

# Computational Algebra

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## Transcript

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# 1 Integer Arithmetic

Topics:

- Addition and Multiplication
- GCD computation
- Primality testing
- Factorization

## 1.1 Addition and Multiplication

Agreement:

- $a, x \in \mathbb{N}$  represented as  $x = \sum_{i=0}^{n-1} a_i \cdot B^i$   $B \in \mathbb{N}_{>1}$  fixed Base ( $a_i \in \{0, \dots, B-1\}$ )
- if  $x \neq 0$ , assume  $a_{n-1} \neq 0$  then define:  
length of  $x := l(x) = n$  = number of digits =  $\lfloor \log_B(x) \rfloor + 1$   
(mnemonic:  $\log_B(B) + 1 = 2$ )
- $l(0) = 1$   
(Amount of memory required to store  $x = 0$ )
- $l(x) := l(|x|)$
- for  $x \in \mathbb{Z}$  represent if as  $x = \text{sgn}(x) * |x|$

### 1.1.1 Algorithm 1 (Simple addition)

input :  $x = \sum_{i=0}^{n-1} a_i \cdot B^i$ ,  $y = \sum_{i=0}^{n-1} b_i \cdot B^i$ ,  $x, y \in \mathbb{N}$

output:  $x + y = \sum_{i=0}^n c_i \cdot B^i$

- (1)  $\sigma = 0$
- (2) for  $i = 0, \dots, (n-1)$  :
- (3)     set  $c_i := a_i + b_i + \sigma_i$   
        $\sigma := 0$
- (4)     if  $(c_i \geq B)$
- (5)         set  $c_i = c_i - B$
- (6)         set  $\sigma = 1$
- (7) set  $c_n = \sigma$

If  $B = 2$  then (3) - (6) can be realized by logic gates:

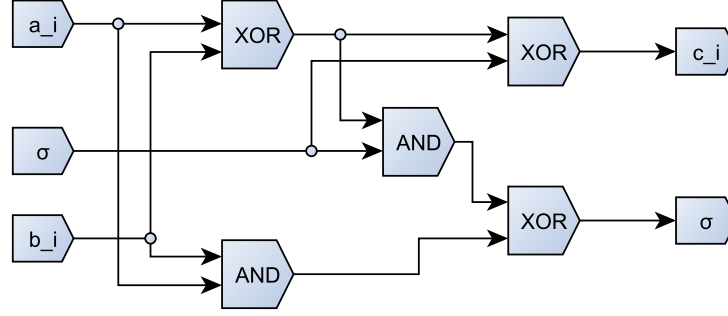


Figure 1: Logic circuit for addition

### 1.1.2 Definition 2 (Bit-Operation)

A bit operation is an operation that can be performed by a logic gate or by searching or writing a bit from / into memory.

### 1.1.3 Definition 3 (Big $O$ )

Let  $M$  be a set (usually  $M = \mathbb{N}$ ),  $f, g : M \mapsto \mathbb{R}_{>0}$   
we write  $f \in O(g)$  if  $\exists c \in \mathbb{R} : f(x) \leq c \cdot g(x) \forall x \in M$

### 1.1.4 Theorem 4 (Lower bound for addition)

Let  $f : \mathbb{N} \mapsto \mathbb{R}$ ,  $n \mapsto$  maximal number of bit operations required by Algorithm 1 to add  $x, y \in \mathbb{N}$  with  $l(x), l(y) \leq n$

Let  $g = id_{\mathbb{N}}$  Then  $f \in O(g)$

We say Algorithm 1 requires  $O(n)$  bit operations for adding two numbers of length  $\leq n$ .  
 $\Rightarrow$  "linear complexity"

Set  $M := \{\text{Set of all algorithms for addition in } \mathbb{N}\}$

For  $A \in M$  define  $f_A : \mathbb{N} \mapsto \mathbb{R}$  as above.

We would like to find  $f_{odd} : \mathbb{N} \mapsto \mathbb{R}$ ,  $n \mapsto \inf\{f_A(n) | A \in M\}$

Since one needs to read  $x, y$  (and write the result) we can not do any better than linear complexity for addition.

#### Subtraction

let  $x, y$  as Algorithm 1,  $x \geq y$

For  $\bar{y} := \sum_{i=0}^{n-1} (B - 1 - b_i) B^i$  (digitwise / bitwise complement)

$\Rightarrow x + \bar{y} = x - y + B^n - 1$

$\Rightarrow x - y = x + \bar{y} + 1 - B^n$  (initially set  $\sigma = 1$ )

**Conclusion:** Addition and Subtraction have cost  $O(n)$

### 1.1.5 Algorithm 5 (Multiplication by "grid method")

input :  $x = \sum_{i=0}^{n-1} a_i \cdot 2^i, \quad y = \sum_{i=0}^{m-1} b_i \cdot 2^i$

output:  $z = x \cdot y$

- (1)  $z := 0$
- (2) for  $i = 0, \dots, (n-1)$
- (3) if  $(a_i \neq 0)$ 

$$\text{set } z := z + \sum_{j=0}^{m-1} b_j 2^{i+j}$$

### 1.1.6 Theorem 6 (Runtime of Algorithm 5)

Algorithm 5 requires  $O(n \cdot m)$  bit operations.

As of the total input length  $n + m$ :

$$n \cdot m \leq \frac{1}{2}(n + m)^2 \rightarrow O((n + m)^2)$$

$\Rightarrow$  Quadratic complexity

### Karatsuba-multiplication:

Observation for polynomials:

$$a + bx, c + dx \text{ have } (a + bx)(c + dx) = ac + (ac + db - (a - b)(c - d))x + bdx^2$$

The point: only used 3 multiplications instead of 4.

Specialize  $x = B$  "large" such that  $x = a + bB$  partition into two blocks. Then multiply the blocks by a recursive call.

### 1.1.7 Algorithm 7 (Karatsuba)

input :  $x, y \in \mathbb{N}$

output:  $z = x \cdot y$

- (1) Choose  $k \in \mathbb{N}$  minimal such that  $l(x), l(y) \leq 2^k$ .  
Set  $B = 2^{2^{k-1}}$
- (2) if  $(k = 0)$   
return  $x \cdot y$  (by bit-operation AND)
- (3) write  $x = x_0 + x_1 B$   
 $y = y_0 + y_1 B$  with  $l(x_i), l(y_i) \leq 2^{k-1}$
- (4) compute  $x_0 \cdot y_0, x_1 \cdot y_1, (x_0 - x_1) \cdot (y_0 - y_1)$  by a recursive call
- (5) return  $z = x_0 y_0 + (x_0 y_0 + x_1 y_1 - (x_0 - x_1)(y_0 - y_1))B + x_1 y_1 B^2$

### 1.1.8 Theorem 8 (Runtime of Algorithm 7)

For multiplying two numbers of length  $\leq n$  Algorithm 7 requires  $O(n^{\log_2 3}) \approx O(n^{1.59})$  bit operations.

**Proof:**

Set  $\Theta(k) :=$  maximal numbers of bit operations for  $l(x), l(y) \leq 2^k$

We have for  $k > 0$  :  $\Theta(k) \leq 3 \underbrace{\Theta(k-1)}_{\text{recursive calls}} + c \underbrace{2^k}_{\text{additions}}$  with ( $c$  some constant)

**Claim:**  $\Theta(k) \leq 3^k + 2c(3^k - 2^k)$

**Proof by Induction on  $k$ :**

$k = 0$  :  $\Theta(k) = 1$

$$\begin{aligned} k-1 \rightarrow k : \Theta(k) &= 3\Theta(k-1) + c2^{k-1} \\ &\leq 3(3^{k-1} + 2c(3^{k-1} - 2^{k-1})) + c2^k \\ &= 3^k + 2c(3^k - 2^k) \end{aligned}$$

So  $\Theta(k) \leq (2c+1)3^k$

Now  $l(x) \leq n$  hence  $2^{k-1} < n$  by minimality of  $k$

So  $k-1 < \log_2 n$

$$\begin{aligned} \Rightarrow \Theta(k) &\leq 3(2c+1)3^{\log_2(n)} \\ &= 3(2c+1)2^{\log_2(3) \log_2(n)} \\ &= 3(2c+1)n^{\log_2(3)} \quad \square \end{aligned}$$

One can modify the terminal condition of Karatsuba to switch to Grid-Multiplication, which is faster for small numbers.

### Fast-Fourier Transformation

Reminder: For a function  $f : \mathbb{R} \mapsto \mathbb{C}$  define:

$\hat{f} : \mathbb{R} \mapsto \mathbb{C}$  by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt \quad (\text{if it exists})$$

Think of  $\omega$  as frequency.

### Definition (Convolution)

Let  $f, g : \mathbb{R} \mapsto \mathbb{C}$

$$(f * g)(x) = \int_{\mathbb{R}} f(t) g(x-t) dt$$

Convolution is analogous to polynomial multiplication

**Formula:**  $\underbrace{(f * g)}_{\text{(Cauchy formula)}} = \hat{f} \cdot \hat{g}$

For a function  $M \mapsto C$  with  $|M| < \infty$  we need the discrete Fourier transform (DFT)



### 1.1.9 Definition 9 (Root of unity)

Let  $R$  be a commutative ring with 1. An element  $\mu \in R$  is called an  $n$ -th root of unity (= root of 1) if  $\mu^n = 1$ .

It is called primitive if  $\mu^i \neq 1$  for  $(0 < i < n)$  i.e.  $\text{ord}(\mu) = n$

Let  $\mu$  be a primitive  $n$ -th root of 1 (e.g.  $e^{2\pi \frac{i}{n}} \in \mathbb{C}$ )

Then the map  $DFT_\mu : R^n \mapsto R^n$

$$(\hat{a}_0, \dots, \hat{a}_n) \mapsto (\hat{a}_0, \dots, \hat{a}_n) \quad \text{with } \hat{a}_i = \sum_{j=0}^{n-1} \mu^{ij} a_j$$

is called discrete Fourier transformation

For polynomials:

$$DFT_\mu : R[x] \mapsto R^n$$

$$f \mapsto (f(\mu^0), \dots, f(\mu^{n-1}))$$

Convolution rule: (from  $f(\mu^i)g(\mu^i) = (f * g)(\mu^i)$ )

$$DFT_\mu(f * g) = DFT_\mu(f) \cdot DFT_\mu(g) \quad (\text{component wise product})$$

Addition of two polynomials in  $R[x]$  of  $\deg(n)$  require  $O(n)$  ring operations.

Multiplication require  $O(n^l)$ .

With Karatsuba have  $O(n^{\log_2(3)})$  ring operations.

Cost  $DFT_\mu(f) \cdot DFT_\mu(g) : O(n)$  ring operations (with  $\mu$  as  $2n$ -th root of 1)

Want: Cheap way of doing  $DFT$  and back-transformation.

### 1.1.10 Algorithm 10 (Fast Fourier transformation FFT)

input :  $f \in R[x]$ ,  $\mu \in R$  primitive  $2^k$ -th root of 1, such that  $\mu^{2^{k-1}} = -1$

output:  $DFT_\mu(f)$

(1) Write  $f(x) = g(x^2) + xh(x^2)$  with  $f, g, h \in R[x]$

(2) if  $(k = 1)$   $// (\Rightarrow \mu = 1)$   
return  $DFT_\mu(f) = (g(1) + h(1), g(1) - h(1))$

(3) Recursive call: compute  $DFT_{\mu^2}(g) = \hat{g}, DFT_{\mu^2}(h) = \hat{h} \in R^{2^{k-1}}$

(4) return  $DFT_\mu(f) = (\hat{f}_0, \dots, \hat{f}_{2^k-1})$  with  $\hat{f}_i = \hat{g}_i + \mu \hat{h}_i$   
where  $\hat{g}_i = \hat{g}_{i-2^{k-1}}$  for  $i \geq 2^{k-1}$

Note: Components of  $\hat{g}$  and  $\hat{h}$  are:

$$\hat{g} = g(\mu^{2^i}), \quad \hat{h}_i = h(\mu^{2^i}) \quad \text{so}$$

$$\hat{f}_i := f(\mu^i) = \hat{g}_i(\mu^{2^i}) + \mu \hat{h}_i(\mu^{2^i}) = \hat{g}_i + \mu \hat{h}_i$$

**Convention:**  $\lg(x) = \log_2(x)$

### 1.1.11 Theorem 11 (Runtime of Algorithm 10)

Let  $n = 2^k$ ,  $f \in R[x]$  with  $\deg(\psi) < n$

Then Algorithm 10 requires  $O(n \cdot \lg(n))$  ring operations.

Better than  $O(n^{1+\epsilon})$ ,  $\forall \epsilon > 0$ !

**Proof:**

Set  $\Theta(k) = \max$  number of ring operations required. By counting obtain for  $k > 1$ :

$$\Theta(k) \leq 2\Theta(k-1) + \underbrace{(\text{compute } \mu^i (i \leq 2^{k-1}))}_{2^{k-1}} + \underbrace{(\mu^i \hat{k}_i)}_{2^{k-1}} + \underbrace{(\text{sums and differences})}_{2^k}$$

$$= 2\Theta(k-1) + 2^{k+1}$$

**Claim:**  $\Theta(k) \leq (2k-1)2^k$

$$k=1 : f = a_0 + a_1 \cdot x \quad DFT_\mu(f) = (a_0 + a_1 \cdot a_0 - a_1) \Rightarrow \Theta(a) = 2$$

$$k-1 \rightarrow k : \Theta(k) \leq 2 \cdot \Theta(k-1) + 2^{k+1} \leq 2 \cdot (2k-3) \cdot 2^{k-1} + 2^{k+1} = (2k-1) \cdot 2^k$$

since  $k = \lg(n)$  obtain  $O(k) \leq (2 \cdot \lg(n) - 1) \cdot n \in O(n \cdot \lg(n)) \quad \square$

**Back-transformation?**

### 1.1.12 Definition 12 (Good root of unity)

A primitive  $n$ -th root of unity is called good (caveat: this is ad-hoc terminology) if:

$$\sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \text{for } (0 < i < n)$$

**Example:**

- (1)  $\mu = e^{2\pi \frac{i}{n}}$  is a good primitive root of unity
- (2)  $R = \mathbb{Z}/(8)$ ,  $\mu = \bar{3} \Rightarrow \mu \cdot B$  is primitive  $2^{nd}$  root of unity  
But  $\bar{B}^0 + \bar{3}^1 = \bar{u} \neq \bar{0}$  so  $\mu$  is not good.

### 1.1.13 Proposition 13 ( $DFT_{\mu^{-1}}$ )

Let  $\mu \in R$  be a good root of 1

$$(a) = (a_0, \dots, a_{n-1}) \in R^n \Rightarrow DFT_\mu^{-1}(DFT_\mu(a)) = n \cdot (a) \quad \text{where } n = 1 + \dots + 1 \in R$$

**Proof:**

$$DFT_\mu(a) = (\hat{a}) = (\hat{a}_0, \dots, \hat{a}_{n-1})$$

$$\text{with } \hat{a}_j = \sum_{k=0}^{n-1} \mu^{jk} a_k$$

$$DFT_{\mu^{-1}}(\hat{a}) = (\hat{\hat{a}}_0, \dots, \hat{\hat{a}}_1)$$

$$\text{with } \hat{\hat{a}}_i = \sum_{j=0}^{n-1} \mu^{-ij} \sum_{k=0}^{n-1} \mu^{jk} a_k = \sum_{k=0}^{n-1} \left( a_k \cdot \underbrace{\sum_{j=0}^{n-1} \mu^{j(k-i)}}_{=0 \text{ if } n \neq k-i \text{ (i.e. } k=i)} \right) = a_i \cdot n \quad \square$$

### 1.1.14 Proposition 14 (Finding good roots of unity)

let  $\mu \in R, n \in \mathbb{N}$

Assume:

- a)  $R$  is an integral Domain and  $\mu$  is a primitive or  $n$ -th root of 1  
(Integral Domain: nonzero commutative ring in which the product of two nonzero elements is nonzero)  
 $\Rightarrow$  Granted by FFT
- b)  $n = 2^b, \mu^{\frac{n}{2}} = -1$ , then  $h > 0 \wedge \text{char}(R) \neq 2$   
 $\rightarrow \mu$  is a good primitive  $n$ -th root of 1 ("root of unity")

**Proof:**

- a) for  $0 < i < n$

$$\underbrace{(\mu^i - 1)}_{\neq 0} \underbrace{\left(\sum_{j=0}^{n-1} \mu^{ij}\right)}_{=0} = \mu^{in} - 1 = 0$$

$\Rightarrow \mu$  is a good root of unity

- \* Let  $0 < i < n$ , write  $i = 2^{k-s} \cdot r$  with  $r$  odd  $\wedge s > 0$

$$\sum_{j=0}^{2^k-1} \mu^{ij} = \sum_{l=0}^{2^{k-s}-1} \sum_{j=0}^{2^s-1} \mu^{i(l \cdot 2^s + j)}$$

$$\mu^{i \cdot 2^s} = 1$$

$$i \cdot 2^s = 2^{k-s} \sum_{j=0}^{2^s-1} \mu^{ij} = 2^{k-s} \sum_{j=0}^{2^{s-1}-1} (\mu^{ij} + \mu^{i(2^{s-1}+j)})$$

$$\text{But } \mu^{i \cdot 2^{s-1}} = \mu^{2^{k-s} \cdot r \cdot 2^{s-1}} = \mu^{2^{k-1} \cdot r} = (-1)^r = -1$$

$$\text{So } \sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \square$$

- b)  $\mu^n = 1, n = 2^k \Rightarrow \text{ord}(\mu) | n \Rightarrow \text{ord}(\mu)$  is power of 2

### 1.1.15 Algorithm 15 (Polynomial multiplication using DFT)

input :  $f, g \in R[x]$  with  $\deg(f) + \deg(g) < 2^k =: n$   
 $\mu \in R$  as a good root of unity; Assume  $2 \in R$  is invertible

output:  $h = f \cdot g$

- (1) compute  $\hat{f} = DFT_{\mu}(f), \hat{g} = DFT_{\mu}(g)$  with  $f, g \in R^n$
- (2) compute  $\hat{h} = \hat{f} \cdot \hat{g}$
- (3) compute  $(h_0, \dots, h_{n-1}) = DFT_{\mu^{-1}}(\hat{h})$  (same as  $DFT_{\mu}(\hat{h})$  but with different order)  
= Back-transformation  $\cdot 2^k$   
set  $h = \frac{1}{2^k} \sum_{i=0}^{n-1} h_i x^i$

### 1.1.16 Theorem 16 (Runtime of Algorithm 15)

Algorithm 15 uses  $O(n \cdot \log(n))$  ring operations for polynomials of  $\deg < n$

**Proof:**

- Choose  $k$  minimal so that  $\deg(f) \cdot \deg(g) < 2^k$   
 $\Rightarrow 2^{k-1} \leq 2n \Rightarrow k \leq \log(n) + 2$
- $\underbrace{O(2k \cdot 2^k)}_{\text{Step 1}} + \underbrace{2^k}_{\text{Step 2}} + \underbrace{O(k \cdot 2^k) + 2^k}_{\text{Step 3}} \in O(2k \cdot 2^k) = O(n(g(n))) \quad \square$

Goal: Multiplication in  $\mathbb{N}$  using DFT

Idea: find roots of 1 in a suitable  $\mathbb{Z}/(m)$

Choose  $m = 2^l + 1, \mu = \bar{2} \in R$

### 1.1.17 Proposition 17 (Add and mul in $O(l)$ )

Let  $m = 2^l + 1, R = \mathbb{Z}/(m)$

Addition in  $R$  and multiplication by  $\bar{2}^i \in R$  ( $0 \leq i < 2l$ ) can be done in  $O(l)$  bit operations

**Proof:**

- Let  $\bar{x} \in R$  with  $0 \leq x \leq 2^l$
- Addition:  $x + \bar{y}$ 
    - (1) compute  $x + y \in \mathbb{N}$ :  $O(l)$
    - (2) if  $x + y > 2^l + 1$  subtract  $2^l + 1$ :  $O(l)$
  - Multiplication by  $\bar{2}^i$  ( $0 \leq i < l$ )
    - (1) Bit-shift  $i$  Bits to the left by relocating in memory:  

$$\underbrace{O(\text{length}(i))}_{\text{compute addr. of new first bit}} + \underbrace{l}_{\text{copying}} = O(\log(l)) + l \in O(l)$$
  - Multiplication by  $\bar{2}^i$  ( $l \leq i < 2l - 1$ )
    - (1) Multiplication by  $\bar{2}^{i-l}$ :  $O(l)$
    - (2) take negative  $\bar{2}^i \cdot \bar{x} = -\bar{2}^{i-l} \cdot \bar{x}$ :  $O(l)$

### 1.1.18 Proposition 18 (Sort of summary)

Let  $k, r \in \mathbb{N}, r > 0, m = 2^{2^k \cdot r} + 1, R = \mathbb{Z}/(m), \mu = \bar{2}^r \in R$

$\Rightarrow 2 \in R$  is invertible,  $\mu$  is a good primitive  $2^{k+1}$ -th root of 1

$\Rightarrow \mu^{2^k} = 1$

**Proof:**  $\rightarrow$  from above

### 1.1.19 Algorithm 19 (Multiplication using FFT)

input :  $x, y \in \mathbb{N}$

output:  $Z = x \cdot y$

- (1) Choose  $k \in \mathbb{N}$  minimal such that  $l(x), l(y) \leq 2^{2^k}$
- (2) if  $k \leq 3$ , compute  $z = x \cdot y$  by Algorithm 5
- (3) set  $B = 2^{2^k}$ ,  $m = 2^{2^k \cdot 4} + 1$ ,  $R = \mathbb{Z}/(m)$ ,  $\mu = \bar{2}^4 \in R$   
 $(\Rightarrow \text{so } \mu \text{ is a good primitive } 2^{k+1}\text{-th root of 1})$
- (4) write  $x = \sum_{i=0}^{2^k-1} x_i \cdot B^i$ , same for  $y$  with  $(0 \leq x_i, y_i < B)$   
 possible since  $x, y < 2^{2^{2k}} = 2^{2^k \cdot 2^k} = B^{2^k}$
- (5) compute:  $\hat{x} = DFT_\mu(\bar{x}_0, \dots, \bar{x}_{2^k-1}, \underbrace{0, \dots, 0}_{2^k \text{ zeros}}) \in R^{2^{k+1}}$   
 same for  $y$   
 $\rightarrow$  use FFT
- (6) compute:  $\hat{z} = \hat{x} \cdot \hat{b} \in R^{2^{k+1}}$  (component wise multiplication)  
 Perform multiplication in  $R$  as follows:  
 Multiply representatives (non negative and  $< m$ ) by recursive call.  
 Then reduce modulo  $m$  by "negative bit shift" (see proof of Proposition 17)
- (7) compute:  $(\bar{z}_0, \dots, \bar{z}_{2^{k+1}-1}) = \frac{1}{2^{k+1}} DFT_{\mu^{-1}}(\hat{z}) \in R$  with  $0 \leq z < m$
- (8) set  $z := \sum_{j=0}^{2^{k+1}-1} z_j \cdot B^j$

### 1.1.20 Theorem 20 (Runtime of Algorithm 19)

Algorithm 19 correctly computes  $t = x \cdot y$  and requires  $O(n \cdot (\log n)^4)$  bit operations for  $l(x), l(y) \leq n$

**Proof:** Correctness

write  $x(t) \sum_{i=0}^{2^k-i} x_i t^i \in \mathbb{Z}[t]$ ,  $y(t)$ ,  $\bar{x}(t) \in R[t], \bar{y}(t), \bar{z}(t)$

by Proposition 18 and Proposition 13 we have  $\bar{z}(t) = \bar{x}(t) \cdot \bar{y}(t)$

The  $l$ -th coefficient of  $x(t) \cdot y(t)$  is  $0 \leq \sum_{i+j=l} x_i \cdot y_j < 2^k \cdot B^2 = 2^{k+2 \cdot 2^k} \leq 2^{2^{k+2}} < m$

So  $z(t) = x(t) \cdot y(t) \Rightarrow z = z(B) = x(B) \cdot y(B) = x \cdot y$  Cost:

Write  $\Theta(k) := \max$  number of bit operations

Analyze Steps:

- (1) compute  $\max \{l(x), l(y)\} : O(l(n)) = O(k)$
- (2)  $O(1)$
- (3) no bit operations
- (4) compute starting addresses of  $x_i, y_i$  in memory:  $2 \cdot 2^k$  increments of the address:  
 $2 \cdot 2 \cdot 2^k = 2^{k+2}$  bit ops  
 $\Rightarrow O(2^k)$
- (5) By Theorem 11 need  $O(2 \cdot 2^{k+1} \cdot (k+1))$  operations in  $R$  which are additions and multiplications by powers of  $\bar{z}$  costing  $O(2^{k+2})$  bit operations.  
Total for (5):  $O(k \cdot 2^{2 \cdot k})$
- (6)  $2^{k+1}$  multiplications of numbers  $< m$ , i.e. of length  $\leq 2^{k+2}$ .  
So  $k' \leq \frac{k+3}{2}$  for  $k'$ : the "new"  $k$  used in the next recursion level.  
For  $\alpha \in R_{>0}$  define  $\Theta(\alpha) := \Theta(|\alpha|)$   
Total for (6):  $2^{k+1}(\Theta(\frac{k+3}{2}) + \underbrace{O(2^{k+2})}_{\text{reduction (mod } m)})$
- (7) For  $DFT_{\mu^{-1}}(\hat{z}) : O(k \cdot 2^{2 \cdot k})$  as (5) Since  $\bar{z}$  is a  $n$  root of 1, multiplication by  $\bar{2}^{-k-1}$  is multiplication by a positive power of  $\bar{2}$ , which costs  $O(2^{k+2})$   
Total for (7):  $O(k \cdot 2^{2 \cdot k})$
- (8) For  $j \leq 2^{k+1}$  have  $\sum_{i=0}^{j-1} z_i \cdot B^i \leq (m-1) \sum_{i=0}^{j-1} B^i = (m-1) \frac{B^j-1}{B-1} < 2(m-1) \frac{B^j}{B} = 2^{1+2^{k+2}+(j-1)2^k}$  so the sum has length  $(j+3) \cdot 2 + 1$   
Adding  $z_j \cdot B^j$  to this sum happens at  $(j \cdot 2^k)$ -th bit and higher  $\Rightarrow$  cost is  $O(2^k)$   
Total for (8):  $O(2^{2 \cdot k})$

Grad total: For  $k \geq 4$ :

$\Theta(k) \leq 2^{k+1} \cdot \Theta(\frac{k+3}{2}) + c \cdot k \cdot 2^{2 \cdot k}$  with  $c$  constant

Also for  $k \in \mathbb{R}_{\geq 4}$

**Define**  $\Lambda(k) := \frac{\Theta(k)}{2^{2 \cdot k}} \Rightarrow \Lambda(k) \leq \frac{2^{k+1} \Theta(\frac{k+3}{2})}{2^{2 \cdot k}} + c \cdot k = 16 \cdot \Lambda(\frac{k+3}{2}) + c \cdot k$

**Define**  $\Omega(k) := \Lambda(k+3)$  So for  $k \in \mathbb{R}_{>1}$

$$\Omega(k) \leq 16 \cdot \Lambda(\frac{k}{2} + 3) + c \cdot (k+3) = \underbrace{16\Omega(\frac{k}{2})}_{*} + c \cdot (k+3)$$

**Claim:** For  $i \in \mathbb{N}$  with  $2^{i-1} \leq k-3$  have:

$$\Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k+3)(1+8+\dots+8^{i-1}) + 3 \cdot c \cdot (1+16+\dots+16^{i-1})$$

**Proof** by induction:

$$i = 0: \Lambda(k) = \Omega(k-3)$$

$$i \rightarrow i+1: \Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k-3)(1+\dots+8^{i-1}) + 3 \cdot c \cdot (1+\dots+16^{i-1}) \leq 2^i \leq k-3 \quad *$$

$$\leq 16^i (16\Omega(\frac{k-3}{2^{i+1}})) + c(\frac{k-1}{2^i} + 3) + c(k-3)\dots = \text{claimed result}$$

Take  $u \in \mathbb{N}$  minimal with  $2^u > k-3 \Rightarrow \Omega(\frac{k-3}{2^u}) \leq \Omega(\lfloor \frac{k-3}{2^u} \rfloor) = \Omega(0) =: D$  (constant)

Note:  $u$  roughly is recursion depth

$$\text{Have } 2^{u-1} \leq k-3 \xRightarrow{\text{claim}} \Lambda(k) \leq 16^u \cdot D + c \cdot \underbrace{(k-3)}_{< 2^u} \cdot \frac{8^u-1}{7} + 3c \cdot \frac{16^u-1}{15} \in O(16^u)$$

$$\text{Have } 2^{u-1} \leq k-3 \Rightarrow u \leq \lg(k-3) + 1$$

$$\Rightarrow \Lambda(k) \in O(16^{\lg(k-3)}) = O((k-3)^4)$$

$$\Rightarrow \Theta(k) = 2^{2 \cdot k} \cdot \Lambda(k) \in O(2^{2k} \cdot (k-3)^4)$$

$$\text{Have } 2^{2(k-1)} < \underbrace{n}_{\max\{l(x) \cdot l(y)\}} \Rightarrow k \leq \frac{\lg(n)}{2} + 1$$

$$\text{So } \Theta(k) \in O(n \cdot (\lg(n))^4) \quad \square$$

### 1.1.21 Theorem 21 (Schönhage-Strassen 1971)

Multiplication of integers of length  $\leq n$  can be done in  $O(n \cdot \lg(n) \cdot \lg(\lg(n)))$  bit operations. Schönhage-Strassen is used for integers of length  $\geq 100.000$ .

Asymptotically faster: Fürer's algorithm.

### Comments on Bit complexity

1. Memory requirement may explode!  
 $\Rightarrow$  No Problem as bit complexity is upper bound for memory requirements, since memory access is included in bit operations  
 $(\rightarrow$  only store what is calculated)
2. Computation of addresses in memory take time  
 $\Rightarrow$  length of addresses  $\approx \lg(\text{memory space})$  computations of addresses  $\approx \lg(\text{memory space})^2$
3. As memory requirement gets larger access times will get longer.  
 $\Rightarrow$  transportation time for data  $\geq \frac{\text{diameter of physical storage}}{2 \cdot \text{speed of light}}$

## 1.2 Division with remainder, Euclidean algorithm

### 1.2.1 Algorithm 1 (Division with remainder)

input :  $b = \sum_{i=0}^{n-1} b_i 2^i$     $a = \sum_{i=0}^{n+m-1} a_i 2^i$    with  $a_i, b_i \in 0, 1$ ,    $b_{n-1} = 1$

output:  $r, q \in \mathbb{N}$    such that  $a = q \cdot b + r$ ,    $0 \leq r < b$

- (1)  $r = a$   
 $q = 0$
- (2) for  $i = m, m-1, \dots, 0$  do
- (3)    if  $r \geq 2^i \cdot b$   
           set  $r := r - 2^i \cdot b$   
            $q := q + 2^i$

### 1.2.2 Proposition 2 (Runtime of Algorithm 1)

Algorithm 1 is correct and requires  $O(n \cdot (m+1))$  bit operations.

**Proof:**

Always have  $a = q \cdot b + r$

**Claim:**

before step (3), have  $0 \leq 2^{i+1} \cdot b$

$i = m$ ;    $0 \leq r = a < 2^{m+n} = 2^{m+1} \cdot 2^{n-1} \leq 2^{m-1} \cdot b$     $i < m$  By step (3)

So after last passage through the loop  $0 \leq r < b$

**Running Time:** In step(3), have comparison and (possibly) subtraction. Only  $n$  bits involved  $\Rightarrow O(n)$

Total:  $O(b \cdot (m+1))$

**Remarks:**

- (1) Division with remainder can be reduced to multiplication.  
 Precisely: given an algorithm for multiplication that requires  $M(n)$  bit operations, there exists an algorithm for division with remainder that requires  $O(M(n))$  bit operations.
- (2) Practically relevant:  
 Jebelean's algorithm (1997):  $O(n^{\lg 3})$
- (3) Alternatively, may choose  $r \in \mathbb{Z}$    such that  $\lfloor \frac{-b}{2} \rfloor < r \leq \lfloor \frac{b}{2} \rfloor$
- (4) Algorithm 1 extends to  $\mathbb{Z}$ .
- (5) All Euclidean rings have division with remainder (by definition).  
 (e.g.,  $R = K[x] \rightarrow$  polynomial ring over field,  
 $R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \subseteq \mathbb{C}, \quad i^2 = -1$ )



### 1.2.3 Algorithm 3 (Euclidean algorithm)

input :  $a, b \in \mathbb{N}$

output:  $\gcd(a, b)$  "greatest common divisor"

- (1) set  $r_0 := a$   
 $r_1 := b$
- (2) for  $i = 1, 2, 3, \dots$  perform steps (3) and (4)
- (3) if  $r_i = 0$   
 $\gcd(a, b) = |r_{i-1}|$
- (4) Division with remainder:  $r_{i-1} = q \cdot r_i + r_{i+1}$   $r_{i+1} \in \mathbb{Z}$   
 $(|r_{i+1}| \leq \frac{1}{2}|r_i|)$

**Example:**

$$a = 287, \quad b = 126$$

$$287 = 2 \cdot 126 + 35 \tag{1}$$

$$126 = 4 \cdot 35 - 14 \tag{2}$$

$$35 = (-2) \cdot (-14) + 7 \tag{3}$$

$$-14 = (-2) \cdot 7 + 0 \tag{4}$$

$$\begin{aligned} \text{So: } 7|(-14) &\xRightarrow{(3)} 7|35 \\ &\xRightarrow{(2)} 7|126 \\ &\xRightarrow{(1)} 7|287 \end{aligned}$$

On the other hand take a common divisor  $d$ ;  $d|287$ ;  $d|126$   
 $\xRightarrow{(1)} d|d \xRightarrow{(2)} d|14 \xRightarrow{(3)} d|7$

### 1.2.4 Theorem 4 (Correctness of Algorithm 3)

Algorithm 3 is correct.

**Proof:**

Since  $r_{i-1} = q \cdot r_i + r_{i+1}$  every integer  $x \in \mathbb{Z}$  satisfies the equivalence  $x|r_{i-1}$  and  $x|r_i \Leftrightarrow x|r_{i+1}$  and  $x|r_i$  so  $\gcd(r_{i-1}, r_i) = \gcd(r_i, r_{i+1}) = \gcd(a, b)$   
when terminating have  $\gcd(a, b) = \gcd(r_{i-1}, 0) = |r_{i-1}| \quad \square$

### 1.2.5 Theorem 5 (Runtime of Algorithm 3)

Algorithm 3 requires  $O(m \cdot n)$  bit operations for  $n = l(a), m = l(b)$

**Proof:**

If  $a < b$  then the first passage yields  $r_2 = a, r_1 = b$ . Cost:  $O(n)$

May assume:  $a \geq b$ . Write  $n_i = l(r_i)$

By Proposition 2  $\exists c$  constant such that the total time is  $\leq c \cdot \underbrace{\sum_{i=1}^k n_i \cdot (n_{i-1} - n_i + 1)}_{=: \sigma(n_0, \dots, n_k)}$

For  $i > 2$ :  $n_i = n_{i-1} - 1$

Special Case:  $n_i = n_{i-1} - 1$  for  $i \geq 2$

$\Rightarrow n_i = n_i - i + 1, n_i = m, k = m + 1$

Obtain  $\sigma(n_0, \dots, n_k) = m \cdot (n - m + 1) + \sum_{i=2}^{m+1} (m - i + 1) \cdot 2 = m \cdot n - m^2 + m + m(m - 1) = m \cdot n$ .

**Claim:** The special case is the worst (most expensive)!

From any sequence  $n_1 > n_2 > \dots > n_k$  get to the special case by iteratively inserting numbers in the gaps. Insert  $s$  with  $n_{j-1} > s > n_j$ .

$\sigma(n_0, \dots, n_{j-1}, s, n_j, \dots, n_k) - \sigma(n_0, \dots, n_k) = \dots = s + (n_{j-1} - s) \cdot (s - n_j)$

$sp\sigma(n_0, \dots, n_k) \leq \sigma(n, m, m - 1, \dots, 2, 1, 0) = n \cdot m \quad \square$

Complexity is quadratic  $\rightarrow$  cheap

### 1.2.6 Algorithm 6 (Extended Euclidean Algorithm)

input :  $a, b \in \mathbb{N}$

output:  $d = \gcd(a, b)$  and  $s, t \in \mathbb{Z}$  such that  $d = s \cdot a + t \cdot b$

(1)  $r_0 := a, r_1 := b, s_0 := 1, t_0 := 0, s_1 := 0, t_1 = 1$

(2) for  $i = 1, 2, \dots$  perform steps (3) - (5)

(3) if  $r_i = 0$   
     set  $d = |r_{i-1}|$   
      $s := \text{sgn}(r_{i-1}) \cdot s_{i-1},$   
      $t := \text{sgn}(r_{i-1}) \cdot t_{i-1}$

(4) division with remainder:  
 $r_{i+1} = r_{i-1} - q_i \cdot r_i, \quad \text{with } |r_{i+1}| \leq \frac{1}{2}|r_i|$

(5) set  $s_{i+1} := s_{i-1} - q_i \cdot s_i,$   
 $t_{i+1} := t_{i-1} - q_i \cdot t_i$

Justification :  $r_i = s_i \cdot a + t_i \cdot b$  throughout

**Application:**  $m, x \in \mathbb{N}$  such that  $m, x$  co-prime (i.e.  $\gcd(x, m) = 1$ )

Algorithm 6 yields:  $1 = s \cdot x + t \cdot m \Rightarrow s \cdot x \equiv 1 \pmod{m}$

So obtain inverse of  $\bar{x} \in \mathbb{Z}/(m)$

### 1.3 Primality testing

Let  $\mathbb{P} \subseteq \mathbb{N}$  be the set of prime numbers.

Challenge: Given  $n \in \mathbb{N}$  decide if  $n \in \mathbb{P}$

**Naive Method:** Trivial division by  $m \leq \lfloor \sqrt{n} \rfloor$ .

Running time is exponential in  $l(n)$ . Even when restricted to division by prime numbers, need approximately  $\frac{\sqrt{n}}{|n|^{1/\sqrt{n}}}$  trivial divisions (prime number theorem)  
 $\rightarrow$  hardly any better!

**Reminder:** (arithmetic modulo  $m$ )

$G$  finite group  $\Rightarrow \forall a \in G \quad a^{|G|} = 1$  Fermat's little theorem

For  $G = (\mathbb{Z}/(p))^\times \quad a^{p-1} \equiv 1 \pmod{p} \in \mathbb{P} \quad \forall a \in \mathbb{Z} \quad \text{with } p \nmid a$

In fact  $(\mathbb{Z}/(p))^\times \cong Z_{p-1}$  is cyclic

For  $m = p_1^{e_1} \dots p_r^{e_r}$  with  $p_i \in \mathbb{P}, e_i \in \mathbb{N}_{>0}$ :

$\mathbb{Z}_{(m)} \cong \mathbb{Z}_{(p_1^{e_1})} \oplus \dots \oplus \mathbb{Z}_{(p_r^{e_r})} \Rightarrow \mathbb{Z}_{(m)}^\times \cong \mathbb{Z}_{(p_1^{e_1})}^\times \times \dots \times \mathbb{Z}_{(p_r^{e_r})}^\times$

what is  $\mathbb{Z}_{(p^e)}$  for  $p \in \mathbb{P}, e \in \mathbb{N}_{>0}$ ?

#### 1.3.1 Theorem 1 (Cyclic group)

Let  $p \in \mathbb{P}$  odd  $e \in \mathbb{N}_{>0} \Rightarrow (\mathbb{Z}_{(p^e)})^\times = Z_{(p-1) \cdot p^{e-1}}$  cyclic

**Proof:**

$(\mathbb{Z}_{(p^e)})^\times \cong Z_{p-1} \Rightarrow \exists z \in \mathbb{Z} : \text{order}(z + p\mathbb{Z}) = p-1$

Set  $a = \bar{z}^{p^{e-1}} \in (\mathbb{Z}_{(p^e)})^\times =: G$

$$a^{p-1} = \bar{z}^{(p-1) \cdot p^{e-1}} = \bar{z}^{|a|} = 1$$

On the other hand, take  $i \in \mathbb{Z}$  such that

$$a^i = 1 \Rightarrow \bar{z}^{i \cdot p^{e-1}} \equiv 1 \pmod{p} \Rightarrow (p-1) \mid (i \cdot p^{e-1}) \Rightarrow (p-1) \mid i.$$

So  $\text{ord}(a) = p-1$ .

Now consider  $b = (p+1) \in G$

**Claim:**  $\text{ord}(b) = p^{e-1}$

**Proof** by induction on  $k \in \mathbb{N}_{>0}$  that  $(p+1)^{p^{k-1}} \equiv p^k + 1 \pmod{p^{k+1}}$

$k=1$  ✓

$k \rightarrow k+1$ : By induction have  $(p+1)^{p^{k-1}} = 1 + p^k + x \cdot p^{k+1}, \quad x \in \mathbb{Z}$

$$\text{Compute: } (p+1)^{p^k} = ((1+p^k) + x \cdot p^{k+1})^p = \sum_{i=0}^p \binom{p}{i} (i+p^k)^{p-i} \cdot x^i \cdot p^{i \cdot (k+1)}$$

$$\stackrel{\text{Only 0-th summand}}{\equiv} (i+p^k) = \sum_{i=0}^p \binom{p}{i} p^{i \cdot k} \stackrel{p \text{ odd}}{\equiv} 1 + p^{k+1} \pmod{p^{k+2}} \quad \checkmark$$

For  $k=e$ :  $(p+1)^{p^{e-1}} \equiv 1 \pmod{p^e} \Rightarrow b^{p^e} = 1 \Rightarrow \text{ord}(b) \mid p^{e-1}$

But  $(p+1)^{p^{e-2}} \equiv p^{e-1} + 1 \pmod{p^e} \Rightarrow b^{p^{e-2}} \neq 1 \in G$

So  $\text{ord}(b) = p^{e-1}$

**Claim:**  $\text{ord}(a \cdot b) = (p-1)p^{e-1} \quad (\Rightarrow \text{Theorem})$

Let  $(a \cdot b)^i = 1 \in G$  with  $i \in \mathbb{Z}$

$$\text{Then } 1 = (a \cdot b)^{i \cdot (p-1)} = (a^{p-1})^i \cdot b^{i \cdot (p-1)} = b^{i \cdot (p-1)} \Rightarrow p^{e-1} \mid i \cdot i(p-1) \Rightarrow p^{e-1} \mid i$$

$$\text{Also } 1 = (a \cdot b)^{p^{e-1} \cdot i} = a^{p^{e-1} \cdot i} \Rightarrow (p-1) \mid p^{e-1} \cdot i \Rightarrow (p-1) \mid i \rightarrow (p-1) \cdot p^{e-1} \mid i \quad \square$$

**Reminder:**  $(\mathbb{Z}/(2^e))^\times \cong Z_2 \times Z_2^{e-2} \quad (e \geq 2)$

### 1.3.2 Algorithm 2 (Fermat Test)

input :  $n \in \mathbb{N}_{>0 \text{ odd}}$

output: " $n \notin \mathbb{P}$ " or "probably  $n \in \mathbb{P}$ "

- (1) Choose  $a \in 2, \dots, n-1$  randomly
- (2) Compute  $a^{n-1} \bmod n$
- (3) If  $a^{n-1} \not\equiv 1 \pmod{n}$  return " $n \notin \mathbb{P}$ "  
else return "probably  $n \in \mathbb{P}$ "

Not very satisfying. Is this fast?

### 1.3.3 Algorithm 3 (Fast exponentiation)

input :  $a \in G$   $G$  is a monoid,  $e \in \mathbb{N}$   $e = \sum_{i=0}^{n-1} e_i 2^i$   $e_i \in \{0, 1\}$

output:  $a^e \in G$

- (1) Set  $b := a, y := 1$
- (2) For  $i = 0, \dots, n-1$  perform (3) - (4)
- (3) if  $e_i = 1$  set  $y := y \cdot b$
- (4) set  $b := b^2$
- (5) return  $y$

this requires  $O(l(e))$  operations in  $G$

For  $G = (\mathbb{Z}/(n)_i)$ , each multiplication requires  $O(l(n)^2)$  bit operations

$\Rightarrow$  Fermat test requires  $O(l(n)^3)$  bit operations  $\rightarrow$  cubic complexity  $\rightarrow$  "fast"!

#### Example:

$n = 561 = 3 \cdot 11 \cdot 17$  For  $a \in \mathbb{Z}$  with  $\gcd(a, n) = 1 \Rightarrow$  have  $a^{n-1} = (a^2)^{280} \equiv 1 \pmod{3}$

$a^{n-1} \equiv 1 \pmod{n}$  Fermat's test says "probably  $n \in \mathbb{P}$ " in 57% of cases.

$n = 2207 \cdot 6619 \cdot 15443$  : output "probably  $n \in \mathbb{P}$ " in 99,93% of cases.

### 1.3.4 Definition 4 (Pseudo-prime, witness, Carmichael numbers)

Let  $n \in N_{>1} \text{ odd}$ ,  $a \in 1, \dots, n-1$

- (a)  $n$  is pseudo-prime to base  $a$  if  $a^{n-1} \equiv 1 \pmod{n}$
- (b) otherwise  $a$  is called a witness of composition of  $n$
- (c) If  $n \notin \mathbb{P}$  but  $a^{n-1} \equiv 1 \pmod{n} \quad \forall a$  with  $\gcd(n, a) = 1$   
then  $n$  is called a Carmichael number.  
There are  $\infty$  Carmichael numbers

### 1.3.5 Proposition 5 (Number of witnesses)

Let  $n \in N_{>1}$ ,  $\text{odd} \wedge \notin \mathbb{P} \wedge \text{not Carmichael}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n, a \text{ is witness of composite of } n\}| > \frac{n-1}{2}$

**Proof:** Consider

$\phi : (\mathbb{Z}/(n))^\times =: G \rightarrow G, \quad \bar{a} \mapsto \bar{a}^{n-1}$

group homomorphism. By assumption,

$|\text{Im}(\phi)| > 1 \Rightarrow |\text{Ker}(\phi)| \leq \frac{|G|}{2} < \frac{n-1}{2}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n \text{ a witness of composite of } n\}| > \frac{n-1}{2} \quad \square$

### Miller-Rabin Test

### 1.3.6 Proposition 6 (Inference from Fermat)

Let  $p \in \mathbb{P} \text{ odd}$ ,  $a \in \{1, \dots, (p-1)\}$  write  $p-1 = 2^k \cdot m$  with  $m \text{ odd}$  Then:  
 $a^m \equiv 1 \pmod{p}$  or  $\exists i \in \{0, \dots, k-1\} : a^{2^i \cdot m} \equiv -1 \pmod{p}$

**Proof:**

Little Fermat:  $\bar{a}^{2^k \cdot m} = 1 \in \mathbb{F}_p$

Assume  $\bar{a}^m \neq 1$  take  $i$  maximal such that:

$\bar{b} = \bar{a}^{2^i \cdot m} \neq 1 \Rightarrow \bar{b}^2 = 1 \Rightarrow \bar{b} \in \mathbb{F}_p$  is a zero of  $x^2 - 1 \in \mathbb{F}_p[x] \Rightarrow \bar{b} = -1$

### 1.3.7 Algorithm 7 (Miller-Rabin-test)

input :  $n \in \mathbb{N}_{>1}, \text{odd}$

output: either " $n \notin \mathbb{P}$ " or "probably  $n \in \mathbb{P}$ "  $\rightarrow$  Monte Carlo Algorithm.

- (1) write  $n - 1 = 2^k \cdot m$  with  $m$  odd
- (2) Choose  $a \in \{2, \dots, n - 1\}$  randomly
- (3) Compute  $b := a^m \pmod n$
- (4) if  $(b \equiv \pm 1 \pmod n)$   
return "probably  $n \in \mathbb{P}$ "
- (5) for  $(i = 0, \dots, k - 1)$  do steps (6) - (7)
- (6) set  $b := b^2 \pmod n$
- (7) if  $(b \equiv -1 \pmod n)$   
return "probably  $n \in \mathbb{P}$ "
- (8) return  $n \notin \mathbb{P}$

### 1.3.8 Definition 8 (strong pseudo-prime / witness)

Let  $n \in \mathbb{N}_{>1}, \text{odd}$   $a \in \{1, \dots, n - 1\}$

- (a)  $n$  is called a strongly pseudo-prime to base  $a$  if Proposition 6 holds for  $a$  and  $p$  replaced by  $n$ .
- (b) Otherwise  $a$  is called a strong witness of composition of  $n$ .

#### Example

Let  $n \in \mathbb{N}_{>1}, \mathbb{P} \text{ odd}$

$a = 2$  strong witness if  $n < 2047$  (including 561)

2 or 3 strong witness if  $n < 1373653$

2,3 or 5 strong witness if  $n < 25326001$

### 1.3.9 Theorem 9 (Bit-complexity of Algorithm 7)

- (a) Algorithm 7 requires  $O(l(n)^3)$  bit operations.  $\rightarrow$  "qubic complexity"  $\rightarrow$  fast!
- (b) if  $b \in \mathbb{P}$  then Algorithm 7 returns "probably  $b \in \mathbb{P}$ "  $\rightarrow$  no false positives.
- (c) if  $n \notin \mathbb{P}$  then more than half of the numbers in  $\{1, \dots, n - 1\}$  are strong witnesses.

**Proof:**

- (a) Step 1 takes  $O(l(n))$  bit operations:  
Using Algorithm 3, we need  $O(l(n-1))$  multiplications in  $\mathbb{Z}/(n)$  each requiring  $O(l(n)^2)$  bit operations.
- (b) Proposition 6
- (c) split in three cases:

**Case 1:**  $n$  is not a Carmichael number.  $\xRightarrow{\text{Prop 5}}$  more than half of all numbers are.

Fermat witness thus also strong witness.

**Case 2:**  $n = p^r \cdot l$  with  $p \in \mathbb{P}$   $r > 1$   $l \in \mathbb{N}_{>0}$   $p \nmid l$

Theorem 1  $\exists x \in \mathbb{Z}$  such that  $x^p \equiv 1 \pmod{p^r}$   $x \not\equiv 1 \pmod{p^r}$

Chinese remainder theorem:  $\exists a \in \mathbb{Z}$  such that  $a \equiv x \pmod{p^r}$   $a \equiv 1 \pmod{l}$

So  $\bar{a}^p = 1 \in \mathbb{Z}/(n) \Rightarrow \bar{a}^n = 1 \Rightarrow \bar{a} \in (\mathbb{Z}/(n))^\times$

i.e.  $\gcd(n, a) = 1$  if  $\bar{a}^{n-1} = 1$  then  $\bar{a} = 1$

But  $a \equiv x \not\equiv 1 \pmod{p^r}$  so  $\bar{a}^{n-1} \neq 1$  hence  $n$  is not Carmichael  $\rightarrow$  Case 1.

**Case 3:**  $n$  is a Carmichael number. By Case 2 have  $n = p \cdot l$  with  $p \in \mathbb{P}$   $p \nmid l$   $l \geq 3$   
 $n$  Carmichael:  $\forall a \in \mathbb{Z}$  with  $\gcd(a, n) = 1$

have  $a^{2^k \cdot m} \equiv 1 \pmod{n}$  (where  $n-1 = 2^k \cdot m$ )

$a^{2^k \cdot m} \equiv 1 \pmod{p}$  Take  $j$  minimal such that

$a^{2^j \cdot m} \equiv 1 \pmod{p}$   $\forall a \in \mathbb{Z}$  such that  $\gcd(a, n) = 1$

so  $0 \leq j \leq l$  in fact,  $j > 0$  since  $(-1)^{2^0 \cdot m} = -1$  with  $m$  odd.

Consider the subgroup  $H := \{\bar{a} \in \mathbb{Z}/(n) \mid \bar{a}^{2^{j-1} \cdot m} \in \{1, -1\} \subseteq (\mathbb{Z}/(n))^\times\}$

Let  $a \in \{1, \dots, n-1\}$   $\gcd(n, a) = 1$   $a$  not a strong witness.

**Claim 1:**  $\bar{a} \in H$

**Case 3.1:**  $\bar{a}^{2^{j-1} \cdot m} = 1 \Rightarrow \bar{a} \in H$

**Case 3.2:**  $a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{n}$   $a^m \not\equiv 1 \pmod{n}$

$\xRightarrow{\text{a nonwitness}}$   $\exists i$  such that  $\underbrace{a^{2^i \cdot m} \equiv -1 \pmod{n}}_*$

$\Rightarrow a^{2^i \cdot m} \equiv -1 \pmod{p} \xRightarrow{\text{def of } j} i < j$

if  $i < j-1$  then  $a^{2^{j-1} \cdot m} = (a^{2^i \cdot m})^{2^{j-1-i}} \equiv (-1)^{2^{j-1-i}} = 1 \pmod{n}$

$\xRightarrow{\text{with } *}$  not in case 3.2

**Claim 2:**  $H \subseteq (\mathbb{Z}/(n))^\times$  proper subgroup.

By definition of  $j$   $\exists x \in \mathbb{Z}$  such that  $x^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p}$

Chinese remainder:  $\exists a \in \mathbb{Z}$  such that

$a \equiv x \pmod{p}$   $a \equiv 1 \pmod{l}$

$\Rightarrow a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p} \equiv 1 \pmod{l} \Rightarrow \bar{a} \notin H$

Claim 2  $\checkmark$

It follows that  $|H| \leq \frac{|(\mathbb{Z}/(n))^\times|}{2} < \frac{n-1}{2}$

so the number of witnesses is  $\geq n-1-|H| > \frac{n-1}{2}$   $\square$

### Remarks:

- (a) A more careful analysis shows that  $2\frac{3}{4}$  of all candidates are strong witnesses
- (b) Calling Algorithm 7 repeatedly decreases the probability of false positives. Running time for prescribed error probability  $p$  is  $O(\lg(p^{-1} \cdot l(n)^3))$   
(Independence assumptions!)

### Connection with Riemann hypothesis

Let  $n \in \mathbb{N}_{>0}$   $\bar{X} : (\mathbb{Z}/(n))^\times \rightarrow \mathbb{C}^\times$  group homomorphism

$$X : \mathbb{Z} \rightarrow \mathbb{C}, a \mapsto \begin{cases} \bar{X}(\bar{a}) & \text{if } \gcd(a, n) = 1 \text{ for } (\bar{a} = a + n\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$$

"residue class character (mod  $n$ )

$$Ex : n = 1 \Rightarrow X(a) = 1 \forall a \in \mathbb{Z}$$

Dirichlet L-series:

$$L_X(s) = \sum_{n=1}^{\infty} \frac{X(n)}{n^s} \text{ converges for } s \in \mathbb{C} \text{ until } Re(s) > 1$$

$L_X(s)$  extends to a meromorphic function on  $\mathbb{C} \mapsto$  "Dirichlet L-function".

For  $n = 1 : L_X(s) = \zeta(s)$  Riemann Zeta-function.

Euler Product:

$$\text{From } (1 - X(p) \cdot p^{-s})^{-1} = \sum_{i=0}^{\infty} (X(p) \cdot p^{-s})^i = \sum_{i=0}^{\infty} \frac{X(p^i)}{p^{is}} \quad \text{derive } L_X(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - X(p) \cdot p^{-s}}$$

Generalized Riemann hypothesis (GRH):

For  $X$  residue class character,  $s \in \mathbb{C}$

with  $L_X(s) = 0$ ,  $0 < Re(s) < 1$  ("critical strip")

then  $Re(s) = \frac{1}{2}$

For  $X = 1 \rightarrow$  ordinary Riemann hypothesis.

### 1.3.10 Theorem (Ankeny & Bach)

GRH  $\Rightarrow \forall X \neq 1$  residue class character

$$\exists p \in \mathbb{P} : X(p) \neq 1, p < 2 \ln(n)^2$$

Let  $H \subsetneq (\mathbb{Z}/(n))^\times =: G$  proper subgroup.

Choose  $N \subsetneq G$  maximal proper subgroup such that  $H \subseteq N \Rightarrow G/N$  cyclic.

$$\bar{X} : G \mapsto \mathbb{C}^\times \text{ with } N = \text{Ker}(\bar{X}) \Rightarrow H \subseteq \text{Ker}(\bar{X})$$

$$\xrightarrow{\text{GRH, Thm1}} \exists p \in \mathbb{P} : p + n\mathbb{Z} \not\subseteq H, p < 2 \cdot \ln(n)^2$$

**Corollary:** Assume GRH.

Let  $n \in \mathbb{N}_{>1}$   $\mathbb{P}$  odd Then there is a strong witness  $a$  of compositeness of  $n$  with  $a < 2 \cdot \ln(n)^2$ .

$\rightarrow$  Obtain deterministic primality test with time  $O(\ln(n)^5)$  bit operations.

### AKS-test

A deterministic polynomial time primality test  $\rightarrow$  "holy grail"

Agrawal, Kayal, Saxena: PRIMES is in P, Annals of Mathematics, 2004.



### 1.3.11 Proposition 10 (Modulo over ideals)

Let  $n \in \mathbb{P}$   $a \in \mathbb{Z} \Rightarrow (x + a)^n \equiv x^n + a \pmod{n}$

where  $x$  is a indeterminate and for  $r \in \mathbb{N}$ :

$$(x + a)^n \equiv (x^n + a) \pmod{n, x^r - 1} \quad (1)$$

(i.e.  $(x + a)^n - (x^n + a) = n \cdot f + (x^r - 1) \cdot g$  with  $f, g \in \mathbb{Z}[x]$ )

**Proof:**

$$(x + a)^n = \sum_{i=0}^n \binom{n}{i} \cdot a^{n-i} \cdot x^i \quad (\text{where } \binom{n}{i} \text{ is a multiple of } n \text{ for } 0 < i < n)$$

$$\equiv x^n + a^n \quad (\leftarrow \text{little Fermat})$$

$$\equiv x^n + a \quad (1) \text{ follows by weakening this.}$$

**Cost** analysis for checking (1) with  $l = \text{length}(n)$ .

Using Algorithm 3, need  $O(l)$  multiplications in  $\mathbb{Z}[x]/(n, x^r - 1) =: R$

Elements of  $R$  are represented as polynomials of degree  $< r$ ,  
coefficients between 0 and  $n$ .

Multiply polynomials:  $O(r^2)$  operation in  $\mathbb{Z}/(n) : O(r^2 \cdot l^2)$

since  $x^{r+k} \equiv x^k \pmod{x^r - 1}$ ,

add coefficients of  $x^{r+k}$  of product polynomial to coefficients  $x^k : O(r \cdot l)$

Total for checking (1):  $O(r^2 \cdot l^3)$  bit operations.

Reduction  $\pmod{x^r - 1}$  is just for keeping the cost under control.

The following is part of AKS-test:

### 1.3.12 Algorithm 11 (Test for perfect power)

input :  $n \in \mathbb{N}_{>1}$

output:  $m, e \in \mathbb{N}$   $e > 1$  such that  $n = m^e$  or "n is not a perfect power"

(1) for  $(e = 2, \dots, \lfloor \lg(n) \rfloor)$  perform (2) - (7) //possible exponents

(2) set  $m_1 = 2, m_2 = n$  //initialize interval  $[m_1, m_2]$  for searching  $\sqrt[e]{n}$

(3) while( $m_1 \leq m_2$ ) do (4) - (7)

(4) set  $m = \lfloor \frac{m_1 + m_2}{2} \rfloor$  // bisect interval

(5) if  $m^e = n$  return  $m, e$

(6) if  $m^e > n$  set  $m_2 = m - 1$

(7) if  $m^e < n$  set  $m_1 = m + 1$

(8) return "not a perfect power"

**Cost:** (for  $l = \text{length}(n)$ )

Compute  $m^e : O(\lg(l) \cdot l^2)$  (abort computation once the result exceeds  $n$ )

Number of passages through inner loops  $\leq \lg(n)$

Number of passages through outer loops  $\leq \lg(n)$

Total cost of Algorithm 11:  $O(l^4 \cdot \lg(l))$

### 1.3.13 Algorithm 12 (AKS-test)

input :  $n \in \mathbb{N}_{>1}$  of length  $l = \text{length}(n) = \lfloor \lg(n) \rfloor + 1$

output: " $n \in \mathbb{P}$ " or " $n \notin \mathbb{P}$ "

- (1) if ( $n$  is a perfect power)  
return " $n \notin \mathbb{P}$ "
- (2) find  $r \in \mathbb{N}_{>1}$  minimal such that  $r|n \vee n^i \not\equiv 1 \pmod{r} \quad \forall i = 1, \dots, l^2$   
//exhaustive search (we will show that  $r \leq l^5$ )
- (3) if  $r|n$   
if ( $r = n$ ) return " $n \in \mathbb{P}$ "  
if ( $r < n$ ) return " $n \notin \mathbb{P}$ "
- (4) for  $a = 1, 2, \dots, \lfloor \sqrt{r} \cdot l \rfloor$  do (5)
- (5) if  $((x+a)^n \not\equiv (x^n + a) \pmod{(n, x^r - 1)})$   
return " $n \notin \mathbb{P}$ "
- (6) return " $n \in \mathbb{P}$ "

### 1.3.14 Lemma 13 (Least common multiple)

For  $n \in \mathbb{N}_{>0}$  have  $\lambda(n) := \text{lcm}(1, 2, \dots, n) \geq 2^{n-2}$

**Proof:** For  $f = \sum_{i=0}^m a \cdot x^i \in \mathbb{Z}(x) \quad a_i \in \mathbb{Z}$

$$\Rightarrow \int_0^1 f(x) dx = \sum_{i=0}^m \frac{a_i}{i+1} = \frac{k}{\lambda(m+1)}$$

with  $k \in \mathbb{Z}$ . Consider  $f_m = x^m \cdot (1-x)^m$

For  $0 < x < 1$ :

$$0 < f_m(x) \leq 4^{-m}$$

$$\Rightarrow 0 < \int_0^1 \underbrace{f_m(x)}_{\frac{k_m}{\lambda(2m+1)}} dx \leq 4^{-1}$$

$$\lambda(2 \cdot m + 1) \geq k_m \cdot 4^m \geq 4^m$$

$$\text{For } n \in \mathbb{N}_{>0} \lambda(n) \geq \lambda(2 \cdot \lfloor \frac{n-1}{2} \rfloor + 1) \geq 4^{\lfloor \frac{n-1}{2} \rfloor} \geq 4^{\frac{n-1}{2}} = 2^{n-2} \quad \square$$

**Corollary:** (not related to AKS)

For  $n \in \mathbb{N}$

$$\pi(n) := |\{p \in \mathbb{P} | p \leq n\}| \geq \frac{n-2}{\lg(n)}$$

**Proof:**

$$2^{n-2} \leq \lambda(n) = \prod_{p \in \mathbb{P}, p \leq n} p^{\lfloor \log_p(n) \rfloor} \leq \prod_{p \leq n} p^{\log_p(n)} = n^{\pi(n)} = 2^{\lg(n)\pi(n)} \quad \square$$

**Prime number theorem:**

$$\lim_{n \rightarrow \infty} \frac{\pi(n)}{n/\ln(n)} = 1$$

**Interpretation:**

The average distance of two primes around some value  $x \in \mathbb{R}_{>1}$  is  $\ln(x)$

### 1.3.15 Lemma 14 (Property of $r$ in Algorithm 12)

In Algorithm 12, have  $r \leq l^5$

**Proof:**

if  $r < l^5 \Rightarrow \forall k \in \{2, \dots, l^5\} : \exists i \in \{1, \dots, l^2\}$

$$n^i \equiv 1 \pmod{k}$$

$$\Rightarrow k \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\Rightarrow \lambda(l^5) \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\xrightarrow[\text{Lemma 13}]{=} 2^{l^5-2} < \prod_{i=1}^{l^2} n^i = n^{\frac{l^2(l^2+1)}{2}}$$

$$\Rightarrow l^5 - l^3 < 4 \quad \text{not true since } l \geq 2 \quad \square$$

### 1.3.16 Theorem 15 (Bit-Complexity of Algorithm 12)

Algorithm 12 requires  $O(l^{16.5})$  bit operations ("polynomial complexity")

**Proof:**

Step(1):  $O(l^4 \cdot \lg(l)) \checkmark$

Step(2): For each  $r$  need:

- test  $r \mid n : O(l^2)$
- compute all  $n^i \pmod{r} : O(l^2 \cdot \lg(r)^2) \xrightarrow[\text{Lemma 14}]{} O(l^2 \cdot \lg(l)^2)$

Step(3):  $O(1)$

$$\text{Step(4): } O(\sqrt{r} \cdot l \cdot r^2 \cdot l^3) \xrightarrow[\text{Lemma 14}]{} O(l^{16.5}) \quad \square$$

**Reminder:** There is a variant of Algorithm 12 with running time  $\tilde{O}(l^6)$ , i.e.,  $O(l^6 \cdot \lg(l)^m)$  with  $m \in \mathbb{N}$ .

**Correctness:**

For  $r \in \mathbb{N}_{>0}$  and  $p \in \mathbb{P}$  write  $I(r, p) := \{m, f\} \in \mathbb{N} \times \mathbb{F}_p[x] \mid f(x)^m \equiv f(x^m) \pmod{x^r - 1}\}$   
 "m is introspective for  $f$  and  $r$ ".

**Example:** Proposition 10 says that:

$$(p, x + \bar{a}) \in I(r, p) \text{ for } a \in \mathbb{Z} \quad r \in \mathbb{N}_{>0} \quad p \in \mathbb{P} \quad (1)$$

### 1.3.17 Lemma 16 (Rules for ideals)

- (a)  $(m, f), (m', f) \in I(r, p) \Rightarrow (m \cdot m', f) \in I(r, p)$
- (b)  $(m, f), (m, g) \in I(r, p) \Rightarrow (m, f \cdot g) \in I(r, p)$
- (c)  $(m \cdot p, f) \in I(r, p), p \nmid r \Rightarrow (m, f) \in I(r, p)$

**Proof:**

- (a)  $f(x)^{m \cdot m'} \equiv f(x^m)^{m'} \pmod{(x^r - 1)}$   
 $f(x^m)^{m'} \equiv f(x^{m \cdot r}) \pmod{(x^{m \cdot r} - 1)}$   
 But  $(x^r - 1) \mid (x^{m \cdot r} - 1)$
- (b)  $(f \cdot g)(x)^m = f(x)^m \cdot g(x)^m \equiv f(x^m) \cdot g(x^m) = (f \cdot g)(x^m) \pmod{(x^r - 1)}$
- (c)  $(f(x)^m)^p \equiv f((x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x^m))^p \pmod{(x^r - 1)}$   
 $\Rightarrow (x^r - 1) \mid ((f(x)^m)^p - f(x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x)^m - f(x^m))^p$   
 $p \nmid r \Rightarrow x^r - 1$  is square free. So  
 $(x^r - 1) \mid (f(x)^m - f(x^m)) \Rightarrow (m, f) \in I(r, p) \quad \square$

### 1.3.18 Theorem 17 (Correctness of Algorithm 12)

Algorithm 12 is correct.

**Proof:**

If the algorithm terminates in step(1),(3) or (5), it is correct. To show: If it terminates in step(6) it is correct, i. e.  $n \in \mathbb{P}$

**Claim 1:**  $\exists p \in \mathbb{P} : p \mid n \quad p \not\equiv 1 \pmod{r} \quad p > r$

Indeed if all prime divisors of  $n$  were  $\equiv 1 \pmod{r}$  then  $n \equiv 1 \pmod{r}$

Contradiction to step(2). All prime divisors of  $n$  are  $> r$  by step (2) and (3) ✓

Steps(2) and (3) imply that  $\gcd(n, r) = 1 \Rightarrow G := \langle \bar{n}, \underbrace{\bar{p}}_{p \bmod r} \rangle \subseteq (\mathbb{Z}/(r))^\times$

Step(2):  $\text{ord}(\bar{n}) > l^2 \Rightarrow l^2 < |G| < r$  (2)

Set  $s := \text{ord}(\bar{p} \in G) \Rightarrow r \mid (p^s - 1)$  with  $q := p^s \Rightarrow r \mid |\mathbb{F}_q^\times| \Rightarrow \exists \zeta \in \mathbb{F}_q$   $r$ -th root of 1

Set  $k := \lfloor \sqrt{r} \cdot l \rfloor \quad m := \left(\frac{n}{p}\right)$

By (1)  $(p, x + \bar{a}) \in I(r, p)$  with  $\bar{a} \in \mathbb{F}_p$

By step(4), have  $(n, x + \bar{a}) \in I(r, p)$

For  $\underline{e} = e_0, \dots, e_k \in \mathbb{N}_0$  set  $f_{\underline{e}} := \prod_{a=0}^k (x + \bar{a})^{e_a}$

Lemma 16 (b):  $(p, f_{\underline{e}}) \in I(r, p)$

$(n, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(c)}} (m, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(a)}} \forall s, t \in \mathbb{N}_0 : (p^s \cdot m^t, f_{\underline{e}}) \in I(r, p)$

$\Rightarrow f_{\underline{e}}(\zeta^{p^s \cdot m^t}) = f_{\underline{e}}(\zeta)^{p^s \cdot m^t}$  (3)

Set  $H := \langle \zeta + \bar{a} | a \in \{0, \dots, k\} \rangle \subseteq \mathbb{F}_q^\times$   
 $(\zeta \notin \mathbb{F}_p \text{ since } r \nmid (p-1) \text{ by Claim 1})$

Consider:  $T := \{(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1} \mid \sum_{a=0}^k e_a < |G|\}$

$\Phi : T \mapsto H, (e_0, \dots, e_k) \mapsto f_{\underline{e}}(\zeta) = \prod_a (\zeta + \bar{a})^{e_a} \in H$

**Claim 2:**  $\Phi$  is injective.

Indeed, take  $(\underline{e}), (\underline{\hat{e}}) \in T$  such that  $\Phi(\underline{e}) = \Phi(\underline{\hat{e}})$

$$\Rightarrow \forall s, t \in \mathbb{N}_0 : f_{\underline{e}}(\zeta^{p^s \cdot m^t}) \stackrel{(3)}{=} f_{\underline{e}}(\zeta)^{p^s \cdot m^t} = f_{\underline{\hat{e}}}(\zeta)^{p^s \cdot m^t} \stackrel{(3)}{=} f_{\underline{\hat{e}}}(\zeta^{p^s \cdot m^t})$$

$f_{\underline{e}} - f_{\underline{\hat{e}}}$  has roots  $\zeta^e$  with  $e \in G$  since  $G = \langle \bar{p}, \bar{m} \rangle$

These are all distinct (since  $\zeta$  is primitive)

But  $\deg(f_{\underline{e}} - f_{\underline{\hat{e}}}) < |G|$  So  $f_{\underline{e}} - f_{\underline{\hat{e}}} = 0$

Since  $k \leq \sqrt{r} \cdot l < r < p$  the  $(x + \bar{a})$  with  $a \in \{0 \dots k\}$  are primitive distinct.

So  $(\underline{e}) = (\underline{\hat{e}})$  ✓

So is  $|H| \geq |T|$  ?

Let  $M$  be the set of all  $\{x_0, \dots, x_k\} \subseteq \{1, \dots, |G| + k\}$

with  $x_0 < x_1 < \dots < x_k$

For  $\{x_0, \dots, x_k\} \in M$  define  $(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1}$  by  $e_a = x_a - x_{a-1}$  with  $x_{-1} := 0$

$$\Rightarrow \sum_{a=0}^k e_a = \sum_{a=0}^k (x_a - x_{a-1} - 1) = x_k - (k+1) < |G|$$

Obtain injection  $M \Leftrightarrow T$

$$\text{So } |H| \geq |T| \geq |M| = \binom{|G|+k}{k+1} \stackrel{(2)}{\geq} \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{k+1} = \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{\lfloor l\sqrt{|a|} \rfloor} \stackrel{(2)}{\geq} \binom{2 \cdot \lfloor l\sqrt{|a|} \rfloor + 1}{\lfloor l\sqrt{|a|} \rfloor}$$

### 1.3.19 Lemma 18 (Property of binomial coefficients)

$$\forall n \in \mathbb{N}_{>1} : \binom{2 \cdot n + 1}{n} > 2^{n+1}$$

**Proof:**

$n = 2 :$

$$\binom{5}{2} = 10 > 2^3$$

$n - 1 \rightarrow n :$

$$\binom{2 \cdot n + 1}{n} = \binom{2 \cdot n}{n-1} + \binom{2 \cdot n}{n} = \binom{2 \cdot n - 1}{n-2} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n} \geq 2 \cdot \binom{2 \cdot n - 1}{n-1} \stackrel{ind.}{>} 2 \cdot 2^n = 2^{n+1}$$

### Continuation of Proof of Theorem 17

$$|H| > 2^{\lfloor l \cdot \sqrt{|a|} \rfloor + 1} \geq 2^{l \cdot \sqrt{|a|}} \geq 2^{\lg(n) \cdot \sqrt{|a|}} = n \sqrt{|a|} \quad (4)$$

Assume  $n \notin \mathbb{P}$  By step (1)  $m$  is not a perfect power

$\Rightarrow$  the map  $\mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{N} \quad (s, t) \mapsto p^s m^t$  is injective.

Set  $A := \{p^s m^t \mid s, t \in \{0, \dots, \lfloor \sqrt{a} \rfloor\}\} \subseteq \mathbb{N}$

$$\Rightarrow |A| = (\lfloor \sqrt{|a|} \rfloor + 1)^2 > |G|$$

Since  $G = \langle \bar{p}, \bar{m} \rangle \subseteq (\mathbb{Z}/(r))^\times$  this implies that  $\exists n, \hat{n} \in A$

such that  $n \neq \hat{n}$  but  $b \equiv \hat{n} \pmod{r}$ .

$$\text{Let } h \in H \Rightarrow h = f_{\underline{e}}(\zeta) \text{ with } (\underline{e}) \in \mathbb{N}_0^{k+1} \Rightarrow h^n \stackrel{(3)}{=} f_{\underline{e}}(\zeta^n) \stackrel{n \equiv \hat{n} \pmod{r}}{=} f_{\underline{e}}(\zeta^{\hat{n}}) \stackrel{(3)}{=} h^{\hat{n}}$$

So the polynomial  $Y^n - Y^{\hat{n}} \in \mathbb{F}_q[Y]$  has all elements of  $H$  as zeros.  
 But  $\deg(Y^n - Y^{\hat{n}}) \leq \max\{n, \hat{n}\} \leq (p \cdot m) \lfloor \sqrt{|G|} \rfloor \leq n \sqrt{|G|} < |H|$   
 $\Rightarrow$  contradiction since  $Y^n - Y^{\hat{n}} \neq 0$   $\square$

## 1.4 Cryptology

A ("Alice") wants to send a message to B ("Bob") such that an eavesdropper E ("Eve") can not read the clear message. So A and B encrypt the message.

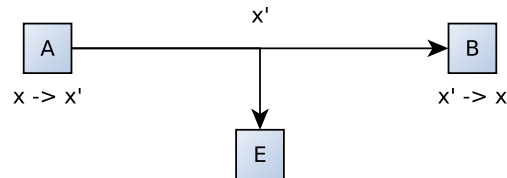


Figure 2: Scheme of eavesdropping

### Symmetric-key cryptography

A and B share secret keys for encryption ( $x \mapsto x'$ ) and decryption ( $x' \mapsto x$ ). Only A and B know the keys.

Example: AES approved by the US government in 2002

Application:

- sending messages
- encrypt files (A=B)

Problem: Key exchange between A and B

### Public-key cryptography

Encryption-map  $\phi : x \mapsto x'$  is made public by B, but decryption  $\phi : x' \mapsto x$  is kept secret.

Advantage: No confidential key exchange.

Disadvantages:

- more costly than symmetric key cryptography
- doubt whether E can reconstruct  $\phi^{-1}$  from  $\phi$  with enough computing power

Applications:

- sending messages
- exchange of symmetric keys
- authentication: Together with  $x$ , B sends  $\phi^{-1}(x)$  (or  $\phi^{-1}|$  Part of  $x$  together with date). A verifies by applying  $\phi$ .  
 Better: challenge-response-protocol.

Examples: RSA, elliptic curve

### 1.4.1 Algorithm (RSA)

- (1) B chooses  $p, q \in \mathbb{P}$  large ( $> 100$  digits)  
with  $p \neq q$   $n := p \cdot q$
- (2) B chooses  $e, f \in \mathbb{N}$  large such that  $e \cdot f \equiv 1 \pmod{\phi(n)}$   
with  $\phi(n) = (p-1)(q-1)$
- (3) B makes  $n, e$  public, keep  $f$  secret
- (4) The message is encoded as an element  $x \in \mathbb{Z}/(n)$
- (5) A computes  $\phi(x) = x^e = y \in \mathbb{Z}/(n)$  and sends  $y$
- (6) B receives  $y$  and computes  $y^f = x \in \mathbb{Z}/(n)$

Comments on steps of RSA:

- (6) Have  $e \cdot f = a \cdot (p-1) \cdot (q-1) + 1$  with  $a \in \mathbb{N}_{>0}$   
 $y^f = x^{e \cdot f}$

Case 1:  $q \nmid f, q \nmid x \Rightarrow x^{a(p-1)(q-1)} = (x^{\phi(n)})^a \stackrel{\text{Little Fermat}}{\equiv} 1^a = 1 \Rightarrow x^{e \cdot f} = x \quad \checkmark$

Case 2:  $p|x, q \nmid x \Rightarrow x^{e \cdot f} \equiv 0 \equiv x \pmod{p}$   
 $x^{e \cdot f} \equiv x \pmod{q}$  as above.

Case 3:  $q|x$  As Case 2

$\Rightarrow$  Correctness of decryption

**Cost:**

- (1) Finding  $p, q$  of length approximately  $l$ . Prime-number theorem: Gap between two primes of length  $\approx l$  is  $O(l)$   
Using Miller Rabin with error probability  $2^{-m}$ . Expected cost of (1) is  $O(m \cdot l^4)$  bit operations.
- (2) Choose  $e$  co-prime to  $\phi(n)$  obtain  $f = \text{inverse} \pmod{\phi(n)}$  by extended euclidean Algorithm:  $O(l^2)$
- (5)(6) Fast exponentiation:  $O(l^3)$

Security of RSA:  $p$  and  $q$  must be so large that factorization of  $n$  is "impossible". Assumption that factorization is expensive could not be shown! But could  $f$  be obtained without knowing  $p$  and  $q$ ? The following algorithm gives a negative answer. It shows that the problem of breaking RSA is always basically factorization.

Remember:  $\phi(n) | (e \cdot f - 1) =: m \leq n^2$

### 1.4.2 Algorithm 1 (Finding a divisor)

Input :  $n \in \mathbb{N}_{>2}$  odd squarefree  $\notin \mathbb{P}$  and  $m \in \mathbb{N}_{>0}$  such that  $\phi(n) \mid m$   $m \leq n^2$

Output:  $d \in \mathbb{N}$  with  $d \mid n$   $1 < d < n$

- (1) Choose  $a \in \{2, \dots, (n-2)\}$  randomly  
set  $k := m$
- (2) If  $d := \gcd(a, n) \neq 1$   
return d
- (3) Repeat steps (4) - (8) //while(true)
- (4) compute  $d := \gcd(n, a^k - 1)$
- (5) If  $d = 1$  go to (1)
- (6) If  $d < n$  return d
- (7) if  $k$  is odd go to (1)
- (8) set  $k := \frac{k}{2}$

Correctness is clear. What about termination and running time?

### 1.4.3 Proposition 2 (Complexity of Algorithm 1)

Algorithm 1 terminates in expected time  $O(l(n)^4)$  bit operations (Las Vegas Algorithm).

**Proof:**

Set  $l := \text{length}(n)$

Have  $n = \prod_{i=1}^r p_i$  with  $p_i \in \mathbb{P}$  distinct.

$\phi(n) = \prod_{i=1}^r (p_i - 1) \mid m$  So initially all  $(p_i - 1)$  divide  $k$ .

At some iteration it happens for the first time that  $(p_i - 1) \nmid k$

Then  $k \equiv \frac{p_1-1}{2} \pmod{(p_1-1)} \Rightarrow a^k \equiv \pm 1 \pmod{p_i}$  -1 occurs for some  $a$

For those  $j$  with  $(p_j - 1) \mid k$  have  $a^k \equiv 1 \pmod{p_j}$

Consider the group homomorphism:  $\phi_i(\mathbb{Z}/(n))^\times \mapsto (\mathbb{Z}/(p_1))^\times \times \dots \times (\mathbb{Z}/(p_r))^\times$   
 $\bar{a} \mapsto (a^k \pmod{p_1}, \dots, a^k \pmod{p_r})$

The image of  $\phi$  is a product of groups  $\{\pm 1\}$  or  $\{1\}$  depending whether  $(p_i - 1) \nmid k$  or  $(p_i - 1) \mid k$



**Conclusion:**

For at least half of all  $a$ 's,  $\phi(\bar{a})$  is neither  $(1, \dots, 1)$  nor  $(-1, \dots, -1)$

If  $a^k \equiv 1 \pmod{p_j}$  then  $p_j \mid (a^k - 1) \Rightarrow p_j \mid d$

If  $a^k \equiv -1 \pmod{p_j}$  then  $p_j \nmid (a^k - 1) \Rightarrow p_j \nmid d$

So for these  $a$  the algorithm is successful.

This means that the expected number of  $a$ 's that need to be tested is  $\leq 2$

(Since  $\sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = 2$  More generally for  $0 < p < 1 : p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} = \frac{1}{p}$ )

Analysis of running time (in bit operations) for each  $a$  (using gcd is quadratic) leads to the claim.  $\square$

**Problems of RSA:**

- How difficult is factorization of integers (lower bound?)
- decryption of some or all messages without having  $f$ ?

**1.4.4 Diffie-Hellmann Key Exchange**

Goal: A, B want to exchange a symmetric key via a public channel

- (1) A and B agree on a  $p \in \mathbb{P}$  (should be large) and  $g \in (\mathbb{Z}/(p))^{\times}$  public
- (2) A chooses  $a \in \{2, \dots, (p-2)\}$  randomly and sends  $u := g^a$  to B
- (3) B chooses  $b \in \{2, \dots, (p-2)\}$  randomly and sends  $v := g^b$  to A
- (4) A computes  $v^a = (g^b)^a = g^{a \cdot b}$   
B computes  $u^b = (g^a)^b = g^{a \cdot b}$

$\Rightarrow$  A and B share  $g^{a \cdot b}$

**Example:**

A chooses  $a = 7$

$$\bar{3}^7 = \bar{11} \in \mathbb{Z}/(17)$$

$$\bar{13}^7 = \bar{4}$$

B chooses  $b = 4$

$$\bar{3}^4 = \bar{13} \in \mathbb{Z}/(17)$$

$$\bar{11}^4 = \bar{4}$$

If Eve reconstructs  $a, b$  from  $g^a$  and  $g^b$  she can compute  $g^{a \cdot b}$

The Security of Diffie-Hellmann depends on the difficulty of the discrete logarithm problem (DLP):

Given  $g \in G$  element of a group or monoid and given  $g^a \in G$ , determine  $a$  (or determine  $a' \in \mathbb{Z}$  such that  $g^a = g^{a'}$ )

**1.4.5 Elliptic curve cryptography (ECC)**

ECC uses elliptic curves as groups.

$$y^2 = x^3 + a \cdot x + b \rightsquigarrow y^2 z = x^3 + axz^2 + bz^3$$

ECC uses suitable elliptic curves on  $\mathbb{F}_a$

## 1.5 Factorization

Let  $m \in \mathbb{N}_{>1}$   $n \notin \mathbb{P}$  Find a divisor  $d$  with  $1 < d < n$ . From this we obtain the factorization of  $n$  by recursion.

Naive method: Trial division. Cost essentially exponential in  $l(n)$

### 1.5.1 Algorithm 1 (Sieve of Eratosthenes)

Input :  $n \in \mathbb{N}_{>1}$

Output: All primes  $\leq n$

- (1) Create a list of all numbers  $\leq n$
- (2)  $p := 2$
- (3) Mark all multiples of  $p$  in the List
- (4) if all numbers are marked  
return
- (5) Let  $p$  be the smallest number that is not marked
- (6)  $p \in \mathbb{P}$  Go to (3)

Running time of Algorithm 1 is exponential.

### Pollard's rho ( $\rho$ ) algorithm:

Idea: Choose a function  $\mathbb{Z}/(n) \mapsto \mathbb{Z}/(n)$  e.g.  $f(x) = x^2 + 1$

Choose  $x_0 \in \mathbb{Z}/(n)$  set  $x_i := f^i(x_0)$  iterative application.

Let  $p \mid n$  be a prime. Since  $|\mathbb{Z}/(p)| < \infty$  then  $\exists i < j : x_i \equiv x_j \pmod{p}$

Starting at  $x_i$  the sequence of  $x_j$  will be periodic mod  $p$ .

$p \mid x_i - x_j \quad p \mid n \Rightarrow p \mid \gcd(n, x_i - x_j) =: d$

If  $x_i \not\equiv x_j \pmod{n}$  (which is not guaranteed) then  $d$  is a proper divisor of  $n$ .

- Recall that gcd computation is cheap
- Testing all pairs is a lot
- Proposition 2 helps with this

### 1.5.2 Proposition 2 (length of periods)

Let  $M$  be a set.  $f : M \mapsto M$  and  $x_0 \in M \quad x_i := f^i(x_0)$

If  $x_{t+l} = x_t$  for  $l, t \in \mathbb{N}l > 0 \quad (\rightarrow t \text{ "off-period", } l \text{ "length of period"})$

$\Rightarrow \exists j \in \mathbb{N}$  with  $0 < j \leq t + l$  such that  $x_j = x_{2j}$

**Proof:**

$f^l(x_t) = x_t \Rightarrow \forall a \in \mathbb{N} \quad f^{a \cdot l}(x_t) = x_t \quad \text{Assume } j = a \cdot l \geq t \quad a \in \mathbb{N}$

$x_{2j} = x_{t+(j-t)+a \cdot l} = f^{(j-t)}(x_{t+a \cdot l}) = f^{(j-t)}(f^{al}(x_t)) = f^{(j-t)}(f^l(x_t)) = x_j$

**Case 1**  $t = 0 \quad j = l \quad \checkmark$

**Case 2**  $t > 0 \quad j = t + \underbrace{(-t \bmod l)}_{\in 0, \dots, (l-1)} \quad \checkmark$

### 1.5.3 Algorithm 3 (Pollard's $\rho$ - Algorithm)

Input :  $n \in \mathbb{N}_{>1}, n \notin \mathbb{P}$

Output: a proper divisor of  $n$  or "FAIL"

- (1) Choose  $x \in \{0, \dots, (n-1)\}$  randomly  
set  $y := x$
- (2) repeat (3)-(6)
- (3)  $x := x^2 + 1 \pmod{n}$      $y := (y^2 + 1)^2 + 1 \pmod{n}$      $// x := x_j y := x_{2j}$
- (4)  $d := \gcd(n, x - y)$
- (5) if  $(1 < d < n)$   
return  $d$
- (6) if  $d = n$   
return "FAIL"

One "FAIL" includes no conclusion so you might want to repeat the Algorithm with a different  $x$ .

Running time? Assume the  $x_i := f^i(x_0)$  are randomly distributed.

When can we expect that a match ( $x_i \equiv x_j \pmod{p}$ ) occurs?  $\rightarrow$  "Birthday Problem"

#### Lemma (Birthday Problem):

We iteratively choose numbers in  $\{1, \dots, n\}$  at random. The expected numbers of choices (if we keep choosing until a number has been chosen twice) is  $< \sqrt{\frac{\pi \cdot n}{2}} + 2$

#### Proof:

Let  $s \geq 2$  be the numbers of choices until a match occurs. For  $k \in \mathbb{N}$  with  $P(s > k)$  as probability

$$P(s > k) = \prod_{i=1}^k \left(1 - \frac{i-1}{n}\right) \leq \prod_{i=1}^k e^{-\frac{i-1}{n}} = e^{\sum_{i=1}^k -\frac{i-1}{n}} = e^{-\frac{k(k-1)}{2n}} \leq e^{-\frac{(k-1)^2}{2n}}$$

\* since  $f(x) = e^x - (1-x) \geq 0$  for  $x \geq 0$

$$f(0) = 0$$

$$f'(x) \geq 0 \text{ if } x \geq 0$$

$$\sum_{k=0}^{\infty} P(s > k) = 2 + \sum_{k=2}^{\infty} P(s > k) \leq 2 + \sum_{k=2}^{\infty} e^{-\frac{(k-1)^2}{2n}} \leq 2 + \int_1^{\infty} e^{-\frac{(x-1)^2}{2n}} dx$$

$$\stackrel{x:=x-1}{=} 2 + \int_0^{\infty} e^{-\frac{x^2}{2n}} dx = 2 + \int_0^{\infty} e^{-\left(\frac{x}{\sqrt{2n}}\right)^2} dx$$

$$\stackrel{x:=\frac{x}{\sqrt{2n}}}{=} 2 + \sqrt{2n} \int_0^{\infty} e^{-x^2} dx = 2 + \sqrt{2n} \cdot \frac{\sqrt{\pi}}{2} = 2 + \sqrt{\frac{n \cdot \pi}{2}}$$

#### Example:

People arrive at a party. When can you expect to have two that share their birthday?

$\rightarrow$  when 26 have arrived!

### 1.5.4 Theorem 4 (Bit-complexity of Algorithm 3)

under suitable assumptions on the distribution  $f^i(x)$  for  $f(x) = x^2 + 1$  Algorithm 3 has the expected running time of  $O(\sqrt[4]{n} \lg(n)^2)$  bit operations

**Proof:**

By Proposition 2 and the Lemma the expected number of runs through the loop is

$$O(\sqrt{p}) = O(\sqrt[4]{n}) \text{ as } p \leq \sqrt{n}$$

Each run through the loop takes  $O(\lg(n)^2)$  bit operations.  $\square$

### Pollard's p-1 Algorithm

Motivation: Let  $p \mid n$  prime divisor

$$\Rightarrow \forall a \in \mathbb{Z} : a^{p-1} \equiv 1 \pmod{p} \quad \text{with } \gcd(a, p) = 1$$

$$\Rightarrow \forall m \in \mathbb{Z} \text{ with } (p-1) \mid m : a^m \equiv 1 \pmod{p}$$

$$p \mid \gcd(a^m - 1, n)$$

Let  $B$  be an upper-bound for the prime powers dividing  $p-1$ .

" $p-1$  is  $B$ -power-smooth".

$$\text{Then } (p-1) \mid \prod_{(q \leq B) \in \mathbb{P}} q^{\lfloor \log_q(B) \rfloor}$$

Neither  $p$  nor  $B$  are known! But guess and try  $B$  and hope for the best.

### 1.5.5 Algorithm 5 (Pollard's $\rho$ - 1 method)

Input :  $n \in \mathbb{N}_{>1} \setminus \mathbb{P}$

Output:  $d \in \mathbb{N}$  with  $d \mid n$   $1 < d < n$  or "FAIL"

- (1) Choose a "smoothness bound"  $B$
- (2) Choose  $a \in \{2, \dots, (n-2)\}$  randomly
- (3) Use Algorithm 1 to find all  $q \in \mathbb{P}$  with  $q \leq B$   
For every  $q$  perform steps (4) - (5)
- (4)  $k := q^{\lfloor \log_q(B) \rfloor}$   
set  $a := a^k \pmod{n}$   
compute  $d := \gcd(n, a - 1)$
- (5) if  $1 < d < n$   
return  $d$
- (6) return "FAIL" //or increase  $B$  and go to (1)

**Consequence:** when setting up RSA  $p, q$  should be chosen such that  $p-1$  and  $q-1$  have large prime divisors.

**The quadratic sieve** (State of the art factorization algorithm)

Observation: if  $n = x^2 - y^2$  then  $n = (x - y) \cdot (x + y)$

Conversely if  $n = a \cdot b$  then  $n = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$

**1-st Idea:** Find  $x, y \in \mathbb{Z}$  such that  $x^2 \equiv y^2 \pmod{n} \quad \wedge \quad x \not\equiv \pm y \pmod{n}$

Then  $n \mid (x - y) \cdot (x + y)$

$\Rightarrow$  for every  $p \in \mathbb{P}$  with  $p \mid n : p \mid (x - y) \vee p \mid (x + y)$

$\Rightarrow p \mid \gcd(x - y, n) \vee p \mid \gcd(x + y, n)$

Since both gcd are  $< n$  receive a non-trivial divisor of  $n$

If  $x^2 \equiv y^2 \pmod{n}$  how probable is it that  $x \equiv \pm y \pmod{n}$  ?

Let  $n = \prod_{i=1}^r p_i^{k_i}$  odd with  $p_i \in \mathbb{P}$  distinct.

Assume  $p_i \nmid x \forall i = 1 \dots r$  Since  $(\mathbb{Z}/(p_i^{k_i}))^\times$  is cyclic there are  $2^r$  classes  $y \pmod{n}$  such that  $x^2 \equiv y^2 \pmod{n}$

[Reason: These classes are given by  $y \equiv \pm x \pmod{p_i^{k_i}}$  These are the only solutions since  $\mathbb{Z}/(p_i^{k_i})^\times$  is cyclic of even order.

$G = \langle \sigma \rangle$  cyclic of order  $2m$

$x = \sigma^i$  Find  $l \in \mathbb{Z}$  such that  $x^2 = (\sigma^l)^2$

$\Leftrightarrow 2j \equiv 2l \pmod{2m} \Leftrightarrow j \equiv l \pmod{m}$

$\Leftrightarrow l \equiv j \pmod{2m}$  or  $l \equiv j + m \pmod{2m}$  ]

But have  $x \equiv \pm y$  only for  $2y$ 's.

Failure probability:  $2^{1-r}$

Handle case  $r = 1$  by Algorithm 11 in 1.3

**Example 1**

$n = 91$  Search  $x, y \in \mathbb{Z} \quad k \in \mathbb{Z}$  such that  $x^2 = k \cdot n + y^2$

Good chance if  $x$  is slightly bigger than  $\sqrt{k \cdot n}$

$k := 1 \Rightarrow \sqrt{91} \approx 9,54 \Rightarrow x := 10 \Rightarrow 10^2 = 100 \equiv 3^2 \pmod{91}$

$n = 10^2 - 3^2 = (10 - 3) \cdot (10 + 3) = 7 \cdot 13 \quad \checkmark$

Another try:

$k := 8 \Rightarrow \sqrt{8 \cdot 91} \approx 26,98 \Rightarrow 27^2 \equiv 1^2 \pmod{91} \quad \gcd(26, 91) = 13$

**Example 2**

$n = 4633 \quad k := 3$

$\sqrt{3 \cdot n} \approx 117,89 \Rightarrow x^2 = 118^2 \equiv 5^2 \pmod{n}$

$\gcd(118 - 5, n) = 113$

$\gcd(118 + 5, n) = 41 \quad \checkmark$

**2-nd Idea:** Choose  $B \in \mathbb{N}$  "smoothness bound" suitable.

Let  $p_2, \dots, p_r \in \mathbb{P}$  be all primes  $\leq B$  (Algorithm 1) set  $p_1 := -1$

The  $p_i$  form a "factor basis".

For  $a \in \mathbb{Z}$  write  $(a \pmod{n})$

for the  $x \in \mathbb{Z}$  with  $x \equiv a \pmod{n}$  and  $-\frac{n}{2} < x \leq \frac{n}{2}$

**Procedure:**

Search numbers  $a_1, \dots, a_m \in \mathbb{Z}$  such that  $(a_i^2 \pmod{n}) = \prod_{j=1}^r p_j^{e_{ij}}$

with  $e_{ij} \in \mathbb{Z}$  ("B numbers")

So for  $\mu_1, \dots, \mu_m \in \mathbb{N}_0$  have  $\left( \prod_{i=1}^m a_i^{\mu_i} \right)^2 \equiv \prod_{i=1}^m \prod_{j=1}^r p_j^{\mu_i \cdot e_{ij}} \pmod{n} = \prod_{j=1}^r p_j^{\sum_{i=1}^m \mu_i \cdot e_{ij}} \pmod{n}$

If the vectors  $(e_{i1}, \dots, e_{ir})$  become linearly dependent mod 2 (guaranteed if  $m > r$ ) then

$\exists \mu_1, \dots, \mu_m \in \{0, 1\}$  not all 0 such that:

$$\sum_{i=1}^m \mu_i \cdot e_{ij} = 2 \cdot k_j \quad k_j \in \mathbb{N}_0$$

with  $x := \prod_{i=1}^m a_i^{\mu_i}$   $y := \prod_{j=1}^r p_j^{k_j}$  obtain  $x^2 \equiv y^2 \pmod{n}$

**Example:**  $n = 4633$  choose  $B = 3 \Rightarrow$  factor basis  $-1, 2, 3$

Search  $a \in \mathbb{Z}$  such that  $|a_i^2 \pmod{n}|$  is small. Idea:  $a \approx \sqrt{n} = 68.06\dots$

$$a_1 := 68 : 68^2 = n - 9 \equiv (-1) \cdot 3^2 \pmod{n}$$

$$\rightarrow e_1 = (1, 0, 2) \rightarrow (1, 0, 0) \in \mathbb{F}_2^3$$

$$a_2 := 69 : 69^2 = n + 128 \equiv 2^7 \pmod{n}$$

$$\rightarrow e_2 = (0, 7, 0) \rightarrow (0, 1, 0) \in \mathbb{F}_2^3$$

$$a_3 := 67 : 67^2 = n - 144 \equiv (-1) \cdot 2^4 \cdot 3^2$$

$$\rightarrow e_3 = (1, 4, 2) \rightarrow (1, 0, 0) \in \mathbb{F}_2^3$$

$$e_1 + e_3 \equiv 0 \pmod{2} \quad \text{In fact:}$$

$$e_1 + e_3 = 2 \cdot \underbrace{(1, 2, 2)}_{(k_1, k_2, k_3)} \rightarrow \mu_1 = 1 \quad \mu_2 = 0 \quad \mu_3 = 1$$

$$x := a_1 \cdot a_3 \equiv -77 \pmod{n}$$

$$y := (-1) \cdot 2^2 \cdot 3^2 = -36$$

$$x - y = -41 \quad x + y = -113$$

$$\gcd(n, x - y) = 41 \quad \gcd(n, x + y) = 113 \quad \checkmark$$

**3rd Idea:** Look for  $a_i$  of the form  $t + \lfloor \sqrt{n} \rfloor$  with  $t$  in "suitable" sieve Interval:  $[-s, s] \cap \mathbb{Z}$

As it turns out if  $s \leq \frac{\sqrt{5}-2}{2} \lfloor \sqrt{n} \rfloor \approx 0.118 \lfloor \sqrt{n} \rfloor$  then

$$(t + \lfloor \sqrt{n} \rfloor)^2 \pmod{n} = (t + \lfloor \sqrt{n} \rfloor)^2 - n =: f(t)$$

When does  $p_j^{e_j}$  divide  $f(t)$  (with  $j \geq 2$ )? Precisely if  $(t + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_j^{e_j}}$

If this holds for some  $t$  then it also holds for all  $t + k \cdot p_j^{e_j}$  with  $k \in \mathbb{Z}$  Moreover if it holds then  $\bar{n} \in \mathbb{F}_{p_j}$  is square. So may remove all  $p_j$  such that  $\bar{n} \in \mathbb{F}_{p_j}$  is a non-square from the factor basis.

Obtain a sieving procedure:

For  $t \in [-s, s] \cap \mathbb{Z}$  with  $p_j^{e_j} \mid f(t)$  "mark" all elements  $t + k \cdot p_j^{e_j} \in [-s, s]$

### 1.5.6 Algorithm 6 (Quadratic sieve, simplified version)

Input :  $n \in \mathbb{N}_{>1} \setminus \mathbb{P}$  odd

Output: A non trivial divisor of  $n$  or "FAIL"

- (1) if  $(n = m^e)$  with  $m, e \in \mathbb{N}_{>1}$   
return  $m$  // can be done with Algorithm 11 § 3
- (2) Choose a "smoothness bound"  $B \in \mathbb{N}$  and a "sieve bound"  $s \in \mathbb{N}$  suitably
- (3) Let  $p_1 = -1$   $p_2, \dots, p_r$  be the factor basis given by  $B$ . Delete those  $p_j$  such that  $\bar{n} \in \mathbb{F}_{p_j}$  is a non-square
- (4) for  $(t = -s, -s + 1, \dots, s - 1)$   
compute  $f_t := |(t + \lfloor \sqrt{n} \rfloor)^2 - n| \in \mathbb{N}_{>0}$
- (5) for  $(t = -s, \dots, s)$   
set  $e_t := (0, \dots, 0) \in \mathbb{N}_0^r$  // initialize exponent vectors
- (6) for  $(t = -s, \dots, 0)$   
set  $e_{t,1} := 1$  //  $\rightarrow$  first entry of each  $e_t$  is the exponent of  $p_1 = -1$  in  $f(t)$
- (7) for  $(j = 2, \dots, r)$  repeat (8) - (10)
- (8) for  $(e = 1, \dots, \lfloor \log_{p_j}(B) \rfloor)$  repeat (9) - (10) // or maybe a bit larger
- (9) solve  $(t + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_j^e}$   
Let  $(t_1 \pmod{p_j^e}), \dots, (t_m \pmod{p_j^e})$  be the solutions.  
// We will see that  $m \in \{0, 2, 4\}$  with  $m = 2$  most frequent.
- (10) for all  $t = t_i + k \cdot p_j^e \in [-s, s]$  with  $k \in \mathbb{Z}, i = 1, \dots, m$   
set  $e_{t,j} := e_{t,j} + 1$   
 $f_t := \frac{f_t}{p_j}$
- (11) let  $t, \dots, t_m$  be those  $t \in [-s, s] \cap \mathbb{Z}$  for which  $f_t = 1$   
/\* So the  $a_i = t_i + \lfloor \sqrt{n} \rfloor$  are  $B$ -numbers and the factorization  
\* of  $a_i^2 \pmod{n} = a_i^2 - n = f(t)$  is given by the exponent  
\* vectors  $e_t$  \*/
- (12) if the  $(e_{t_i} \pmod{2}) \in \mathbb{F}_2^r (i = 1, \dots, m)$  are not linearly dependent.  
return "FAIL"
- (13) compute  $\mu_1, \dots, \mu_m \in \{0, 1\}, k_1, \dots, k_r \in \mathbb{N}_0$  such that  $\sum_{i=1}^m \mu_i e_{t_i} = 2 \cdot (k_1, \dots, k_r)$
- (14) set  $x := \prod_{i=1}^m (t_i + \lfloor \sqrt{n} \rfloor)^{\mu_i} \pmod{n}$   
 $y := \prod_{j=1}^r p_j^{k_j} \pmod{n}$  // Now  $x^2 \equiv y^2 \pmod{n}$

(15) if  $\gcd(n, x - y)$  or  $\gcd(n, x + y)$  is a non-trivial divisor  
     return the non-trivial divisor  
 else  
     return "FAIL"

With good heuristics it will almost certainly never return FAIL.

**Example:**  $n = 20437$

Choose  $B := 10$      $s := 3$

Factor basis:  $p_1 = -1$     $p_2 = 2$     $p_3 = 3$     $p_4 = 7$

(5 omitted as:  $n \equiv 2 \pmod{5}$  non-square)

$\lfloor \sqrt{n} \rfloor = 142$

Solve  $(t + 142)^2 \equiv n \pmod{p_j^e}$

$p_2 = 2$  : Compute modulo 2,4,8.  $n \equiv 5 \pmod{8}$

$t$  odd  $\Rightarrow (t + 142)^2 \equiv 1 \pmod{8} \Rightarrow (t + 142)^2 \equiv n \pmod{4}$  but not  $\pmod{8}$

$t$  even  $\Rightarrow (t + 142)^2 \equiv 0 \pmod{2} \not\equiv n \pmod{2}$

$$\Rightarrow e_{t,2} = \begin{cases} 2 & t \text{ odd} \\ 0 & t \text{ even} \end{cases}$$

$p_3 = 2$  :  $n \equiv 1 \pmod{3}$     $\lfloor \sqrt{n} \rfloor \equiv 1 \pmod{3}$

So  $3 \mid f(t) \Leftrightarrow t + 1 \equiv \pm 1 \pmod{3} \Leftrightarrow t \equiv 0 \text{ or } 1 \pmod{3}$

$e = 2$     $n \equiv 7 \equiv (\pm 4)^2 \pmod{9}$     $\lfloor \sqrt{n} \rfloor \equiv 7 \pmod{9}$

So  $9 \mid f(t) \Leftrightarrow t + 7 \equiv \pm 4 \pmod{9} \Leftrightarrow t \equiv -3, -2 \pmod{9}$

$p_4 = 7$     $n \equiv 4 \pmod{7}$     $4 = (\pm 2)^2$     $\lfloor \sqrt{n} \rfloor \equiv 2 \pmod{7}$

So  $7 \mid f(t) \Leftrightarrow t + 2 \equiv \pm 2 \pmod{7} \Leftrightarrow t \equiv 0 \text{ or } 3 \pmod{7}$

$t$	-3	-2	-1	0	1	2	3
$f_t =  f(t) $	1116	837	556	273	12	295	588
$p_1$ component of $e_t$	1	1	1	1	0	0	0
$p_2$ component	2	0	2	0	2	0	2
$f_t$ divided by 2-power	279	837	139	273	3	299	147
$p_3$ component	2	2	0	1	1	0	1
$f_t$	31	93	139	91	1	299	49
$p_4$ component	0	0	0	1	0	0	2
$f_t$	31	93	139	13	1	299	1

Obtain  $m = 2$  :  $t_1 = 1$     $t_2 = 3$     $e_1 = (0, 2, 1, 0)$     $e_3 = (0, 2, 1, 2)$

They are linear dependent  $\pmod{2}$

$e_1 + e_3 = 2 \cdot (0, 2, 1, 1)$

$x = (142 + 1) \cdot (142 + 3) \equiv 298 \pmod{n}$

$y = p_2^2 \cdot p_3 \cdot p_4 = 2^2 \cdot 3 \cdot 7 = 84$

$\gcd(n, x - y) = \gcd(n, 214) = 107$

$\gcd(n, x + y) = 191$

Indeed  $n = 107 \cdot 191$



**Computing square roots (mod  $p^e$ )****Case 1:  $p$  odd**

Find  $x$  with  $x^2 \equiv n \pmod{p}$  by trying  $x \pmod{p}$  (exactly two solutions). Suppose we have found  $x$  with  $x^2 \equiv n \pmod{p^e}$

So  $x^2 - n = p^e \cdot r \quad r \in \mathbb{Z}$

New  $x$  should be  $x + y \cdot p^e$

Compute modulo  $p^{e+1}$ :  $(x + y \cdot p^e)^2 - n = x^2 + 2yxp^e + y^2p^{2e} - n \equiv p^e \cdot (r + 2xy) \pmod{p^{e+1}}$

So  $(x + y \cdot p^e)^2 \equiv n \pmod{p^{e+1}} \Leftrightarrow 2xy \equiv -r \pmod{p}$  uniquely and easily solvable

$\rightarrow$  Obtain two solutions (mod  $p^e$ )

$\Rightarrow$  special case of "Hensel lifting"

**Case 2:  $p = 2$** 

Find  $x \in \mathbb{Z}$  with  $x^2 \equiv n \pmod{8}$  (0 or 4 solutions since  $n$  odd)

Assume we have  $x^2 \equiv n \pmod{2^e} \quad e \geq 3$

So  $x^2 - n = r \cdot 2^e$

$\Rightarrow (x + y \cdot 2^{e-1})^2 - n = x^2 + xy \cdot 2^e + y^2 2^{2e-2} - n \equiv 2^e(r + xy) \pmod{2^{e+1}}$

So  $(x + y \cdot 2^{e-1})^2 \equiv n \pmod{2^{e+1}} \Leftrightarrow y \equiv r \pmod{2}$

$\rightarrow$  0 or 4 solutions

**Running time of quadratic sieve**

Choose  $B \approx \exp\left(\sqrt{\frac{1}{2} \ln(n) \cdot \ln(\ln(n))}\right)$

If  $s \approx B$  then running time is:  $O\left(\exp\left(\sqrt{\ln(n) \cdot \ln(\ln(n))}\right)\right)$

which is "slightly" sub-exponential

**Factorization algorithm with best complexity (known to date):**

Number field sieve

This also uses ideas 1 and 2, but an algebraic number field is used for generating  $B$ -numbers.

Heuristic Running time (modulo some conjectures):  $O\left(\exp\left(\ln(n)^{\frac{1}{3}} \cdot \ln(\ln(n))\right)^{\frac{2}{3}}\right)$

## 2 Systems of equations

### 2.6 Linear Algebra

Tasks:

- solving systems of linear equations (= linear systems)
- inversions of matrices
- rank determination
- determinants
- matrix products

$K$  field,  $K^{m \times n}$  = set of  $m \times n$  matrices

$GL_n(K)$  = field of  $n \times n$  matrices

Count the cost of algorithms in terms of field operations. If  $K$  is a finite field this translates directly to bit operations.

#### 2.6.1 Proposition 1 (Complexity of usual algorithms)

- Solving an  $m \times n$ -linear system by Gaussian elimination requires  $O(\max\{m, n\}^3)$  field operations
- For  $A \in GL_n(K)$  computing  $A^{-1}$  by usual method requires  $O(n^3)$  field operations.
- Computing  $\det(A)$  "as usual" requires  $O(n^3)$  bit operations.
- Computing  $A \cdot B$  for  $A \in K^{m \times n}$   $B \in K^{n \times l}$  requires  $O(m \cdot n \cdot l)$  field operations.

→ all cubic!

**Proof:**

- Cost of treating the  $k$ -th row with Gauss algorithm:  
 $\leq 1$  inversion,  $\leq (n - k)$  multiplications  
 $\leq (m - k)(n - k)$  multiplications and additions  
 (clearing column below pivot element)  
 Back substitution (i.e. clearing columns above pivot element):  
 Let  $r = rk(A) \leq (k - 1)(n - r)$  multiplications and additions  
 Total cost  $\leq \sum_{k=1}^r (1 + n - k + 2(m - k)(n - k) + 2(k - 1)(n - r))$   
 $= 2mnr - mr^2 - \frac{1}{3}r^3 - nr + \frac{3}{2}r^2 + \frac{5}{6}r - mr$   
 $\in O(\max\{m, n\}^3)$
- Inversion is Gaussian elimination of  $n \times 2n$ -matrix of rank  $n$   
 cost  $\leq \frac{8}{3}n^3 - \frac{3}{2}n^2 + \frac{5}{6}n \in O(n^3)$
- reduced to (a)

(d) obvious

### Strassen-multiplication

let  $A, B \in K^{2n \times 2n}$  Write:  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$  with  $A_{ij} B_{ij} \in K^{n \times n}$

Then  $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$  with  $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} \rightarrow 8$  multiplications.

Set:

$$M_1 := (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$$

$$M_2 := (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$$

$$M_3 := (A_{11} - A_{21}) \cdot (B_{11} + B_{12})$$

$$M_4 := (A_{11} + A_{12}) \cdot B_{22}$$

$$M_5 := A_{11} \cdot (B_{12} - B_{22})$$

$$M_6 := A_{22} \cdot (B_{21} - B_{11})$$

$$M_7 := (A_{21} + A_{22}) \cdot B_{11}$$

Then:

$$C_{11} = M_1 + M_2 - M_4 + M_6$$

$$C_{12} = M_4 + M_5$$

$$C_{21} = M_6 + M_7$$

$$C_{22} = M_2 - M_3 + M_5 - M_7$$

$\rightarrow 7$  Multiplications!

### 2.6.2 Algorithm 2 (Strassen-multiplication)

Input :  $A \in K^{m \times n} B \in K^{n \times l}$

Output:  $A, B \in K^{m \times l}$

- (1) Let  $k$  be minimal such that  $m, n, l \leq 2^k$
- (2) if  $(k = 0)$   $//(\Leftrightarrow A, B \in K^{1 \times 1})$   
return  $A \cdot B$
- (3) Enlarge  $A, B$  by adding zeros such that  $A, B \in K^{2^k \times 2^k}$
- (4) write  $A \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$  with  $A_{ij}B_{ij} \in K^{2^{k-1} \times 2^{k-1}}$
- (5) compute  $M_1 \dots M_7$  as above, do multiplications by recursive call
- (6) compute  $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$  by above formulas
- (7) Output: the upper left  $m \times l$  - part of  $A \cdot B$

### 2.6.3 Theorem 3 (Running time of Algorithm 2)

If  $m, n, l \leq r$  Algorithm 3 requires  $O(r^{\lg(7)})$  field operations

**Proof:**

Set  $\Theta(k)$  = number of field operations.

Step 5:  $7 \cdot \Theta(k-1) + 10 \cdot (2^{k-1})^2$

Step 6:  $8 \cdot (2^{k-1})^2$

Obtain:

$$\Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1} \quad (*)$$

Claim:  $\Theta(k) = 7^{k+1} - 6 \cdot 4^k$

Induction on  $k$

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$k-1 \rightarrow k : \Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1}$

$= 7(7^k - 6 \cdot 4^{k-1}) + 18 \cdot 4^{k-1}$

*induction*

$= 7^{k+1} - 4 \cdot 6 \cdot 4^{k-1} \quad \checkmark$

Have  $2^{k-1} < r \Rightarrow k < \lg(r) + 1 \Rightarrow \Theta(k) < 7^{\lg(r)+2} = 49 \cdot 2^{\lg(7) \cdot \lg(r)} = 49^{\lg(17)} \quad \checkmark$

**Remarks:**

(a)  $\lg(7) = 2.8074\dots$

(b) Coppersmith-Winograd:  $O(r^{2.3754\dots})$

Improved by Stothes (2010), Williams(2011), Le Gall(2014):  $O(r^{2.3729\dots})$

(c) The cost of the best possible algorithm is unknown, even for  $r = 3$

Let  $M : \mathbb{N}_{>0} \mapsto \mathbb{R}_{>0}$  be a function such that two matrices in  $K^{n \times n}$  can be multiplied in  $\leq M(n)$  field operations. Assume  $\exists \epsilon > 0 : \forall n :$

$$2^{2+\epsilon} M(n) \leq M(2n) \leq 8 \cdot M(n) \quad (1)$$

**Example:**  $M(n) = 49 \cdot n^{\lg(7)}$

Recall:  $A = (a_{ij})$  is upper (lower) triangular  $\Leftrightarrow a_{ij} = 0$  for  $i > j$  ( $i < j$ )

### 2.6.4 Proposition 4 (Complexity of matrix inversion)

An upper or lower triangular matrix  $A \in GL_n(K)$  can be inverted in  $O(M(n))$  field operations.

**Proof:**

Let  $k \in \mathbb{N}$  be minimal such that  $n \leq 2^k$

write  $B = \begin{pmatrix} A & 0 \\ 0 & I_{2^k-n} \end{pmatrix} \in GL_{2^k}(K) \Rightarrow B^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & I_{2^k-n} \end{pmatrix}$

Assume  $B$  upper triangular:

$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$   $B_{11}, B_{22} \in GL_{2^{k-1}}(K), B_{12} \in K^{2^{k-1} \times 2^{k-1}}$

$B^{-1} = \begin{pmatrix} B_{11}^{-1} & -B_{11}^{-1} \cdot B_{12} \cdot B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{pmatrix}$

Let  $\Theta(k)$  = computation cost depending on  $k$ .

$$\Theta(k) \leq 2 \cdot \Theta(k-1) + 2M(2^{k-1}) \stackrel{(1)}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2} \cdot M(2^k) \quad (**)$$

Claim:  $\Theta(k) \leq 2^k + M(2^k)$

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$k-1 \rightarrow k : \Theta(k) \stackrel{**}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2}M(2^k) \stackrel{\text{induction}}{\leq} 2(2^{k-1} + M(2^{k-1})) + \frac{1}{2}M(2^k) \stackrel{(1)}{\leq}$

$2^k + \frac{1}{2}M(2^k) + \frac{1}{2}M(2^k) \quad \checkmark$

Have  $n > 2^{k-1} \Rightarrow k < \lg(n) + 1 \Rightarrow \Theta(k) < 2 \cdot n + M(2n) \stackrel{(1)}{\leq} 2 \cdot n + 8 \cdot M(n) \quad \square$

**Project:** Reduce (most) tasks of linear algebra to multiplication.

The following algorithm transforms a matrix such that all tasks become easy.

### 2.6.5 Algorithm 5 (Transforming a matrix)

Input :  $A \in K^{m \times n}$

Output: Matrices  $L, Q, P, U$

such that:  $LQAP = \begin{pmatrix} U & \\ & 0 \end{pmatrix} \begin{matrix} r \\ m-r \end{matrix}$  ( $\leftarrow$  in row-echelon form)  $\in K^{m \times n}$

such that:

- $L \in K^{m \times m}$  lower triangular with 1's on the diagonal
- $Q \in K^{m \times m}$   $P \in K^{n \times n}$  permutation matrices
- $U \in K^{m \times m}$  upper triangular with non-zero diagonal entries ( $r = 0$  if  $A = 0$ )
- If  $r = m$  then  $Q = I_m$

(1) if  $(A = (0...0))$   
 return  $L = Q = (1)$   $P = I_n$   $r = 0$

(2) if  $(A = a_1, \dots, a_n)$   
 let  $i$  be minimal with  $a_i \neq 0$   
 $P :=$  matrix exchanging 1st and  $i$ -th position in  $A$   
 return  $L = Q = (1)$   $P$   $U = A \cdot P$

(3) let  $m_1 = \lfloor \frac{m}{2} \rfloor$   $m_2 = \lceil \frac{m}{2} \rceil$   
 write  $A = \begin{bmatrix} B \\ C \end{bmatrix}_{\substack{m_1 \\ m_2}}$   $B \in K^{m_1 \times n}$   $C \in K^{m_2 \times n}$

(4) Applying the algorithm recursively on  $B$

$$\text{obtain } L_1 \cdot Q_1 \cdot B \cdot P = \begin{bmatrix} U_1 \\ 0 \end{bmatrix}_{\substack{r_1 \\ m_1 - r_1 \\ n}} \quad \text{with } U_1 \in K^{r_1 \times n}$$

(5) write  $L_1 = \begin{bmatrix} L_t & 0 \\ L_l & L_r \end{bmatrix}_{\substack{r_1 \\ m_1 - r_1}}$   $Q_1 = \begin{bmatrix} Q_t \\ Q_b \end{bmatrix}_{\substack{r_1 \\ m_1 - r_1}}$   $U_1 = \begin{bmatrix} E & U'_1 \end{bmatrix}_{\substack{r_1 \\ n - r_1}}$

$$\text{form } D := C \cdot P_1 = \begin{bmatrix} F & D' \end{bmatrix}_{\substack{r_1 \\ n - r_1}}$$

$$\text{and } G := D' - FE^{-1}U' \in K^{m_2 \times (n - r_1)}$$

(6) Apply the algorithm recursively to  $G$ :

$$L_2 \cdot Q_2 \cdot G \cdot P_2 = \begin{bmatrix} U_2 \\ 0 \end{bmatrix}_{\substack{r_2 \\ m_2 - r_2 \\ n - r_1}}$$

(7) return

$$L := \begin{bmatrix} L_t & 0 & 0 \\ -L_2 Q_2 F E^{-1} L_t & L_2 & 0 \\ L_l & 0 & L_r \end{bmatrix}_{\substack{r_1 \\ m_2 \\ m_2 - r_1}}$$

$$Q := \begin{bmatrix} Q_t & 0 \\ 0 & Q_2 \\ Q_b & 0 \end{bmatrix}_{\substack{r_1 \\ m_2 \\ m_1 - r_1}}$$

$$P := P_1 \cdot \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix}_{\substack{r_1 \\ n - r_1}}$$

$$U := \begin{bmatrix} E & U'_1 P_2 \\ 0 & U_2 \end{bmatrix}_{\substack{r_1 \\ n - r_1 \\ r_2}}$$

### 2.6.6 Theorem 6 (Correctness and running time of Algorithm 5)

Algorithm 5 is correct and requires  $O((\frac{n}{m} + 1) \cdot M(m))$  field operations

**Proof:**

Correctness by induction on  $m$

$m = 1$  ✓

$m > 1$  :

$m_1, m_2 < m$  so recursive calls are correct by induction.

By step (7)  $L, Q, P, U$  have desired form.

Compute:

$$\begin{aligned}
 LQAP &= \begin{array}{cc} m_1 & m_2 \\ \begin{array}{cc} L_t Q_t & 0 \\ -L_2 Q_2 F E^{-1} L_t Q_t & L_2 Q_2 \\ L_l Q_t + L_r Q_b & 0 \end{array} \end{array} \begin{array}{c} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \cdot \begin{array}{c} \begin{array}{cc} B \cdot P_1 & \\ & D \end{array} \\ m_1 \\ m_2 \end{array} \cdot \begin{array}{cc} \begin{array}{cc} I_{r_1} & 0 \\ 0 & P_2 \end{array} \\ r_1 \\ n - r_2 \end{array} \\
 &\stackrel{(4)}{=} \begin{array}{cc} n \\ \begin{array}{cc} U_1 & \\ L_2 Q_2 (-F E^{-1} U_1 + D) & \\ 0 & \end{array} \end{array} \begin{array}{c} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \cdot \begin{array}{cc} \begin{array}{cc} I_{r_1} & 0 \\ 0 & P_2 \end{array} \\ r_1 \\ n - r_2 \end{array} \\
 &= \begin{array}{cc} r_1 & n - r_1 \\ \begin{array}{cc} E & U'_1 \\ 0 & L_2 Q_2 G \\ 0 & 0 \end{array} \end{array} \begin{array}{c} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \cdot \begin{array}{cc} \begin{array}{cc} I_{r_1} & 0 \\ 0 & P_2 \end{array} \\ r_1 \\ m_2 \end{array} \stackrel{(6)}{=} \begin{array}{cc} r_1 & n - r_1 \\ \begin{array}{cc} E & U'_1 P_2 \\ 0 & U_2 \\ 0 & 0 \\ 0 & 0 \end{array} \end{array} \begin{array}{c} r_1 \\ r_2 \\ m_2 \cdot r_2 \\ m_2 \cdot r \end{array} = \begin{array}{c} U \\ 0 \end{array}
 \end{aligned}$$

Suppose  $r = m \Rightarrow r_1 = m_1 \quad r_2 = m_2 \Rightarrow Q_1 = I_{m_2} \quad Q_2 = I_{m_2} \Rightarrow Q = I_m$

**Cost:**

Fix  $n \in \mathbb{N}$  and set  $\Theta(k) :=$  maximal cost for a matrix  $A \in K^{m \times n'}$  with  $m \leq 2^k \quad n' \leq n$

Choose  $A \in K^{m \times n'}$  with cost  $= \Theta(k)$

Step(4) and (6):  $\leq \Theta(k - 1)$  each

Step(5):  $E^{-1}$ : by Proposition 4:  $O(M(r_1)) \leq O(M(2^{k-1}))$

$F \cdot E^{-1} \leq M(2^{k-1})$

$F \cdot E^{-1} \cdot U'$ : at most the cost of multiplying a  $2^{k-1} \times 2^{k-1}$  matrix by a  $2^{k-1} \times n$  matrix.

Split right matrix into square parts.

$\Rightarrow$  cost  $\leq \lceil \frac{n}{2^{k-1}} \rceil \cdot M(2^{k-1}) \leq (2^{1-k} \cdot n + 1) \cdot M(2^{k-1})$

$G$ : subtraction:  $m_2 \cdot (n - r_1) \leq 2^{k-1} \cdot n \leq 2^{1-k} \cdot n \cdot M(2^{k-1})$

Step (7):  $F \cdot E^{-1}$  already computed,  $L_2 Q_2$ : permuting rows. Cost:  $\leq 2 \cdot M(2^{k-1})$

Obtain:  $\Theta(k) \leq 2 \cdot \Theta(k - 1) + (2^{-k} \cdot n + c) \cdot M(2^k) \quad c \text{ constant}$

From this obtain by induction:

$\Theta(k) \leq \left( 2^{-k} \cdot n \cdot \frac{1-2^{-k\epsilon}}{1-2^{-\epsilon}} + 2 \cdot c \cdot (1 - 2^{-k}) \right) \cdot M(2^k) \leq \left( \frac{1}{1-2^{-\epsilon}} \cdot \frac{n}{2^k} + 2c \right) \cdot M(2^l)$

Finally obtain: Cost  $\leq 8 \cdot \max \left\{ \frac{1}{1-2^{-\epsilon}} \cdot c \right\} \cdot \left( \frac{n}{m} + 1 \right) \cdot M(m) \quad \square$

$\begin{bmatrix} U \\ 0 \end{bmatrix}$  is in row echelon form. It's convenient to write  $U = \begin{bmatrix} E & U' \end{bmatrix} \begin{matrix} r \\ m-r \end{matrix}$   $U' \in K^{r \times (n-r)}$

Also write  $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{matrix} m \\ m-r \end{matrix}$

### 2.6.7 Theorem 7

Let  $A \in K^{m \times n}$ ,  $LQAP = \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \begin{matrix} r \\ m-r \end{matrix}$  as in Algorithm 5 then

(a)  $rk(A) = r$

(b) The columns of  $P \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}$  form a basis of  $\ker(A)$

(c) A linear system  $Ax = b$   $b \in K^m$  is solvable iff  $L_2Q \cdot b = 0$

(d) if  $Ax = b$  is solvable then  $x = P \cdot \begin{bmatrix} E^{-1}L_1 \\ 0 \end{bmatrix} \begin{matrix} r \\ n-r \end{matrix} \cdot Q \cdot b$  is a solution

(e) if  $A \in GL_n(K)$  then  $\det(A) = \det(P) \cdot \underbrace{\det(E)}_{=\text{prod of diags}}$

and  $A^{-1} = P \cdot E^{-1} \cdot L$

**Proof:**

(a), (e) :  $\checkmark$

(b) :  $LQAP \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix} = \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix} = 0$

$\Rightarrow$  the columns lie in  $\ker(A)$

The columns of  $\begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}$  are linear independent.

$\Rightarrow rk(P \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}) = n - r$

$\Rightarrow$  the columns form a basis.

The space they generate has dimension  $n - r = \dim(\ker(A))$

(c), (d) : If  $A \cdot x = b$  then  $\begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot P^{-1} \cdot x = LQb = \begin{bmatrix} L_1Qb \\ L_2Qb \end{bmatrix}$

$\Rightarrow L_2Qb = 0$

if  $L_2Qb = 0$  then  $A \cdot P \cdot \begin{bmatrix} E^{-1} \cdot L_1 \\ 0 \end{bmatrix} \cdot Q \cdot b = Q^{-1} \cdot L^{-1} \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} E^{-1} \cdot L_1 \\ 0 \end{bmatrix} \cdot Qb =$

$Q^{-1} \cdot L^{-1} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} Q \cdot b = Q^{-1} \cdot L^{-1} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \cdot Q \cdot b = b$



### 2.6.8 Corollary 8

For  $A \in K^{m \times n}$  the determination of  $rk(A)$  and solving linear systems with coefficient matrix  $A$  require  $O((\frac{n}{m} + 1) \cdot M(m))$  field operations.

If  $A \in K^{n \times n}$  then computing  $\det(A)$  and  $A^{-1}$  (if  $A \in GL_n(K)$ ) require  $O(M(n))$  field operations.

From  $LQAP = \begin{bmatrix} U \\ 0 \end{bmatrix}$  get  $A = Q^{-1} \cdot \underbrace{L^{-1}}_{\text{lower triangular}} \begin{bmatrix} U \\ 0 \end{bmatrix} \cdot P^{-1}$

Generally  $Q = I_m \Rightarrow A = L^{-1} \cdot \begin{bmatrix} U \\ 0 \end{bmatrix} P^{-1}$  "LUP decomposition"

If also  $P = I_n$  obtain  $A = L^{-1} \cdot \begin{bmatrix} U \\ 0 \end{bmatrix}$  "LU decomposition"

## 2.7 Algebraic Systems of Equations, Gröbner bases

Given:  $f_1 \dots f_m \in K[x_1 \dots x_n]$  multivariate polynomials.

Wanted: solution set of the algebraic system  $f_1 = f_2 = \dots = f_m = 0$

The solution set  $\mathcal{V}(f_1 \dots f_m) \subseteq K^n$  is called an affine variety.

Often assume  $K = \bar{K}$   $K$  algebraically closed (e.g.  $K = \mathbb{C}$ )

Questions:

1.  $\mathcal{V}(f_1 \dots f_m) \neq \emptyset$  ?
2.  $|\mathcal{V}(f_1 \dots f_m)| < \infty$  ?
3.  $\dim \mathcal{V}(f_1 \dots f_m) = ?$

**Examples:**

$$\begin{aligned} (1) \quad & f_1 = x_1 x_3 x_4^2 - 2x_2 x_4^2 + x_1 x_3 - 2x_2 \\ & f_2 = x_1 x_3 x_4 - 2x_2 x_4 - 1 \\ & f_3 = x_1 x_4^2 + x_1 + 2 \\ & \text{One has } (-x_1 x_4) \cdot f_1 + (x_1 x_4^2 + x_1 9f_2 + f_3) = 2 \\ & \Rightarrow \mathcal{V}(f_1 \dots f_3) = \emptyset \end{aligned}$$

$$\begin{aligned} (2) \quad & f_1 = x^3 + x^2 y + xy + y^2 \\ & f_2 = x^2 y^2 + x^2 + y^3 + y \\ & f_3 = x^3 + xy \\ & (x^2 + y) \mid f_i \quad \forall i = 1, 2, 3 \Rightarrow |\mathcal{V}(f_1, f_2, f_3)| = \infty \end{aligned}$$

**Univariate case (n=1):**

$K[x]$  Euclidean so have Euclidean algorithm for computing  $\gcd(f, g)$ .  $\gcd$  is unique if required to be monic (i.e. the highest coefficient is 1)

Also get  $h_1, h_2 \in K[x]$  such that  $\gcd(f, g) = h_1 f + h_2 g$

Let  $f_1 \dots f_m \in K[x]$  Obtain

$$g := \gcd(f_1 \dots f_m) = \sum_{i=1}^n h_i \cdot f_i \quad \text{with } h_i \in K[x] \quad (*)$$

For  $\xi \in K$  :

$$f_1(\xi) = f_2(\xi) = \dots = f_m(\xi) = 0 \Rightarrow \underset{*}{g(\xi)} = 0 \Rightarrow \underset{g|f_i}{f_i(\xi)} = 0 \quad \forall i$$

$\Rightarrow$  Only need to get zeros of one polynomial!

### Resultant method:

Reminder: For  $f, g \in K[x] : \gcd(f, g) \neq 1 \Leftrightarrow \text{res}(f, g) = 0$

Let  $f_1, f_2 \in K[x_1 \dots x_n]$

$$(\xi_1 \dots \xi_{n-1}) \in K^{n-1} \quad \text{assume } K = \bar{K}$$

Then  $\exists \xi_n \in K$  such that  $f_1(\xi_1 \dots \xi_n) = 0 = f_2(\xi_1 \dots \xi_n)$

$$\Leftrightarrow \text{res}_{x_n}(f_1(\xi_1 \dots \xi_{n-1}, x_n), f_2(\xi_1 \dots \xi_{n-1}, x_n)) = 0$$

Suppose  $\deg_{x_n} f_1(\xi_1 \dots \xi_{n-1}, x_n) = \deg_{x_n}(f_i) \quad (i = 1, 2)$

Set  $h = \text{res}_{x_n}(f_1, f_2) \in K[x_1 \dots x_{n-1}]$

Then  $\text{res}_{x_n} f_1(\xi_1 \dots \xi_{n-1}, x_n), f_2(\xi_1 \dots \xi_{n-1}, x_n) = h(\xi_1 \dots \xi_{n-1})$

Search zeros of  $h \rightarrow$  one variable, one equation fewer.

Limitation: Only for pairs of polynomials ( $m = 2$ ).

Good case:  $m = n = 2$

Given  $f_1 \dots f_m \in K[x_1 \dots x_n]$  form the ideal

$$I = (f_1 \dots f_m) = \left\{ \sum_{i=1}^n g_i f_i \mid g_i \in K[x_1 \dots x_n] \right\}$$

Clearly  $\mathcal{V}(I) = \mathcal{V}(f_1 \dots f_m)$

$f_1 \dots f_m$  are called a ideal basis of  $I$ . They are not unique, not even their size is unique.

### Example:

In Example (1)  $I$  has an alternative basis  $I = (1) \leftarrow$  constant polynomial  $1 \in K[x_1 \dots x_4]$

In Example (2) it turns out that  $I = (x^2 + y)$

### 2.7.1 Theorem 1 (Hilbert's Nullstellensatz)

#### Hilbert's Nullstellensatz (1st version):

Assume  $K = \bar{K}$  let  $I \subseteq K[x_1 \dots x_n]$  be an ideal

Then  $\mathcal{V}(I) = \emptyset \Leftrightarrow 1 \in I$

( $\Leftrightarrow I = K[x_1 \dots x_n] \Leftrightarrow I = (1)$ )

without proof.

For  $I \subseteq R$  ideal in a commutative ring  $R$  the radical ideal of  $I$  is

$$\sqrt{I} = \{a \in R \mid \exists n \in \mathbb{N} : a^n \in I\}$$

$I$  is called a radical ideal if  $I = \sqrt{I}$

Let  $S \subseteq K^n$  set of points

$$\Rightarrow Id(S) := \{f \in K[\underline{x}] \mid f(v) = 0 \forall v \in S\} \subseteq K[\underline{x}] \quad \text{where } K[\underline{x}] := K[x_1, \dots, x_n]$$

is a radical ideal (called vanishing ideal)

**Hilbert's Nullstellensatz (2nd version):**

Let  $K = \bar{K}$   $I \subseteq K[x]$  ideal. Then  $\sqrt{I} = Id(\mathcal{V}(I))$

Obtain bijection:  $\{\text{radical ideals in } K[x]\} \Leftrightarrow \{\text{affine varieties}\}$

This bijection is inclusion-reversing.

**Monomial orderings:****2.7.2 Definition 2 (Monomial, monomial ordering, LM, LT, LC)**

A monomial is a polynomial of the form  $t = x_1^{e_1} \cdot x_2^{e_2} \cdot \dots \cdot x_n^{e_n} =: \underline{x}^{\underline{e}}$  where  $e_i \in \mathbb{N}$

A term is a polynomial of the form  $c \cdot t$   $t$  monomial,  $c \in K \setminus \{0\}$

$M :=$  set of all monomials.

For  $f \in K[x]$ ;  $M(f) :=$  set of all monomials occurring in  $f$ .

$T(f) :=$  set of all terms ...

A monomial ordering is an ordering (= order relation) " $\leq$ " on  $M$  such that:

1. " $\leq$ " is total i.e.  $\forall s, t \in M : s \leq t \vee t \leq s$
2.  $1 \leq t \quad \forall t \in M$
3.  $\forall s, t_1, t_2 \in M : t_1 \leq t_2 \Rightarrow s \cdot t_1 \leq s \cdot t_2$

(This implies:  $s \mid t \Rightarrow s \leq t$ )

For  $f \in K[x] \setminus \{0\}$  we write

$LM(f) =: t$  for the largest monomial in  $M(f)$  ("leading monomial"),

$LT(f) =: c \cdot t$  for the largest term if  $t$  in  $f$  ("leading term")

$LC(f) =: c$  ("leading coefficient")

$LM(0) = LT(0) = LC(0) = 0$

**Example 1:** Lexicographic ordering (lex)

for  $t = x_1^{e_1} \cdot \dots \cdot x_n^{e_n} \quad t' = x_1^{e'_1} \cdot \dots \cdot x_n^{e'_n}$

define  $t \leq t' \Leftarrow t = t' \vee e_i < e'_i$  for the smallest  $i$  with  $e_i \neq e'_i$

**Example 2:** graded reverse lexicographic ordering (grevlex)

$t \leq t' \Leftarrow t = t' \vee \deg(t) < \deg(t') \vee (\deg(t) = \deg(t') \wedge e_i > e'_i)$

for the largest  $i$  such that  $e_i \neq e'_i$

where  $\deg(t) := \sum e_i$

For both lex and grevlex have

$x_1 > x_2 > \dots > x_n$  but  $x_1 \cdot x_3 >_{\text{lex}} x_2^2$

$x_1 > x_2 > \dots > x_n$  but  $x_1 \cdot x_3 <_{\text{grevlex}} x_2^2$

### 2.7.3 Proposition 3 (Sum and product of LM / LT)

Let " $\leq$ " be a monomial ordering  $f, g \in K[x] \Rightarrow$

- (a)  $LT(f \cdot g) = LT(f) \cdot LT(g)$  same for  $LM$
- (b)  $LM(f + g) \leq \max\{LM(f), LM(g)\}$

**Proof:**

(b) ✓

- (a) write  $c \cdot t = LT(f)$   $d \cdot s = LT(g)$   
For  $t' \in M(f)$   $s' \in M(g)$  have  $\underbrace{t's' \leq t \cdot s' \leq t \cdot s}_{=?}$  with equality iff  $s' = s$   $t' = t$

This implies (a)  $\square$

### 2.7.4 Lemma 4 (Dickson-Lemma)

Every subset  $S \subseteq M$  has a finite subset  $B \subseteq S$  ("basis") such that  $\forall s \in S \exists t \in B : t \mid s$

**Proof:** Identify  $M$  with  $\mathbb{N}^n$

Given  $S \subseteq \mathbb{N}^n$  need to show that:

$\exists B \subseteq S, B$  finite such that  $\forall (e_1, \dots, e_n) \in S$

$\exists (d_1, \dots, d_n) \in B$  such that  $\forall i : d_i \leq e_i$

We will write  $(\underline{d}) \leq (\underline{e})$  for this. (This defines a partial ordering in  $\mathbb{N}^n$ )

Induction:

$n = 1$  : if  $\emptyset \neq S \subseteq \mathbb{N}$  then  $\exists d \in S$  such that  $d \leq e \quad \forall e \in S$  ( $\mathbb{N}$  is well-ordered)

$n > 1$  : For  $k \in \mathbb{N}$  write  $S_k := \{(e_2, \dots, e_n) \in \mathbb{N}^{n-1} \mid (k, e_2, \dots, e_n) \in S\} \subseteq \mathbb{N}^{n-1}$

By induction  $\exists B_k \subseteq S_k$  finite such that  $\forall (\underline{e}) \in S_k \exists (\underline{d}) \in B_k$  such that  $(\underline{d}) \leq (\underline{e})$

$\bigcup_{k \in \mathbb{N}} B_k \subseteq \mathbb{N}^{n-1}$  has finite "basis"  $C$

$C$  finite  $\exists r \in \mathbb{N} : C \subseteq \bigcup_{k=0}^r B_k$  (\*)

From  $B := \{(e_1, \dots, e_n) \in \mathbb{N}^n \mid e_1 \leq r, (e_2, \dots, e_n) \in B_{e_1}\} \Rightarrow |B| < \infty, B \subseteq S$

**Claim:**  $B$  basis of  $S$

Let  $(e_1, \dots, e_n) \in S \Rightarrow (e_2, \dots, e_n) \in S_{e_1} \Rightarrow \exists (d_2, \dots, d_n) \in B_{e_1}$  such that  $d_i \leq e_i \forall i \geq 2$

**Case 1:**  $e_1 \leq r$

$\Rightarrow (e_1, d_2, \dots, d_n) \in B$  have  $(e_1, d_2, \dots, d_n) \leq (e_1, \dots, e_n)$

**Case 2:**  $e_1 > r$

$B_{e_1} \subseteq \bigcup_{k \in \mathbb{N}} B_k \Rightarrow \exists (c_2, \dots, c_n) \in C$  such that  $c_i \leq d_i \forall i \geq 2$

By (\*)  $\exists k \leq r : (\underline{c}) \in B_k \Rightarrow (k, c_2, \dots, c_n) \in B$

$(k, c_2, \dots, c_n) \leq (e_1, d_2, \dots, d_n) \leq (e_1, e_2, \dots, e_n)$  ✓  $\square$

### 2.7.5 Corollary 5 (Well-ordering of monomial sets)

Every monomial ordering is a well-ordering i.e. every monomial set  $S \subseteq M$  has an element  $t \in S$  such that  $\forall s \in S : t \leq s$  ( $t$  is a "least element")

**Proof:**

Let  $\emptyset \neq S \subseteq M$  By Lemma 4  $\exists B \subseteq S$  finite such that  $\forall s \in S' \exists t \in B : t \mid s$

Since " $\leq$ " is total and  $B$  is finite  $\exists t \in B$  least element.

Let  $s \in S \Rightarrow \exists t' \in B$  such that  $t' \mid s \Rightarrow t' \leq s$  so  $t \leq t' \leq s$   $\square$

**Gröbner bases:** Let " $\leq$ " be a fixed monomial ordering

### 2.7.6 Definition 6 (Leading ideal, Gröbner bases)

- (a) For  $S \in K[x]$  subset define  $L(S) := (LM(f) \mid f \in S) \subseteq K[x]$   
(ideal generated by all leading monomials of elements of  $S$ ) is called the leading ideal
- (b) Let  $I \subseteq K[x]$  ideal. A finite subset  $G \subseteq I$  is called a Gröbner basis if  $L(I) = L(G)$   
i.e.  $\forall f \in I \exists g \in G : LM(g) \mid LM(f)$

### 2.7.7 Proposition 7 (Ideality of Gröbner bases)

$G$  Gröbner basis of  $I \Rightarrow I = (G)$  i.e.  $G$  is an ideal basis.

**Proof:**  $G \subseteq I \Rightarrow (G) \subseteq I$

Assume this inclusion is strict. Let  $f \in I \setminus (G)$  By Corollary 5 my assume  $LM(f)$  is minimal (among all leading monomials of elements from  $I \setminus (G)$ )

$LM(f) \in L(I) = L(G) \Rightarrow \exists g \in G : LM(g) \mid LM(f)$

Form  $\tilde{f} = f - \frac{LT(f)}{LT(g)}g, \tilde{f} \in I \Rightarrow LM(\tilde{f}) < LM(f)$

by minimality  $\tilde{f} \in (G) \Rightarrow f = \tilde{f} + \frac{LT(f)}{LT(g)}g \in (G)$  contradiction!  $\square$

$G \subseteq I \quad L(G) = L(I) \quad \Rightarrow I = (G)$

**Example:**

$I = (1) \in K[x] \quad S = \{x+1, x\}$  ideal basis but  $I(S) = (x) \neq L(I) = (1)$

$S$  is not a Gröbner basis.

### 2.7.8 Theorem 8 (Gröbner basis of Ideals)

Every ideal  $I \subseteq K[x]$  has a Gröbner basis. In particular  $I$  has a finite basis ( $\rightarrow$  Hilbert's basis theorem) In other words  $K[x]$  is Noetherian.

**Proof:**

For  $\{LM(f) \mid f \in I\}$  there exists (by Dickson lemma) a finite subset  $\{LM(f_1), \dots, LM(f_m)\}, f_i \in I$  such that  $(LM(f_1) \dots LM(f_m)) = L(I) \Rightarrow G = \{f_1 \dots f_m\}$

Gröbner basis  $\square$

First application: Let  $G$  Gröbner basis of  $I$

Then  $\mathcal{V}(I) = \emptyset \Leftrightarrow_{K=\bar{K}} 1 \in I \Leftrightarrow G$  contains a non-zero constant

### 2.7.9 Definition 9 (Normal form)

Let  $S = \{g_1 \dots g_m\} \subseteq K[x]$   $f \in K[x]$

- (a)  $f$  is a normal form with respect to  $S$  if  $\forall t \in M(f) \quad \forall i = 1 \dots m : LM(g_i) \nmid t$
- (b)  $f^* \in K[x]$  is called a normal form of  $f$  with respect to  $S$  if
  - (i)  $f^*$  is in normal form with respect to  $S$
  - (ii)  $\exists h_1 \dots h_m \in K[x]$  such that  $f - f^* = \sum_{i=1}^m h_i g_i$  and  $\forall i : LM(h_i g_i) \leq LM(f)$

#### Example:

$S = \{x, x+1\}$   $f = 1 \Rightarrow f \equiv 0 \pmod{(S)}$

but 0 is not a normal form of  $f$

If  $f = x$  then 0 and  $-1$  are normal forms of  $x$

### 2.7.10 Algorithm 10 (Normal form)

Input :  $S = \{g_1 \dots g_m\} \subseteq K[x]$   $f \in K[x]$

Output : A normal form  $f^*$  of  $f$  with respect to  $S$  and if desired  $h_1 \dots h_m$  satisfying (\*)

- (1) Set  $f^* := f$   
for  $(i = 1 \dots m)$   
     $h_i := 0$
- (2) repeat (3) - (6)
- (3)  $\mathcal{M} := \{(t, i) \mid t \in M(f^*), i \in \{1, \dots, m\} \text{ such that } LM(g_i) \mid t\}$
- (4) if  $(\mathcal{M} = \emptyset)$   
    return  $f^*$  and  $h$
- (5) Choose  $(t, i) \in \mathcal{M}$  such that  $t$  is maximal.  
let  $c \in K$  be the coefficient of  $t$  in  $f^*$
- (6) Set  $f^* := f^* - \frac{c \cdot t}{LT(g_i)} \cdot g_i$   
     $h_i := h_i + \frac{c \cdot t}{LT(g_i)}$

Step (6) cancels the term  $c \cdot t$  from  $f^*$  and may add only monomials smaller than  $t$ . So the  $t$ 's form a strictly descending sequence of monomials  $\Rightarrow$  Algorithm 10 terminates.

Correctness

✓

### 2.7.11 Theorem 11 (Normal form of Gröbner bases)

Let  $G \subseteq K[x]$  be a Gröbner basis of an ideal  $I \subseteq K[x]$

- (a) Every polynomial  $f \in K[x]$  has a unique normal form with respect to  $G$ .  
Write  $NF_G(f)$
- (b)  $NF_G : K[x] \mapsto K[x]$  is  $K$ -linear,  
 $\ker(NF_G) = I$
- (c) if  $\tilde{G} \subseteq K[x]$  is another Gröbner basis (with respect to same monomial ordering)  
then  $NF_G = NF_{\tilde{G}}$

**Proof:**

(a), (c):

Let  $f \in K[x]$   $f^*, \tilde{f} \in K[x]$  be normal forms of  $f$  with respect to  $G$  and  $\tilde{G}$  respectively.

**Claim:**  $f^* = \tilde{f}$

$f^* - f \in I$ ,  $\tilde{f} - f \in I \Rightarrow f^* - \tilde{f} \in I \Rightarrow LM(f^* - \tilde{f}) \in L(G) \in L(\tilde{G})$

if  $f^* \neq \tilde{f} \Rightarrow LM(f^* - \tilde{f}) \in M(f^*)$  or  $\in M(\tilde{f})$

But  $\exists g \in G : LM(g) \mid LM(f^* - \tilde{f})$ ,  $\exists \tilde{g} \in \tilde{G} : LM(\tilde{g}) \mid LM(f^* - \tilde{f})$

This is a contradiction to:

$f^*$  is in normal form with respect to  $G$  and

$\tilde{f}$  is in normal form with respect to  $\tilde{G}$

So  $f^* = \tilde{f}$

(b):

Let  $f, g \in K[x]$   $c \in K$ . Set  $h := NF_G(f + cg) - NF_G(f) - c \cdot NF_G(g)$

To show:  $h = 0$   $h \equiv f + cg - f - cg = 0 \pmod{I}$

$\Rightarrow h \in I \Rightarrow LM(h) \in L(G)$

$h$  is in normal form with respect to  $G$

$\Rightarrow h = 0$

Remains to show:  $\ker(NF_G) = I$

let  $NF_G(f) = 0 \Rightarrow f \equiv 0 \pmod{I} \Rightarrow f \in I$  conversely, let  $f \in I$

$\Rightarrow f^* = NF_G(f) \in I \Rightarrow \exists g \in G : LM(g) \mid LM(f^*)$   $f^*$  in normal form. So  $f^* = 0$   $\square$

Further applications of Gröbner bases:

- Membership test:  $f \in I \Leftrightarrow NF_G(f) = 0$
- Computation in  $A := K[x]/I$  :  $NF_G$  includes an embedding  $A \hookrightarrow K[x]$

## Buchberger's Algorithm

### 2.7.12 Definition 12 ( $S$ -polynomials)

Let  $f, g \in K[\underline{x}] \setminus \{0\}$   $t := \gcd(LM(f), LM(g))$   
 Then  $s_{pol}(f, g) := \frac{LT(g)}{t} \cdot f - \frac{LT(f)}{t} \cdot g$  is the  $S$ -polynomial.  
 The leading monomials of the summands cancel!

**Example:**

$$\begin{aligned} f &= x^2 + y^2, \quad g = x \cdot y \quad " \leq " = lex \\ \Rightarrow LM(f) &= x^2 \quad LM(g) = xy \\ s_{pol}(f, g) &= y \cdot f - x \cdot g = y^3 \end{aligned}$$

### 2.7.13 Theorem 13 (Buchberger's criterion)

For any finite set  $G \subseteq K[\underline{x}]$  the following statements are equivalent:

- (a)  $G$  is a Gröbner basis of  $\langle G \rangle$
- (b) For polynomials  $g, h \in G$ ,  $0$  is a normal form of  $s_{pol}(g, h)$  with respect to  $G$   
 $\rightarrow$  finite test for Gröbner basis!

**Proof:**

"(a)  $\Rightarrow$  (b)":

For  $g, h \in G$  :  $s_{pol}(g, h) \in \langle G \rangle \subseteq \langle G \rangle =: I \xRightarrow{\text{Theorem 13 (b)}} s_{pol}(g, h)$  has normal form 0

$$\Leftrightarrow NF_G(s_{pol}(g, h)) = 0$$

"(a)  $\Leftarrow$  (b)":

Assume  $G$  is not a Gröbner basis  $\Rightarrow \exists f \in I \setminus G$  such that  $LM(f) \notin L(G)$ .

$$\text{Write } G = \{g_1 \dots g_m\}. \text{ Since } f \in \langle G \rangle \exists h_1 \dots h_m \in K[\underline{x}] \text{ have } f = \sum_{i=1}^m h_i \cdot g_i \quad (1)$$

By Corollary 5 may choose  $h_i$  such that

$t := \max\{LM(h_i g_i) \mid i = 1 \dots m\}$  becomes minimal.

$\exists i : LM(f) \in M(h_i g_i)$ . Since  $LM(f) \notin L(G) \wedge LM(f) \neq LM(h_i g_i)$

$$\Rightarrow LM(f) < LM(h_i g_i) \leq t$$

$\Rightarrow$  the coefficient of  $t$  in  $\sum h_i g_i$  is zero.  
 (1)

$$\text{Set } c_1 := \begin{cases} LC(h_i) & \text{if } LM(h_i g_i) = t \\ 0 & \text{otherwise} \end{cases} \text{ Then } \sum_{i=1}^m c_i \cdot LC(g_i) = 0 \quad (2)$$

Without loss assume  $c_1 \neq 0$

Let  $i \in \{2, \dots, m\}$  such that  $c_i \neq 0 \Rightarrow LM(g_i) \mid t$

So  $t_i = \text{lcm}(LM(g_i), LM(g_1) \mid t)$

$$\text{Have } s_{pol}(g_i, g_1) = \frac{LC(g_1) \cdot t_i}{LM(g_i)} g_i - \frac{LC(g_i) \cdot t_i}{LM(g_1)} g_1 \quad LM(s_{pol}(g_i, g_1)) < 0$$

$$\Rightarrow s_i := \frac{t}{t_i} \cdot s_{pol}(g_i, g_1) = LC(g_1) \cdot LM(h_i) \cdot g_i - LC(g_i) \cdot LM(h_1) \cdot g_1 \quad (3)$$

$$\text{By (b) have } s_i = \sum_{j=1}^m h_{ij} \cdot g_j \quad \text{with } h_{ij} \in K[\underline{x}] \text{ such that } LM(h_{ij} g_j) \leq LM(s_i) < t \quad (4)$$



$$\begin{aligned}
& \sum_{j=1}^m \left( \sum_{i=2}^m c_i \cdot h_{ij} \right) \cdot g_j = \sum_{i=2}^m c_i s_i \\
& = \sum_{i=2}^m c_i (LC(g_1)LM(h_i)g_i - LC(g_i)LM(h_1)g_1) + \sum_{i=1}^m c_i LM(h_1)g_1 LC(g_i) \\
& \stackrel{(3)}{=} \sum_{i=1}^m c_i LC(g_1)LM(h_i)g_i
\end{aligned}$$

$$\text{Set } \tilde{h}_j := \frac{1}{LC(g_1)} \cdot \sum_{i=2}^m c_i h_{ij} \Rightarrow g := \sum_{i=1}^m c_i LM(h_i)g_i = \sum_{i=1}^m \tilde{h}_i g_i$$

For each  $i$  have:  $LM(\tilde{h}_i g_i) \stackrel{(4)}{<} t$

$$f = (f - g) + g \stackrel{(1)}{=} \sum_{i=1}^m (h_i - c_i LM(h_i))g_i + \sum_{i=1}^m \tilde{h}_i g_i$$

For each  $i$  have:  $LM((h_i - c_i LM(h_i))g_i) < t$  so  $LM((h_i - c_i LM(h_i) + \tilde{h}_i)g_i) < t$   
contradiction to choice of  $h_i$   $\square$

#### 2.7.14 Algorithm 14 (Buchberger)

Input :  $S \subseteq K[x]$  finite "  $\leq$  " monomial ordering

Output: A Gröbner basis  $G$  of  $I = (S)$  with respect to "  $\leq$  "

- (1)  $G := S \setminus \{0\}$
- (2) for  $g, h \in G$  repeat (3),(4)
- (3)     Compute  $s := s_{pol}(g, h)$   
          and a normal form  $s^*$  of  $s$  with respect to  $G$
- (4)     if( $s^* \neq 0$ )  
          set  $G := G \cup \{s^*\}$   
          go to (2)
- (5) return  $G$

#### 2.7.15 Theorem 15 (Correctness of Algorithm 14)

Algorithm 14 terminates after finitely many steps and computes a Gröbner basis.

**Proof:**

**Termination:**

Let  $G_i$  be the set  $G$  obtained after the  $i$ -th run through the loop.  $G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots$

From  $\bar{G} = \bigcup_i G_i$  finite or infinite.

Lemma 4:  $\exists B \subseteq M$  finite set of monomials,  $B \subseteq \{LM(f) \mid f \in \bar{G}\}$  such that

$\forall f \in \bar{G} \quad \exists t \in B$  such that  $t \mid LM(f)$  (\*)

Since  $|B| < \infty \quad \exists r$  such that  $B \subseteq \{LM(f) \mid f \in G_r\}$

Without loss  $B = \{LM(f) \mid f \in G_r\}$

**Claim:**  $G_r$  is the last of the  $G$

If not  $\exists G_{r+1} : G_{r+1} = G_r \cup \{s^*\}$

$s^* \neq 0$  in normal form with respect to  $G_r$ . But by (\*)  $\exists f \in G_r$  such that  $LM(f) \mid LM(s^*)$  contradiction.

**Correctness:** by Theorem 13  $\square$

**Example:**

$S = \{\underbrace{x^2 + g^2}_f, \underbrace{xy}_g\} \subseteq \mathbb{Q}[x, y]$  "  $\leq$  " lex ordering with  $x > y$

$s_{pol}(f, g) = yf - xg = y^3 =: h$  in normal form with respect to  $S$

$G = \{f, g, h\}$

$s_{pol}(f, g) = h \xrightarrow{\text{normal form}} 0$

$s_{pol}(f, h) = y^3 f - x^2 h = y^5 \xrightarrow{\text{normal form}} 0$

$s_{pol}(g, h) = y^2 g - xh = 0$

$\Rightarrow G$  Gröbner basis

**Cost of Buchberger algorithm:**

- no known upper bound for the running time
- with  $d = \max\{\deg(f) \mid f \in S\} : \underbrace{\deg(g_i)}_{\text{polys from } G} \leq 2 \cdot \left(\frac{d^2}{2} + d\right)^{2^{n-1}}$   
with  $n = \text{number of Variables}$   
 $\Rightarrow$  "Doubly exponential" in  $n$   
Ritscher (2011): upper bound for  $\deg(g_i)$  depending  $\dim(\underbrace{\mathcal{V}}_{\text{Variety}}(S))$
- Nonetheless the algorithm often works
- Many possible optimizations

Variant: Extended Buchberger:

Keep track of how the new elements in  $G$  are represented as linear combination of elements of  $S$ .

### 2.7.16 Definition 16 (Reduced Gröbner basis)

A Gröbner basis  $G$  is called reduced if  $\forall g \in G$

- $g$  is in normal form with respect to  $G \setminus \{g\}$
- $LC(g) = 1$

A given Gröbner basis can be turned into a reduced only by replacing every  $g \in G$  by a normal form of  $g$  with respect to  $G \setminus \{g\}$ . Then remove  $0 \in G$ . Then divide each  $g \in G$  by  $LC(g)$

### 2.7.17 Theorem 17 (Uniqueness of reduced Gröbner basis)

From ideal  $I \subseteq K[x]$  and a monomial ordering " $\leq$ ", there exists a unique reduced Gröbner basis.

## 2.8 Application of Gröbner bases

### 2.8.1 Definition 1 (Elimination ideals)

- (a)  $I \subseteq K[X_1, \dots, X_n]$  ideal,  $l \in \{1, \dots, n\} \Rightarrow I_l := K[X_1, \dots, X_n]$  is called the  $l$ -length elimination ideal
- (b) A monomial ordering " $\leq$ " is called an  $l$ -elimination ordering if  $\forall 1 \leq i \leq l < j \leq n \quad \forall k_i \quad X_i^{k_i} < X_j$

**Example:**

- (1) Let " $\leq$ " be a given monomial ordering. Define " $\leq$ " by:  
for  $s = \underline{x}^d \quad t = \underline{x}^e$  define  
 $s \leq' t \Leftrightarrow \sum_{i=l+1}^n d_i < \sum_{i=l+1}^n e_i$  or have equality  $\sum_{i=l+1}^n d_i = \sum_{i=l+1}^n e_i$  and  $s \leq t$   
 $\Rightarrow$  " $\leq$ " is an  $l$ -eliminating ordering.
- (2) The lexicographic ordering with  $x_1 < x_2 < \dots < x_n$  is an  $l$ -eliminating ordering for all  $l$
- (3) Grevlex is not an  $l$ -eliminating ordering (unless  $l = n$ )

### 2.8.2 Theorem 2

Let  $G$  be a Gröbner basis of an ideal  $I \subseteq K[X_1, \dots, X_n]$  with respect to an  $l$ -elimination ordering. Then  $G_l := K[X_1, \dots, X_n] \cap G$  is a Gröbner basis.  $I : l$  with respect to the restricted monomial ordering.

**Proof:**

$$G_l \subseteq I_l \text{ Let } f \in I_l \setminus \{0\} \quad f \in I$$

$$\Rightarrow \exists g \in G : LM(g) \mid LM(f)$$

To show:  $g \in G$ . Clearly  $LM(g) \in K[X_1, \dots, X_l]$

If  $g \notin K[X_1, \dots, X_n]$  then  $\exists t \in M(g)$  such that

$$X_j \mid t \text{ with } j > l \Rightarrow t \geq X_j \underset{\text{"}\leq\text{" elim ord}}{>} LM(g) \text{ contradiction.} \quad \square$$

**Example:**

$$I = (\underbrace{X_1^2 + X_2^2}_f, \underbrace{x_1 x_2}_g), G = \{f, g, X^3\}$$

Gröbner basis with respect to lex ordering  $X_1 < X_2$

$$I_1 = (X_1^3)$$

**Geometric interpretation:**

Let  $K = \bar{K}$  algebraically closed. On  $K^n$  define the Zariski topology.

By saying that the sets  $\mathcal{V}(I)$  with  $I \subseteq K[X_1, \dots, X_n]$  are the closed sets.

Why is this a topology?

Reminder: A topological space is a set  $X$  together with a system of subsets, called closed subsets, such that three axioms hold.

- (1)  $\emptyset = \mathcal{V}(K[X_1, \dots, X_n]) \quad K^n \in \mathcal{V}(\{0\})$  closed
- (2) Let  $\mathcal{M}$  be a set of closed subsets corresponding to a set  $\mathcal{N}$  of ideals.

$$\text{Then } \bigcap_{I \in \mathcal{N}} \mathcal{V}(I) = \mathcal{V}\left(\sum_{I \in \mathcal{N}} I\right) \quad \checkmark$$

- (3) The union of two closed subsets is closed.

Let  $I, J \subseteq K[\underline{X}]$  ideals.

**Claim:**  $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J)$

**Proof:** " $\subseteq$ " direction:

Let  $v \in \mathcal{V}(I) \cup \mathcal{V}(J) \quad f \in I \cap J$

If  $v \in \mathcal{V}(I)$  then  $f(v) = 0$

$v \in \mathcal{V}(J)$  then  $f(v) = 0$

" $\supseteq$ " direction:

Let  $v \in \mathcal{V}(I \cap J)$  Assume  $v \notin \mathcal{V}(I) \Rightarrow \exists f \in I : f(v) \neq 0$

Let  $g \in J \Rightarrow f \cdot g \in I \cap J \Rightarrow \underbrace{f(v)}_{\neq 0} \cdot g(v) = 0$

$\Rightarrow g(v) = 0 \quad \text{So } v \in \mathcal{V}(J)$

All points in  $K^n$  are closed so are all finite subsets.

**Closures:** For  $X \subseteq K^n$  the closure  $\bar{X}$  is defined as the smallest closed subset containing  $X$ .  $\bar{X}$  is the variety of the largest ideal  $I$  such that  $X \subseteq \mathcal{V}(I)$  This ideal is  $I = Id(X)$ . So  $\bar{X} = \mathcal{V}(Id(X))$

Let  $\Pi_l : K^n \mapsto K^l, (a_1, \dots, a_n) \mapsto (a_1, \dots, a_l)$  projection.

### 2.8.3 Theorem 3

$$I \subseteq K[X_1, \dots, X_n] \Rightarrow \mathcal{V}(I) = \overline{\Pi_l(\mathcal{V}(I))}$$

**Proof:**

Let  $(a_1, \dots, a_l) \in \Pi_l(\mathcal{V}(I)) \Rightarrow \exists n_{l+1} \dots n_1 \in K$  such that  $(a_1, \dots, a_n) \in \mathcal{V}(I)$

Let  $f \in I_l \Rightarrow f \in I \Rightarrow f(a_1, \dots, a_n) = 0$

But  $0 = (a_1, \dots, a_n) = f((a_1, \dots, a_l))$  So  $(a_1, \dots, a_n) \in \mathcal{V}(I_l)$

So  $\Pi_l(\mathcal{V}(I)) \subseteq \mathcal{V}(I_l) \xRightarrow{\mathcal{V}(I_l) \text{ closed}} \overline{\Pi_l(\mathcal{V}(I))} \subseteq \mathcal{V}(I_l)$ .

To show:  $\mathcal{V}(I_l) \subseteq \overline{\Pi_l(\mathcal{V}(I))} \xLeftrightarrow{\text{H. Nullstellensatz}} Id(\Pi_l(\mathcal{V}(I))) \subseteq \sqrt{I_l}$  Take  $f \in Id(\Pi_l(\mathcal{V}(I))) \Rightarrow$

$f \in K[X_1, \dots, X_l] \forall (a_1, \dots, a_n) \in \mathcal{V}(I)$ .

$f(a_1, \dots, a_l) = f(a_1, \dots, a_n) = 0 \Rightarrow f \in Id(\mathcal{V}(I)) = \sqrt{I} \Rightarrow \exists k : f^k \in I \cap K[X_1, \dots, X_l] = I_l$

$\Rightarrow f \in \sqrt{I_l} \quad \square$

**Example:**

(1)  $I = (xy - 1)$

$\Pi_1(\mathcal{V}(I)) = K \setminus \{0\}$  not closed.  $\overline{\Pi_1(\mathcal{V}(I))} = K.I_1 = \{0\}$

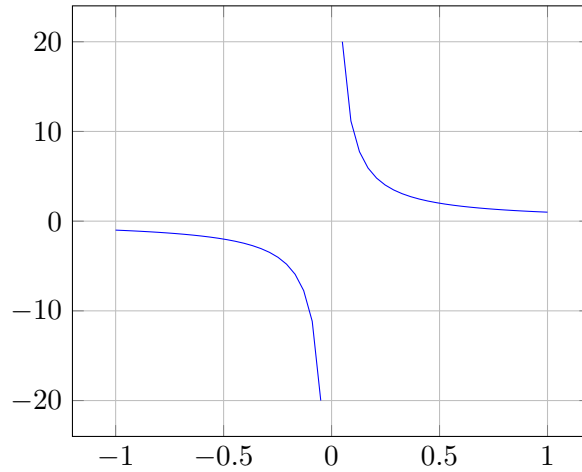


Figure 3: Plot of  $I$  in Example (1)

(2)  $I = (xy - 2x + y - 2, kx^2 - 4xy + y^2)$  has reduced lex Gröbner basis:

$$\{x^2 - 2x + \frac{y^2}{4} + y - 2, xy - 2x + y - 2, (y - 2)(y + 2)^2\}$$

$$\Rightarrow I_1 = ((y - 2)(y + 2)^2)$$

$$\Rightarrow \overline{\Pi_1(\mathcal{V}(I))} = \{2, -2\}$$

$$\Rightarrow \Pi_1(\mathcal{V}(I)) = \{2, -2\}$$

Substitute  $y = 2$ :

$$x^2 - 2x + 1, 2x - 2x + 2 - 2 = 0$$

$$y = -2 :$$

$$x^2 - 2x - 3 = 0, -4x - 4 = 0$$

$$\rightarrow \mathcal{V}(I) = \{(1, 2), (-1, -2)\}$$

### 2.8.4 Algorithm 4 (Solving systems of algebraic equations)

Input :  $f_1, \dots, f_m \in K[X_1, \dots, X_n]$

Output:  $\mathcal{V}(f_1, \dots, f_m)$  if finite otherwise " $\infty$ "

- (1) Compute a Gröbner basis of  $I = (f_1, \dots, f_m)$  with respect to the lex ordering with  $X_1 < X_2 < \dots < X_n$
- (2) for  $(l = 1, \dots, n)$   
     set  $G_l := [X_1, \dots, X_l] \cap G$
- (3)  $M := \{()\} \subseteq K^0$
- (4) for  $(l = 1, \dots, n)$  repeat (5) - (10)
- (5)  $S = \emptyset$
- (6) for  $(a_1, \dots, a_{l-1}) \in M$  repeat (7) - (9)
- (7)  $g := \gcd\{f(a_1, \dots, a_{l-1} \mid f \in G_l)\}$
- (8) if  $(g = 0)$   
     return " $\infty$ "
- (9)  $S := S \cup \{(a_1, \dots, a_{l-1}, a_l) \mid g(a_l) = 0\}$
- (10)  $M := S$
- (11) return  $M$

### Intersections of ideals

### 2.8.5 Proposition 5

Let  $I, J \subseteq K[\underline{x}]$  ideals,  $y$  additional variable.

Form  $L \subseteq K[x_1, \dots, x_n, y]$  generated by  $I \cdot y$  and  $J \cdot (1 - y)$  Then  $I \cap J = K[\underline{x}] \cap L$  (elimination ideal!)

**Proof:**

Let  $f \in J \cap I \Rightarrow f = f \cdot y + f \cdot (1 - y) \in L \Rightarrow f \in K[\underline{x}] \cap L$

Conversely let  $f \in K[\underline{x}] \cap L \Rightarrow f = \sum_{i=1}^r g_i f_i \cdot y_i + \sum_{i=r}^m g_i \cdot f_i (1 - y)$

with  $f_1, \dots, f_r \in I$   $f_{r+1}, \dots, f_m \in J$   $g_i \in K[\underline{x}, y]$

Specialize  $y = 0 \Rightarrow f = \sum_{i=r}^m \underbrace{g_i(y=0)}_{\in K[\underline{x}]} f_i \in J$

Specialize  $y = 1 \Rightarrow f = \sum_{i=1}^r g_i(y=1) f_i \in I$

$\Rightarrow f \in I \cap J$   $\square$

## Dimension

### 2.8.6 Definition 6 (independence modulo $I$ )

Let  $I \subseteq K[\underline{x}]$  ideal. Then polynomials  $f_1, \dots, f_r \in K[\underline{x}]$  are called independent modulo  $I$  if for every polynomial  $F \in K[y_1, \dots, y_r]$  have:  $F(f_1, \dots, f_r) \in I \Rightarrow F = 0$

(So the classes  $\bar{f}_i \in A := K[\underline{x}]/I$  are algebraically independent)

For  $1 \leq i_1 < i_2 < \dots < i_r \leq n : x_{i_1}, \dots, x_{i_r}$  are independent modulo  $I \Leftarrow K[x_{i_1}, \dots, x_{i_r}] \cap I = \{0\}$

(elimination ideal!)

$\dim(I) := \sup\{r \in \mathbb{N} \mid \exists f_1, \dots, f_r \in K[\underline{x}] \text{ independent mod } I\}$  dimension of  $I$

In other words  $\dim(I) =: \text{trdeg}(A)$  "transcendence degree"

If  $I = K[\underline{x}] \Rightarrow \dim(I) = -1$

For  $X = \mathcal{V}(I) : \dim(X) = \dim(I)$

Well defined? Clearly  $\dim(I) = \dim(\sqrt{I})$

### Geometric interpretation:

$x_{i_1}, \dots, x_{i_r}$  are independent mod  $I \Leftrightarrow \Pi_{i_1, \dots, i_r} : K^n \mapsto K^r$  maps  $\mathcal{V}(I)$  to a dense subset of  $K^r$

More generally for  $f_1, \dots, f_r$  have:

$f_1, \dots, f_r$  are independent modulo  $I \Leftrightarrow$  The image of  $\mathcal{V}(I)$  under the "morphism"  $K^n \mapsto K^r, v \mapsto (f_1(v), \dots, f_r(v))$  is dense.

### 2.8.7 Theorem 7

Let  $I \subseteq K[\underline{x}]$  ideal

$\Rightarrow \dim(I) = \max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ are independent modulo } I\}$

### 2.8.8 Lemma 8

A non-empty set  $M$  of ideals in  $K[\underline{x}]$  has an element that is maximal with respect to inclusion.

### Proof:

If not obtain a sequence  $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$  of ideals in  $M$

Set  $I := \bigcup_{j \in \mathbb{N}} I_j$  Check:  $I \subseteq K[\underline{x}]$  ideal

$\Rightarrow I = (f_1, \dots, f_r)$  Each  $f_i$  lies in some  $I_j$

$\xRightarrow{\text{Thm 7.8}} \Rightarrow \exists m : f_i \in I_m \forall i \Rightarrow I \subseteq I_m \subseteq I \Rightarrow I = I_m$

$I_m \subseteq I_{m+1} \subseteq I = I_m \Rightarrow I_{m+1} = I_m$

Recall that an ideal  $I \subseteq K[\underline{x}]$  is called a prime ideal if  $a \cdot b \in I$  for  $a, b \in K[\underline{x}]$  implies  $a \in I$  or  $b \in I$

Fact:  $I$  prime ideal  $\Leftrightarrow I = \sqrt{I}$  and  $\mathcal{V}(I)$  is irreducible, i.e.  $I$  can't be the written as union of two proper closed subsets.

### 2.8.9 Lemma 9

Every radical ideal  $I \subsetneq K[x]$  is a finite intersection of prime ideals.

**Proof:**

If no by Lemma 8 there is a maximal exception  $I \subsetneq K[x]$ .  $I$  not prime ideal  $\Rightarrow \exists a_1, a_2 \in K[x] : a_1 a_2 \in I, a_j \notin I \Rightarrow I \subsetneq I + (a_j) \subseteq \sqrt{I + (a_j)} =: I_j$  (\*)  
 $\Rightarrow I \subseteq I_1 \cap I_2$

**Claim:**  $I = I_1 \cap I_2$  ✓

Let  $g \in I_1 \cap I_2 \Rightarrow \exists r : g^r \in I + (a_j) (j = 1, 2)$   
 $\Rightarrow g^{2r} \in (I + (a_1)) \cdot (I + (a_2)) \subseteq I \xrightarrow{I=\sqrt{I}} g \in I$

(\*) with maximality of  $I$  shows that the  $I_j$  are finite intersections of prime ideals

$\Rightarrow$  same holds for  $I$  contradiction  $\square$

Reminder: If  $L | K$  is a field extension  $L = K[\alpha_1, \dots, \alpha_n]$ .  $\text{trdeg}_K(L)$  the size of every maximally algebraically independent subset of  $\{\alpha_1, \dots, \alpha_n\}$

(Exchange argument similar to linear algebra)

### Proof of Theorem 7

Clearly  $\max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ independent modulo } I\} \leq \dim(I)$

For reverse inequality take  $f_1, \dots, f_r \in K[x]$  independent mod  $I$ . By Lemma 9:  $\sqrt{I} = P_1 \cap \dots \cap P_s$  with  $P_i$  prime ideals.

Assume that the  $f_i$  are dependent modulo every  $P_i \Rightarrow \exists F_i \in K[y_1, \dots, y_r] \setminus \{0\}$  such that

$F_i(f_1, \dots, f_r) \in P_i \Rightarrow$  For  $F := \prod_{i=1}^s F_i$

$F(f_1, \dots, f_r) \in P_1 \cap \dots \cap P_s = \sqrt{I} \Rightarrow \exists k$

$F^k(f_1, \dots, f_r) \in I$ . But  $F^k \neq 0$  contradiction

So  $\exists i : f_1, \dots, f_r$  independent mod  $P_i$

$\bar{f}_1, \dots, \bar{f}_r \in A := K[x]/P$  algebraically independent.

$A$  integral domain  $\Rightarrow L = \text{Quot}(A)$  exists  $\bar{f}_i \in A \subseteq L$ .  $L/K$  is field extension,  $L = K(\bar{x}_1, \dots, \bar{x}_n)$  Since  $\text{trdeg}_K(L) \geq r$  have  $i_1, \dots, i_r$  such that  $\bar{x}_{i_1}, \dots, \bar{x}_{i_r} \in L$  are algebraically independent over  $K$ .

$\Rightarrow x_{i_1}, \dots, x_{i_r}$  are independent modulo  $P_i$

Since  $I \subseteq P_i$ ,  $x_{i_1}, \dots, x_{i_r}$  are independent modulo  $I$

$\Rightarrow r \leq \max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ independent modulo } I\}$

Now we can compute  $\dim(I)$  by using elimination ideals. Have to determine a subset of  $\{x_1, \dots, x_n\}$  of maximal size that is independent modulo  $I \rightarrow$  Gröbner bases.

**Example:**

$I = \underbrace{(xy, xz)}_{\text{Gröbner bases}}$  Independent subsets modulo  $I : \emptyset, \{x\}, \{y\}, \{z\}, \{y, z\}$

$\dim(I) = 2$



**Hilbert series** For  $X \subseteq K^n$  affine variety,  $I = Id(X)$ . Then  $A := K[x]/I$  is the set of polynomial function ("regular functions")  $X \mapsto K$

How large is  $A$ ?  $A$  is a vector-space but it is not finite-dimensional (as vector-space) unless  $\dim(X) \leq 0$

Idea: measure the size of  $A$  by studying the growth of the dimensions of parts of  $A$  given by a filtration  $A = \bigcup_{d \in \mathbb{N}} A_d$

### 2.8.10 Definition 10 (Hilbert series)

Let  $I \subseteq K[x]$  ideal,  $A := K[x]/I$

For  $d \in \mathbb{N}$  set  $A_d := \{f + I \mid f \in K[x], \underbrace{\deg(f)}_{=\max\{\deg(t) \mid t \in M(f)\}} \leq d\} \subseteq A$  subspace

Hilbert function:  $h_I(d) := \dim_K(A_d) < \infty$

Hilbert series of  $I$ :  $H_I(t) = \sum_{d=0}^{\infty} h_I(d) \cdot t^d \in \mathbb{Z}[[t]]$  (formal power series)

**Example:**

$$(1) \quad I = (x_1, \dots, x_n) \Rightarrow A = K \Rightarrow A_d = K \quad \forall d$$

$$h_I(d) = 1 \Rightarrow H_I(t) = \sum_{d=0}^{\infty} t^d = \frac{1}{1-t}$$

$$(2) \quad I = (x_1 - x_2^2) \subseteq K[x_1, x_2]$$

The classes of  $1, x_1, x_1^2, \dots, x_1^d, x_2, x_2x_1, \dots, x_2x_1^{d-1}$  form a basis of  $A_d$

$$\Rightarrow h_I(d) = 2d + 1$$

$$H_I(t) = \frac{1+t}{(1-t)^2}$$

$$(3) \quad I = \{0\} \Rightarrow A = \underbrace{K[x]}_{=x_1, \dots, x_n}$$

write  $H_n(t)$  for Hilbert series  $H_0(t) = \frac{1}{1-t}$

$$\text{For } n > 0 : K[x_1, \dots, x_n]_d = \bigoplus_{i+j=d} K[x_1, \dots, x_{n-1}] \cdot x_n^j$$

$$\Rightarrow H_n(t) = H_{n-1}(t) \left( \sum_{j=0}^{\infty} t^j \right) = H_{n-1}(t) \cdot \frac{1}{1-t} = \frac{1}{(1-t)^{n+1}}$$

Hilbert function:

$$H_n(t) = (1-t)^{-n-1} = \sum_{d=0}^{\infty} \binom{-n-1}{d} (-t)^d$$

$$\Rightarrow h_n(d) = \binom{-n-1}{d} \cdot (-1)^d = \binom{n+d}{d} = \binom{d+n}{n}$$

A total degree monomial ordering is a monomial ordering such that for  $t, t' \in M$  have:  $t \leq t'$  implies  $\deg(t) \leq \deg(t')$

**Example:** grevlex

### 2.8.11 Theorem 11

Let " $\leq$ " be a total degree monomial ordering.  $I \subseteq K[x]$  ideal  $\Rightarrow H_I(t) = H_{L(I)}(t)$

**Proof:**

Let  $G$  be a Gröbner basis of  $I$ ,  $d \in \mathbb{N}$

$NF_G$  induces an injection from  $\phi : A \mapsto K[\underline{x}]$ . Consider  $\phi_d : A_d \mapsto K[\underline{x}]$  restriction

**Claim:**  $im(\phi_d)$  is the space  $V_d \subseteq K[\underline{x}]$  generated by all monomials  $m \in M$

with  $\deg(m) \leq d$  such that  $LM(g) \nmid m \ \forall g \in G$

Let  $f \in V_d \Rightarrow f$  is in normal form with respect to  $G \Rightarrow f = NF_G(f) = \phi(\underbrace{f + I}_{\in A_d}) \in im(\phi_d)$

Conversely let  $f \in im(\phi_d) \Rightarrow \exists g \in K[\underline{x}] :$

$$\deg(g) \leq d$$

$$f = NF_G(g)$$

$$\Rightarrow f = g - \sum_{i=1}^m h_i g_i \quad \text{with } g_i \in G$$

$$h_i \in K[\underline{x}]$$

$$LM(h_i g_i) \leq LM(f)$$

$$\Rightarrow \forall t \in M(h_i g_i) : t \text{ "}\leq\text{" tot. deg. mon. ord.} \Rightarrow \deg(1) \leq \deg(LM(g))$$

$$\text{So } \deg(h_i g_i) \leq d \Rightarrow \deg(f) \leq d$$

(\*)

Since  $f$  is in normal form this implies  $f \in V_d$ .

$$\text{So } V_d = im(\phi_d)$$

$$\Rightarrow h_I(d) = \dim_K(A_d) = \dim_K(im(\phi_d)) = \dim(V_d)$$

$V_d$  only depends on  $d$  and on  $(LM(g) \mid g \in G) = L(I) \Rightarrow h_I$  only depends on  $L(I)$

Since  $L(L(I)) = L(I)$  the theorem follows  $\square$

How to compute  $H_I(t)$  for  $I$  monomial ideal:

Let  $I = (m_1, \dots, m_l) \quad m_i \in M$  Set  $J = (m_1, \dots, m_{l-1})$

Then the map  $J \xrightarrow{\text{surjective}} I/(m_l)$  has the kernel  $J \cap (m_l)$

$$\Rightarrow J/(J \cap (m_l)) \cong I/(m_l)$$

This isomorphism restricts to all homogeneous components

$$\Rightarrow H_I(t) = H_{(m_l)}(t) + H_J(t) - H_{J \cap (m_l)}(t) \quad (*)$$

$$\text{Have } J \cap (m_l) = (lcm(m_1, m_l), \dots, lcm(m_{l-1}, m_l))$$

**2.8.12 Theorem 12**

Let  $I = (m_1, \dots, m_l) \subseteq K[x_1, \dots, x_n]$  with  $m : i \in M$

$$\Rightarrow H_I(t) = \frac{1}{(1-t)^{n+1}} \sum_{S \subseteq \{1, \dots, l\}} (-1)^{|S|} \cdot t^{\deg(lcm\{m_i \mid i \in S\})}$$

**Proof:**

Use induction on  $l$ ,  $(*)$  and bookkeeping.

$l = 0$  : Example (3) ✓

$l = 1$  :  $I = (m_1)$

$$\sum_{d=0}^{\infty} \dim(\underbrace{I_{\leq d}}_{\text{all polys in } I \text{ of deg } \leq d}) \cdot t^d = t^{\deg(m_1)} \cdot H_{\{0\}}(t)$$

$$\Rightarrow H_{(m_1)}(t) = H_{\{0\}}(t) - \sum_d \dim(I_{\leq d}) = \frac{1-t^{\deg(m_1)}}{(1-t)^{n+1}} \quad \checkmark$$

$l - a \rightarrow l$  (with  $l \geq l$ ): By  $(*)$  with  $I = (m_1, \dots, m_l)$  :

$$(1-t)^{n+1} \cdot H_I(t) = 1 - t^{\deg(m_l)} + \sum_{S \subseteq \{1, \dots, l-1\}} (-1)^{|S|} \cdot t^{\deg(\text{lcm}\{m_i | i \in S\})} \\ - \sum_{S \subseteq \{1, \dots, l-1\}} (-1)^{|S|} \cdot t^{\deg(\text{lcm}\{\text{lcm}(m_i, m_l) | i \in S\})}$$

For  $S \neq \emptyset$  then  $\text{lcm}\{\text{lcm}(m_i, m_l) | i \in S\} = \text{lcm}\{m_i | i \in S \cup \{l\}\}$

So the formula is correct  $\square$

$\rightarrow$  may compute Hilbert series by Gröbner bases!

**2.8.13 Corollary 13 (Hilbert-Serre theorem)**

Let  $I \subseteq K[\underline{x}]$  ideal.

Then  $H_I(t) = \frac{a_0 + a_1 t + \dots + a_k t^k}{(1-t)^{n+1}}$   $a_i \in \mathbb{Z}$

For  $d \gg 0$   $h_I(d)$  is a polynomial. More precisely with

$$p_I := \sum_{i=0}^k a_i \binom{x+n-i}{n} \in \mathbb{Q}[x]$$

have  $h_I(d) = p_I(d)$  for  $d \gg 0$

$p_I$  is called the Hilbert polynomial.

**2.8.14 Definition 14 (Algebra, homo-, isomorphism)**

An algebra over a field  $K$  is a commutative ring  $A$  containing  $K$ .

(Usually  $A = \{0\}$  is also considered as an algebra)

$A$  is called finitely generated if

$$\exists a_1, \dots, a_n \in A : A = \{f(a_1, \dots, a_n) \mid f \in K[x_1, \dots, x_n]\} = K[a_1, \dots, a_n]$$

A homomorphism  $\phi : A \rightarrow B$  of  $k$ -algebras is a ring homomorphism that fixes  $k$  element-wise.

$\phi$  is an isomorphism it is bijective.

So if  $A = K[a_1, \dots, a_n]$  we have a surjective homomorphism

$$\phi : K[x_1, \dots, x_n] \rightarrow A, \quad f \mapsto f(a_1, \dots, a_n). \quad \text{If } I := \ker(\phi) \text{ then } K[x_1, \dots, x_n]/I \cong A$$

Given a finitely generated algebra  $A$  (such as  $A = K[x] = K[\underline{x}]/I$  with  $x = \mathcal{V}(I)$  affine variety), then each choice of finite set of generators affords an isomorphism

$K[x_1, \dots, x_n]/I \cong A$ . Then  $p_I$  will depend on the choice of the generators. But might the degree of  $p_I$  be an invariant of  $A$ ?

### 2.8.15 Lemma 15

Let  $I \subseteq K[x_1, \dots, x_n]$ ,  $J \subseteq K[y_1, \dots, y_n]$  ideals in polynomial rings such that  $\underbrace{K[x_1, \dots, x_n]/I}_A \cong \underbrace{K[y_1, \dots, y_n]/J}_B$ . Then  $\deg(p_I) = \deg(p_J)$ .

**Proof:**

Have  $\phi A \xrightarrow{\sim} B \Rightarrow \exists g_1, \dots, g_m \in K[x_1, \dots, x_n]$  such that  $\phi(g_i + I) = y_i + J$

Choose  $e \in \mathbb{N}$  such that  $\deg(g_i) \leq e$

Then  $B_d \subseteq \phi(A_{de}) \Rightarrow h_J(d) = \dim(B_d) \leq \dim(A_{de}) = h_I(de) \Rightarrow \deg(p_J) \leq \deg(p_I)$

By symmetry  $\deg(p_I) \leq \deg(p_J)$ , have equality  $\deg(p_J) = \deg(p_I)$   $\square$

Goal:  $\deg(p_I) = \dim(I)$

### 2.8.16 Theorem 16 (Noether normalization)

Let  $A \neq \{0\}$  be a finitely generated algebra. Then there exist algebraically independent elements  $c_1, \dots, c_m \in A$  such that  $A$  is finitely generated as a module (= vector space, but over a ring rather than a field) over  $C = K[c_1, \dots, c_m] \underset{\text{subalgebra}}{\subseteq} A$ ,

i.e.  $\exists b_1, \dots, b_l \in A$  such that  $A = \sum_{i=1}^l C \cdot b_i$

Moreover if  $A \cong K[x]/I$  then  $m = \dim(I)$  ( $= \text{trdeg}(A)$ )

If  $m = 0$  then  $C = K$

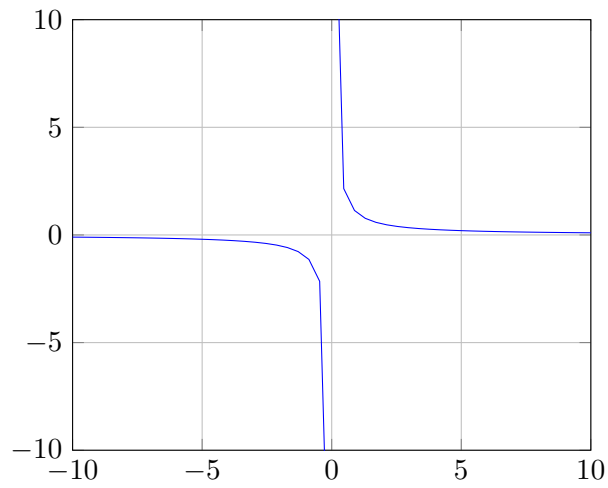
**Proof:**

Omitted  $\rightarrow$  commutative algebra  $\square$

**Geometric interpretation:** Let  $A = K[X]$ ,  $X$  affine variety. Choosing  $c_1, \dots, c_m \in A$  amounts to choosing a morphism  $f : X \mapsto K^m$ ,  $v \mapsto (c_1(v), \dots, c_m(v))$ . It can be shown that if  $C = K[c_1, \dots, c_m]$  is Noether norm, then the morphism is surjective and with finite fibers i.e.  $\forall w \in K^m$  the set  $f^{-1}(\{w\})$  is finite.

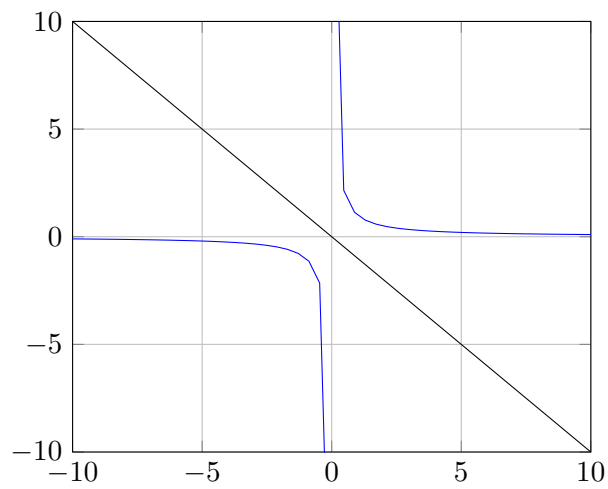
**Example:**

$X = \mathcal{V}(xy - 1) \subseteq \mathbb{C}^2$



The projection to the x- and y-axis are not surjective.

Try  $c = \bar{x} - \bar{y} \in A = \mathbb{C}[x, y]/(xy - 1)$



Have  $\bar{x}^2 - c\bar{x} - 1 = 0 \Rightarrow \forall i : \bar{x}^i \in C \cdot 1 + C\bar{x}$

Similarly  $f \cap \bar{y} \Rightarrow C = \mathcal{C}[c]$  satisfies Theorem 16

### 2.8.17 Theorem 17

Let  $I \subseteq K[x_1, \dots, x_n]$  ideal  $\Rightarrow \dim(I) = \deg(p_I)$

**Proof:**

If  $I = K[x] \Rightarrow \checkmark$

So assume  $I \not\subseteq K[x]$ . Then by Theorem 16  $\exists C = K[c_1, \dots, c_m] \subseteq A := K[x]/I$  and

$\exists b_1, \dots, b_l \in A : A = \sum_{i=1}^l C \cdot b_i, \quad c_1, \dots, c_m$  algebraic independent. Without loss  $b_1 = 1$

Let  $y_1, \dots, y_m, z-1, \dots, z_l$  be new indeterminates and consider the map

$K[y_1, \dots, y_m, z-1, \dots, z_l] \mapsto A, \quad y_i \mapsto c_i, \quad z_j \mapsto b_j$

This is surjective. Let  $J$  be the Kernel.

By Lemma 15 have  $\deg(p_I) = \deg(p_J)$ .

Need to show:  $\deg(p_J) = m$

Write  $B : K[y_1, \dots, y_m, z-1, \dots, z_l]/J$ , consider  $B_d, C_d := \{f+J \mid f \in K[y_1, \dots, y_m], \deg(f) \leq d\} \subseteq B_d$  (for  $d \in \mathbb{N}$ ).

Since  $C_d \subseteq B_d$ , obtain  $h_J(d) = \dim(B_d) \geq \dim(C_d) = \dim(K[y_1, \dots, y_m])_d$   
 $c_1, \dots, c_m \text{ alg. ind.} = \binom{d+m}{m}$

$\Rightarrow \deg(p_J) \geq m$

For  $1 \leq i \leq j \leq l$  have  $b_i b_j = \sum_{k=1}^l a_{ijk} \cdot b_k$  with  $a_{ijk} \in C$  Choose  $e \in \mathbb{N}_{>0}$  such that

$\forall i, j, k : \deg(a_{ijk}) \leq e \Rightarrow b_i b_j \subseteq \sum_{k=1}^l C_{(s-1)e} \cdot b_k$ . So the product of  $s$  of the  $b_i$  lies in  $\sum_{k=1}^l C_{(s-1)e} \cdot b_k$ .

This implies (for  $d \in \mathbb{N}$ )  $B_d \subseteq C_d \cdot b_1 + \sum_{s=1}^d \sum_{k=1}^l C_{d-s} \cdot C_{(s-1)e} \cdot b_k \subseteq \sum_{k=1}^l C_{de} \cdot b_k =: V_d$

$\Rightarrow h_J(d) = \dim(B_d) \leq \dim(V_d) \leq l \cdot \dim(C_{de}) = l \cdot \binom{d \cdot e + m}{m} = \deg(p_J) \leq m \quad \square$

(or substitute  $e$  by  $2e$ )

### 2.8.18 Corollary 18

Let " $\leq$ " be a graded monomial ordering  $I \subseteq K[x]$  ideal  $\Rightarrow \dim(I) = \dim(L(I))$

For computing  $\dim(L(I))$  use the transcendence degree!

Let  $L(I) = (m_1, \dots, m_l)$ ,  $m_i$  monomials. For  $S \subseteq \{x_1, \dots, x_n\}$ ,

$S$  is dependent modulo  $L(I) \Leftrightarrow \exists i : m_i \in K[S]$

$S$  is independent modulo  $L(I) \Leftrightarrow \forall i : m_i \notin K[S]$

i.e.  $m_i$  involves a variable not in  $S$ .

$T := \{1, \dots, n\} \setminus S$  is independent modulo  $L(I)$

$\Leftrightarrow$  Every  $m_i$  involves a variable that is in  $S$

### 2.8.19 Algorithm 19 (Dimension of $I$ )

Input :  $I \subseteq K[\underline{x}]$  ideal

Output:  $\dim(I)$

- (1) Compute a Gröbner basis of  $I$  with respect to a graded monomial ordering.  
Let  $m_i, \dots, m_l$  be the leading monomials of the polynomials in  $G$
- (2) if  $(\exists i : m_i = 1)$   
 $\dim(I) = -1$
- (3) Find  $T \subseteq \{x_1, \dots, x_n\}$  of minimal size,  
such that every  $m_i$  involves a variable from  $T$   
// don't worry about exponential time for this step as (1) is doubly exp anyway
- (4)  $\dim(I) = n - |T|$

## 3 Notes

### 3.1 Notation

- $\mathbb{N} := \mathbb{N}_0$
- $\lg(x) := \log_2(x)$
- $a \mid b$        $a$  is divisible by  $b$        $\Leftrightarrow$        $b \bmod a = 0$   
 $a \nmid b$        $a$  is not divisible by  $b$   $\Leftrightarrow$        $b \bmod a \neq 0$
- $ord(a)$       order of a group element  
 $n > 0$  minimal such that  $a^n = e$       with neutral element  $e$   
if no such  $n$  can be found,  $ord(a) = \infty$
- $char(A)$       Characteristic: the smallest positive  $n$  such that  
 $\underbrace{1 + \dots + 1}_n = 0$       with 1 as the multiplicative identity element  
 $n$  summands
- $im(f)$  image of  $f$  ("Wertebereich")
- $Ker(\phi)$  kernel of  $\phi$ ; Number of elements that map to zero  
 $\Rightarrow$  degree to which the homomorphism  $\phi$  fails to be injective
- $\mathbb{Z}/(m)$       Ring modulo  $m$   
polynomial rings measure for " $<$ " relations not the absolute value but max power.
- $lcm(a_1, \dots, a_n)$       "least common multiple of all  $a_i$ "
- $\underline{e}$  = vector of  $e$ 's
- $K[\underline{x}] := K[x_1, \dots, x_n]$
- $\phi(n) := |\{x \in \mathbb{N} : x < n \wedge \gcd(x, n) = 1\}| = |(\mathbb{Z}/(n))^x|$   
Euler's totient function
- $rk(A)$       Rank of matrix  $A$
- $\left(\frac{n}{p}\right) := \begin{cases} 1 & \text{if } p \mid n \\ -1 & \text{if } n \text{ is a square (mod } p) \\ 0 & \text{otherwise} \end{cases}$   
Legendre symbol (this is not a fraction)
- $\left(\frac{n}{p}\right) = 1 \Leftrightarrow n^{\frac{p-1}{2}} = \left(\frac{n}{p}\right) \equiv 1 \pmod{p}$   
Eulers criterion
- $res(f, g)$  resultant.  $\Rightarrow$  det of Sylvester-Matrix
- $\mathbb{A} :=$  Affine space
- $Id(S) =$  Ideal of  $S$



### 3.2 Various stuff

- Lagrange's theorem  
Every element in a finite group has finite order
- Average number of bit operations for an increment:  
One operation for the last bit + 50% chance for one on the next bit + 25% on the following etc.  $\Rightarrow$  Geometrical row  
 $\Rightarrow$  on average two bit operations
- "Monte Carlo Algorithm"  
Always terminates in reasonable time but might yield false result.
- "Las Vegas Algorithm"  
If it terminates the result is correct. No deterministic running time.
- Chinese remainder theorem  
Given a system of congruences  $x \equiv a_i \pmod{m_i}$  with  $i = 1, \dots, r$   
 $m_i$  pairwise co-prime. Then the unique solution is:  
$$x \equiv a_1 \cdot b \cdot \frac{N}{m_i} + \dots + a_r \cdot b_r \cdot \frac{N}{m_r} \pmod{N} \quad \text{with } b_i \cdot \frac{N}{m_i} \equiv 1 \pmod{m_i}$$
- distance between two square numbers:  
 $(n+1)^2 - n^2 = 2n + 1$   
 $\Rightarrow$  Squares are much more scarce than primes!
- $ax + by = c$  has solutions in  $\mathbb{Z}$  iff  $\Leftrightarrow \gcd(a, b) \mid c$  with  $a, x, b, y \in \mathbb{Z}$
- $S_{f,g} = \begin{pmatrix} f_m & \dots & f_0 & 0 & \dots & 0 \\ 0 & f_m & \dots & f_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dots & \ddots & 0 \\ 0 & \dots & 0 & f_m & \dots & f_0 \\ g_n & \dots & g_0 & 0 & \dots & 0 \\ 0 & g_n & \dots & g_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dots & \ddots & 0 \\ 0 & \dots & 0 & g_n & \dots & g_0 \end{pmatrix}$  Sylvester-Matrix for  $f(x), g(x)$
- finitely generated  
A set  $M$  is finitely generated if  $\exists R^n \mapsto M$  surjective with  $R^n$  finite
- monomial ideal  $I \Leftrightarrow \exists$  a set of generators that are monomials only

### 3.3 Algebraic structures

- Group  
- one inner operation  $(*)$ :  $(G, *)$   
- associativity:  $G \times G \mapsto G$   
 $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$   
- neutral element  $e \in G$ :  $a * e = e * a = a \quad \forall a \in G$   
- inverse element  $a^{-1} \in G$ :  $a * a^{-1} = a^{-1} * a = e \quad \forall a \in G$
- Abelian group  
 $(G, *)$

- $(G, *)$  is a group
- commutativity:  $a * b = b * a \quad \forall a, b \in G$
- Finite group  $(G, *)$ 
  - associativity:  $(a * b) * c = a * (b * c)$
  - unambiguity of reduction:  $(a * x = a * x') \wedge (x * a = x' * a) \Rightarrow x = x'$   
 $\Rightarrow x \mapsto x * a$  and  $x \mapsto a * x$  is bijective  
 $\Rightarrow \exists x : a * x = a \Rightarrow$  neutral element  
 $\exists x : a * x = x \Rightarrow$  inverse element
- Cyclic group  $(G, *)$ 
  - $G$  is a group
  - $G$  is generated by one Element:  $G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$
  - not necessarily finite.
- Semi group  $(S, *)$ 
  - one inner operation  $(*)$ :  $S \times S \mapsto S$
  - associativity:  $(a * b) * c = a * (b * c) \quad \forall a, b, c \in S$
- Field  $(K, +, \cdot)$ 
  - two inner operations  $(+, \cdot)$  such that:
    - $(K, +)$  is an abelian group with neutral element 0
    - $(K \setminus \{0\}, \cdot)$  is an abelian group with neutral element 1
  - distributivity:  $a \cdot (b + c) = a \cdot b + a \cdot c$   
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in K$
- General linear group  $GL_n(K)$ 
  - $K$  is a field
  - $GL_n(K)$  is the set of  $n \times n$  invertible matrices with ordinary matrix multiplication
- Ring  $(R, +, \cdot)$ 
  - $(R, +)$  is an abelian group
  - $(R, \cdot)$  is a semi group
  - distributivity:  $a \cdot (b + c) = a \cdot b + a \cdot c$   
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in R$
- Commutative ring  $(R, +, \cdot)$  is a ring
  - commutativity for  $(\cdot)$   $a \cdot b = b \cdot a \quad \forall a, b \in R$
- Unitary ring (ring with 1)  $(R, +, \cdot)$ 
  - $(R, \cdot)$  is a semi group
  - $(R, \cdot)$  has a neutral element "1"
- Euclidean ring  $R$ 
  - $\exists F : R \mapsto \mathbb{N}_0 \cup \{0\}$   
such that if  $\exists q, r \in R \quad a = b \cdot q + r$  and  $r = 0$  or  $a, b \in R \quad F(r) < F(b)$
- Polynomial ring  $R[X]$ 
  - $R$  is a commutative unitary ring
  - set of all polynomials with coefficients  $\in R$
  - Variables  $X_1 \dots X_n$
- Noetherian Ring  $R$

The following definitions are equal:

- for  $I_1 \subseteq I_2 \subseteq \dots \quad \exists n : I_n = I_{n+1} = \dots$  (the chain of ideals "stabilizes")
- every ideal of  $R$  is finitely generated

### 3.4 Invertible elements

- Let  $(\mathbb{Z}/(n), +)$  be a group or  $(\mathbb{Z}/(n))^\times$  be a group with multiplication.
- $|(\mathbb{Z}/(n))^\times| = \phi(n)$
- $n \in \mathbb{P}$   
 $\Rightarrow (\mathbb{Z}/(n))^\times = \{\bar{0}, \dots, \bar{p-1}\} \cong (\mathbb{Z}/(p-1), +) = Z_{p-1}$  (cyclic Group  $Z$ )
- $n$  is a power of 2  
 $\Rightarrow (\mathbb{Z}/(2^e))^\times \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2^{e-2})$
- $n$  is a power of an odd Prime  
 $\Rightarrow (\mathbb{Z}/(p^k))^\times \cong \mathbb{Z}/(p^{k-1} \cdot (p-1)) \cong Z_{(p^{k-1} \cdot (p-1))}$
- $n = p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$   
 $\Rightarrow (\mathbb{Z}/(n))^\times \cong (\mathbb{Z}/(p_1^{k_1}))^\times \times \dots \times (\mathbb{Z}/(p_r^{k_r}))^\times$