Computational Algebra
-
Transcript

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Summer term 2015

Contents

1 Integer Arithmetic

Topics:

- Addition and Multiplication
- GCD computation
- Primality testing
- Factorization

1.1 Addition and Multiplication

Agreement:

- $a, x \in \mathbb{N}$ represented as $x = \sum_{i=0}^{n-1} a_i \cdot B^i \quad B \in \mathbb{N}_{>1}$ fixed Base ($a_i \in \{0,, B-1\}$)
- if $x \neq 0$, assume $a_{n-1} \neq 0$ then define:
  length of $x := l(x) = n$ = number of digits = $\lfloor \log_B(x) \rfloor + 1$
  (mnemonic: $\log_B(B) + 1 = 2$)
- $l(0) = 1$
  (Amount of memory required to store $x = 0$)
- $l(x) := l(|x|)$
- for $x \in \mathbb{Z}$ represent it as $x = sgn(x) * |x|$

1.1.1 Algorithm 1 (Simple addition)

input : $x = \sum_{i=0}^{n-1} a_i \cdot B^i, \quad y = \sum_{i=0}^{n-1} b_i \cdot B^i, \quad x, y \in \mathbb{N}$

output: $x + y = \sum_{i=0}^{n} c_i \cdot B^i$

(1) $\sigma = 0$

(2) for $i = 0, ..., (n-1)$:

(3) set $c_i := a_i + b_i + \sigma_i$
$\sigma := 0$

(4) if ($c_i \geq B$)

(5) set $c_i = c_i - B$

(6) set $\sigma = 1$

(7) set $c_n = \sigma$

If $B = 2$ then (3) - (6) can be realized by logic gates:

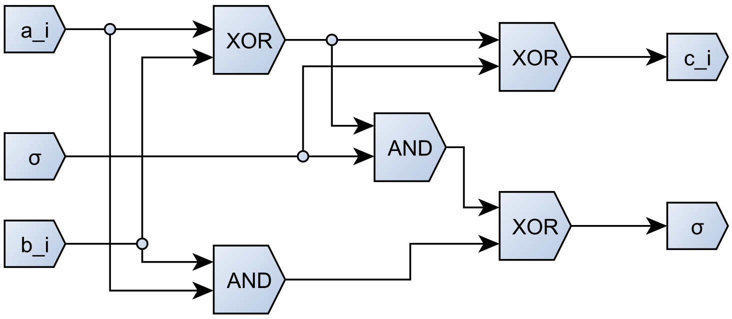


Figure 1: Logic circuit for addition

1.1.2 Definition 2 (Bit-Operation)

A bit operation is an operation that can be performed by a logic gate or by searching or writing a bit from / into memory.

1.1.3 Definition 3 (Big O)

Let M be a set (usually $M = \mathbb{N}$), $\quad f, g : M \mapsto \mathbb{R}_{>}0$
we write $f \in O(g) \quad$ if $\quad \exists c \in \mathbb{R} : f(x) \leq c \cdot g(x) \forall x \in M$

1.1.4 Theorem 4 (Lower bound for addition)

Let $f : \mathbb{N} \mapsto \mathbb{R}, \quad n \mapsto$ maximal number of bit operations required by Algorithm 1 to add $x_y \in \mathbb{N}$ with $l(x), l(y) \leq n$
Let $g = id_{\mathbb{N}}$ Then $f \in O(g)$
We say Algorithm 1 requires $O(n)$ bit operations for adding two numbers of length $\leq n$.
$\Rightarrow$ "linear complexity"
Set $M := \{$Set of all algorithms for addition in $\mathbb{N}\}$
For $A \in M$ define $f_A : \mathbb{N} \mapsto \mathbb{R}$ as above.
We would like to find $f_{odd} : \mathbb{N} \mapsto \mathbb{R}, \quad n \mapsto \inf\{f_A(n) | A \in M\}$
Since one needs to read x, y (and write the result) we can not do any better than linear complexity for addition.
Subtraction
let x, y as Algorithm 1, $x \geq y$
For $\bar{y} := \sum_{i=0}^{n-1} (B - 1 - b_i) B^i \qquad$ (digitwise / bitwise complement)
$\Rightarrow x + \bar{y} = x - y + B^n - 1$
$\Rightarrow x - y = x + \bar{y} + 1 - B^n \qquad$ (initially set $\sigma = 1$)
Conclusion: Addition and Subtraction have cost $O(n)$

1.1.5 Algorithm 5 (Multiplication by "grid method")

input : $x = \sum_{i=0}^{n-1} a_i \cdot 2^i, \quad y = \sum_{i=0}^{m-1} b_i \cdot 2^i$

output: $z = x \cdot y$

(1) $z := 0$

(2) for $i = 0, ..., (n-1)$

(3) if $(a_i \neq 0)$

set $z := z + \sum_{j=1}^{m-1} b_j 2^{i+j}$

1.1.6 Theorem 6 (Runtime of Algorithm 5)

Algorithm 5 requires $O(n \cdot m)$ bit operations.
As of the total input length $n + m$:
$n \cdot m \leq \frac{1}{2}(n+m)^2 \rightarrow O((n+m)^2)$
$\Rightarrow$ Quadratic complexity

Karatsuba-multiplication:
Observation for polynomials:
$a + bx, c + dx$ have $(a+bx)(c+dx) = ac + (ac + db - (a-b)(c-d))x + bdx^2$
The point: only used 3 multiplications instead of 4.
Specialize $x = B$ "large" such that $x = a + bB$ partition into two blocks. Then multiply the blocks by a recursive call.

1.1.7 Algorithm 7 (Karatsuba)

input : $x, y \in \mathbb{N}$

output: $z = x \cdot y$

(1) Choose $k \in \mathbb{N}$ minimal such that $l(x), l(y) \leq 2^k$.
Set $B = 2^{2^{k-1}}$

(2) if $(k = 0)$
return $x \cdot y$ (by bit-operation AND)

(3) write $x = x_0 + x_1 B$
$y = y_0 + y_1 B$ with $l(x_i), l(y_i) \leq 2^{k-1}$

(4) compute $x_0 \cdot y_0,\ x_1 \cdot y_1,\ (x_0 - x_1) \cdot (y_0 - y_1)$ by a recursive call

(5) return $z = x_0 y_0 + (x_0 y_0 + x_1 y_1 - (x_0 - x_1)(y_0 - y_1))B + x_1 y_1 B^2$

1.1.8 Theorem 8 (Runtime of Algorithm 7)

For multiplying two numbers of length $\leq n$ Algorithm 7 requires $O(n^{\log_2 3}) \approx O(n^{1.59})$ bit operations.

Proof:

Set $\Theta(k) :=$ maximal numbers of bit operations for $l(x), l(y) \leq 2^k$

We have for $k > 0 : \Theta(k) \leq 3\Theta\underbrace{(k-1)}_{\text{recursive calls}} + c\underbrace{2^k}_{\text{additions}}$ with (c some constant)

Claim: $\Theta(k) \leq 3^k + 2c(3^k - 2^k)$

Proof by Induction on k:

$k = 0 : \Theta(k) = 1$

$$\begin{aligned} k-1 \to k = \Theta(k) &= 3\Theta(k-1) + c2^{k-1} \\ &\leq 3(3^{k-1} + 2c(3^{k-r} - 2^{k-1})) + c2^k \\ &= 3^k + 2c(3^k - 2^k) \end{aligned}$$

So $\Theta(k) \leq (2c+1)3^k$

Now $l(x) \leq n$ hence $2^{k-1} < n$ by minimality of k

So $k - 1 < \log_2 n$

$$\begin{aligned} \Rightarrow \Theta(k) &\leq 3(2c+1)3^{\log_2(n)} \\ &= 3(2c+1)2^{\log_2(3)\log_2(n)} \\ &= 3(2c+1)n^{\log_2(3)} \qquad \square \end{aligned}$$

One can modify the terminal condition of Karatsuba to switch to Grid-Multiplication, which is faster for small numbers.

Fast-Fourier Transformation

Reminder: For a function $f : \mathbb{R} \mapsto \mathbb{C}$ define:

$\hat{f} : \mathbb{R} \mapsto \mathbb{C}$ by

$\hat{f}(\omega) = \int\limits_{\mathbb{R}} f(t)e^{-i\omega t}dt$ (if it exists)

Think of ω as frequency.

Definition (Convolution)

Let $f, g : \mathbb{R} \mapsto \mathbb{C}$

$(f * g)(x) = \int\limits_{\mathbb{R}} f(t)g(x-t)dt$

Convolution is analogous to polynomial multiplication

Formula: $\underbrace{(f * g)}_{\text{(Cauchy formula)}} = \hat{f} \cdot \hat{g}$

For a function $M \mapsto C$ with $|M| < \infty$ we need the discrete Fourier transform (DFT)

1.1.9 Definition 9 (Root of unity)

Let R be a commutative ring with 1. An element $\mu \in R$ is called an $n-th$ root of unity (= root of 1) if $\mu^n = 1$.
It is called primitive if $\mu^i \neq 1$ for $(0 < i < n)$ i.e. $ord(\mu) = n$

Let μ be a primitive n-th root of 1 (e.g. $e^{2\pi \frac{i}{n}} \quad \in \mathbb{C}$)
Then the map $DFT_\mu : R^n \mapsto R^n$
$(a_0, ..., a_n) \mapsto (\hat{a}_0, ..., \hat{a}_n)$ with $\hat{a}_i = \sum_{j=0}^{n-1} \mu^{ij} a_j$
is called discrete Fourier transformation
For polynomials:
$DFT_\mu : R[x] \mapsto R^n$
$\quad f \mapsto (f(\mu^0), ..., f(\mu^{n-1}))$
Convolution rule: (from $f(\mu^i)g(\mu^i) = (f * g)(\mu^i)$)
$DFT_\mu(f * g) = DFT_\mu(f) \cdot DFT_\mu(g)$ (component wise product)

Addition of two polynomials in R[x] of deg(n) require $O(n)$ ring operations.
Multiplication require $O(n^l)$.
With Karatsuba have $O(n^{\log_2(3)})$ ring operations.
Cost $DFT_\mu(f) \cdot DFT_\mu(g) : O(n)$ ring operations (with μ as 2n-th root of 1)
Want: Cheap way of doing DFT and back-transformation.

1.1.10 Algorithm 10 (Fast Fourier transformation FFT)

input : $f \in R[x]$, $\mu \in R$ primitive 2^k-th root of 1, such that $\mu^{2^{k-1}} = -1$

output: $DFT_\mu(f)$

(1) Write $f(x) = g(x^2) + xh(x^2)$ with $f, g, h \in R[x]$

(2) if $(k = 1)$ $//(\Rightarrow \mu = -1)$
return $DFT_\mu(f) = (g(1) + h(1), g(1) - h(1))$

(3) Recursive call: compute $DFT_{\mu^2}(g) = \hat{g}, DFT_{\mu^2}(h) = \hat{h} \quad \in R^{2^{k-1}}$

(4) return $DFT_\mu(f) = (\hat{f}_0, ..., \hat{f}_{2^k-1})$ with $\hat{f}_i = \hat{g}_i + \mu\hat{h}_i$
where $\hat{g}_i = \hat{g}_{i-2^{k-1}}$ for $i \geq 2^{k-1}$

Note: Components of $\hat{g}$ and $\hat{h}$ are:
$\hat{g} = g(\mu^{2i})$, $\hat{h}_i = h(\mu^{2i})$ so
$\hat{f}_i := f(\mu^i) = \hat{g}_i(\mu^{2i}) + \mu^i h(\mu^{2i}) = \hat{g}_i + \mu^i \hat{h}_i$

Convention: $\lg(x) = \log_2(x)$

1.1.11 Theorem 11 (Runtime of Algorithm 10)

Let $n = 2^k, \quad f \in R[x] \quad$ with $\deg(\psi) < n$
Then Algorithm 10 requires $O(n \cdot \lg(n))$ ring operations.
Better than $O(n^{1+\epsilon}), \quad \forall \epsilon > 0$!
Proof:
Set $\Theta(k)$ = max number of ring operations required. By counting obtain for $k > 1$:
$\Theta(k) \leq 2\Theta(k-1) + \underbrace{(\text{compute } \mu^i (i \leq 2^{k-1}))}_{2^{k-1}} + \underbrace{(\mu^i \hat{k}_i)}_{2^{k-1}} + \underbrace{(\text{sums and differences})}_{2^k}$
$= 2\Theta(k-1) + 2^{k+1}$
Claim: $\Theta(k) \leq (2k-1)2^k$
$k = 1 : f = a_0 + a_1 \cdot x \quad DFT_\mu(f) = (a_0 + a_1 \cdot a_0 - a_1) \Rightarrow \Theta(a) = 2$
$k - 1 \to k : \Theta(k) \leq 2 \cdot \Theta(k-1) + 2^{k+1} \leq 2 \cdot (2k-3) \cdot 2^{k-1} + 2^{k+1} = (2k-1) \cdot 2^k$
since $k = \lg(n)$ obtain $O(k) \leq (2 \cdot \lg(n) - 1) \cdot n \in O(n \cdot \lg(n)) \qquad \square$

Back-transformation?

1.1.12 Definition 12 (Good root of unity)

A primitive n-th root of unity is called good (caveat: this is ad-hoc terminology) if:
$\sum_{j=0}^{n-1} \mu^{ij} = 0 \quad$ for $(0 < i < n)$

Example:

(1) $\mu = e^{2\pi \frac{i}{n}}$ is a good primitive root of unity

(2) $R = \mathbb{Z}/(8), \quad \mu = \bar{3} \Rightarrow \mu \cdot B$ is primitive 2^{nd} root of unity
But $\bar{3}^0 + \bar{3}^1 = \bar{u} \neq \bar{0} \qquad$ so μ is not good.

1.1.13 Proposition 13 ($DFT_{\mu^{-1}}$)

Let $\mu \in R$ be a good root of 1
$(a) = (a_0, ..., a_{n-1}) \in R^n \Rightarrow DFT_{\mu^{-1}}(DFT_\mu(a)) = n \cdot (a) \quad$ where $n = 1 + ... + 1 \in R$
Proof:
$DFT_\mu(a) = (\hat{a}) = (\hat{a}_0, ..., \hat{a}_{n-1})$
with $\hat{a}_j = \sum_{k=0}^{n-1} \mu^{jk} a_k$
$DFT_{\mu^{-1}}(\hat{a}) = (\hat{\hat{a}}_0, ..., \hat{\hat{a}}_1)$

with $\hat{\hat{a}}_i = \sum_{i=0}^{n-1} \mu^{-ij} \sum_{k=0}^{n-1} \mu^{jk} a_k = \sum_{k=0}^{n-1} \left(a_k \cdot \underbrace{\sum_{i=0}^{n-1} \mu^{j(k-i)}}_{=0 \text{ if } n \nmid k-i \ \ (\text{i.e.} k=i)} \right) = a_i \cdot n \qquad \square$

1.1.14 Proposition 14 (Finding good roots of unity)

let $\mu \in R, n \in \mathbb{N}$
Assume:

a) R is an integral Domain and μ is a primitive or n-th root of 1
(Integral Domain: nonzero commutative ring in which the product of two nonzero elements is nonzero)
$\Rightarrow$ Granted by FFT

b) $n = 2^b$, $\mu^{\frac{n}{2}} = -1$, then $k > 0 \wedge char(R) \neq 2$
$\rightarrow \mu$ is a good primitive n-th root of 1 ("root of unity")

Proof:

a) for $0 < i < n$

$$\underbrace{(\mu^i - 1)}_{\neq 0} \underbrace{(\sum_{j=0}^{n-1} \mu^{ij})}_{=0 \ *} = \mu^{in} - 1 = 0$$

$\Rightarrow \mu$ is a good root of unity

b) $\mu^n = 1, \quad n = 2^k \quad \Rightarrow \quad ord(\mu)|n \quad \Rightarrow \quad ord(\mu)$ is power of 2
Since $\mu^{\frac{n}{2}} \neq 1$, this implies $ord(\mu) = n$

* Let $0 < i < n$, write $i = 2^{k-s} \cdot r \quad$ with r odd $\wedge$ $s > 0$

$$\sum_{j=0}^{2^k-1} \mu^{ij} = \sum_{l=0}^{2^{k-s}-1} \sum_{j=0}^{2^s-1} \mu^{i(l \cdot 2^s + j)} \underset{(\mu^{i \cdot 2^s} = 1)}{=} 2^{k-s} \sum_{j=0}^{2^s-1} \mu^{ij} =$$

$$2^{k-s} \sum_{j=0}^{2^{s-1}-1} (\mu^{ij} + \mu^{i(2^{s-1}+j)})$$

But $\mu^{i \cdot 2^{s-1}} = \mu^{2^{k-s} \cdot r \cdot 2^{s-1}} = \mu^{2^{k-1} \cdot r} = (-1)^r = -1$

So $\sum_{j=0}^{n-1} \mu^{ij} = 0 \qquad \square$

1.1.15 Algorithm 15 (Polynomial multiplication using DFT)

input : $f, g \in R[x] \quad$ with $\deg(f) + \deg(g) < 2^k =: n$
$\mu \in R$ as a good root of unity; Assume $2 \in R$ is invertible

output: $h = f \cdot g$

(1) compute $\hat{f} = DFT_\mu(f)$, $\hat{g} = DFT_\mu(g) \quad$ with $f, g \in R^n$

(2) compute $\hat{h} = \hat{f} \cdot \hat{g}$

(3) compute $(h_0, ..., h_{n-1}) = DFT_{\mu^{-1}} \hat{h} \quad$ (same as $DFT_\mu(\hat{h})$ but with different order)
$=$ Back-transformation $\cdot 2^k$
set $h = \frac{1}{2^k} \sum_{i=0}^{n-1} h_i x^i$

1.1.16 Theorem 16 (Runtime of Algorithm 15)

Algorithm 15 uses $O(n \cdot \log(n))$ ring operations for polynomials of deg $< n$
Proof:

- Choose k minimal so that $\deg(f) \cdot \deg(g) < 2^k$
  $\Rightarrow 2^{k-1} \leq 2n \quad \Rightarrow k \leq \log(n) + 2$
- $\underbrace{O(2k \cdot 2^k)}_{\text{Step 1}} + \underbrace{2^k}_{\text{Step 2}} + \underbrace{O(k \cdot 2^k) + 2^k}_{\text{Step 3}} \in O(2k \cdot 2^k) = O(n(g(n)))$ $\square$

Goal: Multiplication in $\mathbb{N}$ using DFT
Idea: find roots of 1 in a suitable $\mathbb{Z}/(m)$
Choose $m = 2^l + 1, \mu = \bar{2} \in R$

1.1.17 Proposition 17 (Add and mul in $O(l)$)

Let $m = 2^l + 1,\ R = \mathbb{Z}/(m) \qquad \Leftrightarrow 2^l \equiv -1$
Addition in R and multiplication by $\bar{2}^i \in R$ $(0 \leq i < 2l)$ can be done in $O(l)$ bit operations
Proof:

Let $\bar{x} \in R \quad$ with $0 \leq x \leq 2^l$

- Addition: $x \bar{+} \bar{y}$
  (1) compute $x + y \in \mathbb{N}: \quad O(l)$
  (2) if $x + y > 2^l + 1 \quad$ subtract $2^l + 1: \quad O(l)$
- Multiplication by $\bar{2}^i \quad (0 \leq i < l)$
  (1) Bit-shift i Bits to the left by relocating in memory:
  $$\underbrace{O(length(i))}_{\text{compute addr. of new first bit}} + \underbrace{l}_{\text{copying}} = O(\log(l)) + l \in O(l)$$
  (2) Subtract bits $\geq 2^l \qquad$ (since $\bar{2}^l = -\bar{1}$) : $O(l)$
  (3) If the result is negative add $2^l + 1 : O(l)$
- Multiplication by $\bar{2}^i \quad (l \leq i < 2l - 1)$
  (1) Multiplication by $\bar{2}^{i-l}: \quad O(l)$
  (2) take negative $\bar{2}^i \cdot \bar{x} = -\bar{2}^{i-l} \cdot \bar{x}: \quad O(l)$

1.1.18 Proposition 18 (Sort of summary)

Let $k, r \in \mathbb{N}, \quad r > 0, \quad m = 2^{2^k \cdot r} + 1, \quad R = \mathbb{Z}/(m), \quad \mu = \bar{2}^r \in R$
$\Rightarrow 2 \in R$ is invertible, μ is a good primitive 2^{k+1}-th root of 1
$\Rightarrow \mu^{2^k} = 1$
Proof: $\rightarrow$ from above

1.1.19 Algorithm 19 (Multiplication using FFT)

input : $x, y \in \mathbb{N}$

output: $Z = x \cdot y$

(1) Choose $k \in \mathbb{N}$ minimal such that $l(x), l(y) \leq 2^{2k}$

(2) if $k \leq 3$, compute $z = x \cdot y$ by Algorithm 5

(3) set $B = 2^{2^k}, \quad m = 2^{2^k \cdot 4} + 1, \quad R = \mathbb{Z}/(m), \quad \mu = \bar{2}^4 \in R$
($\Rightarrow$ so μ is a good primitive 2^{k+1}-th root of 1)

(4) write $x = \sum_{i=0}^{2^k-1} x_i \cdot B^i$, same for y with $(0 \leq x_i, y_i < B)$
possible since $x, y < 2^{2^{2k}} = 2^{2^k \cdot 2^k} = B^{2^k}$

(5) compute: $\hat{x} = DFT_\mu(\bar{x}_0, ..., \bar{x}_{2^k-1}, \underbrace{0, ..., 0}_{2^k \text{zeros}}) \in R^{2^{k+1}}$
same for y
$\rightarrow$ use FFT

(6) compute: $\hat{z} = \hat{x} \cdot \hat{y} \quad \in R^{2^{k+1}}$ (component-wise multiplication)
Perform multiplication in R as follows:
Multiply representatives (non negative and $< m$) by recursive call.
Then reduce modulo m by "negative bit shift" (see proof of Proposition 17)

(7) compute: $(\bar{z}_0, ..., \bar{z}_{2^{k+1}}) = \frac{1}{2^{k+1}} DFT_{\mu^{-1}}(\hat{z}) \quad \in R \quad$ with $0 \leq z < m$

(8) set $z := \sum_{j=0}^{2^{k+1}-1} z_i \cdot B^i$

1.1.20 Theorem 20 (Runtime of Algorithm 19)

Algorithm 19 correctly computes $t = x \cdot y$ and requires $O(n \cdot (\log n)^4)$ bit operations for $l(x), l(y) \leq n$

Proof: Correctness

write $x(t) \sum_{i=0}^{2^k-i} x_i t^i \quad \in \mathbb{Z}[t], \quad y(t), \quad \bar{x}(t) \in R[t], \bar{y}(t), \bar{z}(t)$

by Proposition 18 and Proposition 13 we have $\bar{z}(t) = \bar{x}(t) \cdot \bar{y}(t)$

The l-th coefficient of $x(t) \cdot y(t)$ is $0 \leq \sum_{i+j=l} x_i \cdot y_j < 2^k \cdot B^2 = 2^{k+2 \cdot 2^k} \leq 2^{2^{k+2}} < m$

So $z(t) = x(t) \cdot y(t) \Rightarrow z = z(B) = x(B) \cdot y(B) = x \cdot y$ Cost:

Write $\Theta(k) :=$ max number of bit operations

Analyze Steps:

(1) compute max $\{l(x), l(y)\}: \quad O(l(n)) = O(k)$

(2) $O(1)$

(3) no bit operations

(4) compute starting addresses of x_i, y_i in memory: $2 * 2^k$ increments of the address: $2 \cdot 2 \cdot 2^k = 2^{k+2}$ bit ops
$\Rightarrow O(2^k)$

(5) By Theorem 11 need $O(2 \cdot 2^{k+1} \cdot (k+1))$ operations in R which are additions and multiplications by powers of $\bar{z}$ costing $O(2^{k+2})$ bit operations.
Total for (5): $O(k \cdot 2^{2 \cdot k})$

(6) 2^{k+1} multiplications of numbers $< m$, i.e. of length $\leq 2^{k+2}$.
So $k' \leq \frac{k+3}{2}$ for k': the "new" k used in the next recursion level.
For $\alpha \in R_{>0}$ define $\Theta(\alpha) := \Theta(\lfloor \alpha \rfloor)$
Total for (6): $2^{k+1}(\Theta(\frac{k+3}{2}) + \underbrace{O(2^{k+2})}_{\text{reduction} \pmod m}$

(7) For $DFT_{\mu^{-1}}(\hat{z}) : O(k \cdot 2^{2 \cdot k})$ as (5) Since $\bar{z}$ is a n root of 1, multiplication by $\bar{2}^{-k-1}$ is multiplication by a positive power of $\bar{2}$, which costs $O(2^{k+2})$
Total for (7): $O(k \cdot 2^{2 \cdot k})$

(8) For $j \leq 2^{k+1}$ have $\sum_{i=0}^{j-1} z_i \cdot B^i \leq (m-1) \sum_{i=0}^{j-1} B^i = (m-1)\frac{B^j-1}{B-1} < 2(m-1)\frac{B^j}{B} = 2^{1+2^{k+2}+(j-1)2^k}$ so the sum has length $(j+3) \cdot 2 + 1$
Adding $z_j \cdot B^j$ to this sum happens at $(j \cdot 2^k)$-th bit and higher $\Rightarrow$ cost is $O(2^k)$
Total for (8): $O(2^{2 \cdot k})$

Grad total: For $k \geq 4$:
$\Theta(k) \leq 2^{k+1} \cdot \Theta(\frac{k+3}{2}) + c \cdot k \cdot 2^{2 \cdot k} \quad$ with c constant
Also for $k \in \mathbb{R}_{\geq 4}$

Define $\Lambda(k) := \frac{\Theta(k)}{2^{2\cdot k}} \Rightarrow \Lambda(k) \leq \frac{2^{k+1}\Theta(\frac{k+3}{2})}{2^{2\cdot k}} + c\cdot k = 16\cdot\Lambda(\frac{k+3}{2}) + c\cdot k$
Define $\Omega(k) := \Lambda(k+3)$ So for $k \in \mathbb{R}_{\geq}1$
$\Omega(k) \leq 16\cdot\Lambda(\frac{k}{2}+3) + c\cdot(k+3) = \underbrace{16\Omega(\frac{k}{2}) + c\cdot(k+3)}_{*}$

Claim: For $i \in \mathbb{N}$ with $2^{i-1} \leq k-3$ have:
$\Lambda(k) \leq 16^i\Omega(\frac{k-3}{2^i}) + c\cdot(k+3)(1+8+...+8^{i-1}) + 3\cdot c\cdot(1+16+...+16^{i-1})$
Proof by induction:
$i = 0 \Lambda(k) = \Omega(k-3)$
$i \to i+1 : \Lambda(k) \leq 16^i\Omega(\frac{k-3}{2^i}) + c\cdot(k-3)(1+...+8^{i-1}) + 3\cdot c\cdot(1+...+16^{i-1}) \leq 2^i \leq k-3 \quad *$
$\leq 16^i(16\Omega(\frac{k-3}{2^i+1})) + c(\frac{k-1}{2^i}+3) + c(k-3)... =$ claimed result
Take $u \in \mathbb{N}$ minimal with $2^u > k-3 \Rightarrow \Omega(\frac{k-3}{2^u}) \leq \Omega(\lfloor\frac{k-3}{2^u}\rfloor) = \Omega(0) =: D$ (constant)
Note: u roughly is recursion depth
Have $2^{u-1} \leq k-3 \xRightarrow[\text{claim}]{} \Lambda(k) \leq 16^u\cdot D + c\cdot\underbrace{(k-3)}_{<2^u}\cdot\frac{8^u-1}{7} + 3c\cdot\frac{16^u-1}{15} \quad \in O(16^u)$

Have $2^{u-1} \leq k-3 \Rightarrow u \leq \lg(k-3)+1$
$\Rightarrow \Lambda(k) \in O(16^{\lg(k-3)}) = O((k-3)^4)$
$\Rightarrow \Theta(k) = 2^{2\cdot k}\cdot\Lambda(k) \in O(2^{2k}\cdot(k-3)^4)$
Have $2^{2(k-1)} < \underbrace{n}_{\max\{l(x)\cdot l(y)\}} \Rightarrow k \leq \frac{\lg(n)}{2}+1$
So $\Theta(k) \in O(n\cdot(\lg(n))^4)$ $\square$

1.1.21 Theorem 21 (Schönhage-Strassen 1971)

Multiplication of integers of length $\leq n$ can be done in $O(n\cdot\lg(n)\cdot\lg(\lg(n)))$ bit operations. Schönhage-Strassen is used for integers of length ≥ 100.000.
Asymptotically faster: Fürer's algorithm.

Comments on Bit complexity

1. Memory requirement may explode!
   $\Rightarrow$ No Problem as bit complexity is upper bound for memory requirements, since memory access is included in bit operations
   ($\to$ only store what is calculated)

2. Computation of addresses in memory take time
   $\Rightarrow$ length of addresses $\approx$ lg(memory space) computations of addresses $\approx$ lg(memory space)^2

3. As memory requirement gets larger access times will get longer.
   $\Rightarrow$ transportation time for data $\geq \frac{\text{diameter of physical storage}}{2*\text{speed of light}}$

1.2 Division with remainder, Euclidean algorithm

1.2.1 Algorithm 1 (Division with remainder)

input : $b = \sum_{i=0}^{n-1} b_i 2^i \quad a = \sum_{i=0}^{n+m-1} a_i 2^i \qquad$ with $a_i, b_i, \in 0, 1, \quad b_{n-1} = 1$

output: $r, q \in \mathbb{N} \quad$ such that $a = q \cdot b + r, \quad 0 \leq r < b$

(1) $r = a$
$q = 0$

(2) for $i = m, m-1, ..., 0$ do

(3) if $r \geq 2^i \cdot b$
set $r := r - 2^i \cdot b$
$q := q + 2^i$

1.2.2 Proposition 2 (Runtime of Algorithm 1)

Algorithm 1 is correct and requires $O(n \cdot (m+1))$ bit operations.
Proof:
Always have $a = q \cdot b + r$
Claim:
before step (3), have $0 \leq 2^{i+1} \cdot b$
$i = m : 0 \leq r = a < 2^{m+n} = 2^{m+1} c \cdot 2^{n-1} \underset{b_{n-1}=1}{\leq} 2^{m-1} \cdot b$
$i < m$: By step (3)
So after last passage through the loop $0 \leq r < b$
Running Time: In step(3), have comparison and (possibly) subtraction. Only n bits involved $\Rightarrow O(n)$
Total: $O(n \cdot (m+1))$

Remarks:

(1) Division with remainder can be reduced to multiplication.
Precisely: given an algorithm for multiplication that requires $M(n)$ bit operations, there exists an algorithm for division with remainder that requires $O(M(n))$ bit operations.

(2) Practically relevant:
Jebelean's algorithm (1997): $O(n^{\lg 3})$

(3) Alternatively, may choose $r\mathbb{Z} \quad$ such that $\lfloor \frac{-b}{2} \rfloor < r \leq \lfloor \frac{b}{2} \rfloor$

(4) Algorithm 1 extends to $\mathbb{Z}$.

(5) All Euclidean rings have division with remainder (by definition).
(e.g., $R = K[x] \quad \rightarrow$ polynomial ring over field,
$R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \subseteq \mathbb{C}, \quad i^2 = -1$)

1.2.3 Algorithm 3 (Euclidean algorithm)

input : $a, b \in \mathbb{N}$

output: $\gcd(a, b)$ "greatest common divisor"

(1) set $r_0 := a$
$r_1 := b$

(2) for $i = 1, 2, 3, ...$ perform steps (3) and (4)

(3) if $r_i = 0$
$\gcd(a, b) = |r_{i-1}|$

(4) Division with remainder: $r_{i-1} = q \cdot r_i + r_{i+1}$ $\quad r_{i+1} \in \mathbb{Z}$
$(|r_{i+1}| \leq \frac{1}{2}|r_i|)$

Example:

$$a = 287, \quad b = 126$$

$$287 = 2 \cdot 126 + 35 \tag{1}$$
$$126 = 4 \cdot 35 - 14 \tag{2}$$
$$35 = (-2) \cdot (-14) + 7 \tag{3}$$
$$-14 = (-2) \cdot 7 + 0 \tag{4}$$

$$\begin{aligned} \text{So: } 7|(-14) &\underset{(3)}{\Longrightarrow} 7|35 \\ &\underset{(2)}{\Longrightarrow} 7|126 \\ &\underset{(1)}{\Longrightarrow} 7|287 \end{aligned}$$

On the other hand take a common divisor d; $\quad d|287$; $\quad d|126$
$\underset{(1)}{\Longrightarrow} d|d \underset{(2)}{\Longrightarrow} d|14 \underset{(3)}{\Longrightarrow} d|7$

1.2.4 Theorem 4 (Correctness of Algorithm 3)

Algorithm 3 is correct.
Proof:
Since $r_{i-1} = q \cdot r_i + r_{i+1}$ every integer $x \in \mathbb{Z}$ satisfies the equivalence $x|r_{i-1}$ and $x|r_i \Leftrightarrow x|r_{i+1}$ and $x|r_i$ so $\gcd(r_{i-1}, r_i) = \gcd(r_i, r_{i+1}) = \gcd(a, b)$
when terminating have $\gcd(a, b) = \gcd(r_{i-1}, 0) = |r_{i-1}|$ $\quad\square$

1.2.5 Theorem 5 (Runtime of Algorithm 3)

Algorithm 3 requires $O(m \cdot n)$ bit operations for $n = l(a), m = l(b)$
Proof:
If $a < b$ than the first passage yields $r_2 = a,\ \ r_1 = b$. Cost: $O(n)$
May assume: $a \geq b$. Write $n_i = l(r_i)$
By Proposition 2 $\exists c$ constant such that the total time is $\leq c \cdot \underbrace{\sum_{i=1}^{k} n_i \cdot (n_{i-1} - n_i + 1)}_{=:\sigma(n_0,\ldots,n_k)}$

For $i > 2: \quad n_i = n_{i-1} - 1$
Special Case: $n_i = n_{i-1} - 1 \quad$ for $i \geq 2$
$\Rightarrow n_i = n_i - i + 1, \quad n_i = m, \quad k = m + 1$
Obtain $\sigma(n_0, ..., n_k) = m \cdot (n - m + 1) + \sum_{i=2}^{m+1} (m - i + 1) \cdot 2 = m \cdot n - m^2 + m + m(m-1) = m \cdot n.$
Claim: The special case is the worst (most expensive)!
From any sequence $n_1 > n_2 > ... > n_k$ get to the special case by iteratively inserting numbers in the gaps. Insert s with $n_{j-1} > s > n_j$.
$\sigma(n_0, ..., n_{j-1}, s, n_j, ..., n_k) - \sigma(n_0, ..., n_k) = ... = s + (n_{j-1} - s) \cdot (s - n_j)$
$sp\sigma(n_0, ..., n_k) \leq \sigma(n, m, m-1, ..., 2, 1, 0) = n \cdot m \qquad \square$
Complexity is quadratic $\rightarrow$ cheap

1.2.6 Algorithm 6 (Extended Euclidean Algorithm)

input : $a, b \in \mathbb{N}$

output: $d = \gcd(a, b) \quad$ and $s, t \in \mathbb{Z} \quad$ such that $d = s \cdot a + t \cdot b$

(1) $r_0 := a, r_1 := b, s_0 := 1, t_0 := 0, s_1 := 0, t_1 = 1$

(2) for $i = 1, 2, ... \quad$ perform steps (3) - (5)

(3) if $r_i = 0$
set $d = |r_{i-1}|$
$s := sgn(r_{i-1}) \cdot s_{i-1}$,
$t := sgn(r_{i-1}) \cdot t_{i-1}$

(4) division with remainder:
$r_{i+1} = r_{i-1} - q_i \cdot r_i, \quad$ with $|r_{i+1}| \leq \frac{1}{2}|r_i|$

(5) set $s_{i+1} := s_{i-1} - q_i \cdot s_i$,
$t_{i+1} := t_{i-1} - q_i \cdot t_i$

Justification : $r_i = s_i \cdot a + t_i \cdot b$ throughout
Application: $m, x \in \mathbb{N}$ such that m, x co-prime (i.e. $\gcd(x, m) = 1$)
Algorithm 6 yields: $1 = s \cdot x + t \cdot m \Rightarrow s \cdot x \equiv 1 \pmod{m}$
So obtain inverse of $\bar{x} \in \mathbb{Z}/(m)$

1.3 Primality testing

Let $\mathbb{P} \subseteq \mathbb{N}$ be the set of prime numbers.
Challenge: Given $n \in \mathbb{N}$ decide if $n \in \mathbb{P}$
Naive Method: Trivial division by $m \leq \lfloor \sqrt{n} \rfloor$.
Running time is exponential in $l(n)$. Even when restricted to division by prime numbers, need approximatily $\frac{\sqrt{n}}{|n|\sqrt{n}}$ trivial divisions (prime number theorem)
$\rightarrow$ hardly any better!
Reminder: (arithmetic modulo m)
G finite group $\Rightarrow \forall a \in G \quad a^{|G|} = 1$ Fermat's little theorem
For $G = (\mathbb{Z}/(p))^\times \quad a^{p-1} \equiv 1 \pmod{p} \in \mathbb{P} \quad \forall a \in \mathbb{Z}$ with $p \nmid a$
Infact $(\mathbb{Z}/(p))^\times \cong Z_{p-1}$ is cyclic
For $m = p_1^e, ...p_r^{e_r}$ with $p_i \in \mathbb{P}, \ e_i \in \mathbb{N}_{>0}$:
$\mathbb{Z}_{(m)} \cong \mathbb{Z}_{(p_i^{e_i})} \oplus ... \oplus \mathbb{Z}_{(p_r^{e_r})} \Rightarrow \mathbb{Z}^x_{(m)} \cong \mathbb{Z}^x_{(p_i^{e_i})} \times ... \times \mathbb{Z}^x_{(p_r^{e_r})}$
what is $\mathbb{Z}_{(p^e)}$for $p \in \mathbb{P}, \ e \in \mathbb{N}_{>0}$?

1.3.1 Theorem 1 (Cyclic group)

Let $p \in \mathbb{P}$ *odd* $e \in \mathbb{N}_{>0} \Rightarrow (\mathbb{Z}_{(p^e)})^\times = Z_{(p-1)\cdot p^{e-1}}$ cyclic
Proof:
$(\mathbb{Z}_{(p^e)})^\times \cong Z_{p-1} \Rightarrow \exists z \in \mathbb{Z} : order(z + p\mathbb{Z}) = p - 1$
Set $a = \bar{z}^{p^{e-1}} \in (\mathbb{Z}_{(p^e)})^\times =: G$
$\quad a^{p-1} = \bar{z}^{(p-1)\cdot p^{e-1}} = \bar{z}^{|a|} = 1$
On the other hand, take $i \in \mathbb{Z}$ such that
$a^i = 1 \Rightarrow z^{i \cdot p^{e-1}} \equiv 1 \pmod{p} \Rightarrow (p-1)|(i - p^{e-1}) \Rightarrow (p-1)|i$.
So $ord(a) = p - 1$.
Now consider $b = (p \bar{+} 1) \in G$
Claim: $ord(b) = p^{e-1}$
Proof by induction on $k \in N_{>0}$ that $(p+1)^{p^{k-1}} \equiv p^k + 1 \pmod{p^{k+1}}$
$k = 1 \quad \checkmark$
$k \rightarrow k+1$: By induction have $(p+1)^{p^{k-1}} = 1 + p^k + x \cdot p^{k+1}, \quad x \in \mathbb{Z}$
Compute: $(p+1)^{p^k} = ((1 + p^k) + x \cdot p^{k+1})^p = \sum\limits_{i=0}^{p} \binom{p}{i} (i + p^k)^{p-i} \cdot x^i \cdot p^{i\cdot(k+1)}$

$$\underset{\text{Only 0-th summand}}{\equiv} (i + p^k) = \sum_{i=0}^{p} \binom{p}{i} p^{i \cdot k} \underset{p \text{ odd}}{\equiv} 1 + p^{k+1} \pmod{p^{k+2}} \quad \checkmark$$

For $k = e : (p+1)^{p^{e-1}} \equiv 1 \quad (mod \, p^e) \Rightarrow b^{p^e} = 1 \Rightarrow ord(b)|p^{e-1}$
But $(p+1)^{p^{e-2}} \equiv p^{e-1} + 1 \quad (mod \, p^e) \Rightarrow b^{p^{e-2}} \neq 1 \in G$
So $ord(b) = p^{e-1}$
Claim: $ord(a \cdot b) = (p-1)p^{e-1}$ $\quad$ ($\Rightarrow$ Theorem)
Let $(a \cdot b)^i = 1 \in G$ with $i \in \mathbb{Z}$
Then $1 = (a \cdot b)^{i \cdot (p-1)} = (a^{p-1})^i \cdot b^{i \cdot (p-1)} = b^{i \cdot (p-1)} \Rightarrow p^{e-1}|i \cdot i(p-1) \Rightarrow p^{e-1}|i$
Also $1 = (a \cdot b)^{p^{e-1} \cdot i} = a^{p^{e-1}} \Rightarrow (p-1)|p^{e-1} \cdot i \Rightarrow (p-1)|i \rightarrow (p-1) \cdot p^{e-1}|i \qquad \square$
Reminder: $(\mathbb{Z}/(2^e))^\times \cong Z_2 \times Z_2^{e-2} \quad (e \geq 2)$

1.3.2 Algorithm 2 (Fermat Test)

input : $n \in \mathbb{N}_{>0} odd$

output: "$n \notin \mathbb{P}$" or "probably $n \in \mathbb{P}$"

(1) Choose $a \in 2, ..., n-1$ randomly

(2) Compute $a^{n-1} \mod n$

(3) If $a^{n-1} \not\equiv 1 \pmod n$ return "$n \notin \mathbb{P}$"
else return "probably $n \in \mathbb{P}$"

Not very satisfying. Is this fast?

1.3.3 Algorithm 3 (Fast exponentiation)

input : $a \in G \quad G$ is a monoid, $e \in \mathbb{N} \quad e = \sum_{i=0}^{n-1} e_i 2^i \quad e_i \in \{0, 1\}$

output: $a^e \in G$

(1) Set $b := a, \quad y := 1$

(2) For $i = 0, ..., n-1$ perform (3) - (4)

(3) if $e_i = 1$ set $y := y \cdot b$

(4) set $b := b^2$

(5) return y

this requires $O(l(e))$ operations in G
For $G = (\mathbb{Z}/(n)_i)$, each multiplication requires $O(l(n)^2)$ bit operations
$\Rightarrow$ Fermat test requires $O(l(n)^3)$ bit operations $\rightarrow$ cubic complexity $\rightarrow$ "fast"!
Example:
$n = 561 = 3 \cdot 11 \cdot 17 \quad$ For $a \in \mathbb{Z} \quad$ with $\gcd(a, n) \Rightarrow$ have $a^{n-1} = (a^2)^{280} \equiv 1 \pmod 3$
$a^{n-1} \equiv 1 \pmod n$ Fermat's test says "probably $n \in \mathbb{P}$" in 57% of cases.
$n = 2207 \cdot 6619 \cdot 15443$: output "probably $n \in \mathbb{P}$" in 99,93% of cases.

1.3.4 Definition 4 (Pseudo-prime, witness, Carmichael numbers)

Let $n \in N_{>1} odd, \quad a \in 1, ..., n-1$

(a) n is pseudo-prime to base a if $a^{n-1} \equiv 1 \pmod n$

(b) otherwise a is called a witness of composition of n

(c) If $n \notin \mathbb{P}$ but $a^{n-1} \equiv 1 \pmod n \quad \forall a$ with $\gcd(n, a) = 1$
then n is called a Carmichael number.
There are ∞ Carmichael numbers

1.3.5 Proposition 5 (Number of witnesses)

Let $n \in N_{>1}, \quad odd \wedge \notin \mathbb{P} \wedge$ not Carmichael
$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n, a \text{ is witness of composite of } n\}| > \frac{n-1}{2}$
Proof: Consider
$\phi : (\mathbb{Z}/(n))^{\times} =: G \to G, \quad \bar{a} \mapsto \bar{a}^{n-1}$
group homomorphism. By assumption,
$|im(\phi)| > 1 \Rightarrow |Ker(\phi)| \leq \frac{|a|}{2} < \frac{n-1}{2}$
$\Rightarrow |\{a \in \mathbb{Z} | 0 < a < n \text{ a witness of composite of } n\}| > \frac{n-1}{2} \qquad \square$

Miller-Rabin Test

1.3.6 Proposition 6 (Inference from Fermat)

Let $p \in \mathbb{P}$ *odd*, $\quad a \in \{1, ..., (p-1)\} \quad$ write $p - 1 = 2^k \cdot m \quad$ with m *odd* Then:
$a^m \equiv 1 \pmod p \quad$ or $\quad \exists i \in \{0, ..., k-1\} : a^{2^i \cdot m} \equiv -1 \pmod p$
Proof:
Little Fermat: $\bar{a}^{2^k \cdot m} = 1 \in \mathbb{F}_p$
Assume $\bar{a}^m \neq 1$ take i maximal such that:
$\bar{b} = \bar{a}^{2^i \cdot m} \neq 1 \Rightarrow \bar{b}^2 = 1 \Rightarrow \bar{b} \in \mathbb{F}_p$ is a zero of $x^2 - 1 \in \mathbb{F}_p[x] \Rightarrow \bar{b} = -1$

1.3.7 Algorithm 7 (Miller-Rabin-test)

input : $n \in \mathbb{N}_{>1}, odd$

output: either "$n \notin \mathbb{P}$" or "probably $n \in \mathbb{P}$" $\rightarrow$ Monte Carlo Algorithm.

(1) write $n-1 = 2^k \cdot m$ with m odd

(2) Choose $a \in \{2, ..., n-1\}$ randomly

(3) Compute $b := a^m \mod n$

(4) if ($b \equiv \pm 1 \pmod n$)
return "probably $n \in \mathbb{P}$"

(5) for ($i = 0, ..., k-1$) do steps (6) - (7)

(6) set $b := b^2 \pmod n$

(7) if ($b \equiv -1 \pmod n$)
return "probably $n \in \mathbb{P}$"

(8) return $n \notin \mathbb{P}$"

1.3.8 Definition 8 (strong pseudo-prime / witness)

Let $n \in \mathbb{N}_{>1},\ odd \quad a \in \{1, ..., n-1\}$

(a) n is called a strongly pseudo-prime to base a if Proposition 6 holds for a and p replaced by n.

(b) Otherwise a is called a strong witness of composition of n.

Example
Let $n \in \mathbb{N}_{>1}, \quad \mathbb{P}\ odd$
$a = 2$ strong witness if $n < 2047$ (including 561)
2 or 3 strong witness if $n < 1373653$
2,3 or 5 strong witness if $n < 25326001$

1.3.9 Theorem 9 (Bit-complexity of Algorithm 7)

(a) Algorithm 7 requires $O(l(n)^3)$ bit operations. $\rightarrow$ "qubic complecity" $\rightarrow$ fast!

(b) if $b \in \mathbb{P}$ then Algorithm 7 returns "probably $b \in \mathbb{P}$" $\rightarrow$ no false positives.

(c) if $n \notin \mathbb{P}$ then more than half of the numbers in $\{1, ..., n-1\}$ are strong witnesses.

Proof:

(a) Step 1 takes $O(l(n))$ bit operations:
Using Algorithm 3, we need $O(l(n-1))$ multiplications in $\mathbb{Z}/(n)$ each requiring $O(l(n)^2)$ bit operations.

(b) Proposition 6

(c) split in three cases:

Case 1: n is not a Carmichael number. $\underset{\text{Prop 5}}{\Longrightarrow}$ more than half of all numbers are. Fermat witness thus also strong witness.
Case 2: $n = p^r \cdot l$ with $p \in \mathbb{P} \quad r > 1 \quad l \in \mathbb{N}_{>0} p \nmid l$
Theorem 1 $\exists x \in Z$ such that $x^p \equiv 1 \pmod{p^r} \quad x \not\equiv 1 \pmod{p^r}$
Chinese remainder theorem: $\exists a \in \mathbb{Z}$ such that $a \equiv x \pmod{p^r} \quad a \equiv 1 \pmod{l}$
So $\bar{a}^p = 1 \in \mathbb{Z}_(n) \Rightarrow \bar{a}^n = 1 \Rightarrow \bar{a} \in (\mathbb{Z}/(n))^\times$
i.e. $\gcd(n, a) = 1$ if $\bar{a}^{n-1} = 1$ then $\bar{a} = 1$
But $a \equiv x \neq 1 \pmod{p^r}$ so $\bar{a}^{n-1} \neq 1$ hence n is not Carmichael $\rightarrow$ Case 1.
Case 3: n is a Carmichael number. By Case 2 have $n = p \cdot l$ with $p \in \mathbb{P} \quad p \nmid l \quad l \geq 3$
n Carmichael: $\forall a \in \mathbb{Z}$ with $\gcd(a, n,) = 1$
have $a^{2^k \cdot m} \equiv 1 \pmod{n} \qquad$ (where $n - 1 = 2^k \cdot m$)
$a^{2^k \cdot m} \equiv 1 \pmod{p}$ Take j minimal such that
$a^{2^j \cdot m} \equiv 1 \pmod{p} \quad \forall a \in \mathbb{Z}$ such that $\gcd(a, n) = 1$
so $0 \leq j \leq l$ in fact, $j > 0$ since $(-1)^{2^0 \cdot m} = -1$ with m *odd.*
Consider the subgroup $H := \{\bar{a} \in \mathbb{Z}/(n) | \bar{a}^{2^{j-1} \cdot m} \in \{1, -1\} \subseteq (\mathbb{Z}/(n))^\times\}$
Let $a \in \{1, ..., n-1\} \quad \gcd(n, a) = 1 \quad a$ not a strong witness.
Claim 1: $\bar{a} \in H$
Case 3.1: $\bar{a}^{2^{j-1} \cdot m} = 1 \Rightarrow \bar{a} \in H$
Case 3.2: $a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{n} \qquad a^m \not\equiv 1 \pmod{n}$
$\underset{a \text{ nonwitness}}{\Longrightarrow} \exists i$ such that $\underbrace{a^{2^i \cdot m} \equiv -1 \pmod{n}}_{*}$
$\Rightarrow a^{2^i \cdot m} \equiv -1 \pmod{p} \underset{\text{def of j}}{\Longrightarrow} i < j$
if $i < j - 1$ then $a^{2^{j-1} \cdot m} = (a^{2^i \cdot m})^{2^{j-1-i}} \equiv (-1)^{2^{j-1-i}} = 1 \pmod{n}$
$\underset{\text{with } *}{\Longrightarrow}$ not in case 3.2
Claim 2: $H \subseteq (\mathbb{Z}_(n))^\times$ proper subgroup.
By definition of j $\exists x \in \mathbb{Z}$ such that $x^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p}$
Chinese remainder: $\exists a \in \mathbb{Z}$ such that
$a \equiv x \pmod{p} \qquad a \equiv 1 \pmod{l}$
$\Rightarrow a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p} \equiv 1 \pmod{l} \Rightarrow \bar{a} \notin H$
Claim 2 ✓
It follows that $|H| \leq \frac{|(\mathbb{Z}/(n))^\times|}{2} < \frac{n-1}{2}$
so the number of witnesses is $\geq n - 1 - |H| > \frac{n-1}{2} \qquad \square$

Remarks:

(a) A more careful analysis shows that $\geq \frac{3}{4}$ of all candidates are strong witnesses

(b) Calling Algorithm 7 repeatedly decreases the probability of false positives. Running time for prescribed error probability p is $O(\lg(p^{-1} \cdot l(n)^3))$
(Independence assumptions!)

Connection with Riemann hypothesis
Let $n \in \mathbb{N}_{>0}$ $\quad \bar{X} : (\mathbb{Z}/(n))^{\times} \to \mathbb{C}^x$ group homomorphism
$X : \mathbb{Z} \to \mathbb{C}, a \mapsto \begin{cases} \bar{X}(\bar{a}) & \text{if } \gcd(a,n) = 1 \quad \text{for } (\bar{a} = a + n\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$
"residue class character $\pmod{n}$
$Ex : n = 1 \Rightarrow X(a) = 1 \forall a \in \mathbb{Z}$
Dirichlet L-series:
$L_X(s) = \sum\limits_{n=1}^{\infty} \frac{X(n)}{n^s}$ converges for $s \in \mathbb{C}$ until $Re(s) > 1$
$L_X(s)$ extends to a meromorphic function on $\mathbb{C} \mapsto$ "Divichlet L-function".
For $n = 1 : L_X(s) = \zeta(s)$ Riemann Zeta-function.
Euler Product:
From $(1 - X(p) \cdot p^{-s})^{-1} = \sum\limits_{i=0}^{\infty} (X(p) \cdot p^{-s})^i = \sum\limits_{i=0}^{\infty} \frac{X(p^i)}{p^{is}}$ $\quad$ derive $L_X(s) = \prod\limits_{p \in \mathbb{P}} \frac{1}{1 - X(p) \cdot^{-} s}$
Generalized Riemann hypothesis (GRH):
For X residue class character, $s \in \mathbb{C}$
with $L_X(s) = 0, \quad 0 < Re(s) < 1$ $\qquad$ ("critical strip")
then $Re(s) = \frac{1}{2}$
For $X = 1 \to$ ordinary Riemann hypothesis.

1.3.10 Theorem (Ankeny & Bach)

GRH $\Rightarrow \forall X \neq 1$ residue class character
$\exists p \in \mathbb{P} : X(p) \neq 1, p < 2 \ln(n)^2$

Let $H \subsetneqq (\mathbb{Z}/(n))^{\times} =: G$ proper subgroup.
Choose $N \subsetneqq G$ maximal proper subgroup such that $H \subseteq N \Rightarrow G/N$ cyclic.
$\bar{X} : G \mapsto \mathbb{C}^x$ with $N = Ker(\bar{X}) \Rightarrow H \subseteq Ker(\mathbb{X})$
$\xRightarrow[\text{GRH, Thm1}]{} \exists p \in \mathbb{P} : p + n\mathbb{Z} \notin H, p < 2 \cdot \ln(n)^2$
Corollary: Assume GRH.
Let $n \in \mathbb{N}_{>1}$ $\;\mathbb{P}$ *odd* Then there is a strong witness a of compositeness of n with $a < 2 \cdot \ln(n)^2$.
$\to$ Obtain deterministic primality test with time $O(\ln(n)^5)$ bit operations.

AKS-test
A deterministic polynomial time primality test $\to$ "holy grail"
Agrawal, Kayal, Saxena: PRIMES is in P, Annals of Mathematics, 2004.

1.3.11 Proposition 10 (Modulo over ideals)

Let $n \in \mathbb{P} \quad a \in \mathbb{Z} \Rightarrow (x+a)^n \equiv x^n + a \quad (mod\, n)$
where x is a indeterminate and for $r \in \mathbb{N}$:

$$(x+a)^n \equiv (x^n + a) \pmod{n, x^r - 1} \tag{1}$$

(i.e. $(x+a)^n - (x^n + a) = n \cdot f + (x^r - 1) \cdot g \quad$ with $f, g \in \mathbb{Z}[x]$)
Proof:
$(x+a)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot a^{n-i} \cdot x^i \qquad$ (where $\binom{n}{i}$ is a multiple of n for $0 < i < n$)
$\equiv x^n + a^n$
$\underset{\text{(by little Fermat)}}{\equiv x^n + a}$ (1) follows by weakening this.
Cost analysis for checking (1) with $l = \text{length(n)}$.
Using Algorithm 3, need $O(l)$ multiplications in $\mathbb{Z}[x]/(n, x^r - 1) =: R$
Elements of R are represented as polynomials of degree $< r$,
coefficients between 0 and n.
Multiply polynomials: $O(r^2)$ operation in $\mathbb{Z}/(n) : O(r^2 \cdot l^2)$
since $x^{r+k} \equiv x^k \pmod{x^r - 1}$,
add coefficients of x^{r+k} of product polynomial to coefficients $x^k : O(r \cdot l)$
Total for checking (1): $O(r^2 \cdot l^3)$ bit operations.
Reduction $(\text{mod } x^r - 1)$ is just for keeping the cost under control.
The following is part of AKS-test:

1.3.12 Algorithm 11 (Test for perfect power)

input : $n \in \mathbb{N}_{>1}$

output: $m, e \in \mathbb{N} \quad e > 1$ such that $n = m^e \quad$ or "n is not a perfect power"

(1) for $(e = 2, ..., \lfloor \lg(n) \rfloor)$ perform (2) - (7) //possible exponents

(2) set $m_1 = 2, m_2 = n$ //initialize interval $[m_1, m_2]$ for searching $\sqrt[e]{n}$

(3) while$(m_1 \leq m_2)$ do (4) - (7)

(4) set $m = \lfloor \frac{m_1 + m_2}{2} \rfloor$ // bisect interval

(5) if $m^e = n$ return m, e

(6) if $m^e > n$ set $m_2 = m - 1$

(7) if $m^e < n$ set $m_1 = m + 1$

(8) return "not a perfect power"

Cost: (for $l = \text{length}(n)$)
Compute $m^e : O(\lg(l) \cdot l^2)$ (abort computation once the result exceeds n)
Number of passages through inner loops $\leq \lg(n)$
Number of passages through outer loops $\leq \lg(n)$
Total cost of Algorithm 11: $O(l^4 \cdot \lg(l))$

1.3.13 Algorithm 12 (AKS-test)

input : $n \in \mathbb{N}_{>1}$ of length $l = \text{length}(n,) = \lfloor \lg(n) \rfloor + 1$

output: "$n \in \mathbb{P}$" or "$n \notin \mathbb{P}$"

(1) if (n is a perfect power)
    return "$n \notin \mathbb{P}$"

(2) find $r \in \mathbb{N}_{>1}$ minimal such that $r|n \quad \vee \quad n^i \not\equiv 1 \pmod r \qquad \forall i = 1, ..., l^2$
//exhaustive search (we will show that $r \leq l^5$)

(3) if $r|n$
    if $(r = n)$ return "$n \in \mathbb{P}$"
    if $(r < n)$ return "$n \notin \mathbb{P}$"

(4) for $a = 1, 2..., \lfloor \sqrt{r} \cdot l \rfloor$ do (5)

(5) if $((x + a)^n \not\equiv (x^n + a) \pmod{(n, x^r - 1)})$
    return "$n \notin \mathbb{P}$"

(6) return "$n \in \mathbb{P}$"

1.3.14 Lemma 13 (Least common multiple)

For $n \in \mathbb{N}_{>0}$ have $\lambda(n) := lcm(1, 2, ...n) \geq 2^{n-2}$

Proof: For $f = \sum\limits_{i=0}^{m} a \cdot x^i \in \mathbb{Z}(x) \quad a_i \in \mathbb{Z}$

$\Rightarrow \int\limits_0^1 f(x)dx = \sum\limits_{i=0}^{m} \frac{a_i}{i+1} = \frac{k}{\lambda(m+1)}$

with $k \in \mathbb{Z}$. Consider $f_m = x^m \cdot (1 - x)^m$

For $0 < xy1$:

$0 < f_m(x) \leq 4^{-m}$

$\Rightarrow 0 < \int\limits_0^1 \underbrace{f_m(x)}_{\frac{k_m}{\lambda(2m+1)}} dx \leq 4^{-1}$

$\lambda(2 \cdot m + 1) \geq k_m \cdot 4^m \geq 4^m$

For $n \in \mathbb{N}_{>0} \lambda(n) \geq \lambda(2 \cdot \lfloor \frac{n-1}{2} \rfloor + 1) \geq 4^{\lfloor \frac{n-1}{2} \rfloor} \geq 4^{\frac{n-1}{2}} = 2^{n-2} \qquad \square$

Corollary: (not related to AKS)

For $n \in \mathbb{M}$

$\pi(n) := |\{p \in \mathbb{P} | p \leq n\}| \geq \frac{n-2}{\lg(n)}$

Proof:

$2^{n-2} \leq \lambda(n) = \prod\limits_{p \in \mathbb{P}, p \leq n} p^{\lfloor \log_p(n) \rfloor} \leq \prod\limits_{p \leq n} p^{\log_p(n)} = n^{\pi(n)} = 2^{\lg(n)\pi(n)} \qquad \square$

Prime number theorem:
$\lim\limits_{n\to\infty} \frac{\pi(n)}{n/\ln(n)} = 1$
Interpretation:
The average distance of two primes around some value $x \in \mathbb{R}_{>1}$ is $\ln(x)$

1.3.15 Lemma 14 (Property of r in Algorithm 12)

In Algorithm 12, have $r \leq l^5$
Proof:
if $r < l^5 \Rightarrow \forall k \in k \in \{2, ..., l^5\} : \exists i \in \{1, ..., l^2\}$
$n^i \equiv 1 \pmod{k}$
$\Rightarrow k | \prod\limits_{i=1}^{l^2} (n^i - 1)$
$\Rightarrow \lambda(l^5) | \prod\limits_{i=1}^{l^2} (n^i - 1)$
$\xRightarrow[Lemma 13]{} 2^{l^5-2} < \prod\limits_{i=1}^{l^2} n^i = n^{\frac{l^2(l^2+1)}{2}}$
$\Rightarrow l^5 - l^3 < 4$ not true since $l \geq 2$ $\square$

1.3.16 Theorem 15 (Bit-Complexity of Algorithm 12)

Algorithm 12 requires $O(l^{16.5})$ bit operations ("polynomial complexity")
Proof:

Step(1): $O(l^4 \cdot \lg(l))$✓

Step(2): For each r need:

- test $r|n : O(l^2)$
- compute all $n^i \mod r : O(l^2 \cdot \lg(r)^2) \underset{Lemma 14}{\leq} O(l^2 \cdot \lg(l)^2)$

Step(3): $O(1)$

Step(4): $O(\sqrt{r} \cdot l \cdot r^2 \cdot l^3) \underset{Lemma 14}{\leq} O(l^{16,5})$ $\square$

Reminder: There is a variant of Algorithm 12 with running time $\tilde{O}(l^6)$,i.e., $O(l^6 \cdot \lg(l)^m)$ with $m \in \mathbb{N}$.
Correctness:
For $r \in \mathbb{N}_{>0}$ and $p \in \mathbb{P}$ write $I(r,p) := \{m, f) \in \mathbb{N} \times \mathbb{F}_p[x] | f(x)^m \equiv f(x^m) \pmod{x^r - 1}\}$
"m is introspective for f and r".
Example: Proposition 10 says that:

$$(p, x + \bar{a}) \in I(r,p) \text{for } a \in \mathbb{Z} \quad r \in \mathbb{N}_{>0} \quad p \in \mathbb{P} \tag{1}$$

1.3.17 Lemma 16 (Rules for ideals)

(a) $(m, f), (m', f) \in I(r,p) \Rightarrow (m \cdot m', f) \in I(r,p)$

(b) $(m, f), (m, g) \in I(r,p) \Rightarrow (m, f \cdot g) \in I(r,p)$

(c) $(m \cdot p, f) \in I(r,p), p \nmid r \Rightarrow (m, f) \in I(r,p)$

Proof:

(a) $f(x)^{m \cdot m'} \equiv f(x^m)^{m'} \pmod{(x^r - 1)}$
$f(x^m)^{m'} \equiv f(x^{m \cdot m'}) \pmod{(x^{m \cdot r} - 1)}$
But $(x^r - 1|(x^{m \cdot r} - 1))$

(b) $(f \cdot g)(x)^m = f(x)^m \cdot g(x)^m \equiv f(x^m) \cdot g(x^m) = (f \cdot g) \cdot (x^m) \pmod{(x^r - 1)}$

(c) $(f(x)^m)^p \equiv f((x^m)^p) \underset{Frobeniushomomorphism}{\equiv} (f(x^m))^p \pmod{(x^r - 1)}$
$\Rightarrow (x^r - 1)|((f(x)^m)^p - f(x^m)^p) \underset{Frobeniushomomorphism}{\equiv} (f(x)^m - f(x^m))^p$
$p \nmid r \Rightarrow x^r - 1$ is square free. So
$(x^r - 1)|(f(x)^m) - f(x^m)) \Rightarrow (m, f) \in I(r,p)$ $\square$

1.3.18 Theorem 17 (Correctness of Algorithm 12)

Algorithm 12 is correct.
Proof:
If the algorithm terminates in step(1),(3) or (5), it is correct. To show: If it terminates in step(6) it is correct, i. e. $n \in \mathbb{P}$
Claim 1: $\exists p \in \mathbb{P} : p|n \quad p \not\equiv 1 \pmod{r} \quad p > r$
Indeed if all prime divisors of n were $\equiv 1 \pmod{r}$ then $n \equiv 1 \pmod{r}$
Contradiction to step(2). All prime divisors of n are $> r$ by step (2) and (3) ✓
Steps(2) and (3) imply that $\gcd(n, r) = 1 \Rightarrow G :=< \bar{n}, \underbrace{\bar{p}}_{p \mod r} >\subseteq (\mathbb{Z}/(r))^\times$

Step(2): $ord(\bar{n}) > l^2 \Rightarrow l^2 < |G| < r$ (2)

Set $s := ord(\bar{p} \in G) \Rightarrow r|(p^s - 1)$ with $q := p^s \Rightarrow r||\mathbb{F}_q^\times| \Rightarrow \exists \zeta \in \mathbb{F}_q$ r-th root of 1
Set $k := \lfloor \sqrt{r} \cdot l \rfloor \quad m := (\frac{n}{p})$
By (1) $(p, x + \bar{a}) \in I(r,p)$ with $\bar{a} \in \mathbb{F}_p$
By step(4), have $(n, x + \bar{a}) \in I(r,p)$
For $\underline{e} = e_0, ..., e_k \in \mathbb{N}_0$ set $f_{\underline{e}} := \prod_{a=0}^{k} (x + \bar{a})^{e_a}$
Lemma 16 (b): $(p, f_{\underline{e}}) \in I(r,p)$
$(n, f_{\underline{e}}) \in I(r,p)$
$\xRightarrow[Lemma16(c)]{} (m, f_{\underline{e}}) \in I(r,p)$
$\xRightarrow[Lemma16(a)]{} \forall s, t \in \mathbb{N}_0 : (p^s \cdot m^t, f_{\underline{e}}) \in I(r,p)$

$\Rightarrow f_{\underline{e}}(\zeta^{p^s \cdot m^t}) = f_{\underline{e}}(\zeta)^{p^s \cdot m^t}$ (3)

Set $H := <\zeta + \bar{a} | a \in \{0, ..., k\}> \subseteq \mathbb{F}_q^\times$
($\zeta \notin \mathbb{F}_p$ since $r \nmid (p-1)$ by Claim 1)

Consider: $T := \{(e_0, ..., e_k) \in \mathbb{N}_0^{k+1} | \sum_{a=0}^{k} e_a < |G|\}$

$\Phi : T \mapsto H, (e_0, ..., e_k) \mapsto f_{\underline{e}}(\zeta) = \prod_a (\zeta + \bar{a})^{e_a} \in H$

Claim 2: Φ is injective.
Indeed, take $(\underline{e}), (\underline{\hat{e}}) \in T$ such that $\Phi(\underline{e} = \Phi(\underline{\hat{e}}))$
$\Rightarrow \forall s, t \in \mathbb{N}_0 : f_{\underline{e}}(\zeta^{p^s \cdot m^t}) \underset{(3)}{=} f_{\underline{e}}(\zeta)^{p^s \cdot m^t} = f_{\underline{\hat{e}}}(\zeta)^{p^s \cdot m^t} \underset{(3)}{=} f_{\underline{\hat{e}}}(\zeta^{p^s \cdot m^t})$
$f_{\underline{e}} - f_{\underline{\hat{e}}}$ has roots ζ^e with $e \in G$ since $G = <\bar{p}, \bar{m}>$
These are all distinct (since ζ is primitive)
But $\deg(f_{\underline{e}} - f_{\underline{\hat{e}}}) < |G|$ So $f_{\underline{e}} - f_{\underline{\hat{e}}} = 0$
Since $k \le \sqrt{r} \cdot l \underset{(2)}{<} r < p$ the $(x + \bar{a})$ with $a \in \{0...k\}$ are primitive distinct.
So $(\underline{e}) = (\underline{\hat{e}})$ ✓
So is $|H| \ge |T|$?
Let M be the set of all $\{x_0, ..., x_k\} \subseteq \{1, ..., |G| + k\}$
with $x_0 < x_1 < ... < x_k$
For $\{x_0, ..., x_k\} \in M$ define $(e_0, ..., e_k) \in \mathbb{N}_0^{k+1}$ by $e_a = x_a - x_{a-1}$ with $x_{-1} := 0$
$\Rightarrow \sum_{a=0}^{k} e_a = \sum_{a=0}^{k} (x_a - x_{a-1} - 1) = x_k - (k+1) < |G|$
Obtain injection $M \Leftrightarrow T$
So $|H| \ge |T| \ge |M| = \binom{|G|+k}{k+1} \underset{(2)}{\ge} \binom{\lfloor l\sqrt{|a|}\rfloor+1+k}{k+1} = \binom{\lfloor l\sqrt{|a|}\rfloor+1+k}{\lfloor l\sqrt{|a|}\rfloor} \underset{(2)}{\ge} \binom{2\cdot\lfloor l\sqrt{|a|}\rfloor+1}{\lfloor l\sqrt{|a|}\rfloor}$

1.3.19 Lemma 18 (Property of binomial coefficients)

$\forall n \in \mathbb{N}_{>1} : \binom{2\cdot n+1}{n} > 2^{n+1}$
Proof:
$n = 2 :$
$\binom{5}{2} = 10 > 2^3$
$n - 1 \to n :$
$\binom{2\cdot n+1}{n} = \binom{2\cdot n}{n-1} + \binom{2\cdot n}{n} = \binom{2\cdot n-1}{n-2} + \binom{2\cdot n-1}{n-1} + \binom{2\cdot n-1}{n-1} + \binom{2\cdot n-1}{n} \ge 2 \cdot \binom{2\cdot n-1}{n-1} \underset{ind.}{>} 2 \cdot 2^n = 2^{n+1}$

Continuation of Proof of Theorem 17

$$|H| > 2^{\lfloor l\cdot\sqrt{|a|}\rfloor+1} \ge 2^{l\cdot\sqrt{|a|}} \ge 2^{\lg(n)\cdot\sqrt{|a|}} = n^{\sqrt{|a|}} \tag{4}$$

Assume $n \notin \mathbb{P}$ By step (1) m is not a perfect power
$\Rightarrow$ the map $\mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{N}$ $(s, t) \mapsto p^s m^t$ is injective.
Set $A := \{p^s m^t | s, t \in \{0, .., \lfloor\sqrt{a}\rfloor\}\} \subseteq \mathbb{N}$
$\Rightarrow |A| = (\lfloor\sqrt{|a|}\rfloor + 1)^2 > |G|$
Since $G = <\bar{p}, \bar{m}> \subseteq (\mathbb{Z}/(r))^\times$ this implies that $\exists n, \hat{n} \in A$
such that $n \ne \hat{n}$ but $b \equiv \hat{n} \pmod r$.
Let $h \in H \Rightarrow h = f_{\underline{e}}(\zeta)$ with $(\underline{e}) \in \mathbb{N}_0^{k+1} \Rightarrow h^n \underset{(3)}{=} f_{\underline{e}}(\zeta^n) \underset{n\equiv\hat{n} \pmod r}{=} f_{\underline{e}}(\zeta^{\hat{n}}) \underset{(3)}{=} h^{\hat{n}}$

So the polynomial $Y^n - Y^{\hat{n}} \in \mathbb{F}_q[Y]$ has all elements of H as zeros.
But $\deg(Y^n - Y^{\hat{n}}) \leq max\{n, \hat{n}\} \leq (p \cdot m)^{\lfloor\sqrt{|G|}\rfloor} \leq n^{\sqrt{|G|}} < |H|$
$\Rightarrow$ contradiction since $Y^n - Y^{\hat{n}} \neq 0$ $\square$

1.4 Cryptology

A ("Alice") wants to send a message to B ("Bob") such that an eavesdropper E ("Eve") can not read the clear message. So A and B encrypt the message.

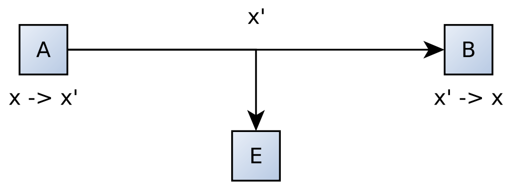


Figure 2: Scheme of eavesdropping

Symmetric-key cryptography
A and B share secret keys for encryption ($x \mapsto x'$) and decryption ($x' \mapsto x$) Only A and B know the keys.
Example: AES approved by the US government in 2002
Application:

- sending messages
- encrypt files (A=B)

Problem: Key exchange between A and B

Public-key cryptography
Encryption-map $\phi : x \mapsto x'$ is made public by B, but decryption $\phi : x' \mapsto x$ is kept secret.
Advantage: No confidential key exchange.
Disadvantages:

- more costly than symmetric key cryptography
- doubt weather E can reconstruct ϕ^{-1} from ϕ with enough computing power

Applications:

- sending messages
- exchange of symmetric keys
- authentication: Together with x, B sends $\phi^{-1}(x)$
  (or $\phi^{-1}|$ Part of x together with date). A verifies by applying ϕ.
  Better: challenge-response-protocol.

Examples: RSA, elliptic curve

1.4.1 Algorithm (RSA)

(1) B chooses $p, q \in \mathbb{P}$ large ($>$ 100 digits)
with $p \neq q \quad n := p \cdot q$

(2) B chooses $e, f \in \mathbb{N}$ large such that $e \cdot f \equiv 1 \pmod{\phi(n)}$
with $\phi(n) = (p-1)(q-1)$

(3) B makes n, e public, keep f secret

(4) The message is encoded as an element $x \in \mathbb{Z}/(n)$

(5) A computes $\phi(x) = x^e = y \in \mathbb{Z}/(n)$ and sends y

(6) B receives y and computes $y^f = x \in \mathbb{Z}/(n)$

Comments on steps of RSA:
(6) Have $e \cdot f = a \cdot (p-1) \cdot (q-1) + 1 \quad$ with $a \in N_{>0}$
$y^f = x^{e \cdot f}$

Case 1: $q \nmid f, q \nmid x \Rightarrow x^{a(p-1)(q-1)} = (x^{\phi(n)})^a \underset{LittleFermat}{\equiv} 1^a = 1 \Rightarrow x^{e \cdot f} = x \qquad \checkmark$

Case 2: $p|x, q \nmid x \Rightarrow x^{e \cdot f} \equiv 0 \equiv x \pmod{p}$
$x^{e \cdot f} \equiv x \pmod{q}$ as above.

Case 3: $q|x$ As Case 2

$\Rightarrow$ Correctness of decryption
Cost:

(1) Finding p, q of length approximately l. Prime-number theorem: Gap between two primes of length $\approx l$ is $O(l)$
Using Miller Rabin with error probability 2^{-m}. Expected cost of (1) is $O(m \cdot l^4)$ bit operations.

(2) Choose e co-prime to $\phi(n)$ obtain $f =$ inverse $\pmod{\phi(n)}$ by extended euclidean Algorithm: $O(l^2)$

(5)(6) Fast exponentiation: $O(l^3)$

Security of RSA: p and q must be so large that factorization of n is "impossible". Assumption that factorization is expensive could not be shown! But could f be obtained without knowing p and q? The following algorithm gives a negative answer. It shows that the problem of breaking RSA is always basically factorization.
Remember: $\phi(n)|(e \cdot f - 1) =: m \leq n^2$

1.4.2 Algorithm 1 (Finding a divisor)

Input : $n \in \mathbb{N}_{>2}$ *odd squarefree* $\notin \mathbb{P}$ and $m \in \mathbb{N}_{>0}$ such that $\phi(n)|m \quad m \leq n^2$

Output: $d \in \mathbb{N}$ with $d|n \quad 1 < d < n$

(1) Choose $a \in \{2, ..., (n-2)\}$ randomly
set $k := m$

(2) If $d := \gcd(a, n) \neq 1$
return d

(3) Repeat steps (4) - (8) //while(true)

(4) compute $d := \gcd(n, a^k - 1)$

(5) If $d = 1$ go to (1)

(6) If $d < n$ return d

(7) if k is *odd* go to (1)

(8) set $k := \frac{k}{2}$

Correctness is clear. What about termination and running time?

1.4.3 Proposition 2 (Complexity of Algorithm 1)

Algorithm 1 terminates in expected time $O(l(n)^4)$ bit operations (Las Vegas Algorithm).
Proof:
Set $l := \text{length}(n)$
Have $n = \prod_{0=1}^{r} p_i$ with $p_i \in \mathbb{P}$ distinct.
$\phi(n) = \prod_{i=1}^{r} (p_i - 1) \mid m$ So initially all $(p_i - 1)$ divide k.
At some iteration it happens for the first time that $(p_i - 1) \nmid k$
Then $k \equiv \frac{p_1 - 1}{2} \pmod{(p_1 - 1)} \Rightarrow a^k \equiv \pm 1 \pmod{p_i}$ -1 occurs fore some a
For those j with $(p_j - 1) \mid k$ have $n^k \equiv 1 \pmod{p_j}$
Consider the group homomorphism: $\phi_i (\mathbb{Z}/(n))^\times \mapsto (\mathbb{Z}/(p_1))^\times \times ... \times (\mathbb{Z}/(p_r))^\times$
$\bar{a} \mapsto (a^k \mod p_1, ..., a^k \mod p_r)$
The image of ϕ is a product of groups $\{\pm\}$ or $\{1\}$ depending whether $(p_i - 1) \nmid k$ or $(p_i - 1)|k$

Conclusion:
For at least half of all a's, $\phi(\bar{a})$ is neither (1,...,1) nor (-1,...,-1)
If $a^k \equiv \ \ 1 \pmod{p_j}$ then $p_j \mid (a^k - 1) \Rightarrow p_j \mid d$
If $a^k \equiv -1 \pmod{p_j}$ then $p_j \nmid (a^k - 1) \Rightarrow p_j \nmid d$
So for these a the algorithm is successful.
This means that the expected number of a's that need to be tested is ≤ 2
(Since $\sum\limits_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = 2$ More generally for $0 < p < 1 : p \cdot \sum\limits_{i=1}^{\infty} i \cdot (1-p)^{i-1} = \frac{1}{p}$)
Analysis of running time (in bit operations) for each a (using gcd is quadratic) leads to the claim. $\square$

Problems of RSA:

- How difficult is factorization of integers (lower bound?)
- decryption of some or all messages without having f?

1.4.4 Diffie-Hellmann Key Exchange

Goal: A, B want to exchange a symmetric key via a public channel

(1) A and B agree on a $p \in \mathbb{P}$ (should be large) and $g \in (\mathbb{Z}/(p))^{\times}$ public

(2) A chooses $a \in \{2, ..., (p-2)\}$ randomly and sends $u := g^a$ to B

(3) B chooses $b \in \{2, ..., (p-2)\}$ randomly and sends $v := g^b$ to A

(4) A computes $v^a = (g^b)^a = g^{a \cdot b}$
B computes $u^b = (g^b)^a = g^{a \cdot b}$

$\Rightarrow$ A and B share $g^{a \cdot b}$

Example:

A chooses a = 7	B chooses b = 4
$\bar{3}^7 = \bar{11} \in \mathbb{Z}/(17)$	$\bar{3}^4 = \bar{13} \in \mathbb{Z}/(17)$
$\bar{13}^7 = \bar{4}$	$\bar{11}^4 = \bar{4}$

If Eve reconstructs a, b from g^a and g^b she can compute $g^{a \cdot b}$

The Security of Diffie-Hellmann depends on the difficulty of the discrete logarithm problem (DLP):
Given $g \in G$ element of a group or monoid and given $g^a \in G$, determine a (or determine $a' \in \mathbb{Z}$ such that $g^a = g^{a'}$)

1.4.5 Elliptic curve cryptography (ECC)

ECC uses elliptic curves as groups.
$y^2 = x^3 + a \cdot x + b \rightsquigarrow y^2 z = x^3 + axz^2 + bz^3$
ECC uses suitable elliptic curves on $\mathbb{F}_a$

1.5 Factorization

Let $m \in \mathbb{N}_{>1}$ $\quad n \notin \mathbb{P}$ Find a divisor d with $1 < d < n$. From this we obtain the factorization of n by recursion.
Naive method: Trial division. Cost essentially exponential in $l(n)$

1.5.1 Algorithm 1 (Sieve of Eratosthenes)

Input : $n \in \mathbb{N}_{>1}$

Output: All primes $\leq n$

(1) Create a list of all numbers $\leq n$

(2) $p := 2$

(3) Mark all multiples of p in the List

(4) if all numbers are marked
    return

(5) Let p be the smallest number that is not marked

(6) $p \in \mathbb{P}$ Go to (3)

Running time of Algorithm 1 is exponential.

Pollard's rho (ρ) algorithm:
Idea: Choose a function $\mathbb{Z}/(n) \mapsto \mathbb{Z}/(n)$ e.g. $f(x) = x^2 + 1$
Choose $x_0 \in \mathbb{Z}/(n)$ set $x_i := f^i(x_0)$ iterative application.
Let $p \mid n$ be a prime. Since $|\mathbb{Z}/(p)| < \infty$ then $\exists i < j : x_i \equiv x_j \pmod p$
Starting at x_i the sequence of x_j will be periodic mod p.
$p \mid x_i - x_j \qquad p \mid n \Rightarrow p \mid \gcd(n, x_i - x_j) =: d$
If $x_i \not\equiv x_j \pmod n$ (which is not guaranteed) then d is a proper divisor of n.

- Recall that gcd computation is cheap
- Testing all pairs is a lot
- Proposition 2 helps with this

1.5.2 Proposition 2 (length of periods)

Let M be a set. $f : M \mapsto M$ and $x_0 \in M \quad x_i := f^i(x_0)$
If $x_{t+l} = x_t$ for $l, t \in \mathbb{N} l > 0 \qquad$ ($\rightarrow$ t "off-period", l "length of period")
$\Rightarrow \exists j \in \mathbb{N}$ with $0 < j \leq t + l$ such that $x_j = x_{2j}$
Proof:
$f^l(x_t) = x_t \Rightarrow \forall a \in \mathbb{N} \quad f^{a \cdot l}(x_t) = x_t \qquad$ Assume $j = a \cdot l \geq t \quad a \in \mathbb{N}$
$x_{2j} = x_{t+(j-t)+a\cdot l} = f^{(j-t)}(x_{t+a\cdot l}) = f^{(j-t)}(f^{al}(x_t)) = f^{(j-t)}(x_t) = x_j$
Case 1 $t = 0 \quad j = l \qquad \checkmark$
Case 2 $t > 0 \quad j = t + \underbrace{(-t \mod l)}_{\in 0,\ldots,(l-1)} \qquad \checkmark$

1.5.3 Algorithm 3 (Pollard's ρ - Algorithm)

Input : $n \in \mathbb{N}_{>1}, n \notin \mathbb{P}$

Output: a proper divisor of n or "FAIL"

(1) Choose $x \in \{0, ..., (n-1)\}$ randomly
set $y := x$

(2) repeat (3)-(6)

(3) $x := x^2 + 1 \pmod n$ $\quad y := (y^2+1)^2 + 1 \pmod n$ $\quad //x := x_j y := x_{2j}$

(4) $d := \gcd(n, x - y)$

(5) if $(1 < d < n)$
return d

(6) if $d = n$
return "FAIL"

One "FAIL" includes no conclusion so you might want to repeat the Algorithm with a different x.
Running time? Assume the $x_i := f^i(x_0)$ are randomly distributed.
When can we expect that a match ($x_i \equiv x_j \pmod p$) occurs? $\rightarrow$"Birthday Problem"

Lemma (Birthday Problem):
We iteratively choose numbers in $\{1, ..., n\}$ at random. The expected numbers of choices (if we keep choosing until a number has been chosen twice) is $< \sqrt{\frac{\pi \cdot n}{2}} + 2$
Proof:
Let $s \geq 2$ be the numbers of choices until a match occurs. For $k \in \mathbb{N}$ with $P(s > k)$ as probability

$$P(s>k) = \prod_{i=1}^{k} \left(1 - \tfrac{i-1}{n}\right) \underset{*}{\leq} \prod_{i=1}^{k} e^{-\frac{i-1}{n}} = e^{\sum_{i=1}^{k} -\frac{i-1}{n}} = e^{\frac{k(1-k)}{2n}} \leq e^{-\frac{(k-1)^2}{2n}}$$

* since $f(x) = e^x - (1-x) \geq 0$ for $x \geq 0$
$f(0) = 0$
$f'(x) \geq 0$ if $x \geq 0$

$$\sum_{k=0}^{\infty} P(s>k) = 2 + \sum_{k=2}^{\infty} P(s>k) \leq 2 + \sum_{k=2}^{\infty} e^{-\frac{(k-1)^2}{2n}} \leq 2 + \int_1^{\infty} e^{-\frac{(x-1)^2}{2n}} dx$$

$$\underset{x:=x-1}{=} 2 + \int_0^{\infty} e^{-\frac{x^2}{2n}} dx = 2 + \int_0^{\infty} e^{-\left(\frac{x}{\sqrt{2n}}\right)^2} dx$$

$$\underset{x:=\frac{x}{\sqrt{2n}}}{=} 2 + \sqrt{2n} \int_0^{\infty} e^{-x^2} dx = 2 + \sqrt{2n} \cdot \frac{\sqrt{\pi}}{2} = 2 + \sqrt{\frac{n \cdot \pi}{2}}$$

Example:
People arrive at a party. When can you expect to have two that share their birthday?
$\rightarrow$ when 26 have arrived!

1.5.4 Theorem 4 (Bit-complexity of Algorithm 3)

under suitable assumptions on the distribution $f^i(x)$ for $f(x) = x^2 + 1$ Algorithm 3 has the expected running time of $O(\sqrt[4]{n}\lg(n)^2)$ bit operations

Proof:
By Proposition 2 and the Lemma the expected number of runs through the loop is $O(\sqrt{p}) = O(\sqrt[4]{n})$ as $p \leq \sqrt{n}$
Each run through the loop takes $O(\lg(n)^2)$ bit operations. $\square$

Pollard's p-1 Algorithm
Motivation: Let $p \mid n$ prime divisor
$\Rightarrow \forall a \in \mathbb{Z} : a^{p-1} \equiv 1 \pmod p$ with $\gcd(a, p) = 1$
$\Rightarrow \forall m \in \mathbb{Z}$ with $(p-1) \mid m :\ a^m \equiv 1 \pmod p$
$p \mid \gcd(a^m - 1, n)$
Let B be an upper-bound for the prime powers dividing $p-1$.
"$p-1$ is $B-$power-smooth".
Then $(p-1) \mid \prod_{(q \leq B) \in \mathbb{P}} q^{\lfloor \log_q(B) \rfloor}$
Neither p nor B are known! But guess and try B and hope for the best.

1.5.5 Algorithm 5 (Pollard's ρ - 1 method)

Input : $n \in \mathbb{N}_{>1} \backslash \mathbb{P}$

Output: $d \in \mathbb{N}$ with $d \mid n \quad 1 < d < n$ or "FAIL"

(1) Choose a "smoothness bound" B

(2) Choose $a \in \{2, ...(n-2)\}$ randomly

(3) Use Algorithm 1 to find all $q \in \mathbb{P}$ with $q \leq B$
For every q perform steps (4) - (5)

(4) $k := q^{\lfloor \log_q(B) \rfloor}$
set $a := a^k \pmod n$
compute $d := \gcd(n, a-1)$

(5) if $1 < d < n$
return d

(6) return "FAIL" //or increase B and go to (1)

Consequence: when setting up RSA p, q should be chosen such that $p-1$ and $q-1$ have large prime divisors.

The quadratic sieve (State of the art factorization algorithm)
Observation: if $n = x^2 - y^2$ then $n = (x-y)\cdot(x+y)$
Conversely if $n = a \cdot b$ then $n = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$

1-st Idea: Find $x, y \in \mathbb{Z}$ such that $x^2 \equiv y^2 \pmod{n} \quad \wedge \quad x \not\equiv \pm y \pmod{n}$
Then $n \mid (x-y)\cdot(x+y)$
$\Rightarrow$ for every $p \in \mathbb{P}$ with $p \mid n : p \mid (x-y) \vee p \mid (x+y)$
$\Rightarrow p \mid \gcd(x-y, n) \vee p \mid \gcd(x+y, n)$
Since both gcd are $< n$ receive a non-trivial divisor of n

If $x^2 \equiv y^2 \pmod{n}$ how probable is it that $x \equiv \pm y \pmod{n}$?
Let $n = \prod\limits_{i=1}^{r} p_i^{k_i}$ *odd* with $p_i \in \mathbb{P}$ distinct.
Assume $p_i \nmid x \forall i = 1...r$ Since $(\mathbb{Z}/(p_i^{k_i}))^\times$ is cyclic there are 2^r classes $y \mod n$ such that $x^2 \equiv y^2 \pmod{n}$
[Reason: These classes are given by $y \equiv \pm x \pmod{p_i^{k_i}}$ These are the only solutions since $\mathbb{Z}/(p_i^{k_i})^\times$ is cyclic of even order.
$G =< \sigma >$ cyclic of order $2m$
$x = \sigma^i$ Find $l \in \mathbb{Z}$ such that $x^2 = (\sigma^l)^2$
$\Leftrightarrow 2j \equiv 2l \pmod{2m} \Leftrightarrow j \equiv l \pmod{m}$
$\Leftrightarrow l \equiv j \pmod{2m}$ or $l \equiv j + m \pmod{2m}$]
But have $x \equiv \pm y$ only for $2y$'s.
Failure probability: 2^{1-r}
Handle case $r = 1$ by Algorithm 11 in 1.3
Example 1
$n = 91$ Search $x, y \in \mathbb{Z} \quad k \in \mathbb{Z}$ such that $x^2 = k \cdot n + y^2$
Good chance if x is slightly bigger than $\sqrt{k \cdot n}$
$k := 1 \Rightarrow \sqrt{91} \approx 9,54 \Rightarrow x := 10 \Rightarrow 10^2 = 100 \equiv 3^2 \pmod{91}$
$n = 10^2 - 3^2 = (10-3)\cdot(10+3) = 7 \cdot 13 \qquad \checkmark$
Another try:
$k := 8 \Rightarrow \sqrt{8 \cdot 91} \approx 26,98 \Rightarrow 27^2 \equiv 1^2 \pmod{91} \quad \gcd(26, 91) = 13$
Example 2
$n = 4633 \quad k := 3$
$\sqrt{3 \cdot n} \approx 117,89 \Rightarrow x^2 = 118^2 \equiv 5^2 \pmod{n}$
$\gcd(118 - 5, n) = 113$
$\gcd(118 + 5, n) = 41 \qquad \checkmark$
2-nd Idea: Choose $B \in \mathbb{N}$ "smoothness bound" suitable.
Let $p_2, ...p_r \in \mathbb{P}$ be all primes $\leq B$ (Algorithm 1) set $p_1 := -1$
The p_i form a "factor basis".
For $a \in \mathbb{Z}$ write $(a \mod n)$
for the $x \in \mathbb{Z}$ with $x \equiv a \pmod{n}$ and $-\frac{n}{2} < x \leq \frac{n}{2}$
Procedure:
Search numbers $a_1, ..., a_m \in \mathbb{Z}$ such that $(a_i^2 \mod n) = \prod\limits_{j=1}^{r} p_j^{e_{ij}}$

with $e_{ij} \in \mathbb{Z}$ ("B numbers")

So for $\mu_i, ..., \mu_m \in \mathbb{N}_0$ have $\left(\prod_{i=1}^{m} a_i^{\mu_i}\right)^2 \equiv \prod_{i=1}^{m}\prod_{j=1}^{r} p_j^{\mu_i \cdot e_{ij}} \pmod{n} = \prod_{j=1}^{r} p_j^{\sum_{i=1}^{m} \mu_i \cdot e_{ij}} \pmod{n}$

If the vectors $(e_{i1}, ..., e_{ir})$ become linearly dependent mod 2 (guaranteed if $m > r$) then $\exists \mu_1, ...\mu_m \in \{0,1\}$ not all 0 such that:

$\sum_{i=1}^{m} \mu_i \cdot e_{ij} = 2 \cdot k_j \qquad k_j \in \mathbb{N}_0$

with $x := \prod_{i=1}^{m} a_i^{\mu_i} \quad y := \prod_{j=1}^{r} p_j^{k_j} \quad$ obtain $x^2 \equiv y^2 \pmod{n}$

Example: $n = 4633 \quad$ choose $B = 3 \quad \Rightarrow$ factor basis $-1, 2, 3$

Search $a \in \mathbb{Z}$ such that $|a_i^2 \mod n|$ is small. Idea: $a \approx \sqrt{n} = 68.06...$

$$
\begin{aligned}
a_1 := 68 \quad &: \quad 68^2 = n - 9 \equiv (-1) \cdot 3^2 \pmod{n} \\
&\quad\ \rightarrow e_1 = (1,0,2) \rightarrow (1,0,0) \in \mathbb{F}_2^3 \\
a_2 := 69 \quad &: \quad 69^2 = n + 128 \equiv 2^7 \pmod{n} \\
&\quad\ \rightarrow e_2 = (0,7,0) \rightarrow (0,1,0) \in \mathbb{F}_2^3 \\
a_3 := 67 \quad &: \quad 67^2 = n - 144 \equiv (-1) \cdot 2^4 \cdot 3^2 \\
&\quad\ \rightarrow e_3 = (1,4,2) \rightarrow (1,0,0) \in \mathbb{F}_2^3
\end{aligned}
$$

$e_1 + e_3 \equiv 0 \pmod{2} \qquad$ In fact:

$e_1 + e_3 = 2 \cdot \underbrace{(1,2,2)}_{(k_1,k_2,k_3)} \rightarrow \mu_1 = 1 \quad \mu_2 = 0 \quad \mu_3 = 1$

$x := a_1 \cdot a_3 \equiv -77 \pmod{n}$

$y := (-1) \cdot 2^2 \cdot 3^2 = -36$

$x - y = -41 \qquad x + y = -113$

$\gcd(n, x-y) = 41 \qquad \gcd(n, x+y) = 113 \qquad \checkmark$

3rd Idea: Look for a_i of the form $t + \lfloor\sqrt{n}\rfloor$ with t in "suitable" sieve Interval: $[-s, s] \cap \mathbb{Z}$

As it turns out if $s \leq \frac{\sqrt{5}-2}{2}\lfloor\sqrt{n}\rfloor \approx 0.118\lfloor\sqrt{n}\rfloor$ then

$(t + \lfloor\sqrt{n}\rfloor)^2 \mod n = (t + \lfloor\sqrt{n}\rfloor)^2 - n =: f(t)$

When does $p_j^{e_j}$ divide $f(t)$ (with $j \geq 2$)? Precisely if $(t + \lfloor\sqrt{n}\rfloor)^2 \equiv n \pmod{p_j^{e_j}}$

If this holds for some t then it also holds for all $t + k \cdot p_j^{e_j}$ with $k \in \mathbb{Z}$ Moreover if it holds then $\bar{n} \in \mathbb{F}_{p_j}$ is square. So may remove all p_j such that $\bar{n} \in \mathbb{F}_{p_j}$ is a non-square from the factor basis.

Obtain a sieving procedure:

For $t \in [-s, s] \cap \mathbb{Z}$ with $p_j^{e_j} \mid f(t)$ "mark" all elements $t + k \cdot p_j^{e_j} \in [-s, s]$

1.5.6 Algorithm 6 (Quadratic sieve, simplified version)

Input : $n \in \mathbb{N}_{>1} \backslash \mathbb{P}$ *odd*

Output: A non trivial divisor of n or "FAIL"

(1) if $(n = m^e)$ with $m, e \in \mathbb{N}_{>1}$
    return m // can be done with Algorithm 11 § 3

(2) Choose a "smoothness bound" $B \in \mathbb{N}$ and a "sieve bound" $s \in \mathbb{N}$ suitably

(3) Let $p_1 = -1 \quad p_2, ..., p_r$ be the factor basis given by B.
Delete those p_j such that $\bar{n} \in \mathbb{F}_{p_j}$ is a non-square

(4) for $(t = -s, -s+1, ..., s-1)$
    compute $f_t := |(t + \lfloor\sqrt{n}\rfloor)^2 - n| \quad \in \mathbb{N}_{>0}$

(5) for $(t = -s, ..., s)$
    set $e_t := (0, ..., 0) \in \mathbb{N}_0^r$ // initialize exponent vectors

(6) for $(t = -s, ..., 0)$
    set $e_{t,1} := 1$ //$\rightarrow$ first entry of each e_t is the exponent of $p_1 = -1$ in $f(t)$

(7) for $(j = 2, ..., r)$ repeat (8) - (10)

(8) for $(e = 1, ...\lfloor\log_{p_j}(B)\rfloor)$ repeat (9) - (10) // or maybe a bit larger

(9) solve $(t + \lfloor\sqrt{n}\rfloor)^2 \equiv n \pmod{p_j^e}$ for t
Let $(t_1 \mod p_j^e), ..., (t_m \mod p_j^e)$ be the solutions.
// We will see that $m \in \{0, 2, 4\}$ with $m = 2$ most frequent.

(10) for all $t = t_i + k \cdot p_j^e \in [-s, s]$ with $k \in \mathbb{Z}$, $i = 1, ..., m$
set $e_{t,j} := e_{t,j} + 1$
    $f_t := \frac{f_t}{p_j}$

(11) let $t, ..., t_m$ be those $t \in [-s, s] \cap \mathbb{Z}$ for which $f_t = 1$
/* So the $a_i = t_i + \lfloor\sqrt{n}\rfloor$ are B-numbers and the factorization
 * of $a_i^2 \mod n = a_i^2 - n = f(t)$ is given by the exponent
 * vectors e_t */

(12) if the $(e_{t_i} \mod 2) \in \mathbb{F}_2^r (i = 1, ..., m)$ are not linearly dependent.
    return "FAIL"

(13) compute $\mu_1, ..., \mu_m \in \{0, 1\}, k_1, ..., k_r \in \mathbb{N}_0$ such that $\sum_{i=1}^{m} \mu_i e_{t_i} = 2 \cdot (k_1, ..., k_r)$

(14) set $x := \prod_{i=1}^{m} (t_i + \lfloor\sqrt{n}\rfloor)^{\mu_i} \mod n$
    $y := \prod_{j=1}^{r} p_j^{k_j} \mod n$ //Now $x^2 \equiv y^2 \pmod{n}$

(15) if $\gcd(n, x-y)$ or $\gcd(n, x+y)$ is a non-trivial divisor
    return the non-trivial divisor
  else
    return "FAIL"

With good heuristics it will almost certainly never return FAIL.

Example: $n = 20437$
Choose $B := 10 \quad s := 3$
Factor basis: $p_1 = -1 \quad p_2 = 2 \quad p_3 = 3 \quad p_4 = 7$
(5 omitted as: $n \equiv 2 \pmod 5$ non-square)
$\lfloor\sqrt{n}\rfloor = 142$
Solve $(t+142)^2 \equiv n \pmod{p_j^e}$
$p_2 = 2$: Compute modulo 2,4,8. $n \equiv 5 \pmod 8$
t *odd* $\Rightarrow (t+142)^2 \equiv 1 \pmod 8 \Rightarrow (t+142)^2 \equiv n \pmod 4$ but not $\pmod 8$
t *even* $\Rightarrow (t+142)^2 \equiv 0 \pmod 2 \not\equiv n \pmod 2$
$\Rightarrow e_{t,2} = \begin{cases} 2 & t \ odd \\ 0 & t \ even \end{cases}$
$p_3 = 2 : \ n \equiv 1 \pmod 3 \qquad \lfloor\sqrt{n}\rfloor \equiv 1 \pmod 3$
So $3 \mid f(t) \Leftrightarrow t+1 \equiv \pm 1 \pmod 3 \Leftrightarrow t \equiv 0$ or $1 \pmod 3$
$e = 2 \quad n \equiv 7 \equiv (\pm 4)^2 \pmod 9 \quad \lfloor\sqrt{n}\rfloor \equiv 7 \pmod 9$
So $9 \mid f(t) \Leftrightarrow t+7 \equiv \pm 4 \pmod 9 \Leftrightarrow t \equiv -3, -2 \pmod 9$
$p_4 = 7 \quad n \equiv 4 \pmod 7 \quad 4 = (\pm 2)^2 \quad \lfloor\sqrt{n}\rfloor \equiv 2 \pmod 7$
So $7 \mid f(t) \Leftrightarrow t+2 \equiv \pm 2 \pmod 7 \Leftrightarrow t \equiv 0$ or $3 \pmod 7$

t	-3	-2	-1	0	1	2	3
$f_t = \lvert f(t)\rvert$	1116	837	556	273	12	295	588
p_1 component of e_t	1	1	1	1	0	0	0
p_2 component	2	0	2	0	2	0	2
f_t divided by 2-power	279	837	139	273	3	299	147
p_3 component	2	2	0	1	1	0	1
f_t	31	93	139	91	1	299	49
p_4 component	0	0	0	1	0	0	2
f_t	31	93	139	13	1	299	1

Obtain $m = 2 : \ t_1 = 1 \quad t_2 = 3 \quad e_1 = (0,2,1,0) \quad e_3 = (0,2,1,2)$
They are linear dependent $\pmod 2$
$e_1 + e_3 = 2 \cdot (0,2,1,1)$
$x = (142+1)\cdot(142+3) \equiv 298 \pmod n$
$y = p_2^2 \cdot p_3 \cdot p_4 = 2^2 \cdot 3 \cdot 7 = 84$
$\gcd(n, x-y) = \gcd(n, 214) = 107$
$\gcd(n, x+y) = 191$
Indeed $n = 107 \cdot 191$

Computing square roots $\pmod{p^e}$

Case 1: p *odd*

Find x with $x^2 \equiv n \pmod{p}$ by trying $x \mod p$ (exactly two solutions). Suppose we have found x with $x^2 \equiv n \pmod{p^e}$

So $x^2 - n = p^e \cdot r \quad r \in \mathbb{Z}$

New x should be $x + y \cdot p^e$

Compute modulo p^{e+1} : $(x+y\cdot^e)^2 - n = x^2 + 2yxp^e + y^2p^{2e} - n \equiv p^e \cdot (r + 2xy) \pmod{p^{e+1}}$

So $(x + y \cdot p^e)^2 \equiv n \pmod{p^{e+1}} \Leftrightarrow 2xy \equiv -r \pmod{p}$ uniquely and easily solvable

$\rightarrow$ Obtain two solutions $\pmod{p^e}$

$\Rightarrow$ special case of "Hensel lifting"

Case 2: $p = 2$

Find $x \in \mathbb{Z}$ with $x^2 \equiv n \pmod{8}$ (0 or 4 solutions since n *odd*)

Assume we have $x^2 \equiv n \pmod{2^e} \quad e \geq 3$

So $x^2 - n = r \cdot 2^e$

$\Rightarrow (x + y \cdot 2^{e-1})^2 - n = x^2 + xy \cdot 2^e + y^2 2^{2e-2} - n \equiv 2^e(r + xy) \pmod{2^{e+1}}$

So $(x + y \cdot 2^{e-1})^2 \equiv n \pmod{2}^{e+1} \Leftrightarrow y \equiv r \pmod{2}$

$\rightarrow$ 0 or 4 solutions

Running time of quadratic sieve

Choose $B \approx \exp\left(\sqrt{\frac{1}{2}\ln(n) \cdot \ln(\ln(n))}\right)$

If $s \approx B$ then running time is: $O\left(\exp\left(\sqrt{\ln(n) \cdot \ln(\ln(n))}\right)\right)$

which is "slightly" sub-exponential

Factorization algorithm with best complexity (known to date):

Number field sieve

This also uses ideas 1 and 2, but an algebraic number field is used for generating B-numbers.

Heuristic Running time (modulo some conjectures): $O\left(\exp\left(\ln(n)^{\frac{1}{3}} \cdot \ln(\ln(n))\right)^{\frac{2}{3}}\right)$

2 Systems of equations

2.6 Linear Algebra

Tasks:

- solving systems of linear equations (= linear systems)
- inversions of matrices
- rank determination
- determinants
- matrix products

K field, $K^{m\times n}$ = set of $m \times n$ matrices
$GL_n(K)$ = field of $n \times n$ matrices
Count the cost of algorithms in terms of field operations. If K is a finite field this translates directly to bit operations.

2.6.1 Proposition 1 (Complexity of usual algorithms)

(a) Solving an $m \times n$-linear system by Gaussian elimination requires $O\left(\max\{m,n\}^3\right)$ field operations

(b) For $A \in GL_n(K)$ computing A^{-1} by usual method requires $O(n^3)$ field operations.

(c) Computing $\det(A)$ "as usual" requires $O(n^3)$ bit operations.

(d) Computing $A \cdot B$ for $A \in K^{m\times n} \quad B \in K^{n\times l}$ requires $O(m \cdot n \cdot l)$ field operations.

$\rightarrow$ all cubic!

Proof:

(a) Cost of treating the k-th row with Gauss algorithm:
≤ 1 inversion, $\leq (n-k)$ multiplications
$\leq (m-k)(n-k)$ multiplications and additions
(clearing column below pivot element)
Back substitution (i.e. clearing columns above pivot element):
Let $r = rk(A) \leq (k-1)(n-r)$ multiplications and additions
Total cost $\leq \sum\limits_{k=1}^{r} (1 + n - k + 2(m-k)(n-k) + 2(k-1)(n-r))$
$= 2mnr - mr^2 - \frac{1}{3}r^3 - nr + \frac{3}{2}r^2 + \frac{5}{6}r - mr$
$\in O(\max\{m,n\}^3)$

(b) Inversion is Gaussian elimination of $n \times 2n$-matrix of rank n
cost $\leq \frac{8}{3}n^3 - \frac{3}{2}n^2 + \frac{5}{6}n \qquad \in O(n^3)$

(c) reduced to (a)

(d) obvious

Strassen-multiplication

let $A, B \in K^{2n\times 2n}$ Write: $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with $A_{ij}\;\; B_{ij} \in K^{n\times n}$

Then $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ with $C_{ij} = A_{i1}B_{aj} + A_{i2}B_{2j} \to 8$ multiplications.

Set:
$M_1 := (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$
$M_2 := (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$
$M_3 := (A_{11} - A_{21}) \cdot (B_{11} + B_{12})$
$M_4 := (A_{11} + A_{12}) \cdot B_{22}$
$M_5 := A_{11} \cdot (B_{12} - B_{22})$
$M_6 := A_{22} \cdot (B_{21} - B_{11})$
$M_7 := (A_{21} + A_{22}) \cdot B_{11}$

Then:
$C_{11} = M_1 + M_2 - M_4 + M_6$
$C_{12} = M_4 + M_5$
$C_{21} = M_6 + M_7$
$C_{22} = M_2 - M_3 + M_5 - M_7$

$\to$ 7 Multiplications!

2.6.2 Algorithm 2 (Strassen-multiplication)

Input : $A \in K^{m\times n} B \in K^{n\times l}$

Output: $A, B \in K^{m\times l}$

(1) Let k be minimal such that $m, n, l \leq 2^k$

(2) if $(k = 0)$ $\qquad$ $//(\Leftrightarrow A, B \in K^{1\times 1})$
return $A \cdot B$

(3) Enlarge A, B by adding zeros such that $A, B \in K^{2^k \times 2^k}$

(4) write $A \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with $A_{ij}B_{ij} \in K^{2^{k-1}\times 2^{k-1}}$

(5) compute $M_1 ... M_7$ as above, do multiplications by recursive call

(6) compute $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ by above formulas

(7) Output: the upper left $m \times l$ - part of $A \cdot B$

2.6.3 Theorem 3 (Running time of Algorithm 2)

If $m, n, l \leq r$ Algorithm 3 requires $O(r^{\lg(7)})$ field operations

Proof:
Set $\Theta(k)$ = number of field operations.

Step 5: $7 \cdot \Theta(k-1) + 10 \cdot (2^{k-1})^2$

Step 6: $8 \cdot (2^{k-1})^2$

Obtain:

$$\Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1} \tag{$*$}$$

Claim: $\Theta(k) = 7^{k+1} - 6 \cdot 4^k$

Induction on k

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$k - 1 \to k : \Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1}$

$\underset{induction}{=} 7(7^k - 6 \cdot 4^{k-1}) + 18 \cdot 4^{k-1}$

$= 7^{k+1} - 4 \cdot 6 \cdot 4^{k-1} \quad \checkmark$

Have $2^{k-1} < r \Rightarrow k < \lg(r) + 1 \Rightarrow \Theta(k) < 7^{\lg(r)+2} = 49 \cdot 2^{\lg(7) \cdot \lg(r)} = 49^{\lg(17)} \quad \checkmark$

Remarks:

(a) $\lg(7) = 2.8074...$

(b) Coppersmith-Winograd: $O(r^{2.3754...})$
Improved by Stothes (2010), Williams(2011), Le Gall(2014): $O(r^{2.3729...})$

(c) The cost of the best possible algorithm is unknown, even for $r = 3$

Let $M : \mathbb{N}_{>0} \mapsto R_{>0}$ be a function such that two matrices in $K^{n \times n}$ can be multiplied in $\leq M(n)$ field operations. Assume $\exists \epsilon > 0 : \forall n :$

$$2^{2+\epsilon} M(n) \leq M(2n) \leq 8 \cdot M(n) \tag{1}$$

Example: $M(n) = 49 \cdot n^{\lg(7)}$

Recall: $A = (a_{ij})$ is upper (lower) triangular $\Leftrightarrow a_{ij} = 0$ for $i > j \qquad (i < j)$

2.6.4 Proposition 4 (Complexity of matrix inversion)

An upper or lower triangular matrix $A \in GL_n(K)$ can be inverted in $O(M(n))$ field operations.

Proof:
Let $k \in \mathbb{N}$ be minimal such that $n \leq 2^k$
write $B = \begin{pmatrix} A & 0 \\ 0 & I_{2^k-n} \end{pmatrix} \in GL_{2^k}(K) \Rightarrow B^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & I_{2^k-n} \end{pmatrix}$
Assume B upper triangular:
$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \qquad B_{11}, B_{22} \in GL_{2^{k-1}}(K), B_{12} \in K^{2^{k-1} \times 2^{k-1}}$
$B^{-1} = \begin{pmatrix} B_{11}^{-1} & -B_{11}^{-1} \cdot B_{12} \cdot B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{pmatrix}$

Let $\Theta(k)$ = computation cost for inversion depending on k.

$$\Theta(k) \leq 2 \cdot \Theta(k-1) + 2 \cdot M(2^{k-1}) \underset{(1)}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2} \cdot M(2^k) \qquad (**)$$

Claim: $\Theta(k) \leq 2^k + M(2^k)$
$k = 0 : \Theta(k) = 1 \qquad \checkmark$
$k-1 \to k : \Theta(k) \underset{**}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2}M(2^k) \underset{induction}{\leq} 2(2^{k-1} + M(2^{k-1})) + \frac{1}{2}M(2^k) \underset{(1)}{\leq}$
$2^k + \frac{1}{2}M(2^k) + \frac{1}{2}M(2^k) \qquad \checkmark$
Have $n > 2^{k-1} \Rightarrow k < \lg(n) + 1 \Rightarrow \Theta(k) < 2 \cdot n + M(2n) \underset{(1)}{\leq} 2 \cdot n + 8 \cdot M(n) \qquad \square$

Project: Reduce (most) tasks of linear algebra to multiplication.
The following algorithm transforms a matrix such that all tasks become easy.

2.6.5 Algorithm 5 (Transforming a matrix)

Input : $A \in K^{m \times n}$

Output: Matrices L, Q, P, U
such that: $LQAP = \begin{array}{|c|}\hline U \\ \hline 0 \\ \hline \end{array} \begin{array}{l} r \quad (\leftarrow \text{ in row-echelon form}) \\ m-r \end{array} \in K^{m \times n}$
such that:

- $L \in K^{m \times m}$ lower triangular with 1's on the diagonal
- $Q \in K^{m \times m} \quad P \in K^{n \times n}$ permutation matrices
- $U \in K^{m \times m}$ upper triangular with non-zero diagonal entries ($r = 0$ if $A = 0$)
- If $r = m$ then $Q = I_m$

(1) if $(A = (0...0))$

return $L = Q = (1) \quad P = I_n \quad r = 0$

(2) if $(A = (a_1, ..., a_n))$

let i be minimal with $a_i \neq 0$

$P :=$ matrix exchanging 1st and i-th position in A

return $L = Q = (1) \quad P \quad U = A \cdot P$

(3) let $m_1 = \lfloor \frac{m}{2} \rfloor \qquad m_2 = \lceil \frac{m}{2} \rceil$

write $A = \begin{array}{|c|l} \cline{1-1} B & m_1 \\ \cline{1-1} C & m_2 \\ \cline{1-1} \end{array} \quad B \in K^{m_1 \times n} \quad C \in K^{m_2 \times n}$

(4) Applying the algorithm recursively on B

obtain $L_1 \cdot Q_1 \cdot B \cdot P = \begin{array}{|c|l} \cline{1-1} U_1 & r_1 \\ \cline{1-1} 0 & m_1 - r_1 \\ \cline{1-1} \multicolumn{1}{c}{n} & \end{array} \quad$ with $U_1 \in K^{r_1 \times n}$

(5) write $L_1 = \begin{array}{|c|c|l} \cline{1-2} L_t & 0 & r_1 \\ \cline{1-2} L_l & L_r & m_1 - r_1 \\ \cline{1-2} \multicolumn{1}{c}{r_1} & \multicolumn{1}{c}{m_1 - r_1} & \end{array} \qquad Q_1 = \begin{array}{|c|l} \cline{1-1} Q_t & r_1 \\ \cline{1-1} Q_b & m_1 - r_1 \\ \cline{1-1} \multicolumn{1}{c}{m_1} & \end{array} \qquad U_1 = \begin{array}{|c|c|l} \cline{1-2} E & U_1' & r_1 \\ \cline{1-2} \multicolumn{1}{c}{r_1} & \multicolumn{1}{c}{n - r_1} & \end{array}$

form $D := C \cdot P_1 = \begin{array}{|c|c|l} \cline{1-2} F & D' & m_2 \\ \cline{1-2} \multicolumn{1}{c}{r_1} & \multicolumn{1}{c}{n - r_1} & \end{array}$

and $G := D' - FE^{-1}U' \in K^{m_2 \times (n - r_1)}$

(6) Apply the algorithm recursively to G :

$L_2 \cdot Q_2 \cdot G \cdot P_2 = \begin{array}{|c|l} \cline{1-1} U_2 & r_2 \\ \cline{1-1} 0 & m_2 - r_2 \\ \cline{1-1} \multicolumn{1}{c}{n - r_1} & \end{array}$

(7) return

$$L := \begin{array}{|ccc|l} \multicolumn{1}{c}{r_1} & \multicolumn{1}{c}{m_2} & \multicolumn{1}{c}{m_2 - r_1} & \\ \cline{1-3} L_t & 0 & 0 & r_1 \\ -L_2Q_2FE^{-1}L_t & L_2 & 0 & m_2 \\ L_l & 0 & L_r & m_1 - r_1 \\ \cline{1-3} \end{array}$$

$$Q := \begin{array}{|cc|l} \multicolumn{1}{c}{m_1} & \multicolumn{1}{c}{m_2} & \\ \cline{1-2} Q_t & 0 & r_1 \\ 0 & Q_2 & m_2 \\ Q_b & 0 & m_1 - r_1 \\ \cline{1-2} \end{array}$$

$$P := P_1 \cdot \begin{array}{|cc|l} \multicolumn{1}{c}{r_1} & \multicolumn{1}{c}{n - r} & \\ \cline{1-2} I_{r_1} & 0 & r_1 \\ 0 & P_2 & n - r_1 \\ \cline{1-2} \end{array}$$

$$U := \begin{array}{|cc|l} \multicolumn{1}{c}{r_1} & \multicolumn{1}{c}{n - r_1} & \\ \cline{1-2} E & U_1'P_2 & r - 1 \\ 0 & U_2 & r_2 \\ \cline{1-2} \end{array}$$

2.6.6 Theorem 6 (Correctness and running time of Algorithm 5)

Algorithm 5 is correct and requires $O((\frac{n}{m}+1)\cdot M(m))$ field operations

Proof:
Correctness by induction on m
$m = 1 \quad \checkmark$
$m > 1$:
$m_1, m_2 < m$ so recursive calls are correct by induction.
By step (7) L, Q, P, U have desired form.
Compute:

$$LQAP = \begin{array}{cc} \begin{matrix} \quad m_1 \quad & \quad m_2 \end{matrix} & \\ \begin{bmatrix} L_tQ_t & 0 \\ -L_2Q_2FE^{-1}L_tQ_t & L_2Q_2 \\ L_lQ_t + L_rQ_b & 0 \end{bmatrix} & \begin{matrix} r_1 \\ m_2 \\ m_1 - r_1 \end{matrix} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} B\cdot P_1 \\ D \end{bmatrix} & \begin{matrix} m_1 \\ m_2 \end{matrix} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} & \begin{matrix} r_1 \\ n - r_2 \end{matrix} \end{array}$$

$$\underset{(4)}{=} \begin{array}{cc} n & \\ \begin{bmatrix} U_1 \\ L_2Q_2(-FE^{-1}U_1 + D) \\ 0 \end{bmatrix} & \begin{matrix} r_1 \\ m_2 \\ m_1 - r_1 \end{matrix} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} & \begin{matrix} r_1 \\ n - r_2 \end{matrix} \end{array}$$

$$= \begin{array}{cc} \begin{matrix} r_1 & \quad n - r_1 \end{matrix} & \\ \begin{bmatrix} E & U_1' \\ 0 & L_2Q_2G \\ 0 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ m_2 \\ m_1 - r_1 \end{matrix} \end{array} \cdot \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} \underset{(6)}{=} \begin{array}{cc} \begin{matrix} r_1 & \quad n - r_1 \end{matrix} & \\ \begin{bmatrix} E & U_1'P_2 \\ 0 & U_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ m_2 \cdot r_2 \\ m_2 \cdot r \end{matrix} \end{array} = \begin{bmatrix} U \\ 0 \end{bmatrix}$$

Suppose $r = m \Rightarrow r_1 = m_1 \quad r_2 = m_2 \Rightarrow Q_1 = I_{m_2} \quad Q_2 = I_{m_2} \Rightarrow Q = I_m$

Cost:
Fix $n \in \mathbb{N}$ and set $\Theta(k) :=$ maximal cost for a matrix $A \in K^{m\times n'}$ with $m \leq 2^k \quad n' \leq n$
Choose $A \in K^{m\times n'}$ with cost $= \Theta(k)$
Step(4) and (6): $\leq \Theta(k-1)$ each
Step(5): E^{-1}: by Proposition 4: $O(M(r_1)) \leq O(M(2^{k-1}))$
$F \cdot E^{-1} :\leq M(2^{k-1})$
$F \cdot E^{-1} \cdot U'$: at most the cost of multiplying a $2^{k1} \times 2^{k-1}-$ matrix by a $2^{k-1} \times n$ matrix.
Split right matrix into square parts.
$\Rightarrow$ cost $\leq \lceil \frac{n}{2^{k-1}} \rceil \cdot M(2^{k-1}) \leq (2^{1-k}\cdot n + 1)\cdot M(2^{k-1})$
G: subtraction: $m_2 \cdot (n - r_1) \leq 2^{k-1}\cdot n \leq 2^{1-k}\cdot n \cdot M(2^{k-1})$
Step (7): $F \cdot E^{-1}$ already computed, L_2Q_2: permuting rows. Cost: $\leq 2 \cdot M(2^{k-1})$
Obtain: $\Theta(k) \leq 2\cdot\Theta(k-1) + (2^{-k}\cdot n + c)\cdot M(2^k) \qquad c$ constant
From this obtain by induction:
$\Theta(k) \leq \left(2^{-k}\cdot n \cdot \frac{1-2^{-k\epsilon}}{1-2^{-\epsilon}} + 2\cdot c\cdot(1-2^{-k})\right)\cdot M(2^k) \leq \left(\frac{1}{1-2^{-\epsilon}}\cdot\frac{n}{2^k} + 2c\right)\cdot M(2^l)$
Finally obtain: Cost $\leq 8 \cdot \max\left\{\frac{1}{1-2^{-\epsilon}}, c\right\}\cdot\left(\frac{n}{m}+1\right)\cdot M(m) \qquad \square$

$\begin{array}{|c|}\hline U \\ \hline 0 \\ \hline \end{array}$ is in row echelon form. It's convenient to write $U = \begin{array}{|c|c|}\hline E & U' \\ \hline \end{array} \; r \qquad U' \in K^{r\times(n-r)}$

Also write $L = \begin{array}{c}m \\ \begin{array}{|c|}\hline L_1 \\ \hline L_2 \\ \hline \end{array}\end{array} \begin{array}{l} r \\ m-r \end{array}$

2.6.7 Theorem 7 (Benefit of matrices of Algorithm 5)

Let $A \in K^{m\times n}$, $LQAP = \begin{array}{cc} r & n-r \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c|}{U'} \\ \hline \multicolumn{2}{|c|}{0} \\ \hline \end{array} \begin{array}{l} \\ r \\ m-r \end{array}$ as in Algorithm 5 then

(a) $rk(A) = r$

(b) The columns of $P\cdot \begin{array}{c} u-r \\ \begin{array}{|c|}\hline E^{-1}\cdot U' \\ \hline -I_{n-r} \\ \hline \end{array}\end{array}$ form a basis of $ker(A)$

(c) A linear system $Ax = b \qquad b \in K^m$ is solvable iff $L_2Q \cdot b = 0$

(d) if $Ax = b$ is solvable then $x = P\cdot \begin{array}{|c|}\hline E^{-1}L_1 \\ \hline 0 \\ \hline \end{array} \begin{array}{l} r \\ n-r \end{array} \cdot Q \cdot b$ is a solution

(e) if $A \in GL_n(K)$ then $\det(A) = \det(P) \cdot \underbrace{\det(E)}_{=\text{prod of diags}}$

and $A^{-1} = P \cdot E^{-1} \cdot L$

Proof:

(a), (e) : ✓

(b) : $LQAP \begin{array}{|c|}\hline E^{-1}\cdot U' \\ \hline -I_{n-r} \\ \hline \end{array} = \begin{array}{|c|c|}\hline E & U' \\ \hline \multicolumn{2}{|c|}{0} \\ \hline \end{array} \cdot \begin{array}{|c|}\hline E^{-1}\cdot U' \\ \hline -I_{n-r} \\ \hline \end{array} = 0$

$\Rightarrow$ the columns lie in $ker(A)$

The columns of $\begin{array}{|c|}\hline E^{-1}\cdot U' \\ \hline -I_{n-r} \\ \hline \end{array}$ are linear independent.

$\Rightarrow rk(P\cdot \begin{array}{|c|}\hline E^{-1}\cdot U' \\ \hline -I_{n-r} \\ \hline \end{array}) = n-r$

$\Rightarrow$ the columns form a basis.

The space they generate has dimension $n - r = dim(ker(A))$

(c), (d) : If $A \cdot x = b$ then $\begin{array}{|c|c|}\hline E & U' \\ \hline \multicolumn{2}{|c|}{0} \\ \hline \end{array} \cdot P^{-1} \cdot x = LQb = \begin{array}{|c|}\hline L_1Qb \\ \hline L_2Qb \\ \hline \end{array}$

$\Rightarrow L_2Qb = 0$

if $L_2Qb = 0$ then $A \cdot P\cdot \begin{array}{|c|}\hline E^{-1}\cdot L_1 \\ \hline 0 \\ \hline \end{array} \cdot Q \cdot b = Q^{-1} \cdot L^{-1} \begin{array}{|c|c|}\hline E & U' \\ \hline \multicolumn{2}{|c|}{0} \\ \hline \end{array} \cdot \begin{array}{|c|}\hline E^{-1}\cdot L_1 \\ \hline 0 \\ \hline \end{array} \cdot Qb =$

$Q^{-}1 \cdot L^{-1} \begin{array}{|c|}\hline L_1 \\ \hline 0 \\ \hline \end{array} Q \cdot b = Q^{-1} \cdot L^{-1} \begin{array}{|c|}\hline L_1 \\ \hline L_2 \\ \hline \end{array} \cdot Q \cdot b = b$

2.6.8 Corollary 8 (Complexity of benefits)

For $A \in K^{m\times n}$ the determination of $rk(A)$ and solving linear systems with coefficient matrix A require $O((\frac{n}{m}+1)\cdot M(m))$ field operations.
If $A \in K^{n\times n}$ then computing $\det(A)$ and A^{-1} (if $A \in GL_n(K)$) require $O(M(n))$ field operations.

From $LQAP = \boxed{\begin{array}{c} U \\ \hline 0 \end{array}}$ get $A = Q^{-1} \cdot \underbrace{L^{-1}}_{\text{lower triangular}} \boxed{\begin{array}{c} U \\ \hline 0 \end{array}} \cdot P^{-1}$

Generally $Q = I_m \Rightarrow A = L^{-1} \cdot \boxed{\begin{array}{c} U \\ \hline 0 \end{array}}\ P^{-1}$ ”LUP decomposition”

If also $P = I_n$ obtain $A = L^{-1} \cdot \boxed{\begin{array}{c} U \\ \hline 0 \end{array}}$ ”LU decomposition”

2.7 Algebraic Systems of Equations, Gröbner bases

Given: $f_1 ... f_m \in K[x_1 ... x_n]$ multivariate polynomials.
Wanted: solution set of the algebraic system $f_1 = f_2 = ... = f_m = 0$
The solution set $\mathcal{V}(f_1 ... f_m) \subseteq K^n$ is called an affine variety.
Often assume $K = \bar{K}$ K algebraically closed (e.g. $K = \mathbb{C}$)
Questions:

1. $\mathcal{V}(f_1 ... f_m) \neq \emptyset$?
2. $|\mathcal{V}(f_1 ... f_m)| < \infty$?
3. $\dim \mathcal{V}(f_1 ... f_m) =$?

Examples:

(1) $f_1 = x_1x_3x_4^2 - 2x_2x_4^2 + x_1x_3 - 2x_2$
$f_2 = x_1x_3x_4 - 2x_2x_4 - 1$
$f_3 = x_1x_4^2 + x_1 + 2$
One has $(-x_1x_4) \cdot f_1 + (x_1x_4^2 + x_19f_2 + f_3) = 2$
$\Rightarrow \mathcal{V}(f_1 ... f_3) = \emptyset$

(2) $f_1 = x^3 + x^2y + xy + y^2$
$f_2 = x^2y^2 + x^2 + y^3 + y$
$f_3 = x^3 + xy$
$(x^2 + y) \mid f_i \quad \forall i = 1, 2, 3 \Rightarrow |\mathcal{V}(f_1, f_2, f_3)| = \infty$

Univariate case (n=1):
$K[x]$ Euclidean so have Euclidean algorithm for computing $\gcd(f, g)$. gcd is unique if required to be monic (i.e. the highest coefficient is 1)
Also get $h_1, h_2 \in K[x]$ such that $\gcd(f, g) = h_1f + h_2g$
Let $f_1 ... f_m \in K[x]$ Obtain

$$g := \gcd(f_1...f_m) = \sum_{i=1}^{n} h_i \cdot f_i \qquad \text{with } h_i \in K[x] \tag{$*$}$$

For $\xi \in K$:
$f_1(\xi) = f_2(\xi) = ... = f_m(\xi) = 0 \underset{*}{\Rightarrow} g(\xi) = 0 \underset{g|f_i}{\Rightarrow} f_i(\xi) = 0 \qquad \forall i$
$\Rightarrow$ Only need to get zeros of one polynomial!

Resultant method:
Reminder: For $f, g \in K[x]$: $\gcd(f, g) \neq 1 \Leftrightarrow res(f, g) = 0$
Let $f_1, f_2 \in K[x_1...x_n]$
$(\xi_1...\xi_{n-1}) \in K^{n-1}$ assume $K = \bar{K}$
Then $\exists \xi_n \in K$ such that $f_1(\xi_1...\xi_n) = 0 = f_2(\xi_1...\xi_n)$
$\Leftrightarrow res_{x_n}(f_1(\xi_1...\xi_{n-1}, x_n)), f_2(\xi_1...\xi_{n-1}, x_n)) = 0$
Suppose $\deg_{x_n} f_1(\xi_1...\xi_{n-1}, x_n)) = \deg_{x_n}(f_i) \qquad (i = 1, 2)$
Set $h = res_{x_n}(f_1, f_2) \in K[x_1...x_{n-1}]$
Then $res_{x_n} f_1(\xi_1...\xi_{n-1}, x_n)), f_2(\xi_1...\xi_{n-1}, x_n)) = h(\xi_1...\xi_{n-1})$
Search zeros of $h \rightarrow$ one variable, one equation fewer.
Limitation: Only for pairs of polynomials ($m = 2$).
Good case: $m = n = 2$

Given $f_1...f_m \in K[x_1...x_n]$ form the ideal
$I = (f_1...f_m) = \left\{ \sum\limits_{i=0}^{n} g_i f_i \mid g_i \in K[x_1...x_n] \right\}$
Clearly $\mathcal{V}(I) = \mathcal{V}(f_1...f_m)$
$f_1...f_m$ are called a ideal basis of I. They are not unique, not even their size is unique.
Example:
In Example (1) I has an alternative basis $I = (1) \quad \leftarrow$ constant polynomial $1 \in K[x_1...x_4]$
In Example (2) it turns out that $I = (x^2 + y)$

2.7.1 Theorem 1 (Hilbert's Nullstellensatz)

Hilbert's Nullstellensatz (1st version):
Assume $K = \bar{K}$ let $I \subseteq K[x_1...x_n]$ be an ideal
Then $\mathcal{V}(I) = \emptyset \quad \Leftrightarrow \quad 1 \in I$
$(\Leftrightarrow I = K[x_1...x_n] \Leftrightarrow I = (1))$
without proof.

For $I \subseteq R$ ideal in a commutative ring R the radical ideal of I is
$\sqrt{I} = \{a \in R \mid \exists n \in \mathbb{N} : a^n \in I\}$
I is called a radical ideal if $I = \sqrt{I}$

Let $S \subseteq K^n$ set of points
$\Rightarrow Id(S) := \{f \in K[\underline{x}] \mid f(v) = 0 \;\; \forall v \in S\} \subseteq K[\underline{x}]$
where $K[\underline{x}] := K[x_1, ..., x_n]$ and $Id(S)$ is a radical ideal (called vanishing ideal)

Hilbert's Nullstellensatz (2nd version):
Let $K = \bar{K} \quad I \subseteq K[\underline{x}]$ ideal. Then $\sqrt{I} = Id(\mathcal{V}(I))$

Obtain bijection: {radical ideals in $K[\underline{x}]$} $\Leftrightarrow$ {affine varieties}
This bijection is inclusion-reversing.

Monomial orderings:

2.7.2 Definition 2 (Monomial, monomial ordering, LM, LT, LC)

A monomial is a polynomial of the form $t = x_1^{e_1} \cdot x_2^{e_2}, ..., x_n^{e_n} =: \underline{x}^{\underline{e}}$ where $e_i \in \mathbb{N}$
A term is a polynomial of the form $c \cdot t \qquad t$ monomial, $c \in K \backslash \{0\}$
$M :=$ set of all monomials.
For $f \in K[\underline{x}]; M(f) :=$ set of all monomials occurring in f.
$T(f) :=$ set of all terms ...
A monomial ordering is an ordering (= order relation) " $\leq$ " on M such that:

1. " $\leq$ " is total i.e. $\forall s, t \in M : s \leq t \vee t \leq s$

2. $1 \leq t \qquad \forall t \in M$

3. $\forall s, t_1, t_2 \in M : t_1 \leq t_2 \Rightarrow s \cdot t_1 \leq s \cdot t_2$

(This implies: $s \mid t \Rightarrow s \leq t$)
For $f \in K[\underline{x}] \backslash \{0\}$ we write
$LM(f) =: t$ for the largest monomial in $M(f)$ ("leading monomial"),
$LT(f) =: c \cdot t$ for the largest term if t is in f ("leading term")
$LC(f) =: c$ ("leading coefficient")
$LM(0) = LT(0) = LC(0) = 0$

Example 1: Lexicographic ordering (lex)
for $t = x_1^{e_1} \cdot ... \cdot x_n^{e_n} \quad t' = x_1^{e'_1} \cdot ... \cdot x_n^{e'_n}$
define $t \leq t' \Leftarrow \quad t = t' \ \vee \ e_i < e'_i$ for the smallest i with $e_i \neq e'_i$

Example 2: graded reverse lexicographic ordering (grevlex)
$t \leq t' \Leftarrow \quad t = t' \ \vee \ \deg(t) < \deg(t') \ \vee \ (\deg(t) = \deg(t') \ \wedge \ e_i > e'_i)$
for the largest i such that $e_i \neq e'_i$
where $\deg(t) := \sum e_i$

For both lex and grevlex have
$x_1 > x_2 > ... > x_n$ but $x_1 \cdot x_3 >_{\text{lex}} x_2^2$
$x_1 > x_2 > ... > x_n$ but $x_1 \cdot x_3 <_{\text{grevlex}} x_2^2$

2.7.3 Proposition 3 (Sum and product of LM / LT)

Let " $\leq$ " be a monomial ordering $f, g \in K[\underline{x}] \Rightarrow$

(a) $LT(f \cdot g) = LT(f) \cdot LT(g)$ same for LM

(b) $LM(f+g) \leq \max\{LM(f), LM(g)\}$

Proof:

(b) ✓

(a) write $c \cdot t = LT(f) \quad d \cdot s = LT(g)$
For $t' \in M(f) \quad s' \in M(g)$ have $\underbrace{t's' \leq t \cdot s' \leq t \cdot s}_{=?}$ with equality iff $s' = s \quad t' = t$

This implies (a) $\square$

2.7.4 Lemma 4 (Dickson-Lemma)

Every subset $S \subseteq M$ has a finite subset $B \subseteq S$ ("basis") such that $\forall s \in S \; \exists t \in B : t \mid s$

Proof: Identify M with $\mathbb{N}^n$
Given $S \subseteq \mathbb{N}^n$ need to show that:
$\exists B \subseteq S, B$ finite such that $\forall (e_1, ..., e_n) \in S$
$\exists (d_1, ..., d_n) \in B$ such that $\forall i : d_i \leq e_i$
We will write $(\underline{d}) \leq (\underline{e})$ for this. (This defines a partial ordering in $\mathbb{N}^n$)
Induction:
$n = 1$: if $\emptyset \neq S \subseteq \mathbb{N}$ then $\exists d \in S$ such that $d \leq e \quad \forall e \in S$ ($\mathbb{N}$ is well-ordered)
$n > 1$: For $k \in \mathbb{N}$ write $S_k := \{(e_2, ..., e_n) \in \mathbb{N}^{n-1} \mid (k, e_2, ..., e_n) \in S\} \subseteq \mathbb{N}^{n-1}$
By induction $\exists B_k \subseteq S_k$ finite such that $\forall (\underline{e}) \in S_k \quad \exists (\underline{d}) \in B_k$ such that $(\underline{d}) \leq (\underline{e})$
$\bigcup\limits_{k \in \mathbb{N}} B_k \subseteq \mathbb{N}^{n-1}$ has finite "basis" C
C finite: $\exists r \in \mathbb{N} : C \subseteq \bigcup\limits_{k=0}^{r} B_k$ $(*)$
From $B := \{(e_1, ..., e_n) \in \mathbb{N}^n \mid e_1 \leq r, (e_2, ..., e_n) \in B_{e_1}\} \Rightarrow |B| < \infty, B \subseteq S$
Claim: B basis of S
Let $(e_1, ..., e_n) \in S \Rightarrow (e_2, ..., e_n) \in S_{e_i} \Rightarrow \exists (d_2, ..., d_n) \in B_{e_1}$ such that $d_i \leq e_i \quad \forall i \geq 2$
Case 1: $e_i \leq r$
$\Rightarrow (e_1, d_2, ..., d_n) \in B$ have $(e_1, d_2, ..., d_n) \leq (e_1, ...e_n)$
Case 2: $e_i > r$
$B_{e_i} \subseteq \bigcup\limits_{k \in \mathbb{N}} B_k \Rightarrow \exists (c_2, ..., c_n) \in C$ such that $c_i \leq d_i \quad \forall i \geq 2$
By $(*) \exists k \leq r : (\underline{c}) \in B_k \Rightarrow (k, c_2, ..., c_n) \in B$
$(k, c_2, ..., c_n) \leq (e_1, d_2, ..., d_n) \leq (e_1, e_2, ..., e_n)$ ✓ $\square$

2.7.5 Corollary 5 (Well-ordering of monomial sets)

Every monomial ordering is a well-ordering i.e. every monomial set $S \subseteq M$ has an element $t \in S$ such that $\forall s \in S : t \leq s$ (t is a "least element")

Proof:
Let $\emptyset \neq S \subseteq M$. By Lemma 4 $\exists B \subseteq S$ finite such that $\forall s \in S' \quad \exists t \in B : t \mid s$
Since " $\leq$ " is total and B is finite $\exists t \in B$ least element.
Let $s \in S \Rightarrow \exists t' \in B$ such that $t' \mid s \Rightarrow t' \leq s$ so $t \leq t' \leq s$ $\square$

Gröbner bases: Let " $\leq$ " be a fixed monomial ordering

2.7.6 Definition 6 (Leading ideal, Gröbner bases)

(a) For $S \in K[\underline{x}]$ subset define $L(S) := (LM(f) \mid f \in S) \subseteq K[\underline{x}]$
(ideal generated by all leading monomials of elements of S) is called the leading ideal

(b) Let $I \subseteq K[\underline{x}]$ ideal. A finite subset $G \subseteq I$ is called a Gröbner basis if $L(I) = L(G)$
i.e. $\forall f \in I \;\; \exists g \in G : LM(g)|LM(f)$

2.7.7 Proposition 7 (Ideality of Gröbner bases)

G Gröbner basis of $I \Rightarrow I = (G)$ i.e. G is an ideal basis.

Proof: $G \subseteq I \Rightarrow (G) \subseteq I$
Assume this inclusion is strict. Let $f \in I \backslash (G)$
By Corollary 5 may assume $LM(f)$ is minimal
(among all leading monomials of elements from $I \backslash (G)$)
$LM(f) \in L(I) = L(G) \Rightarrow \exists g \in G : LM(g) \mid LM(f)$
Form $\tilde{f} = f - \frac{LT(f)}{LT(g)} g, \tilde{f} \in I \Rightarrow LM(\tilde{f}) < LM(f)$
by minimality $\tilde{f} \in (G) \Rightarrow f = \tilde{f} + \frac{LT(f)}{LT(g)} g \in (G)$ contradiction! $\square$

$G \subseteq I \qquad L(G) = L(I) \qquad \Rightarrow I = (G)$
Example:
$I = (1) \in K[x] \quad S = \{x+1, x\}$ ideal basis but $L(S) = (x) \neq L(I) = (1)$
S is not a Gröbner basis.

2.7.8 Theorem 8 (Gröbner basis of Ideals)

Every ideal $I \subseteq K[\underline{x}]$ has a Gröbner basis. In particular I has a finite basis ($\rightarrow$ Hilbert's basis theorem) In other words $K[\underline{x}]$ is Noetherian.

Proof:
For $\{LM(f) \mid f \in I\}$ there exists (by Dickson lemma) a finite subset $\{LM(f_1), ..., LM(f_m)\}, f_i \in I$ such that $(LM(f_1)...LM(f_m)) = L(I) \Rightarrow G = \{f_1 ... f_m\}$
Gröbner basis $\square$
First application: Let G Gröbner basis of I
Then $\mathcal{V}(I) = \emptyset \underset{K=\bar{K}}{\Leftrightarrow} 1 \in I \Leftrightarrow G$ contains a non-zero constant.

2.7.9 Definition 9 (Normal form)

Let $S = \{g_1 ... g_m\} \subseteq K[x] \quad f \in K[x]$

(a) f is a normal form with respect to S if $\forall t \in M(f) \quad \forall i = 1...m : LM(g_i) \nmid t$

(b) $f^* \in K[x]$ is called a normal form of f with respect to S if

  (i) f^* is in normal form with respect to S

  (ii) $\exists h_1 ... h_m \in K[x]$ such that $f - f^* = \sum_{i=1}^{m} h_i g_i$ and $\forall i : LM(h_i g_i) \leq LM(f)$

Example:
$S = \{x, x+1\} \quad f = 1 \Rightarrow f \equiv 0 \pmod{(S)}$
but 0 is not a normal form of f
If $f = x$ then 0 an -1 are normal forms of x

2.7.10 Algorithm 10 (Normal form)

Input : $S = \{g_1 ... g_m\} \subseteq K[x] \quad f \in K[x]$

Output : A normal form f^* of f with respect to S and if desired $h_1 ... h_m$ satisfying $(*)$

(1) Set $f^* := f$
  for $(i = 1...m)$
    $h_i := 0$

(2) repeat (3) - (6)

(3) $\mathcal{M} := \{(t, i) \mid t \in M(f^*), i \in \{1, ..., m\}$ such that $LM(g_i) \mid t\}$

(4)   if$(\mathcal{M} = \emptyset)$
      return f^* and h

(5)   Choose $(t, i) \in \mathcal{M}$ such that t is maximal.
    let $c \in K$ be the coefficient of t in f^*

(6)   Set $f^* := f^* - \frac{c \cdot t}{LT(g_i)} \cdot g_i$
      $h_i := h_i + \frac{c \cdot t}{LT(g_i)}$

Step (6) cancels the term $c \cdot t$ from f^* and may add only monomials smaller than t. So the $t's$ form a strictly descending sequence of monomials $\underset{\text{Cor 5}}{\Rightarrow}$ Algorithm 10 terminates.
Correctness ✓

2.7.11 Theorem 11 (Normal form of Gröbner bases)

Let $G \subseteq K[x]$ be a Gröbner basis of an ideal $I \subseteq K[x]$

(a) Every polynomial $f \in K[x]$ has a unique normal form with respect to G.
Write $NF_G(f)$

(b) $NF_G : K[\underline{x}] \mapsto K[\underline{x}]$ is $K-$linear,
$ker(NF_G) = I$

(c) if $\tilde{G} \subseteq K[x]$ is another Gröbner basis (with respect to same monomial ordering) then $NF_G = NF_{\tilde{G}}$

Proof:
(a), (c):
Let $f \in K[x]$ $\quad f^*, \tilde{f} \in K[x]$ be normal forms of f with respect to G and $\tilde{G}$ respectively.
Claim: $f^* = \tilde{f}$
$f^* - f \in I, \quad \tilde{f} - f \in I \Rightarrow f^* - \tilde{f} \in I \Rightarrow LM(f^* - \tilde{f}) \in L(G) \in L(\tilde{G})$
if $f^* \neq \tilde{f} \Rightarrow LM(f^* - \tilde{f}) \in M(f^*)$ or $\in M(\tilde{f})$
But $\exists g \in G : LM(g) \mid LM(f^* - \tilde{f}), \quad \exists \tilde{g} \in \tilde{G} : LM(\tilde{g}) \mid LM(f^* - \tilde{f})$
This is a contradiction to:
f^* is in normal form with respect to G and
$\tilde{f}$ is in normal form with respect to $\tilde{G}$
So $f^* = \tilde{f}$

(b):
Let $f, g \in K[\underline{x}]$ $\quad c \in K$. Set $h := NF_G(f + cg) - NF_G(f) - c \cdot NF_G(g)$
To show: $h = 0 \qquad h \equiv f + cg - f - cg = 0 \pmod I$
$\Rightarrow h \in I \Rightarrow LM(h) \in L(G)$
h is in normal form with respect to G
$\Rightarrow h = 0$

Remains to show: $ker(NF_G) = I$
let $NF_G(f) = 0 \Rightarrow f \equiv 0 \pmod I \Rightarrow f \in I$ conversely , let $f \in I$
$\Rightarrow f^* = NF_G(f) \in I \Rightarrow \exists g \in G : LM(s) \mid LM(f^*)$ f^* in normal form. So $f^* = 0$ $\square$

Further applications of Gröbner bases:

- Membership test: $f \in I \Leftrightarrow NF_G(f) = 0$
- Computation in $A := K[\underline{x}]/I$: NF_G includes an embedding $A \leftrightarrow K[\underline{x}]$

Buchberger's Algorithm

2.7.12 Definition 12 (S-polynomials)

Let $f, g \in K[\underline{x}]\backslash\{0\}$ $\quad t := \gcd(LM(f), LM(g))$
Then $s_{pol}(f,g) := \frac{LT(g)}{t} \cdot f - \frac{LT(f)}{t} \cdot g$ $\quad$ is the $S-$polynomial.
The leading monomials of the summands cancel!
Example:
$f = x^2 + y^2, \quad g = x \cdot y \quad " \leq " = lex$
$\Rightarrow LM(f) = x^2 \quad LM(g)xy$
$s_{pol}(f,g) = y \cdot f - x \cdot y = y^3$

2.7.13 Theorem 13 (Buchberger's criterion)

For any finite set $G \subseteq K[\underline{x}]$ the following statements are equivalent:

(a) G is a Gröbner basis of (G)

(b) For polynomials $g, h \in G$, 0 is a normal form of $s_{pol}(g,h)$ with respect to G
$\rightarrow$ finite test for Gröbner basis!

Proof:
"(a) $\Rightarrow$ (b)":
For $g, h \in G : s_{pol}(g,h) \in (g,h) \subseteq (G) =: I \underset{\text{Theorem 13 (b)}}{\Rightarrow} s_{pol}(g,h)$ has normal form 0
$\Leftrightarrow NF_G(s_{pol}(g,h)) = 0$
"(a) $\Leftarrow$ (b)":
Assume G is not a Gröbner basis $\Rightarrow \exists f \in I \subset G$ such that $LM(f) \notin L(G)$.
Write $G = \{g_1...g_m\}$. Since $f \in (G)$ $\exists h_1...h_m \in K[\underline{x}]$ have $f = \sum\limits_{i=0}^{m} h_i \cdot g_i$ (1)
By Corollary 5 may choose h_i such that
$t := \max\{LM(h_i g_i) \mid i = 1...m\}$ becomes minimal.
$\exists i : LM(f) \in M(h_i g_i)$. Since $LM(f) \notin L(G) \wedge LM(f) \neq LM(h_i g_i)$
$\Rightarrow LM(f) < LM(h_i g_i) \leq t$
$\underset{(1)}{\Rightarrow}$ the coefficient of t in $\sum h_i g_i$ is zero.
Set $c_1 := \begin{cases} LC(h_i) & \text{if } LM(h_i g_i) = t \\ 0 & \text{otherwise} \end{cases}$ Then $\sum\limits_{i=1}^{m} c_i \cdot LC(g_i) = 0$ (2)
Without loss assume $c_1 \neq 0$
Let $i \in \{2, ..., m\}$ such that $c_i \neq 0 \Rightarrow LM(g_i) \mid t$
So $t_i = lcm(LM(g_i), LM(g_1) \mid t)$
Have $s_{pol}(g_i, g_1) = \frac{LC(g_1) \cdot t_i}{LM(g_i)} g_i - \frac{LC(g_i) \cdot t_i}{LM(g_1)} g_1 \qquad LM(s_{pol}(g_i, g_1)) < 0$
$\Rightarrow s_i := \frac{t}{t_i} \cdot s_{pol}(g_i, g_1) = LC(g_1) \cdot LM(h_i) \cdot g_i - LC(g_i) \cdot LM(h_1) \cdot g_1$ (3)
By (b) have $s_i = \sum\limits_{j=1}^{m} h_{ij} \cdot g_j$ $\quad$ with $h_{ij} \in K[\underline{x}]$ such that $LM(h_i g_i) \leq LM(s_i) < t$ (4)

$\sum\limits_{j=1}^{m}\left(\sum\limits_{i=2}^{m} c_i \cdot h_{ij}\right) \cdot g_j = \sum\limits_{i=2}^{m} c_i s_i$
$\underset{(3)}{=} \sum\limits_{i=2}^{m} c_i \left(LC(g_1)LM(h_i)g_i - LC(g_i)LM(h_1)g_1\right) \underset{(2)}{+} \sum\limits_{i=1}^{m} c_i LM(h_1)g_1 LC(g_i)$
$= \sum\limits_{i=1}^{m} c_i LC(g_1)LM(h_i)g_i$
Set $\tilde{h}_j := \frac{1}{LC(g_1)} \cdot \sum\limits_{i=2}^{m} c_i h_{ij} \Rightarrow g := \sum\limits_{i=1}^{m} c_i LM(h_i)g_i = \sum\limits_{i=1}^{m} \tilde{h}_i g_i$
For each i have: $LM(\tilde{h}_i g_i) \underset{(4)}{<} t$
$f = (f-g) + g \underset{(1)}{=} \sum\limits_{i=1}^{m}(h_i - c_i LM(h_i))g_i + \sum\limits_{i=1}^{m} \tilde{h}_i g_i$
For each i have: $LM((h_i - c_i LM(h_i))g_i) < t$ so $LM((h_i - c_i LM(h_i) + \tilde{h}_i)g_1) < t$ contradiction to choice of h_i $\square$

2.7.14 Algorithm 14 (Buchberger)

Input : $S \subseteq K[\underline{x}]$ finite " $\leq$ " monomial ordering

Output: A Gröbner basis G of $I = (S)$ with respect to " $\leq$ "

(1) $G := S \backslash \{0\}$

(2) for $g, h \in G$ repeat (3),(4)

(3) Compute $s := s_{pol}(g, h)$
and a normal form s^* of s with respect to G

(4) if($s^* \neq 0$)
set $G := G \cup \{s^*\}$
go to (2)

(5) return G

2.7.15 Theorem 15 (Correctness of Algorithm 14)

Algorithm 14 terminates after finitely many steps and computes a Gröbner basis.
Proof:
Termination:
Let G_i be the set G obtained after the $i-$th run through the loop. $G_1 \subseteq G_2 \subseteq G_3 \subseteq ...$
From $\bar{G} = \bigcup\limits_i G_i$ finite or infinite.
Lemma 4: $\exists B \subseteq M$ finite set of monomials, $B \subseteq \{LM(f) \mid f \in \bar{G}\}$ such that
$\forall f \in \bar{G} \quad \exists t \in B$ such that $t \mid LM(f)$ $(*)$
Since $|B| < \infty \quad \exists r$ such that $B \subseteq \{LM(f) \mid f \in G_r\}$
Without loss $B = \{LM(f) \mid f \in G_r\}$

Claim: G_r is the last of the G
If not $\exists G_{r+1} : G_{r+1} = G_r \cup \{s^*\}$
$s^* \neq 0$ in normal form with respect to G_r But by $(*)$ $\exists f \in G_r$ such that $LM(f) \mid LM(s^*)$ contradiction.
Correctness: by Theorem 13 $\square$

Example:
$S = \{\underbrace{x^2 + g^2}_{f}, \underbrace{xy}_{g}\} \subseteq \mathbb{Q}[x, y]$ " $\leq$ " lex ordering with $x > y$
$s_{pol}(f, g) = yf - xg = y^3 =: h$ in normal form with respect to S
$G = \{f, g, h\}$
$s_{pol}(f, g) = h \xrightarrow{\text{normal form}} 0$
$s_{pol}(f, h) = y^3 f - x^2 h = y^5 \xrightarrow{\text{normal form}} 0$
$s_{pol}(g, h) = y^2 g - xh = 0$
$\Rightarrow G$ Gröbner basis

Cost of Buchberger algorithm:

- no known upper bound for the running time
- with $d = \max\{deg(f) \mid f \in S\}$: $\underbrace{deg(g_i)}_{\text{polys from } G} \leq 2 \cdot \left(\frac{d^2}{2} + d\right)^{2^{n-1}}$
  with n = number of Variables
  $\Rightarrow$ "Doubly exponential" in n
  Ritscher (2011): upper bound for deg(g_i) depending $dim(\underbrace{\mathcal{V}}_{\text{Variety}}(S))$
- Nonetheless the algorithm often works
- Many possible optimizations

Variant: Extended Buchberger:
Keep track of how the new elements in G are represented as linear combination of elements of S.

2.7.16 Definition 16 (Reduced Gröbner basis)

A Gröbner basis G is called reduced if $\forall g \in G$

(a) g is in normal form with respect to $G \backslash \{g\}$

(b) $LC(g) = 1$

A given Gröbner basis can be turned into a reduced only by replacing every $g \in G$ by a normal form of g with respect to $G \backslash \{g\}$. Then remove $0 \in G$. Then divide each $g \in G$ by $LC(g)$

2.7.17 Theorem 17 (Uniqueness of reduced Gröbner basis)

From ideal $I \subseteq K[\underline{x}]$ and a monomial ordering " $\leq$ ", there exists a unique reduced Gröbner basis.

2.8 Application of Gröbner bases

2.8.1 Definition 1 (Elimination ideals)

(a) $I \subseteq K[X_1, ...X_n]$ ideal, $l \in \{1, ..., n\} \Rightarrow I_l := K[X_1, ..., X_n]$ is called the $l-$length elimination ideal

(b) A monomial ordering " $\leq$ " is called an $l-$elimination ordering if
$\forall 1 \leq i \leq l < j \leq n \ \forall k_i \quad X_i^k < X_j$

Example:

(1) Let " $\leq$ " be a given monomial ordering. Define " $\leq$ " by:
for $s = \underline{x}^{\underline{d}} \quad t = \underline{x}^{\underline{e}}$ define
$s \leq' t \Leftrightarrow \sum\limits_{i=l+1}^{n} d_i < \sum\limits_{i=l+1}^{n} e_i$ or have equality $\sum\limits_{i=l+1}^{n} d_i = \sum\limits_{i=l+1}^{n} e_i$ and $s \leq t$
$\Rightarrow$ " $\leq$ " is an $l-$eliminating ordering.

(2) The lexicographic ordering with $x_1 < x_2 < ... < x_n$ is an $l-$eliminating ordering for all l

(3) Grevlex is not an $l-$eliminating ordering (unless $l = n$)

2.8.2 Theorem 2

Let G be a Gröbner basis of an ideal $I \subseteq K[X_1, ..., X_n]$ with respect to an $l-$elimination ordering. Then $G_l := K[X_1, ..., X_n] \cap G$ is a Gröbner basis. $I : l$ with respect to the restricted monomial ordering.
Proof:
$G_l \subseteq I_l$ Let $f \in I_l \backslash \{0\} \quad f \in I$
$\Rightarrow \exists g \in G : LM(g) \mid LM(f)$
To show: $g \in G$. Clearly $LM(g) \in K[X_1, ..., X_l]$
If $g \notin K[X_1, ..., X_n]$ then $\exists t \in M(g)$ such that
$X_j \mid t$ with $j > l \Rightarrow t \geq X_j \underset{\text{"}\leq\text{"elim ord}}{>} LM(g)$ contradiction. $\square$
Example:
$I = (\underbrace{X_1^2 + X_2^2}_{f}, \underbrace{x_1 x_2}_{g}), G = \{f, g, X^3\}$
Gröbner basis with respect to lex ordering $X_1 < X_2$
$I_1 = (X_1^3)$

Geometric interpretation:
Let $K = \bar{K}$ algebraically closed. On K^n define the Zariski topology.
By saying that the sets $\mathcal{V}(I)$ with $I \subseteq K[X_1, ..., X_n]$ are the closed sets.
Why is this a topology?
Reminder: A topological space is a set X together with a system of subsets, called closed subsets, such that three axioms hold.

(1) $\emptyset = \mathcal{V}(K[X_1, ..., X_n]) \quad K^n \in \mathcal{V}(\{0\})$ closed

(2) Let $\mathcal{M}$ be a set of closed subsets corresponding to a set $\mathcal{N}$ of ideals.
Then $\bigcap\limits_{I \in \mathcal{N}} \mathcal{V}(I) = \mathcal{V}\left(\sum\limits_{I \in \mathcal{N}} I\right) \qquad \checkmark$

(3) The union of two closed subsets is closed.
Let $I, J \subseteq K[\underline{X}]$ ideals.
Claim: $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J)$
Proof: " $\subseteq$ " direction:
Let $v \in \mathcal{V}(I) \cup \mathcal{V}(J) \quad f \in I \cap J$
If $v \in \mathcal{V}(I)$ then $f(v) = 0$
$v \in \mathcal{V}(J)$ then $f(v) = 0$
" $\supseteq$ " direction:
Let $v \in \mathcal{V}(I \cap J)$ Assume $v \notin \mathcal{V}(I) \Rightarrow \exists f \in I : f(v) \neq 0$
Let $g \in J \Rightarrow f \cdot g \in I \cap J \Rightarrow \underbrace{f(v)}_{\neq 0} \cdot g(v) = 0$
$\Rightarrow g(v) = 0 \qquad$ So $v \in \mathcal{V}(J)$

All points in K^n are closed so are all finite subsets.
Closures: For $X \subseteq K^n$ the closure $\bar{X}$ is defined as the smallest closed subset containing X. $\bar{X}$ is the variety of the largest ideal I such that $X \subseteq \mathcal{V}(I)$ This ideal is $I = Id(X)$. So $\bar{X} = \mathcal{V}(Id(X))$

Let $\Pi_l : K^n \mapsto K^l, (a_1, ..., a_n) \mapsto (a_1, ..., a_l)$ projection.

2.8.3 Theorem 3

$I \subseteq K[X_1, ..., X_n] \Rightarrow \mathcal{V}(I_l) = \overline{\Pi_l(\mathcal{V}(I))}$
Proof:
Let $(a_1, ..., a_l) \in \Pi_l(\mathcal{V}(I)) \Rightarrow \exists n_{l+1}...n_1 \in K$ such that $(a_1, ..., a_n) \in \mathcal{V}(I)$
Let $f \in I_l \Rightarrow f \in I \Rightarrow f(a_1, ..., a_n) = 0$
But $0 = (a_1, ..., a_n) = f((a_1, ..., a_l))$ So $\overline{(a_1, ..., a_n)} \in \mathcal{V}(I_l)$
So $\Pi_l(\mathcal{V}(I)) \subseteq \mathcal{V}(I_l) \underset{\mathcal{V}(I_l) \text{ closed}}{\Rightarrow} \overline{\Pi_l(\mathcal{V}(I))} \subseteq \mathcal{V}(I_l)$.
To show: $\mathcal{V}(I_l) \subseteq \overline{\Pi_l(\mathcal{V}(I_l))} \underset{\text{H. Nullstellensatz}}{\Leftrightarrow} Id(\Pi_l(\mathcal{V}(I_l))) \subseteq \sqrt{I_l}$ Take $f \in Id(\Pi_l(\mathcal{V}(I_l))) \Rightarrow$
$f \in K[X_1, ..., X_l] \forall (a_1, ..., a_n) \in \mathcal{V}(\mathcal{I})$.
$f(a_1, ..., a_l) = f(a_1, ..., a_n) = 0 \Rightarrow f \in Id(\mathcal{V}(I)) = \sqrt{I} \Rightarrow \exists k : f^k \in I \cap K[X_1, ..., X_l] = I_l$
$\Rightarrow f \in \sqrt{I_l} \qquad \square$
Example:

(1) $I = (xy - 1)$
$\Pi_1(\mathcal{V}(I)) = K \backslash \{0\}$ not closed. $\overline{\Pi_1(\mathcal{V}(I))} = K.I_1 = \{0\}$

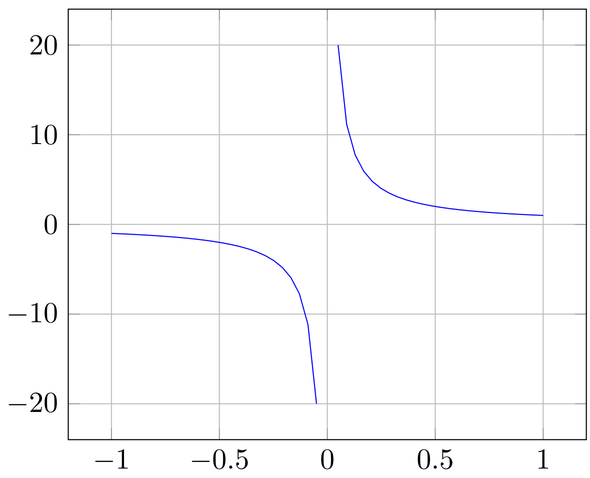


Figure 3: Plot of I in Example (1)

(2) $I = (xy - 2x + y - 2, kx^2 - 4xy + y^2)$ has reduced lex Gröbner basis:
$\{x^2 - 2x + \frac{y^2}{4} + y - 2, xy - 2x + y - 2, (y-2)(y+2)^2\}$
$\Rightarrow I_1 = \big((y-2)(y+2)^2\big)$
$\Rightarrow \overline{\Pi_1(\mathcal{V}(I))} = \{2, -2\}$
$\Rightarrow \Pi_1(\mathcal{V}(I)) = \{2, -2\}$
Substitute $y = 2$:
$x^2 - 2x + 1, 2x - 2x + 2 - 2 = 0$
$y = -2:$
$x^2 - 2x - 3 = 0, -4x - 4 = 0$
$\rightarrow \mathcal{V}(I) = \{(1, 2), (-1, -2)\}$

2.8.4 Algorithm 4 (Solving systems of algebraic equations)

Input : $f_1, ..., f_m \in K[X_1, ..., X_n]$

Output: $\mathcal{V}(f_1, ..., f_m)$ if finite otherwise "∞"

(1) Compute a Gröbner basis of $I = (f_1, ..., f_m)$ with respect to the lex ordering with $X_1 < X_2 < ... < X_n$

(2) for $(l = 1, ...n)$
    set $G_l := [X_1, ..., X_l] \cap G$

(3) $M := \{()\} \subseteq K^0$

(4) for $(l = 1, ..., n)$ repeat (5) - (10)

(5) $\quad S = \emptyset$

(6) $\quad$ for $(a_1, ..., a_{l-1}) \in M$ repeat (7) - (9)

(7) $\quad\quad g := \gcd\{f(a_1, ..., a_{l-1} \mid f \in G_l)\}$

(8) $\quad\quad$ if $(g = 0)$
        return "∞"

(9) $\quad\quad S := S \cup \{(a_1, ..., a_{l-1}, a_l) \mid g(a_l) = 0\}$

(10) $\quad M := S$

(11) return M

Intersections of ideals

2.8.5 Proposition 5

Let $I, J \subseteq K[\underline{x}]$ ideals, y additional variable.
Form $L \subseteq K[x_1, ..., x_n, y]$ generated by $I \cdot y$ and $J \cdot (1 - y)$ Then $I \cap J = K[\underline{x}] \cap L$ (elimination ideal!)
Proof:
Let $f \in J \cap J \Rightarrow f = f \cdot y + f \cdot (1 - y) \in L \Rightarrow f \in K[\underline{x}] \cap L$
Conversely let $f \in K[\underline{x}] \cap L \Rightarrow f = \sum\limits_{i=1}^{r} g_i f_i \cdot y_i + \sum\limits_{i=r}^{m} g_i \cdot f_i(1 - y)$
with $f_1, ...f_r \in I \quad f_r + 1, ..., f_m \in J \quad g_i \in K[\underline{x}, y]$
Specialize $y = 0 \Rightarrow f = \sum\limits_{i=r}^{m} \underbrace{g_i(y = 0)}_{\in K[\underline{x}]} f_i \in J$
Specialize $y = 1 \Rightarrow f = \sum\limits_{i=r}^{m} g_i(y = 1) f_i \in I$
$\Rightarrow f \in I \cap J \qquad \square$

Dimension

2.8.6 Definition 6 (independence modulo I)

Let $I \subseteq K[\underline{x}]$ ideal. Then polynomials $f_1, ..., f_r \in K[\underline{x}]$ are called independent modulo I if for every polynomial $F \in K[y_i, ..., y_r]$ have: $F(f_1, ..., f_r) \in I \Rightarrow F = 0$
(So the classes $\bar{f}_i \in A := K[\underline{x}]/I$ are algebraically independent)
For $1 \leq i_1 < i_2 < ... < i_r \leq n : x_i, ..., x_r$ are independent modulo $I \Leftarrow K[x_{i_1}, ..., x_{i_r}] \cap I = \{0\}$
(elimination ideal!)
$\dim(I) := \sup\{r \in \mathbb{N} \mid \exists f_1, ..., f_r \in K[\underline{x}]$ independent mod $I+\}$ dimension of I
In other words $\dim(I) =: trdeg(A)$ "transcendence degree"
If $I = K[\underline{x}] \Rightarrow \dim(I) = -1$
For $X = \mathcal{V}(I) : \dim(X) = \dim(I)$
Well defined? Clearly $\dim(I) = \dim(\sqrt{I})$

Geometric interpretation:
$x_{i_1}, ..., x_{i_r}$ are independent mod $I \Leftrightarrow \Pi_{i_1,...i_r} : K^n \mapsto K^r$ maps $\mathcal{V}(I)$ to a dense subset of K^r
More generally for $f_1, ..., f_r$ have:
$f_1, ...f_r$ are independent modulo $I \Leftrightarrow$ The image of $\mathcal{V}(I)$ under the "morphism" $K^n \mapsto K^r, v \mapsto (f_1(v), ..., f_r(v))$ is dense.

2.8.7 Theorem 7

Let $I \subseteq K[\underline{x}]$ ideal
$\Rightarrow \dim(I) = \max\{r \mid \exists i_1, ...i_r$ such that $x_{i_1}, ..., x_{i_r}$ are independent modulo $I\}$

2.8.8 Lemma 8

A non-empty set M of ideals in $K[\underline{x}]$ has an element that is maximal with respect to inclusion.

Proof:
If not obtain a sequence $I_1 \subsetneqq I_2 \subsetneqq I_3 \subsetneqq ...$ of ideals in M
Set $I := \bigcup\limits_{j \in \mathbb{N}} I_j$ Check: $I \subseteq K[\underline{x}]$ ideal
$\underset{\text{Thm 7.8}}{\Rightarrow} I = (f_1, ..., f_r)$ Each f_i lies in some I_j
$\Rightarrow \exists m : f_i \in I_m \; \forall i \Rightarrow I \subseteq I_m \subseteq I \Rightarrow I = I_m$
$I_m \subseteq I_{m+1} \subseteq I = I_m \Rightarrow I_{m+1} = I_m$
Recall that an ideal $I \subseteq K[\underline{x}]$ is called a prime ideal if $a \cdot b \in I$ for $a, b \in K[\underline{x}]$ implies $a \in I$ or $b \in I$
Fact: I prime ideal $\Leftrightarrow I = \sqrt{I}$ and $\mathcal{V}(I)$ is irreducible, i.e. I can't be the written as union of two proper closed subsets.

2.8.9 Lemma 9

Every radical ideal $I \subsetneqq K[\underline{x}]$ is a finite intersection of prime ideals.

Proof:
If no by Lemma 8 there is a maximal exception $I \subsetneqq K[\underline{x}]$. I not prime ideal $\Rightarrow \exists a_1, a_2 \in K[\underline{x}] : a_1 a_2 \in I_1 a_j \notin I_1 \Rightarrow I \subsetneqq I + (a_j) \subseteq \sqrt{I + (a_j)} =: I_j \qquad (*)$
$\Rightarrow I \subseteq I_1 \cap I_2$
Claim: $I = I_1 \cap I_2 \qquad \checkmark$
Let $g \in I_1 \cap I_2 \Rightarrow \exists r : g^r \in I + (a_j)(j = 1, 2)$
$\Rightarrow g^{2r} \in (I + (a_1)) \cdot (I + (a_2)) \subseteq I \underset{I=\sqrt{I}}{\Rightarrow} g \in I$

$(*)$ with maximality of I shows that the I_j are finite intersections of prime ideals
$\Rightarrow$ same holds for I contradiction $\square$
Reminder: If $L \mid K$ is a field extension $L = K[\alpha_q, ..., \alpha_n]$. $trdeg_K(L)$ the size of every maximally algebraically independent subset of $\{\alpha_1, ..., \alpha_n\}$
(Exchange argument similar to linear algebra)

Proof of Theorem 7
Clearly $\max\{r \mid \exists i_1, ..., i_r$ such that $x_{i_1}, ..., x_{i_r}$ independent modulo $I\} \leq \dim(I)$
For reverse inequality take $f_1, ..., f_r \in K[\underline{x}]$ independent mod I. By Lemma 9: $\sqrt{I} = P_1 \cap ... \cap P_s$ with P_i prime ideals.
Assume that the f_i are dependent modulo every $P_i \Rightarrow \exists F_i \in K[y_1, ...,_r] \backslash \{0\}$ such that $F_i(f_1, ..., f_r) \in P_i \Rightarrow$ For $F := \prod_{i=1}^{s} F_i i$
$F(f_1, ... f_r) \in P_1 \cap ... \cap P_s = \sqrt{I} \Rightarrow \exists k$
$F^k(f_1, ..., f_r) \in I$. But $F^k \neq 0$ contradiction
So $\exists i : f_1, ..., f_r$ independent mod P_i
$\bar{f}_1, ..., \bar{f}_r \in A := K[\underline{x}]/P$ algebraically independent.
A integral domain $\Rightarrow L = Quot(A)$ exists $\bar{f}_i \in A \subseteq L$. L/K is field extension, $L = K(\bar{x}_1, ..., \bar{x}_n)$ Since $trdeg_K(L) \geq r$ have $i_1, ..., i_r$ such that $\bar{x}_{i_1}, ..., \bar{x}_{i_r} \in L$ are algebraically independent over K.
$\Rightarrow x_{i_1}, ..., x_{i_r}$ are independent modulo P_i
Since $I \subseteq P_i$, $x_{i_1}, ..., x_{i_r}$ are independent modulo I
$\Rightarrow r \leq \max\{r \mid \exists i_1, ..., i_r$ such that $x_{i_1}, ..., x_{i_r}$ independent modulo $I\}$
Now we can compute $\dim(I)$ by using elimination ideals. Have to determine a subset of $\{x_1, ..., x_n\}$ of maximal size that is independent modulo $I \rightarrow$ Gröbner bases.
Example:
$I = (\underbrace{xy, xz}_{\text{Gröbner bases}})$ Independent subsets modulo $I : \emptyset, \{x\}, \{y\}, \{z\}, \{y, z\}$
$\dim(I) = 2$

Hilbert series For $X \subseteq K^n$ affine variety, $I = Id(X)$. Then $A := K[\underline{x}]/I$ is the set of polynomial function ("regular functions") $X \mapsto K$
How large is A? A is a vector-space but it is not finite-dimensional (as vector-space) unless $\dim(X) \leq 0$
Idea: measure the size of A by studying the growth of the dimensions of parts of A given by a filtration $A = \bigcup\limits_{d \in \mathbb{N}} A_d$

2.8.10 Definition 10 (Hilbert series)

Let $I \subseteq K[\underline{x}]$ ideal, $A := K[\underline{x}]/I$
For $d \in \mathbb{N}$ set $A_d := \{f + I \mid f \in K[\underline{x}],\ \underbrace{\deg(f)}_{=\max\{\deg(t) \mid t \in M(f)\}} \leq d\} \subseteq A$ subspace
Hilbert function: $h_I(d) := dim_K(A_d) < \infty$
Hilbert series of I: $H_I(t) = \sum\limits_{d=0}^{\infty} h_I(d) \cdot t^d \in \mathbb{Z}[[t]]$ (formal power series)

Example:

(1) $I = (x_1, ..., x_n) \Rightarrow A = K \Rightarrow A_d = K\ \forall d$
$h_I(d) = 1 \Rightarrow H_I(t) = \sum\limits_{d=0}^{\infty} t^d = \frac{1}{1-t}$

(2) $I = (x_1 - x_2^2) \subseteq K[x_1, x_2]$
The classes of $1, x_1, x_1^2, ..., x_1^d, x_2, x_2x_1, ..., x_2x_1^{d-1}$ form a basis of A_d
$\Rightarrow h_I(d) = 2d + 1$
$H_I(t) = \frac{1+t}{(1-t)^2}$

(3) $I = \{0\} \Rightarrow A = \underbrace{K[\underline{x}]}_{=x_1,...,x_n}$
write $H_n(t)$ for Hilbert series $H_0(t) = \frac{1}{1-t}$
For $n > 0 : K[x_1, ..., x_n]_d = \bigoplus\limits_{i+j=d} K[x_1, ..., x_{n-1}] \cdot x_n^j$
$\Rightarrow H_n(t) = H_{n-1}(t)(\sum\limits_{j=0}^{\infty} t^d) = H_{n-1}(t) \cdot \frac{1}{1-t} = \frac{1}{(1-t)^{n+1}}$
Hilbert function:
$H_n(t) = (1-t)^{-n-1} = \sum\limits_{d=0}^{\infty} \binom{-n-1}{d}(-t)^d$
$\Rightarrow h_n(d) = \binom{-n-1}{d} \cdot (-1)^d = \binom{n+d}{d} = \binom{d+a}{n}$

A total degree monomial ordering is a monomial ordering such that
for $t, t' \in M$ have: $t \leq t'$ implies $\deg(t) \leq \deg(t')$
Example: grevlex

2.8.11 Theorem 11

Let "$\leq$" be a total degree monomial ordering. $I \subseteq K[\underline{x}]$ ideal $\Rightarrow H_I(t) = H_{L(I)}(t)$

Proof:
Let G be a Gröbner basis of I, $d \in \mathbb{N}$
NF_G induces an injection from $\phi : A \mapsto K[\underline{x}]$. Consider $\phi_d : A_d \mapsto K[\underline{x}]$ restriction
Claim: $im(\phi_d)$ is the space $V_d \subseteq K[\underline{x}]$ generated by all monomials $m \in M$ with $\deg(m) \leq d$ such that $LM(g) \nmid m \ \forall g \in G$
Let $f \in V_d \Rightarrow f$ is in normal form with respect to $G \Rightarrow f = NF_G(f) = \phi(\underbrace{f + I}_{\in A_d}) \in im(\phi_d)$

Conversely let $f \in im(\phi_d) \Rightarrow \exists g \in K[\underline{x}]$:
$\deg(g) \leq d$
$f = NF_G(g)$
$\Rightarrow f = g - \sum\limits_{i=1}^{m} h_i g_i \qquad$ with $g_i \in G$
$h_i \in K[\underline{x}]$
$LM(h_i g_i) \leq LM(f)$
$\Rightarrow \forall t \in M(h_i g_i) : t \underset{\text{"}\leq\text{" tot. deg. mon. ord.}}{\Rightarrow} \deg(1) \leq \deg(LM(g))$
So $\deg(h_i g_i) \leq d \underset{(*)}{\Rightarrow} \deg(f) \leq d$
Since f is in normal form this implies $f \in V_d$.
So $V_d = im(\phi_d)$
$\Rightarrow h_I(d) = \dim_K(A_d) = \dim_K(im(\phi_d)) = \dim(V_d)$
V_d only depends on d and on $(LM(g) \mid g \in G) = L(I) \Rightarrow h_I$ only depends on $L(I)$
Since $L(L(I)) = L(I)$ the theorem follows $\qquad \square$

How to compute $H_I(t)$ for I monomial ideal:
Let $I = (m1, ..., m_l) \quad m_i \in M$ Set $J = (m_1, ..., m_{l-1})$
Then the map $J \underset{surjective}{\mapsto} I/(m_l)$ has the kernel $J \cap (m_l)$
$\Rightarrow J/(J \cap (m_l)) \cong I/(m_l)$
This isomorphism restricts to all homogeneous components
$$\Rightarrow H_I(t) = H_{(m_l)}(t) + H_J(t) - H_{J\cap(m_l)}(t) \qquad (*)$$

Have $J \cap (m_l) = (lcm(m_1, m_l), ..., lcm(m_{l-1}, m_l)$

2.8.12 Theorem 12

Let $I = (m_1, ..., m_l) \subseteq K[x_1, ..., x_n]$ with $m : i \in M$
$\Rightarrow H_I(t) = \frac{1}{(1-t)^{n+1}} \sum\limits_{S\subseteq\{1,...,l\}} (-1)^{|S|} \cdot t^{\deg(lcm\{m_i | i\in S\})}$

Proof:
Use induction on l, $(*)$ and bookkeeping.
$l = 0$: Example (3) $\quad\checkmark$
$l = 1 : I = (m_1)$
$\sum\limits_{d=0}^{\infty} \dim(\underbrace{I_{\leq d}}_{\text{all polys in } I \text{ of deg } \leq d}) \cdot t^d = t^{\deg(m_1)} \cdot H_{\{0\}}(t)$
$\Rightarrow H_{(m_1)}(t) = H_{\{0\}}(t) - \sum\limits_d \dim(I_{\leq d}) = \frac{1-t^{\deg(m_1)}}{(1-t)^{n+1}} \quad\checkmark$
$l - a \to l$ (with $l \geq l$): By $(*)$ with $I = (m_1, ..., m_l)$:
$(1-t)^{n+1} \cdot H_I(t) = 1 - t^{\deg(m_l)} + \sum\limits_{S \subseteq \{1,...,l-1\}} (-1)^{|S|} \cdot t^{\deg(lcm\{m_i | i \in S\})}$
$- \sum\limits_{S \subseteq \{1,...,l-1\}} (-1)^{|S|} \cdot t^{\deg(lcm\{lcm(m_i, m_l) | i \in S\})}$
For $S \neq \emptyset$ then $lcm\{lcm(m_i, m_l) \mid i \in S\} = lcm\{m_i \mid i \in S \cup \{l\}\}$
So the formula is correct $\quad\square$
$\rightarrow$ may compute Hilbert series by Gröbner bases!

2.8.13 Corollary 13 (Hilbert-Serre theorem)

Let $I \subseteq K[\underline{x}]$ ideal.
Then $H_I(t) = \frac{a_0 + a_1 t + ... + a_k t^k}{(1-t)^{n+1}} \quad a_i \in \mathbb{Z}$
For $d \gg 0$ $h_I(d)$ is a polynomial. More precisely with
$p_I := \sum\limits_{i=0}^{k} a_i \binom{x+n-i}{n} \in \mathbb{Q}[x]$
have $h_I(d) = p_I(d)$ for $d \gg 0$
p_I is called the Hilbert polynomial.

2.8.14 Definition 14 (Algebra, homo-, isomorphism)

An algebra over a field K is a commutative ring A containing K.
(Usually $A = \{0\}$ is also considered as an algebra)
A is called finitely generated if
$\exists a_1, ..., a_n \in A : A = \{f(a_1, ..., a_n) \mid f \in K[x_1, ..., x_n]\} = K[a_1, ..., a_n]$
A homomorphism $\phi A \mapsto B$ of $k-$algebras is a ring homomorphism that fixes k element-wise.
ϕ is an isomorphism it is bijective.
So if $A = K[a_1, ..., a_n]$ we have a surjective homomorphism
$\phi : K[x_1, ..., x_n] \mapsto A, \;\; f \mapsto f(a_1, ..., a_n)$. If $I := ker(\phi)$ then $K[x_1, ..., x_n]/I \cong A$

Given a finitely generated algebra A (such as $A = K[x] = K[\underline{x}]/I$ with $x = \mathcal{V}(I)$ affine variety), then each choice of finite set of generators affords an isomorphism
$K[x_1, ..., x_n]/I \cong A$. Then p_I will depend on the choice of the generators. But might the degree of p_I be an invariant of A?

2.8.15 Lemma 15

Let $I \subseteq K[x_1, ..., x_n]$, $J \subseteq K[y_1, ..., y_n]$ ideals in polynomial rings such that $\underbrace{K[x_1, ..., x_n]/I}_{A} \cong \underbrace{K[y_1, ..., y_n]/J}_{B}$. Then $\deg(p_I) = \deg(p_J)$.

Proof:
Have $\phi A \xrightarrow{\sim} B \Rightarrow \exists g_1, ..., g_m \in K[x_1, ..., x_n]$ such that $\phi(g_i + I) = y_i + J$
Choose $e \in \mathbb{N}$ such that $\deg(g_i) \leq e$
Then $B_d \subseteq \phi(A_{de}) \Rightarrow h_J(d) = \dim(B_d) \leq \dim(A_d e) = h_I(de) \Rightarrow \deg(p_J) \leq \deg(p_I)$
By symmetry $\deg(p_I) \leq \deg(p_J)$, have equality $\deg(p_J) = \deg(p_I)$ $\square$

Goal: $\deg(p_I) = \dim(I)$

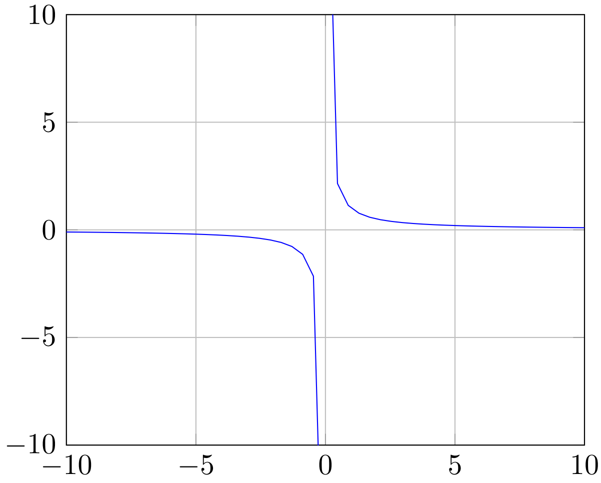
2.8.16 Theorem 16 (Noether normalization)

Let $A \neq \{0\}$ be a finitely generated algebra. Then there exist algebraically independent elements $c_1, ..., c_m \in A$ such that A is finitely generated as a module (= vector space, but ove a ring rather than a field) over $C = K[c_1, ..., c_m] \underset{\text{subalgebra}}{\subseteq} A$,

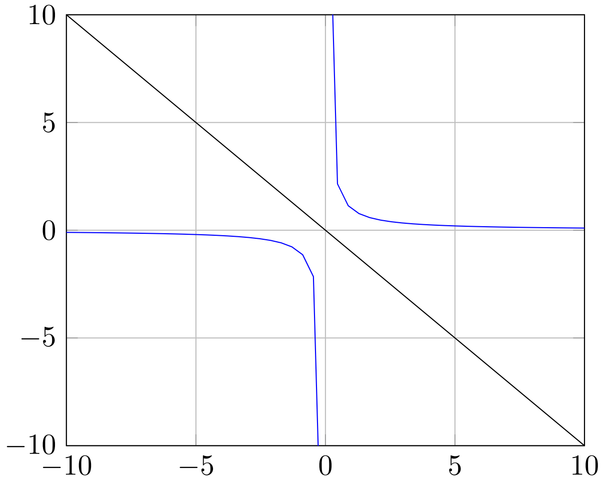
i.e. $\exists b_1, ..., b_l \in A$ such that $A = \sum_{i=1}^{l} C \cdot b_i$
Moreover if $A \cong K[\underline{x}]/I$ then $m = \dim(I)$ $(= trdeg(A))$
If $m ==$ then $C = K$
Proof:
Omitted $\rightarrow$ commutative algebra $\square$

Geometric interpretation: Let $A = K[X]$, X affine variety. Choosing $c_1, ..., c_m \in A$ amounts to choosing a morphism $f : X \mapsto K^m$, $v \mapsto (c_1(v), ..., c_m(v))$. It can be shown that if $C = K[c_1, ..., c_m]$ is Noether norm, then the morphism is surjective and with finite fibers i.e. $\forall w \in K^m$ the set $f^{-1}(\{w\})$ is finite.

Example:
$X = \mathcal{V}(xy - 1) \subseteq \mathbb{C}^2$



The projection to the x- and y-axis are not surjective.
Try $c = \bar{x} - \bar{y} \in A = \mathbb{C}[x, y]/(xy - 1)$



Have $\bar{x}^2 - c\bar{x} - 1 = 0 \Rightarrow \forall i : \bar{x}^i \in C \cdot 1 + C\bar{x}$
Similarly $f \cap \bar{y} \Rightarrow C = \mathcal{C}[c]$ satisfies Theorem 16

2.8.17 Theorem 17

Let $I \subseteq K[x_1, ..., x_n]$ ideal $\Rightarrow \dim(I) = \deg(p_I)$

Proof:

If $I = K[\underline{x}] \Rightarrow \checkmark$

So assume $I \subsetneqq K[\underline{x}]$. Then by Theorem 16 $\exists C = K[c_1, ..., c_m] \subseteq A := K[\underline{x}]/I$ and $\exists b_1, ..., b_l \in A : A = \sum\limits_{i=1}^{l} C \cdot b_i, \quad c_1, ..., c_m$ algebraic independent. Without loss $b_1 = 1$

Let $y_1, ..., y_m, z-1, ..., z_l$ be new indeterminates and consider the map

$K[y_1, ..., y_m, z-1, ..., z_l] \mapsto A, \;\; y_i \mapsto c_i, \;\; z_j \mapsto b_j$

This is surjective. Let J be the Kernel.

By Lemma 15 have $\deg(p_I) = \deg(p_J)$.

Need to show: $\deg(p_J) = m$

Write $B : K[y_1, ..., y_m, z-1, ..., z_l]/J$, consider $B_d, C_d := \{f+J \mid f \in K[y_1, ..., y_m], \deg(f) \leq d\} \subseteq B_d$ (for $d \in \mathbb{N}$).

Since $C_d \subseteq B_d$, obtain $h_J(d) = \dim(B_d) \geq \dim(C_d) \underset{c_1,...,c_m \text{alg.ind.}=\binom{d+m}{m}}{=} \dim(K[y_1, ..., y_m])_d$

$\Rightarrow \deg(p_J) \geq m$

For $1 \leq i \leq j \leq l$ have $b_i b_j = \sum\limits_{k=1}^{l} a_{ijk} \cdot b_k$ with $a_{ijk} \in C$ Choose $e \in \mathbb{N}_{>0}$ such that $\forall i,j,k : \deg(a_ijk) \leq e \Rightarrow b_i b_j \subseteq \sum\limits_{k=1}^{l}$ So the product of s of the b_i lies in $\sum\limits_{k=1}^{l} C_{(s-1)\cdot e} \cdot b_k$.

This implies (for $d \in \mathbb{N}$) $B_d \subseteq C_d \cdot b_1 + \sum\limits_{s=1}^{d} \sum\limits_{k=1}^{l} C_{d-s} \cdot C_{(s-1)e} \cdot b_k \subseteq \sum\limits_{k=1}^{l} C_{de} \cdot b_k =: V_d$

$\Rightarrow h_j(d) = \dim(B_d) \leq \dim(V_d) \leq l \cdot \dim(C_{de}) = l \cdot \binom{d \cdot e + m}{m} = \deg(p_J) \leq m \qquad \square$

(or substitute e by $2e$)

2.8.18 Corollary 18

Let " $\leq$ " be a graded monomial ordering $I \subseteq K[\underline{x}]$ ideal $\Rightarrow \dim(I) = \dim(L(I))$

For computing $\dim(L(I))$ use the transcendence degree!

Let $L(I) = (m_1, ..., m_l)$, m_i monomials. For $S \subseteq \{x_1, ..., x_n\}$,

S is dependent modulo $L(I) \Leftrightarrow \exists i : m_i \in K[S]$

S is independent modulo $L(I) \Leftrightarrow \forall i : m_i \notin K[S]$

i.e. m_i involves a variable not in S.

$T := \{1, ..., n\} \backslash S$ is independent modulo $L(I)$

$\Leftrightarrow$ Every m_i involves a variable that is in S

2.8.19 Algorithm 19 (Dimension of I)

Input : $I \subseteq K[\underline{x}]$ ideal

Output: $\dim(I)$

(1) Compute a Gröbner basis of I with respect to a graded monomial ordering.
Let $m_i, ..., m_l$ be the leading monomials of the polynomials in G

(2) if $(\exists i : m_i = 1)$
$\dim(I) = -1$

(3) Find $T \subseteq \{x_1, ..., x_n\}$ of minimal size,
such that every m_i involves a variable from T
// don't worry about exponential time for this step as (1) is doubly exp anyway

(4) $\dim(I) = n - |T|$

3 Notes

3.1 Notation

- $\mathbb{N} := \mathbb{N}_0$
- $\lg(x) := \log_2(x)$
- $a \mid b$ a is divisible by b $\Leftrightarrow$ $b \mod a = 0$
  $a \nmid b$ a is not divisible by b $\Leftrightarrow$ $b \mod a \neq 0$
- $ord(a)$ order of a group element
  $n > 0$ minimal such that $a^n = e$ with neutral element e
  if no such n can be found, $ord(a) = \infty$
- $char(A)$ Characteristic: the smallest positive n such that
  $\underbrace{1 + ... + 1}_{n\ summands} = 0$ with 1 as the multiplicative identity element
- $im(f)$ image of f ("Wertebereich")
- $Ker(\phi)$ kernel of ϕ; Number of elements that map to zero
  $\Rightarrow$ degree to which the homomorphism ϕ fails to be injective
- $\mathbb{Z}/(m)$ Ring modulo m
  polynomial rings measure for "$<$" relations not the absolute value but max power.
- $lcm(a_1, ..., a_n)$ "least common multiple of all a_i"
- $\underline{e}$ = vector of e's
- $K[\underline{x}] := K[x_1, ..., x_n]$
- $\phi(n) := |\{x \in \mathbb{N} : x < n \wedge \gcd(x, n) = 1\}| = |(\mathbb{Z}/(n))^x|$
  Euler's totient function
- $rk(A)$ Rank of matrix A
  number of linear independent rows
- $$\left(\frac{n}{p}\right) := \begin{cases} 1 & \text{if } p \mid n \\ -1 & \text{if } n \text{ is a square } \pmod{p} \\ 0 & \text{otherwise} \end{cases}$$
  Legendre symbol (this is not a fraction)
- $$\left(\frac{n}{p}\right) = 1 \quad \Leftrightarrow \quad n^{\frac{p-1}{2}} = \left(\frac{n}{p}\right) \equiv 1 \pmod{p}$$
  Eulers criterion
- $res(f, g)$ resultant. $\Rightarrow$ det of Sylvester-Matrix
- $\mathbb{A}$:= Affine space
- $Id(S)$ = Ideal of S

3.2 Various stuff

- Lagrange's theorem
  Every element in a finite group has finite order
- Average number of bit operations for an increment:
  One operation for the last bit + 50% chance for one on the next bit + 25% on the following etc. $\Rightarrow$ Geometrical row
  $\Rightarrow$ on average two bit operations
- "Monte Carlo Algorithm"
  Always terminates in reasonable time but might yield false result.
- "Las Vegas Algorithm"
  If it terminates the result is correct. No deterministic running time.
- Chinese remainder theorem
  Given a system of congruences $x \equiv a_i \pmod{m_i}$ with $i = 1, ..., r$
  m_i pairwise co-prime. Then the unique solution is:
  $x \equiv a_1 \cdot b \cdot \frac{N}{m_i} + ... + a_r \cdot b_r \cdot \frac{N}{m_r} \pmod N$ with $b_i \cdot \frac{N}{m_i} \equiv 1 \pmod{m_i}$
- distance between two square numbers:
  $(n+1)^2 - n^2 = 2n + 1$
  $\Rightarrow$ Squares are much more scarce than primes!
- $ax + by = c$ has solutions in $\mathbb{Z}$ iff $\Leftrightarrow \gcd(a, b) \mid c$ with $a, x, b, y \in \mathbb{Z}$
- $$S_{f,g} = \begin{pmatrix} f_m & & \cdots & & f_0 & 0 & \cdots & 0 \\ 0 & f_m & & \cdots & & f_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \cdots & & \ddots & 0 \\ 0 & \cdots & 0 & f_m & & \cdots & & f_0 \\ g_n & & \cdots & & g_0 & 0 & \cdots & 0 \\ 0 & g_n & & \cdots & & g_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \cdots & & \ddots & 0 \\ 0 & \cdots & 0 & g_n & & \cdots & & g_0 \end{pmatrix}$$ Sylvester-Matrix for $f(x), g(x)$
- finitely generated
  A set M is finitely generated if $\exists R^n \mapsto M$ surjective with R^n finite
- monomial ideal $I \Leftrightarrow \exists$ a set of generators that are monomials only

3.3 Algebraic structures

• Group	$(G, *)$	
- one inner operation $(*)$:	$G \times G \mapsto G$	
- associativity:	$(a * b) * c = a * (b * c)$	$\forall a, b, c \in G$
- neutral element $e \in G$:	$a * e = e * a = a$	$\forall a \in G$
- inverse element $a^{-1} \in G$:	$a * a^{-1} = a^{-1} * a = e$	$\forall a \in G$
• Abelian group	$(G, *)$	

- $(G, *)$ is a group
- commutativity: $a * b = b * a \qquad \forall a, b \in G$

- Finite group $(G, *)$
  - associativity: $(a * b) * c = a * (b * c)$
  - unambiguity of reduction: $(a * x = a * x') \wedge (x * a = x' * a) \Rightarrow x = x'$
    $\Rightarrow x \mapsto x * a$ and $x \mapsto a * x$ is bijective
    $\Rightarrow \exists x : a * x = a \Rightarrow$ neutral element
    $\exists x : a * x = x \Rightarrow$ inverse element
- Cyclic group $(G, *)$
  - G is a group
  - G is generated by one Element: $G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$
  - not necessarily finite.
- Semi group $(S, *)$
  - one inner operation $(*)$: $S \times S \mapsto S$
  - associativity: $(a * b) * c = a * (b * c) \qquad \forall a, b, c \in S$
- Field $(K, +, \cdot)$
  - two inner operations $(+, \cdot)$ such that:
    - $(K, +)$ is an abelian group with neutral element 0
    - $(K \backslash (0), \cdot)$ is an abelian group with neutral element 1
  - distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$
    $(a + b) \cdot c = a \cdot c + b \cdot c \qquad \forall a, b, c \in K$
- General linear group $GL_n(K)$
  - K is a field
  - $GL_n(K)$ is the set of $n \times n$ invertible matrices with ordinary matrix multiplication
- Ring $(R, +, \cdot)$
  - $(R, +)$ is an abelian group
  - $(R, \cdot)$ is a semi group
  - distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$
    $(a + b) \cdot c = a \cdot c + b \cdot c \qquad \forall a, b, c \in R$
- Commutative ring $(R, +, \cdot)$
  -$(R, +, \cdot)$ is a ring
  -commutativity for $(\cdot)$ $a \cdot b = b \cdot a \qquad \forall a, b \in R$
- Unitary ring (ring with 1) $(R, +, \cdot)$
  - $(R, \cdot)$ is a semi group
  - $(R, \cdot)$ has a neutral element "1"
- Euclidean ring R
  - $\exists F : R \mapsto \mathbb{N}_0 \cup \{0\}$
    such that if $\exists q, r \in R \quad a = b \cdot q + r$ and $r = 0$ or $a, b \in R \quad F(r) < F(b)$

- Polynomial ring $R[\underline{X}]$
  - R is a commutative unitary ring
  - set of all polynomials with coefficients $\in R$
  - Variables $X_1 ... X_n$
- Noetherian Ring R

The following definitions are equal:
- for $I_1 \subseteq I_2 \subseteq ...\quad \exists n : I_n = I_{n+1} = ...$ (the chain of ideals "stabilizes")
- every ideal of R is finitely generated

3.4 Invertible elements

- Let $(\mathbb{Z}/(n), +)$ be a group or $(\mathbb{Z}/(n))^{\times}$ be a group with multiplication.

- $|(\mathbb{Z}/(n))^{\times}| = \phi(n)$
- $n \in \mathbb{P}$
  $\Rightarrow (\mathbb{Z}/(n))^{\times} = \{\bar{0}, ..., p \bar{-} 1\} \cong (\mathbb{Z}/(p-1), +) = Z_{p-1}$ (cyclic Group Z)
- n is a power of 2
  $\Rightarrow (\mathbb{Z}/(2^e))^{\times} \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2^{e-2})$
- n is a power of an odd Prime
  $\Rightarrow (\mathbb{Z}/(p^k))^{\times} \cong \mathbb{Z}/(p^{k-1} \cdot (p-1)) \cong Z_{(p^{k-1} \cdot (p-1))}$
- $n = p_1^{k_1}, ..., p_r^{k_r}$
  $\Rightarrow (\mathbb{Z}/(n))^{\times} \cong (\mathbb{Z}/(p_1^{k_1}))^{\times} \times ... \times (\mathbb{Z}/(p_r^{k_r}))^{\times}$