x - 4(C\cos 2x + D\sin 2x) = \cos 2x$$

or

$$-8C\cos 2x - 8D\sin 2x = \cos 2x$$

Therefore $-8C = 1$, $-8D = 0$, and

$$y_{p_2}(x) = -\tfrac{1}{8}\cos 2x$$

By the superposition principle, the general solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1e^{2x} + c_2e^{-2x} - \left(\tfrac{1}{3}x + \tfrac{2}{9}\right)e^x - \tfrac{1}{8}\cos 2x \qquad \blacksquare$$

In Figure 3 we show the particular solution $y_p = y_{p_1} + y_{p_2}$ of the differential equation in Example 4. The other solutions are given in terms of $f(x) = e^{2x}$ and $g(x) = e^{-2x}$.

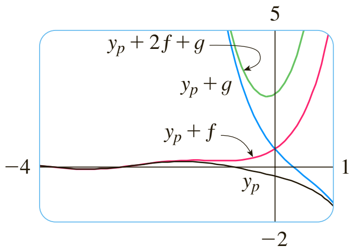


FIGURE 3

Finally we note that the recommended trial solution y_p sometimes turns out to be a solution of the complementary equation and therefore can't be a solution of the nonhomogeneous equation. In such cases we multiply the recommended trial solution by x (or by x^2 if necessary) so that no term in $y_p(x)$ is a solution of the complementary equation.

EXAMPLE 5 Solve $y'' + y = \sin x$.

SOLUTION The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of the complementary equation is

$$y_c(x) = c_1\cos x + c_2\sin x$$

Ordinarily, we would use the trial solution

$$y_p(x) = A\cos x + B\sin x$$

but we observe that it is a solution of the complementary equation, so instead we try

$$y_p(x) = Ax\cos x + Bx\sin x$$

Then

$$y_p'(x) = A\cos x - Ax\sin x + B\sin x + Bx\cos x$$

$$y_p''(x) = -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x$$

Substitution in the differential equation gives

$$y_p'' + y_p = -2A\sin x + 2B\cos x = \sin x$$

The graphs of four solutions of the differential equation in Example 5 are shown in Figure 4.

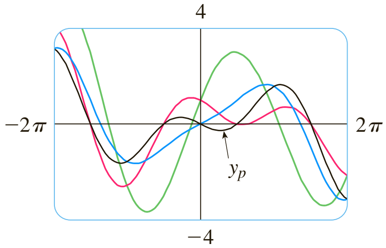


FIGURE 4

so $A = -\frac{1}{2}$, $B = 0$, and

$$y_p(x) = -\tfrac{1}{2}x \cos x$$

The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \tfrac{1}{2}x \cos x$$ ■

We summarize the method of undetermined coefficients as follows:

Summary of the Method of Undetermined Coefficients

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x) \cos mx$ or $G(x) = e^{kx}P(x) \sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x) \cos mx + e^{kx}R(x) \sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

EXAMPLE 6 Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x} \cos 3x$.

SOLUTION Here $G(x)$ has the form of part 2 of the summary, where $k = 2$, $m = 3$, and $P(x) = 1$. So, at first glance, the form of the trial solution would be

$$y_p(x) = e^{2x}(A \cos 3x + B \sin 3x)$$

But the auxiliary equation is $r^2 - 4r + 13 = 0$, with roots $r = 2 \pm 3i$, so the solution of the complementary equation is

$$y_c(x) = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

This means that we have to multiply the suggested trial solution by x. So, instead, we use

$$y_p(x) = xe^{2x}(A \cos 3x + B \sin 3x)$$ ■

The Method of Variation of Parameters

Suppose we have already solved the homogeneous equation $ay'' + by' + cy = 0$ and written the solution as

$$\boxed{4} \qquad y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 are linearly independent solutions. Let's replace the constants (or parameters) c_1 and c_2 in Equation 4 by arbitrary functions $u_1(x)$ and $u_2(x)$. We look for a particular solution of the nonhomogeneous equation $ay'' + by' + cy = G(x)$ of the form

$$\boxed{5} \qquad y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

(This method is called **variation of parameters** because we have varied the parameters c_1 and c_2 to make them functions.) Differentiating Equation 5, we get

$$\boxed{6} \qquad y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2')$$

Since u_1 and u_2 are arbitrary functions, we can impose two conditions on them. One condition is that y_p is a solution of the differential equation; we can choose the other condition so as to simplify our calculations. In view of the expression in Equation 6, let's impose the condition that

$$\boxed{7} \qquad u_1'y_1 + u_2'y_2 = 0$$

Then $$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$$

Substituting in the differential equation, we get

$$a(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) = G$$

or

$$\boxed{8} \qquad u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1'y_1' + u_2'y_2') = G$$

But y_1 and y_2 are solutions of the complementary equation, so

$$ay_1'' + by_1' + cy_1 = 0 \qquad \text{and} \qquad ay_2'' + by_2' + cy_2 = 0$$

and Equation 8 simplifies to

$$\boxed{9} \qquad a(u_1'y_1' + u_2'y_2') = G$$

Equations 7 and 9 form a system of two equations in the unknown functions u_1' and u_2'. After solving this system we may be able to integrate to find u_1 and u_2 and then the particular solution is given by Equation 5.

EXAMPLE 7 Solve the equation $y'' + y = \tan x$, $0 < x < \pi/2$.

SOLUTION The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of $y'' + y = 0$ is $y(x) = c_1 \sin x + c_2 \cos x$. Using variation of parameters, we seek a solution of the form

$$y_p(x) = u_1(x)\sin x + u_2(x)\cos x$$

Then $$y_p' = (u_1'\sin x + u_2'\cos x) + (u_1\cos x - u_2\sin x)$$

Set

$$\boxed{10} \qquad u_1'\sin x + u_2'\cos x = 0$$

Then $$y_p'' = u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x$$

For y_p to be a solution we must have

$$\boxed{11} \qquad y_p'' + y_p = u_1' \cos x - u_2' \sin x = \tan x$$

Solving Equations 10 and 11, we get

$$u_1'(\sin^2 x + \cos^2 x) = \cos x \tan x$$

$$u_1' = \sin x \qquad u_1(x) = -\cos x$$

(We seek a particular solution, so we don't need a constant of integration here.) Then, from Equation 10, we obtain

$$u_2' = -\frac{\sin x}{\cos x} u_1' = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

So $$u_2(x) = \sin x - \ln(\sec x + \tan x)$$

(Note that $\sec x + \tan x > 0$ for $0 < x < \pi/2$.) Therefore

$$\begin{aligned} y_p(x) &= -\cos x \sin x + [\sin x - \ln(\sec x + \tan x)] \cos x \\ &= -\cos x \ln(\sec x + \tan x) \end{aligned}$$

and the general solution is

$$y(x) = c_1 \sin x + c_2 \cos x - \cos x \ln(\sec x + \tan x)$$ ■

Figure 5 shows four solutions of the differential equation in Example 7.

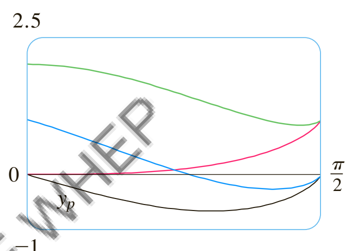


FIGURE 5

17.2 EXERCISES

1–10 Solve the differential equation or initial-value problem using the method of undetermined coefficients.

1. $y'' + 2y' - 8y = 1 - 2x^2$

2. $y'' - 3y' = \sin 2x$

3. $9y'' + y = e^{2x}$

4. $y'' - 2y' + 2y = x + e^x$

5. $y'' - 4y' + 5y = e^{-x}$

6. $y'' - 4y' + 4y = x - \sin x$

7. $y'' - 2y' + 5y = \sin x, \quad y(0) = 1, \quad y'(0) = 1$

8. $y'' - y = xe^{2x}, \quad y(0) = 0, \quad y'(0) = 1$

9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$

10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$

11–12 Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

11. $y'' + 3y' + 2y = \cos x$ **12.** $y'' + 4y = e^{-x}$

13–18 Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

13. $y'' - y' - 2y = xe^x \cos x$

14. $y'' + 4y = \cos 4x + \cos 2x$

15. $y'' - 3y' + 2y = e^x + \sin x$

16. $y'' + 3y' - 4y = (x^3 + x)e^x$

17. $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$

18. $y'' + 4y = e^{3x} + x \sin 2x$

19–22 Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

19. $4y'' + y = \cos x$

20. $y'' - 2y' - 3y = x + 2$

21. $y'' - 2y' + y = e^{2x}$

22. $y'' - y' = e^x$

23–28 Solve the differential equation using the method of variation of parameters.

23. $y'' + y = \sec^2 x,\ 0 < x < \pi/2$

24. $y'' + y = \sec^3 x,\ 0 < x < \pi/2$

25. $y'' - 3y' + 2y = \dfrac{1}{1 + e^{-x}}$

26. $y'' + 3y' + 2y = \sin(e^x)$

27. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

28. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^3}$

17.3 Applications of Second-Order Differential Equations

Second-order linear differential equations have a variety of applications in science and engineering. In this section we explore two of them: the vibration of springs and electric circuits.

Vibrating Springs

We consider the motion of an object with mass m at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level surface (as in Figure 2).

In Section 5.4 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the **spring constant**). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

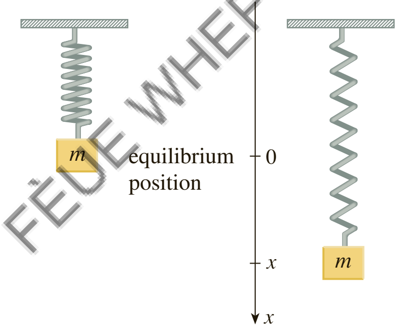


FIGURE 1

$$\boxed{1} \qquad m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx = 0$$

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm\omega i$, where $\omega = \sqrt{k/m}$. Thus the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

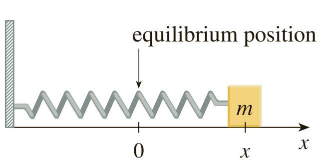


FIGURE 2

which can also be written as

$$x(t) = A\cos(\omega t + \delta)$$

where $\qquad \omega = \sqrt{k/m} \qquad$ (frequency)

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{(amplitude)}$$

$$\cos\delta = \frac{c_1}{A} \qquad \sin\delta = -\frac{c_2}{A} \qquad (\delta \text{ is the phase angle})$$

(See Exercise 17.) This type of motion is called **simple harmonic motion**.

EXAMPLE 1 A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t.

SOLUTION From Hooke's Law, the force required to stretch the spring is

$$k(0.2) = 25.6$$

so $k = 25.6/0.2 = 128$. Using this value of the spring constant k, together with $m = 2$ in Equation 1, we have

$$2\frac{d^2x}{dt^2} + 128x = 0$$

As in the earlier general discussion, the solution of this equation is

2 $$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

We are given the initial condition that $x(0) = 0.2$. But, from Equation 2, $x(0) = c_1$. Therefore $c_1 = 0.2$. Differentiating Equation 2, we get

$$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$$

Since the initial velocity is given as $x'(0) = 0$, we have $c_2 = 0$ and so the solution is

$$x(t) = 0.2 \cos 8t$$ ■

Damped Vibrations

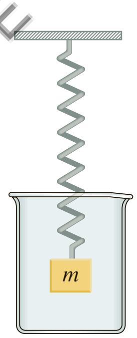


FIGURE 3

Schwinn Cycling and Fitness

We next consider the motion of a spring that is subject to a frictional force (in the case of the horizontal spring of Figure 2) or a damping force (in the case where a vertical spring moves through a fluid as in Figure 3). An example is the damping force supplied by a shock absorber in a car or a bicycle.

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. (This has been confirmed, at least approximately, by some physical experiments.) Thus

$$\text{damping force} = -c\frac{dx}{dt}$$

where c is a positive constant, called the **damping constant**. Thus, in this case, Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} = -kx - c\frac{dx}{dt}$$

or

3 $$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0}$$

Equation 3 is a second-order linear differential equation and its auxiliary equation is $mr^2 + cr + k = 0$. The roots are

$$\boxed{4} \qquad r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

According to Section 17.1 we need to discuss three cases.

CASE I $c^2 - 4mk > 0$ (overdamping)
In this case r_1 and r_2 are distinct real roots and

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

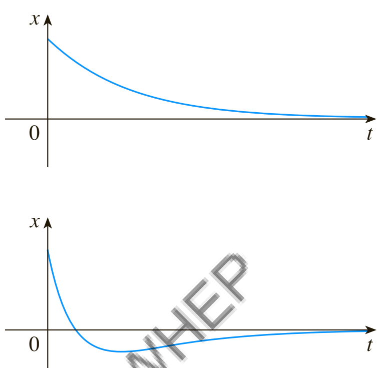


FIGURE 4
Overdamping

Since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < c$, so the roots r_1 and r_2 given by Equations 4 must both be negative. This shows that $x \to 0$ as $t \to \infty$. Typical graphs of x as a function of t are shown in Figure 4. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means that there is a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

CASE II $c^2 - 4mk = 0$ (critical damping)
This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

and the solution is given by

$$x = (c_1 + c_2 t)e^{-(c/2m)t}$$

It is similar to Case I, and typical graphs resemble those in Figure 4 (see Exercise 12), but the damping is just sufficient to suppress vibrations. Any decrease in the viscosity of the fluid leads to the vibrations of the following case.

CASE III $c^2 - 4mk < 0$ (underdamping)
Here the roots are complex:

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\frac{c}{2m} \pm \omega i$$

where
$$\omega = \frac{\sqrt{4mk - c^2}}{2m}$$

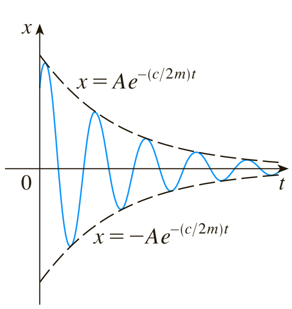


FIGURE 5
Underdamping

The solution is given by

$$x = e^{-(c/2m)t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

We see that there are oscillations that are damped by the factor $e^{-(c/2m)t}$. Since $c > 0$ and $m > 0$, we have $-(c/2m) < 0$ so $e^{-(c/2m)t} \to 0$ as $t \to \infty$. This implies that $x \to 0$ as $t \to \infty$; that is, the motion decays to 0 as time increases. A typical graph is shown in Figure 5.

EXAMPLE 2 Suppose that the spring of Example 1 is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s.

SOLUTION From Example 1, the mass is $m = 2$ and the spring constant is $k = 128$, so the differential equation (3) becomes

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

or

$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0$$

The auxiliary equation is $r^2 + 20r + 64 = (r + 4)(r + 16) = 0$ with roots -4 and -16, so the motion is overdamped and the solution is

$$x(t) = c_1e^{-4t} + c_2e^{-16t}$$

Figure 6 shows the graph of the position function for the overdamped motion in Example 2.

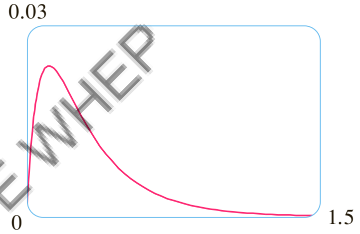


FIGURE 6

We are given that $x(0) = 0$, so $c_1 + c_2 = 0$. Differentiating, we get

$$x'(t) = -4c_1e^{-4t} - 16c_2e^{-16t}$$

so

$$x'(0) = -4c_1 - 16c_2 = 0.6$$

Since $c_2 = -c_1$, this gives $12c_1 = 0.6$ or $c_1 = 0.05$. Therefore

$$x = 0.05(e^{-4t} - e^{-16t})$$

Forced Vibrations

Suppose that, in addition to the restoring force and the damping force, the motion of the spring is affected by an external force $F(t)$. Then Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} + \text{external force}$$

$$= -kx - c\frac{dx}{dt} + F(t)$$

Thus, instead of the homogeneous equation (3), the motion of the spring is now governed by the following nonhomogeneous differential equation:

5 $$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)}$$

The motion of the spring can be determined by the methods of Section 17.2.

A commonly occurring type of external force is a periodic force function

$$F(t) = F_0 \cos \omega_0 t \qquad \text{where} \quad \omega_0 \neq \omega = \sqrt{k/m}$$

In this case, and in the absence of a damping force ($c = 0$), you are asked in Exercise 9 to use the method of undetermined coefficients to show that

$$\boxed{6} \qquad x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$$

If $\omega_0 = \omega$, then the applied frequency reinforces the natural frequency and the result is vibrations of large amplitude. This is the phenomenon of **resonance** (see Exercise 10).

Electric Circuits

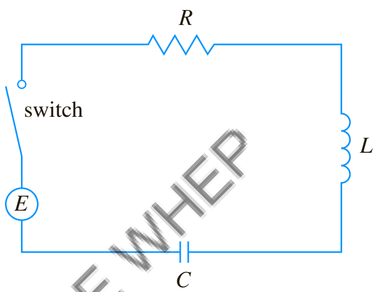


FIGURE 7

In Sections 9.3 and 9.5 we were able to use first-order separable and linear equations to analyze electric circuits that contain a resistor and inductor (see Figure 9.3.5 or Figure 9.5.4) or a resistor and capacitor (see Exercise 9.5.29). Now that we know how to solve second-order linear equations, we are in a position to analyze the circuit shown in Figure 7. It contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t: $I = dQ/dt$. As in Section 9.5, it is known from physics that the voltage drops across the resistor, inductor, and capacitor are

$$RI \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Since $I = dQ/dt$, this equation becomes

$$\boxed{7} \qquad \boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)}$$

which is a second-order linear differential equation with constant coefficients. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions

$$Q(0) = Q_0 \qquad Q'(0) = I(0) = I_0$$

and the initial-value problem can be solved by the methods of Section 17.2.

A differential equation for the current can be obtained by differentiating Equation 7 with respect to t and remembering that $I = dQ/dt$:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)$$

EXAMPLE 3 Find the charge and current at time t in the circuit of Figure 7 if $R = 40\ \Omega$, $L = 1$ H, $C = 16 \times 10^{-4}$ F, $E(t) = 100\cos 10t$, and the initial charge and current are both 0.

SOLUTION With the given values of L, R, C, and $E(t)$, Equation 7 becomes

$$\boxed{8} \qquad \frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100\cos 10t$$

The auxiliary equation is $r^2 + 40r + 625 = 0$ with roots

$$r = \frac{-40 \pm \sqrt{-900}}{2} = -20 \pm 15i$$

so the solution of the complementary equation is

$$Q_c(t) = e^{-20t}(c_1\cos 15t + c_2\sin 15t)$$

For the method of undetermined coefficients we try the particular solution

$$Q_p(t) = A\cos 10t + B\sin 10t$$

Then $$Q_p'(t) = -10A\sin 10t + 10B\cos 10t$$

$$Q_p''(t) = -100A\cos 10t - 100B\sin 10t$$

Substituting into Equation 8, we have

$$(-100A\cos 10t - 100B\sin 10t) + 40(-10A\sin 10t + 10B\cos 10t)$$
$$+ 625(A\cos 10t + B\sin 10t) = 100\cos 10t$$

or $$(525A + 400B)\cos 10t + (-400A + 525B)\sin 10t = 100\cos 10t$$

Equating coefficients, we have

$$525A + 400B = 100 \qquad \text{or} \qquad 21A + 16B = 4$$
$$-400A + 525B = 0 \qquad \text{or} \qquad -16A + 21B = 0$$

The solution of this system is $A = \frac{84}{697}$ and $B = \frac{64}{697}$, so a particular solution is

$$Q_p(t) = \tfrac{1}{697}(84\cos 10t + 64\sin 10t)$$

and the general solution is

$$Q(t) = Q_c(t) + Q_p(t)$$
$$= e^{-20t}(c_1 \cos 15t + c_2 \sin 15t) + \tfrac{4}{697}(21 \cos 10t + 16 \sin 10t)$$

Imposing the initial condition $Q(0) = 0$, we get

$$Q(0) = c_1 + \tfrac{84}{697} = 0 \qquad c_1 = -\tfrac{84}{697}$$

To impose the other initial condition, we first differentiate to find the current:

$$I = \frac{dQ}{dt} = e^{-20t}[(-20c_1 + 15c_2)\cos 15t + (-15c_1 - 20c_2)\sin 15t]$$
$$+ \tfrac{40}{697}(-21 \sin 10t + 16 \cos 10t)$$

$$I(0) = -20c_1 + 15c_2 + \tfrac{640}{697} = 0 \qquad c_2 = -\tfrac{464}{2091}$$

Thus the formula for the charge is

$$Q(t) = \frac{4}{697}\left[\frac{e^{-20t}}{3}(-63 \cos 15t - 116 \sin 15t) + (21 \cos 10t + 16 \sin 10t)\right]$$

and the expression for the current is

$$I(t) = \tfrac{1}{2091}[e^{-20t}(-1920 \cos 15t + 13{,}060 \sin 15t) + 120(-21 \sin 10t + 16 \cos 10t)]$$

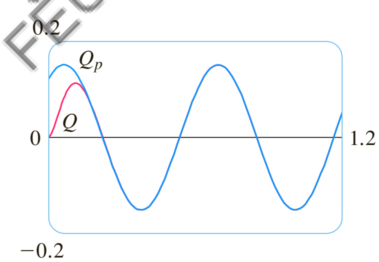


FIGURE 8

NOTE 1 In Example 3 the solution for $Q(t)$ consists of two parts. Since $e^{-20t} \to 0$ as $t \to \infty$ and both $\cos 15t$ and $\sin 15t$ are bounded functions,

$$Q_c(t) = \tfrac{4}{2091}e^{-20t}(-63 \cos 15t - 116 \sin 15t) \to 0 \qquad \text{as } t \to \infty$$

So, for large values of t,

$$Q(t) \approx Q_p(t) = \tfrac{4}{697}(21 \cos 10t + 16 \sin 10t)$$

and, for this reason, $Q_p(t)$ is called the **steady state solution**. Figure 8 shows how the graph of the steady state solution compares with the graph of Q in this case.

$$\boxed{5} \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

$$\boxed{7} \quad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

NOTE 2 Comparing Equations 5 and 7, we see that mathematically they are identical. This suggests the analogies given in the following chart between physical situations that, at first glance, are very different.

Spring system		Electric circuit	
x	displacement	Q	charge
dx/dt	velocity	$I = dQ/dt$	current
m	mass	L	inductance
c	damping constant	R	resistance
k	spring constant	$1/C$	elastance
$F(t)$	external force	$E(t)$	electromotive force

We can also transfer other ideas from one situation to the other. For instance, the steady state solution discussed in Note 1 makes sense in the spring system. And the phenomenon of resonance in the spring system can be usefully carried over to electric circuits as electrical resonance.

17.3 EXERCISES

1. A spring has natural length 0.75 m and a 5-kg mass. A force of 25 N is needed to keep the spring stretched to a length of 1 m. If the spring is stretched to a length of 1.1 m and then released with velocity 0, find the position of the mass after t seconds.

2. A spring with an 8-kg mass is kept stretched 0.4 m beyond its natural length by a force of 32 N. The spring starts at its equilibrium position and is given an initial velocity of 1 m/s. Find the position of the mass at any time t.

3. A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t.

4. A force of 13 N is needed to keep a spring with a 2-kg mass stretched 0.25 m beyond its natural length. The damping constant of the spring is $c = 8$.
(a) If the mass starts at the equilibrium position with a velocity of 0.5 m/s, find its position at time t.
(b) Graph the position function of the mass.

5. For the spring in Exercise 3, find the mass that would produce critical damping.

6. For the spring in Exercise 4, find the damping constant that would produce critical damping.

7. A spring has a mass of 1 kg and its spring constant is $k = 100$. The spring is released at a point 0.1 m above its equilibrium position. Graph the position function for the following values of the damping constant c: 10, 15, 20, 25, 30. What type of damping occurs in each case?

8. A spring has a mass of 1 kg and its damping constant is $c = 10$. The spring starts from its equilibrium position with a velocity of 1 m/s. Graph the position function for the following values of the spring constant k: 10, 20, 25, 30, 40. What type of damping occurs in each case?

9. Suppose a spring has mass m and spring constant k and let $\omega = \sqrt{k/m}$. Suppose that the damping constant is so small that the damping force is negligible. If an external force $F(t) = F_0 \cos \omega_0 t$ is applied, where $\omega_0 \neq \omega$, use the method of undetermined coefficients to show that the motion of the mass is described by Equation 6.

10. As in Exercise 9, consider a spring with mass m, spring constant k, and damping constant $c = 0$, and let $\omega = \sqrt{k/m}$. If an external force $F(t) = F_0 \cos \omega t$ is applied (the applied frequency equals the natural frequency), use the method of undetermined coefficients to show that the motion of the mass is given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t$$

11. Show that if $\omega_0 \neq \omega$, but ω/ω_0 is a rational number, then the motion described by Equation 6 is periodic.

12. Consider a spring subject to a frictional or damping force.
(a) In the critically damped case, the motion is given by $x = c_1 e^{rt} + c_2 t e^{rt}$. Show that the graph of x crosses the t-axis whenever c_1 and c_2 have opposite signs.
(b) In the overdamped case, the motion is given by $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $r_1 > r_2$. Determine a condition on the relative magnitudes of c_1 and c_2 under which the graph of x crosses the t-axis at a positive value of t.

13. A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1$ H, a capacitor with $C = 0.002$ F, and a 12-V battery. If the initial charge and current are both 0, find the charge and current at time t.

14. A series circuit contains a resistor with $R = 24\ \Omega$, an inductor with $L = 2$ H, a capacitor with $C = 0.005$ F, and a 12-V battery. The initial charge is $Q = 0.001$ C and the initial current is 0.
(a) Find the charge and current at time t.
(b) Graph the charge and current functions.

15. The battery in Exercise 13 is replaced by a generator producing a voltage of $E(t) = 12 \sin 10t$. Find the charge at time t.

16. The battery in Exercise 14 is replaced by a generator producing a voltage of $E(t) = 12 \sin 10t$.
(a) Find the charge at time t.
(b) Graph the charge function.

17. Verify that the solution to Equation 1 can be written in the form $x(t) = A \cos(\omega t + \delta)$.

18. The figure shows a pendulum with length L and the angle θ from the vertical to the pendulum. It can be shown that θ, as a function of time, satisfies the nonlinear differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where g is the acceleration due to gravity. For small values of θ we can use the linear approximation $\sin\theta \approx \theta$ and then the differential equation becomes linear.

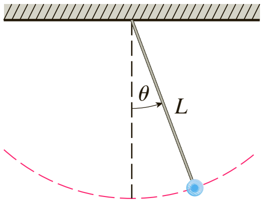
(a) Find the equation of motion of a pendulum with length 1 m if θ is initially 0.2 rad and the initial angular velocity is $d\theta/dt = 1$ rad/s.

(b) What is the maximum angle from the vertical?

(c) What is the period of the pendulum (that is, the time to complete one back-and-forth swing)?

(d) When will the pendulum first be vertical?

(e) What is the angular velocity when the pendulum is vertical?



17.4 Series Solutions

Many differential equations can't be solved explicitly in terms of finite combinations of simple familiar functions. This is true even for a simple-looking equation like

$$\boxed{1} \qquad y'' - 2xy' + y = 0$$

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quantum mechanics. In such a case we use the method of power series; that is, we look for a solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients $c_0, c_1, c_2, \ldots$. This technique resembles the method of undetermined coefficients discussed in Section 17.2.

Before using power series to solve Equation 1, we illustrate the method on the simpler equation $y'' + y = 0$ in Example 1. It's true that we already know how to solve this equation by the techniques of Section 17.1, but it's easier to understand the power series method when it is applied to this simpler equation.

EXAMPLE 1 Use power series to solve the equation $y'' + y = 0$.

SOLUTION We assume there is a solution of the form

$$\boxed{2} \qquad y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

We can differentiate power series term by term, so

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\boxed{3} \qquad y'' = 2c_2 + 2 \cdot 3c_3 x + \cdots = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

By writing out the first few terms of (4), you can see that it is the same as (3). To obtain (4), we replaced n by $n + 2$ and began the summation at 0 instead of 2.

In order to compare the expressions for y and y'' more easily, we rewrite y'' as follows:

$$\boxed{4} \qquad y'' = \sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\boxed{5} \qquad \sum_{n=0}^{\infty} [(n + 2)(n + 1)c_{n+2} + c_n]x^n = 0$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore the coefficients of x^n in Equation 5 must be 0:

$$(n + 2)(n + 1)c_{n+2} + c_n = 0$$

$$\boxed{6} \qquad c_{n+2} = -\frac{c_n}{(n + 1)(n + 2)} \qquad n = 0, 1, 2, 3, \ldots$$

Equation 6 is called a *recursion relation*. If c_0 and c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting $n = 0, 1, 2, 3, \ldots$ in succession.

Put $n = 0$: $\qquad c_2 = -\dfrac{c_0}{1 \cdot 2}$

Put $n = 1$: $\qquad c_3 = -\dfrac{c_1}{2 \cdot 3}$

Put $n = 2$: $\qquad c_4 = -\dfrac{c_2}{3 \cdot 4} = \dfrac{c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \dfrac{c_0}{4!}$

Put $n = 3$: $\qquad c_5 = -\dfrac{c_3}{4 \cdot 5} = \dfrac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \dfrac{c_1}{5!}$

Put $n = 4$: $\qquad c_6 = -\dfrac{c_4}{5 \cdot 6} = -\dfrac{c_0}{4!\,5 \cdot 6} = -\dfrac{c_0}{6!}$

Put $n = 5$: $\qquad c_7 = -\dfrac{c_5}{6 \cdot 7} = -\dfrac{c_1}{5!\,6 \cdot 7} = -\dfrac{c_1}{7!}$

By now we see the pattern:

For the even coefficients, $\quad c_{2n} = (-1)^n \dfrac{c_0}{(2n)!}$

For the odd coefficients, $\quad c_{2n+1} = (-1)^n \dfrac{c_1}{(2n + 1)!}$

Putting these values back into Equation 2, we write the solution as

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$$

$$= c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right)$$

$$+ c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Notice that there are two arbitrary constants, c_0 and c_1. ■

NOTE 1 We recognize the series obtained in Example 1 as being the Maclaurin series for $\cos x$ and $\sin x$. (See Equations 11.10.16 and 11.10.15.) Therefore we could write the solution as

$$y(x) = c_0 \cos x + c_1 \sin x$$

But we are not usually able to express power series solutions of differential equations in terms of known functions.

EXAMPLE 2 Solve $y'' - 2xy' + y = 0$.

SOLUTION We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$$

as in Example 1. Substituting in the differential equation, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - 2x \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2nc_n x^n = \sum_{n=0}^{\infty} 2nc_n x^n$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (2n-1)c_n]x^n = 0$$

This equation is true if the coefficients of x^n are 0:

$$(n+2)(n+1)c_{n+2} - (2n-1)c_n = 0$$

7
$$c_{n+2} = \frac{2n-1}{(n+1)(n+2)} c_n \qquad n = 0, 1, 2, 3, \ldots$$

We solve this recursion relation by putting $n = 0, 1, 2, 3, \ldots$ successively in Equation 7:

$$\text{Put } n = 0: \qquad c_2 = \frac{-1}{1 \cdot 2} c_0$$

$$\text{Put } n = 1: \qquad c_3 = \frac{1}{2 \cdot 3} c_1$$

$$\text{Put } n = 2: \qquad c_4 = \frac{3}{3 \cdot 4} c_2 = -\frac{3}{1 \cdot 2 \cdot 3 \cdot 4} c_0 = -\frac{3}{4!} c_0$$

$$\text{Put } n = 3: \qquad c_5 = \frac{5}{4 \cdot 5} c_3 = \frac{1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} c_1 = \frac{1 \cdot 5}{5!} c_1$$

$$\text{Put } n = 4: \qquad c_6 = \frac{7}{5 \cdot 6} c_4 = -\frac{3 \cdot 7}{4!\, 5 \cdot 6} c_0 = -\frac{3 \cdot 7}{6!} c_0$$

$$\text{Put } n = 5: \qquad c_7 = \frac{9}{6 \cdot 7} c_5 = \frac{1 \cdot 5 \cdot 9}{5!\, 6 \cdot 7} c_1 = \frac{1 \cdot 5 \cdot 9}{7!} c_1$$

$$\text{Put } n = 6: \qquad c_8 = \frac{11}{7 \cdot 8} c_6 = -\frac{3 \cdot 7 \cdot 11}{8!} c_0$$

$$\text{Put } n = 7: \qquad c_9 = \frac{13}{8 \cdot 9} c_7 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1$$

In general, the even coefficients are given by

$$c_{2n} = -\frac{3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n - 5)}{(2n)!} c_0$$

and the odd coefficients are given by

$$c_{2n+1} = \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n - 3)}{(2n + 1)!} c_1$$

The solution is

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \\ &= c_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{3 \cdot 7}{6!}x^6 - \frac{3 \cdot 7 \cdot 11}{8!}x^8 - \cdots\right) \\ &\quad + c_1\left(x + \frac{1}{3!}x^3 + \frac{1 \cdot 5}{5!}x^5 + \frac{1 \cdot 5 \cdot 9}{7!}x^7 + \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!}x^9 + \cdots\right) \end{aligned}$$

or

$$\boxed{8} \qquad \begin{aligned} y &= c_0\left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n - 5)}{(2n)!} x^{2n}\right) \\ &\quad + c_1\left(x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n - 3)}{(2n + 1)!} x^{2n+1}\right) \end{aligned}$$

NOTE 2 In Example 2 we had to *assume* that the differential equation had a series solution. But now we could verify directly that the function given by Equation 8 is indeed a solution.

NOTE 3 Unlike the situation of Example 1, the power series that arise in the solution of Example 2 do not define elementary functions. The functions

$$y_1(x) = 1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{(2n)!} x^{2n}$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

are perfectly good functions but they can't be expressed in terms of familiar functions. We can use these power series expressions for y_1 and y_2 to compute approximate values of the functions and even to graph them. Figure 1 shows the first few partial sums $T_0, T_2, T_4, \ldots$ (Taylor polynomials) for $y_1(x)$, and we see how they converge to y_1. In this way we can graph both y_1 and y_2 as in Figure 2.

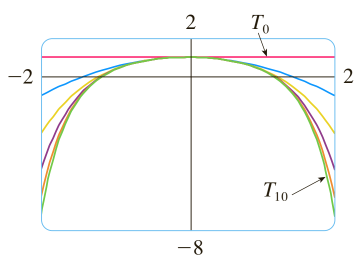


FIGURE 1

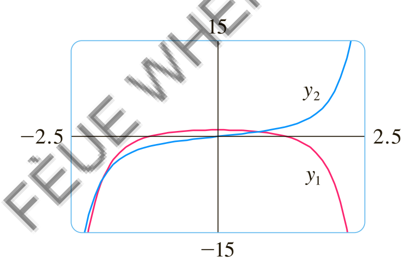


FIGURE 2

NOTE 4 If we were asked to solve the initial-value problem

$$y'' - 2xy' + y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

we would observe from Theorem 11.10.5 that

$$c_0 = y(0) = 0 \qquad c_1 = y'(0) = 1$$

This would simplify the calculations in Example 2, since all of the even coefficients would be 0. The solution to the initial-value problem is

$$y(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

17.4 EXERCISES

1–11 Use power series to solve the differential equation.

1. $y' - y = 0$

2. $y' = xy$

3. $y' = x^2 y$

4. $(x-3)y' + 2y = 0$

5. $y'' + xy' + y = 0$

6. $y'' = y$

7. $(x-1)y'' + y' = 0$

8. $y'' = xy$

9. $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$

10. $y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0$

11. $y'' + x^2 y' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1$

12. The solution of the initial-value problem

$$x^2 y'' + xy' + x^2 y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

is called a Bessel function of order 0.

(a) Solve the initial-value problem to find a power series expansion for the Bessel function.

(b) Graph several Taylor polynomials until you reach one that looks like a good approximation to the Bessel function on the interval $[-5, 5]$.

17 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) Write the general form of a second-order homogeneous linear differential equation with constant coefficients.
   (b) Write the auxiliary equation.
   (c) How do you use the roots of the auxiliary equation to solve the differential equation? Write the form of the solution for each of the three cases that can occur.
2. (a) What is an initial-value problem for a second-order differential equation?
   (b) What is a boundary-value problem for such an equation?
3. (a) Write the general form of a second-order nonhomogeneous linear differential equation with constant coefficients.
   (b) What is the complementary equation? How does it help solve the original differential equation?
   (c) Explain how the method of undetermined coefficients works.
   (d) Explain how the method of variation of parameters works.
4. Discuss two applications of second-order linear differential equations.
5. How do you use power series to solve a differential equation?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If y_1 and y_2 are solutions of $y'' + y = 0$, then $y_1 + y_2$ is also a solution of the equation.
2. If y_1 and y_2 are solutions of $y'' + 6y' + 5y = x$, then $c_1 y_1 + c_2 y_2$ is also a solution of the equation.
3. The general solution of $y'' - y = 0$ can be written as
$$y = c_1 \cosh x + c_2 \sinh x$$
4. The equation $y'' - y = e^x$ has a particular solution of the form
$$y_p = Ae^x$$

EXERCISES

1–10 Solve the differential equation.

1. $4y'' - y = 0$
2. $y'' - 2y' + 10y = 0$
3. $y'' + 3y = 0$
4. $y'' + 8y' + 16y = 0$
5. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = e^{2x}$
6. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x^2$
7. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x\cos x$
8. $\dfrac{d^2y}{dx^2} + 4y = \sin 2x$
9. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 1 + e^{-2x}$
10. $\dfrac{d^2y}{dx^2} + y = \csc x, \quad 0 < x < \pi/2$

11–14 Solve the initial-value problem.

11. $y'' + 6y' = 0, \quad y(1) = 3, \quad y'(1) = 12$
12. $y'' - 6y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = 1$
13. $y'' - 5y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
14. $9y'' + y = 3x + e^{-x}, \quad y(0) = 1, \quad y'(0) = 2$

15–16 Solve the boundary-value problem, if possible.

15. $y'' + 4y' + 29y = 0, \quad y(0) = 1, \quad y(\pi) = -1$
16. $y'' + 4y' + 29y = 0, \quad y(0) = 1, \quad y(\pi) = -e^{-2\pi}$

17. Use power series to solve the initial-value problem
$$y'' + xy' + y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$
18. Use power series to solve the differential equation
$$y'' - xy' - 2y = 0$$
19. A series circuit contains a resistor with $R = 40\ \Omega$, an inductor with $L = 2$ H, a capacitor with $C = 0.0025$ F, and a 12-V battery. The initial charge is $Q = 0.01$ C and the initial current is 0. Find the charge at time t.

20. A spring with a mass of 2 kg has damping constant 16, and a force of 12.8 N keeps the spring stretched 0.2 m beyond its natural length. Find the position of the mass at time t if it starts at the equilibrium position with a velocity of 2.4 m/s.

21. Assume that the earth is a solid sphere of uniform density with mass M and radius $R = 3960$ mi. For a particle of mass m within the earth at a distance r from the earth's center, the gravitational force attracting the particle to the center is

$$F_r = \frac{-GM_r m}{r^2}$$

where G is the gravitational constant and M_r is the mass of the earth within the sphere of radius r.

(a) Show that $F_r = \dfrac{-GMm}{R^3} r$.

(b) Suppose a hole is drilled through the earth along a diameter. Show that if a particle of mass m is dropped from rest at the surface, into the hole, then the distance $y = y(t)$ of the particle from the center of the earth at time t is given by

$$y''(t) = -k^2 y(t)$$

where $k^2 = GM/R^3 = g/R$.

(c) Conclude from part (b) that the particle undergoes simple harmonic motion. Find the period T.

(d) With what speed does the particle pass through the center of the earth?

Appendixes

A Numbers, Inequalities, and Absolute Values

Calculus is based on the real number system. We start with the **integers**:

$$\ldots, \quad -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \ldots$$

Then we construct the **rational numbers**, which are ratios of integers. Thus any rational number r can be expressed as

$$r = \frac{m}{n} \qquad \text{where } m \text{ and } n \text{ are integers and } n \neq 0$$

Examples are

$$\tfrac{1}{2} \qquad -\tfrac{3}{7} \qquad 46 = \tfrac{46}{1} \qquad 0.17 = \tfrac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) Some real numbers, such as $\sqrt{2}$, can't be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that the following are also irrational numbers:

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt[3]{2} \qquad \pi \qquad \sin 1^\circ \qquad \log_{10} 2$$

The set of all real numbers is usually denoted by the symbol $\mathbb{R}$. When we use the word *number* without qualification, we mean "real number."

Every number has a decimal representation. If the number is rational, then the corresponding decimal is repeating. For example,

$$\tfrac{1}{2} = 0.5000\ldots = 0.5\overline{0} \qquad \tfrac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\tfrac{157}{495} = 0.317171717\ldots = 0.3\overline{17} \qquad \tfrac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

(The bar indicates that the sequence of digits repeats forever.) On the other hand, if the number is irrational, the decimal is nonrepeating:

$$\sqrt{2} = 1.414213562373095\ldots \qquad \pi = 3.141592653589793\ldots$$

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol $\approx$ is read "is approximately equal to." The more decimal places we retain, the better the approximation we get.

The real numbers can be represented by points on a line as in Figure 1. The positive direction (to the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number $-x$ is represented by the point x units to the left of the origin. Thus every real number is represented by a point on the line, and every point P on the line corresponds to exactly one real number. The number associated with the point P is called the **coordinate** of P and the line is then called a

coordinate line, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

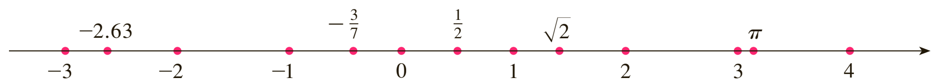


FIGURE 1

The real numbers are ordered. We say a *is less than* b and write $a < b$ if $b - a$ is a positive number. Geometrically this means that a lies to the left of b on the number line. (Equivalently, we say b *is greater than* a and write $b > a$.) The symbol $a \leqslant b$ (or $b \geqslant a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b." For instance, the following are true inequalities:

$$7 < 7.4 < 7.5 \qquad -3 > -\pi \qquad \sqrt{2} < 2 \qquad \sqrt{2} \leqslant 2 \qquad 2 \leqslant 2$$

In what follows we need to use *set notation*. A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $a \notin S$ means that a is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$. If S and T are sets, then their **union** $S \cup T$ is the set consisting of all elements that are in S or T (or in both S and T). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S and T. In other words, $S \cap T$ is the common part of S and T. The empty set, denoted by $\varnothing$, is the set that contains no element.

Some sets can be described by listing their elements between braces. For instance, the set A consisting of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in *set-builder notation* as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of x such that x is an integer and $0 < x < 7$."

Intervals

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\}$$

FIGURE 2
Open interval (a, b)

Notice that the endpoints of the interval—namely, a and b—are excluded. This is indicated by the round brackets ( ) and by the open dots in Figure 2. The **closed interval** from a to b is the set

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}$$

FIGURE 3
Closed interval $[a, b]$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in Figure 3. It is also possible to include only one endpoint in an interval, as shown in Table 1.

We also need to consider infinite intervals such as

$$(a, \infty) = \{x \mid x > a\}$$

This does not mean that ∞ ("infinity") is a number. The notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

1 Table of Intervals

Table 1 lists the nine possible types of intervals. When these intervals are discussed, it is always assumed that $a < b$.

Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$	
$[a, b)$	$\{x \mid a \leqslant x < b\}$	
$(a, b]$	$\{x \mid a < x \leqslant b\}$	
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geqslant a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leqslant b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Inequalities

When working with inequalities, note the following rules.

2 Rules for Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $1/a > 1/b$.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but Rule 4 says that if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality. For example, if we take the inequality $3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

EXAMPLE 1 Solve the inequality $1 + x < 7x + 5$.

SOLUTION The given inequality is satisfied by some values of x but not by others. To *solve* an inequality means to determine the set of numbers x for which the inequality is true. This is called the *solution set*.

First we subtract 1 from each side of the inequality (using Rule 1 with $c = -1$):

$$x < 7x + 4$$

Then we subtract $7x$ from both sides (Rule 1 with $c = -7x$):

$$-6x < 4$$

Now we divide both sides by -6 $\left(\text{Rule 4 with } c = -\frac{1}{6}\right)$:

$$x > -\tfrac{4}{6} = -\tfrac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. ■

EXAMPLE 2 Solve the inequalities $4 \leqslant 3x - 2 < 13$.

SOLUTION Here the solution set consists of all values of x that satisfy both inequalities. Using the rules given in (2), we see that the following inequalities are equivalent:

$$4 \leqslant 3x - 2 < 13$$

$$6 \leqslant 3x < 15 \qquad \text{(add 2)}$$

$$2 \leqslant x < 5 \qquad \text{(divide by 3)}$$

Therefore the solution set is $[2, 5)$. ■

EXAMPLE 3 Solve the inequality $x^2 - 5x + 6 \leqslant 0$.

SOLUTION First we factor the left side:

$$(x - 2)(x - 3) \leqslant 0$$

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty)$$

A visual method for solving Example 3 is to use a graphing device to graph the parabola $y = x^2 - 5x + 6$ (as in Figure 4) and observe that the curve lies on or below the x-axis when $2 \leqslant x \leqslant 3$.

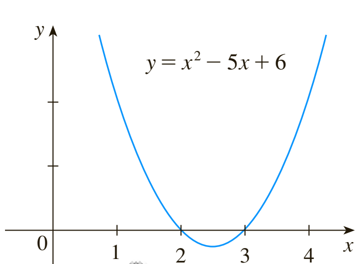


FIGURE 4

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

Another method for obtaining the information in the chart is to use *test values*. For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$, then substitution in $x^2 - 5x + 6$ gives

$$1^2 - 5(1) + 6 = 2$$

The polynomial $x^2 - 5x + 6$ doesn't change sign inside any of the three intervals, so we conclude that it is positive on $(-\infty, 2)$.

Then we read from the chart that $(x - 2)(x - 3)$ is negative when $2 < x < 3$. Thus the solution of the inequality $(x - 2)(x - 3) \leqslant 0$ is

$$\{x \mid 2 \leqslant x \leqslant 3\} = [2, 3]$$

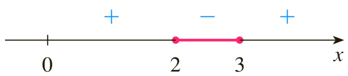


FIGURE 5

Notice that we have included the endpoints 2 and 3 because we are looking for values of x such that the product is either negative or zero. The solution is illustrated in Figure 5. ■

EXAMPLE 4 Solve $x^3 + 3x^2 > 4x$.

SOLUTION First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \qquad \text{or} \qquad x(x - 1)(x + 4) > 0$$

As in Example 3 we solve the corresponding equation $x(x - 1)(x + 4) = 0$ and use the solutions $x = -4$, $x = 0$, and $x = 1$ to divide the real line into four intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 1)$, and $(1, \infty)$. On each interval the product keeps a constant sign, which we list in the following chart:

Interval	x	$x - 1$	$x + 4$	$x(x - 1)(x + 4)$
$x < -4$	−	−	−	−
$-4 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$

FIGURE 6

The solution is illustrated in Figure 6. ■

Absolute Value

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

Remember that if a is negative, then $-a$ is positive.

3
$$\boxed{\begin{aligned} |a| &= a \qquad \text{if } a \geqslant 0 \\ |a| &= -a \qquad \text{if } a < 0 \end{aligned}}$$

EXAMPLE 5 Express $|3x - 2|$ without using the absolute-value symbol.

SOLUTION

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geqslant 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$

$$= \begin{cases} 3x - 2 & \text{if } x \geqslant \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$ ■

Recall that the symbol $\sqrt{\ }$ means "the positive square root of." Thus $\sqrt{r} = s$ means $s^2 = r$ and $s \geqslant 0$. Therefore the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geqslant 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of (3), we then have the equation

4 $$\boxed{\sqrt{a^2} = |a|}$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

5 Properties of Absolute Values Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|} \quad (b \neq 0)$ **3.** $|a^n| = |a|^n$

For solving equations or inequalities involving absolute values, it's often very helpful to use the following statements.

6 Suppose $a > 0$. Then

4. $|x| = a$ if and only if $x = \pm a$

5. $|x| < a$ if and only if $-a < x < a$

6. $|x| > a$ if and only if $x > a$ or $x < -a$

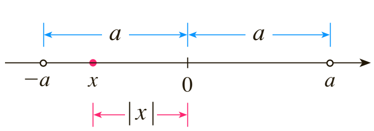


FIGURE 7

For instance, the inequality $|x| < a$ says that the distance from x to the origin is less than a, and you can see from Figure 7 that this is true if and only if x lies between $-a$ and a.

If a and b are any real numbers, then the distance between a and b is the absolute value of the difference, namely, $|a - b|$, which is also equal to $|b - a|$. (See Figure 8.)

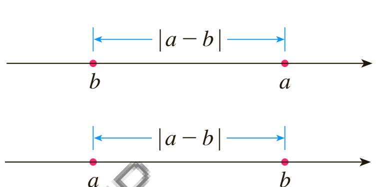


FIGURE 8
Length of a line segment $= |a - b|$

EXAMPLE 6 Solve $|2x - 5| = 3$.

SOLUTION By Property 4 of (6), $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \qquad \text{or} \qquad 2x - 5 = -3$$

So $2x = 8$ or $2x = 2$. Thus $x = 4$ or $x = 1$. ■

EXAMPLE 7 Solve $|x - 5| < 2$.

SOLUTION 1 By Property 5 of (6), $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to each side, we have

$$3 < x < 7$$

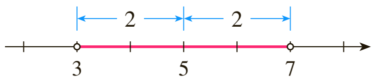


FIGURE 9

and the solution set is the open interval (3, 7).

SOLUTION 2 Geometrically the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 9 we see that this is the interval (3, 7). ■

EXAMPLE 8 Solve $|3x + 2| \geqslant 4$.

SOLUTION By Properties 4 and 6 of (6), $|3x + 2| \geqslant 4$ is equivalent to

$$3x + 2 \geqslant 4 \qquad \text{or} \qquad 3x + 2 \leqslant -4$$

In the first case $3x \geqslant 2$, which gives $x \geqslant \frac{2}{3}$. In the second case $3x \leqslant -6$, which gives $x \leqslant -2$. So the solution set is

$$\left\{x \mid x \leqslant -2 \text{ or } x \geqslant \tfrac{2}{3}\right\} = (-\infty, -2] \cup \left[\tfrac{2}{3}, \infty\right)$$ ■

Another important property of absolute value, called the Triangle Inequality, is used frequently not only in calculus but throughout mathematics in general.

7 The Triangle Inequality If a and b are any real numbers, then

$$|a + b| \leqslant |a| + |b|$$

Observe that if the numbers a and b are both positive or both negative, then the two sides in the Triangle Inequality are actually equal. But if a and b have opposite signs, the left side involves a subtraction and the right side does not. This makes the Triangle Inequality seem reasonable, but we can prove it as follows.

Notice that

$$-|a| \leqslant a \leqslant |a|$$

is always true because a equals either $|a|$ or $-|a|$. The corresponding statement for b is

$$-|b| \leqslant b \leqslant |b|$$

Adding these inequalities, we get

$$-(|a| + |b|) \leqslant a + b \leqslant |a| + |b|$$

If we now apply Properties 4 and 5 (with x replaced by $a + b$ and a by $|a| + |b|$), we obtain

$$|a + b| \leqslant |a| + |b|$$

which is what we wanted to show.

EXAMPLE 9 If $|x - 4| < 0.1$ and $|y - 7| < 0.2$, use the Triangle Inequality to estimate $|(x + y) - 11|$.

SOLUTION In order to use the given information, we use the Triangle Inequality with $a = x - 4$ and $b = y - 7$:

$$\begin{aligned} |(x + y) - 11| &= |(x - 4) + (y - 7)| \\ &\leqslant |x - 4| + |y - 7| \\ &< 0.1 + 0.2 = 0.3 \end{aligned}$$

Thus $$|(x + y) - 11| < 0.3$$ ■

A EXERCISES

1–12 Rewrite the expression without using the absolute-value symbol.

1. $|5 - 23|$

2. $|5| - |-23|$

3. $|-\pi|$

4. $|\pi - 2|$

5. $|\sqrt{5} - 5|$

6. $||-2| - |-3||$

7. $|x - 2|$ if $x < 2$

8. $|x - 2|$ if $x > 2$

9. $|x + 1|$

10. $|2x - 1|$

11. $|x^2 + 1|$

12. $|1 - 2x^2|$

13–38 Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

13. $2x + 7 > 3$

14. $3x - 11 < 4$

15. $1 - x \leqslant 2$

16. $4 - 3x \geqslant 6$

17. $2x + 1 < 5x - 8$

18. $1 + 5x > 5 - 3x$

19. $-1 < 2x - 5 < 7$

20. $1 < 3x + 4 \leqslant 16$

21. $0 \leqslant 1 - x < 1$

22. $-5 \leqslant 3 - 2x \leqslant 9$

23. $4x < 2x + 1 \leqslant 3x + 2$

24. $2x - 3 < x + 4 < 3x - 2$

25. $(x - 1)(x - 2) > 0$

26. $(2x + 3)(x - 1) \geqslant 0$

27. $2x^2 + x \leqslant 1$

28. $x^2 < 2x + 8$

29. $x^2 + x + 1 > 0$

30. $x^2 + x > 1$

31. $x^2 < 3$

32. $x^2 \geqslant 5$

33. $x^3 - x^2 \leqslant 0$

34. $(x + 1)(x - 2)(x + 3) \geqslant 0$

35. $x^3 > x$

36. $x^3 + 3x < 4x^2$

37. $\dfrac{1}{x} < 4$

38. $-3 < \dfrac{1}{x} \leqslant 1$

39. The relationship between the Celsius and Fahrenheit temperature scales is given by $C = \frac{5}{9}(F - 32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range $50 \leqslant F \leqslant 95$?

40. Use the relationship between C and F given in Exercise 39 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leqslant C \leqslant 30$.

41. As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100-m rise, up to about 12 km.
(a) If the ground temperature is 20°C, write a formula for the temperature at height h.
(b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?

42. If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

43–46 Solve the equation for x.

43. $|2x| = 3$

44. $|3x + 5| = 1$

45. $|x + 3| = |2x + 1|$

46. $\left|\dfrac{2x - 1}{x + 1}\right| = 3$

47–56 Solve the inequality.

47. $|x| < 3$

48. $|x| \geqslant 3$

49. $|x - 4| < 1$

50. $|x - 6| < 0.1$

51. $|x + 5| \geqslant 2$

52. $|x + 1| \geqslant 3$

53. $|2x - 3| \leqslant 0.4$

54. $|5x - 2| < 6$

55. $1 \leqslant |x| \leqslant 4$

56. $0 < |x - 5| < \frac{1}{2}$

57–58 Solve for x, assuming a, b, and c are positive constants.

57. $a(bx - c) \geqslant bc$ **58.** $a \leqslant bx + c < 2a$

59–60 Solve for x, assuming a, b, and c are negative constants.

59. $ax + b < c$ **60.** $\dfrac{ax + b}{c} \leqslant b$

61. Suppose that $|x - 2| < 0.01$ and $|y - 3| < 0.04$. Use the Triangle Inequality to show that $|(x + y) - 5| < 0.05$.

62. Show that if $|x + 3| < \frac{1}{2}$, then $|4x + 13| < 3$.

63. Show that if $a < b$, then $a < \dfrac{a + b}{2} < b$.

64. Use Rule 3 to prove Rule 5 of (2).

65. Prove that $|ab| = |a||b|$. [*Hint:* Use Equation 4.]

66. Prove that $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$.

67. Show that if $0 < a < b$, then $a^2 < b^2$.

68. Prove that $|x - y| \geqslant |x| - |y|$. [*Hint:* Use the Triangle Inequality with $a = x - y$ and $b = y$.]

69. Show that the sum, difference, and product of rational numbers are rational numbers.

70. (a) Is the sum of two irrational numbers always an irrational number?

(b) Is the product of two irrational numbers always an irrational number?

B Coordinate Geometry and Lines

Just as the points on a line can be identified with real numbers by assigning them coordinates, as described in Appendix A, so the points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin O on each line. Usually one line is horizontal with positive direction to the right and is called the x-axis; the other line is vertical with positive direction upward and is called the y-axis.

Any point P in the plane can be located by a unique ordered pair of numbers as follows. Draw lines through P perpendicular to the x- and y-axes. These lines intersect the axes in points with coordinates a and b as shown in Figure 1. Then the point P is assigned the ordered pair (a, b). The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P(a, b)$. Several points are labeled with their coordinates in Figure 2.

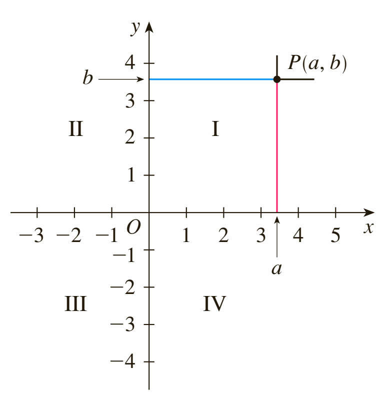


FIGURE 1

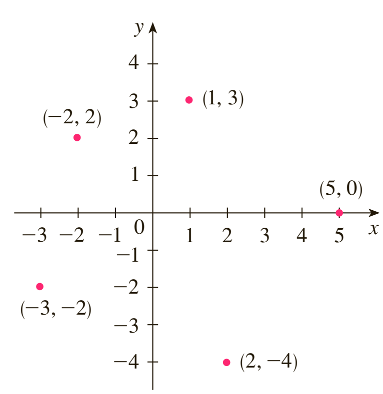


FIGURE 2

By reversing the preceding process we can start with an ordered pair (a, b) and arrive at the corresponding point P. Often we identify the point P with the ordered pair (a, b) and refer to "the point (a, b)." [Although the notation used for an open interval (a, b) is the same as the notation used for a point (a, b), you will be able to tell from the context which meaning is intended.]

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system** in honor of the French mathematician René Descartes (1596–1650), even though another Frenchman, Pierre Fermat (1601–1665), invented the principles of analytic geometry at about the same time as Descartes. The plane supplied with this coordinate system is called the **coordinate plane** or the **Cartesian plane** and is denoted by $\mathbb{R}^2$.

The x- and y-axes are called the **coordinate axes** and divide the Cartesian plane into four quadrants, which are labeled I, II, III, and IV in Figure 1. Notice that the first quadrant consists of those points whose x- and y-coordinates are both positive.

EXAMPLE 1 Describe and sketch the regions given by the following sets.

(a) $\{(x, y) \mid x \geqslant 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

SOLUTION

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it as indicated by the shaded region in Figure 3(a).

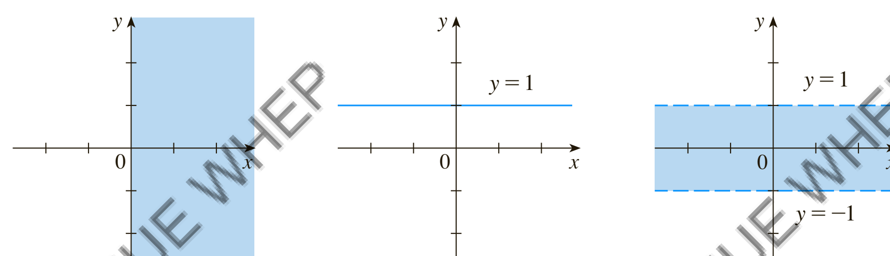


FIGURE 3 (a) $x \geqslant 0$ (b) $y = 1$ (c) $|y| < 1$

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis [see Figure 3(b)].

(c) Recall from Appendix A that

$$|y| < 1 \quad \text{if and only if} \quad -1 < y < 1$$

The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. [These lines are shown as dashed lines in Figure 3(c) to indicate that the points on these lines don't lie in the set.] ■

Recall from Appendix A that the distance between points a and b on a number line is $|a - b| = |b - a|$. Thus the distance between points $P_1(x_1, y_1)$ and $P_3(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$ and the distance between $P_2(x_2, y_2)$ and $P_3(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$. (See Figure 4.)

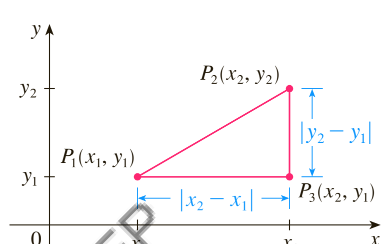


FIGURE 4

To find the distance $|P_1P_2|$ between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we note that triangle $P_1P_2P_3$ in Figure 4 is a right triangle, and so by the Pythagorean Theorem we have

$$|P_1P_2| = \sqrt{|P_1P_3|^2 + |P_2P_3|^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

> **1** **Distance Formula** The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is
>
> $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2 The distance between $(1, -2)$ and $(5, 3)$ is

$$\sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

Lines

We want to find an equation of a given line L; such an equation is satisfied by the coordinates of the points on L and by no other point. To find the equation of L we use its *slope,* which is a measure of the steepness of the line.

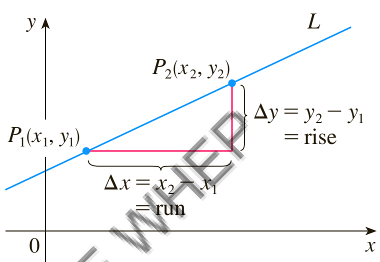


FIGURE 5

> **2** **Definition** The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is
>
> $$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$
>
> The slope of a vertical line is not defined.

Thus the slope of a line is the ratio of the change in y, Δy, to the change in x, Δx. (See Figure 5.) The slope is therefore the rate of change of y with respect to x. The fact that the line is straight means that the rate of change is constant.

Figure 6 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice also that the steepest lines are the ones for which the absolute value of the slope is largest, and a horizontal line has slope 0.

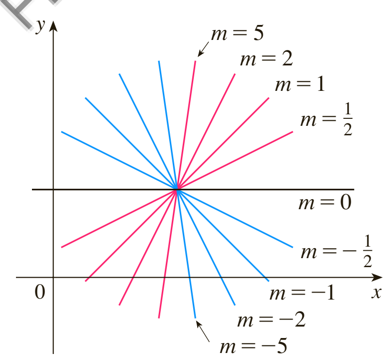


FIGURE 6

Now let's find an equation of the line that passes through a given point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m; that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

and we observe that this equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line.

> **3** **Point-Slope Form of the Equation of a Line** An equation of the line passing through the point $P_1(x_1, y_1)$ and having slope m is
>
> $$y - y_1 = m(x - x_1)$$

EXAMPLE 3 Find an equation of the line through $(1, -7)$ with slope $-\frac{1}{2}$.

SOLUTION Using (3) with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -7$, we obtain an equation of the line as

$$y + 7 = -\tfrac{1}{2}(x - 1)$$

which we can rewrite as

$$2y + 14 = -x + 1 \qquad \text{or} \qquad x + 2y + 13 = 0$$

EXAMPLE 4 Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

SOLUTION By Definition 2 the slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1)$$

which simplifies to $$3x + 2y = 1$$

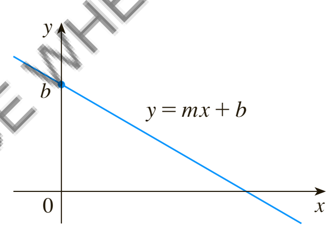


FIGURE 7

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 7.) This means it intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x_1 = 0$ and $y_1 = b$, becomes

$$y - b = m(x - 0)$$

This simplifies as follows.

4 Slope-Intercept Form of the Equation of a Line An equation of the line with slope m and y-intercept b is

$$y = mx + b$$

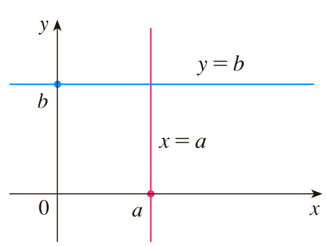


FIGURE 8

In particular, if a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 8). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

Observe that the equation of every line can be written in the form

$$\boxed{Ax + By + C = 0} \tag{5}$$

because a vertical line has the equation $x = a$ or $x - a = 0$ ($A = 1$, $B = 0$, $C = -a$) and a nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$ ($A = -m$, $B = 1$, $C = -b$). Conversely, if we start with a general first-degree equation, that is, an equation of the form (5), where A, B, and C are constants and A and B are not both 0, then we can show that it is the equation of a line. If $B = 0$, the equation becomes $Ax + C = 0$ or $x = -C/A$, which represents a vertical line with x-intercept $-C/A$. If $B \neq 0$, the equation can be rewritten by solving for y:

$$y = -\frac{A}{B}x - \frac{C}{B}$$

and we recognize this as being the slope-intercept form of the equation of a line ($m = -A/B$, $b = -C/B$). Therefore an equation of the form (5) is called a **linear equation** or the **general equation of a line**. For brevity, we often refer to "the line $Ax + By + C = 0$" instead of "the line whose equation is $Ax + By + C = 0$."

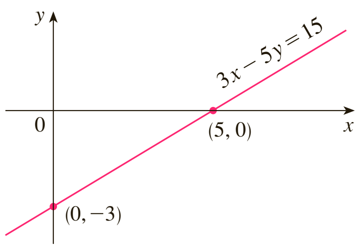


FIGURE 9

EXAMPLE 5 Sketch the graph of the equation $3x - 5y = 15$.

SOLUTION Since the equation is linear, its graph is a line. To draw the graph, we can simply find two points on the line. It's easiest to find the intercepts. Substituting $y = 0$ (the equation of the x-axis) in the given equation, we get $3x = 15$, so $x = 5$ is the x-intercept. Substituting $x = 0$ in the equation, we see that the y-intercept is -3. This allows us to sketch the graph as in Figure 9. ■

EXAMPLE 6 Graph the inequality $x + 2y > 5$.

SOLUTION We are asked to sketch the graph of the set $\{(x, y) \mid x + 2y > 5\}$ and we begin by solving the inequality for y:

$$x + 2y > 5$$
$$2y > -x + 5$$
$$y > -\tfrac{1}{2}x + \tfrac{5}{2}$$

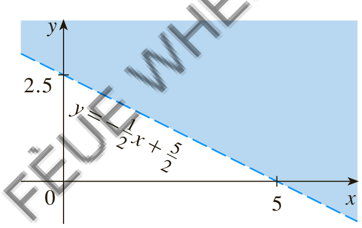


FIGURE 10

Compare this inequality with the equation $y = -\frac{1}{2}x + \frac{5}{2}$, which represents a line with slope $-\frac{1}{2}$ and y-intercept $\frac{5}{2}$. We see that the given graph consists of points whose y-coordinates are *larger* than those on the line $y = -\frac{1}{2}x + \frac{5}{2}$. Thus the graph is the region that lies *above* the line, as illustrated in Figure 10. ■

Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved, for instance, in *Precalculus: Mathematics for Calculus,* Seventh Edition, by Stewart, Redlin, and Watson (Belmont, CA, 2016).

6 Parallel and Perpendicular Lines

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

EXAMPLE 7 Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

SOLUTION The given line can be written in the form

$$y = -\tfrac{2}{3}x - \tfrac{5}{6}$$

which is in slope-intercept form with $m = -\frac{2}{3}$. Parallel lines have the same slope, so the required line has slope $-\frac{2}{3}$ and its equation in point-slope form is

$$y - 2 = -\tfrac{2}{3}(x - 5)$$

We can write this equation as $2x + 3y = 16$. ■

EXAMPLE 8 Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

SOLUTION The equations can be written as

$$y = -\tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{and} \qquad y = \tfrac{3}{2}x - \tfrac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\tfrac{2}{3} \qquad \text{and} \qquad m_2 = \tfrac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular. ■

B EXERCISES

1–6 Find the distance between the points.

1. $(1, 1), \quad (4, 5)$

2. $(1, -3), \quad (5, 7)$

3. $(6, -2), \quad (-1, 3)$

4. $(1, -6), \quad (-1, -3)$

5. $(2, 5), \quad (4, -7)$

6. $(a, b), \quad (b, a)$

7–10 Find the slope of the line through P and Q.

7. $P(1, 5), \quad Q(4, 11)$

8. $P(-1, 6), \quad Q(4, -3)$

9. $P(-3, 3), \quad Q(-1, -6)$

10. $P(-1, -4), \quad Q(6, 0)$

11. Show that the triangle with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$ is isosceles.

12. (a) Show that the triangle with vertices $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$ is a right triangle using the converse of the Pythagorean Theorem.
(b) Use slopes to show that ABC is a right triangle.
(c) Find the area of the triangle.

13. Show that the points $(-2, 9)$, $(4, 6)$, $(1, 0)$, and $(-5, 3)$ are the vertices of a square.

14. (a) Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear (lie on the same line) by showing that $|AB| + |BC| = |AC|$.
(b) Use slopes to show that A, B, and C are collinear.

15. Show that $A(1, 1)$, $B(7, 4)$, $C(5, 10)$, and $D(-1, 7)$ are vertices of a parallelogram.

16. Show that $A(1, 1)$, $B(11, 3)$, $C(10, 8)$, and $D(0, 6)$ are vertices of a rectangle.

17–20 Sketch the graph of the equation.

17. $x = 3$

18. $y = -2$

19. $xy = 0$

20. $|y| = 1$

21–36 Find an equation of the line that satisfies the given conditions.

21. Through $(2, -3)$, slope 6

22. Through $(-1, 4)$, slope -3

23. Through $(1, 7)$, slope $\frac{2}{3}$

24. Through $(-3, -5)$, slope $-\frac{7}{2}$

25. Through $(2, 1)$ and $(1, 6)$

26. Through $(-1, -2)$ and $(4, 3)$

27. Slope 3, y-intercept -2

28. Slope $\frac{2}{5}$, y-intercept 4

29. x-intercept 1, y-intercept -3

30. x-intercept -8, y-intercept 6

31. Through $(4, 5)$, parallel to the x-axis

32. Through $(4, 5)$, parallel to the y-axis

33. Through $(1, -6)$, parallel to the line $x + 2y = 6$

34. y-intercept 6, parallel to the line $2x + 3y + 4 = 0$

35. Through $(-1, -2)$, perpendicular to the line $2x + 5y + 8 = 0$

36. Through $\left(\frac{1}{2}, -\frac{2}{3}\right)$, perpendicular to the line $4x - 8y = 1$

37–42 Find the slope and y-intercept of the line and draw its graph.

37. $x + 3y = 0$

38. $2x - 5y = 0$

39. $y = -2$

40. $2x - 3y + 6 = 0$

41. $3x - 4y = 12$

42. $4x + 5y = 10$

43–52 Sketch the region in the xy-plane.

43. $\{(x, y) \mid x < 0\}$

44. $\{(x, y) \mid y > 0\}$

45. $\{(x, y) \mid xy < 0\}$

46. $\{(x, y) \mid x \geqslant 1 \text{ and } y < 3\}$

47. $\{(x, y) \mid |x| \leqslant 2\}$

48. $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$

49. $\{(x, y) \mid 0 \leqslant y \leqslant 4 \text{ and } x \leqslant 2\}$

50. $\{(x, y) \mid y > 2x - 1\}$

51. $\{(x, y) \mid 1 + x \leqslant y \leqslant 1 - 2x\}$

52. $\{(x, y) \mid -x \leqslant y < \frac{1}{2}(x + 3)\}$

53. Find a point on the y-axis that is equidistant from $(5, -5)$ and $(1, 1)$.

54. Show that the midpoint of the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

55. Find the midpoint of the line segment joining the given points.
(a) $(1, 3)$ and $(7, 15)$ (b) $(-1, 6)$ and $(8, -12)$

56. Find the lengths of the medians of the triangle with vertices $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$. (A median is a line segment from a vertex to the midpoint of the opposite side.)

57. Show that the lines $2x - y = 4$ and $6x - 2y = 10$ are not parallel and find their point of intersection.

58. Show that the lines $3x - 5y + 19 = 0$ and $10x + 6y - 50 = 0$ are perpendicular and find their point of intersection.

59. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

60. (a) Find equations for the sides of the triangle with vertices $P(1, 0)$, $Q(3, 4)$, and $R(-1, 6)$.
(b) Find equations for the medians of this triangle. Where do they intersect?

61. (a) Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be put in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is called the **two-intercept form** of an equation of a line.
(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

62. A car leaves Detroit at 2:00 PM, traveling at a constant speed west along I-96. It passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.
(a) Express the distance traveled in terms of the time elapsed.
(b) Draw the graph of the equation in part (a).
(c) What is the slope of this line? What does it represent?

C Graphs of Second-Degree Equations

In Appendix B we saw that a first-degree, or linear, equation $Ax + By + C = 0$ represents a line. In this section we discuss second-degree equations such as

$$x^2 + y^2 = 1 \qquad y = x^2 + 1 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad x^2 - y^2 = 1$$

which represent a circle, a parabola, an ellipse, and a hyperbola, respectively.

The graph of such an equation in x and y is the set of all points (x, y) that satisfy the equation; it gives a visual representation of the equation. Conversely, given a curve in the xy-plane, we may have to find an equation that represents it, that is, an equation satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the basic principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the geometric problem.

Circles

As an example of this type of problem, let's find an equation of the circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance

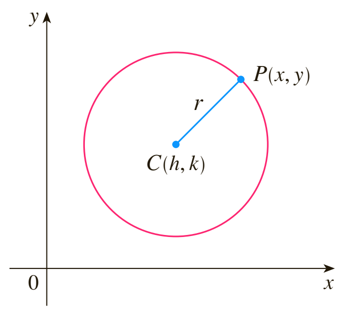


FIGURE 1

from the center $C(h, k)$ is r. (See Figure 1.) Thus P is on the circle if and only if $|PC| = r$. From the distance formula, we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x - h)^2 + (y - k)^2 = r^2$$

This is the desired equation.

1 Equation of a Circle An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2$$

EXAMPLE 1 Find an equation of the circle with radius 3 and center $(2, -5)$.

SOLUTION From Equation 1 with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x - 2)^2 + (y + 5)^2 = 9$$

■

EXAMPLE 2 Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

SOLUTION We first group the x-terms and y-terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

Then we complete the square within each grouping, adding the appropriate constants (the squares of half the coefficients of x and y) to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or

$$(x + 1)^2 + (y - 3)^2 = 3$$

Comparing this equation with the standard equation of a circle (1), we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$. It is sketched in Figure 2.

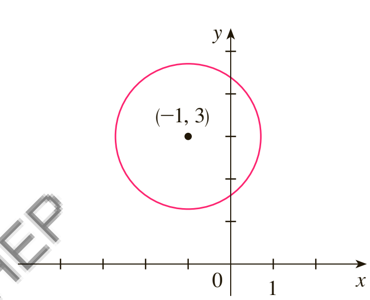


FIGURE 2
$x^2 + y^2 + 2x - 6y + 7 = 0$

■

Parabolas

The geometric properties of parabolas are reviewed in Section 10.5. Here we regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$.

EXAMPLE 3 Draw the graph of the parabola $y = x^2$.

SOLUTION We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure 3.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

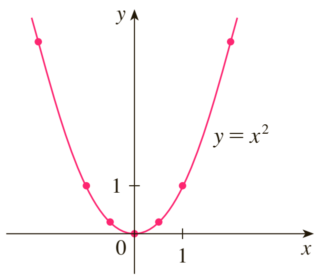


FIGURE 3 ■

Figure 4 shows the graphs of several parabolas with equations of the form $y = ax^2$ for various values of the number a. In each case the *vertex*, the point where the parabola changes direction, is the origin. We see that the parabola $y = ax^2$ opens upward if $a > 0$ and downward if $a < 0$ (as in Figure 5).

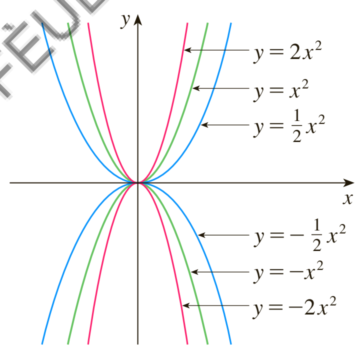
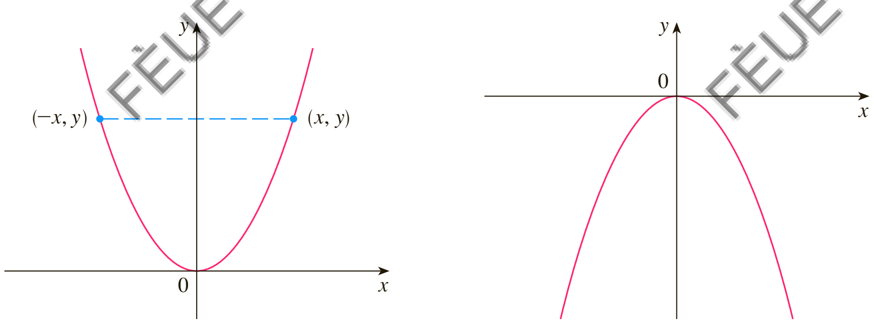


FIGURE 4



(a) $y = ax^2,\ a > 0$

(b) $y = ax^2,\ a < 0$

FIGURE 5

Notice that if (x, y) satisfies $y = ax^2$, then so does $(-x, y)$. This corresponds to the geometric fact that if the right half of the graph is reflected about the y-axis, then the left half of the graph is obtained. We say that the graph is **symmetric with respect to the y-axis**.

> The graph of an equation is symmetric with respect to the y-axis if the equation is unchanged when x is replaced by $-x$.

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See

Figure 6.) This time the parabola is symmetric with respect to the x-axis because if (x, y) satisfies $x = ay^2$, then so does $(x, -y)$.

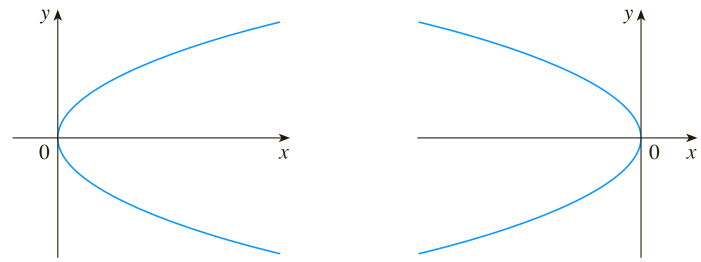


FIGURE 6 (a) $x = ay^2$, $a > 0$ (b) $x = ay^2$, $a < 0$

> The graph of an equation is symmetric with respect to the x-axis if the equation is unchanged when y is replaced by $-y$.

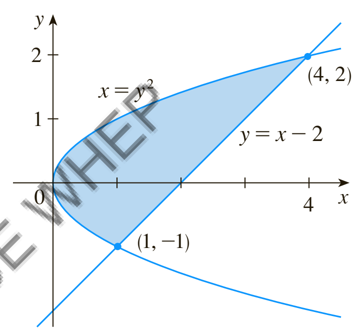


FIGURE 7

EXAMPLE 4 Sketch the region bounded by the parabola $x = y^2$ and the line $y = x - 2$.

SOLUTION First we find the points of intersection by solving the two equations. Substituting $x = y + 2$ into the equation $x = y^2$, we get $y + 2 = y^2$, which gives

$$0 = y^2 - y - 2 = (y - 2)(y + 1)$$

so $y = 2$ or -1. Thus the points of intersection are $(4, 2)$ and $(1, -1)$, and we draw the line $y = x - 2$ passing through these points. We then sketch the parabola $x = y^2$ by referring to Figure 6(a) and having the parabola pass through $(4, 2)$ and $(1, -1)$. The region bounded by $x = y^2$ and $y = x - 2$ means the finite region whose boundaries are these curves. It is sketched in Figure 7. ■

Ellipses

The curve with equation

$$\boxed{2} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive numbers, is called an **ellipse** in standard position. (Geometric properties of ellipses are discussed in Section 10.5.) Observe that Equation 2 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric with respect to both axes. As a further aid to sketching the ellipse, we find its intercepts.

> The ***x*-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph.
>
> The ***y*-intercepts** are the y-coordinates of the points where the graph intersects the y-axis. They are found by setting $x = 0$ in its equation.

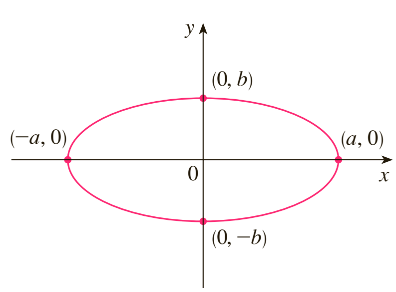


FIGURE 8
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

If we set $y = 0$ in Equation 2, we get $x^2 = a^2$ and so the x-intercepts are $\pm a$. Setting $x = 0$, we get $y^2 = b^2$, so the y-intercepts are $\pm b$. Using this information, together with symmetry, we sketch the ellipse in Figure 8. If $a = b$, the ellipse is a circle with radius a.

EXAMPLE 5 Sketch the graph of $9x^2 + 16y^2 = 144$.

SOLUTION We divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse (2), so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4; the y-intercepts are ± 3. The graph is sketched in Figure 9.

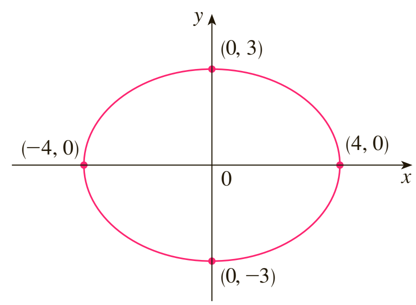


FIGURE 9
$9x^2 + 16y^2 = 144$

Hyperbolas

The curve with equation

3 $$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

is called a **hyperbola** in standard position. Again, Equation 3 is unchanged when x is replaced by $-x$ or y is replaced by $-y$, so the hyperbola is symmetric with respect to both axes. To find the x-intercepts we set $y = 0$ and obtain $x^2 = a^2$ and $x = \pm a$. However, if we put $x = 0$ in Equation 3, we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. In fact, from Equation 3 we obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

which shows that $x^2 \geqslant a^2$ and so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches*. It is sketched in Figure 10.

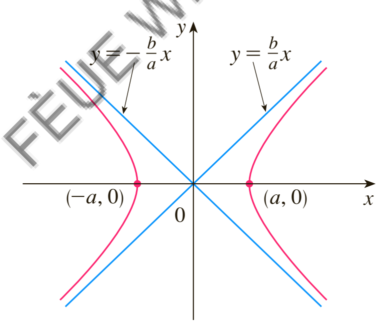


FIGURE 10
The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

In drawing a hyperbola it is useful to draw first its *asymptotes,* which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 10. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. This involves the idea of a limit, which is discussed in Chapter 1. (See also Exercise 3.5.57.)

By interchanging the roles of x and y we get an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

which also represents a hyperbola and is sketched in Figure 11.

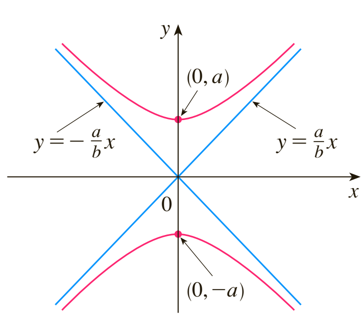


FIGURE 11
The hyperbola $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

EXAMPLE 6 Sketch the curve $9x^2 - 4y^2 = 36$.

SOLUTION Dividing both sides by 36, we obtain

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

which is the standard form of the equation of a hyperbola (Equation 3). Since $a^2 = 4$, the x-intercepts are ± 2. Since $b^2 = 9$, we have $b = 3$ and the asymptotes are $y = \pm\frac{3}{2}x$. The hyperbola is sketched in Figure 12.

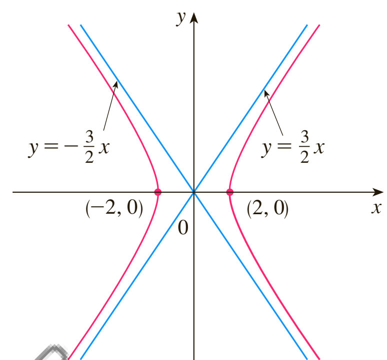
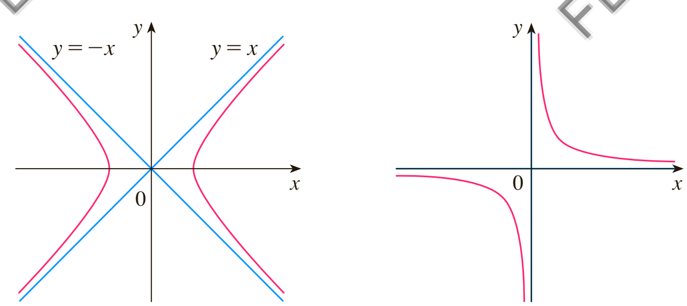


FIGURE 12
The hyperbola $9x^2 - 4y^2 = 36$

If $b = a$, a hyperbola has the equation $x^2 - y^2 = a^2$ (or $y^2 - x^2 = a^2$) and is called an *equilateral hyperbola* [see Figure 13(a)]. Its asymptotes are $y = \pm x$, which are perpendicular. If an equilateral hyperbola is rotated by 45°, the asymptotes become the x- and y-axes, and it can be shown that the new equation of the hyperbola is $xy = k$, where k is a constant [see Figure 13(b)].



(a) $x^2 - y^2 = a^2$

(b) $xy = k \quad (k > 0)$

FIGURE 13
Equilateral hyperbolas

Shifted Conics

Recall that an equation of the circle with center the origin and radius r is $x^2 + y^2 = r^2$, but if the center is the point (h, k), then the equation of the circle becomes

$$(x - h)^2 + (y - k)^2 = r^2$$

Similarly, if we take the ellipse with equation

$$\boxed{4} \qquad \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

and translate it (shift it) so that its center is the point (h, k), then its equation becomes

5
$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

(See Figure 14.)

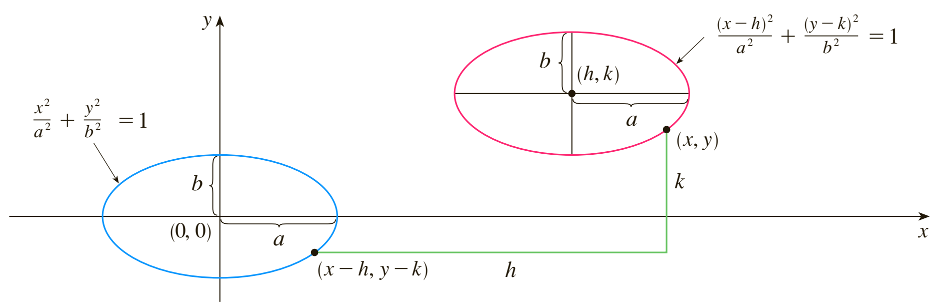


FIGURE 14

Notice that in shifting the ellipse, we replaced x by $x - h$ and y by $y - k$ in Equation 4 to obtain Equation 5. We use the same procedure to shift the parabola $y = ax^2$ so that its vertex (the origin) becomes the point (h, k) as in Figure 15. Replacing x by $x - h$ and y by $y - k$, we see that the new equation is

$$y - k = a(x - h)^2 \qquad \text{or} \qquad y = a(x - h)^2 + k$$

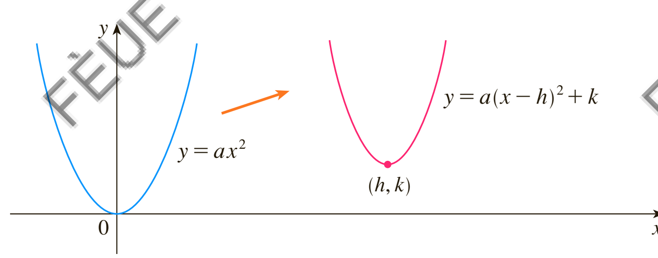


FIGURE 15

EXAMPLE 7 Sketch the graph of the equation $y = 2x^2 - 4x + 1$.

SOLUTION First we complete the square:

$$y = 2(x^2 - 2x) + 1 = 2(x - 1)^2 - 1$$

In this form we see that the equation represents the parabola obtained by shifting $y = 2x^2$ so that its vertex is at the point $(1, -1)$. The graph is sketched in Figure 16. ■

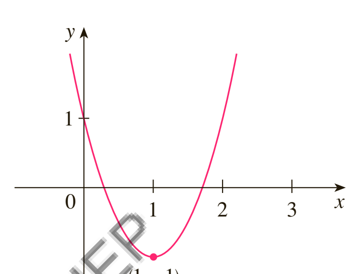


FIGURE 16
$y = 2x^2 - 4x + 1$

EXAMPLE 8 Sketch the curve $x = 1 - y^2$.

SOLUTION This time we start with the parabola $x = -y^2$ (as in Figure 6 with $a = -1$) and shift one unit to the right to get the graph of $x = 1 - y^2$. (See Figure 17.)

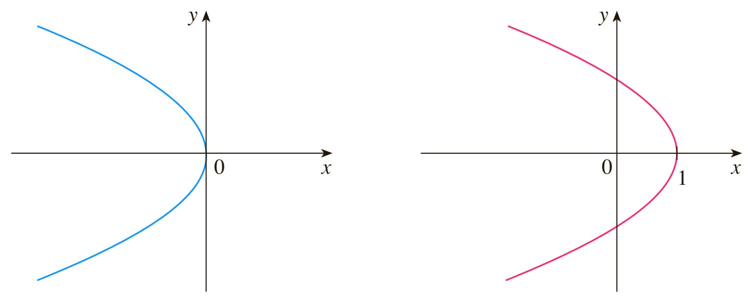


FIGURE 17 (a) $x = -y^2$ (b) $x = 1 - y^2$ ■

C EXERCISES

1–4 Find an equation of a circle that satisfies the given conditions.

1. Center $(3, -1)$, radius 5

2. Center $(-2, -8)$, radius 10

3. Center at the origin, passes through $(4, 7)$

4. Center $(-1, 5)$, passes through $(-4, -6)$

5–9 Show that the equation represents a circle and find the center and radius.

5. $x^2 + y^2 - 4x + 10y + 13 = 0$

6. $x^2 + y^2 + 6y + 2 = 0$

7. $x^2 + y^2 + x = 0$

8. $16x^2 + 16y^2 + 8x + 32y + 1 = 0$

9. $2x^2 + 2y^2 - x + y = 1$

10. Under what condition on the coefficients a, b, and c does the equation $x^2 + y^2 + ax + by + c = 0$ represent a circle? When that condition is satisfied, find the center and radius of the circle.

11–32 Identify the type of curve and sketch the graph. Do not plot points. Just use the standard graphs given in Figures 5, 6, 8, 10, and 11 and shift if necessary.

11. $y = -x^2$

12. $y^2 - x^2 = 1$

13. $x^2 + 4y^2 = 16$

14. $x = -2y^2$

15. $16x^2 - 25y^2 = 400$

16. $25x^2 + 4y^2 = 100$

17. $4x^2 + y^2 = 1$

18. $y = x^2 + 2$

19. $x = y^2 - 1$

20. $9x^2 - 25y^2 = 225$

21. $9y^2 - x^2 = 9$

22. $2x^2 + 5y^2 = 10$

23. $xy = 4$

24. $y = x^2 + 2x$

25. $9(x - 1)^2 + 4(y - 2)^2 = 36$

26. $16x^2 + 9y^2 - 36y = 108$

27. $y = x^2 - 6x + 13$

28. $x^2 - y^2 - 4x + 3 = 0$

29. $x = 4 - y^2$

30. $y^2 - 2x + 6y + 5 = 0$

31. $x^2 + 4y^2 - 6x + 5 = 0$

32. $4x^2 + 9y^2 - 16x + 54y + 61 = 0$

33–34 Sketch the region bounded by the curves.

33. $y = 3x, \quad y = x^2$

34. $y = 4 - x^2, \quad x - 2y = 2$

35. Find an equation of the parabola with vertex $(1, -1)$ that passes through the points $(-1, 3)$ and $(3, 3)$.

36. Find an equation of the ellipse with center at the origin that passes through the points $\left(1, -10\sqrt{2}/3\right)$ and $\left(-2, 5\sqrt{5}/3\right)$.

37–40 Sketch the graph of the set.

37. $\{(x, y) \mid x^2 + y^2 \leqslant 1\}$

38. $\{(x, y) \mid x^2 + y^2 > 4\}$

39. $\{(x, y) \mid y \geqslant x^2 - 1\}$

40. $\{(x, y) \mid x^2 + 4y^2 \leqslant 4\}$

D Trigonometry

Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad. Therefore

1 $$\boxed{\pi \text{ rad} = 180°}$$

and

2 $$1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.3° \qquad 1° = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}$$

EXAMPLE 1

(a) Find the radian measure of 60°. (b) Express $5\pi/4$ rad in degrees.

SOLUTION

(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225° \qquad \blacksquare$$

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

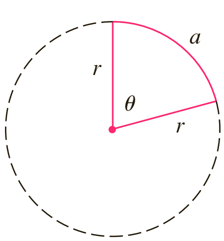


FIGURE 1

Figure 1 shows a sector of a circle with central angle θ and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain

3 $$\boxed{\theta = \frac{a}{r}} \qquad \boxed{a = r\theta}$$

Remember that Equations 3 are valid only when θ is measured in radians.

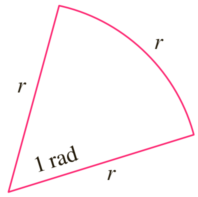


FIGURE 2

In particular, putting $a = r$ in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

EXAMPLE 2

(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
(b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of $3\pi/8$ rad?

SOLUTION

(a) Using Equation 3 with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = \tfrac{6}{5} = 1.2 \text{ rad}$$

(b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8} \text{ cm}$$

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure 3. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4.

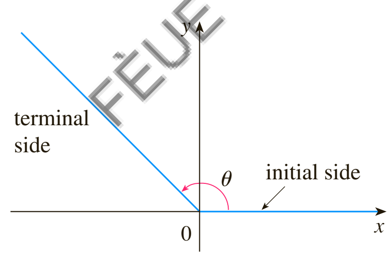


FIGURE 3 $\theta \geqslant 0$

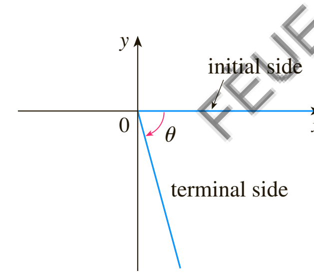


FIGURE 4 $\theta < 0$

Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

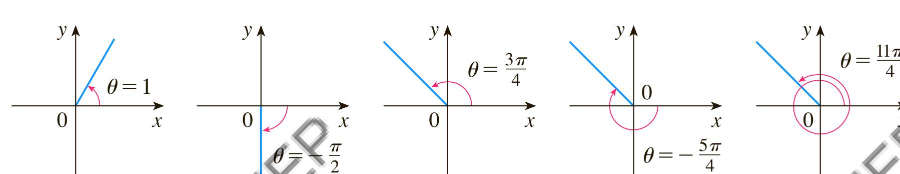


FIGURE 5
Angles in standard position

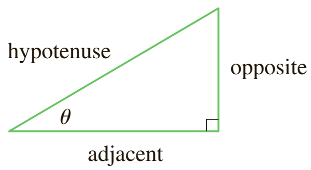


FIGURE 6

The Trigonometric Functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

4

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

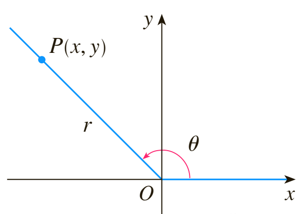


FIGURE 7

This definition doesn't apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then we define

5

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

If we put $r = 1$ in Definition 5 and draw a unit circle with center the origin and label θ as in Figure 8, then the coordinates of P are $(\cos\theta, \sin\theta)$.

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\csc\theta$ and $\cot\theta$ are undefined when $y = 0$. Notice that the definitions in (4) and (5) are consistent when θ is an acute angle.

If θ is a number, the convention is that $\sin\theta$ means the sine of the angle whose *radian* measure is θ. For example, the expression $\sin 3$ implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number, we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

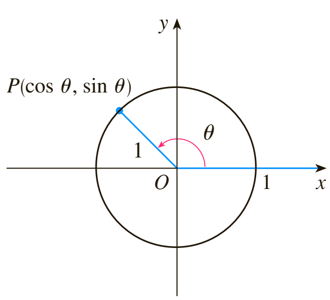


FIGURE 8

If we want to know the sine of the angle 3° we would write $\sin 3°$ and, with our calculator in degree mode, we find that

$$\sin 3° \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 9. For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

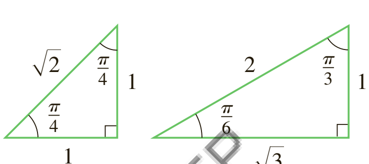


FIGURE 9

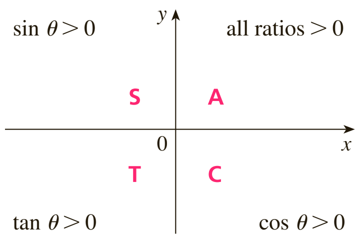


FIGURE 10

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "All Students Take Calculus" shown in Figure 10.

EXAMPLE 3 Find the exact trigonometric ratios for $\theta = 2\pi/3$.

SOLUTION From Figure 11 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \qquad y = \sqrt{3} \qquad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -2 \qquad \cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$

■

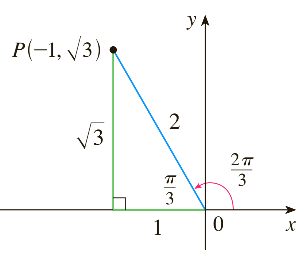


FIGURE 11

The following table gives some values of $\sin\theta$ and $\cos\theta$ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

EXAMPLE 4 If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.

SOLUTION Since $\cos\theta = \frac{2}{5}$, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 12. If the opposite side has length x, then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$. We can now use the diagram to write the other five trigonometric functions:

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2} \qquad \cot\theta = \frac{2}{\sqrt{21}}$$

■

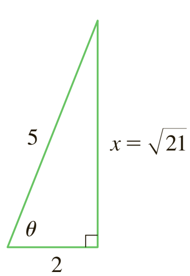


FIGURE 12

EXAMPLE 5 Use a calculator to approximate the value of x in Figure 13.

SOLUTION From the diagram we see that

$$\tan 40° = \frac{16}{x}$$

Therefore

$$x = \frac{16}{\tan 40°} \approx 19.07$$

■

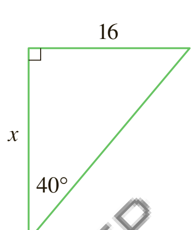


FIGURE 13

Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

6
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

For the next identity we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

7
$$\sin^2\theta + \cos^2\theta = 1$$

If we now divide both sides of Equation 7 by $\cos^2\theta$ and use Equations 6, we get

8
$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if we divide both sides of Equation 7 by $\sin^2\theta$, we get

9
$$1 + \cot^2\theta = \csc^2\theta$$

The identities

10a
$$\sin(-\theta) = -\sin\theta$$

10b
$$\cos(-\theta) = \cos\theta$$

Odd functions and even functions are discussed in Section 1.1.

show that sine is an odd function and cosine is an even function. They are easily proved by drawing a diagram showing θ and $-\theta$ in standard position (see Exercise 39).

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have

11
$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

These identities show that the sine and cosine functions are periodic with period 2π.

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

12a $$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

12b $$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

The proofs of these addition formulas are outlined in Exercises 85, 86, and 87.

By substituting $-y$ for y in Equations 12a and 12b and using Equations 10a and 10b, we obtain the following **subtraction formulas**:

13a $$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

13b $$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for $\tan(x \pm y)$:

14a $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

14b $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put $y = x$ in the addition formulas (12), we get the **double-angle formulas**:

15a $$\sin 2x = 2 \sin x \cos x$$

15b $$\cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for $\cos 2x$:

16a $$\cos 2x = 2 \cos^2 x - 1$$

16b $$\cos 2x = 1 - 2 \sin^2 x$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

17a $$\cos^2 x = \frac{1 + \cos 2x}{2}$$

17b $$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally, we state the **product formulas**, which can be deduced from Equations 12 and 13:

$$\boxed{\begin{aligned} &\textbf{18a} \quad \sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)] \\ &\textbf{18b} \quad \cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)] \\ &\textbf{18c} \quad \sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)] \end{aligned}}$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of the identities 13–18, remember that they can all be deduced from Equations 12a and 12b.

EXAMPLE 6 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

SOLUTION Using the double-angle formula (15a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \qquad \text{or} \qquad \sin x\,(1 - 2 \cos x) = 0$$

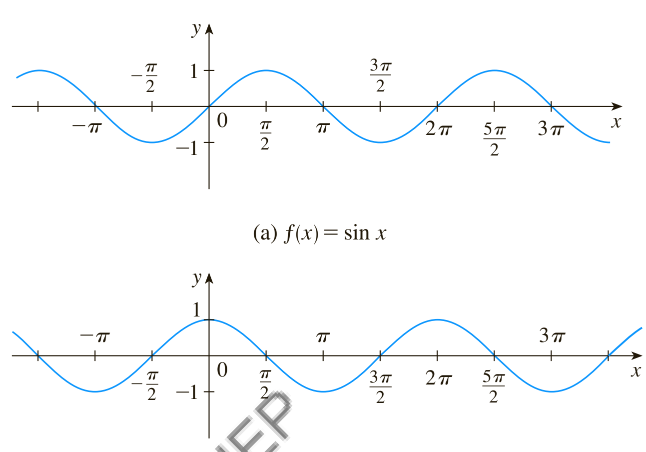
Therefore there are two possibilities:

$$\begin{aligned} \sin x &= 0 & \qquad \text{or} \qquad 1 - 2\cos x &= 0 \\ x &= 0, \pi, 2\pi & \cos x &= \tfrac{1}{2} \\ & & x &= \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

The given equation has five solutions: 0, $\pi/3$, π, $5\pi/3$, and 2π. ■

Graphs of the Trigonometric Functions

The graph of the function $f(x) = \sin x$, shown in Figure 14(a), is obtained by plotting points for $0 \leqslant x \leqslant 2\pi$ and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

FIGURE 14

integer multiples of π, that is,

$$\sin x = 0 \qquad \text{whenever } x = n\pi, \quad n \text{ an integer}$$

Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 12a), the graph of cosine is obtained by shifting the graph of sine by an amount $\pi/2$ to the left [see Figure 14(b)]. Note that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

The graphs of the remaining four trigonometric functions are shown in Figure 15 and their domains are indicated there. Notice that tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π, whereas cosecant and secant have period 2π.

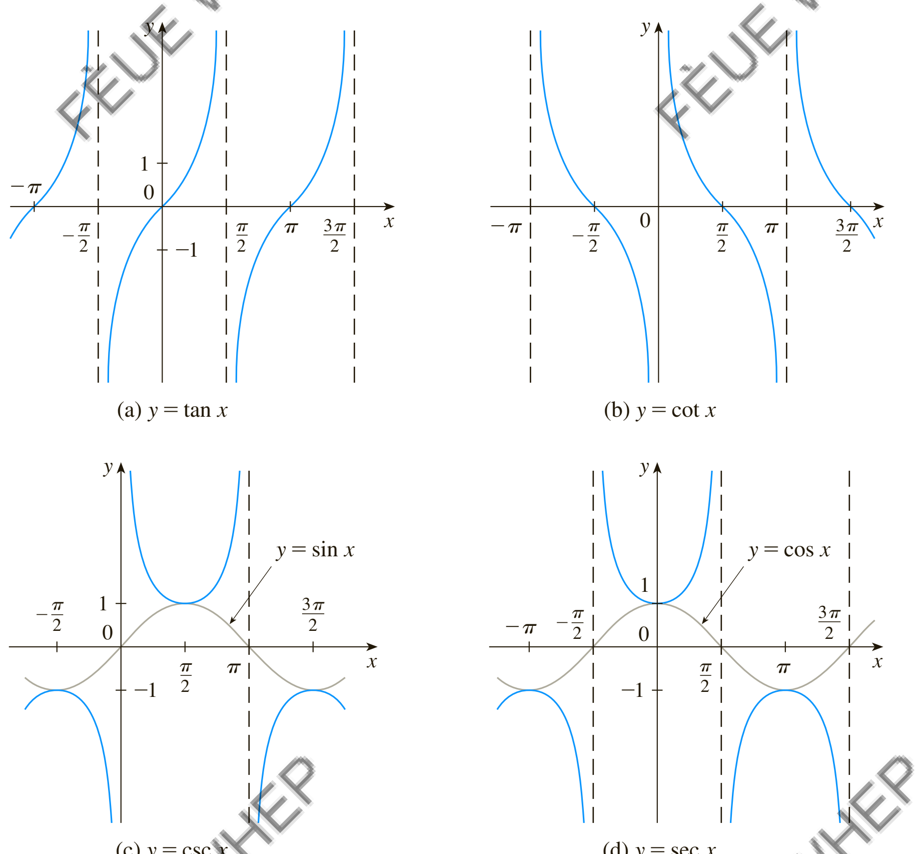


FIGURE 15

D EXERCISES

1–6 Convert from degrees to radians.

1. $210°$ **2.** $300°$ **3.** $9°$

4. $-315°$ **5.** $900°$ **6.** $36°$

7–12 Convert from radians to degrees.

7. 4π **8.** $-\dfrac{7\pi}{2}$ **9.** $\dfrac{5\pi}{12}$

10. $\dfrac{8\pi}{3}$ **11.** $-\dfrac{3\pi}{8}$ **12.** 5

13. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.

14. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of $72°$.

15. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?

16. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

17–22 Draw, in standard position, the angle whose measure is given.

17. $315°$ **18.** $-150°$ **19.** $-\dfrac{3\pi}{4}$ rad

20. $\dfrac{7\pi}{3}$ rad **21.** 2 rad **22.** -3 rad

23–28 Find the exact trigonometric ratios for the angle whose radian measure is given.

23. $\dfrac{3\pi}{4}$ **24.** $\dfrac{4\pi}{3}$ **25.** $\dfrac{9\pi}{2}$

26. -5π **27.** $\dfrac{5\pi}{6}$ **28.** $\dfrac{11\pi}{4}$

29–34 Find the remaining trigonometric ratios.

29. $\sin\theta = \dfrac{3}{5}, \quad 0 < \theta < \dfrac{\pi}{2}$

30. $\tan\alpha = 2, \quad 0 < \alpha < \dfrac{\pi}{2}$

31. $\sec\phi = -1.5, \quad \dfrac{\pi}{2} < \phi < \pi$

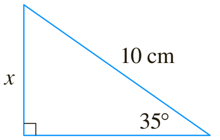
32. $\cos x = -\dfrac{1}{3}, \quad \pi < x < \dfrac{3\pi}{2}$

33. $\cot\beta = 3, \quad \pi < \beta < 2\pi$

34. $\csc\theta = -\dfrac{4}{3}, \quad \dfrac{3\pi}{2} < \theta < 2\pi$

35–38 Find, correct to five decimal places, the length of the side labeled x.

35.



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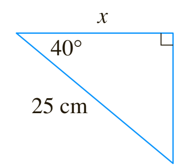

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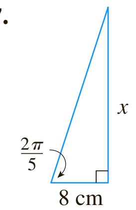
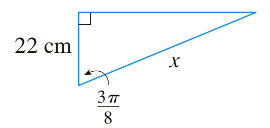


38.



39–41 Prove each equation.

39. (a) Equation 10a (b) Equation 10b

40. (a) Equation 14a (b) Equation 14b

41. (a) Equation 18a (b) Equation 18b
(c) Equation 18c

42–58 Prove the identity.

42. $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

43. $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$ **44.** $\sin(\pi - x) = \sin x$

45. $\sin\theta\,\cot\theta = \cos\theta$ **46.** $(\sin x + \cos x)^2 = 1 + \sin 2x$

47. $\sec y - \cos y = \tan y\,\sin y$

48. $\tan^2\alpha - \sin^2\alpha = \tan^2\alpha\,\sin^2\alpha$

49. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$

50. $2\csc 2t = \sec t\,\csc t$

51. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

52. $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2\sec^2\theta$

53. $\sin x\,\sin 2x + \cos x\,\cos 2x = \cos x$

54. $\sin^2 x - \sin^2 y = \sin(x + y)\,\sin(x - y)$

55. $\dfrac{\sin\phi}{1 - \cos\phi} = \csc\phi + \cot\phi$

56. $\tan x + \tan y = \dfrac{\sin(x + y)}{\cos x\,\cos y}$

57. $\sin 3\theta + \sin \theta = 2 \sin 2\theta \cos \theta$

58. $\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta$

59–64 If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression.

59. $\sin(x + y)$

60. $\cos(x + y)$

61. $\cos(x - y)$

62. $\sin(x - y)$

63. $\sin 2y$

64. $\cos 2y$

65–72 Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

65. $2 \cos x - 1 = 0$

66. $3 \cot^2 x = 1$

67. $2 \sin^2 x = 1$

68. $|\tan x| = 1$

69. $\sin 2x = \cos x$

70. $2 \cos x + \sin 2x = 0$

71. $\sin x = \tan x$

72. $2 + \cos 2x = 3 \cos x$

73–76 Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality.

73. $\sin x \leq \frac{1}{2}$

74. $2 \cos x + 1 > 0$

75. $-1 < \tan x < 1$

76. $\sin x > \cos x$

77–82 Graph the function by starting with the graphs in Figures 14 and 15 and applying the transformations of Section 1.3 where appropriate.

77. $y = \cos\left(x - \dfrac{\pi}{3}\right)$

78. $y = \tan 2x$

79. $y = \dfrac{1}{3}\tan\left(x - \dfrac{\pi}{2}\right)$

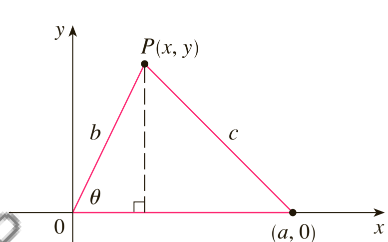
80. $y = 1 + \sec x$

81. $y = |\sin x|$

82. $y = 2 + \sin\left(x + \dfrac{\pi}{4}\right)$

83. Prove the **Law of Cosines**: If a triangle has sides with lengths a, b, and c, and θ is the angle between the sides with lengths a and b, then

$$c^2 = a^2 + b^2 - 2ab \cos\theta$$

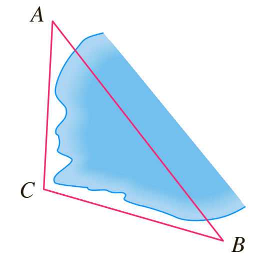


[*Hint:* Introduce a coordinate system so that θ is in standard position, as in the figure. Express x and y in terms of θ and then use the distance formula to compute c.]

84. In order to find the distance $|AB|$ across a small inlet, a point C was located as in the figure and the following measurements were recorded:

$$\angle C = 103° \qquad |AC| = 820 \text{ m} \qquad |BC| = 910 \text{ m}$$

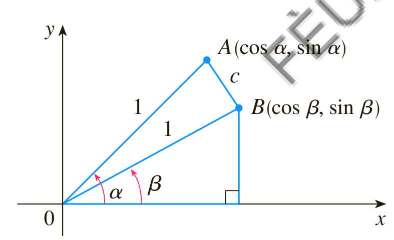
Use the Law of Cosines from Exercise 83 to find the required distance.



85. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

[*Hint:* Compute c^2 in two ways (using the Law of Cosines from Exercise 83 and also using the distance formula) and compare the two expressions.]



86. Use the formula in Exercise 85 to prove the addition formula for cosine (12b).

87. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

to prove the subtraction formula (13a) for the sine function.

88. Show that the area of a triangle with sides of lengths a and b and with included angle θ is

$$A = \tfrac{1}{2}ab \sin \theta$$

89. Find the area of triangle ABC, correct to five decimal places, if

$$|AB| = 10 \text{ cm} \qquad |BC| = 3 \text{ cm} \qquad \angle ABC = 107°$$

E Sigma Notation

A convenient way of writing sums uses the Greek letter Σ (capital sigma, corresponding to our letter S) and is called **sigma notation**.

This tells us to end with $i = n$.

This tells us to add.

$$\sum_{i=m}^{n} a_i$$

This tells us to start with $i = m$.

1 Definition If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leqslant n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

With function notation, Definition 1 can be written as

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus the symbol $\Sigma_{i=m}^{n}$ indicates a summation in which the letter i (called the **index of summation**) takes on consecutive integer values beginning with m and ending with n, that is, $m, m+1, \ldots, n$. Other letters can also be used as the index of summation.

EXAMPLE 1

(a) $\displaystyle\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\displaystyle\sum_{i=3}^{n} i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c) $\displaystyle\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e) $\displaystyle\sum_{i=1}^{3} \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f) $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$ ■

EXAMPLE 2 Write the sum $2^3 + 3^3 + \cdots + n^3$ in sigma notation.

SOLUTION There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$ ■

The following theorem gives three simple rules for working with sigma notation.

2 Theorem If c is any constant (that is, it does not depend on i), then

(a) $\displaystyle\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$ (b) $\displaystyle\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\displaystyle\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

PROOF To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$\begin{aligned}&(a_m + b_m) + (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n)\\ &\qquad = (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n)\end{aligned}$$

Rule (c) is proved similarly. ■

EXAMPLE 3 Find $\displaystyle\sum_{i=1}^{n} 1$.

SOLUTION

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$ ■

EXAMPLE 4 Prove the formula for the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION This formula can be proved by mathematical induction (see page 99) or by the following method used by the German mathematician Karl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum S twice, once in the usual order and once in reverse order:

$$\begin{aligned} S &= 1 + \quad 2 \quad + \quad 3 \quad + \cdots + (n-1) + n\\ S &= n + (n-1) + (n-2) + \cdots + \quad 2 \quad + 1\end{aligned}$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side there are n terms, each of which is $n + 1$, so

$$2S = n(n+1) \qquad \text{or} \qquad S = \frac{n(n+1)}{2}$$ ■

EXAMPLE 5 Prove the formula for the sum of the squares of the first n positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION 1 Let S be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\sum_{i=1}^{n} [(1 + i)^3 - i^3] = (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \cdots + [(n + 1)^3 - \cancel{n^3}]$$
$$= (n + 1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using Theorem 2 and Examples 3 and 4, we have

$$\sum_{i=1}^{n} [(1 + i)^3 - i^3] = \sum_{i=1}^{n} [3i^2 + 3i + 1] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
$$= 3S + 3\frac{n(n + 1)}{2} + n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Thus we have

$$n^3 + 3n^2 + 3n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Solving this equation for S, we obtain

$$3S = n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n$$

or

$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n + 1)(2n + 1)}{6}$$

Principle of Mathematical Induction
Let S_n be a statement involving the positive integer n. Suppose that
1. S_1 is true.
2. If S_k is true, then S_{k+1} is true.

Then S_n is true for all positive integers n.

See pages 99 and 100 for a more thorough discussion of mathematical induction.

SOLUTION 2 Let S_n be the given formula.

1. S_1 is true because $\quad 1^2 = \dfrac{1(1 + 1)(2 \cdot 1 + 1)}{6}$

2. Assume that S_k is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k + 1)(2k + 1)}{6}$$

Then

$$1^2 + 2^2 + 3^2 + \cdots + (k + 1)^2 = (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k + 1)^2$$
$$= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2$$
$$= (k + 1)\frac{k(2k + 1) + 6(k + 1)}{6}$$
$$= (k + 1)\frac{2k^2 + 7k + 6}{6}$$
$$= \frac{(k + 1)(k + 2)(2k + 3)}{6}$$
$$= \frac{(k + 1)[(k + 1) + 1][2(k + 1) + 1]}{6}$$

So S_{k+1} is true.

By the Principle of Mathematical Induction, S_n is true for all n. ■

We list the results of Examples 3, 4, and 5 together with a similar result for cubes (see Exercises 37–40) as Theorem 3. These formulas are needed for finding areas and evaluating integrals in Chapter 4.

3 Theorem Let c be a constant and n a positive integer. Then

(a) $\sum_{i=1}^{n} 1 = n$

(b) $\sum_{i=1}^{n} c = nc$

(c) $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

(d) $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

(e) $\sum_{i=1}^{n} i^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

EXAMPLE 6 Evaluate $\sum_{i=1}^{n} i(4i^2 - 3)$.

SOLUTION Using Theorems 2 and 3, we have

$$\begin{aligned}\sum_{i=1}^{n} i(4i^2 - 3) &= \sum_{i=1}^{n} (4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\ &= 4\left[\frac{n(n+1)}{2}\right]^2 - 3\,\frac{n(n+1)}{2} \\ &= \frac{n(n+1)[2n(n+1) - 3]}{2} \\ &= \frac{n(n+1)(2n^2 + 2n - 3)}{2}\end{aligned}$$

EXAMPLE 7 Find $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.

The type of calculation in Example 7 arises in Chapter 4 when we compute areas.

SOLUTION

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{3}{n^3} i^2 + \frac{3}{n}\right] \\ &= \lim_{n\to\infty} \left[\frac{3}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty} \left[\frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right] \\ &= \lim_{n\to\infty} \left[\frac{1}{2}\cdot\frac{n}{n}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 3\right] \\ &= \lim_{n\to\infty} \left[\frac{1}{2}\cdot 1\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 3\right] \\ &= \tfrac{1}{2}\cdot 1\cdot 1\cdot 2 + 3 = 4\end{aligned}$$

E EXERCISES

1–10 Write the sum in expanded form.

1. $\sum_{i=1}^{5} \sqrt{i}$

2. $\sum_{i=1}^{6} \frac{1}{i+1}$

3. $\sum_{i=4}^{6} 3^i$

4. $\sum_{i=4}^{6} i^3$

5. $\sum_{k=0}^{4} \frac{2k-1}{2k+1}$

6. $\sum_{k=5}^{8} x^k$

7. $\sum_{i=1}^{n} i^{10}$

8. $\sum_{j=n}^{n+3} j^2$

9. $\sum_{j=0}^{n-1} (-1)^j$

10. $\sum_{i=1}^{n} f(x_i)\,\Delta x_i$

11–20 Write the sum in sigma notation.

11. $1 + 2 + 3 + 4 + \cdots + 10$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$

15. $2 + 4 + 6 + 8 + \cdots + 2n$

16. $1 + 3 + 5 + 7 + \cdots + (2n-1)$

17. $1 + 2 + 4 + 8 + 16 + 32$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

19. $x + x^2 + x^3 + \cdots + x^n$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

21–35 Find the value of the sum.

21. $\sum_{i=4}^{8} (3i - 2)$

22. $\sum_{i=3}^{6} i(i+2)$

23. $\sum_{j=1}^{6} 3^{j+1}$

24. $\sum_{k=0}^{8} \cos k\pi$

25. $\sum_{n=1}^{20} (-1)^n$

26. $\sum_{i=1}^{100} 4$

27. $\sum_{i=0}^{4} (2^i + i^2)$

28. $\sum_{i=-2}^{4} 2^{3-i}$

29. $\sum_{i=1}^{n} 2i$

30. $\sum_{i=1}^{n} (2 - 5i)$

31. $\sum_{i=1}^{n} (i^2 + 3i + 4)$

32. $\sum_{i=1}^{n} (3 + 2i)^2$

33. $\sum_{i=1}^{n} (i+1)(i+2)$

34. $\sum_{i=1}^{n} i(i+1)(i+2)$

35. $\sum_{i=1}^{n} (i^3 - i - 2)$

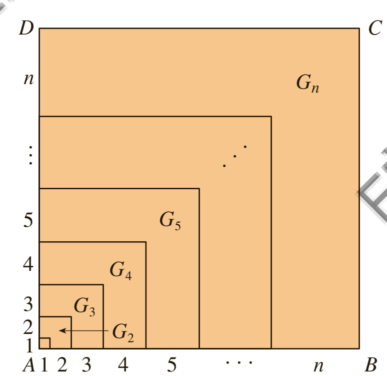
36. Find the number n such that $\sum_{i=1}^{n} i = 78$.

37. Prove formula (b) of Theorem 3.

38. Prove formula (e) of Theorem 3 using mathematical induction.

39. Prove formula (e) of Theorem 3 using a method similar to that of Example 5, Solution 1 [start with $(1+i)^4 - i^4$].

40. Prove formula (e) of Theorem 3 using the following method published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about AD 1010. The figure shows a square $ABCD$ in which sides AB and AD have been divided into segments of lengths $1, 2, 3, \ldots, n$. Thus the side of the square has length $n(n+1)/2$ so the area is $[n(n+1)/2]^2$. But the area is also the sum of the areas of the n "gnomons" $G_1, G_2, \ldots, G_n$ shown in the figure. Show that the area of G_i is i^3 and conclude that formula (e) is true.



41. Evaluate each telescoping sum.

(a) $\sum_{i=1}^{n} [i^4 - (i-1)^4]$

(b) $\sum_{i=1}^{100} (5^i - 5^{i-1})$

(c) $\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

(d) $\sum_{i=1}^{n} (a_i - a_{i-1})$

42. Prove the generalized triangle inequality:

$$\left|\sum_{i=1}^{n} a_i\right| \leqslant \sum_{i=1}^{n} |a_i|$$

43–46 Find the limit.

43. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$

44. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 1\right]$

45. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right]$

46. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(1+\frac{3i}{n}\right)^3-2\left(1+\frac{3i}{n}\right)\right]$

47. Prove the formula for the sum of a finite geometric series with first term a and common ratio $r \neq 1$:

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

48. Evaluate $\displaystyle\sum_{i=1}^{n}\frac{3}{2^{i-1}}$.

49. Evaluate $\displaystyle\sum_{i=1}^{n}(2i + 2^i)$.

50. Evaluate $\displaystyle\sum_{i=1}^{m}\left[\sum_{j=1}^{n}(i + j)\right]$.

F Proofs of Theorems

In this appendix we present proofs of several theorems that are stated in the main body of the text. The sections in which they occur are indicated in the margin.

Section 2.3

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} g(x) = M$$

exist. Then

1. $\displaystyle\lim_{x\to a}[f(x) + g(x)] = L + M$

2. $\displaystyle\lim_{x\to a}[f(x) - g(x)] = L - M$

3. $\displaystyle\lim_{x\to a}[cf(x)] = cL$

4. $\displaystyle\lim_{x\to a}[f(x)g(x)] = LM$

5. $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

PROOF OF LAW 4 Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= |[f(x) - L]g(x) + L[g(x) - M]| \\
&\leqslant |[f(x) - L]g(x)| + |L[g(x) - M]| \qquad \text{(Triangle Inequality)} \\
&= |f(x) - L||g(x)| + |L||g(x) - M|
\end{aligned}$$

We want to make each of these terms less than $\varepsilon/2$.

Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then

$$|g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leqslant |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_3 \quad \text{then} \quad |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then we have $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$, so we can combine the inequalities to obtain

$$\begin{aligned} |f(x)g(x) - LM| &\leqslant |f(x) - L||g(x)| + |L||g(x) - M| \\ &< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L| \frac{\varepsilon}{2(1 + |L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that $\lim_{x \to a} [f(x)g(x)] = LM$. ■

PROOF OF LAW 3 If we take $g(x) = c$ in Law 4, we get

$$\begin{aligned} \lim_{x \to a} [cf(x)] &= \lim_{x \to a} [g(x)f(x)] = \lim_{x \to a} g(x) \cdot \lim_{x \to a} f(x) \\ &= \lim_{x \to a} c \cdot \lim_{x \to a} f(x) \\ &= c \lim_{x \to a} f(x) \qquad \text{(by Law 7)} \end{aligned}$$ ■

PROOF OF LAW 2 Using Law 1 and Law 3 with $c = -1$, we have

$$\begin{aligned} \lim_{x \to a} [f(x) - g(x)] &= \lim_{x \to a} [f(x) + (-1)g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} (-1)g(x) \\ &= \lim_{x \to a} f(x) + (-1) \lim_{x \to a} g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \end{aligned}$$ ■

PROOF OF LAW 5 First let us show that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

To do this we must show that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Observe that
$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|}$$

We know that we can make the numerator small. But we also need to know that the denominator is not small when x is near a. Since $\lim_{x \to a} g(x) = M$, there is a number $\delta_1 > 0$ such that, whenever $0 < |x - a| < \delta_1$, we have

$$|g(x) - M| < \frac{|M|}{2}$$

and therefore
$$\begin{aligned} |M| = |M - g(x) + g(x)| &\leqslant |M - g(x)| + |g(x)| \\ &< \frac{|M|}{2} + |g(x)| \end{aligned}$$

This shows that

$$\text{if} \qquad 0 < |x - a| < \delta_1 \qquad \text{then} \qquad |g(x)| > \frac{|M|}{2}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M||g(x)|} < \frac{1}{|M|} \cdot \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there exists $\delta_2 > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta_2 \qquad \text{then} \qquad |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $0 < |x - a| < \delta$, we have

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

It follows that $\lim_{x \to a} 1/g(x) = 1/M$. Finally, using Law 4, we obtain

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(f(x) \cdot \frac{1}{g(x)}\right) = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$ ■

2 Theorem If $f(x) \leqslant g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M$$

then $L \leqslant M$.

PROOF We use the method of proof by contradiction. Suppose, if possible, that $L > M$. Law 2 of limits says that

$$\lim_{x \to a} [g(x) - f(x)] = M - L$$

Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |[g(x) - f(x)] - (M - L)| < \varepsilon$$

In particular, taking $\varepsilon = L - M$ (noting that $L - M > 0$ by hypothesis), we have a number $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |[g(x) - f(x)] - (M - L)| < L - M$$

Since $b \leqslant |b|$ for any number b, we have

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad [g(x) - f(x)] - (M - L) < L - M$$

which simplifies to

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad g(x) < f(x)$$

But this contradicts $f(x) \leqslant g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \leqslant M$. ■

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then
$$\lim_{x \to a} g(x) = L$$

PROOF Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad |h(x) - L| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$, so

$$L - \varepsilon < f(x) \leqslant g(x) \leqslant h(x) < L + \varepsilon$$

In particular,
$$L - \varepsilon < g(x) < L + \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Therefore $\lim_{x \to a} g(x) = L$. ■

Section 1.8

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b)$$

PROOF Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, we have

$$\lim_{y \to b} f(y) = f(b)$$

and so there exists $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |y - b| < \delta_1 \quad \text{then} \quad |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \to a} g(x) = b$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |g(x) - b| < \delta_1$$

Combining these two statements, we see that whenever $0 < |x - a| < \delta$ we have $|g(x) - b| < \delta_1$, which implies that $|f(g(x)) - f(b)| < \varepsilon$. Therefore we have proved that $\lim_{x \to a} f(g(x)) = f(b)$. ■

Section 2.4

The proof of the following result was promised when we proved that $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$.

Theorem If $0 < \theta < \pi/2$, then $\theta \leqslant \tan \theta$.

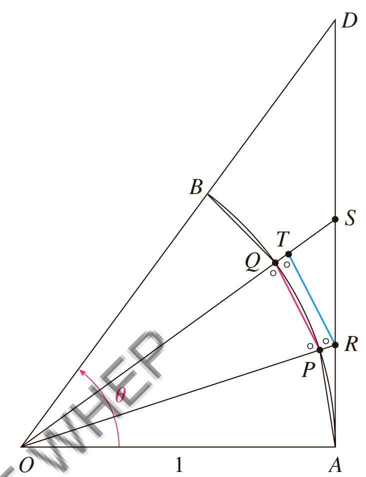


FIGURE 1

PROOF Figure 1 shows a sector of a circle with center O, central angle θ, and radius 1. Then

$$|AD| = |OA| \tan \theta = \tan \theta$$

We approximate the arc AB by an inscribed polygon consisting of n equal line segments and we look at a typical segment PQ. We extend the lines OP and OQ to meet AD in the points R and S. Then we draw $RT \parallel PQ$ as in Figure 2. Observe that

$$\angle RTO = \angle PQO < 90^\circ$$

and so $\angle RTS > 90^\circ$. Therefore we have

$$|PQ| < |RT| < |RS|$$

If we add n such inequalities, we get

$$L_n < |AD| = \tan \theta$$

where L_n is the length of the inscribed polygon. Thus, by Theorem 1.6.2, we have

$$\lim_{n \to \infty} L_n \leqslant \tan \theta$$

But the arc length is defined in Equation 8.1.1 as the limit of the lengths of inscribed polygons, so

$$\theta = \lim_{n \to \infty} L_n \leqslant \tan \theta$$ ■

Section 3.3

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

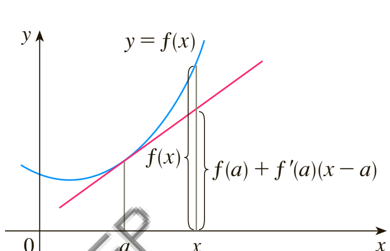


FIGURE 2

PROOF OF (a) Let a be any number in I. We must show that the curve $y = f(x)$ lies above the tangent line at the point $(a, f(a))$. The equation of this tangent is

$$y = f(a) + f'(a)(x - a)$$

So we must show that

$$f(x) > f(a) + f'(a)(x - a)$$

whenever $x \in I$ $(x \neq a)$. (See Figure 2.)

First let us take the case where $x > a$. Applying the Mean Value Theorem to f on the interval $[a, x]$, we get a number c, with $a < c < x$, such that

$$\boxed{1} \qquad f(x) - f(a) = f'(c)(x - a)$$

Since $f'' > 0$ on I, we know from the Increasing/Decreasing Test that f' is increasing on I. Thus, since $a < c$, we have

$$f'(a) < f'(c)$$

and so, multiplying this inequality by the positive number $x - a$, we get

$$\boxed{2} \qquad f'(a)(x - a) < f'(c)(x - a)$$

Now we add $f(a)$ to both sides of this inequality:

$$f(a) + f'(a)(x - a) < f(a) + f'(c)(x - a)$$

But from Equation 1 we have $f(x) = f(a) + f'(c)(x - a)$. So this inequality becomes

$$\boxed{3} \qquad f(x) > f(a) + f'(a)(x - a)$$

which is what we wanted to prove.

For the case where $x < a$ we have $f'(c) < f'(a)$, but multiplication by the negative number $x - a$ reverses the inequality, so we get (2) and (3) as before. ■

Section 6.1

6 Theorem If f is a one-to-one continuous function defined on an interval (a, b), then its inverse function f^{-1} is also continuous.

PROOF First we show that if f is both one-to-one and continuous on (a, b), then it must be either increasing or decreasing on (a, b). If it were neither increasing nor decreasing, then there would exist numbers x_1, x_2, and x_3 in (a, b) with $x_1 < x_2 < x_3$ such that $f(x_2)$ does not lie between $f(x_1)$ and $f(x_3)$. There are two possibilities: either (1) $f(x_3)$ lies between $f(x_1)$ and $f(x_2)$ or (2) $f(x_1)$ lies between $f(x_2)$ and $f(x_3)$. (Draw a picture.) In case (1) we apply the Intermediate Value Theorem to the continuous function f to get a number c between x_1 and x_2 such that $f(c) = f(x_3)$. In case (2) the Intermediate Value Theorem gives a number c between x_2 and x_3 such that $f(c) = f(x_1)$. In either case we have contradicted the fact that f is one-to-one.

Let us assume, for the sake of definiteness, that f is increasing on (a, b). We take any number y_0 in the domain of f^{-1} and we let $f^{-1}(y_0) = x_0$; that is, x_0 is the number in (a, b) such that $f(x_0) = y_0$. To show that f^{-1} is continuous at y_0 we take any $\varepsilon > 0$ such that the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is contained in the interval (a, b). Since f is increasing, it maps the numbers in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ onto the numbers in the interval $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ and f^{-1} reverses the correspondence. If we let δ denote the smaller of the numbers $\delta_1 = y_0 - f(x_0 - \varepsilon)$ and $\delta_2 = f(x_0 + \varepsilon) - y_0$, then the interval $(y_0 - \delta, y_0 + \delta)$ is contained in the interval $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ and so is mapped into the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ by f^{-1}. (See the arrow diagram in Figure 3.) We have therefore found a number $\delta > 0$ such that

$$\text{if} \quad |y - y_0| < \delta \quad \text{then} \quad |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$$

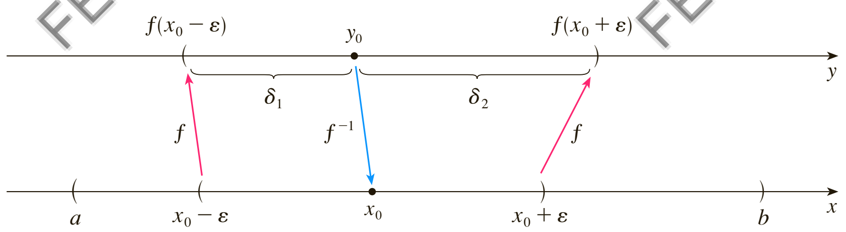


FIGURE 3

This shows that $\lim_{y\to y_0} f^{-1}(y) = f^{-1}(y_0)$ and so f^{-1} is continuous at any number y_0 in its domain. ■

Section 11.8

In order to prove Theorem 11.8.4, we first need the following results.

Theorem

1. If a power series $\sum c_n x^n$ converges when $x = b$ (where $b \neq 0$), then it converges whenever $|x| < |b|$.
2. If a power series $\sum c_n x^n$ diverges when $x = d$ (where $d \neq 0$), then it diverges whenever $|x| > |d|$.

PROOF OF 1 Suppose that $\sum c_n b^n$ converges. Then, by Theorem 11.2.6, we have $\lim_{n\to\infty} c_n b^n = 0$. According to Definition 11.1.2 with $\varepsilon = 1$, there is a positive integer N such that $|c_n b^n| < 1$ whenever $n \geqslant N$. Thus, for $n \geqslant N$, we have

$$|c_n x^n| = \left|\frac{c_n b^n x^n}{b^n}\right| = |c_n b^n|\left|\frac{x}{b}\right|^n < \left|\frac{x}{b}\right|^n$$

If $|x| < |b|$, then $|x/b| < 1$, so $\sum |x/b|^n$ is a convergent geometric series. Therefore, by the Comparison Test, the series $\sum_{n=N}^{\infty} |c_n x^n|$ is convergent. Thus the series $\sum c_n x^n$ is absolutely convergent and therefore convergent. ■

PROOF OF 2 Suppose that $\sum c_n d^n$ diverges. If x is any number such that $|x| > |d|$, then $\sum c_n x^n$ cannot converge because, by part 1, the convergence of $\sum c_n x^n$ would imply the convergence of $\sum c_n d^n$. Therefore $\sum c_n x^n$ diverges whenever $|x| > |d|$. ■

Theorem For a power series $\sum c_n x^n$ there are only three possibilities:

1. The series converges only when $x = 0$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x| < R$ and diverges if $|x| > R$.

PROOF Suppose that neither case 1 nor case 2 is true. Then there are nonzero numbers b and d such that $\sum c_n x^n$ converges for $x = b$ and diverges for $x = d$. Therefore the set $S = \{x \mid \sum c_n x^n \text{ converges}\}$ is not empty. By the preceding theorem, the series diverges if $|x| > |d|$, so $|x| \leqslant |d|$ for all $x \in S$. This says that $|d|$ is an upper bound for the set S. Thus, by the Completeness Axiom (see Section 11.1), S has a least upper bound R. If $|x| > R$, then $x \notin S$, so $\sum c_n x^n$ diverges. If $|x| < R$, then $|x|$ is not an upper bound for S and so there exists $b \in S$ such that $b > |x|$. Since $b \in S$, $\sum c_n x^n$ converges, so by the preceding theorem $\sum c_n x^n$ converges. ■

4 Theorem For a power series $\Sigma\, c_n(x - a)^n$ there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

PROOF If we make the change of variable $u = x - a$, then the power series becomes $\Sigma\, c_n u^n$ and we can apply the preceding theorem to this series. In case 3 we have convergence for $|u| < R$ and divergence for $|u| > R$. Thus we have convergence for $|x - a| < R$ and divergence for $|x - a| > R$. ■

Section 14.3

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

PROOF For small values of h, $h \neq 0$, consider the difference

$$\Delta(h) = [f(a + h, b + h) - f(a + h, b)] - [f(a, b + h) - f(a, b)]$$

Notice that if we let $g(x) = f(x, b + h) - f(x, b)$, then

$$\Delta(h) = g(a + h) - g(a)$$

By the Mean Value Theorem, there is a number c between a and $a + h$ such that

$$g(a + h) - g(a) = g'(c)h = h[f_x(c, b + h) - f_x(c, b)]$$

Applying the Mean Value Theorem again, this time to f_x, we get a number d between b and $b + h$ such that

$$f_x(c, b + h) - f_x(c, b) = f_{xy}(c, d)h$$

Combining these equations, we obtain

$$\Delta(h) = h^2 f_{xy}(c, d)$$

If $h \to 0$, then $(c, d) \to (a, b)$, so the continuity of f_{xy} at (a, b) gives

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \lim_{(c, d) \to (a, b)} f_{xy}(c, d) = f_{xy}(a, b)$$

Similarly, by writing

$$\Delta(h) = [f(a + h, b + h) - f(a, b + h)] - [f(a + h, b) - f(a, b)]$$

and using the Mean Value Theorem twice and the continuity of f_{yx} at (a, b), we obtain

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)$$

It follows that $f_{xy}(a, b) = f_{yx}(a, b)$. ■

Section 14.4

> **8 Theorem** If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

PROOF Let

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

According to (14.4.7), to prove that f is differentiable at (a, b) we have to show that we can write Δz in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Referring to Figure 4, we write

$$\boxed{1}\quad \Delta z = [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)]$$

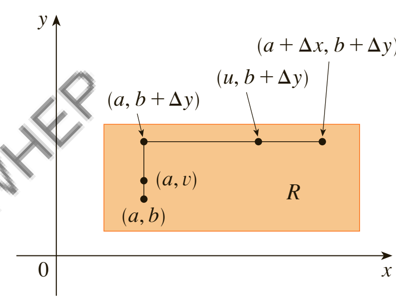


FIGURE 4

Observe that the function of a single variable

$$g(x) = f(x, b + \Delta y)$$

is defined on the interval $[a, a + \Delta x]$ and $g'(x) = f_x(x, b + \Delta y)$. If we apply the Mean Value Theorem to g, we get

$$g(a + \Delta x) - g(a) = g'(u)\,\Delta x$$

where u is some number between a and $a + \Delta x$. In terms of f, this equation becomes

$$f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) = f_x(u, b + \Delta y)\,\Delta x$$

This gives us an expression for the first part of the right side of Equation 1. For the second part we let $h(y) = f(a, y)$. Then h is a function of a single variable defined on the interval $[b, b + \Delta y]$ and $h'(y) = f_y(a, y)$. A second application of the Mean Value Theorem then gives

$$h(b + \Delta y) - h(b) = h'(v)\,\Delta y$$

where v is some number between b and $b + \Delta y$. In terms of f, this becomes

$$f(a, b + \Delta y) - f(a, b) = f_y(a, v)\,\Delta y$$

We now substitute these expressions into Equation 1 and obtain

$$\begin{aligned}\Delta z &= f_x(u, b + \Delta y)\,\Delta x + f_y(a, v)\,\Delta y \\ &= f_x(a, b)\,\Delta x + [f_x(u, b + \Delta y) - f_x(a, b)]\,\Delta x + f_y(a, b)\,\Delta y \\ &\qquad + [f_y(a, v) - f_y(a, b)]\,\Delta y \\ &= f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y\end{aligned}$$

where
$$\varepsilon_1 = f_x(u, b + \Delta y) - f_x(a, b)$$
$$\varepsilon_2 = f_y(a, v) - f_y(a, b)$$

Since $(u, b + \Delta y) \to (a, b)$ and $(a, v) \to (a, b)$ as $(\Delta x,\ \Delta y) \to (0, 0)$ and since f_x and f_y are continuous at (a, b), we see that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x,\ \Delta y) \to (0, 0)$.

Therefore f is differentiable at (a, b). ■

G Complex Numbers

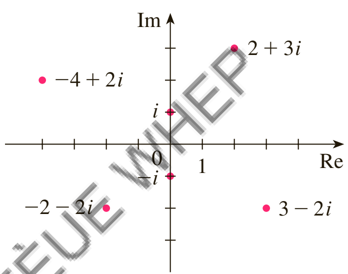


FIGURE 1
Complex numbers as points in the Argand plane

A **complex number** can be represented by an expression of the form $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair (a, b) and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus the complex number $i = 0 + 1 \cdot i$ is identified with the point $(0, 1)$.

The **real part** of the complex number $a + bi$ is the real number a and the **imaginary part** is the real number b. Thus the real part of $4 - 3i$ is 4 and the imaginary part is -3. Two complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$, that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

For instance,
$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + (bi)(c + di) \\ &= ac + adi + bci + bdi^2\end{aligned}$$

Since $i^2 = -1$, this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

EXAMPLE 1

$$\begin{aligned}(-1 + 3i)(2 - 5i) &= (-1)(2 - 5i) + 3i(2 - 5i) \\ &= -2 + 5i + 6i - 15(-1) = 13 + 11i\end{aligned}$$ ■

Division of complex numbers is much like rationalizing the denominator of a rational expression. For the complex number $z = a + bi$, we define its **complex conjugate** to be $\bar{z} = a - bi$. To find the quotient of two complex numbers we multiply numerator and denominator by the complex conjugate of the denominator.

EXAMPLE 2 Express the number $\dfrac{-1 + 3i}{2 + 5i}$ in the form $a + bi$.

SOLUTION We multiply numerator and denominator by the complex conjugate of $2 + 5i$, namely, $2 - 5i$, and we take advantage of the result of Example 1:

$$\frac{-1 + 3i}{2 + 5i} = \frac{-1 + 3i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{13 + 11i}{2^2 + 5^2} = \frac{13}{29} + \frac{11}{29}i$$ ■

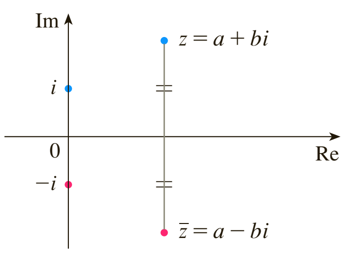


FIGURE 2

The geometric interpretation of the complex conjugate is shown in Figure 2: $\bar{z}$ is the reflection of z in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.

Properties of Conjugates

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \overline{zw} = \bar{z}\,\bar{w} \qquad \overline{z^n} = \bar{z}^n$$

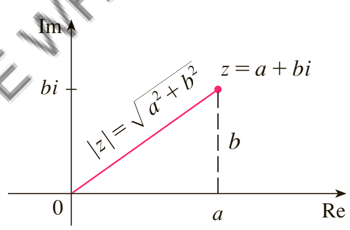


FIGURE 3

The **modulus**, or **absolute value**, $|z|$ of a complex number $z = a + bi$ is its distance from the origin. From Figure 3 we see that if $z = a + bi$, then

$$\boxed{|z| = \sqrt{a^2 + b^2}}$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2 i^2 = a^2 + b^2$$

and so $$\boxed{z\bar{z} = |z|^2}$$

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Since $i^2 = -1$, we can think of i as a square root of -1. But notice that we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of -1. We say that i is the **principal square root** of -1 and write $\sqrt{-1} = i$. In general, if c is any positive number, we write

$$\sqrt{-c} = \sqrt{c}\, i$$

With this convention, the usual derivation and formula for the roots of the quadratic equation $ax^2 + bx + c = 0$ are valid even when $b^2 - 4ac < 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 3 Find the roots of the equation $x^2 + x + 1 = 0$.

SOLUTION Using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$ ■

We observe that the solutions of the equation in Example 3 are complex conjugates of each other. In general, the solutions of any quadratic equation $ax^2 + bx + c = 0$ with real coefficients a, b, and c are always complex conjugates. (If z is real, $\bar{z} = z$, so z is its own conjugate.)

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree at least one has a solution among the complex numbers. This fact is known as the Fundamental Theorem of Algebra and was proved by Gauss.

Polar Form

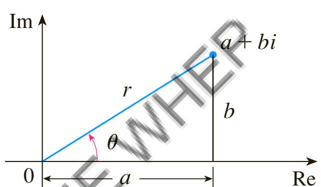


FIGURE 4

We know that any complex number $z = a + bi$ can be considered as a point (a, b) and that any such point can be represented by polar coordinates (r, θ) with $r \geqslant 0$. In fact,

$$a = r\cos\theta \qquad b = r\sin\theta$$

as in Figure 4. Therefore we have

$$z = a + bi = (r\cos\theta) + (r\sin\theta)i$$

Thus we can write any complex number z in the form

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where $$r = |z| = \sqrt{a^2 + b^2} \qquad \text{and} \qquad \tan\theta = \frac{b}{a}$$

The angle θ is called the **argument** of z and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique; any two arguments of z differ by an integer multiple of 2π.

EXAMPLE 4 Write the following numbers in polar form.

(a) $z = 1 + i$ (b) $w = \sqrt{3} - i$

SOLUTION

(a) We have $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = 1$, so we can take $\theta = \pi/4$. Therefore the polar form is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

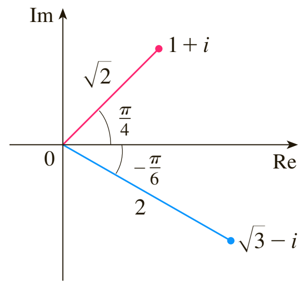


FIGURE 5

(b) Here we have $r = |w| = \sqrt{3+1} = 2$ and $\tan\theta = -1/\sqrt{3}$. Since w lies in the fourth quadrant, we take $\theta = -\pi/6$ and

$$w = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

The numbers z and w are shown in Figure 5. ■

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

1
$$\boxed{z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]}$$

This formula says that *to multiply two complex numbers we multiply the moduli and add the arguments*. (See Figure 6.)

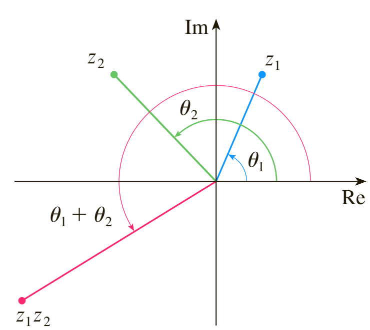


FIGURE 6

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers we divide the moduli and subtract the arguments*.

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad z_2 \neq 0}$$

In particular, taking $z_1 = 1$ and $z_2 = z$ (and therefore $\theta_1 = 0$ and $\theta_2 = \theta$), we have the following, which is illustrated in Figure 7.

$$\boxed{\text{If} \quad z = r(\cos\theta + i\sin\theta), \quad \text{then} \quad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta).}$$

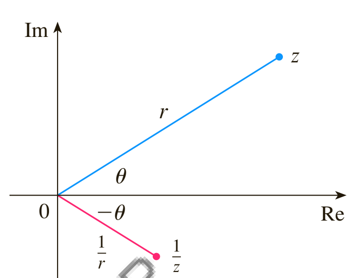


FIGURE 7

EXAMPLE 5 Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in polar form.

SOLUTION From Example 4 we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

and

$$\sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

So, by Equation 1,

$$(1 + i)(\sqrt{3} - i) = 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$$

$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

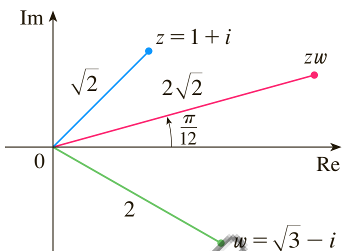


FIGURE 8

This is illustrated in Figure 8. ■

Repeated use of Formula 1 shows how to compute powers of a complex number. If

$$z = r(\cos\theta + i\sin\theta)$$

then

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

and

$$z^3 = zz^2 = r^3(\cos 3\theta + i\sin 3\theta)$$

In general, we obtain the following result, which is named after the French mathematician Abraham De Moivre (1667–1754).

2 De Moivre's Theorem If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

This says that *to take the nth power of a complex number we take the nth power of the modulus and multiply the argument by n.*

EXAMPLE 6 Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

SOLUTION Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 4(a) that $\frac{1}{2} + \frac{1}{2}i$ has the polar form

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by De Moivre's Theorem,

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i \qquad \blacksquare$$

De Moivre's Theorem can also be used to find the nth roots of complex numbers. An nth root of the complex number z is a complex number w such that

$$w^n = z$$

Writing these two numbers in trigonometric form as

$$w = s(\cos\phi + i\sin\phi) \qquad \text{and} \qquad z = r(\cos\theta + i\sin\theta)$$

and using De Moivre's Theorem, we get

$$s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

The equality of these two complex numbers shows that

$$s^n = r \qquad \text{or} \qquad s = r^{1/n}$$

and $$\cos n\phi = \cos\theta \qquad \text{and} \qquad \sin n\phi = \sin\theta$$

From the fact that sine and cosine have period 2π, it follows that

$$n\phi = \theta + 2k\pi \qquad \text{or} \qquad \phi = \frac{\theta + 2k\pi}{n}$$

Thus $$w = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Since this expression gives a different value of w for $k = 0, 1, 2, \ldots, n-1$, we have the following.

3 Roots of a Complex Number Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

Notice that each of the nth roots of z has modulus $|w_k| = r^{1/n}$. Thus all the nth roots of z lie on the circle of radius $r^{1/n}$ in the complex plane. Also, since the argument of each successive nth root exceeds the argument of the previous root by $2\pi/n$, we see that the nth roots of z are equally spaced on this circle.

EXAMPLE 7 Find the six sixth roots of $z = -8$ and graph these roots in the complex plane.

SOLUTION In trigonometric form, $z = 8(\cos \pi + i \sin \pi)$. Applying Equation 3 with $n = 6$, we get

$$w_k = 8^{1/6}\left(\cos \frac{\pi + 2k\pi}{6} + i \sin \frac{\pi + 2k\pi}{6}\right)$$

We get the six sixth roots of -8 by taking $k = 0, 1, 2, 3, 4, 5$ in this formula:

$$w_0 = 8^{1/6}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$$

$$w_1 = 8^{1/6}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = \sqrt{2}\, i$$

$$w_2 = 8^{1/6}\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$$

$$w_3 = 8^{1/6}\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$$

$$w_4 = 8^{1/6}\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right) = -\sqrt{2}\, i$$

$$w_5 = 8^{1/6}\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$$

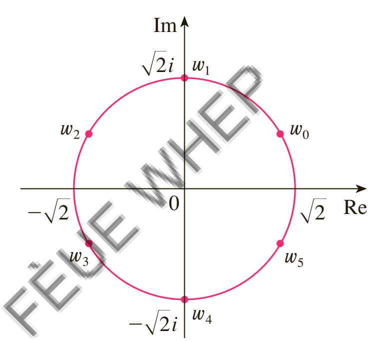


FIGURE 9
The six sixth roots of $z = -8$

All these points lie on the circle of radius $\sqrt{2}$ as shown in Figure 9. ■

Complex Exponentials

We also need to give a meaning to the expression e^z when $z = x + iy$ is a complex number. The theory of infinite series as developed in Chapter 11 can be extended to the case where the terms are complex numbers. Using the Taylor series for e^x (11.10.11) as our guide, we define

$$\boxed{4} \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

$$\boxed{5} \qquad e^{z_1 + z_2} = e^{z_1} e^{z_2}$$

If we put $z = iy$, where y is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = 1, \quad i^5 = i, \quad \ldots$$

we get

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y \end{aligned}$$

Here we have used the Taylor series for $\cos y$ and $\sin y$ (Equations 11.10.16 and 11.10.15). The result is a famous formula called **Euler's formula**:

6 $$\boxed{e^{iy} = \cos y + i \sin y}$$

Combining Euler's formula with Equation 5, we get

7 $$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

EXAMPLE 8 Evaluate: (a) $e^{i\pi}$ (b) $e^{-1+i\pi/2}$

SOLUTION

(a) From Euler's equation (6) we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

(b) Using Equation 7 we get

$$e^{-1+i\pi/2} = e^{-1}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = \frac{1}{e}[0 + i(1)] = \frac{i}{e}$$ ■

We could write the result of Example 8(a) as

$$e^{i\pi} + 1 = 0$$

This equation relates the five most famous numbers in all of mathematics: 0, 1, e, i, and π.

Finally, we note that Euler's equation provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos\theta + i \sin\theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

G EXERCISES

1–14 Evaluate the expression and write your answer in the form $a + bi$.

1. $(5 - 6i) + (3 + 2i)$
2. $(4 - \frac{1}{2}i) - (9 + \frac{5}{2}i)$
3. $(2 + 5i)(4 - i)$
4. $(1 - 2i)(8 - 3i)$
5. $\overline{12 + 7i}$
6. $\overline{2i(\frac{1}{2} - i)}$
7. $\dfrac{1 + 4i}{3 + 2i}$
8. $\dfrac{3 + 2i}{1 - 4i}$
9. $\dfrac{1}{1 + i}$
10. $\dfrac{3}{4 - 3i}$
11. i^3
12. i^{100}
13. $\sqrt{-25}$
14. $\sqrt{-3}\sqrt{-12}$

15–17 Find the complex conjugate and the modulus of the number.

15. $12 - 5i$
16. $-1 + 2\sqrt{2}\, i$
17. $-4i$

18. Prove the following properties of complex numbers.
(a) $\overline{z + w} = \bar{z} + \bar{w}$ (b) $\overline{zw} = \bar{z}\,\bar{w}$
(c) $\overline{z^n} = \bar{z}^n$, where n is a positive integer
[*Hint:* Write $z = a + bi$, $w = c + di$.]

19–24 Find all solutions of the equation.

19. $4x^2 + 9 = 0$ **20.** $x^4 = 1$

21. $x^2 + 2x + 5 = 0$ **22.** $2x^2 - 2x + 1 = 0$

23. $z^2 + z + 2 = 0$ **24.** $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$

25–28 Write the number in polar form with argument between 0 and 2π.

25. $-3 + 3i$ **26.** $1 - \sqrt{3}\,i$

27. $3 + 4i$ **28.** $8i$

29–32 Find polar forms for zw, z/w, and $1/z$ by first putting z and w into polar form.

29. $z = \sqrt{3} + i, \quad w = 1 + \sqrt{3}\,i$

30. $z = 4\sqrt{3} - 4i, \quad w = 8i$

31. $z = 2\sqrt{3} - 2i, \quad w = -1 + i$

32. $z = 4(\sqrt{3} + i), \quad w = -3 - 3i$

33–36 Find the indicated power using De Moivre's Theorem.

33. $(1 + i)^{20}$ **34.** $(1 - \sqrt{3}\,i)^5$

35. $(2\sqrt{3} + 2i)^5$ **36.** $(1 - i)^8$

37–40 Find the indicated roots. Sketch the roots in the complex plane.

37. The eighth roots of 1 **38.** The fifth roots of 32

39. The cube roots of i **40.** The cube roots of $1 + i$

41–46 Write the number in the form $a + bi$.

41. $e^{i\pi/2}$ **42.** $e^{2\pi i}$

43. $e^{i\pi/3}$ **44.** $e^{-i\pi}$

45. $e^{2+i\pi}$ **46.** $e^{\pi+i}$

47. Use De Moivre's Theorem with $n = 3$ to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

48. Use Euler's formula to prove the following formulas for $\cos x$ and $\sin x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

49. If $u(x) = f(x) + ig(x)$ is a complex-valued function of a real variable x and the real and imaginary parts $f(x)$ and $g(x)$ are differentiable functions of x, then the derivative of u is defined to be $u'(x) = f'(x) + ig'(x)$. Use this together with Equation 7 to prove that if $F(x) = e^{rx}$, then $F'(x) = re^{rx}$ when $r = a + bi$ is a complex number.

50. (a) If u is a complex-valued function of a real variable, its indefinite integral $\int u(x)\,dx$ is an antiderivative of u. Evaluate

$$\int e^{(1+i)x}\,dx$$

(b) By considering the real and imaginary parts of the integral in part (a), evaluate the real integrals

$$\int e^x \cos x\,dx \qquad \text{and} \qquad \int e^x \sin x\,dx$$

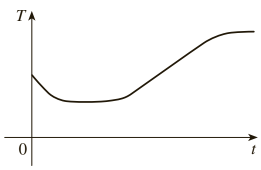
(c) Compare with the method used in Example 7.1.4.

H Answers to Odd-Numbered Exercises

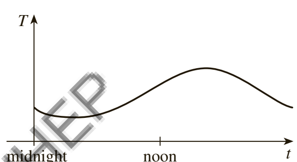
CHAPTER 1

EXERCISES 1.1 ■ PAGE 19

1. Yes
3. (a) 3 (b) -0.2 (c) 0, 3 (d) -0.8
(e) $[-2, 4], [-1, 3]$ (f) $[-2, 1]$
5. $[-85, 115]$ **7.** No
9. Yes, $[-3, 2], [-3, -2) \cup [-1, 3]$
11. (a) 13.8°C (b) 1990 (c) 1910, 2005 (d) $[13.5, 14.5]$
13.



15. (a) 500 MW; 730 MW (b) 4 AM; noon; yes
17.



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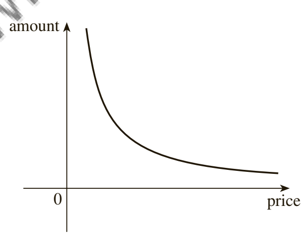
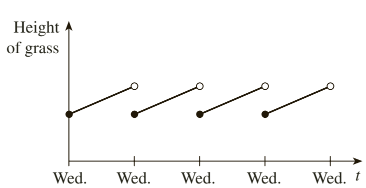
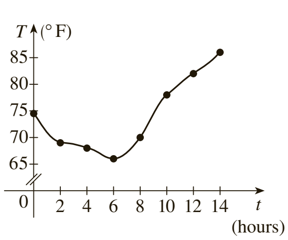


21.



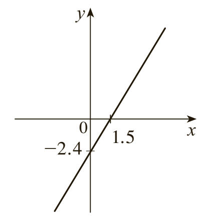
23. (a)



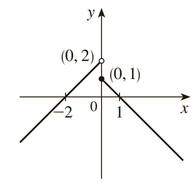
(b) 74°F

25. 12, 16, $3a^2 - a + 2$, $3a^2 + a + 2$, $3a^2 + 5a + 4$,
$6a^2 - 2a + 4$, $12a^2 - 2a + 2$, $3a^4 - a^2 + 2$,
$9a^4 - 6a^3 + 13a^2 - 4a + 4$, $3a^2 + 6ah + 3h^2 - a - h + 2$
27. $-3 - h$ **29.** $-1/(ax)$
31. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **33.** $(-\infty, \infty)$
35. $(-\infty, 0) \cup (5, \infty)$ **37.** $[0, 4]$

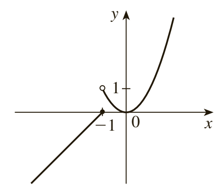
39. $(-\infty, \infty)$



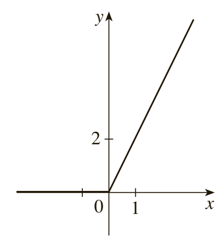
41. $-1, 1, -1$



43. $-2, 0, 4$



45.



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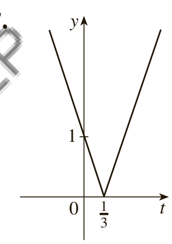

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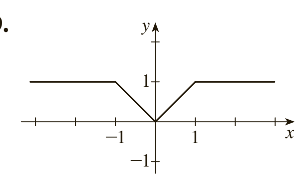
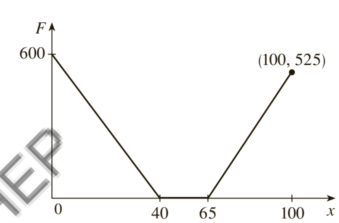


51. $f(x) = \frac{5}{2}x - \frac{11}{2}, 1 \leqslant x \leqslant 5$ **53.** $f(x) = 1 - \sqrt{-x}$

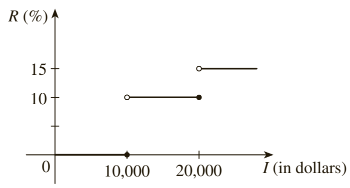
55. $f(x) = \begin{cases} -x + 3 & \text{if } 0 \leqslant x \leqslant 3 \\ 2x - 6 & \text{if } 3 < x \leqslant 5 \end{cases}$

57. $A(L) = 10L - L^2, 0 < L < 10$
59. $A(x) = \sqrt{3}x^2/4, x > 0$ **61.** $S(x) = x^2 + (8/x), x > 0$
63. $V(x) = 4x^3 - 64x^2 + 240x, 0 < x < 6$

65. $F(x) = \begin{cases} 15(40 - x) & \text{if } 0 \leqslant x < 40 \\ 0 & \text{if } 40 \leqslant x \leqslant 65 \\ 15(x - 65) & \text{if } x > 65 \end{cases}$

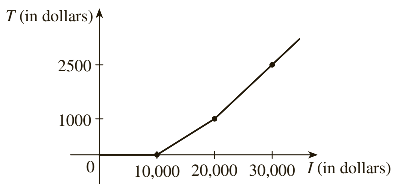


67. (a)



(b) \$400, \$1900

(c)

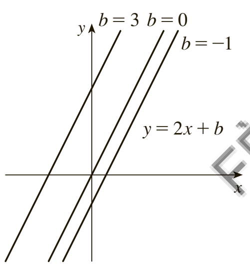


69. f is odd, g is even **71.** (a) $(-5, 3)$ (b) $(-5, -3)$
73. Odd **75.** Neither **77.** Even
79. Even; odd; neither (unless $f = 0$ or $g = 0$)

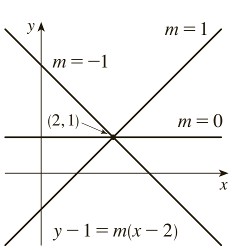
EXERCISES 1.2 ■ PAGE 33

1. (a) Logarithmic (b) Root (c) Rational
(d) Polynomial, degree 2 (e) Exponential (f) Trigonometric
3. (a) h (b) f (c) g
5. $\{x \mid x \neq \pi/2 + 2n\pi,\ n \text{ an integer}\}$

7. (a) $y = 2x + b$,
where b is the y-intercept.

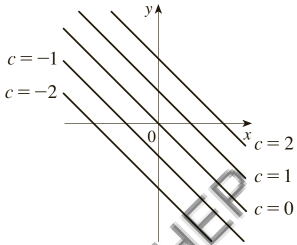


(b) $y = mx + 1 - 2m$,
where m is the slope.



(c) $y = 2x - 3$

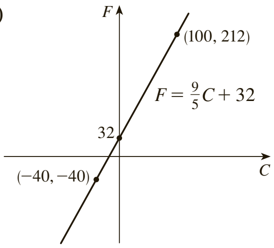
9. Their graphs have slope -1.



11. $f(x) = -3x(x + 1)(x - 2)$

13. (a) 8.34, change in mg for every 1 year change
(b) 8.34 mg

15. (a)



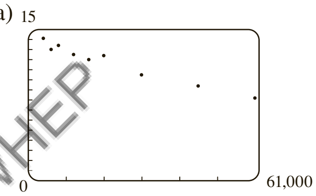
(b) $\frac{9}{5}$, change in °F for every 1°C change; 32, Fahrenheit temperature corresponding to 0°C

17. (a) $T = \frac{1}{6}N + \frac{307}{6}$ (b) $\frac{1}{6}$, change in °F for every chirp per minute change (c) 76°F

19. (a) $P = 0.434d + 15$ (b) 196 ft

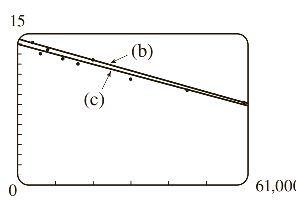
21. (a) Cosine (b) Linear

23. (a)



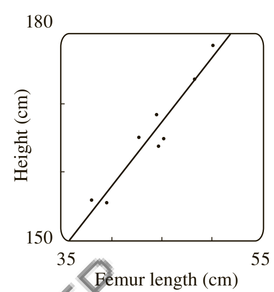
A linear model is appropriate.

(b) $y = -0.000105x + 14.521$



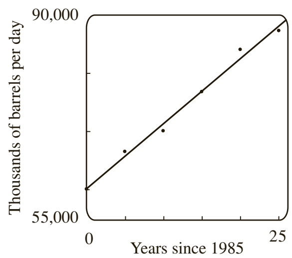
(c) $y = -0.00009979x + 13.951$
(d) About 11.5 per 100 population
(e) About 6% (f) No

25. (a) See graph in part (b).
(b) $y = 1.88074x + 82.64974$



(c) 182.3 cm

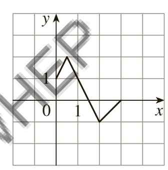
27. (a) A linear model is appropriate. See graph in part (b).
(b) $y = 1116.64x + 60{,}188.33$



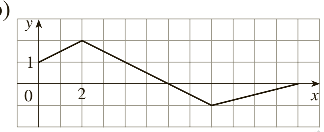
(c) In thousands of barrels per day: 79,171 and 90,338
29. Four times as bright
31. (a) $N = 3.1046A^{0.308}$ (b) 18

EXERCISES 1.3 ■ PAGE 42

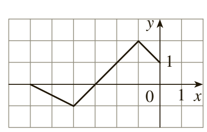
1. (a) $y = f(x) + 3$ (b) $y = f(x) - 3$ (c) $y = f(x - 3)$
(d) $y = f(x + 3)$ (e) $y = -f(x)$ (f) $y = f(-x)$
(g) $y = 3f(x)$ (h) $y = \frac{1}{3}f(x)$
3. (a) 3 (b) 1 (c) 4 (d) 5 (e) 2
5. (a)



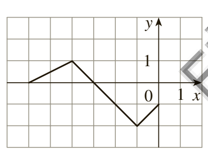
(b)



(c)

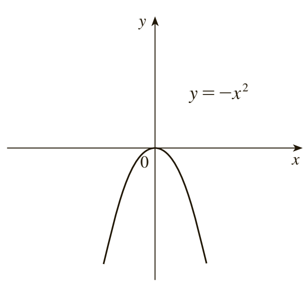


(d)



7. $y = -\sqrt{-x^2 - 5x - 4} - 1$

9.



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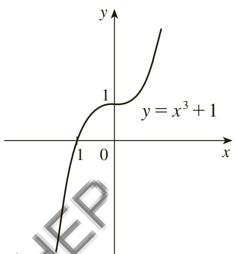

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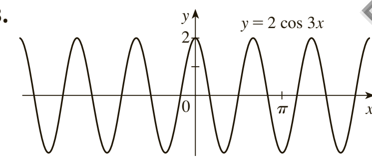
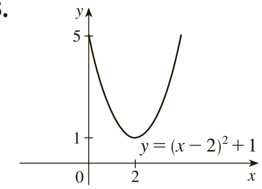
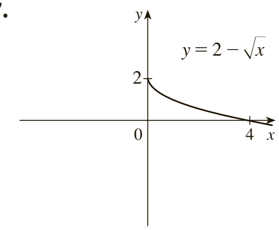


15.



17.



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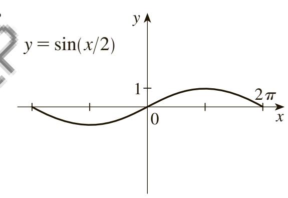

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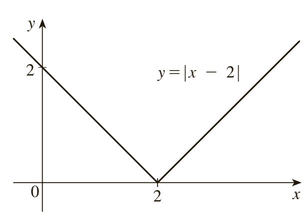
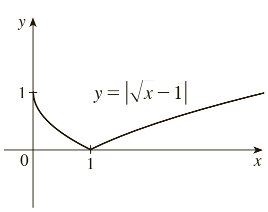
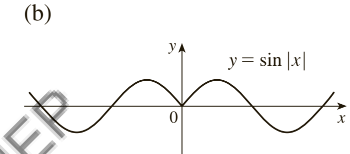


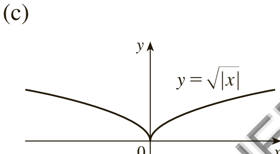
23.



25. $L(t) = 12 + 2\sin\left[\dfrac{2\pi}{365}(t - 80)\right]$
27. $D(t) = 5\cos[(\pi/6)(t - 6.75)] + 7$
29. (a) The portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.
(b)



(c)



31. (a) $(f+g)(x) = x^3 + 5x^2 - 1$, $(-\infty, \infty)$
(b) $(f-g)(x) = x^3 - x^2 + 1$, $(-\infty, \infty)$
(c) $(fg)(x) = 3x^5 + 6x^4 - x^3 - 2x^2$, $(-\infty, \infty)$
(d) $(f/g)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}$, $\left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$

33. (a) $(f \circ g)(x) = 3x^2 + 3x + 5$, $(-\infty, \infty)$
(b) $(g \circ f)(x) = 9x^2 + 33x + 30$, $(-\infty, \infty)$
(c) $(f \circ f)(x) = 9x + 20$, $(-\infty, \infty)$
(d) $(g \circ g)(x) = x^4 + 2x^3 + 2x^2 + x$, $(-\infty, \infty)$

35. (a) $(f \circ g)(x) = \sqrt{4x - 2}$, $\left[\frac{1}{2}, \infty\right)$
(b) $(g \circ f)(x) = 4\sqrt{x+1} - 3$, $[-1, \infty)$
(c) $(f \circ f)(x) = \sqrt{\sqrt{x+1} + 1}$, $[-1, \infty)$
(d) $(g \circ g)(x) = 16x - 15$, $(-\infty, \infty)$

37. (a) $(f \circ g)(x) = \dfrac{2x^2 + 6x + 5}{(x+2)(x+1)}$, $\{x \mid x \neq -2, -1\}$
(b) $(g \circ f)(x) = \dfrac{x^2 + x + 1}{(x+1)^2}$, $\{x \mid x \neq -1, 0\}$
(c) $(f \circ f)(x) = \dfrac{x^4 + 3x^2 + 1}{x(x^2+1)}$, $\{x \mid x \neq 0\}$
(d) $(g \circ g)(x) = \dfrac{2x+3}{3x+5}$, $\{x \mid x \neq -2, -\frac{5}{3}\}$

39. $(f \circ g \circ h)(x) = 3\sin(x^2) - 2$

41. $(f \circ g \circ h)(x) = \sqrt{x^6 + 4x^3 + 1}$

43. $g(x) = 2x + x^2$, $f(x) = x^4$

45. $g(x) = \sqrt[3]{x}$, $f(x) = x/(1+x)$

47. $g(t) = t^2$, $f(t) = \sec t \tan t$

49. $h(x) = \sqrt{x}$, $g(x) = x - 1$, $f(x) = \sqrt{x}$

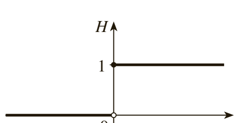
51. $h(t) = \cos t$, $g(t) = \sin t$, $f(t) = t^2$

53. (a) 4 (b) 3 (c) 0 (d) Does not exist; $f(6) = 6$ is not in the domain of g. (e) 4 (f) -2

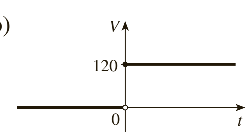
55. (a) $r(t) = 60t$ (b) $(A \circ r)(t) = 3600\pi t^2$; the area of the circle as a function of time

57. (a) $s = \sqrt{d^2 + 36}$ (b) $d = 30t$
(c) $(f \circ g)(t) = \sqrt{900t^2 + 36}$; the distance between the lighthouse and the ship as a function of the time elapsed since noon

59. (a)

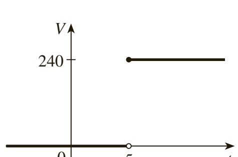


(b)



$V(t) = 120H(t)$

(c)



$V(t) = 240H(t - 5)$

61. Yes; $m_1 m_2$

63. (a) $f(x) = x^2 + 6$ (b) $g(x) = x^2 + x - 1$

65. Yes

EXERCISES 1.4 ■ PAGE 49

1. (a) $-44.4, -38.8, -27.8, -22.2, -16.\overline{6}$
(b) -33.3 (c) $-33\frac{1}{3}$

3. (a) (i) 2 (ii) 1.111111 (iii) 1.010101 (iv) 1.001001 (v) 0.666667 (vi) 0.909091 (vii) 0.990099 (viii) 0.999001 (b) 1 (c) $y = x - 3$

5. (a) (i) -32 ft/s (ii) -25.6 ft/s (iii) -24.8 ft/s (iv) -24.16 ft/s (b) -24 ft/s

7. (a) (i) 29.3 ft/s (ii) 32.7 ft/s (iii) 45.6 ft/s (iv) 48.75 ft/s (b) 29.7 ft/s

9. (a) 0, 1.7321, -1.0847, -2.7433, 4.3301, -2.8173, 0, -2.1651, -2.6061, -5, 3.4202; no (c) -31.4

EXERCISES 1.5 ■ PAGE 59

1. Yes

3. (a) $\lim_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).
(b) $\lim_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

5. (a) 2 (b) 1 (c) 4 (d) Does not exist (e) 3

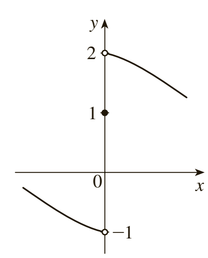
7. (a) -1 (b) -2 (c) Does not exist (d) 2 (e) 0 (f) Does not exist (g) 1 (h) 3

9. (a) $-\infty$ (b) ∞ (c) ∞ (d) $-\infty$ (e) ∞
(f) $x = -7, x = -3, x = 0, x = 6$

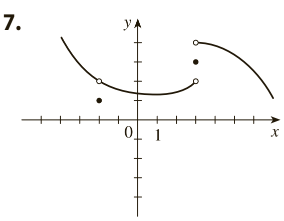
11. $\lim_{x \to a} f(x)$ exists for all a except $a = -1$.

13. (a) 1 (b) 0 (c) Does not exist

15.



17.



19. $\frac{1}{2}$ **21.** $\frac{1}{2}$ **23.** 1.5 **25.** 1 **27.** (a) -1.5

29. ∞ **31.** ∞ **33.** $-\infty$ **35.** $-\infty$ **37.** $-\infty$

39. ∞ **41.** $-\infty$; ∞

43. (a) 0.998000, 0.638259, 0.358484, 0.158680, 0.038851, 0.008928, 0.001465; 0
(b) 0.000572, -0.000614, -0.000907, -0.000978, -0.000993, -0.001000; -0.001

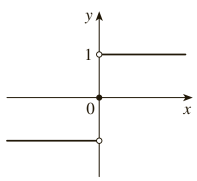
45. No matter how many times we zoom in toward the origin, the graph appears to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.

47. $x \approx \pm 0.90, \pm 2.24$; $x = \pm\sin^{-1}(\pi/4)$, $\pm(\pi - \sin^{-1}(\pi/4))$

49. (a) 6 (b) Within 0.0649 of 1

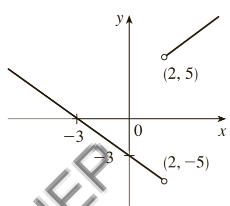
EXERCISES 1.6 ■ PAGE 70

1. (a) -6 (b) -8 (c) 2 (d) -6
(e) Does not exist (f) 0
3. 105 **5.** $\frac{7}{8}$ **7.** 390 **9.** $\frac{3}{2}$ **11.** 4
13. Does not exist **15.** $\frac{6}{5}$ **17.** -10 **19.** $\frac{1}{12}$
21. $\frac{1}{6}$ **23.** $-\frac{1}{9}$ **25.** 1 **27.** $\frac{1}{128}$ **29.** $-\frac{1}{2}$
31. $3x^2$ **33.** (a), (b) $\frac{2}{3}$ **37.** 7 **41.** 6 **43.** -4
45. Does not exist
47. (a)



(b) (i) 1
(ii) -1
(iii) Does not exist
(iv) 1

49. (a) (i) 5 (ii) -5 (b) Does not exist
(c)



51. 7
53. (a) (i) -2 (ii) Does not exist (iii) -3
(b) (i) $n-1$ (ii) n (c) a is not an integer.
59. 8 **65.** 15; -1

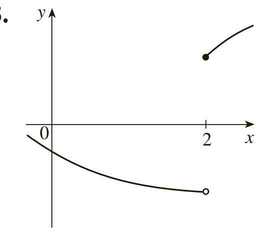
EXERCISES 1.7 ■ PAGE 81

1. 0.1 (or any smaller positive number)
3. 1.44 (or any smaller positive number)
5. 0.0906 (or any smaller positive number)
7. 0.0219 (or any smaller positive number);
0.011 (or any smaller positive number)
9. (a) 0.041 (or any smaller positive number)
(b) $\lim_{x\to 4^+} \dfrac{x^2+4}{\sqrt{x-4}} = \infty$
11. (a) $\sqrt{1000/\pi}$ cm (b) Within approximately 0.0445 cm
(c) Radius; area; $\sqrt{1000/\pi}$; 1000; 5; ≈ 0.0445
13. (a) 0.025 (b) 0.0025
35. (a) 0.093 (b) $\delta = (B^{2/3} - 12)/(6B^{1/3}) - 1$, where
$B = 216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}$
41. Within 0.1

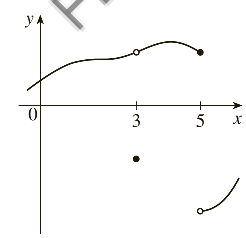
EXERCISES 1.8 ■ PAGE 91

1. $\lim_{x\to 4} f(x) = f(4)$
3. (a) $-4, -2, 2, 4$; $f(-4)$ is not defined and $\lim_{x\to a} f(x)$ does not exist for $a = -2$, 2, and 4
(b) -4, neither; -2, left; 2, right; 4, right

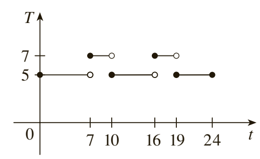
5.



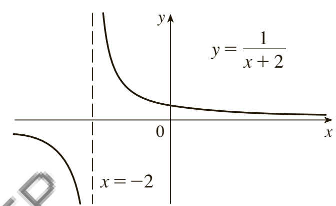
7.



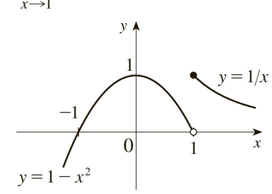
9. (a)



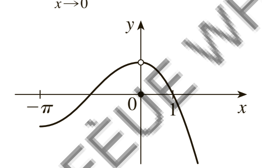
17. $f(-2)$ is undefined.



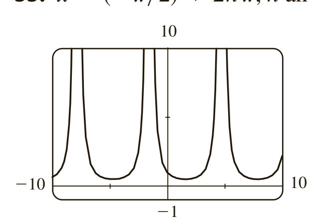
19. $\lim_{x\to 1} f(x)$ does not exist.



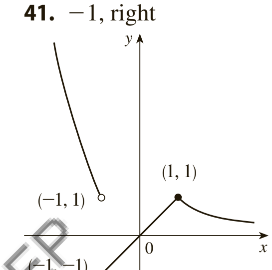
21. $\lim_{x\to 0} f(x) \neq f(0)$



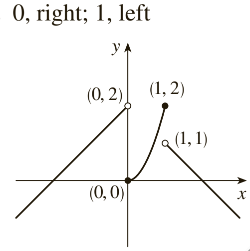
23. Define $f(2) = 3$. **25.** $(-\infty, \infty)$
27. $\left(-\infty, \sqrt[3]{2}\right) \cup \left(\sqrt[3]{2}, \infty\right)$ **29.** $\mathbb{R}$
31. $(-\infty, -1] \cup (0, \infty)$
33. $x = (-\pi/2) + 2n\pi$, n an integer



35. 8 **37.** $\pi^2/16$
41. -1, right



43. 0, right; 1, left



45. $\frac{2}{3}$ **47.** 4

49. (a) $g(x) = x^3 + x^2 + x + 1$ (b) $g(x) = x^2 + x$

57. (b) (0.86, 0.87) **59.** (b) 1.434 **67.** None

69. Yes

CHAPTER 1 REVIEW ■ PAGE 95

True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. True **13.** False **15.** True **17.** True
19. False **21.** True **23.** True **25.** True **27.** False

Exercises

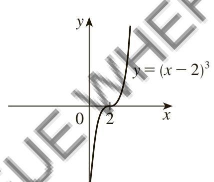
1. (a) 2.7 (b) 2.3, 5.6 (c) $[-6, 6]$ (d) $[-4, 4]$
(e) $[-4, 4]$ (f) Odd; its graph is symmetric about the origin.

3. $2a + h - 2$ **5.** $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$, $(-\infty, 0) \cup (0, \infty)$

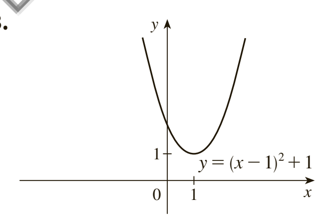
7. $(-\infty, \infty)$, $[0, 2]$

9. (a) Shift the graph 8 units upward.
(b) Shift the graph 8 units to the left.
(c) Stretch the graph vertically by a factor of 2, then shift it 1 unit upward.
(d) Shift the graph 2 units to the right and 2 units downward.
(e) Reflect the graph about the x-axis.
(f) Reflect the graph about the x-axis, then shift 3 units upward.

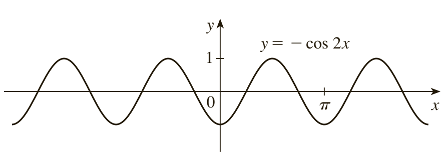
11.



13.



15.



17. (a) Neither (b) Odd (c) Even (d) Neither

19. (a) $(f \circ g)(x) = \sqrt{\sin x}$,
$\{x \mid x \in [2n\pi, \pi + 2n\pi], n \text{ an integer}\}$
(b) $(g \circ f)(x) = \sin \sqrt{x}$, $[0, \infty)$
(c) $(f \circ f)(x) = \sqrt[4]{x}$, $[0, \infty)$
(d) $(g \circ g)(x) = \sin(\sin x)$, $\mathbb{R}$

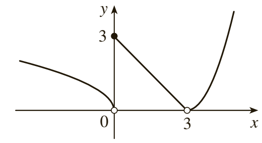
21. $y = 0.2493x - 423.4818$; about 77.6 years

23. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 2
(v) ∞ (vi) $-\infty$
(b) $x = 0$, $x = 2$ (c) $-3, 0, 2, 4$

25. 1 **27.** $\frac{3}{2}$ **29.** 3 **31.** ∞ **33.** $\frac{4}{7}$ **35.** $-\frac{1}{8}$

37. 0 **39.** 1

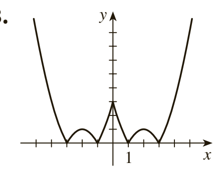
45. (a) (i) 3 (ii) 0 (iii) Does not exist
(iv) 0 (v) 0 (vi) 0
(b) At 0 and 3 (c)



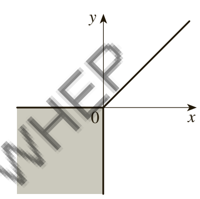
47. $[0, \infty)$ **51.** 0

PRINCIPLES OF PROBLEM SOLVING ■ PAGE 102

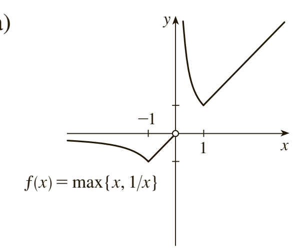
1. $-\frac{7}{3}, 9$ **3.**



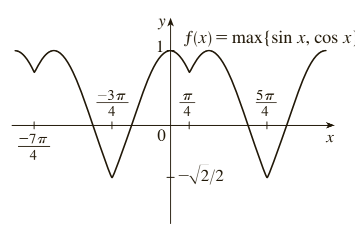
5.



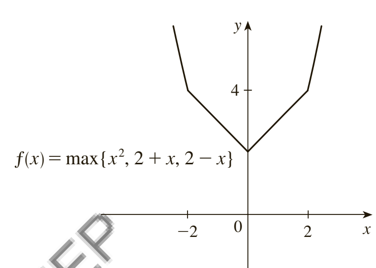
7. (a)



(b)



(c)



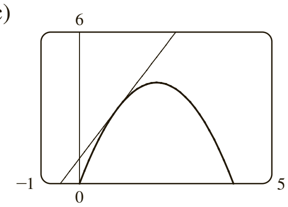
9. 40 mi/h **13.** $f_n(x) = x^{2^{n+1}}$ **15.** $\frac{2}{3}$
17. -4 **19.** (a) Does not exist (b) 1
21. $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ **23.** $\frac{3}{4}$ **25.** (b) Yes (c) Yes; no

CHAPTER 2

EXERCISES 2.1 ■ PAGE 113

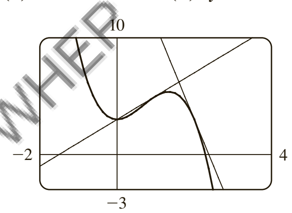
1. (a) $\dfrac{f(x) - f(3)}{x - 3}$ (b) $\displaystyle\lim_{x\to 3} \frac{f(x) - f(3)}{x - 3}$

3. (a) 2 (b) $y = 2x + 1$ (c)



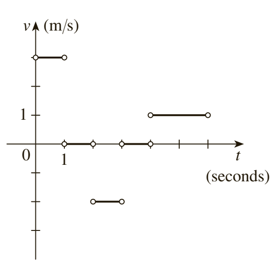
5. $y = -8x + 12$ **7.** $y = \frac{1}{2}x + \frac{1}{2}$

9. (a) $8a - 6a^2$ (b) $y = 2x + 3$, $y = -8x + 19$
(c)

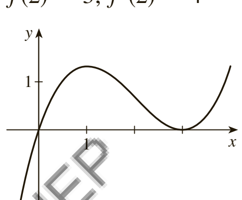


11. (a) Right: $0 < t < 1$ and $4 < t < 6$; left: $2 < t < 3$; standing still: $1 < t < 2$ and $3 < t < 4$

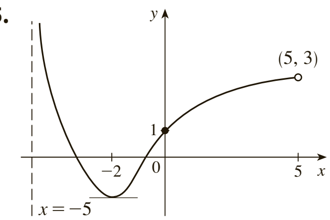
(b)



13. -24 ft/s
15. $-2/a^3$ m/s; -2 m/s; $-\frac{1}{4}$ m/s; $-\frac{2}{27}$ m/s
17. $g'(0)$, 0, $g'(4)$, $g'(2)$, $g'(-2)$
19. (a) 26 (b) No (c) Yes
21. $f(2) = 3$; $f'(2) = 4$
23.

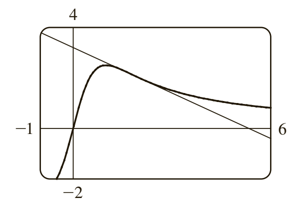


25.

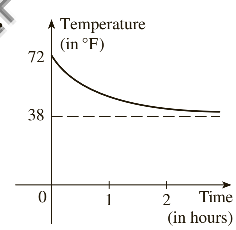


27. 3; $y = 3x - 1$

29. (a) $-\frac{3}{5}$; $y = -\frac{3}{5}x + \frac{16}{5}$ (b)



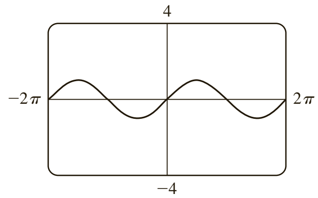
31. $6a - 4$ **33.** $\dfrac{5}{(a + 3)^2}$ **35.** $-\dfrac{1}{\sqrt{1 - 2a}}$
37. $f(x) = \sqrt{x}$, $a = 9$ **39.** $f(x) = x^6$, $a = 2$
41. $f(x) = \cos x$, $a = \pi$ or $f(x) = \cos(\pi + x)$, $a = 0$
43. 32 m/s; 32 m/s
45.



Greater (in magnitude)

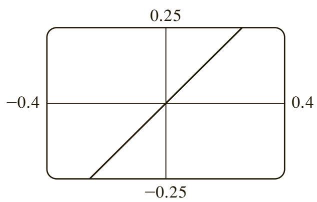
47. (a) In (g/dL)/h: (i) -0.015 (ii) -0.012 (iii) -0.012 (iv) -0.011
(b) -0.012 (g/dL)/h; After 2 hours, the BAC is decreasing at a rate of 0.012 (g/dL)/h.
49. (a) 1169.6 thousands of barrels of oil per day per year; oil consumption rose by an average of 1169.6 thousands of barrels of oil per day each year from 1990 to 2005.
(b) 1397.8 thousands of barrels of oil per day per year
51. (a) (i) \$20.25/unit (ii) \$20.05/unit (b) \$20/unit
53. (a) The rate at which the cost is changing per ounce of gold produced; dollars per ounce
(b) When the 800th ounce of gold is produced, the cost of production is \$17/oz.
(c) Decrease in the short term; increase in the long term
55. (a) The rate at which daily heating costs change with respect to temperature when the temperature is 58°F; dollars/°F
(b) Negative; If the outside temperature increases, the building should require less heating.
57. (a) The rate at which the oxygen solubility changes with respect to the water temperature; (mg/L)/°C
(b) $S'(16) \approx -0.25$; as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of 0.25 (mg/L)/°C.
59. Does not exist

61. (a)



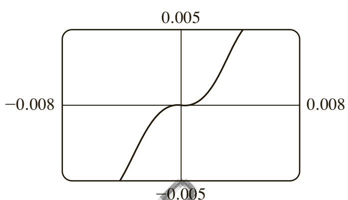
Slope appears to be 1.

(b)



Yes

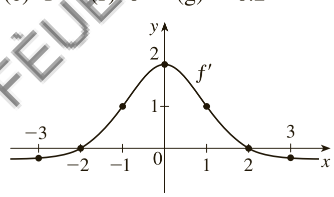
(c)



Yes; 0

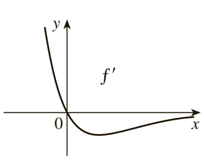
EXERCISES 2.2 ■ PAGE 125

1. (a) -0.2 (b) 0 (c) 1 (d) 2
(e) 1 (f) 0 (g) -0.2

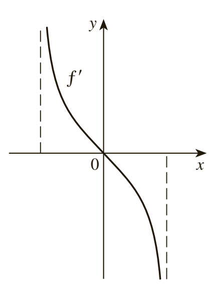


3. (a) II (b) IV (c) I (d) III

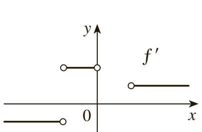
5.



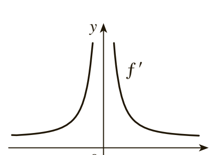
7.



9.

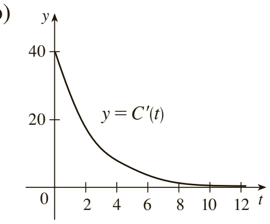


11.



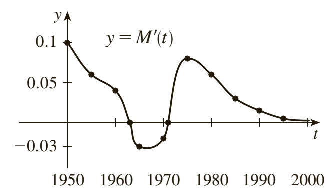
13. (a) The instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours

(b)



The rate of change of percentage of full capacity is decreasing and approaching 0.

15.



1963 to 1971

17. (a) 0, 1, 2, 4 (b) $-1, -2, -4$ (c) $f'(x) = 2x$

19. $f'(x) = 3$, $\mathbb{R}$, $\mathbb{R}$ **21.** $f'(t) = 5t + 6$, $\mathbb{R}$, $\mathbb{R}$

23. $f'(x) = 2x - 6x^2$, $\mathbb{R}$, $\mathbb{R}$

25. $g'(x) = -\dfrac{1}{2\sqrt{9 - x}}$, $(-\infty, 9]$, $(-\infty, 9)$

27. $G'(t) = \dfrac{-7}{(3 + t)^2}$, $(-\infty, -3) \cup (-3, \infty)$, $(-\infty, -3) \cup (-3, \infty)$

29. $f'(x) = 4x^3$, $\mathbb{R}$, $\mathbb{R}$ **31.** (a) $f'(x) = 4x^3 + 2$

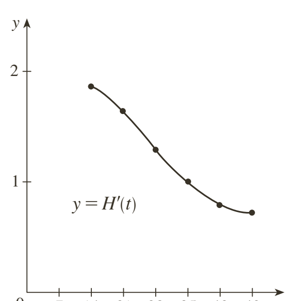
33. (a) The rate at which the unemployment rate is changing, in percent unemployed per year

(b)

t	$U'(t)$	t	$U'(t)$
2003	-0.50	2008	2.35
2004	-0.45	2009	1.90
2005	-0.45	2010	-0.20
2006	-0.25	2011	-0.75
2007	0.60	2012	-0.80

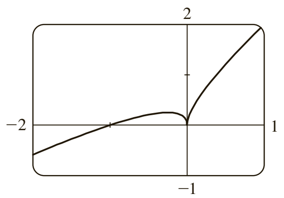
35.

t	14	21	28	35	42	49
$H'(t)$	$\frac{13}{7}$	$\frac{23}{14}$	$\frac{9}{7}$	1	$\frac{11}{14}$	$\frac{5}{7}$



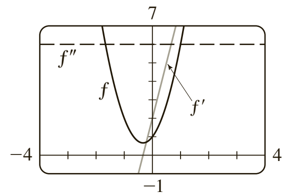
37. (a) The rate at which the percentage of electrical power produced by solar panels is changing, in percentage points per year.
(b) On January 1, 2002, the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

39. -4 (corner); 0 (discontinuity)
41. 1 (not defined); 5 (vertical tangent)
43.

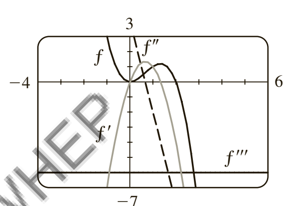


Differentiable at -1; not differentiable at 0

45. $f''(1)$ **47.** $a = f, b = f', c = f''$
49. a = acceleration, b = velocity, c = position
51. $6x + 2$; 6



53.

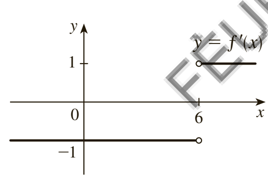


$f'(x) = 4x - 3x^2$,
$f''(x) = 4 - 6x$,
$f'''(x) = -6$,
$f^{(4)}(x) = 0$

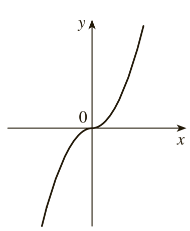
55. (a) $\frac{1}{3}a^{-2/3}$

57. $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$

or $f'(x) = \dfrac{x - 6}{|x - 6|}$

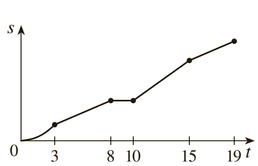


59. (a)

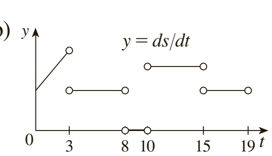


(b) All x
(c) $f'(x) = 2|x|$

63. (a)



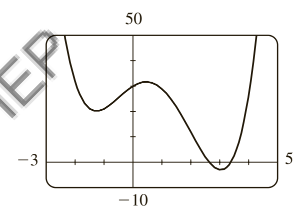
(b)



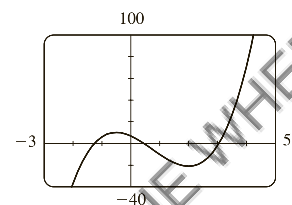
65. 63°

EXERCISES 2.3 ■ PAGE 140

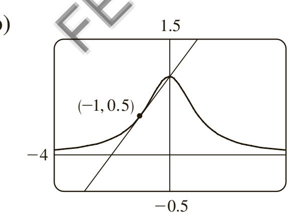
1. $f'(x) = 0$ **3.** $f'(x) = 5.2$ **5.** $f'(t) = 6t^2 - 6t - 4$
7. $g'(x) = 2x - 6x^2$ **9.** $g'(t) = -\frac{3}{2}t^{-7/4}$
11. $F'(r) = -15/r^4$ **13.** $S'(p) = \frac{1}{2}p^{-1/2} - 1$
15. $R'(a) = 18a + 6$ **17.** $y' = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$
19. $G'(q) = -2q^{-2} - 2q^{-3}$ **21.** $u' = -\dfrac{2}{t^3} + \dfrac{3}{t^2\sqrt{t}} - \dfrac{1}{t^2}$
23. $1 - 2x + 6x^2 - 8x^3$ **25.** $f'(x) = 25x^4 + 39x^2 - 6$
27. $F'(y) = 5 + \dfrac{14}{y^2} + \dfrac{9}{y^4}$ **29.** $g'(x) = \dfrac{10}{(3 - 4x)^2}$
31. $y' = \dfrac{x(-x^3 - 3x - 2)}{(x^3 - 1)^2}$ **33.** $y' = \dfrac{t^4 - 8t^3 + 6t^2 + 9}{(t^2 - 4t + 3)^2}$
35. $y' = \dfrac{3 - 2\sqrt{s}}{2s^{5/2}}$ **37.** $f'(t) = \dfrac{-2t - 3}{3t^{2/3}(t - 3)^2}$
39. $F'(x) = 4x + 1 + \dfrac{12}{x^3}$ **41.** $G'(y) = -\dfrac{3ABy^2}{(Ay^3 + B)^2}$
43. $f'(x) = \dfrac{2cx}{(x^2 + c)^2}$
45. $P'(x) = na_nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$
47. $45x^{14} - 15x^2$
49. (a)



(c) $4x^3 - 9x^2 - 12x + 7$

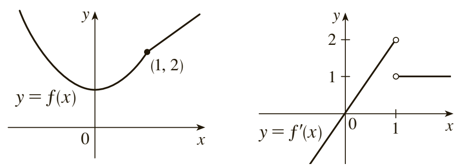


51. $y = \frac{1}{2}x + \frac{1}{2}$
53. (a) $y = \frac{1}{2}x + 1$ (b)



55. $y = \frac{3}{2}x + \frac{1}{2}$, $y = -\frac{2}{3}x + \frac{8}{3}$ **57.** $y = -\frac{1}{2}x + \frac{5}{2}$, $y = 2x$
59. $f'(x) = 0.005x^4 - 0.06x^2$; $f''(x) = 0.02x^3 - 0.12x$
61. $f'(x) = \dfrac{2x^2 + 2x}{(1 + 2x)^2}$; $f''(x) = \dfrac{2}{(1 + 2x)^3}$
63. (a) $v(t) = 3t^2 - 3$, $a(t) = 6t$ (b) 12 m/s^2
(c) $a(1) = 6$ m/s^2
65. 1.718; at 12 years, the length of the fish is increasing at a rate of 1.718 in/year.
67. (a) $V = 5.3/P$
(b) -0.00212; instantaneous rate of change of the volume with respect to the pressure at 25°C; m^3/kPa
69. (a) -16 (b) $-\frac{20}{9}$ (c) 20
71. 16 **73.** (a) 0 (b) $-\frac{2}{3}$
75. (a) $y' = xg'(x) + g(x)$ (b) $y' = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$
(c) $y' = \dfrac{xg'(x) - g(x)}{x^2}$

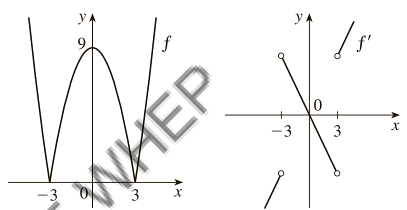
77. $(-2, 21), (1, -6)$ **81.** $y = 3x - 3,\ y = 3x - 7$
83. $y = -2x + 3$ **85.** $(\pm 2, 4)$
87. (c) $3(x^4 + 3x^3 + 17x + 82)^2(4x^3 + 9x^2 + 17)$
89. $P(x) = x^2 - x + 3$ **91.** $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$
93. \$1.627 billion/year
95. $\dfrac{0.0021}{(0.015 + [S])^2}$;
the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.
97. No



99. (a) Not differentiable at 3 or -3

$$f'(x) = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

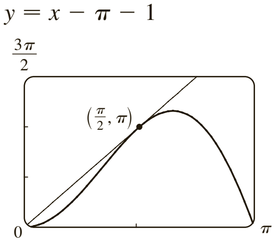
(b)



101. $a = -\frac{1}{2},\ b = 2$ **103.** $c = 3\sqrt{4} = 6$
107. 1000 **109.** 3; 1

EXERCISES 2.4 ■ PAGE 150

1. $f'(x) = x^2 \cos x + 2x \sin x$ **3.** $f'(x) = -3\csc^2 x + 2\sin x$
5. $y' = \sec\theta\,(\sec^2\theta + \tan^2\theta)$
7. $y' = -c\sin t + t(t\cos t + 2\sin t)$
9. $y' = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$ **11.** $f'(\theta) = \dfrac{1}{1 + \cos\theta}$
13. $y' = \dfrac{(t^2 + t)\cos t + \sin t}{(1 + t)^2}$
15. $f'(\theta) = \frac{1}{2}\sin 2\theta + \theta\cos 2\theta$
21. $y = x + 1$ **23.** $y = x - \pi - 1$
25. (a) $y = 2x$ (b)

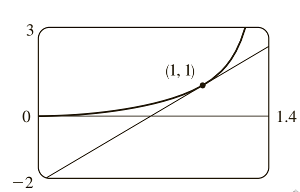


27. (a) $\sec x \tan x - 1$
29. $\theta\cos\theta + \sin\theta;\ 2\cos\theta - \theta\sin\theta$
31. (a) $f'(x) = (1 + \tan x)/\sec x$ (b) $f'(x) = \cos x + \sin x$
33. $(2n + 1)\pi \pm \frac{1}{3}\pi$, n an integer
35. (a) $v(t) = 8\cos t,\ a(t) = -8\sin t$
(b) $4\sqrt{3},\ -4,\ -4\sqrt{3}$; to the left

37. 5 ft/rad **39.** $\frac{5}{3}$ **41.** 3 **43.** $-\frac{3}{4}$
45. $\frac{1}{2}$ **47.** $-\frac{1}{4}$ **49.** $-\sqrt{2}$ **51.** $-\cos x$
53. $A = -\frac{3}{10},\ B = -\frac{1}{10}$
55. (a) $\sec^2 x = \dfrac{1}{\cos^2 x}$ (b) $\sec x \tan x = \dfrac{\sin x}{\cos^2 x}$
(c) $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$
57. 1

EXERCISES 2.5 ■ PAGE 158

1. $\dfrac{4}{3\sqrt[3]{(1 + 4x)^2}}$ **3.** $\pi\sec^2\pi x$ **5.** $\dfrac{\cos x}{2\sqrt{\sin x}}$
7. $F'(x) = 24x^{11}(5x^3 + 2)^3(5x^3 + 1)$
9. $f'(x) = \dfrac{5}{2\sqrt{5x + 1}}$ **11.** $A'(t) = \dfrac{2(\sin t - \sec^2 t)}{(\cos t + \tan t)^3}$
13. $f'(\theta) = -2\theta\sin(\theta^2)$ **15.** $h'(v) = \dfrac{5v^2 + 3}{3(\sqrt[3]{1 + v^2})^2}$
17. $f'(x) = (2x - 3)^3(x^2 + x + 1)^4(28x^2 - 12x - 7)$
19. $h'(t) = \frac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2(20t^2 + 18t - 1)$
21. $g'(u) = \dfrac{48u^2(u^3 - 1)^7}{(u^3 + 1)^9}$ **23.** $y' = \dfrac{1}{2\sqrt{x}\,(x + 1)^{3/2}}$
25. $h'(\theta) = \theta\sec^2(\theta^2\sin\theta)(\theta\cos\theta + 2\sin\theta)$
27. $y' = -\frac{1}{2}\sqrt{1 + \sin x}$
29. $H'(r) = \dfrac{2(r^2 - 1)^2(r^2 + 3r + 5)}{(2r + 1)^6}$
31. $y' = -4\sin(\sec 4x)\sec 4x\tan 4x$
33. $y' = \left(x\cos\sqrt{1 + x^2}\right)/\sqrt{1 + x^2}$
35. $y' = \dfrac{16\sin 2x(1 - \cos 2x)^3}{(1 + \cos 2x)^5}$
37. $y' = -2\cos\theta\cot(\sin\theta)\csc^2(\sin\theta)$
39. $f'(t) = -\sin t\sec^2(\sec(\cos t))\sec(\cos t)\tan(\cos t)$
41. $y' = \dfrac{1 + 1/(2\sqrt{x})}{2\sqrt{x + \sqrt{x}}}$
43. $g'(x) = p(2r\sin rx + n)^{p-1}(2r^2\cos rx)$
45. $y' = \dfrac{-\pi\cos(\tan\pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}}$
47. $y' = -3\cos 3\theta\sin(\sin 3\theta)$;
$y'' = -9\cos^2(3\theta)\cos(\sin 3\theta) + 9(\sin 3\theta)\sin(\sin 3\theta)$
49. $y' = \dfrac{-\sec t\tan t}{2\sqrt{1 - \sec t}}$;
$y'' = \dfrac{\sec t\,(3\sec^3 t - 4\sec^2 t - \sec t + 2)}{4(1 - \sec t)^{3/2}}$
51. $y = 18x + 1$ **53.** $y = -x + \pi$
55. (a) $y = \pi x - \pi + 1$ (b)



57. (a) $f'(x) = \dfrac{2 - 2x^2}{\sqrt{2 - x^2}}$

59. $((\pi/2) + 2n\pi, 3)$, $((3\pi/2) + 2n\pi, -1)$, n an integer

61. 24 **63.** (a) 30 (b) 36

65. (a) $\frac{3}{4}$ (b) Does not exist (c) -2 **67.** $-\frac{1}{6}\sqrt{2}$

69. 120 **71.** 96 **73.** $2^{103}\sin 2x$

75. $v(t) = \frac{5}{2}\pi \cos(10\pi t)$ cm/s

77. (a) $\dfrac{dB}{dt} = \dfrac{7\pi}{54}\cos\dfrac{2\pi t}{5.4}$ (b) 0.16

79. dv/dt is the rate of change of velocity with respect to time; dv/ds is the rate of change of velocity with respect to displacement

81. (b) The factored form **85.** (b) $-n\cos^{n-1}x \sin[(n+1)x]$

EXERCISES 2.6 ■ PAGE 166

1. (a) $y' = 9x/y$ (b) $y = \pm\sqrt{9x^2 - 1}$, $y' = \pm 9x/\sqrt{9x^2 - 1}$

3. (a) $y' = -\sqrt{y}/\sqrt{x}$ (b) $y = (1 - \sqrt{x})^2$, $y' = 1 - 1/\sqrt{x}$

5. $y' = \dfrac{2y - x}{y - 2x}$ **7.** $y' = -\dfrac{2x(2x^2 + y^2)}{y(2x^2 + 3y)}$

9. $y' = \dfrac{x(x + 2y)}{2x^2y + 4xy^2 + 2y^3 + x^2}$ **11.** $y' = \dfrac{2x + y\sin x}{\cos x - 2y}$

13. $y' = \dfrac{1 - 8x^3\sqrt{x + y}}{8y^3\sqrt{x + y} - 1}$ **15.** $y' = \dfrac{y\sec^2(x/y) - y^2}{y^2 + x\sec^2(x/y)}$

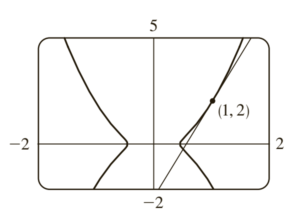
17. $y' = \dfrac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$

19. $y' = -\dfrac{y\cos(xy) + \sin(x + y)}{x\cos(xy) + \sin(x + y)}$ **21.** $-\frac{16}{13}$

23. $\dfrac{dx}{dy} = \dfrac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$ **25.** $y = \frac{1}{2}x$

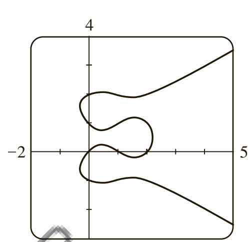
27. $y = \frac{3}{4}x - \frac{1}{2}$ **29.** $y = x + \frac{1}{2}$ **31.** $y = -\frac{9}{13}x + \frac{40}{13}$

33. (a) $y = \frac{9}{2}x - \frac{5}{2}$ (b)



35. $-1/(4y^3)$ **37.** $\dfrac{\cos^2 y\cos x + \sin^2 x\sin y}{\cos^3 y}$ **39.** 0

41. (a)

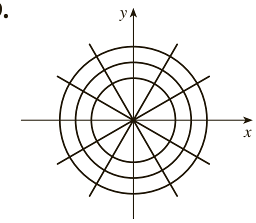


Eight; $x \approx 0.42, 1.58$

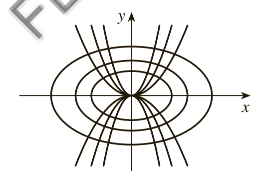
(b) $y = -x + 1$, $y = \frac{1}{3}x + 2$ (c) $1 \mp \frac{1}{3}\sqrt{3}$

43. $(\pm\frac{5}{4}\sqrt{3}, \pm\frac{5}{4})$ **45.** $(x_0x/a^2) - (y_0y/b^2) = 1$

49.



51.



55. (a) $\dfrac{V^3(nb - V)}{PV^3 - n^2aV + 2n^3ab}$ (b) ≈ -4.04 L/atm

57. $(\pm\sqrt{3}, 0)$ **59.** $(-1, -1)$, $(1, 1)$

61. (a) 0 (b) $-\frac{1}{2}$

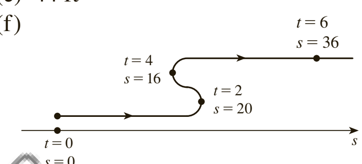
EXERCISES 2.7 ■ PAGE 178

1. (a) $3t^2 - 18t + 24$ (b) 9 ft/s (c) $t = 2$ s, 4 s

(d) $0 \leqslant t < 2$ or $t > 4$

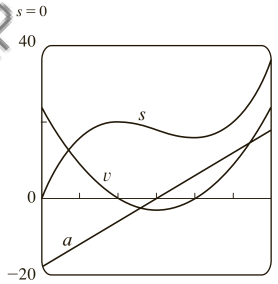
(e) 44 ft

(f)



(g) $6t - 18$; -12 ft/s^2

(h)



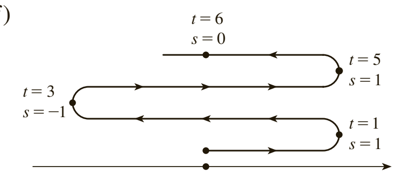
(i) Speeding up when $2 < t < 3$ and $4 < t \leqslant 6$; slowing down when $0 \leqslant t < 2$ and $3 < t < 4$

3. (a) $(\pi/2)\cos(\pi t/2)$ (b) 0 ft/s

(c) $t = 2n + 1$, t a nonnegative integer

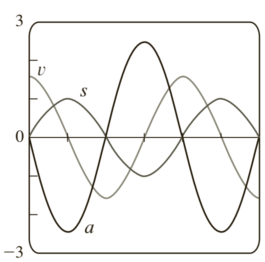
(d) $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on (e) 6 ft

(f)



(g) $(-\pi^2/4)\sin(\pi t/2)$; $-\pi^2/4$ ft/s^2

(h)



(i) Speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$; slowing down when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$

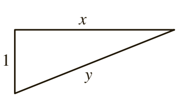
5. (a) Speeding up when $0 < t < 1$ or $2 < t < 3$; slowing down when $1 < t < 2$

(b) Speeding up when $1 < t < 2$ or $3 < t < 4$; slowing down when $0 < t < 1$ or $2 < t < 3$

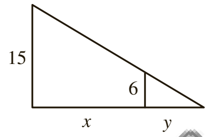
7. (a) 4.9 m/s; -14.7 m/s (b) After 2.5 s (c) $32\frac{5}{8}$ m
(d) ≈ 5.08 s (e) ≈ -25.3 m/s
9. (a) 7.56 m/s (b) ≈ 6.24 m/s; ≈ -6.24 m/s
11. (a) 30 mm^2/mm; the rate at which the area is increasing with respect to side length as x reaches 15 mm
(b) $\Delta A \approx 2x\,\Delta x$
13. (a) (i) 5π (ii) 4.5π (iii) 4.1π
(b) 4π (c) $\Delta A \approx 2\pi r\,\Delta r$
15. (a) 8π ft^2/ft (b) 16π ft^2/ft (c) 24π ft^2/ft
The rate increases as the radius increases.
17. (a) 6 kg/m (b) 12 kg/m (c) 18 kg/m
At the right end; at the left end
19. (a) 4.75 A (b) 5 A; $t = \frac{2}{3}$ s
23. (a) $dV/dP = -C/P^2$ (b) At the beginning
25. (a) 16 million/year; 78.5 million/year
(b) $P(t) = at^3 + bt^2 + ct + d$, where $a \approx -0.0002849$, $b \approx 0.5224331$, $c \approx -6.395641$, $d \approx 1720.586$
(c) $P'(t) = 3at^2 + 2bt + c$
(d) 14.16 million/year (smaller); 71.72 million/year (smaller)
(e) $P'(85) \approx 76.24$ million/year
27. (a) 0.926 cm/s; 0.694 cm/s; 0
(b) 0; -92.6 (cm/s)/cm; -185.2 (cm/s)/cm
(c) At the center; at the edge
29. (a) $C'(x) = 3 + 0.02x + 0.0006x^2$
(b) \$11/pair; the rate at which the cost is changing as the 100th pair of jeans is being produced; the cost of the 101st pair
(c) \$11.07
31. (a) $[xp'(x) - p(x)]/x^2$; the average productivity increases as new workers are added.
33. -0.2436 K/min
35. (a) 0 and 0 (b) $C = 0$
(c) (0, 0), (500, 50); it is possible for the species to coexist.

EXERCISES 2.8 ■ PAGE 185

1. $dV/dt = 3x^2\,dx/dt$ **3.** 48 cm^2/s **5.** $3/(25\pi)$ m/min
7. 128π cm^2/min **9.** (a) 1 (b) 25 **11.** -18
13. (a) The plane's altitude is 1 mi and its speed is 500 mi/h.
(b) The rate at which the distance from the plane to the station is increasing when the plane is 2 mi from the station
(c)



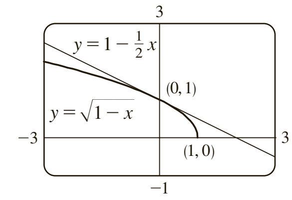
(d) $y^2 = x^2 + 1$
(e) $250\sqrt{3}$ mi/h
15. (a) The height of the pole (15 ft), the height of the man (6 ft), and the speed of the man (5 ft/s)
(b) The rate at which the tip of the man's shadow is moving when he is 40 ft from the pole
(c)



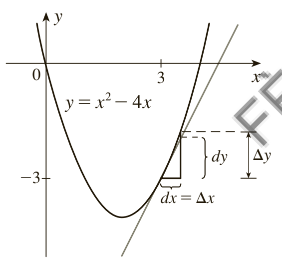
(d) $\dfrac{15}{6} = \dfrac{x + y}{y}$ (e) $\frac{25}{3}$ ft/s
17. 65 mi/h **19.** $837/\sqrt{8674} \approx 8.99$ ft/s
21. -1.6 cm/min **23.** $\frac{720}{13} \approx 55.4$ km/h
25. $(10{,}000 + 800{,}000\pi/9) \approx 2.89 \times 10^5$ cm^3/min
27. $\frac{10}{3}$ cm/min **29.** $6/(5\pi) \approx 0.38$ ft/min
31. $150\sqrt{3}$ cm^2/min **33.** 5 m **35.** πr^2 cm^2/h
37. 80 cm^3/min **39.** $\frac{107}{810} \approx 0.132\ \Omega$/s
41. $\sqrt{7}\,\pi/21 \approx 0.396$ m/min
43. (a) 360 ft/s (b) 0.096 rad/s
45. $\frac{10}{9}\pi$ km/min **47.** $1650/\sqrt{31} \approx 296$ km/h
49. $\frac{7}{4}\sqrt{15} \approx 6.78$ m/s

EXERCISES 2.9 ■ PAGE 192

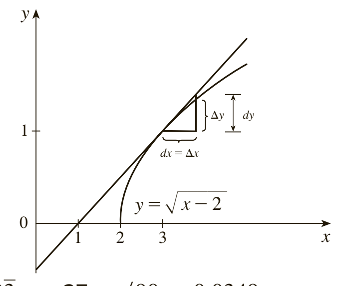
1. $L(x) = 16x + 23$ **3.** $L(x) = \frac{1}{4}x + 1$
5. $\sqrt{1 - x} \approx 1 - \frac{1}{2}x$;
$\sqrt{0.9} \approx 0.95$,
$\sqrt{0.99} \approx 0.995$



7. $-0.368 < x < 0.677$ **9.** $-0.045 < x < 0.055$
11. (a) $dy = -\dfrac{4x}{(x^2 - 3)^3}\,dx$ (b) $dy = -\dfrac{2t^3}{\sqrt{1 - t^4}}\,dt$
13. (a) $dy = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}\,dt$ (b) $dy = \dfrac{-4v}{(1 + v^2)^2}\,dv$
15. (a) $dy = \sec^2 x\,dx$ (b) -0.2
17. (a) $dy = \dfrac{x}{\sqrt{3 + x^2}}\,dx$ (b) -0.05
19. $\Delta y = 1.25$, $dy = 1$



21. $\Delta y \approx 0.34$, $dy = 0.4$



23. 15.968 **25.** $10.00\overline{3}$ **27.** $\pi/90 \approx 0.0349$
31. (a) 270 cm^3, 0.01, 1% (b) 36 cm^2, 0.006, 0.6%
33. (a) $84/\pi \approx 27$ cm^2; $\frac{1}{84} \approx 0.012 = 1.2\%$
(b) $1764/\pi^2 \approx 179$ cm^3; $\frac{1}{56} \approx 0.018 = 1.8\%$
35. (a) $2\pi rh\,\Delta r$ (b) $\pi(\Delta r)^2 h$
41. (a) 4.8, 5.2 (b) Too large

CHAPTER 2 REVIEW ■ PAGE 196

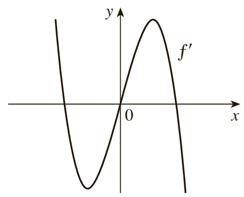
True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. False **13.** True **15.** True

Exercises

1. (a) (i) 3 m/s (ii) 2.75 m/s (iii) 2.625 m/s (iv) 2.525 m/s (b) 2.5 m/s

3.



5. $a = f,\ c = f',\ b = f''$

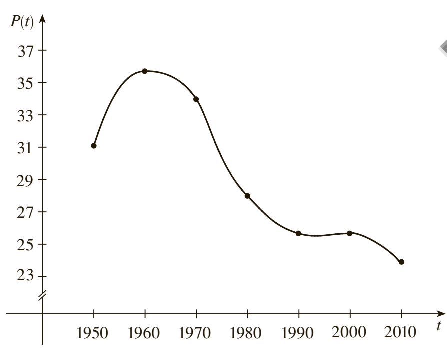
7. (a) The rate at which the cost changes with respect to the interest rate; dollars/(percent per year)
(b) As the interest rate increases past 10%, the cost is increasing at a rate of \$1200/(percent per year).
(c) Always positive

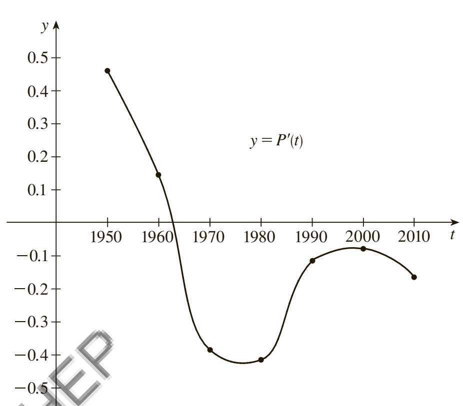
9. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/year).

(b)

t	$P'(t)$
1950	0.460
1960	0.145
1970	−0.385
1980	−0.415
1990	−0.115
2000	−0.085
2010	−0.170

(c)





(d) By obtaining data for the mid-decade years

11. $f'(x) = 3x^2 + 5$ **13.** $4x^7(x+1)^3(3x+2)$

15. $\frac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}$ **17.** $x(\pi x \cos \pi x + 2 \sin \pi x)$

19. $\dfrac{8t^3}{(t^4+1)^2}$ **21.** $-\dfrac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

23. $\dfrac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$ **25.** $\dfrac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1 + \tan 2\theta)^2}$

27. $-(x-1)^{-2}$ **29.** $\dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

31. $-6x\csc^2(3x^2+5)$ **33.** $\dfrac{\cos\sqrt{x} - \sqrt{x}\sin\sqrt{x}}{2\sqrt{x}}$

35. $2\cos\theta\,\tan(\sin\theta)\sec^2(\sin\theta)$

37. $\frac{1}{5}(x\tan x)^{-4/5}(\tan x + x\sec^2 x)$

39. $\cos\left(\tan\sqrt{1+x^3}\right)\left(\sec^2\sqrt{1+x^3}\right)\dfrac{3x^2}{2\sqrt{1+x^3}}$

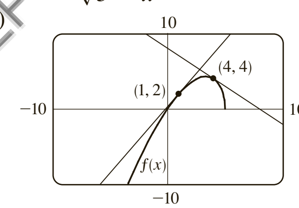
41. $-\frac{4}{27}$ **43.** $-5x^4/y^{11}$ **45.** 1

47. $y = 2\sqrt{3}x + 1 - \pi\sqrt{3}/3$

49. $y = 2x + 1,\ y = -\frac{1}{2}x + 1$

51. (a) $\dfrac{10 - 3x}{2\sqrt{5-x}}$ (b) $y = \frac{7}{4}x + \frac{1}{4},\ y = -x + 8$

(c)



53. $(\pi/4, \sqrt{2}), (5\pi/4, -\sqrt{2})$ **55.** $y = -\frac{2}{3}x^2 + \frac{14}{3}x$

59. (a) 4 (b) 6 (c) $\frac{7}{9}$ (d) 12

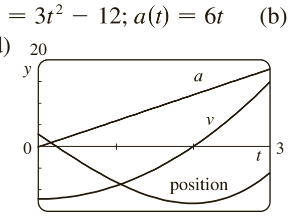
61. $2xg(x) + x^2g'(x)$ **63.** $2g(x)g'(x)$

65. $g'(g(x))g'(x)$ **67.** $g'(\sin x)\cdot\cos x$

69. $\dfrac{f'(x)[g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x) + g(x)]^2}$

71. $f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4)$

73. (a) $v(t) = 3t^2 - 12;\ a(t) = 6t$ (b) $t > 2;\ 0 \le t < 2$
(c) 23 (d)



(e) $t > 2;\ 0 < t < 2$

75. 4 kg/m **77.** $\frac{4}{3}$ cm²/min **79.** 13 ft/s **81.** 400 ft/h

83. (a) $L(x) = 1 + x;\ \sqrt[3]{1+3x} \approx 1 + x;\ \sqrt[3]{1.03} \approx 1.01$
(b) $-0.23 < x < 0.40$

85. $12 + \frac{3}{2}\pi \approx 16.7\text{ cm}^2$ **87.** $\left[\dfrac{d}{dx}\sqrt[4]{x}\right]_{x=16} = \frac{1}{32}$

89. $\frac{1}{4}$ **91.** $\frac{1}{8}x^2$

PROBLEMS PLUS ■ PAGE 200

1. $\left(\pm\sqrt{3}/2, \frac{1}{4}\right)$ **5.** $3\sqrt{2}$ **9.** $\left(0, \frac{5}{4}\right)$

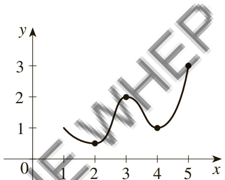
11. 3 lines; (0, 2), $(\frac{4}{3}\sqrt{2}, \frac{2}{3})$ and $(\frac{2}{3}\sqrt{2}, \frac{10}{3})$, $(-\frac{4}{3}\sqrt{2}, \frac{2}{3})$ and $(-\frac{2}{3}\sqrt{2}, \frac{10}{3})$
13. (a) $4\pi\sqrt{3}/\sqrt{11}$ rad/s (b) $40(\cos\theta + \sqrt{8 + \cos^2\theta})$ cm
(c) $-480\pi \sin\theta\,(1 + \cos\theta/\sqrt{8 + \cos^2\theta})$ cm/s
15. $x_T \in (3, \infty)$, $y_T \in (2, \infty)$, $x_N \in (0, \frac{5}{3})$, $y_N \in (-\frac{5}{2}, 0)$
17. (b) (i) 53° (or 127°) (ii) 63° (or 117°)
19. R approaches the midpoint of the radius AO.
21. $-\sin a$ **23.** $(1, -2)$, $(-1, 0)$
25. $\sqrt{29}/58$ **27.** $2 + \frac{375}{128}\pi \approx 11.204$ cm³/min

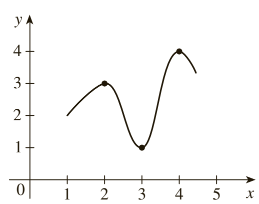
CHAPTER 3

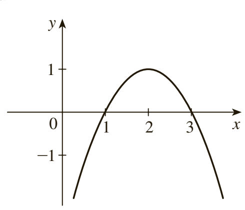
EXERCISES 3.1 ■ PAGE 211

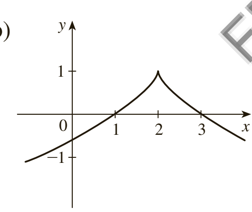
Abbreviations: abs, absolute; loc, local; max, maximum; min, minimum

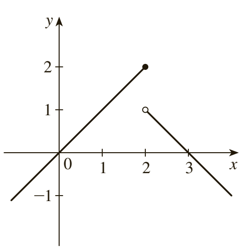
1. Abs min: smallest function value on the entire domain of the function; loc min at c: smallest function value when x is near c
3. Abs max at s, abs min at r, loc max at c, loc min at b and r, neither a max nor a min at a and d
5. Abs max $f(4) = 5$, loc max $f(4) = 5$ and $f(6) = 4$, loc min $f(2) = 2$ and $f(1) = f(5) = 3$

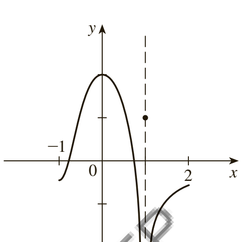
7.


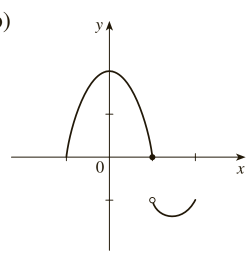
9.


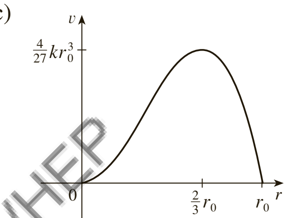
11. (a)


(b)


(c)


13. (a)


(b)


15. Abs max $f(3) = 4$ **17.** Abs max $f(1) = 1$
19. Abs min $f(0) = 0$
21. Abs max $f(\pi/2) = 1$; abs min $f(-\pi/2) = -1$
23. Abs min $f(-1) = 1$; loc min $f(-1) = 1$
25. Abs max $f(0) = 1$
27. Abs min $f(1) = -1$; loc min $f(0) = 0$
29. $\frac{1}{3}$ **31.** $-2, 3$ **33.** 0 **35.** 0, 2 **37.** $0, \frac{4}{9}$
39. $0, \frac{8}{7}, 4$ **41.** $n\pi$ (n an integer) **43.** 10
45. $f(2) = 16$, $f(5) = 7$ **47.** $f(-1) = 8$, $f(2) = -19$
49. $f(-2) = 33$, $f(2) = -31$ **51.** $f(0.2) = 5.2$, $f(1) = 2$
53. $f(4) = 4 - \sqrt[3]{4}$, $f(\sqrt{3}/9) = -2\sqrt{3}/9$
55. $f(\pi/6) = \frac{3}{2}\sqrt{3}$, $f(\pi/2) = 0$
57. $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$
59. (a) 2.19, 1.81 (b) $\frac{6}{25}\sqrt{\frac{3}{5}} + 2$, $-\frac{6}{25}\sqrt{\frac{3}{5}} + 2$
61. (a) 0.32, 0.00 (b) $\frac{3}{16}\sqrt{3}$, 0 **63.** ≈ 3.9665°C
65. About 4.1 months after January 1
67. (a) $r = \frac{2}{3}r_0$ (b) $v = \frac{4}{27}kr_0^3$
(c)


EXERCISES 3.2 ■ PAGE 219

1. 1, 5
3. (a) g is continuous on $[0, 8]$ and differentiable on $(0, 8)$.
(b) 2.2, 6.4 (c) 3.7, 5.5
5. 1 **7.** π
9. f is not differentiable on $(-1, 1)$ **11.** 1
13. $\sqrt{3}/9$ **15.** 1; yes **17.** f is not continous at 3
25. 16 **27.** No **33.** No

EXERCISES 3.3 ■ PAGE 227

Abbreviations: inc, increasing; dec, decreasing; CD, concave downward; CU, concave upward; HA, horizontal asymptote; VA, vertical asymptote; IP, inflection point(s)

1. (a) (1, 3), (4, 6) (b) (0, 1), (3, 4) (c) (0, 2)
(d) (2, 4), (4, 6) (e) (2, 3)
3. (a) I/D Test (b) Concavity Test
(c) Find points at which the concavity changes.
5. (a) Inc on (1, 5); dec on (0, 1) and (5, 6)
(b) Loc max at $x = 5$, loc min at $x = 1$
7. (a) 3, 5 (b) 2, 4, 6 (c) 1, 7
9. (a) Inc on $(-\infty, -1)$, $(3, \infty)$; dec on $(-1, 3)$
(b) Loc max $f(-1) = 9$; loc min $f(3) = -23$
(c) CU on $(1, \infty)$, CD on $(-\infty, 1)$; IP $(1, -7)$
11. (a) Inc on $(-1, 0)$, $(1, \infty)$; dec on $(-\infty, -1)$, $(0, 1)$
(b) Loc max $f(0) = 3$; loc min $f(\pm 1) = 2$
(c) CU on $(-\infty, -\sqrt{3}/3)$, $(\sqrt{3}/3, \infty)$;
CD on $(-\sqrt{3}/3, \sqrt{3}/3)$; IP $(\pm\sqrt{3}/3, \frac{22}{9})$

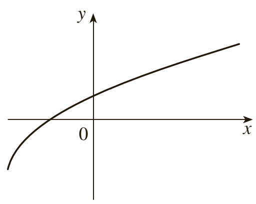
13. (a) Inc on $(0, \pi/4)$, $(5\pi/4, 2\pi)$; dec on $(\pi/4, 5\pi/4)$
(b) Loc max $f(\pi/4) = \sqrt{2}$; loc min $f(5\pi/4) = -\sqrt{2}$
(c) CU on $(3\pi/4, 7\pi/4)$; CD on $(0, 3\pi/4)$, $(7\pi/4, 2\pi)$;
IP $(3\pi/4, 0)$, $(7\pi/4, 0)$

15. Loc max $f(1) = 2$; loc min $f(0) = 1$

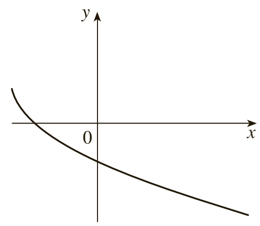
17. Loc min $f\left(\frac{1}{16}\right) = -\frac{1}{4}$

19. (a) f has a local maximum at 2.
(b) f has a horizontal tangent at 6.

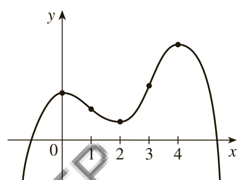
21. (a)



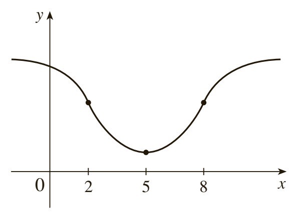
(b)



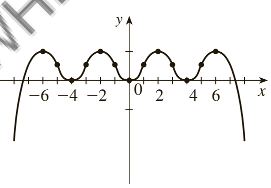
23.



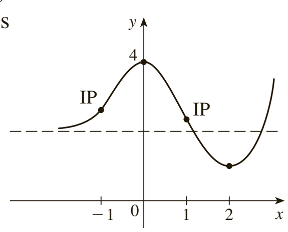
25.



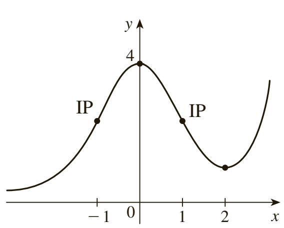
27.



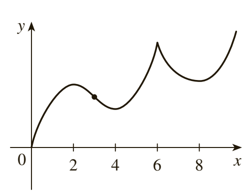
29. (a) No
(b) Yes



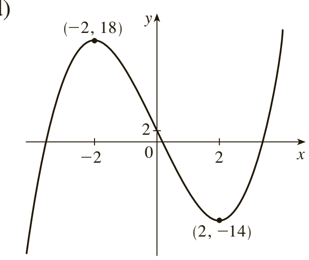
(c)



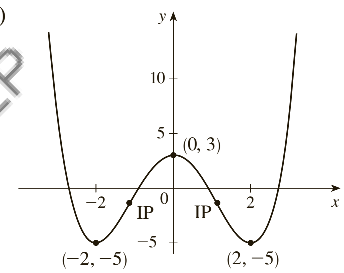
31. (a) Inc on $(0, 2)$, $(4, 6)$, $(8, \infty)$;
dec on $(2, 4)$, $(6, 8)$
(b) Loc max at $x = 2, 6$;
loc min at $x = 4, 8$
(c) CU on $(3, 6)$, $(6, \infty)$;
CD on $(0, 3)$ (d) 3
(e) See graph at right.



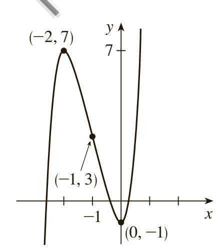
33. (a) Inc on $(-\infty, -2)$, $(2, \infty)$; dec on $(-2, 2)$
(b) Loc max $f(-2) = 18$; loc min $f(2) = -14$
(c) CU on $(0, \infty)$, CD on $(-\infty, 0)$; IP $(0, 2)$
(d)



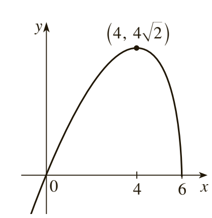
35. (a) Inc on $(-2, 0)$, $(2, \infty)$; dec on $(-\infty, -2)$, $(0, 2)$
(b) Loc max $f(0) = 3$; loc min $f(\pm 2) = -5$
(c) CU on $\left(-\infty, -\dfrac{2}{\sqrt{3}}\right)$, $\left(\dfrac{2}{\sqrt{3}}, \infty\right)$; CD on $\left(-\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}\right)$;
IPs $\left(\pm\dfrac{2}{\sqrt{3}}, -\dfrac{13}{9}\right)$
(d)



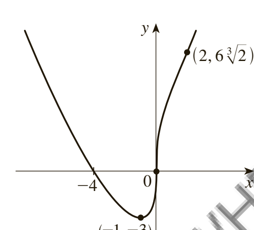
37. (a) Inc on $(-\infty, -2)$, $(0, \infty)$;
dec on $(-2, 0)$
(b) Loc max $h(-2) = 7$;
loc min $h(0) = -1$
(c) CU on $(-1, \infty)$;
CD on $(-\infty, -1)$; IP $(-1, 3)$
(d) See graph at right.



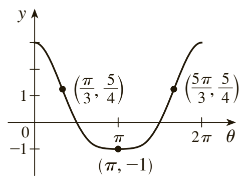
39. (a) Inc on $(-\infty, 4)$;
dec on $(4, 6)$
(b) Loc max $F(4) = 4\sqrt{2}$
(c) CD on $(-\infty, 6)$; No IP
(d) See graph at right.



41. (a) Inc on $(-1, \infty)$;
dec on $(-\infty, -1)$
(b) Loc min $C(-1) = -3$
(c) CU on $(-\infty, 0)$, $(2, \infty)$;
CD on $(0, 2)$;
IPs $(0, 0)$, $(2, 6\sqrt[3]{2})$
(d) See graph at right.



43. (a) Inc on $(\pi, 2\pi)$; dec on $(0, \pi)$
(b) Loc min $f(\pi) = -1$
(c) CU on $(\pi/3, 5\pi/3)$; CD on $(0, \pi/3)$, $(5\pi/3, 2\pi)$; IPs $\left(\pi/3, \frac{5}{4}\right)$, $\left(5\pi/3, \frac{5}{4}\right)$
(d) See graph at right.



45. $(3, \infty)$

47. (a) Loc and abs max $f(1) = \sqrt{2}$, no min (b) $\frac{1}{4}(3 - \sqrt{17})$

49. (b) CD on $(0, 0.85)$, $(1.57, 2.29)$; CU on $(0.85, 1.57)$, $(2.29, \pi)$; IP $(0.85, 0.74)$, $(1.57, 0)$, $(2.29, -0.74)$

51. CU on $(-\infty, -0.6)$, $(0.0, \infty)$; CD on $(-0.6, 0.0)$

53. (a) The rate of increase is initially very small, increases to a maximum at $t \approx 8$ h, then decreases toward 0.
(b) When $t = 8$ (c) CU on $(0, 8)$; CD on $(8, 18)$
(d) $(8, 350)$

55. If $D(t)$ is the size of the deficit as a function of time, then at the time of the speech $D'(t) > 0$ and $D''(t) < 0$.

57. $K(3) - K(2)$; CD

59. $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$

61. (a) $a = 0, b = -1$ (b) $y = -x$ at $(0, 0)$

EXERCISES 3.4 ■ PAGE 241

1. (a) As x becomes large, $f(x)$ approaches 5.
(b) As x becomes large negative, $f(x)$ approaches 3.

3. (a) -2 (b) 2 (c) ∞ (d) $-\infty$
(e) $x = 1, x = 3, y = -2, y = 2$

5. 0 **7.** $\frac{2}{5}$ **9.** $\frac{3}{2}$ **11.** 0 **13.** -1 **15.** 4

17. -2 **19.** $\frac{\sqrt{3}}{4}$ **21.** $\frac{1}{6}$ **23.** $\frac{1}{2}(a - b)$ **25.** ∞

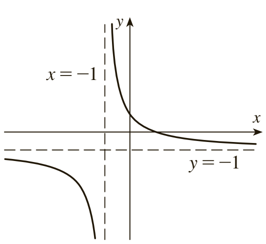
27. $-\infty$ **29.** ∞ **31.** 1 **33.** (a), (b) $-\frac{1}{2}$

35. $y = 4, x = -3$ **37.** $y = 2; x = -2, x = 1$

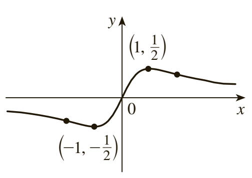
39. $x = 5$ **41.** $y = 3$ **43.** (a) 0 (b) $\pm\infty$

45. $f(x) = \dfrac{2 - x}{x^2(x - 3)}$ **47.** (a) $\frac{5}{4}$ (b) 5

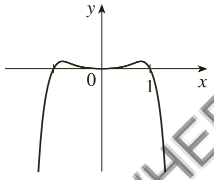
49. $y = -1$



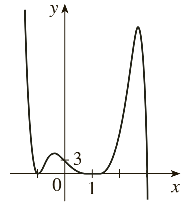
51. $y = 0$



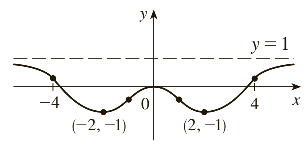
53. $-\infty, -\infty$



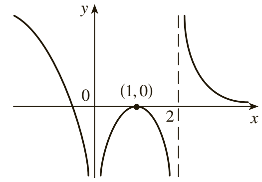
55. $-\infty, \infty$



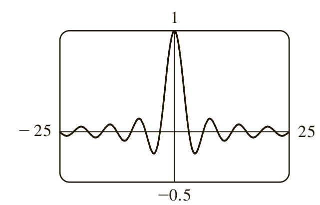
57.



59.



61. (a) 0 (b) An infinite number of times



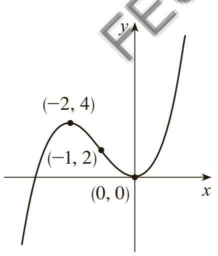
63. 4 **65.** $N \geqslant 15$

67. $N \leqslant -9, N \leqslant -19$ **69.** (a) $x > 100$

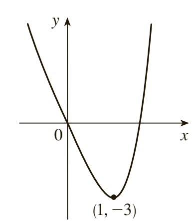
EXERCISES 3.5 ■ PAGE 250

Abbreviation: int, intercept; SA, slant asymptote

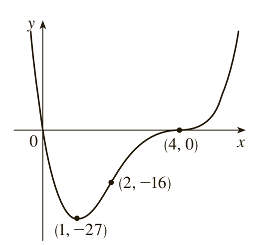
1. A. $\mathbb{R}$ B. y-int 0; x-int $-3, 0$
C. None D. None
E. Inc on $(-\infty, -2)$, $(0, \infty)$; dec on $(-2, 0)$
F. Loc max $f(-2) = 4$; loc min $f(0) = 0$
G. CU on $(-1, \infty)$; CD on $(-\infty, -1)$; IP $(-1, 2)$
H. See graph at right.



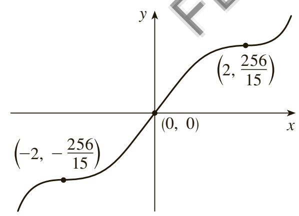
3. A. $\mathbb{R}$ B. y-int 0; x-int 0, $\sqrt[3]{4}$
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = -3$
G. CU on $(-\infty, \infty)$
H. See graph at right.



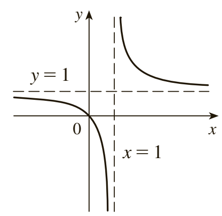
5. A. $\mathbb{R}$ B. y-int 0; x-int 0, 4
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = -27$
G. CU on $(-\infty, 2)$, $(4, \infty)$; CD on $(2, 4)$;
IPs $(2, -16)$, $(4, 0)$
H. See graph at right.



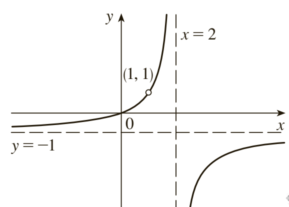
7. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About $(0, 0)$ D. None
E. Inc on $(-\infty, \infty)$
F. None
G. CU on $(-2, 0)$, $(2, \infty)$;
CD on $(-\infty, -2)$, $(0, 2)$;
IPs $\left(-2, -\frac{256}{15}\right)$, $(0, 0)$, $\left(2, \frac{256}{15}\right)$
H. See graph at right.



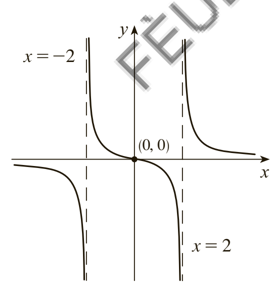
9. A. $\{x \mid x \neq 1\}$
B. y-int 0; x-int 0
C. None D. VA $x = 1$, HA $y = 1$
E. Dec on $(-\infty, 1)$, $(1, \infty)$
F. None
G. CU on $(1, \infty)$; CD on $(-\infty, 1)$
H. See graph at right.



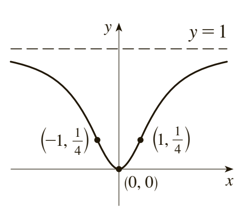
11. A. $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$
B. y-int 0; x-int 0 C. None
D. HA $y = -1$; VA $x = 2$
E. Inc on $(-\infty, 1)$, $(1, 2)$, $(2, \infty)$
F. None
G. CU on $(-\infty, 1)$, $(1, 2)$;
CD on $(2, \infty)$
H. See graph at right.



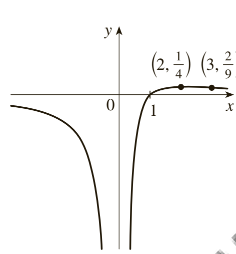
13. A. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
B. y-int 0; x-int 0
C. About $(0, 0)$
D. VA $x = \pm 2$; HA $y = 0$
E. Dec on $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$
F. No local extrema
G. CU on $(-2, 0)$, $(2, \infty)$;
CD on $(-\infty, -2)$, $(0, 2)$; IP $(0, 0)$
H. See graph at right.



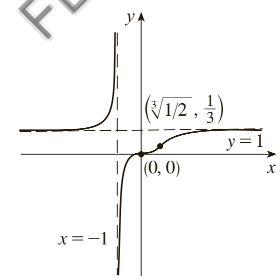
15. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About y-axis D. HA $y = 1$
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$; IPs $\left(\pm 1, \frac{1}{4}\right)$
H. See graph at right.



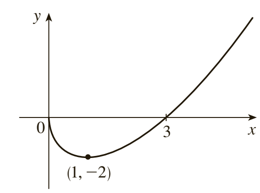
17. A. $(-\infty, 0) \cup (0, \infty)$ B. x-int 1
C. None D. HA $y = 0$; VA $x = 0$
E. Inc on $(0, 2)$;
dec on $(-\infty, 0)$, $(2, \infty)$
F. Loc max $f(2) = \frac{1}{4}$
G. CU on $(3, \infty)$;
CD on $(-\infty, 0)$, $(0, 3)$; IP $\left(3, \frac{2}{9}\right)$
H. See graph at right.



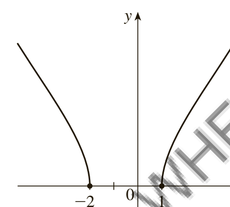
19. A. $(-\infty, -1) \cup (-1, \infty)$
B. y-int 0; x-int 0 C. None
D. VA $x = -1$; HA $y = 1$
E. Inc on $(-\infty, -1)$, $(-1, \infty)$;
F. None
G. CU on $(-\infty, -1)$, $\left(0, \sqrt[3]{\frac{1}{2}}\right)$;
CD on $(-1, 0)$, $\left(\sqrt[3]{\frac{1}{2}}, \infty\right)$;
IPs $(0, 0)$, $\left(\sqrt[3]{\frac{1}{2}}, \frac{1}{3}\right)$
H. See graph at right.



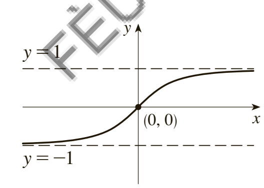
21. A. $[0, \infty)$ B. y-int 0; x-int 0, 3
C. None D. None
E. Inc on $(1, \infty)$; dec on $(0, 1)$
F. Loc min $f(1) = -2$
G. CU on $(0, \infty)$
H. See graph at right.



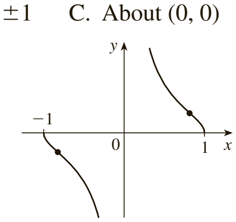
23. A. $(-\infty, -2] \cup [1, \infty)$
B. x-int -2, 1 C. None
D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, -2)$
F. None
G. CD on $(-\infty, -2)$, $(1, \infty)$
H. See graph at right.



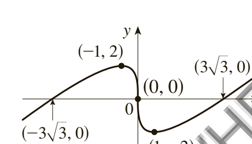
25. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About $(0, 0)$
D. HA $y = \pm 1$
E. Inc on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, 0)$
H. See graph at right.



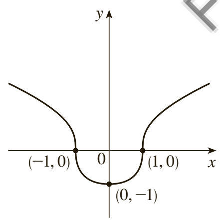
27. A. $[-1, 0) \cup (0, 1]$ B. x-int ± 1 C. About $(0, 0)$
D. VA $x = 0$
E. Dec on $(-1, 0)$, $(0, 1)$
F. None
G. CU on $\left(-1, -\sqrt{2/3}\right)$, $\left(0, \sqrt{2/3}\right)$;
CD on $\left(-\sqrt{2/3}, 0\right)$, $\left(\sqrt{2/3}, 1\right)$;
IPs $\left(\pm\sqrt{2/3}, \pm 1/\sqrt{2}\right)$
H. See graph at right.



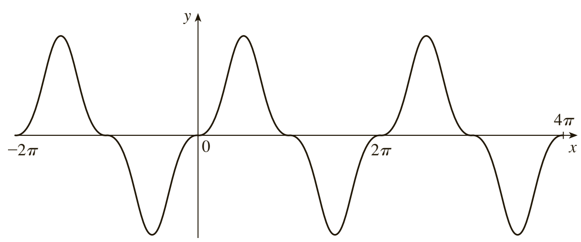
29. A. $\mathbb{R}$ B. y-int 0; x-int $\pm 3\sqrt{3}$, 0 C. About $(0, 0)$
D. None E. Inc on $(-\infty, -1)$, $(1, \infty)$; dec on $(-1, 1)$
F. Loc max $f(-1) = 2$;
loc min $f(1) = -2$
G. CU on $(0, \infty)$;
CD on $(-\infty, 0)$; IP $(0, 0)$
H. See graph at right.



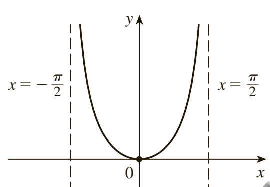
31. A. $\mathbb{R}$ B. y-int -1; x-int ± 1
C. About the y-axis D. None
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = -1$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$; IPs $(\pm 1, 0)$
H. See graph at right.



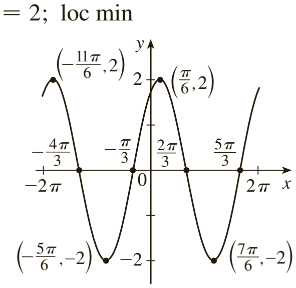
33. A. $\mathbb{R}$ B. y-int 0; x-int $n\pi$ (n an integer)
C. About $(0, 0)$, period 2π D. None
E–G answers for $0 \leqslant x \leqslant \pi$:
E. Inc on $(0, \pi/2)$; dec on $(\pi/2, \pi)$ F. Loc max $f(\pi/2) = 1$
G. Let $\alpha = \sin^{-1}\sqrt{2/3}$; CU on $(0, \alpha)$, $(\pi - \alpha, \pi)$;
CD on $(\alpha, \pi - \alpha)$; IPs at $x = 0, \pi, \alpha, \pi - \alpha$
H.



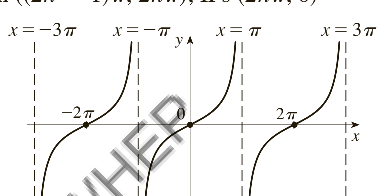
35. A. $(-\pi/2, \pi/2)$ B. y-int 0; x-int 0 C. About y-axis
D. VA $x = \pm\pi/2$
E. Inc on $(0, \pi/2)$;
dec on $(-\pi/2, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-\pi/2, \pi/2)$
H. See graph at right.



37. A. $[-2\pi, 2\pi]$ B. y-int $\sqrt{3}$; x-int $-4\pi/3$, $-\pi/3$, $2\pi/3$, $5\pi/3$ C. Period 2π D. None
E. Inc on $(-2\pi, -11\pi/6)$, $(-5\pi/6, \pi/6)$, $(7\pi/6, 2\pi)$;
dec on $(-11\pi/6, -5\pi/6)$, $(\pi/6, 7\pi/6)$
F. Loc max $f(-11\pi/6) = f(\pi/6) = 2$; loc min
$f(-5\pi/6) = f(7\pi/6) = -2$
G. CU on $(-4\pi/3, -\pi/3)$,
$(2\pi/3, 5\pi/3)$;
CD on $(-2\pi, -4\pi/3)$,
$(-\pi/3, 2\pi/3)$, $(5\pi/3, 2\pi)$;
IPs $(-4\pi/3, 0)$, $(-\pi/3, 0)$,
$(2\pi/3, 0)$, $(5\pi/3, 0)$
H. See graph at right.



39. A. All reals except $(2n + 1)\pi$ (n an integer)
B. y-int 0; x-int $2n\pi$ C. About the origin, period 2π
D. VA $x = (2n + 1)\pi$ E. Inc on $((2n - 1)\pi, (2n + 1)\pi)$
F. None G. CU on $(2n\pi, (2n + 1)\pi)$;
CD on $((2n - 1)\pi, 2n\pi)$; IPs $(2n\pi, 0)$
H.



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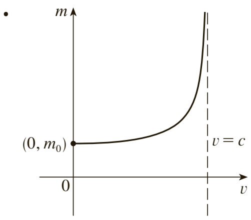

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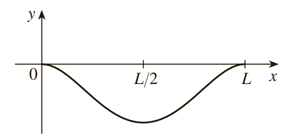
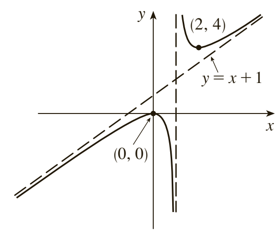
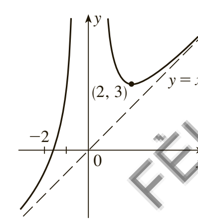


45. $y = x - 1$ **47.** $y = 2x - 3$

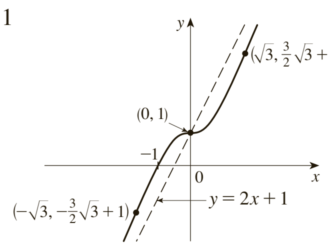
49. A. $(-\infty, 1) \cup (1, \infty)$
B. y-int 0; x-int 0 C. None
D. VA $x = 1$; SA $y = x + 1$
E. Inc on $(-\infty, 0)$, $(2, \infty)$;
dec on $(0, 1)$, $(1, 2)$
F. Loc max $f(0) = 0$;
loc min $f(2) = 4$
G. CU on $(1, \infty)$; CD on $(-\infty, 1)$
H. See graph at right.



51. A. $(-\infty, 0) \cup (0, \infty)$
B. x-int $-\sqrt[3]{4}$ C. None
D. VA $x = 0$; SA $y = x$
E. Inc on $(-\infty, 0)$, $(2, \infty)$;
dec on $(0, 2)$
F. Loc min $f(2) = 3$
G. CU on $(-\infty, 0)$, $(0, \infty)$
H. See graph at right.



53. A. $\mathbb{R}$ B. y-int 1; x-int -1
C. None D. SA $y = 2x + 1$
E. Inc on $(-\infty, \infty)$ F. None
G. CU on $\left(-\infty, -\sqrt{3}\right)$,
$\left(0, \sqrt{3}\right)$;
CD on $\left(-\sqrt{3}, 0\right)$, $\left(\sqrt{3}, \infty\right)$;
IP $\left(\pm\sqrt{3}, 1 \pm \frac{3}{2}\sqrt{3}\right)$, $(0, 1)$
H. See graph at right.



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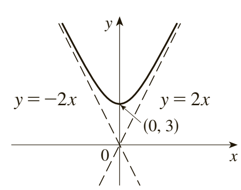
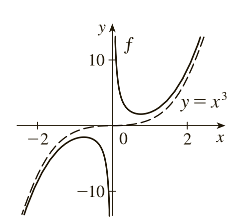
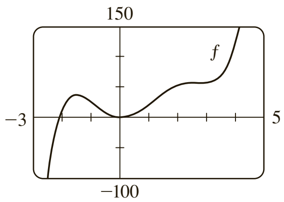


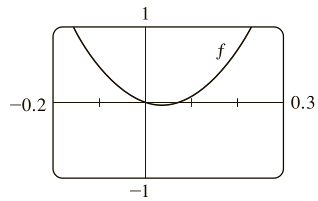
59. VA $x = 0$, asymptotic to $y = x^3$

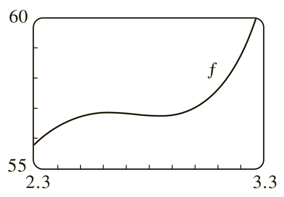


EXERCISES 3.6 ■ PAGE 257

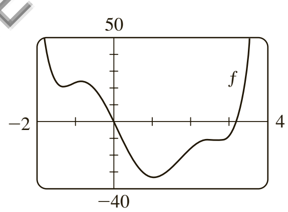
1. Inc on $(-\infty, -1.50)$, $(0.04, 2.62)$, $(2.84, \infty)$; dec on $(-1.50, 0.04)$, $(2.62, 2.84)$; loc max $f(-1.50) \approx 36.47$, $f(2.62) \approx 56.83$; loc min $f(0.04) \approx -0.04$, $f(2.84) \approx 56.73$; CU on $(-0.89, 1.15)$, $(2.74, \infty)$; CD on $(-\infty, -0.89)$, $(1.15, 2.74)$; IPs $(-0.89, 20.90)$, $(1.15, 26.57)$, $(2.74, 56.78)$



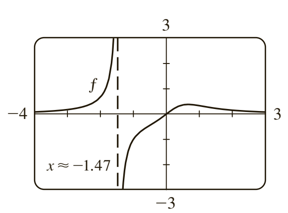




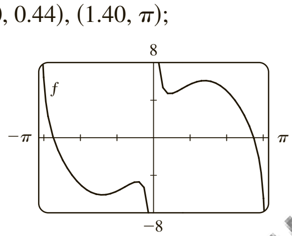
3. Inc on $(-1.31, -0.84)$, $(1.06, 2.50)$, $(2.75, \infty)$; dec on $(-\infty, -1.31)$, $(-0.84, 1.06)$, $(2.50, 2.75)$; loc max $f(-0.84) \approx 23.71$, $f(2.50) \approx -11.02$; loc min $f(-1.31) \approx 20.72$, $f(1.06) \approx -33.12$, $f(2.75) \approx -11.33$; CU on $(-\infty, -1.10)$, $(0.08, 1.72)$, $(2.64, \infty)$; CD on $(-1.10, 0.08)$, $(1.72, 2.64)$; IPs $(-1.10, 22.09)$, $(0.08, -3.88)$, $(1.72, -22.53)$, $(2.64, -11.18)$



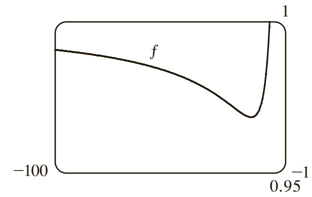
5. Inc on $(-\infty, -1.47)$, $(-1.47, 0.66)$; dec on $(0.66, \infty)$; loc max $f(0.66) \approx 0.38$; CU on $(-\infty, -1.47)$, $(-0.49, 0)$, $(1.10, \infty)$; CD on $(-1.47, -0.49)$, $(0, 1.10)$; IPs $(-0.49, -0.44)$, $(1.10, 0.31)$, $(0, 0)$

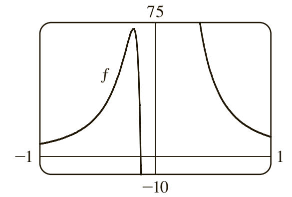


7. Inc on $(-1.40, -0.44)$, $(0.44, 1.40)$; dec on $(-\pi, -1.40)$, $(-0.44, 0)$, $(0, 0.44)$, $(1.40, \pi)$; loc max $f(-0.44) \approx -4.68$, $f(1.40) \approx 6.09$; loc min $f(-1.40) \approx -6.09$, $f(0.44) \approx 4.68$; CU on $(-\pi, -0.77)$, $(0, 0.77)$; CD on $(-0.77, 0)$, $(0.77, \pi)$; IPs $(-0.77, -5.22)$, $(0.77, 5.22)$

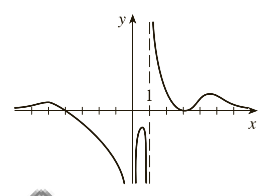


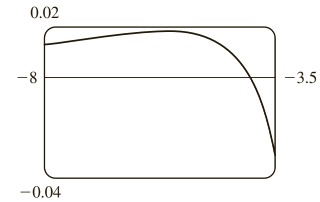
9. Inc on $\left(-8 - \sqrt{61}, -8 + \sqrt{61}\right)$; dec on $\left(-\infty, -8 - \sqrt{61}\right)$, $\left(-8 + \sqrt{61}, 0\right)$, $(0, \infty)$; CU on $\left(-12 - \sqrt{138}, -12 + \sqrt{138}\right)$, $(0, \infty)$; CD on $\left(-\infty, -12 - \sqrt{138}\right)$, $\left(-12 + \sqrt{138}, 0\right)$

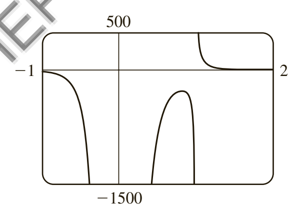


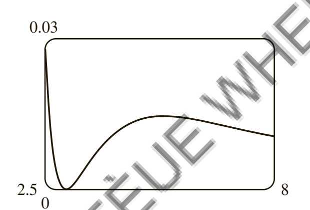


11. Loc max $f(-5.6) \approx 0.018$, $f(0.82) \approx -281.5$, $f(5.2) \approx 0.0145$; loc min $f(3) = 0$







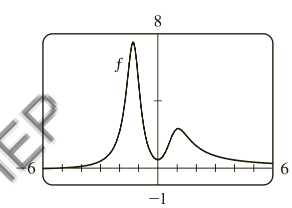


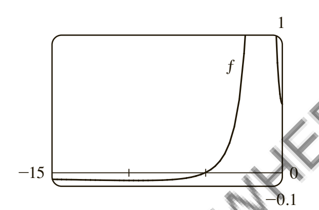
13. $f'(x) = -\dfrac{x(x+1)^2(x^3 + 18x^2 - 44x - 16)}{(x-2)^3(x-4)^5}$

$$f''(x) = 2\frac{(x+1)(x^6 + 36x^5 + 6x^4 - 628x^3 + 684x^2 + 672x + 64)}{(x-2)^4(x-4)^6}$$

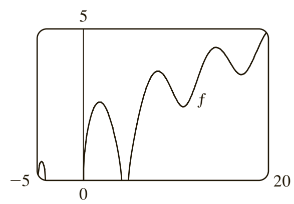
CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$, $(4, \infty)$; CD on $(-\infty, -35.3)$, $(-5.0, -1)$, $(-0.5, -0.1)$; IPs $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, $(-0.1, 0.0000066)$

15. Inc on $(-9.41, -1.29)$, $(0, 1.05)$; dec on $(-\infty, -9.41)$, $(-1.29, 0)$, $(1.05, \infty)$; loc max $f(-1.29) \approx 7.49$, $f(1.05) \approx 2.35$; loc min $f(-9.41) \approx -0.056$, $f(0) = 0.5$; CU on $(-13.81, -1.55)$, $(-1.03, 0.60)$, $(1.48, \infty)$; CD on $(-\infty, -13.81)$, $(-1.55, -1.03)$, $(0.60, 1.48)$; IPs $(-13.81, -0.05)$, $(-1.55, 5.64)$, $(-1.03, 5.39)$, $(0.60, 1.52)$, $(1.48, 1.93)$

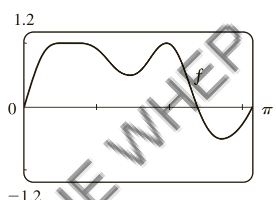


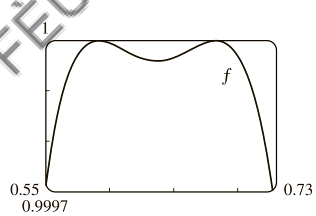


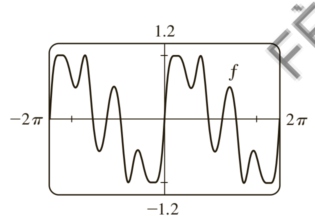
17. Inc on $(-4.91, -4.51)$, $(0, 1.77)$, $(4.91, 8.06)$, $(10.79, 14.34)$, $(17.08, 20)$;
dec on $(-4.51, -4.10)$, $(1.77, 4.10)$, $(8.06, 10.79)$, $(14.34, 17.08)$;
loc max $f(-4.51) \approx 0.62$, $f(1.77) \approx 2.58$, $f(8.06) \approx 3.60$, $f(14.34) \approx 4.39$;
loc min $f(10.79) \approx 2.43$, $f(17.08) \approx 3.49$;
CU on $(9.60, 12.25)$, $(15.81, 18.65)$;
CD on $(-4.91, -4.10)$, $(0, 4.10)$, $(4.91, 9.60)$, $(12.25, 15.81)$, $(18.65, 20)$;
IPs $(9.60, 2.95)$, $(12.25, 3.27)$, $(15.81, 3.91)$, $(18.65, 4.20)$



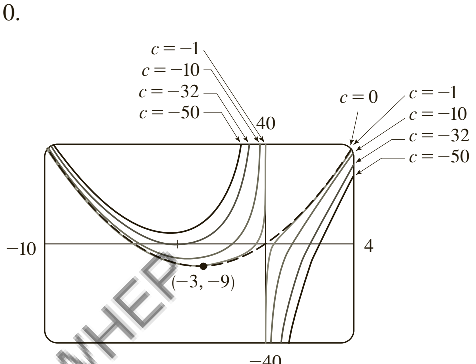
19. Max $f(0.59) \approx 1$, $f(0.68) \approx 1$, $f(1.96) \approx 1$;
min $f(0.64) \approx 0.99996$, $f(1.46) \approx 0.49$, $f(2.73) \approx -0.51$;
IPs $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, $(2.28, 0.34)$



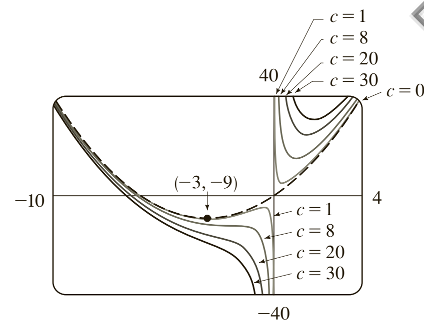




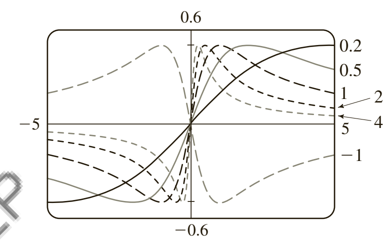
21. For $c < 0$, there is a loc min that moves toward $(-3, -9)$ as c increases. For $0 < c < 8$, there is a loc min that moves toward $(-3, -9)$ and a loc max that moves toward the origin as c decreases. For all $c > 0$, there is a first quadrant loc min that moves toward the origin as c decreases. $c = 0$ is a transitional value that gives the graph of a parabola. For all nonzero c, the y-axis is a VA and there is an IP that moves toward the origin as $|c| \to 0$.
$c \leqslant 0$:



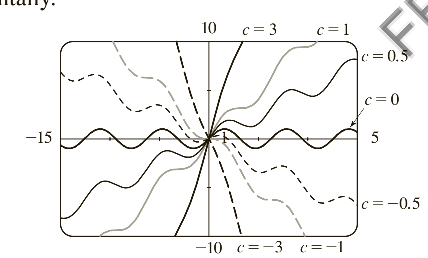
$c \geqslant 0$:



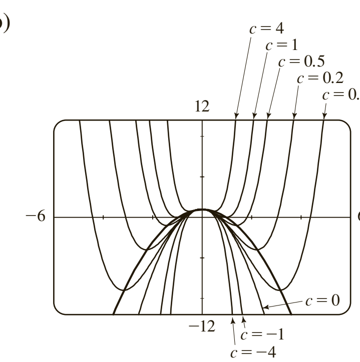
23. For $c > 0$, the maximum and minimum values are always $\pm\frac{1}{2}$, but the extreme points and IPs move closer to the y-axis as c increases. $c = 0$ is a transitional value: when c is replaced by $-c$, the curve is reflected in the x-axis.



25. For $|c| < 1$, the graph has loc max and min values; for $|c| \geqslant 1$ it does not. The function increases for $c \geqslant 1$ and decreases for $c \leqslant -1$. As c changes, the IPs move vertically but not horizontally.



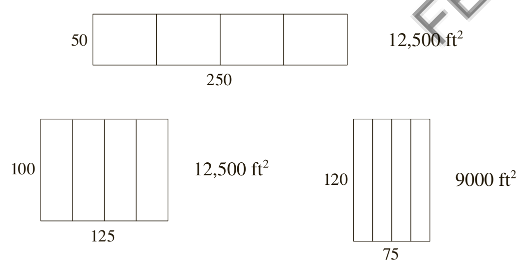
27. (a) Positive (b)



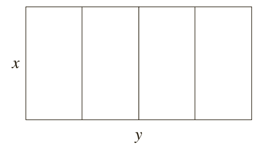
EXERCISES 3.7 ■ PAGE 264

1. (a) 11, 12 (b) 11.5, 11.5 **3.** 10, 10 **5.** $\frac{9}{4}$
7. 25 m by 25 m **9.** $N = 1$

11. (a)



(b)



(c) $A = xy$ (d) $5x + 2y = 750$ (e) $A(x) = 375x - \frac{5}{2}x^2$
(f) 14,062.5 ft^2
13. 1000 ft by 1500 ft **15.** 4000 cm^3 **17.** \$191.28
19. $20\sqrt{30}$ ft by $\frac{40}{3}\sqrt{30}$ ft **21.** $\left(-\frac{6}{5}, \frac{3}{5}\right)$
23. $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$ **25.** Square, side $\sqrt{2}\,r$
27. $L/2, \sqrt{3}\,L/4$ **29.** Base $\sqrt{3}\,r$, height $3r/2$
31. $4\pi r^3/(3\sqrt{3})$ **33.** $\pi r^2(1 + \sqrt{5})$ **35.** 24 cm by 36 cm
37. (a) Use all of the wire for the square
(b) $40\sqrt{3}/(9 + 4\sqrt{3})$ m for the square
39. 16 in. **41.** $V = 2\pi R^3/(9\sqrt{3})$ **45.** $E^2/(4r)$
47. (a) $\frac{3}{2}s^2 \csc\theta\,(\csc\theta - \sqrt{3}\cot\theta)$ (b) $\cos^{-1}(1/\sqrt{3}) \approx 55°$
(c) $6s\left[h + s/(2\sqrt{2})\right]$
49. Row directly to B **51.** $\approx$4.85 km east of the refinery
53. $10\sqrt[3]{3}/(1 + \sqrt[3]{3})$ ft from the stronger source
55. $(a^{2/3} + b^{2/3})^{3/2}$ **57.** $2\sqrt{6}$
59. (b) (i) \$342,491; \$342/unit; \$390/unit (ii) 400
(iii) \$320/unit
61. (a) $p(x) = 19 - \frac{1}{3000}x$ (b) \$9.50
63. (a) $p(x) = 500 - \frac{1}{8}x$ (b) \$250 (c) \$310
69. 9.35 m **73.** $x = 6$ in. **75.** $\pi/6$ **77.** $\frac{1}{2}(L + W)^2$
79. (a) About 5.1 km from B (b) C is close to B; C is close to D; $W/L = \sqrt{25 + x^2}/x$, where $x = |BC|$
(c) $\approx$1.07; no such value (d) $\sqrt{41}/4 \approx 1.6$

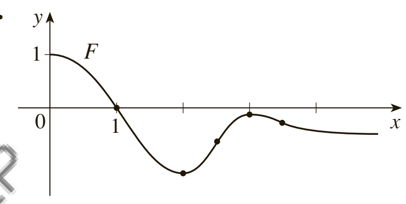
EXERCISES 3.8 ■ PAGE 276

1. (a) $x_2 \approx 7.3, x_3 \approx 6.8$ (b) Yes
3. $\frac{9}{2}$ **5.** a, b, c **7.** 1.5215 **9.** -1.25
11. 2.94283096 **13.** (b) 2.630020 **15.** 0.876726
17. $-3.637958, -1.862365, 0.889470$
19. $-0.549700, 2.629658$ **21.** 0.865474
23. $-1.69312029, -0.74466668, 1.26587094$
25. 0.76682579 **27.** (b) 31.622777
33. (a) $-1.293227, -0.441731, 0.507854$ (b) -2.0212
35. (1.519855, 2.306964) **37.** (0.410245, 0.347810)
39. 0.76286%

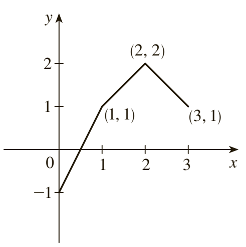
EXERCISES 3.9 ■ PAGE 282

1. $F(x) = 2x^2 + 7x + C$ **3.** $F(x) = \frac{1}{2}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2 + C$
5. $F(x) = 4x^3 + 4x^2 + C$ **7.** $F(x) = 5x^{7/5} + 40x^{1/5} + C$
9. $F(x) = \sqrt{2}\,x + C$ **11.** $F(x) = 2x^{3/2} - \frac{3}{2}x^{4/3} + C$

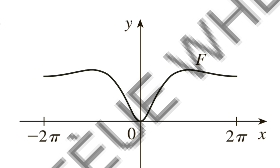
13. $F(x) = \begin{cases} -5/(4x^8) + C_1 & \text{if } x < 0 \\ -5/(4x^8) + C_2 & \text{if } x > 0 \end{cases}$
15. $G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$
17. $H(\theta) = -2\cos\theta - \tan\theta + C_n$ on $(n\pi - \pi/2, n\pi + \pi/2)$, n an integer
19. $F(t) = \frac{16}{3}t^{3/2} - \sec t + C$ **21.** $F(x) = x^5 - \frac{1}{3}x^6 + 4$
23. $f(x) = x^5 - x^4 + x^3 + Cx + D$
25. $f(x) = 2x^2 - \frac{9}{28}x^{7/3} + Cx + D$
27. $f(t) = 2t^3 + \cos t + Ct^2 + Dt + E$
29. $f(x) = x + 2x^{3/2} + 5$ **31.** $f(x) = 4x^{3/2} + 2x^{5/2} + 4$
33. $f(t) = \tan t + \sec t - 2 - \sqrt{2}$
35. $f(x) = -x^2 + 2x^3 - x^4 + 12x + 4$
37. $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$
39. $f(x) = 2x^2 + x^3 + 2x^4 + 2x + 3$
41. $f(t) = \frac{9}{28}t^{7/3} + \cos t + \left(\frac{19}{28} - \cos 1\right)t + 1$
43. 8 **45.** b
47.



49.



51.



53. $s(t) = 1 - \cos t - \sin t$ **55.** $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3$
57. $s(t) = -10\sin t - 3\cos t + (6/\pi)t + 3$
59. (a) $s(t) = 450 - 4.9t^2$ (b) $\sqrt{450/4.9} \approx 9.58$ s
(c) $-9.8\sqrt{450/4.9} \approx -93.9$ m/s (d) About 9.09 s
63. 225 ft **65.** \$742.08 **67.** $\frac{130}{11} \approx 11.8$ s
69. $\frac{88}{15} \approx 5.87\ \text{ft/s}^2$ **71.** 62,500 $\text{km/h}^2 \approx 4.82\ \text{m/s}^2$
73. (a) 22.9125 mi (b) 21.675 mi (c) 30 min 33 s
(d) 55.425 mi

CHAPTER 3 REVIEW ■ PAGE 285

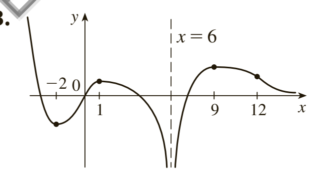
True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. True **13.** False **15.** True **17.** True
19. True

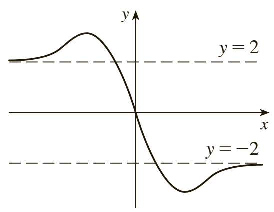
Exercises

1. Abs max $f(2) = f(5) = 18$, abs min $f(0) = -2$, loc max $f(2) = 18$, loc min $f(4) = 14$
3. Abs max $f(2) = \frac{2}{5}$, abs and loc min $f(-\frac{1}{3}) = -\frac{9}{2}$
5. Abs and loc max $f(\pi/6) = \pi/6 + \sqrt{3}$, abs min $f(-\pi) = -\pi - 2$, loc min $f(5\pi/6) = 5\pi/6 - \sqrt{3}$
7. $\frac{1}{2}$ **9.** $-\frac{2}{3}$ **11.** $\frac{3}{4}$

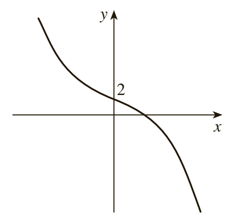
13.



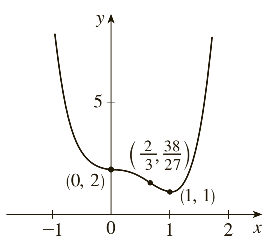
15.



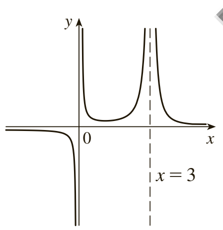
17. A. $\mathbb{R}$ B. y-int 2
C. None D. None
E. Dec on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, 2)$
H. See graph at right.



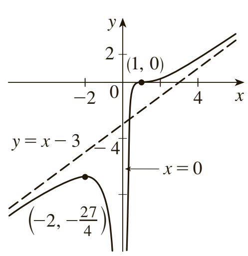
19. A. $\mathbb{R}$ B. y-int 2
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = 1$
G. CU on $(-\infty, 0)$, $(\frac{2}{3}, \infty)$;
CD on $(0, \frac{2}{3})$; IPs $(0, 2)$, $(\frac{2}{3}, \frac{38}{27})$
H. See graph at right.



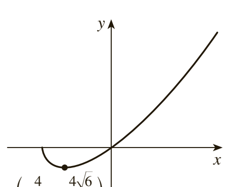
21. A. $\{x \mid x \neq 0, 3\}$
B. None C. None
D. HA $y = 0$; VA $x = 0$, $x = 3$
E. Inc on $(1, 3)$;
dec on $(-\infty, 0)$, $(0, 1)$, $(3, \infty)$
F. Loc min $f(1) = \frac{1}{4}$
G. CU on $(0, 3)$, $(3, \infty)$;
CD on $(-\infty, 0)$
H. See graph at right.



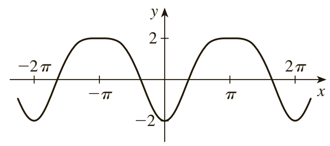
23. A. $(-\infty, 0) \cup (0, \infty)$
B. x-int 1 C. None
D. VA $x = 0$; SA $y = x - 3$
E. Inc on $(-\infty, -2)$, $(0, \infty)$;
dec on $(-2, 0)$
F. Loc max $f(-2) = -\frac{27}{4}$
G. CU on $(1, \infty)$; CD on $(-\infty, 0)$, $(0, 1)$; IP $(1, 0)$
H. See graph at right.



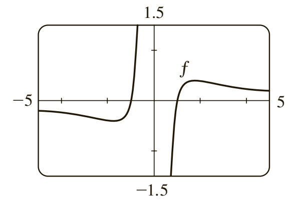
25. A. $[-2, \infty)$
B. y-int 0; x-int -2, 0
C. None D. None
E. Inc on $\left(-\frac{4}{3}, \infty\right)$, dec on $\left(-2, -\frac{4}{3}\right)$
F. Loc min $f\left(-\frac{4}{3}\right) = -\frac{4}{9}\sqrt{6}$
G. CU on $(-2, \infty)$
H. See graph at right.



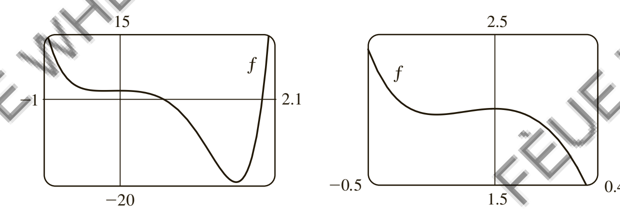
27. A. $\mathbb{R}$ B. y-int -2 C. About y-axis, period 2π
D. None E. Inc on $(2n\pi, (2n + 1)\pi)$, n an integer;
dec on $((2n - 1)\pi, 2n\pi)$
F. Loc max $f((2n + 1)\pi) = 2$; loc min $f(2n\pi) = -2$
G. CU on $(2n\pi - (\pi/3), 2n\pi + (\pi/3))$;
CD on $(2n\pi + (\pi/3), 2n\pi + (5\pi/3))$; IP $\left(2n\pi \pm (\pi/3), -\frac{1}{2}\right)$
H.



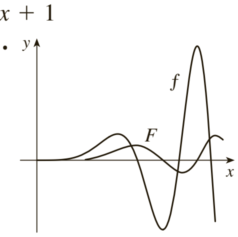
29. Inc on $(-\sqrt{3}, 0)$, $(0, \sqrt{3})$;
dec on $(-\infty, -\sqrt{3})$, $(\sqrt{3}, \infty)$;
loc max $f(\sqrt{3}) = \frac{2}{9}\sqrt{3}$,
loc min $f(-\sqrt{3}) = -\frac{2}{9}\sqrt{3}$;
CU on $(-\sqrt{6}, 0)$, $(\sqrt{6}, \infty)$;
CD on $(-\infty, -\sqrt{6})$, $(0, \sqrt{6})$;
IPs $\left(\sqrt{6}, \frac{5}{36}\sqrt{6}\right)$, $\left(-\sqrt{6}, -\frac{5}{36}\sqrt{6}\right)$



31. Inc on $(-0.23, 0)$, $(1.62, \infty)$; dec on $(-\infty, -0.23)$, $(0, 1.62)$;
loc max $f(0) = 2$; loc min $f(-0.23) \approx 1.96$, $f(1.62) \approx -19.2$; CU on $(-\infty, -0.12)$, $(1.24, \infty)$;
CD on $(-0.12, 1.24)$; IPs $(-0.12, 1.98)$, $(1.24, -12.1)$



37. (a) 0 (b) CU on $\mathbb{R}$ **41.** $3\sqrt{3}\,r^2$
43. $4/\sqrt{3}$ cm from D **45.** $L = C$ **47.** \$11.50
49. 1.297383 **51.** 1.16718557
53. $F(x) = \frac{8}{3}x^{3/2} - 2x^3 + 3x + C$
55. $f(t) = t^2 + 3\cos t + 2$
57. $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$
59. $s(t) = t^2 + \cos t + 2$ **61.**



63. No
65. (b) About 8.5 in. by 2 in. (c) $20/\sqrt{3}$ in. by $20\sqrt{2/3}$ in.
67. (a) $20\sqrt{2} \approx 28$ ft
(b) $\dfrac{dI}{dt} = \dfrac{-480k(h - 4)}{[(h - 4)^2 + 1600]^{5/2}}$, where k is the constant of proportionality

PROBLEMS PLUS ■ PAGE 290

5. $(-2, 4)$, $(2, -4)$ **7.** $\frac{4}{3}$ **11.** $(m/2, m^2/4)$
13. $-3.5 < a < -2.5$
15. (a) $x/(x^2 + 1)$ (b) $\frac{1}{2}$

17. (a) $\dfrac{d\theta}{dt} = -\tan\theta\left[\dfrac{1}{c}\dfrac{dc}{dt} + \dfrac{1}{b}\dfrac{db}{dt}\right]$

(b) $\dfrac{da}{dt} = \dfrac{b\dfrac{db}{dt} + c\dfrac{dc}{dt} - \left(b\dfrac{dc}{dt} + c\dfrac{db}{dt}\right)\sec\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}}$

19. (a) $T_1 = D/c_1$, $T_2 = (2h\sec\theta)/c_1 + (D - 2h\tan\theta)/c_2$, $T_3 = \sqrt{4h^2 + D^2}/c_1$

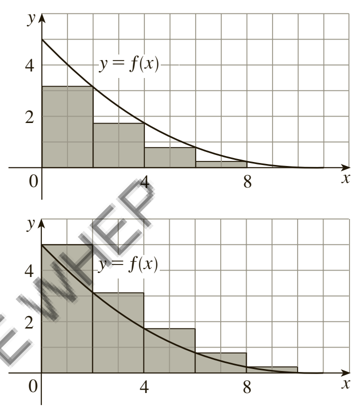
(c) $c_1 \approx 3.85$ km/s, $c_2 \approx 7.66$ km/s, $h \approx 0.42$ km

23. $3/(\sqrt[3]{2} - 1) \approx 11\frac{1}{2}$ h

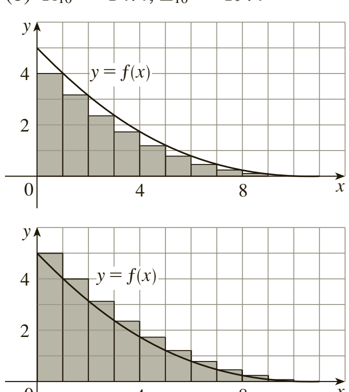
CHAPTER 4

EXERCISES 4.1 ■ PAGE 303

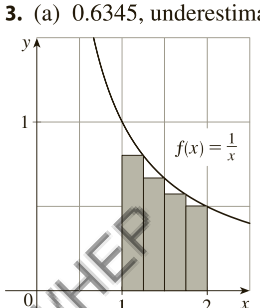
1. (a) $R_5 \approx 12$, $L_5 \approx 22$

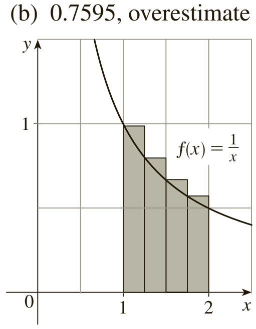


(b) $R_{10} \approx 14.4$, $L_{10} \approx 19.4$

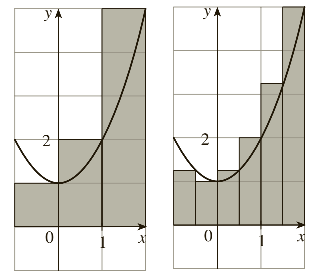


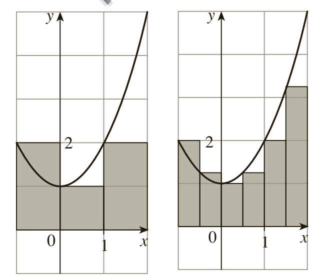
3. (a) 0.6345, underestimate (b) 0.7595, overestimate



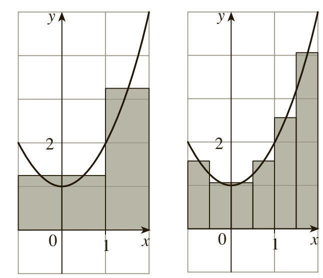


5. (a) 8, 6.875 (b) 5, 5.375



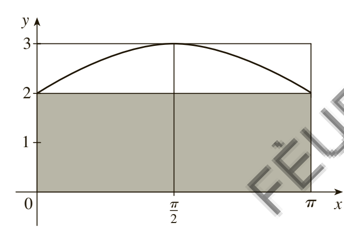


(c) 5.75, 5.9375

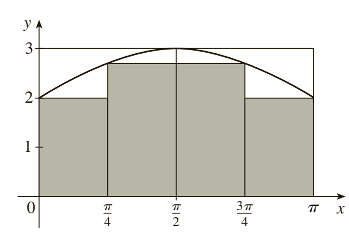


(d) M_6

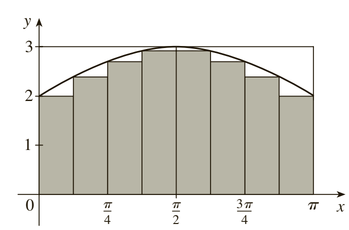
7. $n = 2$: upper $= 3\pi \approx 9.42$, lower $= 2\pi \approx 6.28$



$n = 4$: upper $= (10 + \sqrt{2})(\pi/4) \approx 8.96$,
lower $= (8 + \sqrt{2})(\pi/4) \approx 7.39$



$n = 8$: upper ≈ 8.65, lower ≈ 7.86



9. 0.2533, 0.2170, 0.2101, 0.2050; 0.2

11. (a) Left: 0.8100, 0.7937, 0.7904; right: 0.7600, 0.7770, 0.7804

13. 34.7 ft, 44.8 ft **15.** 63.2 L, 70 L **17.** 155 ft

19. 7840 **21.** $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2(1+2i/n)}{(1+2i/n)^2+1} \cdot \frac{2}{n}$

23. $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin(\pi i/n)} \cdot \frac{\pi}{n}$

25. The region under the graph of $y = \tan x$ from 0 to $\pi/4$

27. (a) $L_n < A < R_n$

29. (a) $\lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5$ (b) $\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

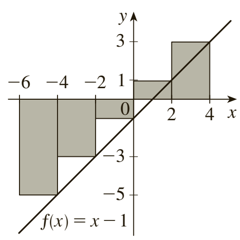
(c) $\frac{32}{3}$

31. $\sin b$, 1

EXERCISES 4.2 ■ PAGE 316

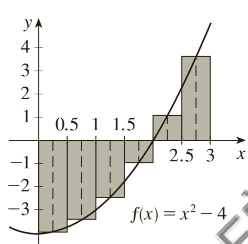
1. -10

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.



3. $-\frac{49}{16}$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the four rectangles below the x-axis.



5. (a) 6 (b) 4 (c) 2

7. Lower, $L_5 = -64$; upper, $R_5 = 16$

9. 6.1820 **11.** 0.9071 **13.** 0.9029, 0.9018

15.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

The values of R_n appear to be approaching 2.

17. $\int_0^{\pi} \frac{\sin x}{1+x}\,dx$ **19.** $\int_2^7 (5x^3 - 4x)\,dx$

21. -9 **23.** $\frac{2}{3}$ **25.** $-\frac{3}{4}$

29. $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + (1+2i/n)^2} \cdot \frac{2}{n}$

31. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\sin \frac{5\pi i}{n}\right) \frac{\pi}{n} = \frac{2}{5}$

33. (a) 4 (b) 10 (c) -3 (d) 2

35. $\frac{3}{2}$ **37.** $3 + \frac{9}{4}\pi$ **39.** $\frac{25}{4}$ **41.** 0 **43.** 3

45. 22.5 **47.** $\int_{-1}^{5} f(x)\,dx$ **49.** 122

51. B < E < A < D < C **53.** 15

59. $0 \leqslant \int_0^1 x^3\,dx \leqslant 1$ **61.** $\frac{\pi}{12} \leqslant \int_{\pi/4}^{\pi/3} \tan x\,dx \leqslant \frac{\pi}{12}\sqrt{3}$

63. $2 \leqslant \int_{-1}^{1} \sqrt{1+x^4}\,dx \leqslant 2\sqrt{2}$ **67.** $\int_1^2 \sqrt{x}\,dx$

73. $\int_0^1 x^4\,dx$ **75.** $\frac{1}{2}$

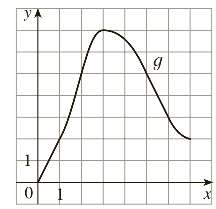
EXERCISES 4.3 ■ PAGE 327

1. One process undoes what the other one does. See the Fundamental Theorem of Calculus, page 326.

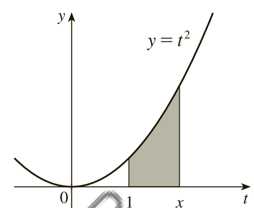
3. (a) 0, 2, 5, 7, 3 (d)

(b) (0, 3)

(c) $x = 3$



5.



(a), (b) x^2

7. $g'(x) = \sqrt{x + x^3}$ **9.** $g'(s) = (s - s^2)^8$

11. $F'(x) = -\sqrt{1 + \sec x}$ **13.** $h'(x) = -\sin^4(1/x)/x^2$

15. $y' = \frac{3(3x+2)}{1+(3x+2)^3}$

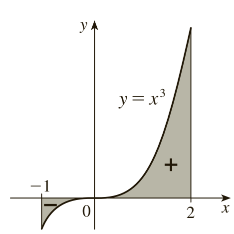
17. $y' = -\frac{1}{2}\tan\sqrt{x}$ **19.** $\frac{26}{3}$ **21.** 2 **23.** $\frac{52}{3}$

25. $1 + \sqrt{3}/2$ **27.** $-\frac{37}{6}$ **29.** $\frac{82}{5}$ **31.** 1 **33.** $\frac{15}{4}$

35. $\frac{17}{2}$ **37.** 0 **39.** $\frac{16}{3}$ **41.** $\frac{32}{3}$

43. $\frac{243}{4}$ **45.** 2

47. 3.75



49. The function $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied.

51. The function $f(\theta) = \sec\theta\,\tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied.

53. $g'(x) = \frac{-2(4x^2-1)}{4x^2+1} + \frac{3(9x^2-1)}{9x^2+1}$

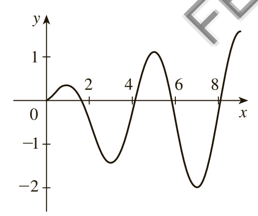
55. $h'(x) = -\frac{1}{2\sqrt{x}}\cos x + 3x^2\cos(x^6)$ **57.** $y = -\frac{1}{\pi}x + 1$

59. $(-4, 0)$ **61.** 29

63. (a) $-2\sqrt{n}$, $\sqrt{4n-2}$, n an integer > 0

(b) $(0, 1)$, $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, and $\left(\sqrt{4n-1}, \sqrt{4n+1}\right)$, n an integer > 0 (c) 0.74

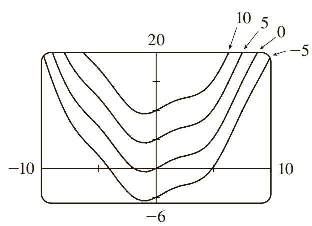
65. (a) Loc max at 1 and 5; loc min at 3 and 7
(b) $x = 9$
(c) $(\frac{1}{2}, 2)$, $(4, 6)$, $(8, 9)$
(d) See graph at right.



67. $\frac{7}{10}$ **75.** $f(x) = x^{3/2}$, $a = 9$
77. (b) Average expenditure over $[0, t]$; minimize average expenditure
79. $\ln 3$ **81.** $\pi/3$ **83.** $e^2 - 1$

EXERCISES 4.4 ■ PAGE 336

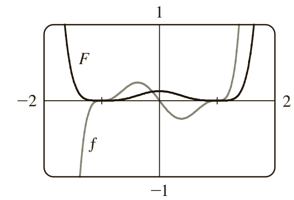
5. $\frac{1}{2.3}x^{2.3} + 2x^{3.5} + C$ **7.** $5x + \frac{2}{9}x^3 + \frac{3}{16}x^4 + C$
9. $\frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C$ **11.** $2\sqrt{x} + x + \frac{2}{3}x^{3/2} + C$
13. $\theta + \tan\theta + C$ **15.** $-\cot t + \cos t + C$
17. $\sin x + \frac{1}{4}x^2 + C$

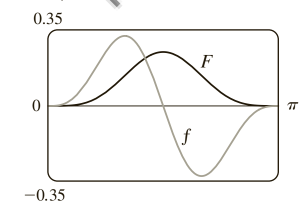


19. $-\frac{10}{3}$ **21.** $\frac{21}{5}$ **23.** -2 **25.** 8 **27.** 36
29. $2\sqrt{5}$ **31.** $\frac{256}{15}$ **33.** $1 + \pi/4$ **35.** $\frac{69}{4}$
37. 1 **39.** $\frac{5}{2}$ **41.** -3.5 **43.** ≈ 1.36 **45.** $\frac{4}{3}$
47. The change in the child's weight (in pounds) between the ages of 5 and 10
49. Number of gallons of oil leaked in the first 2 hours
51. Change in revenue when production is increased from 1000 to 5000 units
53. Newton-meters (or joules) **55.** (a) $-\frac{3}{2}$ m (b) $\frac{41}{6}$ m
57. (a) $v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s (b) $416\frac{2}{3}$ m
59. $46\frac{2}{3}$ kg **61.** 1.4 mi
63. 2,954,707,200 ft^3
65. 39.8 ft/s **67.** 4.75×10^5 megawatt-hours
69. $-\cos x + \cosh x + C$ **71.** $\frac{1}{3}x^3 + x + \tan^{-1}x + C$
73. $\pi/6$

EXERCISES 4.5 ■ PAGE 346

1. $\frac{1}{2}\sin 2x + C$ **3.** $\frac{2}{9}(x^3 + 1)^{3/2} + C$
5. $-\dfrac{1}{4(x^4 - 5)} + C$ **7.** $-\frac{1}{3}(1 - x^2)^{3/2} + C$
9. $-\frac{1}{20}(1 - 2x)^{10} + C$ **11.** $-\frac{3}{2}\cos(2\theta/3) + C$
13. $\frac{1}{3}\sec 3t + C$ **15.** $\frac{1}{5}\sin(1 + 5t) + C$ **17.** $\frac{1}{4}\tan^4\theta + C$
19. $\frac{1}{15}(x^3 + 3x)^5 + C$ **21.** $\frac{2}{3}\sqrt{3ax + bx^3} + C$
23. $\frac{1}{2}(1 + z^3)^{2/3} + C$ **25.** $-\frac{2}{3}(\cot x)^{3/2} + C$
27. $\frac{1}{3}\sec^3 x + C$ **29.** $\frac{1}{40}(2x + 5)^{10} - \frac{5}{36}(2x + 5)^9 + C$
31. $\frac{1}{8}(x^2 - 1)^4 + C$ **33.** $\frac{1}{4}\sin^4 x + C$





35. $2/\pi$ **37.** $\frac{45}{28}$ **39.** $2/\sqrt{3} - 1$
41. 0 **43.** 3 **45.** $\frac{1}{3}(2\sqrt{2} - 1)a^3$ **47.** $\frac{16}{15}$
49. $\frac{1}{2}(\sin 4 - \sin 1)$ **51.** $\frac{1}{6}$ **53.** $\sqrt{3} - \frac{1}{3}$ **55.** 6π
57. $\dfrac{5}{4\pi}\left(1 - \cos\dfrac{2\pi t}{5}\right)$ L **59.** 5
67. $-\frac{1}{3}\ln|5 - 3x| + C$ **69.** $\frac{1}{3}(\ln x)^3 + C$
71. $\frac{2}{3}(1 + e^x)^{3/2} + C$ **73.** $\frac{1}{3}(\arctan x)^3 + C$
75. $\tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$ **77.** $-\ln(1 + \cos^2 x) + C$
79. $\ln|\sin x| + C$ **81.** 2 **83.** $\ln(e + 1)$ **85.** $\pi^2/4$

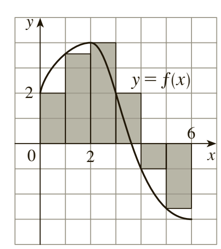
CHAPTER 4 REVIEW ■ PAGE 348

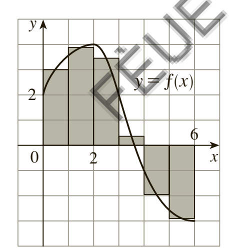
True-False Quiz

1. True **3.** True **5.** False **7.** True **9.** True
11. False **13.** True **15.** False **17.** False

Exercises

1. (a) 8 (b) 5.7





3. $\frac{1}{2} + \pi/4$ **5.** 3 **7.** f is c, f' is b, $\int_0^x f(t)\,dt$ is a
9. $3, 0$ **11.** 37 **13.** $\frac{9}{10}$ **15.** -76 **17.** $\frac{21}{4}$
19. Does not exist **21.** $\frac{1}{3}\sin 1$ **23.** 0
25. $[1/(2\pi)]\sin^2\pi t + C$ **27.** $\frac{1}{2}\sqrt{2} - \frac{1}{2}$ **29.** $\frac{23}{3}$
31. $2\sqrt{1 + \sin x} + C$ **33.** $\frac{64}{5}$ **35.** $F'(x) = x^2/(1 + x^3)$
37. $g'(x) = 4x^3\cos(x^8)$ **39.** $y' = \dfrac{2\cos x - \cos\sqrt{x}}{2x}$
41. $4 \leqslant \int_1^3 \sqrt{x^2 + 3}\,dx \leqslant 4\sqrt{3}$ **45.** 0.280981
47. Number of barrels of oil consumed from January 1, 2000, through January 1, 2008
49. 72,400 **51.** 3 **53.** $(1 + x^2)(x\cos x + \sin x)/x^2$

PROBLEMS PLUS ■ PAGE 353

1. $\pi/2$ **3.** $f(x) = \frac{1}{2}x$ **5.** -1 **7.** $[-1, 2]$
9. (a) $\frac{1}{2}(n - 1)n$
(b) $\frac{1}{2}[\![b]\!](2b - [\![b]\!] - 1) - \frac{1}{2}[\![a]\!](2a - [\![a]\!] - 1)$
15. $y = -\dfrac{2b}{a^2}x^2 + \dfrac{3b}{a}x$ **17.** $2(\sqrt{2} - 1)$

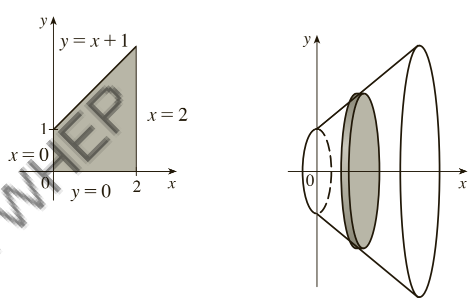
CHAPTER 5

EXERCISES 5.1 ■ PAGE 362

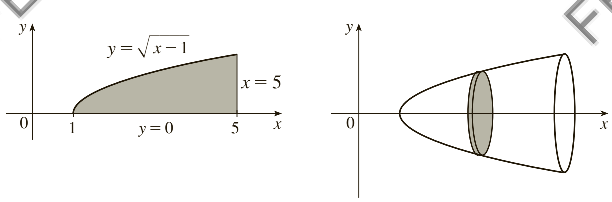
1. $\frac{47}{12}$ **3.** $\frac{4}{3}$ **5.** 19.5 **7.** $\frac{9}{2}$ **9.** $\frac{4}{3}$ **11.** $\frac{8}{3}$
13. 72 **15.** $6\sqrt{3}$ **17.** $\frac{32}{3}$ **19.** $2/\pi + \frac{2}{3}$
21. $2 - \pi/2$ **23.** $\frac{47}{3} - \frac{9}{2}\sqrt[3]{12}$ **25.** $\frac{13}{5}$ **27.** $\frac{3}{4}$
29. (a) 39 (b) 15 **31.** $\frac{4}{3}$ **33.** $\frac{5}{2}$ **35.** $\frac{3}{2}\sqrt{3} - 1$
37. 0, 0.90; 0.04 **39.** -1.11, 1.25, 2.86; 8.38
41. 2.80123 **43.** 0.25142 **45.** $12\sqrt{6} - 9$
47. $117\frac{1}{3}$ ft **49.** 4232 cm^2
51. (a) Twelfth ($t \approx 11.26$) (b) Eighteenth ($t \approx 17.18$)
(c) 706
53. (a) Car A (b) The distance by which A is ahead of B after 1 minute (c) Car A (d) $t \approx 2.2$ min
55. $\frac{24}{5}\sqrt{3}$ **57.** $4^{2/3}$ **59.** ± 6
61. $\ln 2 - \frac{1}{2}$ **63.** $2 - 2\ln 2$

EXERCISES 5.2 ■ PAGE 374

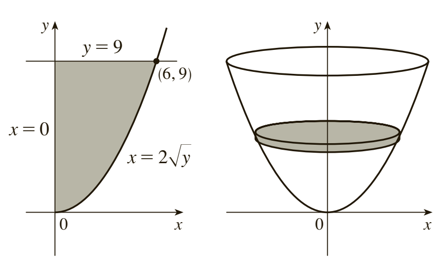
1. $26\pi/3$



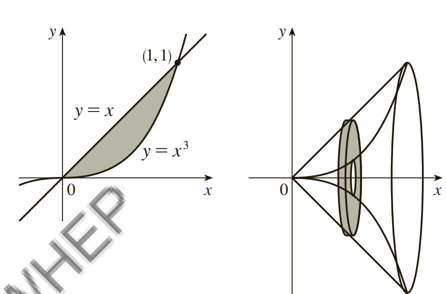
3. 8π



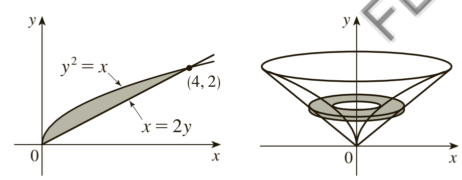
5. 162π



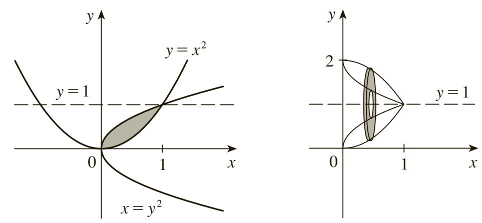
7. $4\pi/21$



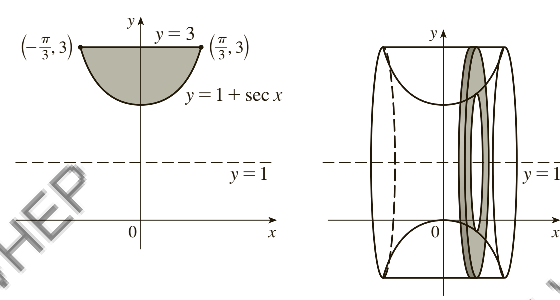
9. $64\pi/15$



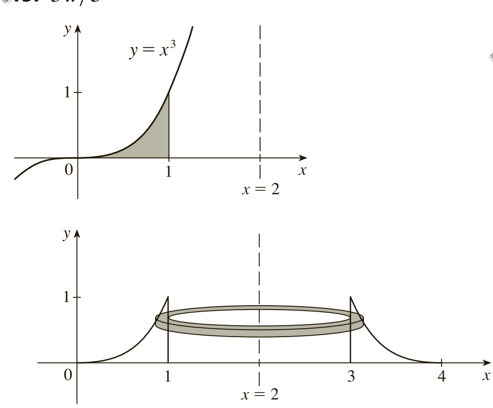
11. $11\pi/30$



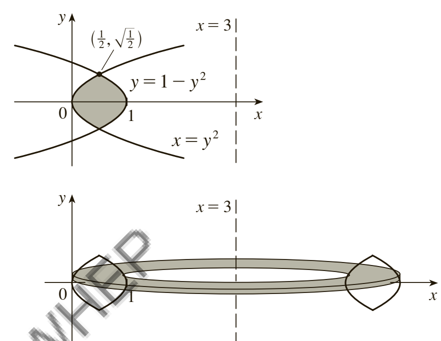
13. $2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$



15. $3\pi/5$



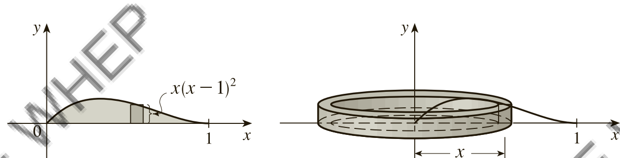
17. $10\sqrt{2}\,\pi/3$



19. $\pi/3$ **21.** $\pi/3$ **23.** $\pi/3$
25. $13\pi/45$ **27.** $\pi/3$ **29.** $17\pi/45$
31. (a) $\pi\int_0^{\pi/4} \tan^2 x\, dx \approx 0.67419$
(b) $\pi\int_0^{\pi/4} (\tan^2 x + 2\tan x)\, dx \approx 2.85178$
33. (a) $2\pi\int_0^2 8\sqrt{1 - x^2/4}\, dx \approx 78.95684$
(b) $2\pi\int_0^1 8\sqrt{4 - 4y^2}\, dy \approx 78.95684$
35. $-1, 0.857; 9.756$ **37.** $\frac{11}{8}\pi^2$
39. Solid obtained by rotating the region $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \sqrt{\sin x}$ about the x-axis
41. Solid obtained by rotating the region above the x-axis bounded by $x = y^2$ and $x = y^4$ about the y-axis
43. $1110\ \text{cm}^3$ **45.** (a) 196 (b) 838
47. $\frac{1}{3}\pi r^2 h$ **49.** $\pi h^2\left(r - \frac{1}{3}h\right)$ **51.** $\frac{2}{3}b^2 h$
53. $10\ \text{cm}^3$ **55.** 24 **57.** $\frac{1}{3}$ **59.** $\frac{8}{15}$ **61.** $4\pi/15$
63. (a) $8\pi R\int_0^r \sqrt{r^2 - y^2}\, dy$ (b) $2\pi^2 r^2 R$
65. (b) $\pi r^2 h$ **67.** $\frac{5}{12}\pi r^3$ **69.** $8\int_0^r \sqrt{R^2 - y^2}\sqrt{r^2 - y^2}\, dy$

EXERCISES 5.3 ■ PAGE 381

1. Circumference $= 2\pi x$, height $= x(x-1)^2$; $\pi/15$



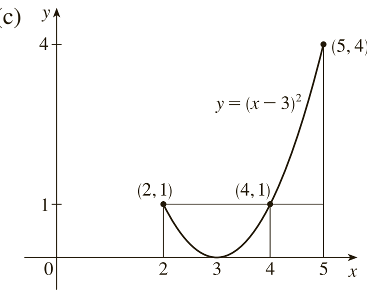
3. $6\pi/7$ **5.** 8π **7.** 8π
9. 4π **11.** 192π **13.** $16\pi/3$
15. $264\pi/5$ **17.** $8\pi/3$ **19.** $13\pi/3$
21. (a) $\int_{2\pi}^{3\pi} 2\pi x \sin x\, dx$ (b) 98.69604
23. (a) $4\pi\int_{-\pi/2}^{\pi/2} (\pi - x)\cos^4 x\, dx$ (b) 46.50942
25. (a) $\int_0^{\pi} 2\pi(4 - y)\sqrt{\sin y}\, dy$ (b) 36.57476
27. 3.68
29. Solid obtained (using shells) by rotating the region $0 \leqslant y \leqslant x^4$, $0 \leqslant x \leqslant 3$ about the y-axis
31. Solid obtained (using shells) by rotating the region $0 \leqslant x \leqslant 1/y^2$, $1 \leqslant y \leqslant 4$ about the line $y = -2$
33. $0, 2.175; 14.450$ **35.** $\frac{1}{32}\pi^3$ **37.** 8π
39. $4\sqrt{3}\,\pi$ **41.** $4\pi/3$
43. $117\pi/5$ **45.** $\frac{4}{3}\pi r^3$ **47.** $\frac{1}{3}\pi r^2 h$

EXERCISES 5.4 ■ PAGE 386

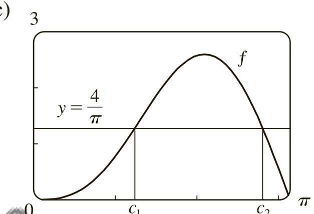
1. (a) 7200 ft-lb (b) 7200 ft-lb
3. 4.5 ft-lb **5.** 180 J **7.** $\frac{15}{4}$ ft-lb
9. (a) $\frac{25}{24} \approx 1.04$ J (b) 10.8 cm **11.** $W_2 = 3W_1$
13. (a) 625 ft-lb (b) $\frac{1875}{4}$ ft-lb **15.** 650,000 ft-lb
17. 3857 J **19.** 62.5 ft-lb **21.** 2450 J
23. $\approx 1.06 \times 10^6$ J **25.** $\approx 1.04 \times 10^5$ ft-lb **27.** 2.0 m
31. (b) 161.3 ft-lb
33. (a) $Gm_1m_2\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$ (b) $\approx 8.50 \times 10^9$ J

EXERCISES 5.5 ■ PAGE 391

1. 7 **3.** $6/\pi$ **5.** 29,524/15 **7.** $2/(5\pi)$
9. (a) 1 (b) 2, 4 (c)



11. (a) $4/\pi$ (b) $\approx 1.24, 2.81$
(c)



15. $\frac{9}{8}$ **17.** $(50 + 28/\pi)$°F ≈ 59°F **19.** 6 kg/m
21. $5/(4\pi) \approx 0.4$ L

CHAPTER 5 REVIEW ■ PAGE 393

Exercises

1. $\frac{8}{3}$ **3.** $\frac{7}{12}$ **5.** $\frac{4}{3} + 4/\pi$ **7.** $64\pi/15$ **9.** $1656\pi/5$
11. $\frac{4}{3}\pi(2ah + h^2)^{3/2}$ **13.** $\int_{-\pi/3}^{\pi/3} 2\pi(\pi/2 - x)\left(\cos^2 x - \frac{1}{4}\right) dx$
15. (a) $2\pi/15$ (b) $\pi/6$ (c) $8\pi/15$
17. (a) 0.38 (b) 0.87
19. Solid obtained by rotating the region $0 \leqslant y \leqslant \cos x$, $0 \leqslant x \leqslant \pi/2$ about the y-axis
21. Solid obtained by rotating the region $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant 2 - \sin x$ about the x-axis
23. 36 **25.** $\frac{125}{3}\sqrt{3}\ \text{m}^3$ **27.** 3.2 J
29. (a) $8000\pi/3 \approx 8378$ ft-lb (b) 2.1 ft
31. $4/\pi$ **33.** $f(x)$

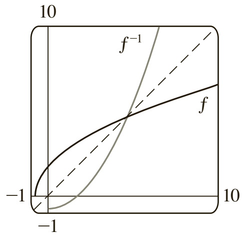
PROBLEMS PLUS ■ PAGE 395

1. (a) $f(t) = 3t^2$ (b) $f(x) = \sqrt{2x/\pi}$ **3.** $\frac{32}{27}$
5. (b) 0.2261 (c) 0.6736 m
(d) (i) $1/(105\pi) \approx 0.003$ in/s (ii) $370\pi/3$ s ≈ 6.5 min
9. $y = \frac{32}{9}x^2$
11. (a) $V = \int_0^h \pi[f(y)]^2\, dy$
(c) $f(y) = \sqrt{kA/(\pi C)}\, y^{1/4}$. Advantage: the markings on the container are equally spaced.
13. $b = 2a$ **15.** $B = 16A$

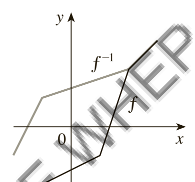
CHAPTER 6

EXERCISES 6.1 ■ PAGE 406

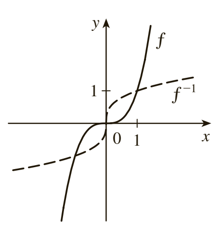
1. (a) See Definition 1.
(b) It must pass the Horizontal Line Test.
3. No **5.** No **7.** Yes **9.** Yes **11.** No **13.** No
15. No **17.** (a) 6 (b) 3 **19.** 4
21. $F = \frac{9}{5}C + 32$; the Fahrenheit temperature as a function of the Celsius temperature; $[-273.15, \infty)$
23. $f^{-1}(x) = \frac{5}{4} - \frac{1}{4}x$ **25.** $f^{-1}(x) = \frac{1}{3}(x-1)^2 - \frac{2}{3}$, $x \geqslant 1$
27. $y = \left(\dfrac{1-x}{1+x}\right)^2$, $-1 < x \leqslant 1$
29. $f^{-1}(x) = \frac{1}{4}(x^2 - 3)$, $x \geqslant 0$



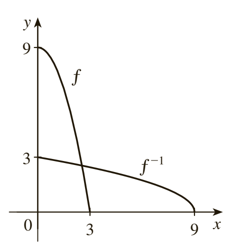
31.



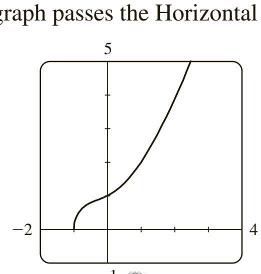
33. (a) $f^{-1}(x) = \sqrt{1-x^2}$, $0 \leqslant x \leqslant 1$; f^{-1} and f are the same function. (b) Quarter-circle in the first quadrant
35. (b) $\frac{1}{12}$
(c) $f^{-1}(x) = \sqrt[3]{x}$,
domain $= \mathbb{R} =$ range
(e)



37. (b) $-\frac{1}{2}$
(c) $f^{-1}(x) = \sqrt{9-x}$,
domain $= [0, 9]$, range $= [0, 3]$
(e)



39. $\frac{1}{6}$ **41.** $2/\pi$ **43.** $\frac{3}{2}$ **45.** $1/\sqrt{28}$
47. The graph passes the Horizontal Line Test.

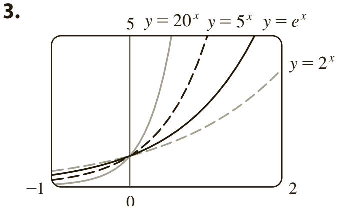


$f^{-1}(x) = -\frac{1}{6}\sqrt[3]{4}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$, where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$; two of the expressions are complex.

49. (a) $g^{-1}(x) = f^{-1}(x) - c$
(b) $h^{-1}(x) = (1/c)f^{-1}(x)$

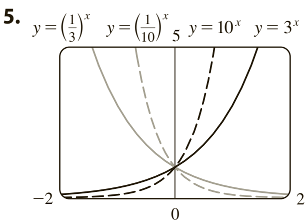
EXERCISES 6.2 ■ PAGE 418

1. (a) $f(x) = b^x$, $b > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$
(d) See Figures 6(c), 6(b), and 6(a), respectively.
3.



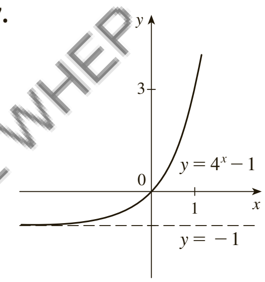
All approach 0 as $x \to -\infty$, all pass through $(0, 1)$, and all are increasing. The larger the base, the faster the rate of increase.

5.

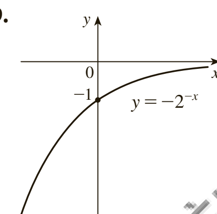


The functions with base greater than 1 are increasing and those with base less than 1 are decreasing. The latter are reflections of the former about the y-axis.

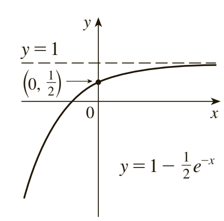
7.



9.



11.

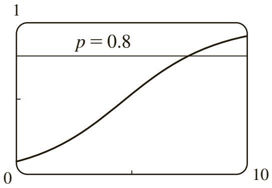


13. (a) $y = e^x - 2$ (b) $y = e^{x-2}$ (c) $y = -e^x$
(d) $y = e^{-x}$ (e) $y = -e^{-x}$
15. (a) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ (b) $(-\infty, \infty)$
17. $f(x) = 3 \cdot 2^x$ **21.** At $x \approx 35.8$ **23.** ∞
25. 1 **27.** 0 **29.** 0 **31.** $f'(x) = 0$
33. $f'(x) = e^x(3x^2 + x - 5)$ **35.** $y' = 3ax^2 e^{ax^3}$
37. $y' = (\sec^2\theta)\, e^{\tan\theta}$ **39.** $f'(x) = \dfrac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$
41. $y' = xe^{-3x}(2 - 3x)$ **43.** $f'(t) = e^{at}(b\cos bt + a\sin bt)$
45. $F'(t) = e^{t\sin 2t}(2t\cos 2t + \sin 2t)$
47. $g'(u) = ue^{\sqrt{\sec u^2}}\sqrt{\sec u^2}\tan u^2$
49. $y' = \dfrac{4e^{2x}}{(1 + e^{2x})^2}\sin\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$ **51.** $y = 2x + 1$
53. $y' = \dfrac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$ **57.** −4, −2

59. $f^{(n)}(x) = 2^n e^{2x}$ **61.** (b) -0.567143 **63.** 3.5 days

65. (a) 1 (b) $kae^{-kt}/(1 + ae^{-kt})^2$

(c) $t \approx 7.4$ h



67. -1 **69.** $f(2) = 2/\sqrt{e}$, $f(-1) = -1/\sqrt[8]{e}$

71. (a) Inc on $(2, \infty)$; dec on $(-\infty, 2)$

(b) CU on $(-\infty, 3)$; CD on $(3, \infty)$ (c) $(3, -2e^{-3})$

73. A. $\{x \mid x \neq -1\}$

B. y-int $1/e$ C. None

D. HA $y = 1$; VA $x = -1$

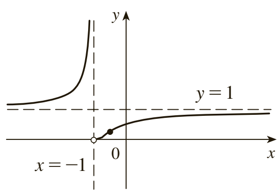
E. Inc on $(-\infty, -1)$, $(-1, \infty)$

F. None

G. CU on $(-\infty, -1)$, $\left(-1, -\frac{1}{2}\right)$;

CD on $\left(-\frac{1}{2}, \infty\right)$; IP $\left(-\frac{1}{2}, 1/e^2\right)$

H. See graph at right.



75. A. $\mathbb{R}$ B. y-int $\frac{1}{2}$ C. None

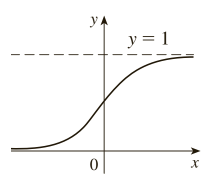
D. HA $y = 0$, $y = 1$

E. Inc on $\mathbb{R}$ F. None

G. CU on $(-\infty, 0)$;

CD on $(0, \infty)$; IP $\left(0, \frac{1}{2}\right)$

H. See graph at right.



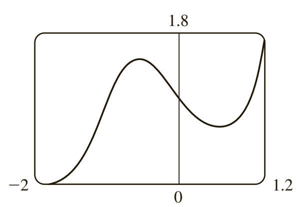
77. 28.57 min, when the rate of increase of drug level in the bloodstream is greatest; 85.71 min, when the rate of decrease is greatest

79. 0.177 mg/mL; 21.4 min

81. Loc max $f\left(-1/\sqrt{3}\right) = e^{2\sqrt{3}/9} \approx 1.5$;

loc min $f\left(1/\sqrt{3}\right) = e^{-2\sqrt{3}/9} \approx 0.7$;

IP $(-0.15, 1.15)$, $(-1.09, 0.82)$



83. $\dfrac{1}{e+1} + e - 1$ **85.** $\dfrac{1}{\pi}(1 - e^{-2\pi})$

87. $\frac{2}{3}(1 + e^x)^{3/2} + C$ **89.** $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

91. $\dfrac{1}{1 - e^u} + C$ **93.** $e - \sqrt{e}$

95. 4.644 **97.** $\pi(e^2 - 1)/2$ **101.** ≈ 4512 L

103. $C_0(1 - e^{-30r/V})$; the total amount of urea removed from the blood in the first 30 minutes of dialysis treatment

105. $\frac{1}{2}$

EXERCISES 6.3 ■ PAGE 426

1. (a) It's defined as the inverse of the exponential function with base b, that is, $\log_b x = y \iff b^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 1.

3. (a) 5 (b) $\frac{1}{3}$

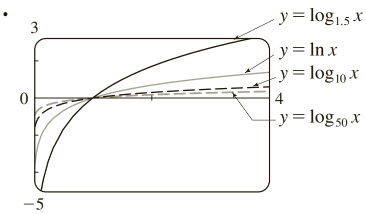
5. (a) 4.5 (b) -4 **7.** (a) 2 (b) $\frac{2}{3}$

9. $\frac{1}{2}\ln a + \frac{1}{2}\ln b$ **11.** $2\ln x - 3\ln y - 4\ln z$

13. $\ln \dfrac{x^2y^3}{z}$ **15.** $\ln 250$ **17.** $\ln \dfrac{\sqrt{x}}{x+1}$

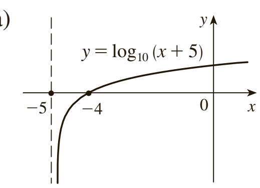
19. (a) 1.430677 (b) 3.680144 (c) 1.651496

21.

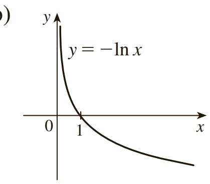


All graphs approach $-\infty$ as $x \to 0^+$, all pass through $(1, 0)$, and all are increasing. The larger the base, the slower the rate of increase.

23. (a)

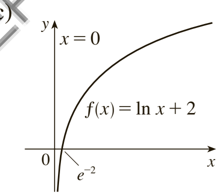


(b)



25. (a) $(0, \infty)$; $(-\infty, \infty)$ (b) e^{-2}

(c)



27. (a) $\frac{1}{4}(7 - \ln 6)$ (b) $\frac{1}{3}(e^2 + 10)$

29. (a) $5 + \log_2 3$ or $5 + (\ln 3)/\ln 2$ (b) $\frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$

31. $-\frac{1}{2}\ln(e - 1)$ **33.** e^e **35.** $\ln 3$

37. (a) 3.7704 (b) 0.3285

39. (a) $0 < x < 1$ (b) $x > \ln 5$

41. About 1,084,588 mi **43.** 8.3

45. (a) $f^{-1}(n) = (3/\ln 2)\ln(n/100)$; the time elapsed when there are n bacteria (b) After about 26.9 hours

47. $-\infty$ **49.** 0 **51.** ∞ **53.** $(-2, 2)$

55. (a) $\left(-\infty, \frac{1}{2}\ln 3\right]$ (b) $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$, $\left[0, \sqrt{3}\right)$

57. (a) $(\ln 3, \infty)$ (b) $f^{-1}(x) = \ln(e^x + 3)$; $\mathbb{R}$

59. $y = e^{x/2} + 1$ **61.** $f^{-1}(x) = \sqrt[3]{\ln x}$

63. $y = 2 + \frac{1}{2}\log_3 x$ **65.** $\left(-\frac{1}{2}\ln 3, \infty\right)$

67. (b) $f^{-1}(x) = \frac{1}{2}(e^x - e^{-x})$

69. f is a constant function.

73. $-1 \leqslant x < 1 - \sqrt{3}$ or $1 + \sqrt{3} < x \leqslant 3$

EXERCISES 6.4 ■ PAGE 436

1. 1. The differentiation formula is simplest.

3. $f'(x) = \dfrac{\cos(\ln x)}{x}$ **5.** $f'(x) = -\dfrac{1}{x}$

7. $f'(x) = \dfrac{-\sin x}{(1 + \cos x)\ln 10}$ **9.** $g'(x) = \dfrac{1}{x} - 2$

11. $F'(t) = \ln t \left(\ln t \cos t + \dfrac{2 \sin t}{t}\right)$

13. $G'(y) = \dfrac{10}{2y+1} - \dfrac{y}{y^2+1}$ **15.** $f'(u) = \dfrac{1 + \ln 2}{u[1 + \ln(2u)]^2}$

17. $f'(x) = 5x^4 + 5^x \ln 5$ **19.** $T'(z) = 2^z\left(\dfrac{1}{z \ln 2} + \ln z\right)$

21. $y' = \dfrac{-x}{1+x}$ **23.** $y' = \sec^2[\ln(ax+b)]\dfrac{a}{ax+b}$

25. $G'(x) = -C(\ln 4)\dfrac{4^{C/x}}{x^2}$

27. $y' = (2 + \ln x)/(2\sqrt{x})$; $y'' = -\ln x/(4x\sqrt{x})$

29. $y' = \tan x$; $y'' = \sec^2 x$

31. $f'(x) = \dfrac{2x - 1 - (x-1)\ln(x-1)}{(x-1)[1 - \ln(x-1)]^2}$;
$(1, 1+e) \cup (1+e, \infty)$

33. $f'(x) = \dfrac{2(x-1)}{x(x-2)}$; $(-\infty, 0) \cup (2, \infty)$ **35.** 2

37. $y = 3x - 9$ **39.** $\cos x + 1/x$ **41.** 7

43. $y' = (x^2+2)^2(x^4+4)^4\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right)$

45. $y' = \sqrt{\dfrac{x-1}{x^4+1}}\left(\dfrac{1}{2x-2} - \dfrac{2x^3}{x^4+1}\right)$

47. $y' = x^x(1 + \ln x)$

49. $y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)$

51. $y' = (\cos x)^x(-x \tan x + \ln \cos x)$

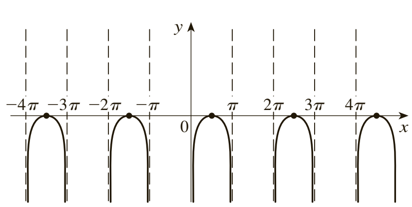
53. $y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x \tan x} - \dfrac{\ln \tan x}{x^2}\right)$

55. $y' = \dfrac{2x}{x^2 + y^2 - 2y}$ **57.** $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(x-1)^n}$

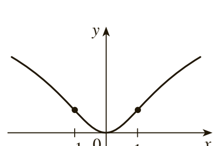
59. 2.958516, 5.290718

61. CU on $(e^{8/3}, \infty)$, CD on $(0, e^{8/3})$, IP $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$

63. A. All x in $(2n\pi, (2n+1)\pi)$ (n an integer)
B. x-int $\pi/2 + 2n\pi$ C. Period 2π D. VA $x = n\pi$
E. Inc on $(2n\pi, \pi/2 + 2n\pi)$; dec on $(\pi/2 + 2n\pi, (2n+1)\pi)$
F. Loc max $f(\pi/2 + 2n\pi) = 0$ G. CD on $(2n\pi, (2n+1)\pi)$
H.



65. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About y-axis D. None
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1), (1, \infty)$;
IP $(\pm 1, \ln 2)$ H. See graph at right.



67. Inc on (0, 2.7), (4.5, 8.2), (10.9, 14.3);
IP (3.8, 1.7), (5.7, 2.1), (10.0, 2.7), (12.0, 2.9)

69. (a) $Q = ab^t$ where $a \approx 100.01244$ and $b \approx 0.000045146$
(b) -670.63 μA

71. $3 \ln 2$ **73.** $\frac{1}{3} \ln \frac{5}{2}$ **75.** $\frac{1}{2}e^2 + e - \frac{1}{2}$

77. $\frac{1}{3}(\ln x)^3 + C$ **79.** $-\ln(1 + \cos^2 x) + C$

81. $\dfrac{15}{\ln 2}$ **85.** $\pi \ln 2$ **87.** 45,974 J **89.** $\frac{1}{3}$

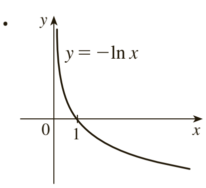
91. $0 < m < 1$; $m - 1 - \ln m$

EXERCISES 6.2* ■ PAGE 445

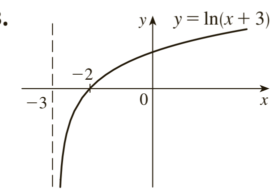
1. $\frac{1}{2} \ln a + \frac{1}{2} \ln b$ **3.** $2 \ln x - 3 \ln y - 4 \ln z$

5. $\ln \dfrac{x^2 y^3}{z}$ **7.** $\ln 250$ **9.** $\ln \dfrac{\sqrt{x}}{x+1}$

11.



13.



15. $-\infty$ **17.** $f'(x) = x^2 + 3x^2 \ln x$

19. $f'(x) = \dfrac{\cos(\ln x)}{x}$ **21.** $f'(x) = -\dfrac{1}{x}$

23. $f'(x) = \dfrac{\sin x}{x} + \cos x \ln(5x)$ **25.** $g'(x) = -\dfrac{2a}{a^2 - x^2}$

27. $G'(y) = \dfrac{10}{2y+1} - \dfrac{y}{y^2+1}$

29. $F'(t) = \ln t \left(\ln t \cos t + \dfrac{2 \sin t}{t}\right)$

31. $f'(u) = \dfrac{1 + \ln 2}{u[1 + \ln(2u)]^2}$ **33.** $y' = \dfrac{10x + 1}{5x^2 + x - 2}$

35. $y' = \sec^2[\ln(ax+b)]\dfrac{a}{ax+b}$

37. $y' = (2 + \ln x)/(2\sqrt{x})$; $y'' = -\ln x/(4x\sqrt{x})$

39. $f'(x) = \dfrac{2x - 1 - (x-1)\ln(x-1)}{(x-1)[1 - \ln(x-1)]^2}$;
$(1, 1+e) \cup (1+e, \infty)$

41. $f'(x) = -\dfrac{1}{2x\sqrt{1 - \ln x}}$; $(0, e]$

43. 2 **45.** $\cos x + 1/x$

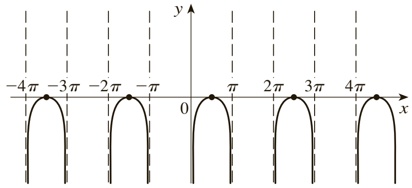
47. $y = 2x - 2$ **49.** $y' = \dfrac{2x}{x^2 + y^2 - 2y}$

51. $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(x-1)^n}$

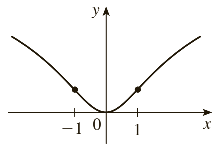
53. 2.958516, 5.290718

55. A. All x in $(2n\pi, (2n+1)\pi)$ (n an integer)
B. x-int $\pi/2 + 2n\pi$ C. Period 2π D. VA $x = n\pi$
E. Inc on $(2n\pi, \pi/2 + 2n\pi)$; dec on $(\pi/2 + 2n\pi, (2n+1)\pi)$
F. Loc max $f(\pi/2 + 2n\pi) = 0$ G. CD on $(2n\pi, (2n+1)\pi)$

H.



57. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About y-axis D. None
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$;
IP $(\pm 1, \ln 2)$ H. See graph at right.



59. Inc on (0, 2.7), (4.5, 8.2), (10.9, 14.3);
IP (3.8, 1.7), (5.7, 2.1), (10.0, 2.7), (12.0, 2.9)

61. $y' = (x^2 + 2)^2(x^4 + 4)^4\left(\dfrac{4x}{x^2 + 2} + \dfrac{16x^3}{x^4 + 4}\right)$

63. $y' = \sqrt{\dfrac{x - 1}{x^4 + 1}}\left(\dfrac{1}{2x - 2} - \dfrac{2x^3}{x^4 + 1}\right)$

65. $3 \ln 2$ **67.** $\frac{1}{3} \ln \frac{5}{2}$ **69.** $\frac{1}{2}e^2 + e - \frac{1}{2}$

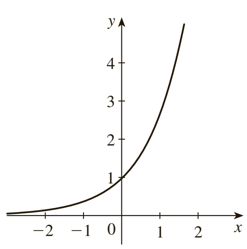
71. $\frac{1}{3}(\ln x)^3 + C$ **73.** $-\ln(1 + \cos^2 x) + C$

77. $\pi \ln 2$ **79.** 45,974 J **81.** $\frac{1}{3}$ **83.** (b) 0.405

87. $0 < m < 1$; $m - 1 - \ln m$

EXERCISES 6.3* ■ PAGE 452

1.



If $f(x) = e^x$, then $f'(0) = 1$.

3. (a) $\frac{1}{2}$ (b) 3

5. (a) $\frac{1}{4}(7 - \ln 6)$ (b) $\frac{1}{3}(e^2 + 10)$

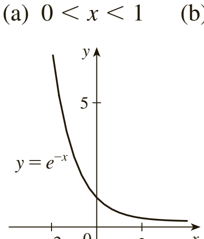
7. (a) $\frac{1}{3}(\ln k - 1)$ (b) $\frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$

9. $-\frac{1}{2}\ln(e - 1)$ **11.** $\ln 3$

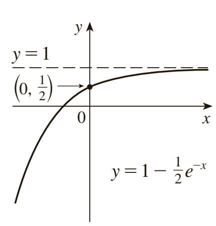
13. (a) 3.7704 (b) 0.3285

15. (a) $0 < x < 1$ (b) $x > \ln 5$

17.



19.



21. (a) $\left(-\infty, \frac{1}{2}\ln 3\right]$ (b) $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$, $\left[0, \sqrt{3}\right)$

23. $y = e^{x/2} + 1$ **25.** $f^{-1}(x) = \sqrt[3]{\ln x}$

27. 1 **29.** 0 **31.** 0

33. $f'(x) = 0$ **35.** $f'(x) = e^x(3x^2 + x - 5)$

37. $y' = 3ax^2 e^{ax^3}$ **39.** $y' = (\sec^2 \theta)\, e^{\tan \theta}$

41. $f'(x) = \dfrac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$ **43.** $y' = xe^{-3x}(2 - 3x)$

45. $f'(t) = e^{at}(b \cos bt + a \sin bt)$

47. $F'(t) = e^{t \sin 2t}(2t \cos 2t + \sin 2t)$

49. $g'(u) = ue^{\sqrt{\sec u^2}}\sqrt{\sec u^2}\,\tan u^2$

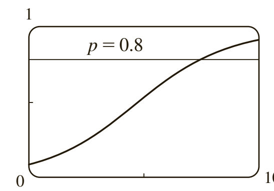
51. $y' = \dfrac{4e^{2x}}{(1 + e^{2x})^2}\sin\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$ **53.** $y = 2x + 1$

55. $y' = \dfrac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$ **59.** $-4, -2$ **61.** $f^{(n)}(x) = 2^n e^{2x}$

63. (b) -0.567143

65. (a) 1 (b) $kae^{-kt}/(1 + ae^{-kt})^2$

(c)

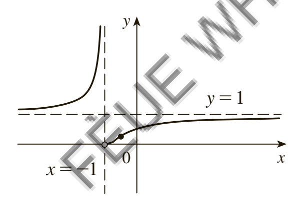


$t \approx 7.4$ h

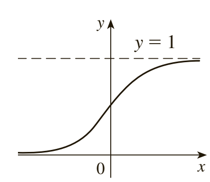
67. -1 **69.** $f(2) = 2/\sqrt{e}$, $f(-1) = -1/\sqrt[8]{e}$

71. (a) Inc on $(2, \infty)$; dec on $(-\infty, 2)$
(b) CU on $(-\infty, 3)$; CD on $(3, \infty)$ (c) $(3, -2e^{-3})$

73. A. $\{x \mid x \neq -1\}$
B. y-int $1/e$ C. None
D. HA $y = 1$; VA $x = -1$
E. Inc on $(-\infty, -1)$, $(-1, \infty)$
F. None
G. CU on $(-\infty, -1)$, $\left(-1, -\frac{1}{2}\right)$;
CD on $\left(-\frac{1}{2}, \infty\right)$; IP $\left(-\frac{1}{2}, 1/e^2\right)$
H. See graph at right.



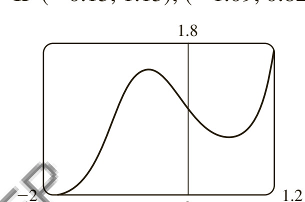
75. A. $\mathbb{R}$ B. y-int $\frac{1}{2}$ C. None
D. HA $y = 0$, $y = 1$
E. Inc on $\mathbb{R}$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $\left(0, \frac{1}{2}\right)$
H. See graph at right.



77. 28.57 min, when the rate of increase of drug level in the bloodstream is greatest; 85.71 min, when the rate of decrease is greatest

79. 0.177 mg/mL; 21.4 min

81. Loc max $f\left(-1/\sqrt{3}\right) = e^{2\sqrt{3}/9} \approx 1.5$;
loc min $f\left(1/\sqrt{3}\right) = e^{-2\sqrt{3}/9} \approx 0.7$;
IP $(-0.15, 1.15)$, $(-1.09, 0.82)$



83. $\dfrac{1}{e + 1} + e - 1$ **85.** $\dfrac{1}{\pi}(1 - e^{-2\pi})$

87. $\frac{2}{3}(1 + e^x)^{3/2} + C$ **89.** $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

91. $\dfrac{1}{1 - e^u} + C$ **93.** $e - \sqrt{e}$ **95.** 4.644

97. $\pi(e^2 - 1)/2$ **101.** ≈ 4512 L

103. $C_0(1 - e^{-30r/V})$; the total amount of urea removed from the blood in the first 30 minutes of dialysis treatment

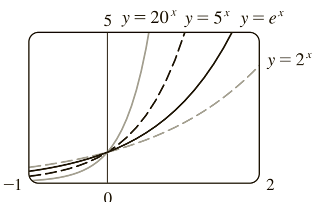
EXERCISES 6.4* ■ PAGE 463

1. (a) $b^x = e^{x \ln b}$ (b) $(-\infty, \infty)$ (c) $(0, \infty)$
(d) See Figures 1, 3, and 2.

3. $e^{-\pi \ln 4}$ **5.** $e^{x^2 \ln 10}$

7. (a) 5 (b) $\frac{1}{3}$ **9.** (a) 2 (b) $\frac{2}{3}$

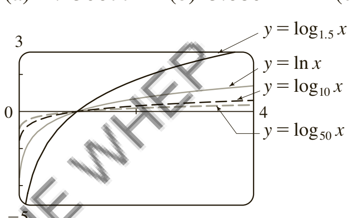
11.



All approach 0 as $x \to -\infty$, all pass through (0, 1), and all are increasing. The larger the base, the faster the rate of increase.

13. (a) 1.430677 (b) 3.680144 (c) 1.651496

15.



All graphs approach $-\infty$ as $x \to 0^+$, all pass through (1, 0), and all are increasing. The larger the base, the slower the rate of increase.

17. $f(x) = 3 \cdot 2^x$ **19.** (b) About 1,084,588 mi

21. ∞ **23.** 0 **25.** $f'(x) = 5x^4 + 5^x \ln 5$

27. $G'(x) = -C(\ln 4)\dfrac{4^{C/x}}{x^2}$ **29.** $L'(v) = 2v \ln 4 \sec^2(4^{v^2}) \cdot 4^{v^2}$

31. $f'(x) = \dfrac{3}{(3x - 1)\ln 2}$ **33.** $y' = \dfrac{x \cot x}{\ln 4} + \log_4 \sin x$

35. $y' = x^x(1 + \ln x)$

37. $y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)$

39. $y' = (\cos x)^x(-x \tan x + \ln \cos x)$

41. $y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x \tan x} - \dfrac{\ln \tan x}{x^2}\right)$

43. $y = (10 \ln 10)x + 10(1 - \ln 10)$ **45.** $\dfrac{15}{\ln 2}$

47. $(\ln x)^2/(2 \ln 10) + C$ $\left[\text{or } \frac{1}{2}(\ln 10)(\log_{10} x)^2 + C\right]$

49. $3^{\sin\theta}/\ln 3 + C$ **51.** $16/(5 \ln 5) - 1/(2 \ln 2)$

53. 0.600967 **55.** $g^{-1}(x) = \sqrt[3]{4^x - 2}$ **57.** 8.3

59. $10^8/\ln 10$ dB/(watt/m^2) **61.** 0.177 mg/mL; 21.4 min

63. 3.5 days

65. (a) $Q = ab^t$ where $a \approx 100.01244$ and $b \approx 0.000045146$
(b) -670.63 μA

EXERCISES 6.5 ■ PAGE 471

1. About 235

3. (a) $100(4.2)^t$ (b) ≈ 7409 (c) $\approx 10{,}632$ bacteria/h
(d) $(\ln 100)/(\ln 4.2) \approx 3.2$ h

5. (a) 1508 million, 1871 million (b) 2161 million
(c) 3972 million; wars in the first half of century, increased life expectancy in second half

7. (a) $Ce^{-0.0005t}$ (b) $-2000 \ln 0.9 \approx 211$ s

9. (a) $100 \times 2^{-t/30}$ mg (b) ≈ 9.92 mg (c) ≈ 199.3 years

11. ≈ 2500 years **13.** Yes; 12.5 billion years

15. (a) ≈ 137°F (b) ≈ 116 min

17. (a) $13.\overline{3}$°C (b) ≈ 67.74 min

19. (a) ≈ 64.5 kPa (b) ≈ 39.9 kPa

21. (a) (i) \$3828.84 (ii) \$3840.25 (iii) \$3850.08
(iv) \$3851.61 (v) \$3852.01 (vi) \$3852.08
(b) $dA/dt = 0.05A$, $A(0) = 3000$

EXERCISES 6.6 ■ PAGE 481

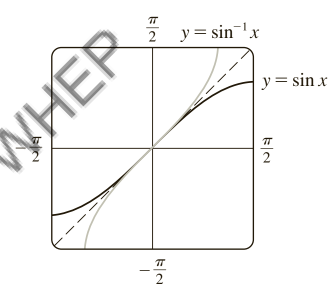
1. (a) $\pi/6$ (b) π **3.** (a) $\pi/4$ (b) $\pi/6$

5. (a) 10 (b) $-\pi/4$

7. $2/\sqrt{5}$ **9.** $\frac{119}{169}$

13. $x/\sqrt{1 + x^2}$

15.



The second graph is the reflection of the first graph about the line $y = x$

23. $y' = \dfrac{2 \tan^{-1} x}{1 + x^2}$ **25.** $y' = \dfrac{1}{\sqrt{-x^2 - x}}$

27. $y' = \sin^{-1} x$ **29.** $F'(x) = \dfrac{3}{\sqrt{x^6 - 1}} + \sec^{-1}(x^3)$

31. $y' = -\dfrac{\sin\theta}{1 + \cos^2\theta}$ **33.** $h'(t) = 0$

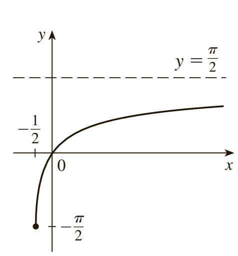
35. $y' = \dfrac{\sqrt{a^2 - b^2}}{a + b \cos x}$

37. $g'(x) = \dfrac{2}{\sqrt{1 - (3 - 2x)^2}}$; [1, 2], (1, 2) **39.** $\pi/6$

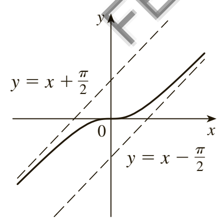
41. $1 - \dfrac{x \arcsin x}{\sqrt{1 - x^2}}$ **43.** $-\pi/2$ **45.** $\pi/2$

47. At a distance $5 - 2\sqrt{5}$ from A **49.** $\frac{1}{4}$ rad/s

51. A. $\left[-\frac{1}{2}, \infty\right)$
B. y-int 0; x-int 0
C. None
D. HA $y = \pi/2$
E. Inc on $\left(-\frac{1}{2}, \infty\right)$
F. None
G. CD on $\left(-\frac{1}{2}, \infty\right)$
H. See graph at right.



53. A. $\mathbb{R}$
B. y-int 0; x-int 0
C. About (0, 0)
D. SA $y = x \pm \pi/2$
E. Inc on $\mathbb{R}$ F. None
G. CU on $(0, \infty)$; CD on $(-\infty, 0)$; IP (0, 0)
H. See graph at right.



55. Max at $x = 0$, min at $x \approx \pm 0.87$, IP at $x \approx \pm 0.52$
57. $F(x) = 2x + 3\tan^{-1}x + C$
59. $4\pi/3$ **61.** $\pi^2/72$
63. $\tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$ **65.** $\ln|\sin^{-1}x| + C$
67. $\frac{1}{3}\sin^{-1}(t^3) + C$ **69.** $2\tan^{-1}\sqrt{x} + C$ **73.** $\pi/2 - 1$

EXERCISES 6.7 ■ PAGE 489

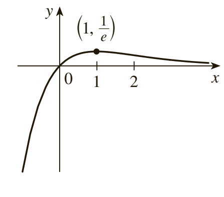
1. (a) 0 (b) 1 **3.** (a) $\frac{13}{5}$ (b) $\frac{1}{2}(e^5 + e^{-5}) \approx 74.20995$
5. (a) 1 (b) 0
21. $\operatorname{sech} x = \frac{3}{5}$, $\sinh x = \frac{4}{3}$, $\operatorname{csch} x = \frac{3}{4}$, $\tanh x = \frac{4}{5}$, $\coth x = \frac{5}{4}$
23. (a) 1 (b) -1 (c) ∞ (d) $-\infty$ (e) 0 (f) 1 (g) ∞ (h) $-\infty$ (i) 0 (j) $\frac{1}{2}$
31. $f'(x) = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$ **33.** $h'(x) = 2x\cosh(x^2)$
35. $G'(t) = \dfrac{t^2 + 1}{2t^2}$
37. $y' = 3e^{\cosh 3x}\sinh 3x$
39. $g'(t) = \coth\sqrt{t^2 + 1} - \dfrac{t^2}{\sqrt{t^2 + 1}}\operatorname{csch}^2\sqrt{t^2 + 1}$
41. $y' = \dfrac{1}{2\sqrt{x(x - 1)}}$
43. $y' = \sinh^{-1}(x/3)$ **45.** $y' = -\csc x$
51. (a) ≈ 0.3572 (b) $\approx 70.34°$
53. (a) ≈ 164.50 m (b) 120 m; ≈ 164.13 m
55. (b) $y = 2\sinh 3x - 4\cosh 3x$
57. $\left(\ln\left(1 + \sqrt{2}\right), \sqrt{2}\right)$
59. $\frac{1}{3}\cosh^3 x + C$ **61.** $2\cosh\sqrt{x} + C$ **63.** $-\operatorname{csch} x + C$
65. $\ln\left(\dfrac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right)$ **67.** $\tanh^{-1}e^x + C$
69. (a) 0, 0.48 (b) 0.04

EXERCISES 6.8 ■ PAGE 499

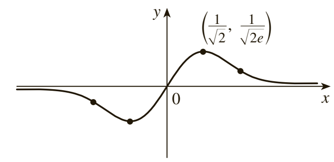
1. (a) Indeterminate (b) 0 (c) 0
(d) ∞, $-\infty$, or does not exist (e) Indeterminate
3. (a) $-\infty$ (b) Indeterminate (c) ∞
5. $\frac{9}{4}$ **7.** 1 **9.** 6 **11.** $-\frac{1}{3}$
13. $-\infty$ **15.** 2 **17.** $\frac{1}{4}$ **19.** 0 **21.** $-\infty$
23. $\frac{8}{5}$ **25.** 3 **27.** $\frac{1}{2}$ **29.** 1 **31.** 1
33. $1/\ln 3$ **35.** 0 **37.** 0 **39.** a/b
41. $\frac{1}{24}$ **43.** π **45.** $\frac{5}{3}$ **47.** 0 **49.** $-2/\pi$
51. $\frac{1}{2}$ **53.** $\frac{1}{2}$ **55.** ∞ **57.** 1 **59.** e^{-2}
61. $1/e$ **63.** 1 **65.** e^4 **67.** e^3 **69.** e^2

71. $\frac{1}{4}$ **75.** 1

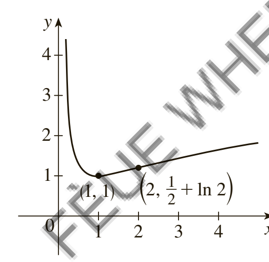
77. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. None D. HA $y = 0$
E. Inc on $(-\infty, 1)$, dec on $(1, \infty)$
F. Loc max $f(1) = 1/e$
G. CU on $(2, \infty)$; CD on $(-\infty, 2)$
IP $(2, 2/e^2)$
H. See graph at right.



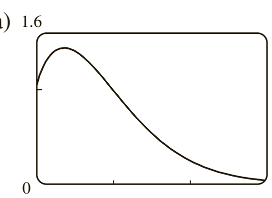
79. A. $\mathbb{R}$ B. y-int 0; x-int 0 C. About (0, 0) D. HA $y = 0$
E. Inc on $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$; dec on $\left(-\infty, -1/\sqrt{2}\right)$, $\left(1/\sqrt{2}, \infty\right)$
F. Loc min $f\left(-1/\sqrt{2}\right) = -1/\sqrt{2e}$; loc max $f\left(1/\sqrt{2}\right) = 1/\sqrt{2e}$
G. CU on $\left(-\sqrt{3/2}, 0\right)$, $\left(\sqrt{3/2}, \infty\right)$;
CD on $\left(-\infty, -\sqrt{3/2}\right)$, $\left(0, \sqrt{3/2}\right)$; IP $\left(\pm\sqrt{3/2}, \pm\sqrt{3/2}e^{-3/2}\right)$, (0, 0)
H.



81. A. $(0, \infty)$ B. None
C. None D. VA $x = 0$
E. Inc on $(1, \infty)$; dec on $(0, 1)$
F. Loc min $f(1) = 1$
G. CU on (0, 2); CD on $(2, \infty)$;
IP $\left(2, \frac{1}{2} + \ln 2\right)$
H. See graph at right.

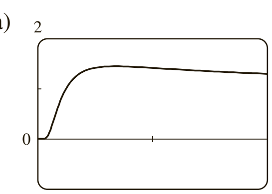


83. (a)



(b) $\lim_{x \to 0^+} x^{-x} = 1$
(c) Max value $f(1/e) = e^{1/e} \approx 1.44$ (d) 1.0

85. (a)



(b) $\lim_{x \to 0^+} x^{1/x} = 0$, $\lim_{x \to \infty} x^{1/x} = 1$
(c) Loc max $f(e) = e^{1/e}$ (d) IPs at $x \approx 0.58, 4.37$
87. f has an absolute minimum for $c > 0$. As c increases, the minimum points get farther away from the origin.
91. (a) M; the population should approach its maximum size as time increases (b) P_0e^{kt}; exponential
93. $\pi/6$ **95.** $\frac{16}{9}a$ **97.** $\frac{1}{2}$ **99.** 56 **103.** (a) 0

CHAPTER 6 REVIEW ■ PAGE 504

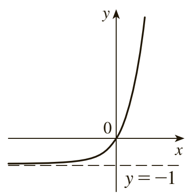
True-False Quiz

1. True **3.** False **5.** True **7.** True **9.** False
11. False **13.** False **15.** True **17.** True **19.** False

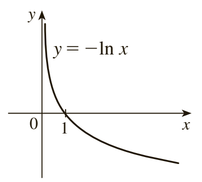
Exercises

1. No **3.** (a) 7 (b) $\frac{1}{8}$

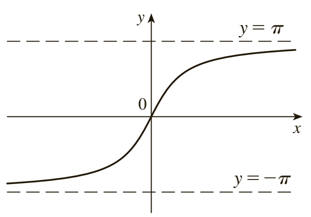
5.



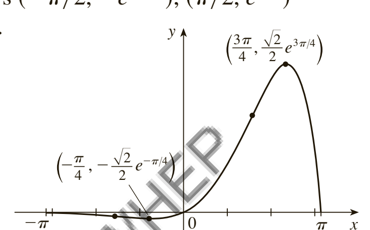
7.



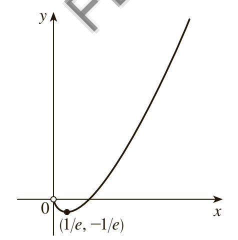
9.



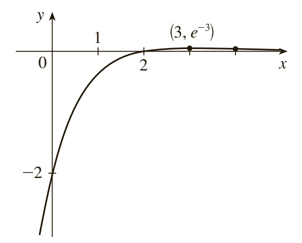
11. (a) 9 (b) 2 **13.** $e^{1/3}$
15. ln ln 17 **17.** $\sqrt{1+e}$
19. tan 1 **21.** $f'(t) = t + 2t \ln t$
23. $h'(\theta) = 2\sec^2(2\theta)e^{\tan 2\theta}$ **25.** $y' = 5 \sec 5x$
27. $y' = \dfrac{4x}{1+16x^2} + \tan^{-1}(4x)$ **29.** $y' = 2 \tan x$
31. $y' = -\dfrac{e^{1/x}(1+2x)}{x^4}$ **33.** $y' = 3^{x \ln x}(\ln 3)(1 + \ln x)$
35. $H'(v) = \dfrac{v}{1+v^2} + \tan^{-1} v$
37. $y' = 2x^2 \cosh(x^2) + \sinh(x^2)$
39. $y' = \cot x - \sin x \cos x$ **41.** $y' = -(1/x)[1 + 1/(\ln x)^2]$
43. $y' = 3 \tanh 3x$ **45.** $y' = (\cosh x)/\sqrt{\sinh^2 x - 1}$
47. $y' = \dfrac{-3\sin\left(e^{\sqrt{\tan 3x}}\right)e^{\sqrt{\tan 3x}}\sec^2(3x)}{2\sqrt{\tan 3x}}$ **49.** $e^{g(x)}g'(x)$
51. $g'(x)/g(x)$ **53.** $2^x(\ln 2)^n$ **57.** $y = -x + 2$
59. $(-3, 0)$ **61.** (a) $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$ (b) $y = ex$
63. 0 **65.** 0 **67.** $-\infty$ **69.** -1
71. 1 **73.** 4 **75.** 0 **77.** $\frac{1}{2}$
79. A. $[-\pi, \pi]$ B. y-int 0; x-int $-\pi, 0, \pi$
C. None D. None
E. Inc on $(-\pi/4, 3\pi/4)$; dec on $(-\pi, -\pi/4)$, $(3\pi/4, \pi)$
F. Loc max $f(3\pi/4) = \frac{1}{2}\sqrt{2}\,e^{3\pi/4}$,
loc min $f(-\pi/4) = -\frac{1}{2}\sqrt{2}\,e^{-\pi/4}$
G. CU on $(-\pi/2, \pi/2)$; CD on $(-\pi, -\pi/2)$, $(\pi/2, \pi)$;
IPs $(-\pi/2, -e^{-\pi/2})$, $(\pi/2, e^{\pi/2})$
H.



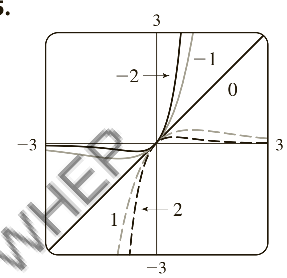
81. A. $(0, \infty)$ B. x-int 1
C. None D. None
E. Inc on $(1/e, \infty)$; dec on $(0, 1/e)$
F. Loc min $f(1/e) = -1/e$
G. CU on $(0, \infty)$
H. See graph at right.



83. A. $\mathbb{R}$
B. y-int -2; x-int 2
C. None D. HA $y = 0$
E. Inc on $(-\infty, 3)$; dec on $(3, \infty)$
F. Loc max $f(3) = e^{-3}$
G. CU on $(4, \infty)$;
CD on $(-\infty, 4)$; IP $(4, 2e^{-4})$
H. See graph at right.



85.



For $c > 0$, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.
For $c < 0$, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = 0$.
As $|c|$ increases, the max and min points and the IPs get closer to the origin.
87. $v(t) = -Ae^{-ct}[c\cos(\omega t + \delta) + \omega \sin(\omega t + \delta)]$,
$a(t) = Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t + \delta) + 2c\omega \sin(\omega t + \delta)]$
89. (a) $200(3.24)^t$ (b) $\approx 22{,}040$
(c) $\approx 25{,}910$ bacteria/h (d) $(\ln 50)/(\ln 3.24) \approx 3.33$ h
91. ≈ 4.32 days **93.** $\frac{1}{4}(1 - e^{-2})$
95. $\arctan e - \pi/4$
97. $2e^{\sqrt{x}} + C$ **99.** $\frac{1}{2}\ln|x^2 + 2x| + C$
101. $-\frac{1}{2}[\ln(\cos x)]^2 + C$ **103.** $2^{\tan\theta}/\ln 2 + C$
105. $-(1/x) - 2\ln|x| + x + C$
109. $e^{\sqrt{x}}/(2x)$
111. $\frac{1}{3}\ln 4$ **113.** $\pi^2/4$ **115.** $\frac{2}{3}$ **117.** $2/e$
121. $e^{2x}(1 + 2x)/(1 - e^{-x})$

PROBLEMS PLUS ■ PAGE 509

3. Abs max $f(-5) = e^{45}$, no abs min **9.** $1/\sqrt{2}$
11. $a = \frac{1}{2}$ **13.** e^{-2} **17.** $2\sqrt{e}$ **19.** $a \leqslant e^{1/e}$

CHAPTER 7

EXERCISES 7.1 ■ PAGE 516

1. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$ **3.** $\frac{1}{5}x \sin 5x + \frac{1}{25}\cos 5x + C$
5. $-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + C$

7. $(x^2 + 2x)\sin x + (2x + 2)\cos x - 2\sin x + C$

9. $x\cos^{-1}x - \sqrt{1 - x^2} + C$ **11.** $\frac{1}{5}t^5\ln t - \frac{1}{25}t^5 + C$

13. $-t\cot t + \ln|\sin t| + C$

15. $x(\ln x)^2 - 2x\ln x + 2x + C$

17. $\frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$

19. $z^3e^z - 3z^2e^z + 6ze^z - 6e^z + C$

21. $\dfrac{e^{2x}}{4(2x + 1)} + C$ **23.** $\dfrac{\pi - 2}{2\pi^2}$

25. $2\cosh 2 - \sinh 2$ **27.** $\frac{4}{5} - \frac{1}{5}\ln 5$ **29.** $-\pi/4$

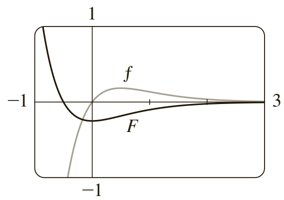
31. $2e^{-1} - 6e^{-5}$ **33.** $\frac{1}{2}\ln 2 - \frac{1}{2}$

35. $\frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$

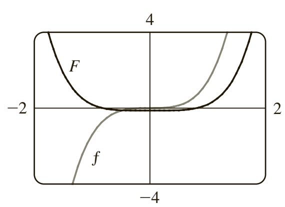
37. $2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$ **39.** $-\frac{1}{2} - \pi/4$

41. $\frac{1}{2}(x^2 - 1)\ln(1 + x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$

43. $-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$



45. $\frac{1}{3}x^2(1 + x^2)^{3/2} - \frac{2}{15}(1 + x^2)^{5/2} + C$



47. (b) $-\frac{1}{4}\cos x\sin^3x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$

49. (b) $\frac{2}{3}, \frac{8}{15}$

55. $x[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6] + C$

57. $\frac{16}{3}\ln 2 - \frac{29}{9}$ **59.** $-1.75119, 1.17210; 3.99926$

61. $4 - 8/\pi$ **63.** $2\pi e$

65. (a) $2\pi\left(2\ln 2 - \frac{3}{4}\right)$ (b) $2\pi[(\ln 2)^2 - 2\ln 2 + 1]$

67. $x\,S(x) + \frac{1}{\pi}\cos\left(\frac{1}{2}\pi x^2\right) + C$

69. $2 - e^{-t}(t^2 + 2t + 2)$ m **71.** 2

EXERCISES 7.2 ■ PAGE 524

1. $\frac{1}{3}\sin^3x - \frac{1}{5}\sin^5x + C$ **3.** $\frac{1}{120}$

5. $-\frac{1}{14}\cos^7(2t) + \frac{1}{5}\cos^5(2t) - \frac{1}{6}\cos^3(2t) + C$

7. $\pi/4$ **9.** $3\pi/8$ **11.** $\pi/16$

13. $\frac{2}{7}(\cos\theta)^{7/2} - \frac{2}{3}(\cos\theta)^{3/2} + C$

15. $\ln|\sin x| - \frac{1}{2}\sin^2x + C$ **17.** $\frac{1}{2}\sin^4x + C$

19. $\frac{1}{4}t^2 - \frac{1}{4}t\sin 2t - \frac{1}{8}\cos 2t + C$ **21.** $\frac{1}{3}\sec^3x + C$

23. $\tan x - x + C$ **25.** $\frac{1}{9}\tan^9x + \frac{2}{7}\tan^7x + \frac{1}{5}\tan^5x + C$

27. $\frac{1}{3}\sec^3x - \sec x + C$ **29.** $\frac{1}{8}\tan^8x + \frac{1}{3}\tan^6x + \frac{1}{4}\tan^4x + C$

31. $\frac{1}{4}\sec^4x - \tan^2x + \ln|\sec x| + C$

33. $x\sec x - \ln|\sec x + \tan x| + C$ **35.** $\sqrt{3} - \frac{1}{3}\pi$

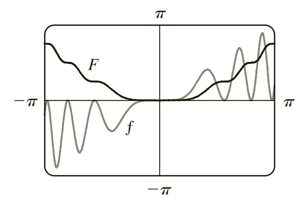
37. $\frac{22}{105}\sqrt{2} - \frac{8}{105}$ **39.** $\ln|\csc x - \cot x| + C$

41. $-\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C$ **43.** $\frac{1}{15}$

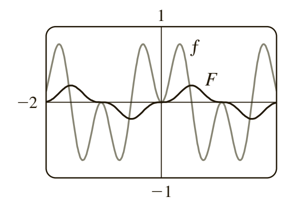
45. $\frac{1}{2}\sqrt{2}$ **47.** $\frac{1}{2}\sin 2x + C$

49. $x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$

51. $\frac{1}{4}x^2 - \frac{1}{4}\sin(x^2)\cos(x^2) + C$



53. $\frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$



55. 0 **57.** $\frac{1}{2}\pi - \frac{4}{3}$ **59.** 0 **61.** $\pi^2/4$

63. $\pi\left(2\sqrt{2} - \frac{5}{2}\right)$ **65.** $s = (1 - \cos^3\omega t)/(3\omega)$

EXERCISES 7.3 ■ PAGE 531

1. $-\dfrac{\sqrt{4 - x^2}}{4x} + C$ **3.** $\sqrt{x^2 - 4} - 2\sec^{-1}\left(\dfrac{x}{2}\right) + C$

5. $\dfrac{1}{3}\dfrac{(x^2 - 1)^{3/2}}{x^3} + C$ **7.** $\dfrac{1}{\sqrt{2}\,a^2}$

9. $\frac{2}{3}\sqrt{3} - \frac{3}{4}\sqrt{2}$ **11.** $\frac{1}{12}$

13. $\frac{1}{6}\sec^{-1}(x/3) - \sqrt{x^2 - 9}/(2x^2) + C$

15. $\frac{1}{16}\pi a^4$ **17.** $\sqrt{x^2 - 7} + C$

19. $\ln\left|\left(\sqrt{1 + x^2} - 1\right)/x\right| + \sqrt{1 + x^2} + C$ **21.** $\frac{9}{500}\pi$

23. $\ln\left|\sqrt{x^2 + 2x + 5} + x + 1\right| + C$

25. $4\sin^{-1}\left(\dfrac{x - 1}{2}\right) + \dfrac{1}{4}(x - 1)^3\sqrt{3 + 2x - x^2}$
$- \frac{2}{3}(3 + 2x - x^2)^{3/2} + C$

27. $\frac{1}{2}(x + 1)\sqrt{x^2 + 2x} - \frac{1}{2}\ln\left|x + 1 + \sqrt{x^2 + 2x}\right| + C$

29. $\frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1 - x^4} + C$

33. $\frac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$ **37.** $\frac{3}{8}\pi^2 + \frac{3}{4}\pi$

41. $2\pi^2Rr^2$ **43.** $r\sqrt{R^2 - r^2} + \pi r^2/2 - R^2\arcsin(r/R)$

EXERCISES 7.4 ■ PAGE 541

1. (a) $\dfrac{A}{1 + 2x} + \dfrac{B}{3 - x}$ (b) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{1 + x}$

3. (a) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{1 + x^2}$ (b) $1 + \dfrac{A}{x} + \dfrac{B}{x - 1} + \dfrac{C}{x - 2}$

5. (a) $x^4 + 4x^2 + 16 + \dfrac{A}{x + 2} + \dfrac{B}{x - 2}$

(b) $\dfrac{Ax + B}{x^2 - x + 1} + \dfrac{Cx + D}{x^2 + 2} + \dfrac{Ex + F}{(x^2 + 2)^2}$

7. $\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x - 1| + C$

9. $\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$ **11.** $2\ln\frac{3}{2}$
13. $a\ln|x-b| + C$ **15.** $\frac{5}{2} - \ln 2 - \ln 3$ $\left(\text{or } \frac{5}{2} - \ln 6\right)$
17. $\frac{27}{5}\ln 2 - \frac{9}{5}\ln 3$ $\left(\text{or } \frac{9}{5}\ln\frac{8}{3}\right)$
19. $\frac{1}{2} - 5\ln 2 + 3\ln 3$ $\left(\text{or } \frac{1}{2} + \ln\frac{27}{32}\right)$
21. $\frac{1}{4}\left[\ln|t+1| - \frac{1}{t+1} - \ln|t-1| - \frac{1}{t-1}\right] + C$
23. $\ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}(x/3) + C$
25. $-2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1}x + C$
27. $\frac{1}{2}\ln(x^2+1) + \tan^{-1}x - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$
29. $\frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$
31. $\frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$
33. $\frac{1}{4}\ln\frac{8}{3}$
35. $2\ln|x| + \frac{3}{2}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)} + C$
37. $\frac{7}{8}\sqrt{2}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3x-8}{4(x^2-4x+6)} + C$
39. $2\tan^{-1}\sqrt{x-1} + C$
41. $-2\ln\sqrt{x} - \frac{2}{\sqrt{x}} + 2\ln\left(\sqrt{x}+1\right) + C$
43. $\frac{3}{10}(x^2+1)^{5/3} - \frac{3}{4}(x^2+1)^{2/3} + C$
45. $2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C$
47. $\ln\left[\frac{(e^x+2)^2}{(e^x+1)}\right] + C$
49. $\ln|\tan t + 1| - \ln|\tan t + 2| + C$
51. $x - \ln(e^x+1) + C$
53. $\left(x - \frac{1}{2}\right)\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + C$
55. $-\frac{1}{2}\ln 3 \approx -0.55$
57. $\frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$ **61.** $\frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$
63. $4\ln\frac{2}{3} + 2$ **65.** $-1 + \frac{11}{3}\ln 2$
67. $t = \ln\frac{10{,}000}{P} + 11\ln\frac{P-9000}{1000}$
69. (a) $\frac{24{,}110}{4879}\frac{1}{5x+2} - \frac{668}{323}\frac{1}{2x+1} - \frac{9438}{80{,}155}\frac{1}{3x-7} + \frac{1}{260{,}015}\frac{22{,}098x+48{,}935}{x^2+x+5}$
(b) $\frac{4822}{4879}\ln|5x+2| - \frac{334}{323}\ln|2x+1| - \frac{3146}{80{,}155}\ln|3x-7| + \frac{11{,}049}{260{,}015}\ln(x^2+x+5) + \frac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\frac{2x+1}{\sqrt{19}} + C$
The CAS omits the absolute value signs and the constant of integration.
75. $\frac{1}{a^n(x-a)} - \frac{1}{a^n x} - \frac{1}{a^{n-1}x^2} - \cdots - \frac{1}{ax^n}$

EXERCISES 7.5 ■ PAGE 547

1. $-\ln(1-\sin x) + C$ **3.** $\frac{32}{3}\ln 2 - \frac{28}{9}$
5. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right) + C$ **7.** $e^{\pi/4} - e^{-\pi/4}$
9. $\frac{4}{5}\ln 2 + \frac{1}{5}\ln 3$ $\left(\text{or } \frac{1}{5}\ln 48\right)$ **11.** $\frac{1}{2}\sec^{-1}x + \frac{\sqrt{x^2-1}}{2x^2} + C$
13. $-\frac{1}{5}\cos^5 t + \frac{2}{7}\cos^7 t - \frac{1}{9}\cos^9 t + C$
15. $x\sec x - \ln|\sec x + \tan x| + C$
17. $\frac{1}{4}\pi^2$ **19.** $e^{e^x} + C$ **21.** $(x+1)\arctan\sqrt{x} - \sqrt{x} + C$
23. $\frac{4097}{45}$ **25.** $4 - \ln 4$ **27.** $x - \ln(1+e^x) + C$
29. $x\ln\left(x + \sqrt{x^2-1}\right) - \sqrt{x^2-1} + C$
31. $\sin^{-1}x - \sqrt{1-x^2} + C$
33. $2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C$
35. 0 **37.** $\frac{1}{4}$ **39.** $\ln|\sec\theta - 1| - \ln|\sec\theta| + C$
41. $\theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C$ **43.** $\frac{2}{3}\tan^{-1}(x^{3/2}) + C$
45. $-\frac{1}{3}(x^3+1)e^{-x^3} + C$
47. $\ln|x-1| - 3(x-1)^{-1} - \frac{3}{2}(x-1)^{-2} - \frac{1}{3}(x-1)^{-3} + C$
49. $\ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$ **51.** $-\ln\left|\frac{\sqrt{4x^2+1}+1}{2x}\right| + C$
53. $\frac{1}{m}x^2\cosh mx - \frac{2}{m^2}x\sinh mx + \frac{2}{m^3}\cosh mx + C$
55. $2\ln\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$
57. $\frac{3}{7}(x+c)^{7/3} - \frac{3}{4}c(x+c)^{4/3} + C$
59. $\frac{1}{32}\ln\left|\frac{x-2}{x+2}\right| - \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + C$
61. $\csc\theta - \cot\theta + C$ or $\tan(\theta/2) + C$
63. $2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C$
65. $-\tan^{-1}(\cos^2 x) + C$ **67.** $\frac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$
69. $\sqrt{2} - 2/\sqrt{3} + \ln\left(2+\sqrt{3}\right) - \ln\left(1+\sqrt{2}\right)$
71. $e^x - \ln(1+e^x) + C$
73. $-\sqrt{1-x^2} + \frac{1}{2}(\arcsin x)^2 + C$ **75.** $\ln|\ln x - 1| + C$
77. $2(x-2)\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C$
79. $\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C$
81. $2\sqrt{1+\sin x} + C$ **83.** $xe^{x^2} + C$

EXERCISES 7.6 ■ PAGE 552

1. $-\frac{5}{21}$ **3.** $\sqrt{13} - \frac{3}{4}\ln\left(4+\sqrt{13}\right) - \frac{1}{2} + \frac{3}{4}\ln 3$
5. $\frac{\pi}{8}\arctan\frac{\pi}{4} - \frac{1}{4}\ln\left(1 + \frac{1}{16}\pi^2\right)$ **7.** $\frac{1}{6}\ln\left|\frac{\sin x - 3}{\sin x + 3}\right| + C$
9. $-\frac{\sqrt{9x^2+4}}{x} + 3\ln\left(3x + \sqrt{9x^2+4}\right) + C$
11. $5\pi/16$ **13.** $2\sqrt{x}\arctan\sqrt{x} - \ln(1+x) + C$
15. $-\ln|\sinh(1/y)| + C$
17. $\frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) - \frac{1}{12}(6+4y-4y^2)^{3/2} + C$

19. $\frac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C$

21. $\dfrac{1}{2\sqrt{3}}\ln\left|\dfrac{e^x + \sqrt{3}}{e^x - \sqrt{3}}\right| + C$

23. $\frac{1}{4}\tan x\sec^3 x + \frac{3}{8}\tan x\sec x + \frac{3}{8}\ln|\sec x + \tan x| + C$

25. $\frac{1}{2}(\ln x)\sqrt{4 + (\ln x)^2} + 2\ln\left[\ln x + \sqrt{4 + (\ln x)^2}\right] + C$

27. $-\frac{1}{2}x^{-2}\cos^{-1}(x^{-2}) + \frac{1}{2}\sqrt{1 - x^{-4}} + C$

29. $\sqrt{e^{2x} - 1} - \cos^{-1}(e^{-x}) + C$

31. $\frac{1}{5}\ln|x^5 + \sqrt{x^{10} - 2}| + C$ **33.** $\frac{3}{8}\pi^2$

37. $\frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\tan x + C$

39. $\frac{1}{4}x(x^2 + 2)\sqrt{x^2 + 4} - 2\ln\left(\sqrt{x^2 + 4} + x\right) + C$

41. $\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}x + \frac{3}{8}\sin x\cos x + C$

43. $-\ln|\cos x| - \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$

45. (a) $-\ln\left|\dfrac{1 + \sqrt{1 - x^2}}{x}\right| + C$;

both have domain $(-1, 0) \cup (0, 1)$

EXERCISES 7.7 ■ PAGE 564

1. (a) $L_2 = 6$, $R_2 = 12$, $M_2 \approx 9.6$

(b) L_2 is an underestimate, R_2 and M_2 are overestimates.

(c) $T_2 = 9 < I$ (d) $L_n < T_n < I < M_n < R_n$

3. (a) $T_4 \approx 0.895759$ (underestimate)

(b) $M_4 \approx 0.908907$ (overestimate); $T_4 < I < M_4$

5. (a) $M_{10} \approx 0.806598$, $E_M \approx -0.001879$

(b) $S_{10} \approx 0.804779$, $E_S \approx -0.000060$

7. (a) 1.506361 (b) 1.518362 (c) 1.511519

9. (a) 2.660833 (b) 2.664377 (c) 2.663244

11. (a) −7.276910 (b) −4.818251 (c) −5.605350

13. (a) −2.364034 (b) −2.310690 (c) −2.346520

15. (a) 0.243747 (b) 0.243748 (c) 0.243751

17. (a) 8.814278 (b) 8.799212 (c) 8.804229

19. (a) $T_8 \approx 0.902333$, $M_8 \approx 0.905620$

(b) $|E_T| \leqslant 0.0078$, $|E_M| \leqslant 0.0039$

(c) $n = 71$ for T_n, $n = 50$ for M_n

21. (a) $T_{10} \approx 1.983524$, $E_T \approx 0.016476$;

$M_{10} \approx 2.008248$, $E_M \approx -0.008248$;

$S_{10} \approx 2.000110$, $E_S \approx -0.000110$

(b) $|E_T| \leqslant 0.025839$, $|E_M| \leqslant 0.012919$, $|E_S| \leqslant 0.000170$

(c) $n = 509$ for T_n, $n = 360$ for M_n, $n = 22$ for S_n

23. (a) 2.8 (b) 7.954926518 (c) 0.2894

(d) 7.954926521 (e) The actual error is much smaller.

(f) 10.9 (g) 7.953789422 (h) 0.0593

(i) The actual error is smaller. (j) $n \geqslant 50$

25.

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	−0.286599	−0.014771	0.007379
10	0.132218	−0.139610	−0.003696	0.001848
20	0.067033	−0.068881	−0.000924	0.000462

Observations are the same as after Example 1.

27.

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	−0.295473	0.147428	−0.003292
12	−0.074023	0.036992	−0.000206

Observations are the same as after Example 1.

29. (a) 19 (b) 18.6 (c) $18.\overline{6}$

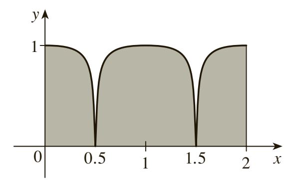
31. (a) 14.4 (b) $\frac{1}{2}$

33. 70.8°F **35.** $37.7\overline{3}$ ft/s **37.** 10,177 megawatt-hours

39. (a) 190 (b) 828

41. 28 **43.** 59.4

45.



EXERCISES 7.8 ■ PAGE 574

Abbreviations: C, convergent; D, divergent

1. (a), (d) Infinite discontinuity (b), (c) Infinite interval

3. $\frac{1}{2} - 1/(2t^2)$; 0.495, 0.49995, 0.4999995; 0.5

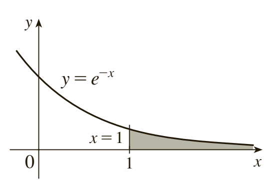
5. 2 **7.** D **9.** $\frac{1}{5}e^{-10}$ **11.** D **13.** 0 **15.** D

17. ln 2 **19.** $-\frac{1}{4}$ **21.** D **23.** $-\pi/8$ **25.** 2

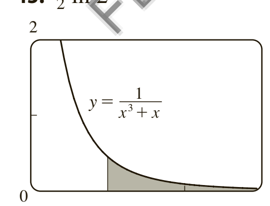
27. D **29.** $\frac{32}{3}$ **31.** D **33.** $\frac{9}{2}$ **35.** D **37.** $-\frac{1}{4}$

39. $-2/e$

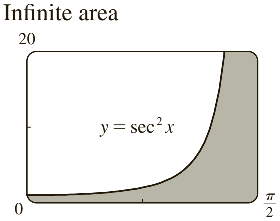
41. $1/e$



43. $\frac{1}{2}\ln 2$



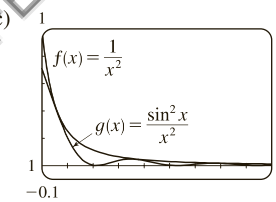
45. Infinite area

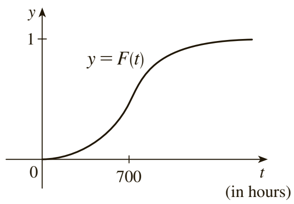


47. (a)

t	$\int_1^t [(\sin^2 x)/x^2]\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1,000	0.672957
10,000	0.673407

It appears that the integral is convergent.

(c)


49. C **51.** D **53.** D **55.** π **57.** $p < 1, 1/(1 - p)$
59. $p > -1, -1/(p + 1)^2$ **63.** π **65.** $\sqrt{2GM/R}$
67. (a)


(b) The rate at which the fraction $F(t)$ increases as t increases
(c) 1; all bulbs burn out eventually

69. $\gamma = \dfrac{cN}{\lambda(k + \lambda)}$ **71.** 1000
73. (a) $F(s) = 1/s, s > 0$ (b) $F(s) = 1/(s - 1), s > 1$
(c) $F(s) = 1/s^2, s > 0$
79. $C = 1$; ln 2 **81.** No

CHAPTER 7 REVIEW ■ PAGE 577

True-False Quiz

1. False **3.** False **5.** False **7.** False
9. (a) True (b) False **11.** False **13.** False

Exercises

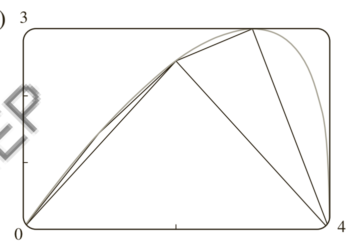
1. $\frac{7}{2} + \ln 2$ **3.** $e^{\sin x} + C$ **5.** $\ln|2t + 1| - \ln|t + 1| + C$
7. $\frac{2}{15}$ **9.** $-\cos(\ln t) + C$ **11.** $\sqrt{3} - \frac{1}{3}\pi$
13. $3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C$
15. $-\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x + 2| + C$
17. $x \sinh x - \cosh x + C$
19. $\frac{1}{18}\ln(9x^2 + 6x + 5) + \frac{1}{9}\tan^{-1}\left[\frac{1}{2}(3x + 1)\right] + C$
21. $\ln\left|x - 2 + \sqrt{x^2 - 4x}\right| + C$
23. $\ln\left|\dfrac{\sqrt{x^2 + 1} - 1}{x}\right| + C$
25. $\frac{3}{2}\ln(x^2 + 1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}(x/\sqrt{2}) + C$
27. $\frac{2}{5}$ **29.** 0 **31.** $6 - \frac{3}{2}\pi$
33. $\dfrac{x}{\sqrt{4 - x^2}} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$
35. $4\sqrt{1 + \sqrt{x}} + C$ **37.** $\frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$
39. $\frac{1}{8}e - \frac{1}{4}$ **41.** $\frac{1}{36}$ **43.** D
45. $4\ln 4 - 8$ **47.** $-\frac{4}{3}$ **49.** $\pi/4$
51. $(x + 1)\ln(x^2 + 2x + 2) + 2\arctan(x + 1) - 2x + C$
53. 0
55. $\frac{1}{4}(2x - 1)\sqrt{4x^2 - 4x - 3} - \ln\left|2x - 1 + \sqrt{4x^2 - 4x - 3}\right| + C$
57. $\frac{1}{2}\sin x\sqrt{4 + \sin^2 x} + 2\ln\left(\sin x + \sqrt{4 + \sin^2 x}\right) + C$
61. No
63. (a) 1.925444 (b) 1.920915 (c) 1.922470
65. (a) 0.01348, $n \geqslant 368$ (b) 0.00674, $n \geqslant 260$
67. 8.6 mi
69. (a) 3.8 (b) 1.7867, 0.000646 (c) $n \geqslant 30$
71. (a) D (b) C
73. 2 **75.** $\frac{3}{16}\pi^2$

PROBLEMS PLUS ■ PAGE 581

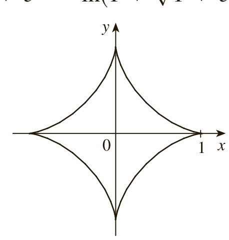
1. About 1.85 inches from the center **3.** 0
7. $f(\pi) = -\pi/2$ **11.** $(b^b a^{-a})^{1/(b-a)}e^{-1}$ **13.** $\frac{1}{8}\pi - \frac{1}{12}$
15. $2 - \sin^{-1}(2/\sqrt{5})$

CHAPTER 8

EXERCISES 8.1 ■ PAGE 588

1. $4\sqrt{5}$ **3.** 3.8202 **5.** 3.4467 **7.** 3.6095
9. $\frac{2}{243}(82\sqrt{82} - 1)$ **11.** $\frac{59}{24}$ **13.** $\frac{32}{3}$
15. $\ln(\sqrt{2} + 1)$ **17.** $\frac{3}{4} + \frac{1}{2}\ln 2$ **19.** $\ln 3 - \frac{1}{2}$
21. $\sqrt{2} + \ln(1 + \sqrt{2})$ **23.** 10.0556
25. 15.498085; 15.374568 **27.** 7.094570; 7.118819
29. (a), (b)


$L_1 = 4$,
$L_2 \approx 6.43$,
$L_4 \approx 7.50$

(c) $\int_0^4 \sqrt{1 + [4(3 - x)/(3(4 - x)^{2/3})]^2}\,dx$ (d) 7.7988
31. $\sqrt{1 + e^4} - \ln\left(1 + \sqrt{1 + e^4}\right) + 2 - \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$
33. 6


35. $s(x) = \frac{2}{27}\left[(1 + 9x)^{3/2} - 10\sqrt{10}\right]$ **37.** $2\sqrt{2}\left(\sqrt{1 + x} - 1\right)$
41. 209.1 m **43.** 29.36 in. **45.** 12.4

EXERCISES 8.2 ■ PAGE 595

1. (a) (i) $\int_0^{\pi/3} 2\pi \tan x\sqrt{1 + \sec^4 x}\,dx$
(ii) $\int_0^{\pi/3} 2\pi x\sqrt{1 + \sec^4 x}\,dx$ (b) (i) 10.5017 (ii) 7.9353
3. (a) (i) $\int_{-1}^{1} 2\pi e^{-x^2}\sqrt{1 + 4x^2 e^{-2x^2}}\,dx$
(ii) $\int_0^1 2\pi x\sqrt{1 + 4x^2 e^{-2x^2}}\,dx$ (b) (i) 11.0753 (ii) 3.9603
5. (a) (i) $\int_0^1 2\pi y\sqrt{1 + (1 + 3y^2)^2}\,dy$
(ii) $\int_0^1 2\pi(y + y^3)\sqrt{1 + (1 + 3y^2)^2}\,dy$
(b) (i) 8.5302 (ii) 13.5134
7. $\frac{1}{27}\pi(145\sqrt{145} - 1)$ **9.** $\frac{1}{6}\pi(27\sqrt{27} - 5\sqrt{5})$
11. $\pi\sqrt{5} + 4\pi\ln\left(\dfrac{1 + \sqrt{5}}{2}\right)$ **13.** $\frac{21}{2}\pi$ **15.** $\frac{3712}{15}\pi$

17. πa^2 **19.** 1,230,507 **21.** 24.145807

23. $\frac{1}{4}\pi\left[4\ln(\sqrt{17}+4) - 4\ln(\sqrt{2}+1) - \sqrt{17} + 4\sqrt{2}\right]$

25. $\frac{1}{6}\pi\left[\ln(\sqrt{10}+3) + 3\sqrt{10}\right]$

29. (a) $\frac{1}{3}\pi a^2$ (b) $\frac{56}{45}\pi\sqrt{3}\,a^2$

31. (a) $2\pi\left[b^2 + \dfrac{a^2 b \sin^{-1}\left(\sqrt{a^2-b^2}/a\right)}{\sqrt{a^2-b^2}}\right]$

(b) $2\pi a^2 + \dfrac{2\pi a b^2}{\sqrt{a^2-b^2}} \ln \dfrac{a+\sqrt{a^2-b^2}}{b}$

33. $\int_a^b 2\pi[c - f(x)]\sqrt{1+[f'(x)]^2}\,dx$ **35.** $4\pi^2 r^2$

37. Both equal $\pi\int_a^b \left(e^{x/2} + e^{-x/2}\right)^2 dx$.

EXERCISES 8.3 ■ PAGE 605

1. (a) 187.5 lb/ft^2 (b) 1875 lb (c) 562.5 lb

3. 7000 lb **5.** 2.36×10^7 N **7.** 9.8×10^3 N

9. 889 lb **11.** $\frac{2}{3}\delta a h^2$ **13.** 5.27×10^5 N

15. (a) 314 N (b) 353 N

17. (a) 5.63×10^3 lb (b) 5.06×10^4 lb
(c) 4.88×10^4 lb (d) 3.03×10^5 lb

19. 4148 lb **21.** 330; 22

23. 10; 14; (1.4, 1) **25.** $\left(\frac{2}{3}, \frac{2}{3}\right)$

27. $\left(\dfrac{1}{e-1}, \dfrac{e+1}{4}\right)$ **29.** $\left(\frac{9}{20}, \frac{9}{20}\right)$

31. $\left(\dfrac{\pi\sqrt{2}-4}{4(\sqrt{2}-1)}, \dfrac{1}{4(\sqrt{2}-1)}\right)$ **33.** $\left(\frac{8}{5}, -\frac{1}{2}\right)$

35. $\left(\dfrac{28}{3(\pi+2)}, \dfrac{10}{3(\pi+2)}\right)$ **37.** $\left(-\frac{1}{5}, -\frac{12}{35}\right)$

41. $\left(0, \frac{1}{12}\right)$ **45.** $\frac{1}{3}\pi r^2 h$ **47.** $\left(\dfrac{8}{\pi}, \dfrac{8}{\pi}\right)$

49. $4\pi^2 rR$

EXERCISES 8.4 ■ PAGE 612

1. \$21,104 **3.** \$140,000; \$60,000 **5.** \$407.25

7. \$166,666.67 **9.** (a) 3800 (b) \$324,900

11. 3727; \$37,753 **13.** $\frac{2}{3}(16\sqrt{2} - 8) \approx$ \$9.75 million

15. \$65,230.48 **17.** $\dfrac{(1-k)(b^{2-k} - a^{2-k})}{(2-k)(b^{1-k} - a^{1-k})}$

19. 1.19×10^{-4} cm^3/s **21.** 6.59 L/min **23.** 5.77 L/min

EXERCISES 8.5 ■ PAGE 619

1. (a) The probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles
(b) The probability that a randomly chosen tire will have a lifetime of at least 25,000 miles

3. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) $\frac{17}{81}$

5. (a) $1/\pi$ (b) $\frac{1}{2}$

7. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) 5

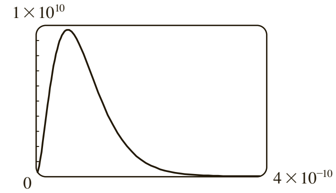
11. (a) $\approx$46.5% (b) $\approx$15.3% (c) About 4.8 s

13. $\approx$59.4% (b) 40 min **15.** $\approx$44%

17. (a) 0.0668 (b) $\approx$5.21% **19.** $\approx$0.9545

21. (b) 0; a_0

(c)



(d) $1 - 41e^{-8} \approx 0.986$ (e) $\frac{3}{2}a_0$

CHAPTER 8 REVIEW ■ PAGE 621

Exercises

1. $\frac{1}{54}\left(109\sqrt{109} - 1\right)$ **3.** $\frac{53}{6}$

5. (a) 3.5121 (b) 22.1391 (c) 29.8522

7. 3.8202 **9.** $\frac{124}{5}$ **11.** $\approx$458 lb **13.** $\left(\frac{8}{5}, 1\right)$

15. $\left(\frac{4}{3}, \frac{4}{3}\right)$ **17.** $2\pi^2$ **19.** \$7166.67

21. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$
(b) $\approx$0.3455 (c) 5; yes

23. (a) $1 - e^{-3/8} \approx 0.31$ (b) $e^{-5/4} \approx 0.29$
(c) $8\ln 2 \approx 5.55$ min

PROBLEMS PLUS ■ PAGE 623

1. $\frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$

3. (a) $2\pi r(r \pm d)$ (b) $\approx 3.36 \times 10^6$ mi^2
(d) $\approx 7.84 \times 10^7$ mi^2

5. (a) $P(z) = P_0 + g\int_0^z \rho(x)\,dx$
(b) $(P_0 - \rho_0 g H)(\pi r^2) + \rho_0 g H e^{L/H} \int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\,dx$

7. Height $\sqrt{2}\,b$, volume $\left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$ **9.** 0.14 m

11. $2/\pi$; $1/\pi$ **13.** $(0, -1)$

CHAPTER 9

EXERCISES 9.1 ■ PAGE 630

3. (a) $\frac{1}{2}, -1$ **5.** (d)

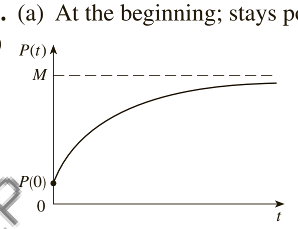
7. (a) It must be either 0 or decreasing
(c) $y = 0$ (d) $y = 1/(x + 2)$

9. (a) $0 < P < 4200$ (b) $P > 4200$
(c) $P = 0$, $P = 4200$

13. (a) III (b) I (c) IV (d) II

15. (a) At the beginning; stays positive, but decreases

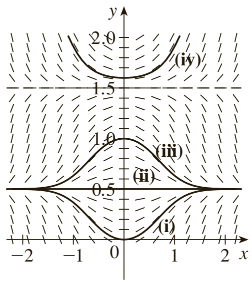
(c)



17. It approaches 0 as c approaches c_s.

EXERCISES 9.2 ■ PAGE 637

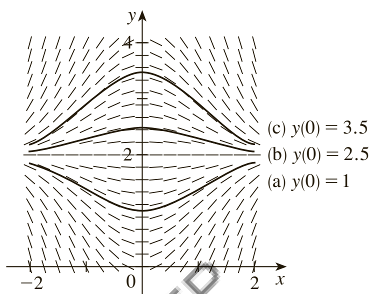
1. (a)



(b) $y = 0.5$, $y = 1.5$

3. III **5.** IV

7.



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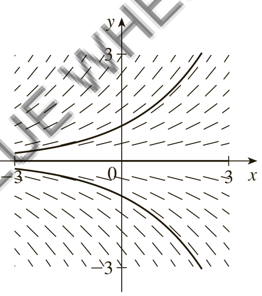

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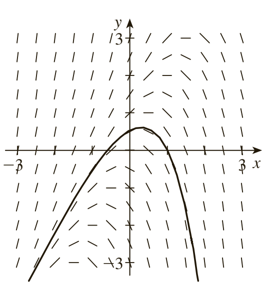

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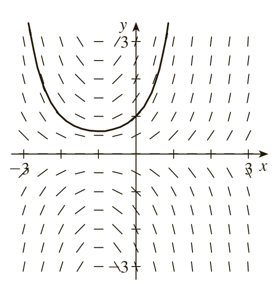

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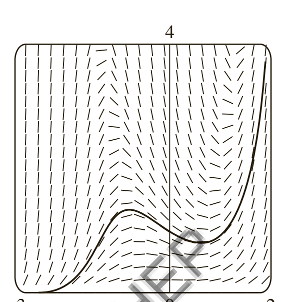
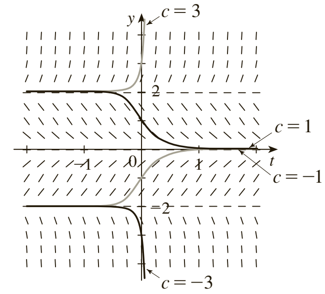


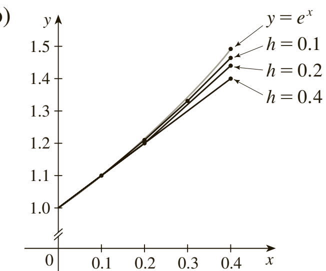
17.



$-2 \leqslant c \leqslant 2$; $-2, 0, 2$

19. (a) (i) 1.4 (ii) 1.44 (iii) 1.4641

(b)



Underestimates

(c) (i) 0.0918 (ii) 0.0518 (iii) 0.0277

It appears that the error is also halved (approximately).

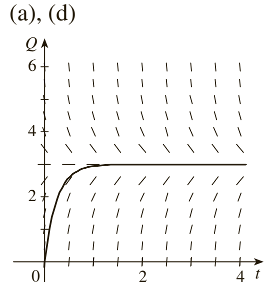
21. $-1, -3, -6.5, -12.25$ **23.** 1.7616

25. (a) (i) 3 (ii) 2.3928 (iii) 2.3701 (iv) 2.3681

(c) (i) -0.6321 (ii) -0.0249 (iii) -0.0022 (iv) -0.0002

It appears that the error is also divided by 10 (approximately).

27. (a), (d)



(b) 3

(c) Yes, $Q = 3$

(e) 2.77 C

EXERCISES 9.3 ■ PAGE 645

1. $y = -1/(x^3 + C)$, $y = 0$

3. $y = \pm\sqrt{x^2 + 2\ln|x| + C}$

5. $e^y - y = 2x + \sin x + C$

7. $\theta \sin\theta + \cos\theta = -\frac{1}{2}e^{-t^2} + C$ **9.** $p = Ke^{t^3/3 - t} - 1$

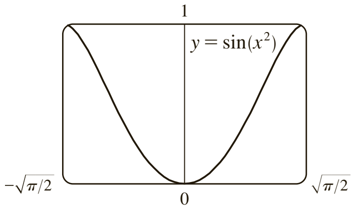
11. $y = -\ln\left(1 - \frac{1}{2}x^2\right)$ **13.** $u = -\sqrt{t^2 + \tan t + 25}$

15. $\frac{1}{2}y^2 + \frac{1}{3}(3 + y^2)^{3/2} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \frac{41}{12}$

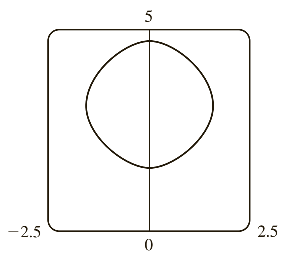
17. $y = \dfrac{4a}{\sqrt{3}}\sin x - a$

19. $y = \sqrt{x^2 + 4}$ **21.** $y = Ke^x - x - 1$

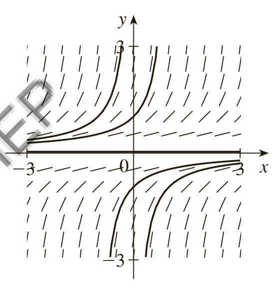
23. (a) $\sin^{-1}y = x^2 + C$
(b) $y = \sin(x^2)$, $-\sqrt{\pi/2} \leqslant x \leqslant \sqrt{\pi/2}$ (c) No



25. $\cos y = \cos x - 1$

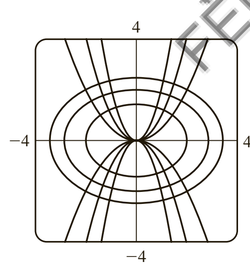


27. (a)

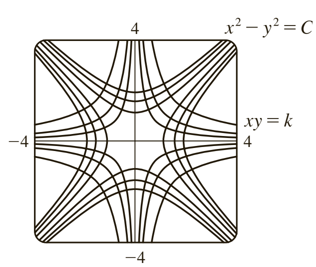


(b) $y = \dfrac{1}{K - x}$

29. $y = Cx^2$



31. $x^2 - y^2 = C$



33. $y = 1 + e^{2 - x^2/2}$ **35.** $y = \left(\frac{1}{2}x^2 + 2\right)^2$

37. $Q(t) = 3 - 3e^{-4t}$; 3 **39.** $P(t) = M - Me^{-kt}$; M

41. (a) $x = a - \dfrac{4}{(kt + 2/\sqrt{a})^2}$

(b) $t = \dfrac{2}{k\sqrt{a - b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a - b}} - \tan^{-1}\sqrt{\dfrac{b - x}{a - b}}\right)$

43. (a) $C(t) = (C_0 - r/k)e^{-kt} + r/k$ (b) r/k; the concentration approaches r/k regardless of the value of C_0

45. (a) $15e^{-t/100}$ kg (b) $15e^{-0.2} \approx 12.3$ kg

47. About 4.9% **49.** g/k

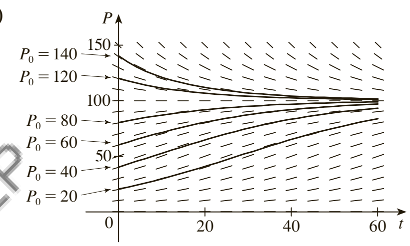
51. (a) $L_1 = KL_2^k$ (b) $B = KV^{0.0794}$

53. (a) $dA/dt = k\sqrt{A}\,(M - A)$ (b) $A(t) = M\left(\dfrac{Ce^{\sqrt{M}\,kt} - 1}{Ce^{\sqrt{M}\,kt} + 1}\right)^2$,
where $C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$ and $A_0 = A(0)$

EXERCISES 9.4 ■ PAGE 657

1. (a) 1200; 0.04 (b) $P(t) = \dfrac{1200}{1 + 19e^{-0.04t}}$ (c) 87

3. (a) 100; 0.05 (b) Where P is close to 0 or 100; on the line $P = 50$; $0 < P_0 < 100$; $P_0 > 100$
(c)



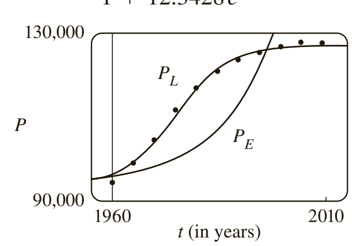
Solutions approach 100; some increase and some decrease, some have an inflection point but others don't; solutions with $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$
(d) $P = 0$, $P = 100$; other solutions move away from $P = 0$ and toward $P = 100$

5. (a) 3.23×10^7 kg (b) $\approx$1.55 years **7.** 9000

9. (a) $\dfrac{dP}{dt} = \dfrac{1}{305}P\left(1 - \dfrac{P}{20}\right)$
(b) 6.24 billion (c) 7.57 billion; 13.87 billion

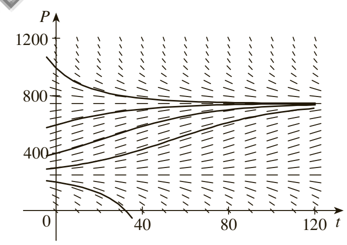
11. (a) $dy/dt = ky(1 - y)$ (b) $y = \dfrac{y_0}{y_0 + (1 - y_0)e^{-kt}}$
(c) 3:36 PM

15. $P_E(t) = 1909.7761\,(1.0796)^t + 94{,}000$;
$P_L(t) = \dfrac{33{,}086.4394}{1 + 12.3428\,e^{-0.1657t}} + 94{,}000$



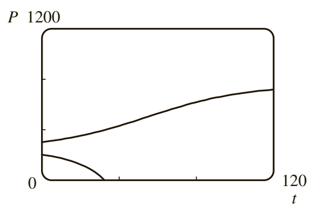
17. (a) $P(t) = \dfrac{m}{k} + \left(P_0 - \dfrac{m}{k}\right)e^{kt}$ (b) $m < kP_0$
(c) $m = kP_0$, $m > kP_0$ (d) Declining

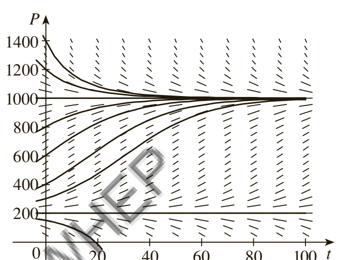
19. (a) Fish are caught at a rate of 15 per week.
(b) See part (d). (c) $P = 250$, $P = 750$

(d) 

$0 < P_0 < 250$: $P \to 0$;
$P_0 = 250$: $P \to 250$;
$P_0 > 250$: $P \to 750$

(e) $P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$
where $k = \frac{1}{11}, -\frac{1}{9}$



21. (b) 

$0 < P_0 < 200$: $P \to 0$
;
$P_0 = 200$: $P \to 200$;
$P_0 > 200$: $P \to 1000$

(c) $P(t) = \dfrac{m(M - P_0) + M(P_0 - m)e^{(M-m)(k/M)t}}{M - P_0 + (P_0 - m)e^{(M-m)(k/M)t}}$

23. (a) $P(t) = P_0 e^{(k/r)[\sin(rt - \phi) + \sin\phi]}$ (b) Does not exist

EXERCISES 9.5 ■ PAGE 665

1. No **3.** Yes **5.** $y = 1 + Ce^{-x}$

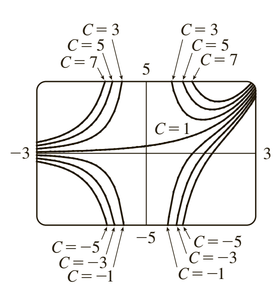
7. $y = x - 1 + Ce^{-x}$ **9.** $y = \frac{2}{3}\sqrt{x} + C/x$

11. $y = x^2(\ln x + C)$ **13.** $y = \frac{1}{3}t^{-3}(1 + t^2)^{3/2} + Ct^{-3}$

15. $y = \dfrac{1}{x}\ln x - \dfrac{1}{x} + \dfrac{3}{x^2}$ **17.** $u = -t^2 + t^3$

19. $y = -x\cos x - x$

21. $y = \dfrac{(x-1)e^x + C}{x^2}$

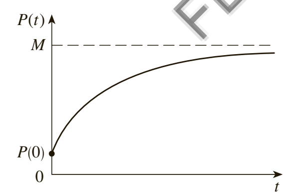


25. $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$

27. (a) $I(t) = 4 - 4e^{-5t}$ (b) $4 - 4e^{-1/2} \approx 1.57$ A

29. $Q(t) = 3(1 - e^{-4t})$, $I(t) = 12e^{-4t}$

31. $P(t) = M + Ce^{-kt}$



33. $y = \frac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}$; 0.2275 kg/L

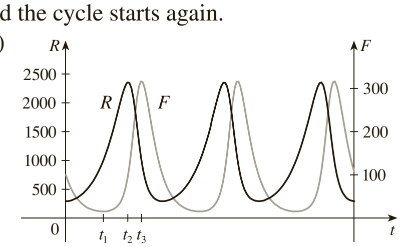
35. (b) mg/c (c) $(mg/c)[t + (m/c)e^{-ct/m}] - m^2g/c^2$

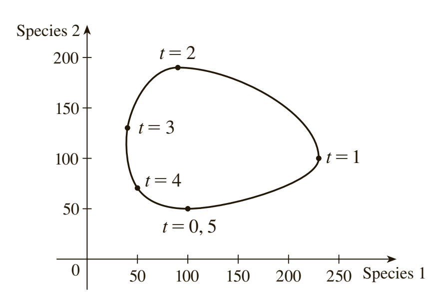
37. (b) $P(t) = \dfrac{M}{1 + MCe^{-kt}}$

EXERCISES 9.6 ■ PAGE 671

1. (a) x = predators, y = prey; growth is restricted only by predators, which feed only on prey.
(b) x = prey, y = predators; growth is restricted by carrying capacity and by predators, which feed only on prey.

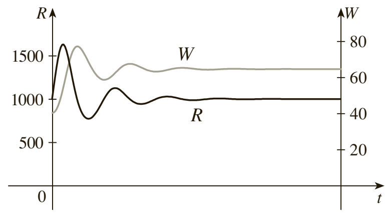
3. (a) Competition
(b) (i) $x = 0$, $y = 0$: zero populations
(ii) $x = 0$, $y = 400$: In the absence of an x-population, the y-population stabilizes at 400.
(iii) $x = 125$, $y = 0$: In the absence of a y-population, the x-population stabilizes at 125.
(iv) $x = 50$, $y = 300$: Both populations are stable.

5. (a) The rabbit population starts at about 300, increases to 2400, then decreases back to 300. The fox population starts at 100, decreases to about 20, increases to about 315, decreases to 100, and the cycle starts again.
(b) 

7. 

11. (a) Population stabilizes at 5000.
(b) (i) $W = 0$, $R = 0$: Zero populations
(ii) $W = 0$, $R = 5000$: In the absence of wolves, the rabbit population is always 5000.
(iii) $W = 64$, $R = 1000$: Both populations are stable.
(c) The populations stabilize at 1000 rabbits and 64 wolves.

(d)



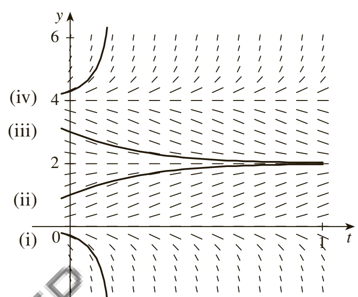
CHAPTER 9 REVIEW ■ PAGE 674

True-False Quiz

1. True **3.** False **5.** True **7.** True

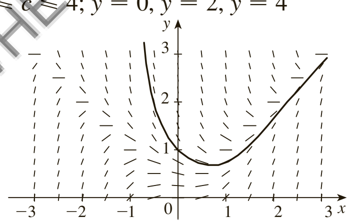
Exercises

1. (a)



(b) $0 \leqslant c \leqslant 4$; $y = 0$, $y = 2$, $y = 4$

3. (a)



$y(0.3) \approx 0.8$

(b) 0.75676

(c) $y = x$ and $y = -x$; there is a loc max or loc min

5. $y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$

7. $y = \pm\sqrt{\ln(x^2 + 2x^{3/2} + C)}$

9. $r(t) = 5e^{t - t^2}$ **11.** $y = \frac{1}{2}x(\ln x)^2 + 2x$

13. $x = C - \frac{1}{2}y^2$

15. (a) $P(t) = \dfrac{2000}{1 + 19e^{-0.1t}}$; ≈ 560 (b) $t = -10 \ln \frac{2}{57} \approx 33.5$

17. (a) $L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$ (b) $L(t) = 53 - 43e^{-0.2t}$

19. 15 days **21.** $k \ln h + h = (-R/V)t + C$

23. (a) Stabilizes at 200,000

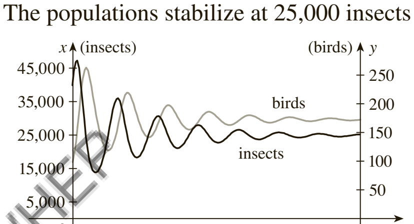
(b) (i) $x = 0$, $y = 0$: Zero populations

(ii) $x = 200{,}000$, $y = 0$: In the absence of birds, the insect population is always 200,000.

(iii) $x = 25{,}000$, $y = 175$: Both populations are stable.

(c) The populations stabilize at 25,000 insects and 175 birds.

(d)



PROBLEMS PLUS ■ PAGE 677

1. $f(x) = \pm 10e^x$ **5.** $y = x^{1/n}$ **7.** 20°C

9. (b) $f(x) = \dfrac{x^2 - L^2}{4L} - \dfrac{L}{2}\ln\left(\dfrac{x}{L}\right)$ (c) No

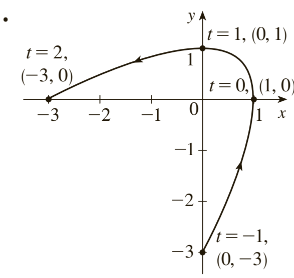
11. (a) 9.8 h (b) $31{,}900\pi$ ft^2; 2000π ft^2/h (c) 5.1 h

13. $x^2 + (y - 6)^2 = 25$ **15.** $y = K/x$, $K \neq 0$

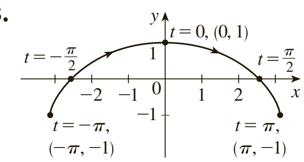
CHAPTER 10

EXERCISES 10.1 ■ PAGE 685

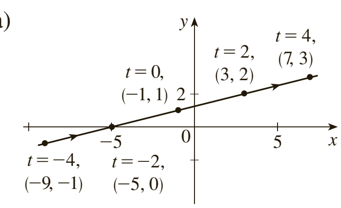
1.



3.

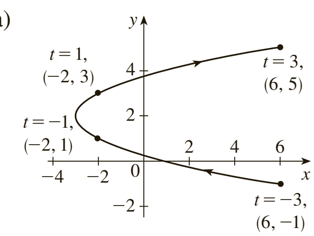


5. (a)



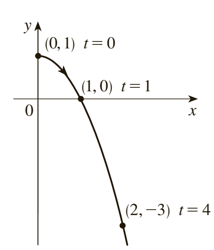
(b) $y = \frac{1}{4}x + \frac{5}{4}$

7. (a)



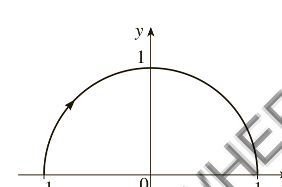
(b) $x = y^2 - 4y + 1$, $-1 \leqslant y \leqslant 5$

9. (a)

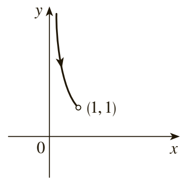


(b) $y = 1 - x^2, x \geqslant 0$

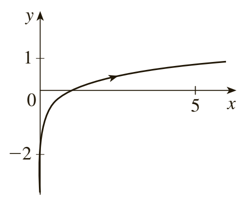
11. (a) $x^2 + y^2 = 1, y \geqslant 0$ (b)



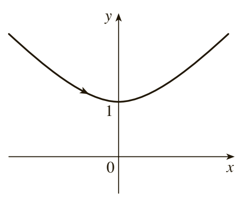
13. (a) $y = 1/x,\ y > 1$ (b)



15. (a) $x = e^{2y}$ (b)



17. (a) $y^2 - x^2 = 1,\ y \geqslant 1$ (b)

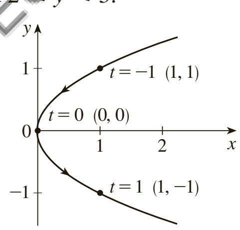


19. Moves counterclockwise along the circle $\left(\dfrac{x-5}{2}\right)^2 + \left(\dfrac{y-3}{2}\right)^2 = 1$ from (3, 3) to (7, 3)

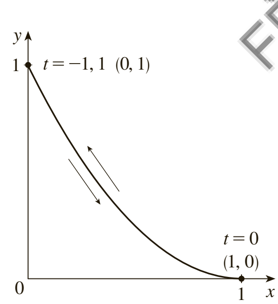
21. Moves 3 times clockwise around the ellipse $(x^2/25) + (y^2/4) = 1$, starting and ending at $(0, -2)$

23. It is contained in the rectangle described by $1 \leqslant x \leqslant 4$ and $2 \leqslant y \leqslant 3$.

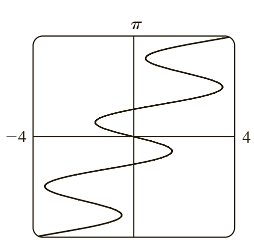
25.



27.



29.



31. (b) $x = -2 + 5t,\ y = 7 - 8t,\ 0 \leqslant t \leqslant 1$

33. (a) $x = 2\cos t,\ y = 1 - 2\sin t,\ 0 \leqslant t \leqslant 2\pi$

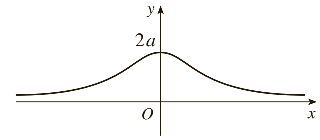
(b) $x = 2\cos t,\ y = 1 + 2\sin t,\ 0 \leqslant t \leqslant 6\pi$

(c) $x = 2\cos t,\ y = 1 + 2\sin t,\ \pi/2 \leqslant t \leqslant 3\pi/2$

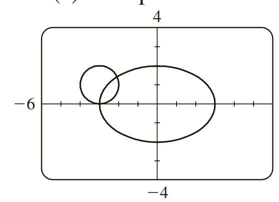
37. The curve $y = x^{2/3}$ is generated in (a). In (b), only the portion with $x \geqslant 0$ is generated, and in (c) we get only the portion with $x > 0$.

41. $x = a\cos\theta,\ y = b\sin\theta;\ (x^2/a^2) + (y^2/b^2) = 1$, ellipse

43.



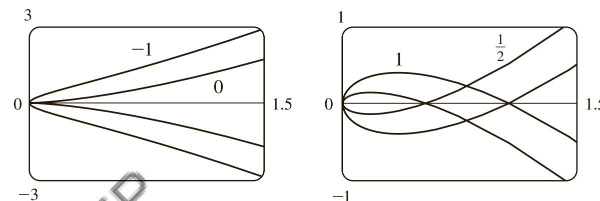
45. (a) Two points of intersection



(b) One collision point at $(-3, 0)$ when $t = 3\pi/2$

(c) There are still two intersection points, but no collision point.

47. For $c = 0$, there is a cusp; for $c > 0$, there is a loop whose size increases as c increases.



49. The curves roughly follow the line $y = x$, and they start having loops when a is between 1.4 and 1.6 (more precisely, when $a > \sqrt{2}$). The loops increase in size as a increases.

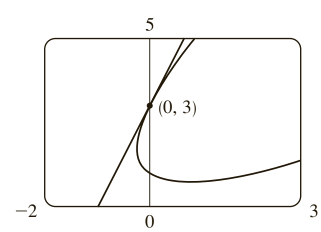
51. As n increases, the number of oscillations increases; a and b determine the width and height.

EXERCISES 10.2 ■ PAGE 695

1. $\frac{1}{2}(1 + t)^{3/2}$ **3.** $y = -x$ **5.** $y = \pi x + \pi^2$

7. $y = 2x + 1$

9. $y = 3x + 3$



11. $\dfrac{2t+1}{2t},\ -\dfrac{1}{4t^3},\ t < 0$ **13.** $e^{-2t}(1 - t),\ e^{-3t}(2t - 3),\ t > \frac{3}{2}$

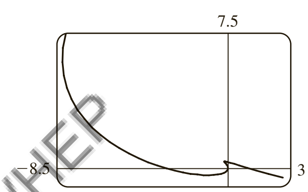
15. $\dfrac{t+1}{t-1},\ \dfrac{-2t}{(t-1)^3},\ 0 < t < 1$

17. Horizontal at $(0, -3)$, vertical at $(\pm 2, -2)$

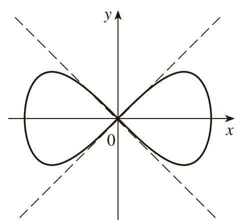
19. Horizontal at $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 1\right)$, no vertical

21. $(0.6, 2)$; $\left(5 \cdot 6^{-6/5},\ e^{6^{-1/5}}\right)$

23.



25. $y = x$, $y = -x$



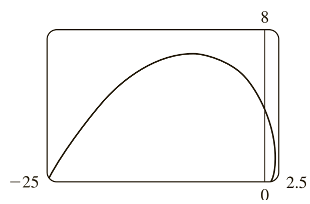
27. (a) $d \sin\theta/(r - d\cos\theta)$ **29.** $(4, 0)$ **31.** πab

33. $\frac{24}{5}$ **35.** $2\pi r^2 + \pi d^2$ **37.** $\int_0^2 \sqrt{2 + 2e^{-2t}}\,dt \approx 3.1416$

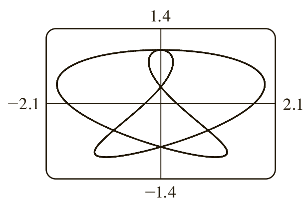
39. $\int_0^{4\pi} \sqrt{5 - 4\cos t}\,dt \approx 26.7298$ **41.** $4\sqrt{2} - 2$

43. $\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1 + \sqrt{2})$

45. $\sqrt{2}\,(e^{\pi} - 1)$

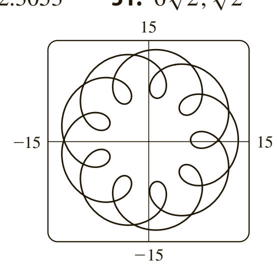


47. 16.7102



49. 612.3053 **51.** $6\sqrt{2}, \sqrt{2}$

55. (a) $t \in [0, 4\pi]$



(b) 294

57. $\int_0^{\pi/2} 2\pi t \cos t \sqrt{t^2 + 1}\,dt \approx 4.7394$

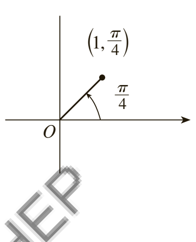
59. $\int_0^1 2\pi e^{-t}\sqrt{1 + 2e^t + e^{2t} + e^{-2t}}\,dt \approx 10.6705$

61. $\frac{2}{1215}\pi(247\sqrt{3} + 64)$ **63.** $\frac{6}{5}\pi a^2$

65. $\frac{24}{5}\pi(949\sqrt{26} + 1)$ **71.** $\frac{1}{4}$

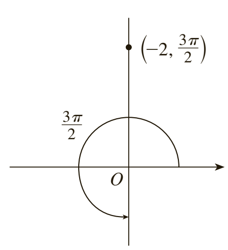
EXERCISES 10.3 ■ PAGE 706

1. (a)



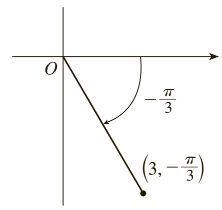
$(1, 9\pi/4)$, $(-1, 5\pi/4)$

(b)



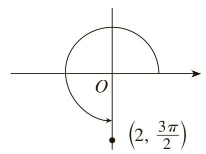
$(2, \pi/2)$, $(-2, 7\pi/2)$

(c)



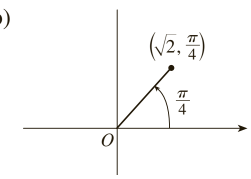
$(3, 5\pi/3)$, $(-3, 2\pi/3)$

3. (a)



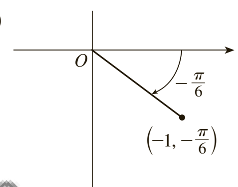
$(0, -2)$

(b)



$(1, 1)$

(c)

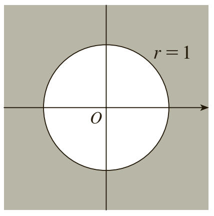


$(-\sqrt{3}/2, 1/2)$

5. (a) (i) $(4\sqrt{2}, 3\pi/4)$ (ii) $(-4\sqrt{2}, 7\pi/4)$

(b) (i) $(6, \pi/3)$ (ii) $(-6, 4\pi/3)$

7.



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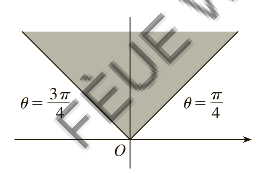

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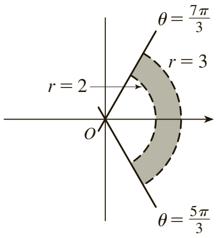


13. $2\sqrt{7}$ **15.** Circle, center O, radius $\sqrt{5}$

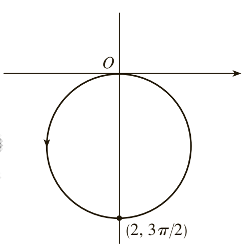
17. Circle, center $(5/2, 0)$, radius $5/2$

19. Hyperbola, center O, foci on x-axis

21. $r = 2\csc\theta$ **23.** $r = 1/(\sin\theta - 3\cos\theta)$

25. $r = 2c\cos\theta$ **27.** (a) $\theta = \pi/6$ (b) $x = 3$

29.



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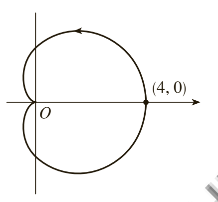

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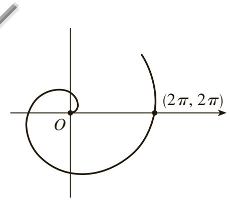

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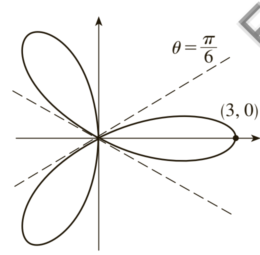

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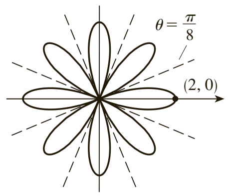

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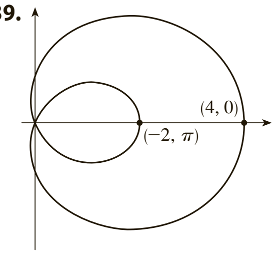

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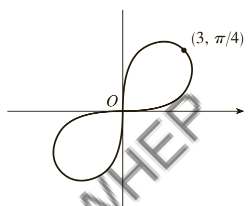

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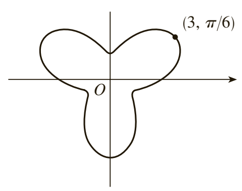

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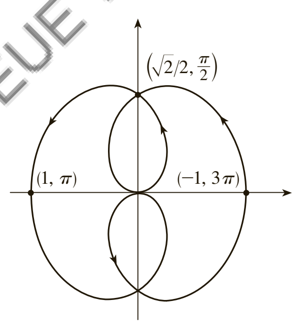

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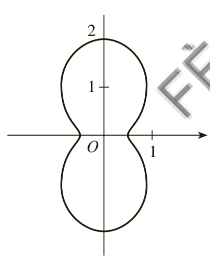

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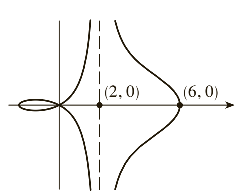

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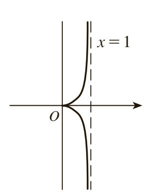


53. (a) For $c < -1$, the inner loop begins at $\theta = \sin^{-1}(-1/c)$ and ends at $\theta = \pi - \sin^{-1}(-1/c)$; for $c > 1$, it begins at $\theta = \pi + \sin^{-1}(1/c)$ and ends at $\theta = 2\pi - \sin^{-1}(1/c)$.

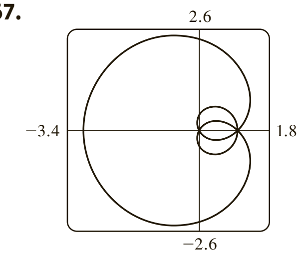
55. $1/\sqrt{3}$ **57.** $-\pi$ **59.** 1

61. Horizontal at $(3/\sqrt{2}, \pi/4)$, $(-3/\sqrt{2}, 3\pi/4)$; vertical at $(3, 0)$, $(0, \pi/2)$

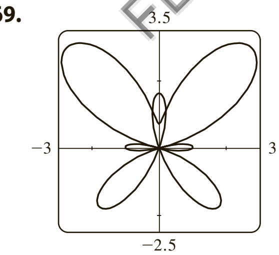
63. Horizontal at $(\frac{3}{2}, \pi/3)$, $(0, \pi)$ [the pole], and $(\frac{3}{2}, 5\pi/3)$; vertical at $(2, 0)$, $(\frac{1}{2}, 2\pi/3)$, $(\frac{1}{2}, 4\pi/3)$

65. Center $(b/2, a/2)$, radius $\sqrt{a^2 + b^2}/2$

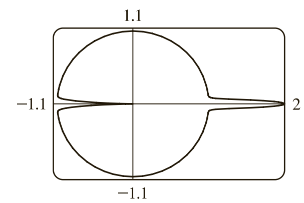
67.



69.



71.



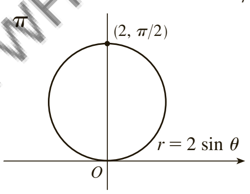
73. By counterclockwise rotation through angle $\pi/6$, $\pi/3$, or α about the origin

75. For $c = 0$, the curve is a circle. As c increases, the left side gets flatter, then has a dimple for $0.5 < c < 1$, a cusp for $c = 1$, and a loop for $c > 1$.

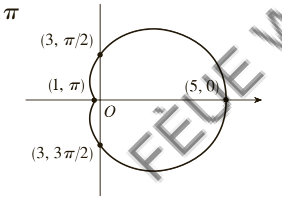
EXERCISES 10.4 ■ PAGE 712

1. $e^{-\pi/4} - e^{-\pi/2}$ **3.** $\pi/2$ **5.** $\frac{1}{2}$ **7.** $\frac{41}{4}\pi$

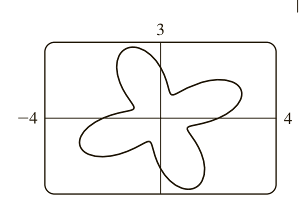
9. π



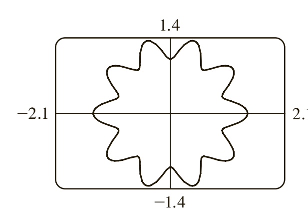
11. 11π



13. $\frac{9}{2}\pi$



15. $\frac{3}{2}\pi$



17. $\frac{4}{3}\pi$ **19.** $\frac{1}{16}\pi$ **21.** $\pi - \frac{3}{2}\sqrt{3}$ **23.** $\frac{4}{3}\pi + 2\sqrt{3}$

25. $4\sqrt{3} - \frac{4}{3}\pi$ **27.** π **29.** $\frac{9}{8}\pi - \frac{9}{4}$ **31.** $\frac{1}{2}\pi - 1$

33. $-\sqrt{3} + 2 + \frac{1}{3}\pi$ **35.** $\frac{1}{4}(\pi + 3\sqrt{3})$

37. $(\frac{1}{2}, \pi/6)$, $(\frac{1}{2}, 5\pi/6)$, and the pole

39. $(1, \theta)$ where $\theta = \pi/12, 5\pi/12, 13\pi/12, 17\pi/12$ and $(-1, \theta)$ where $\theta = 7\pi/12, 11\pi/12, 19\pi/12, 23\pi/12$

41. $(\frac{1}{2}\sqrt{3}, \pi/3)$, $(\frac{1}{2}\sqrt{3}, 2\pi/3)$, and the pole

43. Intersection at $\theta \approx 0.89, 2.25$; area ≈ 3.46

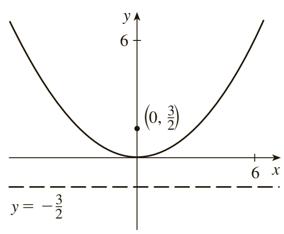
45. 2π **47.** $\frac{8}{3}[(\pi^2+1)^{3/2}-1]$ **49.** $\frac{16}{3}$

51. 2.4221 **53.** 8.0091

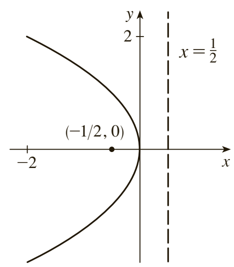
55. (b) $2\pi(2-\sqrt{2})$

EXERCISES 10.5 ■ PAGE 720

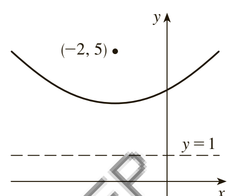
1. $(0, 0), \left(0, \frac{3}{2}\right), y=-\frac{3}{2}$



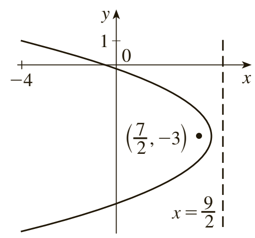
3. $(0, 0), \left(-\frac{1}{2}, 0\right), x=\frac{1}{2}$



5. $(-2, 3)\ (-2, 5), y=1$

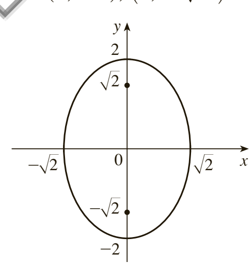


7. $(4, -3), \left(\frac{7}{2}, -3\right), x=\frac{9}{2}$

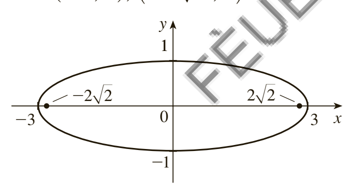


9. $x=-y^2$, focus $\left(\frac{1}{4}, 0\right)$, directrix $x=\frac{1}{4}$

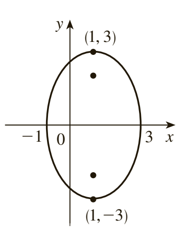
11. $(0, \pm 2), \left(0, \pm\sqrt{2}\right)$



13. $(\pm 3, 0), \left(\pm 2\sqrt{2}, 0\right)$

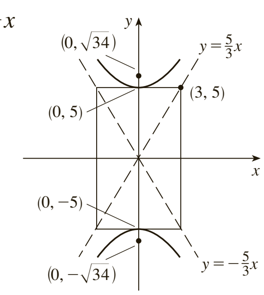


15. $(1, \pm 3), \left(1, \pm\sqrt{5}\right)$

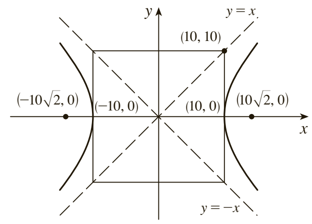


17. $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$, foci $\left(0, \pm\sqrt{5}\right)$

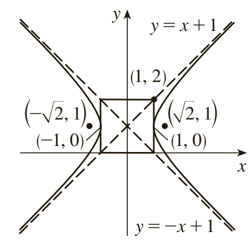
19. $(0, \pm 5)$; $\left(0, \pm\sqrt{34}\right)$; $y=\pm\frac{5}{3}x$



21. $(\pm 10, 0), \left(\pm 10\sqrt{2}, 0\right), y=\pm x$



23. $(\pm 1, 1), \left(\pm\sqrt{2}, 1\right), y-1=\pm x$



25. Hyperbola, $(\pm 1, 0), \left(\pm\sqrt{5}, 0\right)$

27. Ellipse, $\left(\pm\sqrt{2}, 1\right), (\pm 1, 1)$

29. Parabola, $(1, -2), \left(1, -\frac{11}{6}\right)$

31. $y^2=4x$ **33.** $y^2=-12(x+1)$

35. $(y+1)^2=-\frac{1}{2}(x-3)$

37. $\dfrac{x^2}{25}+\dfrac{y^2}{21}=1$ **39.** $\dfrac{x^2}{12}+\dfrac{(y-4)^2}{16}=1$

41. $\dfrac{(x+1)^2}{12}+\dfrac{(y-4)^2}{16}=1$ **43.** $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$

45. $\dfrac{(y-1)^2}{25}-\dfrac{(x+3)^2}{39}=1$ **47.** $\dfrac{x^2}{9}-\dfrac{y^2}{36}=1$

49. $\dfrac{x^2}{3{,}763{,}600}+\dfrac{y^2}{3{,}753{,}196}=1$

51. (a) $\dfrac{121x^2}{1{,}500{,}625}-\dfrac{121y^2}{3{,}339{,}375}=1$ (b) ≈ 248 mi

55. (a) Ellipse (b) Hyperbola (c) No curve

59. 15.9

61. $\dfrac{b^2c}{a}+ab\ln\left(\dfrac{a}{b+c}\right)$ where $c^2=a^2+b^2$

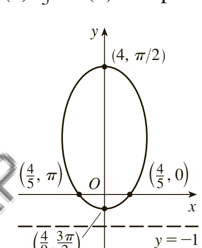
63. $(0, 4/\pi)$

EXERCISES 10.6 ■ PAGE 728

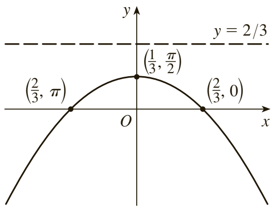
1. $r=\dfrac{4}{2+\cos\theta}$ **3.** $r=\dfrac{6}{2+3\sin\theta}$

5. $r=\dfrac{10}{3-2\cos\theta}$ **7.** $r=\dfrac{6}{1+\sin\theta}$

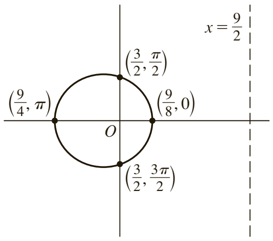
9. (a) $\frac{4}{5}$ (b) Ellipse (c) $y=-1$
(d)



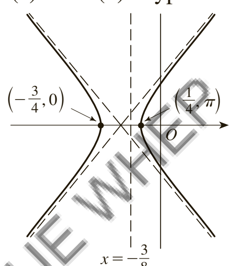
11. (a) 1 (b) Parabola (c) $y = \frac{2}{3}$
(d)



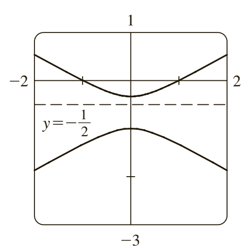
13. (a) $\frac{1}{3}$ (b) Ellipse (c) $x = \frac{9}{2}$
(d)



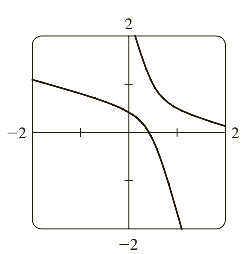
15. (a) 2 (b) Hyperbola (c) $x = -\frac{3}{8}$
(d)



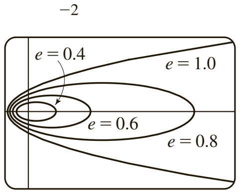
17. (a) $2, y = -\frac{1}{2}$



(b) $r = \dfrac{1}{1 - 2\sin(\theta - 3\pi/4)}$



19. The ellipse is nearly circular when e is close to 0 and becomes more elongated as $e \to 1^-$. At $e = 1$, the curve becomes a parabola.



25. $r = \dfrac{2.26 \times 10^8}{1 + 0.093\cos\theta}$ **27.** $r = \dfrac{1.07}{1 + 0.97\cos\theta}$; 35.64 AU

29. 7.0×10^7 km **31.** 3.6×10^8 km

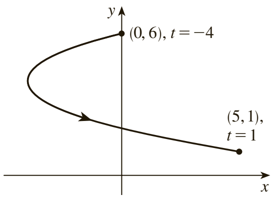
CHAPTER 10 REVIEW ■ PAGE 729

True-False Quiz

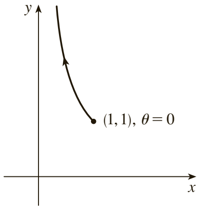
1. False **3.** False **5.** True **7.** False **9.** True

Exercises

1. $x = y^2 - 8y + 12$

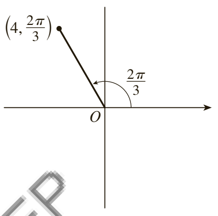


3. $y = 1/x$



5. $x = t, y = \sqrt{t}$; $x = t^4, y = t^2$; $x = \tan^2 t, y = \tan t, 0 \leqslant t < \pi/2$

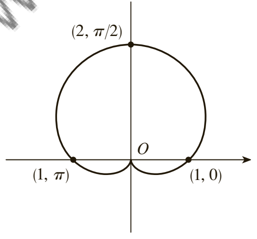
7. (a)



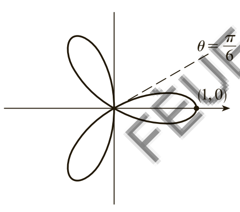
(b) $(3\sqrt{2}, 3\pi/4)$, $(-3\sqrt{2}, 7\pi/4)$

$(-2, 2\sqrt{3})$

9.



11.



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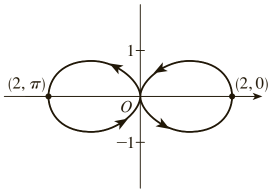
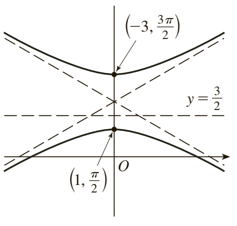
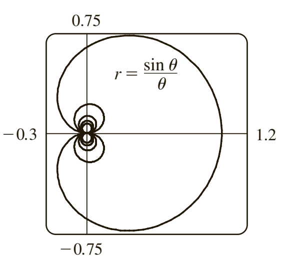


15.



17. $r = \dfrac{2}{\cos\theta + \sin\theta}$

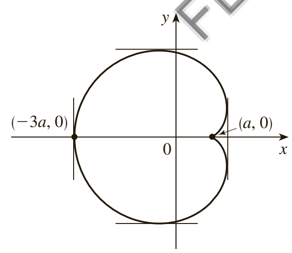
19.



21. 2 **23.** −1

25. $\dfrac{1 + \sin t}{1 + \cos t}, \dfrac{1 + \cos t + \sin t}{(1 + \cos t)^3}$ **27.** $(\frac{11}{8}, \frac{3}{4})$

29. Vertical tangent at $\left(\frac{3}{2}a, \pm\frac{1}{2}\sqrt{3}\,a\right), (-3a, 0)$; horizontal tangent at $(a, 0), \left(-\frac{1}{2}a, \pm\frac{3}{2}\sqrt{3}\,a\right)$



31. 18 **33.** $(2, \pm\pi/3)$ **35.** $\frac{1}{2}(\pi - 1)$

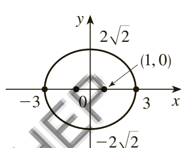
37. $2\left(5\sqrt{5} - 1\right)$

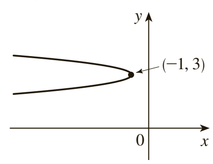
39. $\dfrac{2\sqrt{\pi^2+1} - \sqrt{4\pi^2+1}}{2\pi} + \ln\left(\dfrac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right)$

41. $471{,}295\pi/1024$

43. All curves have the vertical asymptote $x = 1$. For $c < -1$, the curve bulges to the right. At $c = -1$, the curve is the line $x = 1$. For $-1 < c < 0$, it bulges to the left. At $c = 0$ there is a cusp at $(0, 0)$. For $c > 0$, there is a loop.

45. $(\pm1, 0), (\pm3, 0)$ **47.** $\left(-\frac{25}{24}, 3\right), (-1, 3)$





49. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ **51.** $\dfrac{y^2}{72/5} - \dfrac{x^2}{8/5} = 1$

53. $\dfrac{x^2}{25} + \dfrac{(8y-399)^2}{160{,}801} = 1$ **55.** $r = \dfrac{4}{3 + \cos\theta}$

57. $x = a(\cot\theta + \sin\theta\cos\theta),\ y = a(1 + \sin^2\theta)$

PROBLEMS PLUS ■ PAGE 732

1. $\ln(\pi/2)$ **3.** $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$

CHAPTER 11

EXERCISES 11.1 ■ PAGE 744

Abbreviations: C, convergent; D, divergent

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large.
(c) The terms a_n become large as n becomes large.

3. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$ **5.** $\frac{1}{5}, -\frac{1}{25}, \frac{1}{125}, -\frac{1}{625}, \frac{1}{3125}$ **7.** $\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}$

9. 1, 2, 7, 32, 157 **11.** $2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}$ **13.** $a_n = 1/(2n)$

15. $a_n = -3\left(-\frac{2}{3}\right)^{n-1}$ **17.** $a_n = (-1)^{n+1}\dfrac{n^2}{n+1}$

19. 0.4286, 0.4615, 0.4737, 0.4800, 0.4839, 0.4865, 0.4884, 0.4898, 0.4909, 0.4918; yes; $\frac{1}{2}$

21. 0.5000, 1.2500, 0.8750, 1.0625, 0.9688, 1.0156, 0.9922, 1.0039, 0.9980, 1.0010; yes; 1

23. 5 **25.** D **27.** 0 **29.** 1 **31.** 2
33. D **35.** 0 **37.** 0 **39.** D **41.** 0 **43.** 0
45. 1 **47.** e^2 **49.** ln 2 **51.** $\pi/2$ **53.** D **55.** D

57. D **59.** $\pi/4$ **61.** D **63.** 0

65. (a) 1060, 1123.60, 1191.02, 1262.48, 1338.23 (b) D

67. (b) 5734 **69.** $-1 < r < 1$

71. Convergent by the Monotonic Sequence Theorem; $5 \leqslant L < 8$

73. Decreasing; yes **75.** Not monotonic; no

77. Increasing; yes

79. 2 **81.** $\frac{1}{2}\left(3 + \sqrt{5}\right)$ **83.** (b) $\frac{1}{2}\left(1 + \sqrt{5}\right)$

85. (a) 0 (b) 9, 11

EXERCISES 11.2 ■ PAGE 755

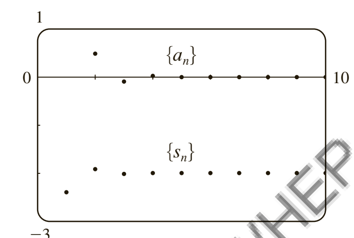
1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

3. 2

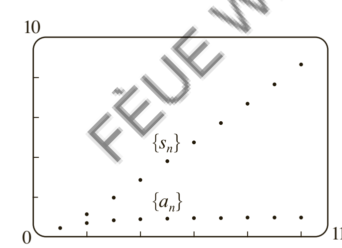
5. 0.5, 0.55, 0.5611, 0.5648, 0.5663, 0.5671, 0.5675, 0.5677; C

7. 1, 1.7937, 2.4871, 3.1170, 3.7018, 4.2521, 4.7749, 5.2749; D

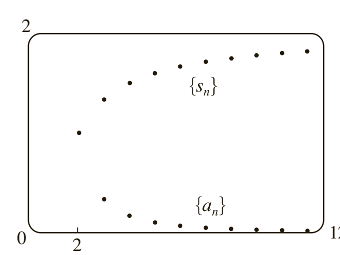
9. −2.40000, −1.92000, −2.01600, −1.99680, −2.00064, −1.99987, −2.00003, −1.99999, −2.00000, −2.00000; convergent, sum = −2



11. 0.44721, 1.15432, 1.98637, 2.88080, 3.80927, 4.75796, 5.71948, 6.68962, 7.66581, 8.64639; divergent



13. 1.00000, 1.33333, 1.50000, 1.60000, 1.66667, 1.71429, 1.75000, 1.77778, 1.80000, 1.81818; convergent, sum = 2



15. (a) Yes (b) No **17.** D **19.** $\frac{25}{3}$ **21.** $\frac{400}{9}$

23. $\frac{1}{7}$ **25.** D **27.** D **29.** D **31.** 9 **33.** D

35. $\dfrac{\sin 100}{1 - \sin 100}$

37. D **39.** D **41.** $e/(e-1)$ **43.** $\frac{3}{2}$ **45.** $\frac{11}{6}$

47. $e - 1$

49. (b) 1 (c) 2 (d) All rational numbers with a terminating decimal representation, except 0

51. $\frac{8}{9}$ **53.** $\frac{838}{333}$ **55.** 45,679/37,000

57. $-\dfrac{1}{5} < x < \dfrac{1}{5};\ \dfrac{-5x}{1+5x}$

59. $-1 < x < 5$; $\dfrac{3}{5-x}$

61. $x > 2$ or $x < -2$; $\dfrac{x}{x-2}$ **63.** $x < 0$; $\dfrac{1}{1-e^x}$

65. 1 **67.** $a_1 = 0$, $a_n = \dfrac{2}{n(n+1)}$ for $n > 1$, sum $= 1$

69. (a) 120 mg; 124 mg
(b) $Q_{n+1} = 100 + 0.20Q_n$ (c) 125 mg

71. (a) 157.875 mg; $\frac{3000}{19}(1 - 0.05^n)$ (b) 157.895 mg

73. (a) $S_n = \dfrac{D(1-c^n)}{1-c}$ (b) 5 **75.** $\frac{1}{2}(\sqrt{3} - 1)$

79. $\dfrac{1}{n(n+1)}$ **81.** The series is divergent.

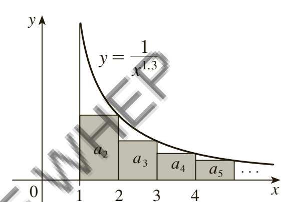
87. $\{s_n\}$ is bounded and increasing.

89. (a) $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$

91. (a) $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}$; $\dfrac{(n+1)! - 1}{(n+1)!}$ (c) 1

EXERCISES 11.3 ■ PAGE 765

1. C



3. C **5.** D **7.** D **9.** C **11.** C **13.** D
15. C **17.** C **19.** D **21.** D **23.** C **25.** C
27. f is neither positive nor decreasing.
29. $p > 1$ **31.** $p < -1$ **33.** $(1, \infty)$
35. (a) $\frac{9}{10}\pi^4$ (b) $\frac{1}{90}\pi^4 - \frac{17}{16}$
37. (a) 1.54977, error $\leqslant 0.1$ (b) 1.64522, error $\leqslant 0.005$
(c) 1.64522 compared to 1.64493 (d) $n > 1000$
39. 0.00145 **45.** $b < 1/e$

EXERCISES 11.4 ■ PAGE 771

1. (a) Nothing (b) C **3.** C **5.** D **7.** C **9.** D
11. C **13.** C **15.** D **17.** D **19.** C **21.** D
23. C **25.** D **27.** C **29.** C **31.** D
33. 0.1993, error $< 2.5 \times 10^{-5}$
35. 0.0739, error $< 6.4 \times 10^{-8}$
45. Yes

EXERCISES 11.5 ■ PAGE 776

1. (a) A series whose terms are alternately positive and negative (b) $0 < b_{n+1} \leqslant b_n$ and $\lim_{n\to\infty} b_n = 0$, where $b_n = |a_n|$ (c) $|R_n| \leqslant b_{n+1}$
3. D **5.** C **7.** D **9.** C **11.** C **13.** D
15. C **17.** C **19.** D **21.** -0.5507 **23.** 5
25. 5 **27.** -0.4597 **29.** -0.1050
31. An underestimate
33. p is not a negative integer. **35.** $\{b_n\}$ is not decreasing.

EXERCISES 11.6 ■ PAGE 782

Abbreviations: AC, absolutely convergent; CC, conditionally convergent
1. (a) D (b) C (c) May converge or diverge
3. CC **5.** AC **7.** AC **9.** D **11.** AC
13. AC **15.** D **17.** AC **19.** AC **21.** AC
23. D **25.** AC **27.** AC **29.** D **31.** CC
33. AC **35.** D **37.** AC **39.** D **41.** AC
43. (a) and (d)
47. (a) $\frac{661}{960} \approx 0.68854$, error < 0.00521
(b) $n \geqslant 11$, 0.693109

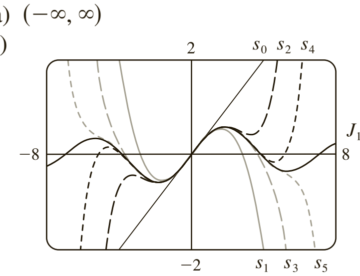
53. (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$; $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

EXERCISES 11.7 ■ PAGE 786

1. D **3.** CC **5.** D **7.** D **9.** C **11.** C
13. C **15.** C **17.** C **19.** C **21.** D **23.** D
25. C **27.** C **29.** C **31.** D
33. C **35.** D **37.** C

EXERCISES 11.8 ■ PAGE 791

1. A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$, where x is a variable and a and the c_n's are constants
3. 1, $(-1, 1)$ **5.** 1, $[-1, 1)$
7. ∞, $(-\infty, \infty)$ **9.** 4, $[-4, 4]$
11. $\frac{1}{4}$, $\left(-\frac{1}{4}, \frac{1}{4}\right]$ **13.** 2, $[-2, 2)$
15. 1, $[1, 3]$ **17.** 2, $[-4, 0)$
19. ∞, $(-\infty, \infty)$ **21.** b, $(a - b, a + b)$ **23.** 0, $\{\frac{1}{2}\}$
25. $\frac{1}{5}$, $\left[\frac{3}{5}, 1\right]$ **27.** ∞, $(-\infty, \infty)$
29. (a) Yes (b) No
31. k^k **33.** No
35. (a) $(-\infty, \infty)$
(b), (c)



37. $(-1, 1)$, $f(x) = (1 + 2x)/(1 - x^2)$ **41.** 2

EXERCISES 11.9 ■ PAGE 797

1. 10 **3.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$, $(-1, 1)$ **5.** $\displaystyle 2\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n$, $(-3, 3)$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{4n+4}}$, $(-2, 2)$ **9.** $\displaystyle -\frac{1}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$, $(-2, 2)$

11. $\displaystyle\sum_{n=0}^{\infty} \left(-1 - \frac{1}{3^{n+1}}\right) x^n$, $(-1, 1)$

13. (a) $\sum_{n=0}^{\infty} (-1)^n(n+1)x^n$, $R = 1$

(b) $\frac{1}{2}\sum_{n=0}^{\infty} (-1)^n(n+2)(n+1)x^n$, $R = 1$

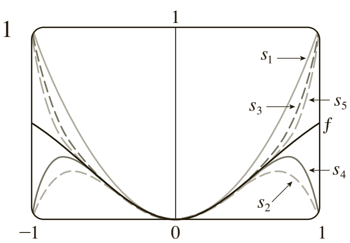
(c) $\frac{1}{2}\sum_{n=2}^{\infty} (-1)^n n(n-1)x^n$, $R = 1$

15. $\ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$, $R = 5$

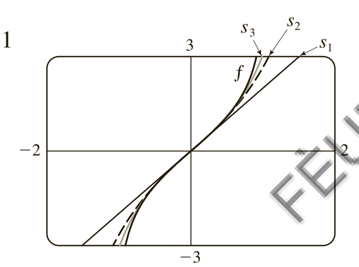
17. $\sum_{n=0}^{\infty} (-1)^n 4^n(n+1)x^{n+1}$, $R = \frac{1}{4}$

19. $\sum_{n=0}^{\infty} (2n+1)x^n$, $R = 1$

21. $\sum_{n=0}^{\infty} (-1)^n x^{2n+2}$, $R = 1$



23. $\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$, $R = 1$



25. $C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$, $R = 1$

27. $C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+3}}{n(n+3)}$, $R = 1$

29. 0.044522 **31.** 0.000395
33. 0.19740
35. (b) 0.920 **39.** $[-1, 1]$, $[-1, 1)$, $(-1, 1)$

EXERCISES 11.10 ■ PAGE 811

1. $b_8 = f^{(8)}(5)/8!$ **3.** $\sum_{n=0}^{\infty} (n+1)x^n$, $R = 1$

5. $x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

7. $2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20{,}736}(x-8)^3$

9. $\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$

11. $\sum_{n=0}^{\infty} (n+1)x^n$, $R = 1$ **13.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, $R = \infty$

15. $\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!}x^n$, $R = \infty$ **17.** $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$, $R = \infty$

19. $50 + 105(x-2) + 92(x-2)^2 + 42(x-2)^3 + 10(x-2)^4 + (x-2)^5$, $R = \infty$

21. $\ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n2^n}(x-2)^n$, $R = 2$

23. $\sum_{n=0}^{\infty} \frac{2^n e^6}{n!}(x-3)^n$, $R = \infty$

25. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$, $R = \infty$

31. $1 - \frac{1}{4}x - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{4^n \cdot n!}x^n$, $R = 1$

33. $\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2^{n+4}}x^n$, $R = 2$

35. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}x^{4n+2}$, $R = 1$

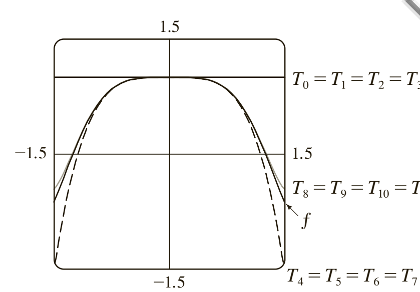
37. $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!}x^{2n+1}$, $R = \infty$

39. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(2n)!}x^{4n+1}$, $R = \infty$

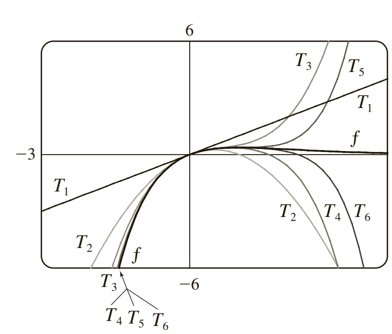
41. $\frac{1}{2}x + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\,2^{3n+1}}x^{2n+1}$, $R = 2$

43. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1}}{(2n)!}x^{2n}$, $R = \infty$

45. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}x^{4n}$, $R = \infty$



47. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!}x^n$, $R = \infty$



49. 0.99619

51. (a) $1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{2n}$

(b) $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n n!} x^{2n+1}$

53. $C + \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{x^{3n+1}}{3n+1}$, $R = 1$

55. $C + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n(2n)!} x^{2n}$, $R = \infty$

57. 0.0059 **59.** 0.40102 **61.** $\frac{1}{2}$ **63.** $\frac{1}{120}$ **65.** $\frac{3}{5}$

67. $1 - \frac{3}{2}x^2 + \frac{25}{24}x^4$ **69.** $1 + \frac{1}{6}x^2 + \frac{7}{360}x^4$

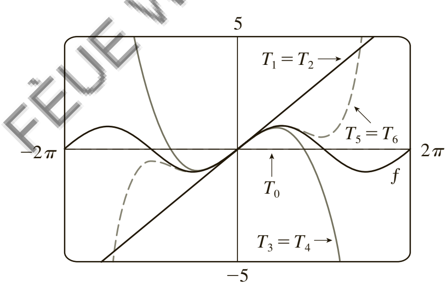
71. $x - \frac{2}{3}x^4 + \frac{23}{45}x^6$

73. e^{-x^4} **75.** $\ln \frac{8}{5}$

77. $1/\sqrt{2}$ **79.** $e^3 - 1$

EXERCISES 11.11 ■ PAGE 820

1. (a) $T_0(x) = 0$, $T_1(x) = T_2(x) = x$, $T_3(x) = T_4(x) = x - \frac{1}{6}x^3$, $T_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

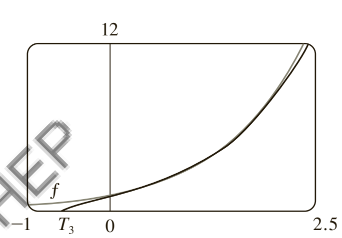


(b)

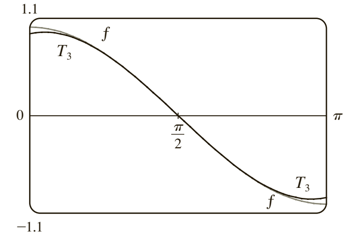
x	f	T_0	$T_1 = T_2$	$T_3 = T_4$	T_5
$\pi/4$	0.7071	0	0.7854	0.7047	0.7071
$\pi/2$	1	0	1.5708	0.9248	1.0045
π	0	0	3.1416	-2.0261	0.5240

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

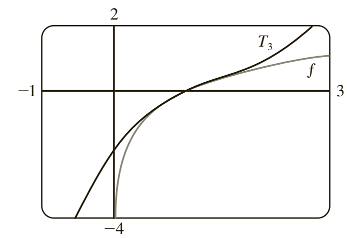
3. $e + e(x-1) + \frac{1}{2}e(x-1)^2 + \frac{1}{6}e(x-1)^3$



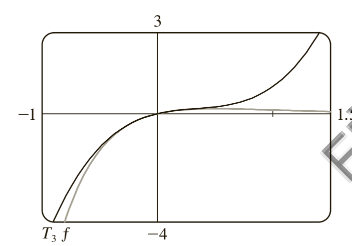
5. $-\left(x - \frac{\pi}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3$



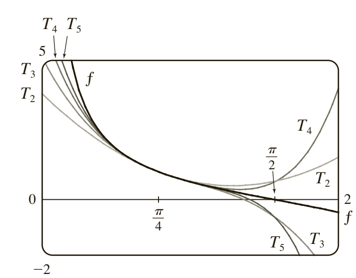
7. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$



9. $x - 2x^2 + 2x^3$



11. $T_5(x) = 1 - 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 - \frac{64}{15}\left(x - \frac{\pi}{4}\right)^5$



13. (a) $1 - (x-1) + (x-1)^2$ (b) 0.006 482 7

15. (a) $1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3$ (b) 0.000 097

17. (a) $1 + \frac{1}{2}x^2$ (b) 0.0015

19. (a) $1 + x^2$ (b) 0.000 06 **21.** (a) $x^2 - \frac{1}{6}x^4$ (b) 0.042

9703

23. 0.17365 **25.** Four **27.** $-1.037 < x < 1.037$
29. $-0.86 < x < 0.86$ **31.** 21 m, no
37. (c) They differ by about 8×10^{-9} km.

CHAPTER 11 REVIEW ■ PAGE 824

True-False Quiz

1. False **3.** True **5.** False **7.** False **9.** False
11. True **13.** True **15.** False **17.** True
19. True **21.** True

Exercises

1. $\frac{1}{2}$ **3.** D **5.** 0 **7.** e^{12} **9.** 2 **11.** C
13. C **15.** D **17.** C **19.** C **21.** C **23.** CC
25. AC **27.** $\frac{1}{11}$ **29.** $\pi/4$ **31.** e^{-e} **35.** 0.9721
37. 0.189 762 24, error $< 6.4 \times 10^{-7}$
41. $4, [-6, 2)$ **43.** $0.5, [2.5, 3.5)$

45. $\displaystyle \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n}+\frac{\sqrt{3}}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}\right]$

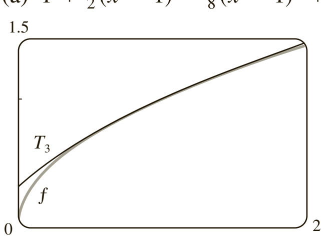
47. $\displaystyle \sum_{n=0}^{\infty}(-1)^n x^{n+2},\ R = 1$ **49.** $\displaystyle \ln 4 - \sum_{n=1}^{\infty}\frac{x^n}{n4^n},\ R = 4$

51. $\displaystyle \sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+4}}{(2n+1)!},\ R = \infty$

53. $\displaystyle \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1\cdot 5\cdot 9\cdot \cdots \cdot(4n-3)}{n!\,2^{6n+1}}x^n,\ R = 16$

55. $\displaystyle C + \ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$

57. (a) $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$
(b)



(c) 0.000 006

59. $-\frac{1}{6}$

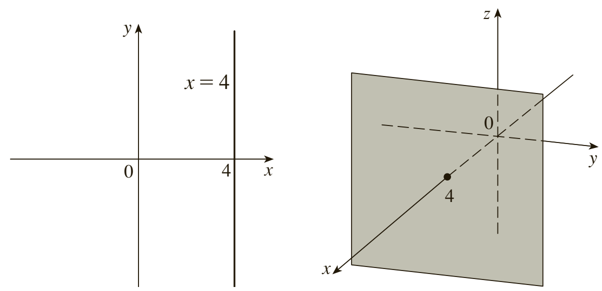
PROBLEMS PLUS ■ PAGE 827

1. $15!/5! = 10{,}897{,}286{,}400$
3. (b) 0 if $x = 0$, $(1/x) - \cot x$ if $x \neq k\pi$, k an integer
5. (a) $s_n = 3 \cdot 4^n$, $l_n = 1/3^n$, $p_n = 4^n/3^{n-1}$ (c) $\frac{2}{5}\sqrt{3}$
9. $\dfrac{3\pi}{4}$ **11.** $(-1, 1)$, $\dfrac{x^3 + 4x^2 + x}{(1-x)^4}$ **13.** $\ln \frac{1}{2}$
17. (a) $\frac{250}{101}\pi\left(e^{-(n-1)\pi/5} - e^{-n\pi/5}\right)$ (b) $\frac{250}{101}\pi$
19. $\dfrac{\pi}{2\sqrt{3}} - 1$
21. $-\left(\dfrac{\pi}{2} - \pi k\right)^2$, where k is a positive integer

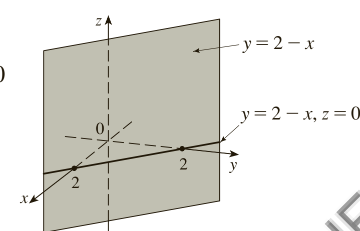
CHAPTER 12

EXERCISES 12.1 ■ PAGE 836

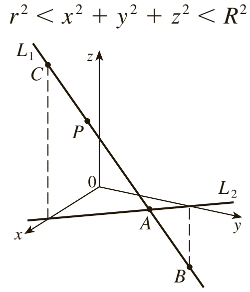
1. $(4, 0, -3)$ **3.** C; A
5. A line parallel to the y-axis and 4 units to the right of it; a vertical plane parallel to the yz-plane and 4 units in front of it.



7. A vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$



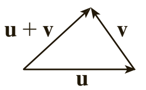
9. (a) $|PQ| = 6$, $|QR| = 2\sqrt{10}$, $|RP| = 6$; isosceles triangle
11. (a) No (b) Yes
13. $(x+3)^2 + (y-2)^2 + (z-5)^2 = 16$;
$(y-2)^2 + (z-5)^2 = 7$, $x = 0$ (a circle)
15. $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$
17. $(1, 2, -4)$, 6 **19.** $(2, 0, -6)$, $9/\sqrt{2}$
21. (b) $\frac{5}{2}, \frac{1}{2}\sqrt{94}, \frac{1}{2}\sqrt{85}$
23. (a) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 36$
(b) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 4$
(c) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 9$
25. A plane parallel to the yz-plane and 5 units in front of it
27. A half-space consisting of all points to the left of the plane $y = 8$
29. All points on or between the horizontal planes $z = 0$ and $z = 6$
31. All points on a circle with radius 2 with center on the z-axis that is contained in the plane $z = -1$
33. All point on a sphere with radius 2 and center $(0, 0, 0)$
35. All points on or between spheres with radii 1 and $\sqrt{5}$ and centers $(0, 0, 0)$
37. All points on or inside a circular cylinder of radius 3 with axis the y-axis
39. $0 < x < 5$ **41.** $r^2 < x^2 + y^2 + z^2 < R^2$
43. (a) $(2, 1, 4)$ (b)



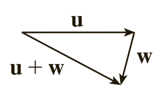
45. $14x - 6y - 10z = 9$, a plane perpendicular to AB
47. $2\sqrt{3} - 3$

EXERCISES 12.2 ■ PAGE 845

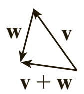
1. (a) Scalar (b) Vector (c) Vector (d) Scalar
3. $\overrightarrow{AB} = \overrightarrow{DC}, \overrightarrow{DA} = \overrightarrow{CB}, \overrightarrow{DE} = \overrightarrow{EB}, \overrightarrow{EA} = \overrightarrow{CE}$
5. (a)



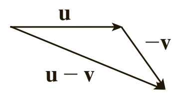
(b)



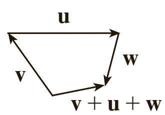
(c)



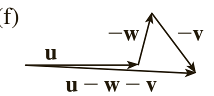
(d)



(e)

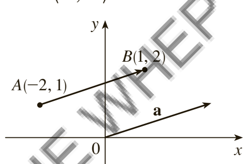


(f)

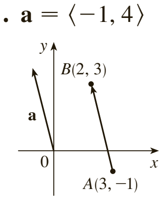


7. $\mathbf{c} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{d} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$

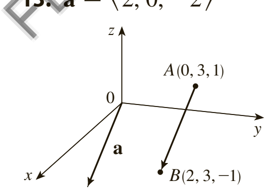
9. $\mathbf{a} = \langle 3, 1 \rangle$



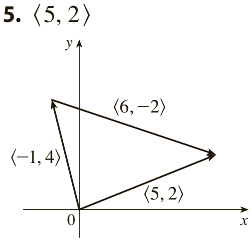
11. $\mathbf{a} = \langle -1, 4 \rangle$



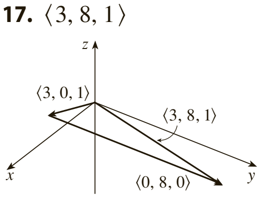
13. $\mathbf{a} = \langle 2, 0, -2 \rangle$



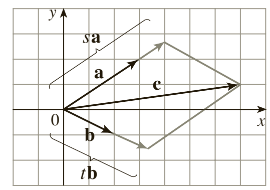
15. $\langle 5, 2 \rangle$



17. $\langle 3, 8, 1 \rangle$



19. $\langle 6, 3 \rangle$, $\langle 6, 14 \rangle$, 5, 13
21. $6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, $20\mathbf{i} - 12\mathbf{j}$, $\sqrt{29}$, 7
23. $\left\langle \dfrac{3}{\sqrt{10}}, -\dfrac{1}{\sqrt{10}} \right\rangle$ **25.** $\frac{8}{9}\mathbf{i} - \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}$ **27.** $60°$
29. $\langle 2, 2\sqrt{3} \rangle$ **31.** ≈ 45.96 ft/s, ≈ 38.57 ft/s
33. $100\sqrt{7} \approx 264.6$ N, $\approx 139.1°$
35. $\sqrt{493} \approx 22.2$ mi/h, N8°W
37. $\approx -177.39\,\mathbf{i} + 211.41\,\mathbf{j}$, $\approx 177.39\,\mathbf{i} + 138.59\,\mathbf{j}$; ≈ 275.97 N, ≈ 225.11 N
39. (a) At an angle of 43.4° from the bank, toward upstream
(b) 20.2 min
41. $\pm(\mathbf{i} + 4\,\mathbf{j})/\sqrt{17}$ **43.** $\mathbf{0}$
45. (a), (b)



(d) $s = \frac{9}{7}$, $t = \frac{11}{7}$

47. A sphere with radius 1, centered at (x_0, y_0, z_0)

EXERCISES 12.3 ■ PAGE 852

1. (b), (c), (d) are meaningful **3.** -3.6 **5.** 19 **7.** 1
9. $14\sqrt{3}$ **11.** $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}$, $\mathbf{u} \cdot \mathbf{w} = -\frac{1}{2}$
15. $\cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right) \approx 63°$ **17.** $\cos^{-1}\left(-\frac{5}{6}\right) \approx 146°$
19. $\cos^{-1}\left(\dfrac{7}{\sqrt{130}}\right) \approx 52°$ **21.** 48°, 75°, 57°
23. (a) Orthogonal (b) Neither
(c) Parallel (d) Orthogonal
25. Yes **27.** $(\mathbf{i} - \mathbf{j} - \mathbf{k})/\sqrt{3}$ $\left[\text{or } (-\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}\right]$
29. 45° **31.** 0° at (0, 0), $\approx 8.1°$ at (1, 1)
33. $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$; 48°, 71°, 48°
35. $1/\sqrt{14}$, $-2/\sqrt{14}$, $-3/\sqrt{14}$; 74°, 122°, 143°
37. $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$; 55°, 55°, 55° **39.** 4, $\left\langle -\frac{20}{13}, \frac{48}{13} \right\rangle$
41. $\frac{1}{9}$, $\left\langle \frac{4}{81}, \frac{7}{81}, -\frac{4}{81} \right\rangle$ **43.** $-7/\sqrt{19}$, $-\frac{21}{19}\mathbf{i} + \frac{21}{19}\mathbf{j} - \frac{7}{19}\mathbf{k}$
47. $\langle 0, 0, -2\sqrt{10} \rangle$ or any vector of the form $\langle s, t, 3s - 2\sqrt{10} \rangle$, $s, t \in \mathbb{R}$
49. 144 J **51.** $2400\cos(40°) \approx 1839$ ft-lb
53. $\frac{13}{5}$ **55.** $\cos^{-1}(1/\sqrt{3}) \approx 55°$

EXERCISES 12.4 ■ PAGE 861

1. $15\,\mathbf{i} - 10\,\mathbf{j} - 3\,\mathbf{k}$ **3.** $14\,\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$
5. $-\frac{3}{2}\mathbf{i} + \frac{7}{4}\mathbf{j} + \frac{2}{3}\mathbf{k}$ **7.** $(1 - t)\,\mathbf{i} + (t^3 - t^2)\,\mathbf{k}$
9. $\mathbf{0}$ **11.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$
13. (a) Scalar (b) Meaningless (c) Vector
(d) Meaningless (e) Meaningless (f) Scalar
15. $96\sqrt{3}$; into the page **17.** $\langle -7, 10, 8 \rangle$, $\langle 7, -10, -8 \rangle$
19. $\left\langle -\dfrac{1}{3\sqrt{3}}, -\dfrac{1}{3\sqrt{3}}, \dfrac{5}{3\sqrt{3}} \right\rangle$, $\left\langle \dfrac{1}{3\sqrt{3}}, \dfrac{1}{3\sqrt{3}}, -\dfrac{5}{3\sqrt{3}} \right\rangle$
27. 20 **29.** (a) $\langle 0, 18, -9 \rangle$ (b) $\frac{9}{2}\sqrt{5}$
31. (a) $\langle 13, -14, 5 \rangle$ (b) $\frac{1}{2}\sqrt{390}$
33. 9 **35.** 16 **39.** $10.8 \sin 80° \approx 10.6$ N·m
41. ≈ 417 N **43.** 60°
45. (b) $\sqrt{97/3}$ **53.** (a) No (b) No (c) Yes

EXERCISES 12.5 ■ PAGE 871

1. (a) True (b) False (c) True (d) False (e) False (f) True (g) False (h) True (i) True (j) False (k) True

3. $\mathbf{r} = (2\,\mathbf{i} + 2.4\,\mathbf{j} + 3.5\,\mathbf{k}) + t(3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k})$;
$x = 2 + 3t,\ y = 2.4 + 2t,\ z = 3.5 - t$

5. $\mathbf{r} = (\mathbf{i} + 6\mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$;
$x = 1 + t,\ y = 3t,\ z = 6 + t$

7. $x = 2 + 2t,\ y = 1 + \frac{1}{2}t,\ z = -3 - 4t$;
$(x - 2)/2 = 2y - 2 = (z + 3)/(-4)$

9. $x = -8 + 11t,\ y = 1 - 3t,\ z = 4;\ \dfrac{x + 8}{11} = \dfrac{y - 1}{-3},\ z = 4$

11. $x = -6 + 2t,\ y = 2 + 3t,\ z = 3 + t$;
$(x + 6)/2 = (y - 2)/3 = z - 3$

13. Yes

15. (a) $(x - 1)/(-1) = (y + 5)/2 = (z - 6)/(-3)$
(b) $(-1, -1, 0),\ \left(-\frac{3}{2}, 0, -\frac{3}{2}\right),\ (0, -3, 3)$

17. $\mathbf{r}(t) = (6\mathbf{i} - \mathbf{j} + 9\mathbf{k}) + t(\mathbf{i} + 7\mathbf{j} - 9\mathbf{k}),\ 0 \leqslant t \leqslant 1$

19. Skew **21.** $(4, -1, -5)$ **23.** $x - 2y + 5z = 0$

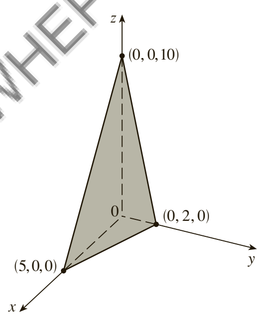
25. $x + 4y + z = 4$ **27.** $5x - y - z = 7$

29. $6x + 6y + 6z = 11$ **31.** $x + y + z = 2$

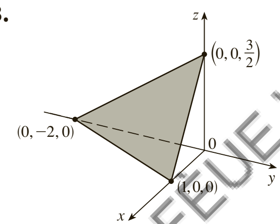
33. $5x - 3y - 8z = -9$ **35.** $8x + y - 2z = 31$

37. $x - 2y + 4z = -3$ **39.** $3x - 8y - z = -38$

41.



43.



45. $(-2, 6, 3)$ **47.** $\left(\frac{2}{5}, 4, 0\right)$ **49.** $1, 0, -1$

51. Perpendicular **53.** Neither, $\cos^{-1}\left(-\dfrac{1}{\sqrt{6}}\right) \approx 114.1°$

55. Parallel

57. (a) $x = 1,\ y = -t,\ z = t$ (b) $\cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right) \approx 15.8°$

59. $x = 1,\ y - 2 = -z$

61. $x + 2y + z = 5$

63. $(x/a) + (y/b) + (z/c) = 1$

65. $x = 3t,\ y = 1 - t,\ z = 2 - 2t$

67. P_2 and P_3 are parallel, P_1 and P_4 are identical

69. $\sqrt{61/14}$ **71.** $\frac{18}{7}$ **73.** $5/\left(2\sqrt{14}\right)$

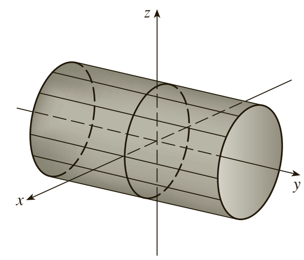
77. $1/\sqrt{6}$ **79.** $13/\sqrt{69}$

81. (a) $x = 325 + 440t,\ y = 810 - 135t,\ z = 561 + 38t$, $0 \leqslant t \leqslant 1$ (b) No

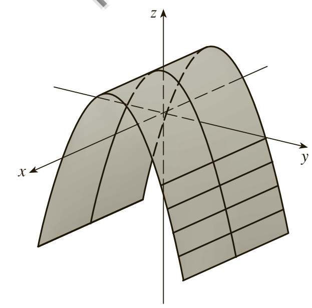
EXERCISES 12.6 ■ PAGE 879

1. (a) Parabola
(b) Parabolic cylinder with rulings parallel to the z-axis
(c) Parabolic cylinder with rulings parallel to the x-axis

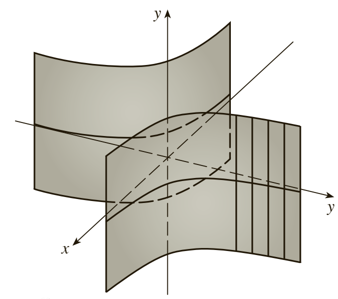
3. Circular cylinder



5. Parabolic cylinder

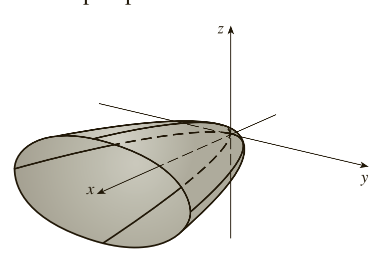


7. Hyperbolic cylinder

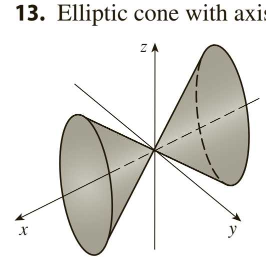


9. (a) $x = k,\ y^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$;
$y = k,\ x^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$;
$z = k,\ x^2 + y^2 = 1 + k^2$, circle
(b) The hyperboloid is rotated so that it has axis the y-axis
(c) The hyperboloid is shifted one unit in the negative y-direction

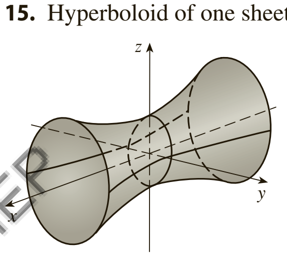
11. Elliptic paraboloid with axis the x-axis



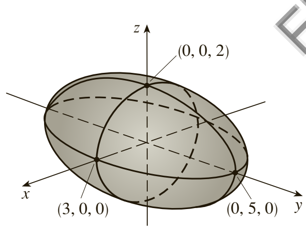
13. Elliptic cone with axis the x-axis



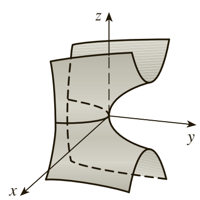
15. Hyperboloid of one sheet with axis the x-axis



17. Ellipsoid

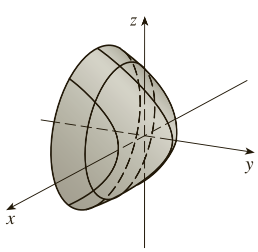


19. Hyperbolic paraboloid



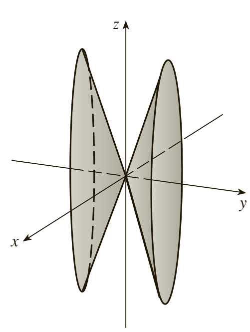
21. VII **23.** II **25.** VI **27.** VIII

29. Circular paraboloid



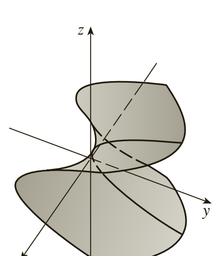
31. $y^2 = x^2 + \dfrac{z^2}{9}$

Elliptic cone with axis the y-axis

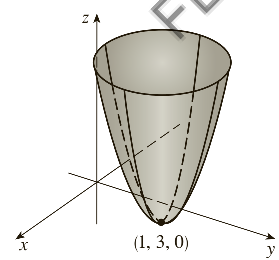


33. $y = z^2 - \dfrac{x^2}{2}$

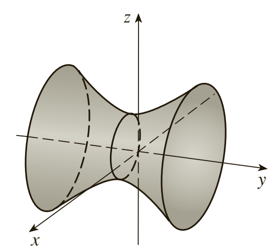
Hyperbolic paraboloid



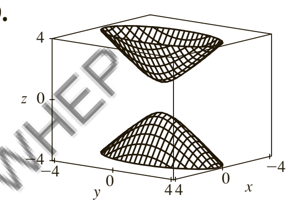
35. $z = (x - 1)^2 + (y - 3)^2$
Circular paraboloid with vertex $(1, 3, 0)$ and axis the vertical line $x = 1$, $y = 3$



37. $\dfrac{(x-2)^2}{5} - \dfrac{y^2}{5} + \dfrac{(z-1)^2}{5} = 1$
Hyperboloid of one sheet with center $(2, 0, 1)$ and axis the horizontal line $x = 2$, $z = 1$



39.



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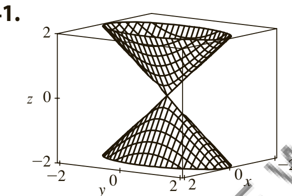
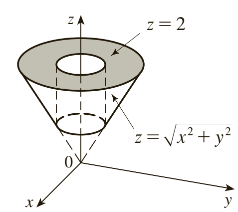


43.

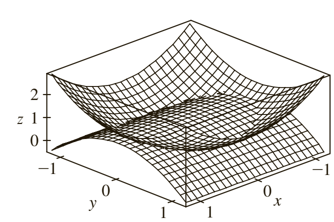


45. $x = y^2 + z^2$ **47.** $-4x = y^2 + z^2$, paraboloid

49. (a) $\dfrac{x^2}{(6378.137)^2} + \dfrac{y^2}{(6378.137)^2} + \dfrac{z^2}{(6356.523)^2} = 1$

(b) Circle (c) Ellipse

53.



CHAPTER 12 REVIEW ■ PAGE 882

True-False Quiz

1. False **3.** False **5.** True **7.** True **9.** True
11. True **13.** True **15.** False **17.** False
19. False **21.** True

Exercises

1. (a) $(x+1)^2 + (y-2)^2 + (z-1)^2 = 69$
(b) $(y-2)^2 + (z-1)^2 = 68, x = 0$
(c) Center $(4, -1, -3)$, radius 5

3. $\mathbf{u} \cdot \mathbf{v} = 3\sqrt{2}$; $|\mathbf{u} \times \mathbf{v}| = 3\sqrt{2}$; out of the page

5. $-2, -4$ **7.** (a) 2 (b) -2 (c) -2 (d) 0

9. $\cos^{-1}\left(\frac{1}{3}\right) \approx 71°$ **11.** (a) $\langle 4, -3, 4 \rangle$ (b) $\sqrt{41}/2$

13. ≈ 166 N, ≈ 114 N

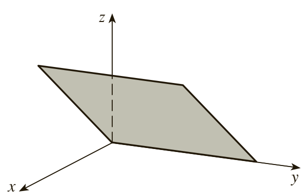
15. $x = 4 - 3t, y = -1 + 2t, z = 2 + 3t$

17. $x = -2 + 2t, y = 2 - t, z = 4 + 5t$

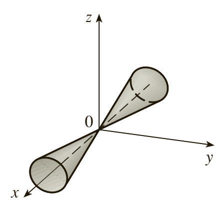
19. $-4x + 3y + z = -14$ **21.** $(1, 4, 4)$ **23.** Skew

25. $x + y + z = 4$ **27.** $22/\sqrt{26}$

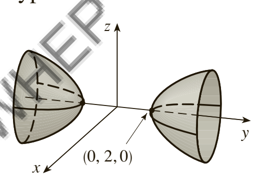
29. Plane



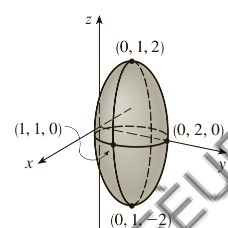
31. Cone



33. Hyperboloid of two sheets



35. Ellipsoid



37. $4x^2 + y^2 + z^2 = 16$

PROBLEMS PLUS ■ PAGE 884

1. $\left(\sqrt{3} - \frac{3}{2}\right)$ m

3. (a) $(x+1)/(-2c) = (y-c)/(c^2-1) = (z-c)/(c^2+1)$
(b) $x^2 + y^2 = t^2 + 1, z = t$ (c) $4\pi/3$

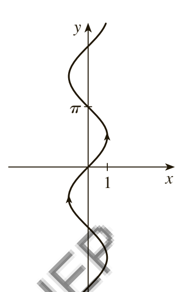
5. 20

CHAPTER 13

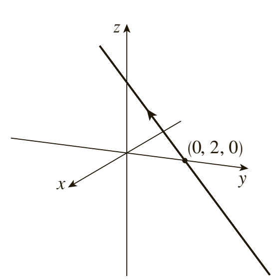
EXERCISES 13.1 ■ PAGE 893

1. $(-1, 3)$ **3.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$ **5.** $\langle -1, \pi/2, 0 \rangle$

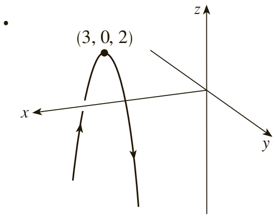
7.



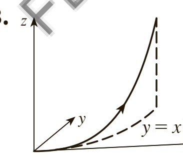
9.



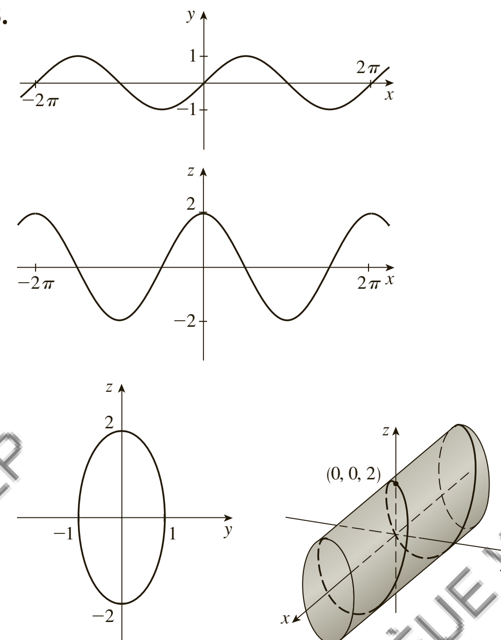
11.



13.



15.

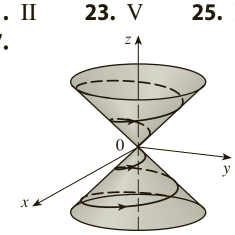


17. $\mathbf{r}(t) = \langle 2 + 4t, 2t, -2t \rangle, 0 \leqslant t \leqslant 1$;
$x = 2 + 4t, y = 2t, z = -2t, 0 \leqslant t \leqslant 1$

19. $\mathbf{r}(t) = \left\langle \frac{1}{2}t, -1 + \frac{4}{3}t, 1 - \frac{3}{4}t \right\rangle, 0 \leqslant t \leqslant 1$;
$x = \frac{1}{2}t, y = -1 + \frac{4}{3}t, z = 1 - \frac{3}{4}t, 0 \leqslant t \leqslant 1$

21. II **23.** V **25.** IV

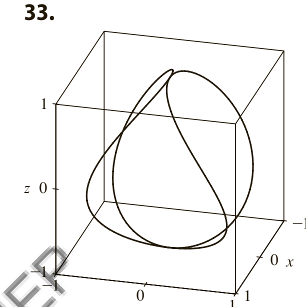
27.



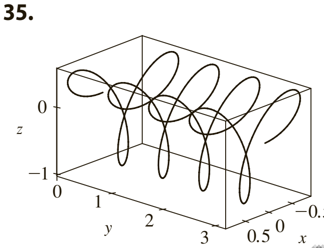
29. $y = e^{x/2}, z = e^x, z = y^2$

31. $(0, 0, 0), (1, 0, 1)$

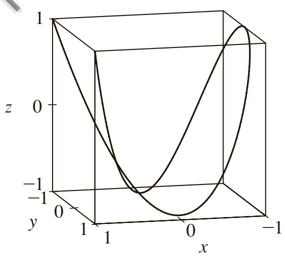
33.



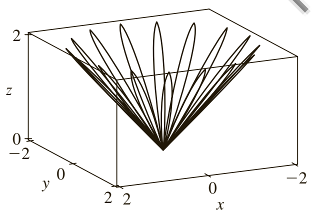
35.



37.



39.

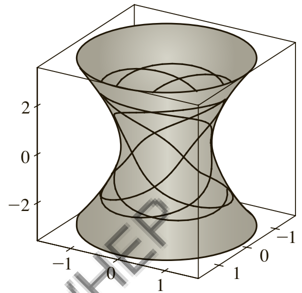


43. $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$

45. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k},\ 0 \leqslant t \leqslant 2\pi$

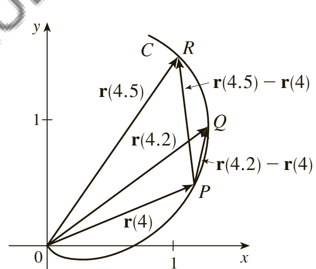
47. $x = 2\cos t,\ y = 2\sin t,\ z = 4\cos^2 t,\ 0 \leqslant t \leqslant 2\pi$ **49.** Yes

51. (a)

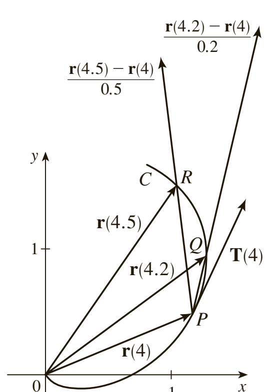


EXERCISES 13.2 ■ PAGE 900

1. (a)

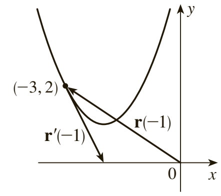


(b), (d)



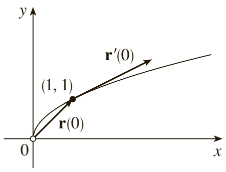
(c) $\mathbf{r}'(4) = \lim_{h\to 0} \dfrac{\mathbf{r}(4+h) - \mathbf{r}(4)}{h}$; $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$

3. (a), (c)



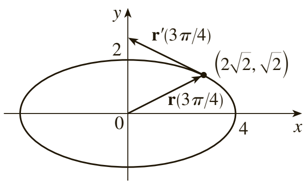
(b) $\mathbf{r}'(t) = \langle 1, 2t \rangle$

5. (a), (c)



(b) $\mathbf{r}'(t) = 2e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$

7. (a), (c)



(b) $\mathbf{r}'(t) = 4\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

9. $\mathbf{r}'(t) = \left\langle \dfrac{1}{2\sqrt{t-2}}, 0, -\dfrac{2}{t^3} \right\rangle$

11. $\mathbf{r}'(t) = 2t\,\mathbf{i} - 2t\sin(t^2)\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}$

13. $\mathbf{r}'(t) = (t\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j} + (\cos^2 t - \sin^2 t)\,\mathbf{k}$

15. $\mathbf{r}'(t) = \mathbf{b} + 2t\mathbf{c}$ **17.** $\left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$ **19.** $\frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$

21. $\langle 1, 2t, 3t^2 \rangle$, $\langle 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14} \rangle$, $\langle 0, 2, 6t \rangle$, $\langle 6t^2, -6t, 2 \rangle$

23. $x = 2 + 2t,\ y = 4 + 2t,\ z = 1 + t$

25. $x = 1 - t,\ y = t,\ z = 1 - t$

27. $\mathbf{r}(t) = (3 - 4t)\,\mathbf{i} + (4 + 3t)\,\mathbf{j} + (2 - 6t)\,\mathbf{k}$

29. $x = t,\ y = 1 - t,\ z = 2t$

31. $x = -\pi - t,\ y = \pi + t,\ z = -\pi t$

33. $66°$ **35.** $2\,\mathbf{i} - 4\,\mathbf{j} + 32\,\mathbf{k}$

37. $(\ln 2)\,\mathbf{i} + (\pi/4)\,\mathbf{j} + \frac{1}{2}\ln 2\,\mathbf{k}$

39. $\tan t\,\mathbf{i} + \frac{1}{8}(t^2+1)^4\,\mathbf{j} + \left(\frac{1}{3}t^3\ln t - \frac{1}{9}t^3\right)\mathbf{k} + \mathbf{C}$

41. $t^2\,\mathbf{i} + t^3\,\mathbf{j} + \left(\frac{2}{3}t^{3/2} - \frac{2}{3}\right)\mathbf{k}$

47. $2t\cos t + 2\sin t - 2\cos t\sin t$ **49.** 35

EXERCISES 13.3 ■ PAGE 908

1. $10\sqrt{10}$ **3.** $e - e^{-1}$ **5.** $\frac{1}{27}(13^{3/2} - 8)$

7. 18.6833 **9.** 10.3311 **11.** 42

13. (a) $s(t) = \sqrt{26}\,(t - 1)$;

$$\mathbf{r}(t(s)) = \left(4 - \frac{s}{\sqrt{26}}\right)\mathbf{i} + \left(\frac{4s}{\sqrt{26}} + 1\right)\mathbf{j} + \left(\frac{3s}{\sqrt{26}} + 3\right)\mathbf{k}$$

(b) $\left(4 - \dfrac{4}{\sqrt{26}}, \dfrac{16}{\sqrt{26}} + 1, \dfrac{12}{\sqrt{26}} + 3\right)$

15. $(3\sin 1, 4, 3\cos 1)$

17. (a) $\langle 1/\sqrt{10}, (-3/\sqrt{10})\sin t, (3/\sqrt{10})\cos t \rangle$, $\langle 0, -\cos t, -\sin t \rangle$ (b) $\frac{3}{10}$

19. (a) $\dfrac{1}{e^{2t}+1}\langle \sqrt{2}e^t, e^{2t}, -1 \rangle$, $\dfrac{1}{e^{2t}+1}\langle 1 - e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t \rangle$

(b) $\sqrt{2}e^{2t}/(e^{2t}+1)^2$

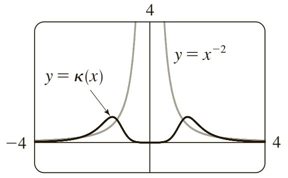
21. $6t^2/(9t^4 + 4t^2)^{3/2}$ **23.** $\dfrac{\sqrt{6}}{2(3t^2 + 1)^2}$

25. $\frac{1}{7}\sqrt{\frac{19}{14}}$ **27.** $12x^2/(1 + 16x^6)^{3/2}$

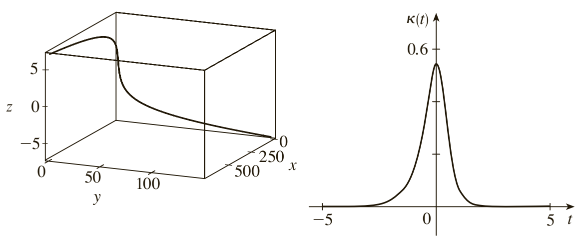
29. $e^x|x + 2|/[1 + (xe^x + e^x)^2]^{3/2}$

31. $\left(-\frac{1}{2}\ln 2, 1/\sqrt{2}\right)$; approaches 0 **33.** (a) P (b) 1.3, 0.7

35.

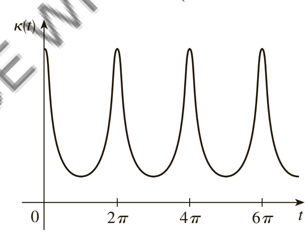


37.



39. a is $y = f(x)$, b is $y = \kappa(x)$

41. $\kappa(t) = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}$



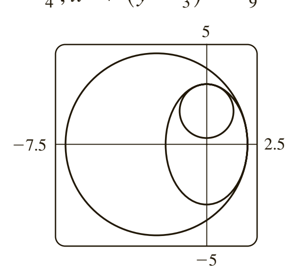
integer multiples of 2π

43. $6t^2/(4t^2 + 9t^4)^{3/2}$

45. $1/\left(\sqrt{2}e^t\right)$ **47.** $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle, \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle, \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

49. $x - 2z = -4\pi$, $2x + z = 2\pi$

51. $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$, $x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$



53. $(-1, -3, 1)$

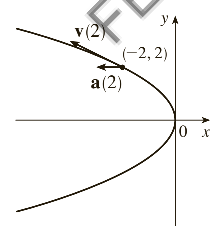
55. $2x + y + 4z = 7$, $6x - 8y - z = -3$

65. $2/(t^4 + 4t^2 + 1)$ **67.** 2.07×10^{10} Å ≈ 2 m

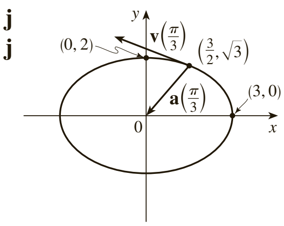
EXERCISES 13.4 ■ PAGE 918

1. (a) $1.8\mathbf{i} - 3.8\mathbf{j} - 0.7\mathbf{k}$, $2.0\mathbf{i} - 2.4\mathbf{j} - 0.6\mathbf{k}$, $2.8\mathbf{i} + 1.8\mathbf{j} - 0.3\mathbf{k}$, $2.8\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k}$
(b) $2.4\mathbf{i} - 0.8\mathbf{j} - 0.5\mathbf{k}$, 2.58

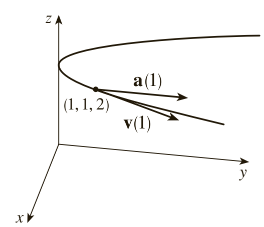
3. $\mathbf{v}(t) = \langle -t, 1 \rangle$
$\mathbf{a}(t) = \langle -1, 0 \rangle$
$|\mathbf{v}(t)| = \sqrt{t^2 + 1}$



5. $\mathbf{v}(t) = -3\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$
$\mathbf{a}(t) = -3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{5\sin^2 t + 4}$



7. $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$
$\mathbf{a}(t) = 2\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$



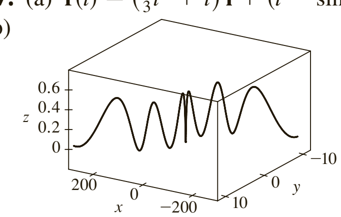
9. $\langle 2t + 1, 2t - 1, 3t^2 \rangle$, $\langle 2, 2, 6t \rangle$, $\sqrt{9t^4 + 8t^2 + 2}$

11. $\sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$, $e^t + e^{-t}$

13. $e^t[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + (t + 1)\mathbf{k}]$, $e^t[-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + (t + 2)\mathbf{k}]$, $e^t\sqrt{t^2 + 2t + 3}$

15. $\mathbf{v}(t) = (2t + 3)\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k}$,
$\mathbf{r}(t) = (t^2 + 3t)\,\mathbf{i} + (1 - t)\,\mathbf{j} + \left(\frac{1}{3}t^3 + 1\right)\mathbf{k}$

17. (a) $\mathbf{r}(t) = \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + (t - \sin t + 1)\,\mathbf{j} + \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right)\mathbf{k}$
(b)



19. $t = 4$

21. $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{25t^2 + 2}$

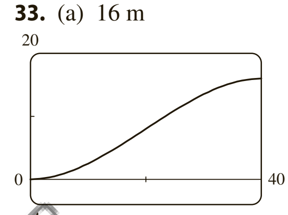
23. (a) ≈ 3535 m (b) ≈ 1531 m (c) 200 m/s

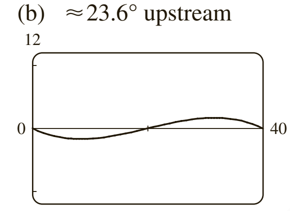
25. ≈ 30 m/s **27.** ≈ 544 ft/s

29. $13.0° < \theta < 36.0°$, $55.4° < \theta < 85.5°$

31. $(250, -50, 0)$; $10\sqrt{93} \approx 96.4$ ft/s

33. (a) 16 m (b) $\approx 23.6°$ upstream





35. The path is contained in a circle that lies in a plane perpendicular to $\mathbf{c}$ with center on a line through the origin in the direction of $\mathbf{c}$.

37. $\dfrac{4+18t^2}{\sqrt{4+9t^2}}, \dfrac{6t}{\sqrt{4+9t^2}}$ **39.** 0, 1 **41.** $\dfrac{7}{\sqrt{30}}, \sqrt{\dfrac{131}{30}}$

43. 4.5 cm/s², 9.0 cm/s² **45.** $t = 1$

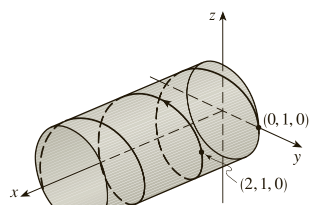
CHAPTER 13 REVIEW ■ PAGE 921

True-False Quiz

1. True **3.** False **5.** False **7.** False
9. True **11.** False **13.** True

Exercises

1. (a)



(b) $\mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\,\mathbf{j} + \pi \cos \pi t\,\mathbf{k}$,
$\mathbf{r}''(t) = -\pi^2 \cos \pi t\,\mathbf{j} - \pi^2 \sin \pi t\,\mathbf{k}$

3. $\mathbf{r}(t) = 4 \cos t\,\mathbf{i} + 4 \sin t\,\mathbf{j} + (5 - 4 \cos t)\mathbf{k}$, $0 \leqslant t \leqslant 2\pi$

5. $\frac{1}{3}\mathbf{i} - (2/\pi^2)\mathbf{j} + (2/\pi)\mathbf{k}$ **7.** 86.631 **9.** 90°

11. (a) $\dfrac{1}{\sqrt{13}}\langle 3 \sin t, -3 \cos t, 2\rangle$ (b) $\langle \cos t, \sin t, 0\rangle$

(c) $\dfrac{1}{\sqrt{13}}\langle -2 \sin t, 2 \cos t, 3\rangle$

(d) $\dfrac{3}{13 \sin t \cos t}$ or $\dfrac{3}{13} \sec t \csc t$

13. $12/17^{3/2}$ **15.** $x - 2y + 2\pi = 0$

17. $\mathbf{v}(t) = (1 + \ln t)\mathbf{i} + \mathbf{j} - e^{-t}\mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}$, $\mathbf{a}(t) = (1/t)\mathbf{i} + e^{-t}\mathbf{k}$

19. $\mathbf{r}(t) = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k}$

21. ≈37.3°, ≈157.4 m

23. (c) $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$

PROBLEMS PLUS ■ PAGE 924

1. (a) $\mathbf{v} = \omega R(-\sin \omega t\,\mathbf{i} + \cos \omega t\,\mathbf{j})$ (c) $\mathbf{a} = -\omega^2\mathbf{r}$

3. (a) 90°, $v_0^2/(2g)$

5. (a) ≈0.94 ft to the right of the table's edge, ≈15 ft/s
(b) ≈7.6° (c) ≈2.13 ft to the right of the table's edge

7. 56°

9. $(a_2b_3 - a_3b_2)(x - c_1) + (a_3b_1 - a_1b_3)(y - c_2) + (a_1b_2 - a_2b_1)(z - c_3) = 0$

CHAPTER 14

EXERCISES 14.1 ■ PAGE 939

1. (a) −27; a temperature of −15°C with wind blowing at 40 km/h feels equivalent to about −27°C without wind.
(b) When the temperature is −20°C, what wind speed gives a wind chill of −30°C? 20 km/h
(c) With a wind speed of 20 km/h, what temperature gives a wind chill of −49°C? −35°C
(d) A function of wind speed that gives wind-chill values when the temperature is −5°C
(e) A function of temperature that gives wind-chill values when the wind speed is 50 km/h

3. ≈94.2; the manufacturer's yearly production is valued at \$94.2 million when 120,000 labor hours are spent and \$20 million in capital is invested.

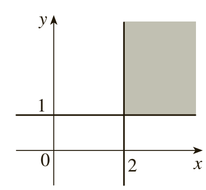
5. (a) ≈20.5; the surface area of a person 70 inches tall who weighs 160 pounds is approximately 20.5 square feet.

7. (a) 25; a 40-knot wind blowing in the open sea for 15 h will create waves about 25 ft high.
(b) $f(30, t)$ is a function of t giving the wave heights produced by 30-knot winds blowing for t hours.
(c) $f(v, 30)$ is a function of v giving the wave heights produced by winds of speed v blowing for 30 hours.

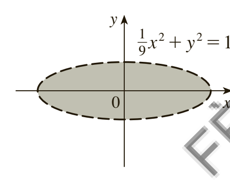
9. (a) 1 (b) $\mathbb{R}^2$ (c) $[-1, 1]$

11. (a) 3
(b) $\{(x, y, z) \mid x^2 + y^2 + z^2 < 4, x \geqslant 0, y \geqslant 0, z \geqslant 0\}$, interior of a sphere of radius 2, center the origin, in the first octant

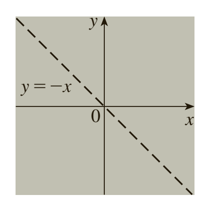
13. $\{(x, y) \mid x \geqslant 2, y \geqslant 1\}$



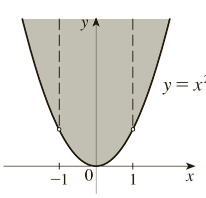
15. $\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\}$



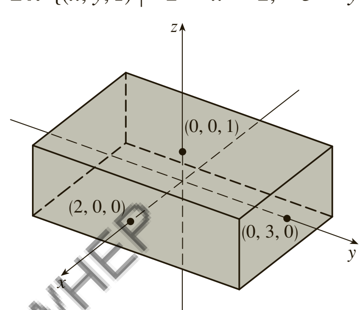
17. $\{(x, y) \mid y \neq -x\}$



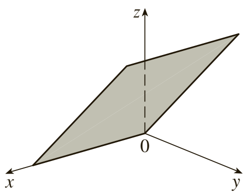
19. $\{(x, y) \mid y \geqslant x^2, x \neq \pm 1\}$



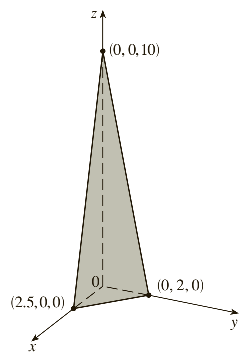
21. $\{(x, y, z) \mid -2 \leqslant x \leqslant 2, -3 \leqslant y \leqslant 3, -1 \leqslant z \leqslant 1\}$



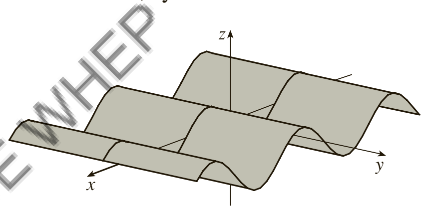
23. $z = y$, plane through the x-axis



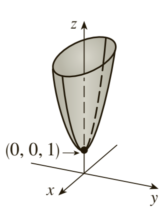
25. $4x + 5y + z = 10$, plane



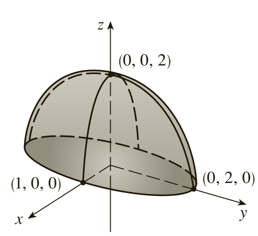
27. $z = \sin x$, cylinder



29. $z = x^2 + 4y^2 + 1$, elliptic paraboloid

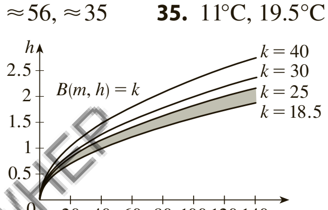


31. $z = \sqrt{4 - 4x^2 - y^2}$, top half of ellipsoid



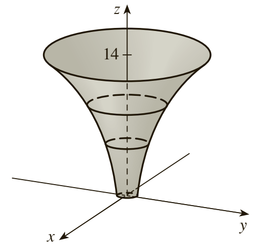
33. $\approx 56, \approx 35$ **35.** 11°C, 19.5°C **37.** Steep; nearly flat

39.

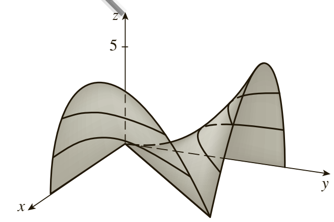


No

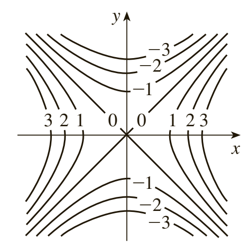
41.



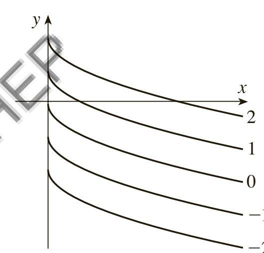
43.



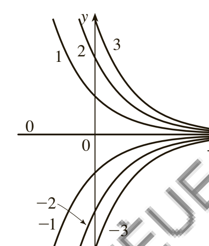
45. $x^2 - y^2 = k$



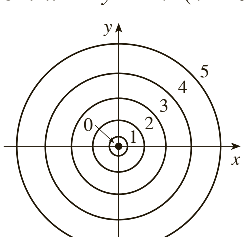
47. $y = -\sqrt{x} + k$



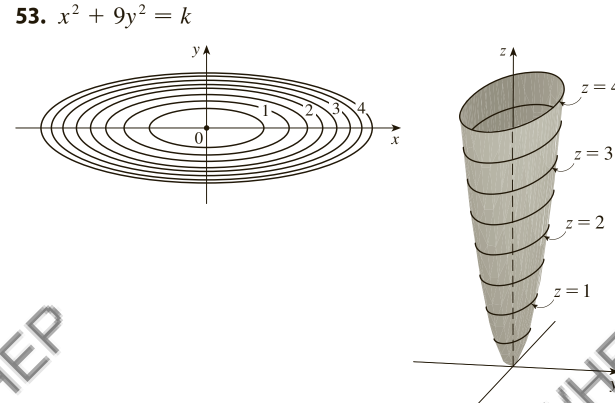
49. $y = ke^{-x}$



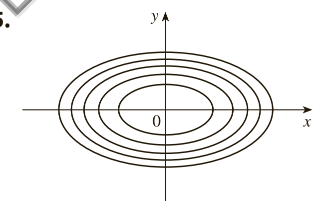
51. $x^2 + y^2 = k^3$ $(k \geqslant 0)$



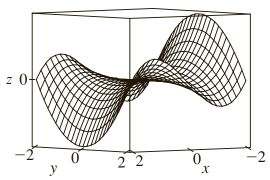
53. $x^2 + 9y^2 = k$



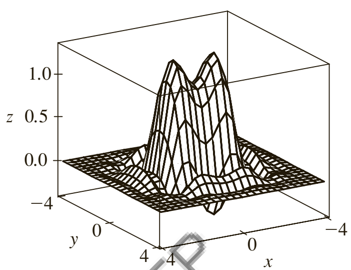
55.



57.

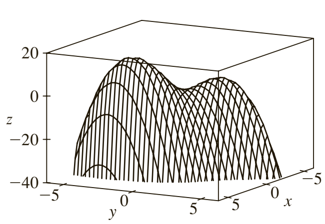


59.



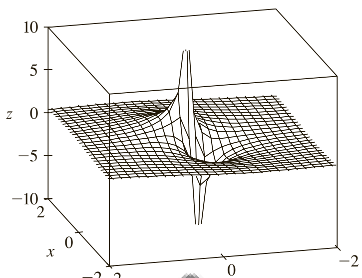
61. (a) C (b) II **63.** (a) F (b) I
65. (a) B (b) VI **67.** Family of parallel planes
69. Family of circular cylinders with axis the x-axis ($k > 0$)
71. (a) Shift the graph of f upward 2 units
(b) Stretch the graph of f vertically by a factor of 2
(c) Reflect the graph of f about the xy-plane
(d) Reflect the graph of f about the xy-plane and then shift it upward 2 units

73.



f appears to have a maximum value of about 15. There are two local maximum points but no local minimum point.

75.

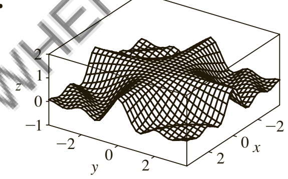


The function values approach 0 as x, y become large; as (x, y) approaches the origin, f approaches $\pm\infty$ or 0, depending on the direction of approach.

77. If $c = 0$, the graph is a cylindrical surface. For $c > 0$, the level curves are ellipses. The graph curves upward as we leave the origin, and the steepness increases as c increases. For $c < 0$, the level curves are hyperbolas. The graph curves upward in the y-direction and downward, approaching the xy-plane, in the x-direction giving a saddle-shaped appearance near (0, 0, 1).
79. $c = -2, 0, 2$ **81.** (b) $y = 0.75x + 0.01$

EXERCISES 14.2 ■ PAGE 950

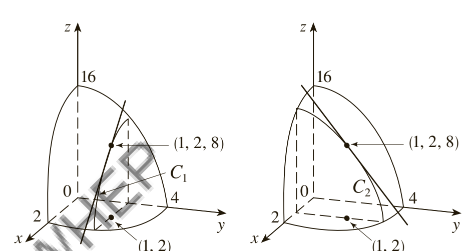
1. Nothing; if f is continuous, $f(3, 1) = 6$ **3.** $-\frac{5}{2}$
5. 56 **7.** $\pi/2$ **9.** Does not exist **11.** Does not exist
13. 0 **15.** Does not exist **17.** 2 **19.** $\sqrt{3}$
21. Does not exist
23. The graph shows that the function approaches different numbers along different lines.
25. $h(x, y) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$; $\{(x, y) \mid 2x + 3y \geqslant 6\}$
27. Along the line $y = x$ **29.** $\mathbb{R}^2$
31. $\{(x, y) \mid x^2 + y^2 \neq 1\}$ **33.** $\{(x, y) \mid x^2 + y^2 \leqslant 1, x \geqslant 0\}$
35. $\{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$
37. $\{(x, y) \mid (x, y) \neq (0, 0)\}$ **39.** 0 **41.** -1
43.



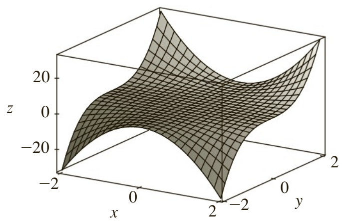
f is continuous on $\mathbb{R}^2$

EXERCISES 14.3 ■ PAGE 963

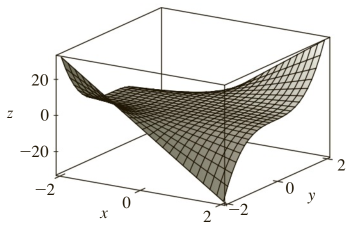
1. (a) The rate of change of temperature as longitude varies, with latitude and time fixed; the rate of change as only latitude varies; the rate of change as only time varies
(b) Positive, negative, positive
3. (a) $f_T(-15, 30) \approx 1.3$; for a temperature of -15°C and wind speed of 30 km/h, the wind-chill index rises by 1.3°C for each degree the temperature increases. $f_v(-15, 30) \approx -0.15$; for a temperature of -15°C and wind speed of 30 km/h, the wind-chill index decreases by 0.15°C for each km/h the wind speed increases.
(b) Positive, negative (c) 0
5. (a) Positive (b) Negative
7. (a) Positive (b) Negative
9. $c = f$, $b = f_x$, $a = f_y$
11. $f_x(1, 2) = -8 =$ slope of C_1, $f_y(1, 2) = -4 =$ slope of C_2



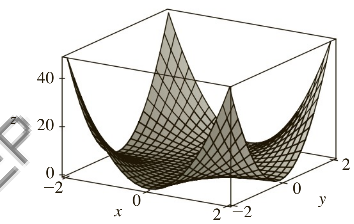
13.



$f(x, y) = x^2y^3$



$f_x(x, y) = 2xy^3$



$f_y(x, y) = 3x^2y^2$

15. $f_x(x, y) = 4x^3 + 5y^3$, $f_y(x, y) = 15xy^2$

17. $f_x(x, t) = -t^2e^{-x}$, $f_t(x, t) = 2te^{-x}$

19. $\dfrac{\partial z}{\partial x} = \dfrac{1}{x + t^2}$, $\dfrac{\partial z}{\partial t} = \dfrac{2t}{x + t^2}$

21. $f_x(x, y) = 1/y$, $f_y(x, y) = -x/y^2$

23. $f_x(x, y) = \dfrac{(ad - bc)y}{(cx + dy)^2}$, $f_y(x, y) = \dfrac{(bc - ad)x}{(cx + dy)^2}$

25. $g_u(u, v) = 10uv(u^2v - v^3)^4$, $g_v(u, v) = 5(u^2 - 3v^2)(u^2v - v^3)^4$

27. $R_p(p, q) = \dfrac{q^2}{1 + p^2q^4}$, $R_q(p, q) = \dfrac{2pq}{1 + p^2q^4}$

29. $F_x(x, y) = \cos(e^x)$, $F_y(x, y) = -\cos(e^y)$

31. $f_x = 3x^2yz^2$, $f_y = x^3z^2 + 2z$, $f_z = 2x^3yz + 2y$

33. $\partial w/\partial x = 1/(x + 2y + 3z)$, $\partial w/\partial y = 2/(x + 2y + 3z)$, $\partial w/\partial z = 3/(x + 2y + 3z)$

35. $\partial p/\partial t = 2t^3/\sqrt{t^4 + u^2\cos v}$,
$\partial p/\partial u = u\cos v/\sqrt{t^4 + u^2\cos v}$,
$\partial p/\partial v = -u^2\sin v/\left(2\sqrt{t^4 + u^2\cos v}\right)$

37. $h_x = 2xy\cos(z/t)$, $h_y = x^2\cos(z/t)$,
$h_z = (-x^2y/t)\sin(z/t)$, $h_t = (x^2yz/t^2)\sin(z/t)$

39. $\partial u/\partial x_i = x_i/\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

41. 1 **43.** $\frac{1}{6}$ **45.** $f_x(x, y) = y^2 - 3x^2y$, $f_y(x, y) = 2xy - x^3$

47. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{3z}$, $\dfrac{\partial z}{\partial y} = -\dfrac{2y}{3z}$

49. $\dfrac{\partial z}{\partial x} = \dfrac{yz}{e^z - xy}$, $\dfrac{\partial z}{\partial y} = \dfrac{xz}{e^z - xy}$

51. (a) $f'(x)$, $g'(y)$ (b) $f'(x + y)$, $f'(x + y)$

53. $f_{xx} = 12x^2y - 12xy^2$, $f_{xy} = 4x^3 - 12x^2y = f_{yx}$, $f_{yy} = -4x^3$

55. $z_{xx} = \dfrac{8y}{(2x + 3y)^3}$, $z_{xy} = \dfrac{6y - 4x}{(2x + 3y)^3} = z_{yx}$,
$z_{yy} = -\dfrac{12x}{(2x + 3y)^3}$

57. $v_{ss} = 2\cos(s^2 - t^2) - 4s^2\sin(s^2 - t^2)$,
$v_{st} = 4st\sin(s^2 - t^2) = v_{ts}$,
$v_{tt} = -2\cos(s^2 - t^2) - 4t^2\sin(s^2 - t^2)$

63. $24xy^2 - 6y$, $24x^2y - 6x$ **65.** $(2x^2y^2z^5 + 6xyz^3 + 2z)e^{xyz^2}$

67. $\frac{3}{4}v(u + v^2)^{-5/2}$ **69.** $4/(y + 2z)^3$, 0 **71.** $6yz^2$

73. ≈ 12.2, ≈ 16.8, ≈ 23.25 **83.** R^2/R_1^2

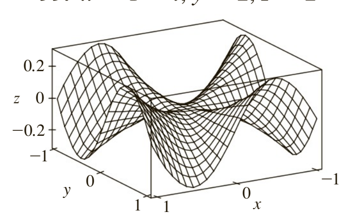
87. $\dfrac{\partial T}{\partial P} = \dfrac{V - nb}{nR}$, $\dfrac{\partial P}{\partial V} = \dfrac{2n^2a}{V^3} - \dfrac{nRT}{(V - nb)^2}$

91. (a) ≈ 0.0545; for a person 70 inches tall who weighs 160 pounds, an increase in weight causes the surface area to increase at a rate of about 0.0545 square feet per pound.
(b) ≈ 0.213; for a person 70 inches tall who weighs 160 pounds, an increase in height (with no change in weight) causes the surface area to increase at a rate of about 0.213 square feet per inch of height.

93. $\partial P/\partial v = 3Av^2 - \dfrac{B(mg/x)^2}{v^2}$ is the rate of change of the power needed during flapping mode with respect to the bird's velocity when the mass and fraction of flapping time remain constant; $\partial P/\partial x = -\dfrac{2Bm^2g^2}{x^3v}$ is the rate at which the power changes when only the fraction of time spent in flapping mode varies; $\partial P/\partial m = \dfrac{2Bmg^2}{x^2v}$ is the rate of change of the power when only the mass varies.

97. No **99.** $x = 1 + t$, $y = 2$, $z = 2 - 2t$ **103.** -2

105. (a)



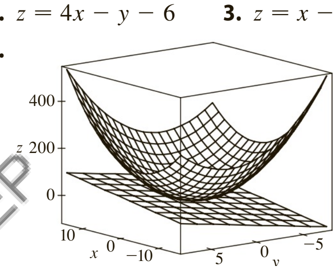
(b) $f_x(x, y) = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$, $f_y(x, y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}$

(c) 0, 0 (e) No, since f_{xy} and f_{yx} are not continuous.

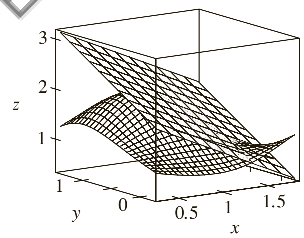
EXERCISES 14.4 ■ PAGE 974

1. $z = 4x - y - 6$ **3.** $z = x - y + 1$ **5.** $x + y + z = 0$

7.



9.



11. $6x + 4y - 23$ **13.** $2x + y - 1$
15. $2x + 2y + \pi - 4$ **19.** 6.3
21. $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$; 6.9914 **23.** $4T + H - 329$; $129°F$
25. $dz = -2e^{-2x}\cos 2\pi t\,dx - 2\pi e^{-2x}\sin 2\pi t\,dt$
27. $dm = 5p^4q^3\,dp + 3p^5q^2\,dq$
29. $dR = \beta^2\cos\gamma\,d\alpha + 2\alpha\beta\cos\gamma\,d\beta - \alpha\beta^2\sin\gamma\,d\gamma$
31. $\Delta z = 0.9225$, $dz = 0.9$ **33.** 5.4 cm^2 **35.** 16 cm^3
37. $\approx -0.0165mg$; decrease **39.** $\frac{1}{17} \approx 0.059\ \Omega$
41. (a) $0.8264m - 34.56h + 38.02$ (b) 18.801
43. $\varepsilon_1 = \Delta x$, $\varepsilon_2 = \Delta y$

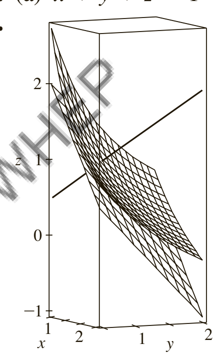
EXERCISES 14.5 ■ PAGE 983

1. $2t(y^3 - 2xy + 3xy^2 - x^2)$
3. $\dfrac{1}{2\sqrt{t}}\cos x\cos y + \dfrac{1}{t^2}\sin x\sin y$
5. $e^{y/z}[2t - (x/z) - (2xy/z^2)]$
7. $\partial z/\partial s = 5(x - y)^4(2st - t^2)$, $\partial z/\partial t = 5(x - y)^4(s^2 - 2st)$
9. $\dfrac{\partial z}{\partial s} = \dfrac{3\sin t - 2t\sin s}{3x + 2y}$, $\dfrac{\partial z}{\partial t} = \dfrac{3s\cos t + 2\cos s}{3x + 2y}$
11. $\dfrac{\partial z}{\partial s} = e^r\left(t\cos\theta - \dfrac{s}{\sqrt{s^2 + t^2}}\sin\theta\right)$,
$\dfrac{\partial z}{\partial t} = e^r\left(s\cos\theta - \dfrac{t}{\sqrt{s^2 + t^2}}\sin\theta\right)$
13. 42 **15.** $7, 2$
17. $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r}$, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s}$,
$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}$
19. $\dfrac{\partial T}{\partial x} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial x} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial x} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial x}$,
$\dfrac{\partial T}{\partial y} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial y} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial y} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial y}$,
$\dfrac{\partial T}{\partial z} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial z} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial z} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial z}$
21. $1582, 3164, -700$ **23.** $2\pi, -2\pi$
25. $\frac{5}{144}, -\frac{5}{96}, \frac{5}{144}$ **27.** $\dfrac{2x + y\sin x}{\cos x - 2y}$
29. $\dfrac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}$
31. $-\dfrac{x}{3z}, -\dfrac{2y}{3z}$ **33.** $\dfrac{yz}{e^z - xy}, \dfrac{xz}{e^z - xy}$
35. $2°\text{C/s}$ **37.** ≈ -0.33 m/s per minute
39. (a) $6\text{ m}^3/\text{s}$ (b) $10\text{ m}^2/\text{s}$ (c) 0 m/s
41. ≈ -0.27 L/s **43.** $-1/(12\sqrt{3})$ rad/s

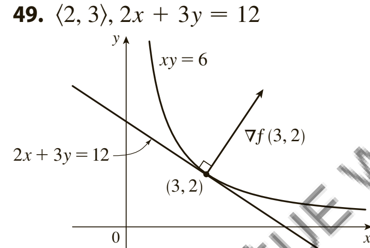
45. (a) $\partial z/\partial r = (\partial z/\partial x)\cos\theta + (\partial z/\partial y)\sin\theta$,
$\partial z/\partial\theta = -(\partial z/\partial x)\,r\sin\theta + (\partial z/\partial y)\,r\cos\theta$
51. $4rs\,\partial^2z/\partial x^2 + (4r^2 + 4s^2)\partial^2z/\partial x\,\partial y + 4rs\,\partial^2z/\partial y^2 + 2\,\partial z/\partial y$

EXERCISES 14.6 ■ PAGE 996

1. ≈ -0.08 mb/km **3.** ≈ 0.778 **5.** $\sqrt{2}/2$
7. (a) $\nabla f(x, y) = (1/y)\mathbf{i} - (x/y^2)\mathbf{j}$ (b) $\mathbf{i} - 2\mathbf{j}$ (c) -1
9. (a) $\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\rangle$
(b) $\langle -3, 2, 2\rangle$ (c) $\frac{2}{5}$
11. $\dfrac{4 - 3\sqrt{3}}{10}$ **13.** $7/(2\sqrt{5})$ **15.** 1 **17.** $\frac{23}{42}$
19. $\frac{2}{5}$ **21.** $\sqrt{65}, \langle 1, 8\rangle$ **23.** $1, \langle 0, 1\rangle$
25. $\frac{3}{4}, \langle 1, -2, -2\rangle$ **27.** (b) $\langle -12, 92\rangle$
29. All points on the line $y = x + 1$ **31.** (a) $-40/(3\sqrt{3})$
33. (a) $32/\sqrt{3}$ (b) $\langle 38, 6, 12\rangle$ (c) $2\sqrt{406}$
35. $\frac{327}{13}$ **39.** $\frac{774}{25}$
41. (a) $x + y + z = 11$ (b) $x - 3 = y - 3 = z - 5$
43. (a) $x + 2y + 6z = 12$ (b) $x - 2 = \dfrac{y - 2}{2} = \dfrac{z - 1}{6}$
45. (a) $x + y + z = 1$ (b) $x = y = z - 1$
47.



49. $\langle 2, 3\rangle$, $2x + 3y = 12$



55. No **59.** $\left(-\frac{5}{4}, -\frac{5}{4}, \frac{25}{8}\right)$
63. $x = -1 - 10t$, $y = 1 - 16t$, $z = 2 - 12t$
65. $(-1, 0, 1)$; $\approx 7.8°$
69. If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, then $af_x + bf_y$ and $cf_x + df_y$ are known, so we solve linear equations for f_x and f_y.

EXERCISES 14.7 ■ PAGE 1007

1. (a) f has a local minimum at (1, 1).
(b) f has a saddle point at (1, 1).
3. Local minimum at (1, 1), saddle point at (0, 0)
5. Minimum $f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$
7. Saddle points at $(1, 1)$, $(-1, -1)$
9. Minima $f\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right) = f\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = -\dfrac{1}{4}$,
saddle point at (0, 0)
11. Maximum $f(-1, 0) = 2$, minimum $f(1, 0) = -2$,
saddle points at $(0, \pm 1)$
13. Maximum $f(0, -1) = 2$, minima $f(\pm 1, 1) = -3$,
saddle points at $(0, 1)$, $(\pm 1, -1)$
15. None
17. Minima $f(x, y) = 1$ at all points (x, y) on x- and y-axes
19. Minima $f(0, 1) = f(\pi, -1) = f(2\pi, 1) = -1$,
saddle points at $(\pi/2, 0)$, $(3\pi/2, 0)$

23. Minima $f(1, \pm 1) = f(-1, \pm 1) = 3$

25. Maximum $f(\pi/3, \pi/3) = 3\sqrt{3}/2$, minimum $f(5\pi/3, 5\pi/3) = -3\sqrt{3}/2$, saddle point at (π, π)

27. Minima $f(0, -0.794) \approx -1.191$, $f(\pm 1.592, 1.267) \approx -1.310$, saddle points $(\pm 0.720, 0.259)$, lowest points $(\pm 1.592, 1.267, -1.310)$

29. Maximum $f(0.170, -1.215) \approx 3.197$, minima $f(-1.301, 0.549) \approx -3.145$, $f(1.131, 0.549) \approx -0.701$, saddle points $(-1.301, -1.215)$, $(0.170, 0.549)$, $(1.131, -1.215)$, no highest or lowest point

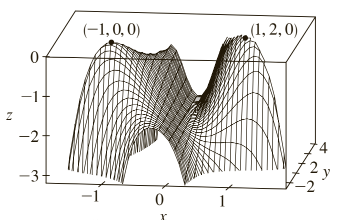
31. Maximum $f(0, \pm 2) = 4$, minimum $f(1, 0) = -1$

33. Maximum $f(\pm 1, 1) = 7$, minimum $f(0, 0) = 4$

35. Maximum $f(0, 3) = f(2, 3) = 7$, minimum $f(1, 1) = -2$

37. Maximum $f(1, 0) = 2$, minimum $f(-1, 0) = -2$

39.



41. $2/\sqrt{3}$ **43.** $(2, 1, \sqrt{5})$, $(2, 1, -\sqrt{5})$ **45.** $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$

47. $8r^3/(3\sqrt{3})$ **49.** $\frac{4}{3}$ **51.** Cube, edge length $c/12$

53. Square base of side 40 cm, height 20 cm **55.** $L^3/(3\sqrt{3})$

57. (a) $H = -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2) \ln(1 - p_1 - p_2)$
(b) $\{(p_1, p_2) \mid 0 < p_1 < 1, p_2 < 1 - p_1\}$
(c) $\ln 3$; $p_1 = p_2 = p_3 = \frac{1}{3}$

EXERCISES 14.8 ■ PAGE 1017

1. $\approx 59, 30$

3. Maximum $f(\pm 1, 0) = 1$, minimum $f(0, \pm 1) = -1$

5. Maximum $f(1, 2) = f(-1, -2) = 2$, minimum $f(1, -2) = f(-1, 2) = -2$

7. Maximum $f(2, 2, 1) = 9$, minimum $f(-2, -2, -1) = -9$

9. Maximum $f(1, \pm\sqrt{2}, 1) = f(-1, \pm\sqrt{2}, -1) = 2$, minimum $f(1, \pm\sqrt{2}, -1) = f(-1, \pm\sqrt{2}, 1) = -2$

11. Maximum $\sqrt{3}$, minimum 1

13. Maximum $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 2$, minimum $f(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) = -2$

15. Minimum $f(1, 1) = f(-1, -1) = 2$

17. Maximum $f(0, 1, \sqrt{2}) = 1 + \sqrt{2}$, minimum $f(0, 1, -\sqrt{2}) = 1 - \sqrt{2}$

19. Maximum $\frac{3}{2}$, minimum $\frac{1}{2}$

21. Maximum $f(3/\sqrt{2}, -3/\sqrt{2}) = 9 + 12\sqrt{2}$, minimum $f(-2, 2) = -8$

23. Maximum $f(\pm 1/\sqrt{2}, \mp 1/(2\sqrt{2})) = e^{1/4}$, minimum $f(\pm 1/\sqrt{2}, \pm 1/(2\sqrt{2})) = e^{-1/4}$

31–43. See Exercises 41–55 in Section 14.7.

45. Nearest $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, farthest $(-1, -1, 2)$

47. Maximum ≈ 9.7938, minimum ≈ -5.3506

49. (a) c/n (b) When $x_1 = x_2 = \cdots = x_n$

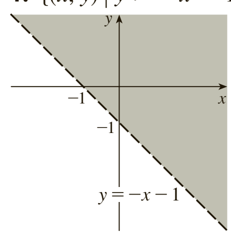
CHAPTER 14 REVIEW ■ PAGE 1022

True-False Quiz

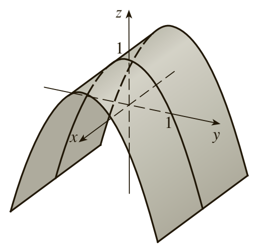
1. True **3.** False **5.** False **7.** True **9.** False **11.** True

Exercises

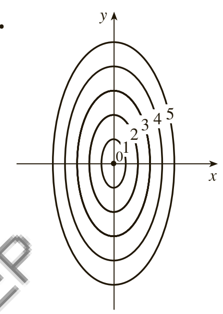
1. $\{(x, y) \mid y > -x - 1\}$



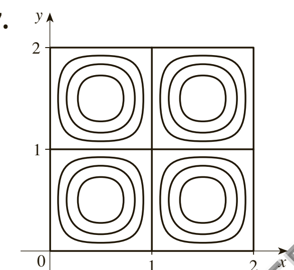
3.



5.



7.



9. $\frac{2}{3}$

11. (a) $\approx 3.5°C/m$, $-3.0°C/m$
(b) $\approx 0.35°C/m$ by Equation 14.6.9 (Definition 14.6.2 gives $\approx 1.1°C/m$.)
(c) -0.25

13. $f_x = 32xy(5y^3 + 2x^2y)^7$, $f_y = (16x^2 + 120y^2)(5y^3 + 2x^2y)^7$

15. $F_\alpha = \dfrac{2\alpha^3}{\alpha^2 + \beta^2} + 2\alpha \ln(\alpha^2 + \beta^2)$, $F_\beta = \dfrac{2\alpha^2\beta}{\alpha^2 + \beta^2}$

17. $S_u = \arctan(v\sqrt{w})$, $S_v = \dfrac{u\sqrt{w}}{1 + v^2 w}$, $S_w = \dfrac{uv}{2\sqrt{w}(1 + v^2 w)}$

19. $f_{xx} = 24x$, $f_{xy} = -2y = f_{yx}$, $f_{yy} = -2x$

21. $f_{xx} = k(k-1)x^{k-2}y^l z^m$, $f_{xy} = klx^{k-1}y^{l-1}z^m = f_{yx}$, $f_{xz} = kmx^{k-1}y^l z^{m-1} = f_{zx}$, $f_{yy} = l(l-1)x^k y^{l-2}z^m$, $f_{yz} = lmx^k y^{l-1}z^{m-1} = f_{zy}$, $f_{zz} = m(m-1)x^k y^l z^{m-2}$

25. (a) $z = 8x + 4y + 1$ (b) $\dfrac{x - 1}{8} = \dfrac{y + 2}{4} = \dfrac{z - 1}{-1}$

27. (a) $2x - 2y - 3z = 3$ (b) $\dfrac{x - 2}{4} = \dfrac{y + 1}{-4} = \dfrac{z - 1}{-6}$

29. (a) $x + 2y + 5z = 0$
(b) $x = 2 + t$, $y = -1 + 2t$, $z = 5t$

31. $(2, \frac{1}{2}, -1)$, $(-2, -\frac{1}{2}, 1)$

33. $60x + \frac{24}{5}y + \frac{32}{5}z - 120$; 38.656

35. $2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

37. $-47, 108$

43. $\langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2}\rangle$ **45.** $-\frac{4}{5}$

47. $\sqrt{145}/2$, $\langle 4, \frac{9}{2}\rangle$ **49.** $\approx \frac{5}{8}$ knots/mi

51. Minimum $f(-4, 1) = -11$

53. Maximum $f(1, 1) = 1$; saddle points $(0, 0)$, $(0, 3)$, $(3, 0)$

55. Maximum $f(1, 2) = 4$, minimum $f(2, 4) = -64$

57. Maximum $f(-1, 0) = 2$, minima $f(1, \pm 1) = -3$, saddle points $(-1, \pm 1)$, $(1, 0)$

59. Maximum $f\left(\pm\sqrt{2/3}, 1/\sqrt{3}\right) = 2/\left(3\sqrt{3}\right)$, minimum $f\left(\pm\sqrt{2/3}, -1/\sqrt{3}\right) = -2/\left(3\sqrt{3}\right)$
61. Maximum 1, minimum -1
63. $\left(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right), \left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$
65. $P\left(2-\sqrt{3}\right), P\left(3-\sqrt{3}\right)/6, P\left(2\sqrt{3}-3\right)/3$

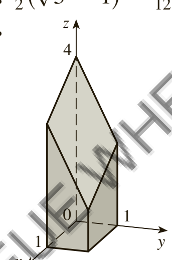
PROBLEMS PLUS ■ PAGE 1025

1. $L^2W^2, \frac{1}{4}L^2W^2$ **3.** (a) $x = w/3$, base $= w/3$ (b) Yes
7. $\sqrt{3/2}, 3/\sqrt{2}$

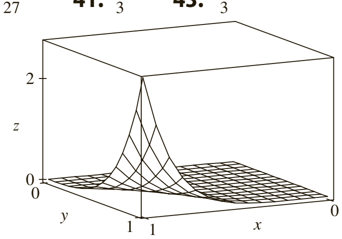
CHAPTER 15

EXERCISES 15.1 ■ PAGE 1039

1. (a) 288 (b) 144 **3.** (a) 0.990 (b) 1.151
5. $U < V < L$ **7.** (a) ≈ 248 (b) ≈ 15.5
9. $24\sqrt{2}$ **11.** 3 **13.** $2 + 8y^2, 3x + 27x^2$
15. 222 **17.** $\frac{5}{2} - e^{-1}$ **19.** 18
21. $\frac{15}{2}\ln 2 + \frac{3}{2}\ln 4$ or $\frac{21}{2}\ln 2$ **23.** 6
25. $\frac{31}{30}$ **27.** 2 **29.** $9\ln 2$
31. $\frac{1}{2}\left(\sqrt{3}-1\right) - \frac{1}{12}\pi$ **33.** $\frac{1}{2}e^{-6} + \frac{5}{2}$
35.



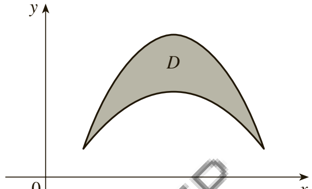
37. 51 **39.** $\frac{166}{27}$ **41.** $\frac{8}{3}$ **43.** $\frac{64}{3}$
45. $21e - 57$



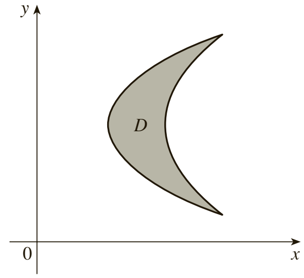
47. $\frac{5}{6}$ **49.** 0
51. Fubini's Theorem does not apply. The integrand has an infinite discontinuity at the origin.

EXERCISES 15.2 ■ PAGE 1048

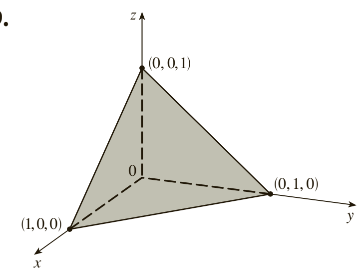
1. $\frac{868}{3}$ **3.** $\frac{1}{6}(e-1)$ **5.** $\frac{1}{3}\sin 1$
7. $\frac{1}{4}\ln 17$ **9.** $\frac{1}{2}\left(1 - e^{-9}\right)$
11. (a)



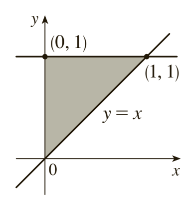
(b)



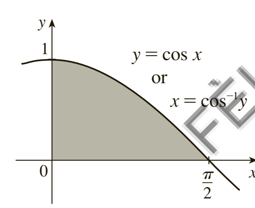
13. Type I: $D = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x\}$, type II: $D = \{(x, y) \mid 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 1\}$; $\frac{1}{3}$
15. $\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\,dy\,dx = \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy = \frac{9}{4}$
17. $\frac{1}{2}(1 - \cos 1)$ **19.** $\frac{11}{3}$ **21.** 0 **23.** $\frac{3}{4}$
25. $\frac{31}{8}$ **27.** $\frac{16}{3}$ **29.** $\frac{128}{15}$ **31.** $\frac{1}{3}$
33. 0, 1.213; 0.713 **35.** $\frac{64}{3}$
37. $\dfrac{10}{3\sqrt{2}}$ or $\dfrac{5\sqrt{2}}{3}$
39.



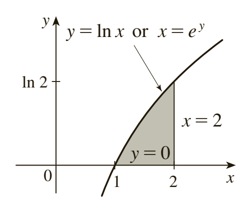
41. 13,984,735,616/14,549,535 **43.** $\pi/2$
45. $\int_0^1 \int_x^1 f(x, y)\,dy\,dx$



47. $\int_0^1 \int_0^{\cos^{-1}y} f(x, y)\,dx\,dy$



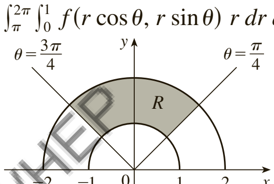
49. $\int_0^{\ln 2} \int_{e^y}^2 f(x, y)\,dx\,dy$



51. $\frac{1}{6}\left(e^9 - 1\right)$ **53.** $\frac{2}{9}\left(2\sqrt{2} - 1\right)$
55. $\frac{1}{3}\left(2\sqrt{2} - 1\right)$ **57.** 1
59. $\dfrac{\sqrt{3}}{2}\pi \leqslant \iint_S \sqrt{4 - x^2y^2}\,dA \leqslant \pi$
61. $\frac{3}{4}$ **65.** 9π **67.** $a^2b + \frac{3}{2}ab^2$ **69.** $\pi a^2 b$

EXERCISES 15.3 ■ PAGE 1054

1. $\int_0^{2\pi} \int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$
3. $\int_\pi^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$
5. $3\pi/4$



7. $\frac{1250}{3}$ **9.** $(\pi/4)(\cos 1 - \cos 9)$
11. $(\pi/2)(1 - e^{-4})$ **13.** $\frac{3}{64}\pi^2$ **15.** $\pi/12$
17. $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$ **19.** $\frac{625}{2}\pi$ **21.** 4π **23.** $\frac{4}{3}\pi a^3$
25. $(\pi/3)(2 - \sqrt{2})$ **27.** $(8\pi/3)(64 - 24\sqrt{3})$
29. $(\pi/4)(1 - e^{-4})$ **31.** $\frac{1}{120}$ **33.** 4.5951
35. $1800\pi\ \text{ft}^3$ **37.** $2/(a + b)$ **39.** $\frac{15}{16}$
41. (a) $\sqrt{\pi}/4$ (b) $\sqrt{\pi}/2$

EXERCISES 15.4 ■ PAGE 1064

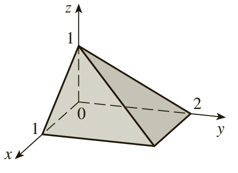
1. 285 C **3.** $42k$, $(2, \frac{85}{28})$ **5.** 6, $(\frac{3}{4}, \frac{3}{2})$ **7.** $\frac{8}{15}k$, $(0, \frac{4}{7})$
9. $\frac{1}{8}(1 - 3e^{-2})$, $\left(\dfrac{e^2 - 5}{e^2 - 3}, \dfrac{8(e^3 - 4)}{27(e^3 - 3e)}\right)$
11. $(\frac{3}{8}, 3\pi/16)$ **13.** $(0, 45/(14\pi))$
15. $(2a/5, 2a/5)$ if vertex is (0, 0) and sides are along positive axes
17. $409.2k$, $182k$, $591.2k$
19. $7ka^6/180$, $7ka^6/180$, $7ka^6/90$ if vertex is (0, 0) and sides are along positive axes
21. $\rho bh^3/3$, $\rho b^3h/3$; $b/\sqrt{3}$, $h/\sqrt{3}$
23. $\rho a^4\pi/16$, $\rho a^4\pi/16$; $a/2$, $a/2$
25. $m = 3\pi/64$, $(\bar{x}, \bar{y}) = \left(\dfrac{16384\sqrt{2}}{10395\pi}, 0\right)$,
$I_x = \dfrac{5\pi}{384} - \dfrac{4}{105}$, $I_y = \dfrac{5\pi}{384} + \dfrac{4}{105}$, $I_0 = \dfrac{5\pi}{192}$
27. (a) $\frac{1}{2}$ (b) 0.375 (c) $\frac{5}{48} \approx 0.1042$
29. (b) (i) $e^{-0.2} \approx 0.8187$
(ii) $1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$ (c) 2, 5
31. (a) ≈ 0.500 (b) ≈ 0.632
33. (a) $\iint_D k\left[1 - \frac{1}{20}\sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$, where D is the disk with radius 10 mi centered at the center of the city
(b) $200\pi k/3 \approx 209k$, $200(\pi/2 - \frac{8}{9})k \approx 136k$, on the edge

EXERCISES 15.5 ■ PAGE 1068

1. $12\sqrt{35}$ **3.** $3\sqrt{14}$ **5.** $(\pi/6)(13\sqrt{13} - 1)$
7. $(\pi/6)(17\sqrt{17} - 5\sqrt{5})$ **9.** $(2\pi/3)(2\sqrt{2} - 1)$
11. $a^2(\pi - 2)$ **13.** 3.6258
15. (a) ≈ 1.83 (b) ≈ 1.8616
17. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[(11\sqrt{5} + 3\sqrt{70})/(3\sqrt{5} + \sqrt{70})\right]$
19. 3.3213 **23.** $(\pi/6)(101\sqrt{101} - 1)$

EXERCISES 15.6 ■ PAGE 1077

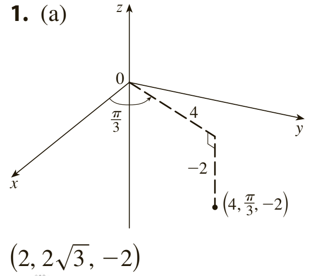
1. $\frac{27}{4}$ **3.** $\frac{16}{15}$ **5.** $\frac{5}{3}$ **7.** $\frac{2}{3}$ **9.** $\frac{27}{2}$ **11.** $9\pi/8$
13. $\frac{65}{28}$ **15.** $\frac{8}{15}$ **17.** $16\pi/3$ **19.** $\frac{16}{3}$ **21.** $\frac{8}{15}$
23. (a) $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx$ (b) $\frac{1}{4}\pi - \frac{1}{3}$
25. ≈ 0.985 **27.**



29. $\int_{-2}^{2} \int_0^{4-x^2} \int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dy\, dx$
$= \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dx\, dy$
$= \int_{-1}^{1} \int_0^{4-4z^2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dy\, dz$
$= \int_0^4 \int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dz\, dy$
$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_0^{4-x^2-4z^2} f(x, y, z)\, dy\, dz\, dx$
$= \int_{-1}^{1} \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} \int_0^{4-x^2-4z^2} f(x, y, z)\, dy\, dx\, dz$
31. $\int_{-2}^{2} \int_{x^2}^{4} \int_0^{2-y/2} f(x, y, z)\, dz\, dy\, dx$
$= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{2-y/2} f(x, y, z)\, dz\, dx\, dy$
$= \int_0^2 \int_0^{4-2z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$
$= \int_0^4 \int_0^{2-y/2} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dz\, dy$
$= \int_{-2}^{2} \int_0^{2-x^2/2} \int_{x^2}^{4-2z} f(x, y, z)\, dy\, dz\, dx$
$= \int_0^2 \int_{-\sqrt{4-2z}}^{\sqrt{4-2z}} \int_{x^2}^{4-2z} f(x, y, z)\, dy\, dx\, dz$
33. $\int_0^1 \int_{\sqrt{x}}^{1} \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z)\, dz\, dx\, dy$
$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x, y, z)\, dx\, dy\, dz = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\, dx\, dz\, dy$
$= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dz\, dx = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dx\, dz$
35. $\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy = \int_0^1 \int_0^x \int_0^y f(x, y, z)\, dz\, dy\, dx$
$= \int_0^1 \int_z^1 \int_y^1 f(x, y, z)\, dx\, dy\, dz = \int_0^1 \int_0^y \int_y^1 f(x, y, z)\, dx\, dz\, dy$
$= \int_0^1 \int_0^x \int_z^x f(x, y, z)\, dy\, dz\, dx = \int_0^1 \int_z^1 \int_z^x f(x, y, z)\, dy\, dx\, dz$
37. 64π **39.** $\frac{3}{2}\pi$, $(0, 0, \frac{1}{3})$
41. a^5, $(7a/12, 7a/12, 7a/12)$
43. $I_x = I_y = I_z = \frac{2}{3}kL^5$ **45.** $\frac{1}{2}\pi kha^4$
47. (a) $m = \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} \sqrt{x^2 + y^2}\, dz\, dy\, dx$
(b) $(\bar{x}, \bar{y}, \bar{z})$, where
$\bar{x} = (1/m) \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} x\sqrt{x^2 + y^2}\, dz\, dy\, dx$,
$\bar{y} = (1/m) \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} y\sqrt{x^2 + y^2}\, dz\, dy\, dx$,
and $\bar{z} = (1/m) \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} z\sqrt{x^2 + y^2}\, dz\, dy\, dx$
(c) $\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} (x^2 + y^2)^{3/2}\, dz\, dy\, dx$
49. (a) $\frac{3}{32}\pi + \frac{11}{24}$
(b) $\left(\dfrac{28}{9\pi + 44}, \dfrac{30\pi + 128}{45\pi + 220}, \dfrac{45\pi + 208}{135\pi + 660}\right)$
(c) $\frac{1}{240}(68 + 15\pi)$
51. (a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{1}{5760}$ **53.** $L^3/8$
55. (a) The region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$
(b) $4\sqrt{6}\pi/45$

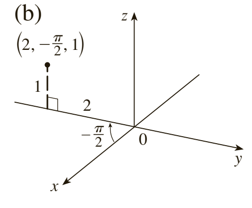
EXERCISES 15.7 ■ PAGE 1083

1. (a)



$(2, 2\sqrt{3}, -2)$

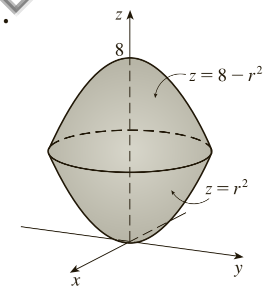
(b)



$(0, -2, 1)$

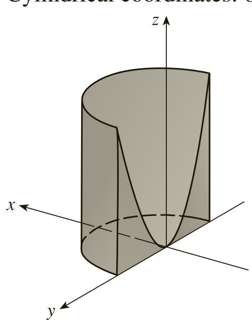
3. (a) $(\sqrt{2}, 3\pi/4, 1)$ (b) $(4, 2\pi/3, 3)$
5. Circular cylinder with radius 2 and axis the z-axis
7. Sphere, radius 2, centered at the origin
9. (a) $z^2 = 1 + r\cos\theta - r^2$ (b) $z = r^2\cos 2\theta$

11.



13. Cylindrical coordinates: $6 \leqslant r \leqslant 7$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant 20$

15. 4π



17. 384π **19.** $\frac{8}{3}\pi + \frac{128}{15}$ **21.** $2\pi/5$ **23.** $\frac{4}{3}\pi(\sqrt{2} - 1)$

25. (a) $\frac{512}{3}\pi$ (b) $(0, 0, \frac{23}{2})$

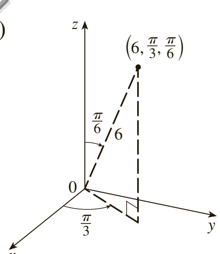
27. $\pi K a^2/8$, $(0, 0, 2a/3)$ **29.** 0

31. (a) $\iiint_C h(P)g(P)\,dV$, where C is the cone

(b) $\approx 3.1 \times 10^{19}$ ft-lb

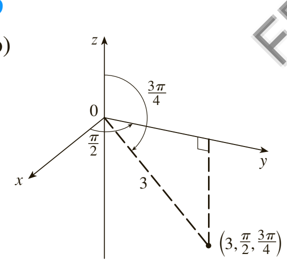
EXERCISES 15.8 ■ PAGE 1089

1. (a)



$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3}\right)$

(b)



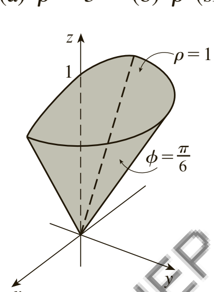
$\left(0, \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

3. (a) $(2, 3\pi/2, \pi/2)$ (b) $(2, 3\pi/4, 3\pi/4)$

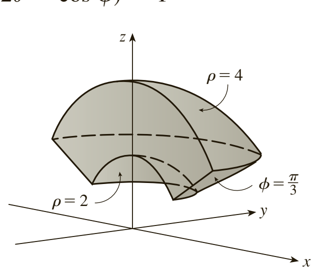
5. Half-cone **7.** Horizontal plane

9. (a) $\rho = 3$ (b) $\rho^2(\sin^2\phi \cos 2\theta - \cos^2\phi) = 1$

11.

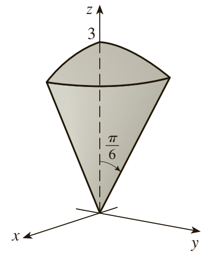


13.



15. $0 \leqslant \phi \leqslant \pi/4$, $0 \leqslant \rho \leqslant \cos\phi$

17. $(9\pi/4)(2 - \sqrt{3})$



19. $\int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$

21. $312{,}500\pi/7$ **23.** $1688\pi/15$ **25.** $\pi/8$

27. $(\sqrt{3} - 1)\pi a^3/3$ **29.** (a) 10π (b) $(0, 0, 2.1)$

31. (a) $\left(0, 0, \frac{7}{12}\right)$ (b) $11K\pi/960$

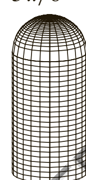
33. (a) $\left(0, 0, \frac{3}{8}a\right)$ (b) $4K\pi a^5/15$ (K is the density)

35. $\frac{1}{3}\pi(2 - \sqrt{2})$, $\left(0, 0, 3/\left[8(2 - \sqrt{2})\right]\right)$

37. (a) $\pi K a^4 h/2$ (K is the density) (b) $\pi K a^2 h(3a^2 + 4h^2)/12$

39. $5\pi/6$ **41.** $(4\sqrt{2} - 5)/15$ **43.** $4096\pi/21$

45.



47. $136\pi/99$

EXERCISES 15.9 ■ PAGE 1100

1. -6 **3.** s **5.** $2uvw$

7. The parallelogram with vertices (0, 0), (6, 3), (12, 1), (6, −2)

9. The region bounded by the line $y = 1$, the y-axis, and $y = \sqrt{x}$

11. $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(u + 2v)$ is one possible transformation, where $S = \{(u, v) \mid -1 \leqslant u \leqslant 1, 1 \leqslant v \leqslant 3\}$

13. $x = u\cos v$, $y = u\sin v$ is one possible transformation, where $S = \{(u, v) \mid 1 \leqslant u \leqslant \sqrt{2}, 0 \leqslant v \leqslant \pi/2\}$

15. -3 **17.** 6π **19.** $2\ln 3$

21. (a) $\frac{4}{3}\pi abc$ (b) $1.083 \times 10^{12}\ \text{km}^3$ (c) $\frac{4}{15}\pi(a^2 + b^2)abck$

23. $\frac{8}{5}\ln 8$ **25.** $\frac{3}{2}\sin 1$ **27.** $e - e^{-1}$

CHAPTER 15 REVIEW ■ PAGE 1101

True-False Quiz

1. True **3.** True **5.** True **7.** True **9.** False

Exercises

1. ≈ 64.0 **3.** $4e^2 - 4e + 3$ **5.** $\frac{1}{2}\sin 1$ **7.** $\frac{2}{3}$

9. $\int_0^{\pi}\int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

11. $(\sqrt{3}, 3, 2)$, $(4, \pi/3, \pi/3)$

13. $(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$, $(4, \pi/4, 4\sqrt{3})$

15. (a) $r^2 + z^2 = 4$, $\rho = 2$ (b) $r = 2$, $\rho\sin\phi = 2$

17. The region inside the loop of the four-leaved rose $r = \sin 2\theta$ in the first quadrant

19. $\frac{1}{2}\sin 1$ **21.** $\frac{1}{2}e^6 - \frac{7}{2}$ **23.** $\frac{1}{4}\ln 2$ **25.** 8

27. $81\pi/5$ **29.** $\frac{81}{2}$ **31.** $\pi/96$ **33.** $\frac{64}{15}$

35. 176 **37.** $\frac{2}{3}$ **39.** $2ma^3/9$

41. (a) $\frac{1}{4}$ (b) $\left(\frac{1}{3}, \frac{8}{15}\right)$

(c) $I_x = \frac{1}{12}$, $I_y = \frac{1}{24}$; $\bar{\bar{y}} = 1/\sqrt{3}$, $\bar{\bar{x}} = 1/\sqrt{6}$

43. (a) $(0, 0, h/4)$ (b) $\pi a^5 h/15$

45. $\ln(\sqrt{2} + \sqrt{3}) + \sqrt{2}/3$ **47.** $\frac{486}{5}$ **49.** 0.0512

51. (a) $\frac{1}{15}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$

53. $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$ **55.** $-\ln 2$ **57.** 0

PROBLEMS PLUS ■ PAGE 1105

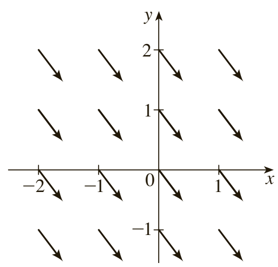
1. 30 **3.** $\frac{1}{2}\sin 1$ **7.** (b) 0.90

13. $abc\pi\left(\dfrac{2}{3} - \dfrac{8}{9\sqrt{3}}\right)$

CHAPTER 16

EXERCISES 16.1 ■ PAGE 1113

1.



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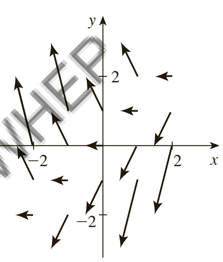

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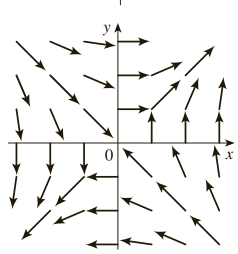
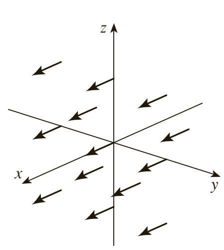
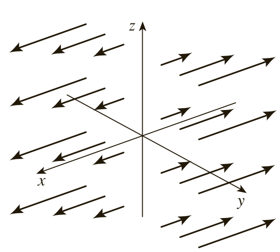


7.

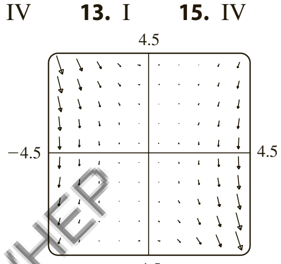


9.



11. IV **13.** I **15.** IV **17.** III

19.

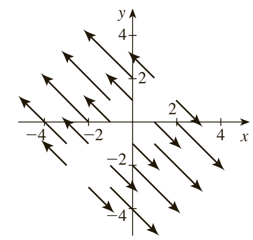


The line $y = 2x$

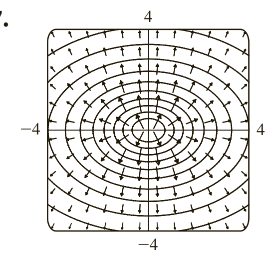
21. $\nabla f(x, y) = y^2 \cos(xy)\,\mathbf{i} + [xy\cos(xy) + \sin(xy)]\,\mathbf{j}$

23. $\nabla f(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{k}$

25. $\nabla f(x, y) = (x - y)\,\mathbf{i} + (y - x)\,\mathbf{j}$

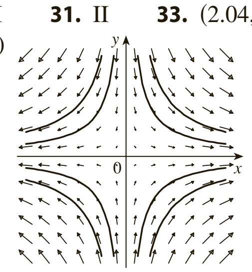


27.



29. III **31.** II **33.** (2.04, 1.03)

35. (a)



$y = C/x$

(b) $y = 1/x,\ x > 0$

EXERCISES 16.2 ■ PAGE 1124

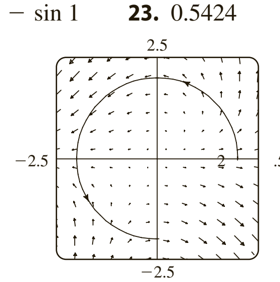
1. $\frac{4}{3}(10^{3/2} - 1)$ **3.** 1638.4 **5.** $\frac{1}{3}\pi^6 + 2\pi$ **7.** $\frac{5}{2}$

9. $\sqrt{2}/3$ **11.** $\frac{1}{12}\sqrt{14}(e^6 - 1)$ **13.** $\frac{2}{5}(e - 1)$ **15.** $\frac{35}{3}$

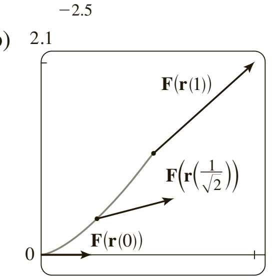
17. (a) Positive (b) Negative **19.** $\frac{1}{20}$

21. $\frac{6}{5} - \cos 1 - \sin 1$ **23.** 0.5424 **25.** 94.8231

27. $3\pi + \frac{2}{3}$



29. (a) $\frac{11}{8} - 1/e$ (b)



31. $\frac{172{,}704}{5{,}632{,}705}\sqrt{2}\,(1 - e^{-14\pi})$ **33.** $2\pi k,\ (4/\pi, 0)$

35. (a) $\bar{x} = (1/m)\int_C x\rho(x, y, z)\,ds$,
$\bar{y} = (1/m)\int_C y\rho(x, y, z)\,ds$,
$\bar{z} = (1/m)\int_C z\rho(x, y, z)\,ds$, where $m = \int_C \rho(x, y, z)\,ds$
(b) $(0, 0, 3\pi)$

37. $I_x = k\left(\frac{1}{2}\pi - \frac{4}{3}\right), I_y = k\left(\frac{1}{2}\pi - \frac{2}{3}\right)$ **39.** $2\pi^2$ **41.** $\frac{7}{3}$

43. (a) $2ma\,\mathbf{i} + 6mbt\,\mathbf{j},\ 0 \leqslant t \leqslant 1$ (b) $2ma^2 + \frac{9}{2}mb^2$

45. $\approx 1.67 \times 10^4$ ft-lb **47.** (b) Yes **51.** ≈ 22 J

EXERCISES 16.3 ■ PAGE 1134

1. 40 **3.** Not conservative
5. $f(x, y) = ye^{xy} + K$ **7.** $f(x, y) = ye^x + x\sin y + K$
9. $f(x, y) = y^2\sin x + x\cos y + K$
11. (b) 16 **13.** (a) $f(x, y) = \frac{1}{3}x^3y^3$ (b) -9
15. (a) $f(x, y, z) = xyz + z^2$ (b) 77
17. (a) $f(x, y, z) = ye^{xz}$ (b) 4 **19.** $4/e$
21. It doesn't matter which curve is chosen.
23. $\frac{31}{4}$ **25.** No **27.** Conservative
31. (a) Yes (b) Yes (c) Yes
33. (a) No (b) Yes (c) Yes

EXERCISES 16.4 ■ PAGE 1141

1. 120 **3.** $\frac{2}{3}$ **5.** $4(e^3 - 1)$ **7.** $\frac{1}{3}$
9. -24π **11.** $-\frac{16}{3}$ **13.** 4π
15. $\frac{1}{15}\pi^4 - \frac{4144}{1125}\pi^2 + \frac{7{,}578{,}368}{253{,}125} \approx 0.0779$
17. $-\frac{1}{12}$ **19.** 3π **21.** (c) $\frac{9}{2}$
23. $(4a/3\pi, 4a/3\pi)$ if the region is the portion of the disk $x^2 + y^2 = a^2$ in the first quadrant
27. 0

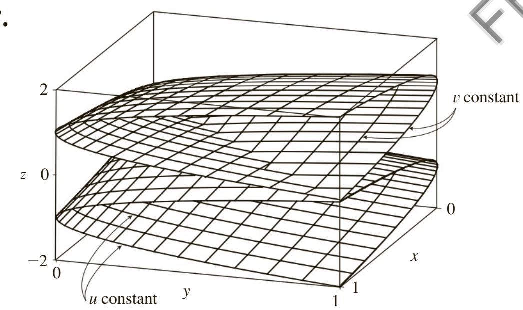
EXERCISES 16.5 ■ PAGE 1149

1. (a) $\mathbf{0}$ (b) $y^2z^2 + x^2z^2 + x^2y^2$
3. (a) $ze^x\,\mathbf{i} + (xye^z - yze^x)\,\mathbf{j} - xe^z\,\mathbf{k}$ (b) $y(e^z + e^x)$
5. (a) $-\dfrac{\sqrt{z}}{(1 + y)^2}\,\mathbf{i} - \dfrac{\sqrt{x}}{(1 + z)^2}\,\mathbf{j} - \dfrac{\sqrt{y}}{(1 + x)^2}\,\mathbf{k}$
(b) $\dfrac{1}{2\sqrt{x}\,(1 + z)} + \dfrac{1}{2\sqrt{y}\,(1 + x)} + \dfrac{1}{2\sqrt{z}\,(1 + y)}$
7. (a) $\langle -e^y\cos z, -e^z\cos x, -e^x\cos y\rangle$
(b) $e^x\sin y + e^y\sin z + e^z\sin x$
9. (a) Negative (b) curl $\mathbf{F} = \mathbf{0}$
11. (a) Zero (b) curl $\mathbf{F}$ points in the negative z-direction.
13. $f(x, y, z) = xy^2z^3 + K$ **15.** Not conservative
17. $f(x, y, z) = xe^{yz} + K$ **19.** No

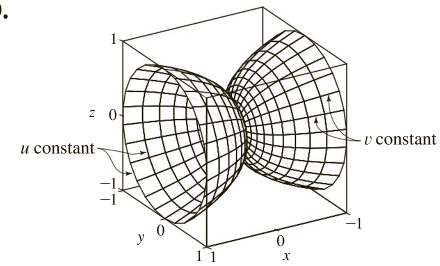
EXERCISES 16.6 ■ PAGE 1160

1. P: yes; Q: no
3. Plane through $(0, 3, 1)$ containing vectors $\langle 1, 0, 4\rangle, \langle 1, -1, 5\rangle$
5. Circular cone with axis the z-axis

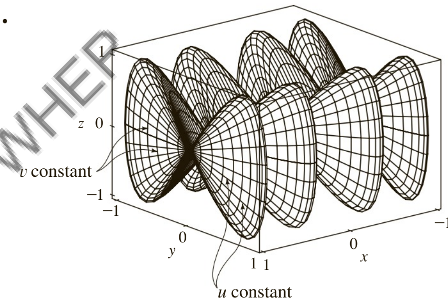
7.



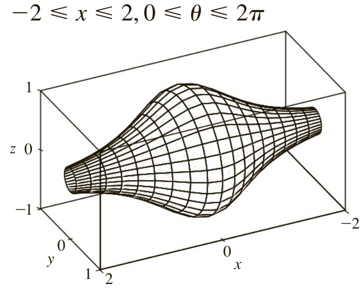
9.



11.

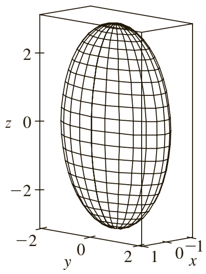


13. IV **15.** I **17.** III
19. $x = u, y = v - u, z = -v$
21. $y = y, z = z, x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$
23. $x = 2\sin\phi\cos\theta, y = 2\sin\phi\sin\theta$, $z = 2\cos\phi, 0 \leqslant \phi \leqslant \pi/4, 0 \leqslant \theta \leqslant 2\pi$ $\left[\text{or } x = x, y = y, z = \sqrt{4 - x^2 - y^2}, x^2 + y^2 \leqslant 2\right]$
25. $x = 6\sin\phi\cos\theta, y = 6\sin\phi\sin\theta, z = 6\cos\phi$, $\pi/6 \leqslant \phi \leqslant \pi/2, 0 \leqslant \theta \leqslant 2\pi$
29. $x = x, y = \dfrac{1}{1 + x^2}\cos\theta, y = \dfrac{1}{1 + x^2}\sin\theta$, $-2 \leqslant x \leqslant 2, 0 \leqslant \theta \leqslant 2\pi$



31. (a) Direction reverses (b) Number of coils doubles
33. $3x - y + 3z = 3$ **35.** $\dfrac{\sqrt{3}}{2}x - \dfrac{1}{2}y + z = \dfrac{\pi}{3}$
37. $-x + 2z = 1$ **39.** $3\sqrt{14}$ **41.** $\sqrt{14}\,\pi$

43. $\frac{4}{15}(3^{5/2} - 2^{7/2} + 1)$ **45.** $(2\pi/3)(2\sqrt{2} - 1)$
47. $(\pi/6)(65^{3/2} - 1)$ **49.** 4 **51.** $\pi R^2 \leqslant A(S) \leqslant \sqrt{3}\,\pi R^2$
53. 3.5618 **55.** (a) ≈ 24.2055 (b) 24.2476
57. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[(11\sqrt{5} + 3\sqrt{70})/(3\sqrt{5} + \sqrt{70})\right]$
59. (b)



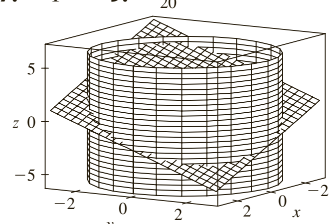
(c) $\int_0^{2\pi}\int_0^{\pi}\sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\;du\,dv$
61. 4π **63.** $2a^2(\pi - 2)$

EXERCISES 16.7 ■ PAGE 1172

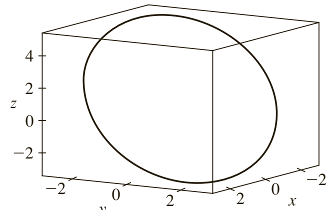
1. ≈ -6.93 **3.** 900π **5.** $11\sqrt{14}$ **7.** $\frac{2}{3}(2\sqrt{2} - 1)$
9. $171\sqrt{14}$ **11.** $\sqrt{21}/3$ **13.** $(\pi/120)(25\sqrt{5} + 1)$
15. $\frac{7}{4}\sqrt{21} - \frac{17}{12}\sqrt{17}$ **17.** 16π **19.** 0 **21.** 4
23. $\frac{713}{180}$ **25.** $\frac{8}{3}\pi$ **27.** 0 **29.** 48 **31.** $2\pi + \frac{8}{3}$
33. 4.5822 **35.** 3.4895
37. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D [P(\partial h/\partial x) - Q + R(\partial h/\partial z)]\,dA$, where D = projection of S onto xz-plane
39. $(0, 0, a/2)$
41. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\,dS$ (b) $4329\sqrt{2}\,\pi/5$
43. 0 kg/s **45.** $\frac{8}{3}\pi a^3\varepsilon_0$ **47.** 1248π

EXERCISES 16.8 ■ PAGE 1179

3. 16π **5.** 0 **7.** -1 **9.** $-\frac{17}{20}$
11. (a) $81\pi/2$ (b)



(c) $x = 3\cos t$, $y = 3\sin t$, $z = 1 - 3(\cos t + \sin t)$, $0 \leqslant t \leqslant 2\pi$



13. -32π **15.** $-\pi$ **17.** 3

EXERCISES 16.9 ■ PAGE 1185

1. $\frac{9}{2}$ **3.** $256\pi/3$ **5.** $\frac{9}{2}$ **7.** $9\pi/2$ **9.** 0
11. π **13.** 2π **15.** $341\sqrt{2}/60 + \frac{81}{20}\arcsin(\sqrt{3}/3)$
17. $13\pi/20$ **19.** Negative at P_1, positive at P_2
21. div $\mathbf{F} > 0$ in quadrants I, II; div $\mathbf{F} < 0$ in quadrants III, IV

CHAPTER 16 REVIEW ■ PAGE 1188

True-False Quiz

1. False **3.** True **5.** False **7.** False
9. True **11.** True **13.** False

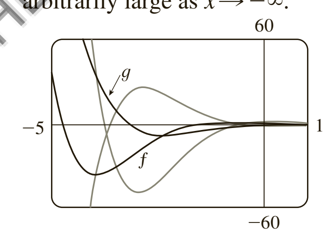
Exercises

1. (a) Negative (b) Positive **3.** $6\sqrt{10}$ **5.** $\frac{4}{15}$ **7.** $\frac{110}{3}$
9. $\frac{11}{12} - 4/e$ **11.** $f(x, y) = e^y + xe^{xy} + K$ **13.** 0
15. 0 **17.** -8π **25.** $\frac{1}{6}(27 - 5\sqrt{5})$
27. $(\pi/60)(391\sqrt{17} + 1)$ **29.** $-64\pi/3$ **31.** 0
33. $-\frac{1}{2}$ **35.** 4π **37.** -4 **39.** 21

CHAPTER 17

EXERCISES 17.1 ■ PAGE 1200

1. $y = c_1e^{3x} + c_2e^{-2x}$ **3.** $y = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)$
5. $y = c_1e^{-x/2} + c_2xe^{-x/2}$ **7.** $y = c_1 + c_2e^{4x/3}$
9. $y = e^{2x}(c_1\cos 3x + c_2\sin 3x)$
11. $y = c_1e^{(\sqrt{3}-1)t/2} + c_2e^{-(\sqrt{3}+1)t/2}$
13. $V = e^{-2t/3}\left[c_1\cos\left(\frac{\sqrt{5}}{3}t\right) + c_2\sin\left(\frac{\sqrt{5}}{3}t\right)\right]$
15. $f(x) = e^{-x}\cos x$, $g(x) = e^{-x}\sin x$. All solution curves approach 0 as $x \to \infty$ and oscillate with amplitudes that become arbitrarily large as $x \to -\infty$.

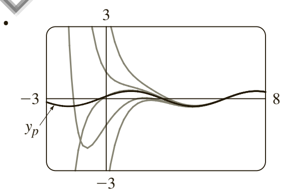


17. $y = \cos(\sqrt{3}x) + \sqrt{3}\sin(\sqrt{3}x)$ **19.** $y = e^{-2x/3} + \frac{2}{3}xe^{-2x/3}$
21. $y = e^{3x}(2\cos x - 3\sin x)$
23. $y = \frac{1}{7}e^{4x-4} - \frac{1}{7}e^{3-3x}$ **25.** $y = -3\cos 4x + 2\sin 4x$
27. $y = 2e^{-2x} - 2xe^{-2x}$ **29.** $y = \dfrac{e-2}{e-1} + \dfrac{e^x}{e-1}$
31. No solution
33. (b) $\lambda = n^2\pi^2/L^2$, n a positive integer; $y = C\sin(n\pi x/L)$
35. (a) $b - a \neq n\pi$, n any integer
(b) $b - a = n\pi$ and $\dfrac{c}{d} \neq e^{a-b}\dfrac{\cos a}{\cos b}$ unless $\cos b = 0$, then $\dfrac{c}{d} \neq e^{a-b}\dfrac{\sin a}{\sin b}$
(c) $b - a = n\pi$ and $\dfrac{c}{d} = e^{a-b}\dfrac{\cos a}{\cos b}$ unless $\cos b = 0$, then $\dfrac{c}{d} = e^{a-b}\dfrac{\sin a}{\sin b}$

EXERCISES 17.2 ■ PAGE 1207

1. $y = c_1e^{2x} + c_2e^{-4x} + \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}$
3. $y = c_1\cos(\frac{1}{3}x) + c_2\sin(\frac{1}{3}x) + \frac{1}{37}e^{2x}$
5. $y = e^{2x}(c_1\cos x + c_2\sin x) + \frac{1}{10}e^{-x}$
7. $y = e^x(\frac{9}{10}\cos 2x - \frac{1}{20}\sin 2x) + \frac{1}{10}\cos x + \frac{1}{5}\sin x$
9. $y = e^x(\frac{1}{2}x^2 - x + 2)$

11.



The solutions are all asymptotic to $y_p = \frac{1}{10}\cos x + \frac{3}{10}\sin x$ as $x \to \infty$. Except for y_p, all solutions approach either ∞ or $-\infty$ as $x \to -\infty$.

13. $y_p = (Ax + B)e^x \cos x + (Cx + D)e^x \sin x$

15. $y_p = Axe^x + B\cos x + C\sin x$

17. $y_p = xe^{-x}[(Ax^2 + Bx + C)\cos 3x + (Dx^2 + Ex + F)\sin 3x]$

19. $y = c_1\cos\left(\frac{1}{2}x\right) + c_2\sin\left(\frac{1}{2}x\right) - \frac{1}{3}\cos x$

21. $y = c_1e^x + c_2xe^x + e^{2x}$

23. $y = c_1\sin x + c_2\cos x + \sin x\ln(\sec x + \tan x) - 1$

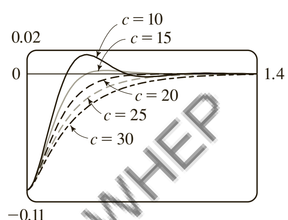
25. $y = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}$

27. $y = e^x\left[c_1 + c_2x - \frac{1}{2}\ln(1 + x^2) + x\tan^{-1}x\right]$

EXERCISES 17.3 ■ PAGE 1215

1. $x = 0.35\cos\left(2\sqrt{5}\,t\right)$ **3.** $x = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$ **5.** $\frac{49}{12}$ kg

7.



13. $Q(t) = (-e^{-10t}/250)(6\cos 20t + 3\sin 20t) + \frac{3}{125}$, $I(t) = \frac{3}{5}e^{-10t}\sin 20t$

15. $Q(t) = e^{-10t}\left[\frac{3}{250}\cos 20t - \frac{3}{500}\sin 20t\right] - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t$

EXERCISES 17.4 ■ PAGE 1220

1. $c_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = c_0e^x$ **3.** $c_0\sum_{n=0}^{\infty}\frac{x^{3n}}{3^n n!} = c_0e^{x^3/3}$

5. $c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n} + c_1\sum_{n=0}^{\infty}\frac{(-2)^n n!}{(2n+1)!}x^{2n+1}$

7. $c_0 + c_1\sum_{n=1}^{\infty}\frac{x^n}{n} = c_0 - c_1\ln(1 - x)$ for $|x| < 1$

9. $\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!} = e^{x^2/2}$

11. $x + \sum_{n=1}^{\infty}\frac{(-1)^n 2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!}x^{3n+1}$

CHAPTER 17 REVIEW ■ PAGE 1221

True-False Quiz

1. True **3.** True

Exercises

1. $y = c_1e^{x/2} + c_2e^{-x/2}$

3. $y = c_1\cos\left(\sqrt{3}x\right) + c_2\sin\left(\sqrt{3}x\right)$

5. $y = e^{2x}(c_1\cos x + c_2\sin x + 1)$

7. $y = c_1e^x + c_2xe^x - \frac{1}{2}\cos x - \frac{1}{2}(x + 1)\sin x$

9. $y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}$

11. $y = 5 - 2e^{-6(x-1)}$ **13.** $y = (e^{4x} - e^x)/3$

15. No solution **17.** $\sum_{n=0}^{\infty}\frac{(-2)^n n!}{(2n+1)!}x^{2n+1}$

19. $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$

21. (c) $2\pi/k \approx 85$ min (d) $\approx$17,600 mi/h

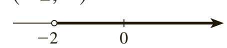
APPENDIXES

EXERCISES A ■ PAGE A9

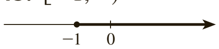
1. 18 **3.** π **5.** $5 - \sqrt{5}$ **7.** $2 - x$

9. $|x + 1| = \begin{cases} x + 1 & \text{for } x \geqslant -1 \\ -x - 1 & \text{for } x < -1 \end{cases}$ **11.** $x^2 + 1$

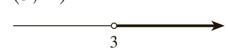
13. $(-2, \infty)$



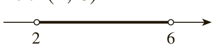
15. $[-1, \infty)$



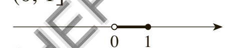
17. $(3, \infty)$



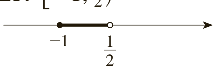
19. $(2, 6)$



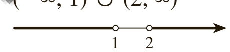
21. $(0, 1]$



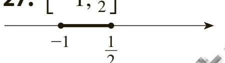
23. $\left[-1, \frac{1}{2}\right)$



25. $(-\infty, 1) \cup (2, \infty)$

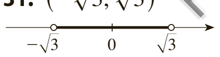


27. $\left[-1, \frac{1}{2}\right]$

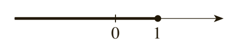


29. $(-\infty, \infty)$

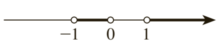
31. $\left(-\sqrt{3}, \sqrt{3}\right)$



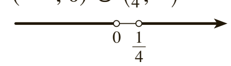
33. $(-\infty, 1]$



35. $(-1, 0) \cup (1, \infty)$



37. $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$



39. $10 \leqslant C \leqslant 35$ **41.** (a) $T = 20 - 10h$, $0 \leqslant h \leqslant 12$ (b) $-30°\text{C} \leqslant T \leqslant 20°\text{C}$ **43.** $\pm\frac{3}{2}$ **45.** $2, -\frac{4}{3}$

47. $(-3, 3)$ **49.** $(3, 5)$ **51.** $(-\infty, -7] \cup [-3, \infty)$

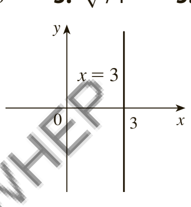
53. $[1.3, 1.7]$ **55.** $[-4, -1] \cup [1, 4]$

57. $x \geqslant (a + b)c/(ab)$ **59.** $x > (c - b)/a$

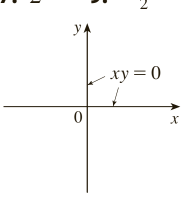
EXERCISES B ■ PAGE A15

1. 5 **3.** $\sqrt{74}$ **5.** $2\sqrt{37}$ **7.** 2 **9.** $-\frac{9}{2}$

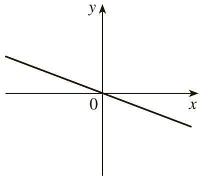
17.

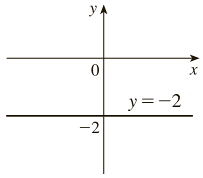


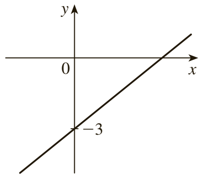
19.



21. $y = 6x - 15$ **23.** $2x - 3y + 19 = 0$
25. $5x + y = 11$ **27.** $y = 3x - 2$ **29.** $y = 3x - 3$
31. $y = 5$ **33.** $x + 2y + 11 = 0$ **35.** $5x - 2y + 1 = 0$
37. $m = -\frac{1}{3}$, $b = 0$ **39.** $m = 0$, $b = -2$ **41.** $m = \frac{3}{4}$, $b = -3$







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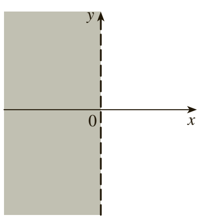

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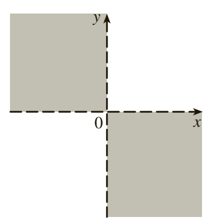

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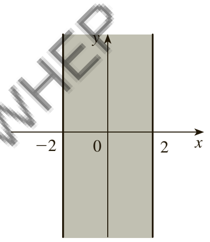

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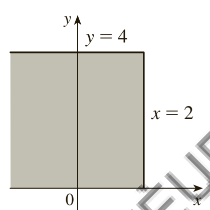

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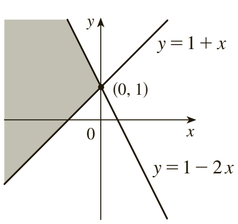
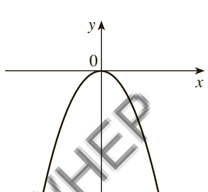


53. $(0, -4)$ **55.** (a) $(4, 9)$ (b) $(3.5, -3)$ **57.** $(1, -2)$
59. $y = x - 3$ **61.** (b) $4x - 3y - 24 = 0$

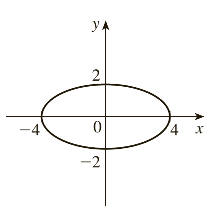
EXERCISES C ■ PAGE A23

1. $(x - 3)^2 + (y + 1)^2 = 25$ **3.** $x^2 + y^2 = 65$
5. $(2, -5)$, 4 **7.** $\left(-\frac{1}{2}, 0\right)$, $\frac{1}{2}$ **9.** $\left(\frac{1}{4}, -\frac{1}{4}\right)$, $\sqrt{10}/4$

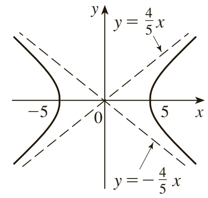
11. Parabola



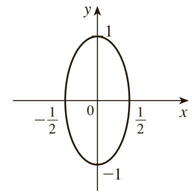
13. Ellipse



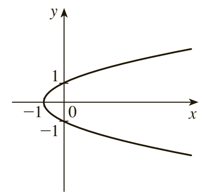
15. Hyperbola



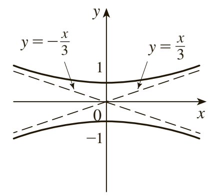
17. Ellipse



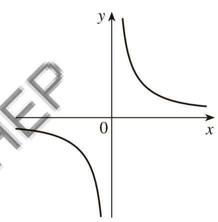
19. Parabola



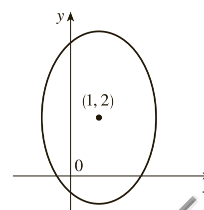
21. Hyperbola



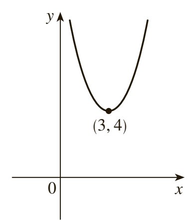
23. Hyperbola



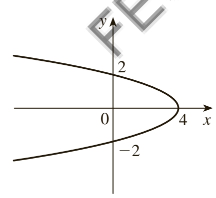
25. Ellipse



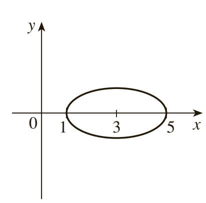
27. Parabola



29. Parabola



31. Ellipse



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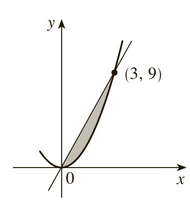
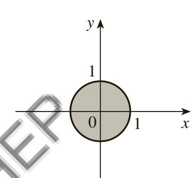


35. $y = x^2 - 2x$

37.



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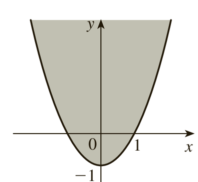
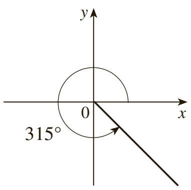


EXERCISES D ■ PAGE A32

1. $7\pi/6$ **3.** $\pi/20$ **5.** 5π **7.** $720°$ **9.** $75°$
11. $-67.5°$ **13.** 3π cm **15.** $\frac{2}{3}$ rad $= (120/\pi)°$

17.



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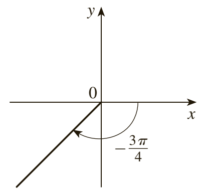

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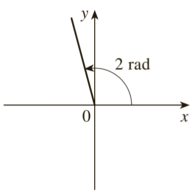
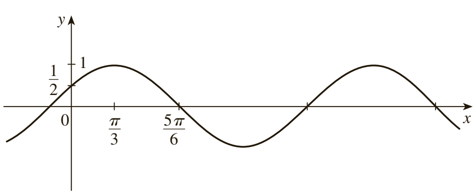


23. $\sin(3\pi/4) = 1/\sqrt{2}$, $\cos(3\pi/4) = -1/\sqrt{2}$, $\tan(3\pi/4) = -1$, $\csc(3\pi/4) = \sqrt{2}$, $\sec(3\pi/4) = -\sqrt{2}$, $\cot(3\pi/4) = -1$
25. $\sin(9\pi/2) = 1$, $\cos(9\pi/2) = 0$, $\csc(9\pi/2) = 1$, $\cot(9\pi/2) = 0$, $\tan(9\pi/2)$ and $\sec(9\pi/2)$ undefined
27. $\sin(5\pi/6) = \frac{1}{2}$, $\cos(5\pi/6) = -\sqrt{3}/2$, $\tan(5\pi/6) = -1/\sqrt{3}$, $\csc(5\pi/6) = 2$, $\sec(5\pi/6) = -2/\sqrt{3}$, $\cot(5\pi/6) = -\sqrt{3}$
29. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$
31. $\sin\phi = \sqrt{5}/3$, $\cos\phi = -\frac{2}{3}$, $\tan\phi = -\sqrt{5}/2$, $\csc\phi = 3/\sqrt{5}$, $\cot\phi = -2/\sqrt{5}$
33. $\sin\beta = -1/\sqrt{10}$, $\cos\beta = -3/\sqrt{10}$, $\tan\beta = \frac{1}{3}$, $\csc\beta = -\sqrt{10}$, $\sec\beta = -\sqrt{10}/3$
35. 5.73576 cm **37.** 24.62147 cm **59.** $\frac{1}{15}(4 + 6\sqrt{2})$
61. $\frac{1}{15}(3 + 8\sqrt{2})$ **63.** $\frac{24}{25}$ **65.** $\pi/3, 5\pi/3$
67. $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ **69.** $\pi/6, \pi/2, 5\pi/6, 3\pi/2$
71. $0, \pi, 2\pi$ **73.** $0 \leqslant x \leqslant \pi/6$ and $5\pi/6 \leqslant x \leqslant 2\pi$
75. $0 \leqslant x < \pi/4$, $3\pi/4 < x < 5\pi/4$, $7\pi/4 < x \leqslant 2\pi$
77.



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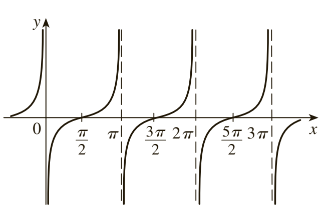
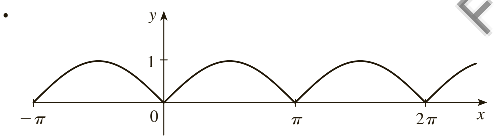


81.



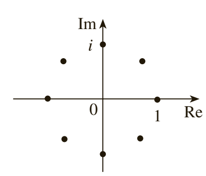
89. 14.34457 cm^2

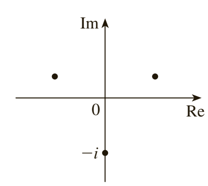
EXERCISES E ■ PAGE A38

1. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$ **3.** $3^4 + 3^5 + 3^6$
5. $-1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$ **7.** $1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$
9. $1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$ **11.** $\sum_{i=1}^{10} i$
13. $\sum_{i=1}^{19} \frac{i}{i+1}$ **15.** $\sum_{i=1}^{n} 2i$ **17.** $\sum_{i=0}^{5} 2^i$ **19.** $\sum_{i=1}^{n} x^i$
21. 80 **23.** 3276 **25.** 0 **27.** 61 **29.** $n(n+1)$
31. $n(n^2 + 6n + 17)/3$ **33.** $n(n^2 + 6n + 11)/3$
35. $n(n^3 + 2n^2 - n - 10)/4$
41. (a) n^4 (b) $5^{100} - 1$ (c) $\frac{97}{300}$ (d) $a_n - a_0$
43. $\frac{1}{3}$ **45.** 14 **49.** $2^{n+1} + n^2 + n - 2$

EXERCISES G ■ PAGE A55

1. $8 - 4i$ **3.** $13 + 18i$ **5.** $12 - 7i$ **7.** $\frac{11}{13} + \frac{10}{13}i$
9. $\frac{1}{2} - \frac{1}{2}i$ **11.** $-i$ **13.** $5i$ **15.** $12 + 5i, 13$
17. $4i, 4$ **19.** $\pm\frac{3}{2}i$ **21.** $-1 \pm 2i$
23. $-\frac{1}{2} \pm (\sqrt{7}/2)i$ **25.** $3\sqrt{2}\,[\cos(3\pi/4) + i\sin(3\pi/4)]$
27. $5\left\{\cos\left[\tan^{-1}\left(\frac{4}{3}\right)\right] + i\sin\left[\tan^{-1}\left(\frac{4}{3}\right)\right]\right\}$
29. $4[\cos(\pi/2) + i\sin(\pi/2)]$, $\cos(-\pi/6) + i\sin(-\pi/6)$, $\frac{1}{2}[\cos(-\pi/6) + i\sin(-\pi/6)]$
31. $4\sqrt{2}\,[\cos(7\pi/12) + i\sin(7\pi/12)]$, $(2\sqrt{2})[\cos(13\pi/12) + i\sin(13\pi/12)]$, $\frac{1}{4}[\cos(\pi/6) + i\sin(\pi/6)]$
33. -1024 **35.** $-512\sqrt{3} + 512i$
37. $\pm 1, \pm i, (1/\sqrt{2})(\pm 1 \pm i)$ **39.** $\pm(\sqrt{3}/2) + \frac{1}{2}i, -i$





41. i **43.** $\frac{1}{2} + (\sqrt{3}/2)i$ **45.** $-e^2$
47. $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$, $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

Index

RP denotes Reference Page numbers.
Numbers followed by asterisks denote pages in Sections 6.2*, 6.3*, and 6.4*

ALGEBRA

Arithmetic Operations

$a(b+c) = ab + ac$

$\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$

$\dfrac{a+c}{b} = \dfrac{a}{b} + \dfrac{c}{b}$

$\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$

Exponents and Radicals

$x^m x^n = x^{m+n}$

$\dfrac{x^m}{x^n} = x^{m-n}$

$(x^m)^n = x^{mn}$

$x^{-n} = \dfrac{1}{x^n}$

$(xy)^n = x^n y^n$

$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

$x^{1/n} = \sqrt[n]{x}$

$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$

$\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$

Factoring Special Polynomials

$x^2 - y^2 = (x+y)(x-y)$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Binomial Theorem

$(x+y)^2 = x^2 + 2xy + y^2$

$(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot \cdots \cdot k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

$|x| > a$ means $x > a$ or $x < -a$

GEOMETRY

Geometric Formulas

Formulas for area A, circumference C, and volume V:

Triangle

$A = \frac{1}{2}bh$

$\quad = \frac{1}{2}ab\sin\theta$

Circle

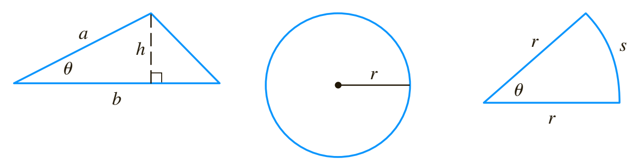
$A = \pi r^2$

$C = 2\pi r$

Sector of Circle

$A = \frac{1}{2}r^2\theta$

$s = r\theta$ (θ in radians)



Sphere

$V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$

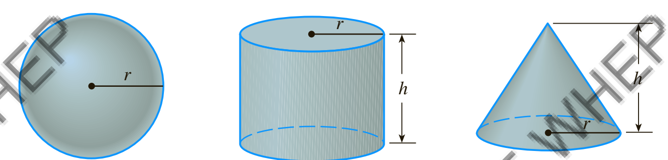
Cylinder

$V = \pi r^2 h$

Cone

$V = \frac{1}{3}\pi r^2 h$

$A = \pi r\sqrt{r^2 + h^2}$



Distance and Midpoint Formulas

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Lines

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Circles

Equation of the circle with center (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

TRIGONOMETRY

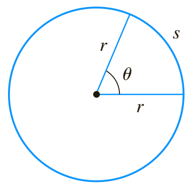
Angle Measurement

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad 1 \text{ rad} = \dfrac{180°}{\pi}$

$s = r\theta$

(θ in radians)

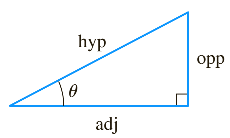


Right Angle Trigonometry

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} \qquad \cot\theta = \dfrac{\text{adj}}{\text{opp}}$

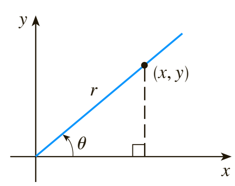


Trigonometric Functions

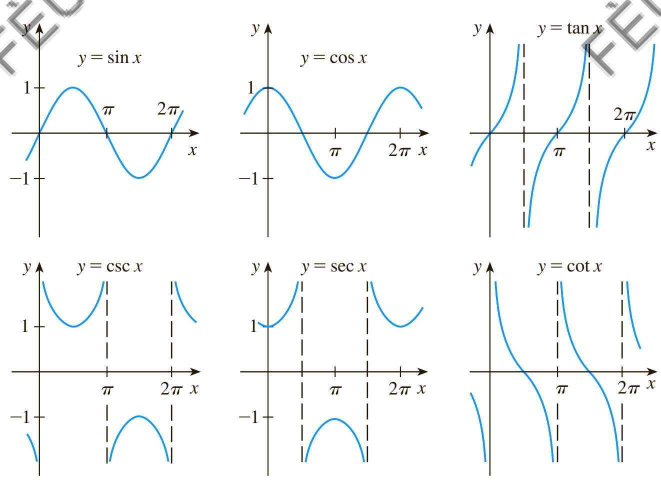
$\sin\theta = \dfrac{y}{r} \qquad \csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r} \qquad \sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$



Graphs of Trigonometric Functions



Trigonometric Functions of Important Angles

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\pi/2$	1	0	—

Fundamental Identities

$\csc\theta = \dfrac{1}{\sin\theta} \qquad \sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} \qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$

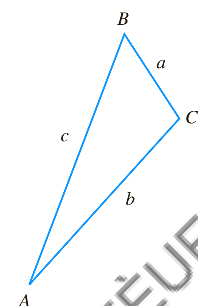
$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta \qquad \sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Addition and Subtraction Formulas

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

Double-Angle Formulas

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

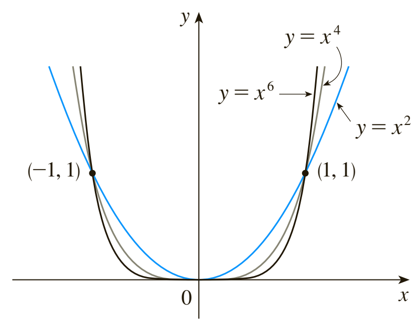
Half-Angle Formulas

$\sin^2 x = \dfrac{1 - \cos 2x}{2} \qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$

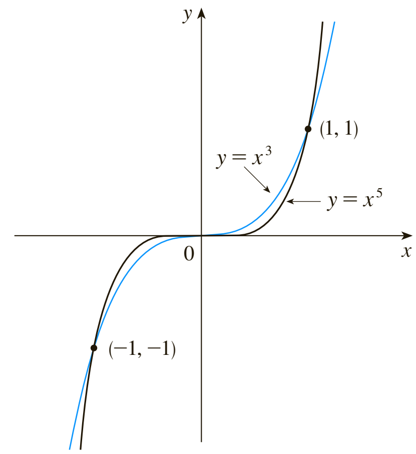
SPECIAL FUNCTIONS

Power Functions $f(x) = x^a$

(i) $f(x) = x^n$, n a positive integer

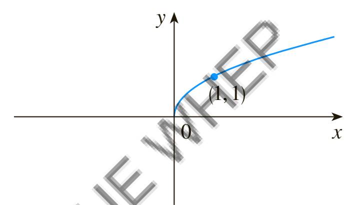


n even

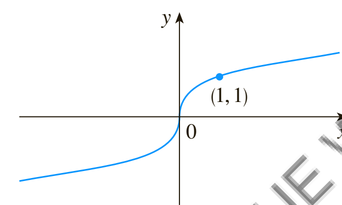


n odd

(ii) $f(x) = x^{1/n} = \sqrt[n]{x}$, n a positive integer

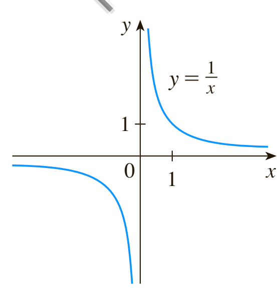


$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$

(iii) $f(x) = x^{-1} = \dfrac{1}{x}$

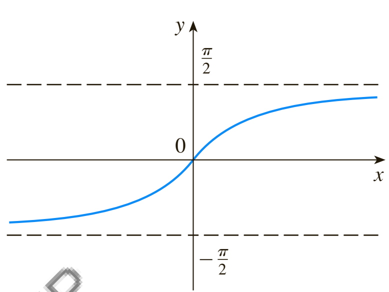


Inverse Trigonometric Functions

$\arcsin x = \sin^{-1}x = y \iff \sin y = x$ and $-\dfrac{\pi}{2} \leqslant y \leqslant \dfrac{\pi}{2}$

$\arccos x = \cos^{-1}x = y \iff \cos y = x$ and $0 \leqslant y \leqslant \pi$

$\arctan x = \tan^{-1}x = y \iff \tan y = x$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$



$y = \tan^{-1}x = \arctan x$

$$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

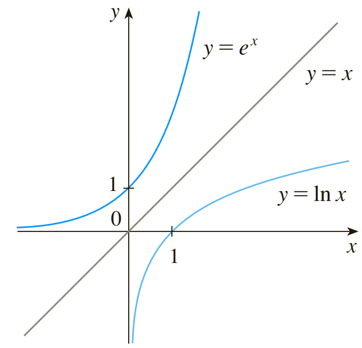
SPECIAL FUNCTIONS

Exponential and Logarithmic Functions

$\log_b x = y \iff b^y = x$

$\ln x = \log_e x$, where $\ln e = 1$

$\ln x = y \iff e^y = x$



Cancellation Equations

$\log_b(b^x) = x \qquad b^{\log_b x} = x$

$\ln(e^x) = x \qquad e^{\ln x} = x$

Laws of Logarithms

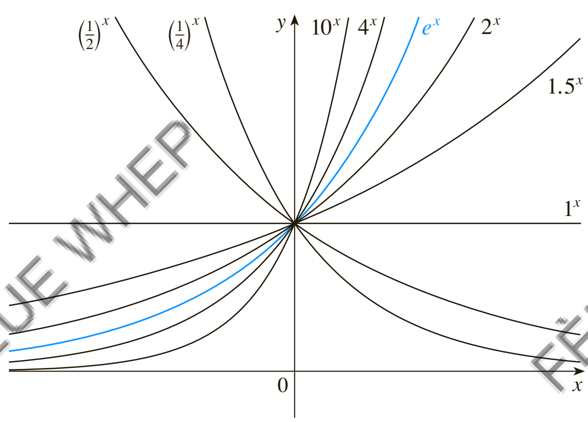
1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

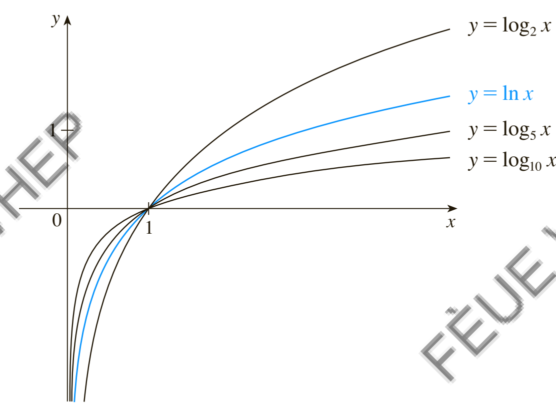
3. $\log_b(x^r) = r\log_b x$

$\lim\limits_{x\to-\infty} e^x = 0 \qquad \lim\limits_{x\to\infty} e^x = \infty$

$\lim\limits_{x\to 0^+} \ln x = -\infty \qquad \lim\limits_{x\to\infty} \ln x = \infty$



Exponential functions



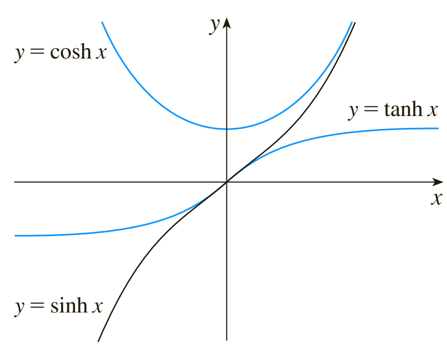
Logarithmic functions

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$



Inverse Hyperbolic Functions

$y = \sinh^{-1}x \iff \sinh y = x \qquad \sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$

$y = \cosh^{-1}x \iff \cosh y = x$ and $y \geqslant 0 \qquad \cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right)$

$y = \tanh^{-1}x \iff \tanh y = x \qquad \tanh^{-1}x = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

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DIFFERENTIATION RULES

General Formulas

1. $\dfrac{d}{dx}(c) = 0$

2. $\dfrac{d}{dx}[cf(x)] = cf'(x)$

3. $\dfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

4. $\dfrac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

5. $\dfrac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

6. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Quotient Rule)

7. $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (Chain Rule)

8. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule)

Exponential and Logarithmic Functions

9. $\dfrac{d}{dx}(e^x) = e^x$

10. $\dfrac{d}{dx}(b^x) = b^x \ln b$

11. $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$

12. $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$

Trigonometric Functions

13. $\dfrac{d}{dx}(\sin x) = \cos x$

14. $\dfrac{d}{dx}(\cos x) = -\sin x$

15. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

16. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

17. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

18. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trigonometric Functions

19. $\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1 - x^2}}$

20. $\dfrac{d}{dx}(\cos^{-1}x) = -\dfrac{1}{\sqrt{1 - x^2}}$

21. $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1 + x^2}$

22. $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$

23. $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2 - 1}}$

24. $\dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1 + x^2}$

Hyperbolic Functions

25. $\dfrac{d}{dx}(\sinh x) = \cosh x$

26. $\dfrac{d}{dx}(\cosh x) = \sinh x$

27. $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

28. $\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

29. $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

30. $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Inverse Hyperbolic Functions

31. $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$

32. $\dfrac{d}{dx}(\cosh^{-1}x) = \dfrac{1}{\sqrt{x^2 - 1}}$

33. $\dfrac{d}{dx}(\tanh^{-1}x) = \dfrac{1}{1 - x^2}$

34. $\dfrac{d}{dx}(\operatorname{csch}^{-1}x) = -\dfrac{1}{|x|\sqrt{x^2 + 1}}$

35. $\dfrac{d}{dx}(\operatorname{sech}^{-1}x) = -\dfrac{1}{x\sqrt{1 - x^2}}$

36. $\dfrac{d}{dx}(\coth^{-1}x) = \dfrac{1}{1 - x^2}$

TABLE OF INTEGRALS

Basic Forms

1. $\int u\,dv = uv - \int v\,du$
2. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
3. $\int \dfrac{du}{u} = \ln|u| + C$
4. $\int e^u\,du = e^u + C$
5. $\int b^u\,du = \dfrac{b^u}{\ln b} + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C$
14. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
15. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
16. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\dfrac{u}{a} + C, \quad a > 0$
17. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} + C$
18. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 - u^2} = \dfrac{1}{2a}\ln\left|\dfrac{u+a}{u-a}\right| + C$
20. $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u-a}{u+a}\right| + C$

Forms Involving $\sqrt{a^2 + u^2}$, $a > 0$

21. $\int \sqrt{a^2 + u^2}\,du = \dfrac{u}{2}\sqrt{a^2 + u^2} + \dfrac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$
22. $\int u^2\sqrt{a^2 + u^2}\,du = \dfrac{u}{8}(a^2 + 2u^2)\sqrt{a^2 + u^2} - \dfrac{a^4}{8}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$
23. $\int \dfrac{\sqrt{a^2 + u^2}}{u}\,du = \sqrt{a^2 + u^2} - a\ln\left|\dfrac{a + \sqrt{a^2 + u^2}}{u}\right| + C$
24. $\int \dfrac{\sqrt{a^2 + u^2}}{u^2}\,du = -\dfrac{\sqrt{a^2 + u^2}}{u} + \ln\left(u + \sqrt{a^2 + u^2}\right) + C$
25. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \ln\left(u + \sqrt{a^2 + u^2}\right) + C$
26. $\int \dfrac{u^2\,du}{\sqrt{a^2 + u^2}} = \dfrac{u}{2}\sqrt{a^2 + u^2} - \dfrac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$
27. $\int \dfrac{du}{u\sqrt{a^2 + u^2}} = -\dfrac{1}{a}\ln\left|\dfrac{\sqrt{a^2 + u^2} + a}{u}\right| + C$
28. $\int \dfrac{du}{u^2\sqrt{a^2 + u^2}} = -\dfrac{\sqrt{a^2 + u^2}}{a^2 u} + C$
29. $\int \dfrac{du}{(a^2 + u^2)^{3/2}} = \dfrac{u}{a^2\sqrt{a^2 + u^2}} + C$

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TABLE OF INTEGRALS

Forms Involving $\sqrt{a^2 - u^2}$, $a > 0$

30. $\displaystyle \int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

31. $\displaystyle \int u^2\sqrt{a^2 - u^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

32. $\displaystyle \int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

33. $\displaystyle \int \frac{\sqrt{a^2 - u^2}}{u^2}\, du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

34. $\displaystyle \int \frac{u^2\, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

35. $\displaystyle \int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

36. $\displaystyle \int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

37. $\displaystyle \int (a^2 - u^2)^{3/2}\, du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

38. $\displaystyle \int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Forms Involving $\sqrt{u^2 - a^2}$, $a > 0$

39. $\displaystyle \int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

40. $\displaystyle \int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

41. $\displaystyle \int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|} + C$

42. $\displaystyle \int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

43. $\displaystyle \int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

44. $\displaystyle \int \frac{u^2\, du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

45. $\displaystyle \int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

46. $\displaystyle \int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

(continued)

TABLE OF INTEGRALS

Forms Involving $a + bu$

47. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a + bu - a\ln|a+bu|\right) + C$

48. $\displaystyle\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

49. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

50. $\displaystyle\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

51. $\displaystyle\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

52. $\displaystyle\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

53. $\displaystyle\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

54. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$

55. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$

56. $\displaystyle\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a+bu} + C$

57. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$

$$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \qquad \text{if } a < 0$$

58. $\displaystyle\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

59. $\displaystyle\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

60. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

61. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

62. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

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TABLE OF INTEGRALS

Trigonometric Forms

63. $\displaystyle\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$

64. $\displaystyle\int \cos^2 u\,du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$

65. $\displaystyle\int \tan^2 u\,du = \tan u - u + C$

66. $\displaystyle\int \cot^2 u\,du = -\cot u - u + C$

67. $\displaystyle\int \sin^3 u\,du = -\tfrac{1}{3}(2 + \sin^2 u)\cos u + C$

68. $\displaystyle\int \cos^3 u\,du = \tfrac{1}{3}(2 + \cos^2 u)\sin u + C$

69. $\displaystyle\int \tan^3 u\,du = \tfrac{1}{2}\tan^2 u + \ln|\cos u| + C$

70. $\displaystyle\int \cot^3 u\,du = -\tfrac{1}{2}\cot^2 u - \ln|\sin u| + C$

71. $\displaystyle\int \sec^3 u\,du = \tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln|\sec u + \tan u| + C$

72. $\displaystyle\int \csc^3 u\,du = -\tfrac{1}{2}\csc u\cot u + \tfrac{1}{2}\ln|\csc u - \cot u| + C$

73. $\displaystyle\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du$

74. $\displaystyle\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du$

75. $\displaystyle\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\,du$

76. $\displaystyle\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\,du$

77. $\displaystyle\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u\,du$

78. $\displaystyle\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u\,du$

79. $\displaystyle\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

80. $\displaystyle\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

81. $\displaystyle\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

82. $\displaystyle\int u\sin u\,du = \sin u - u\cos u + C$

83. $\displaystyle\int u\cos u\,du = \cos u + u\sin u + C$

84. $\displaystyle\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

85. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

86. $\displaystyle\int \sin^n u\cos^m u\,du = -\frac{\sin^{n-1}u\cos^{m+1}u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}u\cos^m u\,du$

$$= \frac{\sin^{n+1}u\cos^{m-1}u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2}u\,du$$

Inverse Trigonometric Forms

87. $\displaystyle\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + C$

88. $\displaystyle\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + C$

89. $\displaystyle\int \tan^{-1}u\,du = u\tan^{-1}u - \tfrac{1}{2}\ln(1+u^2) + C$

90. $\displaystyle\int u\sin^{-1}u\,du = \frac{2u^2-1}{4}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{4} + C$

91. $\displaystyle\int u\cos^{-1}u\,du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$

92. $\displaystyle\int u\tan^{-1}u\,du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$

93. $\displaystyle\int u^n\sin^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1}u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

94. $\displaystyle\int u^n\cos^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1}u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

95. $\displaystyle\int u^n\tan^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1}u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

(continued)

TABLE OF INTEGRALS

Exponential and Logarithmic Forms

96. $\displaystyle\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$

97. $\displaystyle\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

98. $\displaystyle\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu) + C$

99. $\displaystyle\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu) + C$

100. $\displaystyle\int \ln u\,du = u\ln u - u + C$

101. $\displaystyle\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

102. $\displaystyle\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Forms

103. $\displaystyle\int \sinh u\,du = \cosh u + C$

104. $\displaystyle\int \cosh u\,du = \sinh u + C$

105. $\displaystyle\int \tanh u\,du = \ln\cosh u + C$

106. $\displaystyle\int \coth u\,du = \ln|\sinh u| + C$

107. $\displaystyle\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

108. $\displaystyle\int \operatorname{csch} u\,du = \ln\left|\tanh \tfrac{1}{2}u\right| + C$

109. $\displaystyle\int \operatorname{sech}^2 u\,du = \tanh u + C$

110. $\displaystyle\int \operatorname{csch}^2 u\,du = -\coth u + C$

111. $\displaystyle\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

112. $\displaystyle\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Forms Involving $\sqrt{2au - u^2},\ a > 0$

113. $\displaystyle\int \sqrt{2au - u^2}\,du = \frac{u-a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

114. $\displaystyle\int u\sqrt{2au - u^2}\,du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

115. $\displaystyle\int \frac{\sqrt{2au - u^2}}{u}\,du = \sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a-u}{a}\right) + C$

116. $\displaystyle\int \frac{\sqrt{2au - u^2}}{u^2}\,du = -\frac{2\sqrt{2au - u^2}}{u} - \cos^{-1}\left(\frac{a-u}{a}\right) + C$

117. $\displaystyle\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

118. $\displaystyle\int \frac{u\,du}{\sqrt{2au - u^2}} = -\sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a-u}{a}\right) + C$

119. $\displaystyle\int \frac{u^2\,du}{\sqrt{2au - u^2}} = -\frac{(u+3a)}{2}\sqrt{2au - u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

120. $\displaystyle\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$

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CHAPTER 1 CONCEPT CHECK ANSWERS

1. **(a)** What is a function? What are its domain and range?

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. The domain is the set D and the range is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) What is the graph of a function?

The graph of a function f consists of all points (x, y) such that $y = f(x)$ and x is in the domain of f.

(c) How can you tell whether a given curve is the graph of a function?

Use the Vertical Line Test: a curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

A function can be represented verbally, numerically, visually, or algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between the area A and side length s of a square: $A = s^2$ (an explicit formula)

3. **(a)** What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.

A function f is even if it satisfies $f(-x) = f(x)$ for every number x in its domain. If the graph of a function is symmetric with respect to the y-axis, then f is even. Examples are $f(x) = x^2$, $f(x) = \cos x$, $f(x) = |x|$.

(b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.

A function f is odd if it satisfies $f(-x) = -f(x)$ for every number x in its domain. If the graph of a function is symmetric with respect to the origin, then f is odd. Examples are $f(x) = x^3$, $f(x) = \sin x$, $f(x) = 1/x$.

4. What is an increasing function?

A function f is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. What is a mathematical model?

A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon. (See the discussion on pages 23–24.)

6. Give an example of each type of function.

(a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^n$

(c) Exponential function: $f(x) = 2^x$, $f(x) = b^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

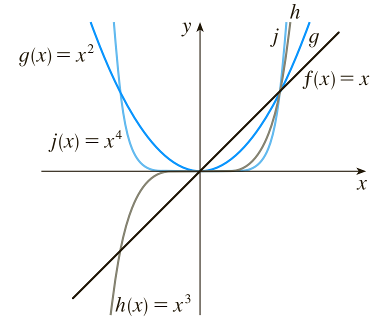
(e) Polynomial of degree 5: $f(x) = x^5 + 2x^4 - 3x^2 + 7$

(f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

7. Sketch by hand, on the same axes, the graphs of the following functions.

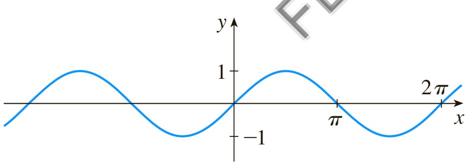
(a) $f(x) = x$ **(b)** $g(x) = x^2$

(c) $h(x) = x^3$ **(d)** $j(x) = x^4$

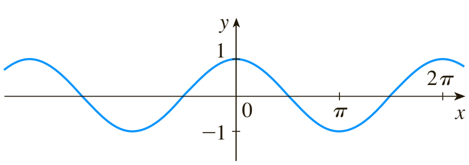


8. Draw, by hand, a rough sketch of the graph of each function.

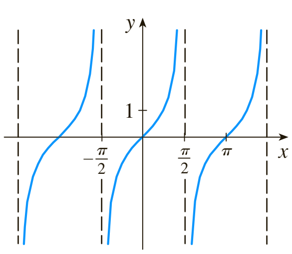
(a) $y = \sin x$



(b) $y = \cos x$



(c) $y = \tan x$



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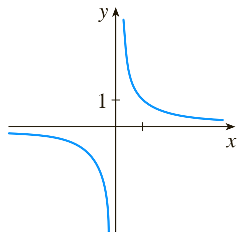
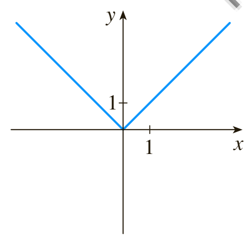
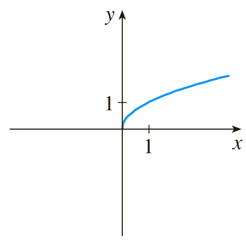
(d) $y = 1/x$



(e) $y = |x|$



(f) $y = \sqrt{x}$



9. Suppose that f has domain A and g has domain B.

(a) What is the domain of $f + g$?

The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) What is the domain of fg?

The domain of fg is also $A \cap B$.

(c) What is the domain of f/g?

The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. How is the composite function $f \circ g$ defined? What is its domain?

The composition of f and g is defined by $(f \circ g)(x) = f(g(x))$. The domain is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.

(a) Shift 2 units upward: $y = f(x) + 2$

(b) Shift 2 units downward: $y = f(x) - 2$

(c) Shift 2 units to the right: $y = f(x - 2)$

(d) Shift 2 units to the left: $y = f(x + 2)$

(e) Reflect about the x-axis: $y = -f(x)$

(f) Reflect about the y-axis: $y = f(-x)$

(g) Stretch vertically by a factor of 2: $y = 2f(x)$

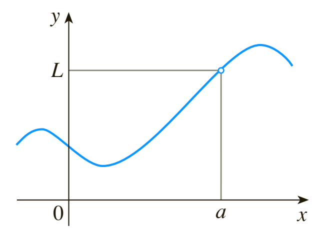
(h) Shrink vertically by a factor of 2: $y = \frac{1}{2}f(x)$

(i) Stretch horizontally by a factor of 2: $y = f\left(\frac{1}{2}x\right)$

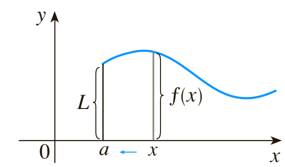
(j) Shrink horizontally by a factor of 2 $y = f(2x)$

12. Explain what each of the following means and illustrate with a sketch.

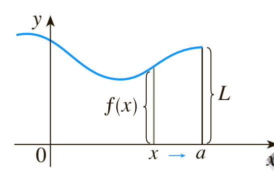
(a) $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a (but $x \neq a$).



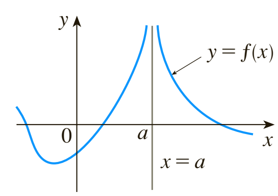
(b) $\lim_{x \to a^+} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a through values greater than a.



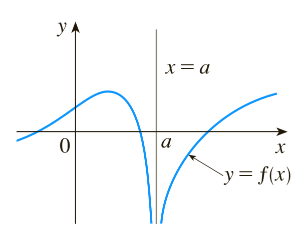
(c) $\lim_{x \to a^-} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a through values less than a.



(d) $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (but not equal to a).



(e) $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a (but not equal to a).



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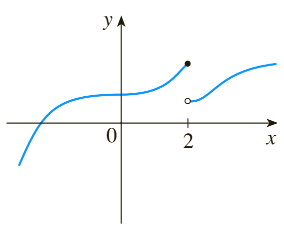
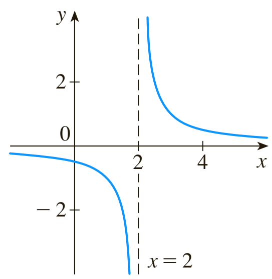
CHAPTER 1 CONCEPT CHECK ANSWERS *(continued)*

13. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

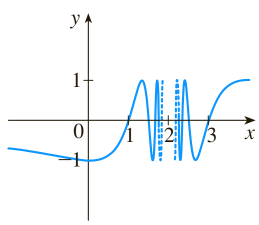
In general, the limit of a function fails to exist when the function values do not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left and right limits are not equal.



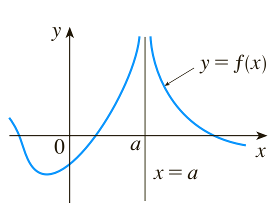
There is an infinite discontinuity.

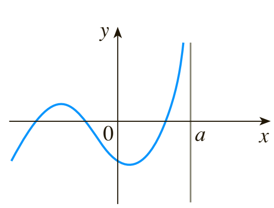


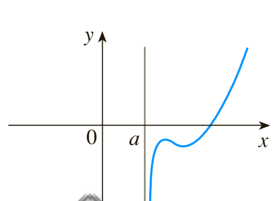
The function values oscillate between 1 and -1 infinitely often.

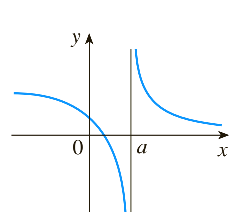
14. What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

It means that the limit of $f(x)$ as x approaches a from one or both sides is positive or negative infinity.









15. State the following Limit Laws.

(a) Sum Law

The limit of a sum is the sum of the limits:
$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(b) Difference Law

The limit of a difference is the difference of the limits:
$\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(c) Constant Multiple Law

The limit of a constant times a function is the constant times the limit of the function: $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$

(d) Product Law

The limit of a product is the product of the limits:
$\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(e) Quotient Law

The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not 0:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$

(f) Power Law

The limit of a power is the power of the limit:
$\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ (for n a positive integer)

(g) Root Law

The limit of a root is the root of the limit:
$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (for n a positive integer)

16. What does the Squeeze Theorem say?

If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$. In other words, if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

17. (a) What does it mean for f to be continuous at a?

A function f is continuous at a number a if the value of the function at $x = a$ is the same as the limit when x approaches a; that is, $\lim_{x\to a} f(x) = f(a)$.

(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

A function f is continuous on the interval $(-\infty, \infty)$ if it is continuous at every real number a.

The graph of such a function has no hole or break in it.

(continued)

18. (a) Give examples of functions that are continuous on $[-1, 1]$.

$f(x) = x^3 - x$, $g(x) = \sqrt{x + 2}$, $y = \sin x$, $y = \tan x$, $y = 1/(x - 3)$, and $h(x) = |x|$ are all continuous on $[-1, 1]$.

(b) Give an example of a function that is not continuous on $[0, 1]$.

$$f(x) = \frac{1}{x - \frac{1}{2}} \qquad \left[f(x) \text{ is not defined at } x = \tfrac{1}{2}\right]$$

19. What does the Intermediate Value Theorem say?

If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$ $[f(a) \neq f(b)]$, Then there exists a number c in (a, b) such that $f(c) = N$. In other words, a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.

CHAPTER 2 CONCEPT CHECK ANSWERS

1. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

The slope of the tangent line is given by

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

2. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

The instantaneous velocity at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

It is equal to the slope of the tangent line to the graph of f at the point $P(a, f(a))$.

3. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.

(a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

(b) The instantaneous rate of change of y with respect to x at $x = x_1$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

4. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

or, equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ (with respect to x) when $x = a$ and also represents the slope of the tangent line to the graph of f at the point $P(a, f(a))$.

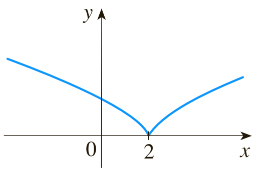
5. **(a)** What does it mean for f to be differentiable at a?

f is differentiable at a if the derivative $f'(a)$ exists.

(b) What is the relation between the differentiability and continuity of a function?

If f is differentiable at a, then f is continuous at a.

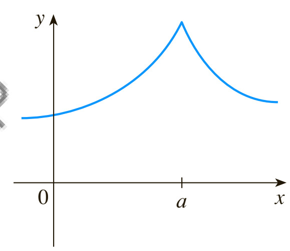
(c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.



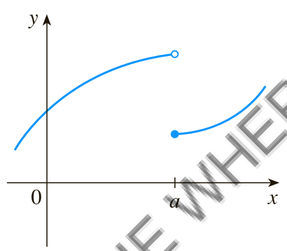
The graph of f changes direction abruptly at $x = 2$, so f has no tangent line there.

6. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

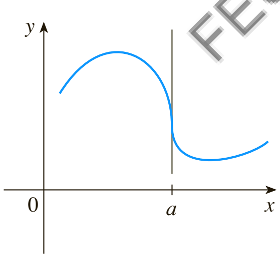
A function is not differentiable at any value where the graph has a "corner," where the graph has a discontinuity, or where it has a vertical tangent line.



A corner



A discontinuity



A vertical tangent

7. What are the second and third derivatives of a function f? If f is the position function of an object, how can you interpret f'' and f'''?

The second derivative f'' is the derivative of f', and the third derivative f''' is the derivative of f''.

If f is the postition function of an object, then f' is the velocity function of the object, f'' is the acceleration function, and f''' is the jerk function (the rate of change of acceleration).

(continued)

8. State each differentiation rule both in symbols and in words.

(a) The Power Rule

If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

To find the derivative of a variable raised to a constant power, we multiply the expression by the exponent and then subtract one from the exponent.

(b) The Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The derivative of a constant times a function is the constant times the derivative of the function.

(c) The Sum Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The derivative of a sum of functions is the sum of the derivatives.

(d) The Difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The derivative of a difference of functions is the difference of the derivatives.

(e) The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

(f) The Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(g) The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

9. State the derivative of each function.

(a) $y = x^n$: $y' = nx^{n-1}$

(b) $y = \sin x$: $y' = \cos x$

(c) $y = \cos x$: $y' = -\sin x$

(d) $y = \tan x$: $y' = \sec^2 x$

(e) $y = \csc x$: $y' = -\csc x \cot x$

(f) $y = \sec x$: $y' = \sec x \tan x$

(g) $y = \cot x$: $y' = -\csc^2 x$

10. Explain how implicit differentiation works.

Implicit differentiation consists of differentiating both sides of an equation with respect to x, treating y as a function of x. Then we solve the resulting equation for y'.

11. Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, or other sciences.

In physics, interpretations of the derivative include velocity, linear density, electrical current, power (the rate of change of work), and the rate of radioactive decay. Chemists can use derivatives to measure reaction rates and the compressibility of a substance under pressure. In biology the derivative measures rates of population growth and blood flow. In economics, the derivative measures marginal cost (the rate of change of cost as more items are produced) and marginal profit. Other examples include the rate of heat flow in geology, the rate of performance improvement in psychology, and the rate at which a rumor spreads in sociology.

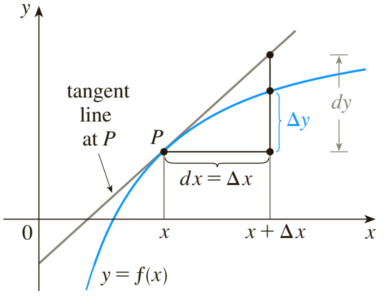
12. (a) Write an expression for the linearization of f at a.

$$L(x) = f(a) + f'(a)(x - a)$$

(b) If $y = f(x)$, write an expression for the differential dy.

$$dy = f'(x)\,dx$$

(c) If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy.



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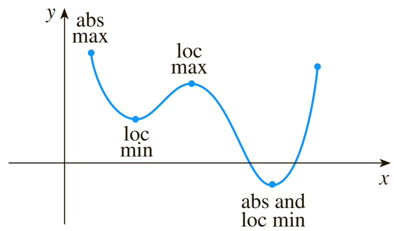
CHAPTER 3 CONCEPT CHECK ANSWERS

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

The function value $f(c)$ is the absolute maximum value of f if $f(c)$ is the largest function value on the entire domain of f, whereas $f(c)$ is a local maximum value if it is the largest function value when x is near c.



2. What does the Extreme Value Theorem say?

If f is a continuous function on a closed interval $[a, b]$, then it always attains an absolute maximum and an absolute minimum value on that interval.

3. (a) State Fermat's Theorem.

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

(b) Define a critical number of f.

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

4. Explain how the Closed Interval Method works.

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$, we follow these three steps:

- Find the critical numbers of f in the interval (a, b) and compute the values of f at these numbers.
- Find the values of f at the endpoints of the interval.
- The largest of the values from the previous two steps is the absolute maximum value; the smallest of these values is the absolute minimum value.

5. (a) State Rolle's Theorem.

Let f be a function that satisfies the following three hypotheses:

- f is continuous on the closed interval $[a, b]$.
- f is differentiable on the open interval (a, b).
- $f(a) = f(b)$

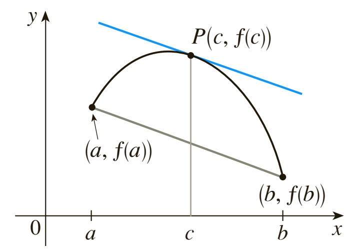
Then there is a number c in (a, b) such that $f'(c) = 0$.

(b) State the Mean Value Theorem and give a geometric interpretation.

If f is continuous on the interval $[a, b]$ and differentiable on (a, b), then there exists a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrically, the theorem says that there is a point $P(c, f(c))$, where $a < c < b$, on the graph of f where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.



6. (a) State the Increasing/Decreasing Test.

If $f'(x) > 0$ on an interval, then f is increasing on that interval.

If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

(b) What does it mean to say that f is concave upward on an interval I?

f is concave upward on an interval if the graph of f lies above all of its tangents on that interval.

(c) State the Concavity Test.

If $f''(x) > 0$ on an interval, then the graph of f is concave upward on that interval.

If $f''(x) < 0$ on an interval, then the graph of f is concave downward on that interval.

(d) What are inflection points? How do you find them?

Inflection points on the graph of a continuous function f are points where the curve changes from concave upward to concave downward or from concave downward to concave upward. They can be found by determining the values at which the second derivative changes sign.

7. (a) State the First Derivative Test.

Suppose that c is a critical number of a continuous function f.

- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

(b) State the Second Derivative Test.

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(continued)

(c) What are the relative advantages and disadvantages of these tests?

The Second Derivative Test is sometimes easier to use, but it is inconclusive when $f''(c) = 0$ and fails if $f''(c)$ does not exist. In either case the First Derivative Test must be used.

8. Explain the meaning of each of the following statements.

(a) $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

(b) $\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

(c) $\lim_{x\to\infty} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by requiring x to be sufficiently large.

(d) The curve $y = f(x)$ has the horizontal asymptote $y = L$.

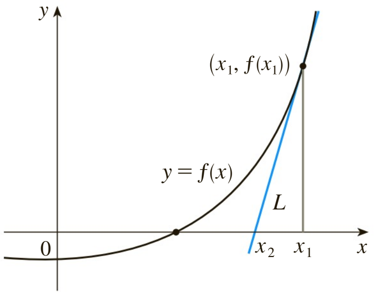
The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

9. If you have a graphing calculator or computer, why do you need calculus to graph a function?

Calculus reveals all the important aspects of a graph, such as local extreme values and inflection points, that can be missed when relying solely on technology. In many cases we can find exact locations of these key points rather than approximations. Using derivatives to identify the behavior of the graph also helps us choose an appropriate viewing window and alerts us to where we may wish to zoom in on a graph.

10. (a) Given an initial approximation x_1 to a root of the equation $f(x) = 0$, explain geometrically, with a diagram, how the second approximation x_2 in Newton's method is obtained.

We find the tangent line L to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$. Then x_2 is the x-intercept of L.



(b) Write an expression for x_2 in terms of x_1, $f(x_1)$, and $f'(x_1)$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

(c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(d) Under what circumstances is Newton's method likely to fail or to work very slowly?

Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0. It also fails when $f'(x_i)$ is undefined.

11. (a) What is an antiderivative of a function f?

A function F is an antiderivative of f if $F'(x) = f(x)$.

(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

They are identical or they differ by a constant.

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CHAPTER 4 CONCEPT CHECK ANSWERS

1. (a) Write an expression for a Riemann sum of a function f on an interval $[a, b]$. Explain the meaning of the notation that you use.

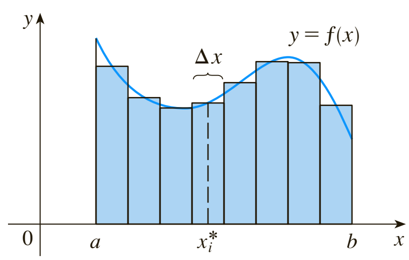
If f is defined for $a \leqslant x \leqslant b$ and we divide the interval $[a, b]$ into n subintervals of equal width Δx, then a Riemann sum of f is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where x_i^* is a point in the ith subinterval.

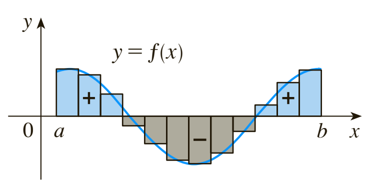
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

If f is positive, then a Riemann sum can be interpreted as the sum of areas of approximating rectangles, as shown in the figure.



(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

If f takes on both positive and negative values then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles minus the areas of the gray rectangles).



2. (a) Write the definition of the definite integral of a continuous function from a to b.

If f is a continuous function on the interval $[a, b]$, then we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots,$ $x_n\ (= b)$ be the endpoints of these subintervals. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

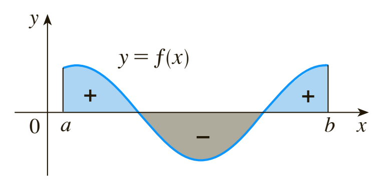
where x_i^* is any sample point in the ith subinterval $[x_{i-1}, x_i]$.

(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?

If f is positive, then $\int_a^b f(x)\,dx$ can be interpreted as the area under the graph of $y = f(x)$ and above the x-axis for $a \leqslant x \leqslant b$.

(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

In this case $\int_a^b f(x)\,dx$ can be interpreted as a "net area," that is, the area of the region above the x-axis and below the graph of f (labeled "+" in the figure) minus the area of the region below the x-axis and above the graph of f (labeled "−").



3. State the Midpoint Rule.

If f is a continuous function on the interval $[a, b]$ and we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$, then

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

where $\bar{x}_i =$ midpoint of $[x_{i-1}, x_i] = \frac{1}{2}(x_{i-1} + x_i)$.

4. State both parts of the Fundamental Theorem of Calculus.

Suppose f is continuous on $[a, b]$.

Part 1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

Part 2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

5. (a) State the Net Change Theorem.

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

$\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

(continued)

6. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.

(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?

$\int_{60}^{120} v(t)\,dt$ represents the net change in position (the displacement) of the particle from $t = 60$ s to $t = 120$ s, in other words, in the second minute.

(b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?

$\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle in the second minute.

(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

$\int_{60}^{120} a(t)\,dt$ represents the change in velocity of the particle in the second minute.

7. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.

The indefinite integral $\int f(x)\,dx$ is another name for an antiderivative of f, so $\int f(x)\,dx = F(x)$ means that $F'(x) = f(x)$.

(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

The connection is given by Part 2 of the Fundamental Theorem:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

if f is continuous on $[a, b]$.

8. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

Part 1 of the Fundamental Theorem of Calculus can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which says that if f is integrated and then the result is differentiated, we arrive back at the original function f.

Since $F'(x) = f(x)$, Part 2 of the theorem (or, equivalently, the Net Change Theorem) states that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function, but in the form $F(b) - F(a)$.

Also, the indefinite integral $\int f(x)\,dx$ represents an antiderivative of f, so

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

9. State the Substitution Rule. In practice, how do you use it?

If $u = g(x)$ is a differentiable function and f is continuous on the range of g, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

In practice, we make the substitutions $u = g(x)$ and $du = g'(x)\,dx$ in the integrand in order to make the integral simpler to evaluate.

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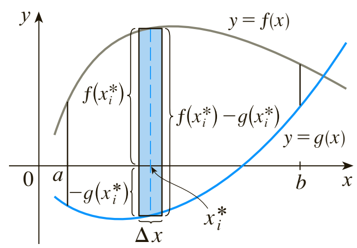
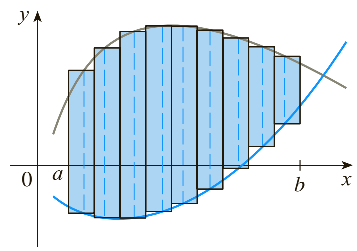
CHAPTER 5 CONCEPT CHECK ANSWERS

1. (a) Draw two typical curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$. Show how to approximate the area between these curves by a Riemann sum and sketch the corresponding approximating rectangles. Then write an expression for the exact area.



A Riemann sum that approximates the area between these curves is $\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$. A sketch of the corresponding approximating rectangles:

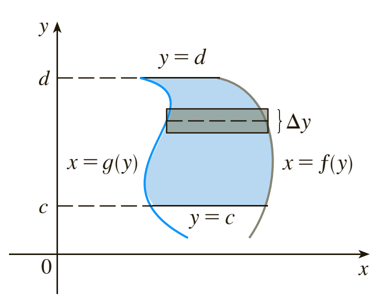


An expression for the exact area is

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x = \int_a^b [f(x) - g(x)]\,dx$$

(b) Explain how the situation changes if the curves have equations $x = f(y)$ and $x = g(y)$, where $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$.

Instead of using "top minus bottom" and integrating from left to right, we use "right minus left" and integrate from bottom to top: $A = \int_c^d [f(y) - g(y)]\,dy$



2. Suppose that Sue runs faster than Kathy throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?

It represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. (a) Suppose S is a solid with known cross-sectional areas. Explain how to approximate the volume of S by a Riemann sum. Then write an expression for the exact volume.

We slice S into n "slabs" of equal width Δx. The volume of the ith slab is approximately $A(x_i^*)\,\Delta x$, where x_i^* is a sample point in the ith slab and $A(x_i^*)$ is the cross-sectional area of S at x_i^*. Then the volume of S is approximately $\sum_{i=1}^{n} A(x_i^*)\,\Delta x$ and the exact volume is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

(b) If S is a solid of revolution, how do you find the cross-sectional areas?

If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi(r_{\text{out}}^2) - \pi(r_{\text{in}}^2)$.

4. (a) What is the volume of a cylindrical shell?

$$V = 2\pi rh\,\Delta r = (\text{circumference})(\text{height})(\text{thickness})$$

(b) Explain how to use cylindrical shells to find the volume of a solid of revolution.

We approximate the region to be revolved by rectangles, oriented so that revolution forms cylindrical shells rather than disks or washers. For a typical shell, find the circumference and height in terms of x or y and calculate

$$V = \int_a^b (\text{circumference})(\text{height})(dx \text{ or } dy)$$

(c) Why might you want to use the shell method instead of slicing?

Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.

5. Suppose that you push a book across a 6-meter-long table by exerting a force $f(x)$ at each point from $x = 0$ to $x = 6$. What does $\int_0^6 f(x)\,dx$ represent? If $f(x)$ is measured in newtons, what are the units for the integral?

$\int_0^6 f(x)\,dx$ represents the amount of work done. Its units are newton-meters, or joules.

6. (a) What is the average value of a function f on an interval $[a, b]$?

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

(b) What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?

If f is continuous on $[a, b]$, then there is a number c in $[a, b]$ at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$. This means that for positive functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.