Math, Problem Set #5, Convex Analysis
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Exercise 7.1

Let V be a vector space and let $S \subseteq V$. We wts that $conv(S)$ is convex.

Let x, $y \in conv(S)$. Then, by the definition of $conv(S)$ we have that, $x = \sum \alpha_i \tilde{x}_i$ and $y = \sum \beta_i \tilde{y}_i$, where $\{\alpha_i\}$ and $\{\beta_i\}$ are sets of coefficients, and $\{\tilde{x}_i\}$ and $\{\tilde{y}_i\}$ are elements of S.

Now, let $\lambda \in (0,1)$. Notice that:

$$\begin{aligned} \lambda x + (1-\lambda y) &= \lambda \sum \alpha_i \tilde{x}_i + (1-\lambda) \sum \beta_i \tilde{y}_i \\ &= \sum \lambda \alpha_i \tilde{x}_i + \sum (1-\lambda) \beta_i \tilde{y}_i \\ &= \sum \delta_j \tilde{z}_j \end{aligned}$$

which is a convex combination of elements of S. Therefore, $\lambda x + (1-\lambda) y \in conv(S)$. Thus $conv(S)$ is convex.

Exercise 7.2

Let V be an inner product space and let $a \in V$ and $b \in \mathbb{R}$.

First, consider the hyperplane:

$$P(a,b) = \{x \in V : \langle a, x \rangle = b\}$$

Let x, $y \in P(a,b)$ and let $\lambda \in (0,1)$. Call $z = \lambda x + (1-\lambda) y$. Notice that:

$$\begin{aligned} \langle a, z \rangle &= \langle a, \lambda x + (1-\lambda) y \rangle \\ &= \lambda \langle a, x \rangle + (1-\lambda) \langle a, y \rangle \\ &= \lambda b + (1-\lambda) b \\ &= b \end{aligned}$$

Hence, $z \in P(a,b)$. Therefore, $P(a,b)$ is a convex set.

Consider now, wlog, the closed halfspace:

$$H(a,b) = \{x \in V : \langle a, x \rangle \leq b\}$$

Let x, $y \in H(a,b)$ and let $\lambda \in (0,1)$. Call $z = \lambda x + (1-\lambda)y$. Notice that:

$$\begin{aligned}\langle a, z\rangle &= \langle a, \lambda x + (1-\lambda)y\rangle \\ &= \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle \\ &\leq \max\{\langle a, x\rangle, \langle a, y\rangle\} \\ &\leq b\end{aligned}$$

Hence, $z \in H(a,b)$. Therefore, $H(a,b)$ is a convex set.

Exercise 7.4

I will first prove the suggested points.

(i)

$$\begin{aligned}\|x-y\|^2 &= \|x-p+p-y\|^2 \\ &= \langle x-p+p-y, x-p+p-y\rangle \\ &= \langle x-p, x-p\rangle + 2\langle x-p, p-y\rangle + \langle p-y, p-y\rangle \\ &= \|x-p\|^2 + \|y-p\|^2 + 2\langle x-p, p-y\rangle\end{aligned}$$

(ii)

If $y \neq p$, then $\|y-p\|^2 > 0$. Moreover, $\langle x-p, p-y\rangle \geq 0$. Therefore:

$$\begin{aligned}&\|x-y\|^2 > \|x-p\|^2 \\ \Rightarrow\ &\|x-y\| > \|x-p\|\end{aligned}$$

(iii)

$$\begin{aligned}\|x-z\|^2 &= \|x-\lambda y-(1-\lambda)p\|^2 \\ &= \|x-p+\lambda(p-y)\|^2 \\ &= \langle x-p+\lambda(p-y), x-p+\lambda(p-y)\rangle \\ &= \langle x-p, x-p\rangle + 2\langle x-p, \lambda(p-y)\rangle + \langle\lambda(p-y), \lambda(p-y)\rangle \\ &= \|x-p\|^2 + \lambda^2\|p-y\|^2 + 2\lambda\langle x-p, \lambda(p-y)\rangle\end{aligned}$$

Exercise 7.6

Call $LC = \{x \in \mathbb{R}^n : f(x) \leq c\} \subseteq \mathbb{R}^n$, the lower contour set at point c of f. Let x, $y \in LC$, let $\lambda \in (0,1)$ and call $z = \lambda x + (1-\lambda)y$. Notice that:

$$\begin{aligned}f(z) &= f(\lambda x + (1-\lambda)y) \\ &\leq \lambda f(x) + (1-\lambda)f(y) \quad \text{(by convexity of f)} \\ &\leq \max\{f(x), f(y)\} \\ &\leq c\end{aligned}$$

Therefore, $z \in LC$ and LC is convex. In other words, the lower contour set of a convex function is convex.

Exercise 7.7

Let $\theta \in (0,1)$ and $x, y \in C$.

$$\begin{aligned} f(\theta x + (1-\theta)y) &= \sum \lambda_i f_i(\theta x + (1-\theta)y) \\ &= \sum \lambda_i(\theta f_i(x) + (1-\theta)f_i(y)) \quad \text{by convexity of each} f_i \\ &= \theta \sum \lambda_i f_i(x) + (1-\theta) \sum \lambda_i f_i(y) \\ &= \theta f(x) + (1-\theta)f(y) \end{aligned}$$

Therefore, f is convex. One direct consequence of the result is that the sum of convex functions is a convex function.

Exercise 7.13

Let f be convex and bounded above. We wts that f is constant.

Suppose not. Then, wlog, take x, y such that $f(x) < f(y)$. Take $\lambda \in (0,1)$ and let $z = \lambda x + (1-\lambda)y$. By convexity of f we have that:

$$\begin{aligned} f(z) &\leq \lambda f(x) + (1-\lambda)f(y) \\ \implies f(x) &\geq \frac{f(z)}{\lambda} + \frac{(1-\lambda)}{\lambda}f(y) \end{aligned}$$

Notice that $\frac{(1-\lambda)}{\lambda} > 0$, $\forall \lambda \in (0,1)$. Now, take the limit as $\lambda \to 0^+$:

$$f(x) \to +\infty$$

so that f is unbounded above, a contradiction. Thus, f must be constant.

Exercise 7.20

Let f be convex and $-f$ be convex. By the definition of convexity it follows that $\forall \lambda \in (0,1)$ and $\forall x, y \in dom(f)$:

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$$

To show that f is affine, we want to show that $\forall \lambda \in \mathbb{R}$ and $\forall x, y \in dom(f)$:

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$$

In other words, we wts that f preserves affine combinations.

We have already studied the case in which $\lambda \in (0, 1)$. Let's now focus on the case in which $\lambda > 0$. Let $z = \lambda x + (1 - \lambda)y$. Notice that:

$$\begin{aligned} x &= \frac{z}{\lambda} - \frac{(1-\lambda}{\lambda}y \\ &= tz + (1-t)y \end{aligned}$$

where $t = 1/\lambda$. Therefore:

$$f(x) = tf(z) + (1-t)f(y)$$

which implies that:

$$\begin{aligned} f(z) &= \frac{f(x)}{t} - \frac{(1-t)}{t}f(y) \\ &= \lambda f(x) + (1-\lambda)f(y) \end{aligned}$$

Let's now focus on the case $\lambda \in (-1, 0)$. Again, let $z = \lambda x + (1 - \lambda)y$ and notice that:

$$\begin{aligned} y &= \frac{z}{(1-\lambda)} - \frac{\lambda}{(1-\lambda}x \\ &= tz + (1-t)x \end{aligned}$$

where $t = \frac{1}{(1-\lambda)} \in (0, 1)$. Therefore we have that:

$$f(y) = tf(z) + (1-t)f(x)$$

which implies that:

$$\begin{aligned} f(z) &= \frac{f(y)}{t} - \frac{(1-t)}{t}f(x) \\ &= \lambda f(x) + (1-\lambda)f(y) \end{aligned}$$

The final case in which $\lambda < -1$ follows directly from the previous two. Therefore, f is affine.