Math Reference

Oliver Bösing

May 2021

1 Equations of Motion

Let (X,Y,Z) be EME2000 fixed frame Cartesian co-ordinates. The orbit of an object is governed by the following equations of motion:

$$
\begin{aligned}
&\ddot{X} = f_{Kep,X}(X,Y,Z) + f_{J2,X}(X,Y,Z) + f_{C22,X}(X,Y,Z,t) + f_{S22,X}(X,Y,Z,t) \\
&+f_{Moon,X}(X,Y,Z,t) + f_{Sun,X}(X,Y,Z,t) + f_{SRP,X}(X,Y,Z,t) \\
&\ddot{Y} = f_{Kep,Y}(X,Y,Z) + f_{J2,Y}(X,Y,Z) + f_{C22,Y}(X,Y,Z,t) + f_{S22,Y}(X,Y,Z,t) \\
&+f_{Moon,Y}(X,Y,Z,t) + f_{Sun,Y}(X,Y,Z,t) + f_{SRP,Y}(X,Y,Z,t) \\
&\ddot{Z} = f_{Kep,Z}(X,Y,Z) + f_{J2,Z}(X,Y,Z) + f_{C22,Z}(X,Y,Z,t) + f_{S22,Z}(X,Y,Z,t) \\
&+f_{Moon,Z}(X,Y,Z,t) + f_{Sun,Z}(X,Y,Z,t) + f_{SRP,Z}(X,Y,Z,t)
\end{aligned} \tag{1}
$$

1.1 KepComponent

$$
\begin{aligned}
f_{Kep,X}(X,Y,Z) &= -\tfrac{GM_E X}{(X^2+Y^2+Z^2)^{3/2}} \\
f_{Kep,Y}(X,Y,Z) &= -\tfrac{GM_E Y}{(X^2+Y^2+Z^2)^{3/2}} \\
f_{Kep,Z}(X,Y,Z) &= -\tfrac{GM_E Z}{(X^2+Y^2+Z^2)^{3/2}}
\end{aligned} \tag{2}
$$

1.1.1 Implementation details

First calculate shared factor

$$
d = \tfrac{1}{(X^2+Y^2+Z^2)^{3/2}} \tag{3}
$$

After that calculate

$$
\begin{aligned}
f_{Kep,X}(X,Y,Z) &= -GM_E dX \\
f_{Kep,Y}(X,Y,Z) &= -GM_E dY \\
f_{Kep,Z}(X,Y,Z) &= -GM_E dZ
\end{aligned} \tag{4}
$$

1.2 J2Component

$$\begin{aligned} f_{J2,X}(X,Y,Z) &= \frac{GM_E R_E^2 \sqrt{5} C_{20} X}{2(X^2+Y^2+Z^2)^{1/2}} \left(\frac{3}{(X^2+Y^2+Z^2)^2} - \frac{15Z^2}{(X^2+Y^2+Z^2)^3} \right) \\ f_{J2,Y}(X,Y,Z) &= \frac{GM_E R_E^2 \sqrt{5} C_{20} Y}{2(X^2+Y^2+Z^2)^{1/2}} \left(\frac{3}{(X^2+Y^2+Z^2)^2} - \frac{15Z^2}{(X^2+Y^2+Z^2)^3} \right) \\ f_{J2,Z}(X,Y,Z) &= \frac{GM_E R_E^2 \sqrt{5} C_{20} Z}{2(X^2+Y^2+Z^2)^{1/2}} \left(\frac{9}{(X^2+Y^2+Z^2)^2} - \frac{15Z^2}{(X^2+Y^2+Z^2)^3} \right) \end{aligned} \tag{5}$$

1.2.1 Implementation details

Precalculate constant factor

$$f = \frac{GM_E R_E^2 \sqrt{5} C_{20}}{2} \tag{6}$$

Per (X,Y,Z) calculate

$$\begin{aligned} d_1 &= f \frac{1}{\sqrt{X^2+Y^2+Z^2}} \\ d_2 &= \frac{1}{(X^2+Y^2+Z^2)^2} \\ s &= \frac{15Z^2}{(X^2+Y^2+Z^2)^3} \end{aligned} \tag{7}$$

And

$$\begin{aligned} f_{J2,X}(X,Y,Z) &= d_1 X \,(3d_2 - s) \\ f_{J2,Y}(X,Y,Z) &= d_1 Y \,(3d_2 - s) \\ f_{J2,Z}(X,Y,Z) &= d_1 Z \,(9d_2 - s) \end{aligned} \tag{8}$$

1.3 C22S22Component

$$\begin{aligned} f_{C22,X}(X,Y,Z,t) &= f_{C22,x}(x,y,z)\cos(\theta_G+\nu_E t) - f_{C22,y}(x,y,z)\sin(\theta_G+\nu_E t) \\ f_{C22,Y}(X,Y,Z,t) &= f_{C22,x}(x,y,z)\sin(\theta_G+\nu_E t) + f_{C22,y}(x,y,z)\cos(\theta_G+\nu_E t) \\ f_{C22,Z}(X,Y,Z,t) &= f_{C22,z}(x,y,z) \\ f_{S22,X}(X,Y,Z,t) &= f_{S22,x}(x,y,z)\cos(\theta_G+\nu_E t) - f_{S22,y}(x,y,z)\sin(\theta_G+\nu_E t) \\ f_{S22,Y}(X,Y,Z,t) &= f_{S22,x}(x,y,z)\sin(\theta_G+\nu_E t) + f_{S22,y}(x,y,z)\cos(\theta_G+\nu_E t) \\ f_{S22,Z}(X,Y,Z,t) &= f_{S22,z}(x,y,z) \end{aligned} \tag{9}$$

with

$$\begin{aligned} x &= X\cos(\theta_G+\nu_E t) + Y\sin(\theta_G+\nu_E t) \\ y &= -X\sin(\theta_G+\nu_E t) + Y\cos(\theta_G+\nu_E t) \\ z &= Z \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
f_{C22,x}(x,y,z) &= \frac{5GM_ER_E^2\sqrt{15}C_{22}x(y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}} + \frac{GM_ER_E^2\sqrt{15}C_{22}x}{(x^2+y^2+z^2)^{5/2}} \\
f_{C22,y}(x,y,z) &= \frac{5GM_ER_E^2\sqrt{15}C_{22}y(y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}} - \frac{GM_ER_E^2\sqrt{15}C_{22}y}{(x^2+y^2+z^2)^{5/2}} \\
f_{C22,z}(x,y,z) &= \frac{5GM_ER_E^2\sqrt{15}C_{22}z(y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}} \\
f_{S22,x}(x,y,z) &= -\frac{5GM_ER_E^2\sqrt{15}S_{22}x^2y}{(x^2+y^2+z^2)^{7/2}} + \frac{GM_ER_E^2\sqrt{15}S_{22}y}{(x^2+y^2+z^2)^{5/2}} \\
f_{S22,y}(x,y,z) &= -\frac{5GM_ER_E^2\sqrt{15}S_{22}xy^2}{(x^2+y^2+z^2)^{7/2}} + \frac{GM_ER_E^2\sqrt{15}S_{22}x}{(x^2+y^2+z^2)^{5/2}} \\
f_{S22,z}(x,y,z) &= -\frac{5GM_ER_E^2\sqrt{15}S_{22}xyz}{(x^2+y^2+z^2)^{7/2}}
\end{aligned} \tag{11}
$$

1.3.1 Implementation details

Precalculate constant factors

$$
f = GM_ER_E^2\sqrt{15} \tag{12}
$$

$$
\begin{aligned}
f_{C22_2} &= GM_ER_E^2\sqrt{15}C_{22} \\
&= fC_{22} \\
f_{C22_1} &= \frac{5GM_ER_E^2\sqrt{15}C_{22}}{2} \\
&= f\left(\frac{5}{2}\right)
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
f_{S22_2} &= GM_ER_E^2\sqrt{15}S_{22} \\
&= fS_{22} \\
f_{S22_1} &= -5GM_ER_E^2\sqrt{15}S_{22} \\
&= -5f_{S22_2}
\end{aligned} \tag{14}
$$

Per time step calculate

$$
\begin{aligned}
f_{sin} &= \sin(\theta_G + \nu_E t) \\
f_{cos} &= \cos(\theta_G + \nu_E t)
\end{aligned} \tag{15}
$$

Per (X,Y,Z) calculate

$$
\begin{aligned}
x &= Xf_{cos} + Yf_{sin} \\
y &= -Xf_{sin} + Yf_{cos} \\
z &= Z
\end{aligned} \tag{16}
$$

Shared terms

$$
\begin{aligned}
n_{C_{22}} &= f_{C_{22_1}}(y^2 - x^2) \\
n_{S_{22}} &= f_{S_{22_1}}xy
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
d_1 &= \frac{1}{\sqrt{(x^2+y^2+z^2)^7}} \\
d_2 &= \frac{1}{\sqrt{(x^2+y^2+z^2)^5}}
\end{aligned} \tag{18}
$$

Calculate

$$\begin{aligned}
f_{C22,x}(x,y,z) &= n_{C_{22}} x d_1 + f_{C_{22_2}} x d_2 \\
f_{C22,y}(x,y,z) &= n_{C_{22}} y d_1 + f_{C_{22_2}} y d_2 \\
f_{C22,z}(x,y,z) &= n_{C_{22}} z d_1 \\
f_{S22,x}(x,y,z) &= n_{S_{22}} x d_1 + f_{S_{22_2}} y d_2 \\
f_{S22,y}(x,y,z) &= n_{S_{22}} y d_1 + f_{S_{22_2}} x d_2 \\
f_{S22,z}(x,y,z) &= n_{S_{22}} z d_1
\end{aligned} \tag{19}$$

$$\begin{aligned}
f_{C22,X}(X,Y,Z,t) &= f_{C22,x}(x,y,z) f_{cos} - f_{C22,y}(x,y,z) f_{sin} \\
f_{C22,Y}(X,Y,Z,t) &= f_{C22,x}(x,y,z) f_{sin} + f_{C22,y}(x,y,z) f_{cos} \\
f_{C22,Z}(X,Y,Z,t) &= f_{C22,z}(x,y,z) \\
f_{S22,X}(X,Y,Z,t) &= f_{S22,x}(x,y,z) f_{cos} - f_{S22,y}(x,y,z) f_{sin} \\
f_{S22,Y}(X,Y,Z,t) &= f_{S22,x}(x,y,z) f_{sin} + f_{S22,y}(x,y,z) f_{cos} \\
f_{S22,Z}(X,Y,Z,t) &= f_{S22,z}(x,y,z)
\end{aligned} \tag{20}$$

1.4 SolComponent

$$\begin{aligned}
f_{Sun,X}(X,Y,Z,t) &= -GM_\odot \left(\frac{(X-X_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \frac{X_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}} \right) \\
f_{Sun,Y}(X,Y,Z,t) &= -GM_\odot \left(\frac{(Y-Y_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \frac{Y_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}} \right) \\
f_{Sun,Z}(X,Y,Z,t) &= -GM_\odot \left(\frac{(Z-Z_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \frac{Z_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}} \right)
\end{aligned} \tag{21}$$

where

$$\begin{pmatrix} X_\odot \\ Y_\odot \\ Z_\odot \end{pmatrix} = \begin{pmatrix} r_\odot \cos\lambda_\odot \\ r_\odot \sin\lambda_\odot \cos\varepsilon \\ r_\odot \sin\lambda_\odot \sin\varepsilon \end{pmatrix} \tag{22}$$

with

$$\begin{aligned}
\lambda_\odot &= \Omega_\odot + \omega_\odot + \ell_\odot + \tfrac{\pi}{180} \left(\tfrac{6892}{3600} \sin\ell_\odot + \tfrac{72}{3600} \sin 2\ell_\odot \right) \\
r_\odot[10^6\text{km}] &= 149.619 - 2.499 \cos\ell_\odot - 0.021 \cos 2\ell_\odot \\
\ell_\odot &= \varphi_{\odot,0} + \nu_\odot t
\end{aligned} \tag{23}$$

1.4.1 Implementation details

1.5 LunComponent

$$\begin{aligned}
f_{Moon,X}(X,Y,Z,t) &= -GM_\mathcal{M} \left(\frac{(X-X_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{X_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}} \right) \\
f_{Moon,Y}(X,Y,Z,t) &= -GM_\mathcal{M} \left(\frac{(Y-Y_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{Y_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}} \right) \\
f_{Moon,Z}(X,Y,Z,t) &= -GM_\mathcal{M} \left(\frac{(Z-Z_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{Z_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}} \right)
\end{aligned} \tag{24}$$

where

$$\begin{pmatrix} X_{\mathcal{M}} \\ Y_{\mathcal{M}} \\ Z_{\mathcal{M}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \cdot \begin{pmatrix} r_{\mathcal{M}}\cos\lambda_{\mathcal{M}}\cos\beta_{\mathcal{M}} \\ r_{\mathcal{M}}\sin\lambda_{\mathcal{M}}\cos\beta_{\mathcal{M}} \\ r_{\mathcal{M}}\sin\beta_{\mathcal{M}} \end{pmatrix} \tag{25}$$

with

$$\begin{aligned} r_{\mathcal{M}}[\text{km}] \quad &= 385000 - 20905\cos(l_{\mathcal{M}}) - 3699\cos(2D_{\mathcal{M}} - l_{\mathcal{M}}) \\ &-2956\cos(2D_{\mathcal{M}}) - 570\cos(2l_{\mathcal{M}}) \\ &+246\cos(2l_{\mathcal{M}} - 2D_{\mathcal{M}}) - 205\cos(l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \\ &-171\cos(l_{\mathcal{M}} + 2D_{\mathcal{M}}) \\ &-152\cos(l_{\mathcal{M}} + l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \end{aligned}$$

$$\begin{aligned} \lambda_{\mathcal{M}} \quad &= L_0 + \tfrac{\pi}{180}(\tfrac{22640}{3600}\sin(l_{\mathcal{M}}) + \tfrac{769}{3600}\sin(2l_{\mathcal{M}}) \\ &- \tfrac{4856}{3600}\sin(l_{\mathcal{M}} - 2D_{\mathcal{M}}) + \tfrac{2370}{3600}\sin(2D_{\mathcal{M}}) \\ &- \tfrac{668}{3600}\sin(l'_{\mathcal{M}}) - \tfrac{412}{3600}\sin(2F_{\mathcal{M}}) \\ &- \tfrac{212}{3600}\sin(2l_{\mathcal{M}} - 2D_{\mathcal{M}}) - \tfrac{206}{3600}\sin(l_{\mathcal{M}} + l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \\ &+ \tfrac{192}{3600}\sin(l_{\mathcal{M}} + 2D_{\mathcal{M}}) - \tfrac{165}{3600}\sin(l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \\ &+ \tfrac{148}{3600}\sin(l_{\mathcal{M}} - l'_{\mathcal{M}}) - \tfrac{125}{3600}\sin(D_{\mathcal{M}}) \\ &- \tfrac{110}{3600}\sin(l_{\mathcal{M}} + l'_{\mathcal{M}}) - \tfrac{55}{3600}\sin(2F_{\mathcal{M}} - 2D_{\mathcal{M}})) \end{aligned} \tag{26}$$

$$\begin{aligned} \beta_{\mathcal{M}} \quad &= \tfrac{\pi}{180}(\tfrac{18520}{3600}\sin\left(F_{\mathcal{M}} + \lambda_{\mathcal{M}} - L_0 + \tfrac{\pi}{180}(\tfrac{412}{3600}\sin(2F_{\mathcal{M}}) + \tfrac{541}{3600}\sin(l'_{\mathcal{M}}))\right) \\ &- \tfrac{526}{3600}\sin(F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\ &+ \tfrac{44}{3600}\sin(l_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) - \tfrac{31}{3600}\sin(-l_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\ &- \tfrac{25}{3600}\sin(-2l_{\mathcal{M}} + F_{\mathcal{M}}) - \tfrac{23}{3600}\sin(l'_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\ &+ \tfrac{21}{3600}\sin(-l_{\mathcal{M}} + F_{\mathcal{M}}) + \tfrac{11}{3600}\sin(-l'_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}})) \end{aligned} \tag{27}$$

and

$$\begin{aligned} \varphi_M &= \nu_{\odot} t \\ \varphi_{M_a} &= \nu_{M_a} t \\ \varphi_{M_p} &= \nu_{M_p} t \\ \varphi_{M_S} &= \nu_{M_s} t \\ L_0 &= \varphi_{M_p} + \varphi_{M_a} + \tfrac{\pi}{180}(218.31617) \\ l_{\mathcal{M}} &= \varphi_{M_a} + \tfrac{\pi}{180}(134.96292) \\ l'_{\mathcal{M}} &= \ell_{\odot} = \varphi_M + \tfrac{\pi}{180}(357.52543) \\ F_{\mathcal{M}} &= \varphi_{M_p} + \varphi_{M_a} + \varphi_{M_S} + \tfrac{\pi}{180}(93.27283) \\ D_{\mathcal{M}} &= \varphi_{M_p} + \varphi_{M_a} - \varphi_M + \tfrac{\pi}{180}(297.85027) \end{aligned}$$

1.5.1 Implementation details

1.6 SRPComponent

$$\begin{aligned} f_{SRP,X}(X,Y,Z,t) \quad &= AOM\frac{P_{SRP}a_{\odot}^2(X-X_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}} \\ f_{SRP,Y}(X,Y,Z,t) \quad &= AOM\frac{P_{SRP}a_{\odot}^2(Y-Y_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}} \\ f_{SRP,Z}(X,Y,Z,t) \quad &= AOM\frac{P_{SRP}a_{\odot}^2(Z-Z_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}} \end{aligned} \tag{28}$$

1.6.1 Implementation details

1.7 DragComponent

1.7.1 Implementation details

2 Constants

$$GM_E = 3.986004407799724 \times 10^5 km^3 sec^{-2} \tag{29}$$

$$GM_\odot = 1.32712440018 \times 10^{11} km^3 sec^{-2} \tag{30}$$

$$GM_\mathcal{M} = 4.9028 \times 10^3 km^3 sec^{-2} \tag{31}$$

$$R_E = 6378.1363 km \tag{32}$$

$$C_{20} = -4.84165371736 \times 10^{-4} \tag{33}$$

$$C_{22} = 2.43914352398 \times 10^{-6} \tag{34}$$

$$S_{22} = -1.40016683654 \times 10^{-6} \tag{35}$$

$$\theta_G = \frac{\pi}{180}(280.4606) \tag{36}$$

$$\nu_E = \frac{\pi}{180}(4.178074622024230 \times 10^{-3}) \tag{37}$$

$$\nu_\odot = \frac{\pi}{180}(1.1407410259335311 \times 10^{-5}) \tag{38}$$

$$\nu_{M_a} = \frac{\pi}{180}(1.512151961904581 \times 10^{-4}) \tag{39}$$

$$\nu_{M_p} = \frac{\pi}{180}(1.2893925235125941 \times 10^{-6}) \tag{40}$$

$$\nu_{M_s} = \frac{\pi}{180}(6.128913003523574 \times 10^{-7}) \tag{41}$$

$$a_\odot = 1.49619 \times 10^8 km \tag{42}$$

$$\varepsilon = \frac{\pi}{180}(23.4392911) \tag{43}$$

$$\varphi_{\odot,0} = \frac{\pi}{180}(357.5256) \tag{44}$$

$$\Omega_\odot + \omega_\odot = \frac{\pi}{180}(282.94) \tag{45}$$

$$P_{SRP} = 4.56 \times 10^{-6} \tag{46}$$