

Computational Algebra

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Transcript

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1 Integer Arithmetic

Topics:

- Addition and Multiplication
- GCD computation
- Primality testing
- Factorization

1.1 Addition and Multiplication

Agreement:

- $a, x \in \mathbb{N}$ represented as $x = \sum_{i=0}^{n-1} a_i \cdot B^i$ $B \in \mathbb{N}_{>1}$ fixed Base ($a_i \in \{0, \dots, B-1\}$)
- if $x \neq 0$, assume $a_{n-1} \neq 0$ then define:
length of $x := l(x) = n$ = number of digits = $\lfloor \log_B(x) \rfloor + 1$
(mnemonic: $\log_B(B) + 1 = 2$)
- $l(0) = 1$
(Amount of memory required to store $x = 0$)
- $l(x) := l(|x|)$
- for $x \in \mathbb{Z}$ represent if as $x = \text{sgn}(x) * |x|$

1.1.1 Algorithm 1 (Simple addition)

input : $x = \sum_{i=0}^{n-1} a_i \cdot B^i$, $y = \sum_{i=0}^{n-1} b_i \cdot B^i$, $x, y \in \mathbb{N}$

output: $x + y = \sum_{i=0}^n c_i \cdot B^i$

- (1) $\sigma = 0$
- (2) for $i = 0, \dots, (n-1)$:
- (3) set $c_i := a_i + b_i + \sigma_i$
 $\sigma := 0$
- (4) if $(c_i \geq B)$
- (5) set $c_i = c_i - B$
- (6) set $\sigma = 1$
- (7) set $c_n = \sigma$

If $B = 2$ then (3) - (6) can be realized by logic gates:

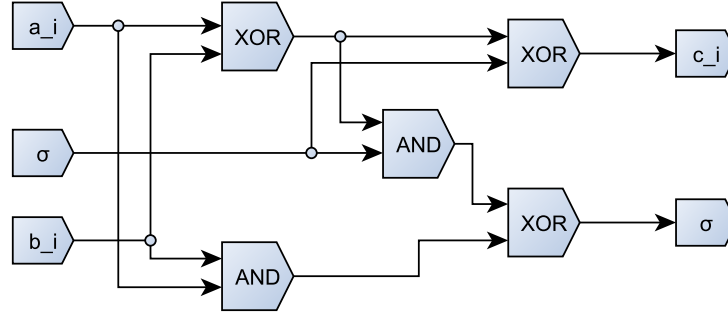


Figure 1: Logic circuit for addition

1.1.2 Definition 2 (Bit-Operation)

A bit operation is an operation that can be performed by a logic gate or by searching or writing a bit from / into memory.

1.1.3 Definition 3 (Big O)

Let M be a set (usually $M = \mathbb{N}$), $f, g : M \mapsto \mathbb{R}_{>0}$
we write $f \in O(g)$ if $\exists c \in \mathbb{R} : f(x) \leq c \cdot g(x) \forall x \in M$

1.1.4 Theorem 4 (Lower bound for addition)

Let $f : \mathbb{N} \mapsto \mathbb{R}$, $n \mapsto$ maximal number of bit operations required by Algorithm 1 to add $x, y \in \mathbb{N}$ with $l(x), l(y) \leq n$

Let $g = id_{\mathbb{N}}$ Then $f \in O(g)$

We say Algorithm 1 requires $O(n)$ bit operations for adding two numbers of length $\leq n$.
 \Rightarrow "linear complexity"

Set $M := \{\text{Set of all algorithms for addition in } \mathbb{N}\}$

For $A \in M$ define $f_A : \mathbb{N} \mapsto \mathbb{R}$ as above.

We would like to find $f_{odd} : \mathbb{N} \mapsto \mathbb{R}$, $n \mapsto \inf\{f_A(n) | A \in M\}$

Since one needs to read x, y (and write the result) we can not do any better than linear complexity for addition.

Subtraction

let x, y as Algorithm 1, $x \geq y$

For $\bar{y} := \sum_{i=0}^{n-1} (B - 1 - b_i) B^i$ (digitwise / bitwise complement)

$\Rightarrow x + \bar{y} = x - y + B^n - 1$

$\Rightarrow x - y = x + \bar{y} + 1 - B^n$ (initially set $\sigma = 1$)

Conclusion: Addition and Subtraction have cost $O(n)$

1.1.5 Algorithm 5 (Multiplication by "grid method")

input : $x = \sum_{i=0}^{n-1} a_i \cdot 2^i, \quad y = \sum_{i=0}^{m-1} b_i \cdot 2^i$

output: $z = x \cdot y$

- (1) $z := 0$
- (2) for $i = 0, \dots, (n-1)$
- (3) if $(a_i \neq 0)$

$$\text{set } z := z + \sum_{j=0}^{m-1} b_j 2^{i+j}$$

1.1.6 Theorem 6 (Runtime of Algorithm 5)

Algorithm 5 requires $O(n \cdot m)$ bit operations.

As of the total input length $n + m$:

$$n \cdot m \leq \frac{1}{2}(n + m)^2 \rightarrow O((n + m)^2)$$

\Rightarrow Quadratic complexity

Karatsuba-multiplication:

Observation for polynomials:

$$a + bx, c + dx \text{ have } (a + bx)(c + dx) = ac + (ac + db - (a - b)(c - d))x + bdx^2$$

The point: only used 3 multiplications instead of 4.

Specialize $x = B$ "large" such that $x = a + bB$ partition into two blocks. Then multiply the blocks by a recursive call.

1.1.7 Algorithm 7 (Karatsuba)

input : $x, y \in \mathbb{N}$

output: $z = x \cdot y$

- (1) Choose $k \in \mathbb{N}$ minimal such that $l(x), l(y) \leq 2^k$.
Set $B = 2^{2^{k-1}}$
- (2) if $(k = 0)$
return $x \cdot y$ (by bit-operation AND)
- (3) write $x = x_0 + x_1 B$
 $y = y_0 + y_1 B$ with $l(x_i), l(y_i) \leq 2^{k-1}$
- (4) compute $x_0 \cdot y_0, x_1 \cdot y_1, (x_0 - x_1) \cdot (y_0 - y_1)$ by a recursive call
- (5) return $z = x_0 y_0 + (x_0 y_0 + x_1 y_1 - (x_0 - x_1)(y_0 - y_1))B + x_1 y_1 B^2$

1.1.8 Theorem 8 (Runtime of Algorithm 7)

For multiplying two numbers of length $\leq n$ Algorithm 7 requires $O(n^{\log_2 3}) \approx O(n^{1.59})$ bit operations.

Proof:

Set $\Theta(k) :=$ maximal numbers of bit operations for $l(x), l(y) \leq 2^k$

We have for $k > 0$: $\Theta(k) \leq 3 \underbrace{\Theta(k-1)}_{\text{recursive calls}} + c \underbrace{2^k}_{\text{additions}}$ with (c some constant)

Claim: $\Theta(k) \leq 3^k + 2c(3^k - 2^k)$

Proof by Induction on k :

$k = 0$: $\Theta(k) = 1$

$$\begin{aligned} k-1 \rightarrow k : \Theta(k) &= 3\Theta(k-1) + c2^{k-1} \\ &\leq 3(3^{k-1} + 2c(3^{k-1} - 2^{k-1})) + c2^k \\ &= 3^k + 2c(3^k - 2^k) \end{aligned}$$

So $\Theta(k) \leq (2c+1)3^k$

Now $l(x) \leq n$ hence $2^{k-1} < n$ by minimality of k

So $k-1 < \log_2 n$

$$\begin{aligned} \Rightarrow \Theta(k) &\leq 3(2c+1)3^{\log_2(n)} \\ &= 3(2c+1)2^{\log_2(3) \log_2(n)} \\ &= 3(2c+1)n^{\log_2(3)} \quad \square \end{aligned}$$

One can modify the terminal condition of Karatsuba to switch to Grid-Multiplication, which is faster for small numbers.

Fast-Fourier Transformation

Reminder: For a function $f : \mathbb{R} \mapsto \mathbb{C}$ define:

$\hat{f} : \mathbb{R} \mapsto \mathbb{C}$ by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt \quad (\text{if it exists})$$

Think of ω as frequency.

Definition (Convolution)

Let $f, g : \mathbb{R} \mapsto \mathbb{C}$

$$(f * g)(x) = \int_{\mathbb{R}} f(t) g(x-t) dt$$

Convolution is analogous to polynomial multiplication

Formula: $\underbrace{(f * g)}_{\text{(Cauchy formula)}} = \hat{f} \cdot \hat{g}$

For a function $M \mapsto C$ with $|M| < \infty$ we need the discrete Fourier transform (DFT)

1.1.9 Definition 9 (Root of unity)

Let R be a commutative ring with 1. An element $\mu \in R$ is called an n -th root of unity (= root of 1) if $\mu^n = 1$.

It is called primitive if $\mu^i \neq 1$ for $(0 < i < n)$ i.e. $\text{ord}(\mu) = n$

Let μ be a primitive n -th root of 1 (e.g. $e^{2\pi \frac{i}{n}} \in \mathbb{C}$)

Then the map $DFT_\mu : R^n \mapsto R^n$

$$(\hat{a}_0, \dots, \hat{a}_n) \mapsto (\hat{a}_0, \dots, \hat{a}_n) \quad \text{with } \hat{a}_i = \sum_{j=0}^{n-1} \mu^{ij} a_j$$

is called discrete Fourier transformation

For polynomials:

$$DFT_\mu : R[x] \mapsto R^n$$

$$f \mapsto (f(\mu^0), \dots, f(\mu^{n-1}))$$

Convolution rule: (from $f(\mu^i)g(\mu^i) = (f * g)(\mu^i)$)

$$DFT_\mu(f * g) = DFT_\mu(f) \cdot DFT_\mu(g) \quad (\text{component wise product})$$

Addition of two polynomials in $R[x]$ of $\deg(n)$ require $O(n)$ ring operations.

Multiplication require $O(n^l)$.

With Karatsuba have $O(n^{\log_2(3)})$ ring operations.

Cost $DFT_\mu(f) \cdot DFT_\mu(g) : O(n)$ ring operations (with μ as $2n$ -th root of 1)

Want: Cheap way of doing DFT and back-transformation.

1.1.10 Algorithm 10 (Fast Fourier transformation FFT)

input : $f \in R[x]$, $\mu \in R$ primitive 2^k -th root of 1, such that $\mu^{2^{k-1}} = -1$

output: $DFT_\mu(f)$

(1) Write $f(x) = g(x^2) + xh(x^2)$ with $f, g, h \in R[x]$

(2) if $(k = 1)$ $// (\Rightarrow \mu = 1)$
return $DFT_\mu(f) = (g(1) + h(1), g(1) - h(1))$

(3) Recursive call: compute $DFT_{\mu^2}(g) = \hat{g}, DFT_{\mu^2}(h) = \hat{h} \in R^{2^{k-1}}$

(4) return $DFT_\mu(f) = (\hat{f}_0, \dots, \hat{f}_{2^k-1})$ with $\hat{f}_i = \hat{g}_i + \mu \hat{h}_i$
where $\hat{g}_i = \hat{g}_{i-2^{k-1}}$ for $i \geq 2^{k-1}$

Note: Components of \hat{g} and \hat{h} are:

$$\hat{g} = g(\mu^{2^i}), \quad \hat{h}_i = h(\mu^{2^i}) \quad \text{so}$$

$$\hat{f}_i := f(\mu^i) = \hat{g}_i(\mu^{2^i}) + \mu \hat{h}_i(\mu^{2^i}) = \hat{g}_i + \mu \hat{h}_i$$

Convention: $\lg(x) = \log_2(x)$

1.1.11 Theorem 11 (Runtime of Algorithm 10)

Let $n = 2^k$, $f \in R[x]$ with $\deg(\psi) < n$

Then Algorithm 10 requires $O(n \cdot \lg(n))$ ring operations.

Better than $O(n^{1+\epsilon})$, $\forall \epsilon > 0$!

Proof:

Set $\Theta(k) = \max$ number of ring operations required. By counting obtain for $k > 1$:

$$\Theta(k) \leq 2\Theta(k-1) + \underbrace{(\text{compute } \mu^i (i \leq 2^{k-1}))}_{2^{k-1}} + \underbrace{(\mu^i \hat{k}_i)}_{2^{k-1}} + \underbrace{(\text{sums and differences})}_{2^k}$$

$$= 2\Theta(k-1) + 2^{k+1}$$

Claim: $\Theta(k) \leq (2k-1)2^k$

$$k=1 : f = a_0 + a_1 \cdot x \quad DFT_\mu(f) = (a_0 + a_1 \cdot a_0 - a_1) \Rightarrow \Theta(a) = 2$$

$$k-1 \rightarrow k : \Theta(k) \leq 2 \cdot \Theta(k-1) + 2^{k+1} \leq 2 \cdot (2k-3) \cdot 2^{k-1} + 2^{k+1} = (2k-1) \cdot 2^k$$

since $k = \lg(n)$ obtain $O(k) \leq (2 \cdot \lg(n) - 1) \cdot n \in O(n \cdot \lg(n))$ \square

Back-transformation?

1.1.12 Definition 12 (Good root of unity)

A primitive n -th root of unity is called good (caveat: this is ad-hoc terminology) if:

$$\sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \text{for } (0 < i < n)$$

Example:

- (1) $\mu = e^{2\pi \frac{i}{n}}$ is a good primitive root of unity
- (2) $R = \mathbb{Z}/(8)$, $\mu = \bar{3} \Rightarrow \mu \cdot B$ is primitive 2^{nd} root of unity
But $\bar{B}^0 + \bar{3}^1 = \bar{u} \neq \bar{0}$ so μ is not good.

1.1.13 Proposition 13 ($DFT_{\mu^{-1}}$)

Let $\mu \in R$ be a good root of 1

$$(a) = (a_0, \dots, a_{n-1}) \in R^n \Rightarrow DFT_\mu^{-1}(DFT_\mu(a)) = n \cdot (a) \quad \text{where } n = 1 + \dots + 1 \in R$$

Proof:

$$DFT_\mu(a) = (\hat{a}) = (\hat{a}_0, \dots, \hat{a}_{n-1})$$

$$\text{with } \hat{a}_j = \sum_{k=0}^{n-1} \mu^{jk} a_k$$

$$DFT_{\mu^{-1}}(\hat{a}) = (\hat{\hat{a}}_0, \dots, \hat{\hat{a}}_1)$$

$$\text{with } \hat{\hat{a}}_i = \sum_{j=0}^{n-1} \mu^{-ij} \sum_{k=0}^{n-1} \mu^{jk} a_k = \sum_{k=0}^{n-1} \left(a_k \cdot \underbrace{\sum_{j=0}^{n-1} \mu^{j(k-i)}}_{=0 \text{ if } n \neq k-i \text{ (i.e. } k=i)} \right) = a_i \cdot n \quad \square$$

1.1.14 Proposition 14 (Finding good roots of unity)

let $\mu \in R, n \in \mathbb{N}$

Assume:

- a) R is an integral Domain and μ is a primitive or n -th root of 1
(Integral Domain: nonzero commutative ring in which the product of two nonzero elements is nonzero)
 \Rightarrow Granted by FFT
- b) $n = 2^b, \mu^{\frac{n}{2}} = -1$, then $h > 0 \wedge \text{char}(R) \neq 2$
 $\rightarrow \mu$ is a good primitive n -th root of 1 ("root of unity")

Proof:

- a) for $0 < i < n$

$$\underbrace{(\mu^i - 1)}_{\neq 0} \underbrace{\left(\sum_{j=0}^{n-1} \mu^{ij}\right)}_{=0} = \mu^{in} - 1 = 0$$

$\Rightarrow \mu$ is a good root of unity

- * Let $0 < i < n$, write $i = 2^{k-s} \cdot r$ with r odd $\wedge s > 0$

$$\sum_{j=0}^{2^k-1} \mu^{ij} = \sum_{l=0}^{2^{k-s}-1} \sum_{j=0}^{2^s-1} \mu^{i(l \cdot 2^s + j)}$$

$$\mu^{i \cdot 2^s} = 1$$

$$i \cdot 2^s = 2^{k-s} \sum_{j=0}^{2^s-1} \mu^{ij} = 2^{k-s} \sum_{j=0}^{2^{s-1}-1} (\mu^{ij} + \mu^{i(2^{s-1}+j)})$$

$$\text{But } \mu^{i \cdot 2^{s-1}} = \mu^{2^{k-s} \cdot r \cdot 2^{s-1}} = \mu^{2^{k-1} \cdot r} = (-1)^r = -1$$

$$\text{So } \sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \square$$

- b) $\mu^n = 1, n = 2^k \Rightarrow \text{ord}(\mu) | n \Rightarrow \text{ord}(\mu)$ is power of 2

1.1.15 Algorithm 15 (Polynomial multiplication using DFT)

input : $f, g \in R[x]$ with $\deg(f) + \deg(g) < 2^k =: n$
 $\mu \in R$ as a good root of unity; Assume 2 $\in R$ is invertible

output: $h = f \cdot g$

- (1) compute $\hat{f} = DFT_{\mu}(f), \hat{g} = DFT_{\mu}(g)$ with $f, g \in R^n$
- (2) compute $\hat{h} = \hat{f} \cdot \hat{g}$
- (3) compute $(h_0, \dots, h_{n-1}) = DFT_{\mu^{-1}}(\hat{h})$ (same as $DFT_{\mu}(\hat{h})$ but with different order)
= Back-transformation $\cdot 2^k$
set $h = \frac{1}{2^k} \sum_{i=0}^{n-1} h_i x^i$

1.1.16 Theorem 16 (Runtime of Algorithm 15)

Algorithm 15 uses $O(n \cdot \log(n))$ ring operations for polynomials of $\deg < n$

Proof:

- Choose k minimal so that $\deg(f) \cdot \deg(g) < 2^k$
 $\Rightarrow 2^{k-1} \leq 2n \Rightarrow k \leq \log(n) + 2$
- $\underbrace{O(2k \cdot 2^k)}_{\text{Step 1}} + \underbrace{2^k}_{\text{Step 2}} + \underbrace{O(k \cdot 2^k) + 2^k}_{\text{Step 3}} \in O(2k \cdot 2^k) = O(n(g(n))) \quad \square$

Goal: Multiplication in \mathbb{N} using DFT

Idea: find roots of 1 in a suitable $\mathbb{Z}/(m)$

Choose $m = 2^l + 1, \mu = \bar{2} \in R$

1.1.17 Proposition 17 (Add and mul in $O(l)$)

Let $m = 2^l + 1, R = \mathbb{Z}/(m)$

Addition in R and multiplication by $\bar{2}^i \in R$ ($0 \leq i < 2l$) can be done in $O(l)$ bit operations

Proof:

- Let $\bar{x} \in R$ with $0 \leq x \leq 2^l$
- Addition: $x + \bar{y}$
 - (1) compute $x + y \in \mathbb{N}$: $O(l)$
 - (2) if $x + y > 2^l + 1$ subtract $2^l + 1$: $O(l)$
 - Multiplication by $\bar{2}^i$ ($0 \leq i < l$)
 - (1) Bit-shift i Bits to the left by relocating in memory:

$$\underbrace{O(\text{length}(i))}_{\text{compute addr. of new first bit}} + \underbrace{l}_{\text{copying}} = O(\log(l)) + l \in O(l)$$
 - Multiplication by $\bar{2}^i$ ($l \leq i < 2l - 1$)
 - (1) Multiplication by $\bar{2}^{i-l}$: $O(l)$
 - (2) take negative $\bar{2}^i \cdot \bar{x} = -\bar{2}^{i-l} \cdot \bar{x}$: $O(l)$

1.1.18 Proposition 18 (Sort of summary)

Let $k, r \in \mathbb{N}, r > 0, m = 2^{2^k \cdot r} + 1, R = \mathbb{Z}/(m), \mu = \bar{2}^r \in R$

$\Rightarrow 2 \in R$ is invertible, μ is a good primitive 2^{k+1} -th root of 1

$\Rightarrow \mu^{2^k} = 1$

Proof: \rightarrow from above

1.1.19 Algorithm 19 (Multiplication using FFT)

input : $x, y \in \mathbb{N}$

output: $Z = x \cdot y$

- (1) Choose $k \in \mathbb{N}$ minimal such that $l(x), l(y) \leq 2^{2^k}$
- (2) if $k \leq 3$, compute $z = x \cdot y$ by Algorithm 5
- (3) set $B = 2^{2^k}$, $m = 2^{2^k \cdot 4} + 1$, $R = \mathbb{Z}/(m)$, $\mu = \bar{2}^4 \in R$
 $(\Rightarrow \text{so } \mu \text{ is a good primitive } 2^{k+1}\text{-th root of 1})$
- (4) write $x = \sum_{i=0}^{2^k-1} x_i \cdot B^i$, same for y with $(0 \leq x_i, y_i < B)$
 possible since $x, y < 2^{2^{2^k}} = 2^{2^k \cdot 2^k} = B^{2^k}$
- (5) compute: $\hat{x} = DFT_\mu(\bar{x}_0, \dots, \bar{x}_{2^k-1}, \underbrace{0, \dots, 0}_{2^k \text{ zeros}}) \in R^{2^{k+1}}$
 same for y
 \rightarrow use FFT
- (6) compute: $\hat{z} = \hat{x} \cdot \hat{b} \in R^{2^{k+1}}$ (component wise multiplication)
 Perform multiplication in R as follows:
 Multiply representatives (non negative and $< m$) by recursive call.
 Then reduce modulo m by "negative bit shift" (see proof of Proposition 17)
- (7) compute: $(\bar{z}_0, \dots, \bar{z}_{2^{k+1}-1}) = \frac{1}{2^{k+1}} DFT_{\mu^{-1}}(\hat{z}) \in R$ with $0 \leq z < m$
- (8) set $z := \sum_{j=0}^{2^{k+1}-1} z_j \cdot B^j$

1.1.20 Theorem 20 (Runtime of Algorithm 19)

Algorithm 19 correctly computes $t = x \cdot y$ and requires $O(n \cdot (\log n)^4)$ bit operations for $l(x), l(y) \leq n$

Proof: Correctness

write $x(t) \sum_{i=0}^{2^k-i} x_i t^i \in \mathbb{Z}[t]$, $y(t)$, $\bar{x}(t) \in R[t], \bar{y}(t), \bar{z}(t)$

by Proposition 18 and Proposition 13 we have $\bar{z}(t) = \bar{x}(t) \cdot \bar{y}(t)$

The l -th coefficient of $x(t) \cdot y(t)$ is $0 \leq \sum_{i+j=l} x_i \cdot y_j < 2^k \cdot B^2 = 2^{k+2 \cdot 2^k} \leq 2^{2^{k+2}} < m$

So $z(t) = x(t) \cdot y(t) \Rightarrow z = z(B) = x(B) \cdot y(B) = x \cdot y$ Cost:

Write $\Theta(k) := \max$ number of bit operations

Analyze Steps:

- (1) compute $\max \{l(x), l(y)\} : O(l(n)) = O(k)$
- (2) $O(1)$
- (3) no bit operations
- (4) compute starting addresses of x_i, y_i in memory: $2 \cdot 2^k$ increments of the address:
 $2 \cdot 2 \cdot 2^k = 2^{k+2}$ bit ops
 $\Rightarrow O(2^k)$
- (5) By Theorem 11 need $O(2 \cdot 2^{k+1} \cdot (k+1))$ operations in R which are additions and multiplications by powers of \bar{z} costing $O(2^{k+2})$ bit operations.
Total for (5): $O(k \cdot 2^{2 \cdot k})$
- (6) 2^{k+1} multiplications of numbers $< m$, i.e. of length $\leq 2^{k+2}$.
So $k' \leq \frac{k+3}{2}$ for k' : the "new" k used in the next recursion level.
For $\alpha \in R_{>0}$ define $\Theta(\alpha) := \Theta(|\alpha|)$
Total for (6): $2^{k+1}(\Theta(\frac{k+3}{2}) + \underbrace{O(2^{k+2})}_{\text{reduction (mod } m)})$
- (7) For $DFT_{\mu^{-1}}(\hat{z}) : O(k \cdot 2^{2 \cdot k})$ as (5) Since \bar{z} is a n root of 1, multiplication by $\bar{2}^{-k-1}$ is multiplication by a positive power of $\bar{2}$, which costs $O(2^{k+2})$
Total for (7): $O(k \cdot 2^{2 \cdot k})$
- (8) For $j \leq 2^{k+1}$ have $\sum_{i=0}^{j-1} z_i \cdot B^i \leq (m-1) \sum_{i=0}^{j-1} B^i = (m-1) \frac{B^j-1}{B-1} < 2(m-1) \frac{B^j}{B} = 2^{1+2^{k+2}+(j-1)2^k}$ so the sum has length $(j+3) \cdot 2 + 1$
Adding $z_j \cdot B^j$ to this sum happens at $(j \cdot 2^k)$ -th bit and higher \Rightarrow cost is $O(2^k)$
Total for (8): $O(2^{2 \cdot k})$

Grad total: For $k \geq 4$:

$\Theta(k) \leq 2^{k+1} \cdot \Theta(\frac{k+3}{2}) + c \cdot k \cdot 2^{2 \cdot k}$ with c constant

Also for $k \in \mathbb{R}_{\geq 4}$

Define $\Lambda(k) := \frac{\Theta(k)}{2^{2 \cdot k}} \Rightarrow \Lambda(k) \leq \frac{2^{k+1} \Theta(\frac{k+3}{2})}{2^{2 \cdot k}} + c \cdot k = 16 \cdot \Lambda(\frac{k+3}{2}) + c \cdot k$

Define $\Omega(k) := \Lambda(k+3)$ So for $k \in \mathbb{R}_{>1}$

$$\Omega(k) \leq 16 \cdot \Lambda(\frac{k}{2} + 3) + c \cdot (k+3) = \underbrace{16\Omega(\frac{k}{2})}_{*} + c \cdot (k+3)$$

Claim: For $i \in \mathbb{N}$ with $2^{i-1} \leq k-3$ have:

$$\Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k+3)(1+8+\dots+8^{i-1}) + 3 \cdot c \cdot (1+16+\dots+16^{i-1})$$

Proof by induction:

$$i = 0: \Lambda(k) = \Omega(k-3)$$

$$i \rightarrow i+1: \Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k-3)(1+\dots+8^{i-1}) + 3 \cdot c \cdot (1+\dots+16^{i-1}) \leq 2^i \leq k-3 \quad *$$

$$\leq 16^i (16\Omega(\frac{k-3}{2^{i+1}})) + c(\frac{k-1}{2^i} + 3) + c(k-3)\dots = \text{claimed result}$$

Take $u \in \mathbb{N}$ minimal with $2^u > k-3 \Rightarrow \Omega(\frac{k-3}{2^u}) \leq \Omega(\lfloor \frac{k-3}{2^u} \rfloor) = \Omega(0) =: D$ (constant)

Note: u roughly is recursion depth

$$\text{Have } 2^{u-1} \leq k-3 \xRightarrow{\text{claim}} \Lambda(k) \leq 16^u \cdot D + c \cdot \underbrace{(k-3)}_{< 2^u} \cdot \frac{8^u-1}{7} + 3c \cdot \frac{16^u-1}{15} \in O(16^u)$$

$$\text{Have } 2^{u-1} \leq k-3 \Rightarrow u \leq \lg(k-3) + 1$$

$$\Rightarrow \Lambda(k) \in O(16^{\lg(k-3)}) = O((k-3)^4)$$

$$\Rightarrow \Theta(k) = 2^{2 \cdot k} \cdot \Lambda(k) \in O(2^{2k} \cdot (k-3)^4)$$

$$\text{Have } 2^{2(k-1)} < \underbrace{n}_{\max\{l(x) \cdot l(y)\}} \Rightarrow k \leq \frac{\lg(n)}{2} + 1$$

$$\text{So } \Theta(k) \in O(n \cdot (\lg(n))^4) \quad \square$$

1.1.21 Theorem 21 (Schönhage-Strassen 1971)

Multiplication of integers of length $\leq n$ can be done in $O(n \cdot \lg(n) \cdot \lg(\lg(n)))$ bit operations. Schönhage-Strassen is used for integers of length ≥ 100.000 .

Asymptotically faster: Fürer's algorithm.

Comments on Bit complexity

1. Memory requirement may explode!
 \Rightarrow No Problem as bit complexity is upper bound for memory requirements, since memory access is included in bit operations
 $(\rightarrow$ only store what is calculated)
2. Computation of addresses in memory take time
 \Rightarrow length of addresses $\approx \lg(\text{memory space})$ computations of addresses $\approx \lg(\text{memory space})^2$
3. As memory requirement gets larger access times will get longer.
 \Rightarrow transportation time for data $\geq \frac{\text{diameter of physical storage}}{2 \cdot \text{speed of light}}$

1.2 Division with remainder, Euclidean algorithm

1.2.1 Algorithm 1 (Division with remainder)

input : $b = \sum_{i=0}^{n-1} b_i 2^i$ $a = \sum_{i=0}^{n+m-1} a_i 2^i$ with $a_i, b_i \in 0, 1$, $b_{n-1} = 1$

output: $r, q \in \mathbb{N}$ such that $a = q \cdot b + r$, $0 \leq r < b$

- (1) $r = a$
 $q = 0$
- (2) for $i = m, m-1, \dots, 0$ do
- (3) if $r \geq 2^i \cdot b$
 set $r := r - 2^i \cdot b$
 $q := q + 2^i$

1.2.2 Proposition 2 (Runtime of Algorithm 1)

Algorithm 1 is correct and requires $O(n \cdot (m+1))$ bit operations.

Proof:

Always have $a = q \cdot b + r$

Claim:

before step (3), have $0 \leq 2^{i+1} \cdot b$

$i = m$; $0 \leq r = a < 2^{m+n} = 2^{m+1} \cdot 2^{n-1} \leq 2^{m-1} \cdot b$ $i < m$ By step (3)

So after last passage through the loop $0 \leq r < b$

Running Time: In step(3), have comparison and (possibly) subtraction. Only n bits involved $\Rightarrow O(n)$

Total: $O(b \cdot (m+1))$

Remarks:

- (1) Division with remainder can be reduced to multiplication.
 Precisely: given an algorithm for multiplication that requires $M(n)$ bit operations, there exists an algorithm for division with remainder that requires $O(M(n))$ bit operations.
- (2) Practically relevant:
 Jebelean's algorithm (1997): $O(n^{\lg 3})$
- (3) Alternatively, may choose $r \in \mathbb{Z}$ such that $\lfloor \frac{-b}{2} \rfloor < r \leq \lfloor \frac{b}{2} \rfloor$
- (4) Algorithm 1 extends to \mathbb{Z} .
- (5) All Euclidean rings have division with remainder (by definition).
 (e.g., $R = K[x] \rightarrow$ polynomial ring over field,
 $R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \subseteq \mathbb{C}, \quad i^2 = -1$)

1.2.3 Algorithm 3 (Euclidean algorithm)

input : $a, b \in \mathbb{N}$

output: $\gcd(a, b)$ "greatest common divisor"

- (1) set $r_0 := a$
 $r_1 := b$
- (2) for $i = 1, 2, 3, \dots$ perform steps (3) and (4)
- (3) if $r_i = 0$
 $\gcd(a, b) = |r_{i-1}|$
- (4) Division with remainder: $r_{i-1} = q \cdot r_i + r_{i+1}$ $r_{i+1} \in \mathbb{Z}$
 $(|r_{i+1}| \leq \frac{1}{2}|r_i|)$

Example:

$$a = 287, \quad b = 126$$

$$287 = 2 \cdot 126 + 35 \tag{1}$$

$$126 = 4 \cdot 35 - 14 \tag{2}$$

$$35 = (-2) \cdot (-14) + 7 \tag{3}$$

$$-14 = (-2) \cdot 7 + 0 \tag{4}$$

$$\begin{aligned} \text{So: } 7|(-14) &\xRightarrow{(3)} 7|35 \\ &\xRightarrow{(2)} 7|126 \\ &\xRightarrow{(1)} 7|287 \end{aligned}$$

On the other hand take a common divisor d ; $d|287$; $d|126$
 $\xRightarrow{(1)} d|d \xRightarrow{(2)} d|14 \xRightarrow{(3)} d|7$

1.2.4 Theorem 4 (Correctness of Algorithm 3)

Algorithm 3 is correct.

Proof:

Since $r_{i-1} = q \cdot r_i + r_{i+1}$ every integer $x \in \mathbb{Z}$ satisfies the equivalence $x|r_{i-1}$ and $x|r_i \Leftrightarrow x|r_{i+1}$ and $x|r_i$ so $\gcd(r_{i-1}, r_i) = \gcd(r_i, r_{i+1}) = \gcd(a, b)$
when terminating have $\gcd(a, b) = \gcd(r_{i-1}, 0) = |r_{i-1}|$ \square

1.2.5 Theorem 5 (Runtime of Algorithm 3)

Algorithm 3 requires $O(m \cdot n)$ bit operations for $n = l(a), m = l(b)$

Proof:

If $a < b$ then the first passage yields $r_2 = a, r_1 = b$. Cost: $O(n)$

May assume: $a \geq b$. Write $n_i = l(r_i)$

By Proposition 2 $\exists c$ constant such that the total time is $\leq c \cdot \underbrace{\sum_{i=1}^k n_i \cdot (n_{i-1} - n_i + 1)}_{=: \sigma(n_0, \dots, n_k)}$

For $i > 2$: $n_i = n_{i-1} - 1$

Special Case: $n_i = n_{i-1} - 1$ for $i \geq 2$

$\Rightarrow n_i = n_i - i + 1, n_i = m, k = m + 1$

Obtain $\sigma(n_0, \dots, n_k) = m \cdot (n - m + 1) + \sum_{i=2}^{m+1} (m - i + 1) \cdot 2 = m \cdot n - m^2 + m + m(m - 1) = m \cdot n$.

Claim: The special case is the worst (most expensive)!

From any sequence $n_1 > n_2 > \dots > n_k$ get to the special case by iteratively inserting numbers in the gaps. Insert s with $n_{j-1} > s > n_j$.

$\sigma(n_0, \dots, n_{j-1}, s, n_j, \dots, n_k) - \sigma(n_0, \dots, n_k) = \dots = s + (n_{j-1} - s) \cdot (s - n_j)$

$sp\sigma(n_0, \dots, n_k) \leq \sigma(n, m, m - 1, \dots, 2, 1, 0) = n \cdot m \quad \square$

Complexity is quadratic \rightarrow cheap

1.2.6 Algorithm 6 (Extended Euclidean Algorithm)

input : $a, b \in \mathbb{N}$

output: $d = \gcd(a, b)$ and $s, t \in \mathbb{Z}$ such that $d = s \cdot a + t \cdot b$

(1) $r_0 := a, r_1 := b, s_0 := 1, t_0 := 0, s_1 := 0, t_1 = 1$

(2) for $i = 1, 2, \dots$ perform steps (3) - (5)

(3) if $r_i = 0$
 set $d = |r_{i-1}|$
 $s := \text{sgn}(r_{i-1}) \cdot s_{i-1},$
 $t := \text{sgn}(r_{i-1}) \cdot t_{i-1}$

(4) division with remainder:
 $r_{i+1} = r_{i-1} - q_i \cdot r_i, \quad \text{with } |r_{i+1}| \leq \frac{1}{2}|r_i|$

(5) set $s_{i+1} := s_{i-1} - q_i \cdot s_i,$
 $t_{i+1} := t_{i-1} - q_i \cdot t_i$

Justification : $r_i = s_i \cdot a + t_i \cdot b$ throughout

Application: $m, x \in \mathbb{N}$ such that m, x co-prime (i.e. $\gcd(x, m) = 1$)

Algorithm 6 yields: $1 = s \cdot x + t \cdot m \Rightarrow s \cdot x \equiv 1 \pmod{m}$

So obtain inverse of $\bar{x} \in \mathbb{Z}/(m)$

1.3 Primality testing

Let $\mathbb{P} \subseteq \mathbb{N}$ be the set of prime numbers.

Challenge: Given $n \in \mathbb{N}$ decide if $n \in \mathbb{P}$

Naive Method: Trivial division by $m \leq \lfloor \sqrt{n} \rfloor$.

Running time is exponential in $l(n)$. Even when restricted to division by prime numbers, need approximately $\frac{\sqrt{n}}{|n|^{1/\sqrt{n}}}$ trivial divisions (prime number theorem)
 \rightarrow hardly any better!

Reminder: (arithmetic modulo m)

G finite group $\Rightarrow \forall a \in G \quad a^{|G|} = 1$ Fermat's little theorem

For $G = (\mathbb{Z}/(p))^\times \quad a^{p-1} \equiv 1 \pmod{p} \in \mathbb{P} \quad \forall a \in \mathbb{Z} \quad \text{with } p \nmid a$

In fact $(\mathbb{Z}/(p))^\times \cong Z_{p-1}$ is cyclic

For $m = p_1^{e_1} \dots p_r^{e_r}$ with $p_i \in \mathbb{P}, e_i \in \mathbb{N}_{>0}$:

$\mathbb{Z}_{(m)} \cong \mathbb{Z}_{(p_1^{e_1})} \oplus \dots \oplus \mathbb{Z}_{(p_r^{e_r})} \Rightarrow \mathbb{Z}_{(m)}^\times \cong \mathbb{Z}_{(p_1^{e_1})}^\times \times \dots \times \mathbb{Z}_{(p_r^{e_r})}^\times$

what is $\mathbb{Z}_{(p^e)}$ for $p \in \mathbb{P}, e \in \mathbb{N}_{>0}$?

1.3.1 Theorem 1 (Cyclic group)

Let $p \in \mathbb{P}$ odd $e \in \mathbb{N}_{>0} \Rightarrow (\mathbb{Z}_{(p^e)})^\times = Z_{(p-1) \cdot p^{e-1}}$ cyclic

Proof:

$(\mathbb{Z}_{(p^e)})^\times \cong Z_{p-1} \Rightarrow \exists z \in \mathbb{Z} : \text{order}(z + p\mathbb{Z}) = p-1$

Set $a = \bar{z}^{p^{e-1}} \in (\mathbb{Z}_{(p^e)})^\times =: G$

$$a^{p-1} = \bar{z}^{(p-1) \cdot p^{e-1}} = \bar{z}^{|a|} = 1$$

On the other hand, take $i \in \mathbb{Z}$ such that

$$a^i = 1 \Rightarrow \bar{z}^{i \cdot p^{e-1}} \equiv 1 \pmod{p} \Rightarrow (p-1) \mid (i \cdot p^{e-1}) \Rightarrow (p-1) \mid i.$$

So $\text{ord}(a) = p-1$.

Now consider $b = (p+1) \in G$

Claim: $\text{ord}(b) = p^{e-1}$

Proof by induction on $k \in \mathbb{N}_{>0}$ that $(p+1)^{p^{k-1}} \equiv p^k + 1 \pmod{p^{k+1}}$

$k=1$ ✓

$k \rightarrow k+1$: By induction have $(p+1)^{p^{k-1}} = 1 + p^k + x \cdot p^{k+1}, \quad x \in \mathbb{Z}$

$$\text{Compute: } (p+1)^{p^k} = ((1+p^k) + x \cdot p^{k+1})^p = \sum_{i=0}^p \binom{p}{i} (i+p^k)^{p-i} \cdot x^i \cdot p^{i \cdot (k+1)}$$

$$\stackrel{\text{Only 0-th summand}}{\equiv} (i+p^k) = \sum_{i=0}^p \binom{p}{i} p^{i \cdot k} \stackrel{p \text{ odd}}{\equiv} 1 + p^{k+1} \pmod{p^{k+2}} \quad \checkmark$$

For $k=e$: $(p+1)^{p^{e-1}} \equiv 1 \pmod{p^e} \Rightarrow b^{p^e} = 1 \Rightarrow \text{ord}(b) \mid p^{e-1}$

But $(p+1)^{p^{e-2}} \equiv p^{e-1} + 1 \pmod{p^e} \Rightarrow b^{p^{e-2}} \neq 1 \in G$

So $\text{ord}(b) = p^{e-1}$

Claim: $\text{ord}(a \cdot b) = (p-1)p^{e-1} \quad (\Rightarrow \text{Theorem})$

Let $(a \cdot b)^i = 1 \in G$ with $i \in \mathbb{Z}$

$$\text{Then } 1 = (a \cdot b)^{i \cdot (p-1)} = (a^{p-1})^i \cdot b^{i \cdot (p-1)} = b^{i \cdot (p-1)} \Rightarrow p^{e-1} \mid i \cdot i(p-1) \Rightarrow p^{e-1} \mid i$$

$$\text{Also } 1 = (a \cdot b)^{p^{e-1} \cdot i} = a^{p^{e-1} \cdot i} \Rightarrow (p-1) \mid p^{e-1} \cdot i \Rightarrow (p-1) \mid i \rightarrow (p-1) \cdot p^{e-1} \mid i \quad \square$$

Reminder: $(\mathbb{Z}/(2^e))^\times \cong Z_2 \times Z_2^{e-2} \quad (e \geq 2)$

1.3.2 Algorithm 2 (Fermat Test)

input : $n \in \mathbb{N}_{>0 \text{ odd}}$

output: " $n \notin \mathbb{P}$ " or "probably $n \in \mathbb{P}$ "

- (1) Choose $a \in 2, \dots, n-1$ randomly
- (2) Compute $a^{n-1} \bmod n$
- (3) If $a^{n-1} \not\equiv 1 \pmod{n}$ return " $n \notin \mathbb{P}$ "
else return "probably $n \in \mathbb{P}$ "

Not very satisfying. Is this fast?

1.3.3 Algorithm 3 (Fast exponentiation)

input : $a \in G$ G is a monoid, $e \in \mathbb{N}$ $e = \sum_{i=0}^{n-1} e_i 2^i$ $e_i \in \{0, 1\}$

output: $a^e \in G$

- (1) Set $b := a, y := 1$
- (2) For $i = 0, \dots, n-1$ perform (3) - (4)
- (3) if $e_i = 1$ set $y := y \cdot b$
- (4) set $b := b^2$
- (5) return y

this requires $O(l(e))$ operations in G

For $G = (\mathbb{Z}/(n)_i)$, each multiplication requires $O(l(n)^2)$ bit operations

\Rightarrow Fermat test requires $O(l(n)^3)$ bit operations \rightarrow cubic complexity \rightarrow "fast"!

Example:

$n = 561 = 3 \cdot 11 \cdot 17$ For $a \in \mathbb{Z}$ with $\gcd(a, n) = 1 \Rightarrow$ have $a^{n-1} = (a^2)^{280} \equiv 1 \pmod{3}$

$a^{n-1} \equiv 1 \pmod{n}$ Fermat's test says "probably $n \in \mathbb{P}$ " in 57% of cases.

$n = 2207 \cdot 6619 \cdot 15443$: output "probably $n \in \mathbb{P}$ " in 99,93% of cases.

1.3.4 Definition 4 (Pseudo-prime, witness, Carmichael numbers)

Let $n \in N_{>1} \text{ odd}$, $a \in 1, \dots, n-1$

- (a) n is pseudo-prime to base a if $a^{n-1} \equiv 1 \pmod{n}$
- (b) otherwise a is called a witness of composition of n
- (c) If $n \notin \mathbb{P}$ but $a^{n-1} \equiv 1 \pmod{n} \quad \forall a$ with $\gcd(n, a) = 1$
then n is called a Carmichael number.
There are ∞ Carmichael numbers

1.3.5 Proposition 5 (Number of witnesses)

Let $n \in N_{>1}$, $\text{odd} \wedge \notin \mathbb{P} \wedge \text{not Carmichael}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n, a \text{ is witness of composite of } n\}| > \frac{n-1}{2}$

Proof: Consider

$\phi : (\mathbb{Z}/(n))^{\times} =: G \rightarrow G, \quad \bar{a} \mapsto \bar{a}^{n-1}$

group homomorphism. By assumption,

$|\text{Im}(\phi)| > 1 \Rightarrow |\text{Ker}(\phi)| \leq \frac{|G|}{2} < \frac{n-1}{2}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n \text{ a witness of composite of } n\}| > \frac{n-1}{2} \quad \square$

Miller-Rabin Test

1.3.6 Proposition 6 (Inference from Fermat)

Let $p \in \mathbb{P} \text{ odd}$, $a \in \{1, \dots, (p-1)\}$ write $p-1 = 2^k \cdot m$ with $m \text{ odd}$ Then:
 $a^m \equiv 1 \pmod{p}$ or $\exists i \in \{0, \dots, k-1\} : a^{2^i \cdot m} \equiv -1 \pmod{p}$

Proof:

Little Fermat: $\bar{a}^{2^k \cdot m} = 1 \in \mathbb{F}_p$

Assume $\bar{a}^m \neq 1$ take i maximal such that:

$\bar{b} = \bar{a}^{2^i \cdot m} \neq 1 \Rightarrow \bar{b}^2 = 1 \Rightarrow \bar{b} \in \mathbb{F}_p$ is a zero of $x^2 - 1 \in \mathbb{F}_p[x] \Rightarrow \bar{b} = -1$

1.3.7 Algorithm 7 (Miller-Rabin-test)

input : $n \in \mathbb{N}_{>1}, \text{odd}$

output: either " $n \notin \mathbb{P}$ " or "probably $n \in \mathbb{P}$ " \rightarrow Monte Carlo Algorithm.

- (1) write $n - 1 = 2^k \cdot m$ with m odd
- (2) Choose $a \in \{2, \dots, n - 1\}$ randomly
- (3) Compute $b := a^m \pmod n$
- (4) if $(b \equiv \pm 1 \pmod n)$
return "probably $n \in \mathbb{P}$ "
- (5) for $(i = 0, \dots, k - 1)$ do steps (6) - (7)
- (6) set $b := b^2 \pmod n$
- (7) if $(b \equiv -1 \pmod n)$
return "probably $n \in \mathbb{P}$ "
- (8) return $n \notin \mathbb{P}$

1.3.8 Definition 8 (strong pseudo-prime / witness)

Let $n \in \mathbb{N}_{>1}, \text{odd}$ $a \in \{1, \dots, n - 1\}$

- (a) n is called a strongly pseudo-prime to base a if Proposition 6 holds for a and p replaced by n .
- (b) Otherwise a is called a strong witness of composition of n .

Example

Let $n \in \mathbb{N}_{>1}, \mathbb{P} \text{ odd}$

$a = 2$ strong witness if $n < 2047$ (including 561)

2 or 3 strong witness if $n < 1373653$

2,3 or 5 strong witness if $n < 25326001$

1.3.9 Theorem 9 (Bit-complexity of Algorithm 7)

- (a) Algorithm 7 requires $O(l(n)^3)$ bit operations. \rightarrow "qubic complexity" \rightarrow fast!
- (b) if $b \in \mathbb{P}$ then Algorithm 7 returns "probably $b \in \mathbb{P}$ " \rightarrow no false positives.
- (c) if $n \notin \mathbb{P}$ then more than half of the numbers in $\{1, \dots, n - 1\}$ are strong witnesses.

Proof:

- (a) Step 1 takes $O(l(n))$ bit operations:
Using Algorithm 3, we need $O(l(n-1))$ multiplications in $\mathbb{Z}/(n)$ each requiring $O(l(n)^2)$ bit operations.
- (b) Proposition 6
- (c) split in three cases:

Case 1: n is not a Carmichael number. $\xRightarrow{\text{Prop 5}}$ more than half of all numbers are.

Fermat witness thus also strong witness.

Case 2: $n = p^r \cdot l$ with $p \in \mathbb{P}$ $r > 1$ $l \in \mathbb{N}_{>0}$ $p \nmid l$

Theorem 1 $\exists x \in \mathbb{Z}$ such that $x^p \equiv 1 \pmod{p^r}$ $x \not\equiv 1 \pmod{p^r}$

Chinese remainder theorem: $\exists a \in \mathbb{Z}$ such that $a \equiv x \pmod{p^r}$ $a \equiv 1 \pmod{l}$

So $\bar{a}^p = 1 \in \mathbb{Z}/(n) \Rightarrow \bar{a}^n = 1 \Rightarrow \bar{a} \in (\mathbb{Z}/(n))^\times$

i.e. $\gcd(n, a) = 1$ if $\bar{a}^{n-1} = 1$ then $\bar{a} = 1$

But $a \equiv x \not\equiv 1 \pmod{p^r}$ so $\bar{a}^{n-1} \neq 1$ hence n is not Carmichael \rightarrow Case 1.

Case 3: n is a Carmichael number. By Case 2 have $n = p \cdot l$ with $p \in \mathbb{P}$ $p \nmid l$ $l \geq 3$
 n Carmichael: $\forall a \in \mathbb{Z}$ with $\gcd(a, n) = 1$

have $a^{2^k \cdot m} \equiv 1 \pmod{n}$ (where $n-1 = 2^k \cdot m$)

$a^{2^k \cdot m} \equiv 1 \pmod{p}$ Take j minimal such that

$a^{2^j \cdot m} \equiv 1 \pmod{p}$ $\forall a \in \mathbb{Z}$ such that $\gcd(a, n) = 1$

so $0 \leq j \leq l$ in fact, $j > 0$ since $(-1)^{2^0 \cdot m} = -1$ with m odd.

Consider the subgroup $H := \{\bar{a} \in \mathbb{Z}/(n) \mid \bar{a}^{2^{j-1} \cdot m} \in \{1, -1\} \subseteq (\mathbb{Z}/(n))^\times\}$

Let $a \in \{1, \dots, n-1\}$ $\gcd(n, a) = 1$ a not a strong witness.

Claim 1: $\bar{a} \in H$

Case 3.1: $\bar{a}^{2^{j-1} \cdot m} = 1 \Rightarrow \bar{a} \in H$

Case 3.2: $a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{n}$ $a^m \not\equiv 1 \pmod{n}$

$\xRightarrow{\text{a nonwitness}} \exists i$ such that $\underbrace{a^{2^i \cdot m} \equiv -1 \pmod{n}}_*$

$\Rightarrow a^{2^i \cdot m} \equiv -1 \pmod{p} \xRightarrow{\text{def of } j} i < j$

if $i < j-1$ then $a^{2^{j-1} \cdot m} = (a^{2^i \cdot m})^{2^{j-1-i}} \equiv (-1)^{2^{j-1-i}} = 1 \pmod{n}$

$\xRightarrow{\text{with } *} \text{not in case 3.2}$

Claim 2: $H \subseteq (\mathbb{Z}/(n))^\times$ proper subgroup.

By definition of j $\exists x \in \mathbb{Z}$ such that $x^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p}$

Chinese remainder: $\exists a \in \mathbb{Z}$ such that

$a \equiv x \pmod{p}$ $a \equiv 1 \pmod{l}$

$\Rightarrow a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p} \equiv 1 \pmod{l} \Rightarrow \bar{a} \notin H$

Claim 2 \checkmark

It follows that $|H| \leq \frac{|(\mathbb{Z}/(n))^\times|}{2} < \frac{n-1}{2}$

so the number of witnesses is $\geq n-1-|H| > \frac{n-1}{2}$ \square

Remarks:

- (a) A more careful analysis shows that $2\frac{3}{4}$ of all candidates are strong witnesses
- (b) Calling Algorithm 7 repeatedly decreases the probability of false positives. Running time for prescribed error probability p is $O(\lg(p^{-1} \cdot l(n)^3))$
(Independence assumptions!)

Connection with Riemann hypothesis

Let $n \in \mathbb{N}_{>0}$ $\bar{X} : (\mathbb{Z}/(n))^\times \rightarrow \mathbb{C}^\times$ group homomorphism

$$X : \mathbb{Z} \rightarrow \mathbb{C}, a \mapsto \begin{cases} \bar{X}(\bar{a}) & \text{if } \gcd(a, n) = 1 \text{ for } (\bar{a} = a + n\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$$

"residue class character (mod n)

$$Ex : n = 1 \Rightarrow X(a) = 1 \forall a \in \mathbb{Z}$$

Dirichlet L-series:

$$L_X(s) = \sum_{n=1}^{\infty} \frac{X(n)}{n^s} \text{ converges for } s \in \mathbb{C} \text{ until } Re(s) > 1$$

$L_X(s)$ extends to a meromorphic function on $\mathbb{C} \mapsto$ "Dirichlet L-function".

For $n = 1 : L_X(s) = \zeta(s)$ Riemann Zeta-function.

Euler Product:

$$\text{From } (1 - X(p) \cdot p^{-s})^{-1} = \sum_{i=0}^{\infty} (X(p) \cdot p^{-s})^i = \sum_{i=0}^{\infty} \frac{X(p^i)}{p^{is}} \quad \text{derive } L_X(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - X(p) \cdot p^{-s}}$$

Generalized Riemann hypothesis (GRH):

For X residue class character, $s \in \mathbb{C}$

with $L_X(s) = 0$, $0 < Re(s) < 1$ ("critical strip")

then $Re(s) = \frac{1}{2}$

For $X = 1 \rightarrow$ ordinary Riemann hypothesis.

1.3.10 Theorem (Ankeny & Bach)

GRH $\Rightarrow \forall X \neq 1$ residue class character

$$\exists p \in \mathbb{P} : X(p) \neq 1, p < 2 \ln(n)^2$$

Let $H \subsetneq (\mathbb{Z}/(n))^\times =: G$ proper subgroup.

Choose $N \subsetneq G$ maximal proper subgroup such that $H \subseteq N \Rightarrow G/N$ cyclic.

$$\bar{X} : G \mapsto \mathbb{C}^\times \text{ with } N = \text{Ker}(\bar{X}) \Rightarrow H \subseteq \text{Ker}(\bar{X})$$

$$\xrightarrow{\text{GRH, Thm1}} \exists p \in \mathbb{P} : p + n\mathbb{Z} \not\subseteq H, p < 2 \cdot \ln(n)^2$$

Corollary: Assume GRH.

Let $n \in \mathbb{N}_{>1}$ \mathbb{P} odd Then there is a strong witness a of compositeness of n with $a < 2 \cdot \ln(n)^2$.

\rightarrow Obtain deterministic primality test with time $O(\ln(n)^5)$ bit operations.

AKS-test

A deterministic polynomial time primality test \rightarrow "holy grail"

Agrawal, Kayal, Saxena: PRIMES is in P, Annals of Mathematics, 2004.

1.3.11 Proposition 10 (Modulo over ideals)

Let $n \in \mathbb{P}$ $a \in \mathbb{Z} \Rightarrow (x + a)^n \equiv x^n + a \pmod{n}$

where x is a indeterminate and for $r \in \mathbb{N}$:

$$(x + a)^n \equiv (x^n + a) \pmod{n, x^r - 1} \quad (1)$$

(i.e. $(x + a)^n - (x^n + a) = n \cdot f + (x^r - 1) \cdot g$ with $f, g \in \mathbb{Z}[x]$)

Proof:

$$(x + a)^n = \sum_{i=0}^n \binom{n}{i} \cdot a^{n-i} \cdot x^i \quad (\text{where } \binom{n}{i} \text{ is a multiple of } n \text{ for } 0 < i < n)$$

$$\equiv x^n + a^n \quad (\leftarrow \text{little Fermat})$$

$$\equiv x^n + a \quad (1) \text{ follows by weakening this.}$$

Cost analysis for checking (1) with $l = \text{length}(n)$.

Using Algorithm 3, need $O(l)$ multiplications in $\mathbb{Z}[x]/(n, x^r - 1) =: R$

Elements of R are represented as polynomials of degree $< r$,
coefficients between 0 and n .

Multiply polynomials: $O(r^2)$ operation in $\mathbb{Z}/(n) : O(r^2 \cdot l^2)$

since $x^{r+k} \equiv x^k \pmod{x^r - 1}$,

add coefficients of x^{r+k} of product polynomial to coefficients $x^k : O(r \cdot l)$

Total for checking (1): $O(r^2 \cdot l^3)$ bit operations.

Reduction $\pmod{x^r - 1}$ is just for keeping the cost under control.

The following is part of AKS-test:

1.3.12 Algorithm 11 (Test for perfect power)

input : $n \in \mathbb{N}_{>1}$

output: $m, e \in \mathbb{N}$ $e > 1$ such that $n = m^e$ or "n is not a perfect power"

(1) for $(e = 2, \dots, \lfloor \lg(n) \rfloor)$ perform (2) - (7) //possible exponents

(2) set $m_1 = 2, m_2 = n$ //initialize interval $[m_1, m_2]$ for searching $\sqrt[e]{n}$

(3) while($m_1 \leq m_2$) do (4) - (7)

(4) set $m = \lfloor \frac{m_1 + m_2}{2} \rfloor$ // bisect interval

(5) if $m^e = n$ return m, e

(6) if $m^e > n$ set $m_2 = m - 1$

(7) if $m^e < n$ set $m_1 = m + 1$

(8) return "not a perfect power"

Cost: (for $l = \text{length}(n)$)

Compute $m^e : O(\lg(l) \cdot l^2)$ (abort computation once the result exceeds n)

Number of passages through inner loops $\leq \lg(n)$

Number of passages through outer loops $\leq \lg(n)$

Total cost of Algorithm 11: $O(l^4 \cdot \lg(l))$

1.3.13 Algorithm 12 (AKS-test)

input : $n \in \mathbb{N}_{>1}$ of length $l = \text{length}(n) = \lfloor \lg(n) \rfloor + 1$

output: " $n \in \mathbb{P}$ " or " $n \notin \mathbb{P}$ "

- (1) if (n is a perfect power)
return " $n \notin \mathbb{P}$ "
- (2) find $r \in \mathbb{N}_{>1}$ minimal such that $r|n \vee n^i \not\equiv 1 \pmod{r} \quad \forall i = 1, \dots, l^2$
//exhaustive search (we will show that $r \leq l^5$)
- (3) if $r|n$
if ($r = n$) return " $n \in \mathbb{P}$ "
if ($r < n$) return " $n \notin \mathbb{P}$ "
- (4) for $a = 1, 2, \dots, \lfloor \sqrt{r} \cdot l \rfloor$ do (5)
- (5) if $((x+a)^n \not\equiv (x^n + a) \pmod{(n, x^r - 1)})$
return " $n \notin \mathbb{P}$ "
- (6) return " $n \in \mathbb{P}$ "

1.3.14 Lemma 13 (Least common multiple)

For $n \in \mathbb{N}_{>0}$ have $\lambda(n) := \text{lcm}(1, 2, \dots, n) \geq 2^{n-2}$

Proof: For $f = \sum_{i=0}^m a \cdot x^i \in \mathbb{Z}(x) \quad a_i \in \mathbb{Z}$

$$\Rightarrow \int_0^1 f(x) dx = \sum_{i=0}^m \frac{a_i}{i+1} = \frac{k}{\lambda(m+1)}$$

with $k \in \mathbb{Z}$. Consider $f_m = x^m \cdot (1-x)^m$

For $0 < x < 1$:

$$0 < f_m(x) \leq 4^{-m}$$

$$\Rightarrow 0 < \int_0^1 \underbrace{f_m(x)}_{\frac{k_m}{\lambda(2m+1)}} dx \leq 4^{-1}$$

$$\lambda(2 \cdot m + 1) \geq k_m \cdot 4^m \geq 4^m$$

$$\text{For } n \in \mathbb{N}_{>0} \lambda(n) \geq \lambda(2 \cdot \lfloor \frac{n-1}{2} \rfloor + 1) \geq 4^{\lfloor \frac{n-1}{2} \rfloor} \geq 4^{\frac{n-1}{2}} = 2^{n-2} \quad \square$$

Corollary: (not related to AKS)

For $n \in \mathbb{N}$

$$\pi(n) := |\{p \in \mathbb{P} | p \leq n\}| \geq \frac{n-2}{\lg(n)}$$

Proof:

$$2^{n-2} \leq \lambda(n) = \prod_{p \in \mathbb{P}, p \leq n} p^{\lfloor \log_p(n) \rfloor} \leq \prod_{p \leq n} p^{\log_p(n)} = n^{\pi(n)} = 2^{\lg(n)\pi(n)} \quad \square$$

Prime number theorem:

$$\lim_{n \rightarrow \infty} \frac{\pi(n)}{n/\ln(n)} = 1$$

Interpretation:

The average distance of two primes around some value $x \in \mathbb{R}_{>1}$ is $\ln(x)$

1.3.15 Lemma 14 (Property of r in Algorithm 12)

In Algorithm 12, have $r \leq l^5$

Proof:

if $r < l^5 \Rightarrow \forall k \in \{2, \dots, l^5\} : \exists i \in \{1, \dots, l^2\}$

$n^i \equiv 1 \pmod{k}$

$$\Rightarrow k \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\Rightarrow \lambda(l^5) \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\xrightarrow[\text{Lemma 13}]{=} 2^{l^5-2} < \prod_{i=1}^{l^2} n^i = n^{\frac{l^2(l^2+1)}{2}}$$

$$\Rightarrow l^5 - l^3 < 4 \quad \text{not true since } l \geq 2 \quad \square$$

1.3.16 Theorem 15 (Bit-Complexity of Algorithm 12)

Algorithm 12 requires $O(l^{16.5})$ bit operations ("polynomial complexity")

Proof:

Step(1): $O(l^4 \cdot \lg(l)) \checkmark$

Step(2): For each r need:

- test $r \mid n : O(l^2)$
- compute all $n^i \pmod{r} : O(l^2 \cdot \lg(r)^2) \xrightarrow[\text{Lemma 14}]{} O(l^2 \cdot \lg(l)^2)$

Step(3): $O(1)$

$$\text{Step(4): } O(\sqrt{r} \cdot l \cdot r^2 \cdot l^3) \xrightarrow[\text{Lemma 14}]{} O(l^{16.5}) \quad \square$$

Reminder: There is a variant of Algorithm 12 with running time $\tilde{O}(l^6)$, i.e., $O(l^6 \cdot \lg(l)^m)$ with $m \in \mathbb{N}$.

Correctness:

For $r \in \mathbb{N}_{>0}$ and $p \in \mathbb{P}$ write $I(r, p) := \{m, f\} \in \mathbb{N} \times \mathbb{F}_p[x] \mid f(x)^m \equiv f(x^m) \pmod{x^r - 1}\}$
 "m is introspective for f and r ".

Example: Proposition 10 says that:

$$(p, x + \bar{a}) \in I(r, p) \text{ for } a \in \mathbb{Z} \quad r \in \mathbb{N}_{>0} \quad p \in \mathbb{P} \quad (1)$$

1.3.17 Lemma 16 (Rules for ideals)

- (a) $(m, f), (m', f) \in I(r, p) \Rightarrow (m \cdot m', f) \in I(r, p)$
- (b) $(m, f), (m, g) \in I(r, p) \Rightarrow (m, f \cdot g) \in I(r, p)$
- (c) $(m \cdot p, f) \in I(r, p), p \nmid r \Rightarrow (m, f) \in I(r, p)$

Proof:

- (a) $f(x)^{m \cdot m'} \equiv f(x^m)^{m'} \pmod{(x^r - 1)}$
 $f(x^m)^{m'} \equiv f(x^{m \cdot r}) \pmod{(x^{m \cdot r} - 1)}$
 But $(x^r - 1) \mid (x^{m \cdot r} - 1)$
- (b) $(f \cdot g)(x)^m = f(x)^m \cdot g(x)^m \equiv f(x^m) \cdot g(x^m) = (f \cdot g)(x^m) \pmod{(x^r - 1)}$
- (c) $(f(x)^m)^p \equiv f((x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x^m))^p \pmod{(x^r - 1)}$
 $\Rightarrow (x^r - 1) \mid ((f(x)^m)^p - f(x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x)^m - f(x^m))^p$
 $p \nmid r \Rightarrow x^r - 1$ is square free. So
 $(x^r - 1) \mid (f(x)^m - f(x^m)) \Rightarrow (m, f) \in I(r, p) \quad \square$

1.3.18 Theorem 17 (Correctness of Algorithm 12)

Algorithm 12 is correct.

Proof:

If the algorithm terminates in step(1),(3) or (5), it is correct. To show: If it terminates in step(6) it is correct, i. e. $n \in \mathbb{P}$

Claim 1: $\exists p \in \mathbb{P} : p \mid n \quad p \not\equiv 1 \pmod{r} \quad p > r$

Indeed if all prime divisors of n were $\equiv 1 \pmod{r}$ then $n \equiv 1 \pmod{r}$

Contradiction to step(2). All prime divisors of n are $> r$ by step (2) and (3) ✓

Steps(2) and (3) imply that $\gcd(n, r) = 1 \Rightarrow G := \langle \bar{n}, \underbrace{\bar{p}}_{p \bmod r} \rangle \subseteq (\mathbb{Z}/(r))^\times$

Step(2): $\text{ord}(\bar{n}) > l^2 \Rightarrow l^2 < |G| < r$ (2)

Set $s := \text{ord}(\bar{p} \in G) \Rightarrow r \mid (p^s - 1)$ with $q := p^s \Rightarrow r \mid |\mathbb{F}_q^\times| \Rightarrow \exists \zeta \in \mathbb{F}_q$ r -th root of 1

Set $k := \lfloor \sqrt{r} \cdot l \rfloor \quad m := \left(\frac{n}{p}\right)$

By (1) $(p, x + \bar{a}) \in I(r, p)$ with $\bar{a} \in \mathbb{F}_p$

By step(4), have $(n, x + \bar{a}) \in I(r, p)$

For $\underline{e} = e_0, \dots, e_k \in \mathbb{N}_0$ set $f_{\underline{e}} := \prod_{a=0}^k (x + \bar{a})^{e_a}$

Lemma 16 (b): $(p, f_{\underline{e}}) \in I(r, p)$

$(n, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(c)}} (m, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(a)}} \forall s, t \in \mathbb{N}_0 : (p^s \cdot m^t, f_{\underline{e}}) \in I(r, p)$

$\Rightarrow f_{\underline{e}}(\zeta^{p^s \cdot m^t}) = f_{\underline{e}}(\zeta)^{p^s \cdot m^t}$ (3)

Set $H := \langle \zeta + \bar{a} | a \in \{0, \dots, k\} \rangle \subseteq \mathbb{F}_q^\times$
 $(\zeta \notin \mathbb{F}_p \text{ since } r \nmid (p-1) \text{ by Claim 1})$

Consider: $T := \{(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1} \mid \sum_{a=0}^k e_a < |G|\}$

$\Phi : T \mapsto H, (e_0, \dots, e_k) \mapsto f_{\underline{e}}(\zeta) = \prod_a (\zeta + \bar{a})^{e_a} \in H$

Claim 2: Φ is injective.

Indeed, take $(\underline{e}), (\underline{\hat{e}}) \in T$ such that $\Phi(\underline{e}) = \Phi(\underline{\hat{e}})$

$$\Rightarrow \forall s, t \in \mathbb{N}_0 : f_{\underline{e}}(\zeta^{p^s \cdot m^t}) \stackrel{(3)}{=} f_{\underline{e}}(\zeta)^{p^s \cdot m^t} = f_{\underline{\hat{e}}}(\zeta)^{p^s \cdot m^t} \stackrel{(3)}{=} f_{\underline{\hat{e}}}(\zeta^{p^s \cdot m^t})$$

$f_{\underline{e}} - f_{\underline{\hat{e}}}$ has roots ζ^e with $e \in G$ since $G = \langle \bar{p}, \bar{m} \rangle$

These are all distinct (since ζ is primitive)

But $\deg(f_{\underline{e}} - f_{\underline{\hat{e}}}) < |G|$ So $f_{\underline{e}} - f_{\underline{\hat{e}}} = 0$

Since $k \leq \sqrt{r} \cdot l < r < p$ the $(x + \bar{a})$ with $a \in \{0 \dots k\}$ are primitive distinct.

So $(\underline{e}) = (\underline{\hat{e}})$ ✓

So is $|H| \geq |T|$?

Let M be the set of all $\{x_0, \dots, x_k\} \subseteq \{1, \dots, |G| + k\}$

with $x_0 < x_1 < \dots < x_k$

For $\{x_0, \dots, x_k\} \in M$ define $(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1}$ by $e_a = x_a - x_{a-1}$ with $x_{-1} := 0$

$$\Rightarrow \sum_{a=0}^k e_a = \sum_{a=0}^k (x_a - x_{a-1} - 1) = x_k - (k+1) < |G|$$

Obtain injection $M \Leftrightarrow T$

$$\text{So } |H| \geq |T| \geq |M| = \binom{|G|+k}{k+1} \stackrel{(2)}{\geq} \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{k+1} = \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{\lfloor l\sqrt{|a|} \rfloor} \stackrel{(2)}{\geq} \binom{2 \cdot \lfloor l\sqrt{|a|} \rfloor + 1}{\lfloor l\sqrt{|a|} \rfloor}$$

1.3.19 Lemma 18 (Property of binomial coefficients)

$$\forall n \in \mathbb{N}_{>1} : \binom{2 \cdot n + 1}{n} > 2^{n+1}$$

Proof:

$n = 2 :$

$$\binom{5}{2} = 10 > 2^3$$

$n - 1 \rightarrow n :$

$$\binom{2 \cdot n + 1}{n} = \binom{2 \cdot n}{n-1} + \binom{2 \cdot n}{n} = \binom{2 \cdot n - 1}{n-2} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n} \geq 2 \cdot \binom{2 \cdot n - 1}{n-1} \stackrel{ind.}{>} 2 \cdot 2^n = 2^{n+1}$$

Continuation of Proof of Theorem 17

$$|H| > 2^{\lfloor l \cdot \sqrt{|a|} \rfloor + 1} \geq 2^{l \cdot \sqrt{|a|}} \geq 2^{\lg(n) \cdot \sqrt{|a|}} = n \sqrt{|a|} \quad (4)$$

Assume $n \notin \mathbb{P}$ By step (1) m is not a perfect power

\Rightarrow the map $\mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{N} \quad (s, t) \mapsto p^s m^t$ is injective.

Set $A := \{p^s m^t \mid s, t \in \{0, \dots, \lfloor \sqrt{a} \rfloor\}\} \subseteq \mathbb{N}$

$$\Rightarrow |A| = (\lfloor \sqrt{|a|} \rfloor + 1)^2 > |G|$$

Since $G = \langle \bar{p}, \bar{m} \rangle \subseteq (\mathbb{Z}/(r))^\times$ this implies that $\exists n, \hat{n} \in A$

such that $n \neq \hat{n}$ but $b \equiv \hat{n} \pmod{r}$.

$$\text{Let } h \in H \Rightarrow h = f_{\underline{e}}(\zeta) \text{ with } (\underline{e}) \in \mathbb{N}_0^{k+1} \Rightarrow h^n \stackrel{(3)}{=} f_{\underline{e}}(\zeta^n) \stackrel{n \equiv \hat{n} \pmod{r}}{=} f_{\underline{e}}(\zeta^{\hat{n}}) \stackrel{(3)}{=} h^{\hat{n}}$$

So the polynomial $Y^n - Y^{\hat{n}} \in \mathbb{F}_q[Y]$ has all elements of H as zeros.
 But $\deg(Y^n - Y^{\hat{n}}) \leq \max\{n, \hat{n}\} \leq (p \cdot m) \lfloor \sqrt{|G|} \rfloor \leq n \sqrt{|G|} < |H|$
 \Rightarrow contradiction since $Y^n - Y^{\hat{n}} \neq 0$ \square

1.4 Cryptology

A ("Alice") wants to send a message to B ("Bob") such that an eavesdropper E ("Eve") can not read the clear message. So A and B encrypt the message.

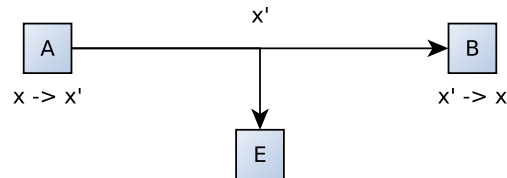


Figure 2: Scheme of eavesdropping

Symmetric-key cryptography

A and B share secret keys for encryption ($x \mapsto x'$) and decryption ($x' \mapsto x$). Only A and B know the keys.

Example: AES approved by the US government in 2002

Application:

- sending messages
- encrypt files (A=B)

Problem: Key exchange between A and B

Public-key cryptography

Encryption-map $\phi : x \mapsto x'$ is made public by B, but decryption $\phi : x' \mapsto x$ is kept secret.

Advantage: No confidential key exchange.

Disadvantages:

- more costly than symmetric key cryptography
- doubt whether E can reconstruct ϕ^{-1} from ϕ with enough computing power

Applications:

- sending messages
- exchange of symmetric keys
- authentication: Together with x , B sends $\phi^{-1}(x)$ (or $\phi^{-1}|$ Part of x together with date). A verifies by applying ϕ .
 Better: challenge-response-protocol.

Examples: RSA, elliptic curve

1.4.1 Algorithm (RSA)

- (1) B chooses $p, q \in \mathbb{P}$ large (> 100 digits)
with $p \neq q$ $n := p \cdot q$
- (2) B chooses $e, f \in \mathbb{N}$ large such that $e \cdot f \equiv 1 \pmod{\phi(n)}$
with $\phi(n) = (p-1)(q-1)$
- (3) B makes n, e public, keep f secret
- (4) The message is encoded as an element $x \in \mathbb{Z}/(n)$
- (5) A computes $\phi(x) = x^e = y \in \mathbb{Z}/(n)$ and sends y
- (6) B receives y and computes $y^f = x \in \mathbb{Z}/(n)$

Comments on steps of RSA:

- (6) Have $e \cdot f = a \cdot (p-1) \cdot (q-1) + 1$ with $a \in \mathbb{N}_{>0}$
 $y^f = x^{e \cdot f}$

$$1: q \nmid f, q \nmid x \Rightarrow x^{a(p-1)(q-1)} = (x^{\phi(n)})^a \stackrel{\text{Little Fermat}}{\equiv} 1^a = 1 \Rightarrow x^{e \cdot f} = x \quad \checkmark$$

- Case 2: $p|x, q \nmid x \Rightarrow x^{e \cdot f} \equiv 0 \equiv x \pmod{p}$
 $x^{e \cdot f} \equiv x \pmod{q}$ as above.

Case 3: $q|x$ As Case 2

\Rightarrow Correctness of decryption

Cost:

- (1) Finding p, q of length approximately l . Prime-number theorem: Gap between two primes of length $\approx l$ is $O(l)$
Using Miller Rabin with error probability 2^m . Expected cost of (1) is $O(m \cdot l^4)$ bit operations.
- (2) Choose e co-prime to $\phi(n)$ obtain $f = \text{inverse} \pmod{\phi(n)}$ by extended euclidean Algorithm: $O(l^2)$
- (5)(6) Fast exponentiation: $O(l^3)$

Security of RSA: p and q must be so large that factorization of n is "impossible". Assumption that factorization is expensive could not be shown! But could f be obtained without knowing p and q ? The following algorithm gives a negative answer. It shows that the problem of breaking RSA is always basically factorization.

Remember: $\phi(n)|(e \cdot f - 1) =: m \leq n^2$

1.4.2 Algorithm 1 (Finding a divisor)

Input : $n \in \mathbb{N}_{>2}$ odd squarefree $\notin \mathbb{P}$ and $m \in \mathbb{N}_{>0}$ such that $\phi(n) \mid m$ $m \leq n^2$

Output: $d \in \mathbb{N}$ with $d \mid n$ $1 < d < n$

- (1) Choose $a \in \{2, \dots, (n-2)\}$ randomly
set $k := m$
- (2) If $d := \gcd(a, n) \neq 1$
return d
- (3) Repeat steps (4) - (8) //while(true)
- (4) compute $d := \gcd(n, a^k - 1)$
- (5) If $d = 1$ go to (1)
- (6) If $d < n$ return d
- (7) if k is odd go to (1)
- (8) set $k := \frac{k}{2}$

Correctness is clear. What about termination and running time?

1.4.3 Proposition 2 (Complexity of Algorithm 1)

Algorithm 1 terminates in expected time $O(l(n)^4)$ bit operations (Las Vegas Algorithm).

Proof:

Set $l := \text{length}(n)$

Have $n = \prod_{i=1}^r p_i$ with $p_i \in \mathbb{P}$ distinct.

$\phi(n) = \prod_{i=1}^r (p_i - 1) \mid m$ So initially all $(p_i - 1)$ divide k .

At some iteration it happens for the first time that $(p_i - 1) \nmid k$

Then $k \equiv \frac{p_1-1}{2} \pmod{(p_1-1)} \Rightarrow a^k \equiv \pm 1 \pmod{p_i}$ -1 occurs for some a

For those j with $(p_j - 1) \mid k$ have $a^k \equiv 1 \pmod{p_j}$

Consider the group homomorphism: $\phi: (\mathbb{Z}/(n))^{\times} \mapsto (\mathbb{Z}/(p_1))^{\times} \times \dots \times (\mathbb{Z}/(p_r))^{\times}$
 $\bar{a} \mapsto (a^k \pmod{p_1}, \dots, a^k \pmod{p_r})$

The image of ϕ is a product of groups $\{\pm 1\}$ or $\{1\}$ depending whether $(p_i - 1) \nmid k$ or $(p_i - 1) \mid k$

Conclusion:

For at least half of all a 's, $\phi(\bar{a})$ is neither $(1, \dots, 1)$ nor $(-1, \dots, -1)$

If $a^k \equiv 1 \pmod{p_j}$ then $p_j \mid (a^k - 1) \Rightarrow p_j \mid d$

If $a^k \equiv -1 \pmod{p_j}$ then $p_j \nmid (a^k - 1) \Rightarrow p_j \nmid d$

So for these a the algorithm is successful.

This means that the expected number of a 's that need to be tested is ≤ 2

(Since $\sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = 2$ More generally for $0 < p < 1 : p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} = \frac{1}{p}$)

Analysis of running time (in bit operations) for each a (using gcd is quadratic) leads to the claim. \square

Problems of RSA:

- How difficult is factorization of integers (lower bound?)
- decryption of some or all messages without having f ?

1.4.4 Diffie-Hellmann Key Exchange

Goal: A, B want to exchange a symmetric key via a public channel

- (1) A and B agree on a $p \in \mathbb{P}$ (should be large) and $g \in (\mathbb{Z}/(p))^{\times}$ public
- (2) A chooses $a \in \{2, \dots, (p-2)\}$ randomly and sends $u := g^a$ to B
- (3) B chooses $b \in \{2, \dots, (p-2)\}$ randomly and sends $v := g^b$ to A
- (4) A computes $v^a = (g^b)^a = g^{a \cdot b}$
B computes $u^b = (g^a)^b = g^{a \cdot b}$

\Rightarrow A and B share $g^{a \cdot b}$

Example:

A chooses $a = 7$

$$\bar{3}^7 = \bar{11} \in \mathbb{Z}/(17)$$

$$\bar{13}^7 = \bar{4}$$

B chooses $b = 4$

$$\bar{3}^4 = \bar{13} \in \mathbb{Z}/(17)$$

$$\bar{11}^4 = \bar{4}$$

If Eve reconstructs a, b from g^a and g^b she can compute $g^{a \cdot b}$

The Security of Diffie-Hellmann depends on the difficulty of the discrete logarithm problem (DLP):

Given $g \in G$ element of a group or monoid and given $g^a \in G$, determine a (or determine $a' \in \mathbb{Z}$ such that $g^a = g^{a'}$)

1.4.5 Elliptic curve cryptography (ECC)

ECC uses elliptic curves as groups.

$$y^2 = x^3 + a \cdot x + b \rightsquigarrow y^2 z = x^3 + axz^2 + bz^3$$

ECC uses suitable elliptic curves on \mathbb{F}_a

1.5 Factorization

Let $m \in \mathbb{N}_{>1}$ $n \notin \mathbb{P}$ Find a divisor d with $1 < d < n$. From this we obtain the factorization of n by recursion.

Naive method: Trial division. Cost essentially exponential in $l(n)$

1.5.1 Algorithm 1 (Sieve of Eratosthenes)

Input : $n \in \mathbb{N}_{>1}$

Output: All primes $\leq n$

- (1) Create a list of all numbers $\leq n$
- (2) $p := 2$
- (3) Mark all multiples of p in the List
- (4) if all numbers are marked
return
- (5) Let p be the smallest number that is not marked
- (6) $p \in \mathbb{P}$ Go to (3)

Running time of Algorithm 1 is exponential.

Pollard's rho (ρ) algorithm:

Idea: Choose a function $\mathbb{Z}/(n) \mapsto \mathbb{Z}/(n)$ e.g. $f(x) = x^2 + 1$

Choose $x_0 \in \mathbb{Z}/(n)$ set $x_i := f^i(x_0)$ iterative application.

Let $p \mid n$ be a prime. Since $|\mathbb{Z}/(p)| < \infty$ then $\exists i < j : x_i \equiv x_j \pmod{p}$

Starting at x_i the sequence of x_j will be periodic mod p .

$p \mid x_i - x_j \quad p \mid n \Rightarrow p \mid \gcd(n, x_i - x_j) =: d$

If $x_i \not\equiv x_j \pmod{n}$ (which is not guaranteed) then d is a proper divisor of n .

- Recall that gcd computation is cheap
- Testing all pairs is a lot
- Proposition 2 helps with this

1.5.2 Proposition 2 (length of periods)

Let M be a set. $f : M \mapsto M$ and $x_0 \in M \quad x_i := f^i(x_0)$

If $x_{t+l} = x_t$ for $l, t \in \mathbb{N}l > 0 \quad (\rightarrow t \text{ "off-period", } l \text{ "length of period"})$

$\Rightarrow \exists j \in \mathbb{N}$ with $0 < j \leq t + l$ such that $x_j = x_{2j}$

Proof:

$f^l(x_t) = x_t \Rightarrow \forall a \in \mathbb{N} \quad f^{a \cdot l}(x_t) = x_t \quad \text{Assume } j = a \cdot l \geq t \quad a \in \mathbb{N}$

$x_{2j} = x_{t+(j-t)+a \cdot l} = f^{(j-t)}(x_{t+a \cdot l}) = f^{(j-t)}(f^{al}(x_t)) = f^{(j-t)}(f^l(x_t)) = x_j$

Case 1 $t = 0 \quad j = l \quad \checkmark$

Case 2 $t > 0 \quad j = t + \underbrace{(-t \bmod l)}_{\in 0, \dots, (l-1)} \quad \checkmark$

1.5.3 Algorithm 3 (Pollard's ρ - Algorithm)

Input : $n \in \mathbb{N}_{>1}, n \notin \mathbb{P}$

Output: a proper divisor of n or "FAIL"

- (1) Choose $x \in \{0, \dots, (n-1)\}$ randomly
set $y := x$
- (2) repeat (3)-(6)
- (3) $x := x^2 + 1 \pmod{n}$ $y := (y^2 + 1)^2 + 1 \pmod{n}$ $// x := x_j y := x_{2j}$
- (4) $d := \gcd(n, x - y)$
- (5) if $(1 < d < n)$
return d
- (6) if $d = n$
return "FAIL"

One "FAIL" includes no conclusion so you might want to repeat the Algorithm with a different x .

Running time? Assume the $x_i := f^i(x_0)$ are randomly distributed.

When can we expect that a match ($x_i \equiv x_j \pmod{p}$) occurs? \rightarrow "Birthday Problem"

Lemma (Birthday Problem):

We iteratively choose numbers in $\{1, \dots, n\}$ at random. The expected numbers of choices (if we keep choosing until a number has been chosen twice) is $< \sqrt{\frac{\pi \cdot n}{2}} + 2$

Proof:

Let $s \geq 2$ be the numbers of choices until a match occurs. For $k \in \mathbb{N}$ with $P()$ as probability

$$P(s > k) = \prod_{i=1}^k \left(1 - \frac{i-1}{n}\right) \leq \prod_{i=1}^k e^{-\frac{i-1}{n}} = e^{\sum_{i=1}^k -\frac{i-1}{n}} = e^{\frac{k(1-k)}{2n}} \leq e^{-\frac{(k-1)^2}{2n}}$$

* since $f(x) = e^x - (1-x) \geq 0$ for $x \geq 0$

$$f(0) = 0$$

$$f'(x) \geq 0 \text{ if } x \geq 0$$

$$\sum_{k=0}^{\infty} P(s > k) = 2 + \sum_{k=2}^{\infty} P(s > k) \leq 2 + \sum_{k=2}^{\infty} e^{-\frac{(k-1)^2}{2n}} \leq 2 + \int_1^{\infty} e^{-\frac{(x-1)^2}{2n}} dx$$

$$\stackrel{x:=x-1}{=} 2 + \int_0^{\infty} e^{-\frac{x^2}{2n}} dx = 2 + \int_0^{\infty} e^{-\left(\frac{x}{\sqrt{2n}}\right)^2} dx$$

$$\stackrel{x:=\frac{x}{\sqrt{2n}}}{=} 2 + \sqrt{2n} \int_0^{\infty} e^{-x^2} dx = 2 + \sqrt{2n} \cdot \frac{\sqrt{\pi}}{2} = 2 + \sqrt{\frac{n \cdot \pi}{2}}$$

Example:

People arrive at a party. When can you expect to have two that share their birthday?

\rightarrow when 26 have arrived!

1.5.4 Theorem 4 (Bit-complexity of Algorithm 3)

under suitable assumptions on the distribution $f^i(x)$ for $f(x) = x^2 + 1$ Algorithm 3 has the expected running time of $O(\sqrt[4]{n} \lg(n)^2)$ bit operations

Proof:

By Proposition 2 and the Lemma the expected number of runs through the loop is

$$O(\sqrt{p}) = O(\sqrt[4]{n}) \text{ as } p \leq \sqrt{n}$$

Each run through the loop takes $O(\lg(n)^2)$ bit operations. \square

Pollard's p-1 Algorithm

Motivation: Let $p \mid n$ prime divisor

$$\Rightarrow \forall a \in \mathbb{Z} : a^{p-1} \equiv 1 \pmod{p} \quad \text{with } \gcd(a, p) = 1$$

$$\Rightarrow \forall m \in \mathbb{Z} \text{ with } (p-1) \mid m : a^m \equiv 1 \pmod{p}$$

$$p \mid \gcd(a^m - 1, n)$$

Let B be an upper-bound for the prime powers dividing $p-1$.

" $p-1$ is B -power-smooth".

$$\text{Then } (p-1) \mid \prod_{(q \leq B) \in \mathbb{P}} q^{\lfloor \log_q(B) \rfloor}$$

Neither p nor B are known! But guess and try B and hope for the best.

1.5.5 Algorithm 5 (Pollard's ρ - 1 method)

Input : $n \in \mathbb{N}_{>1} \setminus \mathbb{P}$

Output: $d \in \mathbb{N}$ with $d \mid n$ $1 < d < n$ or "FAIL"

- (1) Choose a "smoothness bound" B
- (2) Choose $a \in \{2, \dots, (n-2)\}$ randomly
- (3) Use Algorithm 1 to find all $q \in \mathbb{P}$ with $q \leq B$
For every q perform steps (4) - (5)
- (4) $k := q^{\lfloor \log_q(B) \rfloor}$
set $a := a^k \pmod{n}$
compute $d := \gcd(n, a - 1)$
- (5) if $1 < d < n$
return d
- (6) return "FAIL" //or increase B and go to (1)

Consequence: when setting up RSA p, q should be chosen such that $p-1$ and $q-1$ have large prime divisors.

The quadratic sieve (State of the art factorization algorithm)

Observation: if $n = x^2 - y^2$ then $n = (x - y) \cdot (x + y)$

Conversely if $n = a \cdot b$ then $n = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$

1-st Idea: Find $x, y \in \mathbb{Z}$ such that $x^2 \equiv y^2 \pmod{n} \quad \wedge \quad x \not\equiv \pm y \pmod{n}$

Then $n \mid (x - y) \cdot (x + y)$

\Rightarrow for every $p \in \mathbb{P}$ with $p \mid n : p \mid (x - y) \vee p \mid (x + y)$

$\Rightarrow p \mid \gcd(x - y, n) \vee p \mid \gcd(x + y, n)$

Since both gcd are $< n$ receive a non-trivial divisor of n

If $x^2 \equiv y^2 \pmod{n}$ how probable is it that $x \equiv \pm y \pmod{n}$?

Let $n = \prod_{i=1}^r p_i^{k_i}$ odd with $p_i \in \mathbb{P}$ distinct.

Assume $p_i \nmid x \forall i = 1 \dots r$ Since $(\mathbb{Z}/(p_i^{k_i}))^\times$ is cyclic there are 2^r classes $y \pmod{n}$ such that $x^2 \equiv y^2 \pmod{n}$

[Reason: These classes are given by $y \equiv \pm x \pmod{p_i^{k_i}}$ These are the only solutions since $\mathbb{Z}/(p_i^{k_i})^\times$ is cyclic of even order.

$G = \langle \sigma \rangle$ cyclic of order $2m$

$x = \sigma^i$ Find $l \in \mathbb{Z}$ such that $x^2 = (\sigma^l)^2$

$\Leftrightarrow 2j \equiv 2l \pmod{2m} \Leftrightarrow j \equiv l \pmod{m}$

$\Leftrightarrow l \equiv j \pmod{2m}$ or $l \equiv j + m \pmod{2m}$]

But have $x \equiv \pm y$ only for $2y$'s.

Failure probability: 2^{1-r}

Handle case $r = 1$ by Algorithm 11 in 1.3

Example 1

$n = 91$ Search $x, y \in \mathbb{Z} \quad k \in \mathbb{Z}$ such that $x^2 = k \cdot n + y^2$

Good chance if x is slightly bigger than $\sqrt{k \cdot n}$

$k := 1 \Rightarrow \sqrt{91} \approx 9,54 \Rightarrow x := 10 \Rightarrow 10^2 = 100 \equiv 3^2 \pmod{91}$

$n = 10^2 - 3^2 = (10 - 3) \cdot (10 + 3) = 7 \cdot 13 \quad \checkmark$

Another try:

$k := 8 \Rightarrow \sqrt{8 \cdot 91} \approx 26,98 \Rightarrow 27^2 \equiv 1^2 \pmod{91} \quad \gcd(26, 91) = 13$

Example 2

$n = 4633 \quad k := 3$

$\sqrt{3 \cdot n} \approx 117,89 \Rightarrow x^2 = 118^2 \equiv 5^2 \pmod{n}$

$\gcd(118 - 5, n) = 113$

$\gcd(118 + 5, n) = 41 \quad \checkmark$

2-nd Idea: Choose $B \in \mathbb{N}$ "smoothness bound" suitable.

Let $p_2, \dots, p_r \in \mathbb{P}$ be all primes $\leq B$ (Algorithm 1) set $p_1 := -1$

The p_i form a "factor basis".

For $a \in \mathbb{Z}$ write $(a \pmod{n})$

for the $x \in \mathbb{Z}$ with $x \equiv a \pmod{n}$ and $-\frac{n}{2} < x \leq \frac{n}{2}$

Procedure:

Search numbers $a_1, \dots, a_m \in \mathbb{Z}$ such that $(a_i^2 \pmod{n}) = \prod_{j=1}^r p_j^{e_{ij}}$

with $e_{ij} \in \mathbb{Z}$ ("B numbers")

So for $\mu_1, \dots, \mu_m \in \mathbb{N}_0$ have $\left(\prod_{i=1}^m a_i^{\mu_i} \right)^2 \equiv \prod_{i=1}^m \prod_{j=1}^r p_j^{\mu_i \cdot e_{ij}} \pmod{n} = \prod_{j=1}^r p_j^{\sum_{i=1}^m \mu_i \cdot e_{ij}} \pmod{n}$

If the vectors (e_{i1}, \dots, e_{ir}) become linearly dependent mod 2 (guaranteed if $m > r$) then

$\exists \mu_1, \dots, \mu_m \in \{0, 1\}$ not all 0 such that:

$$\sum_{i=1}^m \mu_i \cdot e_{ij} = 2 \cdot k_j \quad k_j \in \mathbb{N}_0$$

with $x := \prod_{i=1}^m a_i^{\mu_i}$ $y := \prod_{j=1}^r p_j^{k_j}$ obtain $x^2 \equiv y^2 \pmod{n}$

Example: $n = 4633$ choose $B = 3 \Rightarrow$ factor basis $-1, 2, 3$

Search $a \in \mathbb{Z}$ such that $|a_i^2 \pmod{n}|$ is small. Idea: $a \approx \sqrt{n} = 68.06\dots$

$$a_1 := 68 : 68^2 = n - 9 \equiv (-1) \cdot 3^2 \pmod{n}$$

$$\rightarrow e_1 = (1, 0, 2) \rightarrow (1, 0, 0) \in \mathbb{F}_2^3$$

$$a_2 := 69 : 69^2 = n + 128 \equiv 2^7 \pmod{n}$$

$$\rightarrow e_2 = (0, 7, 0) \rightarrow (0, 1, 0) \in \mathbb{F}_2^3$$

$$a_3 := 67 : 67^2 = n - 144 \equiv (-1) \cdot 2^4 \cdot 3^2$$

$$\rightarrow e_3 = (1, 4, 2) \rightarrow (1, 0, 0) \in \mathbb{F}_2^3$$

$$e_1 + e_3 \equiv 0 \pmod{2} \quad \text{In fact:}$$

$$e_1 + e_3 = 2 \cdot \underbrace{(1, 2, 2)}_{(k_1, k_2, k_3)} \rightarrow \mu_1 = 1 \quad \mu_2 = 0 \quad \mu_3 = 1$$

$$x := a_1 \cdot a_3 \equiv -77 \pmod{n}$$

$$y := (-1) \cdot 2^2 \cdot 3^2 = -36$$

$$x - y = -41 \quad x + y = -113$$

$$\gcd(n, x - y) = 41 \quad \gcd(n, x + y) = 113 \quad \checkmark$$

3rd Idea: Look for a_i of the form $t + \lfloor \sqrt{n} \rfloor$ with t in a "suitable".

Sieve Interval: $[-s, s] \cap \mathbb{Z}$

As it turns out if $s \leq \frac{\sqrt{5}-2}{2} \lfloor \sqrt{n} \rfloor$ then $(t + \lfloor \sqrt{n} \rfloor)^2 \pmod{n} = (t + \lfloor \sqrt{n} \rfloor)^2 - n =: f(t)$

When does $p_j^{e_j}$ divide $f(t)$ (with $j \geq 2$)? Precisely if $(t + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_j^{e_j}}$

If this holds for some t then it also holds for all $t + k \cdot p_j^{e_j}$ with $k \in \mathbb{Z}$ Moreover if it holds then $\bar{n} \in \mathbb{F}_{p_j}$ is square. So may remove all p_j such that $\bar{n} \in \mathbb{F}_{p_j}$ is a non-square from the factor basis.

Obtain a sieving procedure:

For $t \in [-s, s] \cap \mathbb{Z}$ with $p_j^{e_j} \mid f(t)$ "mark" all elements $t + k \cdot p_j^{e_j} \in [-s, s]$

1.5.6 Algorithm 6 (Quadratic sieve, simplified version)

Input : $n \in \mathbb{N}_{>1} \setminus \mathbb{P}$ odd

Output: A non trivial divisor of n or "FAIL"

- (1) if $(n = m^e)$ with $m, e \in \mathbb{N}_{>1}$
return m // can be done with Algorithm 11 § 3
- (2) Choose a "smoothness bound" $B \in \mathbb{N}$ and a "sieve bound" $s \in \mathbb{N}$ suitably
- (3) Let $p_1 = -1$ p_2, \dots, p_r be the factor basis given by B . Delete those p_j such that $\bar{n} \in \mathbb{F}_{p_j}$ is a non-square
- (4) for $(t = -s, -s + 1, \dots, s - 1)$
compute $f_t := |(t + \lfloor \sqrt{n} \rfloor)^2 - n| \in \mathbb{N}_{>0}$
- (5) for $(t = -s, \dots, s)$
set $e_t := (0, \dots, 0) \in \mathbb{N}_0^r$ // initialize exponent vectors
- (6) for $(t = -s, \dots, 0)$
set $e_{t,1} := 1$ // \rightarrow first entry of each e_t is the exponent of $p_1 = -1$ in $f(t)$
- (7) for $(j = 2, \dots, r)$ repeat (8) - (10)
- (8) for $(e = 1, \dots, \lfloor \log_{p_j}(B) \rfloor)$ repeat (9) - (10) // or maybe a bit larger
- (9) solve $(t + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_j^e}$
Let $(t_1 \pmod{p_j^e}), \dots, (t_m \pmod{p_j^e})$ be the solutions.
// We will see that $m \in \{0, 2, 4\}$ with $m = 2$ most frequent.
- (10) for all $t = t_i + k \cdot p_j^e \in [-s, s]$ with $k \in \mathbb{Z}, i = 1, \dots, m$
set $e_{t,j} := e_{t,j} + 1$
 $f_t := \frac{f_t}{p_j}$
- (11) let t, \dots, t_m be those $t \in [-s, s] \cap \mathbb{Z}$ for which $f_t = 1$
/* So the $a_i = t_i + \lfloor \sqrt{n} \rfloor$ are B -numbers and the factorization
* of $a_i^2 \pmod{n} = a_i^2 - n = f(t)$ is given by the exponent
* vectors e_t */
- (12) if the $(e_{t_i} \pmod{2}) \in \mathbb{F}_2^r (i = 1, \dots, m)$ are not linearly dependent.
return "FAIL"
- (13) compute $\mu_1, \dots, \mu_m \in \{0, 1\}, k_1, \dots, k_r \in \mathbb{N}_0$ such that $\sum_{i=1}^m \mu_i e_{t_i} = 2 \cdot (k_1, \dots, k_r)$
- (14) set $x := \prod_{i=1}^m (t_i + \lfloor \sqrt{n} \rfloor)^{\mu_i} \pmod{n}$
 $y := \prod_{j=1}^r p_j^{k_j} \pmod{n}$ // Now $x^2 \equiv y^2 \pmod{n}$

(15) if $\gcd(n, x - y)$ or $\gcd(n, x + y)$ is a non-trivial divisor
 return the non-trivial divisor
 else
 return "FAIL"

With good heuristics it will almost certainly never return FAIL.

Example: $n = 20437$

Choose $B := 10$ $s := 3$

Factor basis: $p_1 = -1$ $p_2 = 2$ $p_3 = 3$ $p_4 = 7$

(5 omitted as: $n \equiv 2 \pmod{5}$ non-square)

$\lfloor \sqrt{n} \rfloor = 142$

Solve $(t + 142)^2 \equiv n \pmod{p_j^e}$

$p_2 = 2$: Compute modulo 2,4,8. $n \equiv 5 \pmod{8}$

t odd $\Rightarrow (t + 142)^2 \equiv 1 \pmod{8} \Rightarrow (t + 142)^2 \equiv n \pmod{4}$ but not $\pmod{8}$

t even $\Rightarrow (t + 142)^2 \equiv 0 \pmod{2} \not\equiv n \pmod{2}$

$$\Rightarrow e_{t,2} = \begin{cases} 2 & t \text{ odd} \\ 0 & t \text{ even} \end{cases}$$

$p_3 = 2$: $n \equiv 1 \pmod{3}$ $\lfloor \sqrt{n} \rfloor \equiv 1 \pmod{3}$

So $3 \mid f(t) \Leftrightarrow t + 1 \equiv \pm 1 \pmod{3} \Leftrightarrow t \equiv 0 \text{ or } 1 \pmod{3}$

$e = 2$ $n \equiv 7 \equiv (\pm 4)^2 \pmod{9}$ $\lfloor \sqrt{n} \rfloor \equiv 7 \pmod{9}$

So $9 \mid f(t) \Leftrightarrow t + 7 \equiv \pm 4 \pmod{9} \Leftrightarrow t \equiv -3, -2 \pmod{9}$

$p_4 = 7$ $n \equiv 4 \pmod{7}$ $4 = (\pm 2)^2$ $\lfloor \sqrt{n} \rfloor \equiv 2 \pmod{7}$

So $7 \mid f(t) \Leftrightarrow t + 2 \equiv \pm 2 \pmod{7} \Leftrightarrow t \equiv 0 \text{ or } 3 \pmod{7}$

t	-3	-2	-1	0	1	2	3
$f_t = f(t) $	1116	837	556	273	12	295	588
p_1 component of e_t	1	1	1	1	0	0	0
p_2 component	2	0	2	0	2	0	2
f_t divided by 2-power	279	837	139	273	3	299	147
p_3 component	2	2	0	1	1	0	1
f_t	31	93	139	91	1	299	49
p_4 component	0	0	0	1	0	0	2
f_t	31	93	139	13	1	299	1

Obtain $m = 2$: $t_1 = 1$ $t_2 = 3$ $e_1 = (0, 2, 1, 0)$ $e_3 = (0, 2, 1, 2)$

They are linear dependent $\pmod{2}$

$e_1 + e_3 = 2 \cdot (0, 2, 1, 1)$

$x = (142 + 1) \cdot (142 + 3) \equiv 298 \pmod{n}$

$y = p_2^2 \cdot p_3 \cdot p_4 = 2^2 \cdot 3 \cdot 7 = 84$

$\gcd(n, x - y) = \gcd(n, 214) = 107$

$\gcd(n, x + y) = 191$

Indeed $n = 107 \cdot 191$

Computing square roots (mod p^e)

Case 1: p odd

Find x with $x^2 \equiv n \pmod{p}$ by trying $x \pmod{p}$ (exactly two solutions). Suppose we have found x with $x^2 \equiv n \pmod{p^e}$

So $x^2 - n = p^e \cdot r \quad r \in \mathbb{Z}$

New x should be $x + y \cdot p^e$

Compute modulo p^{e+1} : $(x + y \cdot p^e)^2 - n = x^2 + 2yxp^e + y^2p^{2e} - n \equiv p^e \cdot (r + 2xy) \pmod{p^{e+1}}$

So $(x + y \cdot p^e)^2 \equiv n \pmod{p^{e+1}} \Leftrightarrow 2xy \equiv -r \pmod{p}$ uniquely and easily solvable

\rightarrow Obtain two solutions (mod p^e)

\Rightarrow special case of "Hensel lifting"

Case 2: $p = 2$

Find $x \in \mathbb{Z}$ with $x^2 \equiv n \pmod{8}$ (0 or 4 solutions since n odd)

Assume we have $x^2 \equiv n \pmod{2^e} \quad e \geq 3$

So $x^2 - n = r \cdot 2^e$

$\Rightarrow (x + y \cdot 2^{e-1})^2 - n = x^2 + xy \cdot 2^e + y^2 2^{2e-2} - n \equiv 2^e(r + xy) \pmod{2^{e+1}}$

So $(x + y \cdot 2^{e-1})^2 \equiv n \pmod{2^{e+1}} \Leftrightarrow y \equiv r \pmod{2}$

\rightarrow 0 or 4 solutions

Running time of quadratic sieve

Choose $B \approx \exp\left(\sqrt{\frac{1}{2} \ln(n) \cdot \ln(\ln(n))}\right)$

If $s \approx B$ then running time is: $O\left(\exp\left(\sqrt{\ln(n) \cdot \ln(\ln(n))}\right)\right)$

which is "slightly" sub-exponential

Factorization algorithm with best complexity (known to date):

Number field sieve

This also uses ideas 1 and 2, but an algebraic number field is used for generating B -numbers.

Heuristic Running time (modulo some conjectures): $O\left(\exp\left(\ln(n)^{\frac{1}{3}} \cdot \ln(\ln(n))\right)^{\frac{2}{3}}\right)$

2 Systems of equations

2.6 Linear Algebra

Tasks:

- solving systems of linear equations (= linear systems)
- inversions of matrices
- rank determination
- determinants
- matrix products

K field, $K^{m \times n}$ = set of $m \times n$ matrices

$GL_n(K)$ = field of $n \times n$ matrices

Count the cost of algorithms in terms of field operations. If K is a finite field this translates directly to bit operations.

2.6.1 Proposition 1 (Complexity of usual algorithms)

- Solving an $m \times n$ -linear system by Gaussian elimination requires $O(\max\{m, n\}^3)$ field operations
- For $A \in GL_n(K)$ computing A^{-1} by usual method requires $O(n^3)$ field operations.
- Computing $\det(A)$ "as usual" requires $O(n^3)$ bit operations.
- Computing $A \cdot B$ for $A \in K^{m \times n}$ $B \in K^{n \times l}$ requires $O(m \cdot n \cdot l)$ field operations.

→ all cubic!

Proof:

- Cost of treating the k -th row with Gauss algorithm:
 ≤ 1 inversion, $\leq (n - k)$ multiplications
 $\leq (m - k)(n - k)$ multiplications and additions
 (clearing column below pivot element)
 Back substitution (i.e. clearing columns above pivot element):
 Let $r = rk(A) \leq (k - 1)(n - r)$ multiplications and additions
 Total cost $\leq \sum_{k=1}^r (1 + n - k + 2(m - k)(n - k) + 2(k - 1)(n - r))$
 $= 2mnr - mr^2 - \frac{1}{3}r^3 - nr + \frac{3}{2}r^2 + \frac{5}{6}r - mr$
 $\in O(\max\{m, n\}^3)$
- Inversion is Gaussian elimination of $n \times 2n$ -matrix of rank n
 cost $\leq \frac{8}{3}n^3 - \frac{3}{2}n^2 + \frac{5}{6}n \in O(n^3)$
- reduced to (a)

(d) obvious

Strassen-multiplication

let $A, B \in K^{2n \times 2n}$ Write: $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with $A_{ij} B_{ij} \in K^{n \times n}$

Then $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ with $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} \rightarrow 8$ multiplications.

Set:

$$M_1 := (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$$

$$M_2 := (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$$

$$M_3 := (A_{11} - A_{21}) \cdot (B_{11} + B_{12})$$

$$M_4 := (A_{11} + A_{12}) \cdot B_{22}$$

$$M_5 := A_{11} \cdot (B_{12} - B_{22})$$

$$M_6 := A_{22} \cdot (B_{21} - B_{11})$$

$$M_7 := (A_{21} + A_{22}) \cdot B_{11}$$

Then:

$$C_{11} = M_1 + M_2 - M_4 + M_6$$

$$C_{12} = M_4 + M_5$$

$$C_{21} = M_6 + M_7$$

$$C_{22} = M_2 - M_3 + M_5 - M_7$$

$\rightarrow 7$ Multiplications!

2.6.2 Algorithm 2 (Strassen-multiplication)

Input : $A \in K^{m \times n} B \in K^{n \times l}$

Output: $A, B \in K^{m \times l}$

- (1) Let k be minimal such that $m, n, l \leq 2^k$
- (2) if $(k = 0)$ $//(\Leftrightarrow A, B \in K^{1 \times 1})$
return $A \cdot B$
- (3) Enlarge A, B by adding zeros such that $A, B \in K^{2^k \times 2^k}$
- (4) write $A \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with $A_{ij}B_{ij} \in K^{2^{k-1} \times 2^{k-1}}$
- (5) compute $M_1 \dots M_7$ as above, do multiplications by recursive call
- (6) compute $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ by above formulas
- (7) Output: the upper left $m \times l$ - part of $A \cdot B$

2.6.3 Theorem 3 (Running time of Algorithm 2)

If $m, n, l \leq r$ Algorithm 3 requires $O(r^{\lg(7)})$ field operations

Proof:

Set $\Theta(k)$ = number of field operations.

Step 5: $7 \cdot \Theta(k-1) + 10 \cdot (2^{k-1})^2$

Step 6: $8 \cdot (2^{k-1})^2$

Obtain:

$$\Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1} \quad (*)$$

Claim: $\Theta(k) = 7^{k+1} - 6 \cdot 4^k$

Induction on k

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$k-1 \rightarrow k : \Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1}$

$= 7(7^k - 6 \cdot 4^{k-1}) + 18 \cdot 4^{k-1}$

induction

$= 7^{k+1} - 4 \cdot 6 \cdot 4^{k-1} \quad \checkmark$

Have $2^{k-1} < r \Rightarrow k < \lg(r) + 1 \Rightarrow \Theta(k) < 7^{\lg(r)+2} = 49 \cdot 2^{\lg(7) \cdot \lg(r)} = 49^{\lg(17)} \quad \checkmark$

Remarks:

(a) $\lg(7) = 2.8074\dots$

(b) Coppersmith-Winograd: $O(r^{2.3754\dots})$

Improved by Stothes (2010), Williams(2011), Le Gall(2014): $O(r^{2.3729\dots})$

(c) The cost of the best possible algorithm is unknown, even for $r = 3$

Let $M : \mathbb{N}_{>0} \mapsto \mathbb{R}_{>0}$ be a function such that two matrices in $K^{n \times n}$ can be multiplied in $\leq M(n)$ field operations. Assume $\exists \epsilon > 0 : \forall n :$

$$2^{2+\epsilon} M(n) \leq M(2n) \leq 8 \cdot M(n) \quad (1)$$

Example: $M(n) = 49 \cdot n^{\lg(7)}$

Recall: $A = (a_{ij})$ is upper (lower) triangular $\Leftrightarrow a_{ij} = 0$ for $i > j$ ($i < j$)

2.6.4 Proposition 4 (Complexity of matrix inversion)

An upper or lower triangular matrix $A \in GL_n(K)$ can be inverted in $O(M(n))$ field operations.

Proof:

Let $k \in \mathbb{N}$ be minimal such that $n \leq 2^k$

write $B = \begin{pmatrix} A & 0 \\ 0 & I_{2^k-n} \end{pmatrix} \in GL_{2^k}(K) \Rightarrow B^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & I_{2^k-n} \end{pmatrix}$

Assume B upper triangular:

$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$ $B_{11}, B_{22} \in GL_{2^{k-1}}(K), B_{12} \in K^{2^{k-1} \times 2^{k-1}}$

$B^{-1} = \begin{pmatrix} B_{11}^{-1} & -B_{11}^{-1} \cdot B_{12} \cdot B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{pmatrix}$

Let $\Theta(k)$ = computation cost depending on k .

$$\Theta(k) \leq 2 \cdot \Theta(k-1) + 2M(2^{k-1}) \stackrel{(1)}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2} \cdot M(2^k) \quad (**)$$

Claim: $\Theta(k) \leq 2^k + M(2^k)$

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$k-1 \rightarrow k : \Theta(k) \stackrel{**}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2}M(2^k) \stackrel{\text{induction}}{\leq} 2(2^{k-1} + M(2^{k-1})) + \frac{1}{2}M(2^k) \stackrel{(1)}{\leq}$

$2^k + \frac{1}{2}M(2^k) + \frac{1}{2}M(2^k) \quad \checkmark$

Have $n > 2^{k-1} \Rightarrow k < \lg(n) + 1 \Rightarrow \Theta(k) < 2 \cdot n + M(2n) \stackrel{(1)}{\leq} 2 \cdot n + 8 \cdot M(n) \quad \square$

Project: Reduce (most) tasks of linear algebra to multiplication.

The following algorithm transforms a matrix such that all tasks become easy.

2.6.5 Algorithm 5 (Transforming a matrix)

Input : $A \in K^{m \times n}$

Output: Matrices L, Q, P, U

such that: $LQAP = \begin{pmatrix} U \\ 0 \end{pmatrix} \begin{matrix} r \\ m-r \end{matrix}$ (\leftarrow in row-echelon form) $\in K^{m \times n}$

such that:

- $L \in K^{m \times m}$ lower triangular with 1's on the diagonal
- $Q \in K^{m \times m}$ $P \in K^{n \times n}$ permutation matrices
- $U \in K^{m \times m}$ upper triangular with non-zero diagonal entries ($r = 0$ if $A = 0$)
- If $r = m$ then $Q = I_m$

2.6.6 Theorem 6 (Correctness and running time of Algorithm 5)

Algorithm 5 is correct and requires $O((\frac{n}{m} + 1) \cdot M(m))$ field operations

Proof:

Correctness by induction on m

$m = 1$ ✓

$m > 1$:

$m_1, m_2 < m$ so recursive calls are correct by induction.

By step (7) L, Q, P, U have desired form.

Compute:

$$\begin{aligned}
 LQAP &= \begin{array}{cc} m_1 & m_2 \\ \begin{bmatrix} L_t Q_t & 0 \\ -L_2 Q_2 F E^{-1} L_t Q_t & L_2 Q_2 \\ L_l Q_t + L_r Q_b & 0 \end{bmatrix} & \begin{array}{l} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \end{array} \cdot \begin{array}{c} \begin{bmatrix} B \cdot P_1 \\ D \end{bmatrix} \\ m_1 \\ m_2 \end{array} \cdot \begin{array}{cc} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} & \begin{array}{l} r_1 \\ n - r_2 \end{array} \end{array} \\
 &\stackrel{(4)}{=} \begin{array}{cc} n & \\ \begin{bmatrix} U_1 \\ L_2 Q_2 (-F E^{-1} U_1 + D) \\ 0 \end{bmatrix} & \begin{array}{l} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} & \begin{array}{l} r_1 \\ n - r_2 \end{array} \end{array} \\
 &= \begin{array}{cc} r_1 & n - r_1 \\ \begin{bmatrix} E & U'_1 \\ 0 & L_2 Q_2 G \\ 0 & 0 \end{bmatrix} & \begin{array}{l} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} & \begin{array}{l} r_1 \\ m_2 \end{array} \end{array} \stackrel{(6)}{=} \begin{array}{cc} r_1 & n - r_1 \\ \begin{bmatrix} E & U'_1 P_2 \\ 0 & U_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{l} r_1 \\ r_2 \\ m_2 \cdot r_2 \\ m_2 \cdot r \end{array} \end{array} = \begin{array}{c} U \\ 0 \end{array}
 \end{aligned}$$

Suppose $r = m \Rightarrow r_1 = m_1 \quad r_2 = m_2 \Rightarrow Q_1 = I_{m_2} \quad Q_2 = I_{m_2} \Rightarrow Q = I_m$

Cost:

Fix $n \in \mathbb{N}$ and set $\Theta(k) :=$ maximal cost for a matrix $A \in K^{m \times n'}$ with $m \leq 2^k \quad n' \leq n$

Choose $A \in K^{m \times n'}$ with cost = $\Theta(k)$

Step(4) and (6): $\leq \Theta(k - 1)$ each

Step(5): E^{-1} : by Proposition 4: $O(M(r_1)) \leq O(M(2^{k-1}))$

$F \cdot E^{-1} \leq M(2^{k-1})$

$F \cdot E^{-1} \cdot U'$: at most the cost of multiplying a $2^{k-1} \times 2^{k-1}$ matrix by a $2^{k-1} \times n$ matrix.

Split right matrix into square parts.

\Rightarrow cost $\leq \lceil \frac{n}{2^{k-1}} \rceil \cdot M(2^{k-1}) \leq (2^{1-k} \cdot n + 1) \cdot M(2^{k-1})$

G : subtraction: $m_2 \cdot (n - r_1) \leq 2^{k-1} \cdot n \leq 2^{1-k} \cdot n \cdot M(2^{k-1})$

Step (7): $F \cdot E^{-1}$ already computed, $L_2 Q_2$: permuting rows. Cost: $\leq 2 \cdot M(2^{k-1})$

Obtain: $\Theta(k) \leq 2 \cdot \Theta(k - 1) + (2^{-k} \cdot n + c) \cdot M(2^k) \quad c \text{ constant}$

From this obtain by induction:

$\Theta(k) \leq \left(2^{-k} \cdot n \cdot \frac{1-2^{-k\epsilon}}{1-2^{-\epsilon}} + 2 \cdot c \cdot (1 - 2^{-k}) \right) \cdot M(2^k) \leq \left(\frac{1}{1-2^{-\epsilon}} \cdot \frac{n}{2^k} + 2c \right) \cdot M(2^l)$

Finally obtain: Cost $\leq 8 \cdot \max \left\{ \frac{1}{1-2^{-\epsilon}} \cdot c \right\} \cdot \left(\frac{n}{m} + 1 \right) \cdot M(m) \quad \square$

$\begin{bmatrix} U \\ 0 \end{bmatrix}$ is in row echelon form. It's convenient to write $U = \begin{bmatrix} E & U' \end{bmatrix} \begin{matrix} r \\ m-r \end{matrix}$ $U' \in K^{r \times (n-r)}$

Also write $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{matrix} m \\ m-r \end{matrix}$

2.6.7 Theorem 7

Let $A \in K^{m \times n}$, $LQAP = \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \begin{matrix} r \\ m-r \end{matrix}$ as in Algorithm 5 then

(a) $rk(A) = r$

(b) The columns of $P \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}$ form a basis of $ker(A)$

(c) A linear system $Ax = b$ $b \in K^m$ is solvable iff $L_2Q \cdot b = 0$

(d) if $Ax = b$ is solvable then $x = P \cdot \begin{bmatrix} E^{-1}L_1 \\ 0 \end{bmatrix} \begin{matrix} r \\ n-r \end{matrix} \cdot Q \cdot b$ is a solution

(e) if $A \in GL_n(K)$ then $\det(A) = \det(P) \cdot \underbrace{\det(E)}_{=\text{prod of diags}}$

and $A^{-1} = P \cdot E^{-1} \cdot L$

Proof:

(a), (e) : \checkmark

(b) : $LQAP \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix} = \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix} = 0$

\Rightarrow the columns lie in $ker(A)$

The columns of $\begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}$ are linear independent.

$\Rightarrow rk(P \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}) = n - r$

\Rightarrow the columns form a basis.

The space they generate has dimension $n - r = \dim(ker(A))$

(c), (d) : If $A \cdot x = b$ then $\begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot P^{-1} \cdot x = LQb = \begin{bmatrix} L_1Qb \\ L_2Qb \end{bmatrix}$

$\Rightarrow L_2Qb = 0$

if $L_2Qb = 0$ then $A \cdot P \cdot \begin{bmatrix} E^{-1} \cdot L_1 \\ 0 \end{bmatrix} \cdot Q \cdot b = Q^{-1} \cdot L^{-1} \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} E^{-1} \cdot L_1 \\ 0 \end{bmatrix} \cdot Qb =$

$Q^{-1} \cdot L^{-1} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} Q \cdot b = Q^{-1} \cdot L^{-1} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \cdot Q \cdot b = b$

2.6.8 Corollary 8

For $A \in K^{m \times n}$ the determination of $rk(A)$ and solving linear systems with coefficient matrix A require $O((\frac{n}{m} + 1) \cdot M(m))$ field operations.

If $A \in K^{n \times n}$ then computing $\det(A)$ and A^{-1} (if $A \in GL_n(K)$) require $O(M(n))$ field operations.

From $LQAP = \begin{bmatrix} U \\ 0 \end{bmatrix}$ get $A = Q^{-1} \cdot \underbrace{L^{-1}}_{\text{lower triangular}} \begin{bmatrix} U \\ 0 \end{bmatrix} \cdot P^{-1}$

Generally $Q = I_m \Rightarrow A = L^{-1} \cdot \begin{bmatrix} U \\ 0 \end{bmatrix} P^{-1}$ "LUP decomposition"

If also $P = I_n$ obtain $A = L^{-1} \cdot \begin{bmatrix} U \\ 0 \end{bmatrix}$ "LU decomposition"

2.7 Algebraic Systems of Equations, Gröbner bases

Given: $f_1 \dots f_m \in K[x_1 \dots x_n]$ multivariate polynomials.

Wanted: solution set of the algebraic system $f_1 = f_2 = \dots = f_m = 0$

The solution set $\mathcal{V}(f_1 \dots f_m) \subseteq K^n$ is called an affine variety.

Often assume $K = \bar{K}$ K algebraically closed (e.g. $K = \mathbb{C}$)

Questions:

1. $\mathcal{V}(f_1 \dots f_m) \neq \emptyset$?
2. $|\mathcal{V}(f_1 \dots f_m)| < \infty$?
3. $\dim \mathcal{V}(f_1 \dots f_m) = ?$

Examples:

$$\begin{aligned}
 (1) \quad & f_1 = x_1 x_3 x_4^2 - 2x_2 x_4^2 + x_1 x_3 - 2x_2 \\
 & f_2 = x_1 x_3 x_4 - 2x_2 x_4 - 1 \\
 & f_3 = x_1 x_4^2 + x_1 + 2 \\
 & \text{One has } (-x_1 x_4) \cdot f_1 + (x_1 x_4^2 + x_1 9f_2 + f_3) = 2 \\
 & \Rightarrow \mathcal{V}(f_1 \dots f_3) = \emptyset
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & f_1 = x^3 + x^2 y + xy + y^2 \\
 & f_2 = x^2 y^2 + x^2 + y^3 + y \\
 & f_3 = x^3 + xy \\
 & (x^2 + y) \mid f_i \quad \forall i = 1, 2, 3 \Rightarrow |\mathcal{V}(f_1, f_2, f_3)| = \infty
 \end{aligned}$$

Univariate case (n=1):

$K[x]$ Euclidean so have Euclidean algorithm for computing $\gcd(f, g)$. \gcd is unique if required to be monic (i.e. the highest coefficient is 1)

Also get $h_1, h_2 \in K[x]$ such that $\gcd(f, g) = h_1 f + h_2 g$

Let $f_1 \dots f_m \in K[x]$ Obtain

$$g := \gcd(f_1 \dots f_m) = \sum_{i=1}^n h_i \cdot f_i \quad \text{with } h_i \in K[x] \quad (*)$$

For $\xi \in K$:

$$f_1(\xi) = f_2(\xi) = \dots = f_m(\xi) = 0 \Rightarrow \underset{*}{g(\xi)} = 0 \Rightarrow \underset{g|f_i}{f_i(\xi)} = 0 \quad \forall i$$

\Rightarrow Only need to get zeros of one polynomial!

Resultant method:

Reminder: For $f, g \in K[x] : \gcd(f, g) \neq 1 \Leftrightarrow \text{res}(f, g) = 0$

Let $f_1, f_2 \in K[x_1 \dots x_n]$

$$(\xi_1 \dots \xi_{n-1}) \in K^{n-1} \quad \text{assume } K = \bar{K}$$

Then $\exists \xi_n \in K$ such that $f_1(\xi_1 \dots \xi_n) = 0 = f_2(\xi_1 \dots \xi_n)$

$$\Leftrightarrow \text{res}_{x_n}(f_1(\xi_1 \dots \xi_{n-1}, x_n), f_2(\xi_1 \dots \xi_{n-1}, x_n)) = 0$$

Suppose $\deg_{x_n} f_1(\xi_1 \dots \xi_{n-1}, x_n) = \deg_{x_n}(f_i) \quad (i = 1, 2)$

Set $h = \text{res}_{x_n}(f_1, f_2) \in K[x_1 \dots x_{n-1}]$

Then $\text{res}_{x_n} f_1(\xi_1 \dots \xi_{n-1}, x_n), f_2(\xi_1 \dots \xi_{n-1}, x_n) = h(\xi_1 \dots \xi_{n-1})$

Search zeros of $h \rightarrow$ one variable, one equation fewer.

Limitation: Only for pairs of polynomials ($m = 2$).

Good case: $m = n = 2$

Given $f_1 \dots f_m \in K[x_1 \dots x_n]$ form the ideal

$$I = (f_1 \dots f_m) = \left\{ \sum_{i=1}^n g_i f_i \mid g_i \in K[x_1 \dots x_n] \right\}$$

Clearly $\mathcal{V}(I) = \mathcal{V}(f_1 \dots f_m)$

$f_1 \dots f_m$ are called a ideal basis of I . They are not unique, not even their size is unique.

Example:

In Example (1) I has an alternative basis $I = (1) \leftarrow$ constant polynomial $1 \in K[x_1 \dots x_4]$

In Example (2) it turns out that $I = (x^2 + y)$

2.7.1 Theorem 1 (Hilbert's Nullstellensatz)

Hilbert's Nullstellensatz (1st version):

Assume $K = \bar{K}$ let $I \subseteq K[x_1 \dots x_n]$ be an ideal

Then $\mathcal{V}(I) = \emptyset \Leftrightarrow 1 \in I$

($\Leftrightarrow I = K[x_1 \dots x_n] \Leftrightarrow I = (1)$)

without proof.

For $I \subseteq R$ ideal in a commutative ring R the radical ideal of I is

$$\sqrt{I} = \{a \in R \mid \exists n \in \mathbb{N} : a^n \in I\}$$

I is called a radical ideal if $I = \sqrt{I}$

Let $S \subseteq K^n$ set of points

$$\Rightarrow Id(S) := \{f \in K[\underline{x}] \mid f(v) = 0 \forall v \in S\} \subseteq K[\underline{x}] \quad \text{where } K[\underline{x}] := K[x_1, \dots, x_n]$$

is a radical ideal (called vanishing ideal)

Hilbert's Nullstellensatz (2nd version):

Let $K = \bar{K}$ $I \subseteq K[x]$ ideal. Then $\sqrt{I} = Id(\mathcal{V}(I))$

Obtain bijection: $\{\text{radical ideals in } K[x]\} \Leftrightarrow \{\text{affine varieties}\}$

This bijection is inclusion-reversing.

Monomial orderings:**2.7.2 Definition 2 (Monomial, monomial ordering, LM, LT, LC)**

A monomial is a polynomial of the form $t = x_1^{e_1} \cdot x_2^{e_2} \cdot \dots \cdot x_n^{e_n} =: \underline{x}^{\underline{e}}$ where $e_i \in \mathbb{N}$

A term is a polynomial of the form $c \cdot t$ t monomial, $c \in K \setminus \{0\}$

$M :=$ set of all monomials.

For $f \in K[x]$; $M(f) :=$ set of all monomials occurring in f .

$T(f) :=$ set of all terms ...

A monomial ordering is an ordering (= order relation) " \leq " on M such that:

1. " \leq " is total i.e. $\forall s, t \in M : s \leq t \vee t \leq s$
2. $1 \leq t \quad \forall t \in M$
3. $\forall s, t_1, t_2 \in M : t_1 \leq t_2 \Rightarrow s \cdot t_1 \leq s \cdot t_2$

(This implies: $s \mid t \Rightarrow s \leq t$)

For $f \in K[x] \setminus \{0\}$ we write

$LM(f) =: t$ for the largest monomial in $M(f)$ ("leading monomial"),

$LT(f) =: c \cdot t$ for the largest term if t in f ("leading term")

$LC(f) =: c$ ("leading coefficient")

$LM(0) = LT(0) = LC(0) = 0$

Example 1: Lexicographic ordering (lex)

for $t = x_1^{e_1} \cdot \dots \cdot x_n^{e_n} \quad t' = x_1^{e'_1} \cdot \dots \cdot x_n^{e'_n}$

define $t \leq t' \Leftarrow t = t' \vee e_i < e'_i$ for the smallest i with $e_i \neq e'_i$

Example 2: graded reverse lexicographic ordering (grevlex)

$t \leq t' \Leftarrow t = t' \vee \deg(t) < \deg(t') \vee (\deg(t) = \deg(t') \wedge e_i > e'_i)$

for the largest i such that $e_i \neq e'_i$

where $\deg(t) := \sum e_i$

For both lex and grevlex have

$x_1 > x_2 > \dots > x_n$ but $x_1 \cdot x_3 >_{\text{lex}} x_2^2$

$x_1 > x_2 > \dots > x_n$ but $x_1 \cdot x_3 <_{\text{grevlex}} x_2^2$

2.7.3 Proposition 3 (Sum and product of LM / LT)

Let " \leq " be a monomial ordering $f, g \in K[x] \Rightarrow$

- (a) $LT(f \cdot g) = LT(f) \cdot LT(g)$ same for LM
- (b) $LM(f + g) \leq \max\{LM(f), LM(g)\}$

Proof:

(b) ✓

- (a) write $c \cdot t = LT(f)$ $d \cdot s = LT(g)$
For $t' \in M(f)$ $s' \in M(g)$ have $\underbrace{t's' \leq t \cdot s' \leq t \cdot s}_{=?}$ with equality iff $s' = s$ $t' = t$

This implies (a) \square

2.7.4 Lemma 4 (Dickson-Lemma)

Every subset $S \subseteq M$ has a finite subset $B \subseteq S$ ("basis") such that $\forall s \in S \exists t \in B : t \mid s$

Proof: Identify M with \mathbb{N}^n

Given $S \subseteq \mathbb{N}^n$ need to show that:

$\exists B \subseteq S, B$ finite such that $\forall (e_1, \dots, e_n) \in S$

$\exists (d_1, \dots, d_n) \in B$ such that $\forall i : d_i \leq e_i$

We will write $(\underline{d}) \leq (\underline{e})$ for this. (This defines a partial ordering in \mathbb{N}^n)

Induction:

$n = 1$: if $\emptyset \neq S \subseteq \mathbb{N}$ then $\exists d \in S$ such that $d \leq e \quad \forall e \in S$ (\mathbb{N} is well-ordered)

$n > 1$: For $k \in \mathbb{N}$ write $S_k := \{(e_2, \dots, e_n) \in \mathbb{N}^{n-1} \mid (k, e_2, \dots, e_n) \in S\} \subseteq \mathbb{N}^{n-1}$

By induction $\exists B_k \subseteq S_k$ finite such that $\forall (\underline{e}) \in S_k \exists (\underline{d}) \in B_k$ such that $(\underline{d}) \leq (\underline{e})$

$\bigcup_{k \in \mathbb{N}} B_k \subseteq \mathbb{N}^{n-1}$ has finite "basis" C

C finite $\exists r \in \mathbb{N} : C \subseteq \bigcup_{k=0}^r B_k$ (*)

From $B := \{(e_1, \dots, e_n) \in \mathbb{N}^n \mid e_1 \leq r, (e_2, \dots, e_n) \in B_{e_1}\} \Rightarrow |B| < \infty, B \subseteq S$

Claim: B basis of S

Let $(e_1, \dots, e_n) \in S \Rightarrow (e_2, \dots, e_n) \in S_{e_1} \Rightarrow \exists (d_2, \dots, d_n) \in B_{e_1}$ such that $d_i \leq e_i \forall i \geq 2$

Case 1: $e_1 \leq r$

$\Rightarrow (e_1, d_2, \dots, d_n) \in B$ have $(e_1, d_2, \dots, d_n) \leq (e_1, \dots, e_n)$

Case 2: $e_1 > r$

$B_{e_1} \subseteq \bigcup_{k \in \mathbb{N}} B_k \Rightarrow \exists (c_2, \dots, c_n) \in C$ such that $c_i \leq d_i \forall i \geq 2$

By (*) $\exists k \leq r : (\underline{c}) \in B_k \Rightarrow (k, c_2, \dots, c_n) \in B$

$(k, c_2, \dots, c_n) \leq (e_1, d_2, \dots, d_n) \leq (e_1, e_2, \dots, e_n)$ ✓ \square

2.7.5 Corollary 5 (Well-ordering of monomial sets)

Every monomial ordering is a well-ordering i.e. every monomial set $S \subseteq M$ has an element $t \in S$ such that $\forall s \in S : t \leq s$ (t is a "least element")

Proof:

Let $\emptyset \neq S \subseteq M$ By Lemma 4 $\exists B \subseteq S$ finite such that $\forall s \in S' \exists t \in B : t \mid s$

Since " \leq " is total and B is finite $\exists t \in B$ least element.

Let $s \in S \Rightarrow \exists t' \in B$ such that $t' \mid s \Rightarrow t' \leq s$ so $t \leq t' \leq s$ \square

Gröbner bases: Let " \leq " be a fixed monomial ordering

2.7.6 Definition 6 (Leading ideal, Gröbner bases)

- (a) For $S \in K[x]$ subset define $L(S) := (LM(f) \mid f \in S) \subseteq K[x]$
(ideal generated by all leading monomials of elements of S) is called the leading ideal
- (b) Let $I \subseteq K[x]$ ideal. A finite subset $G \subseteq I$ is called a Gröbner basis if $L(I) = L(G)$
i.e. $\forall f \in I \exists g \in G : LM(g) \mid LM(f)$

2.7.7 Proposition 7 (Ideality of Gröbner bases)

G Gröbner basis of $I \Rightarrow I = (G)$ i.e. G is an ideal basis.

Proof: $G \subseteq I \Rightarrow (G) \subseteq I$

Assume this inclusion is strict. Let $f \in I \setminus (G)$ By Corollary 5 my assume $LM(f)$ is minimal (among all leading monomials of elements from $I \setminus (G)$)

$LM(f) \in L(I) = L(G) \Rightarrow \exists g \in G : LM(g) \mid LM(f)$

Form $\tilde{f} = f - \frac{LT(f)}{LT(g)}g, \tilde{f} \in I \Rightarrow LM(\tilde{f}) < LM(f)$

by minimality $\tilde{f} \in (G) \Rightarrow f = \tilde{f} + \frac{LT(f)}{LT(g)}g \in (G)$ contradiction! \square

$G \subseteq I \quad L(G) = L(I) \quad \Rightarrow I = (G)$

Example:

$I = (1) \in K[x] \quad S = \{x+1, x\}$ ideal basis but $I(S) = (x) \neq L(I) = (1)$

S is not a Gröbner basis.

2.7.8 Theorem 8 (Gröbner basis of Ideals)

Every ideal $I \subseteq K[x]$ has a Gröbner basis. In particular I has a finite basis (\rightarrow Hilbert's basis theorem) In other words $K[x]$ is Noetherian.

Proof:

For $\{LM(f) \mid f \in I\}$ there exists (by Dickson lemma) a finite subset $\{LM(f_1), \dots, LM(f_m)\}, f_i \in I$ such that $(LM(f_1) \dots LM(f_m)) = L(I) \Rightarrow G = \{f_1 \dots f_m\}$

Gröbner basis \square

First application: Let G Gröbner basis of I

Then $\mathcal{V}(I) = \emptyset \Leftrightarrow_{K=\bar{K}} 1 \in I \Leftrightarrow G$ contains a non-zero constant

2.7.9 Definition 9 (Normal form)

Let $S = \{g_1 \dots g_m\} \subseteq K[x]$ $f \in K[x]$

- (a) f is a normal form with respect to S if $\forall t \in M(f) \quad \forall i = 1 \dots m : LM(g_i) \nmid t$
- (b) $f^* \in K[x]$ is called a normal form of f with respect to S if
 - (i) f^* is in normal form with respect to S
 - (ii) $\exists h_1 \dots h_m \in K[x]$ such that $f - f^* = \sum_{i=1}^m h_i g_i$ and $\forall i : LM(h_i g_i) \leq LM(f)$

Example:

$S = \{x, x+1\}$ $f = 1 \Rightarrow f \equiv 0 \pmod{(S)}$

but 0 is not a normal form of f

If $f = x$ then 0 and -1 are normal forms of x

2.7.10 Algorithm 10 (Normal form)

Input : $S = \{g_1 \dots g_m\} \subseteq K[x]$ $f \in K[x]$

Output : A normal form f^* of f with respect to S and if desired $h_1 \dots h_m$ satisfying (*)

- (1) Set $f^* := f$
for $(i = 1 \dots m)$
 $h_i := 0$
- (2) repeat (3) - (6)
- (3) $\mathcal{M} := \{(t, i) \mid t \in M(f^*), i \in \{1, \dots, m\} \text{ such that } LM(g_i) \mid t\}$
- (4) if $(\mathcal{M} = \emptyset)$
return f^* and h
- (5) Choose $(t, i) \in \mathcal{M}$ such that t is maximal.
let $c \in K$ be the coefficient of t in f^*
- (6) Set $f^* := f^* - \frac{c \cdot t}{LT(g_i)} \cdot g_i$
 $h_i := h_i + \frac{c \cdot t}{LT(g_i)}$

Step (6) cancels the term $c \cdot t$ from f^* and may add only monomials smaller than t . So the t 's form a strictly descending sequence of monomials \Rightarrow Algorithm 10 terminates.

Correctness

✓

2.7.11 Theorem 11 (Normal form of Gröbner bases)

Let $G \subseteq K[x]$ be a Gröbner basis of an ideal $I \subseteq K[x]$

- (a) Every polynomial $f \in K[x]$ has a unique normal form with respect to G .
Write $NF_G(f)$
- (b) $NF_G : K[x] \mapsto K[x]$ is K -linear,
 $\ker(NF_G) = I$
- (c) if $\tilde{G} \subseteq K[x]$ is another Gröbner basis (with respect to same monomial ordering)
then $NF_G = NF_{\tilde{G}}$

Proof:

(a), (c):

Let $f \in K[x]$ $f^*, \tilde{f} \in K[x]$ be normal forms of f with respect to G and \tilde{G} respectively.

Claim: $f^* = \tilde{f}$

$f^* - f \in I$, $\tilde{f} - f \in I \Rightarrow f^* - \tilde{f} \in I \Rightarrow LM(f^* - \tilde{f}) \in L(G) \in L(\tilde{G})$

if $f^* \neq \tilde{f} \Rightarrow LM(f^* - \tilde{f}) \in M(f^*)$ or $\in M(\tilde{f})$

But $\exists g \in G : LM(g) \mid LM(f^* - \tilde{f})$, $\exists \tilde{g} \in \tilde{G} : LM(\tilde{g}) \mid LM(f^* - \tilde{f})$

This is a contradiction to:

f^* is in normal form with respect to G and

\tilde{f} is in normal form with respect to \tilde{G}

So $f^* = \tilde{f}$

(b):

Let $f, g \in K[x]$ $c \in K$. Set $h := NF_G(f + cg) - NF_G(f) - c \cdot NF_G(g)$

To show: $h = 0$ $h \equiv f + cg - f - cg = 0 \pmod{I}$

$\Rightarrow h \in I \Rightarrow LM(h) \in L(G)$

h is in normal form with respect to G

$\Rightarrow h = 0$

Remains to show: $\ker(NF_G) = I$

let $NF_G(f) = 0 \Rightarrow f \equiv 0 \pmod{I} \Rightarrow f \in I$ conversely, let $f \in I$

$\Rightarrow f^* = NF_G(f) \in I \Rightarrow \exists g \in G : LM(g) \mid LM(f^*)$ f^* in normal form. So $f^* = 0$ \square

Further applications of Gröbner bases:

- Membership test: $f \in I \Leftrightarrow NF_G(f) = 0$
- Computation in $A := K[x]/I$: NF_G includes an embedding $A \hookrightarrow K[x]$

Buchberger's Algorithm

2.7.12 Definition 12 (S -polynomials)

Let $f, g \in K[\underline{x}] \setminus \{0\}$ $t := \gcd(LM(f), LM(g))$
 Then $s_{pol}(f, g) := \frac{LT(g)}{t} \cdot f - \frac{LT(f)}{t} \cdot g$ is the S -polynomial.
 The leading monomials of the summands cancel!

Example:

$$\begin{aligned} f &= x^2 + y^2, \quad g = x \cdot y \quad " \leq " = lex \\ \Rightarrow LM(f) &= x^2 \quad LM(g) = xy \\ s_{pol}(f, g) &= y \cdot f - x \cdot g = y^3 \end{aligned}$$

2.7.13 Theorem 13 (Buchberger's criterion)

For any finite set $G \subseteq K[\underline{x}]$ the following statements are equivalent:

- (a) G is a Gröbner basis of $\langle G \rangle$
- (b) For polynomials $g, h \in G$, 0 is a normal form of $s_{pol}(g, h)$ with respect to G
 \rightarrow finite test for Gröbner basis!

Proof:

"(a) \Rightarrow (b)":

For $g, h \in G$: $s_{pol}(g, h) \in \langle g, h \rangle \subseteq \langle G \rangle =: I \xRightarrow{\text{Theorem 13 (b)}} s_{pol}(g, h)$ has normal form 0

$$\Leftrightarrow NF_G(s_{pol}(g, h)) = 0$$

"(a) \Leftarrow (b)":

Assume G is not a Gröbner basis $\Rightarrow \exists f \in I \setminus G$ such that $LM(f) \notin L(G)$.

$$\text{Write } G = \{g_1 \dots g_m\}. \text{ Since } f \in \langle G \rangle \exists h_1 \dots h_m \in K[\underline{x}] \text{ have } f = \sum_{i=1}^m h_i \cdot g_i \quad (1)$$

By Corollary 5 may choose h_i such that

$t := \max\{LM(h_i g_i) \mid i = 1 \dots m\}$ becomes minimal.

$\exists i : LM(f) \in M(h_i g_i)$. Since $LM(f) \notin L(G) \wedge LM(f) \neq LM(h_i g_i)$

$$\Rightarrow LM(f) < LM(h_i g_i) \leq t$$

\Rightarrow the coefficient of t in $\sum h_i g_i$ is zero.
 (1)

$$\text{Set } c_1 := \begin{cases} LC(h_i) & \text{if } LM(h_i g_i) = t \\ 0 & \text{otherwise} \end{cases} \text{ Then } \sum_{i=1}^m c_i \cdot LC(g_i) = 0 \quad (2)$$

Without loss assume $c_1 \neq 0$

Let $i \in \{2, \dots, m\}$ such that $c_i \neq 0 \Rightarrow LM(g_i) \mid t$

So $t_i = \text{lcm}(LM(g_i), LM(g_1) \mid t)$

$$\text{Have } s_{pol}(g_i, g_1) = \frac{LC(g_1) \cdot t_i}{LM(g_i)} g_i - \frac{LC(g_i) \cdot t_i}{LM(g_1)} g_1 \quad LM(s_{pol}(g_i, g_1)) < 0$$

$$\Rightarrow s_i := \frac{t}{t_i} \cdot s_{pol}(g_i, g_1) = LC(g_1) \cdot LM(h_i) \cdot g_i - LC(g_i) \cdot LM(h_1) \cdot g_1 \quad (3)$$

$$\text{By (b) have } s_i = \sum_{j=1}^m h_{ij} \cdot g_j \quad \text{with } h_{ij} \in K[\underline{x}] \text{ such that } LM(h_{ij} g_j) \leq LM(s_i) < t \quad (4)$$

$$\begin{aligned}
& \sum_{j=1}^m \left(\sum_{i=2}^m c_i \cdot h_{ij} \right) \cdot g_j = \sum_{i=2}^m c_i s_i \\
& = \sum_{i=2}^m c_i (LC(g_1)LM(h_i)g_i - LC(g_i)LM(h_1)g_1) + \sum_{i=1}^m c_i LM(h_1)g_1 LC(g_i) \\
& \stackrel{(3)}{=} \sum_{i=1}^m c_i LC(g_1)LM(h_i)g_i
\end{aligned}$$

$$\text{Set } \tilde{h}_j := \frac{1}{LC(g_1)} \cdot \sum_{i=2}^m c_i h_{ij} \Rightarrow g := \sum_{i=1}^m c_i LM(h_i)g_i = \sum_{i=1}^m \tilde{h}_i g_i$$

For each i have: $LM(\tilde{h}_i g_i) \stackrel{(4)}{<} t$

$$f = (f - g) + g \stackrel{(1)}{=} \sum_{i=1}^m (h_i - c_i LM(h_i))g_i + \sum_{i=1}^m \tilde{h}_i g_i$$

For each i have: $LM((h_i - c_i LM(h_i))g_i) < t$ so $LM((h_i - c_i LM(h_i) + \tilde{h}_i)g_i) < t$
contradiction to choice of h_i \square

2.7.14 Algorithm 14 (Buchberger)

Input : $S \subseteq K[x]$ finite " \leq " monomial ordering

Output: A Gröbner basis G of $I = (S)$ with respect to " \leq "

- (1) $G := S \setminus \{0\}$
- (2) for $g, h \in G$ repeat (3),(4)
- (3) Compute $s := s_{pol}(g, h)$
 and a normal form s^* of s with respect to G
- (4) if($s^* \neq 0$)
 set $G := G \cup \{s^*\}$
 go to (2)
- (5) return G

2.7.15 Theorem 15 (Correctness of Algorithm 14)

Algorithm 14 terminates after finitely many steps and computes a Gröbner basis.

Proof:

Termination:

Let G_i be the set G obtained after the i -th run through the loop. $G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots$

From $\bar{G} = \bigcup_i G_i$ finite or infinite.

Lemma 4: $\exists B \subseteq M$ finite set of monomials, $B \subseteq \{LM(f) \mid f \in \bar{G}\}$ such that

$\forall f \in \bar{G} \quad \exists t \in B$ such that $t \mid LM(f)$ (*)

Since $|B| < \infty \quad \exists r$ such that $B \subseteq \{LM(f) \mid f \in G_r\}$

Without loss $B = \{LM(f) \mid f \in G_r\}$

Claim: G_r is the last of the G

If not $\exists G_{r+1} : G_{r+1} = G_r \cup \{s^*\}$

$s^* \neq 0$ in normal form with respect to G_r . But by (*) $\exists f \in G_r$ such that $LM(f) \mid LM(s^*)$ contradiction.

Correctness: by Theorem 13 \square

Example:

$S = \{\underbrace{x^2 + g^2}_f, \underbrace{xy}_g\} \subseteq \mathbb{Q}[x, y]$ " \leq " lex ordering with $x > y$

$s_{pol}(f, g) = yf - xg = y^3 =: h$ in normal form with respect to S

$G = \{f, g, h\}$

$s_{pol}(f, g) = h \xrightarrow{\text{normal form}} 0$

$s_{pol}(f, h) = y^3 f - x^2 h = y^5 \xrightarrow{\text{normal form}} 0$

$s_{pol}(g, h) = y^2 g - xh = 0$

$\Rightarrow G$ Gröbner basis

Cost of Buchberger algorithm:

- no known upper bound for the running time
- with $d = \max\{\deg(f) \mid f \in S\} : \underbrace{\deg(g_i)}_{\text{polys from } G} \leq 2 \cdot \left(\frac{d^2}{2} + d\right)^{2^{n-1}}$
with $n =$ number of Variables
 \Rightarrow "Doubly exponential" in n
Ritscher (2011): upper bound for $\deg(g_i)$ depending $\dim(\underbrace{\mathcal{V}}_{\text{Variety}}(S))$
- Nonetheless the algorithm often works
- Many possible optimizations

Variant: Extended Buchberger:

Keep track of how the new elements in G are represented as linear combination of elements of S .

2.7.16 Definition 16 (Reduced Gröbner basis)

A Gröbner basis G is called reduced if $\forall g \in G$

- g is in normal form with respect to $G \setminus \{g\}$
- $LC(g) = 1$

A given Gröbner basis can be turned into a reduced only by replacing every $g \in G$ by a normal form of g with respect to $G \setminus \{g\}$. Then remove $0 \in G$. Then divide each $g \in G$ by $LC(g)$

2.7.17 Theorem 17 (Uniqueness of reduced Gröbner basis)

From ideal $I \subseteq K[x]$ and a monomial ordering " \leq ", there exists a unique reduced Gröbner basis.

2.8 Application of Gröbner bases

2.8.1 Definition 1 (Elimination ideals)

- (a) $I \subseteq K[X_1, \dots, X_n]$ ideal, $l \in \{1, \dots, n\} \Rightarrow I_l := K[X_1, \dots, X_n]$ is called the l -length elimination ideal
- (b) A monomial ordering " \leq " is called an l -elimination ordering if $\forall 1 \leq i \leq l < j \leq n \quad \forall k_i \quad X_i^{k_i} < X_j$

Example:

- (1) Let " \leq " be a given monomial ordering. Define " \leq " by:
for $s = \underline{x}^d \quad t = \underline{x}^e$ define
 $s \leq' t \Leftrightarrow \sum_{i=l+1}^n d_i < \sum_{i=l+1}^n e_i$ or have equality $\sum_{i=l+1}^n d_i = \sum_{i=l+1}^n e_i$ and $s \leq t$
 \Rightarrow " \leq " is an l -eliminating ordering.
- (2) The lexicographic ordering with $x_1 < x_2 < \dots < x_n$ is an l -eliminating ordering for all l
- (3) Grevlex is not an l -eliminating ordering (unless $l = n$)

2.8.2 Theorem 2

Let G be a Gröbner basis of an ideal $I \subseteq K[X_1, \dots, X_n]$ with respect to an l -elimination ordering. Then $G_l := K[X_1, \dots, X_n] \cap G$ is a Gröbner basis. $I : l$ with respect to the restricted monomial ordering.

Proof:

$$G_l \subseteq I_l \text{ Let } f \in I_l \setminus \{0\} \quad f \in I$$

$$\Rightarrow \exists g \in G : LM(g) \mid LM(f)$$

To show: $g \in G$. Clearly $LM(g) \in K[X_1, \dots, X_l]$

If $g \notin K[X_1, \dots, X_n]$ then $\exists t \in M(g)$ such that

$$X_j \mid t \text{ with } j > l \Rightarrow t \geq X_j \underset{\text{"}\leq\text{" elim ord}}{>} LM(g) \text{ contradiction.} \quad \square$$

Example:

$$I = (\underbrace{X_1^2 + X_2^2}_f, \underbrace{x_1 x_2}_g), G = \{f, g, X^3\}$$

Gröbner basis with respect to lex ordering $X_1 < X_2$

$$I_1 = (X_1^3)$$

Geometric interpretation:

Let $K = \bar{K}$ algebraically closed. On K^n define the Zariski topology.

By saying that the sets $\mathcal{V}(I)$ with $I \subseteq K[X_1, \dots, X_n]$ are the closed sets.

Why is this a topology?

Reminder: A topological space is a set X together with a system of subsets, called closed subsets, such that three axioms hold.

- (1) $\emptyset = \mathcal{V}(K[X_1, \dots, X_n]) \quad K^n \in \mathcal{V}(\{0\})$ closed
- (2) Let \mathcal{M} be a set of closed subsets corresponding to a set \mathcal{N} of ideals.

$$\text{Then } \bigcap_{I \in \mathcal{N}} \mathcal{V}(I) = \mathcal{V}\left(\sum_{I \in \mathcal{N}} I\right) \quad \checkmark$$

- (3) The union of two closed subsets is closed.

Let $I, J \subseteq K[X]$ ideals.

Claim: $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J)$

Proof: " \subseteq " direction:

Let $v \in \mathcal{V}(I) \cup \mathcal{V}(J) \quad f \in I \cap J$

If $v \in \mathcal{V}(I)$ then $f(v) = 0$

$v \in \mathcal{V}(J)$ then $f(v) = 0$

" \supseteq " direction:

Let $v \in \mathcal{V}(I \cap J)$ Assume $v \notin \mathcal{V}(I) \Rightarrow \exists f \in I : f(v) \neq 0$

Let $g \in J \Rightarrow f \cdot g \in I \cap J \Rightarrow \underbrace{f(v)}_{\neq 0} \cdot g(v) = 0$

$\Rightarrow g(v) = 0 \quad \text{So } v \in \mathcal{V}(J)$

All points in K^n are closed so are all finite subsets.

Closures: For $X \subseteq K^n$ the closure \bar{X} is defined as the smallest closed subset containing X . \bar{X} is the variety of the largest ideal I such that $X \subseteq \mathcal{V}(I)$ This ideal is $I = Id(X)$.

So $\bar{X} = \mathcal{V}(Id(X))$

Let $\Pi_l : K^n \mapsto K^l, (a_1, \dots, a_n) \mapsto (a_1, \dots, a_l)$ projection.

2.8.3 Theorem 3

$$I \subseteq K[X_1, \dots, X_n] \Rightarrow \mathcal{V}(I) = \overline{\Pi_l(\mathcal{V}(I))}$$

Proof:

Let $(a_1, \dots, a_l) \in \Pi_l(\mathcal{V}(I)) \Rightarrow \exists n_{l+1} \dots n_1 \in K$ such that $(a_1, \dots, a_n) \in \mathcal{V}(I)$

Let $f \in I_l \Rightarrow f \in I \Rightarrow f(a_1, \dots, a_n) = 0$

But $0 = (a_1, \dots, a_n) = f((a_1, \dots, a_l))$ So $(a_1, \dots, a_n) \in \mathcal{V}(I_l)$

So $\Pi_l(\mathcal{V}(I)) \subseteq \mathcal{V}(I_l) \xRightarrow{\mathcal{V}(I_l) \text{ closed}} \overline{\Pi_l(\mathcal{V}(I))} \subseteq \mathcal{V}(I_l)$.

To show: $\mathcal{V}(I_l) \subseteq \overline{\Pi_l(\mathcal{V}(I))} \xLeftrightarrow{\text{H. Nullstellensatz}} Id(\Pi_l(\mathcal{V}(I))) \subseteq \sqrt{I_l}$ Take $f \in Id(\Pi_l(\mathcal{V}(I))) \Rightarrow$

$f \in K[X_1, \dots, X_l] \forall (a_1, \dots, a_n) \in \mathcal{V}(I)$.

$f(a_1, \dots, a_l) = f(a_1, \dots, a_n) = 0 \Rightarrow f \in Id(\mathcal{V}(I)) = \sqrt{I} \Rightarrow \exists k : f^k \in I \cap K[X_1, \dots, X_l] = I_l$

$\Rightarrow f \in \sqrt{I_l} \quad \square$

Example:

(1) $I = (xy - 1)$

$\Pi_1(\mathcal{V}(I)) = K \setminus \{0\}$ not closed. $\overline{\Pi_1(\mathcal{V}(I))} = K.I_1 = \{0\}$

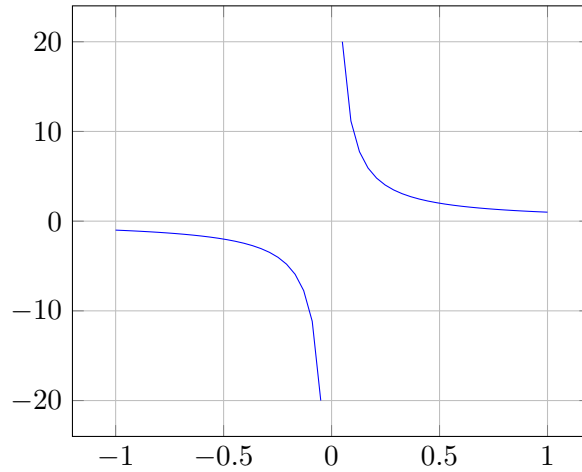


Figure 3: Plot of I in Example (1)

(2) $I = (xy - 2x + y - 2, kx^2 - 4xy + y^2)$ has reduced lex Gröbner basis:

$$\{x^2 - 2x + \frac{y^2}{4} + y - 2, xy - 2x + y - 2, (y - 2)(y + 2)^2\}$$

$$\Rightarrow I_1 = ((y - 2)(y + 2)^2)$$

$$\Rightarrow \overline{\Pi_1(\mathcal{V}(I))} = \{2, -2\}$$

$$\Rightarrow \Pi_1(\mathcal{V}(I)) = \{2, -2\}$$

Substitute $y = 2$:

$$x^2 - 2x + 1, 2x - 2x + 2 - 2 = 0$$

$$y = -2 :$$

$$x^2 - 2x - 3 = 0, -4x - 4 = 0$$

$$\rightarrow \mathcal{V}(I) = \{(1, 2), (-1, -2)\}$$

2.8.4 Algorithm 4 (Solving systems of algebraic equations)

Input : $f_1, \dots, f_m \in K[X_1, \dots, X_n]$

Output: $\mathcal{V}(f_1, \dots, f_m)$ if finite otherwise " ∞ "

- (1) Compute a Gröbner basis of $I = (f_1, \dots, f_m)$ with respect to the lex ordering with $X_1 < X_2 < \dots < X_n$
- (2) for $(l = 1, \dots, n)$
 set $G_l := [X_1, \dots, X_l] \cap G$
- (3) $M := \{()\} \subseteq K^0$
- (4) for $(l = 1, \dots, n)$ repeat (5) - (10)
- (5) $S = \emptyset$
- (6) for $(a_1, \dots, a_{l-1}) \in M$ repeat (7) - (9)
- (7) $g := \gcd\{f(a_1, \dots, a_{l-1} \mid f \in G_l)\}$
- (8) if $(g = 0)$
 return " ∞ "
- (9) $S := S \cup \{(a_1, \dots, a_{l-1}, a_l) \mid g(a_l) = 0\}$
- (10) $M := S$
- (11) return M

Intersections of ideals

2.8.5 Proposition 5

Let $I, J \subseteq K[\underline{x}]$ ideals, y additional variable.

Form $L \subseteq K[x_1, \dots, x_n, y]$ generated by $I \cdot y$ and $J \cdot (1 - y)$ Then $I \cap J = K[\underline{x}] \cap L$ (elimination ideal!)

Proof:

Let $f \in J \cap I \Rightarrow f = f \cdot y + f \cdot (1 - y) \in L \Rightarrow f \in K[\underline{x}] \cap L$

Conversely let $f \in K[\underline{x}] \cap L \Rightarrow f = \sum_{i=1}^r g_i f_i \cdot y_i + \sum_{i=r}^m g_i \cdot f_i (1 - y)$

with $f_1, \dots, f_r \in I$ $f_{r+1}, \dots, f_m \in J$ $g_i \in K[\underline{x}, y]$

Specialize $y = 0 \Rightarrow f = \sum_{i=r}^m \underbrace{g_i(y=0)}_{\in K[\underline{x}]} f_i \in J$

Specialize $y = 1 \Rightarrow f = \sum_{i=1}^r g_i(y=1) f_i \in I$

$\Rightarrow f \in I \cap J$ \square

Dimension

2.8.6 Definition 6 (independence modulo I)

Let $I \subseteq K[\underline{x}]$ ideal. Then polynomials $f_1, \dots, f_r \in K[\underline{x}]$ are called independent modulo I if for every polynomial $F \in K[y_1, \dots, y_r]$ have: $F(f_1, \dots, f_r) \in I \Rightarrow F = 0$

(So the classes $\bar{f}_i \in A := K[\underline{x}]/I$ are algebraically independent)

For $1 \leq i_1 < i_2 < \dots < i_r \leq n : x_{i_1}, \dots, x_{i_r}$ are independent modulo $I \Leftarrow K[x_{i_1}, \dots, x_{i_r}] \cap I = \{0\}$

(elimination ideal!)

$\dim(I) := \sup\{r \in \mathbb{N} \mid \exists f_1, \dots, f_r \in K[\underline{x}] \text{ independent mod } I\}$ dimension of I

In other words $\dim(I) =: \text{trdeg}(A)$ "transcendence degree"

If $I = K[\underline{x}] \Rightarrow \dim(I) = -1$

For $X = \mathcal{V}(I) : \dim(X) = \dim(I)$

Well defined? Clearly $\dim(I) = \dim(\sqrt{I})$

Geometric interpretation:

x_{i_1}, \dots, x_{i_r} are independent mod $I \Leftrightarrow \Pi_{i_1, \dots, i_r} : K^n \mapsto K^r$ maps $\mathcal{V}(I)$ to a dense subset of K^r

More generally for f_1, \dots, f_r have:

f_1, \dots, f_r are independent modulo $I \Leftrightarrow$ The image of $\mathcal{V}(I)$ under the "morphism" $K^n \mapsto K^r, v \mapsto (f_1(v), \dots, f_r(v))$ is dense.

2.8.7 Theorem 7

Let $I \subseteq K[\underline{x}]$ ideal

$\Rightarrow \dim(I) = \max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ are independent modulo } I\}$

2.8.8 Lemma 8

A non-empty set M of ideals in $K[\underline{x}]$ has an element that is maximal with respect to inclusion.

Proof:

If not obtain a sequence $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$ of ideals in M

Set $I := \bigcup_{j \in \mathbb{N}} I_j$ Check: $I \subseteq K[\underline{x}]$ ideal

$\Rightarrow I = (f_1, \dots, f_r)$ Each f_i lies in some I_j

$\xRightarrow{\text{Thm 7.8}} \exists m : f_i \in I_m \forall i \Rightarrow I \subseteq I_m \subseteq I \Rightarrow I = I_m$

$I_m \subseteq I_{m+1} \subseteq I = I_m \Rightarrow I_{m+1} = I_m$

Recall that an ideal $I \subseteq K[\underline{x}]$ is called a prime ideal if $a \cdot b \in I$ for $a, b \in K[\underline{x}]$ implies $a \in I$ or $b \in I$

Fact: I prime ideal $\Leftrightarrow I = \sqrt{I}$ and $\mathcal{V}(I)$ is irreducible, i.e. I can't be the written as union of two proper closed subsets.

2.8.9 Lemma 9

Every radical ideal $I \subsetneq K[x]$ is a finite intersection of prime ideals.

Proof:

If no by Lemma 8 there is a maximal exception $I \subsetneq K[x]$. I not prime ideal $\Rightarrow \exists a_1, a_2 \in K[x] : a_1 a_2 \in I, a_j \notin I \Rightarrow I \subsetneq I + (a_j) \subseteq \sqrt{I + (a_j)} =: I_j$ (*)
 $\Rightarrow I \subseteq I_1 \cap I_2$

Claim: $I = I_1 \cap I_2$ ✓

Let $g \in I_1 \cap I_2 \Rightarrow \exists r : g^r \in I + (a_j) (j = 1, 2)$
 $\Rightarrow g^{2r} \in (I + (a_1)) \cdot (I + (a_2)) \subseteq I \xrightarrow{I=\sqrt{I}} g \in I$

(*) with maximality of I shows that the I_j are finite intersections of prime ideals

\Rightarrow same holds for I contradiction \square

Reminder: If $L | K$ is a field extension $L = K[\alpha_1, \dots, \alpha_n]$. $\text{trdeg}_K(L)$ the size of every maximally algebraically independent subset of $\{\alpha_1, \dots, \alpha_n\}$

(Exchange argument similar to linear algebra)

Proof of Theorem 7

Clearly $\max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ independent modulo } I\} \leq \dim(I)$

For reverse inequality take $f_1, \dots, f_r \in K[x]$ independent mod I . By Lemma 9: $\sqrt{I} = P_1 \cap \dots \cap P_s$ with P_i prime ideals.

Assume that the f_i are dependent modulo every $P_i \Rightarrow \exists F_i \in K[y_1, \dots, y_r] \setminus \{0\}$ such that

$F_i(f_1, \dots, f_r) \in P_i \Rightarrow$ For $F := \prod_{i=1}^s F_i$

$F(f_1, \dots, f_r) \in P_1 \cap \dots \cap P_s = \sqrt{I} \Rightarrow \exists k$

$F^k(f_1, \dots, f_r) \in I$. But $F^k \neq 0$ contradiction

So $\exists i : f_1, \dots, f_r$ independent mod P_i

$\bar{f}_1, \dots, \bar{f}_r \in A := K[x]/P$ algebraically independent.

A integral domain $\Rightarrow L = \text{Quot}(A)$ exists $\bar{f}_i \in A \subseteq L$. L/K is field extension, $L = K(\bar{x}_1, \dots, \bar{x}_n)$ Since $\text{trdeg}_K(L) \geq r$ have i_1, \dots, i_r such that $\bar{x}_{i_1}, \dots, \bar{x}_{i_r} \in L$ are algebraically independent over K .

$\Rightarrow x_{i_1}, \dots, x_{i_r}$ are independent modulo P_i

Since $I \subseteq P_i$, x_{i_1}, \dots, x_{i_r} are independent modulo I

$\Rightarrow r \leq \max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ independent modulo } I\}$

Now we can compute $\dim(I)$ by using elimination ideals. Have to determine a subset of $\{x_1, \dots, x_n\}$ of maximal size that is independent modulo $I \rightarrow$ Gröbner bases.

Example:

$I = \underbrace{(xy, xz)}_{\text{Gröbner bases}}$ Independent subsets modulo $I : \emptyset, \{x\}, \{y\}, \{z\}, \{y, z\}$

$\dim(I) = 2$

Hilbert series For $X \subseteq K^n$ affine variety, $I = Id(X)$. Then $A := K[x]/I$ is the set of polynomial function ("regular functions") $X \mapsto K$

How large is A ? A is a vector-space but it is not finite-dimensional (as vector-space) unless $\dim(X) \leq 0$

Idea: measure the size of A by studying the growth of the dimensions of parts of A given by a filtration $A = \bigcup_{d \in \mathbb{N}} A_d$

2.8.10 Definition 10 (Hilbert series)

Let $I \subseteq K[x]$ ideal, $A := K[x]/I$

For $d \in \mathbb{N}$ set $A_d := \{f + I \mid f \in K[x], \underbrace{\deg(f)}_{=\max\{\deg(t) \mid t \in M(f)\}} \leq d\} \subseteq A$ subspace

Hilbert function: $h_I(d) := \dim_K(A_d) < \infty$

Hilbert series of I : $H_I(t) = \sum_{d=0}^{\infty} h_I(d) \cdot t^d \in \mathbb{Z}[[t]]$ (formal power series)

Example:

$$(1) \quad I = (x_1, \dots, x_n) \Rightarrow A = K \Rightarrow A_d = K \quad \forall d$$

$$h_I(d) = 1 \Rightarrow H_I(t) = \sum_{d=0}^{\infty} t^d = \frac{1}{1-t}$$

$$(2) \quad I = (x_1 - x_2^2) \subseteq K[x_1, x_2]$$

The classes of $1, x_1, x_1^2, \dots, x_1^d, x_2, x_2x_1, \dots, x_2x_1^{d-1}$ form a basis of A_d

$$\Rightarrow h_I(d) = 2d + 1$$

$$H_I(t) = \frac{1+t}{(1-t)^2}$$

$$(3) \quad I = \{0\} \Rightarrow A = \underbrace{K[x]}_{=x_1, \dots, x_n}$$

write $H_n(t)$ for Hilbert series $H_0(t) = \frac{1}{1-t}$

$$\text{For } n > 0 : K[x_1, \dots, x_n]_d = \bigoplus_{i+j=d} K[x_1, \dots, x_{n-1}] \cdot x_n^j$$

$$\Rightarrow H_n(t) = H_{n-1}(t) \left(\sum_{j=0}^{\infty} t^j \right) = H_{n-1}(t) \cdot \frac{1}{1-t} = \frac{1}{(1-t)^{n+1}}$$

Hilbert function:

$$H_n(t) = (1-t)^{-n-1} = \sum_{d=0}^{\infty} \binom{-n-1}{d} (-t)^d$$

$$\Rightarrow h_n(d) = \binom{-n-1}{d} \cdot (-1)^d = \binom{n+d}{d} = \binom{d+n}{n}$$

A total degree monomial ordering is a monomial ordering such that for $t, t' \in M$ have: $t \leq t'$ implies $\deg(t) \leq \deg(t')$

Example: grevlex

2.8.11 Theorem 11

Let " \leq " be a total degree monomial ordering. $I \subseteq K[x]$ ideal $\Rightarrow H_I(t) = H_{L(I)}(t)$

Proof:

Let G be a Gröbner basis of I , $d \in \mathbb{N}$

NF_G induces an injection from $\phi : A \mapsto K[\underline{x}]$. Consider $\phi_d : A_d \mapsto K[\underline{x}]$ restriction

Claim: $im(\phi_d)$ is the space $V_d \subseteq K[\underline{x}]$ generated by all monomials $m \in M$

with $\deg(m) \leq d$ such that $LM(g) \nmid m \ \forall g \in G$

Let $f \in V_d \Rightarrow f$ is in normal form with respect to $G \Rightarrow f = NF_G(f) = \phi(\underbrace{f + I}_{\in A_d}) \in im(\phi_d)$

Conversely let $f \in im(\phi_d) \Rightarrow \exists g \in K[\underline{x}] :$

$$\deg(g) \leq d$$

$$f = NF_G(g)$$

$$\Rightarrow f = g - \sum_{i=1}^m h_i g_i \quad \text{with } g_i \in G$$

$$h_i \in K[\underline{x}]$$

$$LM(h_i g_i) \leq LM(f)$$

$$\Rightarrow \forall t \in M(h_i g_i) : t \text{ "}\leq\text{" tot. deg. mon. ord.} \Rightarrow \deg(1) \leq \deg(LM(g))$$

$$\text{So } \deg(h_i g_i) \leq d \Rightarrow \deg(f) \leq d$$

(*)

Since f is in normal form this implies $f \in V_d$.

$$\text{So } V_d = im(\phi_d)$$

$$\Rightarrow h_I(d) = \dim_K(A_d) = \dim_K(im(\phi_d)) = \dim(V_d)$$

V_d only depends on d and on $(LM(g) \mid g \in G) = L(I) \Rightarrow h_I$ only depends on $L(I)$

Since $L(L(I)) = L(I)$ the theorem follows \square

How to compute $H_I(t)$ for I monomial ideal:

Let $I = (m_1, \dots, m_l) \quad m_i \in M$ Set $J = (m_1, \dots, m_{l-1})$

Then the map $J \xrightarrow{\text{surjective}} I/(m_l)$ has the kernel $J \cap (m_l)$

$$\Rightarrow J/(J \cap (m_l)) \cong I/(m_l)$$

This isomorphism restricts to all homogeneous components

$$\Rightarrow H_I(t) = H_{(m_l)}(t) + H_J(t) - H_{J \cap (m_l)}(t) \quad (*)$$

$$\text{Have } J \cap (m_l) = (lcm(m_1, m_l), \dots, lcm(m_{l-1}, m_l))$$

2.8.12 Theorem 12

Let $I = (m_1, \dots, m_l) \subseteq K[x_1, \dots, x_n]$ with $m : i \in M$

$$\Rightarrow H_I(t) = \frac{1}{(1-t)^{n+1}} \sum_{S \subseteq \{1, \dots, l\}} (-1)^{|S|} \cdot t^{\deg(lcm\{m_i \mid i \in S\})}$$

Proof:

Use induction on l , $(*)$ and bookkeeping.

$l = 0$: Example (3) ✓

$l = 1$: $I = (m_1)$

$$\sum_{d=0}^{\infty} \dim(\underbrace{I_{\leq d}}_{\text{all polys in } I \text{ of deg } \leq d}) \cdot t^d = t^{\deg(m_1)} \cdot H_{\{0\}}(t)$$

$$\Rightarrow H_{(m_1)}(t) = H_{\{0\}}(t) - \sum_d \dim(I_{\leq d}) = \frac{1-t^{\deg(m_1)}}{(1-t)^{n+1}} \quad \checkmark$$

$l - a \rightarrow l$ (with $l \geq l$): By $(*)$ with $I = (m_1, \dots, m_l)$:

$$(1-t)^{n+1} \cdot H_I(t) = 1 - t^{\deg(m_l)} + \sum_{S \subseteq \{1, \dots, l-1\}} (-1)^{|S|} \cdot t^{\deg(\text{lcm}\{m_i | i \in S\})} \\ - \sum_{S \subseteq \{1, \dots, l-1\}} (-1)^{|S|} \cdot t^{\deg(\text{lcm}\{\text{lcm}(m_i, m_l) | i \in S\})}$$

For $S \neq \emptyset$ then $\text{lcm}\{\text{lcm}(m_i, m_l) | i \in S\} = \text{lcm}\{m_i | i \in S \cup \{l\}\}$

So the formula is correct \square

\rightarrow may compute Hilbert series by Gröbner bases!

2.8.13 Corollary 13 (Hilbert-Serre theorem)

Let $I \subseteq K[\underline{x}]$ ideal.

Then $H_I(t) = \frac{a_0 + a_1 t + \dots + a_k t^k}{(1-t)^{n+1}}$ $a_i \in \mathbb{Z}$

For $d \gg 0$ $h_I(d)$ is a polynomial. More precisely with

$$p_I := \sum_{i=0}^k a_i \binom{x+n-i}{n} \in \mathbb{Q}[x]$$

have $h_I(d) = p_I(d)$ for $d \gg 0$

p_I is called the Hilbert polynomial.

2.8.14 Definition 14 (Algebra, homo-, isomorphism)

An algebra over a field K is a commutative ring A containing K .

(Usually $A = \{0\}$ is also considered as an algebra)

A is called finitely generated if

$$\exists a_1, \dots, a_n \in A : A = \{f(a_1, \dots, a_n) \mid f \in K[x_1, \dots, x_n]\} = K[a_1, \dots, a_n]$$

A homomorphism $\phi A \mapsto B$ of k -algebras is a ring homomorphism that fixes k element-wise.

ϕ is an isomorphism it is bijective.

So if $A = K[a_1, \dots, a_n]$ we have a surjective homomorphism

$$\phi : K[x_1, \dots, x_n] \mapsto A, \quad f \mapsto f(a_1, \dots, a_n). \quad \text{If } I := \ker(\phi) \text{ then } K[x_1, \dots, x_n]/I \cong A$$

Given a finitely generated algebra A (such as $A = K[x] = K[\underline{x}]/I$ with $x = \mathcal{V}(I)$ affine variety), then each choice of finite set of generators affords an isomorphism

$K[x_1, \dots, x_n]/I \cong A$. Then p_I will depend on the choice of the generators. But might the degree of p_I be an invariant of A ?

2.8.15 Lemma 15

Let $I \subseteq K[x_1, \dots, x_n]$, $J \subseteq K[y_1, \dots, y_n]$ ideals in polynomial rings such that $\underbrace{K[x_1, \dots, x_n]/I}_A \cong \underbrace{K[y_1, \dots, y_n]/J}_B$. Then $\deg(p_I) = \deg(p_J)$.

Proof:

Have $\phi A \xrightarrow{\sim} B \Rightarrow \exists g_1, \dots, g_m \in K[x_1, \dots, x_n]$ such that $\phi(g_i + I) = y_i + J$

Choose $e \in \mathbb{N}$ such that $\deg(g_i) \leq e$

Then $B_d \subseteq \phi(A_{de}) \Rightarrow h_J(d) = \dim(B_d) \leq \dim(A_{de}) = h_I(de) \Rightarrow \deg(p_J) \leq \deg(p_I)$

By symmetry $\deg(p_I) \leq \deg(p_J)$, have equality $\deg(p_J) = \deg(p_I)$ \square

Goal: $\deg(p_I) = \dim(I)$

2.8.16 Theorem 16 (Noether normalization)

Let $A \neq \{0\}$ be a finitely generated algebra. Then there exist algebraically independent elements $c_1, \dots, c_m \in A$ such that A is finitely generated as a module (= vector space, but over a ring rather than a field) over $C = K[c_1, \dots, c_m] \underset{\text{subalgebra}}{\subseteq} A$,

i.e. $\exists b_1, \dots, b_l \in A$ such that $A = \sum_{i=1}^l C \cdot b_i$

Moreover if $A \cong K[x]/I$ then $m = \dim(I)$ ($= \text{trdeg}(A)$)

If $m = 0$ then $C = K$

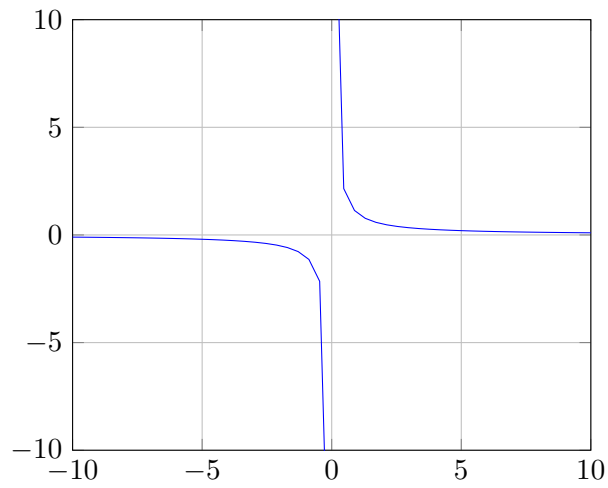
Proof:

Omitted \rightarrow commutative algebra \square

Geometric interpretation: Let $A = K[X]$, X affine variety. Choosing $c_1, \dots, c_m \in A$ amounts to choosing a morphism $f : X \mapsto K^m$, $v \mapsto (c_1(v), \dots, c_m(v))$. It can be shown that if $C = K[c_1, \dots, c_m]$ is Noether norm, then the morphism is surjective and with finite fibers i.e. $\forall w \in K^m$ the set $f^{-1}(\{w\})$ is finite.

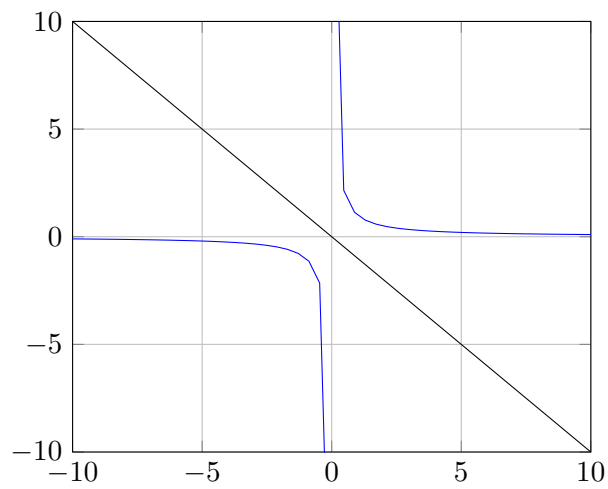
Example:

$X = \mathcal{V}(xy - 1) \subseteq \mathbb{C}^2$



The projection to the x- and y-axis are not surjective.

Try $c = \bar{x} - \bar{y} \in A = \mathbb{C}[x, y]/(xy - 1)$



Have $\bar{x}^2 - c\bar{x} - 1 = 0 \Rightarrow \forall i : \bar{x}^i \in C \cdot 1 + C\bar{x}$

Similarly $f \cap \bar{y} \Rightarrow C = \mathcal{C}[c]$ satisfies Theorem 16

2.8.17 Theorem 17

Let $I \subseteq K[x_1, \dots, x_n]$ ideal $\Rightarrow \dim(I) = \deg(p_I)$

Proof:

If $I = K[x] \Rightarrow \checkmark$

So assume $I \not\subseteq K[x]$. Then by Theorem 16 $\exists C = K[c_1, \dots, c_m] \subseteq A := K[x]/I$ and

$\exists b_1, \dots, b_l \in A : A = \sum_{i=1}^l C \cdot b_i, \quad c_1, \dots, c_m$ algebraic independent. Without loss $b_1 = 1$

Let $y_1, \dots, y_m, z-1, \dots, z_l$ be new indeterminates and consider the map

$K[y_1, \dots, y_m, z-1, \dots, z_l] \mapsto A, \quad y_i \mapsto c_i, \quad z_j \mapsto b_j$

This is surjective. Let J be the Kernel.

By Lemma 15 have $\deg(p_I) = \deg(p_J)$.

Need to show: $\deg(p_J) = m$

Write $B : K[y_1, \dots, y_m, z-1, \dots, z_l]/J$, consider $B_d, C_d := \{f+J \mid f \in K[y_1, \dots, y_m], \deg(f) \leq d\} \subseteq B_d$ (for $d \in \mathbb{N}$).

Since $C_d \subseteq B_d$, obtain $h_J(d) = \dim(B_d) \geq \dim(C_d) = \dim(K[y_1, \dots, y_m])_d$
 $c_1, \dots, c_m \text{ alg. ind.} = \binom{d+m}{m}$

$\Rightarrow \deg(p_J) \geq m$

For $1 \leq i \leq j \leq l$ have $b_i b_j = \sum_{k=1}^l a_{ijk} \cdot b_k$ with $a_{ijk} \in C$ Choose $e \in \mathbb{N}_{>0}$ such that

$\forall i, j, k : \deg(a_{ijk}) \leq e \Rightarrow b_i b_j \subseteq \sum_{k=1}^l C_{(s-1) \cdot e} \cdot b_k$. So the product of s of the b_i lies in $\sum_{k=1}^l C_{(s-1) \cdot e} \cdot b_k$.

This implies (for $d \in \mathbb{N}$) $B_d \subseteq C_d \cdot b_1 + \sum_{s=1}^d \sum_{k=1}^l C_{d-s} \cdot C_{(s-1)e} \cdot b_k \subseteq \sum_{k=1}^l C_{de} \cdot b_k =: V_d$

$\Rightarrow h_J(d) = \dim(B_d) \leq \dim(V_d) \leq l \cdot \dim(C_{de}) = l \cdot \binom{d \cdot e + m}{m} = \deg(p_J) \leq m \quad \square$

(or substitute e by $2e$)

2.8.18 Corollary 18

Let " \leq " be a graded monomial ordering $I \subseteq K[x]$ ideal $\Rightarrow \dim(I) = \dim(L(I))$

For computing $\dim(L(I))$ use the transcendence degree!

Let $L(I) = (m_1, \dots, m_l)$, m_i monomials. For $S \subseteq \{x_1, \dots, x_n\}$,

S is dependent modulo $L(I) \Leftrightarrow \exists i : m_i \in K[S]$

S is independent modulo $L(I) \Leftrightarrow \forall i : m_i \notin K[S]$

i.e. m_i involves a variable not in S .

$T := \{1, \dots, n\} \setminus S$ is independent modulo $L(I)$

\Leftrightarrow Every m_i involves a variable that is in S

2.8.19 Algorithm 19 (Dimension of I)

Input : $I \subseteq K[\underline{x}]$ ideal

Output: $\dim(I)$

- (1) Compute a Gröbner basis of I with respect to a graded monomial ordering.
Let m_i, \dots, m_l be the leading monomials of the polynomials in G
- (2) if $(\exists i : m_i = 1)$
 $\dim(I) = -1$
- (3) Find $T \subseteq \{x_1, \dots, x_n\}$ of minimal size,
such that every m_i involves a variable from T
// don't worry about exponential time for this step as (1) is doubly exp anyway
- (4) $\dim(I) = n - |T|$

3 Notes

3.1 Notation

- $\mathbb{N} := \mathbb{N}_0$
- $\lg(x) := \log_2(x)$
- $a \mid b$ a is divisible by b \Leftrightarrow $b \bmod a = 0$
 $a \nmid b$ a is not divisible by b \Leftrightarrow $b \bmod a \neq 0$
- $ord(a)$ order of a group element
 $n > 0$ minimal such that $a^n = e$ with neutral element e
if no such n can be found, $ord(a) = \infty$
- $char(A)$ Characteristic: the smallest positive n such that
 $\underbrace{1 + \dots + 1}_n = 0$ with 1 as the multiplicative identity element
 n summands
- $im(f)$ image of f ("Wertebereich")
- $Ker(\phi)$ kernel of ϕ ; Number of elements that map to zero
 \Rightarrow degree to which the homomorphism ϕ fails to be injective
- $\mathbb{Z}/(m)$ Ring modulo m
polynomial rings measure for " $<$ " relations not the absolute value but max power.
- $lcm(a_1, \dots, a_n)$ "least common multiple of all a_i "
- \underline{e} = vector of e 's
- $K[\underline{x}] := K[x_1, \dots, x_n]$
- $\phi(n) := |\{x \in \mathbb{N} : x < n \wedge \gcd(x, n) = 1\}| = |(\mathbb{Z}/(n))^x|$
Euler's totient function
- $rk(A)$ Rank of matrix A
- $\left(\frac{n}{p}\right) := \begin{cases} 1 & \text{if } p \mid n \\ -1 & \text{if } n \text{ is a square (mod } p) \\ 0 & \text{otherwise} \end{cases}$
Legendre symbol (this is not a fraction)
- $\left(\frac{n}{p}\right) = 1 \Leftrightarrow n^{\frac{p-1}{2}} = \left(\frac{n}{p}\right) \equiv 1 \pmod{p}$
Eulers criterion
- $res(f, g)$ resultant. \Rightarrow det of Sylvester-Matrix
- $\mathbb{A} :=$ Affine space
- $Id(S) =$ Ideal of S

3.2 Various stuff

- Lagrange's theorem
Every element in a finite group has finite order
- Average number of bit operations for an increment:
One operation for the last bit + 50% chance for one on the next bit + 25% on the following etc. \Rightarrow Geometrical row
 \Rightarrow on average two bit operations
- "Monte Carlo Algorithm"
Always terminates in reasonable time but might yield false result.
- "Las Vegas Algorithm"
If it terminates the result is correct. No deterministic running time.
- Chinese remainder theorem
Given a system of congruences $x \equiv a_i \pmod{m_i}$ with $i = 1, \dots, r$
 m_i pairwise co-prime. Then the unique solution is:
$$x \equiv a_1 \cdot b \cdot \frac{N}{m_1} + \dots + a_r \cdot b_r \cdot \frac{N}{m_r} \pmod{N} \quad \text{with } b_i \cdot \frac{N}{m_i} \equiv 1 \pmod{m_i}$$
- distance between two square numbers:
 $(n+1)^2 - n^2 = 2n + 1$
 \Rightarrow Squares are much more scarce than primes!
- $ax + by = c$ has solutions in \mathbb{Z} iff $\Leftrightarrow \gcd(a, b) \mid c$ with $a, x, b, y \in \mathbb{Z}$
- $S_{f,g} = \begin{pmatrix} f_m & \dots & f_0 & 0 & \dots & 0 \\ 0 & f_m & \dots & f_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dots & \ddots & 0 \\ 0 & \dots & 0 & f_m & \dots & f_0 \\ g_n & \dots & g_0 & 0 & \dots & 0 \\ 0 & g_n & \dots & g_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dots & \ddots & 0 \\ 0 & \dots & 0 & g_n & \dots & g_0 \end{pmatrix}$ Sylvester-Matrix for $f(x), g(x)$
- finitely generated
A set M is finitely generated if $\exists R^n \mapsto M$ surjective with R^n finite
- monomial ideal $I \Leftrightarrow \exists$ a set of generators that are monomials only

3.3 Algebraic structures

- Group $(G, *)$
 - one inner operation $(*)$: $G \times G \mapsto G$
 - associativity: $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
 - neutral element $e \in G$: $a * e = e * a = a \quad \forall a \in G$
 - inverse element $a^{-1} \in G$: $a * a^{-1} = a^{-1} * a = e \quad \forall a \in G$
- Abelian group $(G, *)$

- $(G, *)$ is a group
- commutativity: $a * b = b * a \quad \forall a, b \in G$
- Finite group $(G, *)$
 - associativity: $(a * b) * c = a * (b * c)$
 - unambiguity of reduction: $(a * x = a * x') \wedge (x * a = x' * a) \Rightarrow x = x'$
 $\Rightarrow x \mapsto x * a$ and $x \mapsto a * x$ is bijective
 $\Rightarrow \exists x : a * x = a \Rightarrow$ neutral element
 $\exists x : a * x = x \Rightarrow$ inverse element
- Cyclic group $(G, *)$
 - G is a group
 - G is generated by one Element: $G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$
 - not necessarily finite.
- Semi group $(S, *)$
 - one inner operation $(*)$: $S \times S \mapsto S$
 - associativity: $(a * b) * c = a * (b * c) \quad \forall a, b, c \in S$
- Field $(K, +, \cdot)$
 - two inner operations $(+, \cdot)$ such that:
 - $(K, +)$ is an abelian group with neutral element 0
 - $(K \setminus \{0\}, \cdot)$ is an abelian group with neutral element 1
 - distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in K$
- General linear group $GL_n(K)$
 - K is a field
 - $GL_n(K)$ is the set of $n \times n$ invertible matrices with ordinary matrix multiplication
- Ring $(R, +, \cdot)$
 - $(R, +)$ is an abelian group
 - (R, \cdot) is a semi group
 - distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in R$
- Commutative ring $(R, +, \cdot)$ is a ring
 - commutativity for (\cdot) $a \cdot b = b \cdot a \quad \forall a, b \in R$
- Unitary ring (ring with 1) $(R, +, \cdot)$
 - (R, \cdot) is a semi group
 - (R, \cdot) has a neutral element "1"
- Euclidean ring R
 - $\exists F : R \mapsto \mathbb{N}_0 \cup \{0\}$
such that if $\exists q, r \in R \quad a = b \cdot q + r$ and $r = 0$ or $a, b \in R \quad F(r) < F(b)$
- Polynomial ring $R[X]$
 - R is a commutative unitary ring
 - set of all polynomials with coefficients $\in R$
 - Variables $X_1 \dots X_n$
- Noetherian Ring R

The following definitions are equal:

- for $I_1 \subseteq I_2 \subseteq \dots \quad \exists n : I_n = I_{n+1} = \dots$ (the chain of ideals "stabilizes")
- every ideal of R is finitely generated

3.4 Invertible elements

- Let $(\mathbb{Z}/(n), +)$ be a group or $(\mathbb{Z}/(n))^\times$ be a group with multiplication.
- $|(\mathbb{Z}/(n))^\times| = \phi(n)$
- $n \in \mathbb{P}$
 $\Rightarrow (\mathbb{Z}/(n))^\times = \{\bar{0}, \dots, \bar{p-1}\} \cong (\mathbb{Z}/(p-1), +) = Z_{p-1}$ (cyclic Group Z)
- n is a power of 2
 $\Rightarrow (\mathbb{Z}/(2^e))^\times \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2^{e-2})$
- n is a power of an odd Prime
 $\Rightarrow (\mathbb{Z}/(p^k))^\times \cong \mathbb{Z}/(p^{k-1} \cdot (p-1)) \cong Z_{(p^{k-1} \cdot (p-1))}$
- $n = p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$
 $\Rightarrow (\mathbb{Z}/(n))^\times \cong (\mathbb{Z}/(p_1^{k_1}))^\times \times \dots \times (\mathbb{Z}/(p_r^{k_r}))^\times$