Università degli Studi di Ferrara

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On the Design of Quantum Communication Systems with non-Gaussian States

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Chapter 1

Quantum Mechanics Abstract

In this chapter, a brief overview of quantum mechanics postulates, of the notation and of the essential concept used in this thesis is given. The target of that is to explain to the reader the essential concept, in order to give him the possibility to understand the obtained result.

1.1 Postulates

Like every phisics theory, quantum mechanics is builded from few essential postulates. In this section are briefly introduced the six Dirac-Von Newman postulates of Quantum Mechanics [1, 2].

1.1.1 First postulate

Postulate 1 (State Representation) *The state of an isolated quantum system is represented by a complex unitary vector in an Hilbert space:*

$$|\psi\rangle \in \mathcal{H}$$

The space of possible states of the system is called state space and it is a separable complex Hilbert space.

Observation *Differently from the classical physics, in quantum mechanics the concept of state of system is introduced. In classical mechanics a system is described by his observables, like position or four-wheeled.*

1.1.2 Second postulate

Postulate 2 (Observables) *Every observables of the system is represented by an Hermitian operator acting on the state space:*

$$\mathcal{M} : \mathcal{H} \to \mathcal{H}$$

The outcomes of the measurement can only be one of the eigenvalue of the operator $\mathcal{M}$.

Observation *The possible outcomes of the measurement are real number because $\mathcal{M}$ is self-andjoint.*

1.1.3 Third postulate

Postulate 3 (Born's Rule) *The probability to get the measurement λ_i from the observable $\mathcal{M}$ in the system in state $|\psi\rangle$ is:*

$$\mathbb{P}(\lambda_i) = \langle\psi| \mathcal{P}_i |\psi\rangle$$

where $\langle\psi|$ is the correspondent vector of $|\psi\rangle$ in the dual space of $\mathcal{H}$ and where $\mathcal{P}_i$ is the projection operator of λ_i in the correspondent space.

1.1.4 Fourth postulate

Postulate 4 (Wavefunction Collapse) *The state after measurement of λ_i is $\mathcal{P}_i |\psi\rangle$ (with the necessary normalization):*

$$|\psi'\rangle = \frac{\mathcal{P}_i |\psi\rangle}{\langle\psi| \mathcal{P}_i |\psi\rangle}.$$

1.1.5 Fifth postulate

Postulate 5 (Time Evolution) *The time evolution of an isolated quantum system is given by an unitary operator $\mathcal{U}$:*

$$|\psi(t)\rangle = \mathcal{U}(t_0, t) |\psi(t_0)\rangle .$$

Observation (Time dependent Shrodinger Equation) *From postulate 5, it is possible to obtain the time dependent Shrodinger Equation:*

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

where $H(t)$ is the Hemiltonian matrix.

1.1.6 Sixth postulate

Postulate 6 (Composite System) *The state space of a system composed of $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by*

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

1.2 Combining Systems

The last postulate 6 has very important consequences for composite system. It is possible to describe two tipes of combined systems:

Definition 1 (Product states) *A state $|\psi\rangle \in \mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a pure state if exists $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ such that:*

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle .$$

A product state represents two states which do not interact; an operation on one of them does not perturb the other.

Definition 2 (Entengled states) *A system that is not in a product state (1), is in an entengled state.*

When a system is in an entengled state it is not possible to characterize the two subsystems with the states vector, although the state vector of the composite system is known.

1.2.1 Density operator

For a more general treatment, the following representation of states is given:

Definition 3 *The state of quantum system is described by a linear operator, called density operator such that:*

$$\Xi : \mathcal{H} \to \mathcal{H};\ \Xi^\dagger = \Xi;\ tr\{\Xi\} = 1.$$

According to the definition 3, the postulates 3, 4, 5 can be reformulate as following.

$$\mathbb{P}(\lambda_i) = tr\{\Xi \mathcal{P}_i\} \tag{1.1}$$

$$\Xi' = \frac{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger}{tr\{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger\}} \tag{1.2}$$

$$\Xi(t) = \mathcal{U}\Xi(t_0)\mathcal{U}^\dagger \tag{1.3}$$

1.3 Quantized Electromagnetic Field

Electromagnetic field is the main means of communication for contemporary application. It is important therefor, to give a quantum representation. In this section the representation of quantized electromagnetic field is initially given, so the Fock representation of a state is introduced.

1.3.1 Classical electromagnetic field

In a volume $\mathcal{V} \in \mathbb{R}^3$ classical electromagnetic field is determinated from Maxwell's equations as a superposition of the cavity modes ([3] quoting [4, 5]). Electric field is given by the well-known expression:

$$\boldsymbol{e}(\boldsymbol{r}, t) = \sum_n p_n(t)\boldsymbol{u}_n(\boldsymbol{r}) \tag{1.4}$$

where

$$\boldsymbol{u}_n(\boldsymbol{r}) = \boldsymbol{u}_{n0}\ e^{i\boldsymbol{k}_n \cdot \boldsymbol{r}}$$

and $\boldsymbol{u}_{n0}$ is determinated by the initial condition. The corresponding magnetic field is determinated by:

$$\boldsymbol{h}(\boldsymbol{r},t) = \sum_n q_n(t)\nabla \times \boldsymbol{u}_n(\boldsymbol{r}) \tag{1.5}$$

and

$$p_n(t) = \frac{\mathrm{d}q_n(t)}{\mathrm{d}t}. \tag{1.6}$$

The Hemiltonian associated to the n-th mode is given by

$$H_n = \frac{1}{2}[p_n^2(t) + \omega_n^2 q_n^2(t)]. \tag{1.7}$$

Equivalently, it is possible to define the complex variable $a_n(t)$ as

$$a_n(t) = \frac{\omega_n q_n(t) + ip_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.8}$$

and, using 1.8 in 1.7, it is possible to obtain the following expression of the Hemiltonian:

$$H_n = \hbar\omega_n |a_n(t)|^2. \tag{1.9}$$

1.3.2 Quantized electromagnetic field

The quantization of electromagnetic field is obtained replacing the two quantities $p_n(t)$ and $q_n(t)$ with the Hermitian operators $\boldsymbol{P}_n(t)$, $\boldsymbol{Q}_n(t) : \mathcal{H}_n \to \mathcal{H}_n$ and by imposing the following commutation conditions ([3] quoting [4, 5]):

$$[\boldsymbol{Q}_n, \boldsymbol{P}_m] = i\hbar\delta_{n,m}\boldsymbol{I} \tag{1.10}$$

$$[\boldsymbol{Q}_n, \boldsymbol{Q}_m] = 0 \tag{1.11}$$

$$[\boldsymbol{P}_n, \boldsymbol{P}_m] = 0. \tag{1.12}$$

Defining the annihilation operator $\boldsymbol{A}_n$ as

$$\boldsymbol{A}_n(t) = \frac{\omega_n \boldsymbol{Q}_n(t) + i\boldsymbol{P}_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.13}$$

and the adjoint of $\boldsymbol{A}_n$, the creation operator $\boldsymbol{A}_n^\dagger$ as

$$\boldsymbol{A}_n(t) = \frac{\omega_n \boldsymbol{Q}_n(t) - i\boldsymbol{P}_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.14}$$

it is possible to describe the Hemiltonian of the system as

$$H_n = \hbar\omega_n \boldsymbol{A}_n^\dagger \boldsymbol{A}_n. \tag{1.15}$$

1.3.3 Fock states

In a single mode cavity, it is possible to define the number operator $\boldsymbol{N}$ as

$$\boldsymbol{N} = \boldsymbol{A}^{\dagger}\boldsymbol{A}. \tag{1.16}$$

Single mode Fock states are the eigenvector of N, i.e the solution of equation:

$$\boldsymbol{N}\left|n\right\rangle = n\left|n\right\rangle. \tag{1.17}$$

The Fock state $\left|n\right\rangle$ represent the quantum state with exactly n photons. It is important to evidence that the set of all Fock states forms an orthonormal basis of the Hilbert space $\mathcal{H}$, so every state Ξ can be expressed as

$$\Xi = \sum_{n,m} c_{n,m}\left|n\right\rangle\left\langle m\right| \tag{1.18}$$

with

$$c_{n,m} = \left\langle n\right|\Xi\left|m\right\rangle.$$

Using the representation in Fock basis, it is possible to charaterize some type of quantum states of the quantum electromagnetic field. In the following section the states studied are briefly described.

1.4 QEF States

In this section, some quantum states of electromagnetic field useful for quantum communication are characterized. A brief introduction to another one tool for the description of quantum systems is initially given, so some Gaussian and non-Gaussian states are characterized.

1.4.1 Phase-space description

As seen before, in 1.18, quantum system can be completely described by a density operator Ξ defined in an infinite-dimensional Hilbert space $\mathcal{H}$. This operator can be expressed by the Fock representation (3). Sometimes, however, it is convenient to give another representation of state Ξ by means of a complex function introduced by Wigner [6]: the quasi-probability distribution. In this thesis, this representation will be introduced and it will be used to classify the possible states.

Definition 4 (Quantum characteristic function) *The s-order characteristic function $\chi(\xi, s)$, with $\xi, s \in \mathbb{C}$, associated to the quantum state Ξ is defined as:*

$$\chi(\xi, s) = \exp\left\{\frac{s}{2}|\xi|^2\right\} tr\{\Xi \boldsymbol{D}_\xi\} \tag{1.19}$$

where $\boldsymbol{D}_\xi$ is the displacement operator of parameter ξ, defined as:

$$\boldsymbol{D}_\xi = \exp\left\{\xi \boldsymbol{A}^{\dagger} - \xi^* \boldsymbol{A}\right\}. \tag{1.20}$$

Definition 5 (Quasi-probability distribution) *The s-order quasi-probability distribution $W(\alpha, s)$, with $s \in \mathbb{C}$, associated to the quantum state Ξ is given by:*

$$W(\alpha, s) = \frac{1}{\pi^2} \int_{\mathbb{R}^2} \chi(\xi, s) e^{\alpha\xi^* - \alpha^*\xi} d\xi^2. \tag{1.21}$$

The quasi-probability distribution, for $s = 0$ ($W(\alpha) = W(\alpha, 0)$) is called Wigner W-function.

1.4.2 Gaussian states

With the Wigner W-function $W(\alpha)$, it is possible to define the concept of Gaussian state ([3] quoting [7, 8, 9, 10, 11]).

Definition 6 (Gaussian state) *A quantum state $\Xi_{\boldsymbol{G}}$ is a Gaussian state if its Wigner W-function $W_G(\alpha)$ is Gaussian, i.e*

$$W_G(\alpha) = \frac{1}{\pi\sqrt{det\check{\boldsymbol{C}}_0}} \exp\left\{-\frac{1}{2}(\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\mu}})^H \check{\boldsymbol{C}}_0^{-1}(\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\mu}})\right\}. \tag{1.22}$$

where $\check{\boldsymbol{\mu}}$ is the augmented displacement vector, and $\check{\boldsymbol{C}}_0$ is the augmented covariance matrix.

Coherent states

A coherent state is the state of a quantum armonic oscillator of amplitude μ. It is defined ([3] seen [12, 13]) as the eigenvector $|\mu\rangle$ of $\boldsymbol{A}$ associated to the eigenvalue μ; i.e

$$\boldsymbol{A} |\mu\rangle = \mu |\mu\rangle . \tag{1.23}$$

It is possible to obtain a coherent state of parameter μ, from the ground state as

$$|\mu\rangle = \boldsymbol{D}_\mu |0\rangle . \tag{1.24}$$

As mentioned before, it is possible to characterize a state with the Fock representation and, equivalently, with the Wigner W-function. The last one is given, for a coherent state, by [14]:

$$W(\alpha) = \frac{2}{\pi} \exp\left\{-2|\alpha - \mu|^2\right\}. \tag{1.25}$$

It is easy to proof that $W(\alpha)$ is gaussian, with $\check{\boldsymbol{\mu}} = [\mu \ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \frac{1}{2}\boldsymbol{I}.$$

The Fock representation is given by [15]:

$$|\mu\rangle = e^{-\frac{|\mu|^2}{2}} \sum_{n=0}^{\infty} \frac{\mu^n}{\sqrt{n}} |n\rangle . \tag{1.26}$$

Noisy coherent states
It is possible to characterize the state of a noisy armonic oscillator introducing the thermal state, i.e the state of a electromagnetic cavity system. The Fock representation of the thermal state Ξ_{th} is given by [3]

$$\Xi_{th} = (1-v)\sum_{n=0}^{\infty} v^n |n\rangle \langle n| \tag{1.27}$$

where

$$v = \frac{\bar{n}}{\bar{n}+1}$$

and $\bar{n}$ is the well-known Plank distribution

$$\bar{n} = \left(\exp\left\{-\frac{\hbar\omega}{k_B T} - 1\right\}\right)^{-1}.$$

A noisy coherent states $\Xi_{th}(\mu)$ of parameter μ can be obtained by appling the displacement operator $\boldsymbol{D}_\mu$ to the thermal state Ξ_{th}, as follow:

$$\Xi_{th}(\mu) = \boldsymbol{D}_\mu^\dagger \Xi_{th} \boldsymbol{D}_\mu. \tag{1.28}$$

The Wigner W-function is given by

$$W_{th}(\alpha) = \frac{1}{\pi(\bar{n}+\frac{1}{2})}\exp\left\{-\frac{|\alpha-\mu|^2}{\bar{n}+\frac{1}{2}}\right\} \tag{1.29}$$

and it can be proved that it is a Gaussian function with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \left(\bar{n}+\frac{1}{2}\right)\boldsymbol{I}.$$

The Fock representation is given by

$$\langle n|\,\Xi_{th}(\mu)\,|m\rangle = (1-v)e^{-(1-v)|\mu|^2}\sqrt{\frac{n!}{m!}}v^n[(1-v)\mu^*]^{m-n}L_n^{m-n}\left(\frac{-(1-v)^2|\mu|^2}{v}\right) \tag{1.30}$$

Squeezed states
A squeezed state with amplitude μ and squeezing parameter ζ, is a defined as []

$$|\mu,\zeta\rangle = \boldsymbol{D}_\mu \boldsymbol{S}_\zeta |0\rangle \tag{1.31}$$

where $\boldsymbol{S}_\zeta$ is the squeezing operator, defined as

$$\boldsymbol{S}_\zeta = \exp\left\{\frac{1}{2}\left(\zeta\left(\boldsymbol{A}^\dagger\right)^2 + \zeta^*\boldsymbol{A}^2\right)\right\}. \tag{1.32}$$

It can be proven that a squeezed state is a Gaussian state with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \frac{1}{2}\begin{bmatrix}\cosh(2r) & \sinh(2r)e^{-i\phi} \\ \sinh(2r)e^{-i\phi} & \cosh(2r)\end{bmatrix}$$

with $\zeta = re^{i\phi}$. The Wigner W-function of a squeezed state, unlike that of a coherent state, has not a circular symmetry.

Noisy squeezed states
The representation of a noisy squeezed state $\Xi_{th}(\mu, \zeta)$ is obtained, similary to a noisy coherent state, as:

$$\Xi_{th}(\mu, \zeta) = \boldsymbol{D}_\mu \boldsymbol{S}_\zeta \Xi_{th} \boldsymbol{S}_\zeta^\dagger \boldsymbol{D}_\mu^\dagger. \tag{1.33}$$

The Gaussian Wigner function is obtained with $\check{\boldsymbol{\mu}} = [\mu \ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \left(\bar{n} + \frac{1}{2}\right) \begin{bmatrix} \cosh(2r) & \sinh(2r)e^{-i\phi} \\ \sinh(2r)e^{-i\phi} & \cosh(2r) \end{bmatrix}.$$

The Fock representation is given by [16]

$$\begin{aligned} \langle n| \Xi_{th}(\mu, \zeta) |m\rangle = \frac{\pi Q(0)}{(n!m!)^{1/2}} \sum_{k=0}^{min(n,m)} k! \binom{n}{k} \binom{m}{k} \tilde{A}^k \left(\frac{1}{2}\tilde{B}\right)^{(n-k)/2} \\ \left(\frac{1}{2}\tilde{B}^*\right)^{(m-k)/2} H_{n-k}((2\tilde{B})^{-1/2}\tilde{C}) H_{m-k}((2\tilde{B}^*)^{-1/2}\tilde{C}^*) \end{aligned} \tag{1.34}$$

where H_n is the Hermite polynomial with parameter n,

$$Q(0) = \frac{1}{\pi}[(1+A)^2 - |B|^2]^{-1/2} \exp\left\{ -\frac{(1+A)|C|^2 + \frac{1}{2}[B(C^*)^2 + B^*C^2]}{(1+A)^2 - |B|^2} \right\},$$

$$\tilde{A} = \frac{A(1+A) - |B|^2}{(1+A)^2 - |B|^2},$$

$$\tilde{B} = \frac{B}{(1+A)^2 - |B|^2},$$

$$\tilde{C} = \frac{(1+A)C + BC^*}{(1+A)^2 - |B|^2};$$

and

$$A = \bar{n} + (2\bar{n}+1)(\sinh(r))^2, \ B = -(2\bar{n}+1)e^{i\phi}\sinh(r)\cosh(r), \ C = \mu.$$

1.4.3 Non-Gaussian states

A state that does not meet the definition 6 is a non-Gaussian state. An important class of non-Gaussian states, useful for communications are the photon added states, examined in this thesis.

Photon added states
The photon added state $\Xi^{(1)}$, obtained from the quantum state Ξ, is given by:

$$\Xi^{(1)} = \frac{\boldsymbol{A}^\dagger \Xi \boldsymbol{A}}{tr\{\boldsymbol{A}^\dagger \Xi \boldsymbol{A}\}}. \tag{1.35}$$

The name *photon addition*, despite can be thought, does not mean that the mean photon number of the photon added state is one more then the non-photon added state. In general, its mean number of photon could be the same, more or less then the starting state. Only if $\Xi = |n\rangle\langle n|$, i.e Ξ is the density operator corresponding to the Fock state $|n\rangle$, the result of the photon addition is a state with one more photon.

Logically, the photon added state $\Xi^{(k)}$ (with k photon addition) is given by

$$\Xi^{(k)} = \frac{(\boldsymbol{A}^\dagger)^k \Xi \boldsymbol{A}^k}{tr\{(\boldsymbol{A}^\dagger)^k \Xi \boldsymbol{A}^k\}}. \tag{1.36}$$

The Fock representation of a photon added state, can be obteined as:

$$\Xi^{(k)} = \frac{\tilde{\Xi}^{(k)}}{tr\{\tilde{\Xi}^{(k)}\}} \tag{1.37}$$

and

$$\langle n|\tilde{\Xi}^{(k)}|m\rangle = \begin{cases} \sqrt{\frac{n!m!}{(n-k)!(m-k)!}}\langle n-k|\Xi|m-k\rangle \;\; if \;\; n,m \geq k \\ 0 \;\; otherwise \end{cases}$$

If Ξ is a noisy coherent state of amplitude μ ($\Xi = \Xi_{th}(\mu)$), the photon added state $\Xi_{th}^{(k)}(\mu)$ is called noisy photon added coherent state (PACS); if Ξ is a noisy squeezed state with amplitude μ and squeezing factor ζ ($\Xi = \Xi_{th}(\mu, \zeta)$), the photon added state $\Xi_{th}^{(k)}(\mu, \zeta)$ is called noisy photon added squeezed state (PASS).

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