Math Reference

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1 Equations of Motion

Let (X,Y,Z) be EME2000 fixed frame Cartesian co-ordinates. The orbit of an object is governed by the following equations of motion:

$$\begin{aligned}
&\ddot{X} = f_{Kep,X}(X,Y,Z) + f_{J2,X}(X,Y,Z) + f_{C22,X}(X,Y,Z,t) + f_{S22,X}(X,Y,Z,t) \\
&+f_{Moon,X}(X,Y,Z,t) + f_{Sun,X}(X,Y,Z,t) + f_{SRP,X}(X,Y,Z,t) \\
&\ddot{Y} = f_{Kep,Y}(X,Y,Z) + f_{J2,Y}(X,Y,Z) + f_{C22,Y}(X,Y,Z,t) + f_{S22,Y}(X,Y,Z,t) \\
&+f_{Moon,Y}(X,Y,Z,t) + f_{Sun,Y}(X,Y,Z,t) + f_{SRP,Y}(X,Y,Z,t) \\
&\ddot{Z} = f_{Kep,Z}(X,Y,Z) + f_{J2,Z}(X,Y,Z) + f_{C22,Z}(X,Y,Z,t) + f_{S22,Z}(X,Y,Z,t) \\
&+f_{Moon,Z}(X,Y,Z,t) + f_{Sun,Z}(X,Y,Z,t) + f_{SRP,Z}(X,Y,Z,t)
\end{aligned} \tag{1}$$

1.1 KepComponent

$$\begin{aligned}
f_{Kep,X}(X,Y,Z) &= -\tfrac{GM_E X}{(X^2+Y^2+Z^2)^{3/2}} \\
f_{Kep,Y}(X,Y,Z) &= -\tfrac{GM_E Y}{(X^2+Y^2+Z^2)^{3/2}} \\
f_{Kep,Z}(X,Y,Z) &= -\tfrac{GM_E Z}{(X^2+Y^2+Z^2)^{3/2}}
\end{aligned} \tag{2}$$

1.1.1 Implementation details

First calculate shared factor

$$d = \tfrac{1}{(X^2+Y^2+Z^2)^{3/2}} \tag{3}$$

After that calculate

$$\begin{aligned}
f_{Kep,X}(X,Y,Z) &= -GM_E dX \\
f_{Kep,Y}(X,Y,Z) &= -GM_E dY \\
f_{Kep,Z}(X,Y,Z) &= -GM_E dZ
\end{aligned} \tag{4}$$

1.2 J2Component

$$
\begin{aligned}
f_{J2,X}(X,Y,Z) &= \tfrac{GM_E R_E^2 \sqrt{5} C_{20} X}{2(X^2+Y^2+Z^2)^{1/2}} \left(\tfrac{3}{(X^2+Y^2+Z^2)^2} - \tfrac{15Z^2}{(X^2+Y^2+Z^2)^3} \right) \\
f_{J2,Y}(X,Y,Z) &= \tfrac{GM_E R_E^2 \sqrt{5} C_{20} Y}{2(X^2+Y^2+Z^2)^{1/2}} \left(\tfrac{3}{(X^2+Y^2+Z^2)^2} - \tfrac{15Z^2}{(X^2+Y^2+Z^2)^3} \right) \\
f_{J2,Z}(X,Y,Z) &= \tfrac{GM_E R_E^2 \sqrt{5} C_{20} Z}{2(X^2+Y^2+Z^2)^{1/2}} \left(\tfrac{9}{(X^2+Y^2+Z^2)^2} - \tfrac{15Z^2}{(X^2+Y^2+Z^2)^3} \right)
\end{aligned} \tag{5}
$$

1.2.1 Implementation details

Precalculate constant factor

$$
f = \tfrac{GM_E R_E^2 \sqrt{5} C_{20}}{2} \tag{6}
$$

Per (X,Y,Z) calculate

$$
\begin{aligned}
d_1 &= f \tfrac{1}{\sqrt{X^2+Y^2+Z^2}} \\
d_2 &= \tfrac{1}{(X^2+Y^2+Z^2)^2} \\
s &= \tfrac{15Z^2}{(X^2+Y^2+Z^2)^3}
\end{aligned} \tag{7}
$$

After that calculate

$$
\begin{aligned}
f_{J2,X}(X,Y,Z) &= d_1 X \left(3d_2 - s\right) \\
f_{J2,Y}(X,Y,Z) &= d_1 Y \left(3d_2 - s\right) \\
f_{J2,Z}(X,Y,Z) &= d_1 Z \left(9d_2 - s\right)
\end{aligned} \tag{8}
$$

1.3 C22S22Component

$$
\begin{aligned}
f_{C22,X}(X,Y,Z,t) &= f_{C22,x}(x,y,z)\cos(\theta_G+\nu_E t) - f_{C22,y}(x,y,z)\sin(\theta_G+\nu_E t) \\
f_{C22,Y}(X,Y,Z,t) &= f_{C22,x}(x,y,z)\sin(\theta_G+\nu_E t) + f_{C22,y}(x,y,z)\cos(\theta_G+\nu_E t) \\
f_{C22,Z}(X,Y,Z,t) &= f_{C22,z}(x,y,z) \\
f_{S22,X}(X,Y,Z,t) &= f_{S22,x}(x,y,z)\cos(\theta_G+\nu_E t) - f_{S22,y}(x,y,z)\sin(\theta_G+\nu_E t) \\
f_{S22,Y}(X,Y,Z,t) &= f_{S22,x}(x,y,z)\sin(\theta_G+\nu_E t) + f_{S22,y}(x,y,z)\cos(\theta_G+\nu_E t) \\
f_{S22,Z}(X,Y,Z,t) &= f_{S22,z}(x,y,z)
\end{aligned} \tag{9}
$$

with

$$
\begin{aligned}
x &= X\cos(\theta_G+\nu_E t) + Y\sin(\theta_G+\nu_E t) \\
y &= -X\sin(\theta_G+\nu_E t) + Y\cos(\theta_G+\nu_E t) \\
z &= Z
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
f_{C22,x}(x,y,z) &= \frac{5GM_E R_E^2\sqrt{15}C_{22}x(y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}} + \frac{GM_E R_E^2\sqrt{15}C_{22}x}{(x^2+y^2+z^2)^{5/2}}\\
f_{C22,y}(x,y,z) &= \frac{5GM_E R_E^2\sqrt{15}C_{22}y(y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}} - \frac{GM_E R_E^2\sqrt{15}C_{22}y}{(x^2+y^2+z^2)^{5/2}}\\
f_{C22,z}(x,y,z) &= \frac{5GM_E R_E^2\sqrt{15}C_{22}z(y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}}\\
f_{S22,x}(x,y,z) &= -\frac{5GM_E R_E^2\sqrt{15}S_{22}x^2y}{(x^2+y^2+z^2)^{7/2}} + \frac{GM_E R_E^2\sqrt{15}S_{22}y}{(x^2+y^2+z^2)^{5/2}}\\
f_{S22,y}(x,y,z) &= -\frac{5GM_E R_E^2\sqrt{15}S_{22}xy^2}{(x^2+y^2+z^2)^{7/2}} + \frac{GM_E R_E^2\sqrt{15}S_{22}x}{(x^2+y^2+z^2)^{5/2}}\\
f_{S22,z}(x,y,z) &= -\frac{5GM_E R_E^2\sqrt{15}S_{22}xyz}{(x^2+y^2+z^2)^{7/2}}
\end{aligned}
\tag{11}
$$

1.3.1 Implementation details

1.4 SolComponent

$$
\begin{aligned}
f_{Sun,X}(X,Y,Z,t) &= -GM_\odot\left(\frac{(X-X_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \frac{X_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}}\right)\\
f_{Sun,Y}(X,Y,Z,t) &= -GM_\odot\left(\frac{(Y-Y_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \frac{Y_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}}\right)\\
f_{Sun,Z}(X,Y,Z,t) &= -GM_\odot\left(\frac{(Z-Z_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \frac{Z_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}}\right)
\end{aligned}
\tag{12}
$$

where

$$
\begin{pmatrix} X_\odot \\ Y_\odot \\ Z_\odot \end{pmatrix} = \begin{pmatrix} r_\odot \cos\lambda_\odot \\ r_\odot \sin\lambda_\odot \cos\varepsilon \\ r_\odot \sin\lambda_\odot \sin\varepsilon \end{pmatrix}
\tag{13}
$$

with

$$
\begin{aligned}
\lambda_\odot &= \Omega_\odot + \omega_\odot + \ell_\odot + \tfrac{\pi}{180}\left(\tfrac{6892}{3600}\sin\ell_\odot + \tfrac{72}{3600}\sin 2\ell_\odot\right)\\
r_\odot[10^6\mathrm{km}] &= 149.619 - 2.499\cos\ell_\odot - 0.021\cos 2\ell_\odot\\
\ell_\odot &= \varphi_{\odot,0} + \nu_\odot t
\end{aligned}
\tag{14}
$$

1.4.1 Implementation details

1.5 LunComponent

$$
\begin{aligned}
f_{Moon,X}(X,Y,Z,t) &= -GM_\mathcal{M}\left(\frac{(X-X_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{X_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}}\right)\\
f_{Moon,Y}(X,Y,Z,t) &= -GM_\mathcal{M}\left(\frac{(Y-Y_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{Y_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}}\right)\\
f_{Moon,Z}(X,Y,Z,t) &= -GM_\mathcal{M}\left(\frac{(Z-Z_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{Z_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}}\right)
\end{aligned}
\tag{15}
$$

where

$$
\begin{pmatrix} X_\mathcal{M} \\ Y_\mathcal{M} \\ Z_\mathcal{M} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \cdot \begin{pmatrix} r_\mathcal{M}\cos\lambda_\mathcal{M}\cos\beta_\mathcal{M} \\ r_\mathcal{M}\sin\lambda_\mathcal{M}\cos\beta_\mathcal{M} \\ r_\mathcal{M}\sin\beta_\mathcal{M} \end{pmatrix}
\tag{16}
$$

with

$$
\begin{aligned}
r_{\mathcal{M}}[\mathrm{km}] \quad &= 385000 - 20905\cos(l_{\mathcal{M}}) - 3699\cos(2D_{\mathcal{M}} - l_{\mathcal{M}}) \\
&-2956\cos(2D_{\mathcal{M}}) - 570\cos(2l_{\mathcal{M}}) \\
&+246\cos(2l_{\mathcal{M}} - 2D_{\mathcal{M}}) - 205\cos(l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
&-171\cos(l_{\mathcal{M}} + 2D_{\mathcal{M}}) \\
&-152\cos(l_{\mathcal{M}} + l'_{\mathcal{M}} - 2D_{\mathcal{M}})
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{\mathcal{M}} \quad &= L_0 + \tfrac{\pi}{180}\big(\tfrac{22640}{3600}\sin(l_{\mathcal{M}}) + \tfrac{769}{3600}\sin(2l_{\mathcal{M}}) \\
&-\tfrac{4856}{3600}\sin(l_{\mathcal{M}} - 2D_{\mathcal{M}}) + \tfrac{2370}{3600}\sin(2D_{\mathcal{M}}) \\
&-\tfrac{668}{3600}\sin(l'_{\mathcal{M}}) - \tfrac{412}{3600}\sin(2F_{\mathcal{M}}) \\
&-\tfrac{212}{3600}\sin(2l_{\mathcal{M}} - 2D_{\mathcal{M}}) - \tfrac{206}{3600}\sin(l_{\mathcal{M}} + l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
&+\tfrac{192}{3600}\sin(l_{\mathcal{M}} + 2D_{\mathcal{M}}) - \tfrac{165}{3600}\sin(l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
&+\tfrac{148}{3600}\sin(l_{\mathcal{M}} - l'_{\mathcal{M}}) - \tfrac{125}{3600}\sin(D_{\mathcal{M}}) \\
&-\tfrac{110}{3600}\sin(l_{\mathcal{M}} + l'_{\mathcal{M}}) - \tfrac{55}{3600}\sin(2F_{\mathcal{M}} - 2D_{\mathcal{M}})\big)
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\beta_{\mathcal{M}} \quad &= \tfrac{\pi}{180}\big(\tfrac{18520}{3600}\sin\left(F_{\mathcal{M}} + \lambda_{\mathcal{M}} - L_0 + \tfrac{\pi}{180}\big(\tfrac{412}{3600}\sin(2F_{\mathcal{M}}) + \tfrac{541}{3600}\sin(l'_{\mathcal{M}})\big)\right) \\
&-\tfrac{526}{3600}\sin(F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
&+\tfrac{44}{3600}\sin(l_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) - \tfrac{31}{3600}\sin(-l_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
&-\tfrac{25}{3600}\sin(-2l_{\mathcal{M}} + F_{\mathcal{M}}) - \tfrac{23}{3600}\sin(l'_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
&+\tfrac{21}{3600}\sin(-l_{\mathcal{M}} + F_{\mathcal{M}}) + \tfrac{11}{3600}\sin(-l'_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}})\big)
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\varphi_M \quad &= \nu_{\odot} t \\
\varphi_{M_a} \quad &= \nu_{M_a} t \\
\varphi_{M_p} \quad &= \nu_{M_p} t \\
\varphi_{M_S} \quad &= \nu_{M_s} t \\
L_0 \quad &= \varphi_{M_p} + \varphi_{M_a} + \tfrac{\pi}{180}(218.31617) \\
l_{\mathcal{M}} \quad &= \varphi_{M_a} + \tfrac{\pi}{180}(134.96292) \\
l'_{\mathcal{M}} \quad &= \ell_{\odot} = \varphi_M + \tfrac{\pi}{180}(357.52543) \\
F_{\mathcal{M}} \quad &= \varphi_{M_p} + \varphi_{M_a} + \varphi_{M_S} + \tfrac{\pi}{180}(93.27283) \\
D_{\mathcal{M}} \quad &= \varphi_{M_p} + \varphi_{M_a} - \varphi_M + \tfrac{\pi}{180}(297.85027)
\end{aligned}
$$

1.5.1 Implementation details

1.6 SRPComponent

$$
\begin{aligned}
f_{SRP,X}(X,Y,Z,t) \quad &= AOM\,\tfrac{P_{SRP}a_{\odot}^2(X-X_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}} \\
f_{SRP,Y}(X,Y,Z,t) \quad &= AOM\,\tfrac{P_{SRP}a_{\odot}^2(Y-Y_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}} \\
f_{SRP,Z}(X,Y,Z,t) \quad &= AOM\,\tfrac{P_{SRP}a_{\odot}^2(Z-Z_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}}
\end{aligned}
\tag{19}
$$

1.6.1 Implementation details

1.7 DragComponent

1.7.1 Implementation details

2 Constants

$$GM_E = 3.986004407799724 \times 10^5 km^3 sec^{-2} \tag{20}$$

$$GM_\odot = 1.32712440018 \times 10^{11} km^3 sec^{-2} \tag{21}$$

$$GM_\mathcal{M} = 4.9028 \times 10^3 km^3 sec^{-2} \tag{22}$$

$$R_E = 6378.1363 km \tag{23}$$

$$C_{20} = -4.84165371736 \times 10^{-4} \tag{24}$$

$$C_{22} = 2.43914352398 \times 10^{-6} \tag{25}$$

$$S_{22} = -1.40016683654 \times 10^{-6} \tag{26}$$

$$\theta_G = \frac{\pi}{180}(280.4606) \tag{27}$$

$$\nu_E = \frac{\pi}{180}(4.178074622024230 \times 10^{-3}) \tag{28}$$

$$\nu_\odot = \frac{\pi}{180}(1.1407410259335311 \times 10^{-5}) \tag{29}$$

$$\nu_{M_a} = \frac{\pi}{180}(1.512151961904581 \times 10^{-4}) \tag{30}$$

$$\nu_{M_p} = \frac{\pi}{180}(1.2893925235125941 \times 10^{-6}) \tag{31}$$

$$\nu_{M_s} = \frac{\pi}{180}(6.128913003523574 \times 10^{-7}) \tag{32}$$

$$a_\odot = 1.49619 \times 10^8 km \tag{33}$$

$$\varepsilon = \frac{\pi}{180}(23.4392911) \tag{34}$$

$$\varphi_{\odot,0} = \frac{\pi}{180}(357.5256) \tag{35}$$

$$\Omega_\odot + \omega_\odot = \frac{\pi}{180}(282.94) \tag{36}$$

$$P_{SRP} = 4.56 \times 10^{-6} \tag{37}$$