Università degli Studi di Ferrara

**Università degli Studi di Ferrara**

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# On the Design of Quantum Communication Systems with non-Gaussian States


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# Chapter 1

# Quantum Mechanics Abstract

In this chapter, a bief overview of quantum mechanics postulates, of the notation and of the essential concept used in this thesis is given. The target of that is to explain to the reader the essential concept, in order to give him the possibility to understand the obtained result.

## 1.1 Postulates

Like every phisics theory, quantum mechanics is builded from few essential postulates. In this section are briefly introduced the six Dirac-Von Newman postulates of Quantum Mechanics [1][2].

### 1.1.1 First postulate

**Postulate 1 (State Representation)** *The state of an isolated quantum system is represented by a complex unitary vector in an Hilbert space:*

$$|\psi\rangle \in \mathcal{H}$$

*The space of possible states of the system is called state space and it is a separable complex Hilbert space.*

**Observation** *Differently from the classical physics, in quantum mechanics the concept of state of system is introduced. In classical mechanics a system is described by his observables, like position or four-wheeled.*

### 1.1.2 Second postulate

**Postulate 2 (Observables)** *Every observables of the system is represented by an Hermitian operator acting on the state space:*

$$\mathcal{M} : \mathcal{H} \to \mathcal{H}$$

*The outcomes of the measurement can only be one of the eigenvalue of the operator $\mathcal{M}$.*

**Observation** *The possible outcomes of the measurement are real number because $\mathcal{M}$ is self-andjoint.*

### 1.1.3 Third postulate

**Postulate 3 (Born's Rule)** *The probability to get the measurement $\lambda_i$ from the observable $\mathcal{M}$ in the system in state $|\psi\rangle$ is:*

$$\mathbb{P}(\lambda_i) = \langle\psi| \mathcal{P}_i |\psi\rangle$$

*where $\langle\psi|$ is the correspondent vector of $|\psi\rangle$ in the dual space of $\mathcal{H}$ and where $\mathcal{P}_i$ is the projection operator of $\lambda_i$ in the correspondent space.*

### 1.1.4 Fourth postulate

**Postulate 4 (Wavefunction Collapse)** *The state after measurement of $\lambda_i$ is $\mathcal{P}_i |\psi\rangle$ (with the necessary normalization):*

$$|\psi'\rangle = \frac{\mathcal{P}_i |\psi\rangle}{\langle\psi| \mathcal{P}_i |\psi\rangle}.$$

### 1.1.5 Fifth postulate

**Postulate 5 (Time Evolution)** *The time evolution of an isolated quantum system is given by an unitary operator $\mathcal{U}$:*

$$|\psi(t)\rangle = \mathcal{U}(t_0, t) |\psi(t_0)\rangle .$$

**Observation (Time dependent Shrodinger Equation)** *From postulate 5, is possible to found the time dependent Shrodinger Equation:*

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

*where $H(t)$ is the Hemiltonian matrix.*

### 1.1.6 Sixth postulate

**Postulate 6 (Composite System)** *The state space of a system composite from $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by*

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

## 1.2 Combining Systems

The last postulate 6 has very important consequences for composite system. It is possible to descrive two tipe of combined systems:

**Definition 1 (Product states)** *A state $|\psi\rangle \in \mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a pure state if exists $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ such that:*

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle .$$

A product state represent two states which do not interact; an operation on one of them don't perturb the other.

**Definition 2 (Entengled states)** *A system that is not in a product state (1), is an entengled state.*

When a system is in an entengled state is not possible to charaterize the two subsystems with the states vector, as the state vector of the composite system is known.

### 1.2.1 Density operator

For a more general treatment, the following representation of states is given:

**Definition 3** *The state of quantum system is described by a linear operator, called density operator such that:*

$$\Xi : \mathcal{H} \to \mathcal{H};\ \Xi^\dagger = \Xi;\ tr\{\Xi\} = 1.$$

According to the definition 3, the postulates 3, 4, 5 can be reformulate as following.

$$\mathbb{P}(\lambda_i) = tr\{\Xi \mathcal{P}_i\} \tag{1.1}$$

$$\Xi' = \frac{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger}{tr\{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger\}} \tag{1.2}$$

$$\Xi(t) = \mathcal{U}\Xi(t_0)\mathcal{U}^\dagger \tag{1.3}$$

## 1.3 Quantized Electromagnetic Field

Electromagnetic field is the main means of communication for contemporary application. It is important therefor, to give a quantum representation. In this section the representation of quantized electromagnetic field is initially given, so some relevant states of this are briefly charaterized.

### 1.3.1 Classical electromagnetic field

In a volume $\mathcal{V} \in \mathbb{R}^3$ classical magnetic field is determinated from Maxwell's equations as a superposition of the cavity modes

$$\mathbf{e}(\mathbf{r}, t) = \sum_n p_n(t)\mathbf{u}_n(\mathbf{r}) \tag{1.4}$$

where

$$\mathbf{u}_n(\mathbf{r}) = \mathbf{u}_{n0}\ exp^{i\mathbf{k}_n \cdot \mathbf{r}}$$

and $\mathbf{u}_{n0}$ is determinated by the initial condition. Magnetic field is determinated by:

$$\mathbf{h}(\mathbf{r}, t) = \sum_n q_n(t)\nabla \times \mathbf{u}_n(\mathbf{r}) \tag{1.5}$$

and

$$p_n(t) = \frac{\mathrm{d}q_n(t)}{\mathrm{d}t}. \tag{1.6}$$

The Hemiltonian associated to the n-th mode is given by

$$H_n = \frac{1}{2}[p_n^2(t) + \omega_n^2 q_n^2(t)] \tag{1.7}$$

or, equivalently, it is possible to define the complex variable $a_n(t)$ as

$$a_n(t) = \frac{\omega_n q_n(t) + ip_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.8}$$

and, using 1.8 in 1.7

$$H_n = \hbar\omega_n |a_n(t)|^2. \tag{1.9}$$

### 1.3.2 Quantized electromagnetic field

The quantization of electromagnetic field is obtained replacing the two quantities $p_n(t)$ and $q_n(t)$ with the Hermitian operators $\mathbf{P}_n(t),\ \mathbf{Q}_n(t) : \mathcal{H}_n \to \mathcal{H}_n$ and by imposing the following commutation condition:

$$[\mathbf{Q}_n, \mathbf{P}_m] = i\hbar\delta_{n,m}\mathbf{I} \tag{1.10}$$

$$[\mathbf{Q}_n, \mathbf{Q}_m] = 0 \tag{1.11}$$

$$[\mathbf{P}_n, \mathbf{P}_m] = 0. \tag{1.12}$$

Defining the annihilation operator $\mathbf{A}_n$ as

$$\mathbf{A}_n(t) = \frac{\omega_n \mathbf{Q}_n(t) + i\mathbf{P}_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.13}$$

and the adjoint of $\mathbf{A}_n$, the creation operator $\mathbf{A}_n^\dagger$ as

$$\mathbf{A}_n(t) = \frac{\omega_n \mathbf{Q}_n(t) - i\mathbf{P}_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.14}$$

it is possible to describe the Hamiltonian of system as

$$H_n = \hbar\omega_n \mathbf{A}_n^\dagger \mathbf{A}_n. \tag{1.15}$$

It is possible to charaterize some type of quantum states of the quantum electromagnetic field; in the following section the states studied are briefly described.

# Bibliography

[1] P.A.M. Dirac. *The Principles of Quantum Mechanics.* Oxford University Press, 1981.

[2] J. Von Neumann. *Mathematical foundations of quantum mechanics.* Princeton University Press, 1995.