# Computational Algebra
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# Transcript

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# 1 Integer Arithmetic

Topics:

- Addition and Multiplication
- GCD computation
- Primality testing
- Factorization

## 1.1 Addition and Multiplication

Agreement:

- $a, x \in \mathbb{N}$ represented as $x = \sum_{i=0}^{n-1} a_i \cdot B^i \quad B \in \mathbb{N}_{>1}$ fixed Base
  $(a_i \in \{0,, B-1\})$
- if $x \neq 0$, assume $a_{n-1} \neq$ then define:
  length of $x := l(x) = n$ = number of digits = $\lfloor \log_B(x) \rfloor + 1$
  (mnemonic: $\log_B(B) + 1 = 2$)
- $l(0) = 1$
  (Amount of memory required to store $x = 0$)
- $l(x) := l(|x|)$
- for $x \in \mathbb{Z}$ represent if as $x = sgn(x) * |x|$

### 1.1.1 Algorithm 1 (Simple addition)

input : $x = \sum_{i=0}^{n-1} a_i \cdot B^i, \quad y = \sum_{i=0}^{n-1} b_i \cdot B^i, \quad x, y \in \mathbb{N}$

output: $x + y = \sum_{i=0}^{n} c_i \cdot B^i$

(1) $\sigma = 0$

(2) for $i = 0, ..., (n-1)$ :

(3) set $c_i := a_i + b_i + \sigma_i$ and $\sigma := 0$

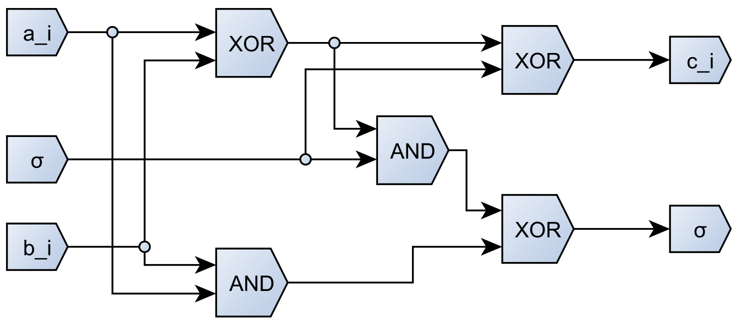
(4) if $(c_i \geq B)$

(5) set $c_i = c_i - B$

(6) set $\sigma = 1$

(7) set $c_n = \sigma$

If $B = 2$ then (3) - (6) can be realized by logic gates:




### 1.1.2 Definition 2 (Bit-Operation)

A bit operation is an operation that can be performed by a logic gate or by searching or writing a bit from / into memory.

### 1.1.3 Definition 3 (Big O)

Let $M$ be a set (usually $M = \mathbb{N}$), $\quad f, g : M \mapsto \mathbb{R}_{>}0$
we write $f \in O(g) \quad$ if $\quad \exists c \in \mathbb{R} : f(x) \leq c \cdot g(x) \forall x \in M$

### 1.1.4 Theorem 4 (Lower bound for addition)

Let $f : \mathbb{N} \mapsto \mathbb{R}$, $\quad n \mapsto$ maximal number of bit operations required by Algorithm 1 to add $x_y \in \mathbb{N}$ with $l(x), l(y) \leq n$
Let $g = id_{\mathbb{N}}$ Then $f \in O(g)$
We say Algorithm 1 requires $O(n)$ bit operations for adding two numbers of length $\leq n$.
$\Rightarrow$ "linear complexity"
Set $M := \{$Set of all algorithms for addition in $\mathbb{N}\}$
For $A \in M$ define $f_A : \mathbb{N} \mapsto \mathbb{R}$ as above.
We would like to find $f_{odd} : \mathbb{N} \mapsto \mathbb{R}$, $\quad n \mapsto \inf\{f_A(n) | A \in M\}$
Since one needs to read $x, y$ (and write the result) we can not do any better than linear complexity for addition.
**Subtraction**
let $x, y$ as Algorithm 1, $x \geq y$
For $\bar{y} := \sum_{i=0}^{n-1} (B - 1 - b_i) B^i \qquad$ (digitwise / bitwise complement)
$\Rightarrow x + \bar{y} = x - y + B^n - 1$
$\Rightarrow x - y = x + \bar{y} + 1 - B^n \qquad$ (initially set $\sigma = 1$)
**Conclusion:** Addition and Subtraction have cost $O(n)$

### 1.1.5 Algorithm 5 (Multiplication by "grid method")

input : $x = \sum_{i=0}^{n-1} a_i \cdot 2^i, \quad y = \sum_{i=0}^{m-1} b_i \cdot 2^i$

output: $z = x \cdot y$

(1) $z := 0$

(2) for $i = 0, ..., (n-1)$

(3) if $(a_i \neq 0)$ set $z := z + \sum_{j=1}^{m-1} b_j 2^{i+j}$

### 1.1.6 Theorem 6 (Runtime of Algorithm 5)

Algorithm 5 requires $O(n * m)$ bit operations.
As of the total input length $n + m$:
$n \cdot m \leq \frac{1}{2}(n+m)^2 \to O((n+m)^2)$
$\Rightarrow$ Quadratic complexity

**Karatsuba-multiplication:**
Observation for polynomials:
$a + bx, c + dx$ have $(a+bx)(c+dx) = ac + (ac + db - (a-b)(c-d))x + bdx^2$
The point: only used 3 multiplications instead of 4.
Specialize $x = B$ "large" such that $x = a + bB$ partition into two blocks. Then multiply the blocks by a recursive call.

### 1.1.7 Algorithm 7 (Karatsuba)

input : $x, y \in \mathbb{N}$

output: $z = x \cdot y$

(1) Choose $k \in \mathbb{N}$ minimal such that $l(x), l(y) \leq 2^k$.
Set $B = 2^{2^{k-1}}$

(2) if $(k = 0)$ return $x \cdot y$ (by bit-operation AND)

(3) write $x = x_0 + x_1 B, \quad y = y_0 + y_1 B$ with $l(x_i), l(y_i) \leq 2^{k-1}$

(4) compute $x_0 \cdot y_0, \ x_1 \cdot y_1, \ (x_0 - x_1) \cdot (y_0 - y_1)$ by a recursive call

(5) return $z = x_0 y_0 + (x_0 y_0 + x_1 y_1 - (x_0 - x_1)(y_0 - y_1))B + x_1 y_1 B^2$

### 1.1.8 Theorem 8 (Runtime of Algorithm 7)

For multiplying two numbers of length $\leq n$ Algorithm 7 requires $O(n^{\log_2 3}) \approx O(n^{1.59})$ bit operations.
**Proof:**
Set $\Theta(k) :=$ maximal numbers of bit operations for $l(x), l(y) \leq 2^k$
We have for $k > 0 : \Theta(k) \leq 3\Theta \underbrace{(k-1)}_{\text{recursive calls}} +c \underbrace{2^k}_{\text{additions}}$ with ($c$ some constant)
**Claim:** $\Theta(k) \leq 3^k + 2c(3^k - 2^k)$
**Proof by Induction on $k$:**
$k = 0 : \Theta(k) = 1$
$k-1 \to k = \Theta(k) = 3\Theta(k-1) + c2^{k-1}$
$\leq 3(3^{k-1} + 2c(3^{k-r} - 2^{k-1})) + c2^k$
$= 3^k + 2c(3^k - 2^k)$
So $\Theta(k) \leq (2c+1)3^k$
Now $l(x) \leq n$ hence $2^{k-1} < n$ by minimality of $k$
So $k - 1 < \log_2 n$
$\Rightarrow \Theta(k) \leq 3(2c+1)3^{\log_2(n)}$
$= 3(2c+1)2^{\log_2(3)\log_2(n)}$
$= 3(2c+1)n^{\log_2(3)}$ $\square$
One can modify the terminal condition of Karatsuba to switch to Grid-Multiplication, which is faster for small numbers.

**Fast-Fourier Transform**
Reminder: For a function $f : \mathbb{R} \mapsto \mathbb{C}$ define:
$\hat{f} : \mathbb{R} \mapsto \mathbb{C}$ by
$\hat{f}(\omega) = \int\limits_{\mathbb{R}} f(t)e^{-i\omega t}dt$ (if it exists)
Think of $\omega$ as frequency.

**Definition (Convolution)**
Let $f, g : \mathbb{R} \mapsto \mathbb{C}$
$(f * g)(x) = \int\limits_{\mathbb{R}} f(t)g(x-t)dt$
Convolution is analogous to polynomial multiplication **Formula:** $\underbrace{(f * g)}_{\text{(Cauchy formula)}} = \hat{f} \cdot \hat{g}$
For a function $M \mapsto C$ with $|M| < \infty$ we need the discrete Fourier transform (DFT)

### 1.1.9 Definition 9 (Root of unity)

Let $R$ be a commutative ring with 1. An element $\mu \in R$ is called an $n-th$ root of unity (= root of 1) if $\mu^n = 1$.
It is called primitive if $\mu^i \neq 1$ for $(0 < i < n)$ i.e. $ord(\mu) = n$
let $\mu$ be a primitive n-th root of 1 (e.g. $e^{2\pi \frac{i}{n}} \quad \in \mathbb{C}$)
Then the map $DFT_\mu : R^n \mapsto R^n$
$(\hat{a}_0, ..., \hat{a}_n) \mapsto (\hat{a}_0, ..., \hat{a}_n) \quad$ with $\hat{a}_i = \sum\limits_{j=0}^{n-1} \mu^{ij} a_j$
is called discrete Fourier transformation
For polynomials:
$DFT_\mu : R[x] \mapsto R^n$
$\qquad f \ \mapsto (f(\mu^0), ..., f(\mu^{n-1})$
Convolution rule: (from $f(\mu^i)g(\mu^i) = (f * g)(\mu^i))$
$DFT_\mu(f * g) = DFT_\mu(f) \cdot DFT_\mu(g) \qquad$ (component wise product)

Addition of two polynomials in R[x] of $deg(n)$ require $O(n)$ ring operations. Multiplication require $O(n^l)$.
With Karatsuba have $O(n^{\log_2(3)})$ ring operations.
Cost $DFT_\mu(f) \cdot DFT_\mu(g) : O(n)$ ring operations $\quad$ (with $\mu$ as 2n-th root of 1)
Want: Cheap way of doing $DFT$ and back-transformation.

### 1.1.10 Algorithm 10 (Fast Fourier transformation FFT)

input : $f \in R[x], \quad \mu \in R$ primitive $2^k$-th root of 1, such that $\mu^{2^{k-1}} = -1$

output: $DFT_\mu(f)$

(1) Write $f(x) = g(x^2) + xh(x^2) \quad$ with $f, g, h \in R[x]$

(2) if $k = 1 \quad (\Rightarrow \mu = 1) \quad$ return $DFT_\mu(f) = (g(1) + h(1), g(1) - h(1))$

(3) Recursive call: compute $DFT_{\mu^2}(g) = \hat{g}, DFT_{\mu^2}(h) = \hat{h} \quad \in R^{2^{k-1}}$

(4) return $DFT_\mu(f) = (\hat{f}_0, ..., \hat{f}_{2^k-1}) \quad$ with $\hat{f}_i = \hat{g}_i + \mu\hat{h}_i$
where $\hat{g}_i = \hat{g}_{i-2^{k-1}} \quad$ for $i \geq 2^{k-1}$

Note: Components of $\hat{g}$ and $\hat{h}$ are:
$\hat{g} = g(\mu^{2i}), \quad \hat{h}_i = h(\mu^{2i}) \quad$ so
$\hat{f}_i := f(\mu^i) = \hat{g}_i(\mu^{2i}) + \mu h(\mu^{2i}) = \hat{g}_i + \mu\hat{h}_i$

**Convention:** $\lg(x) = \log_2(x)$

### 1.1.11 Theorem 11 (Runtime of Algorithm 10)

Let $n = 2^k, \quad f \in R[x] \quad$ with $deg(\psi) < n$ Then Algorithm 10 requires $O(n \cdot \lg(n))$ ring operations.
Better than $O(n^{1+\epsilon}), \quad \forall \epsilon > 0!$
**Proof:**
Set $\Theta(k)$ = max number of ring operations required. By counting obtain for $k > 1$:
$\Theta(k) \leq 2\Theta(k-1) + \underbrace{(\text{compute } \mu^i (i \leq 2^{k-1}))}_{2^{k-1}} + \underbrace{(\mu^i \hat{k}_i)}_{2^{k-1}} + \underbrace{(\text{sums and differences})}_{2^k}$
$= 2\Theta(k-1) + 2^{k+1}$
**Claim:** $\Theta(k) \leq (2k-1)2^k$
$k = 1 : f = a_0 + a_1 \cdot x \quad DFT_\mu(f) = (a_0 + a_1 \cdot a_0 - a_1) \Rightarrow \Theta(a) = 2$
$k - 1 \to k : \Theta(k) \leq 2 \cdot \Theta(k-1) + 2^{k+1} \leq 2 \cdot (2k-3) \cdot 2^{k-1} + 2^{k+1} = (2k-1) \cdot 2^k$
since $k = \lg(n)$ obtain $O(k) \leq (2 \cdot \lg(n) - 1) \cdot n \in O(n \cdot \lg(n)) \qquad \square$

**Back-transformation?**

### 1.1.12 Definition 12 (Good root of unity)

A primitive n-th root of unity is called good (caveat: this is ad-hoc terminology) if:
$\sum_{j=0}^{n-1} \mu^{ij} = 0 \quad$ for $(0 < i < n)$
**example:**

(1) $\mu = e^{2\pi \frac{i}{n}}$ is a good primitive root of unity

(2) $R = \mathbb{Z}/(8), \quad \mu = \bar{3} \Rightarrow \mu \cdot B$ is primitive $2^{nd}$ root of unity
But $\bar{B}^0 + \bar{3}^1 = \bar{u} \neq \bar{0} \qquad$ so $\mu$ is not good.

### 1.1.13 Proposition 13 ($DFT_{\mu^{-1}}$)

let $\mu \in R$ be a good root of 1
$(a) = (a_0, ..., a_{n-1}) \in R^n \Rightarrow DFT_\mu^{-1}(DFT_\mu(a)) = n \cdot (a) \quad$ where $n = 1 + ... + 1 \in R$
**Proof:**
$DFT_\mu(a) = (\hat{a}) = (\hat{a}_0, ..., \hat{a}_{n-1})$
with $\hat{a}_j = \sum_{k=0}^{n-1} \mu^{jk} a_k$
$DFT_{\mu^{-1}}(\hat{a}) = (\hat{\hat{a}}_0, ..., \hat{\hat{a}}_1)$
with $\hat{\hat{a}}_i \sum_{i=0}^{n-1} \mu^{-ij} \sum_{k=0}^{n-i} \mu^{jk} a_k = \sum_{k=0}^{n-1} (a_k \cdot \underbrace{\sum_{i=0}^{n-1} \mu^{j(k-i)}}_{=0 \text{ if } n \neq k-i \text{ (i.e.} k=i)}) = a_i \cdot n \qquad \square$

### 1.1.14 Proposition 14 (Finding good roots of unity)

let $\mu \in R, n \in \mathbb{N}$
Assume:

a) $R$ is an integral Domain and $\mu$ is a primitive or n-th root of 1
(Integral Domain: nonzero commutative ring in which the product of two nonzero elements is nonzero)
$\Rightarrow$ Granted by FFT

b) $n = 2^b$, $\mu^{\frac{n}{2}} = -1$, then $h > 0 \wedge char(R) \neq 2$
$\rightarrow \mu$ is a good primitive n-th root of 1 ("root of unity")

**Proof:**

a) for $0 < i < n$
$\underbrace{(\mu^i - 1)}_{\neq 0} \underbrace{(\sum_{j=0}^{n-1} \mu^{ij})}_{=0\ *} = \mu^{in} - 1 = 0$
$\Rightarrow \mu$ is a good root of unity

* Let $0 < i < n$, write $i = 2^{k-s} \cdot r$ with $r$ odd $\wedge\ s > 0$
$\sum_{j=0}^{2^k-1} \mu^{ij} = \sum_{l=0}^{2^{k-s}-1} \sum_{j=0}^{2^s-1} \mu^{i(l \cdot 2^s + j)}$
$\mu^{i \cdot 2^s} = 1$
$i \cdot 2^s = 2^{k-s} \sum_{j=0}^{2^s-1} \mu^{ij} = 2^{k-s} \sum_{j=0}^{2^{s-1}-1} (\mu^{ij} + \mu^{i(2^{s-1}+j)})$
But $\mu^{i \cdot 2^{s-1}} = \mu^{2^{k-s} \cdot r \cdot 2^{s-1}} = \mu^{2^{k-1} \cdot r} = (-1)^r = -1$
So $\sum_{j=0}^{n-1} \mu^{ij} = 0$ $\square$

b) $\mu^n = 1, n = 2^k \Rightarrow ord(\mu) | n \Rightarrow ord(\mu)$ is power of 2

### 1.1.15 Algorithm 15 (Polynomial multiplication using DFT)

input : $f, g \in R[x]$ with $deg(f) + deg(g) < 2^k =: n$
$\mu \in R$ as a good root of unity; Assume $2 \in R$ is invertible

output: $h = f \cdot g$

(1) compute $\hat{f} = DFT_\mu(f)$, $\hat{g} = DFT_\mu(g)$ with $f, g \in R^n$

(2) compute $\hat{h} = \hat{f} \cdot \hat{g}$

(3) compute $(h_0, ..., h_{n-1}) = DFT_{\mu^{-1}} \hat{h}$ (same as $DFT_\mu(\hat{h})$ but with different order)
$=$ Back-transformation $\cdot 2^k$
set $h = \frac{1}{2^k} \sum_{i=0}^{n-1} h_i x^i$

### 1.1.16 Theorem 16 (Runtime of Algorithm 15)

Algorithm 15 uses $O(n \cdot \log(n))$ ring operations for polynomials of $deg < n$
**Proof:**

- Choose k minimal so that $deg(f) \cdot deg(g) < 2^k$
  $\Rightarrow 2^{k-1} \leq 2n \quad \Rightarrow k \leq \log(n) + 2$
- $\underbrace{O(2k \cdot 2^k)}_{\text{Step 1}} + \underbrace{2^k}_{\text{Step 2}} + \underbrace{O(k \cdot 2^k) + 2^k}_{\text{Step 3}} \in O(2k \cdot 2^k) = O(n(g(n)))$ $\square$

Goal: Multiplication in $\mathbb{N}$ using DFT
Idea: find roots of 1 in a suitable $\mathbb{Z}/(m)$
Choose $m = 2^l + 1, \mu = \bar{2} \in R$

### 1.1.17 Proposition 17 (Add and mul in $O(l)$)

Let $m = 2^l + 1,\ R = \mathbb{Z}/(m)$
Addition in $R$ and multiplication by $\bar{2}^i \in R$ $(0 \leq i < 2l)$ can be done in $O(l)$ bit operations
**Proof:**

Let $\bar{x} \in R$ with $0 \leq x \leq 2^l$

- Addition: $x \bar{+} \bar{y}$
  (1) compute $x + y \in \mathbb{N}$: $O(l)$
  (2) if $x + y > 2^l + 1$ subtract $2^l + 1$: $O(l)$
- Multiplication by $\bar{2}^i$ $(0 \leq i < l)$
  (1) Bit-shift $i$ Bits to the left by relocating in memory:
  $$\underbrace{O(length(i))}_{\text{compute addr. of new first bit}} + \underbrace{l}_{\text{copying}} = O(\log(l)) + l \in O(l)$$
- Multiplication by $\bar{2}^i$ $(l \leq i < 2l - 1)$
  (1) Multiplication by $\bar{2}^{i-l}$: $O(l)$
  (2) take negative $\bar{2}^i \cdot \bar{x} = -\bar{2}^{i-l} \cdot \bar{x}$: $O(l)$

### 1.1.18 Proposition 18 (Sort of summary)

Let $k, r \in \mathbb{N}$, $r > 0$, $m = 2^{2^k \cdot r} + 1$, $R = \mathbb{Z}/(m)$, $\mu = \bar{2}^r \in R$
$\Rightarrow 2 \in R$ is invertible, $\mu$ is a good primitive $2^{k+1}$-th root of 1
$\Rightarrow \mu^{2^k} = 1$
**Proof:** $\rightarrow$ from above

### 1.1.19 Algorithm 19 (Multiplication using FFT)

input : $x, y \in \mathbb{N}$

output: $Z = x \cdot y$

(1) Choose $k \in \mathbb{N}$ minimal such that $l(x), l(y) \leq 2^{2k}$

(2) if $k \leq 3$, compute $z = x \cdot y$ by Algortithm 5

(3) set $B = 2^{2^k}, \quad m = 2^{2^k \cdot 4} + 1, \quad R = \mathbb{Z}/(m), \quad \mu = \bar{2}^4 \in R$
($\Rightarrow$ so $\mu$ is a good primitive $2^{k+1}$-th root of 1)

(4) write $x = \sum_{i=0}^{2^k-1} x_i \cdot B^i$, same for $y$ with $(0 \leq x_i, y_i < B)$
possible since $x, y < 2^{2^{2k}} = 2^{2^k \cdot 2^k} = B^{2^k}$

(5) compute: $\hat{x} = DFT_\mu(\bar{x}_0, ..., \hat{x}_{2^k-1}, \underbrace{0, ..., 0}_{2^k \text{zeros}}) \in R^{2^{k+1}}$
same for $y$
$\rightarrow$ use FFT

(6) compute: $\hat{z} = \hat{x} \cdot \hat{b} \quad \in R^{2^{k+1}}$ (component wise multiplication)
Perform multiplication in $R$ as follows:
Multiply representatives (non negative and $< m$) by recursive call.
Then reduce modulo $m$ by "negative bit shift" (see proof of Proposition 17)

(7) compute: $(\bar{z}_0, ..., \bar{z}_{2^{k+1}}) = \frac{1}{2^{k+1}} DFT_{\mu^{-1}}(\hat{z}) \quad \in R$ with $0 \leq z < m$

(8) set $z := \sum_{j=0}^{2^{k+1}-1} z_i \cdot B^i$

### 1.1.20 Theorem 20 (Runtime of Algorithm 19)

Algorithm 19 correctly computes $t = x \cdot y$ and requires $O(n \cdot (\log n)^4)$ bit operations for $l(x), l(y) \leq n$

**Proof:** Correctness

write $x(t) \sum_{i=0}^{2^k-i} x_i t^i \quad \in \mathbb{Z}[t], \quad y(t), \quad \bar{x}(t) \in R[t], \bar{y}(t), \bar{z}(t)$

by Proposition 18 and Proposition 13 we have $\bar{z}(t) = \bar{x}(t) \cdot \bar{y}(t)$

The l-th coefficient of $x(t) \cdot y(t)$ is $0 \leq \sum_{i+j=l} x_i \cdot y_j < 2^k \cdot B^2 = 2^{k+2\cdot 2^k} \leq 2^{2^{k+2}} < m$

So $z(t) = x(t) \cdot y(t) \Rightarrow z = z(B) = x(B) \cdot y(B) = x \cdot y$ Cost:

Write $\Theta(k) :=$ max number of bit operations

Analyze Steps:

(1) compute max $\{l(x), l(y)\}: \quad O(l(n)) = O(k)$

(2) $O(1)$

(3) no bit operations

(4) compute starting addresses of $x_i, y_i$ in memory: $2 * 2^k$ increments of the address: $2 \cdot 2 \cdot 2^k = 2^{k+2}$ bit ops
$\Rightarrow O(2^k)$

(5) By Theorem 11 need $O(2 \cdot 2^{k+1} \cdot (k+1))$ operations in $R$ which are additions and multiplications by powers of $\bar{z}$ costing $O(2^{k+2})$ bit operations.
Total for (5): $O(k \cdot 2^{2\cdot k})$

(6) $2^{k+1}$ multiplications of numbers $< m$, i.e. of length $\leq 2^{k+2}$.
So $k' \leq \frac{k+3}{2}$ for $k'$: the "new" $k$ used in the next recursion level.
For $\alpha \in R_{>0}$ define $\Theta(\alpha) := \Theta(\lfloor \alpha \rfloor)$
Total for (6): $2^{k+1}(\Theta(\frac{k+3}{2}) + \underbrace{O(2^{k+2})}_{\text{reduction mod m}})$

(7) For $DFT_{\mu^{-1}}(\hat{z}) : O(k \cdot 2^{2\cdot k}) as (5)$ Since $\bar{z}$ is a $n$ root of 1, multiplication by $\bar{2}^{-k-1}$ is multiplication by a positive power of $\bar{2}$, which costs $O(2^{k+2})$
Total for (7): $O(k \cdot 2^{2\cdot k})$

(8) For $j \leq 2^{k+1}$ have $\sum_{i=0}^{j-1} z_i \cdot B^i \leq (m-1) \sum_{i=0}^{j-1} B^i = (m-1)\frac{B^j-1}{B-1} < 2(m-1)\frac{B^j}{B} = 2^{1+2^{k+2}+(j-1)2^k}$ so the sum has length $(j+3) \cdot 2 + 1$
Adding $z_j \cdot B^j$ to this sum happens at $(j \cdot 2^k)$-th bit and higher $\Rightarrow$ cost is $O(2^k)$
Total for (8): $O(2^{2\cdot k})$

Grad total: For $k \geq 4$ :
$\Theta(k) \leq 2^{k+1} \cdot \Theta(\frac{k+3}{2}) + c \cdot k \cdot 2^{2\cdot k} \quad$ with $c$ constant
Also for $k \in \mathbb{R}_{\geq 4}$

**Define** $\Lambda(k) := \frac{\Theta(k)}{2^{2\cdot k}} \Rightarrow \Lambda(k) \leq \frac{2^{k+1}\Theta(\frac{k+3}{2})}{2^{2\cdot k}} + c\cdot k = 16\cdot\Lambda(\frac{k+3}{2}) + c\cdot k$
**Define** $\Omega(k) := \Lambda(k+3)$ So for $k \in \mathbb{R}_{\geq}1$
$\Omega(k) \leq 16\cdot\Lambda(\frac{k}{2}+3) + c\cdot(k+3) = \underbrace{16\Omega(\frac{k}{2}) + c\cdot(k+3)}_{*}$

**Claim**: For $i\ in \mathbb{N}$ with $2^{i-1} \leq k-3$ have:
$\Lambda(k) \leq 16^i\Omega(\frac{k-3}{2^i}) + c\cdot(k+3)(1+8+...+8^{i-1}) + 3\cdot c\cdot(1+16+...+16^{i-1})$
**Proof** by induction:
$i = 0 \Lambda(k) = \Omega(k-3)$
$i \to i+1 : \Lambda(k) \leq 16^i\Omega(\frac{k-3}{2^i}) + c\cdot(k-3)(1+...+8^{i-1}) + 3\cdot c\cdot(1+...+16^{i-1}) \leq 2^i \leq k-3 \quad *$
$\leq 16^i(16\Omega(\frac{k-3}{2^i+1})) + c(\frac{k-1}{2^i}+3) + c(k-3)... =$ claimed result
Take $u \in \mathbb{N}$ minimal with $2^u > k-3 \Rightarrow \Omega(\frac{k-3}{2^u}) \leq \Omega(\lfloor\frac{k-3}{2^u}\rfloor) = \Omega(0) =: D$ (constant)
Note: u rouhly is recursion depth
Have $2^{u-1} \leq k-3 \xRightarrow[\text{claim}]{} \Lambda(k) \leq 16^u\cdot D + c\cdot\underbrace{(k-3)}_{<2^u}\cdot\frac{8^u-1}{7} + 3c\cdot\frac{16^u-1}{15} \quad \in O(16^u)$

Have $2^{u-1} \leq k-3 \Rightarrow u \leq \lg(k-3)+1$
$\Rightarrow \Lambda(k) \in O(16^{\lg(k-3)}) = O((k-3)^4)$
$\Rightarrow \Theta(k) = 2^{2\cdot k}\cdot\Lambda(k) \in O(2^{2k}\cdot(k-3)^4)$
Have $2^{2(k-1)} < \underbrace{n}_{\max\{l(x)\cdot l(y)\}} \Rightarrow k \leq \frac{\lg(n)}{2}+1$
So $\Theta(k) \in O(n\cdot(\lg(n))^4)$ $\square$

### 1.1.21 Theorem 21 (Schönhage-Strassen 1971)

Multiplication of integers of length $\leq n$ can be done in $O(n\cdot\lg(n)\cdot\lg(\lg(n)))$ bit operations. Schönhage-Strassen is used for integers of length $\geq 100.000$.
Asymptotically faster: Fürer's algorithm.

**Comments on Bit complexity**

1. Memory requirement may explode!
   $\Rightarrow$ No Problem as bit complexity is upper bound for memory requirements, since memory access is included in bit operations
   ($\to$ only store what is calculated)

2. Computation of addresses in memory take time
   $\Rightarrow$ length of addresses $\approx$ lg(memory space) computations of addresses $\approx$ lg(memory space)$^2$

3. As memory requirement gets larger access times will get longer.
   $\Rightarrow$ transportation time for data $\geq \frac{\text{diameter of physical storage}}{2*\text{speed of light}}$

## 1.2 Division with remainder, Euclidean algorithm

### 1.2.1 Algorithm 1 (Division with remainder)

input : $b = \sum_{i=0}^{n-1} b_i 2^i \quad a = \sum_{i=0}^{n+m-1} a_i 2^i \qquad$ with $a_i, b_i, \in 0, 1, \quad b_{n-1} = 1$

output: $r, q \in \mathbb{N}$ such that $a = q \cdot qb + r, \quad 0 \leq r < b$

(1) $r = a_i \quad q = 0$

(2) for $i = m, m-1, ..., 0$ do

(3) if $r \leq 2^i \cdot b$ then set $r := r - 2^i \cdot b, \quad q = q + 2^i$

### 1.2.2 Proposition 2 (Runtime of Algorithm 1)

Algorithm 1 is correct and requires $O(n \cdot (m+1))$ bit operations.
**Proof:**
Always have $a = q \cdot b + r$
**Claim:**
before setp (3), have $0 \leq 2^{i+1} \cdot b$
$i = m; \quad 0 \leq r = a < 2^{m+n} = 2^{m+1} c \cdot 2^{n-1} \leq 2^{m-1} \cdot b \quad i < m$ By step (3)
So after last passage through the loop $0 \leq r < b$
**Running Time:** In step(3), have comparison and (possiby) subtraction. Only n bits involved $\Rightarrow O(n)$
Total: $O(b \cdot (m+1))$

**Remarks:**

(1) Division with remainder can be reduced to multiplication.
Precisely: given an algorithm for multiplication that requires $M(n)$ bit operations, there exists an algorithm for division with remainder that requires O(M(n)) bit operations.

(2) Practically relevant:
Jebelean's algorithm (1997): $O(n^{\lg 3})$

(3) Alternatively, may choose $r\mathbb{Z}$ such that $\lfloor \frac{-b}{2} \rfloor < r \leq \lfloor \frac{b}{2} \rfloor$

(4) Algorithm 1 extends to $\mathbb{Z}$.

(5) All Euclidean rings have division with remainder (by definition).
(e.g., $R = K[x] \quad \rightarrow$ polynomial ring over field,
$R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \subseteq \mathbb{C}, \quad i^2 = -1$)

### 1.2.3 Algorithm 3 (Euclidean algorithm)

input : $a, b \in \mathbb{N}$

output: $gcd(a, b)$ "greatest common divisor"

(1) set $r_0 := a, \quad r_i := b$

(2) for $i = 1, 2, 3, ...$ perform steps (3) and (4)

(3) if $r_i = 0$ then $gcd(a, b) = |r_{i-1}|$

(4) Division with remainder: $r_{i-1} = q \cdot r_i + r_{i+1} \qquad r_{i+1} \in \mathbb{Z}$
$|r_{i+1}| \leq \frac{1}{2}|r_i|$

**Example:**

$$a = 287, \quad b = 126$$

$$287 = 2 \cdot 126 + 35 \tag{1}$$
$$126 = 4 \cdot 35 - 14 \tag{2}$$
$$35 = (-2) \cdot (-14) + 7 \tag{3}$$
$$-14 = (-2) \cdot 7 + 0 \tag{4}$$

$$\begin{aligned} \text{So: } 7|(-14) &\underset{(3)}{\Longrightarrow} 7|35 \\ &\underset{(2)}{\Longrightarrow} 7|126 \\ &\underset{(1)}{\Longrightarrow} 7|287 \end{aligned}$$

On the other hand take a common divisor $d$; $\quad d|287; \quad d|126$
$\underset{(1)}{\Longrightarrow} d|d \underset{(2)}{\Longrightarrow} d|14 \underset{(3)}{\Longrightarrow} d|7$

### 1.2.4 Theorem 4 (Correctness of Algorithm 3)

Algorithm 3 is correct.
**Proof:**
Since $r_{i-1} = q \cdot r_i + r_{i+1}$ every integer $x \in \mathbb{Z}$ satisfies the equivalcence $x|r_{i-1}$ and $x|r_i \Leftrightarrow x|r_{i+1}$ and $x|r_i$ so $gcd(r_{i-1}, r_i) = gcd(r_i, r_{i+1} = gcd(a, b)$
when terminating have $gcd(a, b) = gcd(r_{i-1}, 0) = |r_{i-1}| \qquad \square$

### 1.2.5 Theorem 5 (Runtime of Algorithm 3)

Algorithm 3 requires $O(m \cdot n)$ bit operations for $n = l(a), m = l(b)$
**Proof:**
If $a < b$ than the first passage yields $r_2 = a, \;\; r_1 = b$. Cost: $O(n)$
May assume: $a \geq b$. Write $n_i = l(r_i)$
By Proposition 2 $\exists c$ constant such that the total time is $\leq c \cdot \underbrace{\sum_{i=1}^{k} n_i \cdot (n_{i-1} - n_i + 1)}_{=:\sigma(n_0,...,n_k)}$

For $i > 2: \quad n_i = n_{i-1} - 1$
Special Case: $n_i = n_{i-1} - 1 \quad$ for $i \geq 2$
$\Rightarrow n_i = n_i - i + 1, \quad n_i = m, \quad k = m + 1$
Obtain $\sigma(n_0, ..., n_k) = m\cdot(n-m+1) + \sum_{i=2}^{m+1} (m-i+1)\cdot 2 = m\cdot n - m^2 + m + m(m-1) = m*n.$
**Claim:** The special case is the worst (most expensive)!
From any sequence $n_1 > n_2 > ... > n_k$ get to the special case by iteratively inserting numbers in the gaps. Insert $s$ with $n_{j-1} > s > n_j$.
$\sigma(n_0, ..., n_{j-1}, s, n_j, ..., n_k) - \sigma(n_0, ..., n_k) = ... = s + (n_{j-1} - s) \cdot (s - n_j)$
$sp\sigma(n_0, ..., n_k) \leq \sigma(n, m, m-1, ..., 2, 1, 0) = n \cdot m \qquad \square$
Complexity is quadratic $\rightarrow$ cheap

### 1.2.6 Algorithm 6 (Extended Euclidean Algorithm)

input : $a, b \in \mathbb{N}$

output: $d = gcd(a, b) \quad$ and $s, t \in \mathbb{Z} \quad$ such that $d = s \cdot a + t \cdot b$

(1) $r_0 := a, r_1 := b, s_0 := 1, t_0 := 0, s_1 := 0, t_1 = 1$

(2) for $i = 1, 2, ... \quad$ perform steps (3) - (5)

(3) if $r_i = 0 \quad$ set $d = |r_{i-1}|$
$s := sgn(r_{i-1}) \cdot s_{i-1}$,
$t := sgn(r_{i-1}) \cdot t_{i-1}$

(4) division with remainder:
$r_{i+1} = r_{i-1} - q_i \cdot r_i, \quad$ with $|r_{i+1}| \leq \frac{1}{2}|r_i|$

(5) set $s_{i+1} := s_{i-1} - q_i \cdot s_i$,
$t_{i+1} := t_{i-1} - q_i \cdot t_i$

Justification : $r_i = s_i \cdot a + t_i \cdot b$ throughout
**Application:** $m, x \in \mathbb{N}$ such that $m, x$ co-prime (i.e. $gcd(x, m) = 1$)
Algorithm 6 yields: $1 = s \cdot x + t \cdot m \Rightarrow s \cdot x \equiv 1 \; mod \; m$. So obtain inverse of $\bar{x} \in \mathbb{Z}/(m)$

## 1.3 Primality testing

Let $\mathbb{P} \subseteq \mathbb{N}$ be the set of prime numbers.
Challenge: Given $n \in \mathbb{N}$ decide if $n \in \mathbb{P}$
**Naive Method:** Trivial division by $m \leq \lfloor \sqrt{n} \rfloor$.
Running time is exponential in $l(n)$. Even when restricted to division by prime numbers, need approximatily $\frac{\sqrt{n}}{|n|\sqrt{n}}$ trivial divisions (prime number theorem)
$\rightarrow$ hardly any better!
**Reminder:** (arithmetic modulo m)
$G$ finite group $\Rightarrow \forall a \in G \quad a^{|G|} = 1$ Fermat's little theorem
For $G = (\mathbb{Z}/(p)^x \quad a^{p-1} \equiv 1 \; mod \; p \in \mathbb{P} \quad \forall a \in \mathbb{Z}$ with $p \nmid a$
Infact $(\mathbb{Z}/(p))^x \cong Z_{p-1}$ is cyclic
For $m = p_1^e, ...p_r^{e_r}$ with $p_i \in \mathbb{P}, \; e_i \in \mathbb{N}_{>0}$:
$\mathbb{Z}_{(m)} \cong \mathbb{Z}_{(p_i^{e_i})} \oplus ... \oplus \mathbb{Z}_{(p_r^{e_r})} \Rightarrow \mathbb{Z}^x_{(m)} \cong \mathbb{Z}^x_{(p_i^{e_i})} \times ... \times \mathbb{Z}^x_{(p_r^{e_r})}$
what is $\mathbb{Z}_{(p^e)}$for $p \in \mathbb{P}, \; e \in \mathbb{N}_{>0}$ ?

### 1.3.1 Theorem 1 (Cyclic)

Let $p \in \mathbb{P} off \;\; e \in \mathbb{N}_{>0} \Rightarrow (\mathbb{Z}_{(p^e)})^x = Z_{(p-1)\cdot p^{e-1}}$ cyclic
**Proof:**
$(\mathbb{Z}_{(p^e)})^x \cong Z_{p-1} \Rightarrow \exists z \in \mathbb{Z} : order(z + p\mathbb{Z}) = p - 1$
Set $a = \bar{z}^{p^{e-1}} \in (\mathbb{Z}_{(p^e)})^x =: G$
$\quad a^{p-1} = \bar{z}^{(p-1)\cdot p^{e-1}} = \bar{z}^{|a|} = 1$
On the other hand, take $i \in \mathbb{Z}$ such that
$a^i = 1 \Rightarrow z^{i \cdot p^{e-1}} \equiv 1 mod p \Rightarrow (p-1)|(i - p^{e-1}) \Rightarrow (p-1)|i.$
So $ord(a) = p - 1$.
Now consider $b = (p \bar{+} 1) \in G$
**Claim:** $ord(b) = p^{e-1}$
**Proof** by induction on $k \in N_{>0}$ that $(p+1)^{p^{k-1}} \equiv p^k + 1 \; mod \; p^{k+1}$
$k = 1 \quad \checkmark$
$k \rightarrow k + 1$ : By induction have $(p+1)^{p^{k-1}} = 1 + p^k + x \cdot p^{k+1}, \quad x \in \mathbb{Z}$
Compute: $(p+1)^{p^k} = ((1+p^k) + x \cdot p^{k+1})^p = \sum_{i=0}^{p} (p \; over \; i)(i+p^k)^{p-i} \cdot x^i \cdot p^{i \cdot (k+1)}$

$$\underbrace{\equiv}_{\text{Only 0-th summand}} (i + p^k) = \sum_{i=0}^{p} (p \; over \; i) p^{i \cdot k} \underbrace{\equiv}_{p \; odd} 1 + p^{k+1} \; mod \; p^{k+2} \quad \checkmark$$

For $k = e : (p+1)^{p^{e-1}} \equiv | \; mod \; p^e \Rightarrow b^{p^e} = 1 \Rightarrow ord(b)|p^{e-1}$
But $(p+1)^{p^{e-2}} \equiv p^{e-1} + 1 \; mod \; p^e \Rightarrow b^{p^{e-2}} \neq 1 \in G$
So $ord(b) = p^{e-1}$
**Claim:** $ord(a \cdot b) = (p-1)p^{e-1}$ $\quad$ ($\Rightarrow$ Theorem)
Let $(a \cdot b)^i = 1 \in G$ with $i \in \mathbb{Z}$
Then $1 = (a \cdot b)^{i \cdot (p-1)} = (a^{p-1})^i \cdot b^{i \cdot (p-1)} = b^{i \cdot (p-1)} \Rightarrow p^{e-1}|i \cdot i(p-1) \Rightarrow p^{e-1}|i$
Also $1 = (a \cdot b)^{p^{e-1} \cdot i} = a^{p^{e-1}} \Rightarrow (p-1)|p^{e-1} \cdot i \Rightarrow (p-1)|i \rightarrow (p-1) \cdot p^{e-1}|i \qquad \square$

**Reminder:** $(\mathbb{Z}/(2^e))^x \cong Z_2 \times Z_2^{e-2} \quad (e \geq 2)$

### 1.3.2 Algorithm 2 (Fermat Test)

input : $n \in \mathbb{N}_{>0} odd$

output: "$n \notin \mathbb{P}$" or "probably $n \in \mathbb{P}$"

(1) Choose $a \in 2, ..., n-1$ randomly

(2) Compute $a^{n-1}$ $mod\ n$

(3) If $a^{n-1} \neq 1\ mod\ n$ then return "$n \notin \mathbb{P}$"
otherwise return "probably $n \in \mathbb{P}$"

Not very satisfying. Is this fast

### 1.3.3 Algorithm 3 (fast exponentiation)

input : $a \in G \quad G$ is a monoid, $e \in \mathbb{N}, e = \sum_{i_0}^{n-1} e_i 2^i, e_i \in \{0, 1\}$

output: $a^e \in G$

(1) Set $b := a, \quad y := 1$

(2) For $i = 0, ..., n-1$ perform (3) - (4)

(3) if $e_i = i$ set $y := y \cdot b$

(4) set $b := b^2$

(5) return y

this requires $O(l(e))$ operations in $G$
For $G = (\mathbb{Z}/(n)_i)$, each multiplication requires $O(l(n)^2)$ bit operations
$\Rightarrow$ Fermat test requires $O(l(n)^3)$ bit operations $\rightarrow$ cubic complexity $\rightarrow$ "fast"!
**Example:**
$n = 561 = 3 \cdot 11 \cdot 17 \quad$ For $a \in \mathbb{Z} \quad$ with $gcd(a, n) \Rightarrow$ have $a^{n-1} = (a^2)^{280} \equiv 1\ mod\ 3$
$a^{n-1} \equiv 1\ mod\ n$ Fermat's test says "probably $n \in \mathbb{P}$" in 57% of cases.
$n = 2207 \cdot 6619 \cdot 15443$ : output "probably $n \in \mathbb{P}$" in 99,93% of cases.

### 1.3.4 Definition 4 (pseudo-prime, witness, Carmichael numbers)

Let $n \in N_{>1} odd, \quad a \in 1, ..., n-1$

(a) $n$ is pseudo-prime to base $a$ if $a^{n-1} \equiv 1 \; mod \; n$

(b) otherwise a is called a witness of composition of $n$

(c) If $n \notin \mathbb{P}$ but $a^{n-1} \equiv 1 \; mod \; n \quad \forall a$ with $gcd(n, a) = 1$
then n is called a Carmichael number.
There are $\infty$ Carmichael numbers

### 1.3.5 Proposition 5:

Let $n \in N_{>1}, \quad odd \notin \mathbb{P}$ not Carmichael
$\Rightarrow |\{a \in \mathbb{Z} | 0 < a < n$ a witness of composite of $n\} > \frac{n-1}{2}$
**Proof:** Consider
$\phi : (\mathbb{Z}/(n))^x =: G \to G, \quad \bar{a} \mapsto \bar{a}^{n-1}$
group homomorphism. By assumption,
$|im(\phi| > 1 \Rightarrow |Ker(\phi)| \leq \frac{|a|}{2} < \frac{n-1}{2}$
$\Rightarrow |\{a \in \mathbb{Z} | 0 < a < n$ a witness of composite of $n\} > \frac{n-1}{2}$ $\square$

**Miller-Rabin Test**

### 1.3.6 Proposition 6

Let $p \in \mathbb{P} \; odd, \quad a \in \{1, ..., p-1\}$ write $p - 1 = 2^k$ with $m \; odd$ Then:
$a^m \equiv 1 \; mod \; p$ or $\exists i \in \{0, ..., k-1\}$;
$a^{2^i \cdot m} \equiv -1 \; mod \; p_i$
**Proof:**
Little Fermat: $\bar{a}^{2^k \cdot m} = 1 \in \mathbb{F}_p$
Assume $\bar{a}^m \neq 1$ take $i$ maximal such that:
$\bar{b} = \bar{a}^{2^i \cdot m} \neq 1 \Rightarrow \bar{b}^2 = 1 \Rightarrow \bar{b} \in \mathbb{F}_p$ is a zero of $x^2 - 1 \in \mathbb{F}_p[x] \Rightarrow \bar{b} = -1$

### 1.3.7 Algorithm 7 (Miller -Rabin)

input : $n \in \mathbb{N}_{>1}, odd$

output: either "$n \notin \mathbb{P}$" or "probably $n \in \mathbb{P}$" $\rightarrow$ Monte Carlo Algorithm.

(1) write $n - 1 = 2^k \cdot m$ with $m$ odd

(2) Choose $s \in \{2, ..., n-1\}$ randomly

(3) Compute $b := a^m \; mod \; n$

(4) if ($b \equiv \pm 1 \; mod \; n$) return "probably $n \in \mathbb{P}$"

(5) for ($i = 0, ..., k-1$) do steps (6) - (7)

(6) set $b := b^2 \; mod \; n$

(7) if $(b \equiv -1 \; mod \; n)$ return "probably $n \in \mathbb{P}$"

(8) return $n \notin \mathbb{P}$"

### 1.3.8 Definition 8

Let $n \in \mathbb{N}_{>1}, \; odd \quad a \in \{1, ..., n-1\}$

(a) n is called a strongly pseudo-prime to base $a$ if Proposition 6 holds for $a$ and $p$ replaced by n.

(b) Otherwise a is called a strong witness of composition of $n$.

**Example**
Let $n \in \mathbb{N}_{>1}, \quad \mathbb{P} \; odd$
$a = 2$ strong witness if $n < 2047$ (including 561)
2 or 3 strong witness if $n < 1373653$
2,3 or 5 strong witness if $n < 25326001$

### 1.3.9 Theorem 7

(a) Algorithm 7 requires $O(l(n)^3)$ bit operations. $\rightarrow$ "qubic complecity" $\rightarrow$ fast!

(b) if $b \in \mathbb{P}$ then Algorithm 7 returns "probably $b \in \mathbb{P}$" $\rightarrow$ no false positives.

(c) if $n \notin \mathbb{P}$ then more than half of the numbers in $\{1, ..., n-1\}$ are strong witnesses.

**Proof:**

(a) Step 1 takes $O(l(n))$ bit operations:
Using Algorithm 3, we need $O(l(n-1))$ multiplications in $\mathbb{Z}/(n)$ each requiring $O(l(n)^2)$ bit operations.

(b) Proposition 6

(c) split in three cases:

**Case 1:** $n$ is not a Carmichael number. $\underset{Prop5}{\Longrightarrow}$ more than half of all numbers are.
Fermat witness thus also strong witness.
**Case 2:** $n = p^r \cdot l$ with $p \in \mathbb{P} \quad r > 1 \quad l \in \mathbb{N}_{>0} p \nmid l$
Theorem 1 $\exists x \in Z$ such that $x^p \equiv 1 \; mod \; p^r \quad x \neq 1 \; mod \; p^r$
Chinese remainder theorem: $\exists a \in \mathbb{Z}$ such that $a \equiv x \; mod \; p^r \quad a \equiv 1 \; mod \; l$
So $\bar{a}^p = 1 \in \mathbb{Z}_(n) \Rightarrow \bar{a}^n = 1 \Rightarrow \bar{a} \in (\mathbb{Z}/(n))^x$
i.e. $gcd(n, a) = 1$ if $\bar{a}^{n-1} = 1$ then $\bar{a} = 1$
But $a \equiv x \neq 1 \; mod \; p^r$ so $\bar{a}^{n-1} \neq 1$ hence $n$ is not Carmichael $\rightarrow$ Case 1.
**Case 3:** $n$ is a Carmichael number. By Case 2 have $n = p \cdot l$ with $p \in \mathbb{P} \quad p \nmid l \quad l \geq 3$

$n$ Carmichael: $\forall a \in \mathbb{Z}$ with $gcd(a, n, ) = 1$
have $a^{2^k \cdot m} \equiv 1 \; mod \; n$ (where $n - 1 = 2^k \cdot m$)
$a^{2^k \cdot m} \equiv 1 \; mod \; p$ Take j minimal such that
$a^{2^j \cdot m} \equiv 1 \; mod \; p \forall a \in \mathbb{Z}$ such that $gcd(a, n) = 1$
so $0 \leq j \leq l$ in fact, $j > 0$ since $(-1)^{2^0 \cdot m} = -1$ with $m$ *odd*.
Consider the subgroup $H := \{\bar{a} \in \mathbb{Z}/(n) | \bar{a}^{2^{j-1} \cdot m} \in \{1, -1\} \subseteq (\mathbb{Z}/(n))^x\}$
Let $a \in \{1, ..., n-1\}$ $\quad gcd(n, a) = 1 \quad a$ not a strong witness.
**Claim 1:** $\bar{a} \in H$
**Case 3.1:** $\bar{a}^{2^{j-1} \cdot m} = 1 \Rightarrow \bar{a} \in H$
**Case 3.2:** $a^{2^{j-1} \cdot m} \neq 1 \; mod \; n \qquad a^m \neq 1 \; mod \; n$
$\xRightarrow[a \text{ nonwitness}]{} \exists i$ such that $\underbrace{a^{2^i \cdot m} \equiv -1 \; mod \; n}_{*}$
$\Rightarrow a^{2^i \cdot m} \equiv -1 \; mod \; p \xRightarrow[\text{def of j}]{} i < j$
if $i < j - 1$ then $a^{2^{j-1} \cdot m} = (a^{2^i \cdot m})^{2^{j-1-i}} \equiv (-1)^{2^{j-1-i}} = 1 \; mod \, n$
$\xRightarrow[\text{with } *]{}$ not in case 3.2
**Claim 2:** $H \subsetneq (\mathbb{Z}_(n))^x$ proper subgroup.
By definition of $j \exists x \in \mathbb{Z}$ such that $x^{2^{j-1} \cdot m} \neq 1 \; mod \; p$
Chinese remainder: $\exists a \in \mathbb{Z}$ such that
$a \equiv x \; mod \; p \qquad a \equiv 1 \; mod \; l \Rightarrow a^{2^{j-1} \cdot m} \neq 1 \; mod \; p \equiv 1 \; mod \; l \Rightarrow \bar{a} \notin H$
Claim 2 ✓
It follows that $|H| \leq \frac{|(\mathbb{Z}/(n))^x|}{2} < \frac{n-1}{2}$
so the number of witnesses is $\geq n - 1 - |H| > \frac{n-1}{2}$ $\qquad \square$
**Remarks:**

(a) A more careful analysis shows that $2\frac{3}{4}$ of all candidates are strong witnesses

(b) Calling Algorithm 7 repeatedly decreases the probability of false positives. Running time for prescribed error probability $p$ is $O(\lg(p^{-1} \cdot l(n)^3))$
(Independence assumptions!)

**Connection with Riemann hypothesis**
Let $n \in \mathbb{N}_{>0} \quad \bar{X} : (\mathbb{Z}/(n))^x \to \mathbb{C}^x$ group homomorphism
$X : \mathbb{Z} \to \mathbb{C}, a \mapsto \begin{cases} \bar{X}(\bar{a}) & \text{if } gcd(a, n) = 1 \quad \text{for } (\bar{a} = a + n\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$
"residence class character *mod n*
$Ex : n = 1 \Rightarrow X(a) = 1 \forall a \in \mathbb{Z}$
Divichlet L-series:
$L_X(s) = \sum\limits_{n=1}^{\infty} \frac{X(n)}{n^s}$ converges for $s \in \mathbb{C}$ until $Re(s) > 1$
$L_X(s)$ extends to a meromorphic function on $\mathbb{C} \mapsto$ "Divichlet L-function".
For $n = 1 : L_X(s) = \zeta(s)$ Riemann Zeta-function.
Euler Product:
From $(1 - X(p) \cdot p^{-s})^{-1} = \sum\limits_{i=0}^{\infty} (X(p) \cdot p^{-s})^i = \sum\limits_{i=0}^{\infty} \frac{X(p^i)}{p^{is}}$ $\quad$ derive $L_X(s) = \prod\limits_{p \in \mathbb{P}} \frac{1}{1 - X(p) \cdot^{-} s}$

Generalized Riemann hypothesis (GRH):
For X residue class character, $s \in \mathbb{C}$
with $L_X(s) = 0, \quad 0 < Re(s) < 1 \qquad$ ("critical strip")
then $Re(s) = \frac{1}{2}$
For $X = 1 \to$ ordinary Riemann hypothesis.

### 1.3.10 Theorem (Arkeny & Bach)

GRH $\Rightarrow \forall X \neq 1$ residence class character
$\exists p \in \mathbb{P} : X(p) \neq 1, p < 2\ln(n)^2$

Let $H \subsetneqq (\mathbb{Z}/(n))^x =: G$ propper subgroup.
Choose $N \subsetneqq G$ maximal proper subgroup such that $H \subseteq N \Rightarrow G/N$ cyclic.
$\bar{X} : G \mapsto \mathbb{C}^x$ with $N = Ker(\bar{X}) \Rightarrow H \subseteq Ker(\mathbb{X})$
$\xRightarrow[\text{GRH, Thm1}]{} \exists p \in \mathbb{P} : p + n\mathbb{Z} \notin H, p < 2 \cdot \ln(n)^2$
**Corollary:** Assume GRH.
Let $n \in \mathbb{N}_{>1} \quad \mathbb{P}$ *odd* Then there is a strong witness $a$ of compositeness of $n$ with $a < 2 \cdot \ln(n)^2$.
$\to$ Obtain deterministic primality test with time $O(\ln(n)^5)$ bit operations.

**AKS-test**
A deterministic polynomial time primaly test $\to$ "holy grail"
Agrawal, Kayal, Saxena: PRIMES is in P, Annals of Mathematics, 2004.

### 1.3.11 Proposition 10

Let $n \in \mathbb{P} \quad a \in \mathbb{Z} \Rightarrow (x+a)^n \equiv x^n + a \; mod \; n$
where x is a indeterminate) and for $r \in \mathbb{N}$:

$$(x+a)^n \equiv x^n + a \; mod \; (n, x^r - 1) \tag{1}$$

(i.e. $(x+a)^n - (x^n + a) = n \cdot f + (x^r - 1) \cdot g \quad$ with $f, g \in \mathbb{Z}[x]$)
**Proof:**
$(x+a)^n = \sum\limits_{i=0}^{n} \binom{n}{i} \cdot x^i a^{n-1} \qquad$ (where $\binom{n}{i}$ is a multiple of $n$ for $0 < in$)
$\equiv x^n + a^n \qquad (\leftarrow$ little Fermat)
$\equiv x^n + a \qquad$ (1) follows by weakening this.
**Cost** analysis for checing (1) with $l =$ length(n).
Using Algorithm 3, need $O(l)$ multiplications in $\mathbb{Z}[x]/(n, x^r - 1) =: R$
Elements of $R$ are represented as polynomials of degree $< r$,
coefficients between 0 and $n$.
Multiply polynomials: $O(r^2)$ operation in $\mathbb{Z}/(n) : O(r^2 \cdot l^2)$
since $x^{r+k} \equiv x^k \; mod \; x^r - 1$,
add coefficients of $x^{r+k}$ of product polynomial to coefficients $x^k : O(r \cdot l)$
Total for checking (1): $O(r^2 \cdot l^3)$ bit operations.

Reduction $mod\ x^r - 1$ is just for keeping the cost under control.
The following is part of AKS-test:

### 1.3.12 Algorithm 11 (Test for perfect power)

input : $n \in \mathbb{N}_{>1}$

output: $m, e \in \mathbb{N} \quad e > 1$ such that $n = m^e$ or "n is not a perfect power"

(1) for ($e = 2, ..., \lfloor \lg(n) \rfloor$) perform (2) - (7) //possible exponents

(2) set $m_1 = 2, m_2 = n$ //initialize interval $[m_1, m_2]$ for searching $\sqrt[e]{n}$

(3) while($m_1 \leq m_2$) do (4) - (7)

(4) set $m = \lfloor \frac{m_1+m_2}{2}$ // bisect interval

(5) if $m^e = n$ return $m, e$

(6) if $m^e > n$ set $m_2 = m - 1$

(7) if $,^e < n$ set $m_1 = m + 1$

(8) return "not a perfect power"

**Cost:** (for $l$ = length($n$))
Compute $m^e : O(\lg(l) \cdot l^2)$ (abort computation once the result exceeds n)
Number of passages through inner loops $\leq \lg(n)$
Number of passages through outer loops $\leq \lg(n)$
Total cost of Algorithm 11: $O(l^4 \cdot \lg(l))$

### 1.3.13 Algorithm 12 (AKS-test)

input : $n \in \mathbb{N}_{>1}$ of length $l = \text{length}(n) = \lfloor \lg(n) \rfloor + 1$

output: "$n \in \mathbb{P}$" or "$n \notin \mathbb{P}$"

(1) check if n is a perfect power.
if yes, return "$n \notin \mathbb{P}$"

(2) find $r \in \mathbb{N}_{>1}$ minimal such that $r|n \quad \vee \quad n^i \neq 1\ mod\ r \qquad \forall i = 1, ..., l^2$
//exhaustive search (we will show that $r \leq l^5$)

(3) if $r|n$
if $r = n$ return "$n \in \mathbb{P}$"
if $r < n$ return "$n \notin \mathbb{P}$"

(4) for $a = 1, 2..., \lfloor \sqrt{r} \cdot l \rfloor$ do (5)

(5) if $(x + a)^n \neq x^n + a\ mod\ (n, x^r - 1)$ return "$n \notin \mathbb{P}$"

(6) return "$n \in \mathbb{P}$"

polynomial rings measure not abs value but max power

## 2 Notes

- $a|b$
  $a$ is divisible by $b$
- $a \nmid b$
  $a$ is not divisible by $b$
- $ord(a)$
- $char(A)$ the smallest positive n such that
  $\underbrace{1 + ... + 1}_{n \;\; summands} = 0 \quad$ with 1 as the multiplicative identity element
- $\mathbb{Z}/(m)$
  Ring modulo m
- $\lg(x) = \log_2(x)$
- Average number of bit operations for an increment:
  one operation for the last bit + 50% chance for one on the next bit + 25% on the following etc. $\Rightarrow$ Geometrical row
  $\Rightarrow$ on average two bit operations
- "Monte Carlo Algorithm": Always terminates in reasonable time but might yield false result
- "Las Vegas Algorithm"