Università degli Studi di Ferrara

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DEPARTMENT OF ENGINEERING
Bachelor's degree in Electronic and Computer Science Engineering

Communication Networks

On the Design of Quantum Communication Systems with non-Gaussian States

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Academic Year 2020-2021

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Chapter 1

Introduction

Quantum mechanics is opening doors to new ways of information processing and transmitting. The ability to deeply understand the essence of matter will lead, in the foreseeable future, to design and implement innovative technologies with a remarkable increase in performance. That is a very significant fact if we consider the increasingly computerization and necessity to be connected in our society. Communication will play a key role in this scenario, so it is essential to study and optimize an efficient and reliable communication system.

The aim of this thesis is analyzing the performance of an optimal binary quantum discriminator, in presence of information encoded with photon added quantum states. Firstly, the cardinal principles of the quantum mechanics will be introduced (Chapter 2), followed by a brief rewiev of some useful tools aimed at characterizing a quantum state-based communication system. The third chapter will concern a brief presentation of the quantum communication modalities (quantum modulation) and some key aspects in the recognition of information (quantum state discriminator). In this context, the optimal discriminator for binary systems will be introduced, that is the one which minimizes the distribution error probability (DEP) in the recognition of the transmitted symbol. Finally, the behaviour of the latter will be analyzed in the presence of non-Gaussian states of photon added coherent states (PACSs) and photon added squeezed states (PASSs) typologies, with the goal of researching the optimal parameters configuration that minimizes the error probabilty.

Chapter 2

Elements of quantum information with continuos variables

In this chapter is given a brief overview of quantum mechanics postulates, of the notation and of the essential concept used in this thesis. The target of that is to explain to the reader the essential concept, in order to give him the possibility to understand the obtained result.

2.1 Preliminaries on quantum mechanics

For understand the important results about the communication with continuos states, it is essential to give a brief introduction about the main aspects of quantum mechanics theory. This theory is based on a solid mathematic framework presented in this section. It is impossible to discuss about quantum mechanics without its mathematical formalism.

2.1.1 Postulates

Like every phisics theory, quantum mechanics is builded from few essential postulates. In this section are briefly introduced the six Dirac-Von Newman postulates of Quantum Mechanics [**quantumMec_Dirac**, **quantumMec_Neumann**].

Postulate 1 (State Representation) *The state of an isolated quantum system is represented by a complex unitary vector $|\psi\rangle$ in an Hilbert space $\mathcal{H}$:*

$$|\psi\rangle \in \mathcal{H}$$

The space of possible states of the system is called state space and it is a separable complex Hilbert space.

Observation *Differently from the classical physics, in quantum mechanics the concept of state of system is introduced. In classical mechanics a system is described by his observables, like position or four-wheeled.*

Postulate 2 (Observables) *Every observables of the system is represented by an Hermitian operator $\mathcal{M}$ acting on the state space:*

$$\mathcal{M} : \mathcal{H} \to \mathcal{H}$$

The outcomes of the measurement can only be one of the eigenvalue of the operator $\mathcal{M}$.

Observation *The possible outcomes of the measurement are real number because $\mathcal{M}$ is self-andjoint.*

Postulate 3 (Born's Rule) *The probability to get the measurement λ_i from the observable $\mathcal{M}$ in the system in state $|\psi\rangle$ is:*

$$\mathbb{P}(\lambda_i) = \langle\psi| \mathcal{P}_i |\psi\rangle$$

where $\langle\psi|$ is the correspondent vector of $|\psi\rangle$ in the dual space of $\mathcal{H}$ and where $\mathcal{P}_i$ is the projection operator of λ_i in the correspondent space.

Postulate 4 (Wavefunction Collapse) *The state $|\psi'\rangle$ after measurement of λ_i is $\mathcal{P}_i |\psi\rangle$ (with the necessary normalization):*

$$|\psi'\rangle = \frac{\mathcal{P}_i |\psi\rangle}{\langle\psi| \mathcal{P}_i |\psi\rangle}.$$

Postulate 5 (Time Evolution) *The time evolution $|\psi(t)\rangle$ of an isolated quantum system is given by an unitary operator $\mathcal{U}$:*

$$|\psi(t)\rangle = \mathcal{U}(t_0, t) |\psi(t_0)\rangle .$$

Observation (Time dependent Shrodinger Equation) *From postulate 5, it is possible to obtain the time dependent Shrodinger Equation:*

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

where $H(t)$ is the Hemiltonian matrix, $\hbar$ is the reduxed Planck's constant and $i = \sqrt{-1}$ is the immaginary unit.

Postulate 6 (Composite System) *The state space $\mathcal{H}$ of a system composed of $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by*

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

2.1.2 The density operator

The last postulate 6 has very important consequences for composite system. It is possible to describe two tipes of combined systems:

Definition 1 (Product states) *A state $|\psi\rangle \in \mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a pure state if exists $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ such that:*

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle .$$

A product state represents two systems which do not interact; an operation on one of them does not perturb the other.

Definition 2 (Entengled states) *A system that is not in a product state (1), is in an entengled state.*

When a system is in an entengled state it is not possible to characterize the two subsystems with the states vector, although the state vector of the composite system is known.

Density operator
For a more general treatment, the following representation of states is given:

Definition 3 *The state of quantum system is described by a linear operator Ξ, called density operator such that:*

$$\Xi : \mathcal{H} \to \mathcal{H};\ \Xi^\dagger = \Xi;\ tr\{\Xi\} = 1.$$

According to the definition 3, the postulates 3, 4, 5 can be reformulate as following.

$$\mathbb{P}(\lambda_i) = tr\{\Xi \mathcal{P}_i\} \tag{2.1}$$

$$\Xi' = \frac{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger}{tr\{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger\}} \tag{2.2}$$

$$\Xi(t) = \mathcal{U}\Xi(t_0)\mathcal{U}^\dagger \tag{2.3}$$

2.2 Continuos-Variables Quantum Systems

A quantum system is called a continuous-variable system when it has an infinite-dimensional Hilbert space described by observables with continuous eigenspectra [**ContinuousVar**]. Continuous-variables systems play a very important role in communications. this section presents the key aspects for the representation of this systems.

2.2.1 Hilbert space

2.2.2 Phase space

As seen before in 2.1.2, a quantum system can be completely described by a density operator Ξ defined in an infinite-dimensional Hilbert space $\mathcal{H}$, and this operator can be expressed by the Fock representation. Sometimes, however, it is convenient to give another representation of state Ξ by means of a complex function introduced by Wigner [**Wigner**]: the quasi-probability distribution. In this thesis, this representation will be introduced and it will be used to classify the possible states.

Definition 4 (Quantum characteristic function) *The s-order characteristic function $\chi(\xi, s)$, with $\xi, s \in \mathbb{C}$, associated to the quantum state Ξ is defined as:*

$$\chi(\xi, s) = \exp\left\{\frac{s}{2}|\xi|^2\right\} tr\{\Xi \boldsymbol{D}_\xi\} \tag{2.4}$$

where $\boldsymbol{D}_\xi$ is the displacement operator of parameter ξ, defined as:

$$\boldsymbol{D}_\xi = \exp\left\{\xi \boldsymbol{A}^\dagger - \xi^* \boldsymbol{A}\right\}. \tag{2.5}$$

Definition 5 (Quasi-probability distribution) *The s-order quasi-probability distribution $W(\alpha, s)$, with $s \in \mathbb{C}$, associated to the quantum state Ξ is given by:*

$$W(\alpha, s) = \frac{1}{\pi^2} \int_{\mathbb{R}^2} \chi(\xi, s) e^{\alpha \xi^* - \alpha^* \xi} d\xi^2.$$

The quasi-probability distribution, for $s = 0$ ($W(\alpha) = W(\alpha, 0)$) is called Wigner W-function.

2.3 Gaussian States

Gaussian quantum states are an important class of quantum states of continuous-variables systems. They are defined as ([**tesiGuerrini**] quoting [**Gaussian1**, **Gaussian2**, **Gaussian3**, **Gaussian4**, **Gaussian5**]):

Definition 6 (Gaussian state) *A quantum state $\Xi_{\boldsymbol{G}}$ is a Gaussian state if its Wigner W-function $W_G(\alpha)$ is Gaussian, i.e*

$$W_G(\alpha) = \frac{1}{\pi\sqrt{det\check{\boldsymbol{C}}_0}} \exp\left\{-\frac{1}{2}(\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\mu}})^H \check{\boldsymbol{C}}_0{}^{-1} (\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\mu}})\right\}. \tag{2.6}$$

where $\check{\boldsymbol{\mu}}$ is the augmented displacement vector, and $\check{\boldsymbol{C}}_0$ is the augmented covariance matrix.

We remark that if $\boldsymbol{\mu} \in \mathbb{R}^2$ is the displacement vector and $\boldsymbol{C}_0$ is the covariance matrix, the augmented displacement vector and covariance matrix are given by the following transformation:

$$\begin{aligned} \check{\boldsymbol{\mu}} &= \frac{1}{\sqrt{2}} \boldsymbol{J} \boldsymbol{\mu} \\ \check{\boldsymbol{C}}_0 &= \frac{1}{2} \boldsymbol{J} \boldsymbol{C}_0 \boldsymbol{J}^H \end{aligned} \tag{2.7}$$

where

$$\boldsymbol{J} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

Two important types of Gaussian states will be analyzed now: the coherent state and the squeezed state. For each one of this states is presented the noisy version too.

2.3.1 Coherent state

A coherent state is the state of a quantum armonic oscillator of amplitude μ. It is defined ([**tesiGuerrini**] seen [**CohSt_Glauber**, **CohSt_Glauber2**]) as the eigenvector $|\mu\rangle$ of $\boldsymbol{A}$ associated to the eigenvalue μ; i.e

$$\boldsymbol{A}\,|\mu\rangle = \mu\,|\mu\rangle\,. \tag{2.8}$$

It is possible to obtain a coherent state of parameter μ, appliying the displacement operator to the ground state:

$$|\mu\rangle = \boldsymbol{D}_\mu\,|0\rangle\,. \tag{2.9}$$

As mentioned before, it is possible to characterize a state with the Fock representation and, equivalently, with the Wigner W-function. The last one is given, for a coherent state, by [**QuantumNoise**]:

$$W(\alpha) = \frac{2}{\pi}\exp\left\{-2|\alpha-\mu|^2\right\}. \tag{2.10}$$

It is easy to proof that $W(\alpha)$ is gaussian, with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \frac{1}{2}\boldsymbol{I}.$$

The Fock representation is given by [**Dowling**]:

$$|\mu\rangle = e^{-\frac{|\mu|}{2}^2}\sum_{n=0}^{\infty}\frac{\mu^n}{\sqrt{n}}\,|n\rangle\,. \tag{2.11}$$

Noisy coherent states

It is possible to characterize the state of a noisy armonic oscillator introducing the thermal state, i.e the state of an electromagnetic cavity in thermal equilibrium. The Fock representation of the thermal state Ξ_{th} is given by [**tesiGuerrini**]

$$\Xi_{th} = (1-v)\sum_{n=0}^{\infty} v^n\,|n\rangle\,\langle n| \tag{2.12}$$

where

$$v = \frac{\bar{n}}{\bar{n}+1}$$

and $\bar{n}$ is the well-known Plank distribution

$$\bar{n} = \left(\exp\left\{\frac{\hbar\omega}{k_B T}\right\} - 1\right)^{-1}.$$

A noisy coherent states $\Xi_{th}(\mu)$ of parameter μ can be obtained by appling the displacement operator $\boldsymbol{D}_\mu$ to the thermal state Ξ_{th}, as follow:

$$\Xi_{th}(\mu) = \boldsymbol{D}_\mu^\dagger \Xi_{th} \boldsymbol{D}_\mu. \tag{2.13}$$

The Wigner W-function is given by [**QuantumNoise**]

$$W_{th}(\alpha) = \frac{1}{\pi(\bar{n}+\frac{1}{2})} \exp\left\{ -\frac{|\alpha-\mu|^2}{\bar{n}+\frac{1}{2}} \right\} \tag{2.14}$$

and it can be proved that it is a Gaussian function with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \left(\bar{n} + \frac{1}{2}\right) \boldsymbol{I}.$$

The Fock representation is given by

$$\langle n| \, \Xi_{th}(\mu) \, |m\rangle = (1-v) e^{-(1-v)|\mu|^2} \sqrt{\frac{n!}{m!}} v^n [(1-v)\mu^*]^{m-n} L_n^{m-n} \left(\frac{-(1-v)^2|\mu|^2}{v} \right) \tag{2.15}$$

2.3.2 Squeezed state

A squeezed state with amplitude μ and squeezing parameter ζ, is a defined as [**tesiGuerrini**, **YuenRadField**, **QMnoiseInterf**]

$$|\mu, \zeta\rangle = \boldsymbol{D}_\mu \boldsymbol{S}_\zeta \, |0\rangle \tag{2.16}$$

where $\boldsymbol{S}_\zeta$ is the squeezing operator, defined as

$$\boldsymbol{S}_\zeta = \exp\left\{ \frac{1}{2} \left(\zeta \left(\boldsymbol{A}^\dagger\right)^2 + \zeta^* \boldsymbol{A}^2 \right) \right\}. \tag{2.17}$$

It can be proven that a squeezed state is a Gaussian state with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \frac{1}{2} \begin{bmatrix} \cosh(2r) & \sinh(2r)e^{-i\phi} \\ \sinh(2r)e^{-i\phi} & \cosh(2r) \end{bmatrix}$$

with $\zeta = re^{i\phi}$. The Wigner W-function of a squeezed state, differently from the one of a coherent state, has not a circular symmetry.

Noisy squeezed states

The representation of a noisy squeezed state $\Xi_{th}(\mu, \zeta)$ is obtained, similarly to a noisy coherent state, as:

$$\Xi_{th}(\mu, \zeta) = \boldsymbol{D}_\mu \boldsymbol{S}_\zeta \Xi_{th} \boldsymbol{S}_\zeta^\dagger \boldsymbol{D}_\mu^\dagger. \tag{2.18}$$

The Gaussian Wigner W-function is obtained with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \left(\bar{n} + \frac{1}{2}\right) \begin{bmatrix} \cosh(2r) & \sinh(2r)e^{-i\phi} \\ \sinh(2r)e^{-i\phi} & \cosh(2r) \end{bmatrix}.$$

The Fock representation is given by [**MarMar_1993**]

$$\langle n| \Xi_{th}(\mu,\zeta) |m\rangle = \frac{\pi Q(0)}{(n!m!)^{1/2}} \sum_{k=0}^{min(n,m)} k! \binom{n}{k}\binom{m}{k} \tilde{A}^k \left(\frac{1}{2}\tilde{B}\right)^{(n-k)/2} \left(\frac{1}{2}\tilde{B}^*\right)^{(m-k)/2} H_{n-k}((2\tilde{B})^{-1/2}\tilde{C}) H_{m-k}((2\tilde{B}^*)^{-1/2}\tilde{C}^*) \tag{2.19}$$

where H_n is the Hermite polynomial with parameter n,

$$Q(0) = \frac{1}{\pi}[(1+A)^2 - |B|^2]^{-1/2} \exp\left\{ -\frac{(1+A)|C|^2 + \frac{1}{2}[B(C^*)^2 + B^*C^2]}{(1+A)^2 - |B|^2} \right\},$$

$$\tilde{A} = \frac{A(1+A) - |B|^2}{(1+A)^2 - |B|^2},$$

$$\tilde{B} = \frac{B}{(1+A)^2 - |B|^2},$$

$$\tilde{C} = \frac{(1+A)C + BC^*}{(1+A)^2 - |B|^2};$$

and

$$A = \bar{n} + (2\bar{n}+1)(\sinh(r))^2, \; B = -(2\bar{n}+1)e^{i\phi}\sinh(r)\cosh(r), \; C = \mu.$$

2.4 Non-Gaussian States

A state that does not fulfill the definition 2.3 is a non-Gaussian state. An important and useful for communications class of non-Gaussian states, are the photon added states, examined in this thesis. Lastly will be mentioned another type of non-Gaussian state: the photon subtracted state.

2.4.1 Photon added states

The photon added state $\Xi^{(1)}$, obtained from the quantum state Ξ, is given by:

$$\Xi^{(1)} = \frac{\boldsymbol{A}^\dagger \Xi \boldsymbol{A}}{tr\{\boldsymbol{A}^\dagger \Xi \boldsymbol{A}\}}. \tag{2.20}$$

The name *photon addition* could lead to believe that the mean photon number of the photon added state is icreased by one compared to the previous non photon added state. However, that is incorrect. In general, its mean number of photons could be the same, more or less than the starting state. Only if $\Xi = |n\rangle\langle n|$, i.e Ξ is the density operator corresponding to the Fock state $|n\rangle$, the result of the photon addition is a state with one more photon.

Logically, the photon added state $\Xi^{(k)}$ (with k photon addition) is given by

$$\Xi^{(k)} = \frac{(\boldsymbol{A}^\dagger)^k \Xi \boldsymbol{A}^k}{tr\{(\boldsymbol{A}^\dagger)^k \Xi \boldsymbol{A}^k\}}. \tag{2.21}$$

(a) Noisy coherent state

(b) noisy PACS

Figure 2.1: Comparison between Noisy coherent state and noisy PACS Wigner W-function
$\bar{n} = 10^{-2}$ and $k = 2$.

The Fock representation of a photon added state, can be obteined as:

$$\Xi^{(k)} = \frac{\tilde{\Xi}^{(k)}}{tr\{\tilde{\Xi}^{(k)}\}} \tag{2.22}$$

and

$$\langle n|\,\tilde{\Xi}^{(k)}\,|m\rangle = \begin{cases} \sqrt{\frac{n!m!}{(n-k)!(m-k)!}}\,\langle n-k|\,\Xi\,|m-k\rangle \;\; if \;\; n,m \geq k \\ 0 \; otherwise. \end{cases}$$

The Wigner W-function of a photon added state is not Gaussian (2.3).

Noisy photon added coherent states
If Ξ is a noisy coherent state of amplitude μ ($\Xi = \Xi_{th}(\mu)$), the photon added state $\Xi_{th}^{(k)}(\mu)$ is called noisy photon added coherent state (noisy PACS). The Fock representation can be given in closed form by **[PACSDisc]**

$$\langle n|\,\Xi_{th}^{(k)}(\mu)\,|m\rangle = \begin{cases} c_{n,m}^{(k)} \;\; for \; both \; n,m \geq k \\ 0 \;\; otherwise \end{cases} \tag{2.23}$$

where

$$c_{n,m}^{(k)} = \frac{(1-v)^{k+1}e^{-(1-v)|\mu|^2}}{v^k}\sqrt{n!}m!\binom{m}{k}v^n\left[(1-v)\,\mu^*\right]^{m-n}\frac{L_{n-k}^{m-n}\left(\frac{-(1-v)|\mu|^2}{v}\right)}{L_k\left(-|\mu|^2(1-v)\right)}.$$

The Wigner W-function is given by:

$$W(\alpha) = \frac{(-1)^k}{(2\bar{n}+1)^k}\frac{L_k\left(\frac{|2\alpha(\bar{n}+1)-\mu|^2}{(2\bar{n}+1)(\bar{n}+1)}\right)}{L_k}\left(-\frac{|\mu|^2}{\bar{n}+1}\right)W_{th}(\alpha) \tag{2.24}$$

where $W_{th}(\alpha)$ is the Wigner W-function of a noisy coherent state 2.14. In figure 2.1 are plotted the Wigner W-function of a noisy coherent state and a noisy PACS, for $\bar{n} = 10^{-2}$ and $k = 2$. It is evident that the Wigner W-function of the photon added state is not Gaussian.

Noisy photon added squeezed states
if Ξ is a noisy squeezed state with amplitude μ and squeezing factor ζ ($\Xi = \Xi_{th}(\mu,\zeta)$), the photon added state $\Xi_{th}^{(k)}(\mu,\zeta)$ is called noisy photon added squeezed state (PASS).

2.4.2 Photon subtracted states

Chapter 3

Quantum Communication Systems with non-Gaussian States

Thanks to quantum mechanics, it will be possible to overcome the limits of classical communication systems. In the last decades the research in this field has led to very intresting results that could significantly improve the performance of communication systems. This chapter gives a brief overview of quantum communication tools: in the first section we present the equivalents of classical modulation for quantum communication systems; in the second section we report the concept of quantum states discriminator (QSD), included when it can be considered optimal.

3.1 Quantum Modulation

As in a classical system, it is possible to define the concept of modulation for a quantum communication system. The transmitted information will be associated to a quantum state of the electromagnetic field, so it can be transmitted on the communication channel.

It is possible to think about the quantum transmitter as in figure 3.1. The bit source emits a bit sequence $a[n]$, the serial-parallel converter parallelizes a group of l-bit (where if L is the number of quantum states, $l = \log_2(L)$) and sends them to the quantum modulator. This latter associates one quantum state to every group of bit. The operation of quantum state creation, in real cases, is affected by noise.

The sequence of operations is very close to a classical transmitter: the main difference is that the modulator maps the bits into quantum states instead of classical modulation. Therefore, it is possible to achieve the quantum equivalent of classical modulation, with several states. After that, the impact on performance can be tested. This thesis only considers and assesses the binary cases, in the OOK and BPSK configuration.

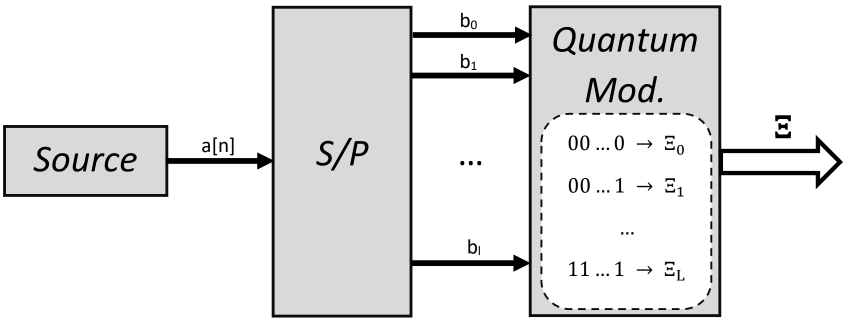


Figure 3.1: Block diagram of a quantum transmitter.

3.1.1 OOK modulation

The OOK (on-off keying) is the most simple possible configuration for a communication system. The quantum implementation of that is realized associating the low-energy state to the ground state $|0\rangle$ and the high-energy state to another state. It is important to consider that the physical realization of these states are not free-noise; this issue will be considered using noisy states 3.1.

$$\Xi_0 = \Xi_{th} \tag{3.1}$$
$$\Xi_1 = \Xi_{th}(\mu)$$

In the equation 3.1, the high-energy state is associated to a coherent state. This configuration has been widely analyzed in [**helstrom1**, **helstrom2**, **coherentComm1**, **coherentComm2**, **coherentComm3**, **coherentComm4**] but this is not the only possible way. The use of PACS states $\Xi_{th}^{(k)}(\mu)$ is analyzed in [**PACSDisc**, **tesiGuerrini**]; the use of PASS are briefly assessed in the following chapter of this thesis.

3.1.2 BPSK modulation

BPSK quantum systems are implemented using two states with opposite amplitude, like

$$\Xi_0 = \Xi_{th}(-\mu) \tag{3.2}$$
$$\Xi_1 = \Xi_{th}(\mu).$$

There is no guarantee that the use of a BPSK solution in a quantum system will improve its performance. The effect depends on which are the used states. In the next chapter some configuration are assessed.

3.2 Quantum Discriminator

The problem of quantum state discrimination (QSD) is one of the most important aspects of quantum communication. As in classical communication,

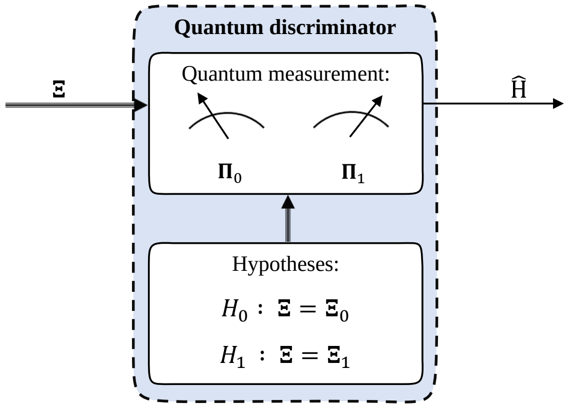


Figure 3.2: Binary quantum state discriminator.

the ability to distinguish between two or more states in presence of noise can be decisive in order to determine the performance of the communication system. However, differently from the classical situation, the discrimination can be done using a custom-designed quantum discriminator, overcoming the classical physics limits.

3.2.1 Binary quantum state discrimination

The problem of discrimination between two quantum states is realized, as every measurement process 2, using an operator or a set of operators. If the state of the system is unknown, as shown in figure 3.2, there are two hypotheses about the state $\boldsymbol{\Xi}$ (the problem is easly generalizable for M different states), given by:

$$\begin{aligned} H_0 &: \boldsymbol{\Xi} = \boldsymbol{\Xi}_0 \\ H_1 &: \boldsymbol{\Xi} = \boldsymbol{\Xi}_1 \end{aligned} \tag{3.3}$$

It is necessary a set of two positive-definites operator (POVM)

$$\mathcal{P} = \{\boldsymbol{\Pi}_0, \boldsymbol{\Pi}_1\} \tag{3.4}$$

for the discrimination process, and the probability that the hypothesis H_j is choosen if H_k is the right choose is given by [**tesiGuerrini**]:

$$\mathbb{P}\{H_j|H_k\} = tr\{\boldsymbol{\Xi}_k\boldsymbol{\Pi}_j\}. \tag{3.5}$$

The distribution error probability (DEP) in the discrimination process, if p_0 and p_1 are respectively the probabilty of symbols 0 and 1, is so given by

$$P_e = 1 - (p_0 tr\{\boldsymbol{\Xi}_0\boldsymbol{\Pi}_0\} + p_1 tr\{\boldsymbol{\Xi}_1\boldsymbol{\Pi}_1\}) \,. \tag{3.6}$$

3.2.2 Optimal discriminator

The issue of finding the optimal POVM that minimizes the DEP was exhaustively discuss by Helstrom in [**helstrom3**, **helstrom4**]. The minimum distribution error probability (MDEP) for a binary communication system is given by the well-known Helstrom bound

$$\breve{P}_e = \frac{1}{2}\left(1 - \left\|p_1\Xi_1 - p_0\Xi_0\right\|_1\right), \tag{3.7}$$

where $p_0,\ p_1$ are the probability that the states $\Xi_0,\ \Xi_1$ are trasmitted and the operator $\|\cdot\|_1$ represents the trace norm. The MDEP 3.7 is obtained with the following POVM:

$$\breve{\mathbf{\Pi}}_0 = \sum_{\substack{i \\ \lambda_i < 0}} \left|\lambda_i\right\rangle\left\langle\lambda_i\right|, \tag{3.8}$$

$$\breve{\mathbf{\Pi}}_1 = 1 - \breve{\mathbf{\Pi}}_0 = \sum_{\substack{i \\ \lambda_i \geq 0}} \left|\lambda_i\right\rangle\left\langle\lambda_i\right|;$$

where $|\lambda_i\rangle$ is the eigenvector of $p_1\Xi_1 - p_0\Xi_0$ associated to the eigenvalue λ_i. For pure states, i.e $\Xi_0 = |\psi_0\rangle\langle\psi_0|$ and $\Xi_1 = |\psi_1\rangle\langle\psi_1|$, the equation 3.7 begin

$$\breve{P}_e = \frac{1}{2}\left(1 - \sqrt{1 - 4p_0p_1|\langle\psi_0|\psi_1\rangle|^2}\right). \tag{3.9}$$

It is possible to observe that, for pure states, the MDEP is equal to 0 if $\langle\psi_0|\psi_1\rangle$, that is $|\psi_0\rangle$ and $|\psi_1\rangle$ are orthogonal states.

Chapter 4

Discrimination of Photon Added States

This chapter characterizes two systems with different types of photon added state in term of their discrimination MDEP 3.7: a photon added coherent state system (from [**PACSDisc**])and a photon added squeezed state. For each system, we consider the OOK configuration and the BPSK. The use of photon added states, as it will be shown, can improve significantly the performance of the communication.

The analyzed situation does not consider the channel effect on the transmitted information: it has been supposed that the noisy states reach the discriminator as they were created. The channel effects for a PACS system are described in [**PACSDisc**].

4.1 PACS Discrimination

The effect of the use of PACS in an OOK communication system was extensively discussed in [**PACSDisc**]. In this section the most important result will be reported and a BPSK system will be tested.

4.1.1 Quantum OOK PACS system

The use of PACS in an OOK system can significantly improve the performance. The MDEP, with thermal noise, is given by the Helstrom bound 3.7, where, for an OOK PACS system,

$$\begin{aligned} \Xi_0 &= \Xi_{th} = \Xi_{th}^{(0)}(0) \\ \Xi_1 &= \Xi_{th}^{(k)}(\mu). \end{aligned} \tag{4.1}$$

It is useful, in order to evaluate the performance of the system, to introduce the mean number of photon n_p in a quantum state Ξ, which is given by:

$$n_p = tr\{\Xi \boldsymbol{A}^{\dagger}\boldsymbol{A}\}. \tag{4.2}$$

For a photon added coherent state, the equation 4.2 becomes [**PACSDisc**]:

$$n_p(\mu, \bar{n}) = \frac{N_{k+1}(\mu, \bar{n})}{N_k(\mu, \bar{n})} - 1, \tag{4.3}$$

where

$$N_k(\mu, \bar{n}) = tr\{(\boldsymbol{A}^\dagger)^k \Xi_{th}(\mu) \boldsymbol{A}^k\}. \quad (4.4)$$

It is possible to observe that the minimum of n_p is given by:

$$n_p(0, \bar{n}) = (k+1)(\bar{n}+1) - 1. \quad (4.5)$$

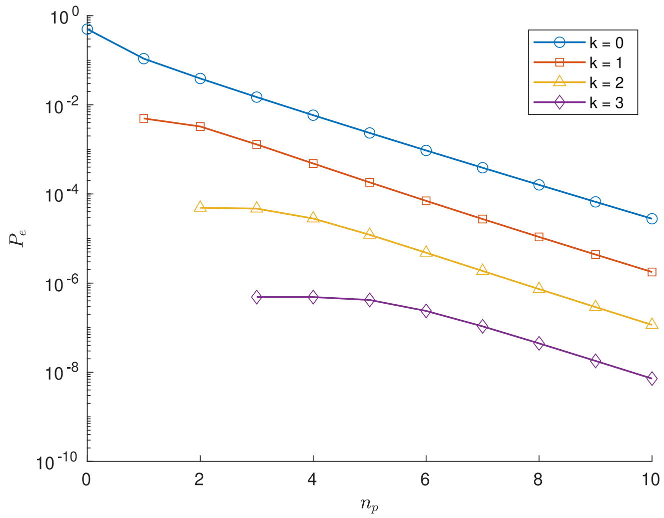


Figure 4.1: MDEP for PACS QOOK with: $k = 0, 1, 2, 3$; $\bar{n} = 10^{-2}$; $p_0 = p_1 = 1/2$

The MDEP of a quantum OOK system with PACS in function of n_p is represented in the figure 4.1, where in the x-axis there are the mean number of photon n_p in the state Ξ_1 (which corresponds to the mean number of photon in the whole system) and in the y-axes there are the MDEP (P_e). The simulation was obtained for equiprobable symbols and mean number of thermal photons $\bar{n} = 10^{-2}$. The argument of the trace norm $\|\cdot\|_1$ in the Helstrom bound 3.7, is an operator in an infinite dimensional Hilbert space; for the simulation, it has been approximated in $N = 30$ dimension. It is possible to observe that the photon addition improves significantly the performance in terms of error probability.

4.1.2 Quantum BPSK PACS system

It can be interesting to assess the effect of photon addition in a quantum BPSK system. The constellation is given, for a PACS BPSK, by:

$$\Xi_0 = \Xi_{th} = \Xi_{th}^{(k)}(-\mu) \quad (4.6)$$
$$\Xi_1 = \Xi_{th}^{(k)}(\mu).$$

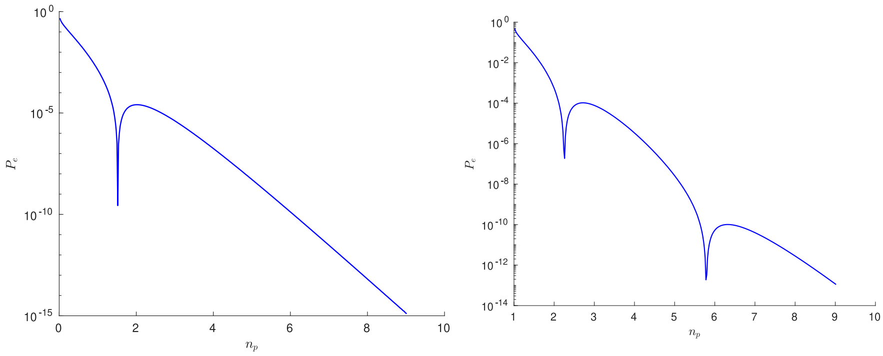
In absence of noise ($\bar{n} = 0$), the MDEP is given by formula 3.9 where

$$|\psi_0\rangle = \left|-\mu^{(k)}\right\rangle, \quad |\psi_1\rangle = \left|\mu^{(k)}\right\rangle. \quad (4.7)$$

The inner product is given, in closed form, by [**PACSDisc**]:

$$\left\langle -\mu^{(k)} \middle| \mu^{(k)} \right\rangle = \frac{L_k(|\mu|^2)}{L_k(-|\mu|^2)} e^{-2|\mu|^2}, \tag{4.8}$$

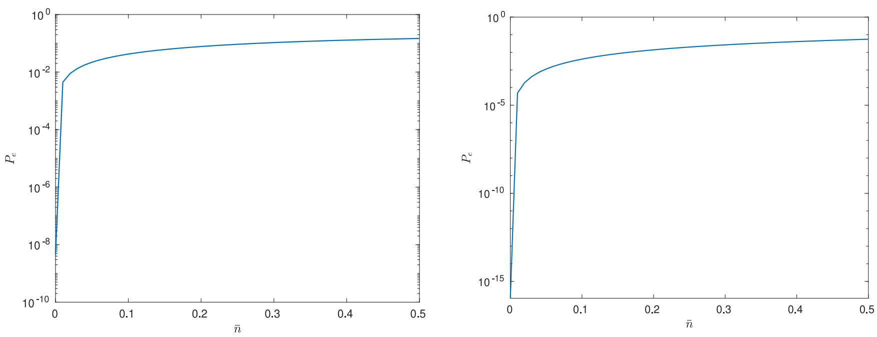
where $L_k(x)$ is the Laguerre polynomial of parameter k, evaluate in x. In figure



(a) $k = 1$ (b) $k = 2$

Figure 4.2: MDEP of quantum BPSK in absence of noise, $N = 30$.

4.2 the MDEP in absence of noise, for QBPSK with PACS, is plotted for $k = 1$ and for $k = 2$, in function of n_p, with $N = 30$. It can be noticed that exist k zeros in the MDEP plot, where k is the number of photon additions. The existence of these zeros is not really useful for the design of a quantum communication system because their selectivity factors are too high for a phisical implementation. It is, nevertheless, possible to use that in order to evaluate the effect of the thermal noise. In figure 4.3, the sluggish performance due to the thermal noise is clear.



(a) $k = 1$ & $\mu = 0.54$ (b) $k = 2$ & $\mu = 1.58$

Figure 4.3: Thermal noise effect, in corrispondence of MDEP zeros ($N = 40$).

The plot shows the trend of MDEP, for zeros value of μ, in function of $\bar{n}$; the used approximation is $N = 30$. The MDEP in presence of noise is given using the expression 3.7.

The comparison between a quantum OOK system and a quantum BPSK system is given in figure 4.4, in function of the mean energy of the system

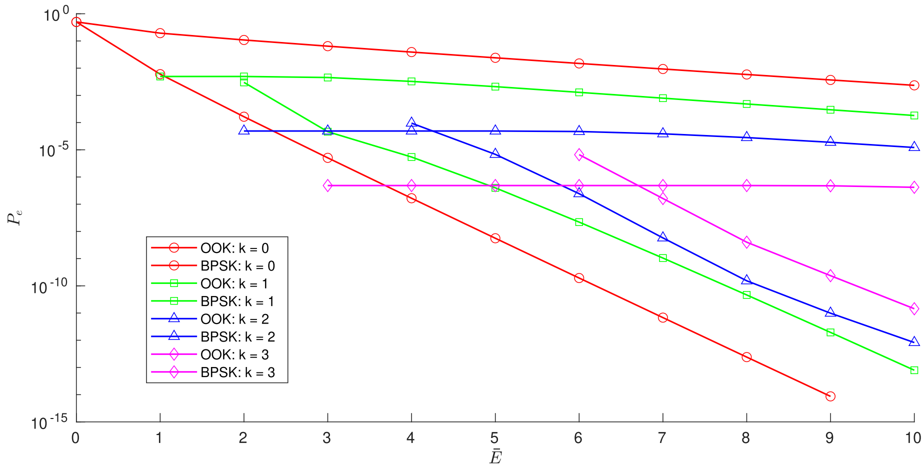


Figure 4.4: BPSK and OOK comparison.
$N = 45$, $\bar{n} = 10^{-2}$, $p_0 = p_1 = 1/2$

$\bar{E}$, that is equal to n_p for the OOK system and the sum of the n_p for each state for the BPSK system. The plots are given with $\bar{n} = 10^{-2}$, $N = 45$ and equiprobable symbols. The obtained result is really intresting: the quantum BPSK has a sluggish performance due two the photon addition process. At an equal level of energy $\bar{E}$, for PACS systems, it is possible to find an OOK configuration that maximizes the performance.

4.2 PASS Discrimination

The use of squeezed states instead of coherent states allows us to overcome the limits of precedent systems. The representation of a noisy squeezed state is given in 2.3.2. This section initially discusses the advantages of using squeezed states without photon addition; then it evaluates the effect of the photon addition in quantum OOK and BPSK systems.

4.2.1 Squeezed states discrimination

At first, we assess the effect of squeezing on the performance in absence of photon addition and thermal noise. As for PACS systems, it can be useful to define the mean number of photon n_p in a squeezed state, which is given by

$$n_p(\mu, r) = |\mu|^2 + (\sinh r)^2 ; \tag{4.9}$$

where μ is the amplitude of the starter coherent state and the squeezing factor is $\zeta = re^{i\theta}$. The minimum value of n_p is given by $n_p(0, r) = \sinh r^2$.

For a quantum squeezed states BPSK system, the MDEP, obtained with the Helstrom bound 3.7, is plotted in figure 4.5 in function of $\bar{E}$, i.e the mean energy of the system (given by the sum of n_p for each state), with: $\theta = \pi$, $N = 30$, equiprobable symbols and $\bar{n} = 0$.

It can be noticed that the optimal configuration of r depends on the energy in the system. For low energy levels the squeezing has not a positive effect.

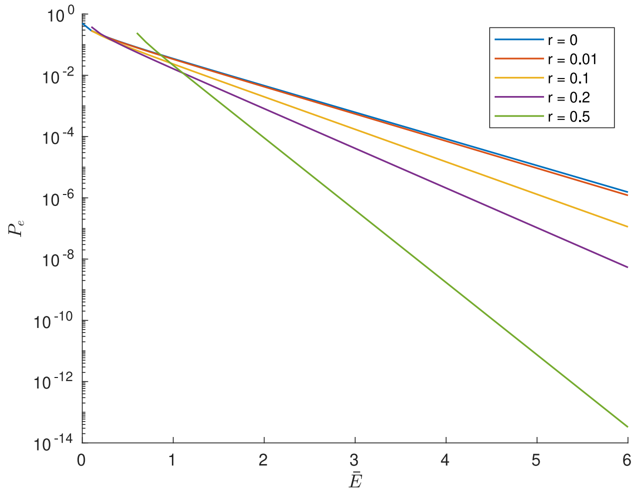


Figure 4.5: MDEP of squeezed state BPSK system. $N = 30$, $\bar{n} = 0$, $\theta = \pi$, $p_0 = p_1 = 1/2$

4.2.2 PASS discrimination

We are analyzing now the performance of a noisy photon added squeezed state system (PASS) 2.4.1, in OOK and BPSK configuration. The MDEP is found again with the Helstrom bound 3.7. As for the other systems, it is useful to define the mean number of photons n_p for noisy photon added squeezed states, that is given by:

$$n_p(\mu, \zeta, \bar{n}) = \frac{N_{k+1}(\mu, \zeta, \bar{n})}{N_k(\mu, \zeta, \bar{n})} - 1, \tag{4.10}$$

where

$$\begin{aligned} N_k(\mu, \zeta, \bar{n}) &= tr\{(\boldsymbol{A}^\dagger)^k \boldsymbol{\Xi}_{th}(\mu, \zeta) \boldsymbol{A}^k\} \\ &= tr\{(\boldsymbol{A}^\dagger)^k \boldsymbol{D}_\mu \boldsymbol{S}_\zeta \boldsymbol{\Xi}_{th} \boldsymbol{S}_\zeta^\dagger \boldsymbol{D}_\mu^\dagger \boldsymbol{A}^k\}. \end{aligned} \tag{4.11}$$

OOK PASS system

The constellation of a quantum OOK system with noisy PASS, is given by:

$$\begin{aligned} \boldsymbol{\Xi}_0 &= \boldsymbol{\Xi}_{th}^{(0)}(0, 0) \\ \boldsymbol{\Xi}_1 &= \boldsymbol{\Xi}_{th}^{(k)}(\mu, \zeta). \end{aligned} \tag{4.12}$$

In figure 4.6, the MDEP of a quantum OOK noisy PASS system is plotted in function of the mean number of photon n_p in the PASS (that is equivalent to the mean energy in the system). For the simulation are used $N = 30$, $\bar{n} = 10^{-2}$, $\theta = \pi$ and equiprobable symbols. It can be noticed that the photon addition significantly improves the performance of the system, at least for the plotted energy level.

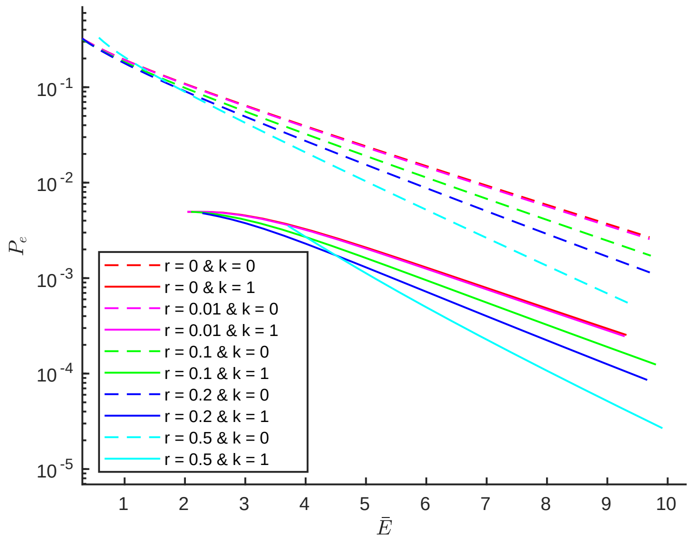


Figure 4.6: MDEP of noisy PASS quantum OOK system.
$N = 30$, $\bar{n} = 10^{-2}$, $\theta = \pi$ and $p_0 = p_1 = 1/2$.

BPSK PASS system

Similary to the PACS BPSK, the constellation of PASS BPSK is given by:

$$\begin{aligned} \mathbf{\Xi}_0 &= \mathbf{\Xi}_{th}^{(k)}(-\mu, \zeta) \\ \mathbf{\Xi}_1 &= \mathbf{\Xi}_{th}^{(k)}(\mu, \zeta). \end{aligned} \tag{4.13}$$

The figure 4.7 shows the effects of the photon addition in a quantum BPSK system, in terms of the mean energy in the system $\bar{E}$, given by the sum of the mean photon number n_p of $\mathbf{\Xi}_0$ and $\mathbf{\Xi}_1$. The parameters used for the simulation are $N = 30$, $\bar{n} = 10^{-2}$, $\theta = \pi$ and equiprobable symbols. As in PACS case, it is evident that the photon addition, in a BPSK system, has not the positive effect that has in an OOK system.

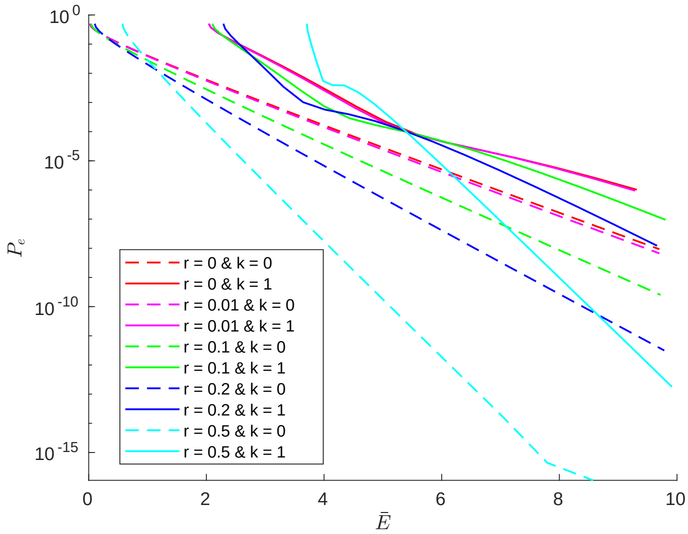


Figure 4.7: MDEP of noisy PASS quantum BPSK system.
$N = 30$, $\bar{n} = 10^{-2}$, $\theta = \pi$ and $p_0 = p_1 = 1/2$.

Chapter 5

Conclusion

Questa tesi si pone come obiettivo di analizzare le prestazioni nella QSD (Quantum states discrimination) per sistemi binari, in termini di probabilità di errore nel riconoscimento dei simboli, con l'utilizzo di un discriminatore ottimo. Sono stati presentati inizialmente i fondamenti della teoria meccanica quantistica nella formulazione di Dirac-Neumann e nella sua generalizzazione tramite l'utilizzo di Density operator. Sono stati quindi descritti i concetti di modulazione quantum e di discriminazione di stati quantistici ed infine sono state analizzate le prestazioni di alcuni sistemi in termini di minimum distribution error probability (MDEP). Tutte le valutazioni sono state fatte supponendo che la comunicazione non risenta di effetti associati al canale di comunicazione, dunque che lo stato emesso dal trasmettitore giunga al discriminatore del ricevitore senza modifiche.

Questa tesi pone in evidenza come l'utilizzo di stati non Gaussiani (non-Gaussian states) photon added in sistemi OOK può apportare un miglioramento nella QSD rispetto all'utilizzo di stati Gaussiani. La combinazione in particolare di squeezing e photon addition (PASSs) risulta estremamante efficace. L'effetto della photon addition si manifesta invece negativo in sistemi quantistici BPSK.

I risultati ottenuti possono essere di notevole importanza nel design di un sistema di comunicazione che sfrutti stati quantistici, permettendo di sfruttare al meglio le potenzialità di questa teoria fisica.