

# Computational Algebra

-

## Transcript

by Fabio Gratl

Summer term 2015

### Contents

<b>1</b>	<b>Integer Arithmetic</b>	<b>5</b>
1.1	Addition and Multiplication . . . . .	5
1.1.1	Algorithm 1 (Simple addition) . . . . .	5
1.1.2	Definition 2 (Bit-Operation) . . . . .	6
1.1.3	Definition 3 (Big $O$ ) . . . . .	6
1.1.4	Theorem 4 (Lower bound for addition) . . . . .	6
1.1.5	Algorithm 5 (Multiplication by "grid method") . . . . .	7
1.1.6	Theorem 6 (Runtime of Algorithm 5) . . . . .	7
1.1.7	Algorithm 7 (Karatsuba) . . . . .	7
1.1.8	Theorem 8 (Runtime of Algorithm 7) . . . . .	8
1.1.9	Definition 9 (Root of unity) . . . . .	9
1.1.10	Algorithm 10 (Fast Fourier transformation FFT) . . . . .	9
1.1.11	Theorem 11 (Runtime of Algorithm 10) . . . . .	10
1.1.12	Definition 12 (Good root of unity) . . . . .	10
1.1.13	Proposition 13 ( $DFT_{\mu^{-1}}$ ) . . . . .	10
1.1.14	Proposition 14 (Finding good roots of unity) . . . . .	11
1.1.15	Algorithm 15 (Polynomial multiplication using DFT) . . . . .	11
1.1.16	Theorem 16 (Runtime of Algorithm 15) . . . . .	12
1.1.17	Proposition 17 (Add and mul in $O(l)$ ) . . . . .	12
1.1.18	Proposition 18 (Sort of summary) . . . . .	12
1.1.19	Algorithm 19 (Multiplication using FFT) . . . . .	13
1.1.20	Theorem 20 (Runtime of Algorithm 19) . . . . .	14
1.1.21	Theorem 21 (Schönhage-Strassen 1971) . . . . .	15
1.2	Division with remainder, Euclidean algorithm . . . . .	16
1.2.1	Algorithm 1 (Division with remainder) . . . . .	16
1.2.2	Proposition 2 (Runtime of Algorithm 1) . . . . .	16

1.2.3	Algorithm 3 (Euclidean algorithm)	17
1.2.4	Theorem 4 (Correctness of Algorithm 3)	17
1.2.5	Theorem 5 (Runtime of Algorithm 3)	18
1.2.6	Algorithm 6 (Extended Euclidean Algorithm)	18
1.3	Primality testing	19
1.3.1	Theorem 1 (Cyclic group)	19
1.3.2	Algorithm 2 (Fermat Test)	20
1.3.3	Algorithm 3 (Fast exponentiation)	20
1.3.4	Definition 4 (Pseudo-prime, witness, Carmichael numbers)	21
1.3.5	Proposition 5 (Number of witnesses)	21
1.3.6	Proposition 6 (Inference from Fermat)	21
1.3.7	Algorithm 7 (Miller-Rabin-test)	22
1.3.8	Definition 8 (strong pseudo-prime / witness)	22
1.3.9	Theorem 9 (Bit-complexity of Algorithm 7)	22
1.3.10	Theorem (Ankeny & Bach)	24
1.3.11	Proposition 10 (Modulo over ideals)	25
1.3.12	Algorithm 11 (Test for perfect power)	25
1.3.13	Algorithm 12 (AKS-test)	26
1.3.14	Lemma 13 (Least common multiple)	26
1.3.15	Lemma 14 (Property of $r$ in Algorithm 12)	27
1.3.16	Theorem 15 (Bit-Complexity of Algorithm 12)	27
1.3.17	Lemma 16 (Rules for ideals)	28
1.3.18	Theorem 17 (Correctness of Algorithm 12)	28
1.3.19	Lemma 18 (Property of binomial coefficients)	29
1.4	Cryptology	30
1.4.1	Algorithm (RSA)	31
1.4.2	Algorithm 1 (Finding a divisor)	32
1.4.3	Proposition 2 (Complexity of Algorithm 1)	32
1.4.4	Diffie-Hellmann Key Exchange	33
1.4.5	Elliptic curve cryptography (ECC)	33
1.5	Factorization	34
1.5.1	Algorithm 1 (Sieve of Eratosthenes)	34
1.5.2	Proposition 2 (length of periods)	34
1.5.3	Algorithm 3 (Pollard's $\rho$ - Algorithm)	35
1.5.4	Theorem 4 (Bit-complexity of Algorithm 3)	36
1.5.5	Algorithm 5 (Pollard's $\rho$ - 1 method)	36
1.5.6	Algorithm 6 (Quadratic sieve, simplified version)	39
<b>2</b>	<b>Systems of equations</b>	<b>42</b>
2.6	Linear Algebra	42
2.6.1	Proposition 1 (Complexity of usual algorithms)	42
2.6.2	Algorithm 2 (Strassen-multiplication)	43
2.6.3	Theorem 3 (Running time of Algorithm 2)	44
2.6.4	Proposition 4 (Complexity of matrix inversion)	45

2.6.5	Algorithm 5 (Transforming a matrix)	45
2.6.6	Theorem 6 (Correctness and running time of Algorithm 5)	47
2.6.7	Theorem 7	48
2.6.8	Corollary 8	49
2.7	Algebraic Systems of Equations, Gröbner bases	49
2.7.1	Theorem 1 (Hilbert's Nullstellensatz)	50
2.7.2	Definition 2 (Monomial, monomial ordering, LM, LT, LC)	51
2.7.3	Proposition 3 (Sum and product of LM / LT)	52
2.7.4	Lemma 4 (Dickson-Lemma)	52
2.7.5	Corollary 5 (Well-ordering of monomial sets)	52
2.7.6	Definition 6 (Leading ideal, Gröbner bases)	53
2.7.7	Proposition 7 (Ideality of Gröbner bases)	53
2.7.8	Theorem 8 (Gröbner basis of Ideals)	53
2.7.9	Definition 9 (Normal form)	54
2.7.10	Algorithm 10 (Normal form)	54
2.7.11	Theorem 11 (Normal form of Gröbner bases)	55
2.7.12	Definition 12 ( $S$ -polynomials)	56
2.7.13	Theorem 13 (Buchberger's criterion)	56
2.7.14	Algorithm 14 (Buchberger)	57
2.7.15	Theorem 15 (Correctness of Algorithm 14)	57
2.7.16	Definition 16 (Reduced Gröbner basis)	58
2.7.17	Theorem 17 (Uniqueness of reduced Gröbner basis)	59
2.8	Application of Gröbner bases	59
2.8.1	Definition 1 (Elimination ideals)	59
2.8.2	Theorem 2	59
2.8.3	Theorem 3	61
2.8.4	Algorithm 4 (Solving systems of algebraic equations)	62
2.8.5	Proposition 5	62
2.8.6	Definition 6 (independence modulo $I$ )	63
2.8.7	Theorem 7	63
2.8.8	Lemma 8	63
2.8.9	Lemma 9	64
2.8.10	Definition 10 (Hilbert series)	65
2.8.11	Theorem 11	65
2.8.12	Theorem 12	66
2.8.13	Corollary 13 (Hilbert-Serre theorem)	67
2.8.14	Definition 14 (Algebra, homo-, isomorphism)	67
2.8.15	Lemma 15	68
2.8.16	Theorem 16 (Noether normalization)	68
2.8.17	Theorem 17	70
2.8.18	Corollary 18	70
2.8.19	Algorithm 19 (Dimension of $I$ )	71

<b>3</b>	<b>Notes</b>	<b>72</b>
3.1	Notation . . . . .	72
3.2	Various stuff . . . . .	73
3.3	Algebraic structures . . . . .	73
3.4	Invertible elements . . . . .	75

# 1 Integer Arithmetic

Topics:

- Addition and Multiplication
- GCD computation
- Primality testing
- Factorization

## 1.1 Addition and Multiplication

Agreement:

- $a, x \in \mathbb{N}$  represented as  $x = \sum_{i=0}^{n-1} a_i \cdot B^i$   $B \in \mathbb{N}_{>1}$  fixed Base ( $a_i \in \{0, \dots, B-1\}$ )
- if  $x \neq 0$ , assume  $a_{n-1} \neq 0$  then define:  
length of  $x := l(x) = n$  = number of digits =  $\lfloor \log_B(x) \rfloor + 1$   
(mnemonic:  $\log_B(B) + 1 = 2$ )
- $l(0) = 1$   
(Amount of memory required to store  $x = 0$ )
- $l(x) := l(|x|)$
- for  $x \in \mathbb{Z}$  represent if as  $x = \text{sgn}(x) * |x|$

### 1.1.1 Algorithm 1 (Simple addition)

input :  $x = \sum_{i=0}^{n-1} a_i \cdot B^i$ ,  $y = \sum_{i=0}^{n-1} b_i \cdot B^i$ ,  $x, y \in \mathbb{N}$

output:  $x + y = \sum_{i=0}^n c_i \cdot B^i$

- (1)  $\sigma = 0$
- (2) for  $i = 0, \dots, (n-1)$  :
- (3)     set  $c_i := a_i + b_i + \sigma_i$   
        $\sigma := 0$
- (4)     if  $(c_i \geq B)$
- (5)         set  $c_i = c_i - B$
- (6)         set  $\sigma = 1$
- (7) set  $c_n = \sigma$

If  $B = 2$  then (3) - (6) can be realized by logic gates:

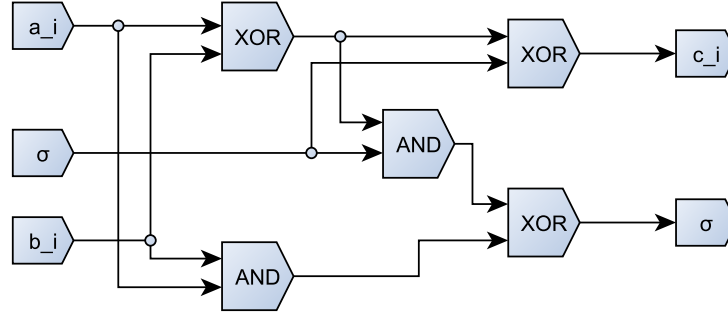


Figure 1: Logic circuit for addition

### 1.1.2 Definition 2 (Bit-Operation)

A bit operation is an operation that can be performed by a logic gate or by searching or writing a bit from / into memory.

### 1.1.3 Definition 3 (Big $O$ )

Let  $M$  be a set (usually  $M = \mathbb{N}$ ),  $f, g : M \mapsto \mathbb{R}_{>0}$   
we write  $f \in O(g)$  if  $\exists c \in \mathbb{R} : f(x) \leq c \cdot g(x) \forall x \in M$

### 1.1.4 Theorem 4 (Lower bound for addition)

Let  $f : \mathbb{N} \mapsto \mathbb{R}$ ,  $n \mapsto$  maximal number of bit operations required by Algorithm 1 to add  $x, y \in \mathbb{N}$  with  $l(x), l(y) \leq n$

Let  $g = id_{\mathbb{N}}$  Then  $f \in O(g)$

We say Algorithm 1 requires  $O(n)$  bit operations for adding two numbers of length  $\leq n$ .  
 $\Rightarrow$  "linear complexity"

Set  $M := \{\text{Set of all algorithms for addition in } \mathbb{N}\}$

For  $A \in M$  define  $f_A : \mathbb{N} \mapsto \mathbb{R}$  as above.

We would like to find  $f_{odd} : \mathbb{N} \mapsto \mathbb{R}$ ,  $n \mapsto \inf\{f_A(n) | A \in M\}$

Since one needs to read  $x, y$  (and write the result) we can not do any better than linear complexity for addition.

#### Subtraction

let  $x, y$  as Algorithm 1,  $x \geq y$

For  $\bar{y} := \sum_{i=0}^{n-1} (B - 1 - b_i) B^i$  (digitwise / bitwise complement)

$\Rightarrow x + \bar{y} = x - y + B^n - 1$

$\Rightarrow x - y = x + \bar{y} + 1 - B^n$  (initially set  $\sigma = 1$ )

**Conclusion:** Addition and Subtraction have cost  $O(n)$

### 1.1.5 Algorithm 5 (Multiplication by "grid method")

input :  $x = \sum_{i=0}^{n-1} a_i \cdot 2^i, \quad y = \sum_{i=0}^{m-1} b_i \cdot 2^i$

output:  $z = x \cdot y$

- (1)  $z := 0$
- (2) for  $i = 0, \dots, (n-1)$
- (3) if  $(a_i \neq 0)$ 

$$\text{set } z := z + \sum_{j=0}^{m-1} b_j 2^{i+j}$$

### 1.1.6 Theorem 6 (Runtime of Algorithm 5)

Algorithm 5 requires  $O(n \cdot m)$  bit operations.

As of the total input length  $n + m$ :

$$n \cdot m \leq \frac{1}{2}(n + m)^2 \rightarrow O((n + m)^2)$$

$\Rightarrow$  Quadratic complexity

### Karatsuba-multiplication:

Observation for polynomials:

$$a + bx, c + dx \text{ have } (a + bx)(c + dx) = ac + (ac + db - (a - b)(c - d))x + bdx^2$$

The point: only used 3 multiplications instead of 4.

Specialize  $x = B$  "large" such that  $x = a + bB$  partition into two blocks. Then multiply the blocks by a recursive call.

### 1.1.7 Algorithm 7 (Karatsuba)

input :  $x, y \in \mathbb{N}$

output:  $z = x \cdot y$

- (1) Choose  $k \in \mathbb{N}$  minimal such that  $l(x), l(y) \leq 2^k$ .  
Set  $B = 2^{2^{k-1}}$
- (2) if  $(k = 0)$   
return  $x \cdot y$  (by bit-operation AND)
- (3) write  $x = x_0 + x_1 B$   
 $y = y_0 + y_1 B$  with  $l(x_i), l(y_i) \leq 2^{k-1}$
- (4) compute  $x_0 \cdot y_0, x_1 \cdot y_1, (x_0 - x_1) \cdot (y_0 - y_1)$  by a recursive call
- (5) return  $z = x_0 y_0 + (x_0 y_0 + x_1 y_1 - (x_0 - x_1)(y_0 - y_1))B + x_1 y_1 B^2$

### 1.1.8 Theorem 8 (Runtime of Algorithm 7)

For multiplying two numbers of length  $\leq n$  Algorithm 7 requires  $O(n^{\log_2 3}) \approx O(n^{1.59})$  bit operations.

**Proof:**

Set  $\Theta(k) :=$  maximal numbers of bit operations for  $l(x), l(y) \leq 2^k$

We have for  $k > 0$  :  $\Theta(k) \leq 3 \underbrace{\Theta(k-1)}_{\text{recursive calls}} + c \underbrace{2^k}_{\text{additions}}$  with ( $c$  some constant)

**Claim:**  $\Theta(k) \leq 3^k + 2c(3^k - 2^k)$

**Proof by Induction on  $k$ :**

$k = 0$  :  $\Theta(k) = 1$

$$\begin{aligned} k-1 \rightarrow k : \Theta(k) &= 3\Theta(k-1) + c2^{k-1} \\ &\leq 3(3^{k-1} + 2c(3^{k-1} - 2^{k-1})) + c2^k \\ &= 3^k + 2c(3^k - 2^k) \end{aligned}$$

So  $\Theta(k) \leq (2c+1)3^k$

Now  $l(x) \leq n$  hence  $2^{k-1} < n$  by minimality of  $k$

So  $k-1 < \log_2 n$

$$\begin{aligned} \Rightarrow \Theta(k) &\leq 3(2c+1)3^{\log_2(n)} \\ &= 3(2c+1)2^{\log_2(3) \log_2(n)} \\ &= 3(2c+1)n^{\log_2(3)} \quad \square \end{aligned}$$

One can modify the terminal condition of Karatsuba to switch to Grid-Multiplication, which is faster for small numbers.

### Fast-Fourier Transformation

Reminder: For a function  $f : \mathbb{R} \mapsto \mathbb{C}$  define:

$\hat{f} : \mathbb{R} \mapsto \mathbb{C}$  by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt \quad (\text{if it exists})$$

Think of  $\omega$  as frequency.

### Definition (Convolution)

Let  $f, g : \mathbb{R} \mapsto \mathbb{C}$

$$(f * g)(x) = \int_{\mathbb{R}} f(t) g(x-t) dt$$

Convolution is analogous to polynomial multiplication

**Formula:**  $\underbrace{(f * g)}_{\text{(Cauchy formula)}} = \hat{f} \cdot \hat{g}$

For a function  $M \mapsto C$  with  $|M| < \infty$  we need the discrete Fourier transform (DFT)



### 1.1.9 Definition 9 (Root of unity)

Let  $R$  be a commutative ring with 1. An element  $\mu \in R$  is called an  $n$ -th root of unity (= root of 1) if  $\mu^n = 1$ .

It is called primitive if  $\mu^i \neq 1$  for  $(0 < i < n)$  i.e.  $\text{ord}(\mu) = n$

Let  $\mu$  be a primitive  $n$ -th root of 1 (e.g.  $e^{2\pi \frac{i}{n}} \in \mathbb{C}$ )

Then the map  $DFT_\mu : R^n \mapsto R^n$

$$(\hat{a}_0, \dots, \hat{a}_n) \mapsto (\hat{a}_0, \dots, \hat{a}_n) \quad \text{with } \hat{a}_i = \sum_{j=0}^{n-1} \mu^{ij} a_j$$

is called discrete Fourier transformation

For polynomials:

$$DFT_\mu : R[x] \mapsto R^n$$

$$f \mapsto (f(\mu^0), \dots, f(\mu^{n-1}))$$

Convolution rule: (from  $f(\mu^i)g(\mu^i) = (f * g)(\mu^i)$ )

$$DFT_\mu(f * g) = DFT_\mu(f) \cdot DFT_\mu(g) \quad (\text{component wise product})$$

Addition of two polynomials in  $R[x]$  of  $\deg(n)$  require  $O(n)$  ring operations.

Multiplication require  $O(n^l)$ .

With Karatsuba have  $O(n^{\log_2(3)})$  ring operations.

Cost  $DFT_\mu(f) \cdot DFT_\mu(g) : O(n)$  ring operations (with  $\mu$  as  $2n$ -th root of 1)

Want: Cheap way of doing  $DFT$  and back-transformation.

### 1.1.10 Algorithm 10 (Fast Fourier transformation FFT)

input :  $f \in R[x]$ ,  $\mu \in R$  primitive  $2^k$ -th root of 1, such that  $\mu^{2^{k-1}} = -1$

output:  $DFT_\mu(f)$

(1) Write  $f(x) = g(x^2) + xh(x^2)$  with  $f, g, h \in R[x]$

(2) if  $(k = 1)$   $// (\Rightarrow \mu = 1)$   
return  $DFT_\mu(f) = (g(1) + h(1), g(1) - h(1))$

(3) Recursive call: compute  $DFT_{\mu^2}(g) = \hat{g}, DFT_{\mu^2}(h) = \hat{h} \in R^{2^{k-1}}$

(4) return  $DFT_\mu(f) = (\hat{f}_0, \dots, \hat{f}_{2^k-1})$  with  $\hat{f}_i = \hat{g}_i + \mu \hat{h}_i$   
where  $\hat{g}_i = \hat{g}_{i-2^{k-1}}$  for  $i \geq 2^{k-1}$

Note: Components of  $\hat{g}$  and  $\hat{h}$  are:

$$\hat{g} = g(\mu^{2^i}), \quad \hat{h}_i = h(\mu^{2^i}) \quad \text{so}$$

$$\hat{f}_i := f(\mu^i) = \hat{g}_i(\mu^{2^i}) + \mu \hat{h}_i(\mu^{2^i}) = \hat{g}_i + \mu \hat{h}_i$$

**Convention:**  $\lg(x) = \log_2(x)$

### 1.1.11 Theorem 11 (Runtime of Algorithm 10)

Let  $n = 2^k$ ,  $f \in R[x]$  with  $\deg(\psi) < n$

Then Algorithm 10 requires  $O(n \cdot \lg(n))$  ring operations.

Better than  $O(n^{1+\epsilon})$ ,  $\forall \epsilon > 0$ !

**Proof:**

Set  $\Theta(k) = \max$  number of ring operations required. By counting obtain for  $k > 1$ :

$$\Theta(k) \leq 2\Theta(k-1) + \underbrace{(\text{compute } \mu^i (i \leq 2^{k-1}))}_{2^{k-1}} + \underbrace{(\mu^i \hat{k}_i)}_{2^{k-1}} + \underbrace{(\text{sums and differences})}_{2^k}$$

$$= 2\Theta(k-1) + 2^{k+1}$$

**Claim:**  $\Theta(k) \leq (2k-1)2^k$

$$k=1 : f = a_0 + a_1 \cdot x \quad DFT_\mu(f) = (a_0 + a_1 \cdot a_0 - a_1) \Rightarrow \Theta(a) = 2$$

$$k-1 \rightarrow k : \Theta(k) \leq 2 \cdot \Theta(k-1) + 2^{k+1} \leq 2 \cdot (2k-3) \cdot 2^{k-1} + 2^{k+1} = (2k-1) \cdot 2^k$$

since  $k = \lg(n)$  obtain  $O(k) \leq (2 \cdot \lg(n) - 1) \cdot n \in O(n \cdot \lg(n)) \quad \square$

**Back-transformation?**

### 1.1.12 Definition 12 (Good root of unity)

A primitive  $n$ -th root of unity is called good (caveat: this is ad-hoc terminology) if:

$$\sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \text{for } (0 < i < n)$$

**Example:**

- (1)  $\mu = e^{2\pi \frac{i}{n}}$  is a good primitive root of unity
- (2)  $R = \mathbb{Z}/(8)$ ,  $\mu = \bar{3} \Rightarrow \mu \cdot B$  is primitive  $2^{nd}$  root of unity  
But  $\bar{B}^0 + \bar{3}^1 = \bar{u} \neq \bar{0}$  so  $\mu$  is not good.

### 1.1.13 Proposition 13 ( $DFT_{\mu^{-1}}$ )

Let  $\mu \in R$  be a good root of 1

$$(a) = (a_0, \dots, a_{n-1}) \in R^n \Rightarrow DFT_\mu^{-1}(DFT_\mu(a)) = n \cdot (a) \quad \text{where } n = 1 + \dots + 1 \in R$$

**Proof:**

$$DFT_\mu(a) = (\hat{a}) = (\hat{a}_0, \dots, \hat{a}_{n-1})$$

$$\text{with } \hat{a}_j = \sum_{k=0}^{n-1} \mu^{jk} a_k$$

$$DFT_{\mu^{-1}}(\hat{a}) = (\hat{\hat{a}}_0, \dots, \hat{\hat{a}}_1)$$

$$\text{with } \hat{\hat{a}}_i = \sum_{j=0}^{n-1} \mu^{-ij} \sum_{k=0}^{n-1} \mu^{jk} a_k = \sum_{k=0}^{n-1} \left( a_k \cdot \underbrace{\sum_{j=0}^{n-1} \mu^{j(k-i)}}_{=0 \text{ if } n \neq k-i \text{ (i.e. } k=i)} \right) = a_i \cdot n \quad \square$$

### 1.1.14 Proposition 14 (Finding good roots of unity)

let  $\mu \in R, n \in \mathbb{N}$

Assume:

- a)  $R$  is an integral Domain and  $\mu$  is a primitive or  $n$ -th root of 1  
(Integral Domain: nonzero commutative ring in which the product of two nonzero elements is nonzero)  
 $\Rightarrow$  Granted by FFT
- b)  $n = 2^b, \mu^{\frac{n}{2}} = -1$ , then  $h > 0 \wedge \text{char}(R) \neq 2$   
 $\rightarrow \mu$  is a good primitive  $n$ -th root of 1 ("root of unity")

**Proof:**

- a) for  $0 < i < n$

$$\underbrace{(\mu^i - 1)}_{\neq 0} \underbrace{\left(\sum_{j=0}^{n-1} \mu^{ij}\right)}_{=0} = \mu^{in} - 1 = 0$$

$\Rightarrow \mu$  is a good root of unity

- \* Let  $0 < i < n$ , write  $i = 2^{k-s} \cdot r$  with  $r$  odd  $\wedge s > 0$

$$\sum_{j=0}^{2^k-1} \mu^{ij} = \sum_{l=0}^{2^{k-s}-1} \sum_{j=0}^{2^s-1} \mu^{i(l \cdot 2^s + j)}$$

$$\mu^{i \cdot 2^s} = 1$$

$$i \cdot 2^s = 2^{k-s} \sum_{j=0}^{2^s-1} \mu^{ij} = 2^{k-s} \sum_{j=0}^{2^{s-1}-1} (\mu^{ij} + \mu^{i(2^{s-1}+j)})$$

$$\text{But } \mu^{i \cdot 2^{s-1}} = \mu^{2^{k-s} \cdot r \cdot 2^{s-1}} = \mu^{2^{k-1} \cdot r} = (-1)^r = -1$$

$$\text{So } \sum_{j=0}^{n-1} \mu^{ij} = 0 \quad \square$$

- b)  $\mu^n = 1, n = 2^k \Rightarrow \text{ord}(\mu) | n \Rightarrow \text{ord}(\mu)$  is power of 2

### 1.1.15 Algorithm 15 (Polynomial multiplication using DFT)

input :  $f, g \in R[x]$  with  $\deg(f) + \deg(g) < 2^k =: n$   
 $\mu \in R$  as a good root of unity; Assume 2  $\in R$  is invertible

output:  $h = f \cdot g$

- (1) compute  $\hat{f} = DFT_{\mu}(f), \hat{g} = DFT_{\mu}(g)$  with  $f, g \in R^n$
- (2) compute  $\hat{h} = \hat{f} \cdot \hat{g}$
- (3) compute  $(h_0, \dots, h_{n-1}) = DFT_{\mu^{-1}}(\hat{h})$  (same as  $DFT_{\mu}(\hat{h})$  but with different order)  
= Back-transformation  $\cdot 2^k$   
set  $h = \frac{1}{2^k} \sum_{i=0}^{n-1} h_i x^i$

### 1.1.16 Theorem 16 (Runtime of Algorithm 15)

Algorithm 15 uses  $O(n \cdot \log(n))$  ring operations for polynomials of  $\deg < n$

**Proof:**

- Choose  $k$  minimal so that  $\deg(f) \cdot \deg(g) < 2^k$   
 $\Rightarrow 2^{k-1} \leq 2n \Rightarrow k \leq \log(n) + 2$
- $\underbrace{O(2k \cdot 2^k)}_{\text{Step 1}} + \underbrace{2^k}_{\text{Step 2}} + \underbrace{O(k \cdot 2^k) + 2^k}_{\text{Step 3}} \in O(2k \cdot 2^k) = O(n(g(n))) \quad \square$

Goal: Multiplication in  $\mathbb{N}$  using DFT

Idea: find roots of 1 in a suitable  $\mathbb{Z}/(m)$

Choose  $m = 2^l + 1, \mu = \bar{2} \in R$

### 1.1.17 Proposition 17 (Add and mul in $O(l)$ )

Let  $m = 2^l + 1, R = \mathbb{Z}/(m)$

Addition in  $R$  and multiplication by  $\bar{2}^i \in R$  ( $0 \leq i < 2l$ ) can be done in  $O(l)$  bit operations

**Proof:**

- Let  $\bar{x} \in R$  with  $0 \leq x \leq 2^l$
- Addition:  $x + \bar{y}$ 
    - (1) compute  $x + y \in \mathbb{N}$ :  $O(l)$
    - (2) if  $x + y > 2^l + 1$  subtract  $2^l + 1$ :  $O(l)$
  - Multiplication by  $\bar{2}^i$  ( $0 \leq i < l$ )
    - (1) Bit-shift  $i$  Bits to the left by relocating in memory:
 
$$\underbrace{O(\text{length}(i))}_{\text{compute addr. of new first bit}} + \underbrace{l}_{\text{copying}} = O(\log(l)) + l \in O(l)$$
  - Multiplication by  $\bar{2}^i$  ( $l \leq i < 2l - 1$ )
    - (1) Multiplication by  $\bar{2}^{i-l}$ :  $O(l)$
    - (2) take negative  $\bar{2}^i \cdot \bar{x} = -\bar{2}^{i-l} \cdot \bar{x}$ :  $O(l)$

### 1.1.18 Proposition 18 (Sort of summary)

Let  $k, r \in \mathbb{N}, r > 0, m = 2^{2^k \cdot r} + 1, R = \mathbb{Z}/(m), \mu = \bar{2}^r \in R$

$\Rightarrow 2 \in R$  is invertible,  $\mu$  is a good primitive  $2^{k+1}$ -th root of 1

$\Rightarrow \mu^{2^k} = 1$

**Proof:**  $\rightarrow$  from above

### 1.1.19 Algorithm 19 (Multiplication using FFT)

input :  $x, y \in \mathbb{N}$

output:  $Z = x \cdot y$

- (1) Choose  $k \in \mathbb{N}$  minimal such that  $l(x), l(y) \leq 2^{2^k}$
- (2) if  $k \leq 3$ , compute  $z = x \cdot y$  by Algorithm 5
- (3) set  $B = 2^{2^k}$ ,  $m = 2^{2^k \cdot 4} + 1$ ,  $R = \mathbb{Z}/(m)$ ,  $\mu = \bar{2}^4 \in R$   
 $(\Rightarrow \text{so } \mu \text{ is a good primitive } 2^{k+1}\text{-th root of 1})$
- (4) write  $x = \sum_{i=0}^{2^k-1} x_i \cdot B^i$ , same for  $y$  with  $(0 \leq x_i, y_i < B)$   
 possible since  $x, y < 2^{2^{2k}} = 2^{2^k \cdot 2^k} = B^{2^k}$
- (5) compute:  $\hat{x} = DFT_\mu(\bar{x}_0, \dots, \bar{x}_{2^k-1}, \underbrace{0, \dots, 0}_{2^k \text{ zeros}}) \in R^{2^{k+1}}$   
 same for  $y$   
 $\rightarrow$  use FFT
- (6) compute:  $\hat{z} = \hat{x} \cdot \hat{b} \in R^{2^{k+1}}$  (component wise multiplication)  
 Perform multiplication in  $R$  as follows:  
 Multiply representatives (non negative and  $< m$ ) by recursive call.  
 Then reduce modulo  $m$  by "negative bit shift" (see proof of Proposition 17)
- (7) compute:  $(\bar{z}_0, \dots, \bar{z}_{2^{k+1}-1}) = \frac{1}{2^{k+1}} DFT_{\mu^{-1}}(\hat{z}) \in R$  with  $0 \leq z < m$
- (8) set  $z := \sum_{j=0}^{2^{k+1}-1} z_j \cdot B^j$

### 1.1.20 Theorem 20 (Runtime of Algorithm 19)

Algorithm 19 correctly computes  $t = x \cdot y$  and requires  $O(n \cdot (\log n)^4)$  bit operations for  $l(x), l(y) \leq n$

**Proof:** Correctness

write  $x(t) \sum_{i=0}^{2^k-i} x_i t^i \in \mathbb{Z}[t]$ ,  $y(t)$ ,  $\bar{x}(t) \in R[t], \bar{y}(t), \bar{z}(t)$

by Proposition 18 and Proposition 13 we have  $\bar{z}(t) = \bar{x}(t) \cdot \bar{y}(t)$

The  $l$ -th coefficient of  $x(t) \cdot y(t)$  is  $0 \leq \sum_{i+j=l} x_i \cdot y_j < 2^k \cdot B^2 = 2^{k+2 \cdot 2^k} \leq 2^{2^{k+2}} < m$

So  $z(t) = x(t) \cdot y(t) \Rightarrow z = z(B) = x(B) \cdot y(B) = x \cdot y$  Cost:

Write  $\Theta(k) := \max$  number of bit operations

Analyze Steps:

- (1) compute  $\max \{l(x), l(y)\} : O(l(n)) = O(k)$
- (2)  $O(1)$
- (3) no bit operations
- (4) compute starting addresses of  $x_i, y_i$  in memory:  $2 \cdot 2^k$  increments of the address:  
 $2 \cdot 2 \cdot 2^k = 2^{k+2}$  bit ops  
 $\Rightarrow O(2^k)$
- (5) By Theorem 11 need  $O(2 \cdot 2^{k+1} \cdot (k+1))$  operations in  $R$  which are additions and multiplications by powers of  $\bar{z}$  costing  $O(2^{k+2})$  bit operations.  
Total for (5):  $O(k \cdot 2^{2 \cdot k})$
- (6)  $2^{k+1}$  multiplications of numbers  $< m$ , i.e. of length  $\leq 2^{k+2}$ .  
So  $k' \leq \frac{k+3}{2}$  for  $k'$ : the "new"  $k$  used in the next recursion level.  
For  $\alpha \in R_{>0}$  define  $\Theta(\alpha) := \Theta(|\alpha|)$   
Total for (6):  $2^{k+1}(\Theta(\frac{k+3}{2}) + \underbrace{O(2^{k+2})}_{\text{reduction (mod } m)})$
- (7) For  $DFT_{\mu^{-1}}(\hat{z}) : O(k \cdot 2^{2 \cdot k})$  as (5) Since  $\bar{z}$  is a  $n$  root of 1, multiplication by  $\bar{2}^{-k-1}$  is multiplication by a positive power of  $\bar{2}$ , which costs  $O(2^{k+2})$   
Total for (7):  $O(k \cdot 2^{2 \cdot k})$
- (8) For  $j \leq 2^{k+1}$  have  $\sum_{i=0}^{j-1} z_i \cdot B^i \leq (m-1) \sum_{i=0}^{j-1} B^i = (m-1) \frac{B^j-1}{B-1} < 2(m-1) \frac{B^j}{B} = 2^{1+2^{k+2}+(j-1)2^k}$  so the sum has length  $(j+3) \cdot 2 + 1$   
Adding  $z_j \cdot B^j$  to this sum happens at  $(j \cdot 2^k)$ -th bit and higher  $\Rightarrow$  cost is  $O(2^k)$   
Total for (8):  $O(2^{2 \cdot k})$

Grad total: For  $k \geq 4$ :

$\Theta(k) \leq 2^{k+1} \cdot \Theta(\frac{k+3}{2}) + c \cdot k \cdot 2^{2 \cdot k}$  with  $c$  constant

Also for  $k \in \mathbb{R}_{\geq 4}$

**Define**  $\Lambda(k) := \frac{\Theta(k)}{2^{2 \cdot k}} \Rightarrow \Lambda(k) \leq \frac{2^{k+1} \Theta(\frac{k+3}{2})}{2^{2 \cdot k}} + c \cdot k = 16 \cdot \Lambda(\frac{k+3}{2}) + c \cdot k$

**Define**  $\Omega(k) := \Lambda(k+3)$  So for  $k \in \mathbb{R}_{>1}$

$$\Omega(k) \leq 16 \cdot \Lambda(\frac{k}{2} + 3) + c \cdot (k+3) = \underbrace{16\Omega(\frac{k}{2})}_{*} + c \cdot (k+3)$$

**Claim:** For  $i \in \mathbb{N}$  with  $2^{i-1} \leq k-3$  have:

$$\Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k+3)(1+8+\dots+8^{i-1}) + 3 \cdot c \cdot (1+16+\dots+16^{i-1})$$

**Proof** by induction:

$$i = 0: \Lambda(k) = \Omega(k-3)$$

$$i \rightarrow i+1: \Lambda(k) \leq 16^i \Omega(\frac{k-3}{2^i}) + c \cdot (k-3)(1+\dots+8^{i-1}) + 3 \cdot c \cdot (1+\dots+16^{i-1}) \leq 2^i \leq k-3 \quad *$$

$$\leq 16^i (16\Omega(\frac{k-3}{2^{i+1}})) + c(\frac{k-1}{2^i} + 3) + c(k-3)\dots = \text{claimed result}$$

Take  $u \in \mathbb{N}$  minimal with  $2^u > k-3 \Rightarrow \Omega(\frac{k-3}{2^u}) \leq \Omega(\lfloor \frac{k-3}{2^u} \rfloor) = \Omega(0) =: D$  (constant)

Note:  $u$  roughly is recursion depth

$$\text{Have } 2^{u-1} \leq k-3 \xRightarrow{\text{claim}} \Lambda(k) \leq 16^u \cdot D + c \cdot \underbrace{(k-3)}_{< 2^u} \cdot \frac{8^u-1}{7} + 3c \cdot \frac{16^u-1}{15} \in O(16^u)$$

$$\text{Have } 2^{u-1} \leq k-3 \Rightarrow u \leq \lg(k-3) + 1$$

$$\Rightarrow \Lambda(k) \in O(16^{\lg(k-3)}) = O((k-3)^4)$$

$$\Rightarrow \Theta(k) = 2^{2 \cdot k} \cdot \Lambda(k) \in O(2^{2k} \cdot (k-3)^4)$$

$$\text{Have } 2^{2(k-1)} < \underbrace{n}_{\max\{l(x) \cdot l(y)\}} \Rightarrow k \leq \frac{\lg(n)}{2} + 1$$

$$\text{So } \Theta(k) \in O(n \cdot (\lg(n))^4) \quad \square$$

### 1.1.21 Theorem 21 (Schönhage-Strassen 1971)

Multiplication of integers of length  $\leq n$  can be done in  $O(n \cdot \lg(n) \cdot \lg(\lg(n)))$  bit operations. Schönhage-Strassen is used for integers of length  $\geq 100.000$ .

Asymptotically faster: Fürer's algorithm.

### Comments on Bit complexity

1. Memory requirement may explode!  
 $\Rightarrow$  No Problem as bit complexity is upper bound for memory requirements, since memory access is included in bit operations  
 $(\rightarrow$  only store what is calculated)
2. Computation of addresses in memory take time  
 $\Rightarrow$  length of addresses  $\approx \lg(\text{memory space})$  computations of addresses  $\approx \lg(\text{memory space})^2$
3. As memory requirement gets larger access times will get longer.  
 $\Rightarrow$  transportation time for data  $\geq \frac{\text{diameter of physical storage}}{2 \cdot \text{speed of light}}$

## 1.2 Division with remainder, Euclidean algorithm

### 1.2.1 Algorithm 1 (Division with remainder)

input :  $b = \sum_{i=0}^{n-1} b_i 2^i$     $a = \sum_{i=0}^{n+m-1} a_i 2^i$    with  $a_i, b_i \in \{0, 1\}$ ,    $b_{n-1} = 1$

output:  $r, q \in \mathbb{N}$    such that  $a = q \cdot b + r$ ,    $0 \leq r < b$

- (1)  $r = a$   
 $q = 0$
- (2) for  $i = m, m-1, \dots, 0$  do
- (3)    if  $r \geq 2^i \cdot b$   
           set  $r := r - 2^i \cdot b$   
            $q := q + 2^i$

### 1.2.2 Proposition 2 (Runtime of Algorithm 1)

Algorithm 1 is correct and requires  $O(n \cdot (m+1))$  bit operations.

**Proof:**

Always have  $a = q \cdot b + r$

**Claim:**

before step (3), have  $0 \leq 2^{i+1} \cdot b$

$i = m$ ;    $0 \leq r = a < 2^{m+n} = 2^{m+1} \cdot 2^{n-1} \leq 2^{m-1} \cdot b$     $i < m$  By step (3)

So after last passage through the loop  $0 \leq r < b$

**Running Time:** In step(3), have comparison and (possibly) subtraction. Only  $n$  bits involved  $\Rightarrow O(n)$

Total:  $O(b \cdot (m+1))$

**Remarks:**

- (1) Division with remainder can be reduced to multiplication.  
 Precisely: given an algorithm for multiplication that requires  $M(n)$  bit operations, there exists an algorithm for division with remainder that requires  $O(M(n))$  bit operations.
- (2) Practically relevant:  
 Jebelean's algorithm (1997):  $O(n^{\lg 3})$
- (3) Alternatively, may choose  $r \in \mathbb{Z}$    such that  $\lfloor \frac{-b}{2} \rfloor < r \leq \lfloor \frac{b}{2} \rfloor$
- (4) Algorithm 1 extends to  $\mathbb{Z}$ .
- (5) All Euclidean rings have division with remainder (by definition).  
 (e.g.,  $R = K[x] \rightarrow$  polynomial ring over field,  
 $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}, \quad i^2 = -1$ )



### 1.2.3 Algorithm 3 (Euclidean algorithm)

input :  $a, b \in \mathbb{N}$

output:  $\gcd(a, b)$  "greatest common divisor"

- (1) set  $r_0 := a$   
 $r_1 := b$
- (2) for  $i = 1, 2, 3, \dots$  perform steps (3) and (4)
- (3) if  $r_i = 0$   
 $\gcd(a, b) = |r_{i-1}|$
- (4) Division with remainder:  $r_{i-1} = q \cdot r_i + r_{i+1}$   $r_{i+1} \in \mathbb{Z}$   
 $(|r_{i+1}| \leq \frac{1}{2}|r_i|)$

**Example:**

$$a = 287, \quad b = 126$$

$$287 = 2 \cdot 126 + 35 \tag{1}$$

$$126 = 4 \cdot 35 - 14 \tag{2}$$

$$35 = (-2) \cdot (-14) + 7 \tag{3}$$

$$-14 = (-2) \cdot 7 + 0 \tag{4}$$

$$\begin{aligned} \text{So: } 7|(-14) &\xRightarrow{(3)} 7|35 \\ &\xRightarrow{(2)} 7|126 \\ &\xRightarrow{(1)} 7|287 \end{aligned}$$

On the other hand take a common divisor  $d$ ;  $d|287$ ;  $d|126$   
 $\xRightarrow{(1)} d|d \xRightarrow{(2)} d|14 \xRightarrow{(3)} d|7$

### 1.2.4 Theorem 4 (Correctness of Algorithm 3)

Algorithm 3 is correct.

**Proof:**

Since  $r_{i-1} = q \cdot r_i + r_{i+1}$  every integer  $x \in \mathbb{Z}$  satisfies the equivalence  $x|r_{i-1}$  and  $x|r_i \Leftrightarrow x|r_{i+1}$  and  $x|r_i$  so  $\gcd(r_{i-1}, r_i) = \gcd(r_i, r_{i+1}) = \gcd(a, b)$   
when terminating have  $\gcd(a, b) = \gcd(r_{i-1}, 0) = |r_{i-1}|$   $\square$

### 1.2.5 Theorem 5 (Runtime of Algorithm 3)

Algorithm 3 requires  $O(m \cdot n)$  bit operations for  $n = l(a), m = l(b)$

**Proof:**

If  $a < b$  then the first passage yields  $r_2 = a, r_1 = b$ . Cost:  $O(n)$

May assume:  $a \geq b$ . Write  $n_i = l(r_i)$

By Proposition 2  $\exists c$  constant such that the total time is  $\leq c \cdot \underbrace{\sum_{i=1}^k n_i \cdot (n_{i-1} - n_i + 1)}_{=:\sigma(n_0, \dots, n_k)}$

For  $i > 2$ :  $n_i = n_{i-1} - 1$

Special Case:  $n_i = n_{i-1} - 1$  for  $i \geq 2$

$\Rightarrow n_i = n_i - i + 1, n_i = m, k = m + 1$

Obtain  $\sigma(n_0, \dots, n_k) = m \cdot (n - m + 1) + \sum_{i=2}^{m+1} (m - i + 1) \cdot 2 = m \cdot n - m^2 + m + m(m - 1) = m \cdot n$ .

**Claim:** The special case is the worst (most expensive)!

From any sequence  $n_1 > n_2 > \dots > n_k$  get to the special case by iteratively inserting numbers in the gaps. Insert  $s$  with  $n_{j-1} > s > n_j$ .

$\sigma(n_0, \dots, n_{j-1}, s, n_j, \dots, n_k) - \sigma(n_0, \dots, n_k) = \dots = s + (n_{j-1} - s) \cdot (s - n_j)$

$sp\sigma(n_0, \dots, n_k) \leq \sigma(n, m, m - 1, \dots, 2, 1, 0) = n \cdot m \quad \square$

Complexity is quadratic  $\rightarrow$  cheap

### 1.2.6 Algorithm 6 (Extended Euclidean Algorithm)

input :  $a, b \in \mathbb{N}$

output:  $d = \gcd(a, b)$  and  $s, t \in \mathbb{Z}$  such that  $d = s \cdot a + t \cdot b$

(1)  $r_0 := a, r_1 := b, s_0 := 1, t_0 := 0, s_1 := 0, t_1 = 1$

(2) for  $i = 1, 2, \dots$  perform steps (3) - (5)

(3) if  $r_i = 0$   
     set  $d = |r_{i-1}|$   
      $s := \text{sgn}(r_{i-1}) \cdot s_{i-1},$   
      $t := \text{sgn}(r_{i-1}) \cdot t_{i-1}$

(4) division with remainder:  
 $r_{i+1} = r_{i-1} - q_i \cdot r_i, \quad \text{with } |r_{i+1}| \leq \frac{1}{2}|r_i|$

(5) set  $s_{i+1} := s_{i-1} - q_i \cdot s_i,$   
 $t_{i+1} := t_{i-1} - q_i \cdot t_i$

Justification :  $r_i = s_i \cdot a + t_i \cdot b$  throughout

**Application:**  $m, x \in \mathbb{N}$  such that  $m, x$  co-prime (i.e.  $\gcd(x, m) = 1$ )

Algorithm 6 yields:  $1 = s \cdot x + t \cdot m \Rightarrow s \cdot x \equiv 1 \pmod{m}$

So obtain inverse of  $\bar{x} \in \mathbb{Z}/(m)$

### 1.3 Primality testing

Let  $\mathbb{P} \subseteq \mathbb{N}$  be the set of prime numbers.

Challenge: Given  $n \in \mathbb{N}$  decide if  $n \in \mathbb{P}$

**Naive Method:** Trivial division by  $m \leq \lfloor \sqrt{n} \rfloor$ .

Running time is exponential in  $l(n)$ . Even when restricted to division by prime numbers, need approximately  $\frac{\sqrt{n}}{|n|^{1/\sqrt{n}}}$  trivial divisions (prime number theorem)  
 $\rightarrow$  hardly any better!

**Reminder:** (arithmetic modulo  $m$ )

$G$  finite group  $\Rightarrow \forall a \in G \quad a^{|G|} = 1$  Fermat's little theorem

For  $G = (\mathbb{Z}/(p))^\times \quad a^{p-1} \equiv 1 \pmod{p} \in \mathbb{P} \quad \forall a \in \mathbb{Z} \quad \text{with } p \nmid a$

In fact  $(\mathbb{Z}/(p))^\times \cong Z_{p-1}$  is cyclic

For  $m = p_1^{e_1} \dots p_r^{e_r}$  with  $p_i \in \mathbb{P}, e_i \in \mathbb{N}_{>0}$ :

$\mathbb{Z}_{(m)} \cong \mathbb{Z}_{(p_1^{e_1})} \oplus \dots \oplus \mathbb{Z}_{(p_r^{e_r})} \Rightarrow \mathbb{Z}_{(m)}^\times \cong \mathbb{Z}_{(p_1^{e_1})}^\times \times \dots \times \mathbb{Z}_{(p_r^{e_r})}^\times$

what is  $\mathbb{Z}_{(p^e)}$  for  $p \in \mathbb{P}, e \in \mathbb{N}_{>0}$ ?

#### 1.3.1 Theorem 1 (Cyclic group)

Let  $p \in \mathbb{P}$  odd  $e \in \mathbb{N}_{>0} \Rightarrow (\mathbb{Z}_{(p^e)})^\times = Z_{(p-1) \cdot p^{e-1}}$  cyclic

**Proof:**

$(\mathbb{Z}_{(p^e)})^\times \cong Z_{p-1} \Rightarrow \exists z \in \mathbb{Z} : \text{order}(z + p\mathbb{Z}) = p-1$

Set  $a = \bar{z}^{p^{e-1}} \in (\mathbb{Z}_{(p^e)})^\times =: G$

$$a^{p-1} = \bar{z}^{(p-1) \cdot p^{e-1}} = \bar{z}^{|a|} = 1$$

On the other hand, take  $i \in \mathbb{Z}$  such that

$$a^i = 1 \Rightarrow \bar{z}^{i \cdot p^{e-1}} \equiv 1 \pmod{p} \Rightarrow (p-1) \mid (i \cdot p^{e-1}) \Rightarrow (p-1) \mid i.$$

So  $\text{ord}(a) = p-1$ .

Now consider  $b = (p+1) \in G$

**Claim:**  $\text{ord}(b) = p^{e-1}$

**Proof** by induction on  $k \in \mathbb{N}_{>0}$  that  $(p+1)^{p^{k-1}} \equiv p^k + 1 \pmod{p^{k+1}}$

$k=1$  ✓

$k \rightarrow k+1$ : By induction have  $(p+1)^{p^{k-1}} = 1 + p^k + x \cdot p^{k+1}, \quad x \in \mathbb{Z}$

$$\text{Compute: } (p+1)^{p^k} = ((1+p^k) + x \cdot p^{k+1})^p = \sum_{i=0}^p \binom{p}{i} (i+p^k)^{p-i} \cdot x^i \cdot p^{i \cdot (k+1)}$$

$$\stackrel{\text{Only 0-th summand}}{\equiv} (i+p^k) = \sum_{i=0}^p \binom{p}{i} p^{i \cdot k} \stackrel{p \text{ odd}}{\equiv} 1 + p^{k+1} \pmod{p^{k+2}} \quad \checkmark$$

For  $k=e$ :  $(p+1)^{p^{e-1}} \equiv 1 \pmod{p^e} \Rightarrow b^{p^e} = 1 \Rightarrow \text{ord}(b) \mid p^{e-1}$

But  $(p+1)^{p^{e-2}} \equiv p^{e-1} + 1 \pmod{p^e} \Rightarrow b^{p^{e-2}} \neq 1 \in G$

So  $\text{ord}(b) = p^{e-1}$

**Claim:**  $\text{ord}(a \cdot b) = (p-1)p^{e-1} \quad (\Rightarrow \text{Theorem})$

Let  $(a \cdot b)^i = 1 \in G$  with  $i \in \mathbb{Z}$

$$\text{Then } 1 = (a \cdot b)^{i \cdot (p-1)} = (a^{p-1})^i \cdot b^{i \cdot (p-1)} = b^{i \cdot (p-1)} \Rightarrow p^{e-1} \mid i \cdot i(p-1) \Rightarrow p^{e-1} \mid i$$

$$\text{Also } 1 = (a \cdot b)^{p^{e-1} \cdot i} = a^{p^{e-1} \cdot i} \Rightarrow (p-1) \mid p^{e-1} \cdot i \Rightarrow (p-1) \mid i \rightarrow (p-1) \cdot p^{e-1} \mid i \quad \square$$

**Reminder:**  $(\mathbb{Z}/(2^e))^\times \cong Z_2 \times Z_2^{e-2} \quad (e \geq 2)$

### 1.3.2 Algorithm 2 (Fermat Test)

input :  $n \in \mathbb{N}_{>0 \text{ odd}}$

output: " $n \notin \mathbb{P}$ " or "probably  $n \in \mathbb{P}$ "

- (1) Choose  $a \in 2, \dots, n-1$  randomly
- (2) Compute  $a^{n-1} \bmod n$
- (3) If  $a^{n-1} \not\equiv 1 \pmod{n}$  return " $n \notin \mathbb{P}$ "  
else return "probably  $n \in \mathbb{P}$ "

Not very satisfying. Is this fast?

### 1.3.3 Algorithm 3 (Fast exponentiation)

input :  $a \in G$   $G$  is a monoid,  $e \in \mathbb{N}$   $e = \sum_{i=0}^{n-1} e_i 2^i$   $e_i \in \{0, 1\}$

output:  $a^e \in G$

- (1) Set  $b := a, y := 1$
- (2) For  $i = 0, \dots, n-1$  perform (3) - (4)
- (3) if  $e_i = 1$  set  $y := y \cdot b$
- (4) set  $b := b^2$
- (5) return  $y$

this requires  $O(l(e))$  operations in  $G$

For  $G = (\mathbb{Z}/(n)_i)$ , each multiplication requires  $O(l(n)^2)$  bit operations

$\Rightarrow$  Fermat test requires  $O(l(n)^3)$  bit operations  $\rightarrow$  cubic complexity  $\rightarrow$  "fast"!

**Example:**

$n = 561 = 3 \cdot 11 \cdot 17$  For  $a \in \mathbb{Z}$  with  $\gcd(a, n) = 1 \Rightarrow$  have  $a^{n-1} = (a^2)^{280} \equiv 1 \pmod{3}$

$a^{n-1} \equiv 1 \pmod{n}$  Fermat's test says "probably  $n \in \mathbb{P}$ " in 57% of cases.

$n = 2207 \cdot 6619 \cdot 15443$  : output "probably  $n \in \mathbb{P}$ " in 99,93% of cases.

### 1.3.4 Definition 4 (Pseudo-prime, witness, Carmichael numbers)

Let  $n \in N_{>1} \text{ odd}$ ,  $a \in 1, \dots, n-1$

- (a)  $n$  is pseudo-prime to base  $a$  if  $a^{n-1} \equiv 1 \pmod{n}$
- (b) otherwise  $a$  is called a witness of composition of  $n$
- (c) If  $n \notin \mathbb{P}$  but  $a^{n-1} \equiv 1 \pmod{n} \quad \forall a$  with  $\gcd(n, a) = 1$   
then  $n$  is called a Carmichael number.  
There are  $\infty$  Carmichael numbers

### 1.3.5 Proposition 5 (Number of witnesses)

Let  $n \in N_{>1}$ ,  $\text{odd} \wedge \notin \mathbb{P} \wedge \text{not Carmichael}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n, a \text{ is witness of composite of } n\}| > \frac{n-1}{2}$

**Proof:** Consider

$\phi : (\mathbb{Z}/(n))^{\times} =: G \rightarrow G, \quad \bar{a} \mapsto \bar{a}^{n-1}$

group homomorphism. By assumption,

$|\text{Im}(\phi)| > 1 \Rightarrow |\text{Ker}(\phi)| \leq \frac{|G|}{2} < \frac{n-1}{2}$

$\Rightarrow |\{a \in \mathbb{Z} \mid 0 < a < n \text{ a witness of composite of } n\}| > \frac{n-1}{2} \quad \square$

### Miller-Rabin Test

### 1.3.6 Proposition 6 (Inference from Fermat)

Let  $p \in \mathbb{P} \text{ odd}$ ,  $a \in \{1, \dots, (p-1)\}$  write  $p-1 = 2^k \cdot m$  with  $m \text{ odd}$  Then:

$a^m \equiv 1 \pmod{p}$  or  $\exists i \in \{0, \dots, k-1\} : a^{2^i \cdot m} \equiv -1 \pmod{p}$

**Proof:**

Little Fermat:  $\bar{a}^{2^k \cdot m} = 1 \in \mathbb{F}_p$

Assume  $\bar{a}^m \neq 1$  take  $i$  maximal such that:

$\bar{b} = \bar{a}^{2^i \cdot m} \neq 1 \Rightarrow \bar{b}^2 = 1 \Rightarrow \bar{b} \in \mathbb{F}_p$  is a zero of  $x^2 - 1 \in \mathbb{F}_p[x] \Rightarrow \bar{b} = -1$

### 1.3.7 Algorithm 7 (Miller-Rabin-test)

input :  $n \in \mathbb{N}_{>1}, \text{odd}$

output: either " $n \notin \mathbb{P}$ " or "probably  $n \in \mathbb{P}$ "  $\rightarrow$  Monte Carlo Algorithm.

- (1) write  $n - 1 = 2^k \cdot m$  with  $m$  odd
- (2) Choose  $a \in \{2, \dots, n - 1\}$  randomly
- (3) Compute  $b := a^m \pmod n$
- (4) if  $(b \equiv \pm 1 \pmod n)$   
return "probably  $n \in \mathbb{P}$ "
- (5) for  $(i = 0, \dots, k - 1)$  do steps (6) - (7)
- (6) set  $b := b^2 \pmod n$
- (7) if  $(b \equiv -1 \pmod n)$   
return "probably  $n \in \mathbb{P}$ "
- (8) return  $n \notin \mathbb{P}$

### 1.3.8 Definition 8 (strong pseudo-prime / witness)

Let  $n \in \mathbb{N}_{>1}, \text{odd}$   $a \in \{1, \dots, n - 1\}$

- (a)  $n$  is called a strongly pseudo-prime to base  $a$  if Proposition 6 holds for  $a$  and  $p$  replaced by  $n$ .
- (b) Otherwise  $a$  is called a strong witness of composition of  $n$ .

#### Example

Let  $n \in \mathbb{N}_{>1}, \mathbb{P} \text{ odd}$

$a = 2$  strong witness if  $n < 2047$  (including 561)

2 or 3 strong witness if  $n < 1373653$

2,3 or 5 strong witness if  $n < 25326001$

### 1.3.9 Theorem 9 (Bit-complexity of Algorithm 7)

- (a) Algorithm 7 requires  $O(l(n)^3)$  bit operations.  $\rightarrow$  "qubic complecity"  $\rightarrow$  fast!
- (b) if  $b \in \mathbb{P}$  then Algorithm 7 returns "probably  $b \in \mathbb{P}$ "  $\rightarrow$  no false positives.
- (c) if  $n \notin \mathbb{P}$  then more than half of the numbers in  $\{1, \dots, n - 1\}$  are strong witnesses.

**Proof:**

- (a) Step 1 takes  $O(l(n))$  bit operations:  
Using Algorithm 3, we need  $O(l(n-1))$  multiplications in  $\mathbb{Z}/(n)$  each requiring  $O(l(n)^2)$  bit operations.
- (b) Proposition 6
- (c) split in three cases:

**Case 1:**  $n$  is not a Carmichael number.  $\xRightarrow{\text{Prop 5}}$  more than half of all numbers are.

Fermat witness thus also strong witness.

**Case 2:**  $n = p^r \cdot l$  with  $p \in \mathbb{P}$   $r > 1$   $l \in \mathbb{N}_{>0}$   $p \nmid l$

Theorem 1  $\exists x \in \mathbb{Z}$  such that  $x^p \equiv 1 \pmod{p^r}$   $x \not\equiv 1 \pmod{p^r}$

Chinese remainder theorem:  $\exists a \in \mathbb{Z}$  such that  $a \equiv x \pmod{p^r}$   $a \equiv 1 \pmod{l}$

So  $\bar{a}^p = 1 \in \mathbb{Z}/(n) \Rightarrow \bar{a}^n = 1 \Rightarrow \bar{a} \in (\mathbb{Z}/(n))^\times$

i.e.  $\gcd(n, a) = 1$  if  $\bar{a}^{n-1} = 1$  then  $\bar{a} = 1$

But  $a \equiv x \not\equiv 1 \pmod{p^r}$  so  $\bar{a}^{n-1} \neq 1$  hence  $n$  is not Carmichael  $\rightarrow$  Case 1.

**Case 3:**  $n$  is a Carmichael number. By Case 2 have  $n = p \cdot l$  with  $p \in \mathbb{P}$   $p \nmid l$   $l \geq 3$   
 $n$  Carmichael:  $\forall a \in \mathbb{Z}$  with  $\gcd(a, n) = 1$

have  $a^{2^k \cdot m} \equiv 1 \pmod{n}$  (where  $n-1 = 2^k \cdot m$ )

$a^{2^k \cdot m} \equiv 1 \pmod{p}$  Take  $j$  minimal such that

$a^{2^j \cdot m} \equiv 1 \pmod{p}$   $\forall a \in \mathbb{Z}$  such that  $\gcd(a, n) = 1$

so  $0 \leq j \leq l$  in fact,  $j > 0$  since  $(-1)^{2^0 \cdot m} = -1$  with  $m$  odd.

Consider the subgroup  $H := \{\bar{a} \in \mathbb{Z}/(n) \mid \bar{a}^{2^{j-1} \cdot m} \in \{1, -1\} \subseteq (\mathbb{Z}/(n))^\times\}$

Let  $a \in \{1, \dots, n-1\}$   $\gcd(n, a) = 1$   $a$  not a strong witness.

**Claim 1:**  $\bar{a} \in H$

**Case 3.1:**  $\bar{a}^{2^{j-1} \cdot m} = 1 \Rightarrow \bar{a} \in H$

**Case 3.2:**  $a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{n}$   $a^m \not\equiv 1 \pmod{n}$

$\xRightarrow{\text{a nonwitness}}$   $\exists i$  such that  $\underbrace{a^{2^i \cdot m} \equiv -1 \pmod{n}}_*$

$\Rightarrow a^{2^i \cdot m} \equiv -1 \pmod{p} \xRightarrow{\text{def of } j} i < j$

if  $i < j-1$  then  $a^{2^{j-1} \cdot m} = (a^{2^i \cdot m})^{2^{j-1-i}} \equiv (-1)^{2^{j-1-i}} = 1 \pmod{n}$

$\xRightarrow{\text{with } *}$  not in case 3.2

**Claim 2:**  $H \subseteq (\mathbb{Z}/(n))^\times$  proper subgroup.

By definition of  $j$   $\exists x \in \mathbb{Z}$  such that  $x^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p}$

Chinese remainder:  $\exists a \in \mathbb{Z}$  such that

$a \equiv x \pmod{p}$   $a \equiv 1 \pmod{l}$

$\Rightarrow a^{2^{j-1} \cdot m} \not\equiv 1 \pmod{p} \equiv 1 \pmod{l} \Rightarrow \bar{a} \notin H$

Claim 2  $\checkmark$

It follows that  $|H| \leq \frac{|(\mathbb{Z}/(n))^\times|}{2} < \frac{n-1}{2}$

so the number of witnesses is  $\geq n-1 - |H| > \frac{n-1}{2}$   $\square$

### Remarks:

- (a) A more careful analysis shows that  $2\frac{3}{4}$  of all candidates are strong witnesses
- (b) Calling Algorithm 7 repeatedly decreases the probability of false positives. Running time for prescribed error probability  $p$  is  $O(\lg(p^{-1} \cdot l(n)^3))$   
(Independence assumptions!)

### Connection with Riemann hypothesis

Let  $n \in \mathbb{N}_{>0}$   $\bar{X} : (\mathbb{Z}/(n))^\times \rightarrow \mathbb{C}^\times$  group homomorphism

$$X : \mathbb{Z} \rightarrow \mathbb{C}, a \mapsto \begin{cases} \bar{X}(\bar{a}) & \text{if } \gcd(a, n) = 1 \text{ for } (\bar{a} = a + n\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$$

"residue class character (mod  $n$ )

$$Ex : n = 1 \Rightarrow X(a) = 1 \forall a \in \mathbb{Z}$$

Dirichlet L-series:

$$L_X(s) = \sum_{n=1}^{\infty} \frac{X(n)}{n^s} \text{ converges for } s \in \mathbb{C} \text{ until } Re(s) > 1$$

$L_X(s)$  extends to a meromorphic function on  $\mathbb{C} \mapsto$  "Dirichlet L-function".

For  $n = 1 : L_X(s) = \zeta(s)$  Riemann Zeta-function.

Euler Product:

$$\text{From } (1 - X(p) \cdot p^{-s})^{-1} = \sum_{i=0}^{\infty} (X(p) \cdot p^{-s})^i = \sum_{i=0}^{\infty} \frac{X(p^i)}{p^{is}} \quad \text{derive } L_X(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - X(p) \cdot p^{-s}}$$

Generalized Riemann hypothesis (GRH):

For  $X$  residue class character,  $s \in \mathbb{C}$

with  $L_X(s) = 0$ ,  $0 < Re(s) < 1$  ("critical strip")

then  $Re(s) = \frac{1}{2}$

For  $X = 1 \rightarrow$  ordinary Riemann hypothesis.

### 1.3.10 Theorem (Ankeny & Bach)

GRH  $\Rightarrow \forall X \neq 1$  residue class character

$$\exists p \in \mathbb{P} : X(p) \neq 1, p < 2 \ln(n)^2$$

Let  $H \subsetneq (\mathbb{Z}/(n))^\times =: G$  proper subgroup.

Choose  $N \subsetneq G$  maximal proper subgroup such that  $H \subseteq N \Rightarrow G/N$  cyclic.

$$\bar{X} : G \mapsto \mathbb{C}^\times \text{ with } N = \text{Ker}(\bar{X}) \Rightarrow H \subseteq \text{Ker}(\bar{X})$$

$$\xrightarrow{\text{GRH, Thm1}} \exists p \in \mathbb{P} : p + n\mathbb{Z} \not\subseteq H, p < 2 \cdot \ln(n)^2$$

**Corollary:** Assume GRH.

Let  $n \in \mathbb{N}_{>1}$   $\mathbb{P}$  odd Then there is a strong witness  $a$  of compositeness of  $n$  with  $a < 2 \cdot \ln(n)^2$ .

$\rightarrow$  Obtain deterministic primality test with time  $O(\ln(n)^5)$  bit operations.

### AKS-test

A deterministic polynomial time primality test  $\rightarrow$  "holy grail"

Agrawal, Kayal, Saxena: PRIMES is in P, Annals of Mathematics, 2004.



### 1.3.11 Proposition 10 (Modulo over ideals)

Let  $n \in \mathbb{P}$   $a \in \mathbb{Z} \Rightarrow (x + a)^n \equiv x^n + a \pmod{n}$

where  $x$  is a indeterminate and for  $r \in \mathbb{N}$ :

$$(x + a)^n \equiv (x^n + a) \pmod{n, x^r - 1} \quad (1)$$

(i.e.  $(x + a)^n - (x^n + a) = n \cdot f + (x^r - 1) \cdot g$  with  $f, g \in \mathbb{Z}[x]$ )

**Proof:**

$$(x + a)^n = \sum_{i=0}^n \binom{n}{i} \cdot a^{n-i} \cdot x^i \quad (\text{where } \binom{n}{i} \text{ is a multiple of } n \text{ for } 0 < i < n)$$

$$\equiv x^n + a^n \quad (\leftarrow \text{little Fermat})$$

$$\equiv x^n + a \quad (1) \text{ follows by weakening this.}$$

**Cost** analysis for checking (1) with  $l = \text{length}(n)$ .

Using Algorithm 3, need  $O(l)$  multiplications in  $\mathbb{Z}[x]/(n, x^r - 1) =: R$

Elements of  $R$  are represented as polynomials of degree  $< r$ ,  
coefficients between 0 and  $n$ .

Multiply polynomials:  $O(r^2)$  operation in  $\mathbb{Z}/(n) : O(r^2 \cdot l^2)$

since  $x^{r+k} \equiv x^k \pmod{x^r - 1}$ ,

add coefficients of  $x^{r+k}$  of product polynomial to coefficients  $x^k : O(r \cdot l)$

Total for checking (1):  $O(r^2 \cdot l^3)$  bit operations.

Reduction  $\pmod{x^r - 1}$  is just for keeping the cost under control.

The following is part of AKS-test:

### 1.3.12 Algorithm 11 (Test for perfect power)

input :  $n \in \mathbb{N}_{>1}$

output:  $m, e \in \mathbb{N}$   $e > 1$  such that  $n = m^e$  or "n is not a perfect power"

(1) for  $(e = 2, \dots, \lfloor \lg(n) \rfloor)$  perform (2) - (7) //possible exponents

(2) set  $m_1 = 2, m_2 = n$  //initialize interval  $[m_1, m_2]$  for searching  $\sqrt[e]{n}$

(3) while( $m_1 \leq m_2$ ) do (4) - (7)

(4) set  $m = \lfloor \frac{m_1 + m_2}{2} \rfloor$  // bisect interval

(5) if  $m^e = n$  return  $m, e$

(6) if  $m^e > n$  set  $m_2 = m - 1$

(7) if  $m^e < n$  set  $m_1 = m + 1$

(8) return "not a perfect power"

**Cost:** (for  $l = \text{length}(n)$ )

Compute  $m^e : O(\lg(l) \cdot l^2)$  (abort computation once the result exceeds  $n$ )

Number of passages through inner loops  $\leq \lg(n)$

Number of passages through outer loops  $\leq \lg(n)$

Total cost of Algorithm 11:  $O(l^4 \cdot \lg(l))$

### 1.3.13 Algorithm 12 (AKS-test)

input :  $n \in \mathbb{N}_{>1}$  of length  $l = \text{length}(n) = \lfloor \lg(n) \rfloor + 1$

output: " $n \in \mathbb{P}$ " or " $n \notin \mathbb{P}$ "

- (1) if ( $n$  is a perfect power)  
return " $n \notin \mathbb{P}$ "
- (2) find  $r \in \mathbb{N}_{>1}$  minimal such that  $r|n \vee n^i \not\equiv 1 \pmod{r} \quad \forall i = 1, \dots, l^2$   
//exhaustive search (we will show that  $r \leq l^5$ )
- (3) if  $r|n$   
if ( $r = n$ ) return " $n \in \mathbb{P}$ "  
if ( $r < n$ ) return " $n \notin \mathbb{P}$ "
- (4) for  $a = 1, 2, \dots, \lfloor \sqrt{r} \cdot l \rfloor$  do (5)
- (5) if  $((x+a)^n \not\equiv (x^n + a) \pmod{(n, x^r - 1)})$   
return " $n \notin \mathbb{P}$ "
- (6) return " $n \in \mathbb{P}$ "

### 1.3.14 Lemma 13 (Least common multiple)

For  $n \in \mathbb{N}_{>0}$  have  $\lambda(n) := \text{lcm}(1, 2, \dots, n) \geq 2^{n-2}$

**Proof:** For  $f = \sum_{i=0}^m a \cdot x^i \in \mathbb{Z}(x) \quad a_i \in \mathbb{Z}$

$$\Rightarrow \int_0^1 f(x) dx = \sum_{i=0}^m \frac{a_i}{i+1} = \frac{k}{\lambda(m+1)}$$

with  $k \in \mathbb{Z}$ . Consider  $f_m = x^m \cdot (1-x)^m$

For  $0 < x < 1$ :

$$0 < f_m(x) \leq 4^{-m}$$

$$\Rightarrow 0 < \int_0^1 \underbrace{f_m(x)}_{\frac{k_m}{\lambda(2m+1)}} dx \leq 4^{-1}$$

$$\lambda(2 \cdot m + 1) \geq k_m \cdot 4^m \geq 4^m$$

$$\text{For } n \in \mathbb{N}_{>0} \lambda(n) \geq \lambda(2 \cdot \lfloor \frac{n-1}{2} \rfloor + 1) \geq 4^{\lfloor \frac{n-1}{2} \rfloor} \geq 4^{\frac{n-1}{2}} = 2^{n-2} \quad \square$$

**Corollary:** (not related to AKS)

For  $n \in \mathbb{N}$

$$\pi(n) := |\{p \in \mathbb{P} | p \leq n\}| \geq \frac{n-2}{\lg(n)}$$

**Proof:**

$$2^{n-2} \leq \lambda(n) = \prod_{p \in \mathbb{P}, p \leq n} p^{\lfloor \log_p(n) \rfloor} \leq \prod_{p \leq n} p^{\log_p(n)} = n^{\pi(n)} = 2^{\lg(n)\pi(n)} \quad \square$$

**Prime number theorem:**

$$\lim_{n \rightarrow \infty} \frac{\pi(n)}{n/\ln(n)} = 1$$

**Interpretation:**

The average distance of two primes around some value  $x \in \mathbb{R}_{>1}$  is  $\ln(x)$

### 1.3.15 Lemma 14 (Property of $r$ in Algorithm 12)

In Algorithm 12, have  $r \leq l^5$

**Proof:**

if  $r < l^5 \Rightarrow \forall k \in \{2, \dots, l^5\} : \exists i \in \{1, \dots, l^2\}$

$$n^i \equiv 1 \pmod{k}$$

$$\Rightarrow k \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\Rightarrow \lambda(l^5) \mid \prod_{i=1}^{l^2} (n^i - 1)$$

$$\xrightarrow[\text{Lemma 13}]{=} 2^{l^5-2} < \prod_{i=1}^{l^2} n^i = n^{\frac{l^2(l^2+1)}{2}}$$

$$\Rightarrow l^5 - l^3 < 4 \quad \text{not true since } l \geq 2 \quad \square$$

### 1.3.16 Theorem 15 (Bit-Complexity of Algorithm 12)

Algorithm 12 requires  $O(l^{16.5})$  bit operations ("polynomial complexity")

**Proof:**

Step(1):  $O(l^4 \cdot \lg(l)) \checkmark$

Step(2): For each  $r$  need:

- test  $r \mid n : O(l^2)$
- compute all  $n^i \pmod{r} : O(l^2 \cdot \lg(r)^2) \underset{\text{Lemma 14}}{\leq} O(l^2 \cdot \lg(l)^2)$

Step(3):  $O(1)$

$$\text{Step(4): } O(\sqrt{r} \cdot l \cdot r^2 \cdot l^3) \underset{\text{Lemma 14}}{\leq} O(l^{16.5}) \quad \square$$

**Reminder:** There is a variant of Algorithm 12 with running time  $\tilde{O}(l^6)$ , i.e.,  $O(l^6 \cdot \lg(l)^m)$  with  $m \in \mathbb{N}$ .

**Correctness:**

For  $r \in \mathbb{N}_{>0}$  and  $p \in \mathbb{P}$  write  $I(r, p) := \{m, f\} \in \mathbb{N} \times \mathbb{F}_p[x] \mid f(x)^m \equiv f(x^m) \pmod{x^r - 1}\}$   
 "m is introspective for  $f$  and  $r$ ".

**Example:** Proposition 10 says that:

$$(p, x + \bar{a}) \in I(r, p) \text{ for } a \in \mathbb{Z} \quad r \in \mathbb{N}_{>0} \quad p \in \mathbb{P} \quad (1)$$

### 1.3.17 Lemma 16 (Rules for ideals)

- (a)  $(m, f), (m', f) \in I(r, p) \Rightarrow (m \cdot m', f) \in I(r, p)$
- (b)  $(m, f), (m, g) \in I(r, p) \Rightarrow (m, f \cdot g) \in I(r, p)$
- (c)  $(m \cdot p, f) \in I(r, p), p \nmid r \Rightarrow (m, f) \in I(r, p)$

**Proof:**

- (a)  $f(x)^{m \cdot m'} \equiv f(x^m)^{m'} \pmod{(x^r - 1)}$   
 $f(x^m)^{m'} \equiv f(x^{m \cdot r}) \pmod{(x^{m \cdot r} - 1)}$   
 But  $(x^r - 1) \mid (x^{m \cdot r} - 1)$
- (b)  $(f \cdot g)(x)^m = f(x)^m \cdot g(x)^m \equiv f(x^m) \cdot g(x^m) = (f \cdot g)(x^m) \pmod{(x^r - 1)}$
- (c)  $(f(x)^m)^p \equiv f((x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x^m))^p \pmod{(x^r - 1)}$   
 $\Rightarrow (x^r - 1) \mid ((f(x)^m)^p - f(x^m)^p) \stackrel{\text{Frobenius homomorphism}}{\equiv} (f(x)^m - f(x^m))^p$   
 $p \nmid r \Rightarrow x^r - 1$  is square free. So  
 $(x^r - 1) \mid (f(x)^m - f(x^m)) \Rightarrow (m, f) \in I(r, p) \quad \square$

### 1.3.18 Theorem 17 (Correctness of Algorithm 12)

Algorithm 12 is correct.

**Proof:**

If the algorithm terminates in step(1),(3) or (5), it is correct. To show: If it terminates in step(6) it is correct, i. e.  $n \in \mathbb{P}$

**Claim 1:**  $\exists p \in \mathbb{P} : p \mid n \quad p \not\equiv 1 \pmod{r} \quad p > r$

Indeed if all prime divisors of  $n$  were  $\equiv 1 \pmod{r}$  then  $n \equiv 1 \pmod{r}$

Contradiction to step(2). All prime divisors of  $n$  are  $> r$  by step (2) and (3) ✓

Steps(2) and (3) imply that  $\gcd(n, r) = 1 \Rightarrow G := \langle \bar{n}, \underbrace{\bar{p}}_{p \bmod r} \rangle \subseteq (\mathbb{Z}/(r))^\times$

Step(2):  $\text{ord}(\bar{n}) > l^2 \Rightarrow l^2 < |G| < r$  (2)

Set  $s := \text{ord}(\bar{p} \in G) \Rightarrow r \mid (p^s - 1)$  with  $q := p^s \Rightarrow r \mid |\mathbb{F}_q^\times| \Rightarrow \exists \zeta \in \mathbb{F}_q$   $r$ -th root of 1

Set  $k := \lfloor \sqrt{r} \cdot l \rfloor \quad m := \left(\frac{n}{p}\right)$

By (1)  $(p, x + \bar{a}) \in I(r, p)$  with  $\bar{a} \in \mathbb{F}_p$

By step(4), have  $(n, x + \bar{a}) \in I(r, p)$

For  $\underline{e} = e_0, \dots, e_k \in \mathbb{N}_0$  set  $f_{\underline{e}} := \prod_{a=0}^k (x + \bar{a})^{e_a}$

Lemma 16 (b):  $(p, f_{\underline{e}}) \in I(r, p)$

$(n, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(c)}} (m, f_{\underline{e}}) \in I(r, p)$

$\xRightarrow{\text{Lemma 16(a)}} \forall s, t \in \mathbb{N}_0 : (p^s \cdot m^t, f_{\underline{e}}) \in I(r, p)$

$\Rightarrow f_{\underline{e}}(\zeta^{p^s \cdot m^t}) = f_{\underline{e}}(\zeta)^{p^s \cdot m^t}$  (3)

Set  $H := \langle \zeta + \bar{a} | a \in \{0, \dots, k\} \rangle \subseteq \mathbb{F}_q^\times$   
 $(\zeta \notin \mathbb{F}_p \text{ since } r \nmid (p-1) \text{ by Claim 1})$

Consider:  $T := \{(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1} \mid \sum_{a=0}^k e_a < |G|\}$

$\Phi : T \mapsto H, (e_0, \dots, e_k) \mapsto f_{\underline{e}}(\zeta) = \prod_a (\zeta + \bar{a})^{e_a} \in H$

**Claim 2:**  $\Phi$  is injective.

Indeed, take  $(\underline{e}), (\underline{\hat{e}}) \in T$  such that  $\Phi(\underline{e}) = \Phi(\underline{\hat{e}})$

$$\Rightarrow \forall s, t \in \mathbb{N}_0 : f_{\underline{e}}(\zeta^{p^s \cdot m^t}) \stackrel{(3)}{=} f_{\underline{e}}(\zeta)^{p^s \cdot m^t} = f_{\underline{\hat{e}}}(\zeta)^{p^s \cdot m^t} \stackrel{(3)}{=} f_{\underline{\hat{e}}}(\zeta^{p^s \cdot m^t})$$

$f_{\underline{e}} - f_{\underline{\hat{e}}}$  has roots  $\zeta^e$  with  $e \in G$  since  $G = \langle \bar{p}, \bar{m} \rangle$

These are all distinct (since  $\zeta$  is primitive)

But  $\deg(f_{\underline{e}} - f_{\underline{\hat{e}}}) < |G|$  So  $f_{\underline{e}} - f_{\underline{\hat{e}}} = 0$

Since  $k \leq \sqrt{r} \cdot l < r < p$  the  $(x + \bar{a})$  with  $a \in \{0 \dots k\}$  are primitive distinct.

So  $(\underline{e}) = (\underline{\hat{e}})$  ✓

So is  $|H| \geq |T|$  ?

Let  $M$  be the set of all  $\{x_0, \dots, x_k\} \subseteq \{1, \dots, |G| + k\}$

with  $x_0 < x_1 < \dots < x_k$

For  $\{x_0, \dots, x_k\} \in M$  define  $(e_0, \dots, e_k) \in \mathbb{N}_0^{k+1}$  by  $e_a = x_a - x_{a-1}$  with  $x_{-1} := 0$

$$\Rightarrow \sum_{a=0}^k e_a = \sum_{a=0}^k (x_a - x_{a-1} - 1) = x_k - (k+1) < |G|$$

Obtain injection  $M \Leftrightarrow T$

$$\text{So } |H| \geq |T| \geq |M| = \binom{|G|+k}{k+1} \stackrel{(2)}{\geq} \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{k+1} = \binom{\lfloor l\sqrt{|a|} \rfloor + 1 + k}{\lfloor l\sqrt{|a|} \rfloor} \stackrel{(2)}{\geq} \binom{2 \cdot \lfloor l\sqrt{|a|} \rfloor + 1}{\lfloor l\sqrt{|a|} \rfloor}$$

### 1.3.19 Lemma 18 (Property of binomial coefficients)

$$\forall n \in \mathbb{N}_{>1} : \binom{2 \cdot n + 1}{n} > 2^{n+1}$$

**Proof:**

$n = 2 :$

$$\binom{5}{2} = 10 > 2^3$$

$n - 1 \rightarrow n :$

$$\binom{2 \cdot n + 1}{n} = \binom{2 \cdot n}{n-1} + \binom{2 \cdot n}{n} = \binom{2 \cdot n - 1}{n-2} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n-1} + \binom{2 \cdot n - 1}{n} \geq 2 \cdot \binom{2 \cdot n - 1}{n-1} \stackrel{ind.}{>} 2 \cdot 2^n = 2^{n+1}$$

### Continuation of Proof of Theorem 17

$$|H| > 2^{\lfloor l \cdot \sqrt{|a|} \rfloor + 1} \geq 2^{l \cdot \sqrt{|a|}} \geq 2^{\lg(n) \cdot \sqrt{|a|}} = n \sqrt{|a|} \quad (4)$$

Assume  $n \notin \mathbb{P}$  By step (1)  $m$  is not a perfect power

$\Rightarrow$  the map  $\mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{N} \quad (s, t) \mapsto p^s m^t$  is injective.

Set  $A := \{p^s m^t \mid s, t \in \{0, \dots, \lfloor \sqrt{a} \rfloor\}\} \subseteq \mathbb{N}$

$$\Rightarrow |A| = (\lfloor \sqrt{|a|} \rfloor + 1)^2 > |G|$$

Since  $G = \langle \bar{p}, \bar{m} \rangle \subseteq (\mathbb{Z}/(r))^\times$  this implies that  $\exists n, \hat{n} \in A$

such that  $n \neq \hat{n}$  but  $b \equiv \hat{n} \pmod{r}$ .

$$\text{Let } h \in H \Rightarrow h = f_{\underline{e}}(\zeta) \text{ with } (\underline{e}) \in \mathbb{N}_0^{k+1} \Rightarrow h^n \stackrel{(3)}{=} f_{\underline{e}}(\zeta^n) \stackrel{n \equiv \hat{n} \pmod{r}}{=} f_{\underline{e}}(\zeta^{\hat{n}}) \stackrel{(3)}{=} h^{\hat{n}}$$

So the polynomial  $Y^n - Y^{\hat{n}} \in \mathbb{F}_q[Y]$  has all elements of  $H$  as zeros.  
 But  $\deg(Y^n - Y^{\hat{n}}) \leq \max\{n, \hat{n}\} \leq (p \cdot m) \lfloor \sqrt{|G|} \rfloor \leq n \sqrt{|G|} < |H|$   
 $\Rightarrow$  contradiction since  $Y^n - Y^{\hat{n}} \neq 0$   $\square$

## 1.4 Cryptology

A ("Alice") wants to send a message to B ("Bob") such that an eavesdropper E ("Eve") can not read the clear message. So A and B encrypt the message.

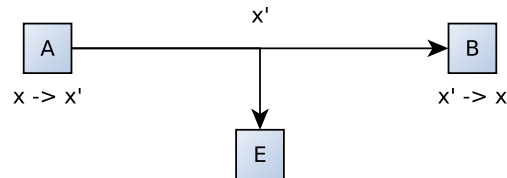


Figure 2: Scheme of eavesdropping

### Symmetric-key cryptography

A and B share secret keys for encryption ( $x \mapsto x'$ ) and decryption ( $x' \mapsto x$ ). Only A and B know the keys.

Example: AES approved by the US government in 2002

Application:

- sending messages
- encrypt files (A=B)

Problem: Key exchange between A and B

### Public-key cryptography

Encryption-map  $\phi : x \mapsto x'$  is made public by B, but decryption  $\phi : x' \mapsto x$  is kept secret.

Advantage: No confidential key exchange.

Disadvantages:

- more costly than symmetric key cryptography
- doubt whether E can reconstruct  $\phi^{-1}$  from  $\phi$  with enough computing power

Applications:

- sending messages
- exchange of symmetric keys
- authentication: Together with  $x$ , B sends  $\phi^{-1}(x)$  (or  $\phi^{-1}|$  Part of  $x$  together with date). A verifies by applying  $\phi$ .  
 Better: challenge-response-protocol.

Examples: RSA, elliptic curve

### 1.4.1 Algorithm (RSA)

- (1) B chooses  $p, q \in \mathbb{P}$  large ( $> 100$  digits)  
with  $p \neq q$   $n := p \cdot q$
- (2) B chooses  $e, f \in \mathbb{N}$  large such that  $e \cdot f \equiv 1 \pmod{\phi(n)}$   
with  $\phi(n) = (p-1)(q-1)$
- (3) B makes  $n, e$  public, keep  $f$  secret
- (4) The message is encoded as an element  $x \in \mathbb{Z}/(n)$
- (5) A computes  $\phi(x) = x^e = y \in \mathbb{Z}/(n)$  and sends  $y$
- (6) B receives  $y$  and computes  $y^f = x \in \mathbb{Z}/(n)$

Comments on steps of RSA:

- (6) Have  $e \cdot f = a \cdot (p-1) \cdot (q-1) + 1$  with  $a \in \mathbb{N}_{>0}$   
 $y^f = x^{e \cdot f}$

$$1: q \nmid f, q \nmid x \Rightarrow x^{a(p-1)(q-1)} = (x^{\phi(n)})^a \stackrel{\text{Little Fermat}}{\equiv} 1^a = 1 \Rightarrow x^{e \cdot f} = x \quad \checkmark$$

- Case 2:  $p|x, q \nmid x \Rightarrow x^{e \cdot f} \equiv 0 \equiv x \pmod{p}$   
 $x^{e \cdot f} \equiv x \pmod{q}$  as above.

Case 3:  $q|x$  As Case 2

$\Rightarrow$  Correctness of decryption

**Cost:**

- (1) Finding  $p, q$  of length approximately  $l$ . Prime-number theorem: Gap between two primes of length  $\approx l$  is  $O(l)$   
Using Miller Rabin with error probability  $2^m$ . Expected cost of (1) is  $O(m \cdot l^4)$  bit operations.
- (2) Choose  $e$  co-prime to  $\phi(n)$  obtain  $f = \text{inverse} \pmod{\phi(n)}$  by extended euclidean Algorithm:  $O(l^2)$
- (5)(6) Fast exponentiation:  $O(l^3)$

Security of RSA:  $p$  and  $q$  must be so large that factorization of  $n$  is "impossible". Assumption that factorization is expensive could not be shown! But could  $f$  be obtained without knowing  $p$  and  $q$ ? The following algorithm gives a negative answer. It shows that the problem of breaking RSA is always basically factorization.

Remember:  $\phi(n) | (e \cdot f - 1) \Rightarrow m \leq n^2$

### 1.4.2 Algorithm 1 (Finding a divisor)

Input :  $n \in \mathbb{N}_{>2}$  odd squarefree  $\notin \mathbb{P}$  and  $m \in \mathbb{N}_{>0}$  such that  $\phi(n) \mid m$   $m \leq n^2$

Output:  $d \in \mathbb{N}$  with  $d \mid n$   $1 < d < n$

- (1) Choose  $a \in \{2, \dots, (n-2)\}$  randomly  
set  $k := m$
- (2) If  $d := \gcd(a, n) \neq 1$   
return d
- (3) Repeat steps (4) - (8) //while(true)
- (4) compute  $d := \gcd(n, a^k - 1)$
- (5) If  $d = 1$  go to (1)
- (6) If  $d < n$  return d
- (7) if  $k$  is odd go to (1)
- (8) set  $k := \frac{k}{2}$

Correctness is clear. What about termination and running time?

### 1.4.3 Proposition 2 (Complexity of Algorithm 1)

Algorithm 1 terminates in expected time  $O(l(n)^4)$  bit operations (Las Vegas Algorithm).

**Proof:**

Set  $l := \text{length}(n)$

Have  $n = \prod_{i=1}^r p_i$  with  $p_i \in \mathbb{P}$  distinct.

$\phi(n) = \prod_{i=1}^r (p_i - 1) \mid m$  So initially all  $(p_i - 1)$  divide  $k$ .

At some iteration it happens for the first time that  $(p_i - 1) \nmid k$

Then  $k \equiv \frac{p_1-1}{2} \pmod{(p_1-1)} \Rightarrow a^k \equiv \pm 1 \pmod{p_i}$  -1 occurs for some  $a$

For those  $j$  with  $(p_j - 1) \mid k$  have  $a^k \equiv 1 \pmod{p_j}$

Consider the group homomorphism:  $\phi: (\mathbb{Z}/(n))^{\times} \mapsto (\mathbb{Z}/(p_1))^{\times} \times \dots \times (\mathbb{Z}/(p_r))^{\times}$   
 $\bar{a} \mapsto (a^k \pmod{p_1}, \dots, a^k \pmod{p_r})$

The image of  $\phi$  is a product of groups  $\{\pm 1\}$  or  $\{1\}$  depending whether  $(p_i - 1) \nmid k$  or  $(p_i - 1) \mid k$



**Conclusion:**

For at least half of all  $a$ 's,  $\phi(\bar{a})$  is neither  $(1, \dots, 1)$  nor  $(-1, \dots, -1)$

If  $a^k \equiv 1 \pmod{p_j}$  then  $p_j \mid (a^k - 1) \Rightarrow p_j \mid d$

If  $a^k \equiv -1 \pmod{p_j}$  then  $p_j \nmid (a^k - 1) \Rightarrow p_j \nmid d$

So for these  $a$  the algorithm is successful.

This means that the expected number of  $a$ 's that need to be tested is  $\leq 2$

(Since  $\sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = 2$  More generally for  $0 < p < 1 : p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} = \frac{1}{p}$ )

Analysis of running time (in bit operations) for each  $a$  (using gcd is quadratic) leads to the claim.  $\square$

**Problems of RSA:**

- How difficult is factorization of integers (lower bound?)
- decryption of some or all messages without having  $f$ ?

**1.4.4 Diffie-Hellmann Key Exchange**

Goal: A, B want to exchange a symmetric key via a public channel

- (1) A and B agree on a  $p \in \mathbb{P}$  (should be large) and  $g \in (\mathbb{Z}/(p))^{\times}$  public
- (2) A chooses  $a \in \{2, \dots, (p-2)\}$  randomly and sends  $u := g^a$  to B
- (3) B chooses  $b \in \{2, \dots, (p-2)\}$  randomly and sends  $v := g^b$  to A
- (4) A computes  $v^a = (g^b)^a = g^{a \cdot b}$   
B computes  $u^b = (g^a)^b = g^{a \cdot b}$

$\Rightarrow$  A and B share  $g^{a \cdot b}$

**Example:**

A chooses  $a = 7$

$$\bar{3}^7 = \bar{11} \in \mathbb{Z}/(17)$$

$$\bar{13}^7 = \bar{4}$$

B chooses  $b = 4$

$$\bar{3}^4 = \bar{13} \in \mathbb{Z}/(17)$$

$$\bar{11}^4 = \bar{4}$$

If Eve reconstructs  $a, b$  from  $g^a$  and  $g^b$  she can compute  $g^{a \cdot b}$

The Security of Diffie-Hellmann depends on the difficulty of the discrete logarithm problem (DLP):

Given  $g \in G$  element of a group or monoid and given  $g^a \in G$ , determine  $a$  (or determine  $a' \in \mathbb{Z}$  such that  $g^a = g^{a'}$ )

**1.4.5 Elliptic curve cryptography (ECC)**

ECC uses elliptic curves as groups.

$$y^2 = x^3 + a \cdot x + b \rightsquigarrow y^2 z = x^3 + axz^2 + bz^3$$

ECC uses suitable elliptic curves on  $\mathbb{F}_a$

## 1.5 Factorization

Let  $m \in \mathbb{N}_{>1}$   $n \notin \mathbb{P}$  Find a divisor  $d$  with  $1 < d < n$ . From this we obtain the factorization of  $n$  by recursion.

Naive method: Trial division. Cost essentially exponential in  $l(n)$

### 1.5.1 Algorithm 1 (Sieve of Eratosthenes)

Input :  $n \in \mathbb{N}_{>1}$

Output: All primes  $\leq n$

- (1) Create a list of all numbers  $\leq n$
- (2)  $p := 2$
- (3) Mark all multiples of  $p$  in the List
- (4) if all numbers are marked  
return
- (5) Let  $p$  be the smallest number that is not marked
- (6)  $p \in \mathbb{P}$  Go to (3)

Running time of Algorithm 1 is exponential.

#### Pollard's rho ( $\rho$ ) algorithm:

Idea: Choose a function  $\mathbb{Z}/(n) \mapsto \mathbb{Z}/(n)$  e.g.  $f(x) = x^2 + 1$

Choose  $x_0 \in \mathbb{Z}/(n)$  set  $x_i := f^i(x_0)$  iterative application.

Let  $p \mid n$  be a prime. Since  $|\mathbb{Z}/(p)| < \infty$  then  $\exists i < j : x_i \equiv x_j \pmod{p}$

Starting at  $x_i$  the sequence of  $x_j$  will be periodic mod  $p$ .

$p \mid x_i - x_j \quad p \mid n \Rightarrow p \mid \gcd(n, x_i - x_j) =: d$

If  $x_i \not\equiv x_j \pmod{n}$  (which is not guaranteed) then  $d$  is a proper divisor of  $n$ .

- Recall that gcd computation is cheap
- Testing all pairs is a lot
- Proposition 2 helps with this

### 1.5.2 Proposition 2 (length of periods)

Let  $M$  be a set.  $f : M \mapsto M$  and  $x_0 \in M \quad x_i := f^i(x_0)$

If  $x_{t+l} = x_t$  for  $l, t \in \mathbb{N}l > 0 \quad (\rightarrow t \text{ "off-period", } l \text{ "length of period"})$

$\Rightarrow \exists j \in \mathbb{N}$  with  $0 < j \leq t + l$  such that  $x_j = x_{2j}$

**Proof:**

$f^l(x_t) = x_t \Rightarrow \forall a \in \mathbb{N} \quad f^{a \cdot l}(x_t) = x_t \quad \text{Assume } j = a \cdot l \geq t \quad a \in \mathbb{N}$

$x_{2j} = x_{t+(j-t)+a \cdot l} = f^{(j-t)}(x_{t+a \cdot l}) = f^{(j-t)}(f^{al}(x_t)) = f^{(j-t)}(f^l(x_t)) = x_j$

**Case 1**  $t = 0 \quad j = l \quad \checkmark$

**Case 2**  $t > 0 \quad j = t + \underbrace{(-t \bmod l)}_{\in 0, \dots, (l-1)} \quad \checkmark$

### 1.5.3 Algorithm 3 (Pollard's $\rho$ - Algorithm)

Input :  $n \in \mathbb{N}_{>1}, n \notin \mathbb{P}$

Output: a proper divisor of  $n$  or "FAIL"

- (1) Choose  $x \in \{0, \dots, (n-1)\}$  randomly  
set  $y := x$
- (2) repeat (3)-(6)
- (3)  $x := x^2 + 1 \pmod{n}$      $y := (y^2 + 1)^2 + 1 \pmod{n}$      $// x := x_j y := x_{2j}$
- (4)  $d := \gcd(n, x - y)$
- (5) if  $(1 < d < n)$   
return  $d$
- (6) if  $d = n$   
return "FAIL"

One "FAIL" includes no conclusion so you might want to repeat the Algorithm with a different  $x$ .

Running time? Assume the  $x_i := f^i(x_0)$  are randomly distributed.

When can we expect that a match ( $x_i \equiv x_j \pmod{p}$ ) occurs?  $\rightarrow$  "Birthday Problem"

#### Lemma (Birthday Problem):

We iteratively choose numbers in  $\{1, \dots, n\}$  at random. The expected numbers of choices (if we keep choosing until a number has been chosen twice) is  $< \sqrt{\frac{\pi \cdot n}{2}} + 2$

#### Proof:

Let  $s \geq 2$  be the numbers of choices until a match occurs. For  $k \in \mathbb{N}$  with  $P()$  as probability

$$P(s > k) = \prod_{i=1}^k \left(1 - \frac{i-1}{n}\right) \leq \prod_{i=1}^k e^{-\frac{i-1}{n}} = e^{\sum_{i=1}^k -\frac{i-1}{n}} = e^{\frac{k(1-k)}{2n}} \leq e^{-\frac{(k-1)^2}{2n}}$$

\* since  $f(x) = e^x - (1-x) \geq 0$  for  $x \geq 0$

$$f(0) = 0$$

$$f'(x) \geq 0 \text{ if } x \geq 0$$

$$\sum_{k=0}^{\infty} P(s > k) = 2 + \sum_{k=2}^{\infty} P(s > k) \leq 2 + \sum_{k=2}^{\infty} e^{-\frac{(k-1)^2}{2n}} \leq 2 + \int_1^{\infty} e^{-\frac{(x-1)^2}{2n}} dx$$

$$\stackrel{x:=x-1}{=} 2 + \int_0^{\infty} e^{-\frac{x^2}{2n}} dx = 2 + \int_0^{\infty} e^{-\left(\frac{x}{\sqrt{2n}}\right)^2} dx$$

$$\stackrel{x:=\frac{x}{\sqrt{2n}}}{=} 2 + \sqrt{2n} \int_0^{\infty} e^{-x^2} dx = 2 + \sqrt{2n} \cdot \frac{\sqrt{\pi}}{2} = 2 + \sqrt{\frac{n \cdot \pi}{2}}$$

#### Example:

People arrive at a party. When can you expect to have two that share their birthday?

$\rightarrow$  when 26 have arrived!

### 1.5.4 Theorem 4 (Bit-complexity of Algorithm 3)

under suitable assumptions on the distribution  $f^i(x)$  for  $f(x) = x^2 + 1$  Algorithm 3 has the expected running time of  $O(\sqrt[4]{n} \lg(n)^2)$  bit operations

**Proof:**

By Proposition 2 and the Lemma the expected number of runs through the loop is

$$O(\sqrt{p}) = O(\sqrt[4]{n}) \text{ as } p \leq \sqrt{n}$$

Each run through the loop takes  $O(\lg(n)^2)$  bit operations.  $\square$

### Pollard's p-1 Algorithm

Motivation: Let  $p \mid n$  prime divisor

$$\Rightarrow \forall a \in \mathbb{Z} : a^{p-1} \equiv 1 \pmod{p} \quad \text{with } \gcd(a, p) = 1$$

$$\Rightarrow \forall m \in \mathbb{Z} \text{ with } (p-1) \mid m : a^m \equiv 1 \pmod{p}$$

$$p \mid \gcd(a^m - 1, n)$$

Let  $B$  be an upper-bound for the prime powers dividing  $p-1$ .

" $p-1$  is  $B$ -power-smooth".

$$\text{Then } (p-1) \mid \prod_{(q \leq B) \in \mathbb{P}} q^{\lfloor \log_q(B) \rfloor}$$

Neither  $p$  nor  $B$  are known! But guess and try  $B$  and hope for the best.

### 1.5.5 Algorithm 5 (Pollard's $\rho$ - 1 method)

Input :  $n \in \mathbb{N}_{>1} \setminus \mathbb{P}$

Output:  $d \in \mathbb{N}$  with  $d \mid n$   $1 < d < n$  or "FAIL"

- (1) Choose a "smoothness bound"  $B$
- (2) Choose  $a \in \{2, \dots, (n-2)\}$  randomly
- (3) Use Algorithm 1 to find all  $q \in \mathbb{P}$  with  $q \leq B$   
For every  $q$  perform steps (4) - (5)
- (4)  $k := q^{\lfloor \log_q(B) \rfloor}$   
set  $a := a^k \pmod{n}$   
compute  $d := \gcd(n, a - 1)$
- (5) if  $1 < d < n$   
return  $d$
- (6) return "FAIL" //or increase  $B$  and go to (1)

**Consequence:** when setting up RSA  $p, q$  should be chosen such that  $p-1$  and  $q-1$  have large prime divisors.

**The quadratic sieve** (State of the art factorization algorithm)

Observation: if  $n = x^2 - y^2$  then  $n = (x - y) \cdot (x + y)$

Conversely if  $n = a \cdot b$  then  $n = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$

**1-st Idea:** Find  $x, y \in \mathbb{Z}$  such that  $x^2 \equiv y^2 \pmod{n} \quad \wedge \quad x \not\equiv \pm y \pmod{n}$

Then  $n \mid (x - y) \cdot (x + y)$

$\Rightarrow$  for every  $p \in \mathbb{P}$  with  $p \mid n : p \mid (x - y) \vee p \mid (x + y)$

$\Rightarrow p \mid \gcd(x - y, n) \vee p \mid \gcd(x + y, n)$

Since both gcd are  $< n$  receive a non-trivial divisor of  $n$

If  $x^2 \equiv y^2 \pmod{n}$  how probable is it that  $x \equiv \pm y \pmod{n}$  ?

Let  $n = \prod_{i=1}^r p_i^{k_i}$  odd with  $p_i \in \mathbb{P}$  distinct.

Assume  $p_i \nmid x \forall i = 1 \dots r$  Since  $(\mathbb{Z}/(p_i^{k_i}))^\times$  is cyclic there are  $2^r$  classes  $y \pmod{n}$  such that  $x^2 \equiv y^2 \pmod{n}$

[Reason: These classes are given by  $y \equiv \pm x \pmod{p_i^{k_i}}$  These are the only solutions since  $\mathbb{Z}/(p_i^{k_i})^\times$  is cyclic of even order.

$G = \langle \sigma \rangle$  cyclic of order  $2m$

$x = \sigma^i$  Find  $l \in \mathbb{Z}$  such that  $x^2 = (\sigma^l)^2$

$\Leftrightarrow 2j \equiv 2l \pmod{2m} \Leftrightarrow j \equiv l \pmod{m}$

$\Leftrightarrow l \equiv j \pmod{2m}$  or  $l \equiv j + m \pmod{2m}$  ]

But have  $x \equiv \pm y$  only for  $2y$ 's.

Failure probability:  $2^{1-r}$

Handle case  $r = 1$  by Algorithm 11 in 1.3

**Example 1**

$n = 91$  Search  $x, y \in \mathbb{Z} \quad k \in \mathbb{Z}$  such that  $x^2 = k \cdot n + y^2$

Good chance if  $x$  is slightly bigger than  $\sqrt{k \cdot n}$

$k := 1 \Rightarrow \sqrt{91} \approx 9,54 \Rightarrow x := 10 \Rightarrow 10^2 = 100 \equiv 3^2 \pmod{91}$

$n = 10^2 - 3^2 = (10 - 3) \cdot (10 + 3) = 7 \cdot 13 \quad \checkmark$

Another try:

$k := 8 \Rightarrow \sqrt{8 \cdot 91} \approx 26,98 \Rightarrow 27^2 \equiv 1^2 \pmod{91} \quad \gcd(26, 91) = 13$

**Example 2**

$n = 4633 \quad k := 3$

$\sqrt{3 \cdot n} \approx 117,89 \Rightarrow x^2 = 118^2 \equiv 5^2 \pmod{n}$

$\gcd(118 - 5, n) = 113$

$\gcd(118 + 5, n) = 41 \quad \checkmark$

**2-nd Idea:** Choose  $B \in \mathbb{N}$  "smoothness bound" suitable.

Let  $p_2, \dots, p_r \in \mathbb{P}$  be all primes  $\leq B$  (Algorithm 1) set  $p_1 := -1$

The  $p_i$  form a "factor basis".

For  $a \in \mathbb{Z}$  write  $(a \pmod{n})$

for the  $x \in \mathbb{Z}$  with  $x \equiv a \pmod{n}$  and  $-\frac{n}{2} < x \leq \frac{n}{2}$

**Procedure:**

Search numbers  $a_1, \dots, a_m \in \mathbb{Z}$  such that  $(a_i^2 \pmod{n}) = \prod_{j=1}^r p_j^{e_{ij}}$

with  $e_{ij} \in \mathbb{Z}$  ("B numbers")

So for  $\mu_1, \dots, \mu_m \in \mathbb{N}_0$  have  $\left( \prod_{i=1}^m a_i^{\mu_i} \right)^2 \equiv \prod_{i=1}^m \prod_{j=1}^r p_j^{\mu_i \cdot e_{ij}} \pmod{n} = \prod_{j=1}^r p_j^{\sum_{i=1}^m \mu_i \cdot e_{ij}} \pmod{n}$

If the vectors  $(e_{i1}, \dots, e_{ir})$  become linearly dependent mod 2 (guaranteed if  $m > r$ ) then

$\exists \mu_1, \dots, \mu_m \in \{0, 1\}$  not all 0 such that:

$$\sum_{i=1}^m \mu_i \cdot e_{ij} = 2 \cdot k_j \quad k_j \in \mathbb{N}_0$$

with  $x := \prod_{i=1}^m a_i^{\mu_i}$   $y := \prod_{j=1}^r p_j^{k_j}$  obtain  $x^2 \equiv y^2 \pmod{n}$

**Example:**  $n = 4633$  choose  $B = 3 \Rightarrow$  factor basis  $-1, 2, 3$

Search  $a \in \mathbb{Z}$  such that  $|a_i^2 \pmod{n}|$  is small. Idea:  $a \approx \sqrt{n} = 68.06\dots$

$$a_1 := 68 : 68^2 = n - 9 \equiv (-1) \cdot 3^2 \pmod{n}$$

$$\rightarrow e_1 = (1, 0, 2) \rightarrow (1, 0, 0) \in \mathbb{F}_2^3$$

$$a_2 := 69 : 69^2 = n + 128 \equiv 2^7 \pmod{n}$$

$$\rightarrow e_2 = (0, 7, 0) \rightarrow (0, 1, 0) \in \mathbb{F}_2^3$$

$$a_3 := 67 : 67^2 = n - 144 \equiv (-1) \cdot 2^4 \cdot 3^2$$

$$\rightarrow e_3 = (1, 4, 2) \rightarrow (1, 0, 0) \in \mathbb{F}_2^3$$

$$e_1 + e_3 \equiv 0 \pmod{2} \quad \text{In fact:}$$

$$e_1 + e_3 = 2 \cdot \underbrace{(1, 2, 2)}_{(k_1, k_2, k_3)} \rightarrow \mu_1 = 1 \quad \mu_2 = 0 \quad \mu_3 = 1$$

$$x := a_1 \cdot a_3 \equiv -77 \pmod{n}$$

$$y := (-1) \cdot 2^2 \cdot 3^2 = -36$$

$$x - y = -41 \quad x + y = -113$$

$$\gcd(n, x - y) = 41 \quad \gcd(n, x + y) = 113 \quad \checkmark$$

**3rd Idea:** Look for  $a_i$  of the form  $t + \lfloor \sqrt{n} \rfloor$  with  $t$  in a "suitable".

Sieve Interval:  $[-s, s] \cap \mathbb{Z}$

As it turns out if  $s \leq \frac{\sqrt{5}-2}{2} \lfloor \sqrt{n} \rfloor$  then  $(t + \lfloor \sqrt{n} \rfloor)^2 \pmod{n} = (t + \lfloor \sqrt{n} \rfloor)^2 - n =: f(t)$

When does  $p_j^{e_j}$  divide  $f(t)$  (with  $j \geq 2$ )? Precisely if  $(t + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_j^{e_j}}$

If this holds for some  $t$  then it also holds for all  $t + k \cdot p_j^{e_j}$  with  $k \in \mathbb{Z}$  Moreover if it holds then  $\bar{n} \in \mathbb{F}_{p_j}$  is square. So may remove all  $p_j$  such that  $\bar{n} \in \mathbb{F}_{p_j}$  is a non-square from the factor basis.

Obtain a sieving procedure:

For  $t \in [-s, s] \cap \mathbb{Z}$  with  $p_j^{e_j} \mid f(t)$  "mark" all elements  $t + k \cdot p_j^{e_j} \in [-s, s]$

### 1.5.6 Algorithm 6 (Quadratic sieve, simplified version)

Input :  $n \in \mathbb{N}_{>1} \setminus \mathbb{P}$  odd

Output: A non trivial divisor of  $n$  or "FAIL"

- (1) if  $(n = m^e)$  with  $m, e \in \mathbb{N}_{>1}$   
return  $m$  // can be done with Algorithm 11 § 3
- (2) Choose a "smoothness bound"  $B \in \mathbb{N}$  and a "sieve bound"  $s \in \mathbb{N}$  suitably
- (3) Let  $p_1 = -1$   $p_2, \dots, p_r$  be the factor basis given by  $B$ . Delete those  $p_j$  such that  $\bar{n} \in \mathbb{F}_{p_j}$  is a non-square
- (4) for  $(t = -s, -s + 1, \dots, s - 1)$   
compute  $f_t := |(t + \lfloor \sqrt{n} \rfloor)^2 - n| \in \mathbb{N}_{>0}$
- (5) for  $(t = -s, \dots, s)$   
set  $e_t := (0, \dots, 0) \in \mathbb{N}_0^r$  // initialize exponent vectors
- (6) for  $(t = -s, \dots, 0)$   
set  $e_{t,1} := 1$  //  $\rightarrow$  first entry of each  $e_t$  is the exponent of  $p_1 = -1$  in  $f(t)$
- (7) for  $(j = 2, \dots, r)$  repeat (8) - (10)
- (8) for  $(e = 1, \dots, \lfloor \log_{p_j}(B) \rfloor)$  repeat (9) - (10) // or maybe a bit larger
- (9) solve  $(t + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_j^e}$   
Let  $(t_1 \pmod{p_j^e}), \dots, (t_m \pmod{p_j^e})$  be the solutions.  
// We will see that  $m \in \{0, 2, 4\}$  with  $m = 2$  most frequent.
- (10) for all  $t = t_i + k \cdot p_j^e \in [-s, s]$  with  $k \in \mathbb{Z}, i = 1, \dots, m$   
set  $e_{t,j} := e_{t,j} + 1$   
 $f_t := \frac{f_t}{p_j}$
- (11) let  $t, \dots, t_m$  be those  $t \in [-s, s] \cap \mathbb{Z}$  for which  $f_t = 1$   
/\* So the  $a_i = t_i + \lfloor \sqrt{n} \rfloor$  are  $B$ -numbers and the factorization  
\* of  $a_i^2 \pmod{n} = a_i^2 - n = f(t)$  is given by the exponent  
\* vectors  $e_t$  \*/
- (12) if the  $(e_{t_i} \pmod{2}) \in \mathbb{F}_2^r (i = 1, \dots, m)$  are not linearly dependent.  
return "FAIL"
- (13) compute  $\mu_1, \dots, \mu_m \in \{0, 1\}, k_1, \dots, k_r \in \mathbb{N}_0$  such that  $\sum_{i=1}^m \mu_i e_{t_i} = 2 \cdot (k_1, \dots, k_r)$
- (14) set  $x := \prod_{i=1}^m (t_i + \lfloor \sqrt{n} \rfloor)^{\mu_i} \pmod{n}$   
 $y := \prod_{j=1}^r p_j^{k_j} \pmod{n}$  // Now  $x^2 \equiv y^2 \pmod{n}$

(15) if  $\gcd(n, x - y)$  or  $\gcd(n, x + y)$  is a non-trivial divisor  
     return the non-trivial divisor  
 else  
     return "FAIL"

With good heuristics it will almost certainly never return FAIL.

**Example:**  $n = 20437$

Choose  $B := 10$      $s := 3$

Factor basis:  $p_1 = -1$     $p_2 = 2$     $p_3 = 3$     $p_4 = 7$

(5 omitted as:  $n \equiv 2 \pmod{5}$  non-square)

$\lfloor \sqrt{n} \rfloor = 142$

Solve  $(t + 142)^2 \equiv n \pmod{p_j^e}$

$p_2 = 2$  : Compute modulo 2,4,8.  $n \equiv 5 \pmod{8}$

$t \text{ odd} \Rightarrow (t + 142)^2 \equiv 1 \pmod{8} \Rightarrow (t + 142)^2 \equiv n \pmod{4}$  but not  $\pmod{8}$

$t \text{ even} \Rightarrow (t + 142)^2 \equiv 0 \pmod{2} \not\equiv n \pmod{2}$

$$\Rightarrow e_{t,2} = \begin{cases} 2 & t \text{ odd} \\ 0 & t \text{ even} \end{cases}$$

$p_3 = 2$  :  $n \equiv 1 \pmod{3}$     $\lfloor \sqrt{n} \rfloor \equiv 1 \pmod{3}$

So  $3 \mid f(t) \Leftrightarrow t + 1 \equiv \pm 1 \pmod{3} \Leftrightarrow t \equiv 0 \text{ or } 1 \pmod{3}$

$e = 2$     $n \equiv 7 \equiv (\pm 4)^2 \pmod{9}$     $\lfloor \sqrt{n} \rfloor \equiv 7 \pmod{9}$

So  $9 \mid f(t) \Leftrightarrow t + 7 \equiv \pm 4 \pmod{9} \Leftrightarrow t \equiv -3, -2 \pmod{9}$

$p_4 = 7$     $n \equiv 4 \pmod{7}$     $4 = (\pm 2)^2$     $\lfloor \sqrt{n} \rfloor \equiv 2 \pmod{7}$

So  $7 \mid f(t) \Leftrightarrow t + 2 \equiv \pm 2 \pmod{7} \Leftrightarrow t \equiv 0 \text{ or } 3 \pmod{7}$

$t$	-3	-2	-1	0	1	2	3
$f_t =  f(t) $	1116	837	556	273	12	295	588
$p_1$ component of $e_t$	1	1	1	1	0	0	0
$p_2$ component	2	0	2	0	2	0	2
$f_t$ divided by 2-power	279	837	139	273	3	299	147
$p_3$ component	2	2	0	1	1	0	1
$f_t$	31	93	139	91	1	299	49
$p_4$ component	0	0	0	1	0	0	2
$f_t$	31	93	139	13	1	299	1

Obtain  $m = 2$  :  $t_1 = 1$     $t_2 = 3$     $e_1 = (0, 2, 1, 0)$     $e_3 = (0, 2, 1, 2)$

They are linear dependent  $\pmod{2}$

$e_1 + e_3 = 2 \cdot (0, 2, 1, 1)$

$x = (142 + 1) \cdot (142 + 3) \equiv 298 \pmod{n}$

$y = p_2^2 \cdot p_3 \cdot p_4 = 2^2 \cdot 3 \cdot 7 = 84$

$\gcd(n, x - y) = \gcd(n, 214) = 107$

$\gcd(n, x + y) = 191$

Indeed  $n = 107 \cdot 191$



### Computing square roots (mod $p^e$ )

#### Case 1: $p$ odd

Find  $x$  with  $x^2 \equiv n \pmod{p}$  by trying  $x \pmod{p}$  (exactly two solutions). Suppose we have found  $x$  with  $x^2 \equiv n \pmod{p^e}$

So  $x^2 - n = p^e \cdot r \quad r \in \mathbb{Z}$

New  $x$  should be  $x + y \cdot p^e$

Compute modulo  $p^{e+1}$ :  $(x + y \cdot p^e)^2 - n = x^2 + 2yxp^e + y^2p^{2e} - n \equiv p^e \cdot (r + 2xy) \pmod{p^{e+1}}$

So  $(x + y \cdot p^e)^2 \equiv n \pmod{p^{e+1}} \Leftrightarrow 2xy \equiv -r \pmod{p}$  uniquely and easily solvable

$\rightarrow$  Obtain two solutions (mod  $p^e$ )

$\Rightarrow$  special case of "Hensel lifting"

#### Case 2: $p = 2$

Find  $x \in \mathbb{Z}$  with  $x^2 \equiv n \pmod{8}$  (0 or 4 solutions since  $n$  odd)

Assume we have  $x^2 \equiv n \pmod{2^e} \quad e \geq 3$

So  $x^2 - n = r \cdot 2^e$

$\Rightarrow (x + y \cdot 2^{e-1})^2 - n = x^2 + xy \cdot 2^e + y^2 2^{2e-2} - n \equiv 2^e(r + xy) \pmod{2^{e+1}}$

So  $(x + y \cdot 2^{e-1})^2 \equiv n \pmod{2^{e+1}} \Leftrightarrow y \equiv r \pmod{2}$

$\rightarrow$  0 or 4 solutions

### Running time of quadratic sieve

Choose  $B \approx \exp\left(\sqrt{\frac{1}{2} \ln(n) \cdot \ln(\ln(n))}\right)$

If  $s \approx B$  then running time is:  $O\left(\exp\left(\sqrt{\ln(n) \cdot \ln(\ln(n))}\right)\right)$

which is "slightly" sub-exponential

### Factorization algorithm with best complexity (known to date):

Number field sieve

This also uses ideas 1 and 2, but an algebraic number field is used for generating  $B$ -numbers.

Heuristic Running time (modulo some conjectures):  $O\left(\exp\left(\ln(n)^{\frac{1}{3}} \cdot \ln(\ln(n))\right)^{\frac{2}{3}}\right)$

## 2 Systems of equations

### 2.6 Linear Algebra

Tasks:

- solving systems of linear equations (= linear systems)
- inversions of matrices
- rank determination
- determinants
- matrix products

$K$  field,  $K^{m \times n}$  = set of  $m \times n$  matrices

$GL_n(K)$  = field of  $n \times n$  matrices

Count the cost of algorithms in terms of field operations. If  $K$  is a finite field this translates directly to bit operations.

#### 2.6.1 Proposition 1 (Complexity of usual algorithms)

- (a) Solving an  $m \times n$ -linear system by Gaussian elimination requires  $O(\max\{m, n\}^3)$  field operations
- (b) For  $A \in GL_n(K)$  computing  $A^{-1}$  by usual method requires  $O(n^3)$  field operations.
- (c) Computing  $\det(A)$  "as usual" requires  $O(n^3)$  bit operations.
- (d) Computing  $A \cdot B$  for  $A \in K^{m \times n}$   $B \in K^{n \times l}$  requires  $O(m \cdot n \cdot l)$  field operations.

→ all cubic!

**Proof:**

- (a) Cost of treating the  $k$ -th row with Gauss algorithm:
  - $\leq 1$  inversion,  $\leq (n - k)$  multiplications
  - $\leq (m - k)(n - k)$  multiplications and additions
  - (clearing column below pivot element)
  - Back substitution (i.e. clearing columns above pivot element):
  - Let  $r = rk(A) \leq (k - 1)(n - r)$  multiplications and additions
  - Total cost  $\leq \sum_{k=1}^r (1 + n - k + 2(m - k)(n - k) + 2(k - 1)(n - r))$
  - $= 2mnr - mr^2 - \frac{1}{3}r^3 - nr + \frac{3}{2}r^2 + \frac{5}{6}r - mr$
  - $\in O(\max\{m, n\}^3)$
- (b) Inversion is Gaussian elimination of  $n \times 2n$ -matrix of rank  $n$ 
  - cost  $\leq \frac{8}{3}n^3 - \frac{3}{2}n^2 + \frac{5}{6}n \in O(n^3)$
- (c) reduced to (a)

(d) obvious

### Strassen-multiplication

let  $A, B \in K^{2n \times 2n}$  Write:  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$  with  $A_{ij} B_{ij} \in K^{n \times n}$

Then  $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$  with  $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} \rightarrow 8$  multiplications.

Set:

$$M_1 := (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$$

$$M_2 := (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$$

$$M_3 := (A_{11} - A_{21}) \cdot (B_{11} + B_{12})$$

$$M_4 := (A_{11} + A_{12}) \cdot B_{22}$$

$$M_5 := A_{11} \cdot (B_{12} - B_{22})$$

$$M_6 := A_{22} \cdot (B_{21} - B_{11})$$

$$M_7 := (A_{21} + A_{22}) \cdot B_{11}$$

Then:

$$C_{11} = M_1 + M_2 - M_4 + M_6$$

$$C_{12} = M_4 + M_5$$

$$C_{21} = M_6 + M_7$$

$$C_{22} = M_2 - M_3 + M_5 - M_7$$

$\rightarrow 7$  Multiplications!

### 2.6.2 Algorithm 2 (Strassen-multiplication)

Input :  $A \in K^{m \times n} B \in K^{n \times l}$

Output:  $A, B \in K^{m \times l}$

- (1) Let  $k$  be minimal such that  $m, n, l \leq 2^k$
- (2) if  $(k = 0)$   $//(\Leftrightarrow A, B \in K^{1 \times 1})$   
return  $A \cdot B$
- (3) Enlarge  $A, B$  by adding zeros such that  $A, B \in K^{2^k \times 2^k}$
- (4) write  $A \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$  with  $A_{ij}B_{ij} \in K^{2^{k-1} \times 2^{k-1}}$
- (5) compute  $M_1 \dots M_7$  as above, do multiplications by recursive call
- (6) compute  $A \cdot B = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$  by above formulas
- (7) Output: the upper left  $m \times l$  - part of  $A \cdot B$

### 2.6.3 Theorem 3 (Running time of Algorithm 2)

If  $m, n, l \leq r$  Algorithm 3 requires  $O(r^{\lg(7)})$  field operations

**Proof:**

Set  $\Theta(k)$  = number of field operations.

Step 5:  $7 \cdot \Theta(k-1) + 10 \cdot (2^{k-1})^2$

Step 6:  $8 \cdot (2^{k-1})^2$

Obtain:

$$\Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1} \quad (*)$$

Claim:  $\Theta(k) = 7^{k+1} - 6 \cdot 4^k$

Induction on  $k$

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$k-1 \rightarrow k : \Theta(k) = 7\Theta(k-1) + 18 \cdot 4^{k-1}$

$= 7(7^k - 6 \cdot 4^{k-1}) + 18 \cdot 4^{k-1}$

*induction*

$= 7^{k+1} - 4 \cdot 6 \cdot 4^{k-1} \quad \checkmark$

Have  $2^{k-1} < r \Rightarrow k < \lg(r) + 1 \Rightarrow \Theta(k) < 7^{\lg(r)+2} = 49 \cdot 2^{\lg(7) \cdot \lg(r)} = 49^{\lg(17)} \quad \checkmark$

**Remarks:**

(a)  $\lg(7) = 2.8074\dots$

(b) Coppersmith-Winograd:  $O(r^{2.3754\dots})$

Improved by Stothes (2010), Williams(2011), Le Gall(2014):  $O(r^{2.3729\dots})$

(c) The cost of the best possible algorithm is unknown, even for  $r = 3$

Let  $M : \mathbb{N}_{>0} \mapsto \mathbb{R}_{>0}$  be a function such that two matrices in  $K^{n \times n}$  can be multiplied in  $\leq M(n)$  field operations. Assume  $\exists \epsilon > 0 : \forall n :$

$$2^{2+\epsilon} M(n) \leq M(2n) \leq 8 \cdot M(n) \quad (1)$$

**Example:**  $M(n) = 49 \cdot n^{\lg(7)}$

Recall:  $A = (a_{ij})$  is upper (lower) triangular  $\Leftrightarrow a_{ij} = 0$  for  $i > j$  ( $i < j$ )

### 2.6.4 Proposition 4 (Complexity of matrix inversion)

An upper or lower triangular matrix  $A \in GL_n(K)$  can be inverted in  $O(M(n))$  field operations.

**Proof:**

Let  $k \in \mathbb{N}$  be minimal such that  $n \leq 2^k$

write  $B = \begin{pmatrix} A & 0 \\ 0 & I_{2^k-n} \end{pmatrix} \in GL_{2^k}(K) \Rightarrow B^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & I_{2^k-n} \end{pmatrix}$

Assume  $B$  upper triangular:

$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$   $B_{11}, B_{22} \in GL_{2^{k-1}}(K), B_{12} \in K^{2^{k-1} \times 2^{k-1}}$

$B^{-1} = \begin{pmatrix} B_{11}^{-1} & -B_{11}^{-1} \cdot B_{12} \cdot B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{pmatrix}$

Let  $\Theta(k)$  = computation cost depending on  $k$ .

$$\Theta(k) \leq 2 \cdot \Theta(k-1) + 2M(2^{k-1}) \stackrel{(1)}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2} \cdot M(2^k) \quad (**)$$

Claim:  $\Theta(k) \leq 2^k + M(2^k)$

$k = 0 : \Theta(k) = 1 \quad \checkmark$

$k-1 \rightarrow k : \Theta(k) \stackrel{**}{\leq} 2 \cdot \Theta(k-1) + \frac{1}{2}M(2^k) \stackrel{induction}{\leq} 2(2^{k-1} + M(2^{k-1})) + \frac{1}{2}M(2^k) \stackrel{(1)}{\leq}$

$2^k + \frac{1}{2}M(2^k) + \frac{1}{2}M(2^k) \quad \checkmark$

Have  $n > 2^{k-1} \Rightarrow k < \lg(n) + 1 \Rightarrow \Theta(k) < 2 \cdot n + M(2n) \stackrel{(1)}{\leq} 2 \cdot n + 8 \cdot M(n) \quad \square$

**Project:** Reduce (most) tasks of linear algebra to multiplication.

The following algorithm transforms a matrix such that all tasks become easy.

### 2.6.5 Algorithm 5 (Transforming a matrix)

Input :  $A \in K^{m \times n}$

Output: Matrices  $L, Q, P, U$

such that:  $LQAP = \begin{pmatrix} U \\ 0 \end{pmatrix} \begin{matrix} r \\ m-r \end{matrix}$  ( $\leftarrow$  in row-echelon form)  $\in K^{m \times n}$

such that:

- $L \in K^{m \times m}$  lower triangular with 1's on the diagonal
- $Q \in K^{m \times m}$   $P \in K^{n \times n}$  permutation matrices
- $U \in K^{m \times m}$  upper triangular with non-zero diagonal entries ( $r = 0$  if  $A = 0$ )
- If  $r = m$  then  $Q = I_m$

(1) if  $(A = (0...0))$   
 return  $L = Q = (1)$   $P = I_n$   $r = 0$

(2) if  $(A = a_1, \dots, a_n)$   
 let  $i$  be minimal with  $a_i \neq 0$   
 $P :=$  matrix exchanging 1st and  $i$ -th position in  $A$   
 return  $L = Q = (1)$   $P$   $U = A \cdot P$

(3) let  $m_1 = \lfloor \frac{m}{2} \rfloor$   $m_2 = \lceil \frac{m}{2} \rceil$   
 write  $A = \begin{bmatrix} B \\ C \end{bmatrix} \begin{matrix} m_1 \\ m_2 \end{matrix}$   $B \in K^{m_1 \times n}$   $C \in K^{m_2 \times n}$

(4) Applying the algorithm recursively on  $B$

obtain  $L_1 \cdot Q_1 \cdot B \cdot P = \begin{bmatrix} U_1 \\ 0 \end{bmatrix} \begin{matrix} r_1 \\ m_1 - r_1 \end{matrix}$  with  $U_1 \in K^{r_1 \times n}$   
 $n$

(5) write  $L_1 = \begin{bmatrix} L_t & 0 \\ L_l & L_r \end{bmatrix} \begin{matrix} r_1 \\ m_1 - r_1 \end{matrix}$   $Q_1 = \begin{bmatrix} Q_t \\ Q_b \end{bmatrix} \begin{matrix} r_1 \\ m_1 - r_1 \end{matrix}$   $U_1 = \begin{bmatrix} E & U'_1 \end{bmatrix} \begin{matrix} r_1 \\ n - r_1 \end{matrix}$   
 $r_1$   $m_1 - r_1$   $m_1$

form  $D := C \cdot P_1 = \begin{bmatrix} F & D' \end{bmatrix} \begin{matrix} r_1 \\ n - r_1 \end{matrix}$   
 $m_2$

and  $G := D' - FE^{-1}U' \in K^{m_2 \times (n-r_1)}$

(6) Apply the algorithm recursively to  $G$ :

$L_2 \cdot Q_2 \cdot G \cdot P_2 = \begin{bmatrix} U_2 \\ 0 \end{bmatrix} \begin{matrix} r_2 \\ m_2 - r_2 \end{matrix}$   
 $n - r_1$

(7) return

$L := \begin{bmatrix} L_t & 0 & 0 \\ -L_2 Q_2 F E^{-1} L_t & L_2 & 0 \\ L_l & 0 & L_r \end{bmatrix} \begin{matrix} r_1 \\ m_2 \\ m_2 - r_1 \end{matrix}$   
 $r_1$   $m_2$   $m_2 - r_1$   $r_1$   $m_2$   $m_1 - r_1$

$Q := \begin{bmatrix} Q_t & 0 \\ 0 & Q_2 \\ Q_b & 0 \end{bmatrix} \begin{matrix} r_1 \\ m_2 \\ m_1 - r_1 \end{matrix}$   
 $m_1$   $m_2$   $r_1$   $m_2$   $m_1 - r_1$

$P := P_1 \cdot \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} \begin{matrix} r_1 \\ n - r_1 \end{matrix}$   
 $r_1$   $n - r_1$   $r_1$   $n - r_1$

$U := \begin{bmatrix} E & U'_1 P_2 \\ 0 & U_2 \end{bmatrix} \begin{matrix} r - 1 \\ r_2 \end{matrix}$   
 $r_1$   $n - r_1$   $r - 1$   $r_2$

### 2.6.6 Theorem 6 (Correctness and running time of Algorithm 5)

Algorithm 5 is correct and requires  $O((\frac{n}{m} + 1) \cdot M(m))$  field operations

**Proof:**

Correctness by induction on  $m$

$m = 1$  ✓

$m > 1$  :

$m_1, m_2 < m$  so recursive calls are correct by induction.

By step (7)  $L, Q, P, U$  have desired form.

Compute:

$$\begin{aligned}
 LQAP &= \begin{array}{cc} m_1 & m_2 \\ \begin{bmatrix} L_t Q_t & 0 \\ -L_2 Q_2 F E^{-1} L_t Q_t & L_2 Q_2 \\ L_l Q_t + L_r Q_b & 0 \end{bmatrix} & \begin{array}{l} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} B \cdot P_1 \\ D \end{bmatrix} & \begin{array}{l} m_1 \\ m_2 \end{array} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} & \begin{array}{l} r_1 \\ n - r_2 \end{array} \end{array} \\
 &\stackrel{(4)}{=} \begin{array}{cc} n \\ \begin{bmatrix} U_1 \\ L_2 Q_2 (-F E^{-1} U_1 + D) \\ 0 \end{bmatrix} & \begin{array}{l} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} & \begin{array}{l} r_1 \\ n - r_2 \end{array} \end{array} \\
 &= \begin{array}{cc} r_1 & n - r_1 \\ \begin{bmatrix} E & U'_1 \\ 0 & L_2 Q_2 G \\ 0 & 0 \end{bmatrix} & \begin{array}{l} r_1 \\ m_2 \\ m_1 - r_1 \end{array} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} I_{r_1} & 0 \\ 0 & P_2 \end{bmatrix} & \begin{array}{l} r_1 \\ m_2 \end{array} \end{array} \stackrel{(6)}{=} \begin{array}{cc} r_1 & n - r_1 \\ \begin{bmatrix} E & U'_1 P_2 \\ 0 & U_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{l} r_1 \\ r_2 \\ m_2 \cdot r_2 \\ m_2 \cdot r \end{array} \end{array} = \begin{array}{c} U \\ 0 \end{array}
 \end{aligned}$$

Suppose  $r = m \Rightarrow r_1 = m_1 \quad r_2 = m_2 \Rightarrow Q_1 = I_{m_2} \quad Q_2 = I_{m_2} \Rightarrow Q = I_m$

**Cost:**

Fix  $n \in \mathbb{N}$  and set  $\Theta(k) :=$  maximal cost for a matrix  $A \in K^{m \times n'}$  with  $m \leq 2^k \quad n' \leq n$

Choose  $A \in K^{m \times n'}$  with cost =  $\Theta(k)$

Step(4) and (6):  $\leq \Theta(k - 1)$  each

Step(5):  $E^{-1}$ : by Proposition 4:  $O(M(r_1)) \leq O(M(2^{k-1}))$

$F \cdot E^{-1} \leq M(2^{k-1})$

$F \cdot E^{-1} \cdot U'$ : at most the cost of multiplying a  $2^{k-1} \times 2^{k-1}$  matrix by a  $2^{k-1} \times n$  matrix.

Split right matrix into square parts.

$\Rightarrow$  cost  $\leq \lceil \frac{n}{2^{k-1}} \rceil \cdot M(2^{k-1}) \leq (2^{1-k} \cdot n + 1) \cdot M(2^{k-1})$

$G$ : subtraction:  $m_2 \cdot (n - r_1) \leq 2^{k-1} \cdot n \leq 2^{1-k} \cdot n \cdot M(2^{k-1})$

Step (7):  $F \cdot E^{-1}$  already computed,  $L_2 Q_2$ : permuting rows. Cost:  $\leq 2 \cdot M(2^{k-1})$

Obtain:  $\Theta(k) \leq 2 \cdot \Theta(k - 1) + (2^{-k} \cdot n + c) \cdot M(2^k) \quad c \text{ constant}$

From this obtain by induction:

$\Theta(k) \leq \left( 2^{-k} \cdot n \cdot \frac{1-2^{-k\epsilon}}{1-2^{-\epsilon}} + 2 \cdot c \cdot (1 - 2^{-k}) \right) \cdot M(2^k) \leq \left( \frac{1}{1-2^{-\epsilon}} \cdot \frac{n}{2^k} + 2c \right) \cdot M(2^l)$

Finally obtain: Cost  $\leq 8 \cdot \max \left\{ \frac{1}{1-2^{-\epsilon}} \cdot c \right\} \cdot \left( \frac{n}{m} + 1 \right) \cdot M(m) \quad \square$

$\begin{bmatrix} U \\ 0 \end{bmatrix}$  is in row echelon form. It's convenient to write  $U = \begin{bmatrix} E & U' \end{bmatrix} \begin{matrix} r \\ m-r \end{matrix}$   $U' \in K^{r \times (n-r)}$

Also write  $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{matrix} m \\ m-r \end{matrix}$

### 2.6.7 Theorem 7

Let  $A \in K^{m \times n}$ ,  $LQAP = \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \begin{matrix} r \\ m-r \end{matrix}$  as in Algorithm 5 then

(a)  $rk(A) = r$

(b) The columns of  $P \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}$  form a basis of  $ker(A)$

(c) A linear system  $Ax = b$   $b \in K^m$  is solvable iff  $L_2Q \cdot b = 0$

(d) if  $Ax = b$  is solvable then  $x = P \cdot \begin{bmatrix} E^{-1}L_1 \\ 0 \end{bmatrix} \begin{matrix} r \\ n-r \end{matrix} \cdot Q \cdot b$  is a solution

(e) if  $A \in GL_n(K)$  then  $\det(A) = \det(P) \cdot \underbrace{\det(E)}_{=\text{prod of diags}}$

and  $A^{-1} = P \cdot E^{-1} \cdot L$

**Proof:**

(a), (e) :  $\checkmark$

(b) :  $LQAP \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix} = \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix} = 0$

$\Rightarrow$  the columns lie in  $ker(A)$

The columns of  $\begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}$  are linear independent.

$\Rightarrow rk(P \cdot \begin{bmatrix} E^{-1} \cdot U' \\ -I_{n-r} \end{bmatrix}) = n - r$

$\Rightarrow$  the columns form a basis.

The space they generate has dimension  $n - r = \dim(ker(A))$

(c), (d) : If  $A \cdot x = b$  then  $\begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot P^{-1} \cdot x = LQb = \begin{bmatrix} L_1Qb \\ L_2Qb \end{bmatrix}$

$\Rightarrow L_2Qb = 0$

if  $L_2Qb = 0$  then  $A \cdot P \cdot \begin{bmatrix} E^{-1} \cdot L_1 \\ 0 \end{bmatrix} \cdot Q \cdot b = Q^{-1} \cdot L^{-1} \begin{bmatrix} E & U' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} E^{-1} \cdot L_1 \\ 0 \end{bmatrix} \cdot Qb =$

$Q^{-1} \cdot L^{-1} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} Q \cdot b = Q^{-1} \cdot L^{-1} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \cdot Q \cdot b = b$



### 2.6.8 Corollary 8

For  $A \in K^{m \times n}$  the determination of  $rk(A)$  and solving linear systems with coefficient matrix  $A$  require  $O((\frac{n}{m} + 1) \cdot M(m))$  field operations.

If  $A \in K^{n \times n}$  then computing  $\det(A)$  and  $A^{-1}$  (if  $A \in GL_n(K)$ ) require  $O(M(n))$  field operations.

From  $LQAP = \begin{bmatrix} U \\ 0 \end{bmatrix}$  get  $A = Q^{-1} \cdot \underbrace{L^{-1}}_{\text{lower triangular}} \begin{bmatrix} U \\ 0 \end{bmatrix} \cdot P^{-1}$

Generally  $Q = I_m \Rightarrow A = L^{-1} \cdot \begin{bmatrix} U \\ 0 \end{bmatrix} P^{-1}$  "LUP decomposition"

If also  $P = I_n$  obtain  $A = L^{-1} \cdot \begin{bmatrix} U \\ 0 \end{bmatrix}$  "LU decomposition"

## 2.7 Algebraic Systems of Equations, Gröbner bases

Given:  $f_1 \dots f_m \in K[x_1 \dots x_n]$  multivariate polynomials.

Wanted: solution set of the algebraic system  $f_1 = f_2 = \dots = f_m = 0$

The solution set  $\mathcal{V}(f_1 \dots f_m) \subseteq K^n$  is called an affine variety.

Often assume  $K = \bar{K}$   $K$  algebraically closed (e.g.  $K = \mathbb{C}$ )

Questions:

1.  $\mathcal{V}(f_1 \dots f_m) \neq \emptyset$  ?
2.  $|\mathcal{V}(f_1 \dots f_m)| < \infty$  ?
3.  $\dim \mathcal{V}(f_1 \dots f_m) = ?$

**Examples:**

$$(1) f_1 = x_1 x_3 x_4^2 - 2x_2 x_4^2 + x_1 x_3 - 2x_2$$

$$f_2 = x_1 x_3 x_4 - 2x_2 x_4 - 1$$

$$f_3 = x_1 x_4^2 + x_1 + 2$$

$$\text{One has } (-x_1 x_4) \cdot f_1 + (x_1 x_4^2 + x_1 9f_2 + f_3) = 2$$

$$\Rightarrow \mathcal{V}(f_1 \dots f_3) = \emptyset$$

$$(2) f_1 = x^3 + x^2 y + xy + y^2$$

$$f_2 = x^2 y^2 + x^2 + y^3 + y$$

$$f_3 = x^3 + xy$$

$$(x^2 + y) \mid f_i \quad \forall i = 1, 2, 3 \Rightarrow |\mathcal{V}(f_1, f_2, f_3)| = \infty$$

**Univariate case (n=1):**

$K[x]$  Euclidean so have Euclidean algorithm for computing  $\gcd(f, g)$ .  $\gcd$  is unique if required to be monic (i.e. the highest coefficient is 1)

Also get  $h_1, h_2 \in K[x]$  such that  $\gcd(f, g) = h_1 f + h_2 g$

Let  $f_1 \dots f_m \in K[x]$  Obtain

$$g := \gcd(f_1 \dots f_m) = \sum_{i=1}^n h_i \cdot f_i \quad \text{with } h_i \in K[x] \quad (*)$$

For  $\xi \in K$  :

$$f_1(\xi) = f_2(\xi) = \dots = f_m(\xi) = 0 \Rightarrow \underset{*}{g(\xi)} = 0 \Rightarrow \underset{g|f_i}{f_i(\xi)} = 0 \quad \forall i$$

$\Rightarrow$  Only need to get zeros of one polynomial!

### Resultant method:

Reminder: For  $f, g \in K[x] : \gcd(f, g) \neq 1 \Leftrightarrow \text{res}(f, g) = 0$

Let  $f_1, f_2 \in K[x_1 \dots x_n]$

$$(\xi_1 \dots \xi_{n-1}) \in K^{n-1} \quad \text{assume } K = \bar{K}$$

Then  $\exists \xi_n \in K$  such that  $f_1(\xi_1 \dots \xi_n) = 0 = f_2(\xi_1 \dots \xi_n)$

$$\Leftrightarrow \text{res}_{x_n}(f_1(\xi_1 \dots \xi_{n-1}, x_n), f_2(\xi_1 \dots \xi_{n-1}, x_n)) = 0$$

Suppose  $\deg_{x_n} f_1(\xi_1 \dots \xi_{n-1}, x_n) = \deg_{x_n}(f_i) \quad (i = 1, 2)$

Set  $h = \text{res}_{x_n}(f_1, f_2) \in K[x_1 \dots x_{n-1}]$

Then  $\text{res}_{x_n} f_1(\xi_1 \dots \xi_{n-1}, x_n), f_2(\xi_1 \dots \xi_{n-1}, x_n) = h(\xi_1 \dots \xi_{n-1})$

Search zeros of  $h \rightarrow$  one variable, one equation fewer.

Limitation: Only for pairs of polynomials ( $m = 2$ ).

Good case:  $m = n = 2$

Given  $f_1 \dots f_m \in K[x_1 \dots x_n]$  form the ideal

$$I = (f_1 \dots f_m) = \left\{ \sum_{i=1}^n g_i f_i \mid g_i \in K[x_1 \dots x_n] \right\}$$

Clearly  $\mathcal{V}(I) = \mathcal{V}(f_1 \dots f_m)$

$f_1 \dots f_m$  are called a ideal basis of  $I$ . They are not unique, not even their size is unique.

### Example:

In Example (1)  $I$  has an alternative basis  $I = (1) \leftarrow$  constant polynomial  $1 \in K[x_1 \dots x_4]$

In Example (2) it turns out that  $I = (x^2 + y)$

### 2.7.1 Theorem 1 (Hilbert's Nullstellensatz)

#### Hilbert's Nullstellensatz (1st version):

Assume  $K = \bar{K}$  let  $I \subseteq K[x_1 \dots x_n]$  be an ideal

Then  $\mathcal{V}(I) = \emptyset \Leftrightarrow 1 \in I$

( $\Leftrightarrow I = K[x_1 \dots x_n] \Leftrightarrow I = (1)$ )

without proof.

For  $I \subseteq R$  ideal in a commutative ring  $R$  the radical ideal of  $I$  is

$$\sqrt{I} = \{a \in R \mid \exists n \in \mathbb{N} : a^n \in I\}$$

$I$  is called a radical ideal if  $I = \sqrt{I}$

Let  $S \subseteq K^n$  set of points

$$\Rightarrow Id(S) := \{f \in K[\underline{x}] \mid f(v) = 0 \forall v \in S\} \subseteq K[\underline{x}] \quad \text{where } K[\underline{x}] := K[x_1, \dots, x_n]$$

is a radical ideal (called vanishing ideal)

**Hilbert's Nullstellensatz (2nd version):**

Let  $K = \bar{K}$   $I \subseteq K[x]$  ideal. Then  $\sqrt{I} = Id(\mathcal{V}(I))$

Obtain bijection:  $\{\text{radical ideals in } K[x]\} \Leftrightarrow \{\text{affine varieties}\}$

This bijection is inclusion-reversing.

**Monomial orderings:****2.7.2 Definition 2 (Monomial, monomial ordering, LM, LT, LC)**

A monomial is a polynomial of the form  $t = x_1^{e_1} \cdot x_2^{e_2} \cdot \dots \cdot x_n^{e_n} =: \underline{x}^e$  where  $e_i \in \mathbb{N}$

A term is a polynomial of the form  $c \cdot t$   $t$  monomial,  $c \in K \setminus \{0\}$

$M :=$  set of all monomials.

For  $f \in K[x]$ ;  $M(f) :=$  set of all monomials occurring in  $f$ .

$T(f) :=$  set of all terms ...

A monomial ordering is an ordering (= order relation) " $\leq$ " on  $M$  such that:

1. " $\leq$ " is total i.e.  $\forall s, t \in M : s \leq t \vee t \leq s$
2.  $1 \leq t \quad \forall t \in M$
3.  $\forall s, t_1, t_2 \in M : t_1 \leq t_2 \Rightarrow s \cdot t_1 \leq s \cdot t_2$

(This implies:  $s \mid t \Rightarrow s \leq t$ )

For  $f \in K[x] \setminus \{0\}$  we write

$LM(f) =: t$  for the largest monomial in  $M(f)$  ("leading monomial"),

$LT(f) =: c \cdot t$  for the largest term if  $t$  in  $f$  ("leading term")

$LC(f) =: c$  ("leading coefficient")

$LM(0) = LT(0) = LC(0) = 0$

**Example 1:** Lexicographic ordering (lex)

for  $t = x_1^{e_1} \cdot \dots \cdot x_n^{e_n} \quad t' = x_1^{e'_1} \cdot \dots \cdot x_n^{e'_n}$

define  $t \leq t' \Leftarrow t = t' \vee e_i < e'_i$  for the smallest  $i$  with  $e_i \neq e'_i$

**Example 2:** graded reverse lexicographic ordering (grevlex)

$t \leq t' \Leftarrow t = t' \vee \deg(t) < \deg(t') \vee (\deg(t) = \deg(t') \wedge e_i > e'_i)$

for the largest  $i$  such that  $e_i \neq e'_i$

where  $\deg(t) := \sum e_i$

For both lex and grevlex have

$x_1 > x_2 > \dots > x_n$  but  $x_1 \cdot x_3 >_{\text{lex}} x_2^2$

$x_1 > x_2 > \dots > x_n$  but  $x_1 \cdot x_3 <_{\text{grevlex}} x_2^2$

### 2.7.3 Proposition 3 (Sum and product of LM / LT)

Let " $\leq$ " be a monomial ordering  $f, g \in K[x] \Rightarrow$

- (a)  $LT(f \cdot g) = LT(f) \cdot LT(g)$  same for  $LM$
- (b)  $LM(f + g) \leq \max\{LM(f), LM(g)\}$

**Proof:**

(b) ✓

- (a) write  $c \cdot t = LT(f)$   $d \cdot s = LT(g)$   
For  $t' \in M(f)$   $s' \in M(g)$  have  $\underbrace{t's' \leq t \cdot s' \leq t \cdot s}_{=?}$  with equality iff  $s' = s$   $t' = t$

This implies (a)  $\square$

### 2.7.4 Lemma 4 (Dickson-Lemma)

Every subset  $S \subseteq M$  has a finite subset  $B \subseteq S$  ("basis") such that  $\forall s \in S \exists t \in B : t \mid s$

**Proof:** Identify  $M$  with  $\mathbb{N}^n$

Given  $S \subseteq \mathbb{N}^n$  need to show that:

$\exists B \subseteq S, B$  finite such that  $\forall (e_1, \dots, e_n) \in S$

$\exists (d_1, \dots, d_n) \in B$  such that  $\forall i : d_i \leq e_i$

We will write  $(\underline{d}) \leq (\underline{e})$  for this. (This defines a partial ordering in  $\mathbb{N}^n$ )

Induction:

$n = 1$  : if  $\emptyset \neq S \subseteq \mathbb{N}$  then  $\exists d \in S$  such that  $d \leq e \quad \forall e \in S$  ( $\mathbb{N}$  is well-ordered)

$n > 1$  : For  $k \in \mathbb{N}$  write  $S_k := \{(e_2, \dots, e_n) \in \mathbb{N}^{n-1} \mid (k, e_2, \dots, e_n) \in S\} \subseteq \mathbb{N}^{n-1}$

By induction  $\exists B_k \subseteq S_k$  finite such that  $\forall (\underline{e}) \in S_k \exists (\underline{d}) \in B_k$  such that  $(\underline{d}) \leq (\underline{e})$

$\bigcup_{k \in \mathbb{N}} B_k \subseteq \mathbb{N}^{n-1}$  has finite "basis"  $C$

$C$  finite  $\exists r \in \mathbb{N} : C \subseteq \bigcup_{k=0}^r B_k$  (\*)

From  $B := \{(e_1, \dots, e_n) \in \mathbb{N}^n \mid e_1 \leq r, (e_2, \dots, e_n) \in B_{e_1}\} \Rightarrow |B| < \infty, B \subseteq S$

**Claim:**  $B$  basis of  $S$

Let  $(e_1, \dots, e_n) \in S \Rightarrow (e_2, \dots, e_n) \in S_{e_1} \Rightarrow \exists (d_2, \dots, d_n) \in B_{e_1}$  such that  $d_i \leq e_i \forall i \geq 2$

**Case 1:**  $e_1 \leq r$

$\Rightarrow (e_1, d_2, \dots, d_n) \in B$  have  $(e_1, d_2, \dots, d_n) \leq (e_1, \dots, e_n)$

**Case 2:**  $e_1 > r$

$B_{e_1} \subseteq \bigcup_{k \in \mathbb{N}} B_k \Rightarrow \exists (c_2, \dots, c_n) \in C$  such that  $c_i \leq d_i \forall i \geq 2$

By (\*)  $\exists k \leq r : (\underline{c}) \in B_k \Rightarrow (k, c_2, \dots, c_n) \in B$

$(k, c_2, \dots, c_n) \leq (e_1, d_2, \dots, d_n) \leq (e_1, e_2, \dots, e_n)$  ✓  $\square$

### 2.7.5 Corollary 5 (Well-ordering of monomial sets)

Every monomial ordering is a well-ordering i.e. every monomial set  $S \subseteq M$  has an element  $t \in S$  such that  $\forall s \in S : t \leq s$  ( $t$  is a "least element")

**Proof:**

Let  $\emptyset \neq S \subseteq M$  By Lemma 4  $\exists B \subseteq S$  finite such that  $\forall s \in S' \exists t \in B : t \mid s$

Since " $\leq$ " is total and  $B$  is finite  $\exists t \in B$  least element.

Let  $s \in S \Rightarrow \exists t' \in B$  such that  $t' \mid s \Rightarrow t' \leq s$  so  $t \leq t' \leq s$   $\square$

**Gröbner bases:** Let " $\leq$ " be a fixed monomial ordering

### 2.7.6 Definition 6 (Leading ideal, Gröbner bases)

- (a) For  $S \in K[x]$  subset define  $L(S) := (LM(f) \mid f \in S) \subseteq K[x]$   
(ideal generated by all leading monomials of elements of  $S$ ) is called the leading ideal
- (b) Let  $I \subseteq K[x]$  ideal. A finite subset  $G \subseteq I$  is called a Gröbner basis if  $L(I) = L(G)$   
i.e.  $\forall f \in I \exists g \in G : LM(g) \mid LM(f)$

### 2.7.7 Proposition 7 (Ideality of Gröbner bases)

$G$  Gröbner basis of  $I \Rightarrow I = (G)$  i.e.  $G$  is an ideal basis.

**Proof:**  $G \subseteq I \Rightarrow (G) \subseteq I$

Assume this inclusion is strict. Let  $f \in I \setminus (G)$  By Corollary 5 my assume  $LM(f)$  is minimal (among all leading monomials of elements from  $I \setminus (G)$ )

$LM(f) \in L(I) = L(G) \Rightarrow \exists g \in G : LM(g) \mid LM(f)$

Form  $\tilde{f} = f - \frac{LT(f)}{LT(g)}g, \tilde{f} \in I \Rightarrow LM(\tilde{f}) < LM(f)$

by minimality  $\tilde{f} \in (G) \Rightarrow f = \tilde{f} + \frac{LT(f)}{LT(g)}g \in (G)$  contradiction!  $\square$

$G \subseteq I \quad L(G) = L(I) \quad \Rightarrow I = (G)$

**Example:**

$I = (1) \in K[x] \quad S = \{x+1, x\}$  ideal basis but  $I(S) = (x) \neq L(I) = (1)$

$S$  is not a Gröbner basis.

### 2.7.8 Theorem 8 (Gröbner basis of Ideals)

Every ideal  $I \subseteq K[x]$  has a Gröbner basis. In particular  $I$  has a finite basis ( $\rightarrow$  Hilbert's basis theorem) In other words  $K[x]$  is Noetherian.

**Proof:**

For  $\{LM(f) \mid f \in I\}$  there exists (by Dickson lemma) a finite subset  $\{LM(f_1), \dots, LM(f_m)\}, f_i \in I$  such that  $(LM(f_1) \dots LM(f_m)) = L(I) \Rightarrow G = \{f_1 \dots f_m\}$

Gröbner basis  $\square$

First application: Let  $G$  Gröbner basis of  $I$

Then  $\mathcal{V}(I) = \emptyset \Leftrightarrow_{K=\bar{K}} 1 \in I \Leftrightarrow G$  contains a non-zero constant

### 2.7.9 Definition 9 (Normal form)

Let  $S = \{g_1 \dots g_m\} \subseteq K[x]$   $f \in K[x]$

- (a)  $f$  is a normal form with respect to  $S$  if  $\forall t \in M(f) \quad \forall i = 1 \dots m : LM(g_i) \nmid t$
- (b)  $f^* \in K[x]$  is called a normal form of  $f$  with respect to  $S$  if
  - (i)  $f^*$  is in normal form with respect to  $S$
  - (ii)  $\exists h_1 \dots h_m \in K[x]$  such that  $f - f^* = \sum_{i=1}^m h_i g_i$  and  $\forall i : LM(h_i g_i) \leq LM(f)$

#### Example:

$S = \{x, x+1\}$   $f = 1 \Rightarrow f \equiv 0 \pmod{(S)}$

but 0 is not a normal form of  $f$

If  $f = x$  then 0 and  $-1$  are normal forms of  $x$

### 2.7.10 Algorithm 10 (Normal form)

Input :  $S = \{g_1 \dots g_m\} \subseteq K[x]$   $f \in K[x]$

Output : A normal form  $f^*$  of  $f$  with respect to  $S$  and if desired  $h_1 \dots h_m$  satisfying (\*)

- (1) Set  $f^* := f$   
for  $(i = 1 \dots m)$   
     $h_i := 0$
- (2) repeat (3) - (6)
- (3)  $\mathcal{M} := \{(t, i) \mid t \in M(f^*), i \in \{1, \dots, m\} \text{ such that } LM(g_i) \mid t\}$
- (4) if  $(\mathcal{M} = \emptyset)$   
    return  $f^*$  and  $h$
- (5) Choose  $(t, i) \in \mathcal{M}$  such that  $t$  is maximal.  
let  $c \in K$  be the coefficient of  $t$  in  $f^*$
- (6) Set  $f^* := f^* - \frac{c \cdot t}{LT(g_i)} \cdot g_i$   
     $h_i := h_i + \frac{c \cdot t}{LT(g_i)}$

Step (6) cancels the term  $c \cdot t$  from  $f^*$  and may add only monomials smaller than  $t$ . So the  $t$ 's form a strictly descending sequence of monomials  $\Rightarrow$  Algorithm 10 terminates.

Correctness

✓

### 2.7.11 Theorem 11 (Normal form of Gröbner bases)

Let  $G \subseteq K[x]$  be a Gröbner basis of an ideal  $I \subseteq K[x]$

- (a) Every polynomial  $f \in K[x]$  has a unique normal form with respect to  $G$ .  
Write  $NF_G(f)$
- (b)  $NF_G : K[x] \mapsto K[x]$  is  $K$ -linear,  
 $\ker(NF_G) = I$
- (c) if  $\tilde{G} \subseteq K[x]$  is another Gröbner basis (with respect to same monomial ordering)  
then  $NF_G = NF_{\tilde{G}}$

**Proof:**

(a), (c):

Let  $f \in K[x]$   $f^*, \tilde{f} \in K[x]$  be normal forms of  $f$  with respect to  $G$  and  $\tilde{G}$  respectively.

**Claim:**  $f^* = \tilde{f}$

$f^* - f \in I$ ,  $\tilde{f} - f \in I \Rightarrow f^* - \tilde{f} \in I \Rightarrow LM(f^* - \tilde{f}) \in L(G) \in L(\tilde{G})$

if  $f^* \neq \tilde{f} \Rightarrow LM(f^* - \tilde{f}) \in M(f^*)$  or  $\in M(\tilde{f})$

But  $\exists g \in G : LM(g) \mid LM(f^* - \tilde{f})$ ,  $\exists \tilde{g} \in \tilde{G} : LM(\tilde{g}) \mid LM(f^* - \tilde{f})$

This is a contradiction to:

$f^*$  is in normal form with respect to  $G$  and

$\tilde{f}$  is in normal form with respect to  $\tilde{G}$

So  $f^* = \tilde{f}$

(b):

Let  $f, g \in K[x]$   $c \in K$ . Set  $h := NF_G(f + cg) - NF_G(f) - c \cdot NF_G(g)$

To show:  $h = 0$   $h \equiv f + cg - f - cg = 0 \pmod{I}$

$\Rightarrow h \in I \Rightarrow LM(h) \in L(G)$

$h$  is in normal form with respect to  $G$

$\Rightarrow h = 0$

Remains to show:  $\ker(NF_G) = I$

let  $NF_G(f) = 0 \Rightarrow f \equiv 0 \pmod{I} \Rightarrow f \in I$  conversely, let  $f \in I$

$\Rightarrow f^* = NF_G(f) \in I \Rightarrow \exists g \in G : LM(g) \mid LM(f^*)$   $f^*$  in normal form. So  $f^* = 0$  □

Further applications of Gröbner bases:

- Membership test:  $f \in I \Leftrightarrow NF_G(f) = 0$
- Computation in  $A := K[x]/I$  :  $NF_G$  includes an embedding  $A \hookrightarrow K[x]$

## Buchberger's Algorithm

### 2.7.12 Definition 12 (*S*-polynomials)

Let  $f, g \in K[\underline{x}] \setminus \{0\}$   $t := \gcd(LM(f), LM(g))$

Then  $s_{pol}(f, g) := \frac{LT(g)}{t} \cdot f - \frac{LT(f)}{t} \cdot g$  is the *S*-polynomial.

The leading monomials of the summands cancel!

**Example:**

$$f = x^2 + y^2, \quad g = x \cdot y \quad " \leq " = lex$$

$$\Rightarrow LM(f) = x^2 \quad LM(g) = xy$$

$$s_{pol}(f, g) = y \cdot f - x \cdot g = y^3$$

### 2.7.13 Theorem 13 (Buchberger's criterion)

For any finite set  $G \subseteq K[\underline{x}]$  the following statements are equivalent:

- (a)  $G$  is a Gröbner basis of  $\langle G \rangle$
- (b) For polynomials  $g, h \in G$ ,  $0$  is a normal form of  $s_{pol}(g, h)$  with respect to  $G$   
 $\rightarrow$  finite test for Gröbner basis!

**Proof:**

"(a)  $\Rightarrow$  (b)":

For  $g, h \in G : s_{pol}(g, h) \in \langle g, h \rangle \subseteq \langle G \rangle =: I \xRightarrow{\text{Theorem 13 (b)}} s_{pol}(g, h)$  has normal form 0

$$\Leftrightarrow NF_G(s_{pol}(g, h)) = 0$$

"(a)  $\Leftarrow$  (b)":

Assume  $G$  is not a Gröbner basis  $\Rightarrow \exists f \in I \setminus G$  such that  $LM(f) \notin L(G)$ .

$$\text{Write } G = \{g_1 \dots g_m\}. \text{ Since } f \in \langle G \rangle \exists h_1 \dots h_m \in K[\underline{x}] \text{ have } f = \sum_{i=1}^m h_i \cdot g_i \quad (1)$$

By Corollary 5 may choose  $h_i$  such that

$t := \max\{LM(h_i g_i) \mid i = 1 \dots m\}$  becomes minimal.

$\exists i : LM(f) \in M(h_i g_i)$ . Since  $LM(f) \notin L(G) \wedge LM(f) \neq LM(h_i g_i)$

$$\Rightarrow LM(f) < LM(h_i g_i) \leq t$$

$\Rightarrow$  the coefficient of  $t$  in  $\sum h_i g_i$  is zero.

$$\text{Set } c_1 := \begin{cases} LC(h_i) & \text{if } LM(h_i g_i) = t \\ 0 & \text{otherwise} \end{cases} \text{ Then } \sum_{i=1}^m c_i \cdot LC(g_i) = 0 \quad (2)$$

Without loss assume  $c_1 \neq 0$

Let  $i \in \{2, \dots, m\}$  such that  $c_i \neq 0 \Rightarrow LM(g_i) \mid t$

So  $t_i = \text{lcm}(LM(g_i), LM(g_1) \mid t)$

$$\text{Have } s_{pol}(g_i, g_1) = \frac{LC(g_1) \cdot t_i}{LM(g_i)} g_i - \frac{LC(g_i) \cdot t_i}{LM(g_1)} g_1 \quad LM(s_{pol}(g_i, g_1)) < 0$$

$$\Rightarrow s_i := \frac{t}{t_i} \cdot s_{pol}(g_i, g_1) = LC(g_1) \cdot LM(h_i) \cdot g_i - LC(g_i) \cdot LM(h_1) \cdot g_1 \quad (3)$$

$$\text{By (b) have } s_i = \sum_{j=1}^m h_{ij} \cdot g_j \quad \text{with } h_{ij} \in K[\underline{x}] \text{ such that } LM(h_{ij} g_j) \leq LM(s_i) < t \quad (4)$$



$$\begin{aligned}
& \sum_{j=1}^m \left( \sum_{i=2}^m c_i \cdot h_{ij} \right) \cdot g_j = \sum_{i=2}^m c_i s_i \\
& = \sum_{i=2}^m c_i (LC(g_1)LM(h_i)g_i - LC(g_i)LM(h_1)g_1) + \sum_{i=1}^m c_i LM(h_1)g_1 LC(g_i) \\
& \stackrel{(3)}{=} \sum_{i=1}^m c_i LC(g_1)LM(h_i)g_i
\end{aligned}$$

$$\text{Set } \tilde{h}_j := \frac{1}{LC(g_1)} \cdot \sum_{i=2}^m c_i h_{ij} \Rightarrow g := \sum_{i=1}^m c_i LM(h_i)g_i = \sum_{i=1}^m \tilde{h}_i g_i$$

For each  $i$  have:  $LM(\tilde{h}_i g_i) \stackrel{(4)}{<} t$

$$f = (f - g) + g \stackrel{(1)}{=} \sum_{i=1}^m (h_i - c_i LM(h_i))g_i + \sum_{i=1}^m \tilde{h}_i g_i$$

For each  $i$  have:  $LM((h_i - c_i LM(h_i))g_i) < t$  so  $LM((h_i - c_i LM(h_i) + \tilde{h}_i)g_i) < t$   
contradiction to choice of  $h_i$   $\square$

#### 2.7.14 Algorithm 14 (Buchberger)

Input :  $S \subseteq K[x]$  finite "  $\leq$  " monomial ordering

Output: A Gröbner basis  $G$  of  $I = (S)$  with respect to "  $\leq$  "

- (1)  $G := S \setminus \{0\}$
- (2) for  $g, h \in G$  repeat (3),(4)
- (3)     Compute  $s := s_{pol}(g, h)$   
          and a normal form  $s^*$  of  $s$  with respect to  $G$
- (4)     if( $s^* \neq 0$ )  
          set  $G := G \cup \{s^*\}$   
          go to (2)
- (5) return  $G$

#### 2.7.15 Theorem 15 (Correctness of Algorithm 14)

Algorithm 14 terminates after finitely many steps and computes a Gröbner basis.

**Proof:**

**Termination:**

Let  $G_i$  be the set  $G$  obtained after the  $i$ -th run through the loop.  $G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots$

From  $\bar{G} = \bigcup_i G_i$  finite or infinite.

Lemma 4:  $\exists B \subseteq M$  finite set of monomials,  $B \subseteq \{LM(f) \mid f \in \bar{G}\}$  such that

$\forall f \in \bar{G} \quad \exists t \in B$  such that  $t \mid LM(f)$  (\*)

Since  $|B| < \infty \quad \exists r$  such that  $B \subseteq \{LM(f) \mid f \in G_r\}$

Without loss  $B = \{LM(f) \mid f \in G_r\}$

**Claim:**  $G_r$  is the last of the  $G$

If not  $\exists G_{r+1} : G_{r+1} = G_r \cup \{s^*\}$

$s^* \neq 0$  in normal form with respect to  $G_r$ . But by (\*)  $\exists f \in G_r$  such that  $LM(f) \mid LM(s^*)$  contradiction.

**Correctness:** by Theorem 13  $\square$

**Example:**

$S = \{\underbrace{x^2 + g^2}_f, \underbrace{xy}_g\} \subseteq \mathbb{Q}[x, y]$  "  $\leq$  " lex ordering with  $x > y$

$s_{pol}(f, g) = yf - xg = y^3 =: h$  in normal form with respect to  $S$

$G = \{f, g, h\}$

$s_{pol}(f, g) = h \xrightarrow{\text{normal form}} 0$

$s_{pol}(f, h) = y^3 f - x^2 h = y^5 \xrightarrow{\text{normal form}} 0$

$s_{pol}(g, h) = y^2 g - xh = 0$

$\Rightarrow G$  Gröbner basis

**Cost of Buchberger algorithm:**

- no known upper bound for the running time
- with  $d = \max\{\deg(f) \mid f \in S\} : \underbrace{\deg(g_i)}_{\text{polys from } G} \leq 2 \cdot \left(\frac{d^2}{2} + d\right)^{2^{n-1}}$   
with  $n = \text{number of Variables}$   
 $\Rightarrow$  "Doubly exponential" in  $n$   
Ritscher (2011): upper bound for  $\deg(g_i)$  depending  $\dim(\underbrace{\mathcal{V}}_{\text{Variety}}(S))$
- Nonetheless the algorithm often works
- Many possible optimizations

Variant: Extended Buchberger:

Keep track of how the new elements in  $G$  are represented as linear combination of elements of  $S$ .

### 2.7.16 Definition 16 (Reduced Gröbner basis)

A Gröbner basis  $G$  is called reduced if  $\forall g \in G$

- $g$  is in normal form with respect to  $G \setminus \{g\}$
- $LC(g) = 1$

A given Gröbner basis can be turned into a reduced only by replacing every  $g \in G$  by a normal form of  $g$  with respect to  $G \setminus \{g\}$ . Then remove  $0 \in G$ . Then divide each  $g \in G$  by  $LC(g)$

### 2.7.17 Theorem 17 (Uniqueness of reduced Gröbner basis)

From ideal  $I \subseteq K[x]$  and a monomial ordering " $\leq$ ", there exists a unique reduced Gröbner basis.

## 2.8 Application of Gröbner bases

### 2.8.1 Definition 1 (Elimination ideals)

- (a)  $I \subseteq K[X_1, \dots, X_n]$  ideal,  $l \in \{1, \dots, n\} \Rightarrow I_l := K[X_1, \dots, X_n]$  is called the  $l$ -length elimination ideal
- (b) A monomial ordering " $\leq$ " is called an  $l$ -elimination ordering if  
 $\forall 1 \leq i \leq l < j \leq n \quad \forall k_i \quad X_i^{k_i} < X_j$

**Example:**

- (1) Let " $\leq$ " be a given monomial ordering. Define " $\leq$ " by:  
for  $s = \underline{x}^d \quad t = \underline{x}^e$  define  
 $s \leq' t \Leftrightarrow \sum_{i=l+1}^n d_i < \sum_{i=l+1}^n e_i$  or have equality  $\sum_{i=l+1}^n d_i = \sum_{i=l+1}^n e_i$  and  $s \leq t$   
 $\Rightarrow$  " $\leq$ " is an  $l$ -eliminating ordering.
- (2) The lexicographic ordering with  $x_1 < x_2 < \dots < x_n$  is an  $l$ -eliminating ordering for all  $l$
- (3) Grevlex is not an  $l$ -eliminating ordering (unless  $l = n$ )

### 2.8.2 Theorem 2

Let  $G$  be a Gröbner basis of an ideal  $I \subseteq K[X_1, \dots, X_n]$  with respect to an  $l$ -elimination ordering. Then  $G_l := K[X_1, \dots, X_n] \cap G$  is a Gröbner basis.  $I : l$  with respect to the restricted monomial ordering.

**Proof:**

$G_l \subseteq I_l$  Let  $f \in I_l \setminus \{0\} \quad f \in I$

$\Rightarrow \exists g \in G : LM(g) \mid LM(f)$

To show:  $g \in G$ . Clearly  $LM(g) \in K[X_1, \dots, X_l]$

If  $g \notin K[X_1, \dots, X_n]$  then  $\exists t \in M(g)$  such that

$X_j \mid t$  with  $j > l \Rightarrow t \geq X_j >_{\text{"}\leq\text{"}_{\text{elim ord}}} LM(g)$  contradiction.  $\square$

**Example:**

$$I = (\underbrace{X_1^2 + X_2^2}_f, \underbrace{x_1 x_2}_g), G = \{f, g, X^3\}$$

Gröbner basis with respect to lex ordering  $X_1 < X_2$

$$I_1 = (X_1^3)$$

**Geometric interpretation:**

Let  $K = \bar{K}$  algebraically closed. On  $K^n$  define the Zariski topology.

By saying that the sets  $\mathcal{V}(I)$  with  $I \subseteq K[X_1, \dots, X_n]$  are the closed sets.

Why is this a topology?

Reminder: A topological space is a set  $X$  together with a system of subsets, called closed subsets, such that three axioms hold.

- (1)  $\emptyset = \mathcal{V}(K[X_1, \dots, X_n]) \quad K^n \in \mathcal{V}(\{0\})$  closed
- (2) Let  $\mathcal{M}$  be a set of closed subsets corresponding to a set  $\mathcal{N}$  of ideals.

$$\text{Then } \bigcap_{I \in \mathcal{N}} \mathcal{V}(I) = \mathcal{V}\left(\sum_{I \in \mathcal{N}} I\right) \quad \checkmark$$

- (3) The union of two closed subsets is closed.

Let  $I, J \subseteq K[X]$  ideals.

**Claim:**  $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J)$

**Proof:** " $\subseteq$ " direction:

Let  $v \in \mathcal{V}(I) \cup \mathcal{V}(J) \quad f \in I \cap J$

If  $v \in \mathcal{V}(I)$  then  $f(v) = 0$

$v \in \mathcal{V}(J)$  then  $f(v) = 0$

" $\supseteq$ " direction:

Let  $v \in \mathcal{V}(I \cap J)$  Assume  $v \notin \mathcal{V}(I) \Rightarrow \exists f \in I : f(v) \neq 0$

Let  $g \in J \Rightarrow f \cdot g \in I \cap J \Rightarrow \underbrace{f(v)}_{\neq 0} \cdot g(v) = 0$

$\Rightarrow g(v) = 0 \quad \text{So } v \in \mathcal{V}(J)$

All points in  $K^n$  are closed so are all finite subsets.

**Closures:** For  $X \subseteq K^n$  the closure  $\bar{X}$  is defined as the smallest closed subset containing  $X$ .  $\bar{X}$  is the variety of the largest ideal  $I$  such that  $X \subseteq \mathcal{V}(I)$  This ideal is  $I = Id(X)$ . So  $\bar{X} = \mathcal{V}(Id(X))$

Let  $\Pi_l : K^n \mapsto K^l, (a_1, \dots, a_n) \mapsto (a_1, \dots, a_l)$  projection.

### 2.8.3 Theorem 3

$$I \subseteq K[X_1, \dots, X_n] \Rightarrow \mathcal{V}(I) = \overline{\Pi_l(\mathcal{V}(I))}$$

**Proof:**

Let  $(a_1, \dots, a_l) \in \Pi_l(\mathcal{V}(I)) \Rightarrow \exists n_{l+1} \dots n_1 \in K$  such that  $(a_1, \dots, a_n) \in \mathcal{V}(I)$

Let  $f \in I_l \Rightarrow f \in I \Rightarrow f(a_1, \dots, a_n) = 0$

But  $0 = (a_1, \dots, a_n) = f((a_1, \dots, a_l))$  So  $(a_1, \dots, a_n) \in \mathcal{V}(I_l)$

So  $\Pi_l(\mathcal{V}(I)) \subseteq \mathcal{V}(I_l) \xRightarrow{\mathcal{V}(I_l) \text{ closed}} \overline{\Pi_l(\mathcal{V}(I))} \subseteq \mathcal{V}(I_l)$ .

To show:  $\mathcal{V}(I_l) \subseteq \overline{\Pi_l(\mathcal{V}(I))} \xLeftrightarrow{\text{H. Nullstellensatz}} Id(\Pi_l(\mathcal{V}(I))) \subseteq \sqrt{I_l}$  Take  $f \in Id(\Pi_l(\mathcal{V}(I))) \Rightarrow$

$f \in K[X_1, \dots, X_l] \forall (a_1, \dots, a_n) \in \mathcal{V}(I)$ .

$f(a_1, \dots, a_l) = f(a_1, \dots, a_n) = 0 \Rightarrow f \in Id(\mathcal{V}(I)) = \sqrt{I} \Rightarrow \exists k : f^k \in I \cap K[X_1, \dots, X_l] = I_l$

$\Rightarrow f \in \sqrt{I_l} \quad \square$

**Example:**

(1)  $I = (xy - 1)$

$\Pi_1(\mathcal{V}(I)) = K \setminus \{0\}$  not closed.  $\overline{\Pi_1(\mathcal{V}(I))} = K.I_1 = \{0\}$

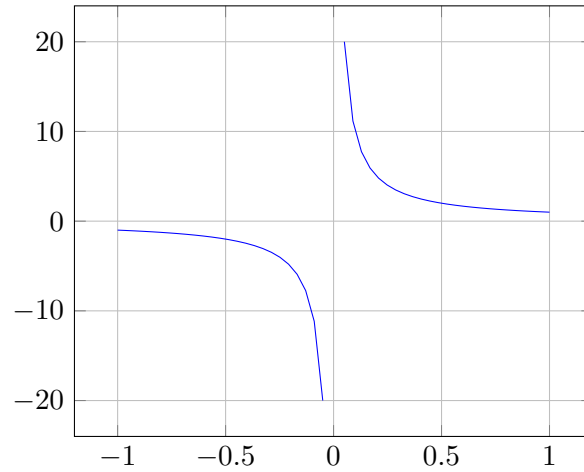


Figure 3: Plot of  $I$  in Example (1)

(2)  $I = (xy - 2x + y - 2, kx^2 - 4xy + y^2)$  has reduced lex Gröbner basis:

$$\{x^2 - 2x + \frac{y^2}{4} + y - 2, xy - 2x + y - 2, (y - 2)(y + 2)^2\}$$

$$\Rightarrow I_1 = ((y - 2)(y + 2)^2)$$

$$\Rightarrow \overline{\Pi_1(\mathcal{V}(I))} = \{2, -2\}$$

$$\Rightarrow \Pi_1(\mathcal{V}(I)) = \{2, -2\}$$

Substitute  $y = 2$ :

$$x^2 - 2x + 1, 2x - 2x + 2 - 2 = 0$$

$$y = -2 :$$

$$x^2 - 2x - 3 = 0, -4x - 4 = 0$$

$$\rightarrow \mathcal{V}(I) = \{(1, 2), (-1, -2)\}$$

### 2.8.4 Algorithm 4 (Solving systems of algebraic equations)

Input :  $f_1, \dots, f_m \in K[X_1, \dots, X_n]$

Output:  $\mathcal{V}(f_1, \dots, f_m)$  if finite otherwise " $\infty$ "

- (1) Compute a Gröbner basis of  $I = (f_1, \dots, f_m)$  with respect to the lex ordering with  $X_1 < X_2 < \dots < X_n$
- (2) for  $(l = 1, \dots, n)$   
     set  $G_l := [X_1, \dots, X_l] \cap G$
- (3)  $M := \{()\} \subseteq K^0$
- (4) for  $(l = 1, \dots, n)$  repeat (5) - (10)
- (5)  $S = \emptyset$
- (6) for  $(a_1, \dots, a_{l-1}) \in M$  repeat (7) - (9)
- (7)  $g := \gcd\{f(a_1, \dots, a_{l-1} \mid f \in G_l)\}$
- (8) if  $(g = 0)$   
     return " $\infty$ "
- (9)  $S := S \cup \{(a_1, \dots, a_{l-1}, a_l) \mid g(a_l) = 0\}$
- (10)  $M := S$
- (11) return  $M$

### Intersections of ideals

### 2.8.5 Proposition 5

Let  $I, J \subseteq K[\underline{x}]$  ideals,  $y$  additional variable.

Form  $L \subseteq K[x_1, \dots, x_n, y]$  generated by  $I \cdot y$  and  $J \cdot (1 - y)$  Then  $I \cap J = K[\underline{x}] \cap L$  (elimination ideal!)

**Proof:**

Let  $f \in J \cap I \Rightarrow f = f \cdot y + f \cdot (1 - y) \in L \Rightarrow f \in K[\underline{x}] \cap L$

Conversely let  $f \in K[\underline{x}] \cap L \Rightarrow f = \sum_{i=1}^r g_i f_i \cdot y_i + \sum_{i=r}^m g_i \cdot f_i (1 - y)$

with  $f_1, \dots, f_r \in I$   $f_{r+1}, \dots, f_m \in J$   $g_i \in K[\underline{x}, y]$

Specialize  $y = 0 \Rightarrow f = \sum_{i=r}^m \underbrace{g_i(y=0)}_{\in K[\underline{x}]} f_i \in J$

Specialize  $y = 1 \Rightarrow f = \sum_{i=1}^r g_i(y=1) f_i \in I$

$\Rightarrow f \in I \cap J$   $\square$

## Dimension

### 2.8.6 Definition 6 (independence modulo $I$ )

Let  $I \subseteq K[\underline{x}]$  ideal. Then polynomials  $f_1, \dots, f_r \in K[\underline{x}]$  are called independent modulo  $I$  if for every polynomial  $F \in K[y_1, \dots, y_r]$  have:  $F(f_1, \dots, f_r) \in I \Rightarrow F = 0$

(So the classes  $\bar{f}_i \in A := K[\underline{x}]/I$  are algebraically independent)

For  $1 \leq i_1 < i_2 < \dots < i_r \leq n : x_{i_1}, \dots, x_{i_r}$  are independent modulo  $I \Leftarrow K[x_{i_1}, \dots, x_{i_r}] \cap I = \{0\}$

(elimination ideal!)

$\dim(I) := \sup\{r \in \mathbb{N} \mid \exists f_1, \dots, f_r \in K[\underline{x}] \text{ independent mod } I\}$  dimension of  $I$

In other words  $\dim(I) =: \text{trdeg}(A)$  "transcendence degree"

If  $I = K[\underline{x}] \Rightarrow \dim(I) = -1$

For  $X = \mathcal{V}(I) : \dim(X) = \dim(I)$

Well defined? Clearly  $\dim(I) = \dim(\sqrt{I})$

### Geometric interpretation:

$x_{i_1}, \dots, x_{i_r}$  are independent mod  $I \Leftrightarrow \Pi_{i_1, \dots, i_r} : K^n \mapsto K^r$  maps  $\mathcal{V}(I)$  to a dense subset of  $K^r$

More generally for  $f_1, \dots, f_r$  have:

$f_1, \dots, f_r$  are independent modulo  $I \Leftrightarrow$  The image of  $\mathcal{V}(I)$  under the "morphism"  $K^n \mapsto K^r, v \mapsto (f_1(v), \dots, f_r(v))$  is dense.

### 2.8.7 Theorem 7

Let  $I \subseteq K[\underline{x}]$  ideal

$\Rightarrow \dim(I) = \max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ are independent modulo } I\}$

### 2.8.8 Lemma 8

A non-empty set  $M$  of ideals in  $K[\underline{x}]$  has an element that is maximal with respect to inclusion.

### Proof:

If not obtain a sequence  $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$  of ideals in  $M$

Set  $I := \bigcup_{j \in \mathbb{N}} I_j$  Check:  $I \subseteq K[\underline{x}]$  ideal

$\Rightarrow I = (f_1, \dots, f_r)$  Each  $f_i$  lies in some  $I_j$

$\Rightarrow \exists m : f_i \in I_m \forall i \Rightarrow I \subseteq I_m \subseteq I \Rightarrow I = I_m$

$I_m \subseteq I_{m+1} \subseteq I = I_m \Rightarrow I_{m+1} = I_m$

Recall that an ideal  $I \subseteq K[\underline{x}]$  is called a prime ideal if  $a \cdot b \in I$  for  $a, b \in K[\underline{x}]$  implies  $a \in I$  or  $b \in I$

Fact:  $I$  prime ideal  $\Leftrightarrow I = \sqrt{I}$  and  $\mathcal{V}(I)$  is irreducible, i.e.  $I$  can't be the written as union of two proper closed subsets.

### 2.8.9 Lemma 9

Every radical ideal  $I \subsetneq K[x]$  is a finite intersection of prime ideals.

**Proof:**

If no by Lemma 8 there is a maximal exception  $I \subsetneq K[x]$ .  $I$  not prime ideal  $\Rightarrow \exists a_1, a_2 \in K[x] : a_1 a_2 \in I, a_j \notin I \Rightarrow I \subsetneq I + (a_j) \subseteq \sqrt{I + (a_j)} =: I_j$  (\*)  
 $\Rightarrow I \subseteq I_1 \cap I_2$

**Claim:**  $I = I_1 \cap I_2$  ✓

Let  $g \in I_1 \cap I_2 \Rightarrow \exists r : g^r \in I + (a_j) (j = 1, 2)$   
 $\Rightarrow g^{2r} \in (I + (a_1)) \cdot (I + (a_2)) \subseteq I \xrightarrow{I=\sqrt{I}} g \in I$

(\*) with maximality of  $I$  shows that the  $I_j$  are finite intersections of prime ideals

$\Rightarrow$  same holds for  $I$  contradiction  $\square$

Reminder: If  $L | K$  is a field extension  $L = K[\alpha_1, \dots, \alpha_n]$ .  $\text{trdeg}_K(L)$  the size of every maximally algebraically independent subset of  $\{\alpha_1, \dots, \alpha_n\}$

(Exchange argument similar to linear algebra)

### Proof of Theorem 7

Clearly  $\max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ independent modulo } I\} \leq \dim(I)$

For reverse inequality take  $f_1, \dots, f_r \in K[x]$  independent mod  $I$ . By Lemma 9:  $\sqrt{I} = P_1 \cap \dots \cap P_s$  with  $P_i$  prime ideals.

Assume that the  $f_i$  are dependent modulo every  $P_i \Rightarrow \exists F_i \in K[y_1, \dots, y_r] \setminus \{0\}$  such that

$F_i(f_1, \dots, f_r) \in P_i \Rightarrow$  For  $F := \prod_{i=1}^s F_i$

$F(f_1, \dots, f_r) \in P_1 \cap \dots \cap P_s = \sqrt{I} \Rightarrow \exists k$

$F^k(f_1, \dots, f_r) \in I$ . But  $F^k \neq 0$  contradiction

So  $\exists i : f_1, \dots, f_r$  independent mod  $P_i$

$\bar{f}_1, \dots, \bar{f}_r \in A := K[x]/P$  algebraically independent.

$A$  integral domain  $\Rightarrow L = \text{Quot}(A)$  exists  $\bar{f}_i \in A \subseteq L$ .  $L/K$  is field extension,  $L = K(\bar{x}_1, \dots, \bar{x}_n)$  Since  $\text{trdeg}_K(L) \geq r$  have  $i_1, \dots, i_r$  such that  $\bar{x}_{i_1}, \dots, \bar{x}_{i_r} \in L$  are algebraically independent over  $K$ .

$\Rightarrow x_{i_1}, \dots, x_{i_r}$  are independent modulo  $P_i$

Since  $I \subseteq P_i$ ,  $x_{i_1}, \dots, x_{i_r}$  are independent modulo  $I$

$\Rightarrow r \leq \max\{r \mid \exists i_1, \dots, i_r \text{ such that } x_{i_1}, \dots, x_{i_r} \text{ independent modulo } I\}$

Now we can compute  $\dim(I)$  by using elimination ideals. Have to determine a subset of  $\{x_1, \dots, x_n\}$  of maximal size that is independent modulo  $I \rightarrow$  Gröbner bases.

**Example:**

$I = \underbrace{(xy, xz)}_{\text{Gröbner bases}}$  Independent subsets modulo  $I : \emptyset, \{x\}, \{y\}, \{z\}, \{y, z\}$

$\dim(I) = 2$



**Hilbert series** For  $X \subseteq K^n$  affine variety,  $I = Id(X)$ . Then  $A := K[x]/I$  is the set of polynomial function ("regular functions")  $X \mapsto K$

How large is  $A$ ?  $A$  is a vector-space but it is not finite-dimensional (as vector-space) unless  $\dim(X) \leq 0$

Idea: measure the size of  $A$  by studying the growth of the dimensions of parts of  $A$  given by a filtration  $A = \bigcup_{d \in \mathbb{N}} A_d$

### 2.8.10 Definition 10 (Hilbert series)

Let  $I \subseteq K[x]$  ideal,  $A := K[x]/I$

For  $d \in \mathbb{N}$  set  $A_d := \{f + I \mid f \in K[x], \underbrace{\deg(f)}_{=\max\{\deg(t) \mid t \in M(f)\}} \leq d\} \subseteq A$  subspace

Hilbert function:  $h_I(d) := \dim_K(A_d) < \infty$

Hilbert series of  $I$ :  $H_I(t) = \sum_{d=0}^{\infty} h_I(d) \cdot t^d \in \mathbb{Z}[[t]]$  (formal power series)

**Example:**

$$(1) \quad I = (x_1, \dots, x_n) \Rightarrow A = K \Rightarrow A_d = K \quad \forall d$$

$$h_I(d) = 1 \Rightarrow H_I(t) = \sum_{d=0}^{\infty} t^d = \frac{1}{1-t}$$

$$(2) \quad I = (x_1 - x_2^2) \subseteq K[x_1, x_2]$$

The classes of  $1, x_1, x_1^2, \dots, x_1^d, x_2, x_2x_1, \dots, x_2x_1^{d-1}$  form a basis of  $A_d$

$$\Rightarrow h_I(d) = 2d + 1$$

$$H_I(t) = \frac{1+t}{(1-t)^2}$$

$$(3) \quad I = \{0\} \Rightarrow A = \underbrace{K[x]}_{=x_1, \dots, x_n}$$

write  $H_n(t)$  for Hilbert series  $H_0(t) = \frac{1}{1-t}$

$$\text{For } n > 0 : K[x_1, \dots, x_n]_d = \bigoplus_{i+j=d} K[x_1, \dots, x_{n-1}] \cdot x_n^j$$

$$\Rightarrow H_n(t) = H_{n-1}(t) \left( \sum_{j=0}^{\infty} t^j \right) = H_{n-1}(t) \cdot \frac{1}{1-t} = \frac{1}{(1-t)^{n+1}}$$

Hilbert function:

$$H_n(t) = (1-t)^{-n-1} = \sum_{d=0}^{\infty} \binom{-n-1}{d} (-t)^d$$

$$\Rightarrow h_n(d) = \binom{-n-1}{d} \cdot (-1)^d = \binom{n+d}{d} = \binom{d+n}{n}$$

A total degree monomial ordering is a monomial ordering such that for  $t, t' \in M$  have:  $t \leq t'$  implies  $\deg(t) \leq \deg(t')$

**Example:** grevlex

### 2.8.11 Theorem 11

Let " $\leq$ " be a total degree monomial ordering.  $I \subseteq K[x]$  ideal  $\Rightarrow H_I(t) = H_{L(I)}(t)$

**Proof:**

Let  $G$  be a Gröbner basis of  $I$ ,  $d \in \mathbb{N}$

$NF_G$  induces an injection from  $\phi : A \mapsto K[\underline{x}]$ . Consider  $\phi_d : A_d \mapsto K[\underline{x}]$  restriction

**Claim:**  $im(\phi_d)$  is the space  $V_d \subseteq K[\underline{x}]$  generated by all monomials  $m \in M$

with  $\deg(m) \leq d$  such that  $LM(g) \nmid m \quad \forall g \in G$

Let  $f \in V_d \Rightarrow f$  is in normal form with respect to  $G \Rightarrow f = NF_G(f) = \phi(\underbrace{f + I}_{\in A_d}) \in im(\phi_d)$

Conversely let  $f \in im(\phi_d) \Rightarrow \exists g \in K[\underline{x}] :$

$$\deg(g) \leq d$$

$$f = NF_G(g)$$

$$\Rightarrow f = g - \sum_{i=1}^m h_i g_i \quad \text{with } g_i \in G$$

$$h_i \in K[\underline{x}]$$

$$LM(h_i g_i) \leq LM(f)$$

$$\Rightarrow \forall t \in M(h_i g_i) : t \stackrel{\text{"}\leq\text{" tot. deg. mon. ord.}}{\Rightarrow} \deg(1) \leq \deg(LM(g))$$

$$\text{So } \deg(h_i g_i) \leq d \stackrel{(*)}{\Rightarrow} \deg(f) \leq d$$

Since  $f$  is in normal form this implies  $f \in V_d$ .

$$\text{So } V_d = im(\phi_d)$$

$$\Rightarrow h_I(d) = \dim_K(A_d) = \dim_K(im(\phi_d)) = \dim(V_d)$$

$V_d$  only depends on  $d$  and on  $(LM(g) \mid g \in G) = L(I) \Rightarrow h_I$  only depends on  $L(I)$

Since  $L(L(I)) = L(I)$  the theorem follows  $\square$

How to compute  $H_I(t)$  for  $I$  monomial ideal:

Let  $I = (m_1, \dots, m_l) \quad m_i \in M$  Set  $J = (m_1, \dots, m_{l-1})$

Then the map  $J \xrightarrow{\text{surjective}} I/(m_l)$  has the kernel  $J \cap (m_l)$

$$\Rightarrow J/(J \cap (m_l)) \cong I/(m_l)$$

This isomorphism restricts to all homogeneous components

$$\Rightarrow H_I(t) = H_{(m_l)}(t) + H_J(t) - H_{J \cap (m_l)}(t) \quad (*)$$

$$\text{Have } J \cap (m_l) = (lcm(m_1, m_l), \dots, lcm(m_{l-1}, m_l))$$

**2.8.12 Theorem 12**

Let  $I = (m_1, \dots, m_l) \subseteq K[x_1, \dots, x_n]$  with  $m : i \in M$

$$\Rightarrow H_I(t) = \frac{1}{(1-t)^{n+1}} \sum_{S \subseteq \{1, \dots, l\}} (-1)^{|S|} \deg(lcm\{m_i \mid i \in S\}) \cdot t$$

**Proof:**

Use induction on  $l$ ,  $(*)$  and bookkeeping.

$l = 0$  : Example (3) ✓

$l = 1$  :  $I = (m_1)$

$$\sum_{d=0}^{\infty} \dim(\underbrace{I_{\leq d}}_{\text{all polys in } I \text{ of } \deg \leq d}) \cdot t^d = t^{\deg(m_1)} \cdot H_{\{0\}}(t)$$

$$\Rightarrow H_{(m_1)}(t) = H_{\{0\}}(t) - \sum_d \dim(I_{\leq d}) = \frac{1-t^{\deg(m_1)}}{(1-t)^{n+1}} \quad \checkmark$$

$l - a \rightarrow l$  (with  $l \geq l$ ): By  $(*)$  with  $I = (m_1, \dots, m_l)$  :

$$(1-t)^{n+1} \cdot H_I(t) = 1 - t^{\deg(m_l)} + \sum_{S \subseteq \{1, \dots, l-1\}} (-1)^{|S|} \cdot t^{\deg(\text{lcm}\{m_i | i \in S\})} \\ - \sum_{S \subseteq \{1, \dots, l-1\}} (-1)^{|S|} \cdot t^{\deg(\text{lcm}\{\text{lcm}(m_i, m_l) | i \in S\})}$$

For  $S \neq \emptyset$  then  $\text{lcm}\{\text{lcm}(m_i, m_l) | i \in S\} = \text{lcm}\{m_i | i \in S \cup \{l\}\}$

So the formula is correct  $\square$

$\rightarrow$  may compute Hilbert series by Gröbner bases!

**2.8.13 Corollary 13 (Hilbert-Serre theorem)**

Let  $I \subseteq K[\underline{x}]$  ideal.

Then  $H_I(t) = \frac{a_0 + a_1 t + \dots + a_k t^k}{(1-t)^{n+1}}$   $a_i \in \mathbb{Z}$

For  $d \gg 0$   $h_I(d)$  is a polynomial. More precisely with

$$p_I := \sum_{i=0}^k a_i \binom{x+n-i}{n} \in \mathbb{Q}[x]$$

have  $h_I(d) = p_I(d)$  for  $d \gg 0$

$p_I$  is called the Hilbert polynomial.

**2.8.14 Definition 14 (Algebra, homo-, isomorphism)**

An algebra over a field  $K$  is a commutative ring  $A$  containing  $K$ .

(Usually  $A = \{0\}$  is also considered as an algebra)

$A$  is called finitely generated if

$$\exists a_1, \dots, a_n \in A : A = \{f(a_1, \dots, a_n) \mid f \in K[x_1, \dots, x_n]\} = K[a_1, \dots, a_n]$$

A homomorphism  $\phi : A \rightarrow B$  of  $k$ -algebras is a ring homomorphism that fixes  $k$  element-wise.

$\phi$  is an isomorphism it is bijective.

So if  $A = K[a_1, \dots, a_n]$  we have a surjective homomorphism

$$\phi : K[x_1, \dots, x_n] \rightarrow A, \quad f \mapsto f(a_1, \dots, a_n). \quad \text{If } I := \ker(\phi) \text{ then } K[x_1, \dots, x_n]/I \cong A$$

Given a finitely generated algebra  $A$  (such as  $A = K[x] = K[\underline{x}]/I$  with  $x = \mathcal{V}(I)$  affine variety), then each choice of finite set of generators affords an isomorphism

$K[x_1, \dots, x_n]/I \cong A$ . Then  $p_I$  will depend on the choice of the generators. But might the degree of  $p_I$  be an invariant of  $A$ ?

### 2.8.15 Lemma 15

Let  $I \subseteq K[x_1, \dots, x_n]$ ,  $J \subseteq K[y_1, \dots, y_n]$  ideals in polynomial rings such that  $\underbrace{K[x_1, \dots, x_n]/I}_A \cong \underbrace{K[y_1, \dots, y_n]/J}_B$ . Then  $\deg(p_I) = \deg(p_J)$ .

**Proof:**

Have  $\phi A \xrightarrow{\sim} B \Rightarrow \exists g_1, \dots, g_m \in K[x_1, \dots, x_n]$  such that  $\phi(g_i + I) = y_i + J$

Choose  $e \in \mathbb{N}$  such that  $\deg(g_i) \leq e$

Then  $B_d \subseteq \phi(A_{de}) \Rightarrow h_J(d) = \dim(B_d) \leq \dim(A_{de}) = h_I(de) \Rightarrow \deg(p_J) \leq \deg(p_I)$

By symmetry  $\deg(p_I) \leq \deg(p_J)$ , have equality  $\deg(p_J) = \deg(p_I)$   $\square$

Goal:  $\deg(p_I) = \dim(I)$

### 2.8.16 Theorem 16 (Noether normalization)

Let  $A \neq \{0\}$  be a finitely generated algebra. Then there exist algebraically independent elements  $c_1, \dots, c_m \in A$  such that  $A$  is finitely generated as a module (= vector space, but over a ring rather than a field) over  $C = K[c_1, \dots, c_m] \underset{\text{subalgebra}}{\subseteq} A$ ,

i.e.  $\exists b_1, \dots, b_l \in A$  such that  $A = \sum_{i=1}^l C \cdot b_i$

Moreover if  $A \cong K[\underline{x}]/I$  then  $m = \dim(I)$  ( $= \text{trdeg}(A)$ )

If  $m = 0$  then  $C = K$

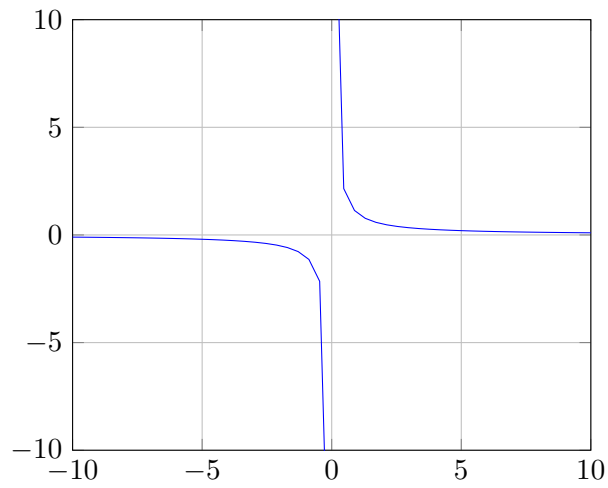
**Proof:**

Omitted  $\rightarrow$  commutative algebra  $\square$

**Geometric interpretation:** Let  $A = K[X]$ ,  $X$  affine variety. Choosing  $c_1, \dots, c_m \in A$  amounts to choosing a morphism  $f : X \mapsto K^m$ ,  $v \mapsto (c_1(v), \dots, c_m(v))$ . It can be shown that if  $C = K[c_1, \dots, c_m]$  is Noether norm, then the morphism is surjective and with finite fibers i.e.  $\forall w \in K^m$  the set  $f^{-1}(\{w\})$  is finite.

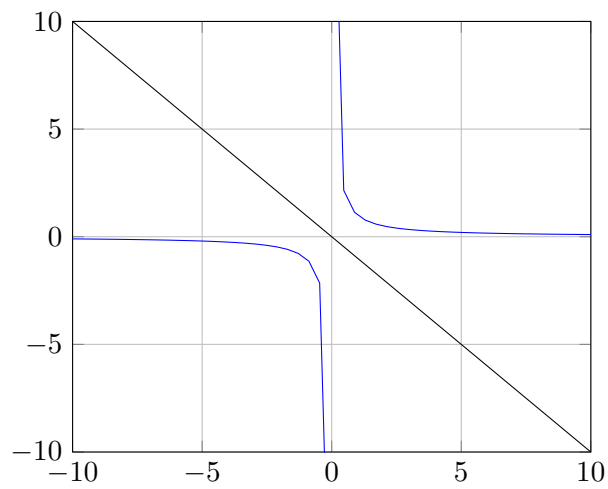
**Example:**

$X = \mathcal{V}(xy - 1) \subseteq \mathbb{C}^2$



The projection to the x- and y-axis are not surjective.

Try  $c = \bar{x} - \bar{y} \in A = \mathbb{C}[x, y]/(xy - 1)$



Have  $\bar{x}^2 - c\bar{x} - 1 = 0 \Rightarrow \forall i : \bar{x}^i \in C \cdot 1 + C\bar{x}$

Similarly  $f \cap \bar{y} \Rightarrow C = \mathcal{C}[c]$  satisfies Theorem 16

### 2.8.17 Theorem 17

Let  $I \subseteq K[x_1, \dots, x_n]$  ideal  $\Rightarrow \dim(I) = \deg(p_I)$

**Proof:**

If  $I = K[x] \Rightarrow \checkmark$

So assume  $I \not\subseteq K[x]$ . Then by Theorem 16  $\exists C = K[c_1, \dots, c_m] \subseteq A := K[x]/I$  and

$\exists b_1, \dots, b_l \in A : A = \sum_{i=1}^l C \cdot b_i, \quad c_1, \dots, c_m$  algebraic independent. Without loss  $b_1 = 1$

Let  $y_1, \dots, y_m, z-1, \dots, z_l$  be new indeterminates and consider the map

$K[y_1, \dots, y_m, z-1, \dots, z_l] \mapsto A, \quad y_i \mapsto c_i, \quad z_j \mapsto b_j$

This is surjective. Let  $J$  be the Kernel.

By Lemma 15 have  $\deg(p_I) = \deg(p_J)$ .

Need to show:  $\deg(p_J) = m$

Write  $B : K[y_1, \dots, y_m, z-1, \dots, z_l]/J$ , consider  $B_d, C_d := \{f+J \mid f \in K[y_1, \dots, y_m], \deg(f) \leq d\} \subseteq B_d$  (for  $d \in \mathbb{N}$ ).

Since  $C_d \subseteq B_d$ , obtain  $h_J(d) = \dim(B_d) \geq \dim(C_d) = \dim(K[y_1, \dots, y_m])_d$   
 $c_1, \dots, c_m \text{ alg. ind.} = \binom{d+m}{m}$

$\Rightarrow \deg(p_J) \geq m$

For  $1 \leq i \leq j \leq l$  have  $b_i b_j = \sum_{k=1}^l a_{ijk} \cdot b_k$  with  $a_{ijk} \in C$  Choose  $e \in \mathbb{N}_{>0}$  such that

$\forall i, j, k : \deg(a_{ijk}) \leq e \Rightarrow b_i b_j \subseteq \sum_{k=1}^l C_{(s-1)e} \cdot b_k$ . So the product of  $s$  of the  $b_i$  lies in  $\sum_{k=1}^l C_{(s-1)e} \cdot b_k$ .

This implies (for  $d \in \mathbb{N}$ )  $B_d \subseteq C_d \cdot b_1 + \sum_{s=1}^d \sum_{k=1}^l C_{d-s} \cdot C_{(s-1)e} \cdot b_k \subseteq \sum_{k=1}^l C_{de} \cdot b_k =: V_d$

$\Rightarrow h_J(d) = \dim(B_d) \leq \dim(V_d) \leq l \cdot \dim(C_{de}) = l \cdot \binom{d \cdot e + m}{m} = \deg(p_J) \leq m \quad \square$

(or substitute  $e$  by  $2e$ )

### 2.8.18 Corollary 18

Let " $\leq$ " be a graded monomial ordering  $I \subseteq K[x]$  ideal  $\Rightarrow \dim(I) = \dim(L(I))$

For computing  $\dim(L(I))$  use the transcendence degree!

Let  $L(I) = (m_1, \dots, m_l)$ ,  $m_i$  monomials. For  $S \subseteq \{x_1, \dots, x_n\}$ ,

$S$  is dependent modulo  $L(I) \Leftrightarrow \exists i : m_i \in K[S]$

$S$  is independent modulo  $L(I) \Leftrightarrow \forall i : m_i \notin K[S]$

i.e.  $m_i$  involves a variable not in  $S$ .

$T := \{1, \dots, n\} \setminus S$  is independent modulo  $L(I)$

$\Leftrightarrow$  Every  $m_i$  involves a variable that is in  $S$

### 2.8.19 Algorithm 19 (Dimension of $I$ )

Input :  $I \subseteq K[\underline{x}]$  ideal

Output:  $\dim(I)$

- (1) Compute a Gröbner basis of  $I$  with respect to a graded monomial ordering.  
Let  $m_i, \dots, m_l$  be the leading monomials of the polynomials in  $G$
- (2) if  $(\exists i : m_i = 1)$   
 $\dim(I) = -1$
- (3) Find  $T \subseteq \{x_1, \dots, x_n\}$  of minimal size,  
such that every  $m_i$  involves a variable from  $T$   
// don't worry about exponential time for this step as (1) is doubly exp anyway
- (4)  $\dim(I) = n - |T|$

## 3 Notes

### 3.1 Notation

- $\mathbb{N} := \mathbb{N}_0$
- $\lg(x) := \log_2(x)$
- $a \mid b$        $a$  is divisible by  $b$        $\Leftrightarrow$        $b \bmod a = 0$   
 $a \nmid b$        $a$  is not divisible by  $b$   $\Leftrightarrow$        $b \bmod a \neq 0$
- $ord(a)$       order of a group element  
 $n > 0$  minimal such that  $a^n = e$       with neutral element  $e$   
if no such  $n$  can be found,  $ord(a) = \infty$
- $char(A)$       Characteristic: the smallest positive  $n$  such that  
 $\underbrace{1 + \dots + 1}_n = 0$       with 1 as the multiplicative identity element  
 $n$  summands
- $im(f)$  image of  $f$  ("Wertebereich")
- $Ker(\phi)$  kernel of  $\phi$ ; Number of elements that map to zero  
 $\Rightarrow$  degree to which the homomorphism  $\phi$  fails to be injective
- $\mathbb{Z}/(m)$       Ring modulo  $m$   
polynomial rings measure for " $<$ " relations not the absolute value but max power.
- $lcm(a_1, \dots, a_n)$       "least common multiple of all  $a_i$ "
- $\underline{e}$  = vector of  $e$ 's
- $K[\underline{x}] := K[x_1, \dots, x_n]$
- $\phi(n) := |\{x \in \mathbb{N} : x < n \wedge \gcd(x, n) = 1\}| = |(\mathbb{Z}/(n))^x|$   
Euler's totient function
- $rk(A)$       Rank of matrix  $A$
- $\left(\frac{n}{p}\right) := \begin{cases} 1 & \text{if } p \mid n \\ -1 & \text{if } n \text{ is a square (mod } p) \\ 0 & \text{otherwise} \end{cases}$   
Legendre symbol (this is not a fraction)
- $\left(\frac{n}{p}\right) = 1 \Leftrightarrow n^{\frac{p-1}{2}} = \left(\frac{n}{p}\right) \equiv 1 \pmod{p}$   
Eulers criterion
- $res(f, g)$  resultant.  $\Rightarrow$  det of Sylvester-Matrix
- $\mathbb{A} :=$  Affine space
- $Id(S) =$  Ideal of  $S$



### 3.2 Various stuff

- Lagrange's theorem  
Every element in a finite group has finite order
- Average number of bit operations for an increment:  
One operation for the last bit + 50% chance for one on the next bit + 25% on the following etc.  $\Rightarrow$  Geometrical row  
 $\Rightarrow$  on average two bit operations
- "Monte Carlo Algorithm"  
Always terminates in reasonable time but might yield false result.
- "Las Vegas Algorithm"  
If it terminates the result is correct. No deterministic running time.
- Chinese remainder theorem  
Given a system of congruences  $x \equiv a_i \pmod{m_i}$  with  $i = 1, \dots, r$   
 $m_i$  pairwise co-prime. Then the unique solution is:  
$$x \equiv a_1 \cdot b \cdot \frac{N}{m_1} + \dots + a_r \cdot b_r \cdot \frac{N}{m_r} \pmod{N} \quad \text{with } b_i \cdot \frac{N}{m_i} \equiv 1 \pmod{m_i}$$
- distance between two square numbers:  
 $(n+1)^2 - n^2 = 2n + 1$   
 $\Rightarrow$  Squares are much more scarce than primes!
- $ax + by = c$  has solutions in  $\mathbb{Z}$  iff  $\Leftrightarrow \gcd(a, b) \mid c$  with  $a, x, b, y \in \mathbb{Z}$
- $S_{f,g} = \begin{pmatrix} f_m & \dots & f_0 & 0 & \dots & 0 \\ 0 & f_m & \dots & f_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dots & \ddots & 0 \\ 0 & \dots & 0 & f_m & \dots & f_0 \\ g_n & \dots & g_0 & 0 & \dots & 0 \\ 0 & g_n & \dots & g_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dots & \ddots & 0 \\ 0 & \dots & 0 & g_n & \dots & g_0 \end{pmatrix}$  Sylvester-Matrix for  $f(x), g(x)$
- finitely generated  
A set  $M$  is finitely generated if  $\exists R^n \mapsto M$  surjective with  $R^n$  finite
- monomial ideal  $I \Leftrightarrow \exists$  a set of generators that are monomials only

### 3.3 Algebraic structures

- Group  $(G, *)$ 
  - one inner operation  $(*)$ :  $G \times G \mapsto G$
  - associativity:  $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
  - neutral element  $e \in G$ :  $a * e = e * a = a \quad \forall a \in G$
  - inverse element  $a^{-1} \in G$ :  $a * a^{-1} = a^{-1} * a = e \quad \forall a \in G$
- Abelian group  $(G, *)$

- $(G, *)$  is a group
- commutativity:  $a * b = b * a \quad \forall a, b \in G$
- Finite group  $(G, *)$ 
  - associativity:  $(a * b) * c = a * (b * c)$
  - unambiguity of reduction:  $(a * x = a * x') \wedge (x * a = x' * a) \Rightarrow x = x'$   
 $\Rightarrow x \mapsto x * a$  and  $x \mapsto a * x$  is bijective  
 $\Rightarrow \exists x : a * x = a \Rightarrow$  neutral element  
 $\exists x : a * x = x \Rightarrow$  inverse element
- Cyclic group  $(G, *)$ 
  - $G$  is a group
  - $G$  is generated by one Element:  $G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$
  - not necessarily finite.
- Semi group  $(S, *)$ 
  - one inner operation  $(*)$ :  $S \times S \mapsto S$
  - associativity:  $(a * b) * c = a * (b * c) \quad \forall a, b, c \in S$
- Field  $(K, +, \cdot)$ 
  - two inner operations  $(+, \cdot)$  such that:
    - $(K, +)$  is an abelian group with neutral element 0
    - $(K \setminus \{0\}, \cdot)$  is an abelian group with neutral element 1
  - distributivity:  $a \cdot (b + c) = a \cdot b + a \cdot c$   
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in K$
- General linear group  $GL_n(K)$ 
  - $K$  is a field
  - $GL_n(K)$  is the set of  $n \times n$  invertible matrices with ordinary matrix multiplication
- Ring  $(R, +, \cdot)$ 
  - $(R, +)$  is an abelian group
  - $(R, \cdot)$  is a semi group
  - distributivity:  $a \cdot (b + c) = a \cdot b + a \cdot c$   
 $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in R$
- Commutative ring  $(R, +, \cdot)$  is a ring
  - commutativity for  $(\cdot)$   $a \cdot b = b \cdot a \quad \forall a, b \in R$
- Unitary ring (ring with 1)  $(R, +, \cdot)$ 
  - $(R, \cdot)$  is a semi group
  - $(R, \cdot)$  has a neutral element "1"
- Euclidean ring  $R$ 
  - $\exists F : R \mapsto \mathbb{N}_0 \cup \{0\}$   
such that if  $\exists q, r \in R \quad a = b \cdot q + r$  and  $r = 0$  or  $a, b \in R \quad F(r) < F(b)$
- Polynomial ring  $R[X]$ 
  - $R$  is a commutative unitary ring
  - set of all polynomials with coefficients  $\in R$
  - Variables  $X_1 \dots X_n$
- Noetherian Ring  $R$

The following definitions are equal:

- for  $I_1 \subseteq I_2 \subseteq \dots \quad \exists n : I_n = I_{n+1} = \dots$  (the chain of ideals "stabilizes")
- every ideal of  $R$  is finitely generated

### 3.4 Invertible elements

- Let  $(\mathbb{Z}/(n), +)$  be a group or  $(\mathbb{Z}/(n))^\times$  be a group with multiplication.
- $|(\mathbb{Z}/(n))^\times| = \phi(n)$
- $n \in \mathbb{P}$   
 $\Rightarrow (\mathbb{Z}/(n))^\times = \{\bar{0}, \dots, \bar{p-1}\} \cong (\mathbb{Z}/(p-1), +) = Z_{p-1}$  (cyclic Group  $Z$ )
- $n$  is a power of 2  
 $\Rightarrow (\mathbb{Z}/(2^e))^\times \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2^{e-2})$
- $n$  is a power of an odd Prime  
 $\Rightarrow (\mathbb{Z}/(p^k))^\times \cong \mathbb{Z}/(p^{k-1} \cdot (p-1)) \cong Z_{(p^{k-1} \cdot (p-1))}$
- $n = p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$   
 $\Rightarrow (\mathbb{Z}/(n))^\times \cong (\mathbb{Z}/(p_1^{k_1}))^\times \times \dots \times (\mathbb{Z}/(p_r^{k_r}))^\times$