

# Autoregressive Moving Average Models

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## Loading required package: ggplot2
## Loading required package: astsa
## Loading required package: ggfortify
## Loading required package: zoo
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## Attaching package: 'zoo'
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## The following objects are masked from 'package:base':
##
##      as.Date, as.Date.numeric
## Loading required package: gmwm
## Loading required package: gridExtra
```

In this chapter we introduce a class of time series models that is flexible and among the most commonly used to describe stationary time series. This class is represented by the Autoregressive Moving Average (ARMA) models which combine and include the autoregressive and moving average models seen in the previous chapter, which we first discuss in further detail before introducing the general ARMA class.

## Autoregressive Models (AR(p))

The class of autoregressive models is based on the idea that previous values in the time series are needed to explain current values in the series. For this class of models, we assume that the  $p$  previous observations are needed for this purpose and we therefore denote this class as AR( $p$ ). In the previous chapter, the model we introduced was an AR(1) in which only the immediately previous observation is needed to explain the following one and therefore represents a particular model which is part of the more general class of AR( $p$ ) models.

The AR( $p$ ) models can be formally represented as follows

$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + W_t,$$

where  $\min_{i=1,\dots,p} |\phi_i| \geq 0$  and  $W_t$  is a (Gaussian) white noise process with variance  $\sigma^2$ . In general, we will assume that the expectation of the process ( $X_t$ ), as well as that of the following ones in this chapter, is zero. The reason for this simplification is that if  $\mathbb{E}[X_t] = \mu$ , we can define an AR process *around*  $\mu$  as follows:

$$X_t - \mu = \sum_{i=1}^p (\phi_i X_{t-i} - \mu) + W_t,$$

which is equivalent to

$$X_t = \mu^* + \sum_{i=1}^p \phi_i X_{t-i} + W_t,$$

where  $\mu^* = \mu(1 - \sum_{i=1}^p \phi_i)$ . Therefore, to simplify the notation we will generally consider only zero mean process, since adding means (as well as other deterministic trends) is easy.

A useful way of representing AR processes is through the backshift operator introduced in the previous section and is as follows

$$\begin{aligned}
X_t &= \phi_1 X_{t-1} + \dots + \phi_p y_{t-p} + w_t \\
&= \phi_1 B X_t + \dots + \phi_p B^p X_t + W_t, \\
&= (\phi_1 B + \dots + \phi_p B^p) X_t + W_t
\end{aligned}$$

which finally yields

$$(1 - \phi_1 B - \dots - \phi_p B^p) X_t = W_t,$$

which, in abbreviated form, can be expressed as

$$\phi(B) X_t = W_t.$$

We will see that  $\phi(B)$  is important to establish the stationarity of these processes and is called the *autoregressive* operator. Moreover, this quantity is closely related to another important property of AR processes, called *causality*. Before formally definition this new property, we consider the following example, which provides intuitive illustration of its importance.

**Example:** Consider a classical AR(1) model with  $|\phi| > 1$ . Such model could be expressed as

$$X_t = \phi^{-1} X_{t+1} - \phi^{-1} W_t = \phi^{-k} X_{t+k} - \sum_{i=1}^{k-1} \phi^{-i} W_{t+i}.$$

Since  $|\phi| > 1$ , we obtain

$$X_t = - \sum_{j=1}^{\infty} \phi^{-j} W_{t-j},$$

which is a linear process and therefore is stationary. Unfortunately, such model is useless because the future is required to predict the future and such processes are called non-causal.