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On the Design of Quantum Communication Systems with non-Gaussian States

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Chapter 1

Quantum Mechanics Abstract

In this chapter, a brief overview of quantum mechanics postulates, of the notation and of the essential concept used in this thesis is given. The target of that is to explain to the reader the essential concept, in order to give him the possibility to understand the obtained result.

1.1 Postulates

Like every phisics theory, quantum mechanics is builded from few essential postulates. In this section are briefly introduced the six Dirac-Von Newman postulates of Quantum Mechanics [1, 5].

1.1.1 First postulate

Postulate 1 (State Representation) *The state of an isolated quantum system is represented by a complex unitary vector in an Hilbert space:*

$$|\psi\rangle \in \mathcal{H}$$

The space of possible states of the system is called state space and it is a separable complex Hilbert space.

Observation *Differently from the classical physics, in quantum mechanics the concept of state of system is introduced. In classical mechanics a system is described by his observables, like position or four-wheeled.*

1.1.2 Second postulate

Postulate 2 (Observables) *Every observables of the system is represented by an Hermitian operator acting on the state space:*

$$\mathcal{M} : \mathcal{H} \to \mathcal{H}$$

The outcomes of the measurement can only be one of the eigenvalue of the operator $\mathcal{M}$.

Observation *The possible outcomes of the measurement are real number because $\mathcal{M}$ is self-andjoint.*

1.1.3 Third postulate

Postulate 3 (Born's Rule) *The probability to get the measurement λ_i from the observable $\mathcal{M}$ in the system in state $|\psi\rangle$ is:*

$$\mathbb{P}(\lambda_i) = \langle\psi| \mathcal{P}_i |\psi\rangle$$

where $\langle\psi|$ is the correspondent vector of $|\psi\rangle$ in the dual space of $\mathcal{H}$ and where $\mathcal{P}_i$ is the projection operator of λ_i in the correspondent space.

1.1.4 Fourth postulate

Postulate 4 (Wavefunction Collapse) *The state after measurement of λ_i is $\mathcal{P}_i |\psi\rangle$ (with the necessary normalization):*

$$|\psi'\rangle = \frac{\mathcal{P}_i |\psi\rangle}{\langle\psi| \mathcal{P}_i |\psi\rangle}.$$

1.1.5 Fifth postulate

Postulate 5 (Time Evolution) *The time evolution of an isolated quantum system is given by an unitary operator $\mathcal{U}$:*

$$|\psi(t)\rangle = \mathcal{U}(t_0, t) |\psi(t_0)\rangle .$$

Observation (Time dependent Shrodinger Equation) *From postulate 5, it is possible to obtain the time dependent Shrodinger Equation:*

$$i\hbar\frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

where $H(t)$ is the Hemiltonian matrix.

1.1.6 Sixth postulate

Postulate 6 (Composite System) *The state space of a system composed of $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by*

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

1.2 Combining Systems

The last postulate 6 has very important consequences for composite system. It is possible to describe two tipes of combined systems:

Definition 1 (Product states) *A state $|\psi\rangle \in \mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a pure state if exists $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ such that:*

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle .$$

A product state represents two states which do not interact; an operation on one of them does not perturb the other.

Definition 2 (Entengled states) *A system that is not in a product state (1), is in an entengled state.*

When a system is in an entengled state it is not possible to characterize the two subsystems with the states vector, although the state vector of the composite system is known.

1.2.1 Density operator

For a more general treatment, the following representation of states is given:

Definition 3 *The state of quantum system is described by a linear operator, called density operator such that:*

$$\Xi : \mathcal{H} \rightarrow \mathcal{H};\ \Xi^\dagger = \Xi;\ tr\{\Xi\} = 1.$$

According to the definition 3, the postulates 3, 4, 5 can be reformulate as following.

$$\mathbb{P}(\lambda_i) = tr\{\Xi \mathcal{P}_i\} \tag{1.1}$$

$$\Xi' = \frac{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger}{tr\{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger\}} \tag{1.2}$$

$$\Xi(t) = \mathcal{U}\Xi(t_0)\mathcal{U}^\dagger \tag{1.3}$$

1.3 Quantized Electromagnetic Field

Electromagnetic field is the main means of communication for contemporary application. It is important therefor, to give a quantum representation. In this section the representation of quantized electromagnetic field is initially given, so the Fock representation of a state is introduced.

1.3.1 Classical electromagnetic field

In a volume $\mathcal{V} \in \mathbb{R}^3$ classical electromagnetic field is determinated from Maxwell's equations as a superposition of the cavity modes ([2] quoting [3, 4]). Electric field is given by the well-known expression:

$$\mathbf{e}(\mathbf{r}, t) = \sum_n p_n(t)\mathbf{u}_n(\mathbf{r}) \tag{1.4}$$

where

$$\mathbf{u}_n(\mathbf{r}) = \mathbf{u}_{n0}\ e^{i\mathbf{k}_n \cdot \mathbf{r}}$$

and $\mathbf{u}_{n0}$ is determinated by the initial condition. The corresponding magnetic field is determinated by:

$$\mathbf{h}(\mathbf{r}, t) = \sum_n q_n(t)\nabla \times \mathbf{u}_n(\mathbf{r}) \tag{1.5}$$

and

$$p_n(t) = \frac{\mathrm{d}q_n(t)}{\mathrm{d}t}. \tag{1.6}$$

The Hemiltonian associated to the n-th mode is given by

$$H_n = \frac{1}{2}[p_n^2(t) + \omega_n^2 q_n^2(t)]. \tag{1.7}$$

Equivalently, it is possible to define the complex variable $a_n(t)$ as

$$a_n(t) = \frac{\omega_n q_n(t) + ip_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.8}$$

and, using 1.8 in 1.7, it is possible to obtain the following expression of the Hemiltonian:

$$H_n = \hbar\omega_n |a_n(t)|^2. \tag{1.9}$$

1.3.2 Quantized electromagnetic field

The quantization of electromagnetic field is obtained replacing the two quantities $p_n(t)$ and $q_n(t)$ with the Hermitian operators $\mathbf{P}_n(t)$, $\mathbf{Q}_n(t) : \mathcal{H}_n \rightarrow \mathcal{H}_n$ and by imposing the following commutation conditions ([2] quoting [3, 4]):

$$[\mathbf{Q}_n, \mathbf{P}_m] = i\hbar\delta_{n,m}\mathbf{I} \tag{1.10}$$

$$[\mathbf{Q}_n, \mathbf{Q}_m] = 0 \tag{1.11}$$

$$[\mathbf{P}_n, \mathbf{P}_m] = 0. \tag{1.12}$$

Defining the annihilation operator $\mathbf{A}_n$ as

$$\mathbf{A}_n(t) = \frac{\omega_n \mathbf{Q}_n(t) + i\mathbf{P}_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.13}$$

and the adjoint of $\mathbf{A}_n$, the creation operator $\mathbf{A}_n^\dagger$ as

$$\mathbf{A}_n(t) = \frac{\omega_n \mathbf{Q}_n(t) - i\mathbf{P}_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.14}$$

it is possible to describe the Hemiltonian of the system as

$$H_n = \hbar\omega_n \mathbf{A}_n^\dagger \mathbf{A}_n. \tag{1.15}$$

1.3.3 Fock states

In a single mode cavity, it is possible to define the number operator $\mathbf{N}$ as

$$\mathbf{N} = \mathbf{A}^{\dagger}\mathbf{A}. \tag{1.16}$$

Single mode Fock states are the eigenvector of N, i.e the solution of equation:

$$\mathbf{N}\left|n\right\rangle = n\left|n\right\rangle. \tag{1.17}$$

The Fock state $|n\rangle$ represent the quantum state with exactly n photons. It is important to evidence that the set of all Fock states forms an orthonormal basis of the Hilbert space $\mathcal{H}$, so every state Ξ can be expressed as

$$\mathbf{\Xi} = \sum_{n,m} c_{n,m}\left|n\right\rangle\left\langle m\right| \tag{1.18}$$

with

$$c_{n,m} = \left\langle n\right|\mathbf{\Xi}\left|m\right\rangle.$$

Using the representation in Fock basis, it is possible to charaterize some type of quantum states of the quantum electromagnetic field. In the following section the states studied are briefly described.

1.4 QEF States

In this section, some quantum states of electromagnetic field useful for quantum communication are characterized. A brief introduction to another one tool for the description of quantum systems is initially given, so some Gaussian and non-Gaussian states are characterized.

1.4.1 Phase-space description

As seen before, in 1.18, quantum system can be completely described by a density operator Ξ defined in an infinite-dimensional Hilbert space $\mathcal{H}$. This operator can be expressed by the Fock representation (3).

Sometimes, however, it is convenient to give another representation of state Ξ by means of a complex function introduced by Wigner [6]: the quasi-probability distribution. In this thesis, this representation will be introduced and it will be used to classify the possible states.

Definition 4 (Quantum characteristic function)

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