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On the Design of Quantum Communication Systems with non-Gaussian States

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Chapter 1

Quantum Mechanics Abstract

In this chapter, a brief overview of quantum mechanics postulates, of the notation and of the essential concept used in this thesis is given. The target of that is to explain to the reader the essential concept, in order to give him the possibility to understand the obtained result.

1.1 Postulates

Like every phisics theory, quantum mechanics is builded from few essential postulates. In this section are briefly introduced the six Dirac-Von Newman postulates of Quantum Mechanics [1, 2].

1.1.1 First postulate

Postulate 1 (State Representation) *The state of an isolated quantum system is represented by a complex unitary vector in an Hilbert space:*

$$|\psi\rangle \in \mathcal{H}$$

The space of possible states of the system is called state space and it is a separable complex Hilbert space.

Observation *Differently from the classical physics, in quantum mechanics the concept of state of system is introduced. In classical mechanics a system is described by his observables, like position or four-wheeled.*

1.1.2 Second postulate

Postulate 2 (Observables) *Every observables of the system is represented by an Hermitian operator acting on the state space:*

$$\mathcal{M} : \mathcal{H} \to \mathcal{H}$$

The outcomes of the measurement can only be one of the eigenvalue of the operator $\mathcal{M}$.

Observation *The possible outcomes of the measurement are real number because $\mathcal{M}$ is self-andjoint.*

1.1.3 Third postulate

Postulate 3 (Born's Rule) *The probability to get the measurement λ_i from the observable $\mathcal{M}$ in the system in state $|\psi\rangle$ is:*

$$\mathbb{P}(\lambda_i) = \langle\psi| \mathcal{P}_i |\psi\rangle$$

where $\langle\psi|$ is the correspondent vector of $|\psi\rangle$ in the dual space of $\mathcal{H}$ and where $\mathcal{P}_i$ is the projection operator of λ_i in the correspondent space.

1.1.4 Fourth postulate

Postulate 4 (Wavefunction Collapse) *The state after measurement of λ_i is $\mathcal{P}_i |\psi\rangle$ (with the necessary normalization):*

$$|\psi'\rangle = \frac{\mathcal{P}_i |\psi\rangle}{\langle\psi| \mathcal{P}_i |\psi\rangle}.$$

1.1.5 Fifth postulate

Postulate 5 (Time Evolution) *The time evolution of an isolated quantum system is given by an unitary operator $\mathcal{U}$:*

$$|\psi(t)\rangle = \mathcal{U}(t_0, t) |\psi(t_0)\rangle .$$

Observation (Time dependent Shrodinger Equation) *From postulate 5, it is possible to obtain the time dependent Shrodinger Equation:*

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

where $H(t)$ is the Hemiltonian matrix.

1.1.6 Sixth postulate

Postulate 6 (Composite System) *The state space of a system composed of $\mathcal{H}_1$ and $\mathcal{H}_2$ is given by*

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

1.2 Combining Systems

The last postulate 6 has very important consequences for composite system. It is possible to describe two tipes of combined systems:

Definition 1 (Product states) *A state $|\psi\rangle \in \mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a pure state if exists $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ such that:*

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle .$$

A product state represents two states which do not interact; an operation on one of them does not perturb the other.

Definition 2 (Entengled states) *A system that is not in a product state (1), is in an entengled state.*

When a system is in an entengled state it is not possible to characterize the two subsystems with the states vector, although the state vector of the composite system is known.

1.2.1 Density operator

For a more general treatment, the following representation of states is given:

Definition 3 *The state of quantum system is described by a linear operator, called density operator such that:*

$$\Xi : \mathcal{H} \to \mathcal{H};\ \Xi^\dagger = \Xi;\ tr\{\Xi\} = 1.$$

According to the definition 3, the postulates 3, 4, 5 can be reformulate as following.

$$\mathbb{P}(\lambda_i) = tr\{\Xi \mathcal{P}_i\} \tag{1.1}$$

$$\Xi' = \frac{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger}{tr\{\mathcal{P}_i \Xi \mathcal{P}_i^\dagger\}} \tag{1.2}$$

$$\Xi(t) = \mathcal{U}\Xi(t_0)\mathcal{U}^\dagger \tag{1.3}$$

1.3 Quantized Electromagnetic Field

Electromagnetic field is the main means of communication for contemporary applications, it is important therefor to give its quantum conception. In this section, the representation of quantized electromagnetic field is firstly given, then the Fock's representation of a quantum state is introduced.

1.3.1 Classical electromagnetic field

In a volume $\mathcal{V} \in \mathbb{R}^3$ classical electromagnetic field is determinated from Maxwell's equations as a superposition of the cavity modes ([3] quoting [4, 5]). Electric field is given by the well-known expression:

$$\boldsymbol{e}(\boldsymbol{r}, t) = -\sum_n p_n(t)\boldsymbol{u}_n(\boldsymbol{r}) \tag{1.4}$$

where

$$\boldsymbol{u}_n(\boldsymbol{r}) = \boldsymbol{u}_{n0}\ e^{i\boldsymbol{k}_n \cdot \boldsymbol{r}}$$

and $\boldsymbol{u}_{n0}$ is determinated by the initial condition. The corresponding magnetic field is determinated by:

$$\boldsymbol{h}(\boldsymbol{r}, t) = \sum_n q_n(t)\nabla \times \boldsymbol{u}_n(\boldsymbol{r}) \tag{1.5}$$

and

$$p_n(t) = \frac{\mathrm{d}q_n(t)}{\mathrm{d}t}. \tag{1.6}$$

The Hemiltonian associated to the n-th mode is given by

$$H_n = \frac{1}{2}[p_n^2(t) + \omega_n^2 q_n^2(t)]. \tag{1.7}$$

Equivalently, it is possible to define the complex variable $a_n(t)$ as

$$a_n(t) = \frac{\omega_n q_n(t) + ip_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.8}$$

and, using 1.8 in 1.7, it is possible to obtain the following expression of the Hemiltonian:

$$H_n = \hbar\omega_n |a_n(t)|^2. \tag{1.9}$$

1.3.2 Quantized electromagnetic field

The quantization of electromagnetic field is obtained replacing the two quantities $p_n(t)$ and $q_n(t)$ with the Hermitian operators $\boldsymbol{P}_n(t)$, $\boldsymbol{Q}_n(t) : \mathcal{H}_n \to \mathcal{H}_n$ and by imposing the following commutation conditions ([3] quoting [4, 5]):

$$[\boldsymbol{Q}_n, \boldsymbol{P}_m] = i\hbar\delta_{n,m}\boldsymbol{I} \tag{1.10}$$

$$[\boldsymbol{Q}_n, \boldsymbol{Q}_m] = 0 \tag{1.11}$$

$$[\boldsymbol{P}_n, \boldsymbol{P}_m] = 0. \tag{1.12}$$

Defining the annihilation operator $\boldsymbol{A}_n$ as

$$\boldsymbol{A}_n(t) = \frac{\omega_n \boldsymbol{Q}_n(t) + i\boldsymbol{P}_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.13}$$

and the adjoint of $\boldsymbol{A}_n$, the creation operator $\boldsymbol{A}_n^\dagger$ as

$$\boldsymbol{A}_n(t) = \frac{\omega_n \boldsymbol{Q}_n(t) - i\boldsymbol{P}_n(t)}{\sqrt{2\hbar\omega_n}} \tag{1.14}$$

it is possible to describe the Hemiltonian of the system as

$$H_n = \hbar\omega_n \boldsymbol{A}_n^\dagger \boldsymbol{A}_n. \tag{1.15}$$

1.3.3 Fock states

In a single mode cavity, it is possible to define the number operator $\boldsymbol{N}$ as

$$\boldsymbol{N} = \boldsymbol{A}^{\dagger}\boldsymbol{A}. \tag{1.16}$$

Single mode Fock states are the eigenvector of N, i.e the solution of equation:

$$\boldsymbol{N}\left|n\right\rangle = n\left|n\right\rangle. \tag{1.17}$$

The Fock state $|n\rangle$ represents the quantum state with exactly n photons. It is important to point out that the set of all Fock states forms an orthonormal basis of the Hilbert space $\mathcal{H}$, so every state Ξ can be expressed as

$$\Xi = \sum_{n,m} c_{n,m}\left|n\right\rangle\left\langle m\right| \tag{1.18}$$

with

$$c_{n,m} = \left\langle n\right|\Xi\left|m\right\rangle.$$

Using the representation in Fock basis, it is possible to characterize different types of quantum state of the quantum electromagnetic field. In the following section the states studied are briefly described.

1.4 QEF States

In this section, some quantum states of electromagnetic field useful for quantum communication are characterized. It is firstly given a brief introduction to another one tool for the description of quantum systems, then some Gaussian and non-Gaussian states are characterized.

1.4.1 Phase-space description

As seen before, in 1.18, quantum system can be completely described by a density operator Ξ defined in an infinite-dimensional Hilbert space $\mathcal{H}$ and this operator can be expressed by the Fock representation (3). Sometimes, however, it is convenient to give another representation of state Ξ by means of a complex function introduced by Wigner [6]: the quasi-probability distribution. In this thesis, this representation will be introduced and it will be used to classify the possible states.

Definition 4 (Quantum characteristic function) *The s-order characteristic function $\chi(\xi, s)$, with $\xi, s \in \mathbb{C}$, associated to the quantum state Ξ is defined as:*

$$\chi(\xi, s) = \exp\left\{\frac{s}{2}|\xi|^2\right\} tr\{\Xi \boldsymbol{D}_{\xi}\} \tag{1.19}$$

where $\boldsymbol{D}_{\xi}$ is the displacement operator of parameter ξ, defined as:

$$\boldsymbol{D}_{\xi} = \exp\left\{\xi \boldsymbol{A}^{\dagger} - \xi^{*}\boldsymbol{A}\right\}. \tag{1.20}$$

Definition 5 (Quasi-probability distribution) *The s-order quasi-probability distribution $W(\alpha, s)$, with $s \in \mathbb{C}$, associated to the quantum state Ξ is given by:*

$$W(\alpha, s) = \frac{1}{\pi^2} \int_{\mathbb{R}^2} \chi(\xi, s) e^{\alpha\xi^* - \alpha^*\xi} d\xi^2. \tag{1.21}$$

The quasi-probability distribution, for $s = 0$ ($W(\alpha) = W(\alpha, 0)$) is called Wigner W-function.

1.4.2 Gaussian states

With the Wigner W-function $W(\alpha)$, it is possible to define the concept of Gaussian state ([3] quoting [7, 8, 9, 10, 11]).

Definition 6 (Gaussian state) *A quantum state $\Xi_{\boldsymbol{G}}$ is a Gaussian state if its Wigner W-function $W_G(\alpha)$ is Gaussian, i.e*

$$W_G(\alpha) = \frac{1}{\pi\sqrt{det\check{\boldsymbol{C}}_0}} \exp\left\{-\frac{1}{2}(\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\mu}})^H \check{\boldsymbol{C}}_0^{-1} (\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\mu}})\right\}. \tag{1.22}$$

where $\check{\boldsymbol{\mu}}$ is the augmented displacement vector, and $\check{\boldsymbol{C}}_0$ is the augmented covariance matrix.

Coherent states

A coherent state is the state of a quantum armonic oscillator of amplitude μ. It is defined ([3] seen [12, 13]) as the eigenvector $|\mu\rangle$ of $\boldsymbol{A}$ associated to the eigenvalue μ; i.e

$$\boldsymbol{A} |\mu\rangle = \mu |\mu\rangle. \tag{1.23}$$

It is possible to obtain a coherent state of parameter μ, from the ground state as

$$|\mu\rangle = \boldsymbol{D}_\mu |0\rangle. \tag{1.24}$$

As mentioned before, it is possible to characterize a state with the Fock representation and, equivalently, with the Wigner W-function. The last one is given, for a coherent state, by [14]:

$$W(\alpha) = \frac{2}{\pi} \exp\left\{-2|\alpha - \mu|^2\right\}. \tag{1.25}$$

It is easy to proof that $W(\alpha)$ is gaussian, with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \frac{1}{2}\boldsymbol{I}.$$

The Fock representation is given by [15]:

$$|\mu\rangle = e^{-\frac{|\mu|^2}{2}} \sum_{n=0}^{\infty} \frac{\mu^n}{\sqrt{n}} |n\rangle. \tag{1.26}$$

Noisy coherent states

It is possible to characterize the state of a noisy armonic oscillator introducing the thermal state, i.e the state of a electromagnetic cavity system. The Fock representation of the thermal state Ξ_{th} is given by [3]

$$\Xi_{th} = (1-v)\sum_{n=0}^{\infty} v^n \left|n\right\rangle \left\langle n\right| \tag{1.27}$$

where

$$v = \frac{\bar{n}}{\bar{n}+1}$$

and $\bar{n}$ is the well-known Plank distribution

$$\bar{n} = \left(\exp\left\{\frac{\hbar\omega}{k_B T}\right\} - 1\right)^{-1}.$$

A noisy coherent states $\Xi_{th}(\mu)$ of parameter μ can be obtained by appling the displacement operator $\boldsymbol{D}_\mu$ to the thermal state Ξ_{th}, as follow:

$$\Xi_{th}(\mu) = \boldsymbol{D}_\mu^\dagger \Xi_{th} \boldsymbol{D}_\mu. \tag{1.28}$$

The Wigner W-function is given by [14]

$$W_{th}(\alpha) = \frac{1}{\pi(\bar{n}+\frac{1}{2})} \exp\left\{-\frac{|\alpha-\mu|^2}{\bar{n}+\frac{1}{2}}\right\} \tag{1.29}$$

and it can be proved that it is a Gaussian function with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \left(\bar{n}+\frac{1}{2}\right)\boldsymbol{I}.$$

The Fock representation is given by

$$\left\langle n\right| \Xi_{th}(\mu) \left|m\right\rangle = (1-v)e^{-(1-v)|\mu|^2}\sqrt{\frac{n!}{m!}}v^n[(1-v)\mu^*]^{m-n}L_n^{m-n}\left(\frac{-(1-v)^2|\mu|^2}{v}\right) \tag{1.30}$$

Squeezed states

A squeezed state with amplitude μ and squeezing parameter ζ, is a defined as [3, 16, 17]

$$\left|\mu,\zeta\right\rangle = \boldsymbol{D}_\mu \boldsymbol{S}_\zeta \left|0\right\rangle \tag{1.31}$$

where $\boldsymbol{S}_\zeta$ is the squeezing operator, defined as

$$\boldsymbol{S}_\zeta = \exp\left\{\frac{1}{2}\left(\zeta\left(\boldsymbol{A}^\dagger\right)^2 + \zeta^*\boldsymbol{A}^2\right)\right\}. \tag{1.32}$$

It can be proven that a squeezed state is a Gaussian state with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \frac{1}{2}\begin{bmatrix} \cosh(2r) & \sinh(2r)e^{-i\phi} \\ \sinh(2r)e^{-i\phi} & \cosh(2r) \end{bmatrix}$$

with $\zeta = re^{i\phi}$. The Wigner W-function of a squeezed state, unlike that of a coherent state, has not a circular symmetry.

Noisy squeezed states

The representation of a noisy squeezed state $\Xi_{th}(\mu,\zeta)$ is obtained, similary to a noisy coherent state, as:

$$\Xi_{th}(\mu,\zeta) = \boldsymbol{D}_{\mu}\boldsymbol{S}_{\zeta}\Xi_{th}\boldsymbol{S}_{\zeta}^{\dagger}\boldsymbol{D}_{\mu}^{\dagger}. \tag{1.33}$$

The Gaussian Wigner function is obtained with $\check{\boldsymbol{\mu}} = [\mu\ \mu^*]^T$ and

$$\check{\boldsymbol{C}}_0 = \left(\bar{n} + \frac{1}{2}\right)\begin{bmatrix} \cosh(2r) & \sinh(2r)e^{-i\phi} \\ \sinh(2r)e^{-i\phi} & \cosh(2r) \end{bmatrix}.$$

The Fock representation is given by [18]

$$\begin{aligned}\langle n|\,\Xi_{th}(\mu,\zeta)\,|m\rangle = \frac{\pi Q(0)}{(n!m!)^{1/2}} \sum_{k=0}^{min(n,m)} k!\binom{n}{k}\binom{m}{k}\tilde{A}^k\left(\frac{1}{2}\tilde{B}\right)^{(n-k)/2} \\ \left(\frac{1}{2}\tilde{B}^*\right)^{(m-k)/2} H_{n-k}((2\tilde{B})^{-1/2}\tilde{C})H_{m-k}((2\tilde{B}^*)^{-1/2}\tilde{C}^*)\end{aligned} \tag{1.34}$$

where H_n is the Hermite polynomial with parameter n,

$$Q(0) = \frac{1}{\pi}[(1+A)^2 - |B|^2]^{-1/2}\exp\left\{-\frac{(1+A)|C|^2 + \frac{1}{2}[B(C^*)^2 + B^*C^2]}{(1+A)^2 - |B|^2}\right\},$$

$$\tilde{A} = \frac{A(1+A) - |B|^2}{(1+A)^2 - |B|^2},$$

$$\tilde{B} = \frac{B}{(1+A)^2 - |B|^2},$$

$$\tilde{C} = \frac{(1+A)C + BC^*}{(1+A)^2 - |B|^2};$$

and

$$A = \bar{n} + (2\bar{n}+1)(\sinh(r))^2,\ B = -(2\bar{n}+1)e^{i\phi}\sinh(r)\cosh(r),\ C = \mu.$$

1.4.3 Non-Gaussian states

A state that does not meet the definition 6 is a non-Gaussian state. An important class of non-Gaussian states, useful for communications are the photon added states, examined in this thesis.

Photon added states

The photon added state $\Xi^{(1)}$, obtained from the quantum state Ξ, is given by:

$$\Xi^{(1)} = \frac{\boldsymbol{A}^{\dagger}\Xi\boldsymbol{A}}{tr\{\boldsymbol{A}^{\dagger}\Xi\boldsymbol{A}\}}. \tag{1.35}$$

The name *photon addition*, despite can be thought, does not mean that the mean photon number of the photon added state is one more then the non-photon added state. In general, its mean number of photon could be the same, more or less then the starting state. Only if $\Xi = |n\rangle\langle n|$, i.e Ξ is the density operator corresponding to the Fock state $|n\rangle$, the result of the photon addition is a state with one more photon.

Logically, the photon added state $\Xi^{(k)}$ (with k photon addition) is given by

$$\Xi^{(k)} = \frac{(\boldsymbol{A}^{\dagger})^k \Xi \boldsymbol{A}^k}{tr\{(\boldsymbol{A}^{\dagger})^k \Xi \boldsymbol{A}^k\}}. \tag{1.36}$$

The Fock representation of a photon added state, can be obteined as:

$$\Xi^{(k)} = \frac{\tilde{\Xi}^{(k)}}{tr\{\tilde{\Xi}^{(k)}\}} \tag{1.37}$$

and

$$\langle n|\,\tilde{\Xi}^{(k)}\,|m\rangle = \begin{cases} \sqrt{\frac{n!m!}{(n-k)!(m-k)!}}\,\langle n-k|\,\Xi\,|m-k\rangle \;\; if\;\; n,m \geq k \\ 0\;\; otherwise \end{cases}$$

If Ξ is a noisy coherent state of amplitude μ ($\Xi = \Xi_{th}(\mu)$), the photon added state $\Xi_{th}^{(k)}(\mu)$ is called noisy photon added coherent state (PACS); if Ξ is a noisy squeezed state with amplitude μ and squeezing factor ζ ($\Xi = \Xi_{th}(\mu, \zeta)$), the photon added state $\Xi_{th}^{(k)}(\mu, \zeta)$ is called noisy photon added squeezed state (PASS).

Chapter 2

Quantum Communication

Quantum mechanics is allowing to overcome the limits of classical comunication systems. In the last decades the research in this field is leading to very intresting results that could be significantly improves the performance of communication systems. In this chapter, a brief overview of quantum communication tools is given: in first section the equivalents of classical modulation for quantum communication systems is presented; in the second section the concept of quantum states discriminator (QSD) is given and the optimal QSD are presented.

2.1 Quantum Modulation

As in a classical system, it is possible to define the concept of modulation for a quantum communication system. The trasmitted information, will be associated to a quantum state of the electromagnetic field, so it can be trasmitted on the communication channel.

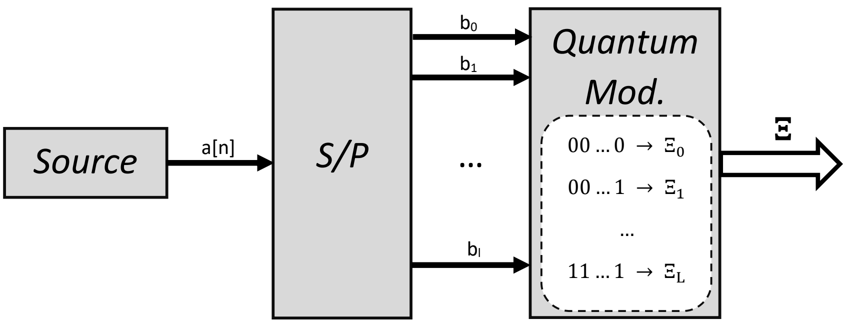


Figure 2.1: Block diagram of a quantum trasmitter.

It is possible to think about the quantum trasmitter as in figure 2.1. The bit source emits a bit sequence $a[n]$, the serial-parallel converter parallelize a group of l-bit (where, if L is the number of quantum states, $l = \log_2(L)$) and sends them to the quantum modulator; this last associate to every group of bit, one quantum state. The operation of quantum state creation, in real cases, is affected by noise.

The sequence of operation is very closer to a classical trasmitter, the main difference is that the modulator map the bits into quantum states instead of classical modulation. It is, therefor, possible to achieve the equivalent of classical modulation, that it is called quantum modulation, with several states and then, assess the impact on performance. In this thesis only the binary cases will be considered and assessed, in the OOK and BPSK configuration.

2.1.1 OOK modulation

The OOK (on-off keying) is the most simple possible configuration for a communication system. The quantum implementation of that is realized associating the low-energy state to the ground state $|0\rangle$ and the high-energy state to another state. It is important to consider that the physical realization of these states are not free-noise; this issue will be considered using noisy states 2.1.

$$\Xi_0 = \Xi_{th} \tag{2.1}$$

$$\Xi_1 = \Xi_{th}(\mu)$$

In the equation 2.1, the high-energy state is associated to a coherent state. This configuration has been widely analyzed in [19, 20, 21, 22, 23, 24] but this is not the only possible way. The use of PACS states $\Xi_{th}^{(k)}(\mu)$ is analyzed in [25, 3]; the use of PASS are briefly assessed in the following chapter of this thesis.

2.1.2 BPSK modulation

BPSK quantum systems are implemented using two states with opposite amplitude, like

$$\Xi_0 = \Xi_{th}(-\mu) \tag{2.2}$$

$$\Xi_1 = \Xi_{th}(\mu).$$

The use of a BPSK solution, in a quantum system, is not said that improves the performance of the system. The effect depends on which are the used states. In the next chapter some configuration are assessed.

2.2 Quantum Discriminator

The problem of quantum state discrimination (QSD) are one of the most important aspect of quantum communication. As in classical communication, the ability to distinguish between two or more states, in presence of noise, can be decisive for improve the performance of the communication system. Unlike the classical situation however, the discrimination can be done using a custom-designed quantum discriminator, overcoming the classical physics limits.

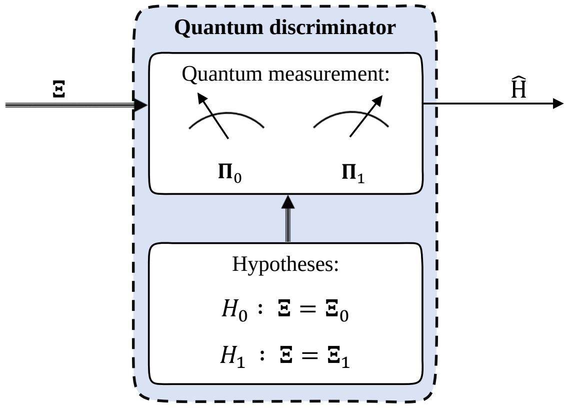


Figure 2.2: Binary quantum state discriminator.

2.2.1 Binary quantum state discrimination

The problem of discrimination between two quantum states is realized, as every measurement process 2, using an operator or with a set of operators. If the state of system is unknown, as shown in figure 2.2, there are two hypotheses about the state Ξ (the problem is generalises easly for M different states), given by:

$$\begin{aligned} H_0 &: \boldsymbol{\Xi} = \boldsymbol{\Xi}_0 \\ H_1 &: \boldsymbol{\Xi} = \boldsymbol{\Xi}_1 \end{aligned} \tag{2.3}$$

It is necessary a set of two positive-definites operator (POVM):

$$\mathcal{P} = \{\boldsymbol{\Pi}_0, \boldsymbol{\Pi}_1\} \tag{2.4}$$

for the discrimination process and the probability that the hypothesis H_j is choosen if H_k is the right choose, is given by [3]:

$$\mathbb{P}\{H_j|H_k\} = tr\{\boldsymbol{\Xi}_k\boldsymbol{\Pi}_j\}. \tag{2.5}$$

The distribution error probability (DEP) in the discrimination process, if p_0 and p_1 are respectively the probabilty of symbols 0 and 1, is so given by

$$P_e = 1 - (p_0 tr\{\boldsymbol{\Xi}_0\boldsymbol{\Pi}_0\} + p_1 tr\{\boldsymbol{\Xi}_0\boldsymbol{\Pi}_0\})\,. \tag{2.6}$$

2.2.2 Optimal discriminator

The issue of finding the optimal POVM that minimize was exhaustively discuss by Helstrom in [26, 27]. The minimum distribution error probability (MDEP), for a binary communication system, is given by the well-known Helstrom bound

$$\breve{P}_e = \frac{1}{2}\left(1 - \|p_1\boldsymbol{\Xi}_1 - p_0\boldsymbol{\Xi}_0\|_1\right), \tag{2.7}$$

where $p_0,\ p_1$ are the probability that the states $\Xi_0,\ \Xi_1$ are trasmitted and the operator $\|\cdot\|_1$ represents the trace norm. The MDEP 2.7 is obtained with the following POVM:

$$\breve{\mathbf{\Pi}}_0 = \sum_{\substack{i \\ \lambda_i < 0}} |\lambda_i\rangle \langle\lambda_i| \,, \tag{2.8}$$

$$\breve{\mathbf{\Pi}}_1 = 1 - \breve{\mathbf{\Pi}}_0 = \sum_{\substack{i \\ \lambda_i \geq 0}} |\lambda_i\rangle \langle\lambda_i| \,;$$

where $|\lambda_i\rangle$ is the eigenvector of $p_1\Xi_1 - p_0\Xi_0$ associated to the eigenvalue λ_i. For pure states, i.e $\Xi_0 = |\psi_0\rangle \langle\psi_0|$ and $\Xi_1 = |\psi_1\rangle \langle\psi_1|$, the equation 2.7 begin

$$\breve{P}_e = \frac{1}{2}\left(1 - \sqrt{1 - 4p_0p_1|\langle\psi_0|\psi_1\rangle|^2}\right) . \tag{2.9}$$

It is possible to observe that, for pure states, the MDEP is equal to 0 if $\langle\psi_0|\psi_1\rangle$, that is $|\psi_0\rangle$ and $|\psi_1\rangle$ are orthogonal states.

Chapter 3

Discrimination of Photon Added States

In this chapter, two tipe system with photon added states are characterized in term of their discrimination MDEP 2.7: a photon added coherent state system (from [25])and a photon added squeezed state. For each system the OOK configuration and the BPSK will be considered. The use of photon added states, as it will be shown, can improve significatly the performance of the communication.

The analyzed situation do not consider the channel effect on the trasmitted information, it was supposed that the noisy states reach the discriminator as it was created. The channel effects for a PACS system are described in [25].

3.1 PACS Discrimination

The effect of the use of PACS in an OOK communication system was extensivly discuss in [25]. In this section the most important result will be reported and a BPSK system will be tested.

3.1.1 Quantum OOK PACS system

The use of PACS in an OOK system can significatly improves the performance. The MDEP, with thermal noise, is given by the Helstrom bound 2.7, where, for an OOK PACS system,

$$\Xi_0 = \Xi_{th} = \Xi_{th}^{(0)}(0) \tag{3.1}$$

$$\Xi_1 = \Xi_{th}^{(k)}(\mu).$$

It is useful, for evaluate the performance of the system, to introduce the mean number of photon n_p in a quantum state Ξ, that is given by:

$$n_p = tr\{\Xi \boldsymbol{A}^\dagger \boldsymbol{A}\}. \tag{3.2}$$

For a photon added coherent state, the equation 3.2 begin [25]:

$$n_p(\mu, \bar{n}) = \frac{N_{k+1}(\mu, \bar{n})}{N_k(\mu, \bar{n})} - 1, \tag{3.3}$$

where

$$N_k(\mu, \bar{n}) = tr\{(\boldsymbol{A}^\dagger)^k \Xi_{th}(\mu) \boldsymbol{A}^k\}. \quad (3.4)$$

It is possible to observe that the minimum of n_p is given by:

$$n_p(0, \bar{n}) = (k+1)(\bar{n}+1) - 1. \quad (3.5)$$

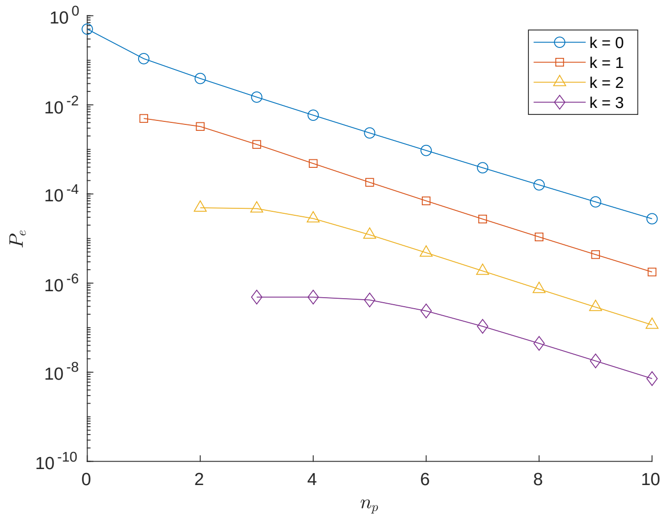


Figure 3.1: MDEP for PACS QOOK with: $k = 0, 1, 2, 3$; $\bar{n} = 10^{-2}$; $p_0 = p_1 = 1/2$

The MDEP of a quantum OOK system with PACS in function of n_p, is represented in the figure 3.1, where in the x-axis there are the mean number of photon n_p in the state Ξ_1 (which corresponds to the mean number of photon in the system) and in the y-axes there are the MDEP (P_e). The simulation was obtained for equiprobable symbols and mean number of thermal photons $\bar{n} = 10^{-2}$. The argument of the trace norm $\|\cdot\|_1$ in the Helstrom bound 2.7, is an operator in an infinite dimensional Hilbert space; for the simulation, it was been approximate in $N = 30$ dimension. It is possible to observe that the photon addition improves significatly the performance in term of error probability.

3.1.2 Quantum BPSK PACS system

It can be interesting to assess the effect of photon addition in a quantum BPSK system. The constellation is given, for a PACS BPSK, by:

$$\Xi_0 = \Xi_{th} = \Xi_{th}^{(k)}(-\mu) \quad (3.6)$$
$$\Xi_1 = \Xi_{th}^{(k)}(\mu).$$

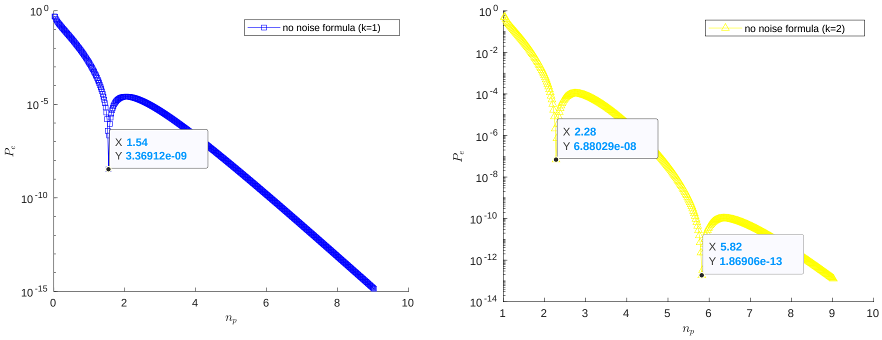
In absence of noise ($\bar{n} = 0$), the MDEP is given by formula 2.9 where

$$|\psi_0\rangle = \left|-\mu^{(k)}\right\rangle, \quad (3.7)$$
$$|\psi_1\rangle = \left|\mu^{(k)}\right\rangle.$$

The inner product is given, in closed form, by [25]:

$$\left\langle -\mu^{(k)} \middle| \mu^{(k)} \right\rangle = \frac{L_k(|\mu|^2)}{L_k(-|\mu|^2)} e^{-2|\mu|^2}, \tag{3.8}$$

where $L_k(x)$ is the Laguerre polynomial of parameter k, evaluate in x. In figure



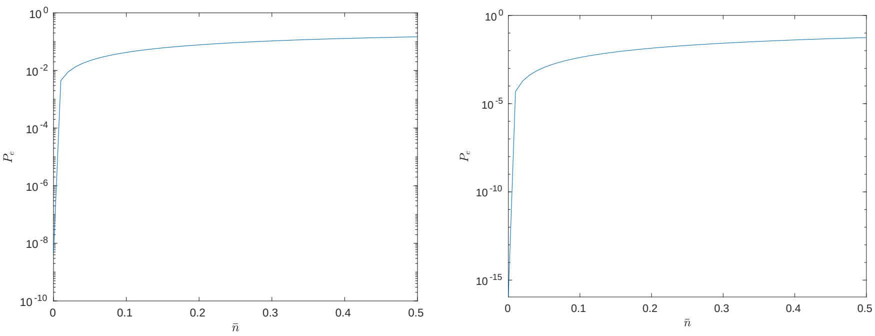
(a) $k = 1$

(b) $k = 2$

Figure 3.2: MDEP of quantum BPSK in absence of noise, $N = 30$.

3.2 the MDEP in absence of noise, for QBPSK with PACS, is plotted for $k = 1$ and for $k = 2$, in function of n_p, with $N = 30$. It can be notice that exist k zeros in the MDEP plot, where k is the number of photon addition. The existence of this zeros is not really useful for the design of a quantum communication system because their selectivity factors is too high for a phisically implementation. It is, nevertheless, possible to use that for evaluate the effect of the thermal noise. In figure 3.3, the sluggish performance because of thermal noise is clear. The

Figure 3.3: Thermal noise effect, in corrispondence of MDEP zeros ($N = 40$).



(a) $k = 1$ & $\mu = 0.54$

(b) $k = 2$ & $\mu = 1.58$

plot show the trend of MDEP, for zeros value of μ, in function of $\bar{n}$; the used approximation is $N = 30$. The MDEP in presence of noise is given using the expression 2.7.

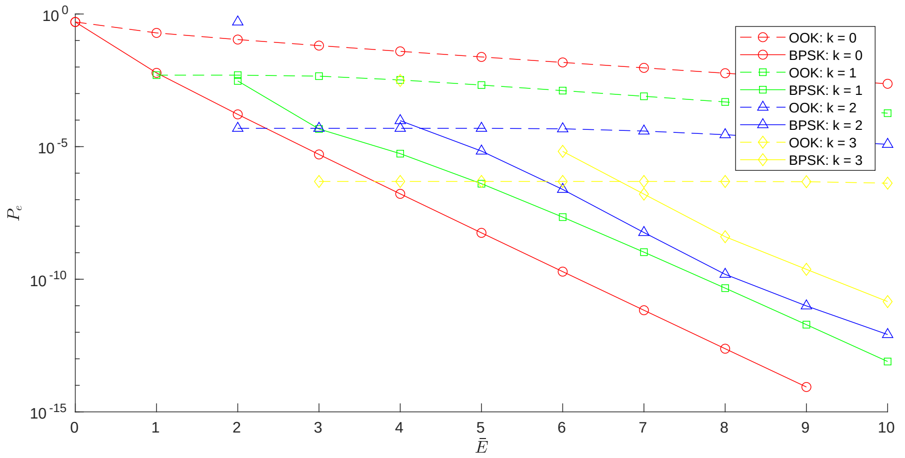


Figure 3.4: BPSK and OOK comparison.
$N = 45$, $\bar{n} = 10^{-2}$, $p_0 = p_1 = 1/2$

The comparison between a quantum OOK system and a quantum BPSK system is given in figure 3.4, in function of the mean energy of the system $\bar{E}$, that is equal to n_p for the OOK system and the sum of the n_p for each state for the BPSK system. The plots are given with $\bar{n} = 10^{-2}$, $N = 45$ and equiprobable symbols. The obtained result is really intresting: the quantum BPSK has a sluggish performance because of the photon addition process. At an equal level of energy $\bar{E}$, for PACS systems, it is possible to find an OOK configuration that maximize the performance.

3.2 PASS Discrimination

The use of squeezed states instead of coherent states can allow to overcome the limits of precedent systems. The representation of a noisy squeezed state is given in 1.4.2. In this section the advantages of the use of squeezed states without photon addition, are initially discuss; then the effect of the photon addition in a quantum OOK and BPSK systems are evaluate.

3.2.1 Squeezed states discrimination

At first, the effect of squeezing, whithout photon addition and thermal noise, on the performance are assessed. As for PACS systems, it can be useful to define the mean number of photon n_p in a squeezed state, that is given by

$$n_p(\mu, r) = |\mu|^2 + (\sinh r)^2 ; \tag{3.9}$$

where μ is the amplitude of the starter coherent state and the squeezing factor is $\zeta = re^{i\theta}$. The minimum value of n_p is given by $n_p(0, r) = \sinh r^2$.

For a quantum squeezed states BPSK system, the MDEP, obtained with the Helstrom bound 2.7, is plotted in figure 3.5 in function of $\bar{E}$, i.e the mean energy

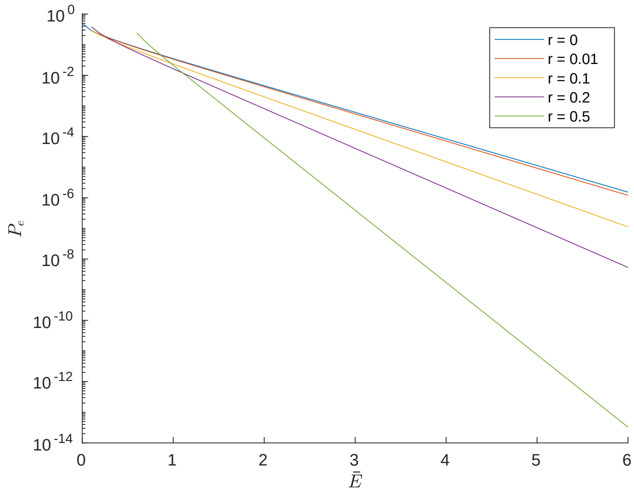


Figure 3.5: MDEP of squeezed state BPSK system. $N = 30$, $\bar{n} = 0$, $\theta = \pi$, $p_0 = p_1 = 1/2$

of the system (given by the sum of n_p for each state), with: $\theta = \pi$, $N = 30$, equiprobable symbols and $\bar{n} = 0$.

It can be notice that the optimal configuration of r depends from the energy in the system. For low energy levels the squeezing has not a positive effects.

3.2.2 PASS discrimination

It is analyzed now, the performance of a noisy photon added squeezed state system (PASS) 1.4.3, in OOK and BPSK configuration. The MDEP is found again with the Helstrom bound 2.7. As for the other systems, it is usefull to define the mean number of photon n_p for noisy photon added squeezed states, that is given by:

$$n_p(\mu, \zeta, \bar{n}) = \frac{N_{k+1}(\mu, \zeta, \bar{n})}{N_k(\mu, \zeta, \bar{n})} - 1, \tag{3.10}$$

where

$$\begin{aligned} N_k(\mu, \zeta, \bar{n}) &= tr\{(\boldsymbol{A}^\dagger)^k \Xi_{th}(\mu, \zeta) \boldsymbol{A}^k\} \\ &= tr\{(\boldsymbol{A}^\dagger)^k \boldsymbol{D}_\mu \boldsymbol{S}_\zeta \Xi_{th} \boldsymbol{S}_\zeta^\dagger \boldsymbol{D}_\mu^\dagger \boldsymbol{A}^k\}. \end{aligned} \tag{3.11}$$

OOK PASS system

The constellation of a quantum OOK system with noisy PASS, is given by:

$$\begin{aligned} \Xi_0 &= \Xi_{th}^{(0)}(0, 0) \\ \Xi_1 &= \Xi_{th}^{(k)}(\mu, \zeta). \end{aligned} \tag{3.12}$$

In figure 3.6 the MDEP of a quantum OOK noisy PASS system is plotted, in function of the mean number of photon n_p in the PASS (that is equivalent to the mean energy in the system). For the simulation are used $N = 30$, $\bar{n} = 10^{-2}$,

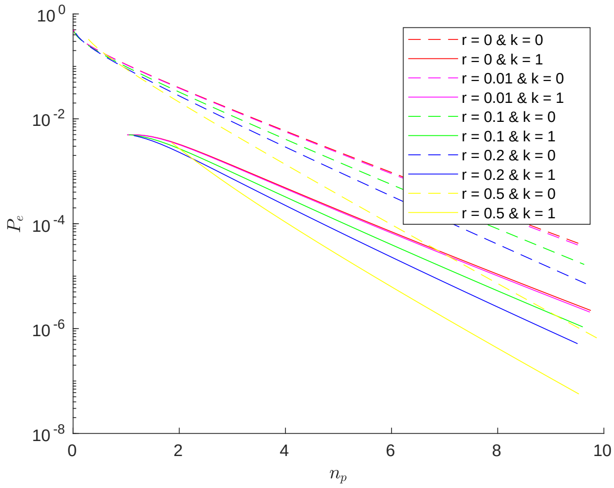


Figure 3.6: MDEP of noisy PASS quantum OOK system.
$N = 30$, $\bar{n} = 10^{-2}$, $\theta = \pi$ and $p_0 = p_1 = 1/2$.

$\theta = \pi$ and equiprobable symbols. It can be notice that the photon addition significatly improves the performance of the system, at least for the plotted energy level.

BPSK PASS system

Similary to the PACS BPSK, the constellation of PASS BPSK is given by:

$$\begin{aligned} \Xi_0 &= \Xi_{th}^{(k)}(-\mu, \zeta) \\ \Xi_1 &= \Xi_{th}^{(k)}(\mu, \zeta). \end{aligned} \tag{3.13}$$

The figure 3.7 shown the effects of the photon addition in a quantum BPSK system, in terms of the mean energy in the system $\bar{E}$, given by the sum of the mean photon number n_p of Ξ_0 and Ξ_1. The parameters used for the simulation are $N = 30$, $\bar{n} = 10^{-2}$, $\theta = \pi$ and equiprobable symbols. As in PACS case, it is evident that the photon addition, in a BPSK system, has not the positive effect that has in an OOK system.

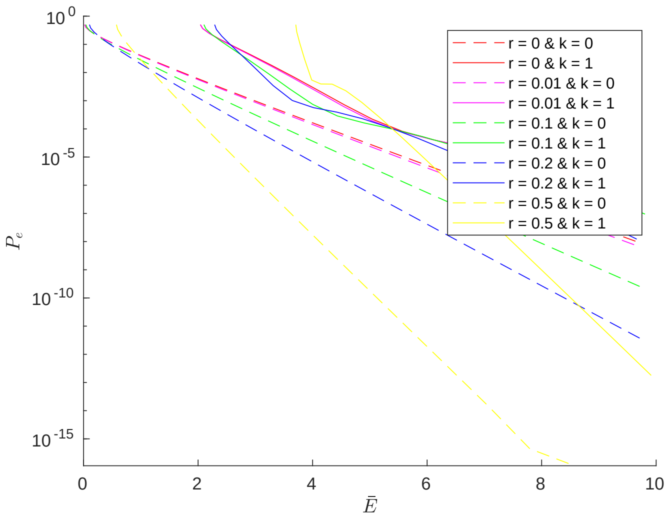


Figure 3.7: MDEP of noisy PASS quantum BPSK system.
$N = 30$, $\bar{n} = 10^{-2}$, $\theta = \pi$ and $p_0 = p_1 = 1/2$.

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