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1. (a) (i) $A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$.

~~eigenvalue~~ eigenvalue: $\det(A-\lambda I)=0$.

$\therefore (0.6-\lambda)(-0.6-\lambda)-0.8^2=0$.

$\lambda^2=1 \Rightarrow \lambda=\pm 1$

when $\lambda_1=1$,

$Av_1=\lambda_1 v_1$. $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} v_1 = v_1 \Rightarrow v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

when $\lambda_2=-1$

$Av_2=\lambda_2 v_2$ $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} v_2 = -v_2 \Rightarrow v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(ii) assume v_1 and λ_1 is the eigenvector and ~~the~~ eigenvalue of matrix A.

~~(iii) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2-2=0$.~~

~~$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are orthogonal.~~

$\because Av_1=\lambda_1 v_1$

~~(iv)~~ $v_1^T A^T = v_1^T \lambda_1$

$\therefore v_1^T A^T A v_1 = \lambda_1^2 v_1^T v_1$.

$\because v_1^T v_1 = \lambda_1^2 v_1^T v_1$

$\therefore \lambda_1^2=1$

$\lambda_1 = \|1\|$.

(iii) $Av_1=\lambda_1 v_1$ $v_2^T A^T = \lambda_2 v_2^T$

$\therefore v_2^T A^T A v_1 = \lambda_1\lambda_2 v_2^T v_1$.

$v_2^T v_1 = \lambda_1\lambda_2 v_2^T v_1$

for the formula to be consist for all time,

$v_2^T v_1 = 0$. i.e, v_2; v_1 are orthogonal to each other.

(iv) the vector may rotate ~~~~ according to the matrix A. counter clockwisely.

(b) (i) The left singular vectors of A is the same as the eigenvectors of AA^T, and the right singular vector of A is the same as the eigenvectors of A^TA.

(ii) the singular values of A are the square root of eigenvalues of AA^T and A^TA.

(c). (i) F.

(ii) T

(iii) T

(iv) F

(v) F

2 (a) (i). $P(H50|T)$

$$= \frac{P(H50)P(T|H50)}{P(T)}$$

$$P(T) = P(H50)P(T|H50) + P(H60)P(T|H60).$$

$$P(H50) = P(H60) = \tfrac{1}{2},\quad P(T|H50) = 0.5,\quad P(T|H60) = 0.4$$

$$\therefore\ P(H50|T) = \frac{\frac{1}{2}\times 0.5}{\frac{1}{2}\times 0.5 + \frac{1}{2}\times 0.4} = \frac{5}{9}$$

(ii) $P(H50|T,H,H,H)$

$$= \frac{P(H50)P(T,H,H,H|H50)}{P(T,H,H,H)} = \frac{0.5^4}{0.5^4 + 0.4\times 0.6^3} = 0.42$$

(iii) $$P(H50|9H1T) = \frac{P(H50)P(9H1T|H50)}{P(9H1T)}$$

$$P(9H1T) = \tfrac{1}{3}\times 0.5^9\times 0.5 + \tfrac{1}{3}\times 0.55^9\times 0.45 + \tfrac{1}{3}\times 0.6^9\times 0.4$$

$$\therefore\ P(H50|9H1T) = \frac{0.5^{10}}{0.5^9\times 0.5 + 0.55^9\times 0.45 + 0.6^9\times 0.4} = 0.138$$

$$P(H55|9H1T) = \frac{0.55^9\times 0.45}{0.5^9\times 0.5 + 0.55^9\times 0.45 + 0.6^9\times 0.4} = 0.293.$$

$$p(H\ 60 \mid 9H1T) = \frac{0.6^9 \times 0.4}{0.5^{10} + 0.55^9 \times 0.45 + 0.6^9 \times 0.4} = 0.569$$

(b) $$p(\text{pregnant} \mid \text{positive}) = \frac{p(\text{positive} \mid \text{pregnant})\, p(\text{pregnant})}{p(\text{positive})}$$

$$= \frac{0.99 \times 0.01}{0.99 \times 0.01 + 0.99 \times 0.1} = \frac{1}{11}$$

The result makes sense because most of the female population (99%) is not pregnant, so even if the ~~test~~ shows positive, the non-pregnancy still takes ~~larger~~ a larger aspect.

(c) $E[Ax+b] = A \cdot E[x] + b.$

(d) $\mathrm{cov}(Ax+b) = A\,\mathrm{cov}(X)A^T$

3. (a) $\nabla_x x^T A y = Ay.$

(b) $\nabla_y x^T A y = A^T x$

(c) $\nabla_A x^T A y = x y^T$

(d) $\nabla_x f = \cancel{Ax+b}.\ (A+A^T)x + b$

(e) $\nabla_A f = B^T$

4. $$\min_W \frac{1}{2}\sum_{i=1}^{n} \|y^{(i)} - Wx^{(i)}\|^2$$

$$= \min_W \frac{1}{2}\|Y - WX\| \quad \text{where } Y = [y^{(1)}\ y^{(2)} \cdots y^{(n)}]$$

$$X = [x^{(1)}\ x^{(2)} \cdots x^{(n)}]$$

$\therefore$ according to pseudo inverse

$$W = (XX^T)^{-1} X^T Y$$

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Linear regression workbook

This workbook will walk you through a linear regression example. It will provide familiarity with Jupyter Notebook and Python. Please print (to pdf) a completed version of this workbook for submission with HW #1.

ECE 239AS, Winter Quarter 2019, Prof. J.C. Kao, TAs M. Kleinman and A. Wickstrom and K. Liang and W. Chuang

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt

#allows matlab plots to be generated in line
%matplotlib inline
```

Data generation

For any example, we first have to generate some appropriate data to use. The following cell generates data according to the model: $y = x - 2x^2 + x^3 + \epsilon$

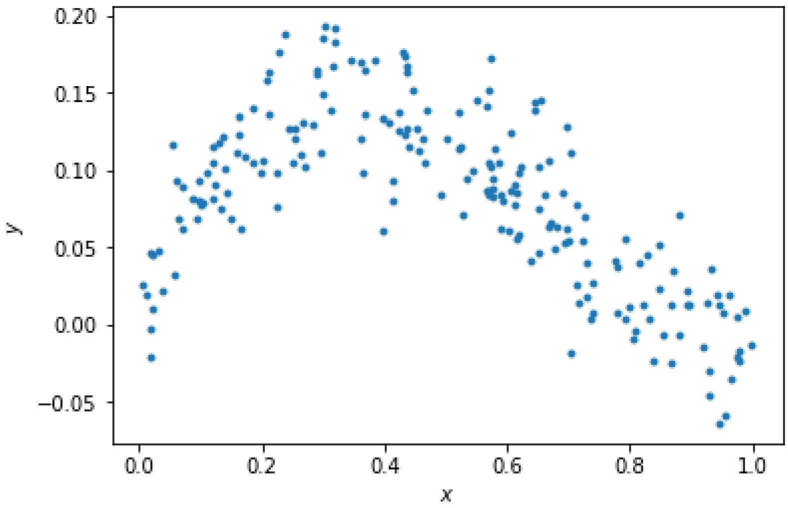
In [2]:

```
np.random.seed(0)  # Sets the random seed.
num_train = 200     # Number of training data points

# Generate the training data
x = np.random.uniform(low=0, high=1, size=(num_train,))
y = x - 2*x**2 + x**3 + np.random.normal(loc=0, scale=0.03, size=(num_train,))
f = plt.figure()
ax = f.gca()
ax.plot(x, y, '.')
ax.set_xlabel('$x$')
ax.set_ylabel('$y$')
```

Out[2]:

```
Text(0,0.5,'$y$')
```



QUESTIONS:

Write your answers in the markdown cell below this one:

(1) What is the generating distribution of x?

(2) What is the distribution of the additive noise ϵ?

ANSWERS:

(1) The distribution of x is uniform distribution.

(2) The distribution of noise is normal distribution.

Fitting data to the model (5 points)

Here, we'll do linear regression to fit the parameters of a model $y = ax + b$.

In [3]:

```
# xhat = (x, 1)
xhat = np.vstack((x, np.ones_like(x)))
xhat = xhat.transpose()
y = y.transpose()
# ==================== #
# START YOUR CODE HERE #
# ==================== #
# GOAL: create a variable theta; theta is a numpy array whose elements are [a, b]

theta = np.matmul(np.matmul(np.linalg.inv(np.matmul(xhat.transpose(),xhat)),xhat.transp
ose()),y)
# ================== #
# END YOUR CODE HERE #
# ================== #
```

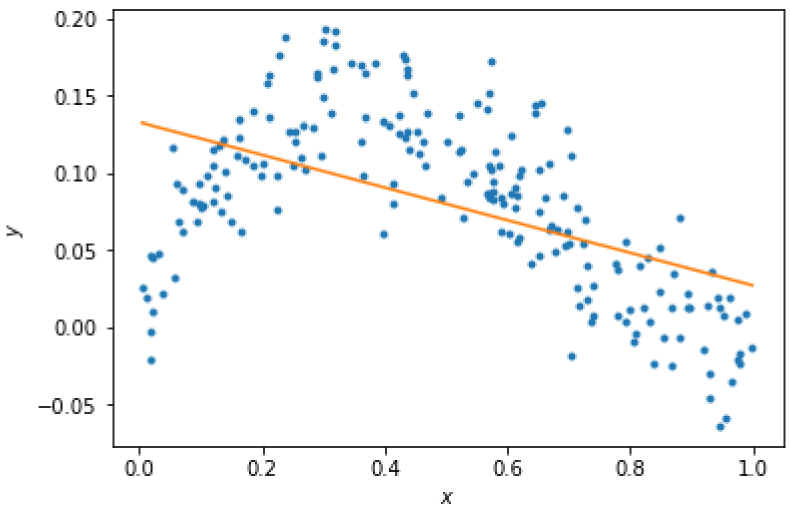
In [4]:

```
# Plot the data and your model fit.
f = plt.figure()
ax = f.gca()
ax.plot(x, y, '.')
ax.set_xlabel('$x$')
ax.set_ylabel('$y$')

# Plot the regression line
xs = np.linspace(min(x), max(x),50)
xs = np.vstack((xs, np.ones_like(xs)))
plt.plot(xs[0,:], theta.dot(xs))
```

Out[4]:

```
[<matplotlib.lines.Line2D at 0x2234766a390>]
```



QUESTIONS

(1) Does the linear model under- or overfit the data?

(2) How to change the model to improve the fitting?

ANSWERS

(1) The linear model under-fit the data.

(2) Use a polinomial model with higher orders.

Fitting data to the model (10 points)

Here, we'll now do regression to polynomial models of orders 1 to 5. Note, the order 1 model is the linear model you prior fit.

In [5]:

```
N = 5
xhats = []
thetas = []

# ==================== #
# START YOUR CODE HERE #
# ==================== #

# GOAL: create a variable thetas.
# thetas is a list, where theta[i] are the model parameters for the polynomial fit of o
rder i+1.
#   i.e., thetas[0] is equivalent to theta above.
#   i.e., thetas[1] should be a length 3 np.array with the coefficients of the x^2, x,
 and 1 respectively.
#   ... etc.

for i in range(N):
    if i == 0:
        xhat = np.vstack((x,np.ones_like(x)))
        xhats.append(xhat)
    else:
        xhat = np.vstack((np.power(x,i+1),xhats[-1]))
        xhats.append(xhat)
    xhat = xhat.transpose()
    theta = np.matmul(np.matmul(np.linalg.inv(np.matmul(xhat.transpose(),xhat)),xhat.tr
anspose()),y)
    thetas.append(theta)

# ================== #
# END YOUR CODE HERE #
# ================== #
```

In [6]:

```
thetas
```

Out[6]:

```
[array([-0.10599633,  0.13315817]),
 array([-0.48023061,  0.36743967,  0.05521084]),
 array([ 0.8843808 , -1.82077417,  0.91178032,  0.00979068]),
 array([ 0.14080037,  0.60466289, -1.64250929,  0.87250485,  0.01175321]),
 array([ 0.52432592, -1.164568  ,  1.76052438, -2.07430275,  0.93373916,
         0.009716  ])]
```

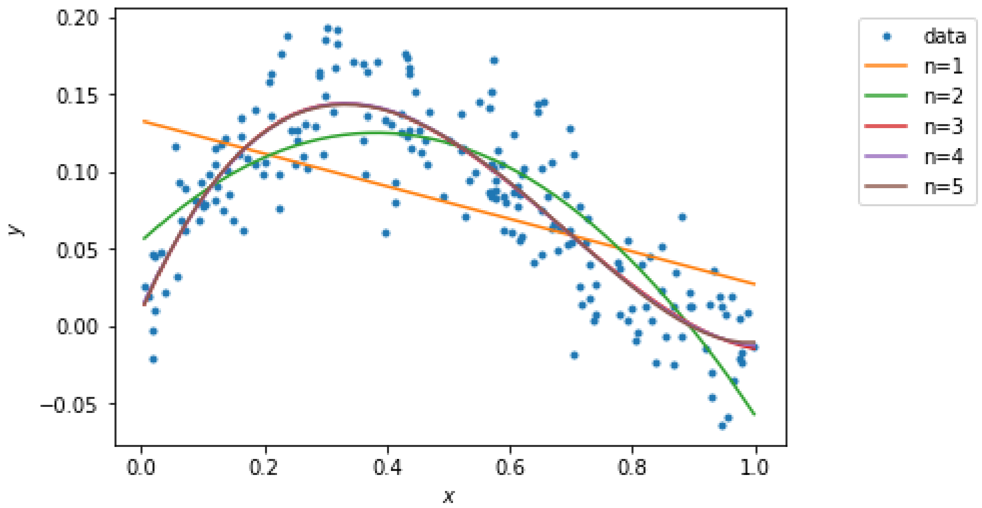
In [7]:

```
# Plot the data
f = plt.figure()
ax = f.gca()
ax.plot(x, y, '.')
ax.set_xlabel('$x$')
ax.set_ylabel('$y$')

# Plot the regression lines
plot_xs = []
for i in np.arange(N):
    if i == 0:
        plot_x = np.vstack((np.linspace(min(x), max(x),50), np.ones(50)))
    else:
        plot_x = np.vstack((plot_x[-2]**(i+1), plot_x))
    plot_xs.append(plot_x)

for i in np.arange(N):
    ax.plot(plot_xs[i][-2,:], thetas[i].dot(plot_xs[i]))

labels = ['data']
[labels.append('n={}'.format(i+1)) for i in np.arange(N)]
bbox_to_anchor=(1.3, 1)
lgd = ax.legend(labels, bbox_to_anchor=bbox_to_anchor)
```



Calculating the training error (10 points)

Here, we'll now calculate the training error of polynomial models of orders 1 to 5.

In [8]:

```
training_errors = []

# ==================== #
# START YOUR CODE HERE #
# ==================== #

# GOAL: create a variable training_errors, a list of 5 elements,
# where training_errors[i] are the training loss for the polynomial fit of order i+1.

for xhat, theta in zip(xhats,thetas):
    error = np.linalg.norm(y - np.matmul(theta,xhat), ord=2)**2
    training_errors.append(error)


# ================== #
# END YOUR CODE HERE #
# ================== #

print ('Training errors are: \n', training_errors)
```

```
Training errors are: 
 [0.4759922176725402, 0.2184984441853706, 0.16339207602210748, 0.163307074
70593958, 0.1632295839105059]
```

QUESTIONS

(1) What polynomial has the best training error?

(2) Why is this expected?

ANSWERS

(1) Polynomial with order 5 has the best training error.

(2) A higher order of polynomial is expectd to fit the training data better than lower order.

Generating new samples and testing error (5 points)

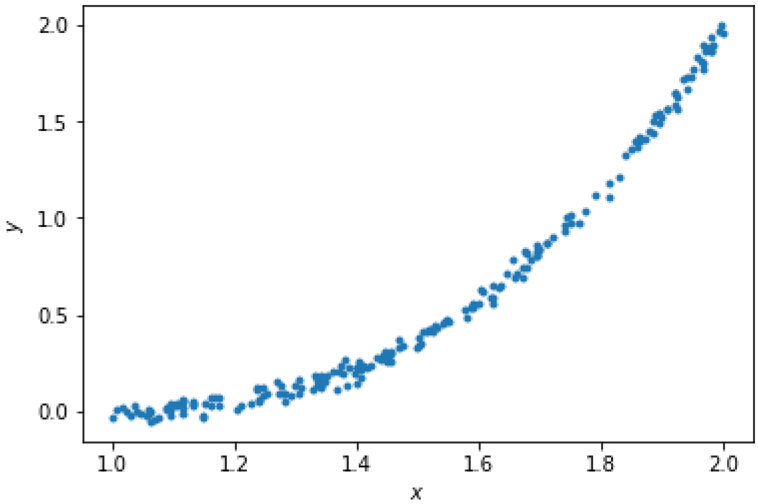
Here, we'll now generate new samples and calculate testing error of polynomial models of orders 1 to 5.

In [9]:

```
x = np.random.uniform(low=1, high=2, size=(num_train,))
y = x - 2*x**2 + x**3 + np.random.normal(loc=0, scale=0.03, size=(num_train,))
f = plt.figure()
ax = f.gca()
ax.plot(x, y, '.')
ax.set_xlabel('$x$')
ax.set_ylabel('$y$')
```

Out[9]:

```
Text(0,0.5,'$y$')
```



In [10]:

```
xhats = []
for i in np.arange(N):
    if i == 0:
        xhat = np.vstack((x, np.ones_like(x)))
        plot_x = np.vstack((np.linspace(min(x), max(x),50), np.ones(50)))
    else:
        xhat = np.vstack((x**(i+1), xhat))
        plot_x = np.vstack((plot_x[-2]**(i+1), plot_x))

    xhats.append(xhat)
```

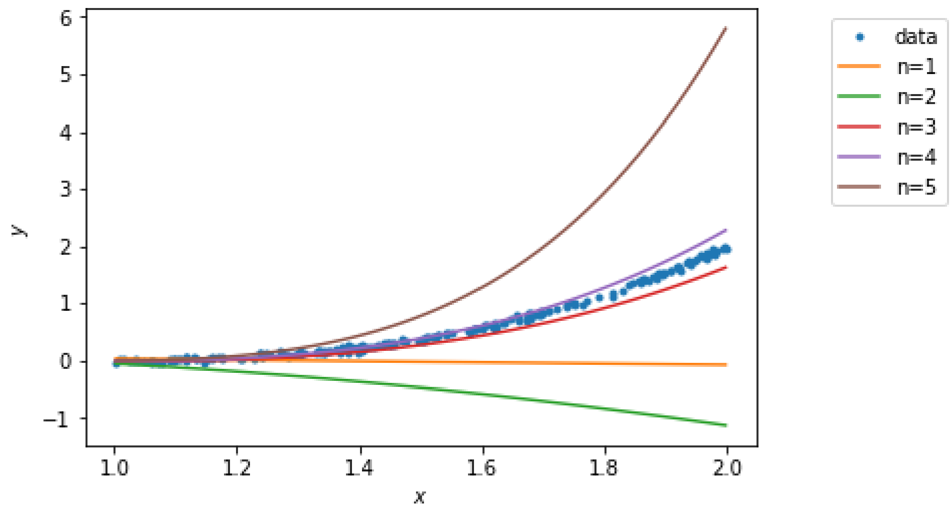
In [11]:

```
# Plot the data
f = plt.figure()
ax = f.gca()
ax.plot(x, y, '.')
ax.set_xlabel('$x$')
ax.set_ylabel('$y$')

# Plot the regression lines
plot_xs = []
for i in np.arange(N):
    if i == 0:
        plot_x = np.vstack((np.linspace(min(x), max(x),50), np.ones(50)))
    else:
        plot_x = np.vstack((plot_x[-2]**(i+1), plot_x))
    plot_xs.append(plot_x)

for i in np.arange(N):
    ax.plot(plot_xs[i][-2,:], thetas[i].dot(plot_xs[i]))

labels = ['data']
[labels.append('n={}'.format(i+1)) for i in np.arange(N)]
bbox_to_anchor=(1.3, 1)
lgd = ax.legend(labels, bbox_to_anchor=bbox_to_anchor)
```



In [12]:

```
testing_errors = []

# ==================== #
# START YOUR CODE HERE #
# ==================== #

# GOAL: create a variable testing_errors, a list of 5 elements,
# where testing_errors[i] are the testing loss for the polynomial fit of order i+1.
for xhat, theta in zip(xhats,thetas):
    error = np.linalg.norm(y - np.matmul(theta,xhat), ord=2)**2
    testing_errors.append(error)

# ================== #
# END YOUR CODE HERE #
# ================== #

print ('Testing errors are: \n', testing_errors)
```

```
Testing errors are:
 [161.72330369101172, 426.38384890115896, 6.251394216547027, 2.37415303785
4201, 429.8204363359659]
```

QUESTIONS

(1) What polynomial has the best testing error?

(2) Why polynomial models of orders 5 does not generalize well?

ANSWERS

(1) Polynomial with order 4 has the best testing error.

(2) Becasue a high order polynomial may overfit the data. It will have low training error but a high testing error.