Math, Problem Set #4, Optimization Theory
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Exercise 6.1

In order to rewrite the problem in the standard form let:

$$\begin{aligned} f(w) &= -e^{w^T x} \\ H(w) &= y^T w - w^T x \\ G(w) &= -w^T(x - Aw + Ay) \end{aligned}$$

Then, we have that:

$$\begin{aligned} \min \quad & f(w) \\ \text{s.t.} \quad & H(w) = b \\ & G(w) \leq a \end{aligned}$$

Exercise 6.5

Given the way the problem is framed, I will assume that the production inputs are perfect complements.

To write the problem in standard form let:

$$\begin{aligned} x &= [K, L, q^k, q^m]' \\ f(x) &= 0.07q^m + 0.05q^k \\ H(x) &= [y^m - \min\{\frac{1}{4}K \quad \frac{1}{2}L\} \quad y^k - \min\{\frac{1}{3}K, L\}]' \\ G(x) &= [L \quad K]' \\ a &= [6000 \quad 240000]' \end{aligned}$$

where K represents grams of plastic, L minutes of labor, q^k the amount of knobs produced, and q^m the amount of milk produced.

Therefore we have:

$$\begin{aligned} \min \quad & f(w) \\ \text{s.t.} \quad & H(w) = 0 \\ & G(w) \leq a \end{aligned}$$

Exercise 6.6

First, let's compute the gradient of the function:

$$(\nabla f)(x,y) = [6xy + 4y^2 + y \quad 3x^2 + 8xy + x]'$$

which can be rewritten as:

$$(\nabla f)(x,y) = [y(6x + 4y + 1) \quad x(3x + 8y + 1)]'$$

and equate it to zero to find all the critical points (notice that all points in the feasibility sets are interior points, so that this identifies all the critical points). It is immediate to see that the critical points are given by $(0,0)$, $(0,-\frac{1}{4})$, $(-\frac{1}{3},0)$, and $(-\frac{1}{3},-\frac{1}{12})$.

To understand the nature of these critical points, we need to evaluate the Hessian matrix, which is given by:

$$(Hf)(x,y) = \begin{bmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix}$$

Evaluating the Hessian matrix at each critical point we obtain:

$$(Hf)(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The Hessian matrix at the point $(0,0)$ is indefinite, so we have a saddle point.

$$(Hf)(0,-\frac{1}{4}) = \begin{bmatrix} -\frac{6}{4} & -1 \\ -1 & 0 \end{bmatrix}$$

The Hessian matrix at the point $(0,-\frac{1}{4})$ is indefinite, so we have a saddle point.

$$(Hf)(-\frac{1}{3},0) = \begin{bmatrix} 0 & -1 \\ -1 & -\frac{8}{3} \end{bmatrix}$$

The Hessian matrix at the point $(-\frac{1}{3},0)$ is indefinite, so we have a saddle point.

$$(Hf)(-\frac{1}{3},-\frac{1}{12}) = \begin{bmatrix} 0 & -\frac{5}{3} \\ -\frac{5}{3} & -\frac{8}{3} \end{bmatrix}$$

The Hessian matrix at the point $(-\frac{1}{3},-\frac{1}{12})$ is indefinite, so we have a saddle point.

Exercise 6.11

Let $f(x) = ax^2 + bx + c$. Notice that:

$$\begin{aligned} f'(x) &= 2ax + b \\ f''(x) &= 2a \end{aligned}$$

Notice that the true minimum of the function is given, for every x, by:

$$x = -\frac{b}{2a}$$

Start with any initial guess x_0. The Newton Algorithm tells us that:

$$\begin{aligned} x_1 &= x_0 - \frac{f'(x)}{f''(x)} \\ &= x_0 - \frac{2ax_0 + b}{2a} \\ &= x_0 - \frac{2ax_0}{2a} - \frac{b}{2a} \\ &= x_0 - x_0 - \frac{b}{2a} \\ &= -\frac{b}{2a} \end{aligned}$$

hence, the algorithm converges after one iteration.

Exercise 6.14

Please see the attached `.ipynb` Jupyter notebook.