

# Computational Algebra

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## Transcript

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June 18, 2015

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# 1 Integer Arithmetic

Topics:

- Addition and Multiplication
- GCD computation
- Primality testing
- Factorization

## 1.1 Addition and Multiplication

Agreement:

- $a, x \in \mathbb{N}$  represented as  $x = \sum_{i=0}^{n-1} a_i \cdot B^i$   $B \in \mathbb{N}_{>1}$  fixed Base ( $a_i \in \{0, \dots, B-1\}$ )
- if  $x \neq 0$ , assume  $a_{n-1} \neq 0$  then define:  
length of  $x := l(x) = n$  = number of digits =  $\lfloor \log_B(x) \rfloor + 1$   
(mnemonic:  $\log_B(B) + 1 = 2$ )
- $l(0) = 1$   
(Amount of memory required to store  $x = 0$ )
- $l(x) := l(|x|)$
- for  $x \in \mathbb{Z}$  represent if as  $x = \text{sgn}(x) * |x|$

### 1.1.1 Algorithm 1 (Simple addition)

input :  $x = \sum_{i=0}^{n-1} a_i \cdot B^i$ ,  $y = \sum_{i=0}^{n-1} b_i \cdot B^i$ ,  $x, y \in \mathbb{N}$

output:  $x + y = \sum_{i=0}^n c_i \cdot B^i$

- (1)  $\sigma = 0$
- (2) for  $i = 0, \dots, (n-1)$  :
- (3)     set  $c_i := a_i + b_i + \sigma_i$    and    $\sigma := 0$
- (4)     if  $(c_i \geq B)$
- (5)         set  $c_i = c_i - B$
- (6)         set  $\sigma = 1$
- (7) set  $c_n = \sigma$

If  $B = 2$  then (3) - (6) can be realized by logic gates:

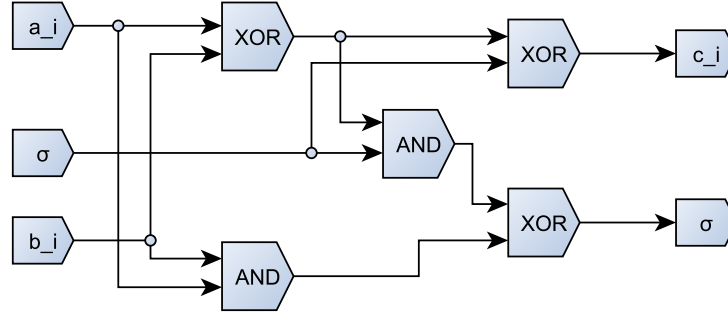


Figure 1: Logic circuit for addition

### 1.1.2 Definition 2 (Bit-Operation)

A bit operation is an operation that can be performed by a logic gate or by searching or writing a bit from / into memory.

### 1.1.3 Definition 3 (Big O)

Let  $M$  be a set (usually  $M = \mathbb{N}$ ),  $f, g : M \mapsto \mathbb{R}_{>0}$   
we write  $f \in O(g)$  if  $\exists c \in \mathbb{R} : f(x) \leq c \cdot g(x) \forall x \in M$

### 1.1.4 Theorem 4 (Lower bound for addition)

Let  $f : \mathbb{N} \mapsto \mathbb{R}$ ,  $n \mapsto$  maximal number of bit operations required by Algorithm 1 to add  $x, y \in \mathbb{N}$  with  $l(x), l(y) \leq n$

Let  $g = id_{\mathbb{N}}$  Then  $f \in O(g)$

We say Algorithm 1 requires  $O(n)$  bit operations for adding two numbers of length  $\leq n$ .  
 $\Rightarrow$  "linear complexity"

Set  $M := \{\text{Set of all algorithms for addition in } \mathbb{N}\}$

For  $A \in M$  define  $f_A : \mathbb{N} \mapsto \mathbb{R}$  as above.

We would like to find  $f_{odd} : \mathbb{N} \mapsto \mathbb{R}$ ,  $n \mapsto \inf\{f_A(n) | A \in M\}$

Since one needs to read  $x, y$  (and write the result) we can not do any better than linear complexity for addition.

#### Subtraction

let  $x, y$  as Algorithm 1,  $x \geq y$

For  $\bar{y} := \sum_{i=0}^{n-1} (B - 1 - b_i) B^i$  (digitwise / bitwise complement)

$\Rightarrow x + \bar{y} = x - y + B^n - 1$

$\Rightarrow x - y = x + \bar{y} + 1 - B^n$  (initially set  $\sigma = 1$ )

**Conclusion:** Addition and Subtraction have cost  $O(n)$

### 1.1.5 Algorithm 5 (Multiplication by "grid method")

input :  $x = \sum_{i=0}^{n-1} a_i \cdot 2^i, \quad y = \sum_{i=0}^{m-1} b_i \cdot 2^i$

output:  $z = x \cdot y$

- (1)  $z := 0$
- (2) for  $i = 0, \dots, (n-1)$
- (3)     if  $(a_i \neq 0)$  set  $z := z + \sum_{j=0}^{m-1} b_j 2^{i+j}$

### 1.1.6 Theorem 6 (Runtime of Algorithm 5)

Algorithm 5 requires  $O(n * m)$  bit operations.

As of the total input length  $n + m$ :

$$n \cdot m \leq \frac{1}{2}(n + m)^2 \rightarrow O((n + m)^2)$$

$\Rightarrow$  Quadratic complexity

### Karatsuba-multiplication:

Observation for polynomials:

$$a + bx, c + dx \text{ have } (a + bx)(c + dx) = ac + (ac + db - (a - b)(c - d))x + bdx^2$$

The point: only used 3 multiplications instead of 4.

Specialize  $x = B$  "large" such that  $x = a + bB$  partition into two blocks. Then multiply the blocks by a recursive call.

### 1.1.7 Algorithm 7 (Karatsuba)

input :  $x, y \in \mathbb{N}$

output:  $z = x \cdot y$

- (1) Choose  $k \in \mathbb{N}$  minimal such that  $l(x), l(y) \leq 2^k$ .  
Set  $B = 2^{2^{k-1}}$
- (2) if  $(k = 0)$  return  $x \cdot y$  (by bit-operation AND)
- (3) write  $x = x_0 + x_1 B, \quad y = y_0 + y_1 B$  with  $l(x_i), l(y_i) \leq 2^{k-1}$
- (4) compute  $x_0 \cdot y_0, \quad x_1 \cdot y_1, \quad (x_0 - x_1) \cdot (y_0 - y_1)$  by a recursive call
- (5) return  $z = x_0 y_0 + (x_0 y_0 + x_1 y_1 - (x_0 - x_1)(y_0 - y_1))B + x_1 y_1 B^2$

### 1.1.8 Theorem 8 (Runtime of Algorithm 7)

For multiplying two numbers of length  $\leq n$  Algorithm 7 requires  $O(n^{\log_2 3}) \approx O(n^{1.59})$  bit operations.

**Proof:**

Set  $\Theta(k) :=$  maximal numbers of bit operations for  $l(x), l(y) \leq 2^k$

We have for  $k > 0$  :  $\Theta(k) \leq 3 \underbrace{\Theta(k-1)}_{\text{recursive calls}} + c \underbrace{2^k}_{\text{additions}}$  with  $(c \text{ some constant})$

**Claim:**  $\Theta(k) \leq 3^k + 2c(3^k - 2^k)$

**Proof by Induction on  $k$ :**

$k = 0$  :  $\Theta(k) = 1$

$$\begin{aligned} k-1 \rightarrow k : \Theta(k) &= 3\Theta(k-1) + c2^{k-1} \\ &\leq 3(3^{k-1} + 2c(3^{k-1} - 2^{k-1})) + c2^k \\ &= 3^k + 2c(3^k - 2^k) \end{aligned}$$

So  $\Theta(k) \leq (2c+1)3^k$

Now  $l(x) \leq n$  hence  $2^{k-1} < n$  by minimality of  $k$

So  $k-1 < \log_2 n$

$$\begin{aligned} \Rightarrow \Theta(k) &\leq 3(2c+1)3^{\log_2(n)} \\ &= 3(2c+1)2^{\log_2(3) \log_2(n)} \\ &= 3(2c+1)n^{\log_2(3)} \quad \square \end{aligned}$$

One can modify the terminal condition of Karatsuba to switch to Grid-Multiplication, which is faster for small numbers.

### Fast-Fourier Transform

Reminder: For a function  $f : \mathbb{R} \mapsto \mathbb{C}$  define:

$\hat{f} : \mathbb{R} \mapsto \mathbb{C}$  by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt \quad (\text{if it exists})$$

Think of  $\omega$  as frequency.

### Definition (Convolution)

Let  $f, g : \mathbb{R} \mapsto \mathbb{C}$

$$(f * g)(x) = \int_{\mathbb{R}} f(t) g(x-t) dt$$

Convolution is analogous to polynomial multiplication **Formula:**  $\underbrace{(f * g)}_{\text{(Cauchy formula)}} = \hat{f} \cdot \hat{g}$

For a function  $M \mapsto C$  with  $|M| < \infty$  we need the discrete Fourier transform (DFT)

### 1.1.9 Definition 9 (Root of unity)

Let  $R$  be a commutative ring with 1. An element  $\mu \in R$  is called an  $n$ -th root of unity (= root of 1) if  $\mu^n = 1$ .

It is called primitive if  $\mu^i \neq 1$  for  $(0 < i < n)$  i.e.  $\text{ord}(\mu) = n$

let  $\mu$  be a primitive  $n$ -th root of 1 (e.g.  $e^{2\pi \frac{i}{n}} \in \mathbb{C}$ )

Then the map  $DFT_\mu : R^n \mapsto R^n$

$$(\hat{a}_0, \dots, \hat{a}_n) \mapsto (\hat{a}_0, \dots, \hat{a}_n) \quad \text{with } \hat{a}_i = \sum_{j=0}^{n-1} \mu^{ij} a_j$$

is called discrete Fourier transformation

For polynomials:

$$DFT_\mu : R[x] \mapsto R^n$$

$$f \mapsto (f(\mu^0), \dots, f(\mu^{n-1}))$$

Convolution rule: (from  $f(\mu^i)g(\mu^i) = (f * g)(\mu^i)$ )

$$DFT_\mu(f * g) = DFT_\mu(f) \cdot DFT_\mu(g) \quad (\text{component wise product})$$

Addition of two polynomials in  $R[x]$  of  $\deg(n)$  require  $O(n)$  ring operations.

Multiplication require  $O(n^l)$ .

With Karatsuba have  $O(n^{\log_2(3)})$  ring operations.

Cost  $DFT_\mu(f) \cdot DFT_\mu(g) : O(n)$  ring operations (with  $\mu$  as  $2n$ -th root of 1)

Want: Cheap way of doing  $DFT$  and back-transformation.

### 1.1.10 Algorithm 10 (Fast Fourier transformation FFT)

input :  $f \in R[x]$ ,  $\mu \in R$  primitive  $2^k$ -th root of 1, such that  $\mu^{2^{k-1}} = -1$

output:  $DFT_\mu(f)$

(1) Write  $f(x) = g(x^2) + xh(x^2)$  with  $f, g, h \in R[x]$

(2) if  $(k = 1)$  // ( $\Rightarrow \mu = 1$ )  
return  $DFT_\mu(f) = (g(1) + h(1), g(1) - h(1))$

(3) Recursive call: compute  $DFT_{\mu^2}(g) = \hat{g}, DFT_{\mu^2}(h) = \hat{h} \in R^{2^{k-1}}$

(4) return  $DFT_\mu(f) = (\hat{f}_0, \dots, \hat{f}_{2^k-1})$  with  $\hat{f}_i = \hat{g}_i + \mu \hat{h}_i$   
where  $\hat{g}_i = \hat{g}_{i-2^{k-1}}$  for  $i \geq 2^{k-1}$

Note: Components of  $\hat{g}$  and  $\hat{h}$  are:

$$\hat{g} = g(\mu^{2^i}), \quad \hat{h}_i = h(\mu^{2^i}) \quad \text{so}$$

$$\hat{f}_i := f(\mu^i) = \hat{g}_i(\mu^{2^i}) + \mu \hat{h}_i(\mu^{2^i}) = \hat{g}_i + \mu \hat{h}_i$$

**Convention:**  $\lg(x) = \log_2(x)$



### 1.1.11 Theorem 11 (Runtime of Algorithm 10)

Let  $n = 2^k$ ,  $f \in R[x]$  with  $\deg(\psi) < n$

Then Algorithm 10 requires  $O(n \cdot \lg(n))$  ring operations.

Better than  $O(n^{1+\epsilon})$ ,  $\forall \epsilon > 0$ !

**Proof:**

Set  $\Theta(k) = \max$  number of ring operations required. By counting obtain for  $k > 1$ :

$$\Theta(k) \leq 2\Theta(k-1) + \underbrace{(\text{compute } \mu^i (i \leq 2^{k-1}))}_{2^{k-1}} + \underbrace{(\mu^i \hat{k}_i)}_{2^{k-1}} + \underbrace{(\text{sums and differences})}_{2^k}$$

$$= 2\Theta(k-1) + 2^{k+1}$$

**Claim:**  $\Theta(k) \leq (2k-1)2^k$

$$k=1 : f = a_0 + a_1 \cdot x \quad DFT_\mu(f) = (a_0 + a_1 \cdot a_0 - a_1) \Rightarrow \Theta(a) = 2$$

$$k-1 \rightarrow k : \Theta(k) \leq 2 \cdot \Theta(k-1) + 2^{k+1} \leq 2 \cdot (2k-3) \cdot 2^{k-1} + 2^{k+1} = (2k-1) \cdot 2^k$$

since  $k = \lg(n)$  obtain  $O(k) \leq (2 \cdot \lg(n) - 1) \cdot n \in O(n \cdot \lg(n)) \quad \square$

**Back-transformation?**

### 1.1.12 Definition 12 (Good root of unity)

A primitive  $n$ -th root of unity is called good (caveat: this is ad-hoc terminology) if:

$$\sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \text{for } (0 < i < n)$$

**example:**

(1)  $\mu = e^{2\pi \frac{i}{n}}$  is a good primitive root of unity

(2)  $R = \mathbb{Z}/(8)$ ,  $\mu = \bar{3} \Rightarrow \mu \cdot B$  is primitive  $2^{nd}$  root of unity  
But  $\bar{B}^0 + \bar{3}^1 = \bar{u} \neq \bar{0}$  so  $\mu$  is not good.

### 1.1.13 Proposition 13 ( $DFT_{\mu^{-1}}$ )

let  $\mu \in R$  be a good root of 1

$$(a) = (a_0, \dots, a_{n-1}) \in R^n \Rightarrow DFT_\mu^{-1}(DFT_\mu(a)) = n \cdot (a) \quad \text{where } n = 1 + \dots + 1 \in R$$

**Proof:**

$$DFT_\mu(a) = (\hat{a}) = (\hat{a}_0, \dots, \hat{a}_{n-1})$$

$$\text{with } \hat{a}_j = \sum_{k=0}^{n-1} \mu^{jk} a_k$$

$$DFT_{\mu^{-1}}(\hat{a}) = (\hat{\hat{a}}_0, \dots, \hat{\hat{a}}_1)$$

$$\text{with } \hat{\hat{a}}_i = \sum_{j=0}^{n-1} \mu^{-ij} \sum_{k=0}^{n-1} \mu^{jk} a_k = \sum_{k=0}^{n-1} \left( a_k \cdot \underbrace{\sum_{j=0}^{n-1} \mu^{j(k-i)}}_{=0 \text{ if } n \neq k-i \text{ (i.e. } k=i)} \right) = a_i \cdot n \quad \square$$

### 1.1.14 Proposition 14 (Finding good roots of unity)

let  $\mu \in R, n \in \mathbb{N}$

Assume:

- a)  $R$  is an integral Domain and  $\mu$  is a primitive or  $n$ -th root of 1  
(Integral Domain: nonzero commutative ring in which the product of two nonzero elements is nonzero)  
 $\Rightarrow$  Granted by FFT
- b)  $n = 2^b, \mu^{\frac{n}{2}} = -1$ , then  $h > 0 \wedge \text{char}(R) \neq 2$   
 $\rightarrow \mu$  is a good primitive  $n$ -th root of 1 ("root of unity")

**Proof:**

- a) for  $0 < i < n$

$$\underbrace{(\mu^i - 1)}_{\neq 0} \underbrace{\left(\sum_{j=0}^{n-1} \mu^{ij}\right)}_{=0} = \mu^{in} - 1 = 0$$

$\Rightarrow \mu$  is a good root of unity

- \* Let  $0 < i < n$ , write  $i = 2^{k-s} \cdot r$  with  $r$  odd  $\wedge s > 0$

$$\sum_{j=0}^{2^k-1} \mu^{ij} = \sum_{l=0}^{2^{k-s}-1} \sum_{j=0}^{2^s-1} \mu^{i(l \cdot 2^s + j)}$$

$$\mu^{i \cdot 2^s} = 1$$

$$i \cdot 2^s = 2^{k-s} \sum_{j=0}^{2^s-1} \mu^{ij} = 2^{k-s} \sum_{j=0}^{2^{s-1}-1} (\mu^{ij} + \mu^{i(2^{s-1}+j)})$$

$$\text{But } \mu^{i \cdot 2^{s-1}} = \mu^{2^{k-s} \cdot r \cdot 2^{s-1}} = \mu^{2^{k-1} \cdot r} = (-1)^r = -1$$

$$\text{So } \sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \square$$

- b)  $\mu^n = 1, n = 2^k \Rightarrow \text{ord}(\mu) | n \Rightarrow \text{ord}(\mu)$  is power of 2

### 1.1.15 Algorithm 15 (Polynomial multiplication using DFT)

input :  $f, g \in R[x]$  with  $\deg(f) + \deg(g) < 2^k =: n$   
 $\mu \in R$  as a good root of unity; Assume  $2 \in R$  is invertible

output:  $h = f \cdot g$

- (1) compute  $\hat{f} = DFT_{\mu}(f), \hat{g} = DFT_{\mu}(g)$  with  $f, g \in R^n$
- (2) compute  $\hat{h} = \hat{f} \cdot \hat{g}$
- (3) compute  $(h_0, \dots, h_{n-1}) = DFT_{\mu^{-1}} \hat{h}$  (same as  $DFT_{\mu}(\hat{h})$  but with different order)  
= Back-transformation  $\cdot 2^k$   
set  $h = \frac{1}{2^k} \sum_{i=0}^{n-1} h_i x^i$

### 1.1.16 Theorem 16 (Runtime of Algorithm 15)

Algorithm 15 uses  $O(n \cdot \log(n))$  ring operations for polynomials of  $\deg < n$

**Proof:**

- Choose  $k$  minimal so that  $\deg(f) \cdot \deg(g) < 2^k$   
 $\Rightarrow 2^{k-1} \leq 2n \Rightarrow k \leq \log(n) + 2$
- $\underbrace{O(2k \cdot 2^k)}_{\text{Step 1}} + \underbrace{2^k}_{\text{Step 2}} + \underbrace{O(k \cdot 2^k) + 2^k}_{\text{Step 3}} \in O(2k \cdot 2^k) = O(n(g(n))) \quad \square$

Goal: Multiplication in  $\mathbb{N}$  using DFT

Idea: find roots of 1 in a suitable  $\mathbb{Z}/(m)$

Choose  $m = 2^l + 1, \mu = \bar{2} \in R$

### 1.1.17 Proposition 17 (Add and mul in $O(l)$ )

Let  $m = 2^l + 1, R = \mathbb{Z}/(m)$

Addition in  $R$  and multiplication by  $\bar{2}^i \in R$  ( $0 \leq i < 2l$ ) can be done in  $O(l)$  bit operations

**Proof:**

- Let  $\bar{x} \in R$  with  $0 \leq x \leq 2^l$
- Addition:  $x + \bar{y}$ 
    - (1) compute  $x + y \in \mathbb{N}$ :  $O(l)$
    - (2) if  $x + y > 2^l + 1$  subtract  $2^l + 1$ :  $O(l)$
  - Multiplication by  $\bar{2}^i$  ( $0 \leq i < l$ )
    - (1) Bit-shift  $i$  Bits to the left by relocating in memory:  

$$\underbrace{O(\text{length}(i))}_{\text{compute addr. of new first bit}} + \underbrace{l}_{\text{copying}} = O(\log(l)) + l \in O(l)$$
  - Multiplication by  $\bar{2}^i$  ( $l \leq i < 2l - 1$ )
    - (1) Multiplication by  $\bar{2}^{i-l}$ :  $O(l)$
    - (2) take negative  $\bar{2}^i \cdot \bar{x} = -\bar{2}^{i-l} \cdot \bar{x}$ :  $O(l)$

### 1.1.18 Proposition 18 (Sort of summary)

Let  $k, r \in \mathbb{N}, r > 0, m = 2^{2^k \cdot r} + 1, R = \mathbb{Z}/(m), \mu = \bar{2}^r \in R$

$\Rightarrow 2 \in R$  is invertible,  $\mu$  is a good primitive  $2^{k+1}$ -th root of 1

$\Rightarrow \mu^{2^k} = 1$

**Proof:**  $\rightarrow$  from above

### 1.1.19 Algorithm 19 (Multiplication using FFT)

input :  $x, y \in \mathbb{N}$

output:  $Z = x \cdot y$

- (1) Choose  $k \in \mathbb{N}$  minimal such that  $l(x), l(y) \leq 2^{2^k}$
- (2) if  $k \leq 3$ , compute  $z = x \cdot y$  by Algorithm 5
- (3) set  $B = 2^{2^k}$ ,  $m = 2^{2^k \cdot 4} + 1$ ,  $R = \mathbb{Z}/(m)$ ,  $\mu = \bar{2}^4 \in R$   
 $(\Rightarrow \text{so } \mu \text{ is a good primitive } 2^{k+1}\text{-th root of 1})$
- (4) write  $x = \sum_{i=0}^{2^k-1} x_i \cdot B^i$ , same for  $y$  with  $(0 \leq x_i, y_i < B)$   
 possible since  $x, y < 2^{2^{2^k}} = 2^{2^k \cdot 2^k} = B^{2^k}$
- (5) compute:  $\hat{x} = DFT_\mu(\bar{x}_0, \dots, \bar{x}_{2^k-1}, \underbrace{0, \dots, 0}_{2^k \text{ zeros}}) \in R^{2^{k+1}}$   
 same for  $y$   
 $\rightarrow$  use FFT
- (6) compute:  $\hat{z} = \hat{x} \cdot \hat{b} \in R^{2^{k+1}}$  (component wise multiplication)  
 Perform multiplication in  $R$  as follows:  
 Multiply representatives (non negative and  $< m$ ) by recursive call.  
 Then reduce modulo  $m$  by "negative bit shift" (see proof of Proposition 17)
- (7) compute:  $(\bar{z}_0, \dots, \bar{z}_{2^{k+1}-1}) = \frac{1}{2^{k+1}} DFT_{\mu^{-1}}(\hat{z}) \in R$  with  $0 \leq z < m$
- (8) set  $z := \sum_{j=0}^{2^{k+1}-1} z_j \cdot B^j$

### 1.1.20 Theorem 20 (Runtime of Algorithm 19)

Algorithm 19 correctly computes  $t = x \cdot y$  and requires  $O(n \cdot (\log n)^4)$  bit operations for  $l(x), l(y) \leq n$

**Proof:** Correctness

write  $x(t) \sum_{i=0}^{2^k-1} x_i t^i \in \mathbb{Z}[t]$ ,  $y(t)$ ,  $\bar{x}(t) \in R[t], \bar{y}(t), \bar{z}(t)$

by Proposition 18 and Proposition 13 we have  $\bar{z}(t) = \bar{x}(t) \cdot \bar{y}(t)$

The  $l$ -th coefficient of  $x(t) \cdot y(t)$  is  $0 \leq \sum_{i+j=l} x_i \cdot y_j < 2^k \cdot B^2 = 2^{k+2 \cdot 2^k} \leq 2^{2^{k+2}} < m$

So  $z(t) = x(t) \cdot y(t) \Rightarrow z = z(B) = x(B) \cdot y(B) = x \cdot y$  Cost:

Write  $\Theta(k) := \max$  number of bit operations

Analyze Steps:

- (1) compute  $\max \{l(x), l(y)\} : O(l(n)) = O(k)$
- (2)  $O(1)$
- (3) no bit operations
- (4) compute starting addresses of  $x_i, y_i$  in memory:  $2 \cdot 2^k$  increments of the address:  
 $2 \cdot 2 \cdot 2^k = 2^{k+2}$  bit ops  
 $\Rightarrow O(2^k)$
- (5) By Theorem 11 need  $O(2 \cdot 2^{k+1} \cdot (k+1))$  operations in  $R$  which are additions and multiplications by powers of  $\bar{z}$  costing  $O(2^{k+2})$  bit operations.  
Total for (5):  $O(k \cdot 2^{2 \cdot k})$
- (6)  $2^{k+1}$  multiplications of numbers  $< m$ , i.e. of length  $\leq 2^{k+2}$ .  
So  $k' \leq \frac{k+3}{2}$  for  $k'$ : the "new"  $k$  used in the next recursion level.  
For  $\alpha \in R_{>0}$  define  $\Theta(\alpha) := \Theta(\lfloor \alpha \rfloor)$   
Total for (6):  $2^{k+1}(\Theta(\frac{k+3}{2}) + \underbrace{O(2^{k+2})}_{\text{reduction (mod } m)})$
- (7) For  $DFT_{\mu^{-1}}(\hat{z}) : O(k \cdot 2^{2 \cdot k})$  as (5) Since  $\bar{z}$  is a  $n$  root of 1, multiplication by  $\bar{2}^{-k-1}$  is multiplication by a positive power of  $\bar{2}$ , which costs  $O(2^{k+2})$   
Total for (7):  $O(k \cdot 2^{2 \cdot k})$
- (8) For  $j \leq 2^{k+1}$  have  $\sum_{i=0}^{j-1} z_i \cdot B^i \leq (m-1) \sum_{i=0}^{j-1} B^i = (m-1) \frac{B^j - 1}{B-1} < 2(m-1) \frac{B^j}{B} = 2^{1+2^{k+2}+(j-1)2^k}$  so the sum has length  $(j+3) \cdot 2 + 1$   
Adding  $z_j \cdot B^j$  to this sum happens at  $(j \cdot 2^k)$ -th bit and higher  $\Rightarrow$  cost is  $O(2^k)$   
Total for (8):  $O(2^{2 \cdot k})$

Grad total: For  $k \geq 4$ :

$\Theta(k) \leq 2^{k+1} \cdot \Theta(\frac{k+3}{2}) + c \cdot k \cdot 2^{2 \cdot k}$  with  $c$  constant

Also for  $k \in \mathbb{R}_{\geq 4}$

**Define**  $\Lambda(k) := \frac{\Theta(k)}{2^{2 \cdot k}} \Rightarrow \Lambda(k) \leq \frac{2^{k+1} \Theta(\frac{k+3}{2})}{2^{2 \cdot k}} + c \cdot k = 16 \cdot \Lambda(\frac{k+3}{2}) + c \cdot k$

**Define**  $\Omega(k) := \Lambda(k+3)$  So for  $k \in \mathbb{R}_{>1}$

$$\Omega(k) \leq 16 \cdot \Lambda(\frac{k}{2} + 3) + c \cdot (k+3) = \underbrace{16\Omega(\frac{k}{2})}_{*} + c \cdot (k+3)$$

**Claim:** For  $i \in \mathbb{N}$  with  $2^{i-1} \leq k-3$  have:

$$\Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k+3)(1+8+\dots+8^{i-1}) + 3 \cdot c \cdot (1+16+\dots+16^{i-1})$$

**Proof** by induction:

$$i = 0: \Lambda(k) = \Omega(k-3)$$

$$i \rightarrow i+1: \Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k-3)(1+\dots+8^{i-1}) + 3 \cdot c \cdot (1+\dots+16^{i-1}) \leq 2^i \leq k-3 \quad *$$

$$\leq 16^i (16\Omega(\frac{k-3}{2^{i+1}})) + c(\frac{k-1}{2^i} + 3) + c(k-3)\dots = \text{claimed result}$$

Take  $u \in \mathbb{N}$  minimal with  $2^u > k-3 \Rightarrow \Omega(\frac{k-3}{2^u}) \leq \Omega(\lfloor \frac{k-3}{2^u} \rfloor) = \Omega(0) =: D$  (constant)

Note:  $u$  roughly is recursion depth

$$\text{Have } 2^{u-1} \leq k-3 \xRightarrow{\text{claim}} \Lambda(k) \leq 16^u \cdot D + c \cdot \underbrace{(k-3)}_{< 2^u} \cdot \frac{8^u-1}{7} + 3c \cdot \frac{16^u-1}{15} \in O(16^u)$$

$$\text{Have } 2^{u-1} \leq k-3 \Rightarrow u \leq \lg(k-3) + 1$$

$$\Rightarrow \Lambda(k) \in O(16^{\lg(k-3)}) = O((k-3)^4)$$

$$\Rightarrow \Theta(k) = 2^{2 \cdot k} \cdot \Lambda(k) \in O(2^{2k} \cdot (k-3)^4)$$

$$\text{Have } 2^{2(k-1)} < \underbrace{n}_{\max\{l(x) \cdot l(y)\}} \Rightarrow k \leq \frac{\lg(n)}{2} + 1$$

$$\text{So } \Theta(k) \in O(n \cdot (\lg(n))^4) \quad \square$$

### 1.1.21 Theorem 21 (Schönhage-Strassen 1971)

Multiplication of integers of length  $\leq n$  can be done in  $O(n \cdot \lg(n) \cdot \lg(\lg(n)))$  bit operations. Schönhage-Strassen is used for integers of length  $\geq 100.000$ .

Asymptotically faster: Fürer's algorithm.

### Comments on Bit complexity

1. Memory requirement may explode!  
 $\Rightarrow$  No Problem as bit complexity is upper bound for memory requirements, since memory access is included in bit operations  
 $(\rightarrow$  only store what is calculated)
2. Computation of addresses in memory take time  
 $\Rightarrow$  length of addresses  $\approx \lg(\text{memory space})$  computations of addresses  $\approx \lg(\text{memory space})^2$
3. As memory requirement gets larger access times will get longer.  
 $\Rightarrow$  transportation time for data  $\geq \frac{\text{diameter of physical storage}}{2 \cdot \text{speed of light}}$

## 1.2 Division with remainder, Euclidean algorithm

### 1.2.1 Algorithm 1 (Division with remainder)

input :  $b = \sum_{i=0}^{n-1} b_i 2^i$   $a = \sum_{i=0}^{n+m-1} a_i 2^i$  with  $a_i, b_i \in 0, 1$ ,  $b_{n-1} = 1$

output:  $r, q \in \mathbb{N}$  such that  $a = q \cdot b + r$ ,  $0 \leq r < b$

(1)  $r = a$   $q = 0$

(2) for  $i = m, m-1, \dots, 0$  do

(3) if  $r \leq 2^i \cdot b$  then set  $r := r - 2^i \cdot b$ ,  $q = q + 2^i$

### 1.2.2 Proposition 2 (Runtime of Algorithm 1)

Algorithm 1 is correct and requires  $O(n \cdot (m+1))$  bit operations.

**Proof:**

Always have  $a = q \cdot b + r$

**Claim:**

before step (3), have  $0 \leq 2^{i+1} \cdot b$

$i = m$ ;  $0 \leq r = a < 2^{m+n} = 2^{m+1} \cdot 2^{n-1} \leq 2^{m-1} \cdot b$   $i < m$  By step (3)

So after last passage through the loop  $0 \leq r < b$

**Running Time:** In step(3), have comparison and (possibly) subtraction. Only  $n$  bits involved  $\Rightarrow O(n)$

Total:  $O(b \cdot (m+1))$

**Remarks:**

(1) Division with remainder can be reduced to multiplication.

Precisely: given an algorithm for multiplication that requires  $M(n)$  bit operations, there exists an algorithm for division with remainder that requires  $O(M(n))$  bit operations.

(2) Practically relevant:

Jebelean's algorithm (1997):  $O(n^{\lg 3})$

(3) Alternatively, may choose  $r \in \mathbb{Z}$  such that  $\lfloor \frac{-b}{2} \rfloor < r \leq \lfloor \frac{b}{2} \rfloor$

(4) Algorithm 1 extends to  $\mathbb{Z}$ .

(5) All Euclidean rings have division with remainder (by definition).

(e.g.,  $R = K[x] \rightarrow$  polynomial ring over field,

$R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ ,  $i^2 = -1$ )

### 1.2.3 Algorithm 3 (Euclidean algorithm)

input :  $a, b \in \mathbb{N}$

output:  $\gcd(a, b)$  "greatest common divisor"

- (1) set  $r_0 := a, \quad r_i := b$
- (2) for  $i = 1, 2, 3, \dots$  perform steps (3) and (4)
- (3) if  $r_i = 0$  then  $\gcd(a, b) = |r_{i-1}|$
- (4) Division with remainder:  $r_{i-1} = q \cdot r_i + r_{i+1} \quad r_{i+1} \in \mathbb{Z}$   
 $|r_{i+1}| \leq \frac{1}{2}|r_i|$

**Example:**

$$a = 287, \quad b = 126$$

$$287 = 2 \cdot 126 + 35 \tag{1}$$

$$126 = 4 \cdot 35 - 14 \tag{2}$$

$$35 = (-2) \cdot (-14) + 7 \tag{3}$$

$$-14 = (-2) \cdot 7 + 0 \tag{4}$$

$$\begin{aligned} \text{So: } 7 | (-14) &\xRightarrow{(3)} 7 | 35 \\ &\xRightarrow{(2)} 7 | 126 \\ &\xRightarrow{(1)} 7 | 287 \end{aligned}$$

On the other hand take a common divisor  $d$ ;  $d | 287$ ;  $d | 126$

$$\xRightarrow{(1)} d | d \xRightarrow{(2)} d | 14 \xRightarrow{(3)} d | 7$$

### 1.2.4 Theorem 4 (Correctness of Algorithm 3)

Algorithm 3 is correct.

**Proof:**

Since  $r_{i-1} = q \cdot r_i + r_{i+1}$  every integer  $x \in \mathbb{Z}$  satisfies the equivalence  $x | r_{i-1}$  and  $x | r_i \Leftrightarrow x | r_{i+1}$  and  $x | r_i$  so  $\gcd(r_{i-1}, r_i) = \gcd(r_i, r_{i+1}) = \gcd(a, b)$  when terminating have  $\gcd(a, b) = \gcd(r_{i-1}, 0) = |r_{i-1}| \quad \square$



### 1.2.5 Theorem 5 (Runtime of Algorithm 3)

Algorithm 3 requires  $O(m \cdot n)$  bit operations for  $n = l(a), m = l(b)$

**Proof:**

If  $a < b$  then the first passage yields  $r_2 = a, r_1 = b$ . Cost:  $O(n)$

May assume:  $a \geq b$ . Write  $n_i = l(r_i)$

By Proposition 2  $\exists c$  constant such that the total time is  $\leq c \cdot \underbrace{\sum_{i=1}^k n_i \cdot (n_{i-1} - n_i + 1)}_{=: \sigma(n_0, \dots, n_k)}$

For  $i > 2$ :  $n_i = n_{i-1} - 1$

Special Case:  $n_i = n_{i-1} - 1$  for  $i \geq 2$

$\Rightarrow n_i = n_i - i + 1, n_i = m, k = m + 1$

Obtain  $\sigma(n_0, \dots, n_k) = m \cdot (n - m + 1) + \sum_{i=2}^{m+1} (m - i + 1) \cdot 2 = m \cdot n - m^2 + m + m(m - 1) = m \cdot n$ .

**Claim:** The special case is the worst (most expensive)!

From any sequence  $n_1 > n_2 > \dots > n_k$  get to the special case by iteratively inserting numbers in the gaps. Insert  $s$  with  $n_{j-1} > s > n_j$ .

$\sigma(n_0, \dots, n_{j-1}, s, n_j, \dots, n_k) - \sigma(n_0, \dots, n_k) = \dots = s + (n_{j-1} - s) \cdot (s - n_j)$

$sp\sigma(n_0, \dots, n_k) \leq \sigma(n, m, m - 1, \dots, 2, 1, 0) = n \cdot m \quad \square$

Complexity is quadratic  $\rightarrow$  cheap

### 1.2.6 Algorithm 6 (Extended Euclidean Algorithm)

input :  $a, b \in \mathbb{N}$

output:  $d = \gcd(a, b)$  and  $s, t \in \mathbb{Z}$  such that  $d = s \cdot a + t \cdot b$

(1)  $r_0 := a, r_1 := b, s_0 := 1, t_0 := 0, s_1 := 0, t_1 = 1$

(2) for  $i = 1, 2, \dots$  perform steps (3) - (5)

(3) if  $r_i = 0$  set  $d = |r_{i-1}|$   
 $s := \text{sgn}(r_{i-1}) \cdot s_{i-1},$   
 $t := \text{sgn}(r_{i-1}) \cdot t_{i-1}$

(4) division with remainder:  
 $r_{i+1} = r_{i-1} - q_i \cdot r_i, \quad \text{with } |r_{i+1}| \leq \frac{1}{2}|r_i|$

(5) set  $s_{i+1} := s_{i-1} - q_i \cdot s_i,$   
 $t_{i+1} := t_{i-1} - q_i \cdot t_i$

Justification :  $r_i = s_i \cdot a + t_i \cdot b$  throughout

**Application:**  $m, x \in \mathbb{N}$  such that  $m, x$  co-prime (i.e.  $\gcd(x, m) = 1$ )

Algorithm 6 yields:  $1 = s \cdot x + t \cdot m \Rightarrow s \cdot x \equiv 1 \pmod{m}$

So obtain inverse of  $\bar{x} \in \mathbb{Z}/(m)$

### 1.3 Primality testing

Let  $\mathbb{P} \subseteq \mathbb{N}$  be the set of prime numbers.

Challenge: Given  $n \in \mathbb{N}$  decide if  $n \in \mathbb{P}$

**Naive Method:** Trivial division by  $m \leq \lfloor \sqrt{n} \rfloor$ .

Running time is exponential in  $l(n)$ . Even when restricted to division by prime numbers, need approximately  $\frac{\sqrt{n}}{|n|^{1/\sqrt{n}}}$  trivial divisions (prime number theorem)  
 $\rightarrow$  hardly any better!

**Reminder:** (arithmetic modulo  $m$ )

$G$  finite group  $\Rightarrow \forall a \in G \quad a^{|G|} = 1$  Fermat's little theorem

For  $G = (\mathbb{Z}/(p))^\times \quad a^{p-1} \equiv 1 \pmod{p} \in \mathbb{P} \quad \forall a \in \mathbb{Z} \quad \text{with } p \nmid a$

In fact  $(\mathbb{Z}/(p))^\times \cong Z_{p-1}$  is cyclic

For  $m = p_1^{e_1} \dots p_r^{e_r}$  with  $p_i \in \mathbb{P}, e_i \in \mathbb{N}_{>0}$ :

$\mathbb{Z}_{(m)} \cong \mathbb{Z}_{(p_1^{e_1})} \oplus \dots \oplus \mathbb{Z}_{(p_r^{e_r})} \Rightarrow \mathbb{Z}_{(m)}^\times \cong \mathbb{Z}_{(p_1^{e_1})}^\times \times \dots \times \mathbb{Z}_{(p_r^{e_r})}^\times$

what is  $\mathbb{Z}_{(p^e)}$  for  $p \in \mathbb{P}, e \in \mathbb{N}_{>0}$ ?

#### 1.3.1 Theorem 1 (Cyclic group)

Let  $p \in \mathbb{P}$  odd  $e \in \mathbb{N}_{>0} \Rightarrow (\mathbb{Z}_{(p^e)})^\times = Z_{(p-1) \cdot p^{e-1}}$  cyclic

**Proof:**

$(\mathbb{Z}_{(p^e)})^\times \cong Z_{p-1} \Rightarrow \exists z \in \mathbb{Z} : \text{order}(z + p\mathbb{Z}) = p-1$

Set  $a = \bar{z}^{p^{e-1}} \in (\mathbb{Z}_{(p^e)})^\times =: G$

$$a^{p-1} = \bar{z}^{(p-1) \cdot p^{e-1}} = \bar{z}^{|a|} = 1$$

On the other hand, take  $i \in \mathbb{Z}$  such that

$$a^i = 1 \Rightarrow \bar{z}^{i \cdot p^{e-1}} \equiv 1 \pmod{p} \Rightarrow (p-1) \mid (i \cdot p^{e-1}) \Rightarrow (p-1) \mid i.$$

So  $\text{ord}(a) = p-1$ .

Now consider  $b = (p+1) \in G$

**Claim:**  $\text{ord}(b) = p^{e-1}$

**Proof** by induction on  $k \in \mathbb{N}_{>0}$  that  $(p+1)^{p^{k-1}} \equiv p^k + 1 \pmod{p^{k+1}}$

$k=1$  ✓

$k \rightarrow k+1$ : By induction have  $(p+1)^{p^{k-1}} = 1 + p^k + x \cdot p^{k+1}, \quad x \in \mathbb{Z}$

Compute:  $(p+1)^{p^k} = ((1+p^k) + x \cdot p^{k+1})^p = \sum_{i=0}^p \binom{p}{i} (i+p^k)^{p-i} \cdot x^i \cdot p^{i \cdot (k+1)}$

Only 0-th summand  $\equiv (i+p^k) = \sum_{i=0}^p \binom{p}{i} p^{i \cdot k} \equiv_{p \text{ odd}} 1 + p^{k+1} \pmod{p^{k+2}}$  ✓

For  $k=e$ :  $(p+1)^{p^{e-1}} \equiv 1 \pmod{p^e} \Rightarrow b^{p^e} = 1 \Rightarrow \text{ord}(b) \mid p^{e-1}$

But  $(p+1)^{p^{e-2}} \equiv p^{e-1} + 1 \pmod{p^e} \Rightarrow b^{p^{e-2}} \neq 1 \in G$

So  $\text{ord}(b) = p^{e-1}$

**Claim:**  $\text{ord}(a \cdot b) = (p-1)p^{e-1} \quad (\Rightarrow \text{Theorem})$

Let  $(a \cdot b)^i = 1 \in G$  with  $i \in \mathbb{Z}$

Then  $1 = (a \cdot b)^{i \cdot (p-1)} = (a^{p-1})^i \cdot b^{i \cdot (p-1)} = b^{i \cdot (p-1)} \Rightarrow p^{e-1} \mid i \cdot i(p-1) \Rightarrow p^{e-1} \mid i$

Also  $1 = (a \cdot b)^{p^{e-1} \cdot i} = a^{p^{e-1}} \Rightarrow (p-1) \mid p^{e-1} \cdot i \Rightarrow (p-1) \mid i \rightarrow (p-1) \cdot p^{e-1} \mid i \quad \square$

**Reminder:**  $(\mathbb{Z}/(2^e))^\times \cong Z_2 \times Z_2^{e-2} \quad (e \geq 2)$

### 1.3.2 Algorithm 2 (Fermat Test)

input :  $n \in \mathbb{N}_{>0 \text{ odd}}$

output: " $n \notin \mathbb{P}$ " or "probably  $n \in \mathbb{P}$ "

- (1) Choose  $a \in 2, \dots, n-1$  randomly
- (2) Compute  $a^{n-1} \bmod n$
- (3) If  $a^{n-1} \not\equiv 1 \pmod{n}$  return " $n \notin \mathbb{P}$ "  
else return "probably  $n \in \mathbb{P}$ "

Not very satisfying. Is this fast?

### 1.3.3 Algorithm 3 (Fast exponentiation)

input :  $a \in G$   $G$  is a monoid,  $e \in \mathbb{N}$ ,  $e = \sum_{i_0}^{n-1} e_i 2^i$ ,  $e_i \in \{0, 1\}$

output:  $a^e \in G$

- (1) Set  $b := a$ ,  $y := 1$
- (2) For  $i = 0, \dots, n-1$  perform (3) - (4)
- (3) if  $e_i = 1$  set  $y := y \cdot b$
- (4) set  $b := b^2$
- (5) return  $y$

this requires  $O(l(e))$  operations in  $G$

For  $G = (\mathbb{Z}/(n)_i)$ , each multiplication requires  $O(l(n)^2)$  bit operations

$\Rightarrow$  Fermat test requires  $O(l(n)^3)$  bit operations  $\rightarrow$  cubic complexity  $\rightarrow$  "fast"!

#### Example:

$n = 561 = 3 \cdot 11 \cdot 17$  For  $a \in \mathbb{Z}$  with  $\gcd(a, n) = 1 \Rightarrow$  have  $a^{n-1} = (a^2)^{280} \equiv 1 \pmod{3}$

$a^{n-1} \equiv 1 \pmod{n}$  Fermat's test says "probably  $n \in \mathbb{P}$ " in 57% of cases.

$n = 2207 \cdot 6619 \cdot 15443$  : output "probably  $n \in \mathbb{P}$ " in 99,93% of cases.

### 1.3.4 Definition 4 (Pseudo-prime, witness, Carmichael numbers)

Let  $n \in N_{>1} \text{ odd}$ ,  $a \in 1, \dots, n-1$

- (a)  $n$  is pseudo-prime to base  $a$  if  $a^{n-1} \equiv 1 \pmod{n}$
- (b) otherwise  $a$  is called a witness of composition of  $n$
- (c) If  $n \notin \mathbb{P}$  but  $a^{n-1} \equiv 1 \pmod{n} \quad \forall a$  with  $\gcd(n, a) = 1$   
then  $n$  is called a Carmichael number.  
There are  $\infty$  Carmichael numbers

### 1.3.5 Proposition 5 (Number of witnesses)

Let  $n \in N_{>1}$ ,  $\text{odd} \wedge \notin \mathbb{P} \wedge \text{not Carmichael}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n, a \text{ is witness of composite of } n\}| > \frac{n-1}{2}$

**Proof:** Consider

$\phi : (\mathbb{Z}/(n))^\times =: G \rightarrow G, \quad \bar{a} \mapsto \bar{a}^{n-1}$

group homomorphism. By assumption,

$|\text{im}(\phi)| > 1 \Rightarrow |\text{Ker}(\phi)| \leq \frac{|G|}{2} < \frac{n-1}{2}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n \text{ a witness of composite of } n\}| > \frac{n-1}{2} \quad \square$

### Miller-Rabin Test

### 1.3.6 Proposition 6 (Inference from Fermat)

Let  $p \in \mathbb{P} \text{ odd}$ ,  $a \in \{1, \dots, (p-1)\}$  write  $p-1 = 2^k \cdot m$  with  $m \text{ odd}$  Then:

$a^m \equiv 1 \pmod{p}$  or  $\exists i \in \{0, \dots, k-1\} : a^{2^i \cdot m} \equiv -1 \pmod{p}$

**Proof:**

Little Fermat:  $\bar{a}^{2^k \cdot m} = 1 \in \mathbb{F}_p$

Assume  $\bar{a}^m \neq 1$  take  $i$  maximal such that:

$\bar{b} = \bar{a}^{2^i \cdot m} \neq 1 \Rightarrow \bar{b}^2 = 1 \Rightarrow \bar{b} \in \mathbb{F}_p$  is a zero of  $x^2 - 1 \in \mathbb{F}_p[x] \Rightarrow \bar{b} = -1$

### 1.3.7 Algorithm 7 (Miller-Rabin-test)

input :  $n \in \mathbb{N}_{>1}, \text{odd}$

output: either " $n \notin \mathbb{P}$ " or "probably  $n \in \mathbb{P}$ "  $\rightarrow$  Monte Carlo Algorithm.

- (1) write  $n - 1 = 2^k \cdot m$  with  $m$  odd
- (2) Choose  $a \in \{2, \dots, n - 1\}$  randomly
- (3) Compute  $b := a^m \pmod n$
- (4) if  $(b \equiv \pm 1 \pmod n)$   
return "probably  $n \in \mathbb{P}$ "
- (5) for  $(i = 0, \dots, k - 1)$  do steps (6) - (7)
- (6) set  $b := b^2 \pmod n$
- (7) if  $(b \equiv -1 \pmod n)$   
return "probably  $n \in \mathbb{P}$ "
- (8) return  $n \notin \mathbb{P}$

### 1.3.8 Definition 8 (strong pseudo-prime / witness)

Let  $n \in \mathbb{N}_{>1}, \text{odd}$   $a \in \{1, \dots, n - 1\}$

- (a)  $n$  is called a strongly pseudo-prime to base  $a$  if Proposition 6 holds for  $a$  and  $p$  replaced by  $n$ .
- (b) Otherwise  $a$  is called a strong witness of composition of  $n$ .

#### Example

Let  $n \in \mathbb{N}_{>1}, \mathbb{P} \text{ odd}$

$a = 2$  strong witness if  $n < 2047$  (including 561)

2 or 3 strong witness if  $n < 1373653$

2,3 or 5 strong witness if  $n < 25326001$

### 1.3.9 Theorem 9 (Bit-complexity of Algorithm 7)

- (a) Algorithm 7 requires  $O(l(n)^3)$  bit operations.  $\rightarrow$  "qubic complexity"  $\rightarrow$  fast!
- (b) if  $b \in \mathbb{P}$  then Algorithm 7 returns "probably  $b \in \mathbb{P}$ "  $\rightarrow$  no false positives.
- (c) if  $n \notin \mathbb{P}$  then more than half of the numbers in  $\{1, \dots, n - 1\}$  are strong witnesses.

**Proof:**

- (a) Step 1 takes  $O(l(n))$  bit operations:  
Using Algorithm 3, we need  $O(l(n-1))$  multiplications in  $\mathbb{Z}/(n)$  each requiring  $O(l(n)^2)$  bit operations.
- (b) Proposition 6
- (c) split in three cases:

**Case 1:**  $n$  is not a Carmichael number.  $\xRightarrow{\text{Prop 5}}$  more than half of all numbers are.

Fermat witness thus also strong witness.

**Case 2:**  $n = p^r \cdot l$  with  $p \in \mathbb{P}$   $r > 1$   $l \in \mathbb{N}_{>0}$   $p \nmid l$

Theorem 1  $\exists x \in \mathbb{Z}$  such that  $x^p \equiv 1 \pmod{p^r}$   $x \not\equiv 1 \pmod{p^r}$

Chinese remainder theorem:  $\exists a \in \mathbb{Z}$  such that  $a \equiv x \pmod{p^r}$   $a \equiv 1 \pmod{l}$

So  $\bar{a}^p = 1 \in \mathbb{Z}/(n) \Rightarrow \bar{a}^n = 1 \Rightarrow \bar{a} \in (\mathbb{Z}/(n))^\times$

i.e.  $\gcd(n, a) = 1$  if  $\bar{a}^{n-1} = 1$  then  $\bar{a} = 1$

But  $a \equiv x \not\equiv 1 \pmod{p^r}$  so  $\bar{a}^{n-1} \neq 1$  hence  $n$  is not Carmichael  $\rightarrow$  Case 1.

**Case 3:**  $n$  is a Carmichael number. By Case 2 have  $n = p \cdot l$  with  $p \in \mathbb{P}$   $p \nmid l$   $l \geq 3$   
 $n$  Carmichael:  $\forall a \in \mathbb{Z}$  with  $\gcd(a, n) = 1$

have  $a^{2^k \cdot m} \equiv 1 \pmod{n}$  (where  $n-1 = 2^k \cdot m$ )

$a^{2^k \cdot m} \equiv 1 \pmod{p}$  Take  $j$  minimal such that

$a^{2^j \cdot m} \equiv 1 \pmod{p}$   $\forall a \in \mathbb{Z}$  such that  $\gcd(a, n) = 1$

so  $0 \leq j \leq l$  in fact,  $j > 0$  since  $(-1)^{2^0 \cdot m} = -1$  with  $m$  odd.

Consider the subgroup  $H := \{\bar{a} \in \mathbb{Z}/(n) \mid \bar{a}^{2^{j-1} \cdot m} \in \{1, -1\} \subseteq (\mathbb{Z}/(n))^\times\}$

Let  $a \in \{1, \dots, n-1\}$   $\gcd(n, a) = 1$   $a$  not a strong witness.

**Claim 1:**  $\bar{a} \in H$

**Case 3.1:**  $\bar{a}^{2^{j-1} \cdot m} = 1 \Rightarrow \bar{a} \in H$

**Case 3.2:**  $a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{n}$   $a^m \not\equiv 1 \pmod{n}$

$\xRightarrow{\text{a nonwitness}} \exists i$  such that  $\underbrace{a^{2^i \cdot m} \equiv -1 \pmod{n}}_*$

$\Rightarrow a^{2^i \cdot m} \equiv -1 \pmod{p} \xRightarrow{\text{def of } j} i < j$

if  $i < j-1$  then  $a^{2^{j-1} \cdot m} = (a^{2^i \cdot m})^{2^{j-1-i}} \equiv (-1)^{2^{j-1-i}} = 1 \pmod{n}$

$\xRightarrow{\text{with } *} \text{not in case 3.2}$

**Claim 2:**  $H \subseteq (\mathbb{Z}/(n))^\times$  proper subgroup.

By definition of  $j$   $\exists x \in \mathbb{Z}$  such that  $x^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p}$

Chinese remainder:  $\exists a \in \mathbb{Z}$  such that

$a \equiv x \pmod{p}$   $a \equiv 1 \pmod{l}$

$\Rightarrow a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p} \equiv 1 \pmod{l} \Rightarrow \bar{a} \notin H$

Claim 2  $\checkmark$

It follows that  $|H| \leq \frac{|(\mathbb{Z}/(n))^\times|}{2} < \frac{n-1}{2}$

so the number of witnesses is  $\geq n-1-|H| > \frac{n-1}{2}$   $\square$

### Remarks:

- (a) A more careful analysis shows that  $2\frac{3}{4}$  of all candidates are strong witnesses
- (b) Calling Algorithm 7 repeatedly decreases the probability of false positives. Running time for prescribed error probability  $p$  is  $O(\lg(p^{-1} \cdot l(n)^3))$   
(Independence assumptions!)

### Connection with Riemann hypothesis

Let  $n \in \mathbb{N}_{>0}$   $\bar{X} : (\mathbb{Z}/(n))^\times \rightarrow \mathbb{C}^\times$  group homomorphism

$$X : \mathbb{Z} \rightarrow \mathbb{C}, a \mapsto \begin{cases} \bar{X}(\bar{a}) & \text{if } \gcd(a, n) = 1 \text{ for } (\bar{a} = a + n\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$$

"residue class character (mod  $n$ )

$$Ex : n = 1 \Rightarrow X(a) = 1 \forall a \in \mathbb{Z}$$

Dirichlet L-series:

$$L_X(s) = \sum_{n=1}^{\infty} \frac{X(n)}{n^s} \text{ converges for } s \in \mathbb{C} \text{ until } Re(s) > 1$$

$L_X(s)$  extends to a meromorphic function on  $\mathbb{C} \mapsto$  "Dirichlet L-function".

For  $n = 1 : L_X(s) = \zeta(s)$  Riemann Zeta-function.

Euler Product:

$$\text{From } (1 - X(p) \cdot p^{-s})^{-1} = \sum_{i=0}^{\infty} (X(p) \cdot p^{-s})^i = \sum_{i=0}^{\infty} \frac{X(p^i)}{p^{is}} \quad \text{derive } L_X(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - X(p) \cdot p^{-s}}$$

Generalized Riemann hypothesis (GRH):

For  $X$  residue class character,  $s \in \mathbb{C}$

with  $L_X(s) = 0$ ,  $0 < Re(s) < 1$  ("critical strip")

then  $Re(s) = \frac{1}{2}$

For  $X = 1 \rightarrow$  ordinary Riemann hypothesis.

### 1.3.10 Theorem (Ankeny & Bach)

GRH  $\Rightarrow \forall X \neq 1$  residue class character

$$\exists p \in \mathbb{P} : X(p) \neq 1, p < 2 \ln(n)^2$$

Let  $H \subsetneq (\mathbb{Z}/(n))^\times =: G$  proper subgroup.

Choose  $N \subsetneq G$  maximal proper subgroup such that  $H \subseteq N \Rightarrow G/N$  cyclic.

$$\bar{X} : G \mapsto \mathbb{C}^\times \text{ with } N = \text{Ker}(\bar{X}) \Rightarrow H \subseteq \text{Ker}(\bar{X})$$

$$\xrightarrow{\text{GRH, Thm1}} \exists p \in \mathbb{P} : p + n\mathbb{Z} \not\subseteq H, p < 2 \cdot \ln(n)^2$$

**Corollary:** Assume GRH.

Let  $n \in \mathbb{N}_{>1}$   $\mathbb{P}$  odd Then there is a strong witness  $a$  of compositeness of  $n$  with  $a < 2 \cdot \ln(n)^2$ .

$\rightarrow$  Obtain deterministic primality test with time  $O(\ln(n)^5)$  bit operations.

### AKS-test

A deterministic polynomial time primality test  $\rightarrow$  "holy grail"

Agrawal, Kayal, Saxena: PRIMES is in P, Annals of Mathematics, 2004.

### 1.3.11 Proposition 10 (Modulo over ideals)

Let  $n \in \mathbb{P}$   $a \in \mathbb{Z} \Rightarrow (x + a)^n \equiv x^n + a \pmod{n}$

where  $x$  is a indeterminate and for  $r \in \mathbb{N}$ :

$$(x + a)^n \equiv (x^n + a) \pmod{n, x^r - 1} \quad (1)$$

(i.e.  $(x + a)^n - (x^n + a) = n \cdot f + (x^r - 1) \cdot g$  with  $f, g \in \mathbb{Z}[x]$ )

**Proof:**

$$(x + a)^n = \sum_{i=0}^n \binom{n}{i} \cdot a^{n-i} \cdot x^i \quad (\text{where } \binom{n}{i} \text{ is a multiple of } n \text{ for } 0 < i < n)$$

$$\equiv x^n + a^n \quad (\leftarrow \text{little Fermat})$$

$$\equiv x^n + a \quad (1) \text{ follows by weakening this.}$$

**Cost** analysis for checking (1) with  $l = \text{length}(n)$ .

Using Algorithm 3, need  $O(l)$  multiplications in  $\mathbb{Z}[x]/(n, x^r - 1) =: R$

Elements of  $R$  are represented as polynomials of degree  $< r$ ,  
coefficients between 0 and  $n$ .

Multiply polynomials:  $O(r^2)$  operation in  $\mathbb{Z}/(n) : O(r^2 \cdot l^2)$

since  $x^{r+k} \equiv x^k \pmod{x^r - 1}$ ,

add coefficients of  $x^{r+k}$  of product polynomial to coefficients  $x^k : O(r \cdot l)$

Total for checking (1):  $O(r^2 \cdot l^3)$  bit operations.

Reduction  $\pmod{x^r - 1}$  is just for keeping the cost under control.

The following is part of AKS-test:

### 1.3.12 Algorithm 11 (Test for perfect power)

input :  $n \in \mathbb{N}_{>1}$

output:  $m, e \in \mathbb{N}$   $e > 1$  such that  $n = m^e$  or "n is not a perfect power"

(1) for  $(e = 2, \dots, \lfloor \lg(n) \rfloor)$  perform (2) - (7) //possible exponents

(2) set  $m_1 = 2, m_2 = n$  //initialize interval  $[m_1, m_2]$  for searching  $\sqrt[e]{n}$

(3) while( $m_1 \leq m_2$ ) do (4) - (7)

(4) set  $m = \lfloor \frac{m_1 + m_2}{2} \rfloor$  // bisect interval

(5) if  $m^e = n$  return  $m, e$

(6) if  $m^e > n$  set  $m_2 = m - 1$

(7) if  $m^e < n$  set  $m_1 = m + 1$

(8) return "not a perfect power"

**Cost:** (for  $l = \text{length}(n)$ )

Compute  $m^e : O(\lg(l) \cdot l^2)$  (abort computation once the result exceeds  $n$ )

Number of passages through inner loops  $\leq \lg(n)$

Number of passages through outer loops  $\leq \lg(n)$

Total cost of Algorithm 11:  $O(l^4 \cdot \lg(l))$



### 1.3.13 Algorithm 12 (AKS-test)

input :  $n \in \mathbb{N}_{>1}$  of length  $l = \text{length}(n) = \lfloor \lg(n) \rfloor + 1$

output: " $n \in \mathbb{P}$ " or " $n \notin \mathbb{P}$ "

- (1) if ( $n$  is a perfect power)  
return " $n \notin \mathbb{P}$ "
- (2) find  $r \in \mathbb{N}_{>1}$  minimal such that  $r|n \vee n^i \not\equiv 1 \pmod{r} \quad \forall i = 1, \dots, l^2$   
//exhaustive search (we will show that  $r \leq l^5$ )
- (3) if  $r|n$   
if ( $r = n$ ) return " $n \in \mathbb{P}$ "  
if ( $r < n$ ) return " $n \notin \mathbb{P}$ "
- (4) for  $a = 1, 2, \dots, \lfloor \sqrt{r} \cdot l \rfloor$  do (5)
- (5) if  $((x+a)^n \not\equiv (x^n + a) \pmod{(n, x^r - 1)})$   
return " $n \notin \mathbb{P}$ "
- (6) return " $n \in \mathbb{P}$ "

### 1.3.14 Lemma 13 (Least common multiple)

For  $n \in \mathbb{N}_{>0}$  have  $\lambda(n) := \text{lcm}(1, 2, \dots, n) \geq 2^{n-2}$

**Proof:** For  $f = \sum_{i=0}^m a \cdot x^i \in \mathbb{Z}(x) \quad a_i \in \mathbb{Z}$

$$\Rightarrow \int_0^1 f(x) dx = \sum_{i=0}^m \frac{a_i}{i+1} = \frac{k}{\lambda(m+1)}$$

with  $k \in \mathbb{Z}$ . Consider  $f_m = x^m \cdot (1-x)^m$

For  $0 < x < 1$ :

$$0 < f_m(x) \leq 4^{-m}$$

$$\Rightarrow 0 < \int_0^1 \underbrace{f_m(x)}_{\frac{k_m}{\lambda(2m+1)}} dx \leq 4^{-1}$$

$$\lambda(2 \cdot m + 1) \geq k_m \cdot 4^m \geq 4^m$$

$$\text{For } n \in \mathbb{N}_{>0} \lambda(n) \geq \lambda(2 \cdot \lfloor \frac{n-1}{2} \rfloor + 1) \geq 4^{\lfloor \frac{n-1}{2} \rfloor} \geq 4^{\frac{n-1}{2}} = 2^{n-2} \quad \square$$

**Corollary:** (not related to AKS)

For  $n \in \mathbb{N}$

$$\pi(n) := |\{p \in \mathbb{P} | p \leq n\}| \geq \frac{n-2}{\lg(n)}$$

**Proof:**

$$2^{n-2} \leq \lambda(n) = \prod_{p \in \mathbb{P}, p \leq n} p^{\lfloor \log_p(n) \rfloor} \leq \prod_{p \leq n} p^{\log_p(n)} = n^{\pi(n)} = 2^{\lg(n)\pi(n)} \quad \square$$

**Prime number theorem:**

$$\lim_{n \rightarrow \infty} \frac{\pi(n)}{n/\ln(n)} = 1$$

**Interpretation:**

The average distance of two primes around some value  $x \in \mathbb{R}_{>1}$  is  $\ln(x)$

### 1.3.15 Lemma 14 (Property of $r$ in Algorithm 12)

In Algorithm 12, have  $r \leq l^5$

**Proof:**

if  $r < l^5 \Rightarrow \forall k \in \{2, \dots, l^5\} : \exists i \in \{1, \dots, l^2\}$

$$n^i \equiv 1 \pmod{k}$$

$$\Rightarrow k \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\Rightarrow \lambda(l^5) \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\xrightarrow[\text{Lemma 13}]{=} 2^{l^5-2} < \prod_{i=1}^{l^2} n^i = n^{\frac{l^2(l^2+1)}{2}}$$

$$\Rightarrow l^5 - l^3 < 4 \quad \text{not true since } l \geq 2 \quad \square$$

### 1.3.16 Theorem 15 (Bit-Complexity of Algorithm 12)

Algorithm 12 requires  $O(l^{16.5})$  bit operations ("polynomial complexity")

**Proof:**

Step(1):  $O(l^4 \cdot \lg(l)) \checkmark$

Step(2): For each  $r$  need:

- test  $r \mid n : O(l^2)$
- compute all  $n^i \pmod{r} : O(l^2 \cdot \lg(r)^2) \xrightarrow[\text{Lemma 14}]{} O(l^2 \cdot \lg(l)^2)$

Step(3):  $O(1)$

$$\text{Step(4): } O(\sqrt{r} \cdot l \cdot r^2 \cdot l^3) \xrightarrow[\text{Lemma 14}]{} O(l^{16.5}) \quad \square$$

**Reminder:** There is a variant of Algorithm 12 with running time  $\tilde{O}(l^6)$ , i.e.,  $O(l^6 \cdot \lg(l)^m)$  with  $m \in \mathbb{N}$ .

**Correctness:**

For  $r \in \mathbb{N}_{>0}$  and  $p \in \mathbb{P}$  write  $I(r, p) := \{m, f\} \in \mathbb{N} \times \mathbb{F}_p[x] \mid f(x)^m \equiv f(x^m) \pmod{x^r - 1}\}$   
 "m is introspective for  $f$  and  $r$ ".

**Example:** Proposition 10 says that:

$$(p, x + \bar{a}) \in I(r, p) \text{ for } a \in \mathbb{Z} \quad r \in \mathbb{N}_{>0} \quad p \in \mathbb{P} \quad (1)$$

### 1.3.17 Lemma 16 (Rules for ideals)

- (a)  $(m, f), (m', f) \in I(r, p) \Rightarrow (m \cdot m', f) \in I(r, p)$
- (b)  $(m, f), (m, g) \in I(r, p) \Rightarrow (m, f \cdot g) \in I(r, p)$
- (c)  $(m \cdot p, f) \in I(r, p), p \nmid r \Rightarrow (m, f) \in I(r, p)$

**Proof:**

- (a)  $f(x)^{m \cdot m'} \equiv f(x^m)^{m'} \pmod{(x^r - 1)}$   
 $f(x^m)^{m'} \equiv f(x^{m \cdot m'}) \pmod{(x^{m \cdot r} - 1)}$   
 But  $(x^r - 1) \mid (x^{m \cdot r} - 1)$
- (b)  $(f \cdot g)(x)^m = f(x)^m \cdot g(x)^m \equiv f(x^m) \cdot g(x^m) = (f \cdot g)(x^m) \pmod{(x^r - 1)}$
- (c)  $(f(x)^m)^p \equiv f((x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x^m))^p \pmod{(x^r - 1)}$   
 $\Rightarrow (x^r - 1) \mid ((f(x)^m)^p - f(x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x)^m - f(x^m))^p$   
 $p \nmid r \Rightarrow x^r - 1$  is square free. So  
 $(x^r - 1) \mid (f(x)^m - f(x^m)) \Rightarrow (m, f) \in I(r, p) \quad \square$

### 1.3.18 Theorem 17 (Correctness of Algorithm 12)

Algorithm 12 is correct.

**Proof:**

If the algorithm terminates in step(1),(3) or (5), it is correct. To show: If it terminates in step(6) it is correct, i. e.  $n \in \mathbb{P}$

**Claim 1:**  $\exists p \in \mathbb{P} : p \mid n \quad p \not\equiv 1 \pmod{r} \quad p > r$

Indeed if all prime divisors of  $n$  were  $\equiv 1 \pmod{r}$  then  $n \equiv 1 \pmod{r}$

Contradiction to step(2). All prime divisors of  $n$  are  $> r$  by step (2) and (3) ✓

Steps(2) and (3) imply that  $\gcd(n, r) = 1 \Rightarrow G := \langle \bar{n}, \underbrace{\bar{p}}_{p \bmod r} \rangle \subseteq (\mathbb{Z}/(r))^\times$

Step(2):  $\text{ord}(\bar{n}) > l^2 \Rightarrow l^2 < |G| < r$  (2)

Set  $s := \text{ord}(\bar{p} \in G) \Rightarrow r \mid (p^s - 1)$  with  $q := p^s \Rightarrow r \mid |\mathbb{F}_q^\times| \Rightarrow \exists \zeta \in \mathbb{F}_q$   $r$ -th root of 1

Set  $k := \lfloor \sqrt{r} \cdot l \rfloor \quad m := \left(\frac{n}{p}\right)$

By (1)  $(p, x + \bar{a}) \in I(r, p)$  with  $\bar{a} \in \mathbb{F}_p$

By step(4), have  $(n, x + \bar{a}) \in I(r, p)$

For  $\underline{e} = e_0, \dots, e_k \in \mathbb{N}_0$  set  $f_{\underline{e}} := \prod_{a=0}^k (x + \bar{a})^{e_a}$

Lemma 16 (b):  $(p, f_{\underline{e}}) \in I(r, p)$

$(n, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(c)}} (m, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(a)}} \forall s, t \in \mathbb{N}_0 : (p^s \cdot m^t, f_{\underline{e}}) \in I(r, p)$

$\Rightarrow f_{\underline{e}}(\zeta^{p^s \cdot m^t}) = f_{\underline{e}}(\zeta)^{p^s \cdot m^t}$  (3)

Set  $H := \langle \zeta + \bar{a} | a \in \{0, \dots, k\} \rangle \subseteq \mathbb{F}_q^\times$   
 $(\zeta \notin \mathbb{F}_p \text{ since } r \nmid (p-1) \text{ by Claim 1})$

Consider:  $T := \{(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1} \mid \sum_{a=0}^k e_a < |G|\}$

$\Phi : T \mapsto H, (e_0, \dots, e_k) \mapsto f_{\underline{e}}(\zeta) = \prod_a (\zeta + \bar{a})^{e_a} \in H$

**Claim 2:**  $\Phi$  is injective.

Indeed, take  $(\underline{e}), (\underline{\hat{e}}) \in T$  such that  $\Phi(\underline{e}) = \Phi(\underline{\hat{e}})$

$$\Rightarrow \forall s, t \in \mathbb{N}_0 : f_{\underline{e}}(\zeta^{p^s \cdot m^t}) \stackrel{(3)}{=} f_{\underline{e}}(\zeta)^{p^s \cdot m^t} = f_{\underline{\hat{e}}}(\zeta)^{p^s \cdot m^t} \stackrel{(3)}{=} f_{\underline{\hat{e}}}(\zeta^{p^s \cdot m^t})$$

$f_{\underline{e}} - f_{\underline{\hat{e}}}$  has roots  $\zeta^e$  with  $e \in G$  since  $G = \langle \bar{p}, \bar{m} \rangle$

These are all distinct (since  $\zeta$  is primitive)

But  $\deg(f_{\underline{e}} - f_{\underline{\hat{e}}}) < |G|$  So  $f_{\underline{e}} - f_{\underline{\hat{e}}} = 0$

Since  $k \leq \sqrt{r} \cdot l < r < p$  the  $(x + \bar{a})$  with  $a \in \{0 \dots k\}$  are primitive distinct.

So  $(\underline{e}) = (\underline{\hat{e}})$  ✓

So is  $|H| \geq |T|$  ?

Let  $M$  be the set of all  $\{x_0, \dots, x_k\} \subseteq \{1, \dots, |G| + k\}$

with  $x_0 < x_1 < \dots < x_k$

For  $\{x_0, \dots, x_k\} \in M$  define  $(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1}$  by  $e_a = x_a - x_{a-1}$  with  $x_{-1} := 0$

$$\Rightarrow \sum_{a=0}^k e_a = \sum_{a=0}^k (x_a - x_{a-1} - 1) = x_k - (k+1) < |G|$$

Obtain injection  $M \Leftrightarrow T$

$$\text{So } |H| \geq |T| \geq |M| = \binom{|G|+k}{k+1} \stackrel{(2)}{\geq} \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{k+1} = \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{\lfloor l\sqrt{|a|} \rfloor} \stackrel{(2)}{\geq} \binom{2 \cdot \lfloor l\sqrt{|a|} \rfloor + 1}{\lfloor l\sqrt{|a|} \rfloor}$$

### 1.3.19 Lemma 18 (Property of binomial coefficients)

$$\forall n \in \mathbb{N}_{>1} : \binom{2 \cdot n + 1}{n} > 2^{n+1}$$

**Proof:**

$n = 2 :$

$$\binom{5}{2} = 10 > 2^3$$

$n - 1 \rightarrow n :$

$$\binom{2 \cdot n + 1}{n} = \binom{2 \cdot n}{n-1} + \binom{2 \cdot n}{n} = \binom{2 \cdot n - 1}{n-2} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n} \geq 2 \cdot \binom{2 \cdot n - 1}{n-1} \stackrel{ind.}{>} 2 \cdot 2^n = 2^{n+1}$$

### Continuation of Proof of Theorem 17

$$|H| > 2^{\lfloor l \cdot \sqrt{|a|} \rfloor + 1} \geq 2^{l \cdot \sqrt{|a|}} \geq 2^{\lg(n) \cdot \sqrt{|a|}} = n \sqrt{|a|} \quad (4)$$

Assume  $n \notin \mathbb{P}$  By step (1)  $m$  is not a perfect power

$\Rightarrow$  the map  $\mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{N} \quad (s, t) \mapsto p^s m^t$  is injective.

Set  $A := \{p^s m^t \mid s, t \in \{0, \dots, \lfloor \sqrt{a} \rfloor\}\} \subseteq \mathbb{N}$

$$\Rightarrow |A| = (\lfloor \sqrt{|a|} \rfloor + 1)^2 > |G|$$

Since  $G = \langle \bar{p}, \bar{m} \rangle \subseteq (\mathbb{Z}/(r))^\times$  this implies that  $\exists n, \hat{n} \in A$

such that  $n \neq \hat{n}$  but  $b \equiv \hat{n} \pmod{r}$ .

$$\text{Let } h \in H \Rightarrow h = f_{\underline{e}}(\zeta) \text{ with } (\underline{e}) \in \mathbb{N}_0^{k+1} \Rightarrow h^n \stackrel{(3)}{=} f_{\underline{e}}(\zeta^n) \stackrel{n \equiv \hat{n} \pmod{r}}{=} f_{\underline{e}}(\zeta^{\hat{n}}) \stackrel{(3)}{=} h^{\hat{n}}$$

So the polynomial  $Y^n - Y^{\hat{n}} \in \mathbb{F}_q[Y]$  has all elements of  $H$  as zeros.  
 But  $\deg(Y^n - Y^{\hat{n}}) \leq \max\{n, \hat{n}\} \leq (p \cdot m)^{\lfloor \sqrt{|G|} \rfloor} \leq n\sqrt{|G|} < |H|$   
 $\Rightarrow$  contradiction since  $Y^n - Y^{\hat{n}} \neq 0$   $\square$

## 1.4 Cryptology

A ("Alice") wants to send a message to B ("Bob") such that an eavesdropper E ("Eve") can not read the clear message. So A and B encrypt the message.



Figure 2: Scheme of eavesdropping

### Symmetric-key cryptography

A and B share secret keys for encryption ( $x \mapsto x'$ ) and decryption ( $x' \mapsto x$ ). Only A and B know the keys.

Example: AES approved by the US government in 2002

Application:

- sending messages
- encrypt files (A=B)

Problem: Key exchange between A and B

### Public-key cryptography

Encryption-map  $\phi : x \mapsto x'$  is made public by B, but decryption  $\phi : x' \mapsto x$  is kept secret.

Advantage: No confidential key exchange.

Disadvantages:

- more costly than symmetric key cryptography
- doubt whether E can reconstruct  $\phi^{-1}$  from  $\phi$  with enough computing power

Applications:

- sending messages
- exchange of symmetric keys
- authentication: Together with  $x$ , B sends  $\phi^{-1}(x)$  (or  $\phi^{-1}|$  Part of  $x$  together with date). A verifies by applying  $\phi$ .  
 Better: challenge-response-protocol.

Examples: RSA, elliptic curve

### 1.4.1 Algorithm (RSA)

- (1) B chooses  $p, q \in \mathbb{P}$  large ( $> 100$  digits)  
with  $p \neq q$   $n := p \cdot q$
- (2) B chooses  $e, f \in \mathbb{N}$  large such that  $e \cdot f \equiv 1 \pmod{\phi(n)}$   
with  $\phi(n) = (p-1)(q-1)$
- (3) B makes  $n, e$  public, keep  $f$  secret
- (4) The message is encoded as an element  $x \in \mathbb{Z}/(n)$
- (5) A computes  $\phi(x) = x^e = y \in \mathbb{Z}/(n)$  and sends  $y$
- (6) B receives  $y$  and computes  $y^f = x \in \mathbb{Z}/(n)$

Comments on steps of RSA:

- (6) Have  $e \cdot f = a \cdot (p-1) \cdot (q-1) + 1$  with  $a \in \mathbb{N}_{>0}$   
 $y^f = x^{e \cdot f}$

$$1: q \nmid f, q \nmid x \Rightarrow x^{a(p-1)(q-1)} = (x^{\phi(n)})^a \stackrel{\text{LittleFermat}}{\equiv} 1^a = 1 \Rightarrow x^{e \cdot f} = x \quad \checkmark$$

- Case 2:  $p|x, q \nmid x \Rightarrow x^{e \cdot f} \equiv 0 \equiv x \pmod{p}$   
 $x^{e \cdot f} \equiv x \pmod{q}$  as above.

Case 3:  $q|x$  As Case 2

$\Rightarrow$  Correctness of decryption

**Cost:**

- (1) Finding  $p, q$  of length approximately  $l$ . Prime-number theorem: Gap between two primes of length  $\approx l$  is  $O(l)$   
Using Miller Rabin with error probability  $2^m$ . Expected cost of (1) is  $O(m \cdot l^4)$  bit operations.
- (2) Choose  $e$  co-prime to  $\phi(n)$  obtain  $f = \text{inverse} \pmod{\phi(n)}$  by extended euclidean Algorithm:  $O(l^2)$
- (5)(6) Fast exponentiation:  $O(l^3)$

Security of RSA:  $p$  and  $q$  must be so large that factorization of  $n$  is "impossible". Assumption that factorization is expensive could not be shown! But could  $f$  be obtained without knowing  $p$  and  $q$ ? The following algorithm gives a negative answer. It shows that the problem of breaking RSA is always basically factorization.

Remember:  $\phi(n) | (e \cdot f - 1) \Rightarrow m \leq n^2$

### 1.4.2 Algorithm 1 (Finding a divisor)

Input :  $n \in \mathbb{N}_{>2}$  odd squarefree  $\notin \mathbb{P}$  and  $m \in \mathbb{N}_{>0}$  such that  $\phi(n) \mid m$   $m \leq n^2$

Output:  $d \in \mathbb{N}$  with  $d \mid n$   $1 < d < n$

- (1) Choose  $a \in \{2, \dots, (n-2)\}$  randomly  
set  $k := m$
- (2) If  $d := \gcd(a, n) \neq 1$   
return d
- (3) Repeat steps (4) - (8) //while(true)
- (4) compute  $d := \gcd(n, a^k - 1)$
- (5) If  $d = 1$  go to (1)
- (6) If  $d < n$  return d
- (7) if  $k$  is odd go to (1)
- (8) set  $k := \frac{k}{2}$

Correctness is clear. What about termination and running time?

### 1.4.3 Proposition 2 (Complexity of Algorithm 1)

Algorithm 1 terminates in expected time  $O(l(n)^4)$  bit operations (Las Vegas Algorithm).

**Proof:**

Set  $l := \text{length}(n)$

Have  $n = \prod_{i=1}^r p_i$  with  $p_i \in \mathbb{P}$  distinct.

$\phi(n) = \prod_{i=1}^r (p_i - 1) \mid m$  So initially all  $(p_i - 1)$  divide  $k$ .

At some iteration it happens for the first time that  $(p_i - 1) \nmid k$

Then  $k \equiv \frac{p_1-1}{2} \pmod{(p_1-1)} \Rightarrow a^k \equiv \pm 1 \pmod{p_i}$  -1 occurs for some  $a$

For those  $j$  with  $(p_j - 1) \mid k$  have  $a^k \equiv 1 \pmod{p_j}$

Consider the group homomorphism:  $\phi_i(\mathbb{Z}/(n))^{\times} \mapsto (\mathbb{Z}/(p_1))^{\times} \times \dots \times (\mathbb{Z}/(p_r))^{\times}$   
 $\bar{a} \mapsto (a^k \pmod{p_1}, \dots, a^k \pmod{p_r})$

The image of  $\phi$  is a product of groups  $\{\pm 1\}$  or  $\{1\}$  depending whether  $(p_i - 1) \nmid k$  or  $(p_i - 1) \mid k$

**Conclusion:**

For at least half of all  $a$ 's,  $\phi(\bar{a})$  is neither  $(1, \dots, 1)$  nor  $(-1, \dots, -1)$

If  $a^k \equiv 1 \pmod{p_j}$  then  $p_j \mid (a^k - 1) \Rightarrow p_j \mid d$

If  $a^k \equiv -1 \pmod{p_j}$  then  $p_j \nmid (a^k - 1) \Rightarrow p_j \nmid d$

So for these  $a$  the algorithm is successful.

This means that the expected number of  $a$ 's that need to be tested is  $\leq 2$

(Since  $\sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = 2$  More generally for  $0 < p < 1 : p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} = \frac{1}{p}$ )

Analysis of running time (in bit operations) for each  $a$  (using gcd is quadratic) leads to the claim.  $\square$

**Problems of RSA:**

- How difficult is factorization of integers (lower bound?)
- decryption of some or all messages without having  $f$ ?

**1.4.4 Diffie-Hellmann Key Exchange**

Goal: A, B want to exchange a symmetric key via a public channel

- (1) A and B agree on a  $p \in \mathbb{P}$  (should be large) and  $g \in (\mathbb{Z}/(p))^{\times}$  public
- (2) A chooses  $a \in \{2, \dots, (p-2)\}$  randomly and sends  $u := g^a$  to B
- (3) B chooses  $b \in \{2, \dots, (p-2)\}$  randomly and sends  $v := g^b$  to A
- (4) A computes  $v^a = (g^b)^a = g^{a \cdot b}$   
B computes  $u^b = (g^a)^b = g^{a \cdot b}$

$\Rightarrow$  A and B share  $g^{a \cdot b}$

**Example:**

A chooses  $a = 7$

$$\bar{3}^7 = \bar{11} \in \mathbb{Z}/(17)$$

$$\bar{13}^7 = \bar{4}$$

B chooses  $b = 4$

$$\bar{3}^4 = \bar{13} \in \mathbb{Z}/(17)$$

$$\bar{11}^4 = \bar{4}$$

If Eve reconstructs  $a, b$  from  $g^a$  and  $g^b$  she can compute  $g^{a \cdot b}$

The Security of Diffie-Hellmann depends on the difficulty of the discrete logarithm problem (DLP):

Given  $g \in G$  element of a group or monoid and given  $g^a \in G$ , determine  $a$  (or determine  $a' \in \mathbb{Z}$  such that  $g^a = g^{a'}$ )

**1.4.5 Elliptic curve cryptography (ECC)**

ECC uses elliptic curves as groups.

$$y^2 = x^3 + a \cdot x + b \rightsquigarrow y^2 z = x^3 + axz^2 + bz^3$$

ECC uses suitable elliptic curves on  $\mathbb{F}_a$



## 1.5 Factorization

Let  $m \in \mathbb{N}_{>1}$   $n \notin \mathbb{P}$  Find a divisor  $d$  with  $1 < d < n$ . From this we obtain the factorization of  $n$  by recursion.

Naive method: Trial division. Cost essentially exponential in  $l(n)$

### 1.5.1 Algorithm 1 (Sieve of Eratosthenes)

Input :  $n \in \mathbb{N}_{>1}$

Output: All primes  $\leq n$

- (1) Create a list of all numbers  $\leq n$
- (2)  $p := 2$
- (3) Mark all multiples of  $p$  in the List
- (4) if all numbers are marked  
return
- (5) Let  $p$  be the smallest number that is not marked
- (6)  $p \in \mathbb{P}$  Go to (3)

Running time of Algorithm 1 is exponential.

#### Pollard's rho ( $\rho$ ) algorithm:

Idea: Choose a function  $\mathbb{Z}/(m) \mapsto \mathbb{Z}/(n)$  e.g.  $f(x) = x^2 + 1$

Choose  $x_0 \in \mathbb{Z}/(n)$  set  $x_i := f^i(x_0)$  iterative application.

Let  $p \mid n$  be a prime. Since  $|\mathbb{Z}/(p)| < \infty$  then  $\exists i < j : x_i \equiv x_j \pmod{p}$

Starting at  $x_i$  the sequence of  $x_j$  will be periodic.

$p \mid x_i - x_j \quad p \mid n \Rightarrow p \mid \gcd(n, x_i - x_j) =: d$

If  $x_i \not\equiv x_j \pmod{n}$  (which is not guaranteed) then  $d$  is a proper divisor of  $n$ .

- Recall that gcd computation is cheap
- Testing all pairs is a lot
- Proposition 2 helps with this

### 1.5.2 Proposition 2 (length of periods)

Let  $M$  be a set of functions  $f : M \mapsto M$  and  $x_0 \in M$   $x_i := f^i(x_0)$

If  $x_{t+l} = x_t$  for  $l, t \in \mathbb{N} > 0$  ( $\rightarrow t$  "off-period",  $l$  "length of period")

$\Rightarrow \exists j \in \mathbb{N}$  with  $0 < j \leq t + l$  such that  $x_j = x_{2j}$

**Proof:**

$f^l(x_t) = x_t \Rightarrow \forall a \in \mathbb{N} \quad f^{a \cdot l}(x_t) = x_t$  Assume  $j = a \cdot l \geq t$   $a \in \mathbb{N}$

$x_{2j} = x_{t+(j-t)+a \cdot l} = f^{(j-t)}(x_{t+a \cdot l}) = f^{(j-t)}(f^{al}(x_t)) = f^{(j-t)}(x_t) = x_j$

**Case 1**  $t = 0$   $j = l$  ✓

**Case 2**  $t > 0$   $j = t + (-t \bmod l) \in 0, \dots, (l-1)$  ✓

### 1.5.3 Algorithm 3 (Pollard's $\rho$ -Algorithm)

Input :  $n \in \mathbb{N}_{>1}, n \notin \mathbb{P}$

Output: a proper divisor of  $n$  or "FAIL"

- (1) Choose  $x \in \{0, \dots, (n-1)\}$  randomly  
set  $y := x$
- (2) repeat (3)-(6)
- (3)  $x := x^2 + 1 \pmod{n}$      $y := (y^2 + 1)^2 + 1 \pmod{n}$      $// x := x_j y := x_{2j}$
- (4)  $d := \gcd(n, x - y)$
- (5) if  $(1 < d < n)$   
return  $d$
- (6) if  $d = n$   
return "FAIL"

One "FAIL" includes no conclusion so you might want to repeat the Algorithm with a different  $x$ .

Running time? Assume the  $x_i := f^i(x_0)$  are randomly distributed.

When can we expect that a match ( $x_i \equiv x_j \pmod{p}$ ) occurs?  $\rightarrow$  "Birthday Problem"

#### Lemma (Birthday Problem):

We iteratively choose numbers in  $\{1, \dots, n\}$  at random. The expected numbers of choices (if we keep choosing until a number has been chosen twice) is  $< \sqrt{\frac{\pi \cdot n}{2}} + 2$

#### Proof:

Let  $s \geq 2$  be the numbers of choices until a match occurs. For  $k \in \mathbb{N}$  with  $P()$  as probability

$$P(s > k) = \prod_{i=1}^k \left(1 - \frac{i-1}{n}\right) \leq \prod_{i=1}^k e^{-\frac{i-1}{n}} = e^{\sum_{i=1}^k -\frac{i-1}{n}} = e^{\frac{k(1-k)}{2n}} \leq e^{-\frac{(k-1)^2}{2n}}$$

\* since  $f(x) = e^x - (1-x) \geq 0$  for  $x \geq 0$

$$f(0) = 0$$

$$f'(x) \geq 0 \text{ if } x \geq 0$$

$$\sum_{k=0}^{\infty} P(s > k) = 2 + \sum_{k=2}^{\infty} P(s > k) \leq 2 + \sum_{k=2}^{\infty} e^{-\frac{(k-1)^2}{2n}} \leq 2 + \int_1^{\infty} e^{-\frac{(x-1)^2}{2n}} dx$$

$$\stackrel{x:=x-1}{=} 2 + \int_0^{\infty} e^{-\frac{x^2}{2n}} dx = 2 + \int_0^{\infty} e^{-\left(\frac{x}{\sqrt{2n}}\right)^2} dx$$

$$\stackrel{x:=\frac{x}{\sqrt{2n}}}{=} 2 + \sqrt{2n} \int_0^{\infty} e^{-x^2} dx = 2 + \sqrt{2n} \cdot \frac{\pi}{2} = 2 + \sqrt{\frac{n \cdot \pi}{2}}$$

#### Example:

People arrive at a party. When can you expect to have two that share their birthday?

$\rightarrow$  when 26 have arrived!

### 1.5.4 Theorem 4 (Bit-complexity of Algorithm 3)

under suitable assumptions on the distribution  $f^i(x)$  for  $f(x) = x^2 + 1$  Algorithm 3 has the expected running time of  $O(\sqrt[n]{n} \lg(n)^2)$  bit operations

**Proof:**

By Proposition 2 and the Lemma the expected number of runs through the loop is

$$O(\sqrt[p]{p}) = O(\sqrt[n]{n}) \text{ as } p \leq \sqrt{n}$$

Each run through the loop takes  $O(\lg(n)^2)$  bit operations.  $\square$

### Pollard's p-1 Algorithm

Motivation: Let  $p \mid n$  prime divisor

$$\Rightarrow \forall a \in \mathbb{Z} : a^{p-1} \equiv 1 \pmod{p} \quad \text{whith } \gcd(a, p) = 1$$

$$\Rightarrow \forall m \in \mathbb{Z} \text{ with } (p-1) \mid m : a^m \equiv 1 \pmod{p}$$

$$p \mid \gcd(a^m - 1, n)$$

Let  $B$  be an upper-bound for the prime powers dividing  $p-1$ .

" $p-1$  is  $B$ -power-smooth".

$$\text{Then } (p-1) \mid \prod_{(q \leq B) \in \mathbb{P}} q^{\lfloor \log_q(B) \rfloor}$$

Neither  $p$  nor  $B$  are known! But guess and try  $B$  and hope for the best.

### 1.5.5 Algorithm 5 (Pollard's p-1 method)

Input :  $n \in \mathbb{N}_{>1} \setminus \mathbb{P}$

Output:  $d \in \mathbb{N}$  with  $d \mid n$   $1 < d < n$  or "FAIL"

- (1) Choose a "smoothness bound"  $B$
- (2) Choose  $a \in \{2, \dots, (n-2)\}$  randomly
- (3) Use Algorithm 1 to find all  $q \in \mathbb{P}$  with  $q \leq B$   
For every  $q$  perform steps (4) - (5)
- (4)  $k := q^{\lfloor \log_q(B) \rfloor}$   
set  $a := a^k \pmod{n}$   
compute  $d := \gcd(n, a - 1)$
- (5) if  $1 < d < n$   
return  $d$
- (6) return "FAIL" //or increase  $B$  and go to (1)

**Consequence:** when setting up RSA  $p, q$  should be chosen such that  $p-1$  and  $q-1$  have large prime divisors.

**The quadratic sieve** (State of the art factorization algorithm)

Observation: if  $n = x^2 - y^2$  then  $n = (x - y) \cdot (x + y)$

Conversely if  $n = a \cdot b$  then  $n = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$

**1-st Idea:** Find  $x, y \in \mathbb{Z}$  such that  $x^2 \equiv y^2 \pmod{n} \quad \wedge \quad x \not\equiv \pm y \pmod{n}$

Then  $n \mid (x - y) \cdot (x + y)$

$\Rightarrow$  for every  $y$ :  $e \in \mathbb{P}$  with  $p \mid n : p \mid (x - y) \vee p \mid (x + y)$

$\Rightarrow p \mid \gcd(x - y, n) \vee p \mid \gcd(x + y, n)$

Since both gcd are  $< a$  receive a non-trivial divisor of  $n$

If  $x^2 \equiv y^2 \pmod{n}$  how probable is it that  $x \equiv \pm y \pmod{n}$  ?

Let  $n = \prod_{i=1}^r p_i^{k_i}$  odd with  $p_i \in \mathbb{P}$  distinct.

Assume  $p_i \nmid x \forall i = 1 \dots r$  Since  $(\mathbb{Z}/(p_i^{k_i}))^\times$  is cyclic there are  $2^r$  classes  $y \pmod{n}$  such that  $x^2 \equiv y^2 \pmod{n}$

[Reason: These classes are given by  $y \equiv \pm x \pmod{p_i^{k_i}}$  These are the only solutions since  $\mathbb{Z}/(p_i^{k_i})^\times$  is cyclic of even order.

$G = \langle \sigma \rangle$  cyclic of order  $2m$

$x = \sigma^i$  Find  $l \in \mathbb{Z}$  such that  $x^2 = (\sigma^l)^2$

$\Leftrightarrow 2j \equiv 2l \pmod{2m} \Leftrightarrow j \equiv l \pmod{m}$

$\Leftrightarrow l \equiv j \pmod{2m}$  or  $l \equiv j + m \pmod{2m}$  ]

But have  $x \equiv \pm y$  only for  $2y$ 's.

Failure probability:  $2^{1-r}$

Handle case  $r = 1$  by Algorithm 11 in 1.3

**Example 1**

$n = 91$  Search  $x, y \in \mathbb{Z} \quad k \in \mathbb{Z}$  such that  $x^2 = k \cdot n + y^2$

Good chance if  $x$  is slightly bigger than  $\sqrt{k \cdot n}$

$k := 1 \Rightarrow \sqrt{91} \approx 9,54 \Rightarrow x := 10 \Rightarrow 10^2 = 100 \equiv 3^2 \pmod{91}$

$n = 10^2 - 3^2 = (10 - 3) \cdot (10 + 3) = 7 \cdot 13 \quad \checkmark$

Another try:

$k := 8 \Rightarrow \sqrt{8 \cdot 91} \approx 26,98 \Rightarrow 27^2 \equiv 1^2 \pmod{91} \quad \gcd(26, 91) = 13$

**Example 2**

$n = 4633 \quad k := 3$

$\sqrt{3 \cdot n} \approx 117,89 \Rightarrow x^2 = 118^2 \equiv 5^2 \pmod{n}$

$\gcd(118 - 5, n) = 113$

$\gcd(118 + 5, n) = 41 \quad \checkmark$

**2-nd Idea:** Choose  $B \in \mathbb{N}$  "smoothness bound" suitable.

Let  $p_2, \dots, p_r \in \mathbb{P}$  be all primes  $\leq B$  (Algorithm 1) set  $p_1 := -1$

The  $p_i$  form a "factor basis".

For  $a \in \mathbb{Z}$  write  $(a \pmod{n})$

for the  $x \in \mathbb{Z}$  with  $x \equiv a \pmod{n}$  and  $-\frac{n}{2} < x \leq \frac{n}{2}$

**Procedure:**

Search numbers  $a_1, \dots, a_m \in \mathbb{Z}$  such that  $(a_i^2 \pmod{n}) = \prod_{j=1}^r p_j^{e_{ij}}$

with  $e_{ij} \in \mathbb{Z}$  ("B numbers")

So for  $\mu_1, \dots, \mu_m \in \mathbb{N}_0$  have  $\left( \prod_{i=1}^m a_i^{\mu_i} \right)^2 \equiv \prod_{i=1}^m \prod_{j=1}^r p_j^{\mu_i \cdot e_{ij}} \pmod{n} = \prod_{j=1}^r p_j^{\sum_{i=1}^m \mu_i \cdot e_{ij}} \pmod{n}$

If the vectors  $(e_{i1}, \dots, e_{ir})$  become linearly dependant mod 2 (guaranteed if  $m > r$ )

then  $\exists \mu_1, \dots, \mu_m \in \{0, 1\}$  not all 0 such that:

$$\sum_{i=1}^m \mu_i \cdot e_{ij} = 2 \cdot k_j \quad k_j \in \mathbb{N}_0$$

with  $x := \prod_{i=1}^m a_i^{\mu_i}$   $y := \prod_{j=1}^r p_j^{k_j}$  obtain  $x^2 \equiv y^2 \pmod{n}$

**Example:**  $n = 4633$  choose  $B = 3 \Rightarrow$  factor basis  $-1, 2, 3$

Search  $a \in \mathbb{Z}$  such that  $|a_i^2 \pmod{n}|$  is small. Idea:  $a \approx \sqrt{n} = 68.06\dots$

$$a_1 := 68 : 68^2 = n - 9 \equiv (-1) \cdot 3^2 \pmod{n}$$

$$\rightarrow e_1 = (1, 0, 2) \rightarrow (1, 0, 0) \in \mathbb{F}_2^3$$

$$a_2 := 69 : 69^2 = n + 128 \equiv 2^7 \pmod{n}$$

$$\rightarrow e_2 = (0, 7, 0) \rightarrow (0, 1, 0) \in \mathbb{F}_2^3$$

$$a_3 := 67 : 67^2 = n - 144 \equiv (-1) \cdot 2^4 \cdot 3^2$$

$$\rightarrow e_3 = (1, 4, 2) \rightarrow (1, 0, 0) \in \mathbb{F}_2^3$$

$$e_1 + e_3 \equiv 0 \pmod{2} \quad \text{In fact:}$$

$$e_1 + e_3 = 2 \cdot \underbrace{(1, 2, 2)}_{(k_1, k_2, k_3)} \rightarrow \mu_1 = 1 \quad \mu_2 = 0 \quad \mu_3 = 1$$

$$x := a_1 \cdot a_3 \equiv -77 \pmod{n}$$

$$y := (-1) \cdot 2^2 \cdot 3^2 = -36$$

$$x - y = -41 \quad x + y = -113$$

$$\gcd(n, x - y) = 41 \quad \gcd(n, x + y) = 113 \quad \checkmark$$

**3rd Idea:** Look for  $a_i$  of the form  $t + \lfloor \sqrt{n} \rfloor$  with  $t$  in a "suitable".

Sieve Interval:  $[-s, s] \cap \mathbb{Z}$

As it turns out if  $s \leq \frac{\sqrt{5}-2}{2} \lfloor \sqrt{n} \rfloor$  then  $(t + \lfloor \sqrt{n} \rfloor)^2 \pmod{n} = (t + \lfloor \sqrt{n} \rfloor)^2 - n =: f(t)$

When does  $p_j^{e_j}$  divide  $f(t)$  (with  $j \geq 2$ )? Precisely if  $(t + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_j^{e_j}}$

If this holds for some  $t$  then it also holds for all  $t + k \cdot p_j^{e_j}$  with  $k \in \mathbb{Z}$  Moreover if it holds then  $\bar{n} \in \mathbb{F}_{p_j}$  is square. So may remove all  $p_j$  such that  $\bar{n} \in \mathbb{F}_{p_j}$  is a non-square from the factor basis.

Obtain a sieving procedure:

For  $t \in [-s, s] \cap \mathbb{Z}$  with  $p_j^{e_j} \mid f(t)$  "mark" all elements  $t + k \cdot p_j^{e_j} \in [-s, s]$

### 1.5.6 Algorithm 6 (Quadratic sieve, simplified version)

Input :  $n \in \mathbb{N}_{>1} \setminus \mathbb{P}$  odd

Output: A non trivial divisor of  $n$  or "FAIL"

- (1) if  $(n = m^e)$  with  $m, e \in \mathbb{N}_{>1}$   
return  $m$  // can be done with Algorithm 11 § 3
- (2) Choose a "smooteness bound"  $B \in \mathbb{N}$  and a "sieve bound"  $s \in \mathbb{N}$  suitably
- (3) Let  $p_1 = -1$   $p_2, \dots, p_r$  be the factor basis given by  $B$ . Delete those  $p_j$  such that  $\bar{n} \in \mathbb{F}_{p_j}$  is a non-square
- (4) for  $(t = -s, -s + 1, \dots, s - 1)$   
compute  $f_t := |(t + \lfloor \sqrt{n} \rfloor)^2 - n| \in \mathbb{N}_{>0}$
- (5) for  $(t = -s, \dots, s)$   
set  $e_t := (0, \dots, 0) \in \mathbb{N}_0^r$  // initialize exponent vectors
- (6) for  $(t = -s, \dots, 0)$   
set  $e_{t,1} := 1$  //  $\rightarrow$  first entry of each  $e_t$  is the exponent of  $p_1 = -1$  in  $f(t)$
- (7) for  $(j = 2, \dots, r)$  repeat (8) - (10)
- (8) for  $(e = 1, \dots, \lfloor \log_{p_j}(B) \rfloor)$  repeat (9) - (10) // or maybe a bit larger
- (9) solve  $(t + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_j^e}$   
Let  $(t_i \pmod{p_j^e}), \dots, (t_m \pmod{p_j^e})$  be the solutions.  
// We will see that  $m \in \{0, 2, 4\}$  with  $m = 2$  most frequent.
- (10) for all  $t = t_i + k \cdot p_j^e \in [-s, s]$  with  $k \in \mathbb{Z}, i = 1, \dots, m$   
set  $e_{t,j} := e_{t,j} + 1$   
 $f_t := \frac{f_t}{p_j}$
- (11) let  $t, \dots, t_m$  be those  $t \in [-s, s] \cap \mathbb{Z}$  for which  $f_t = 1$   
/\* So the  $a_i = t_i + \lfloor \sqrt{n} \rfloor$  are  $B$ -numbers and the factorization  
\* of  $a_i^2 \pmod{n} = a_i^2 - n = f(t)$  is given by the exponent  
\* vectors  $e_t$  \*/
- (12) if the  $(e_{t_i} \pmod{2}) \in \mathbb{F}_2^r (i = 1, \dots, m)$  are not linearly dependent.  
return "FAIL"
- (13) compute  $\mu_1, \dots, \mu_m \in \{0, 1\}, k_1, \dots, k_r \in \mathbb{N}_0$  such that  $\sum_{i=1}^m \mu_i e_{t_i} = 2 \cdot (k_1, \dots, k_r)$
- (14) set  $x := \prod_{i=1}^m (t_i + \lfloor \sqrt{n} \rfloor)^{\mu_i} \pmod{n}$   
 $y := \prod_{j=1}^r p_j^{k_j} \pmod{n}$  // Now  $x^2 \equiv y^2 \pmod{n}$

(15) if  $\gcd(n, x - y)$  or  $\gcd(n, x + y)$  is a non-trivial divisor  
     return the non-trivial divisor  
 else  
     return "FAIL"

With good heuristics it will almost certainly never return FAIL.

**Example:**  $n = 20437$

Choose  $B := 10$      $s := 3$

Factor basis:  $p_1 = -1$     $p_2 = 2$     $p_3 = 3$     $p_4 = 7$

(5 omitted as:  $n \equiv 2 \pmod{5}$  non-square)

$\lfloor \sqrt{n} \rfloor = 142$

Solve  $(t + 142)^2 \equiv n \pmod{p_j^e}$

$p_2 = 2$  : Compute modulo 2,4,8.  $n \equiv 5 \pmod{8}$

$t$  odd  $\Rightarrow (t + 142)^2 \equiv 1 \pmod{8} \Rightarrow (t + 142)^2 \equiv n \pmod{4}$  but not  $\pmod{8}$

$t$  even  $\Rightarrow (t + 142)^2 \equiv 0 \pmod{2} \not\equiv n \pmod{2}$

$$\Rightarrow e_{t,2} = \begin{cases} 2 & t \text{ odd} \\ 0 & t \text{ even} \end{cases}$$

$p_3 = 2$  :  $n \equiv 1 \pmod{3}$     $\lfloor \sqrt{n} \rfloor \equiv 1 \pmod{3}$

So  $3 \mid f(t) \Leftrightarrow t + 1 \equiv \pm 1 \pmod{3} \Leftrightarrow t \equiv 0 \text{ or } 1 \pmod{3}$

$e = 2$     $n \equiv 7 \equiv (\pm 4)^2 \pmod{9}$     $\lfloor \sqrt{n} \rfloor \equiv 7 \pmod{9}$

So  $9 \mid f(t) \Leftrightarrow t + 7 \equiv \pm 4 \pmod{9} \Leftrightarrow t \equiv -3, -2 \pmod{9}$

$p_4 = 7$     $n \equiv 4 \pmod{7}$     $4 = (\pm 2)^2$     $\lfloor \sqrt{n} \rfloor \equiv 2 \pmod{7}$

So  $7 \mid f(t) \Leftrightarrow t + 2 \equiv \pm 2 \pmod{7} \Leftrightarrow t \equiv 0 \text{ or } 3 \pmod{7}$

$t$	-3	-2	-1	0	1	2	3
$f_t =  f(t) $	1116	837	556	273	12	295	588
$p_1$ component of $e_t$	1	1	1	1	0	0	0
$p_2$ component	2	0	2	0	2	0	2
$f_t$ divided by 2-power	279	837	139	273	3	299	147
$p_3$ component	2	2	0	1	1	0	1
$f_t$	31	93	139	91	1	299	49
$p_4$ component	0	0	0	1	0	0	2
$f_t$	31	93	139	13	1	299	1

Obtain  $m = 2$  :  $t_1 = 1$     $t_2 = 3$     $e_1 = (0, 2, 1, 0)$     $e_3 = (0, 2, 1, 2)$

They are lineary dependent  $\pmod{2}$

$e_1 + e_3 = 2 \cdot (0, 2, 1, 1)$

$x = (142 + 1) \cdot (142 + 3) \equiv 298 \pmod{n}$

$y = p_2^2 \cdot p_3 \cdot p_4 = 2^2 \cdot 3 \cdot 7 = 84$

$\gcd(n, x - y) = \gcd(n, 214) = 107$

$\gcd(n, x + y) = 191$

Indeed  $n = 107 \cdot 191$

### Computing square roots (mod $p^e$ )

#### Case 1: $p$ odd

Find  $x$  with  $x^2 \equiv n \pmod{p}$  by trying  $x \pmod{p}$  (exactly two solutions). Suppose we have found  $x$  with  $x^2 \equiv n \pmod{p^e}$

So  $x^2 - n = p^e \cdot r \quad r \in \mathbb{Z}$

New  $x$  should be  $x + y \cdot p^e$

Compute modulo  $p^{e+1}$ :  $(x + y \cdot p^e)^2 - n = x^2 + 2yxp^e + y^2p^{2e} - n \equiv p^e \cdot (r + 2xy) \pmod{p^{e+1}}$

So  $(x + y \cdot p^e)^2 \equiv n \pmod{p^{e+1}} \Leftrightarrow 2xy \equiv -r \pmod{p}$  uniquely and easily solvable

$\rightarrow$  Obtain two solutions (mod  $p^e$ )

$\Rightarrow$  special case of "Hensel lifting"

#### Case 2: $p = 2$

Find  $x \in \mathbb{Z}$  with  $x^2 \equiv n \pmod{8}$  (0 or 4 solutions since  $n$  odd)

Assume we have  $x^2 \equiv n \pmod{2^e} \quad e \geq 3$

So  $x^2 - n = r \cdot 2^e$

$\Rightarrow (x + y \cdot 2^{e-1})^2 - n = x^2 + xy \cdot 2^e + y^2 2^{2e-2} - n \equiv 2^e(r + xy) \pmod{2^{e+1}}$

So  $(x + y \cdot 2^{e-1})^2 \equiv n \pmod{2^{e+1}} \Leftrightarrow y \equiv r \pmod{2}$

$\rightarrow$  0 or 4 solutions

### Running time of quadratic sieve

Choose  $B \approx \exp\left(\sqrt{\frac{1}{2} \ln(n) \cdot \ln(\ln(n))}\right)$

If  $s \approx B$  then running time is:  $O\left(\exp\left(\sqrt{\ln(n) \cdot \ln(\ln(n))}\right)\right)$

which is "slightly" sub-exponential

### Factorization algorithm with best complexity (known to date):

Number field sieve

This also uses ideas 1 and 2, but an algebraic number field is used for generating  $B$ -numbers.

Heuristic Running time (modulo some conjectures):  $O\left(\exp\left(\ln(n)^{\frac{1}{3}} \cdot \ln(\ln(n))\right)^{\frac{2}{3}}\right)$



## 2 Systems of equations

### 2.6 Linear Algebra

Tasks:

- solving systems of linear equations (= linear systems)
- inversions of matrices
- rank determination
- determinants
- matrix products

$K$  field,  $K^{m \times n}$  = set of  $n \times n$  matrices

$GL_n(K)$  ...

Count the cost of algorithms in terms of field operations. If  $K$  is a finite field this translates directly to bit operations.

#### 2.6.1 Proposition 1 (Complexity of usual algorithms)

- (a) Solving an  $m \times n$ -linear system by Gaussian elimination requires  $O(\max\{m, n\}^3)$  field operations
- (b) For  $A \in GL_n(K)$  computing  $A^{-1}$  by usual method requires  $O(n^3)$  field operations.
- (c) Computing  $\det(A)$  "as usual" requires  $O(n^3)$  bit operations.
- (d) Computing  $A \cdot B$  for  $A \in K^{m \times n}$   $B \in K^{n \times l}$  requires  $O(m \cdot n \cdot l)$  field operations.

→ all cubic!

**Proof:**

- (a) Cost of treating the  $k$ -th row with Gauss algorithm:
  - $\leq 1$  inversion,  $\leq (n - k)$  multiplications
  - $\leq (m - k)(n - k)$  multiplications and additions
  - (clearing column below pivot element)
  - Back substitution (i.e. clearing columns above pivot element):
  - Let  $r = rk(A) \leq (k - 1)(n - r)$  multiplications and additions
  - Total cost  $\leq \sum_{k=1}^r (1 + n - k + 2(m - k)(n - k) + 2(k - 1)(n - r))$
  - $= 2mnr - mr^2 - \frac{1}{3}r^3 - nr + \frac{3}{2}r^2 + \frac{5}{6}r - mr$
  - $\in O(\max\{m, n\}^3)$
- (b) Inversion is Gaussian elimination of  $n \times 2n$ -matrix of rank  $n$   
cost  $\leq \frac{8}{3}n^3 - \frac{3}{2}n^2 + \frac{5}{6}n \in O(n^3)$
- (c) reduced to (a)

(d) obvious

### Strassen-multiplication

let  $A, B \in K^{2n \times 2n}$  Write:  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$  with  $A_{ij} B_{ij} \in K^{n \times n}$

Then  $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$  with  $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} \rightarrow 8$  multiplications.

Set:

$$M_1 := (A_{12} - A_{22})(B_{21} - B_{22})$$

$$M_2 := (A_{11} + A_{22})(B_{11} + B_{22})$$

$$M_3 := (A_{11} - A_{21})(B_{11} + B_{12})$$

$$M_4 := (A_{11} + A_{12})B_{22}$$

$$M_5 := A_{11}(B_{12} - B_{22})$$

$$M_6 := (A_{22}(B_{21} - B_{11}))$$

$$M_7 := (A_{21} - A_{22})B_{11}$$

Then:

$$C_{11} = M_1 + M_2 - M_4 + M_6, C_{12} = M_4 + M_5$$

$$C_{21} = M_6 + M_7$$

$$C_{22} = M_2 - M_3 + M_5 - M_7$$

$\rightarrow 7$  Multiplications!

### 2.6.2 Algorithm 2 (Strassen-multiplication)

Input :  $A \in K^{m \times n}, B \in K^{n \times l}$

Output:  $A, B \in K^{m \times l}$

- (1) Let  $k$  be minimal such that  $m, n, l \leq 2^k$
- (2) if  $(k = 0)$   $//(\Leftrightarrow A, B \in K^{1 \times 1})$   
return  $A \cdot B$
- (3) Enlarge  $A, B$  by adding zeros such that  $A, B \in K^{2^k \times 2^k}$
- (4) write  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$  with  $A_{ij}B_{ij} \in K^{2^{k-1} \times 2^{k-1}}$
- (5) compute  $M_1 \dots M_7$  as above, do multiplications by recursive call
- (6) compute  $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$  by above formulas
- (7) Output: the upper left  $m \times l$  - part of  $A \cdot B$

### 2.6.3 Theorem 3 (Running time of Algorithm 2)

If  $m, n, l \leq r$  Algorithm 3 requires  $O(r^{\lg(7)})$  field operations

**Proof:**

Set  $\Theta(k)$  = number of field operations.

Step 5:  $7 \cdot \Theta(k-1) + 10 \cdot (2^{k-1})^2$

Step 6:  $8 \cdot (2^{k-1})^2$

Obtain:

$$\Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1} \quad (*)$$

Claim:  $\Theta(k) = 7^{k+1} - 6 \cdot 4^k$

Induction on  $k$

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$k-1 \rightarrow k : \Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1}$

$= 7(7^k - 6 \cdot 4^{k-1}) + 18 \cdot 4^{k-1}$

*induction*

$= 7^{k+1} - 4 \cdot 6 \cdot 4^{k-1} \quad \checkmark$

Have  $2^{k-1} < r \Rightarrow k < \lg(r) + 1 \Rightarrow \Theta(k) < 7^{\lg(r)+2} = 49 \cdot 2^{\lg(7) \cdot \lg(r)} = 49^{\lg(17)} \quad \checkmark$

**Remarks:**

(a)  $\lg(7) = 2.8074\dots$

(b) Coppersmith-Winograd:  $O(r^{2.3754\dots})$

Improved by Stothes (2010), Williams(2011), LeGall(2014):  $O(r^{2.3729\dots})$

(c) The cost of the best possible algorithm is unknown, even for  $r = 3$

Let  $M : \mathbb{N}_{>0} \mapsto \mathbb{R}_{>0}$  be a function such that two matrices in  $K^{n \times n}$  can be multiplied in  $\leq M(n)$  field operations. Assume  $\exists \epsilon > 0 : \forall n :$

$$2^{2+\epsilon} M(n) \leq M(2n) \leq 8 \cdot M(n) \quad (1)$$

**Example:**  $M(n) = 49 \cdot n^{\lg(7)}$

Recall:  $A = (a_{ij})$  is upper (lower) triangular  $\Leftrightarrow a_{ij} = 0$  for  $i > j$  ( $i < j$ )

### 2.6.4 Proposition 4 (Complexity of matrix inversion)

An upper of lower triangular matrix  $A \in GL_n(K)$  can be inverted in  $O(M(n))$  field operations.

**Proof:**

Let  $k \in \mathbb{N}$  be minimal such that  $n \leq 2^k$

write  $B = \begin{pmatrix} A & 0 \\ 0 & I_{2^k-n} \end{pmatrix} \in GL_{2^k}(K) \Rightarrow B^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & I_{2^k-n} \end{pmatrix}$

Assume  $B$  upper triangular:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \quad B_{11}, B_{22} \in GL_{2^{k-1}}(K), B_{12} \in K^{2^{k-1} \times 2^{k-1}}$$

$$B^{-1} = \begin{pmatrix} B_{11}^{-1} & -B_{11}^{-1} \cdot B_{12} \cdot B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{pmatrix}$$

Let  $\Theta(k)$  = computation cost depending on  $k$ .

$$\Theta(k) \leq 2 \cdot \Theta(k-1) + 2M(2^{k-1}) \stackrel{(1)}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2} \cdot M(2^k) \quad (**)$$

Claim:  $\Theta(k) \leq 2^k + M(2^k)$

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$$k-1 \rightarrow k : \Theta(k) \stackrel{**}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2}M(2^k) \stackrel{induction}{\leq} 2(2^{k-1} + M(2^{k-1})) + \frac{1}{2}M(2^k) \stackrel{(1)}{\leq}$$

$$2^k + \frac{1}{2}M(2^k) + \frac{1}{2}M(2^k) \quad \checkmark$$

$$\text{Have } n > 2^{k-1} \Rightarrow k < \lg(n) + 1 \Rightarrow \Theta(k) < 2 \cdot n + M(2n) \stackrel{(1)}{\leq} 2 \cdot n + 8 \cdot M(n) \quad \square$$

**Project:** Reduce (most) tasks of linear algebra to multiplication.

The following algorithm transforms a matrix such that all tasks become easy.

### 2.6.5 Algorithm 5 (Transforming a matrix)

Input :  $A \in K^{m \times n}$

Output: Matrices  $L, Q, P, U$  such that:  $LQAP = \begin{bmatrix} U \\ 0 \end{bmatrix} \begin{matrix} r \leftarrow \text{in row echelon form} \\ \text{m-r} \end{matrix} \in K^{m \times n}$

such that:

- $L \in K^{m \times m}$  lower triangular with 1's on the diagonal
- $Q \in K^{m \times m} \quad P \in K^{n \times n}$  permutation matrices
- $U \in K^{m \times m}$  upper triangular with non-zero diagonal entries ( $r = 0$  if  $A = 0$ )
- If  $r = m$  then  $Q = I_m$

(1) if  $(A = (...))$   
return  $L = Q = (1) \quad P = I_n \quad r = 0$

(2) if  $(A = a_1, \dots, a_n)$   
let  $i$  be minimal with  $a_i \neq 0$   
 $P :=$  matrix exchanging 1st and  $i$ -th position in  $A$   
return  $L = Q = (1) \quad P \quad U = A \cdot P$

(3) let  $m_1 = \lfloor \frac{m}{2} \rfloor \quad m_2 = \lceil \frac{m}{2} \rceil$   
write  $A = \begin{bmatrix} B \\ C \end{bmatrix} \begin{matrix} m_1 \\ m_2 \end{matrix} \quad B \in K^{m_1 \times n} \quad C \in K^{m_2 \times n}$

(4) Applying the algorithm recursively on  $B$

$$\text{obtain } L, Q, B, P = \begin{bmatrix} U_1 \\ Q \end{bmatrix} \begin{matrix} r_1 \\ m_1 - r_1 \end{matrix} \quad \text{with } U_1 \in K^{r_1 \times n}$$

$$(5) \text{ write } L_1 = \begin{bmatrix} L_t & 0 \\ L_l & L_r \end{bmatrix} \begin{matrix} r_1 \\ m_1 - r_1 \end{matrix} \quad Q_1 = \begin{bmatrix} Q_t \\ Q_b \end{bmatrix} \begin{matrix} r_1 \\ m_1 - r_1 \end{matrix} \quad U_1 = \begin{bmatrix} E & U'_1 \end{bmatrix} \begin{matrix} r_1 \\ m_1 - r_1 \end{matrix}$$

$$\text{Form } D := C \cdot P_1 = \begin{bmatrix} F & D' \end{bmatrix} \begin{matrix} m_2 \\ m_1 - r_1 \end{matrix} \quad \text{and } G := D' - FE^{-1}U' \in K^{m_2 \times (n-r_1)}$$

$$(6) \text{ Apply algorithm recursively to } G := L_2 \cdot Q_2 \cdot G \cdot P_2 = \begin{bmatrix} U_2 \\ 0 \end{bmatrix} \begin{matrix} r_2 \\ m_2 - r_2 \end{matrix}$$

(7) return

$$L := \begin{bmatrix} L_t & 0 & 0 \\ -L_2 Q_2 F E^{-1} L_t & L_2 & 0 \\ L_l & 0 & L_r \end{bmatrix} \begin{matrix} r_1 \\ m_2 \\ m_1 - r_1 \end{matrix}$$

$$Q := \begin{bmatrix} Q_t & 0 \\ 0 & Q_2 \\ Q_b & 0 \end{bmatrix} \begin{matrix} r_1 \\ m_2 \\ m_1 - r_1 \end{matrix}$$

$$P := P_1 \cdot \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} \begin{matrix} r_1 \\ n - r_1 \end{matrix}$$

$$U := \begin{bmatrix} E & U'_1 P_2 \\ 0 & U_2 \end{bmatrix} \begin{matrix} r - 1 \\ r_2 \end{matrix}$$

### 2.6.6 Theorem 6 (Correctness and running time of Algorithm 5)

Algorithm 5 is correct and requires  $O((\frac{n}{m} + 1) \cdot M(m))$  field operations

**Proof:**

Correctness by induction on  $m$

$m = 1$  ✓

$m > 1$  :

$m_1, m_2 < m$  so recursive calls are correct by induction.

By step (7)  $L, Q, P, U$  have desired form.

Compute:

$$LQAP = \begin{bmatrix} L_t Q_t & 0 \\ -L_2 Q_2 F E^{-1} L_t Q_t & L_2 Q_2 \\ L_l Q_t + L_r Q_b & 0 \end{bmatrix} \begin{matrix} m_1 \\ m_2 \\ m_1 - r_1 \end{matrix} \cdot \begin{bmatrix} B \cdot P_1 \\ D \end{bmatrix} \begin{matrix} m_1 \\ m_2 \end{matrix} \cdot \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} \begin{matrix} r_1 \\ n - r_2 \end{matrix}$$

$$\stackrel{(4)}{=} \begin{bmatrix} U_1 \\ L_2 Q_2 (-F E^{-1} U_1 + D) \\ 0 \end{bmatrix} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} \begin{matrix} r_1 \\ n - r_2 \end{matrix}$$

$$= \begin{array}{c|c} r_1 & n-r_1 \\ \hline E & U'_1 \\ 0 & L_2 Q_2 G \\ 0 & 0 \end{array} \begin{array}{c} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \cdot \begin{array}{c|c} I_{r_1} & 0 \\ \hline 0 & P_2 \end{array} \stackrel{(6)}{=} \begin{array}{c|c} r_1 & n-r_1 \\ \hline E & U'_1 P_2 \\ 0 & U_2 \\ 0 & 0 \\ 0 & 0 \end{array} \begin{array}{c} r_1 \\ r_2 \\ m_2 \cdot r_2 \\ m_2 \cdot r \end{array} = \begin{array}{c|c} U \\ \hline 0 \end{array}$$

Suppose  $r = m \Rightarrow r_1 = m_1 \quad r_2 = m_2 \Rightarrow Q_1 = I_{m_2} \quad Q_2 = I_{m_2} \Rightarrow Q = I_m$

**Cost:**

Fix  $n \in \mathbb{N}$  and set  $\Theta(k) :=$  maximal cost for a matrix  $A \in K^{m \times n'}$  with  $m \leq 2^k \quad n' \leq n$

Choose  $A \in K^{m \times n'}$  with cost  $= \Theta(k)$

Step(4) and (6):  $\leq \Theta(k-1)$  each

Step(5):  $E^{-1}$ : by Proposition 4:  $O(M(r_1)) \leq O(M(2^{k-1}))$

$F \cdot E^{-1} \leq M(2^{k-1})$

$F \cdot E^{-1} \cdot U'$ : at most the cost of multiplying a  $2^{k-1} \times 2^{k-1}$ -matrix by a  $2^{k-1} \times n$  matrix.

Split right matrix into square parts.

$\Rightarrow$  cost  $\leq \lceil \frac{n}{2^{k-1}} \rceil \cdot M(2^{k-1}) \leq (2^{1-k} \cdot n + 1) \cdot M(2^{k-1})$

$G$ : subtraction:  $m_2 \cdot (n - r_1) \leq 2^{k-1} \cdot n \leq 2^{1-k} \cdot n \cdot M(2^{k-1})$

Step (7):  $F \cdot E^{-1}$  already computed,  $L_2 Q_2$ : permuting rows. Cost:  $\leq 2 \cdot M(2^{k-1})$

Obtain:  $\Theta(k) \leq 2 \cdot \Theta(k-1) + (2^{-k} \cdot n + c) \cdot M(2^k) \quad c \text{ constant}$

From this obtain by induction:

$$\Theta(k) \leq \left( 2^{-k} \cdot n \cdot \frac{1-2^{-k\epsilon}}{1-2^{-\epsilon}} + 2 \cdot c \cdot (1 - 2^{-k}) \right) \cdot M(2^k) \leq \left( \frac{1}{1-2^{-\epsilon}} \cdot \frac{n}{2^k} + 2c \right) \cdot M(2^l)$$

Finally obtain: Cost  $\leq 8 \cdot \max \left\{ \frac{1}{1-2^{-\epsilon}} \cdot c \right\} \cdot \left( \frac{n}{m} + 1 \right) \cdot M(m) \quad \square$

$\begin{array}{c|c} U \\ \hline 0 \end{array}$  is in row echelon form. It's convenient to write  $U = \begin{array}{c|c} E & U' \end{array} r \quad U' \in K^{r \times (n-r)}$

Also write  $L = \begin{array}{c|c} m \\ \hline L_1 \\ \hline L_2 \end{array} r \quad m-r$

**2.6.7 Theorem 7**

Let  $A \in K^{m \times n}$ ,  $LQAP = \begin{array}{c|c} E & U' \\ \hline 0 \end{array} \begin{array}{c} r \\ m-r \end{array}$  as in Algorithm 5 then

(a)  $rk(A) = r$

(b) The columns of  $P \cdot \begin{array}{c|c} E^{-1} \cdot U' \\ \hline -I_{n-r} \end{array}$  form a basis of  $\ker(A)$

(c) A linear system  $Ax = b \quad b \in K^m$  is solvable iff  $L_2 Q \cdot b = 0$

(d) if  $Ax = b$  is solvable then  $x = P \cdot \begin{array}{c|c} E^{-1} L_1 \\ \hline 0 \end{array} \begin{array}{c} r \\ n-r \end{array} \cdot Q \cdot P$  is a solution

$$\begin{aligned}
\text{(e) if } A \in GL_n(K) \text{ then } \det(A) &= \det(P) \cdot \underbrace{\det(E)}_{=\text{prod of diags}} \\
\text{and } A^{-1} &= P \cdot E^{-1} \cdot L
\end{aligned}$$

## 3 Notes

### 3.1 Notation

- $\mathbb{N} := \mathbb{N}_0$
- $\lg(x) := \log_2(x)$
- $a \mid b$        $a$  is divisible by  $b$        $\Leftrightarrow$        $b \bmod a = 0$   
 $a \nmid b$        $a$  is not divisible by  $b$   $\Leftrightarrow$        $b \bmod a \neq 0$
- $ord(a)$       order of a group element  
 $n > 0$  minimal such that  $a^n = e$       with neutral element  $e$   
if no such  $n$  can be found,  $ord(a) = \infty$
- $char(A)$       Characteristic: the smallest positive  $n$  such that  
 $\underbrace{1 + \dots + 1}_n = 0$       with 1 as the multiplicative identity element  
 $n$  summands
- $\mathbb{Z}/(m)$       Ring modulo  $m$   
polynomial rings measure for " $<$ " relations not the absolute value but max power.
- $lcm(a_1, \dots, a_n)$       "least common multiple of all  $a_i$ "
- $\underline{e}$  = vector of e's
- $\phi(n) := |\{x \in \mathbb{N} : x < n \wedge \gcd(x, n) = 1\}| = |(\mathbb{Z}/(n))^x|$   
Euler's totient function
- $rk(A)$       Rank of matrix A

### 3.2 Various stuff

- Lagrange's theorem  
Every element in a finite group has finite order
- Average number of bit operations for an increment:  
One operation for the last bit + 50% chance for one on the next bit + 25% on the following etc.  $\Rightarrow$  Geometrical row  
 $\Rightarrow$  on average two bit operations
- "Monte Carlo Algorithm"  
Always terminates in reasonable time but might yield false result.
- "Las Vegas Algorithm"  
If it terminates the result is correct. No deterministic running time.
- Chinese remainder theorem  
Given a system of congruences  $x \equiv a_i \pmod{m_i}$  with  $i = 1, \dots, r$   
 $m_i$  pairwise co-prime. Then the unique solution is:  
$$x \equiv a_1 \cdot b \cdot \frac{N}{m_1} + \dots + a_r \cdot b_r \cdot \frac{N}{m_r} \pmod{N} \quad \text{with } b_i \cdot \frac{N}{m_i} \equiv 1 \pmod{m_i}$$
- distance between two square numbers:  
 $(n+1)^2 - n^2 = 2n+1$   
 $\Rightarrow$  Squares are much more scarce than primes!



- $ax + by = c$  has solutions in  $\mathbb{Z}$  iff  $\Leftrightarrow \gcd(a, b) \mid c$  with  $a, x, b, y \in \mathbb{Z}$

### 3.3 Algebraic structures

- Group  $(G, *)$ 
  - one inner operation  $(*)$ :  $G \times G \mapsto G$
  - associativity:  $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
  - neutral element  $e \in G$ :  $a * e = e * a = a \quad \forall a \in G$
  - inverse element  $a^{-1} \in G$ :  $a * a^{-1} = a^{-1} * a = e \quad \forall a \in G$
- Abelian group  $(G, *)$ 
  - $(G, *)$  is a group
  - commutativity:  $a * b = b * a \quad \forall a, b \in G$
- Finite group  $(G, *)$ 
  - associativity:  $(a * b) * c = a * (b * c)$
  - unambiguity of reduction:  $(a * x = a * x') \wedge (x * a = x' * a) \Rightarrow x = x'$   
 $\Rightarrow x \mapsto x * a$  and  $x \mapsto a * x$  is bijective  
 $\Rightarrow \exists x : a * x = a \Rightarrow$  neutral element  
 $\exists x : a * x = x \Rightarrow$  inverse element
- Cyclic group  $(G, *)$ 
  - $G$  is a group
  - $G$  is generated by one Element:  $G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$
  - not necessarily finite.
- Semi group  $(S, *)$ 
  - one inner operation  $(*)$ :  $S \times S \mapsto S$
  - associativity:  $(a * b) * c = a * (b * c) \quad \forall a, b, c \in S$
- Field  $(K, +, \cdot)$ 
  - two inner operations  $(+, \cdot)$  such that:
    - $(K, +)$  is an abelian group with neutral element 0
    - $(K \setminus \{0\}, \cdot)$  is an abelian group with neutral element 1
  - distributivity:  $a \cdot (b + c) = a \cdot b + a \cdot c$   
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in K$
- Ring  $(R, +, \cdot)$ 
  - $(R, +)$  is an abelian group
  - $(R, \cdot)$  is a semi group
  - distributivity:  $a \cdot (b + c) = a \cdot b + a \cdot c$   
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in R$
- Commutative ring  $(R, +, \cdot)$ 
  - $(R, +, \cdot)$  is a ring
  - commutativity for  $(\cdot)$   $a \cdot b = b \cdot a \quad \forall a, b \in R$
- Unitary ring (ring with 1)  $(R, +, \cdot)$ 
  - $(R, \cdot)$  is a semi group
  - $(R, \cdot)$  has a neutral element "1"
- Euclidean ring  $R$ 
  - $\exists F : R \mapsto \mathbb{N}_0 \cup \{0\}$

such that if  $\exists q, r \in R \quad a = b \cdot q + r \quad \text{and} \quad r = 0 \quad \text{or} \quad a, b \in R \quad F(r) < F(b)$

- Polynomial ring  $R[X]$ 
  - $R$  is a commutative unitary ring
  - set of all polynomials with coefficients  $\in R$

### 3.4 Invertible elements

- Let  $(\mathbb{Z}/(n), +)$  be a group or  $(\mathbb{Z}/(n))^\times$  be a group with multiplication.
- $|(\mathbb{Z}/(n))^\times| = \phi(n)$
- $n \in \mathbb{P}$ 
  - $\Rightarrow (\mathbb{Z}/(n))^\times = \{\bar{0}, \dots, \bar{p-1}\} \cong (\mathbb{Z}/(p-1), +) = Z_{p-1}$  (cyclic Group  $Z$ )
- $n$  is a power of 2
  - $\Rightarrow (\mathbb{Z}/(2^e))^\times \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2^{e-2})$
- $n$  is a power of an odd Prime
  - $\Rightarrow (\mathbb{Z}/(p^k))^\times \cong \mathbb{Z}/(p^{k-1} \cdot (p-1)) \cong Z_{(p^{k-1} \cdot (p-1))}$
- $n = p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$ 
  - $\Rightarrow (\mathbb{Z}/(n))^\times \cong (\mathbb{Z}/(p_1^{k_1}))^\times \times \dots \times (\mathbb{Z}/(p_r^{k_r}))^\times$