Math, Problem Set #2, Inner Product Spaces

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Exercise 3.1

(i)

$$\begin{aligned}\langle x, y\rangle &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 - (\|x\|^2 - 2\langle x, y\rangle + \|y\|^2)) \\ &= \frac{1}{4}(4\langle x, y\rangle) = \langle x, y\rangle\end{aligned}$$

(ii)

$$\frac{1}{2}(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle) = \frac{1}{2}(2\|x\|^2 + 2\|y\|^2) = \|x\|^2 + \|y\|^2$$

Excercise 3.2

- $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$
- $-\|x-y\|^2 = \|x\|^2 - \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$
- $i\|x-iy\|^2 = i\|x\|^2 + i\|y\|^2 + \langle x, y\rangle - \langle y, x\rangle$
- $-i\|x+iy\|^2 = -i\|x\|^2 + -i\|y\|^2 + \langle x, y\rangle - \langle y, x\rangle$

Thus,

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2) = \frac{1}{4}(4\langle x, y\rangle) = \langle x, y\rangle$$

Exercise 3.3

(i)

$$cos(\theta) = \frac{\langle x, x^5\rangle}{\|x\|\|x^5\|}$$

$\langle x, x^5\rangle = \int_0^1 x^6 dx = \frac{1}{7}$, and $\|x\|^2 = \int_0^1 x^2 dx = \frac{1}{3}$, and $\|x^5\| = \int_0 1x^10dx = \frac{1}{11}$. Thus,

$$\theta = cos^{-1}(\frac{\sqrt{3}\sqrt{11}}{7})$$

(ii)

$\langle x^2, x^4 \ = \int 0^1 x^6 dx = \frac{1}{7}$, and $\|x^2\|^2 = \int_0 1x^4 dx = \frac{1}{5}$, and $\|x^4\|^2 = \int_0 1x^8 dx = \frac{1}{9}$. Thus,

$$\theta = cos^{-1}(\frac{3\sqrt{5}}{7})$$

.

Exercise 3.8

(i)

Using Integration by Parts,

- $\langle cos(t), sin(t) \rangle = 0$
- $\langle cos(t), cos(2t) \rangle = 0$
- $\langle cos(t), sin(2t) \rangle = 0$
- $\langle sin(t), cos(2t) \rangle = 0$
- $\langle sin(t), sin(2t) \rangle = 0$
- $\langle sin(2t), cos(2t) \rangle = 0$

Also,

- $\|cos(t)\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} cos^2(t) dt = \frac{1}{\pi} [\frac{t}{2} + \frac{sin(2t)}{4}] \Big|_{-\pi}^{\pi} = \frac{1}{\pi}\pi = 1.$

  In the same way,
- $\|sin(t)\|^2 = 1$
- $\|cos(2t)\|^2 = 1$
- $\|sin(2t)\|^2 = 1$

(ii)

$$\|t\|^2 = \frac{1}{\pi} \int_{pi}^{\pi} t^2 dt = \frac{1}{\pi} [\frac{t^3}{3}] \Big|_{-\pi}^{\pi} = \frac{2}{3}\pi^2$$

Thus,

$$\|t\| = \sqrt{\frac{2}{3}}\pi$$

.

(iii)

By (i), we have that $proj_X cos(3t) = 0$.

(iv)

$$proj_X t = \langle t, cos(t)\rangle cos(t)+$$

$$\langle t, sin(t)\rangle sin(t) + \langle t, cos(2t)\rangle cos(2t)$$

$$+\langle t, sin(2t)\rangle sin(2t) = 2sin(t) - sin(2t)$$

Exercise 3.9

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then,

$$R_\theta x = \begin{bmatrix} cos(\theta)x_1 - sin(\theta)x_2 \\ sin(\theta)x_1 + cos(\theta)x_2 \end{bmatrix}$$

and

$$R_\theta x = \begin{bmatrix} cos(\theta)y_1 - sin(\theta)y_2 \\ sin(\theta)y_1 + cos(\theta)y_2 \end{bmatrix}$$

Thus,
$\langle R_\theta x, R_\theta y\rangle = (sin^2(\theta)+cos^2(\theta))x_1y_1+(sin^2(\theta)+cos^2(\theta))x_2y_2 = x_1y_1+x_2y_2 = \langle x, y\rangle$.

Exercise 3.10

(i)

Suppose Q is an orthonormal matrix. Then, $\langle Qx, Qy\rangle = x^H Q^H Qy = x^H y = \langle x, y\rangle$. This can be true if and only if $Q^H Q = I$. Since F^n is finite dimensional, by prop. 3.2.12 we must have Q invertible. Thus, $Q^{-1}Q = I = Q^H Q$, which holds if and only if $Q^{-1} = Q^H$.
Now suppose $Q^H Q = QQ^H = I$. Then, for all $x, yinF^n$, $\langle x, y\rangle = x^H y = x^H Q^H Qy = (Qx)^H Qy = \langle Qx, Qy\rangle$, which can hold iff Q is the matrix representation of an orthonormal operator.

(ii)

Suppose Q is an orthonormal matrix. Then, $\langle Qx, Qx\rangle = x^H Q^H Qx = x^H Ix = x^H x = \langle x, x\rangle = \|x\|^2$. This holds if and only if $\sqrt{\langle Qx, Qx\rangle} = \|Qx\| = \sqrt{\|x\|^2} = \|x\|$.

(iii)

Q is orthonormal, i.e., $Q^HQ = QQ^H = I \iff Q^H = Q^{-1} \iff (Q^H)^H = (Q^{-1})^H = Q$. Thus,

$$(Q^{-1})^H Q^{-1} = QQ^{-1} = I$$

and

$$Q^{-1}(Q^{-1})^H = Q^{-1}Q = I$$

(iv)

Q is an orthonormal matrix. Let $q_i = Qe_i bethei^thcolumnofQ$. Then, $\langle q_j, q_i\rangle = \langle Qe_j, Qe_i\rangle = e_j^H Q^H Q e_i = delta_i j$, where $delta_i j$ is the Kroenecker delta. This shows that Q has orthonormal columns.

(v)

Q is orthonormal. Let $det(Q) = x + iy$. Then,

$$det(QQ^H) = det(Q)det(Q^H) = 1 = (det(Q))^2 = det(Q)\overline{det(Q)} = |det(Q)|$$

.

Consider $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Clearly, since A is upper triangular, $det(A) = 1$ but the inner product of the second and third columns is not zero. Thus, the second and third columns are not orthogonal and A is not orthogonal.

(vi)

Q_1 and Q_2 are orthogonal. Then,

$$(Q_1Q_2)^H Q_1 Q_2 = Q_2^H Q_1^H Q_1 Q_2 = Q_2^H I Q_2 = I$$

. and

$$Q_1Q_2(Q_1Q_2)^H = Q_1Q_2Q_2^HQ_1^H = Q_1 I Q_1^H = I$$

.

Exercise 3.11

Suppose $x_2 = \alpha x_1$. Then, applying G-S we get

$$q_1 = \frac{x_1}{\|x_1\|}$$

and

$$q_2 = \frac{x_2 - p_2}{\|x_2 - p_2\|}$$

where $p_2 = \langle q_1, x_2 \rangle q_1 = \langle \frac{x_1}{\|x_1\|}, \alpha x_1 \rangle \frac{x_1}{\|x_1\|} = x_2$. Hence, $\|x_2 - p_2\| = 0$, which means that applying G-S to a set of linearly dependent vectors implies dividing the residual of one of the vectors by zero, sooner or later.

Exercise 3.16

Suppose $A = Q_1 R_1 = Q_2 R_2$. Then, $P = Q_2^H Q_1 = R_2 R_1^{-1}$. On the left hand side of the last equation we have an orhonormal matrix (orthonormal matrices are closed under multiplication), while on the right hand side we have an upper triangular matrix with all non-negative entries (upper triangular matrices are also closed under multiplication). Hence, the matrix P is an orthonormal, upper triangular matrix with non-negative entries. By the properties of orthonormal matrices $P^H = P^{-1} \iff P$ is a diagonal matrix $\iff P$'s non-zero entries are either $+$ or -1. Since P has all positive entries, then its nonzero entries are $1's$. Thus, $P = Q_2^H Q_1 = R_2 R_1^{-1} = I$. Hence,

$$Q_1 = Q_2$$

and

$$R_2 = R_1$$

Exercise 3.17

$(\hat{Q}\hat{R})^H \hat{Q}\hat{R}x = (\hat{Q}\hat{R}^H b \iff \hat{R}^H \hat{Q}^H \hat{Q}\hat{R}x = (\hat{Q}\hat{R})^H b \iff \hat{R}^H \hat{R}x = \hat{R}^H \hat{Q}^H b \iff \hat{R}x = \hat{Q}^H b$.

Exercise 3.23

Let x, y belong to V. Then,

- $\|x\| = \|x + y - y\| \leq \|x - y\| + \|y\| \iff \|x\| - \|y\| = \|x - y\| \leq \|x - y\|$

  In the same way,

- $\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$

Thus,

$$|\|x\| - \|y\|| \leq \|x - y\|$$

3.24

(i)

- Since $0 \leq |f(t)|$, then
$$0 \leq \int_a^b |f(t)|dt = \|f\|_1.$$
Further, the above integral (that is, $\|f\|_1$) is identically zero if and only if $|f| = 0$ $\Longleftrightarrow f = 0$ almost everywhere on $[a, b]$.

- We have
$$\|\alpha f(t)\|_1 = \int_a^b |\alpha f(t)|dt = |\alpha| \int_a^b |f(t)|dt = |\alpha|\|f\|_1$$
for α in F.

- For any f, g in $C([a, b], F)$, we have that $|f(t) + g(t)| \leq |f(t)| + g(t)|$. Thus,
$$\|f(t)+g(t)\|_1 = \int_a^b |f(t)+g(t)|dt \leq \int_a^b |f(t)|+|g(t)|dt = \int_a^b |f|dt+\int_a^b |g|dt = \|f\|_1+\|g\|_1$$
.

(ii)

- Since $|f|^2 \geq 0$, we get that $\|f\|_2 \geq 0$ and $\|f\|_2 = 0$ if and only if $f = 0$ almost everywhere on $[a, b]$.

- For any α in F, we have
$$\|\alpha f\|_2 = \left(\int_a^b |\alpha f|^2 dt\right)^{1/2} = \alpha \left(\int_a^b |f|^2 dt\right)^{1/2} = \alpha\|f\|_2$$

- For any f, g in $C([a, b], F)$, we have $|f + g| \leq |f| + |g|$. Then,
$$\|f + g\|_2^2 \leq \int_a^b |f|^2 + |g|^2 + 2|fg|dt = \int_a^b |f|^2 dt + \int_a^b |g|^2 dt + 2\int_a^b |fg|dt$$
$$\leq \|f\|_2^2 + \|g\|_2^2 + 2\|f\|_2\|g\|_2$$
Since $2\|f\|_2\|g\|_2 \geq 0$, then the triangle inequality is established.

(iii)

- Since $|f| \geq 0$, then $sup_{[a,b]}|f| \geq 0$ and $sup_{[a,b]}|f| = 0 \Longleftrightarrow |f| = 0$ on $[a, b]$.
- Let α belong to F. Then,

$$\|\alpha f\|_\infty = sup|\alpha f| = |\alpha|sup|f(x)| = |\alpha|\|f\|_\infty$$

- Let f, g be in $C([a, b], F)$. Then,

$$\|f + g\|_\infty = sup|f + g| \leq sup|f| + sup|g| = \|f\|_\infty + \|g\|_\infty$$

Exercise 3.26

- Reflexivity: Let m = 0.5, M=1.5. Then,

$$0.5\|x\|_a \leq \|x\|_a\|1.5\|x\|_a$$

  Thus, $\|.\|_a \sim \|.\|_a$ for all x in X.

- Symmetry: Suppose $\|.\|_a \sim \|.\|_b$. Then, there exist $0 < m \leq M$ such that

$$m\|x\|_a \leq \|x\|_b \leq M\|x\|_a$$

  Therefore,

$$\|x\|_a \leq \frac{1}{m}\|x\|_b \quad \text{and} \quad \frac{1}{M}\|x\|_b \leq \|x\|_a$$

  This implies that

$$\frac{1}{M}\|x\|_b \leq 1\|x\|_a \leq \frac{1}{m}\|x\|_b \quad \text{and thus} \quad \|.\|_b \sim \|.\|_a$$

  The proof for the other direction is done in the same way by interchanging $\|.\|_a$ and $\|.\|_b$.

- Transitivity: suppose $\|.\|_a \sim \|.\|_b$ and $\|.\|_b \sim \|.\|_c$. Then, there exist $0 < m_1 \leq M_1$, $0 < m_2 \leq M_2$ such that for all $x \in X$

$$m_1\|x\|_a \leq \|x\|_b \leq M_1\|x\|_a \quad \text{and} \quad m_2\|x\|_b \leq \|x\|_c \leq M_2\|x\|_b$$

  Hence, given that $0 < m_1m_2 \leq M_1M_2$, we obtain

$$m_1m_2\|x\|_a \leq m_2\|x\|_b \leq \|x\|_c \leq M_1M_2\|x\|_a$$

  Therefore, Topological Equivalence is an Equivalence Relation.

(i)

For all $x \in X$,

$$\begin{aligned}\|x\|_2 &= \|x_1e_1 + x_2e_2 + ... + x_ne_n\|_2 \\ &\leq \|x_1e_1\|_2 + ... + \|x_ne_n\|_2 \\ &= |x_1| + ... + |x_n| = \|x\|_1\end{aligned}$$

where x_i is the i^{th} element of the vector $x \in X$. Now let y be a vector of 1's. Then,

$$\|x\|_1 = |\langle y, x\rangle| = |x_1| + |x_2| + ... + |x_n| \leq \|x\|_2\|y\|_2 = \sqrt{n}\|x\|_2$$

Thus,

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$$

(ii)

First notice that

$$\begin{aligned}\|x\|_\infty &= \max_i\{|x_1|, |x_2|, ..., |x_n|\} \\ &= |x_i| = \sqrt{x_i^2} \qquad \text{(for some i)} \\ &\leq \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = \|x\|_2\end{aligned}$$

Second,

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \leq \sqrt{n\|x\|_\infty^2} = \sqrt{n}\|x\|_\infty$$

Therefore,

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

Exercise 3.28

(i)

Using the first inequality in (3.26), we obtain

$$\frac{1}{\sqrt{n}}\|A\|_2 = \frac{1}{\sqrt{n}}\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sup_{x\neq 0}\frac{\|Ax\|_2}{\sqrt{n}\|x\|_2} \leq \sup_{x\neq 0}\frac{\|Ax\|_1}{\|x\|_1} = \|A\|_1 \leq$$
$$\leq \sup_{x\neq 0}\frac{\sqrt{n}\|Ax\|_2}{\|x\|_1} \leq \sqrt{n}\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sqrt{n}\|A\|_2$$

(ii)

Using the second inequality in (3.26), we have

$$\frac{1}{\sqrt{n}}\|A\|_\infty = \sup_{x\neq 0}\frac{\|Ax\|_\infty}{\sqrt{n}\|x\|_2} \leq \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2 \leq \sqrt{n}\sup_{x\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n}\|A\|_\infty$$

Exercise 3.29

Using the result in (3.10), we have

$$\|Q\| = \sup_{x\neq 0} \frac{\|Qx\|}{\|x\|} = \sup_{x\neq 0} \frac{\|x\|}{\|x\|} = 1$$

For all $A \in M_n(F)$,

$$\|R_x\|_2 = \sup_{A\neq 0} \frac{\|R_x A\|_2}{\|A\|_2} = \sup_{A\neq 0} \frac{\|Ax\|_2}{\|A\|_2} \leq \sup_{A\neq 0} \frac{\|Ax\|_2}{\|Ax\|_2}\|x\|_2 = \|x\|_2$$

where we have used the fact that

$$\|A\|_2 = \sup_{y\neq 0} \frac{\|Ay\|_2}{\|y\|_2} \geq \frac{\|Ay\|_2}{\|y\|_2}$$

for all $y \in F^n$. Thus,

$$\|R_x\|_2 \leq \|x\|_2.$$

Further, when A is orthonormal, $\|Ax\|_2 = \|A\|_2\|x\|_2 = \|x\|_2$, which implies

$$\|R_x\|_2 = \sup_{A\neq 0} \frac{\|Ax\|_2}{\|A\|_2} = \|x\|_2.$$

Exercise 3.30

Using the fact that $\|.\|$ is a norm, we can check

- Positivity: $\|A\|_S = \|SAS^{-1}\| \geq 0$, and $\|A\|_S = 0 \iff SAS^{-1} = 0 \iff A = 0$ (given that S is invertible).
- Scaling: $\|\alpha A\|_S = \|S(\alpha A)S^{-1}\| = |\alpha|\|SAS^{-1}\|$.
- Triangle: $\|A + B\|_S = \|S(A + B)S^{-1}\| = \|SAS^{-1} + SBS^{-1}\| \leq \|SAS^{-1}\| + \|SBS^{-1}\|$.
- Submultiplicative Property: $\|AB\|_S = \|SABS^{-1}\| = \|SABS^{-1}SBS^{-1}\| \leq \|SAS^{-1}\|\|SBS^{-1}\|$.

Hence, $\|.\|_S$ is a matrix norm, as well.

Exercise 3.37

Let $1, x, x^2$ be the standard basis for V. Applying G-S to this basis, we get

- $g_1 = \frac{1}{\|1\|} = 1$
- $g_2 = \frac{x-p_2}{\|x-p_2\|} = 2\sqrt{3}x - \sqrt{3}$, where $p_2 = \langle 1, x\rangle = \int_0^1 x dx = \frac{1}{2}$, and $\|x - p_2\|^2 = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$.

- $g_3 = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$

where $g1, g2, g3$ is an orthonormal basis for V. Then, using (Remark 3.7.2), we can calculate q as

$$q = \sum_{i=1}^{3} \overline{L(g_i)g_i} = 180x^2 - 168x + 24.$$

Exercise 3.38

$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. D^* is such that

$$\langle f, Dg \rangle = \int_0^1 fg'dx = -\int_0^1 f'gdx = -\langle Df, g \rangle = \langle -Df, g \rangle.$$

(I integrated by parts). Thus, $D^* = -D = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

3.39

(i)

$\langle w, (S+T)v \rangle_w = \langle w, Sv+Tv \rangle_w = \langle w, Sv \rangle_w + \langle w, Tv \rangle_w = \langle S^*w, v \rangle_v + \langle T^*w, v \rangle_v = \langle (S^*+T^*)w, v \rangle_v$. Thus, $(S+T)^* = S^* + T^*$. Also, $\langle w, \alpha Tv \rangle_w = \alpha \langle T^*w, v \rangle_v = \langle \overline{\alpha} T^* w, v \rangle_v$. Hence, $(\alpha T)^* = \overline{\alpha} T^*$.

(ii)

$\langle w, Sv \rangle_w = \langle S^*w, v \rangle_v = \langle w, (S^*)^*v \rangle_w$. Thus, $(S^*)^* = S$.

0.1 (iii)

$\langle v', sTv \rangle_v = \langle (ST)^*v', v \rangle_v = \langle v', S(Tv) \rangle_v = \langle S^*v', v \rangle_v = \langle T^*S^*v', v \rangle_v$. Thus, $(ST)^* = T^*S^*$.

(iv)

- $\langle v', v \rangle_v = \langle v', T(T^{-1}v) \rangle_v = \langle (T^{-1})^*T^*v', v \rangle_v \iff (T^{-1})^*T^* = I \iff (T^{-1})^* = (T^*)^{-1}$
- The other direction is similar.

Exercise 3.40

(i)

Let B, B' be in $M_n(F)$. Then,

$$\langle V', AB\rangle = Tr(B'^H AB = Tr((A^H B')^H B) = \langle A^H B', B\rangle = \langle A^* B', B\rangle \iff A^H = A^*.$$

(ii)

$$\langle A_2, A_3 A_1\rangle = Tr(A_1 A_2^H A_3) = Tr((A_2 A_1^H)^H A_3) = \langle A_2 A_1^H, A_3\rangle = \langle A_2 A_1^*, A_3\rangle$$

(from part (i)).

(iii)

$\langle C, AX - XA\rangle = Tr(C^H(AX - XA)) = Tr(C^H AX - C^H XA) = Tr((C^H A - AC^H)X) = \langle A^H C - CA^H, X\rangle = \langle A^* C - CA^*, X\rangle = \langle T_{A^*}(C), X\rangle$. Thus, $(T_A)^* = T_{A^*}$.