

Computational Algebra

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Transcript

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1 Integer Arithmetic

Topics:

- Addition and Multiplication
- GCD computation
- Primality testing
- Factorization

1.1 Addition and Multiplication

Agreement:

- $a, x \in \mathbb{N}$ represented as $x = \sum_{i=0}^{n-1} a_i \cdot B^i$ $B \in \mathbb{N}_{>1}$ fixed Base
($a_i \in \{0, B-1\}$)
- if $x \neq 0$, assume $a_{n-1} \neq 0$ then define:
length of $x := l(x) = n$ = number of digits = $\lfloor \log_B(x) \rfloor + 1$
(mnemonic: $\log_B(B) + 1 = 2$)
- $l(0) = 1$
(Amount of memory required to store $x = 0$)
- $l(x) := l(|x|)$
- for $x \in \mathbb{Z}$ represent if as $x = \text{sgn}(x) * |x|$

1.1.1 Algorithm 1 (Simple addition)

input : $x = \sum_{i=0}^{n-1} a_i \cdot B^i, \quad y = \sum_{i=0}^{n-1} b_i \cdot B^i, \quad x, y \in \mathbb{N}$

output: $x + y = \sum_{i=0}^n c_i \cdot B^i$

- (1) $\sigma = 0$
- (2) for $i = 0, \dots, (n-1)$:
- (3) set $c_i := a_i + b_i + \sigma_i$ and $\sigma := 0$
- (4) if $(c_i \geq B)$
- (5) set $c_i = c_i - B$
- (6) set $\sigma = 1$
- (7) set $c_n = \sigma$

If $B = 2$ then (3) - (6) can be realized by logic gates:



1.1.2 Definition 2 (Bit-Operation)

A bit operation is an operation that can be performed by a logic gate or by searching or writing a bit from / into memory.

1.1.3 Definition 3 (Big O)

Let M be a set (usually $M = \mathbb{N}$), $f, g : M \mapsto \mathbb{R}_{>0}$
we write $f \in O(g)$ if $\exists c \in \mathbb{R} : f(x) \leq c \cdot g(x) \forall x \in M$

1.1.4 Theorem 4 (Lower bound for addition)

Let $f : \mathbb{N} \mapsto \mathbb{R}$, $n \mapsto$ maximal number of bit operations required by Algorithm 1 to add $x, y \in \mathbb{N}$ with $l(x), l(y) \leq n$

Let $g = id_{\mathbb{N}}$ Then $f \in O(g)$

We say Algorithm 1 requires $O(n)$ bit operations for adding two numbers of length $\leq n$.
 \Rightarrow "linear complexity"

Set $M := \{\text{Set of all algorithms for addition in } \mathbb{N}\}$

For $A \in M$ define $f_A : \mathbb{N} \mapsto \mathbb{R}$ as above.

We would like to find $f_{odd} : \mathbb{N} \mapsto \mathbb{R}$, $n \mapsto \inf\{f_A(n) | A \in M\}$

Since one needs to read x, y (and write the result) we can not do any better than linear complexity for addition.

Subtraction

let x, y as Algorithm 1, $x \geq y$

For $\bar{y} := \sum_{i=0}^{n-1} (B - 1 - b_i) B^i$ (digitwise / bitwise complement)

$\Rightarrow x + \bar{y} = x - y + B^n - 1$

$\Rightarrow x - y = x + \bar{y} + 1 - B^n$ (initially set $\sigma = 1$)

Conclusion: Addition and Subtraction have cost $O(n)$

1.1.5 Algorithm 5 (Multiplication by "grid method")

input : $x = \sum_{i=0}^{n-1} a_i \cdot 2^i, \quad y = \sum_{i=0}^{m-1} b_i \cdot 2^i$

output: $z = x \cdot y$

- (1) $z := 0$
- (2) for $i = 0, \dots, (n-1)$
- (3) if $(a_i \neq 0)$ set $z := z + \sum_{j=0}^{m-1} b_j 2^{i+j}$

1.1.6 Theorem 6 (Runtime of Algorithm 5)

Algorithm 5 requires $O(n * m)$ bit operations.

As of the total input length $n + m$:

$$n \cdot m \leq \frac{1}{2}(n + m)^2 \rightarrow O((n + m)^2)$$

\Rightarrow Quadratic complexity

Karatsuba-multiplication:

Observation for polynomials:

$$a + bx, c + dx \text{ have } (a + bx)(c + dx) = ac + (ac + db - (a - b)(c - d))x + bdx^2$$

The point: only used 3 multiplications instead of 4.

Specialize $x = B$ "large" such that $x = a + bB$ partition into two blocks. Then multiply the blocks by a recursive call.

1.1.7 Algorithm 7 (Karatsuba)

input : $x, y \in \mathbb{N}$

output: $z = x \cdot y$

- (1) Choose $k \in \mathbb{N}$ minimal such that $l(x), l(y) \leq 2^k$.
Set $B = 2^{2^{k-1}}$
- (2) if $(k = 0)$ return $x \cdot y$ (by bit-operation AND)
- (3) write $x = x_0 + x_1 B, \quad y = y_0 + y_1 B$ with $l(x_i), l(y_i) \leq 2^{k-1}$
- (4) compute $x_0 \cdot y_0, \quad x_1 \cdot y_1, \quad (x_0 - x_1) \cdot (y_0 - y_1)$ by a recursive call
- (5) return $z = x_0 y_0 + (x_0 y_0 + x_1 y_1 - (x_0 - x_1)(y_0 - y_1))B + x_1 y_1 B^2$

1.1.8 Theorem 8 (Runtime of Algorithm 7)

For multiplying two numbers of length $\leq n$ Algorithm 7 requires $O(n^{\log_2 3}) \approx O(n^{1.59})$ bit operations.

Proof:

Set $\Theta(k) :=$ maximal numbers of bit operations for $l(x), l(y) \leq 2^k$

We have for $k > 0$: $\Theta(k) \leq 3 \underbrace{\Theta(k-1)}_{\text{recursive calls}} + c \underbrace{2^k}_{\text{additions}}$ with (c some constant)

Claim: $\Theta(k) \leq 3^k + 2c(3^k - 2^k)$

Proof by Induction on k :

$k = 0$: $\Theta(k) = 1$

$$\begin{aligned} k-1 \rightarrow k : \Theta(k) &= 3\Theta(k-1) + c2^{k-1} \\ &\leq 3(3^{k-1} + 2c(3^{k-1} - 2^{k-1})) + c2^k \\ &= 3^k + 2c(3^k - 2^k) \end{aligned}$$

So $\Theta(k) \leq (2c+1)3^k$

Now $l(x) \leq n$ hence $2^{k-1} < n$ by minimality of k

So $k-1 < \log_2 n$

$$\begin{aligned} \Rightarrow \Theta(k) &\leq 3(2c+1)3^{\log_2(n)} \\ &= 3(2c+1)2^{\log_2(3) \log_2(n)} \\ &= 3(2c+1)n^{\log_2(3)} \quad \square \end{aligned}$$

One can modify the terminal condition of Karatsuba to switch to Grid-Multiplication, which is faster for small numbers.

Fast-Fourier Transform

Reminder: For a function $f : \mathbb{R} \mapsto \mathbb{C}$ define:

$\hat{f} : \mathbb{R} \mapsto \mathbb{C}$ by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt \quad (\text{if it exists})$$

Think of ω as frequency.

Definition (Convolution)

Let $f, g : \mathbb{R} \mapsto \mathbb{C}$

$$(f * g)(x) = \int_{\mathbb{R}} f(t) g(x-t) dt$$

Convolution is analogous to polynomial multiplication **Formula:** $\underbrace{(f * g)}_{\text{(Cauchy formula)}} = \hat{f} \cdot \hat{g}$

For a function $M \mapsto C$ with $|M| < \infty$ we need the discrete Fourier transform (DFT)

1.1.9 Definition 9 (Root of unity)

Let R be a commutative ring with 1. An element $\mu \in R$ is called an n -th root of unity (= root of 1) if $\mu^n = 1$.

It is called primitive if $\mu^i \neq 1$ for $(0 < i < n)$ i.e. $\text{ord}(\mu) = n$

let μ be a primitive n -th root of 1 (e.g. $e^{2\pi \frac{i}{n}} \in \mathbb{C}$)

Then the map $DFT_\mu : R^n \mapsto R^n$

$$(\hat{a}_0, \dots, \hat{a}_n) \mapsto (\hat{a}_0, \dots, \hat{a}_n) \quad \text{with } \hat{a}_i = \sum_{j=0}^{n-1} \mu^{ij} a_j$$

is called discrete Fourier transformation

For polynomials:

$$DFT_\mu : R[x] \mapsto R^n$$

$$f \mapsto (f(\mu^0), \dots, f(\mu^{n-1}))$$

Convolution rule: (from $f(\mu^i)g(\mu^i) = (f * g)(\mu^i)$)

$$DFT_\mu(f * g) = DFT_\mu(f) \cdot DFT_\mu(g) \quad (\text{component wise product})$$

Addition of two polynomials in $R[x]$ of $\deg(n)$ require $O(n)$ ring operations. Multiplication require $O(n^l)$.

With Karatsuba have $O(n^{\log_2(3)})$ ring operations.

Cost $DFT_\mu(f) \cdot DFT_\mu(g) : O(n)$ ring operations (with μ as $2n$ -th root of 1)

Want: Cheap way of doing DFT and back-transformation.

1.1.10 Algorithm 10 (Fast Fourier transformation FFT)

input : $f \in R[x]$, $\mu \in R$ primitive 2^k -th root of 1, such that $\mu^{2^{k-1}} = -1$

output: $DFT_\mu(f)$

- (1) Write $f(x) = g(x^2) + xh(x^2)$ with $f, g, h \in R[x]$
- (2) if $k = 1$ ($\Rightarrow \mu = 1$) return $DFT_\mu(f) = (g(1) + h(1), g(1) - h(1))$
- (3) Recursive call: compute $DFT_{\mu^2}(g) = \hat{g}, DFT_{\mu^2}(h) = \hat{h} \in R^{2^{k-1}}$
- (4) return $DFT_\mu(f) = (\hat{f}_0, \dots, \hat{f}_{2^k-1})$ with $\hat{f}_i = \hat{g}_i + \mu \hat{h}_i$
where $\hat{g}_i = \hat{g}_{i-2^{k-1}}$ for $i \geq 2^{k-1}$

Note: Components of \hat{g} and \hat{h} are:

$$\hat{g} = g(\mu^{2^i}), \quad \hat{h}_i = h(\mu^{2^i}) \quad \text{so}$$

$$\hat{f}_i := f(\mu^i) = \hat{g}_i(\mu^{2^i}) + \mu \hat{h}_i(\mu^{2^i}) = \hat{g}_i + \mu \hat{h}_i$$

Convention: $\lg(x) = \log_2(x)$

1.1.11 Theorem 11 (Runtime of Algorithm 10)

Let $n = 2^k$, $f \in R[x]$ with $\deg(\psi) < n$ Then Algorithm 10 requires $O(n \cdot \lg(n))$ ring operations.

Better than $O(n^{1+\epsilon})$, $\forall \epsilon > 0$!

Proof:

Set $\Theta(k) = \max$ number of ring operations required. By counting obtain for $k > 1$:

$$\Theta(k) \leq 2\Theta(k-1) + \underbrace{(\text{compute } \mu^i (i \leq 2^{k-1}))}_{2^{k-1}} + \underbrace{(\mu^i \hat{k}_i)}_{2^{k-1}} + \underbrace{(\text{sums and differences})}_{2^k}$$

$$= 2\Theta(k-1) + 2^{k+1}$$

Claim: $\Theta(k) \leq (2k-1)2^k$

$$k=1 : f = a_0 + a_1 \cdot x \quad DFT_\mu(f) = (a_0 + a_1 \cdot a_0 - a_1) \Rightarrow \Theta(a) = 2$$

$$k-1 \rightarrow k : \Theta(k) \leq 2 \cdot \Theta(k-1) + 2^{k+1} \leq 2 \cdot (2k-3) \cdot 2^{k-1} + 2^{k+1} = (2k-1) \cdot 2^k$$

since $k = \lg(n)$ obtain $O(k) \leq (2 \cdot \lg(n) - 1) \cdot n \in O(n \cdot \lg(n)) \quad \square$

Back-transformation?

1.1.12 Definition 12 (Good root of unity)

A primitive n -th root of unity is called good (caveat: this is ad-hoc terminology) if:

$$\sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \text{for } (0 < i < n)$$

example:

- (1) $\mu = e^{2\pi \frac{i}{n}}$ is a good primitive root of unity
- (2) $R = \mathbb{Z}/(8)$, $\mu = \bar{3} \Rightarrow \mu \cdot B$ is primitive 2^{nd} root of unity
But $\bar{B}^0 + \bar{3}^1 = \bar{u} \neq \bar{0}$ so μ is not good.

1.1.13 Proposition 13 ($DFT_{\mu^{-1}}$)

let $\mu \in R$ be a good root of 1

$$(a) = (a_0, \dots, a_{n-1}) \in R^n \Rightarrow DFT_\mu^{-1}(DFT_\mu(a)) = n \cdot (a) \quad \text{where } n = 1 + \dots + 1 \in R$$

Proof:

$$DFT_\mu(a) = (\hat{a}) = (\hat{a}_0, \dots, \hat{a}_{n-1})$$

$$\text{with } \hat{a}_j = \sum_{k=0}^{n-1} \mu^{jk} a_k$$

$$DFT_{\mu^{-1}}(\hat{a}) = (\hat{\hat{a}}_0, \dots, \hat{\hat{a}}_1)$$

$$\text{with } \hat{\hat{a}}_i = \sum_{j=0}^{n-1} \mu^{-ij} \sum_{k=0}^{n-1} \mu^{jk} a_k = \sum_{k=0}^{n-1} (a_k \cdot \underbrace{\sum_{j=0}^{n-1} \mu^{j(k-i)}}_{=0 \text{ if } n \neq k-i \text{ (i.e. } k=i)}) = a_i \cdot n \quad \square$$

1.1.14 Proposition 14 (Finding good roots of unity)

let $\mu \in R, n \in \mathbb{N}$

Assume:

- a) R is an integral Domain and μ is a primitive or n -th root of 1
(Integral Domain: nonzero commutative ring in which the product of two nonzero elements is nonzero)
 \Rightarrow Granted by FFT
- b) $n = 2^b, \mu^{\frac{n}{2}} = -1$, then $h > 0 \wedge \text{char}(R) \neq 2$
 $\rightarrow \mu$ is a good primitive n -th root of 1 ("root of unity")

Proof:

- a) for $0 < i < n$

$$\underbrace{(\mu^i - 1)}_{\neq 0} \underbrace{\left(\sum_{j=0}^{n-1} \mu^{ij}\right)}_{=0} = \mu^{in} - 1 = 0$$

$\Rightarrow \mu$ is a good root of unity

- * Let $0 < i < n$, write $i = 2^{k-s} \cdot r$ with r odd $\wedge s > 0$

$$\sum_{j=0}^{2^k-1} \mu^{ij} = \sum_{l=0}^{2^{k-s}-1} \sum_{j=0}^{2^s-1} \mu^{i(l \cdot 2^s + j)}$$

$$\mu^{i \cdot 2^s} = 1$$

$$i \cdot 2^s = 2^{k-s} \sum_{j=0}^{2^s-1} \mu^{ij} = 2^{k-s} \sum_{j=0}^{2^{s-1}-1} (\mu^{ij} + \mu^{i(2^{s-1}+j)})$$

$$\text{But } \mu^{i \cdot 2^{s-1}} = \mu^{2^{k-s} \cdot r \cdot 2^{s-1}} = \mu^{2^{k-1} \cdot r} = (-1)^r = -1$$

$$\text{So } \sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \square$$

- b) $\mu^n = 1, n = 2^k \Rightarrow \text{ord}(\mu) | n \Rightarrow \text{ord}(\mu)$ is power of 2

1.1.15 Algorithm 15 (Polynomial multiplication using DFT)

input : $f, g \in R[x]$ with $\deg(f) + \deg(g) < 2^k =: n$
 $\mu \in R$ as a good root of unity; Assume $2 \in R$ is invertible

output: $h = f \cdot g$

- (1) compute $\hat{f} = DFT_{\mu}(f), \hat{g} = DFT_{\mu}(g)$ with $f, g \in R^n$
- (2) compute $\hat{h} = \hat{f} \cdot \hat{g}$
- (3) compute $(h_0, \dots, h_{n-1}) = DFT_{\mu^{-1}} \hat{h}$ (same as $DFT_{\mu}(\hat{h})$ but with different order)
= Back-transformation $\cdot 2^k$
set $h = \frac{1}{2^k} \sum_{i=0}^{n-1} h_i x^i$

1.1.16 Theorem 16 (Runtime of Algorithm 15)

Algorithm 15 uses $O(n \cdot \log(n))$ ring operations for polynomials of $\deg < n$

Proof:

- Choose k minimal so that $\deg(f) \cdot \deg(g) < 2^k$
 $\Rightarrow 2^{k-1} \leq 2n \Rightarrow k \leq \log(n) + 2$
- $\underbrace{O(2k \cdot 2^k)}_{\text{Step 1}} + \underbrace{2^k}_{\text{Step 2}} + \underbrace{O(k \cdot 2^k) + 2^k}_{\text{Step 3}} \in O(2k \cdot 2^k) = O(n(g(n))) \quad \square$

Goal: Multiplication in \mathbb{N} using DFT

Idea: find roots of 1 in a suitable $\mathbb{Z}/(m)$

Choose $m = 2^l + 1, \mu = \bar{2} \in R$

1.1.17 Proposition 17 (Add and mul in $O(l)$)

Let $m = 2^l + 1, R = \mathbb{Z}/(m)$

Addition in R and multiplication by $\bar{2}^i \in R$ ($0 \leq i < 2l$) can be done in $O(l)$ bit operations

Proof:

- Let $\bar{x} \in R$ with $0 \leq x \leq 2^l$
- Addition: $x + \bar{y}$
 - (1) compute $x + y \in \mathbb{N}$: $O(l)$
 - (2) if $x + y > 2^l + 1$ subtract $2^l + 1$: $O(l)$
 - Multiplication by $\bar{2}^i$ ($0 \leq i < l$)
 - (1) Bit-shift i Bits to the left by relocating in memory:

$$\underbrace{O(\text{length}(i))}_{\text{compute addr. of new first bit}} + \underbrace{l}_{\text{copying}} = O(\log(l)) + l \in O(l)$$
 - Multiplication by $\bar{2}^i$ ($l \leq i < 2l - 1$)
 - (1) Multiplication by $\bar{2}^{i-l}$: $O(l)$
 - (2) take negative $\bar{2}^i \cdot \bar{x} = -\bar{2}^{i-l} \cdot \bar{x}$: $O(l)$

1.1.18 Proposition 18 (Sort of summary)

Let $k, r \in \mathbb{N}, r > 0, m = 2^{2^k \cdot r} + 1, R = \mathbb{Z}/(m), \mu = \bar{2}^r \in R$

$\Rightarrow 2 \in R$ is invertible, μ is a good primitive 2^{k+1} -th root of 1

$\Rightarrow \mu^{2^k} = 1$

Proof: \rightarrow from above

1.1.19 Algorithm 19 (Multiplication using FFT)

input : $x, y \in \mathbb{N}$

output: $Z = x \cdot y$

- (1) Choose $k \in \mathbb{N}$ minimal such that $l(x), l(y) \leq 2^{2^k}$
- (2) if $k \leq 3$, compute $z = x \cdot y$ by Algorithm 5
- (3) set $B = 2^{2^k}$, $m = 2^{2^k \cdot 4} + 1$, $R = \mathbb{Z}/(m)$, $\mu = \bar{2}^4 \in R$
 $(\Rightarrow \text{so } \mu \text{ is a good primitive } 2^{k+1}\text{-th root of 1})$
- (4) write $x = \sum_{i=0}^{2^k-1} x_i \cdot B^i$, same for y with $(0 \leq x_i, y_i < B)$
 possible since $x, y < 2^{2^{2^k}} = 2^{2^k \cdot 2^k} = B^{2^k}$
- (5) compute: $\hat{x} = DFT_\mu(\bar{x}_0, \dots, \bar{x}_{2^k-1}, \underbrace{0, \dots, 0}_{2^k \text{ zeros}}) \in R^{2^{k+1}}$
 same for y
 \rightarrow use FFT
- (6) compute: $\hat{z} = \hat{x} \cdot \hat{b} \in R^{2^{k+1}}$ (component wise multiplication)
 Perform multiplication in R as follows:
 Multiply representatives (non negative and $< m$) by recursive call.
 Then reduce modulo m by "negative bit shift" (see proof of Proposition 17)
- (7) compute: $(\bar{z}_0, \dots, \bar{z}_{2^{k+1}-1}) = \frac{1}{2^{k+1}} DFT_{\mu^{-1}}(\hat{z}) \in R$ with $0 \leq z < m$
- (8) set $z := \sum_{j=0}^{2^{k+1}-1} z_j \cdot B^j$

1.1.20 Theorem 20 (Runtime of Algorithm 19)

Algorithm 19 correctly computes $t = x \cdot y$ and requires $O(n \cdot (\log n)^4)$ bit operations for $l(x), l(y) \leq n$

Proof: Correctness

write $x(t) \sum_{i=0}^{2^k-i} x_i t^i \in \mathbb{Z}[t]$, $y(t), \bar{x}(t) \in R[t], \bar{y}(t), \bar{z}(t)$

by Proposition 18 and Proposition 13 we have $\bar{z}(t) = \bar{x}(t) \cdot \bar{y}(t)$

The l -th coefficient of $x(t) \cdot y(t)$ is $0 \leq \sum_{i+j=l} x_i \cdot y_j < 2^k \cdot B^2 = 2^{k+2 \cdot 2^k} \leq 2^{2^{k+2}} < m$

So $z(t) = x(t) \cdot y(t) \Rightarrow z = z(B) = x(B) \cdot y(B) = x \cdot y$ Cost:

Write $\Theta(k) := \max$ number of bit operations

Analyze Steps:

- (1) compute $\max \{l(x), l(y)\} : O(l(n)) = O(k)$
- (2) $O(1)$
- (3) no bit operations
- (4) compute starting addresses of x_i, y_i in memory: $2 \cdot 2^k$ increments of the address:
 $2 \cdot 2 \cdot 2^k = 2^{k+2}$ bit ops
 $\Rightarrow O(2^k)$
- (5) By Theorem 11 need $O(2 \cdot 2^{k+1} \cdot (k+1))$ operations in R which are additions and multiplications by powers of \bar{z} costing $O(2^{k+2})$ bit operations.
Total for (5): $O(k \cdot 2^{2 \cdot k})$
- (6) 2^{k+1} multiplications of numbers $< m$, i.e. of length $\leq 2^{k+2}$.
So $k' \leq \frac{k+3}{2}$ for k' : the "new" k used in the next recursion level.
For $\alpha \in R_{>0}$ define $\Theta(\alpha) := \Theta(\lfloor \alpha \rfloor)$
Total for (6): $2^{k+1}(\Theta(\frac{k+3}{2}) + \underbrace{O(2^{k+2})}_{\text{reduction mod } m})$
- (7) For $DFT_{\mu-1}(\hat{z}) : O(k \cdot 2^{2 \cdot k})$ as (5) Since \bar{z} is a n root of 1, multiplication by $\bar{2}^{-k-1}$ is multiplication by a positive power of $\bar{2}$, which costs $O(2^{k+2})$
Total for (7): $O(k \cdot 2^{2 \cdot k})$
- (8) For $j \leq 2^{k+1}$ have $\sum_{i=0}^{j-1} z_i \cdot B^i \leq (m-1) \sum_{i=0}^{j-1} B^i = (m-1) \frac{B^j-1}{B-1} < 2(m-1) \frac{B^j}{B} = 2^{1+2^{k+2}+(j-1)2^k}$ so the sum has length $(j+3) \cdot 2 + 1$
Adding $z_j \cdot B^j$ to this sum happens at $(j \cdot 2^k)$ -th bit and higher \Rightarrow cost is $O(2^k)$
Total for (8): $O(2^{2 \cdot k})$

Grad total: For $k \geq 4$:

$$\Theta(k) \leq 2^{k+1} \cdot \Theta(\frac{k+3}{2}) + c \cdot k \cdot 2^{2 \cdot k} \quad \text{with } c \text{ constant}$$

Also for $k \in \mathbb{R}_{\geq 4}$

Define $\Lambda(k) := \frac{\Theta(k)}{2^{2 \cdot k}} \Rightarrow \Lambda(k) \leq \frac{2^{k+1} \Theta(\frac{k+3}{2})}{2^{2 \cdot k}} + c \cdot k = 16 \cdot \Lambda(\frac{k+3}{2}) + c \cdot k$

Define $\Omega(k) := \Lambda(k+3)$ So for $k \in \mathbb{R}_{>1}$

$$\Omega(k) \leq 16 \cdot \Lambda(\frac{k}{2} + 3) + c \cdot (k+3) = \underbrace{16 \Omega(\frac{k}{2})}_{*} + c \cdot (k+3)$$

Claim: For $i \in \mathbb{N}$ with $2^{i-1} \leq k-3$ have:

$$\Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k+3)(1+8+\dots+8^{i-1}) + 3 \cdot c \cdot (1+16+\dots+16^{i-1})$$

Proof by induction:

$$i = 0: \Lambda(k) = \Omega(k-3)$$

$$i \rightarrow i+1: \Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k-3)(1+\dots+8^{i-1}) + 3 \cdot c \cdot (1+\dots+16^{i-1}) \leq 2^i \leq k-3 \quad *$$

$$\leq 16^i (16 \Omega(\frac{k-3}{2^{i+1}})) + c(\frac{k-1}{2^i} + 3) + c(k-3)\dots = \text{claimed result}$$

Take $u \in \mathbb{N}$ minimal with $2^u > k-3 \Rightarrow \Omega(\frac{k-3}{2^u}) \leq \Omega(\lfloor \frac{k-3}{2^u} \rfloor) = \Omega(0) =: D$ (constant)

Note: u roughly is recursion depth

$$\text{Have } 2^{u-1} \leq k-3 \xRightarrow{\text{claim}} \Lambda(k) \leq 16^u \cdot D + c \cdot \underbrace{(k-3)}_{< 2^u} \cdot \frac{8^u-1}{7} + 3c \cdot \frac{16^u-1}{15} \in O(16^u)$$

$$\text{Have } 2^{u-1} \leq k-3 \Rightarrow u \leq \lg(k-3) + 1$$

$$\Rightarrow \Lambda(k) \in O(16^{\lg(k-3)}) = O((k-3)^4)$$

$$\Rightarrow \Theta(k) = 2^{2 \cdot k} \cdot \Lambda(k) \in O(2^{2k} \cdot (k-3)^4)$$

$$\text{Have } 2^{2(k-1)} < \underbrace{n}_{\max\{l(x) \cdot l(y)\}} \Rightarrow k \leq \frac{\lg(n)}{2} + 1$$

$$\text{So } \Theta(k) \in O(n \cdot (\lg(n))^4) \quad \square$$

1.1.21 Theorem 21 (Schönhage-Strassen 1971)

Multiplication of integers of length $\leq n$ can be done in $O(n \cdot \lg(n) \cdot \lg(\lg(n)))$ bit operations. Schönhage-Strassen is used for integers of length ≥ 100.000 .

Asymptotically faster: Fürer's algorithm.

Comments on Bit complexity

1. Memory requirement may explode!
 \Rightarrow No Problem as bit complexity is upper bound for memory requirements, since memory access is included in bit operations
 $(\rightarrow$ only store what is calculated)
2. Computation of addresses in memory take time
 \Rightarrow length of addresses $\approx \lg(\text{memory space})$ computations of addresses $\approx \lg(\text{memory space})^2$
3. As memory requirement gets larger access times will get longer.
 \Rightarrow transportation time for data $\geq \frac{\text{diameter of physical storage}}{2 \cdot \text{speed of light}}$

1.2 Division with remainder, Euclidean algorithm

1.2.1 Algorithm 1 (Division with remainder)

input : $b = \sum_{i=0}^{n-1} b_i 2^i \quad \sum_{i=0}^{n+m-1} a_i 2^i$ with $a_i, b_i \in 0, 1, \quad b_{n-1} = 1$

output: $r, q \in \mathbb{N}$ such that $a = q \cdot b + r, \quad 0 \leq r < b$

(1) $r = a; \quad q = 0$

(2) for $i = m, m-1, \dots, 0$ do

(3) if $r \leq 2^i \cdot b$ then set $r := r - 2^i \cdot b, \quad q = q + 2^i$

1.2.2 Proposition 2 (Runtime of Algorithm 1)

Algorithm 1 is correct and requires $O(n \cdot (m+1))$ bit operations.

Proof:

Always have $a = q \cdot b + r$

Claim:

before step (3), have $0 \leq 2^{i+1} \cdot b$

$i = m; \quad 0 \leq r = a < 2^{m+n} = 2^{m+1} \cdot 2^{n-1} \leq 2^{m-1} \cdot b \quad i < m$ By step (3)

So after last passage through the loop $0 \leq r < b$

Running Time: In step(3), have comparison and (possibly) subtraction. Only n bits involved $\Rightarrow O(n)$

Total: $O(b \cdot (m+1))$

Remarks:

(1) Division with remainder can be reduced to multiplication.

Precisely: given an algorithm for multiplication that requires $M(n)$ bit operations, there exists an algorithm for division with remainder that requires $O(M(n))$ bit operations.

(2) Practically relevant:

Jebelean's algorithm (1997): $O(n^{\lg 3})$

(3) Alternatively, may choose $r \in \mathbb{Z}$ such that $\lfloor \frac{-b}{2} \rfloor < r \leq \lfloor \frac{b}{2} \rfloor$

(4) Algorithm 1 extends to \mathbb{Z} .

(5) All Euclidean rings have division with remainder (by definition).

(e.g., $R = K[x] \rightarrow$ polynomial ring over field,

$R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \subseteq \mathbb{C}, \quad i^2 = -1$)

1.2.3 Algorithm 3 (Euclidean algorithm)

input : $a, b \in \mathbb{N}$

output: $\gcd(a, b)$ "greatest common divisor"

- (1) set $r_0 := a, \quad r_i := b$
- (2) for $i = 1, 2, 3, \dots$ perform steps (3) and (4)
- (3) if $r_i = 0$ then $\gcd(a, b) = |r_{i-1}|$
- (4) Division with remainder: $r_{i-1} = q \cdot r_i + r_{i+1} \quad r_{i+1} \in \mathbb{Z}$
 $|r_{i+1}| \leq \frac{1}{2}|r_i|$

Example:

$$a = 287, \quad b = 126$$

$$287 = 2 \cdot 126 + 35 \tag{1}$$

$$126 = 4 \cdot 35 - 14 \tag{2}$$

$$35 = (-2) \cdot (-14) + 7 \tag{3}$$

$$-14 = (-2) \cdot 7 + 0 \tag{4}$$

$$\begin{aligned} \text{So: } 7|(-14) &\xRightarrow{(3)} 7|35 \\ &\xRightarrow{(2)} 7|126 \\ &\xRightarrow{(1)} 7|287 \end{aligned}$$

On the other hand take a common divisor d ; $d|287$; $d|126$

$$\xRightarrow{(1)} d|d \xRightarrow{(2)} d|14 \xRightarrow{(3)} d|7$$

1.2.4 Theorem 4 (Correctness of Algorithm 3)

Algorithm 3 is correct.

Proof:

Since $r_{i-1} = q \cdot r_i + r_{i+1}$ every integer $x \in \mathbb{Z}$ satisfies the equivalence $x|r_{i-1}$ and $x|r_i \Leftrightarrow x|r_{i+1}$ and $x|r_i$ so $\gcd(r_{i-1}, r_i) = \gcd(r_i, r_{i+1}) = \gcd(a, b)$ when terminating have $\gcd(a, b) = \gcd(r_{i-1}, 0) = |r_{i-1}| \quad \square$

1.2.5 Theorem 5 (Runtime of Algorithm 3)

Algorithm 3 requires $O(m \cdot n)$ bit operations for $n = l(a), m = l(b)$

Proof:

If $a < b$ then the first passage yields $r_2 = a, r_1 = b$. Cost: $O(n)$

May assume: $a \geq b$. Write $n_i = l(r_i)$

By Proposition 2 $\exists c$ constant such that the total time is $\leq c \cdot \underbrace{\sum_{i=1}^k n_i \cdot (n_{i-1} - n_i + 1)}_{=: \sigma(n_0, \dots, n_k)}$

For $i > 2$: $n_i = n_{i-1} - 1$

Special Case: $n_i = n_{i-1} - 1$ for $i \geq 2$

$\Rightarrow n_i = n_i - i + 1, n_i = m, k = m + 1$

Obtain $\sigma(n_0, \dots, n_k) = m \cdot (n - m + 1) + \sum_{i=2}^{m+1} (m - i + 1) \cdot 2 = m \cdot n - m^2 + m + m(m - 1) = m * n$.

Claim: The special case is the worst (most expensive)!

From any sequence $n_1 > n_2 > \dots > n_k$ get to the special case by iteratively inserting numbers in the gaps. Insert s with $n_{j-1} > s > n_j$.

$\sigma(n_0, \dots, n_{j-1}, s, n_j, \dots, n_k) - \sigma(n_0, \dots, n_k) = \dots = s + (n_{j-1} - s) \cdot (s - n_j)$

$sp\sigma(n_0, \dots, n_k) \leq \sigma(n, m, m - 1, \dots, 2, 1, 0) = n \cdot m \quad \square$

Complexity is quadratic \rightarrow cheap

1.2.6 Algorithm 6 (Extended Euclidean Algorithm)

input : $a, b \in \mathbb{N}$

output: $d = \gcd(a, b)$ and $s, t \in \mathbb{Z}$ such that $d = s \cdot a + t \cdot b$

(1) $r_0 := a, r_1 := b, s_0 := 1, t_0 := 0, s_1 := 0, t_1 := 1$

(2) for $i = 1, 2, \dots$ perform steps (3) - (5)

(3) if $r_i = 0$ set $d = |r_{i-1}|$
 $s := \text{sgn}(r_{i-1}) \cdot s_{i-1},$
 $t := \text{sgn}(r_{i-1}) \cdot t_{i-1}$

(4) division with remainder:
 $r_{i+1} = r_{i-1} - q_i \cdot r_i, \quad \text{with } |r_{i+1}| \leq \frac{1}{2}|r_i|$

(5) set $s_{i+1} := s_{i-1} - q_i \cdot s_i,$
 $t_{i+1} := t_{i-1} - q_i \cdot t_i$

Justification : $r_i = s_i \cdot a + t_i \cdot b$ throughout

Application: $m, x \in \mathbb{N}$ such that m, x co-prime (i.e. $\gcd(x, m) = 1$)

Algorithm 6 yields: $1 = s \cdot x + t \cdot m \Rightarrow s \cdot x \equiv 1 \pmod{m}$. So obtain inverse of $\bar{x} \in \mathbb{Z}/(m)$

1.3 Primality testing

Let $\mathbb{P} \subseteq \mathbb{N}$ be the set of prime numbers.

Challenge: Given $n \in \mathbb{N}$ decide if $n \in \mathbb{P}$

Naive Method: Trivial division by $m \leq \lfloor \sqrt{n} \rfloor$.

Running time is exponential in $l(n)$. Even when restricted to division by prime numbers, need approximately $\frac{\sqrt{n}}{|n|^{1/\sqrt{n}}}$ trivial divisions (prime number theorem)
 \rightarrow hardly any better!

Reminder: (arithmetic modulo m)

G finite group $\Rightarrow \forall a \in G \quad a^{|G|} = 1$ Fermat's little theorem

For $G = (\mathbb{Z}/(p))^x \quad a^{p-1} \equiv 1 \pmod{p} \in \mathbb{P} \quad \forall a \in \mathbb{Z} \quad \text{with } p \nmid a$

In fact $(\mathbb{Z}/(p))^x \cong Z_{p-1}$ is cyclic

For $m = p_1^{e_1} \dots p_r^{e_r}$ with $p_i \in \mathbb{P}, e_i \in \mathbb{N}_{>0}$:

$\mathbb{Z}_{(m)} \cong \mathbb{Z}_{(p_1^{e_1})} \oplus \dots \oplus \mathbb{Z}_{(p_r^{e_r})} \Rightarrow \mathbb{Z}_{(m)}^x \cong \mathbb{Z}_{(p_1^{e_1})}^x \times \dots \times \mathbb{Z}_{(p_r^{e_r})}^x$

what is $\mathbb{Z}_{(p^e)}$ for $p \in \mathbb{P}, e \in \mathbb{N}_{>0}$?

1.3.1 Theorem 1 (Cyclic)

Let $p \in \mathbb{P}$ off $e \in \mathbb{N}_{>0} \Rightarrow (\mathbb{Z}_{(p^e)})^x = Z_{(p-1) \cdot p^{e-1}}$ cyclic

Proof:

$(\mathbb{Z}_{(p^e)})^x \cong Z_{p-1} \Rightarrow \exists z \in \mathbb{Z} : \text{order}(z + p\mathbb{Z}) = p - 1$

Set $a = \bar{z}^{p^{e-1}} \in (\mathbb{Z}_{(p^e)})^x =: G$

$$a^{p-1} = \bar{z}^{(p-1) \cdot p^{e-1}} = \bar{z}^{|a|} = 1$$

On the other hand, take $i \in \mathbb{Z}$ such that

$$a^i = 1 \Rightarrow \bar{z}^{i \cdot p^{e-1}} \equiv 1 \pmod{p} \Rightarrow (p-1) \mid (i - p^{e-1}) \Rightarrow (p-1) \mid i.$$

So $\text{ord}(a) = p - 1$.

Now consider $b = (p + 1) \in G$

Claim: $\text{ord}(b) = p^{e-1}$

Proof by induction on $k \in \mathbb{N}_{>0}$ that $(p + 1)^{p^{k-1}} \equiv p^k + 1 \pmod{p^{k+1}}$

$k = 1 \quad \checkmark$

$k \rightarrow k + 1$: By induction have $(p + 1)^{p^{k-1}} = 1 + p^k + x \cdot p^{k+1}, \quad x \in \mathbb{Z}$

Compute: $(p + 1)^{p^k} = ((1 + p^k) + x \cdot p^{k+1})^p = \sum_{i=0}^p \binom{p}{i} (1 + p^k)^{p-i} \cdot x^i \cdot p^{i(k+1)}$

$$\underbrace{\equiv}_{\text{Only 0-th summand}} (1 + p^k) = \sum_{i=0}^p \binom{p}{i} (1 + p^k)^{p-i} \underbrace{\equiv}_{p \text{ odd}} 1 + p^{k+1} \pmod{p^{k+2}} \quad \checkmark$$

For $k = e : (p + 1)^{p^{e-1}} \equiv 1 \pmod{p^e} \Rightarrow b^{p^e} = 1 \Rightarrow \text{ord}(b) \mid p^{e-1}$

But $(p + 1)^{p^{e-2}} \equiv p^{e-1} + 1 \pmod{p^e} \Rightarrow b^{p^{e-2}} \neq 1 \in G$

So $\text{ord}(b) = p^{e-1}$

Claim: $\text{ord}(a \cdot b) = (p - 1)p^{e-1} \quad (\Rightarrow \text{Theorem})$

Let $(a \cdot b)^i = 1 \in G$ with $i \in \mathbb{Z}$

Then $1 = (a \cdot b)^{i \cdot (p-1)} = (a^{p-1})^i \cdot b^{i \cdot (p-1)} = b^{i \cdot (p-1)} \Rightarrow p^{e-1} \mid i \cdot i(p-1) \Rightarrow p^{e-1} \mid i$

Also $1 = (a \cdot b)^{p^{e-1} \cdot i} = a^{p^{e-1}} \Rightarrow (p-1) \mid p^{e-1} \cdot i \Rightarrow (p-1) \mid i \rightarrow (p-1) \cdot p^{e-1} \mid i \quad \square$

Reminder: $(\mathbb{Z}/(2^e))^x \cong Z_2 \times Z_2^{e-2} \quad (e \geq 2)$

1.3.2 Algorithm 2 (Fermat Test)

input : $n \in \mathbb{N}_{>0 \text{ odd}}$

output: " $n \notin \mathbb{P}$ " or "probably $n \in \mathbb{P}$ "

- (1) Choose $a \in 2, \dots, n-1$ randomly
- (2) Compute $a^{n-1} \bmod n$
- (3) If $a^{n-1} \not\equiv 1 \bmod n$ then return " $n \notin \mathbb{P}$ "
otherwise return "probably $n \in \mathbb{P}$ "

Not very satisfying. Is this fast

1.3.3 Algorithm 3 (fast exponentiation)

input : $a \in G$ G is a monoid, $e \in \mathbb{N}$, $e = \sum_{i=0}^{n-1} e_i 2^i$, $e_i \in \{0, 1\}$

output: $a^e \in G$

- (1) Set $b := a$, $y := 1$
- (2) For $i = 0, \dots, n-1$ perform (3) - (4)
- (3) if $e_i = 1$ set $y := y \cdot b$
- (4) set $b := b^2$
- (5) return y

this requires $O(l(e))$ operations in G

For $G = (\mathbb{Z}/(n)_i)$, each multiplication requires $O(l(n)^2)$ bit operations

\Rightarrow Fermat test requires $O(l(n)^3)$ bit operations \rightarrow cubic complexity \rightarrow "fast"!

Example:

$n = 561 = 3 \cdot 11 \cdot 17$ For $a \in \mathbb{Z}$ with $\gcd(a, n) = 1 \Rightarrow$ have $a^{n-1} = (a^2)^{280} \equiv 1 \bmod 3$
 $a^{n-1} \equiv 1 \bmod n$ Fermat's test says "probably $n \in \mathbb{P}$ " in 57% of cases.

$n = 2207 \cdot 6619 \cdot 15443$: output "probably $n \in \mathbb{P}$ " in 99,93% of cases.

1.3.4 Definition 4 (pseudo-prime, witness, Carmichael numbers)

Let $n \in \mathbb{N}_{>1, odd}$, $a \in 1, \dots, n-1$

- (a) n is pseudo-prime to base a if $a^{n-1} \equiv 1 \pmod{n}$
- (b) otherwise a is called a witness of composition of n
- (c) If $n \notin \mathbb{P}$ but $a^{n-1} \equiv 1 \pmod{n} \quad \forall a$ with $\gcd(n, a) = 1$
then n is called a Carmichael number.
There are ∞ Carmichael numbers

1.3.5 Proposition 5:

Let $n \in \mathbb{N}_{>1}$, $odd \notin \mathbb{P}$ not Carmichael

$\Rightarrow |\{a \in \mathbb{Z} | 0 < a < n \text{ a witness of composite of } n\}| > \frac{n-1}{2}$

Proof: Consider

$\phi : (\mathbb{Z}/(n))^x \rightarrow G, \quad \bar{a} \mapsto \bar{a}^{n-1}$

group homomorphism. By assumption,

$|\text{im}(\phi)| > 1 \Rightarrow |\text{Ker}(\phi)| \leq \frac{|a|}{2} < \frac{n-1}{2}$

$\Rightarrow |\{a \in \mathbb{Z} | 0 < a < n \text{ a witness of composite of } n\}| > \frac{n-1}{2} \quad \square$

Miller-Rabin Test

1.3.6 Proposition 6

Let $p \in \mathbb{P}$ odd, $a \in \{1, \dots, p-1\}$ write $p-1 = 2^k$ with m odd Then:

$a^m \equiv 1 \pmod{p}$ or $\exists i \in \{0, \dots, k-1\};$

$a^{2^i \cdot m} \equiv -1 \pmod{p}$

Proof:

Little Fermat: $\bar{a}^{2^k \cdot m} = 1 \in \mathbb{F}_p$

Assume $\bar{a}^m \neq 1$ take i maximal such that:

$\bar{b} = \bar{a}^{2^i \cdot m} \neq 1 \Rightarrow \bar{b}^2 = 1 \Rightarrow \bar{b} \in \mathbb{F}_p$ is a zero of $x^2 - 1 \in \mathbb{F}_p[x] \Rightarrow \bar{b} = -1$

1.3.7 Algorithm 7 (Miller -Rabin)

input : $n \in \mathbb{N}_{>1, odd}$

output: either " $n \notin \mathbb{P}$ " or "probably $n \in \mathbb{P}$ " \rightarrow Monte Carlo Algorithm.

- (1) write $n-1 = 2^k \cdot m$ with m odd
- (2) Choose $s \in \{2, \dots, n-1\}$ randomly
- (3) Compute $b := a^m \pmod{n}$
- (4) if $(b \equiv \pm 1 \pmod{n})$ return "probably $n \in \mathbb{P}$ "
- (5) for $(i = 0, \dots, k-1)$ do steps (6) - (7)

- (6) set $b := b^2 \bmod n$
- (7) if $(b \equiv -1 \bmod n)$ return "probably $n \in \mathbb{P}$ "
- (8) return $n \notin \mathbb{P}$

1.3.8 Definition 8

Let $n \in \mathbb{N}_{>1}$, $odd \quad a \in \{1, \dots, n-1\}$

- (a) n is called a strongly pseudo-prime to base a if Proposition 6 holds for a and p replaced by n .
- (b) Otherwise a is called a strong witness of composition of n .

Example

Let $n \in \mathbb{N}_{>1}$, $\mathbb{P} \text{ odd}$
 $a = 2$ strong witness if $n < 2047$ (including 561)
 2 or 3 strong witness if $n < 1373653$
 $2, 3$ or 5 strong witness if $n < 25326001$

1.3.9 Theorem 7

- (a) Algorithm 7 requires $O(l(n)^3)$ bit operations. \rightarrow "qubic complexity" \rightarrow fast!
- (b) if $b \in \mathbb{P}$ then Algorithm 7 returns "probably $b \in \mathbb{P}$ " \rightarrow no false positives.
- (c) if $n \notin \mathbb{P}$ then more than half of the numbers in $\{1, \dots, n-1\}$ are strong witnesses.

Proof:

- (a) Step 1 takes $O(l(n))$ bit operations:
Using Algorithm 3, we need $O(l(n-1))$ multiplications in $\mathbb{Z}/(n)$ each requiring $O(l(n)^2)$ bit operations.
- (b) Proposition 6
- (c) split in three cases:

Case 1: n is not a Carmichael number. $\xRightarrow{Prop5}$ more than half of all numbers are.

Fermat witness thus also strong witness.

Case 2: $n = p^r \cdot l$ with $p \in \mathbb{P} \quad r > 1 \quad l \in \mathbb{N}_{>0} p \nmid l$

Theorem 1 $\exists x \in \mathbb{Z}$ such that $x^p \equiv 1 \bmod p^r \quad x \not\equiv 1 \bmod p^r$

Chinese remainder theorem: $\exists a \in \mathbb{Z}$ such that $a \equiv x \bmod p^r \quad a \equiv 1 \bmod l$

So $\bar{a}^p = 1 \in \mathbb{Z}/(n) \Rightarrow \bar{a}^n = 1 \Rightarrow \bar{a} \in (\mathbb{Z}/(n))^x$

i.e. $\gcd(n, a) = 1$ if $\bar{a}^{n-1} = 1$ then $\bar{a} = 1$

But $a \equiv x \not\equiv 1 \bmod p^r$ so $\bar{a}^{n-1} \neq 1$ hence n is not Carmichael \rightarrow Case 1.

Case 3: n is a Carmichael number. By Case 2 have $n = p \cdot l$ with $p \in \mathbb{P} \quad p \nmid l \quad l \geq 3$

n Carmichael: $\forall a \in \mathbb{Z}$ with $\gcd(a, n) = 1$
have $a^{2^k \cdot m} \equiv 1 \pmod n$ (where $n - 1 = 2^k \cdot m$)
 $a^{2^k \cdot m} \equiv 1 \pmod p$ Take j minimal such that
 $a^{2^j \cdot m} \equiv 1 \pmod p \forall a \in \mathbb{Z}$ such that $\gcd(a, n) = 1$
so $0 \leq j \leq l$ in fact, $j > 0$ since $(-1)^{2^0 \cdot m} = -1$ with m odd.
Consider the subgroup $H := \{\bar{a} \in \mathbb{Z}/(n) \mid \bar{a}^{2^{j-1} \cdot m} \in \{1, -1\} \subseteq (\mathbb{Z}/(n))^x\}$
Let $a \in \{1, \dots, n-1\}$ $\gcd(n, a) = 1$ a not a strong witness.

Claim 1: $\bar{a} \in H$

Case 3.1: $\bar{a}^{2^{j-1} \cdot m} = 1 \Rightarrow \bar{a} \in H$

Case 3.2: $a^{2^{j-1} \cdot m} \not\equiv 1 \pmod n$ $a^m \not\equiv 1 \pmod n$

$\xRightarrow{a \text{ nonwitness}} \exists i$ such that $\underbrace{a^{2^i \cdot m} \equiv -1 \pmod n}_*$

$\Rightarrow a^{2^i \cdot m} \equiv -1 \pmod p \xRightarrow{\text{def of } j} i < j$

if $i < j - 1$ then $a^{2^{j-1} \cdot m} = (a^{2^i \cdot m})^{2^{j-1-i}} \equiv (-1)^{2^{j-1-i}} = 1 \pmod p$

\Rightarrow contradiction to minimality of j . So $i = j-1 \xRightarrow{\text{with } *} \bar{a} \in H \Rightarrow$ Claim 1 \square

Claim 2: $H \subseteq (\mathbb{Z}/(n))^x$ proper subgroup.

By definition of $j \exists x \in \mathbb{Z}$ such that $x^{2^{j-1} \cdot m} \not\equiv 1 \pmod p$

Chinese remainder: $\exists a \in \mathbb{Z}$ such that

$a \equiv x \pmod p$ $a \equiv 1 \pmod l \Rightarrow a^{2^{j-1} \cdot m} \not\equiv 1 \pmod p \equiv 1 \pmod l \Rightarrow \bar{a} \notin H$

Claim 2 \checkmark

It follows that $|H| \leq \frac{|\mathbb{Z}/(n)|^x}{2} < \frac{n-1}{2}$

so the number of witnesses is $\geq n-1 - |H| > \frac{n-1}{2}$ \square

Remarks:

- (a) A more careful analysis shows that $2\frac{3}{4}$ of all candidates are strong witnesses
- (b) Calling Algorithm 7 repeatedly decreases the probability of false positives. Running time for prescribed error probability p is $O(\lg(p^{-1} \cdot l(n)^3))$
(Independence assumptions!)

polynomial rings measure not abs value but max power

2 Notes

- $a|b$
 a is divisible by b
- $a \nmid b$
 a is not divisible by b
- $\text{ord}(a)$
- $\text{char}(A)$ the smallest positive n such that
 $\underbrace{1 + \dots + 1}_n = 0$ with 1 as the multiplicative identity element
n summands
- $\mathbb{Z}/(m)$
Ring modulo m
- $\lg(x) = \log_2(x)$
- Average number of bit operations for an increment:
one operation for the last bit + 50% chance for one on the next bit + 25% on the following etc. \Rightarrow Geometrical row
 \Rightarrow on average two bit operations
- "Monte Carlo Algorithm": Always terminates in reasonable time but might yield false result
- "Las Vegas Algorithm"