Math Reference

Oliver Bösing

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1 Equations of Motion

Let (X,Y,Z) be EME2000 fixed frame Cartesian co-ordinates. The orbit of an object is governed by the following equations of motion:

$$\begin{aligned}
&\ddot{X} = f_{Kep,X}(X,Y,Z) + f_{J2,X}(X,Y,Z) + f_{C22,X}(X,Y,Z,t) + f_{S22,X}(X,Y,Z,t) \\
&+f_{Moon,X}(X,Y,Z,t) + f_{Sun,X}(X,Y,Z,t) + f_{SRP,X}(X,Y,Z,AOM) + f_{Drag,X}(X,Y,Z,A,m,v) \\
&\ddot{Y} = f_{Kep,Y}(X,Y,Z) + f_{J2,Y}(X,Y,Z) + f_{C22,Y}(X,Y,Z,t) + f_{S22,Y}(X,Y,Z,t) \\
&+f_{Moon,Y}(X,Y,Z,t) + f_{Sun,Y}(X,Y,Z,t) + f_{SRP,Y}(X,Y,Z,AOM) + f_{Drag,Y}(X,Y,Z,A,m,v) \\
&\ddot{Z} = f_{Kep,Z}(X,Y,Z) + f_{J2,Z}(X,Y,Z) + f_{C22,Z}(X,Y,Z,t) + f_{S22,Z}(X,Y,Z,t) \\
&+f_{Moon,Z}(X,Y,Z,t) + f_{Sun,Z}(X,Y,Z,t) + f_{SRP,Z}(X,Y,Z,AOM) + f_{Drag,Z}(X,Y,Z,A,m,v)
\end{aligned} \tag{1}$$

1.1 KepComponent

$$\begin{aligned}
f_{Kep,X}(X,Y,Z) &= -\frac{GM_E X}{(X^2+Y^2+Z^2)^{3/2}} \\
f_{Kep,Y}(X,Y,Z) &= -\frac{GM_E Y}{(X^2+Y^2+Z^2)^{3/2}} \\
f_{Kep,Z}(X,Y,Z) &= -\frac{GM_E Z}{(X^2+Y^2+Z^2)^{3/2}}
\end{aligned} \tag{2}$$

1.1.1 Implementation details

First calculate shared factor

$$d = \frac{1}{(X^2+Y^2+Z^2)^{3/2}} \tag{3}$$

After that calculate

$$\begin{aligned}
f_{Kep,X}(X,Y,Z) &= -GM_E dX \\
f_{Kep,Y}(X,Y,Z) &= -GM_E dY \\
f_{Kep,Z}(X,Y,Z) &= -GM_E dZ
\end{aligned} \tag{4}$$

1.2 J2Component

$$\begin{aligned}
f_{J2,X}(X,Y,Z) &= \tfrac{GM_E R_E^2 \sqrt{5} C_{20} X}{2(X^2+Y^2+Z^2)^{1/2}} \left(\tfrac{3}{(X^2+Y^2+Z^2)^2} - \tfrac{15Z^2}{(X^2+Y^2+Z^2)^3} \right) \\
f_{J2,Y}(X,Y,Z) &= \tfrac{GM_E R_E^2 \sqrt{5} C_{20} Y}{2(X^2+Y^2+Z^2)^{1/2}} \left(\tfrac{3}{(X^2+Y^2+Z^2)^2} - \tfrac{15Z^2}{(X^2+Y^2+Z^2)^3} \right) \\
f_{J2,Z}(X,Y,Z) &= \tfrac{GM_E R_E^2 \sqrt{5} C_{20} Z}{2(X^2+Y^2+Z^2)^{1/2}} \left(\tfrac{9}{(X^2+Y^2+Z^2)^2} - \tfrac{15Z^2}{(X^2+Y^2+Z^2)^3} \right)
\end{aligned} \tag{5}$$

1.2.1 Implementation details

Precalculate constant factor

$$f = \tfrac{GM_E R_E^2 \sqrt{5} C_{20}}{2} \tag{6}$$

Per (X,Y,Z) calculate

$$\begin{aligned}
d_1 &= f \tfrac{1}{\sqrt{X^2+Y^2+Z^2}} \\
d_2 &= \tfrac{1}{(X^2+Y^2+Z^2)^2} \\
s &= \tfrac{15Z^2}{(X^2+Y^2+Z^2)^3}
\end{aligned} \tag{7}$$

And

$$\begin{aligned}
f_{J2,X}(X,Y,Z) &= d_1 X \left(3d_2 - s\right) \\
f_{J2,Y}(X,Y,Z) &= d_1 Y \left(3d_2 - s\right) \\
f_{J2,Z}(X,Y,Z) &= d_1 Z \left(9d_2 - s\right)
\end{aligned} \tag{8}$$

1.3 C22S22Component

$$\begin{aligned}
f_{C22,X}(X,Y,Z,t) &= f_{C22,x}(x,y,z)\cos(\theta_G + \nu_E t) - f_{C22,y}(x,y,z)\sin(\theta_G + \nu_E t) \\
f_{C22,Y}(X,Y,Z,t) &= f_{C22,x}(x,y,z)\sin(\theta_G + \nu_E t) + f_{C22,y}(x,y,z)\cos(\theta_G + \nu_E t) \\
f_{C22,Z}(X,Y,Z,t) &= f_{C22,z}(x,y,z) \\
f_{S22,X}(X,Y,Z,t) &= f_{S22,x}(x,y,z)\cos(\theta_G + \nu_E t) - f_{S22,y}(x,y,z)\sin(\theta_G + \nu_E t) \\
f_{S22,Y}(X,Y,Z,t) &= f_{S22,x}(x,y,z)\sin(\theta_G + \nu_E t) + f_{S22,y}(x,y,z)\cos(\theta_G + \nu_E t) \\
f_{S22,Z}(X,Y,Z,t) &= f_{S22,z}(x,y,z)
\end{aligned} \tag{9}$$

with

$$\begin{aligned}
x &= X\cos(\theta_G + \nu_E t) + Y\sin(\theta_G + \nu_E t) \\
y &= -X\sin(\theta_G + \nu_E t) + Y\cos(\theta_G + \nu_E t) \\
z &= Z
\end{aligned} \tag{10}$$

and

$$
\begin{aligned}
f_{C22,x}(x,y,z) &= \tfrac{5GM_E R_E^2 \sqrt{15} C_{22} x (y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}} + \tfrac{GM_E R_E^2 \sqrt{15} C_{22} x}{(x^2+y^2+z^2)^{5/2}} \\
f_{C22,y}(x,y,z) &= \tfrac{5GM_E R_E^2 \sqrt{15} C_{22} y (y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}} - \tfrac{GM_E R_E^2 \sqrt{15} C_{22} y}{(x^2+y^2+z^2)^{5/2}} \\
f_{C22,z}(x,y,z) &= \tfrac{5GM_E R_E^2 \sqrt{15} C_{22} z (y^2-x^2)}{2(x^2+y^2+z^2)^{7/2}} \\
f_{S22,x}(x,y,z) &= -\tfrac{5GM_E R_E^2 \sqrt{15} S_{22} x^2 y}{(x^2+y^2+z^2)^{7/2}} + \tfrac{GM_E R_E^2 \sqrt{15} S_{22} y}{(x^2+y^2+z^2)^{5/2}} \\
f_{S22,y}(x,y,z) &= -\tfrac{5GM_E R_E^2 \sqrt{15} S_{22} x y^2}{(x^2+y^2+z^2)^{7/2}} + \tfrac{GM_E R_E^2 \sqrt{15} S_{22} x}{(x^2+y^2+z^2)^{5/2}} \\
f_{S22,z}(x,y,z) &= -\tfrac{5GM_E R_E^2 \sqrt{15} S_{22} xyz}{(x^2+y^2+z^2)^{7/2}}
\end{aligned} \tag{11}
$$

1.3.1 Implementation details

Precalculate constant factors

$$
f = GM_E R_E^2 \sqrt{15} \tag{12}
$$

$$
\begin{aligned}
f_{C22_2} &= GM_E R_E^2 \sqrt{15} C_{22} \\
&= f C_{22} \\
f_{C22_1} &= \tfrac{5GM_E R_E^2 \sqrt{15} C_{22}}{2} \\
&= f_{C22_2} \left(\tfrac{5}{2}\right)
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
f_{S22_2} &= GM_E R_E^2 \sqrt{15} S_{22} \\
&= f S_{22} \\
f_{S22_1} &= -5GM_E R_E^2 \sqrt{15} S_{22} \\
&= -5 f_{S22_2}
\end{aligned} \tag{14}
$$

Per time step calculate

$$
\begin{aligned}
f_{sin} &= \sin(\theta_G + \nu_E t) \\
f_{cos} &= \cos(\theta_G + \nu_E t)
\end{aligned} \tag{15}
$$

Per (X,Y,Z) calculate

$$
\begin{aligned}
x &= X f_{cos} + Y f_{sin} \\
y &= -X f_{sin} + Y f_{cos} \\
z &= Z
\end{aligned} \tag{16}
$$

Shared terms

$$
\begin{aligned}
n_{C_{22}} &= f_{C22_1} (y^2 - x^2) \\
n_{S_{22}} &= f_{S22_1} xy
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
d_1 &= \tfrac{1}{\sqrt{(x^2+y^2+z^2)^7}} \\
d_2 &= \tfrac{1}{\sqrt{(x^2+y^2+z^2)^5}}
\end{aligned} \tag{18}
$$

Calculate

$$
\begin{aligned}
f_{C22,x}(x,y,z) &= n_{C_{22}} x d_1 + f_{C_{22_2}} x d_2 \\
f_{C22,y}(x,y,z) &= n_{C_{22}} y d_1 + f_{C_{22_2}} y d_2 \\
f_{S22,x}(x,y,z) &= n_{S_{22}} x d_1 + f_{S_{22_2}} y d_2 \\
f_{S22,y}(x,y,z) &= n_{S_{22}} y d_1 + f_{S_{22_2}} x d_2
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
f_{C22,X}(X,Y,Z,t) &= f_{C22,x}(x,y,z) f_{cos} - f_{C22,y}(x,y,z) f_{sin} \\
f_{C22,Y}(X,Y,Z,t) &= f_{C22,x}(x,y,z) f_{sin} + f_{C22,y}(x,y,z) f_{cos} \\
f_{C22,Z}(X,Y,Z,t) &= n_{C_{22}} z d_1 \\
f_{S22,X}(X,Y,Z,t) &= f_{S22,x}(x,y,z) f_{cos} - f_{S22,y}(x,y,z) f_{sin} \\
f_{S22,Y}(X,Y,Z,t) &= f_{S22,x}(x,y,z) f_{sin} + f_{S22,y}(x,y,z) f_{cos} \\
f_{S22,Z}(X,Y,Z,t) &= n_{S_{22}} z d_1
\end{aligned} \tag{20}
$$

1.4 SolComponent

$$
\begin{aligned}
f_{Sun,X}(X,Y,Z,t) &= -GM_\odot \left(\tfrac{(X-X_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \tfrac{X_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}} \right) \\
f_{Sun,Y}(X,Y,Z,t) &= -GM_\odot \left(\tfrac{(Y-Y_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \tfrac{Y_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}} \right) \\
f_{Sun,Z}(X,Y,Z,t) &= -GM_\odot \left(\tfrac{(Z-Z_\odot)}{[(X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2]^{3/2}} + \tfrac{Z_\odot}{(X_\odot^2+Y_\odot^2+Z_\odot^2)^{3/2}} \right)
\end{aligned} \tag{21}
$$

where

$$
\begin{pmatrix} X_\odot \\ Y_\odot \\ Z_\odot \end{pmatrix} = \begin{pmatrix} r_\odot \cos\lambda_\odot \\ r_\odot \sin\lambda_\odot \cos\varepsilon \\ r_\odot \sin\lambda_\odot \sin\varepsilon \end{pmatrix} \tag{22}
$$

with

$$
\begin{aligned}
\lambda_\odot &= \Omega_\odot + \omega_\odot + \ell_\odot + \left(\tfrac{6892}{3600}\sin\ell_\odot + \tfrac{72}{3600}\sin 2\ell_\odot\right) \\
r_\odot[10^6\text{km}] &= 149.619 - 2.499\cos\ell_\odot - 0.021\cos 2\ell_\odot \\
\ell_\odot &= \varphi_{\odot,0} + \nu_\odot t
\end{aligned} \tag{23}
$$

1.4.1 Implementation details

Per time step calculate

$$
\begin{aligned}
\lambda_\odot &= \Omega_\odot + \omega_\odot + \ell_\odot + \left(\tfrac{6892}{3600}\sin\ell_\odot + \tfrac{72}{3600}\sin 2\ell_\odot\right) \\
r_\odot[10^6\text{km}] &= 149.619 - 2.499\cos\ell_\odot - 0.021\cos 2\ell_\odot \\
\ell_\odot &= \varphi_{\odot,0} + \nu_\odot t
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
f_{X_\odot} &= \cos\lambda_\odot \\
f_{Y_\odot} &= \sin\lambda_\odot \cos\varepsilon \\
f_{Z_\odot} &= \sin\lambda_\odot \sin\varepsilon
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
X_\odot &= f_{X_\odot} r_\odot \times 10^6 \\
Y_\odot &= f_{Y_\odot} r_\odot \times 10^6 \\
Z_\odot &= f_{Z_\odot} r_\odot \times 10^6
\end{aligned} \tag{26}
$$

$$d_1 = \frac{1}{\sqrt{(X_\odot^2+Y_\odot^2+Z_\odot^2)^3}} \tag{27}$$

$$\begin{aligned} P_{X_\odot} &= d_1 X_\odot \\ P_{Y_\odot} &= d_1 Y_\odot \\ P_{Z_\odot} &= d_1 Z_\odot \end{aligned} \tag{28}$$

Per (X,Y,Z) calculate

$$d_2 = \frac{1}{\sqrt{((X-X_\odot)^2+(Y-Y_\odot)^2+(Z-Z_\odot)^2)^3}} \tag{29}$$

$$\begin{aligned} f_{Sun,X}(X,Y,Z,t) &= -GM_\odot(d_2(X-X_\odot)+P_{X_\odot}) \\ f_{Sun,Y}(X,Y,Z,t) &= -GM_\odot(d_2(Y-Y_\odot)+P_{Y_\odot}) \\ f_{Sun,Z}(X,Y,Z,t) &= -GM_\odot(d_2(Z-Z_\odot)+P_{Z_\odot}) \end{aligned} \tag{30}$$

1.5 LunComponent

$$\begin{aligned} f_{Moon,X}(X,Y,Z,t) &= -GM_\mathcal{M}\left(\frac{(X-X_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{X_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}}\right) \\ f_{Moon,Y}(X,Y,Z,t) &= -GM_\mathcal{M}\left(\frac{(Y-Y_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{Y_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}}\right) \\ f_{Moon,Z}(X,Y,Z,t) &= -GM_\mathcal{M}\left(\frac{(Z-Z_\mathcal{M})}{[(X-X_\mathcal{M})^2+(Y-Y_\mathcal{M})^2+(Z-Z_\mathcal{M})^2]^{3/2}} + \frac{Z_\mathcal{M}}{(X_\mathcal{M}^2+Y_\mathcal{M}^2+Z_\mathcal{M}^2)^{3/2}}\right) \end{aligned} \tag{31}$$

where

$$\begin{pmatrix} X_\mathcal{M} \\ Y_\mathcal{M} \\ Z_\mathcal{M} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \cdot \begin{pmatrix} r_\mathcal{M}\cos\lambda_\mathcal{M}\cos\beta_\mathcal{M} \\ r_\mathcal{M}\sin\lambda_\mathcal{M}\cos\beta_\mathcal{M} \\ r_\mathcal{M}\sin\beta_\mathcal{M} \end{pmatrix} \tag{32}$$

with

$$\begin{aligned} r_\mathcal{M}[\mathrm{km}] &= 385000 - 20905\cos(l_\mathcal{M}) - 3699\cos(2D_\mathcal{M}-l_\mathcal{M}) \\ &-2956\cos(2D_\mathcal{M}) - 570\cos(2l_\mathcal{M}) \\ &+246\cos(2l_\mathcal{M}-2D_\mathcal{M}) - 205\cos(l'_\mathcal{M}-2D_\mathcal{M}) \\ &-171\cos(l_\mathcal{M}+2D_\mathcal{M}) \\ &-152\cos(l_\mathcal{M}+l'_\mathcal{M}-2D_\mathcal{M}) \end{aligned} \tag{33}$$

$$\begin{aligned} \lambda_\mathcal{M} &= L_0 + (\tfrac{22640}{3600}\sin(l_\mathcal{M}) + \tfrac{769}{3600}\sin(2l_\mathcal{M}) \\ &-\tfrac{4856}{3600}\sin(l_\mathcal{M}-2D_\mathcal{M}) + \tfrac{2370}{3600}\sin(2D_\mathcal{M}) \\ &-\tfrac{668}{3600}\sin(l'_\mathcal{M}) - \tfrac{412}{3600}\sin(2F_\mathcal{M}) \\ &-\tfrac{212}{3600}\sin(2l_\mathcal{M}-2D_\mathcal{M}) - \tfrac{206}{3600}\sin(l_\mathcal{M}+l'_\mathcal{M}-2D_\mathcal{M}) \\ &+\tfrac{192}{3600}\sin(l_\mathcal{M}+2D_\mathcal{M}) - \tfrac{165}{3600}\sin(l'_\mathcal{M}-2D_\mathcal{M}) \\ &+\tfrac{148}{3600}\sin(l_\mathcal{M}-l'_\mathcal{M}) - \tfrac{125}{3600}\sin(D_\mathcal{M}) \\ &-\tfrac{110}{3600}\sin(l_\mathcal{M}+l'_\mathcal{M}) - \tfrac{55}{3600}\sin(2F_\mathcal{M}-2D_\mathcal{M})) \end{aligned} \tag{34}$$

$$
\begin{aligned}
\beta_{\mathcal{M}} \quad & = \big(\tfrac{18520}{3600} \sin\big(F_{\mathcal{M}} + \lambda_{\mathcal{M}} - L_0 + (\tfrac{412}{3600} \sin(2F_{\mathcal{M}}) + \tfrac{541}{3600} \sin(l'_{\mathcal{M}}))\big) \\
& - \tfrac{526}{3600} \sin(F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
& + \tfrac{44}{3600} \sin(l_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) - \tfrac{31}{3600} \sin(-l_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
& - \tfrac{25}{3600} \sin(-2l_{\mathcal{M}} + F_{\mathcal{M}}) - \tfrac{23}{3600} \sin(l'_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
& + \tfrac{21}{3600} \sin(-l_{\mathcal{M}} + F_{\mathcal{M}}) + \tfrac{11}{3600} \sin(-l'_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}})\big)
\end{aligned} \tag{35}
$$

and

$$
\begin{aligned}
\varphi_M \quad & = \nu_{\odot} t \\
\varphi_{M_a} \quad & = \nu_{M_a} t \\
\varphi_{M_p} \quad & = \nu_{M_p} t \\
\varphi_{M_S} \quad & = \nu_{M_s} t \\
L_0 \quad & = \varphi_{M_p} + \varphi_{M_a} + (218.31617) \\
l_{\mathcal{M}} \quad & = \varphi_{M_a} + (134.96292) \\
l'_{\mathcal{M}} \quad & = \ell_{\odot} = \varphi_M + (357.52543) \\
F_{\mathcal{M}} \quad & = \varphi_{M_p} + \varphi_{M_a} + \varphi_{M_S} + (93.27283) \\
D_{\mathcal{M}} \quad & = \varphi_{M_p} + \varphi_{M_a} - \varphi_M + (297.85027)
\end{aligned} \tag{36}
$$

1.5.1 Implementation details

Per time step calculate

$$
\begin{aligned}
\varphi_M \quad & = \nu_{\odot} t \\
\varphi_{M_a} \quad & = \nu_{M_a} t \\
\varphi_{M_p} \quad & = \nu_{M_p} t \\
\varphi_{M_S} \quad & = \nu_{M_s} t \\
L_0 \quad & = \varphi_{M_p} + \varphi_{M_a} + (218.31617) \\
l_{\mathcal{M}} \quad & = \varphi_{M_a} + (134.96292) \\
l'_{\mathcal{M}} \quad & = \ell_{\odot} = \varphi_M + (357.52543) \\
F_{\mathcal{M}} \quad & = \varphi_{M_p} + \varphi_{M_a} + \varphi_{M_S} + (93.27283) \\
D_{\mathcal{M}} \quad & = \varphi_{M_p} + \varphi_{M_a} - \varphi_M + (297.85027)
\end{aligned} \tag{37}
$$

Rearrangement of terms clarifies order of executed calculations

$$
\begin{aligned}
r_{\mathcal{M}} \quad & = -152 \cos(l_{\mathcal{M}} + l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
& -171 \cos(l_{\mathcal{M}} + 2D_{\mathcal{M}}) \\
& -205 \cos(l'_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
& +246 \cos(2l_{\mathcal{M}} - 2D_{\mathcal{M}}) \\
& -2956 \cos(2D_{\mathcal{M}}) \\
& -3699 \cos(2D_{\mathcal{M}} - l_{\mathcal{M}}) \\
& -20905 \cos(l_{\mathcal{M}}) \\
& +385000
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
\lambda_{\mathcal{M}} =& -\tfrac{55}{3600}\sin(2F_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&-\tfrac{110}{3600}\sin(l_{\mathcal{M}} + l'_{\mathcal{M}})\\
&-\tfrac{125}{3600}\sin(D_{\mathcal{M}})\\
&+\tfrac{148}{3600}\sin(l_{\mathcal{M}} - l'_{\mathcal{M}})\\
&-\tfrac{165}{3600}\sin(l'_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&+\tfrac{192}{3600}\sin(l_{\mathcal{M}} + 2D_{\mathcal{M}})\\
&-\tfrac{206}{3600}\sin(l_{\mathcal{M}} + l'_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&-\tfrac{212}{3600}\sin(2l_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&-\tfrac{412}{3600}\sin(2F_{\mathcal{M}})\\
&-\tfrac{668}{3600}\sin(l'_{\mathcal{M}})\\
&+\tfrac{769}{3600}\sin(2l_{\mathcal{M}})\\
&+\tfrac{2370}{3600}\sin(2D_{\mathcal{M}})\\
&-\tfrac{4856}{3600}\sin(l_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&+\tfrac{22640}{3600}\sin(l_{\mathcal{M}})\\
&+L_0
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\beta_{\mathcal{M}} =& \tfrac{11}{3600}\sin(-l'_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&+\tfrac{21}{3600}\sin(-l_{\mathcal{M}} + F_{\mathcal{M}}) + F_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&-\tfrac{23}{3600}\sin(l'_{\mathcal{M}}\\
&-\tfrac{25}{3600}\sin(-2l_{\mathcal{M}} + F_{\mathcal{M}})\\
&-\tfrac{31}{3600}\sin(-l_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&+\tfrac{44}{3600}\sin(l_{\mathcal{M}} + F_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&-\tfrac{526}{3600}\sin(F_{\mathcal{M}} - 2D_{\mathcal{M}})\\
&+\tfrac{18520}{3600}\sin\left(F_{\mathcal{M}} + \lambda_{\mathcal{M}} - L_0 + (\tfrac{412}{3600}\sin(2F_{\mathcal{M}} + \tfrac{541}{3600}\sin(l'_{\mathcal{M}}))\right)
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
f_{\mathcal{M},1} &= \cos\lambda_{\mathcal{M}}\cos\beta_{\mathcal{M}}\\
f_{\mathcal{M},2} &= \sin\lambda_{\mathcal{M}}\cos\beta_{\mathcal{M}}\\
f_{\mathcal{M},3} &= \sin\beta_{\mathcal{M}}
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
X_{\mathcal{M}} &= f_{\mathcal{M},1} r_{\mathcal{M}}\\
Y_{\mathcal{M}} &= (f_{\mathcal{M},2}\cos\varepsilon - f_{\mathcal{M},3}\sin\varepsilon) r_{\mathcal{M}}\\
Z_{\mathcal{M}} &= (f_{\mathcal{M},2}\sin\varepsilon + f_{\mathcal{M},3}\cos\varepsilon) r_{\mathcal{M}}
\end{aligned}
\tag{42}
$$

$$
d_1 = \frac{1}{\sqrt{(X_{\mathcal{M}}^2 + Y_{\mathcal{M}}^2 + Z_{\mathcal{M}}^2)^3}}
\tag{43}
$$

$$
\begin{aligned}
P_{X_{\mathcal{M}}} &= d_1 X_{\mathcal{M}}\\
P_{Y_{\mathcal{M}}} &= d_1 Y_{\mathcal{M}}\\
P_{Z_{\mathcal{M}}} &= d_1 Z_{\mathcal{M}}
\end{aligned}
\tag{44}
$$

Per (X,Y,Z) calculate

$$
d_2 = \frac{1}{\sqrt{((X - X_{\mathcal{M}})^2 + (Y - Y_{\mathcal{M}})^2 + (Z - Z_{\mathcal{M}})^2)^3}}
\tag{45}
$$

$$
\begin{aligned}
f_{Moon,X}(X,Y,Z,t) &= -GM_{\mathcal{M}}(d_2(X - X_{\mathcal{M}}) + P_{X_{\mathcal{M}}})\\
f_{Moon,Y}(X,Y,Z,t) &= -GM_{\mathcal{M}}(d_2(Y - Y_{\mathcal{M}}) + P_{Y_{\mathcal{M}}})\\
f_{Moon,Z}(X,Y,Z,t) &= -GM_{\mathcal{M}}(d_2(Z - Z_{\mathcal{M}}) + P_{Z_{\mathcal{M}}})
\end{aligned}
\tag{46}
$$

1.6 SRPComponent

$$\begin{aligned}
f_{SRP,X}(X,Y,Z,t) &= AOM\frac{P_{SRP}a_{\odot}^2(X-X_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}} \\
f_{SRP,Y}(X,Y,Z,t) &= AOM\frac{P_{SRP}a_{\odot}^2(Y-Y_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}} \\
f_{SRP,Z}(X,Y,Z,t) &= AOM\frac{P_{SRP}a_{\odot}^2(Z-Z_{\odot})}{[(X-X_{\odot})^2+(Y-Y_{\odot})^2+(Z-Z_{\odot})^2]^{3/2}}
\end{aligned} \tag{47}$$

1.6.1 Implementation details

1.7 DragComponent

Page 145 (PDF 82) http://farside.ph.utexas.edu/teaching/celestial/Celestialhtml/node94.html

$$\begin{aligned}
f_{Drag,X}(X,Y,Z,C_D,A,m,v) &= -\frac{pC_DAv_{rel,x}^2}{2m} \\
f_{Drag,Y}(X,Y,Z,C_D,A,m,v) &= -\frac{pC_DAv_{rel,y}^2}{2m} \\
f_{Drag,Z}(X,Y,Z,C_D,A,m,v) &= -\frac{pC_DAv_{rel,z}^2}{2m}
\end{aligned} \tag{48}$$

with
http://farside.ph.utexas.edu/teaching/celestial/Celestialhtml/node94.html

$$p = p_0 \exp\left(\frac{\sqrt{X^2+Y^2+Z^2}-R_E}{H}\right) \tag{49}$$

and
Calculate relative velocity v_{rel} in respect to atmosphere:

$$\begin{aligned}
v_{rel,x} &= v_x - v_{a,x} \\
v_{rel,y} &= v_y - v_{a,y} \\
v_{rel,z} &= v_z - v_{a,z}
\end{aligned} \tag{50}$$

where
Atmospheric velocity

$$\begin{aligned}
v_{a,x} &= ? \\
v_{a,y} &= ? \\
v_{a,z} &= ?
\end{aligned} \tag{51}$$

1.7.1 Implementation details

2 Constants

$$GM_E = 3.986004407799724 \times 10^5 km^3 sec^{-2} \tag{52}$$
$$GM_\odot = 1.32712440018 \times 10^{11} km^3 sec^{-2} \tag{53}$$
$$GM_{\mathcal{M}} = 4.9028 \times 10^3 km^3 sec^{-2} \tag{54}$$
$$R_E = 6378.1363 km \tag{55}$$
$$C_{20} = -4.84165371736 \times 10^{-4} \tag{56}$$
$$C_{22} = 2.43914352398 \times 10^{-6} \tag{57}$$
$$S_{22} = -1.40016683654 \times 10^{-6} \tag{58}$$
$$\theta_G = 280.4606 \tag{59}$$
$$\nu_E = 4.178074622024230 \times 10^{-3} \tag{60}$$
$$\nu_\odot = 1.1407410259335311 \times 10^{-5} \tag{61}$$
$$\nu_{M_a} = 1.512151961904581 \times 10^{-4} \tag{62}$$
$$\nu_{M_p} = 1.2893925235125941 \times 10^{-6} \tag{63}$$
$$\nu_{M_s} = 6.128913003523574 \times 10^{-7} \tag{64}$$
$$a_\odot = 1.49619 \times 10^8 km \tag{65}$$
$$\varepsilon = 23.4392911 \tag{66}$$
$$\varphi_{\odot,0} = 357.5256 \tag{67}$$
$$\Omega_\odot + \omega_\odot = (282.94) \tag{68}$$
$$P_{SRP} = 4.56 \times 10^{-6} \tag{69}$$
$$p_0 = 1.3 kgm^{-3} \tag{70}$$
$$H = 8.5 \times 10^3 m \tag{71}$$