

# Computational Algebra

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## Transcript

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# 1 Integer Arithmetic

Topics:

- Addition and Multiplication
- GCD computation
- Primality testing
- Factorization

## 1.1 Addition and Multiplication

Agreement:

- $a, x \in \mathbb{N}$  represented as  $x = \sum_{i=0}^{n-1} a_i \cdot B^i$   $B \in \mathbb{N}_{>1}$  fixed Base ( $a_i \in \{0, \dots, B-1\}$ )
- if  $x \neq 0$ , assume  $a_{n-1} \neq 0$  then define:  
length of  $x := l(x) = n$  = number of digits =  $\lfloor \log_B(x) \rfloor + 1$   
(mnemonic:  $\log_B(B) + 1 = 2$ )
- $l(0) = 1$   
(Amount of memory required to store  $x = 0$ )
- $l(x) := l(|x|)$
- for  $x \in \mathbb{Z}$  represent if as  $x = \text{sgn}(x) * |x|$

### 1.1.1 Algorithm 1 (Simple addition)

input :  $x = \sum_{i=0}^{n-1} a_i \cdot B^i$ ,  $y = \sum_{i=0}^{n-1} b_i \cdot B^i$ ,  $x, y \in \mathbb{N}$

output:  $x + y = \sum_{i=0}^n c_i \cdot B^i$

- (1)  $\sigma = 0$
- (2) for  $i = 0, \dots, (n-1)$  :
- (3)     set  $c_i := a_i + b_i + \sigma_i$    and    $\sigma := 0$
- (4)     if  $(c_i \geq B)$
- (5)         set  $c_i = c_i - B$
- (6)         set  $\sigma = 1$
- (7) set  $c_n = \sigma$

If  $B = 2$  then (3) - (6) can be realized by logic gates:

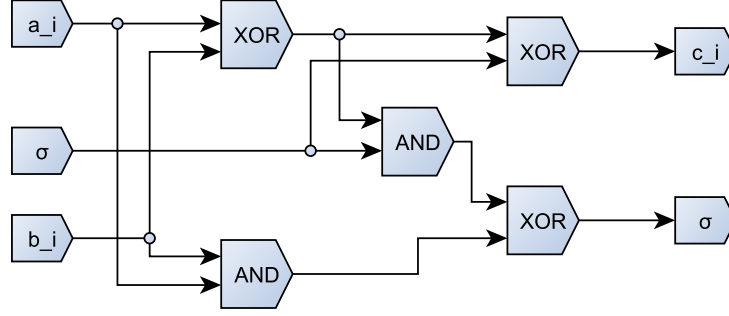


Figure 1: Logic circuit for addition

### 1.1.2 Definition 2 (Bit-Operation)

A bit operation is an operation that can be performed by a logic gate or by searching or writing a bit from / into memory.

### 1.1.3 Definition 3 (Big O)

Let  $M$  be a set (usually  $M = \mathbb{N}$ ),  $f, g : M \mapsto \mathbb{R}_{>0}$   
we write  $f \in O(g)$  if  $\exists c \in \mathbb{R} : f(x) \leq c \cdot g(x) \forall x \in M$

### 1.1.4 Theorem 4 (Lower bound for addition)

Let  $f : \mathbb{N} \mapsto \mathbb{R}$ ,  $n \mapsto$  maximal number of bit operations required by Algorithm 1 to add  $x, y \in \mathbb{N}$  with  $l(x), l(y) \leq n$

Let  $g = id_{\mathbb{N}}$  Then  $f \in O(g)$

We say Algorithm 1 requires  $O(n)$  bit operations for adding two numbers of length  $\leq n$ .  
 $\Rightarrow$  "linear complexity"

Set  $M := \{\text{Set of all algorithms for addition in } \mathbb{N}\}$

For  $A \in M$  define  $f_A : \mathbb{N} \mapsto \mathbb{R}$  as above.

We would like to find  $f_{\text{odd}} : \mathbb{N} \mapsto \mathbb{R}$ ,  $n \mapsto \inf\{f_A(n) | A \in M\}$

Since one needs to read  $x, y$  (and write the result) we can not do any better than linear complexity for addition.

#### Subtraction

let  $x, y$  as Algorithm 1,  $x \geq y$

For  $\bar{y} := \sum_{i=0}^{n-1} (B-1-b_i)B^i$  (digitwise / bitwise complement)

$\Rightarrow x + \bar{y} = x - y + B^n - 1$

$\Rightarrow x - y = x + \bar{y} + 1 - B^n$  (initially set  $\sigma = 1$ )

**Conclusion:** Addition and Subtraction have cost  $O(n)$

### 1.1.5 Algorithm 5 (Multiplication by "grid method")

input :  $x = \sum_{i=0}^{n-1} a_i \cdot 2^i, \quad y = \sum_{i=0}^{m-1} b_i \cdot 2^i$

output:  $z = x \cdot y$

- (1)  $z := 0$
- (2) for  $i = 0, \dots, (n-1)$
- (3)     if  $(a_i \neq 0)$  set  $z := z + \sum_{j=0}^{m-1} b_j 2^{i+j}$

### 1.1.6 Theorem 6 (Runtime of Algorithm 5)

Algorithm 5 requires  $O(n * m)$  bit operations.

As of the total input length  $n + m$ :

$$n \cdot m \leq \frac{1}{2}(n + m)^2 \rightarrow O((n + m)^2)$$

$\Rightarrow$  Quadratic complexity

### Karatsuba-multiplication:

Observation for polynomials:

$$a + bx, c + dx \text{ have } (a + bx)(c + dx) = ac + (ac + db - (a - b)(c - d))x + bdx^2$$

The point: only used 3 multiplications instead of 4.

Specialize  $x = B$  "large" such that  $x = a + bB$  partition into two blocks. Then multiply the blocks by a recursive call.

### 1.1.7 Algorithm 7 (Karatsuba)

input :  $x, y \in \mathbb{N}$

output:  $z = x \cdot y$

- (1) Choose  $k \in \mathbb{N}$  minimal such that  $l(x), l(y) \leq 2^k$ .  
Set  $B = 2^{2^{k-1}}$
- (2) if  $(k = 0)$  return  $x \cdot y$  (by bit-operation AND)
- (3) write  $x = x_0 + x_1 B, \quad y = y_0 + y_1 B$  with  $l(x_i), l(y_i) \leq 2^{k-1}$
- (4) compute  $x_0 \cdot y_0, \quad x_1 \cdot y_1, \quad (x_0 - x_1) \cdot (y_0 - y_1)$  by a recursive call
- (5) return  $z = x_0 y_0 + (x_0 y_0 + x_1 y_1 - (x_0 - x_1)(y_0 - y_1))B + x_1 y_1 B^2$

### 1.1.8 Theorem 8 (Runtime of Algorithm 7)

For multiplying two numbers of length  $\leq n$  Algorithm 7 requires  $O(n^{\log_2 3}) \approx O(n^{1.59})$  bit operations.

**Proof:**

Set  $\Theta(k) :=$  maximal numbers of bit operations for  $l(x), l(y) \leq 2^k$

We have for  $k > 0$  :  $\Theta(k) \leq 3 \underbrace{\Theta(k-1)}_{\text{recursive calls}} + c \underbrace{2^k}_{\text{additions}}$  with  $(c \text{ some constant})$

**Claim:**  $\Theta(k) \leq 3^k + 2c(3^k - 2^k)$

**Proof by Induction on  $k$ :**

$k = 0$  :  $\Theta(k) = 1$

$$\begin{aligned} k-1 \rightarrow k : \Theta(k) &= 3\Theta(k-1) + c2^{k-1} \\ &\leq 3(3^{k-1} + 2c(3^{k-1} - 2^{k-1})) + c2^{k-1} \\ &= 3^k + 2c(3^k - 2^k) \end{aligned}$$

So  $\Theta(k) \leq (2c+1)3^k$

Now  $l(x) \leq n$  hence  $2^{k-1} < n$  by minimality of  $k$

So  $k-1 < \log_2 n$

$$\begin{aligned} \Rightarrow \Theta(k) &\leq 3(2c+1)3^{\log_2(n)} \\ &= 3(2c+1)2^{\log_2(3) \log_2(n)} \\ &= 3(2c+1)n^{\log_2(3)} \quad \square \end{aligned}$$

One can modify the terminal condition of Karatsuba to switch to Grid-Multiplication, which is faster for small numbers.

### Fast-Fourier Transform

Reminder: For a function  $f : \mathbb{R} \mapsto \mathbb{C}$  define:

$\hat{f} : \mathbb{R} \mapsto \mathbb{C}$  by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt \quad (\text{if it exists})$$

Think of  $\omega$  as frequency.

### Definition (Convolution)

Let  $f, g : \mathbb{R} \mapsto \mathbb{C}$

$$(f * g)(x) = \int_{\mathbb{R}} f(t) g(x-t) dt$$

Convolution is analogous to polynomial multiplication **Formula:**  $\underbrace{(f * g)}_{\text{(Cauchy formula)}} = \hat{f} \cdot \hat{g}$

For a function  $M \mapsto C$  with  $|M| < \infty$  we need the discrete Fourier transform (DFT)

### 1.1.9 Definition 9 (Root of unity)

Let  $R$  be a commutative ring with 1. An element  $\mu \in R$  is called an  $n$ -th root of unity (= root of 1) if  $\mu^n = 1$ .

It is called primitive if  $\mu^i \neq 1$  for  $(0 < i < n)$  i.e.  $\text{ord}(\mu) = n$

let  $\mu$  be a primitive  $n$ -th root of 1 (e.g.  $e^{2\pi \frac{i}{n}} \in \mathbb{C}$ )

Then the map  $DFT_\mu : R^n \mapsto R^n$

$$(\hat{a}_0, \dots, \hat{a}_n) \mapsto (\hat{a}_0, \dots, \hat{a}_n) \quad \text{with } \hat{a}_i = \sum_{j=0}^{n-1} \mu^{ij} a_j$$

is called discrete Fourier transformation

For polynomials:

$$DFT_\mu : R[x] \mapsto R^n$$

$$f \mapsto (f(\mu^0), \dots, f(\mu^{n-1}))$$

Convolution rule: (from  $f(\mu^i)g(\mu^i) = (f * g)(\mu^i)$ )

$$DFT_\mu(f * g) = DFT_\mu(f) \cdot DFT_\mu(g) \quad (\text{component wise product})$$

Addition of two polynomials in  $R[x]$  of  $\deg(n)$  require  $O(n)$  ring operations.

Multiplication require  $O(n^l)$ .

With Karatsuba have  $O(n^{\log_2(3)})$  ring operations.

Cost  $DFT_\mu(f) \cdot DFT_\mu(g) : O(n)$  ring operations (with  $\mu$  as  $2n$ -th root of 1)

Want: Cheap way of doing  $DFT$  and back-transformation.

### 1.1.10 Algorithm 10 (Fast Fourier transformation FFT)

input :  $f \in R[x]$ ,  $\mu \in R$  primitive  $2^k$ -th root of 1, such that  $\mu^{2^{k-1}} = -1$

output:  $DFT_\mu(f)$

(1) Write  $f(x) = g(x^2) + xh(x^2)$  with  $f, g, h \in R[x]$

(2) if  $(k = 1)$  // ( $\Rightarrow \mu = 1$ )  
return  $DFT_\mu(f) = (g(1) + h(1), g(1) - h(1))$

(3) Recursive call: compute  $DFT_{\mu^2}(g) = \hat{g}, DFT_{\mu^2}(h) = \hat{h} \in R^{2^{k-1}}$

(4) return  $DFT_\mu(f) = (\hat{f}_0, \dots, \hat{f}_{2^k-1})$  with  $\hat{f}_i = \hat{g}_i + \mu \hat{h}_i$   
where  $\hat{g}_i = \hat{g}_{i-2^{k-1}}$  for  $i \geq 2^{k-1}$

Note: Components of  $\hat{g}$  and  $\hat{h}$  are:

$$\hat{g} = g(\mu^{2^i}), \quad \hat{h}_i = h(\mu^{2^i}) \quad \text{so}$$

$$\hat{f}_i := f(\mu^i) = \hat{g}_i(\mu^{2^i}) + \mu \hat{h}_i(\mu^{2^i}) = \hat{g}_i + \mu \hat{h}_i$$

**Convention:**  $\lg(x) = \log_2(x)$

### 1.1.11 Theorem 11 (Runtime of Algorithm 10)

Let  $n = 2^k$ ,  $f \in R[x]$  with  $\deg(\psi) < n$

Then Algorithm 10 requires  $O(n \cdot \lg(n))$  ring operations.

Better than  $O(n^{1+\epsilon})$ ,  $\forall \epsilon > 0$ !

**Proof:**

Set  $\Theta(k) = \max$  number of ring operations required. By counting obtain for  $k > 1$ :

$$\Theta(k) \leq 2\Theta(k-1) + \underbrace{(\text{compute } \mu^i (i \leq 2^{k-1}))}_{2^{k-1}} + \underbrace{(\mu^i \hat{k}_i)}_{2^{k-1}} + \underbrace{(\text{sums and differences})}_{2^k}$$

$$= 2\Theta(k-1) + 2^{k+1}$$

**Claim:**  $\Theta(k) \leq (2k-1)2^k$

$$k=1 : f = a_0 + a_1 \cdot x \quad DFT_\mu(f) = (a_0 + a_1 \cdot a_0 - a_1) \Rightarrow \Theta(a) = 2$$

$$k-1 \rightarrow k : \Theta(k) \leq 2 \cdot \Theta(k-1) + 2^{k+1} \leq 2 \cdot (2k-3) \cdot 2^{k-1} + 2^{k+1} = (2k-1) \cdot 2^k$$

since  $k = \lg(n)$  obtain  $O(k) \leq (2 \cdot \lg(n) - 1) \cdot n \in O(n \cdot \lg(n)) \quad \square$

**Back-transformation?**

### 1.1.12 Definition 12 (Good root of unity)

A primitive  $n$ -th root of unity is called good (caveat: this is ad-hoc terminology) if:

$$\sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \text{for } (0 < i < n)$$

**example:**

(1)  $\mu = e^{2\pi \frac{i}{n}}$  is a good primitive root of unity

(2)  $R = \mathbb{Z}/(8)$ ,  $\mu = \bar{3} \Rightarrow \mu \cdot B$  is primitive  $2^{nd}$  root of unity  
But  $\bar{B}^0 + \bar{3}^1 = \bar{u} \neq \bar{0}$  so  $\mu$  is not good.

### 1.1.13 Proposition 13 ( $DFT_{\mu^{-1}}$ )

let  $\mu \in R$  be a good root of 1

$$(a) = (a_0, \dots, a_{n-1}) \in R^n \Rightarrow DFT_\mu^{-1}(DFT_\mu(a)) = n \cdot (a) \quad \text{where } n = 1 + \dots + 1 \in R$$

**Proof:**

$$DFT_\mu(a) = (\hat{a}) = (\hat{a}_0, \dots, \hat{a}_{n-1})$$

$$\text{with } \hat{a}_j = \sum_{k=0}^{n-1} \mu^{jk} a_k$$

$$DFT_{\mu^{-1}}(\hat{a}) = (\hat{\hat{a}}_0, \dots, \hat{\hat{a}}_1)$$

$$\text{with } \hat{\hat{a}}_i = \sum_{j=0}^{n-1} \mu^{-ij} \sum_{k=0}^{n-1} \mu^{jk} a_k = \sum_{k=0}^{n-1} \left( a_k \cdot \underbrace{\sum_{j=0}^{n-1} \mu^{j(k-i)}}_{=0 \text{ if } n \neq k-i \text{ (i.e. } k=i)} \right) = a_i \cdot n \quad \square$$



### 1.1.14 Proposition 14 (Finding good roots of unity)

let  $\mu \in R, n \in \mathbb{N}$

Assume:

- a)  $R$  is an integral Domain and  $\mu$  is a primitive or  $n$ -th root of 1  
(Integral Domain: nonzero commutative ring in which the product of two nonzero elements is nonzero)  
 $\Rightarrow$  Granted by FFT
- b)  $n = 2^b, \mu^{\frac{n}{2}} = -1$ , then  $h > 0 \wedge \text{char}(R) \neq 2$   
 $\rightarrow \mu$  is a good primitive  $n$ -th root of 1 ("root of unity")

**Proof:**

- a) for  $0 < i < n$

$$\underbrace{(\mu^i - 1)}_{\neq 0} \underbrace{\left(\sum_{j=0}^{n-1} \mu^{ij}\right)}_{=0} = \mu^{in} - 1 = 0$$

$\Rightarrow \mu$  is a good root of unity

- \* Let  $0 < i < n$ , write  $i = 2^{k-s} \cdot r$  with  $r$  odd  $\wedge s > 0$

$$\sum_{j=0}^{2^k-1} \mu^{ij} = \sum_{l=0}^{2^{k-s}-1} \sum_{j=0}^{2^s-1} \mu^{i(l \cdot 2^s + j)}$$

$$\mu^{i \cdot 2^s} = 1$$

$$i \cdot 2^s = 2^{k-s} \sum_{j=0}^{2^s-1} \mu^{ij} = 2^{k-s} \sum_{j=0}^{2^{s-1}-1} (\mu^{ij} + \mu^{i(2^{s-1}+j)})$$

$$\text{But } \mu^{i \cdot 2^{s-1}} = \mu^{2^{k-s} \cdot r \cdot 2^{s-1}} = \mu^{2^{k-1} \cdot r} = (-1)^r = -1$$

$$\text{So } \sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \square$$

- b)  $\mu^n = 1, n = 2^k \Rightarrow \text{ord}(\mu) | n \Rightarrow \text{ord}(\mu)$  is power of 2

### 1.1.15 Algorithm 15 (Polynomial multiplication using DFT)

input :  $f, g \in R[x]$  with  $\deg(f) + \deg(g) < 2^k =: n$   
 $\mu \in R$  as a good root of unity; Assume  $2 \in R$  is invertible

output:  $h = f \cdot g$

- (1) compute  $\hat{f} = DFT_{\mu}(f), \hat{g} = DFT_{\mu}(g)$  with  $f, g \in R^n$
- (2) compute  $\hat{h} = \hat{f} \cdot \hat{g}$
- (3) compute  $(h_0, \dots, h_{n-1}) = DFT_{\mu^{-1}} \hat{h}$  (same as  $DFT_{\mu}(\hat{h})$  but with different order)  
= Back-transformation  $\cdot 2^k$   
set  $h = \frac{1}{2^k} \sum_{i=0}^{n-1} h_i x^i$

### 1.1.16 Theorem 16 (Runtime of Algorithm 15)

Algorithm 15 uses  $O(n \cdot \log(n))$  ring operations for polynomials of  $\deg < n$

**Proof:**

- Choose  $k$  minimal so that  $\deg(f) \cdot \deg(g) < 2^k$   
 $\Rightarrow 2^{k-1} \leq 2n \Rightarrow k \leq \log(n) + 2$
- $\underbrace{O(2k \cdot 2^k)}_{\text{Step 1}} + \underbrace{2^k}_{\text{Step 2}} + \underbrace{O(k \cdot 2^k) + 2^k}_{\text{Step 3}} \in O(2k \cdot 2^k) = O(n(g(n))) \quad \square$

Goal: Multiplication in  $\mathbb{N}$  using DFT

Idea: find roots of 1 in a suitable  $\mathbb{Z}/(m)$

Choose  $m = 2^l + 1, \mu = \bar{2} \in R$

### 1.1.17 Proposition 17 (Add and mul in $O(l)$ )

Let  $m = 2^l + 1, R = \mathbb{Z}/(m)$

Addition in  $R$  and multiplication by  $\bar{2}^i \in R$  ( $0 \leq i < 2l$ ) can be done in  $O(l)$  bit operations

**Proof:**

- Let  $\bar{x} \in R$  with  $0 \leq x \leq 2^l$
- Addition:  $x + \bar{y}$ 
    - (1) compute  $x + y \in \mathbb{N}$ :  $O(l)$
    - (2) if  $x + y > 2^l + 1$  subtract  $2^l + 1$ :  $O(l)$
  - Multiplication by  $\bar{2}^i$  ( $0 \leq i < l$ )
    - (1) Bit-shift  $i$  Bits to the left by relocating in memory:  

$$\underbrace{O(\text{length}(i))}_{\text{compute addr. of new first bit}} + \underbrace{l}_{\text{copying}} = O(\log(l)) + l \in O(l)$$
  - Multiplication by  $\bar{2}^i$  ( $l \leq i < 2l - 1$ )
    - (1) Multiplication by  $\bar{2}^{i-l}$ :  $O(l)$
    - (2) take negative  $\bar{2}^i \cdot \bar{x} = -\bar{2}^{i-l} \cdot \bar{x}$ :  $O(l)$

### 1.1.18 Proposition 18 (Sort of summary)

Let  $k, r \in \mathbb{N}, r > 0, m = 2^{2^k \cdot r} + 1, R = \mathbb{Z}/(m), \mu = \bar{2}^r \in R$

$\Rightarrow 2 \in R$  is invertible,  $\mu$  is a good primitive  $2^{k+1}$ -th root of 1

$\Rightarrow \mu^{2^k} = 1$

**Proof:**  $\rightarrow$  from above

### 1.1.19 Algorithm 19 (Multiplication using FFT)

input :  $x, y \in \mathbb{N}$

output:  $Z = x \cdot y$

- (1) Choose  $k \in \mathbb{N}$  minimal such that  $l(x), l(y) \leq 2^{2^k}$
- (2) if  $k \leq 3$ , compute  $z = x \cdot y$  by Algorithm 5
- (3) set  $B = 2^{2^k}$ ,  $m = 2^{2^k \cdot 4} + 1$ ,  $R = \mathbb{Z}/(m)$ ,  $\mu = \bar{2}^4 \in R$   
 $(\Rightarrow \text{so } \mu \text{ is a good primitive } 2^{k+1}\text{-th root of 1})$
- (4) write  $x = \sum_{i=0}^{2^k-1} x_i \cdot B^i$ , same for  $y$  with  $(0 \leq x_i, y_i < B)$   
 possible since  $x, y < 2^{2^{2k}} = 2^{2^k \cdot 2^k} = B^{2^k}$
- (5) compute:  $\hat{x} = DFT_\mu(\bar{x}_0, \dots, \bar{x}_{2^k-1}, \underbrace{0, \dots, 0}_{2^k \text{ zeros}}) \in R^{2^{k+1}}$   
 same for  $y$   
 $\rightarrow$  use FFT
- (6) compute:  $\hat{z} = \hat{x} \cdot \hat{b} \in R^{2^{k+1}}$  (component wise multiplication)  
 Perform multiplication in  $R$  as follows:  
 Multiply representatives (non negative and  $< m$ ) by recursive call.  
 Then reduce modulo  $m$  by "negative bit shift" (see proof of Proposition 17)
- (7) compute:  $(\bar{z}_0, \dots, \bar{z}_{2^{k+1}-1}) = \frac{1}{2^{k+1}} DFT_{\mu^{-1}}(\hat{z}) \in R$  with  $0 \leq z < m$
- (8) set  $z := \sum_{j=0}^{2^{k+1}-1} z_j \cdot B^j$

### 1.1.20 Theorem 20 (Runtime of Algorithm 19)

Algorithm 19 correctly computes  $t = x \cdot y$  and requires  $O(n \cdot (\log n)^4)$  bit operations for  $l(x), l(y) \leq n$

**Proof:** Correctness

write  $x(t) \sum_{i=0}^{2^k-i} x_i t^i \in \mathbb{Z}[t]$ ,  $y(t)$ ,  $\bar{x}(t) \in R[t], \bar{y}(t), \bar{z}(t)$

by Proposition 18 and Proposition 13 we have  $\bar{z}(t) = \bar{x}(t) \cdot \bar{y}(t)$

The  $l$ -th coefficient of  $x(t) \cdot y(t)$  is  $0 \leq \sum_{i+j=l} x_i \cdot y_j < 2^k \cdot B^2 = 2^{k+2 \cdot 2^k} \leq 2^{2^{k+2}} < m$

So  $z(t) = x(t) \cdot y(t) \Rightarrow z = z(B) = x(B) \cdot y(B) = x \cdot y$  Cost:

Write  $\Theta(k) := \max$  number of bit operations

Analyze Steps:

- (1) compute  $\max \{l(x), l(y)\} : O(l(n)) = O(k)$
- (2)  $O(1)$
- (3) no bit operations
- (4) compute starting addresses of  $x_i, y_i$  in memory:  $2 \cdot 2^k$  increments of the address:  
 $2 \cdot 2 \cdot 2^k = 2^{k+2}$  bit ops  
 $\Rightarrow O(2^k)$
- (5) By Theorem 11 need  $O(2 \cdot 2^{k+1} \cdot (k+1))$  operations in  $R$  which are additions and multiplications by powers of  $\bar{z}$  costing  $O(2^{k+2})$  bit operations.  
Total for (5):  $O(k \cdot 2^{2 \cdot k})$
- (6)  $2^{k+1}$  multiplications of numbers  $< m$ , i.e. of length  $\leq 2^{k+2}$ .  
So  $k' \leq \frac{k+3}{2}$  for  $k'$ : the "new"  $k$  used in the next recursion level.  
For  $\alpha \in R_{>0}$  define  $\Theta(\alpha) := \Theta(\lfloor \alpha \rfloor)$   
Total for (6):  $2^{k+1}(\Theta(\frac{k+3}{2}) + \underbrace{O(2^{k+2})}_{\text{reduction (mod } m)})$
- (7) For  $DFT_{\mu^{-1}}(\hat{z}) : O(k \cdot 2^{2 \cdot k})$  as (5) Since  $\bar{z}$  is a  $n$  root of 1, multiplication by  $\bar{2}^{-k-1}$  is multiplication by a positive power of  $\bar{2}$ , which costs  $O(2^{k+2})$   
Total for (7):  $O(k \cdot 2^{2 \cdot k})$
- (8) For  $j \leq 2^{k+1}$  have  $\sum_{i=0}^{j-1} z_i \cdot B^i \leq (m-1) \sum_{i=0}^{j-1} B^i = (m-1) \frac{B^j-1}{B-1} < 2(m-1) \frac{B^j}{B} = 2^{1+2^{k+2}+(j-1)2^k}$  so the sum has length  $(j+3) \cdot 2 + 1$   
Adding  $z_j \cdot B^j$  to this sum happens at  $(j \cdot 2^k)$ -th bit and higher  $\Rightarrow$  cost is  $O(2^k)$   
Total for (8):  $O(2^{2 \cdot k})$

Grad total: For  $k \geq 4$ :

$\Theta(k) \leq 2^{k+1} \cdot \Theta(\frac{k+3}{2}) + c \cdot k \cdot 2^{2 \cdot k}$  with  $c$  constant

Also for  $k \in \mathbb{R}_{\geq 4}$

**Define**  $\Lambda(k) := \frac{\Theta(k)}{2^{2 \cdot k}} \Rightarrow \Lambda(k) \leq \frac{2^{k+1} \Theta(\frac{k+3}{2})}{2^{2 \cdot k}} + c \cdot k = 16 \cdot \Lambda(\frac{k+3}{2}) + c \cdot k$

**Define**  $\Omega(k) := \Lambda(k+3)$  So for  $k \in \mathbb{R}_{>1}$

$$\Omega(k) \leq 16 \cdot \Lambda(\frac{k}{2} + 3) + c \cdot (k+3) = \underbrace{16\Omega(\frac{k}{2})}_{*} + c \cdot (k+3)$$

**Claim:** For  $i \in \mathbb{N}$  with  $2^{i-1} \leq k-3$  have:

$$\Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k+3)(1+8+\dots+8^{i-1}) + 3 \cdot c \cdot (1+16+\dots+16^{i-1})$$

**Proof** by induction:

$$i = 0: \Lambda(k) = \Omega(k-3)$$

$$i \rightarrow i+1: \Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k-3)(1+\dots+8^{i-1}) + 3 \cdot c \cdot (1+\dots+16^{i-1}) \leq 2^i \leq k-3 \quad *$$

$$\leq 16^i (16\Omega(\frac{k-3}{2^{i+1}})) + c(\frac{k-1}{2^i} + 3) + c(k-3)\dots = \text{claimed result}$$

Take  $u \in \mathbb{N}$  minimal with  $2^u > k-3 \Rightarrow \Omega(\frac{k-3}{2^u}) \leq \Omega(\lfloor \frac{k-3}{2^u} \rfloor) = \Omega(0) =: D$  (constant)

Note:  $u$  roughly is recursion depth

$$\text{Have } 2^{u-1} \leq k-3 \xRightarrow{\text{claim}} \Lambda(k) \leq 16^u \cdot D + c \cdot \underbrace{(k-3)}_{< 2^u} \cdot \frac{8^u-1}{7} + 3c \cdot \frac{16^u-1}{15} \in O(16^u)$$

$$\text{Have } 2^{u-1} \leq k-3 \Rightarrow u \leq \lg(k-3) + 1$$

$$\Rightarrow \Lambda(k) \in O(16^{\lg(k-3)}) = O((k-3)^4)$$

$$\Rightarrow \Theta(k) = 2^{2 \cdot k} \cdot \Lambda(k) \in O(2^{2k} \cdot (k-3)^4)$$

$$\text{Have } 2^{2(k-1)} < \underbrace{n}_{\max\{l(x) \cdot l(y)\}} \Rightarrow k \leq \frac{\lg(n)}{2} + 1$$

$$\text{So } \Theta(k) \in O(n \cdot (\lg(n))^4) \quad \square$$

### 1.1.21 Theorem 21 (Schönhage-Strassen 1971)

Multiplication of integers of length  $\leq n$  can be done in  $O(n \cdot \lg(n) \cdot \lg(\lg(n)))$  bit operations. Schönhage-Strassen is used for integers of length  $\geq 100.000$ .

Asymptotically faster: Fürer's algorithm.

### Comments on Bit complexity

1. Memory requirement may explode!  
 $\Rightarrow$  No Problem as bit complexity is upper bound for memory requirements, since memory access is included in bit operations  
 $(\rightarrow$  only store what is calculated)
2. Computation of addresses in memory take time  
 $\Rightarrow$  length of addresses  $\approx \lg(\text{memory space})$  computations of addresses  $\approx \lg(\text{memory space})^2$
3. As memory requirement gets larger access times will get longer.  
 $\Rightarrow$  transportation time for data  $\geq \frac{\text{diameter of physical storage}}{2 \cdot \text{speed of light}}$

## 1.2 Division with remainder, Euclidean algorithm

### 1.2.1 Algorithm 1 (Division with remainder)

input :  $b = \sum_{i=0}^{n-1} b_i 2^i$   $a = \sum_{i=0}^{n+m-1} a_i 2^i$  with  $a_i, b_i \in 0, 1$ ,  $b_{n-1} = 1$

output:  $r, q \in \mathbb{N}$  such that  $a = q \cdot b + r$ ,  $0 \leq r < b$

(1)  $r = a$   $q = 0$

(2) for  $i = m, m-1, \dots, 0$  do

(3) if  $r \leq 2^i \cdot b$  then set  $r := r - 2^i \cdot b$ ,  $q = q + 2^i$

### 1.2.2 Proposition 2 (Runtime of Algorithm 1)

Algorithm 1 is correct and requires  $O(n \cdot (m+1))$  bit operations.

**Proof:**

Always have  $a = q \cdot b + r$

**Claim:**

before step (3), have  $0 \leq 2^{i+1} \cdot b$

$i = m$ ;  $0 \leq r = a < 2^{m+n} = 2^{m+1} \cdot 2^{n-1} \leq 2^{m-1} \cdot b$   $i < m$  By step (3)

So after last passage through the loop  $0 \leq r < b$

**Running Time:** In step(3), have comparison and (possibly) subtraction. Only  $n$  bits involved  $\Rightarrow O(n)$

Total:  $O(b \cdot (m+1))$

**Remarks:**

- (1) Division with remainder can be reduced to multiplication.  
Precisely: given an algorithm for multiplication that requires  $M(n)$  bit operations, there exists an algorithm for division with remainder that requires  $O(M(n))$  bit operations.
- (2) Practically relevant:  
Jebelean's algorithm (1997):  $O(n^{\lg 3})$
- (3) Alternatively, may choose  $r \in \mathbb{Z}$  such that  $\lfloor \frac{-b}{2} \rfloor < r \leq \lfloor \frac{b}{2} \rfloor$
- (4) Algorithm 1 extends to  $\mathbb{Z}$ .
- (5) All Euclidean rings have division with remainder (by definition).  
(e.g.,  $R = K[x] \rightarrow$  polynomial ring over field,  
 $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ ,  $i^2 = -1$ )

### 1.2.3 Algorithm 3 (Euclidean algorithm)

input :  $a, b \in \mathbb{N}$

output:  $\gcd(a, b)$  "greatest common divisor"

- (1) set  $r_0 := a, \quad r_i := b$
- (2) for  $i = 1, 2, 3, \dots$  perform steps (3) and (4)
- (3) if  $r_i = 0$  then  $\gcd(a, b) = |r_{i-1}|$
- (4) Division with remainder:  $r_{i-1} = q \cdot r_i + r_{i+1} \quad r_{i+1} \in \mathbb{Z}$   
 $|r_{i+1}| \leq \frac{1}{2}|r_i|$

**Example:**

$$a = 287, \quad b = 126$$

$$287 = 2 \cdot 126 + 35 \tag{1}$$

$$126 = 4 \cdot 35 - 14 \tag{2}$$

$$35 = (-2) \cdot (-14) + 7 \tag{3}$$

$$-14 = (-2) \cdot 7 + 0 \tag{4}$$

$$\begin{aligned} \text{So: } 7|(-14) &\xRightarrow{(3)} 7|35 \\ &\xRightarrow{(2)} 7|126 \\ &\xRightarrow{(1)} 7|287 \end{aligned}$$

On the other hand take a common divisor  $d$ ;  $d|287$ ;  $d|126$

$$\xRightarrow{(1)} d|d \xRightarrow{(2)} d|14 \xRightarrow{(3)} d|7$$

### 1.2.4 Theorem 4 (Correctness of Algorithm 3)

Algorithm 3 is correct.

**Proof:**

Since  $r_{i-1} = q \cdot r_i + r_{i+1}$  every integer  $x \in \mathbb{Z}$  satisfies the equivalence  $x|r_{i-1}$  and  $x|r_i \Leftrightarrow x|r_{i+1}$  and  $x|r_i$  so  $\gcd(r_{i-1}, r_i) = \gcd(r_i, r_{i+1}) = \gcd(a, b)$  when terminating have  $\gcd(a, b) = \gcd(r_{i-1}, 0) = |r_{i-1}| \quad \square$

### 1.2.5 Theorem 5 (Runtime of Algorithm 3)

Algorithm 3 requires  $O(m \cdot n)$  bit operations for  $n = l(a), m = l(b)$

**Proof:**

If  $a < b$  then the first passage yields  $r_2 = a, r_1 = b$ . Cost:  $O(n)$

May assume:  $a \geq b$ . Write  $n_i = l(r_i)$

By Proposition 2  $\exists c$  constant such that the total time is  $\leq c \cdot \underbrace{\sum_{i=1}^k n_i \cdot (n_{i-1} - n_i + 1)}_{=: \sigma(n_0, \dots, n_k)}$

For  $i > 2$ :  $n_i = n_{i-1} - 1$

Special Case:  $n_i = n_{i-1} - 1$  for  $i \geq 2$

$\Rightarrow n_i = n_i - i + 1, n_i = m, k = m + 1$

Obtain  $\sigma(n_0, \dots, n_k) = m \cdot (n - m + 1) + \sum_{i=2}^{m+1} (m - i + 1) \cdot 2 = m \cdot n - m^2 + m + m(m - 1) = m \cdot n$ .

**Claim:** The special case is the worst (most expensive)!

From any sequence  $n_1 > n_2 > \dots > n_k$  get to the special case by iteratively inserting numbers in the gaps. Insert  $s$  with  $n_{j-1} > s > n_j$ .

$\sigma(n_0, \dots, n_{j-1}, s, n_j, \dots, n_k) - \sigma(n_0, \dots, n_k) = \dots = s + (n_{j-1} - s) \cdot (s - n_j)$

$sp\sigma(n_0, \dots, n_k) \leq \sigma(n, m, m - 1, \dots, 2, 1, 0) = n \cdot m \quad \square$

Complexity is quadratic  $\rightarrow$  cheap

### 1.2.6 Algorithm 6 (Extended Euclidean Algorithm)

input :  $a, b \in \mathbb{N}$

output:  $d = \gcd(a, b)$  and  $s, t \in \mathbb{Z}$  such that  $d = s \cdot a + t \cdot b$

(1)  $r_0 := a, r_1 := b, s_0 := 1, t_0 := 0, s_1 := 0, t_1 := 1$

(2) for  $i = 1, 2, \dots$  perform steps (3) - (5)

(3) if  $r_i = 0$  set  $d = |r_{i-1}|$   
 $s := \text{sgn}(r_{i-1}) \cdot s_{i-1},$   
 $t := \text{sgn}(r_{i-1}) \cdot t_{i-1}$

(4) division with remainder:  
 $r_{i+1} = r_{i-1} - q_i \cdot r_i, \quad \text{with } |r_{i+1}| \leq \frac{1}{2}|r_i|$

(5) set  $s_{i+1} := s_{i-1} - q_i \cdot s_i,$   
 $t_{i+1} := t_{i-1} - q_i \cdot t_i$

Justification :  $r_i = s_i \cdot a + t_i \cdot b$  throughout

**Application:**  $m, x \in \mathbb{N}$  such that  $m, x$  co-prime (i.e.  $\gcd(x, m) = 1$ )

Algorithm 6 yields:  $1 = s \cdot x + t \cdot m \Rightarrow s \cdot x \equiv 1 \pmod{m}$

So obtain inverse of  $\bar{x} \in \mathbb{Z}/(m)$



### 1.3 Primality testing

Let  $\mathbb{P} \subseteq \mathbb{N}$  be the set of prime numbers.

Challenge: Given  $n \in \mathbb{N}$  decide if  $n \in \mathbb{P}$

**Naive Method:** Trivial division by  $m \leq \lfloor \sqrt{n} \rfloor$ .

Running time is exponential in  $l(n)$ . Even when restricted to division by prime numbers, need approximately  $\frac{\sqrt{n}}{|n|^{\sqrt{n}}}$  trivial divisions (prime number theorem)  
 $\rightarrow$  hardly any better!

**Reminder:** (arithmetic modulo  $m$ )

$G$  finite group  $\Rightarrow \forall a \in G \quad a^{|G|} = 1$  Fermat's little theorem

For  $G = (\mathbb{Z}/(p))^\times \quad a^{p-1} \equiv 1 \pmod{p} \in \mathbb{P} \quad \forall a \in \mathbb{Z} \quad \text{with } p \nmid a$

In fact  $(\mathbb{Z}/(p))^\times \cong Z_{p-1}$  is cyclic

For  $m = p_1^{e_1} \dots p_r^{e_r}$  with  $p_i \in \mathbb{P}, e_i \in \mathbb{N}_{>0}$ :

$\mathbb{Z}_{(m)} \cong \mathbb{Z}_{(p_1^{e_1})} \oplus \dots \oplus \mathbb{Z}_{(p_r^{e_r})} \Rightarrow \mathbb{Z}_{(m)}^\times \cong \mathbb{Z}_{(p_1^{e_1})}^\times \times \dots \times \mathbb{Z}_{(p_r^{e_r})}^\times$

what is  $\mathbb{Z}_{(p^e)}$  for  $p \in \mathbb{P}, e \in \mathbb{N}_{>0}$ ?

#### 1.3.1 Theorem 1 (Cyclic group)

Let  $p \in \mathbb{P}$  odd  $e \in \mathbb{N}_{>0} \Rightarrow (\mathbb{Z}_{(p^e)})^\times = Z_{(p-1) \cdot p^{e-1}}$  cyclic

**Proof:**

$(\mathbb{Z}_{(p^e)})^\times \cong Z_{p-1} \Rightarrow \exists z \in \mathbb{Z} : \text{order}(z + p\mathbb{Z}) = p-1$

Set  $a = \bar{z}^{p^{e-1}} \in (\mathbb{Z}_{(p^e)})^\times =: G$

$$a^{p-1} = \bar{z}^{(p-1) \cdot p^{e-1}} = \bar{z}^{|a|} = 1$$

On the other hand, take  $i \in \mathbb{Z}$  such that

$$a^i = 1 \Rightarrow \bar{z}^{i \cdot p^{e-1}} \equiv 1 \pmod{p} \Rightarrow (p-1) \mid (i \cdot p^{e-1}) \Rightarrow (p-1) \mid i.$$

So  $\text{ord}(a) = p-1$ .

Now consider  $b = (p+1) \in G$

**Claim:**  $\text{ord}(b) = p^{e-1}$

**Proof** by induction on  $k \in \mathbb{N}_{>0}$  that  $(p+1)^{p^{k-1}} \equiv p^k + 1 \pmod{p^{k+1}}$

$k=1$  ✓

$k \rightarrow k+1$ : By induction have  $(p+1)^{p^{k-1}} = 1 + p^k + x \cdot p^{k+1}, \quad x \in \mathbb{Z}$

$$\text{Compute: } (p+1)^{p^k} = ((1+p^k) + x \cdot p^{k+1})^p = \sum_{i=0}^p \binom{p}{i} (i+p^k)^{p-i} \cdot x^i \cdot p^{i \cdot (k+1)}$$

$$\stackrel{\text{Only 0-th summand}}{\equiv} (i+p^k) = \sum_{i=0}^p \binom{p}{i} p^{i \cdot k} \stackrel{p \text{ odd}}{\equiv} 1 + p^{k+1} \pmod{p^{k+2}} \quad \checkmark$$

For  $k=e$ :  $(p+1)^{p^{e-1}} \equiv 1 \pmod{p^e} \Rightarrow b^{p^e} = 1 \Rightarrow \text{ord}(b) \mid p^{e-1}$

But  $(p+1)^{p^{e-2}} \equiv p^{e-1} + 1 \pmod{p^e} \Rightarrow b^{p^{e-2}} \neq 1 \in G$

So  $\text{ord}(b) = p^{e-1}$

**Claim:**  $\text{ord}(a \cdot b) = (p-1)p^{e-1} \quad (\Rightarrow \text{Theorem})$

Let  $(a \cdot b)^i = 1 \in G$  with  $i \in \mathbb{Z}$

$$\text{Then } 1 = (a \cdot b)^{i \cdot (p-1)} = (a^{p-1})^i \cdot b^{i \cdot (p-1)} = b^{i \cdot (p-1)} \Rightarrow p^{e-1} \mid i \cdot i(p-1) \Rightarrow p^{e-1} \mid i$$

$$\text{Also } 1 = (a \cdot b)^{p^{e-1} \cdot i} = a^{p^{e-1}} \Rightarrow (p-1) \mid p^{e-1} \cdot i \Rightarrow (p-1) \mid i \rightarrow (p-1) \cdot p^{e-1} \mid i \quad \square$$

**Reminder:**  $(\mathbb{Z}/(2^e))^\times \cong Z_2 \times Z_2^{e-2} \quad (e \geq 2)$

### 1.3.2 Algorithm 2 (Fermat Test)

input :  $n \in \mathbb{N}_{>0} \text{ odd}$

output: " $n \notin \mathbb{P}$ " or "probably  $n \in \mathbb{P}$ "

- (1) Choose  $a \in 2, \dots, n-1$  randomly
- (2) Compute  $a^{n-1} \bmod n$
- (3) If  $a^{n-1} \not\equiv 1 \pmod{n}$  return " $n \notin \mathbb{P}$ "  
else return "probably  $n \in \mathbb{P}$ "

Not very satisfying. Is this fast?

### 1.3.3 Algorithm 3 (fast exponentiation)

input :  $a \in G$   $G$  is a monoid,  $e \in \mathbb{N}$ ,  $e = \sum_{i_0}^{n-1} e_i 2^i$ ,  $e_i \in \{0, 1\}$

output:  $a^e \in G$

- (1) Set  $b := a$ ,  $y := 1$
- (2) For  $i = 0, \dots, n-1$  perform (3) - (4)
- (3) if  $e_i = 1$  set  $y := y \cdot b$
- (4) set  $b := b^2$
- (5) return  $y$

this requires  $O(l(e))$  operations in  $G$

For  $G = (\mathbb{Z}/(n)_i)$ , each multiplication requires  $O(l(n)^2)$  bit operations

$\Rightarrow$  Fermat test requires  $O(l(n)^3)$  bit operations  $\rightarrow$  cubic complexity  $\rightarrow$  "fast"!

#### Example:

$n = 561 = 3 \cdot 11 \cdot 17$  For  $a \in \mathbb{Z}$  with  $\gcd(a, n) = 1 \Rightarrow$  have  $a^{n-1} = (a^2)^{280} \equiv 1 \pmod{3}$

$a^{n-1} \equiv 1 \pmod{n}$  Fermat's test says "probably  $n \in \mathbb{P}$ " in 57% of cases.

$n = 2207 \cdot 6619 \cdot 15443$  : output "probably  $n \in \mathbb{P}$ " in 99,93% of cases.

### 1.3.4 Definition 4 (pseudo-prime, witness, Carmichael numbers)

Let  $n \in N_{>1} \text{ odd}$ ,  $a \in 1, \dots, n-1$

- (a)  $n$  is pseudo-prime to base  $a$  if  $a^{n-1} \equiv 1 \pmod{n}$
- (b) otherwise  $a$  is called a witness of composition of  $n$
- (c) If  $n \notin \mathbb{P}$  but  $a^{n-1} \equiv 1 \pmod{n} \quad \forall a$  with  $\gcd(n, a) = 1$   
then  $n$  is called a Carmichael number.  
There are  $\infty$  Carmichael numbers

### 1.3.5 Proposition 5 (Number of witnesses)

Let  $n \in N_{>1}$ ,  $\text{odd} \wedge \notin \mathbb{P} \wedge \text{not Carmichael}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n, a \text{ is witness of composite of } n\}| > \frac{n-1}{2}$

**Proof:** Consider

$\phi : (\mathbb{Z}/(n))^\times \rightarrow G \rightarrow G, \quad \bar{a} \mapsto \bar{a}^{n-1}$

group homomorphism. By assumption,

$|\text{im}(\phi)| > 1 \Rightarrow |\text{Ker}(\phi)| \leq \frac{|a|}{2} < \frac{n-1}{2}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n \text{ a witness of composite of } n\}| > \frac{n-1}{2} \quad \square$

### Miller-Rabin Test

### 1.3.6 Proposition 6 (inference from Fermat)

Let  $p \in \mathbb{P} \text{ odd}$ ,  $a \in \{1, \dots, (p-1)\}$  write  $p-1 = 2^k \cdot m$  with  $m \text{ odd}$  Then:

$a^m \equiv 1 \pmod{p}$  or  $\exists i \in \{0, \dots, k-1\} : a^{2^i \cdot m} \equiv -1 \pmod{p}$

**Proof:**

Little Fermat:  $\bar{a}^{2^k \cdot m} = 1 \in \mathbb{F}_p$

Assume  $\bar{a}^m \neq 1$  take  $i$  maximal such that:

$\bar{b} = \bar{a}^{2^i \cdot m} \neq 1 \Rightarrow \bar{b}^2 = 1 \Rightarrow \bar{b} \in \mathbb{F}_p$  is a zero of  $x^2 - 1 \in \mathbb{F}_p[x] \Rightarrow \bar{b} = -1$

### 1.3.7 Algorithm 7 (Miller-Rabin-test)

input :  $n \in \mathbb{N}_{>1}, \text{odd}$

output: either " $n \notin \mathbb{P}$ " or "probably  $n \in \mathbb{P}$ "  $\rightarrow$  Monte Carlo Algorithm.

- (1) write  $n - 1 = 2^k \cdot m$  with  $m$  odd
- (2) Choose  $a \in \{2, \dots, n - 1\}$  randomly
- (3) Compute  $b := a^m \pmod n$
- (4) if  $(b \equiv \pm 1 \pmod n)$   
return "probably  $n \in \mathbb{P}$ "
- (5) for  $(i = 0, \dots, k - 1)$  do steps (6) - (7)
- (6) set  $b := b^2 \pmod n$
- (7) if  $(b \equiv -1 \pmod n)$   
return "probably  $n \in \mathbb{P}$ "
- (8) return  $n \notin \mathbb{P}$

### 1.3.8 Definition 8 (strong pseudo-prime / witness)

Let  $n \in \mathbb{N}_{>1}, \text{odd}$   $a \in \{1, \dots, n - 1\}$

- (a)  $n$  is called a strongly pseudo-prime to base  $a$  if Proposition 6 holds for  $a$  and  $p$  replaced by  $n$ .
- (b) Otherwise  $a$  is called a strong witness of composition of  $n$ .

#### Example

Let  $n \in \mathbb{N}_{>1}, \mathbb{P} \text{ odd}$

$a = 2$  strong witness if  $n < 2047$  (including 561)

2 or 3 strong witness if  $n < 1373653$

2,3 or 5 strong witness if  $n < 25326001$

### 1.3.9 Theorem 9 (Bit-complexity of Algorithm 7)

- (a) Algorithm 7 requires  $O(l(n)^3)$  bit operations.  $\rightarrow$  "qubic complexity"  $\rightarrow$  fast!
- (b) if  $b \in \mathbb{P}$  then Algorithm 7 returns "probably  $b \in \mathbb{P}$ "  $\rightarrow$  no false positives.
- (c) if  $n \notin \mathbb{P}$  then more than half of the numbers in  $\{1, \dots, n - 1\}$  are strong witnesses.

**Proof:**

(a) Step 1 takes  $O(l(n))$  bit operations:

Using Algorithm 3, we need  $O(l(n-1))$  multiplications in  $\mathbb{Z}/(n)$  each requiring  $O(l(n)^2)$  bit operations.

(b) Proposition 6

(c) split in three cases:

**Case 1:**  $n$  is not a Carmichael number.  $\xRightarrow{\text{Prop 5}}$  more than half of all numbers are.

Fermat witness thus also strong witness.

**Case 2:**  $n = p^r \cdot l$  with  $p \in \mathbb{P}$   $r > 1$   $l \in \mathbb{N}_{>0}$   $p \nmid l$

Theorem 1  $\exists x \in \mathbb{Z}$  such that  $x^p \equiv 1 \pmod{p^r}$   $x \not\equiv 1 \pmod{p^r}$

Chinese remainder theorem:  $\exists a \in \mathbb{Z}$  such that  $a \equiv x \pmod{p^r}$   $a \equiv 1 \pmod{l}$

So  $\bar{a}^p = 1 \in \mathbb{Z}/(n) \Rightarrow \bar{a}^n = 1 \Rightarrow \bar{a} \in (\mathbb{Z}/(n))^\times$

i.e.  $\gcd(n, a) = 1$  if  $\bar{a}^{n-1} = 1$  then  $\bar{a} = 1$

But  $a \equiv x \not\equiv 1 \pmod{p^r}$  so  $\bar{a}^{n-1} \neq 1$  hence  $n$  is not Carmichael  $\rightarrow$  Case 1.

**Case 3:**  $n$  is a Carmichael number. By Case 2 have  $n = p \cdot l$  with  $p \in \mathbb{P}$   $p \nmid l$   $l \geq 3$   
 $n$  Carmichael:  $\forall a \in \mathbb{Z}$  with  $\gcd(a, n) = 1$

have  $a^{2^k \cdot m} \equiv 1 \pmod{n}$  (where  $n-1 = 2^k \cdot m$ )

$a^{2^k \cdot m} \equiv 1 \pmod{p}$  Take  $j$  minimal such that

$a^{2^j \cdot m} \equiv 1 \pmod{p}$   $\forall a \in \mathbb{Z}$  such that  $\gcd(a, n) = 1$

so  $0 \leq j \leq l$  in fact,  $j > 0$  since  $(-1)^{2^0 \cdot m} = -1$  with  $m$  odd.

Consider the subgroup  $H := \{\bar{a} \in \mathbb{Z}/(n) \mid \bar{a}^{2^{j-1} \cdot m} \in \{1, -1\} \subseteq (\mathbb{Z}/(n))^\times\}$

Let  $a \in \{1, \dots, n-1\}$   $\gcd(n, a) = 1$   $a$  not a strong witness.

**Claim 1:**  $\bar{a} \in H$

**Case 3.1:**  $\bar{a}^{2^{j-1} \cdot m} = 1 \Rightarrow \bar{a} \in H$

**Case 3.2:**  $a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{n}$   $a^m \not\equiv 1 \pmod{n}$

$\xRightarrow{\text{a nonwitness}}$   $\exists i$  such that  $\underbrace{a^{2^i \cdot m} \equiv -1 \pmod{n}}_*$

$\Rightarrow a^{2^i \cdot m} \equiv -1 \pmod{p} \xRightarrow{\text{def of } j} i < j$

if  $i < j-1$  then  $a^{2^{j-1} \cdot m} = (a^{2^i \cdot m})^{2^{j-1-i}} \equiv (-1)^{2^{j-1-i}} = 1 \pmod{n}$

$\xRightarrow{\text{with } *}$  not in case 3.2

**Claim 2:**  $H \subseteq (\mathbb{Z}/(n))^\times$  proper subgroup.

By definition of  $j$   $\exists x \in \mathbb{Z}$  such that  $x^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p}$

Chinese remainder:  $\exists a \in \mathbb{Z}$  such that

$a \equiv x \pmod{p}$   $a \equiv 1 \pmod{l}$

$\Rightarrow a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p} \equiv 1 \pmod{l} \Rightarrow \bar{a} \notin H$

Claim 2  $\checkmark$

It follows that  $|H| \leq \frac{|(\mathbb{Z}/(n))^\times|}{2} < \frac{n-1}{2}$

so the number of witnesses is  $\geq n-1-|H| > \frac{n-1}{2}$   $\square$

### Remarks:

- (a) A more careful analysis shows that  $2\frac{3}{4}$  of all candidates are strong witnesses
- (b) Calling Algorithm 7 repeatedly decreases the probability of false positives. Running time for prescribed error probability  $p$  is  $O(\lg(p^{-1} \cdot l(n)^3))$   
(Independence assumptions!)

### Connection with Riemann hypothesis

Let  $n \in \mathbb{N}_{>0}$   $\bar{X} : (\mathbb{Z}/(n))^\times \rightarrow \mathbb{C}^\times$  group homomorphism

$$X : \mathbb{Z} \rightarrow \mathbb{C}, a \mapsto \begin{cases} \bar{X}(\bar{a}) & \text{if } \gcd(a, n) = 1 \text{ for } (\bar{a} = a + n\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$$

"residue class character (mod  $n$ )

$$Ex : n = 1 \Rightarrow X(a) = 1 \forall a \in \mathbb{Z}$$

Dirichlet L-series:

$$L_X(s) = \sum_{n=1}^{\infty} \frac{X(n)}{n^s} \text{ converges for } s \in \mathbb{C} \text{ until } Re(s) > 1$$

$L_X(s)$  extends to a meromorphic function on  $\mathbb{C} \mapsto$  "Dirichlet L-function".

For  $n = 1 : L_X(s) = \zeta(s)$  Riemann Zeta-function.

Euler Product:

$$\text{From } (1 - X(p) \cdot p^{-s})^{-1} = \sum_{i=0}^{\infty} (X(p) \cdot p^{-s})^i = \sum_{i=0}^{\infty} \frac{X(p^i)}{p^{is}} \quad \text{derive } L_X(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - X(p) \cdot p^{-s}}$$

Generalized Riemann hypothesis (GRH):

For  $X$  residue class character,  $s \in \mathbb{C}$

with  $L_X(s) = 0$ ,  $0 < Re(s) < 1$  ("critical strip")

then  $Re(s) = \frac{1}{2}$

For  $X = 1 \rightarrow$  ordinary Riemann hypothesis.

### 1.3.10 Theorem (Arkeny & Bach)

GRH  $\Rightarrow \forall X \neq 1$  residue class character

$$\exists p \in \mathbb{P} : X(p) \neq 1, p < 2 \ln(n)^2$$

Let  $H \subsetneq (\mathbb{Z}/(n))^\times =: G$  proper subgroup.

Choose  $N \subsetneq G$  maximal proper subgroup such that  $H \subseteq N \Rightarrow G/N$  cyclic.

$$\bar{X} : G \mapsto \mathbb{C}^\times \text{ with } N = \text{Ker}(\bar{X}) \Rightarrow H \subseteq \text{Ker}(\bar{X})$$

$$\xrightarrow{\text{GRH, Thm1}} \exists p \in \mathbb{P} : p + n\mathbb{Z} \not\subseteq H, p < 2 \cdot \ln(n)^2$$

**Corollary:** Assume GRH.

Let  $n \in \mathbb{N}_{>1}$   $\mathbb{P}$  odd Then there is a strong witness  $a$  of compositeness of  $n$  with  $a < 2 \cdot \ln(n)^2$ .

$\rightarrow$  Obtain deterministic primality test with time  $O(\ln(n)^5)$  bit operations.

### AKS-test

A deterministic polynomial time primality test  $\rightarrow$  "holy grail"

Agrawal, Kayal, Saxena: PRIMES is in P, Annals of Mathematics, 2004.

### 1.3.11 Proposition 10 (modulo over ideals)

Let  $n \in \mathbb{P}$   $a \in \mathbb{Z} \Rightarrow (x + a)^n \equiv x^n + a \pmod{n}$

where  $x$  is a indeterminate and for  $r \in \mathbb{N}$ :

$$(x + a)^n \equiv (x^n + a) \pmod{n, x^r - 1} \quad (1)$$

(i.e.  $(x + a)^n - (x^n + a) = n \cdot f + (x^r - 1) \cdot g$  with  $f, g \in \mathbb{Z}[x]$ )

**Proof:**

$$(x + a)^n = \sum_{i=0}^n \binom{n}{i} \cdot x^i a^{n-i} \quad (\text{where } \binom{n}{i} \text{ is a multiple of } n \text{ for } 0 < i < n)$$

$$\equiv x^n + a^n \quad (\leftarrow \text{little Fermat})$$

$$\equiv x^n + a \quad (1) \text{ follows by weakening this.}$$

**Cost** analysis for checking (1) with  $l = \text{length}(n)$ .

Using Algorithm 3, need  $O(l)$  multiplications in  $\mathbb{Z}[x]/(n, x^r - 1) =: R$

Elements of  $R$  are represented as polynomials of degree  $< r$ ,  
coefficients between 0 and  $n$ .

Multiply polynomials:  $O(r^2)$  operation in  $\mathbb{Z}/(n) : O(r^2 \cdot l^2)$

since  $x^{r+k} \equiv x^k \pmod{x^r - 1}$ ,

add coefficients of  $x^{r+k}$  of product polynomial to coefficients  $x^k : O(r \cdot l)$

Total for checking (1):  $O(r^2 \cdot l^3)$  bit operations.

Reduction  $\pmod{x^r - 1}$  is just for keeping the cost under control.

The following is part of AKS-test:

### 1.3.12 Algorithm 11 (Test for perfect power)

input :  $n \in \mathbb{N}_{>1}$

output:  $m, e \in \mathbb{N}$   $e > 1$  such that  $n = m^e$  or "n is not a perfect power"

(1) for  $(e = 2, \dots, \lfloor \lg(n) \rfloor)$  perform (2) - (7) //possible exponents

(2) set  $m_1 = 2, m_2 = n$  //initialize interval  $[m_1, m_2]$  for searching  $\sqrt[e]{n}$

(3) while( $m_1 \leq m_2$ ) do (4) - (7)

(4) set  $m = \lfloor \frac{m_1 + m_2}{2} \rfloor$  // bisect interval

(5) if  $m^e = n$  return  $m, e$

(6) if  $m^e > n$  set  $m_2 = m - 1$

(7) if  $m^e < n$  set  $m_1 = m + 1$

(8) return "not a perfect power"

**Cost:** (for  $l = \text{length}(n)$ )

Compute  $m^e : O(\lg(l) \cdot l^2)$  (abort computation once the result exceeds  $n$ )

Number of passages through inner loops  $\leq \lg(n)$

Number of passages through outer loops  $\leq \lg(n)$

Total cost of Algorithm 11:  $O(l^4 \cdot \lg(l))$

### 1.3.13 Algorithm 12 (AKS-test)

input :  $n \in \mathbb{N}_{>1}$  of length  $l = \text{length}(n) = \lfloor \lg(n) \rfloor + 1$

output: " $n \in \mathbb{P}$ " or " $n \notin \mathbb{P}$ "

- (1) if ( $n$  is a perfect power)  
return " $n \notin \mathbb{P}$ "
- (2) find  $r \in \mathbb{N}_{>1}$  minimal such that  $r|n \vee n^i \not\equiv 1 \pmod{r} \quad \forall i = 1, \dots, l^2$   
//exhaustive search (we will show that  $r \leq l^5$ )
- (3) if  $r|n$   
if ( $r = n$ ) return " $n \in \mathbb{P}$ "  
if ( $r < n$ ) return " $n \notin \mathbb{P}$ "
- (4) for  $a = 1, 2, \dots, \lfloor \sqrt{r} \cdot l \rfloor$  do (5)
- (5) if  $((x+a)^n \not\equiv (x^n + a) \pmod{(n, x^r - 1)})$   
return " $n \notin \mathbb{P}$ "
- (6) return " $n \in \mathbb{P}$ "

### 1.3.14 Lemma 13 (Least common multiple)

For  $n \in \mathbb{N}_{>0}$  have  $\lambda(n) := \text{lcm}(1, 2, \dots, n) \geq 2^{n-2}$

**Proof:** For  $f = \sum_{i=0}^m a \cdot x^i \in \mathbb{Z}(x) \quad a_i \in \mathbb{Z}$

$$\Rightarrow \int_0^1 f(x) dx = \sum_{i=0}^m \frac{a_i}{i+1} = \frac{k}{\lambda(m+1)}$$

with  $k \in \mathbb{Z}$ . Consider  $f_m = x^m \cdot (1-x)^m$

For  $0 < x < 1$ :

$$0 < f_m(x) \leq 4^{-m}$$

$$\Rightarrow 0 < \int_0^1 \underbrace{f_m(x)}_{\frac{k_m}{\lambda(2m+1)}} dx \leq 4^{-1}$$

$$\lambda(2 \cdot m + 1) \geq k_m \cdot 4^m \geq 4^m$$

$$\text{For } n \in \mathbb{N}_{>0} \lambda(n) \geq \lambda(2 \cdot \lfloor \frac{n-1}{2} \rfloor + 1) \geq 4^{\lfloor \frac{n-1}{2} \rfloor} \geq 4^{\frac{n-1}{2}} = 2^{n-2} \quad \square$$

**Corollary:** (not related to AKS)

For  $n \in \mathbb{N}$

$$\pi(n) := |\{p \in \mathbb{P} | p \leq n\}| \geq \frac{n-2}{\lg(n)}$$

**Proof:**

$$2^{n-2} \leq \lambda(n) = \prod_{p \in \mathbb{P}, p \leq n} p^{\lfloor \log_p(n) \rfloor} \leq \prod_{p \leq n} p^{\log_p(n)} = n^{\pi(n)} = 2^{\lg(n)\pi(n)} \quad \square$$



**Prime number theorem:**

$$\lim_{n \rightarrow \infty} \frac{\pi(n)}{n/\ln(n)} = 1$$

**Interpretation:**

The average distance of two primes around some value  $x \in \mathbb{R}_{>1}$  is  $\ln(x)$

### 1.3.15 Lemma 14 (Property of $r$ in Algorithm 12)

In Algorithm 12, have  $r \leq l^5$

**Proof:**

if  $r < l^5 \Rightarrow \forall k \in \{2, \dots, l^5\} : \exists i \in \{1, \dots, l^2\}$

$$n^i \equiv 1 \pmod{k}$$

$$\Rightarrow k \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\Rightarrow \lambda(l^5) \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\xrightarrow[\text{Lemma 13}]{=} 2^{l^5-2} < \prod_{i=1}^{l^2} n^i = n^{\frac{l^2(l^2+1)}{2}}$$

$$\Rightarrow l^5 - l^3 < 4 \quad \text{not true since } l \geq 2 \quad \square$$

### 1.3.16 Theorem 15 (Bit-Complexity of Algorithm 12)

Algorithm 12 requires  $O(l^{16.5})$  bit operations ("polynomial complexity")

**Proof:**

Step(1):  $O(l^4 \cdot \lg(l)) \checkmark$

Step(2): For each  $r$  need:

- test  $r \mid n : O(l^2)$
- compute all  $n^i \pmod{r} : O(l^2 \cdot \lg(r)^2) \underset{\text{Lemma 14}}{\leq} O(l^2 \cdot \lg(l)^2)$

Step(3):  $O(1)$

$$\text{Step(4): } O(\sqrt{r} \cdot l \cdot r^2 \cdot l^3) \underset{\text{Lemma 14}}{\leq} O(l^{16.5}) \quad \square$$

**Reminder:** There is a variant of Algorithm 12 with running time  $\tilde{O}(l^6)$ , i.e.,  $O(l^6 \cdot \lg(l)^m)$  with  $m \in \mathbb{N}$ .

**Correctness:**

For  $r \in \mathbb{N}_{>0}$  and  $p \in \mathbb{P}$  write  $I(r, p) := \{m, f\} \in \mathbb{N} \times \mathbb{F}_p[x] \mid f(x)^m \equiv f(x^m) \pmod{x^r - 1}\}$   
 "m is introspective for  $f$  and  $r$ ".

**Example:** Proposition 10 says that:

$$(p, x + \bar{a}) \in I(r, p) \text{ for } a \in \mathbb{Z} \quad r \in \mathbb{N}_{>0} \quad p \in \mathbb{P} \quad (1)$$

### 1.3.17 Lemma 16 (Rules for ideals)

- (a)  $(m, f), (m', f) \in I(r, p) \Rightarrow (m \cdot m', f) \in I(r, p)$
- (b)  $(m, f), (m, g) \in I(r, p) \Rightarrow (m, f \cdot g) \in I(r, p)$
- (c)  $(m \cdot p, f) \in I(r, p), p \nmid r \Rightarrow (m, f) \in I(r, p)$

**Proof:**

- (a)  $f(x)^{m \cdot m'} \equiv f(x^m)^{m'} \pmod{(x^r - 1)}$   
 $f(x^m)^{m'} \equiv f(x^{m \cdot m'}) \pmod{(x^{m \cdot r} - 1)}$   
 But  $(x^r - 1) \mid (x^{m \cdot r} - 1)$
- (b)  $(f \cdot g)(x)^m = f(x)^m \cdot g(x)^m \equiv f(x^m) \cdot g(x^m) = (f \cdot g)(x^m) \pmod{(x^r - 1)}$
- (c)  $(f(x)^m)^p \equiv f((x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x^m))^p \pmod{(x^r - 1)}$   
 $\Rightarrow (x^r - 1) \mid ((f(x)^m)^p - f(x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x)^m - f(x^m))^p$   
 $p \nmid r \Rightarrow x^r - 1$  is square free. So  
 $(x^r - 1) \mid (f(x)^m - f(x^m)) \Rightarrow (m, f) \in I(r, p) \quad \square$

### 1.3.18 Theorem 17 (Correctness of Algorithm 12)

Algorithm 12 is correct.

**Proof:**

If the algorithm terminates in step(1),(3) or (5), it is correct. To show: If it terminates in step(6) it is correct, i. e.  $n \in \mathbb{P}$

**Claim 1:**  $\exists p \in \mathbb{P} : p \mid n \quad p \not\equiv 1 \pmod{r} \quad p > r$

Indeed if all prime divisors of  $n$  were  $\equiv 1 \pmod{r}$  then  $n \equiv 1 \pmod{r}$

Contradiction to step(2). All prime divisors of  $n$  are  $> r$  by step (2) and (3) ✓

Steps(2) and (3) imply that  $\gcd(n, r) = 1 \Rightarrow G := \langle \bar{n}, \underbrace{\bar{p}}_{p \bmod r} \rangle \subseteq (\mathbb{Z}/(r))^\times$

Step(2):  $\text{ord}(\bar{n}) > l^2 \Rightarrow l^2 < |G| < r$  (2)

Set  $s := \text{ord}(\bar{p} \in G) \Rightarrow r \mid (p^s - 1)$  with  $q := p^s \Rightarrow r \mid |\mathbb{F}_q^\times| \Rightarrow \exists \zeta \in \mathbb{F}_q$   $r$ -th root of 1

Set  $k := \lfloor \sqrt{r} \cdot l \rfloor \quad m := \left(\frac{n}{p}\right)$

By (1)  $(p, x + \bar{a}) \in I(r, p)$  with  $\bar{a} \in \mathbb{F}_p$

By step(4), have  $(n, x + \bar{a}) \in I(r, p)$

For  $\underline{e} = e_0, \dots, e_k \in \mathbb{N}_0$  set  $f_{\underline{e}} := \prod_{a=0}^k (x + \bar{a})^{e_a}$

Lemma 16 (b):  $(p, f_{\underline{e}}) \in I(r, p)$

$(n, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(c)}} (m, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(a)}} \forall s, t \in \mathbb{N}_0 : (p^s \cdot m^t, f_{\underline{e}}) \in I(r, p)$

$\Rightarrow f_{\underline{e}}(\zeta^{p^s \cdot m^t}) = f_{\underline{e}}(\zeta)^{p^s \cdot m^t}$  (3)

Set  $H := \langle \zeta + \bar{a} | a \in \{0, \dots, k\} \rangle \subseteq \mathbb{F}_q^\times$   
 $(\zeta \notin \mathbb{F}_p \text{ since } r \nmid (p-1) \text{ by Claim 1})$

Consider:  $T := \{(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1} \mid \sum_{a=0}^k e_a < |G|\}$

$\Phi : T \mapsto H, (e_0, \dots, e_k) \mapsto f_{\underline{e}}(\zeta) = \prod_a (\zeta + \bar{a})^{e_a} \in H$

**Claim 2:**  $\Phi$  is injective.

Indeed, take  $(\underline{e}), (\underline{\hat{e}}) \in T$  such that  $\Phi(\underline{e}) = \Phi(\underline{\hat{e}})$

$$\Rightarrow \forall s, t \in \mathbb{N}_0 : f_{\underline{e}}(\zeta^{p^s \cdot m^t}) \stackrel{(3)}{=} f_{\underline{e}}(\zeta)^{p^s \cdot m^t} = f_{\underline{\hat{e}}}(\zeta)^{p^s \cdot m^t} \stackrel{(3)}{=} f_{\underline{\hat{e}}}(\zeta^{p^s \cdot m^t})$$

$f_{\underline{e}} - f_{\underline{\hat{e}}}$  has roots  $\zeta^e$  with  $e \in G$  since  $G = \langle \bar{p}, \bar{m} \rangle$

These are all distinct (since  $\zeta$  is primitive)

But  $\deg(f_{\underline{e}} - f_{\underline{\hat{e}}}) < |G|$  So  $f_{\underline{e}} - f_{\underline{\hat{e}}} = 0$

Since  $k \leq \sqrt{r} \cdot l < r < p$  the  $(x + \bar{a})$  with  $a \in \{0 \dots k\}$  are primitive distinct.

So  $(\underline{e}) = (\underline{\hat{e}})$  ✓

So is  $|H| \geq |T|$  ?

Let  $M$  be the set of all  $\{x_0, \dots, x_k\} \subseteq \{1, \dots, |G| + k\}$

with  $x_0 < x_1 < \dots < x_k$

For  $\{x_0, \dots, x_k\} \in M$  define  $(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1}$  by  $e_a = x_a - x_{a-1}$  with  $x_{-1} := 0$

$$\Rightarrow \sum_{a=0}^k e_a = \sum_{a=0}^k (x_a - x_{a-1} - 1) = x_k - (k+1) < |G|$$

Obtain injection  $M \Leftrightarrow T$

$$\text{So } |H| \geq |T| \geq |M| = \binom{|G|+k}{k+1} \stackrel{(2)}{\geq} \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{k+1} = \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{\lfloor l\sqrt{|a|} \rfloor} \stackrel{(2)}{\geq} \binom{2 \cdot \lfloor l\sqrt{|a|} \rfloor + 1}{\lfloor l\sqrt{|a|} \rfloor}$$

### 1.3.19 Lemma 18 (Property of binomial coefficients)

$$\forall n \in \mathbb{N}_{>1} : \binom{2 \cdot n + 1}{n} > 2^{n+1}$$

**Proof:**

$n = 2 :$

$$\binom{5}{2} = 10 > 2^3$$

$n - 1 \rightarrow n :$

$$\binom{2 \cdot n + 1}{n} = \binom{2 \cdot n}{n-1} + \binom{2 \cdot n}{n} = \binom{2 \cdot n - 1}{n-2} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n} \geq 2 \cdot \binom{2 \cdot n - 1}{n-1} \stackrel{ind.}{>} 2 \cdot 2^n = 2^{n+1}$$

### Continuation of Proof of Theorem 17

$$|H| > 2^{\lfloor l \cdot \sqrt{|a|} \rfloor + 1} \geq 2^{l \cdot \sqrt{|a|}} \geq 2^{\lg(n) \cdot \sqrt{|a|}} = n \sqrt{|a|} \quad (4)$$

Assume  $n \notin \mathbb{P}$  By step (1)  $m$  is not a perfect power

$\Rightarrow$  the map  $\mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{N} \quad (s, t) \mapsto p^s m^t$  is injective.

Set  $A := \{p^s m^t \mid s, t \in \{0, \dots, \lfloor \sqrt{a} \rfloor\}\} \subseteq \mathbb{N}$

$$\Rightarrow |A| = (\lfloor \sqrt{|a|} \rfloor + 1)^2 > |G|$$

Since  $G = \langle \bar{p}, \bar{m} \rangle \subseteq (\mathbb{Z}/(r))^\times$  this implies that  $\exists n, \hat{n} \in A$

such that  $n \neq \hat{n}$  but  $b \equiv \hat{n} \pmod{r}$ .

$$\text{Let } h \in H \Rightarrow h = f_{\underline{e}}(\zeta) \text{ with } (\underline{e}) \in \mathbb{N}_0^{k+1} \Rightarrow h^n \stackrel{(3)}{=} f_{\underline{e}}(\zeta^n) \stackrel{n \equiv \hat{n} \pmod{r}}{=} f_{\underline{e}}(\zeta^{\hat{n}}) \stackrel{(3)}{=} h^{\hat{n}}$$

So the polynomial  $Y^n - Y^{\hat{n}} \in \mathbb{F}_q[Y]$  has all elements of  $H$  as zeros.  
 But  $\deg(Y^n - Y^{\hat{n}}) \leq \max\{n, \hat{n}\} \leq (p \cdot m)^{\lfloor \sqrt{|G|} \rfloor} \leq n\sqrt{|G|} < |H|$   
 $\Rightarrow$  contradiction since  $Y^n - Y^{\hat{n}} \neq 0$   $\square$

## 1.4 Cryptology

A ("Alice") wants to send a message to B ("Bob") such that an eavesdropper E ("Eve") can not read the clear message. So A and B encrypt the message.

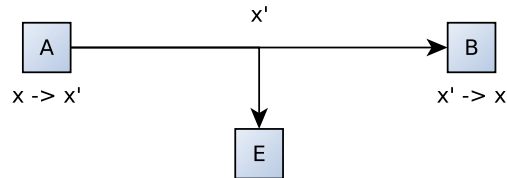


Figure 2: Scheme of eavesdropping

### Symmetric-key cryptography

A and B share secret keys for encryption ( $x \mapsto x'$ ) and decryption ( $x' \mapsto x$ ). Only A and B know the keys.

Example: AES approved by the US government in 2002

Application:

- sending messages
- encrypt files (A=B)

Problem: Key exchange between A and B

### Public-key cryptography

Encryption-map  $\phi : x \mapsto x'$  is made public by B, but decryption  $\phi : x' \mapsto x$  is kept secret.

Advantage: No confidential key exchange.

Disadvantages:

- more costly than symmetric key cryptography
- doubt whether E can reconstruct  $\phi^{-1}$  from  $\phi$  with enough computing power

Applications:

- sending messages
- exchange of symmetric keys
- authentication: Together with  $x$ , B sends  $\phi^{-1}(x)$  (or  $\phi^{-1}|$  Part of  $x$  together with date). A verifies by applying  $\phi$ .  
 Better: challenge-response-protocol.

Examples: RSA, elliptic curve

## RSA

- (1) B chooses  $p, q \in \mathbb{P}$  large ( $> 100\text{digits}$ )  
with  $p \neq q$   $n := p \cdot q$
- (2) B chooses  $e, f \in \mathbb{N}$  large such that  $e \cdot f \equiv 1 \pmod{\phi(n)}$   
with  $\phi(n) = (p-1)(q-1)$
- (3) B makes  $n, e$  public, keep  $f$  secret
- (4) The message is encoded as an element  $x \in \mathbb{Z}/(n)$
- (5) A computes  $\phi(x) = x^e = y \in \mathbb{Z}/(n)$  and sends  $y$
- (6) B receives  $y$  and computes  $y^f = x \in \mathbb{Z}/(n)$

Comments on steps of RSA:

- (6) Have  $e \cdot f = a \cdot (p-1) \cdot (q-1) + 1$  with  $a \in \mathbb{N}_{>0}$   
 $y^f = x^{e \cdot f}$

Case 1:  $q \nmid f, q \nmid x \Rightarrow x^{a(p-1)(q-1)} = (x^{\phi(n)})^a \stackrel{\text{LittleFermat}}{\equiv} 1^a = 1 \Rightarrow x^{e \cdot f} = x \quad \checkmark$

Case 2:  $p|x, q \nmid x \Rightarrow x^{e \cdot f} \equiv 0 \equiv x \pmod{p}$   
 $x^{e \cdot f} \equiv x \pmod{q}$  as above.

Case 3:  $q|x$  As Case 2

$\Rightarrow$  Correctness of decryption

**Cost:**

- (1) Finding  $p, q$  of length approximately  $l$ . Prime-number theorem: Gap between two primes of length  $\approx l$  is  $O(l)$   
Using Miller Rabin with error probability  $2^m$ . Expected cost of (1) is  $O(m \cdot l^4)$  bit operations.
- (2) Choose  $e$  co-prime to  $\phi(n)$  obtain  $f = \text{inverse} \pmod{\phi(n)}$  by extended euclidean Algorithm:  $O(l^2)$
- (5)(6) Fast exponentiation:  $O(l^3)$

Security of RSA:  $p$  and  $q$  must be so large that factorization of  $a$  is "impossible". Assumption that factorization is expensive could not be shown! But could  $f$  be obtained without knowing  $p$  and  $q$ ? The following algorithm gives a negative answer. It shows that the problem of breaking RSA is always basically factorization.

Remember:  $\phi(n)|(e \cdot f - 1) =: m \leq n^2$

### 1.4.1 Algorithm 1 (finding a divisor)

Input :  $n \in \mathbb{N}_{>2}$  odd squarefree  $e \notin \mathbb{P}$  and  $m \in \mathbb{N}_{>0}$  such that  $\phi(n) \mid m$   $m \leq n^2$

Output:  $d \in \mathbb{N}$  with  $d \mid n-1$   $1 < d < n$

- (1) Choose  $a \in \{2, \dots, (n-2)\}$  randomly  
set  $k := m$
- (2) If  $d := \gcd(a, n-1)$   
return d
- (3) Repeat steps (4) - (8) //while(true)
- (4) compute  $d := \gcd(n, a^k - 1)$
- (5) If  $d = 1$  go to (1)
- (6) If  $d < n$  return d
- (7) if  $k$  is odd go to (1)
- (8) set  $k := \frac{k}{2}$

Correctness is clear. What about termination and running time?

### 1.4.2 Proposition 2 (Complexity of Algorithm 1)

Algorithm 1 terminates in expected time  $O(l(n)^4)$  bit operations (Las Vegas Algorithm).

**Proof:**

Set  $l := \text{length}(n)$

Have  $n = \prod_{i=1}^r p_i$  with  $p_i \in \mathbb{P}$  distinct.

$\phi(n) = \prod_{i=1}^r (p_i - 1) \mid m$  So initially all  $(p_i - 1)$  divide  $k$ .

At some iteration it happens for the first time that  $(p_i - 1) \nmid k$

Then  $k \equiv \frac{p_1-1}{2} \pmod{(p_1-1)} \Rightarrow a^k \equiv \pm 1 \pmod{p_i}$  -1 occurs for some  $a$

For those  $j$  with  $(p_j - 1) \mid k$  have  $a^k \equiv 1 \pmod{p_j}$

Consider the group homomorphism:  $\phi_i(\mathbb{Z}/(n))^{\times} \mapsto (\mathbb{Z}/(p_1))^{\times} \times \dots \times (\mathbb{Z}/(p_r))^{\times}$   
 $\bar{a} \mapsto (a^k \pmod{p_1}, \dots, a^k \pmod{p_r})$

The image of  $\phi$  is a product of groups  $\{\pm 1\}$  or  $\{1\}$  depending whether  $(p_i - 1) \nmid k$  or  $(p_i - 1) \mid k$

**Conclusion:**

For at least half of all  $a$ 's,  $\phi(\bar{a})$  is neither  $(1, \dots, 1)$  nor  $(-1, \dots, -1)$

If  $a^k \equiv 1 \pmod{p_j}$  then  $p_j \mid (a^k - 1) \Rightarrow p_j \mid d$

If  $a^k \equiv -1 \pmod{p_j}$  then  $p_j \nmid (a^k - 1) \Rightarrow p_j \nmid d$

So for these  $a$  the algorithm is successful.

This means that the expected number of  $a$ 's that need to be tested is  $\leq 2$

(Since  $\sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = 2$  More generally for  $0 < p < 1 : p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} = \frac{1}{p}$ )

Analysis of running time (in bit operations) for each  $a$  (using gcd is quadratic) leads to the claim.  $\square$

### Problems of RSA:

- How difficult is factorization of integers (lower bound?)
- decryption of some or all messages without having  $f$ ?

## 2 Notes

### 2.1 Notation

- $\mathbb{N} := \mathbb{N}_0$
- $\lg(x) := \log_2(x)$
- $a \mid b$        $a$  is divisible by  $b$        $\Leftrightarrow$        $b \bmod a = 0$   
 $a \nmid b$        $a$  is not divisible by  $b$   $\Leftrightarrow$        $b \bmod a \neq 0$
- $ord(a)$       order of a group element  
 $n > 0$  minimal such that  $a^n = e$       with neutral element  $e$   
if no such  $n$  can be found,  $ord(a) = \infty$
- $char(A)$       Characteristic: the smallest positive  $n$  such that  
 $\underbrace{1 + \dots + 1}_n = 0$       with 1 as the multiplicative identity element  
 $n$  summands
- $\mathbb{Z}/(m)$       Ring modulo  $m$   
polynomial rings measure for " $<$ " relations not the absolute value but max power.
- $lcm(a_1, \dots, a_n)$       "least common multiple of all  $a_i$ "
- $\underline{e}$  = vector of e's
- $\phi(n) := |(\mathbb{Z}/(n))^x|$       Euler's totient function

### 2.2 Various stuff

- Lagrange's theorem: every element in a finite group has finite order
- Average number of bit operations for an increment:  
one operation for the last bit + 50% chance for one on the next bit + 25% on the following etc.  $\Rightarrow$  Geometrical row  
 $\Rightarrow$  on average two bit operations
- "Monte Carlo Algorithm": Always terminates in reasonable time but might yield false result.
- "Las Vegas Algorithm": If it terminates the result is correct. No deterministic running time.

### 2.3 Algebraic structures

- Group       $(G, *)$ 
  - one inner operation  $(*)$ :       $G \times G \mapsto G$
  - associativity:       $(a * b) * c = a * (b * c)$        $\forall a, b, c \in G$
  - neutral element  $e \in G$ :       $a * e = e * a = a$        $\forall a \in G$
  - inverse element  $a^{-1} \in G$ :       $a * a^{-1} = a^{-1} * a = e$        $\forall a \in G$
- Abelian group       $(G, *)$ 
  - $(G, *)$  is a group
  - commutativity:       $a * b = b * a$        $\forall a, b \in G$



- Finite group  $(G, *)$ 
  - associativity:  $(a * b) * c = a * (b * c)$
  - unambiguity of reduction:  $(a * x = a * x') \wedge (x * a = x' * a) \Rightarrow x = x'$   
 $\Rightarrow x \mapsto x * a$  and  $x \mapsto a * x$  is bijective  
 $\Rightarrow \exists x : a * x = a \Rightarrow$  neutral element  
 $\exists x : a * x = x \Rightarrow$  inverse element
- Cyclic group  $(G, *)$ 
  - $G$  is a group
  - $G$  is generated by one Element:  $G = \langle g \rangle = \{g^n | n \in \mathbb{Z}\}$
  - not necessarily finite.
- Semi group  $(S, *)$ 
  - one inner operation  $(*)$ :  $S \times S \mapsto S$
  - associativity:  $(a * b) * c = a * (b * c) \quad \forall a, b, c \in S$
- Field  $(K, +, \cdot)$ 
  - two inner operations  $(+, \cdot)$  such that:
    - $(K, +)$  is an abelian group with neutral element 0
    - $(K \setminus \{0\}, \cdot)$  is an abelian group with neutral element 1
  - distributivity:  $a \cdot (b + c) = a \cdot b + a \cdot c$   
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in K$
- Ring  $(R, +, \cdot)$ 
  - $(R, +)$  is an abelian group
  - $(R, \cdot)$  is a semi group
  - distributivity:  $a \cdot (b + c) = a \cdot b + a \cdot c$   
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in R$
- Commutative ring  $(R, +, \cdot)$ 
  - $(R, +, \cdot)$  is a ring
  - commutativity for  $(\cdot)$   $a \cdot b = b \cdot a \quad \forall a, b \in R$
- Unitary ring (ring with 1)  $(R, +, \cdot)$ 
  - $(R, \cdot)$  is a semi group
  - $(R, \cdot)$  has a neutral element "1"
- Euclidean ring  $R$ 
  - $\exists F : R \mapsto \mathbb{N}_0 \cup \{0\}$   
such that if  $\exists q, r \in R \quad a = b \cdot q + r$  and  $r = 0$  or  $a, b \in R \quad F(r) < F(b)$
- Polynomial ring  $R[X]$ 
  - $R$  is a commutative unitary ring
  - set of all polynomials with coefficients  $\in R$