Symbolic 2D Nested Convolution

Fast-Convolution Python Library

June 20, 2024

$$s = (a_1^t \otimes a_2^t)\{[(q_1 \odot b_1)g(q_2 \odot b_2)^t] \odot (c_1^t \otimes c_2^t)\}d$$

$$d = \begin{bmatrix} d_0 & d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 & d_9 \\ d_{10} & d_{11} & d_{12} & d_{13} & d_{14} \\ d_{15} & d_{16} & d_{17} & d_{18} & d_{19} \\ d_{20} & d_{21} & d_{22} & d_{23} & d_{24} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 & 24 \end{bmatrix}$$

$$g = \begin{bmatrix} g_0 & g_1 & g_2 \\ g_3 & g_4 & g_5 \\ g_6 & g_7 & g_8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

$$G = (q_1 \odot b_1)g(q_2 \odot b_2)^t$$

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{1}{6} & \frac{5}{3} & 2 \\ \frac{3}{2} & -6 & -\frac{2}{3} & \frac{31}{6} & 5 \\ 3 & -\frac{21}{2} & -\frac{7}{6} & \frac{26}{3} & 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \odot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{6} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{6} & \frac{2}{3} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \left(\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{6} \\ 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \right)^t = \begin{bmatrix} g_0 & g_1 & g_2 \\ g_3 & g_4 & g_5 \\ g_6 & g_7 & g_8 \end{bmatrix} \left(\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{6} \\ 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \right)^t$$

$$\begin{bmatrix} G_0 & G_1 & G_2 & G_3 & G_4 \\ G_5 & G_6 & G_7 & G_8 & G_9 \\ G_{10} & G_{11} & G_{12} & G_{13} & G_{14} \\ G_{15} & G_{16} & G_{17} & G_{18} & G_{19} \\ G_{20} & G_{21} & G_{22} & G_{23} & G_{24} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{4} & -\frac{1}{12} & \frac{5}{6} & 1 \\ -\frac{9}{4} & 9 & 1 & -\frac{31}{4} & -\frac{15}{2} \\ -\frac{1}{4} & 1 & \frac{1}{9} & -\frac{31}{36} & -\frac{5}{6} \\ \frac{5}{2} & -\frac{37}{4} & -\frac{37}{36} & \frac{70}{9} & \frac{22}{3} \\ 3 & -\frac{21}{2} & -\frac{7}{6} & \frac{26}{3} & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{1}{6} & \frac{5}{3} & 2 \\ \frac{3}{2} & -6 & -\frac{2}{3} & \frac{31}{6} & 5 \\ 3 & -\frac{21}{2} & -\frac{7}{6} & \frac{26}{3} & 8 \end{bmatrix} = \left(\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{6} \\ 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{1}{6} & \frac{5}{3} & 2 \\ \frac{3}{2} & -6 & -\frac{2}{3} & \frac{31}{6} & 5 \\ 3 & -\frac{21}{2} & -\frac{7}{6} & \frac{26}{3} & 8 \end{bmatrix} = \left(\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{6} \\ 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \end{bmatrix}$$

$$C = c_1^t \otimes c_2^t$$

$$C = \left[\begin{smallmatrix} 4 & -2 & -4 & 2 & 0 & -2 & 1 & 2 & -1 & 0 & -4 & 2 & 4 & -2 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -2 & 2 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 4 & 2 & -2 & 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -6 & 2 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & -4 & 6 & -2 & 0 & 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -4 & 2 & 0 & -2 & 1 & 2 & -1 & 0 & -4 & 2 & 4 & -2 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 2 & 4 & -2 & 0 & -2 & 1 & 2 & -1 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 & -2 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 6 & -2 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 2 & 4 & -2 & 0 & -2 & 1 & 2 & -1 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & -2 & -4 & 2 & 0 & -6 & 3 & 6 & -3 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & -2 & 2 & 0 & 0 & 6 & 3 & -3 & 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -6 & 2 & 0 & 0 & -6 & 9 & -3 & 0 & 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 3 & 0 & -3 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & -4 & 2 & 0 & -6 & 3 & 6 & -3 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & -2 & -4 & 2 & 0 & -2 & 1 & 2 & -1 & 0 & -4 & 2 & 4 & -2 & 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & -2 & 2 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 4 & 2 & -2 & 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -6 & 2 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & -4 & 6 & -2 & 0 & 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & -4 & 2 & 0 & -2 & 1 & 2 & -1 & 0 & -4 & 2 & 4 & -2 & 0 & 2 & -1 & -2 & 1 \end{smallmatrix}\right] = \begin{bmatrix} 2 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & -2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & -2 & 1 \end{bmatrix}$$

$$A = a_1^t \otimes a_2^t$$

$$A = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 & 1 & 0 & 1 & 1 & 4 & 1 & 0 & 1 & 1 & 4 & 1 & 0 & 1 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & -1 & 1 & -2 & 0 & 0 & 2 & -2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 1 & 0 & -1 & -1 & -4 & -1 & 0 & 2 & 2 & 8 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 4 & 4 & 4 & 4 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 4 & -4 & 8 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 1 & 0 & 1 & 1 & 4 & 1 & 0 & 4 & 4 & 16 & 4 & 0 & 1 & 1 & 4 & 1 \end{smallmatrix}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 4 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 4 & 1 \end{bmatrix}$$

$$D = Cd$$

$$\begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \\ D_{13} \\ D_{14} \\ D_{15} \\ D_{16} \\ D_{17} \\ D_{18} \\ D_{19} \\ D_{20} \\ D_{21} \\ D_{22} \\ D_{23} \\ D_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ 0 \\ 0 \\ 0 \\ 4 \\ 12 \\ 2 \\ -4 \\ 4 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20 \\ 0 \\ 0 \\ 0 \\ 0 \\ 20 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{smallmatrix} 4 & -2 & -4 & 2 & 0 & -2 & 1 & 2 & -1 & 0 & -4 & 2 & 4 & -2 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -2 & 2 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 4 & 2 & -2 & 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -6 & 2 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & -4 & 6 & -2 & 0 & 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -4 & 2 & 0 & -2 & 1 & 2 & -1 & 0 & -4 & 2 & 4 & -2 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 2 & 4 & -2 & 0 & -2 & 1 & 2 & -1 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 & -2 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 6 & -2 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 2 & 4 & -2 & 0 & -2 & 1 & 2 & -1 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & -2 & -4 & 2 & 0 & -6 & 3 & 6 & -3 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & -2 & 2 & 0 & 0 & 6 & 3 & -3 & 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -6 & 2 & 0 & 0 & -6 & 9 & -3 & 0 & 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 3 & 0 & -3 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & -4 & 2 & 0 & -6 & 3 & 6 & -3 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & -2 & -4 & 2 & 0 & -2 & 1 & 2 & -1 & 0 & -4 & 2 & 4 & -2 & 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & -2 & 2 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 4 & 2 & -2 & 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -6 & 2 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & -4 & 6 & -2 & 0 & 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & -4 & 2 & 0 & -2 & 1 & 2 & -1 & 0 & -4 & 2 & 4 & -2 & 0 & 2 & -1 & -2 & 1 \end{smallmatrix}\right] \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \end{bmatrix} = C \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \\ d_{10} \\ d_{11} \\ d_{12} \\ d_{13} \\ d_{14} \\ d_{15} \\ d_{16} \\ d_{17} \\ d_{18} \\ d_{19} \\ d_{20} \\ d_{21} \\ d_{22} \\ d_{23} \\ d_{24} \end{bmatrix}$$

$$S = G \odot D$$

$$\begin{bmatrix} S_0 & S_1 & S_2 & S_3 & S_4 \\ S_5 & S_6 & S_7 & S_8 & S_9 \\ S_{10} & S_{11} & S_{12} & S_{13} & S_{14} \\ S_{15} & S_{16} & S_{17} & S_{18} & S_{19} \\ S_{20} & S_{21} & S_{22} & S_{23} & S_{24} \end{bmatrix} = \begin{bmatrix} 0 & -15 & 0 & 0 & 0 \\ -9 & 108 & 2 & 31 & -30 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 185 & 0 & 0 & 0 \\ 0 & -210 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{4} & -\frac{1}{12} & \frac{5}{6} & 1 \\ -\frac{9}{4} & 9 & 1 & -\frac{31}{4} & -\frac{15}{2} \\ -\frac{1}{4} & 1 & \frac{1}{9} & -\frac{31}{36} & -\frac{5}{6} \\ \frac{5}{2} & -\frac{37}{4} & -\frac{37}{36} & \frac{70}{9} & \frac{22}{3} \\ 3 & -\frac{21}{2} & -\frac{7}{6} & \frac{26}{3} & 8 \end{bmatrix} \odot \begin{bmatrix} 0 & 20 & 0 & 0 & 0 \\ 4 & 12 & 2 & -4 & 4 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 \end{bmatrix}$$

$$s = AS$$

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \end{bmatrix} = \begin{bmatrix} 312 \\ 348 \\ 384 \\ 492 \\ 528 \\ 564 \\ 672 \\ 708 \\ 744 \end{bmatrix} = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 & 1 & 0 & 1 & 1 & 4 & 1 & 0 & 1 & 1 & 4 & 1 & 0 & 1 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & -1 & 1 & -2 & 0 & 0 & 2 & -2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 1 & 0 & -1 & -1 & -4 & -1 & 0 & 2 & 2 & 8 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 4 & 4 & 4 & 4 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 4 & -4 & 8 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 1 & 0 & 1 & 1 & 4 & 1 & 0 & 4 & 4 & 16 & 4 & 0 & 1 & 1 & 4 & 1 \end{smallmatrix}\right] \begin{bmatrix} 0 \\ -15 \\ 0 \\ 0 \\ 0 \\ -9 \\ 108 \\ 2 \\ 31 \\ -30 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 185 \\ 0 \\ 0 \\ 0 \\ 0 \\ -210 \\ 0 \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \\ S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \\ S_{15} \\ S_{16} \\ S_{17} \\ S_{18} \\ S_{19} \\ S_{20} \\ S_{21} \\ S_{22} \\ S_{23} \\ S_{24} \end{bmatrix}$$

$$A = \left[\begin{smallmatrix} 2^0 & 2^0 & 2^0 & 2^0 & 0 & 2^0 & 2^0 & 2^0 & 2^0 & 0 & 2^0 & 2^0 & 2^0 & 2^0 & 0 & 2^0 & 2^0 & 2^0 & 2^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^0 & -2^0 & 2^1 & 0 & 0 & 2^0 & -2^0 & 2^1 & 0 & 0 & 2^0 & -2^0 & 2^1 & 0 & 0 & 2^0 & -2^0 & 2^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^0 & 2^0 & 2^2 & 2^0 & 0 & 2^0 & 2^0 & 2^2 & 2^0 & 0 & 2^0 & 2^0 & 2^2 & 2^0 & 0 & 2^0 & 2^0 & 2^2 & 2^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2^0 & 2^0 & 2^0 & 2^0 & 0 & -2^0 & -2^0 & -2^0 & -2^0 & 0 & 2^1 & 2^1 & 2^1 & 2^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2^0 & -2^0 & 2^1 & 0 & 0 & -2^0 & 2^0 & -2^1 & 0 & 0 & 2^1 & -2^1 & 2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2^0 & 2^0 & 2^2 & 2^0 & 0 & -2^0 & -2^0 & -2^2 & -2^0 & 0 & 2^1 & 2^1 & 2^3 & 2^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2^0 & 2^0 & 2^0 & 2^0 & 0 & 2^0 & 2^0 & 2^0 & 2^0 & 0 & 2^2 & 2^2 & 2^2 & 2^2 & 0 & 2^0 & 2^0 & 2^0 & 2^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2^0 & -2^0 & 2^1 & 0 & 0 & 2^0 & -2^0 & 2^1 & 0 & 0 & 2^2 & -2^2 & 2^3 & 0 & 0 & 2^0 & -2^0 & 2^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2^0 & 2^0 & 2^2 & 2^0 & 0 & 2^0 & 2^0 & 2^2 & 2^0 & 0 & 2^2 & 2^2 & 2^4 & 2^2 & 0 & 2^0 & 2^0 & 2^2 & 2^0 \end{smallmatrix}\right]$$

$$C = \left[\begin{smallmatrix} 2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2^1 & -2^2 & 2^2 & -2^1 & 2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2^2 & -2^1 & -2^1-2^2 & 0 & -2^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2^1 & 2^1 & 2^1 & 2^1 & -2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2^1 & 0 & 0 & 0 & 0 & -2^2 & 0 & 0 & 0 & 0 & 2^2 & 0 & 0 & 0 & 0 & -2^1 & 0 & 0 & 0 & 0 & 2^2 & 0 & 0 & 0 & 0 \\ 2^0 & 2^1 & -2^1 & 2^0 & -2^1 & 2^1 & 2^2 & -2^2 & 2^1 & -2^2 & -2^1 & -2^2 & 2^2 & -2^1 & 2^2 & 2^0 & 2^1 & -2^1 & 2^0 & -2^1 & -2^1 & -2^2 & 2^2 & -2^1 & 2^2 \\ 2^1 & 2^0 & 2^0+2^1 & 0 & 2^0 & 2^2 & 2^1 & 2^1+2^2 & 0 & 2^1 & -2^2 & -2^1 & -2^1-2^2 & 0 & -2^1 & 2^1 & 2^0 & 2^0+2^1 & 0 & 2^0 & -2^2 & -2^1 & -2^1-2^2 & 0 & -2^1 \\ -2^0 & -2^0 & -2^0 & -2^0 & 2^1 & -2^1 & -2^1 & -2^1 & -2^1 & 2^2 & 2^1 & 2^1 & 2^1 & 2^1 & -2^2 & -2^0 & -2^0 & -2^0 & -2^0 & 2^1 & 2^1 & 2^1 & 2^1 & 2^1 & -2^2 \\ 0 & 0 & 0 & 0 & -2^0 & 0 & 0 & 0 & 0 & -2^1 & 0 & 0 & 0 & 0 & 2^1 & 0 & 0 & 0 & 0 & -2^0 & 0 & 0 & 0 & 0 & 2^1 \\ -2^2 & 0 & 0 & 0 & 0 & -2^1 & 0 & 0 & 0 & 0 & -2^1-2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2^1 & 0 & 0 & 0 & 0 \\ 2^1 & 2^2 & -2^2 & 2^1 & -2^2 & 2^0 & 2^1 & -2^1 & 2^0 & -2^1 & 2^0+2^1 & 2^1+2^2 & -2^1-2^2 & 2^0+2^1 & -2^1-2^2 & 0 & 0 & 0 & 0 & 0 & 2^0 & 2^1 & -2^1 & 2^0 & -2^1 \\ 2^2 & 2^1 & 2^1+2^2 & 0 & 2^1 & 2^1 & 2^0 & 2^0+2^1 & 0 & 2^0 & 2^1+2^2 & 2^0+2^1 & 2^0+2^3 & 0 & 2^0+2^1 & 0 & 0 & 0 & 0 & 0 & 2^1 & 2^0 & 2^0+2^1 & 0 & 2^0 \\ -2^1 & -2^1 & -2^1 & -2^1 & 2^2 & -2^0 & -2^0 & -2^0 & -2^0 & 2^1 & -2^0-2^1 & -2^0-2^1 & -2^0-2^1 & -2^0-2^1 & 2^1+2^2 & 0 & 0 & 0 & 0 & 0 & -2^0 & -2^0 & -2^0 & -2^0 & 2^1 \\ 0 & 0 & 0 & 0 & -2^1 & 0 & 0 & 0 & 0 & -2^0 & 0 & 0 & 0 & 0 & -2^0-2^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2^0 \\ 2^1 & 0 & 0 & 0 & 0 & 2^1 & 0 & 0 & 0 & 0 & 2^1 & 0 & 0 & 0 & 0 & 2^1 & 0 & 0 & 0 & 0 & -2^2 & 0 & 0 & 0 & 0 \\ -2^0 & -2^1 & 2^1 & -2^0 & 2^1 & -2^0 & -2^1 & 2^1 & -2^0 & 2^1 & -2^0 & -2^1 & 2^1 & -2^0 & 2^1 & -2^0 & -2^1 & 2^1 & -2^0 & 2^1 & 2^1 & 2^2 & -2^2 & 2^1 & -2^2 \\ -2^1 & -2^0 & -2^0-2^1 & 0 & -2^0 & -2^1 & -2^0 & -2^0-2^1 & 0 & -2^0 & -2^1 & -2^0 & -2^0-2^1 & 0 & -2^0 & -2^1 & -2^0 & -2^0-2^1 & 0 & -2^0 & 2^2 & 2^1 & 2^1+2^2 & 0 & 2^1 \\ 2^0 & 2^0 & 2^0 & 2^0 & -2^1 & 2^0 & 2^0 & 2^0 & 2^0 & -2^1 & 2^0 & 2^0 & 2^0 & 2^0 & -2^1 & 2^0 & 2^0 & 2^0 & 2^0 & -2^1 & -2^1 & -2^1 & -2^1 & -2^1 & 2^2 \\ 0 & 0 & 0 & 0 & 2^0 & 0 & 0 & 0 & 0 & 2^0 & 0 & 0 & 0 & 0 & 2^0 & 0 & 0 & 0 & 0 & 2^0 & 0 & 0 & 0 & 0 & -2^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2^0 & -2^1 & 2^1 & -2^0 & 2^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2^1 & -2^0 & -2^0-2^1 & 0 & -2^0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^0 & 2^0 & 2^0 & 2^0 & -2^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^0 \end{smallmatrix}\right]$$