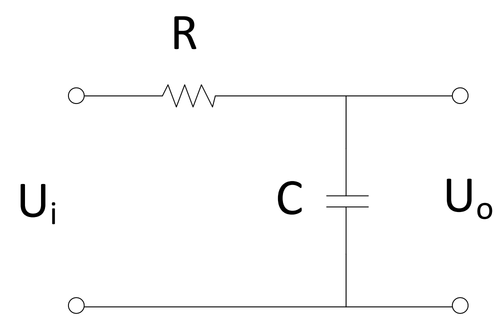
Control Laboratory

Chapter 8 Frequency Response Methods

8.1 Introduction

Example:

If $U_i = A\sin \omega t$, determine $y(t) = U_o(t)$



$$\frac{Y(s)}{R(s)} = \frac{U_o(s)}{U_i(s)} = \frac{1}{RCs+1}$$

$$Y(s) = \frac{1}{RCs+1}R(s) = \frac{1}{RCs+1} \cdot \frac{A\omega}{s^2+\omega^2}$$

$$y(t) = \frac{A\tau\omega}{1+\tau^2\omega^2}e^{-\frac{t}{\tau}} + \frac{A}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t - \arctan\tau\omega) \quad (\tau = RC)$$

$$y(\infty) = \lim_{t\to\infty} y(t) = \frac{A}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t - \arctan\tau\omega)$$

8.1 Introduction

$$\frac{A(\omega)}{A} = \frac{1}{\sqrt{1+\tau^2\omega^2}}, \quad \Phi(\omega) = \text{-}\arctan\tau\omega$$

amplitude **phase**

Both are the function of ω

$G(s) = \dfrac{1}{\tau s+1}$ Let $s = j\omega$, we get frequency transfer function

$$\boxed{G(j\omega) = \frac{1}{j\tau\omega+1}}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\tau^2\omega^2+1}}$$

amplitude character

$$\angle G(j\omega) = -\arctan\omega\tau$$

phase character

8.1 Introduction

Concepts:

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal

Note:

① the sinusoidal is a unique input signal;

② For a linear system, output signal is sinusoidal in the steady state.
(Frequency method is for steady, not dynamic process)

③ Compared with the input sinusoidal signal $A\sin(\omega t)$,
the magnitude increases $|T(j\omega)|$, and the phase changes $\angle T(j\omega)$

Advantages:

① Frequency characteristic plot can be measured by experiment;

② Plot is convenient for analysis and design.

8.2 Frequency Response Plots

System output response

$$Y(jw) = T(jw)R(jw) = \frac{G(jw)}{1+G(jw)}R(jw)$$

System close-loop transfer function

$$T(jw) = \frac{G(jw)}{1+G(jw)}$$

System open-loop transfer function

$$T(jw) = \frac{G(jw)}{1+G(jw)} \Rightarrow G(jw)$$

8.2 Frequency Response Plots

System open-loop transfer function

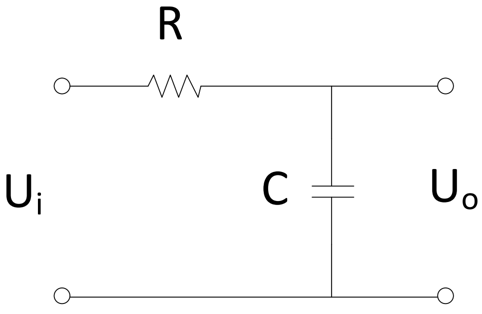
$$G(s) \overset{s=j\omega}{\Longrightarrow} G(j\omega) = R(\omega) + jX(\omega) = \left|G(j\omega)\right| e^{\angle G(j\omega)}$$

$$R(\omega) = \mathrm{Re}\left[G(j\omega)\right],\ jX(\omega) = \mathrm{Im}\left[G(j\omega)\right]$$

- **polar plot**

$$G(s) = \frac{1}{RCs+1} \Rightarrow G(j\omega) = \frac{1}{RC\omega j+1}$$

$$\text{let}\,\omega_1 = \frac{1}{RC} \Rightarrow G(j\omega) = \frac{1}{\frac{\omega}{\omega_1} j+1}$$



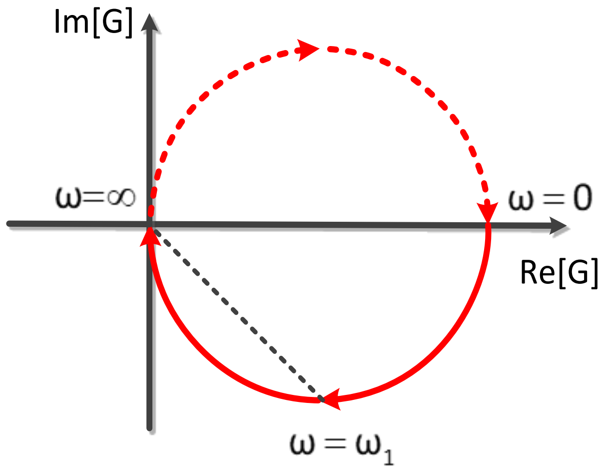
8.2 Frequency Response Plots

$$G(j\omega)=\frac{1}{1+(\frac{\omega}{\omega_1})^2}-j\frac{\frac{\omega}{\omega_1}}{1+(\frac{\omega}{\omega_1})^2}$$

$$|G(j\omega)|=\frac{1}{\sqrt{1+(\frac{\omega}{\omega_1})^2}}$$

$$\angle G(j\omega)=-\arctan\frac{\omega}{\omega_1}$$

ω	$R(\omega)$	$X(\omega)$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0	1	0
ω_1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$-45°$
∞	0	0	0	$-90°$



8.2 Frequency Response Plots

Tips:

① $R(\omega), X(\omega)$ or $|G(j\omega)|, \angle G(j\omega)$, choose one method for drawing

② $\omega(-\infty \to +\infty)$ so the plot is symmetric about the real axis;

the regular polar plot is from 0 to +∞, we use the plot of

$\omega(-\infty \to +\infty)$ to verify the stability of system.

Example:

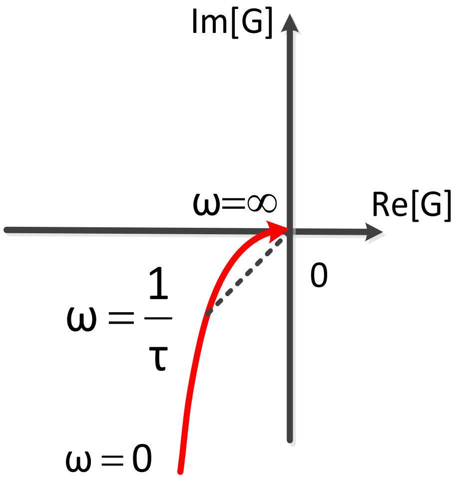
$$G(s) = \frac{1}{s(s+1)} \Rightarrow G(j\omega) = \frac{1}{jw(jw\tau+1)}$$

$$|G(j\omega)| = \frac{1}{|jw||jw\tau+1|} = \frac{1}{w\sqrt{(w\tau)^2+1}};$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\tau\omega$$

8.2 Frequency Response Plots

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	$-90°$
$\frac{1}{\tau}$	$\frac{1}{\sqrt{2}}$	$-135°$
∞	0	$-180°$



The limitations of polar plot:

① The addition of poles and zeros to an existing system requires the recalculation of the frequency response.

② Calculating the frequency response in this manner is tedious and does not indicate the effect of the individual poles and zeros.

8.2 Frequency Response Plots

Unit: decibels(dB)

Ⅱ logarithmic plots (Bode diagram)

Draw the diagram of $G(j\omega) \Rightarrow 20\lg|G(j\omega)|$ and $\angle G(j\omega)$

Eg: $G(j\omega) = \dfrac{1}{j\omega\tau + 1}$

$$20\lg|G(j\omega)| = 20\lg\frac{1}{\sqrt{\tau^2\omega^2 + 1}} = -10\lg(\tau^2\omega^2 + 1)$$

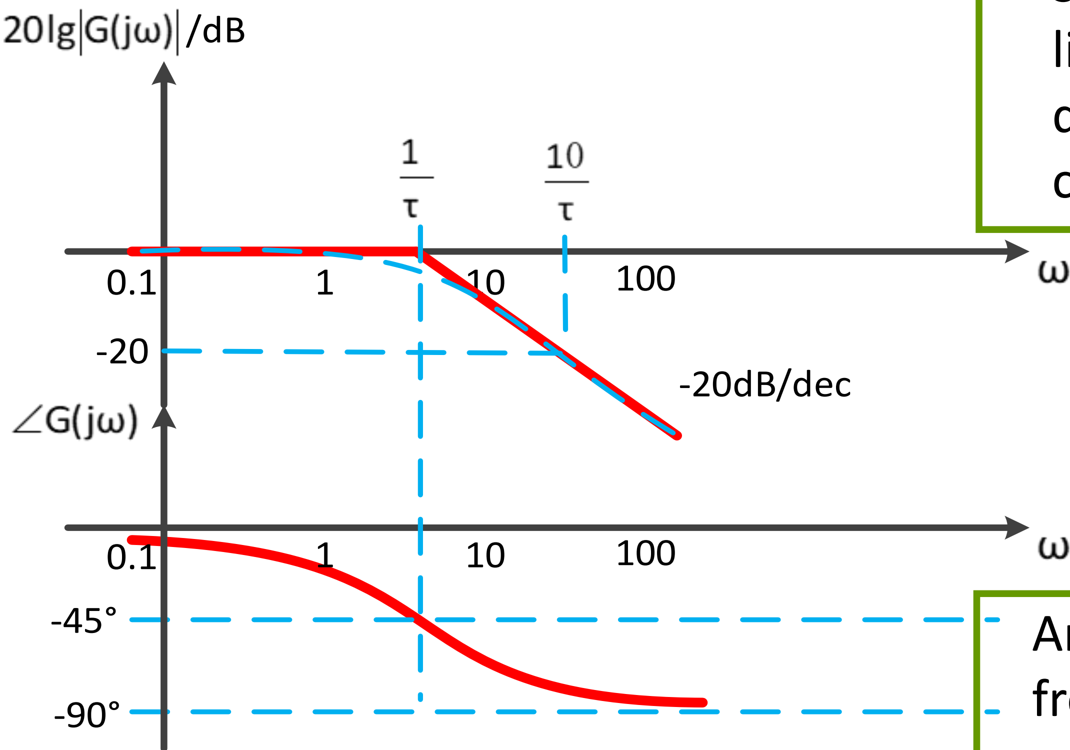
$$\omega << \frac{1}{\tau} \qquad 20\lg|G(j\omega)| = 0$$

$$\omega >> \frac{1}{\tau} \qquad 20\lg|G(j\omega)| = -20\lg\omega\tau$$

$$\omega = \frac{1}{\tau} \qquad 20\lg|G(j\omega)| = -20\lg\frac{1}{\sqrt{2}} = -3\text{dB}$$

Break frequency or corner frequency

8.2 Frequency Response Plots



Semilog paper with a linear coordinate for dB and a logarithmic coordinate for ω .

An interval of two frequencies with a ratio equal to 10 is called a decade.

8.2 Frequency Response Plots

Advantages of the logarithmic coordinate:

① The extension of the range of frequencies;

② Multiplicative factors are converted into additive factors.

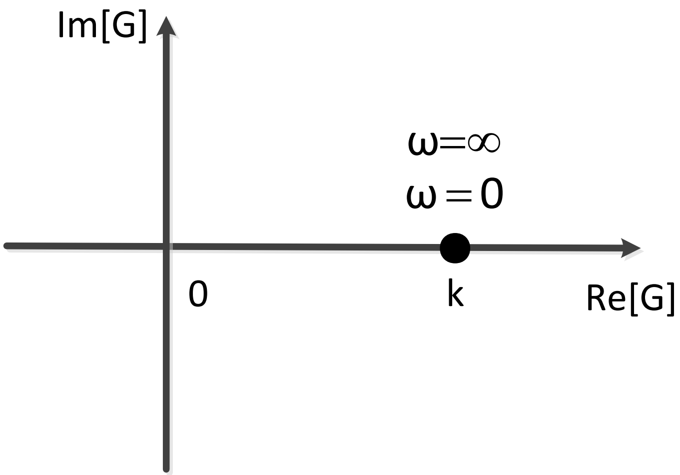
Note: $\lg 0 = -\infty$, so there is no $\omega = 0$ point on the horizontal axis

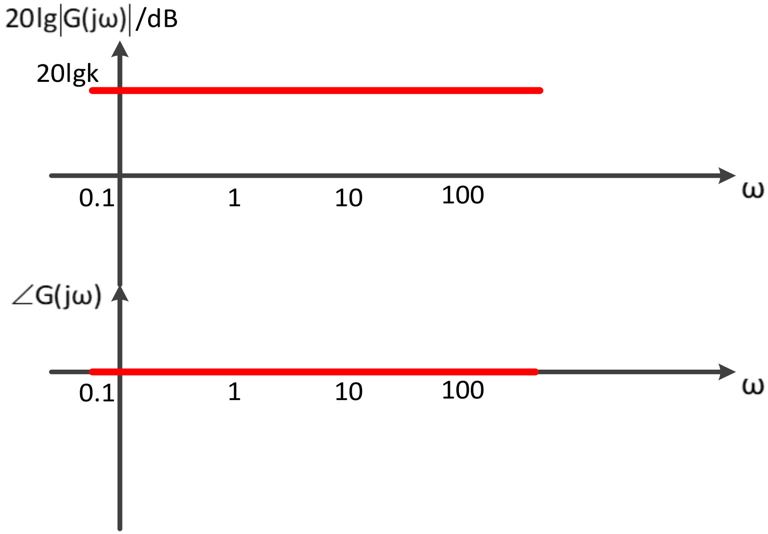
8.2 Frequency Response Plots

Typical elements :

①**proportional element**

$$G(s)=k, G(j\omega)=k, |G(j\omega)|=k, \angle G(j\omega)=0$$

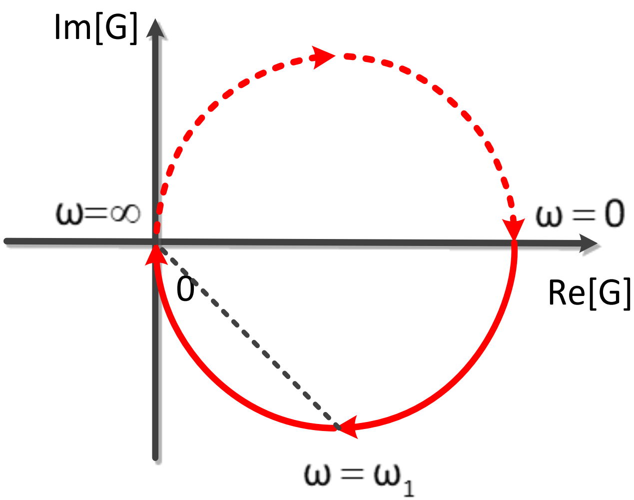


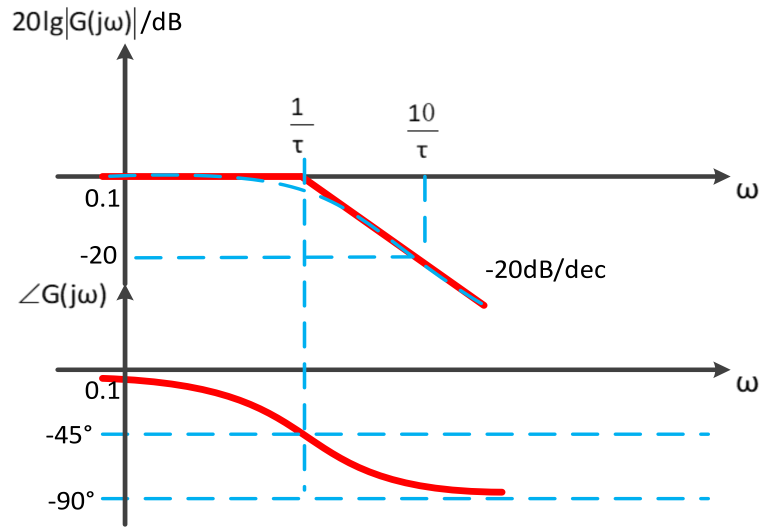


8.2 Frequency Response Plots

②**inertial element**

$$G(s)=\frac{1}{\tau s+1},G(j\omega)=\frac{1}{j\omega\tau+1}=\frac{1}{\omega^2\tau^2+1}-j\frac{\omega\tau}{\omega^2\tau^2+1}$$





The magnitude curve is asymptotic, the biggest error is 3dB at $\frac{1}{\tau}$

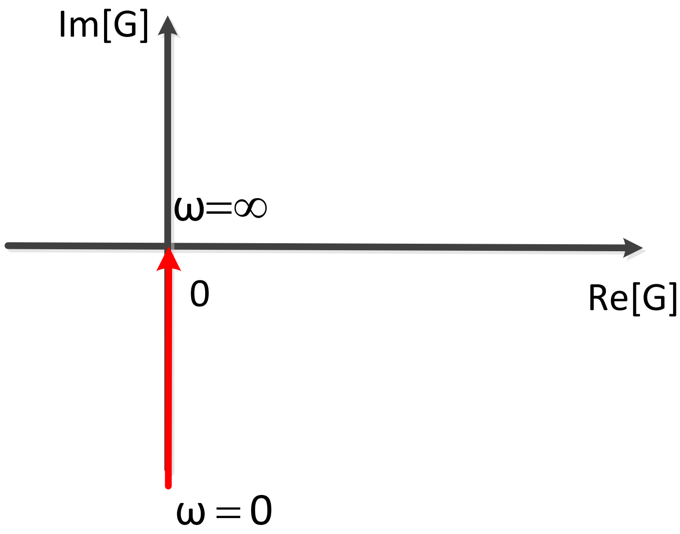
$\omega=\frac{1}{\tau}$:break frequency or corner frequency

8.2 Frequency Response Plots

③**integral element**

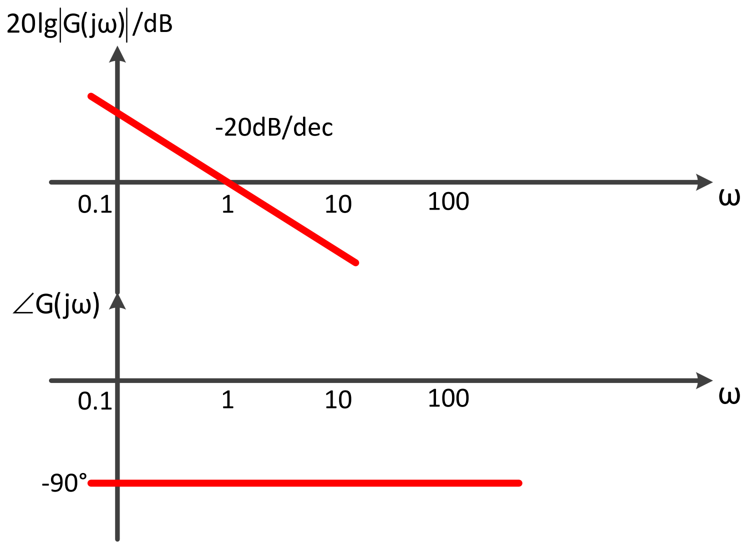
$$G(s)=\frac{1}{s},G(j\omega)=\frac{1}{j\omega}=-j\frac{1}{\omega},\left|G(j\omega)\right|=\frac{1}{\omega},\angle G(j\omega)=-90^\circ$$

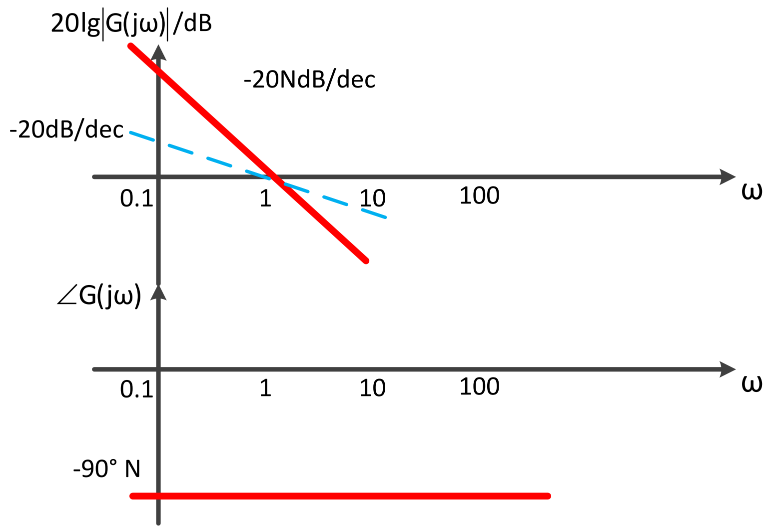
ω	$\left\|G(j\omega)\right\|$	$\angle G(j\omega)$
0	∞	-90°
∞	0	-90°



$$20\lg\left|G(j\omega)\right|=20\lg\frac{1}{\omega}=-20\lg\omega$$

8.2 Frequency Response Plots





$$G(s)=\frac{1}{s^N},G(j\omega)=\frac{1}{(j\omega)^N},\angle G(j\omega)=\angle j\omega+\angle j\omega+\cdots=-90^\circ\times N$$

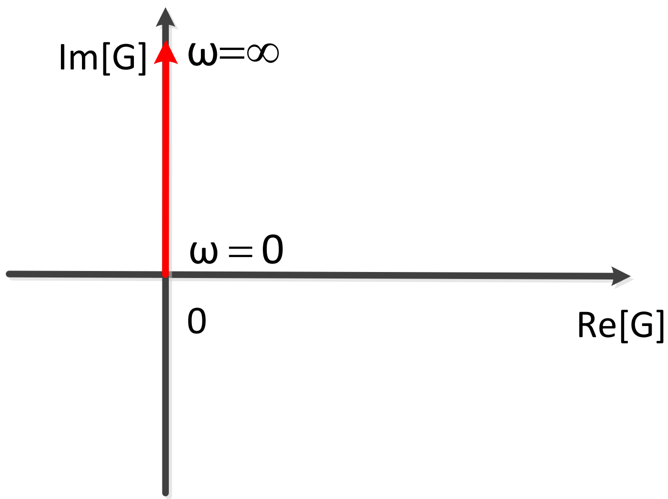
$$|G(j\omega)|=\frac{1}{\omega^N},20\lg|G(j\omega)|=-20N\lg\omega$$

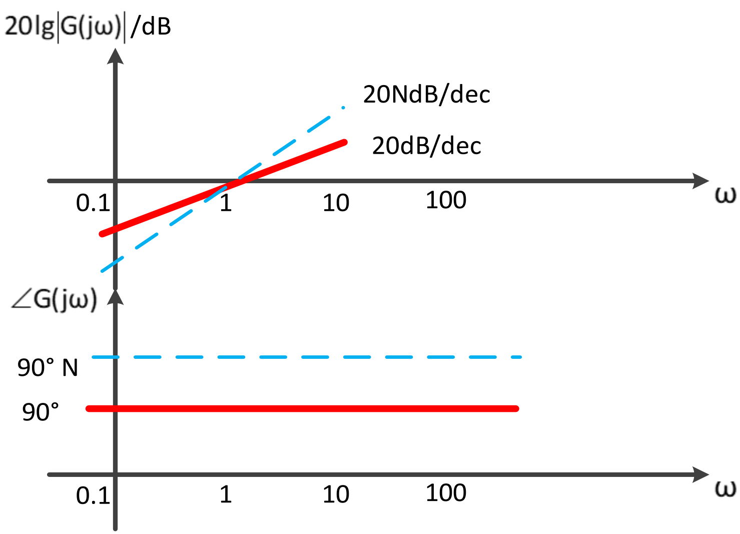
8.2 Frequency Response Plots

④**differential element**

$$G(s)=s, G(j\omega)=j\omega, \left|G(j\omega)\right|=\omega, \angle G(j\omega)=90^\circ$$

$$20\lg\left|G(j\omega)\right|=20\lg\omega$$





8.2 Frequency Response Plots

⑤**first-order differential element**

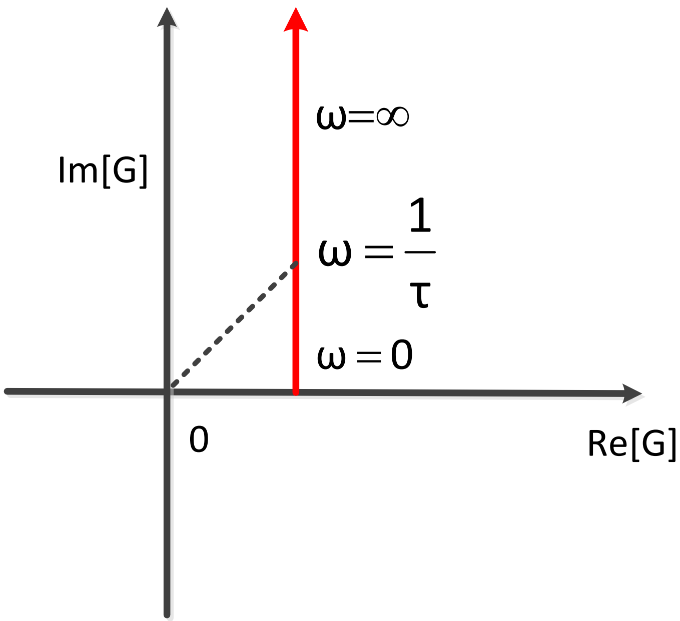
$$G(s)=\tau s+1, G(j\omega)=j\omega\tau+1 \overset{\omega_1=\frac{1}{\tau}}{\Rightarrow} G(j\omega)=j\frac{\omega}{\omega_1}+1, |G(j\omega)|=\sqrt{1+(\frac{\omega}{\omega_1})^2}$$

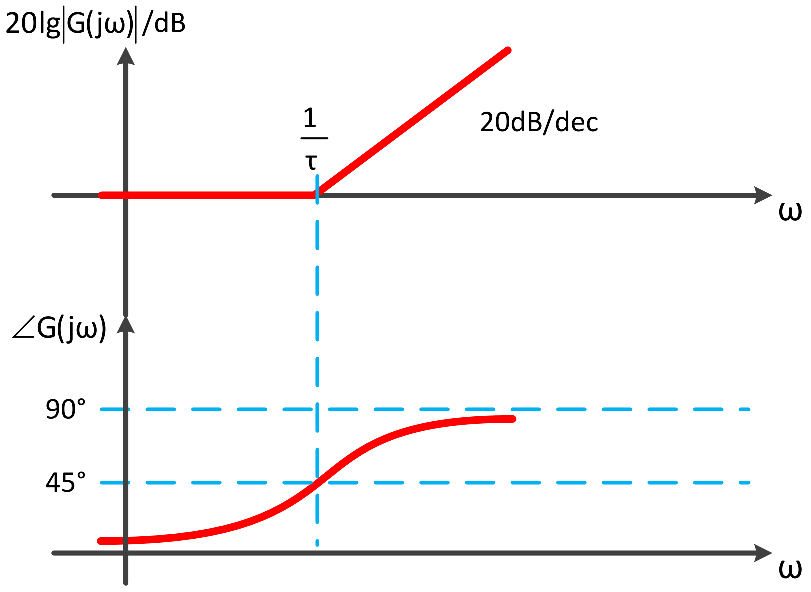
$$\angle G(j\omega)=\arctan\frac{\omega}{\omega_1},\ 20\lg|G(j\omega)|=20\lg\sqrt{1+(\frac{\omega}{\omega_1})^2}$$

ω	$20\lg\sqrt{1+(\frac{\omega}{\omega_1})^2}$
$\omega \ll \omega_1=\frac{1}{\tau}$	0
$\omega \gg \omega_1=\frac{1}{\tau}$	∞

ω	$\|G(j\omega)\|$	$\angle G(j\omega)$
0	1	0
$\frac{1}{\tau}$	$\sqrt{2}$	45°
∞	∞	90°

8.2 Frequency Response Plots





8.2 Frequency Response Plots

⑥**oscillating element**

$$G(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\xRightarrow{\omega_n=\frac{1}{\tau}}G(s)=\frac{1}{\tau^2 s^2+2\xi\tau s+1}$$

$$G(j\omega)=\frac{1}{-\tau^2\omega^2+2\xi\tau\omega j+1}$$

$$=\frac{1-\tau^2\omega^2}{(1-\tau^2\omega^2)^2+(2\xi\tau\omega)^2}-j\frac{2\xi\tau\omega}{(1-\tau^2\omega^2)^2+(2\xi\tau\omega)^2}$$

$$\left|G(j\omega)\right|=\frac{1}{\sqrt{(1-\tau^2\omega^2)^2+(2\xi\tau\omega)^2}}$$

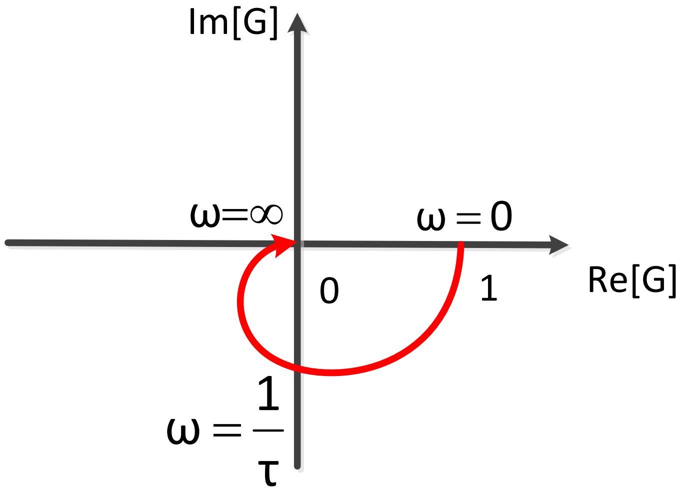
$$\angle G(j\omega)=-\arctan\frac{2\xi\tau\omega}{1-\tau^2\omega^2}$$

8.2 Frequency Response Plots

$$|G(j\omega)| = \frac{1}{\sqrt{(1-\tau^2\omega^2)^2 + (2\xi\tau\omega)^2}}$$

$$\angle G(j\omega) = -\arctan\frac{2\xi\tau\omega}{1-\tau^2\omega^2}$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0
$\omega=\omega_n=\frac{1}{\tau}$	$\frac{1}{2\xi}$	-90°
∞	0	-180°

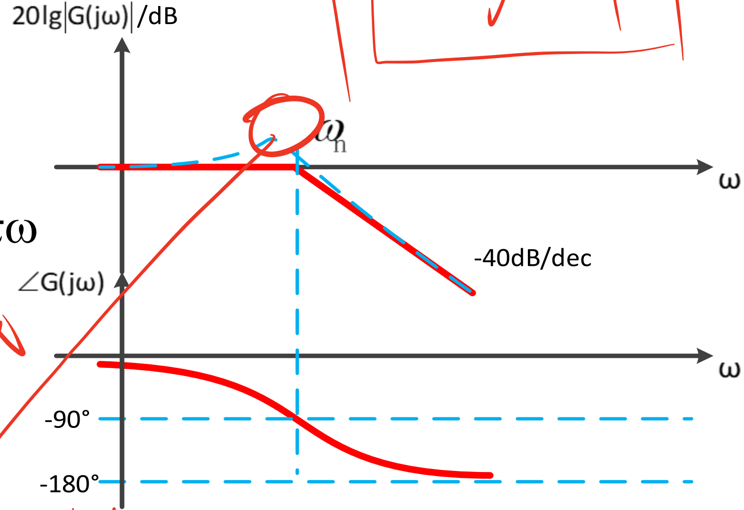


8.2 Frequency Response Plots

$$20\lg|G(j\omega)| = -20\lg[(1-\tau^2\omega^2)^2 + (2\xi\tau\omega)^2]^{\frac{1}{2}}$$

条件 $\boxed{0<\xi<1}$

ω	$20\lg\lvert G(j\omega)\rvert$
$\omega \ll \omega_n = \dfrac{1}{\tau}$	0
$\omega \gg \omega_n = \dfrac{1}{\tau}$	$-10\lg(\tau^2\omega^2)^2 = -40\lg\tau\omega$



$$\frac{d\left[|G(j\omega)|\right]}{d\omega} = 0$$

求出时域

条件

性能

resonant frequency

$$\omega_r = \omega_n\sqrt{1-2\xi^2}\,(\xi < 0.707)$$

resonant peak

$$M_r = |G(j\omega_r)| = (2\xi\sqrt{1-\xi^2})^{-1}$$

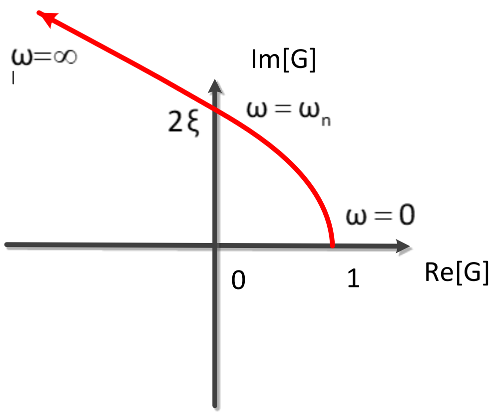
8.2 Frequency Response Plots

⑦**second-order differential element**

$$G(s)=\frac{s^2}{\omega_n^2}+2\xi\frac{s}{\omega_n}+1=\frac{s^2+2\xi\omega_n s+\omega_n^2}{\omega_n^2}\xrightarrow{s=j\omega}$$

$$G(j\omega)=\frac{-\omega^2+2\xi\omega_n\omega j+\omega_n^2}{\omega_n^2}=1-\frac{\omega^2}{\omega_n^2}+j\frac{2\xi\omega}{\omega_n}$$

ω	Re	Im
0	1	0
ω_n	0	2ξ
∞	$-\infty$	∞

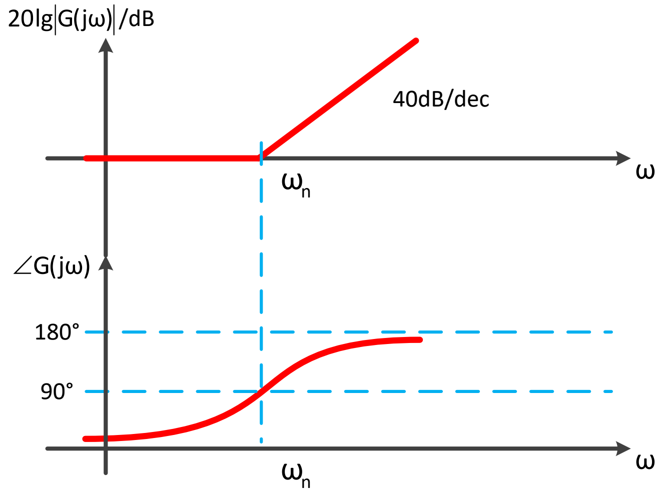


8.2 Frequency Response Plots

$$|G(j\omega)| = \sqrt{(1-\frac{\omega^2}{\omega_n^2})^2 + (\frac{2\xi\omega}{\omega_n})^2}$$

$$20\lg|G(j\omega)| = 20\lg\sqrt{(1-\frac{\omega^2}{\omega_n^2})^2 + (\frac{2\xi\omega}{\omega_n})^2}$$

$$\angle G(j\omega) = \arctan\frac{2\xi\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

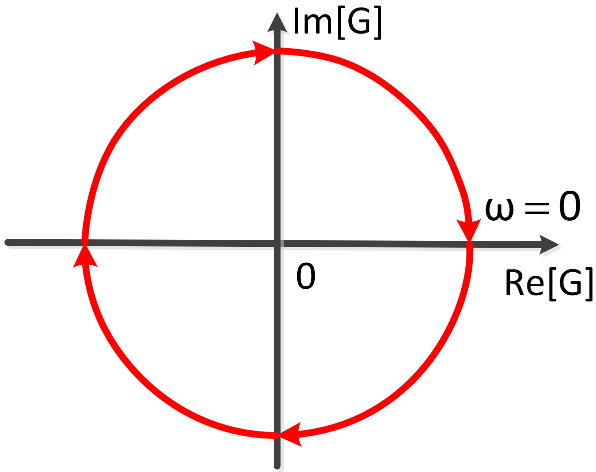


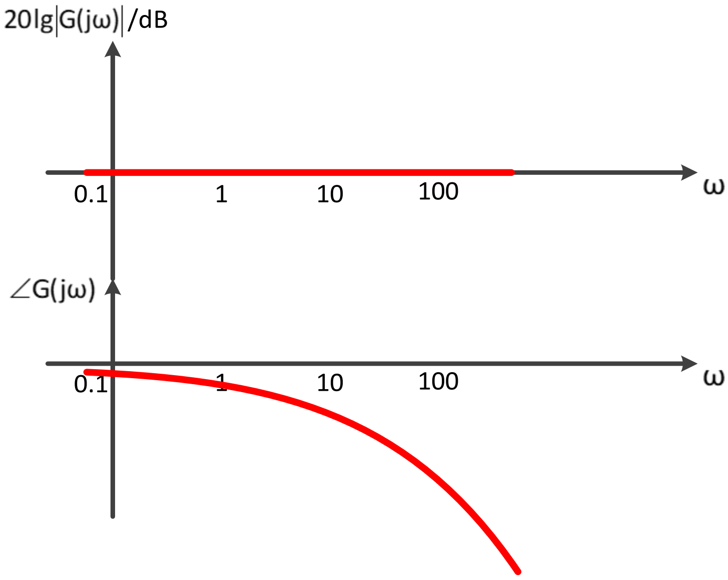
8.2 Frequency Response Plots

⑧**delay element**

$$G(s)=e^{-\tau s}\overset{s=j\omega}{\Rightarrow}G(j\omega)=e^{-\tau j\omega},\left|G(j\omega)\right|=1,\angle G(j\omega)=-\tau\omega$$

$$20\lg\left|G(j\omega)\right|=0$$





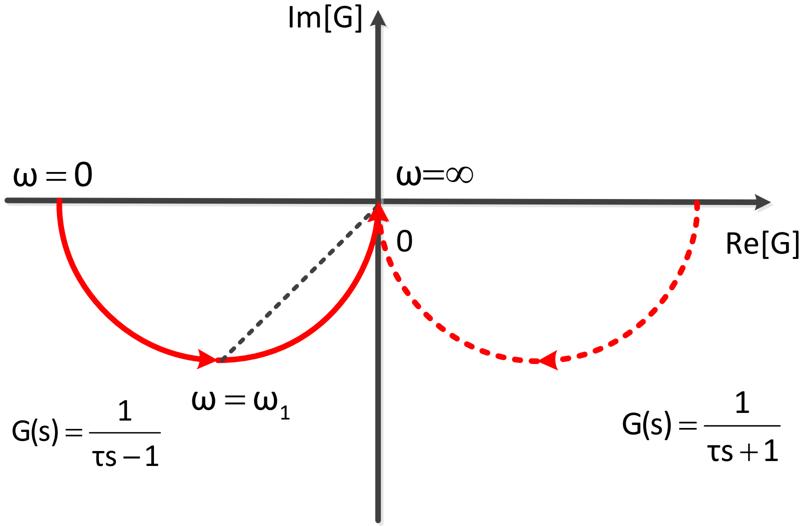
8.2 Frequency Response Plots

⑨ nonminimum phase element

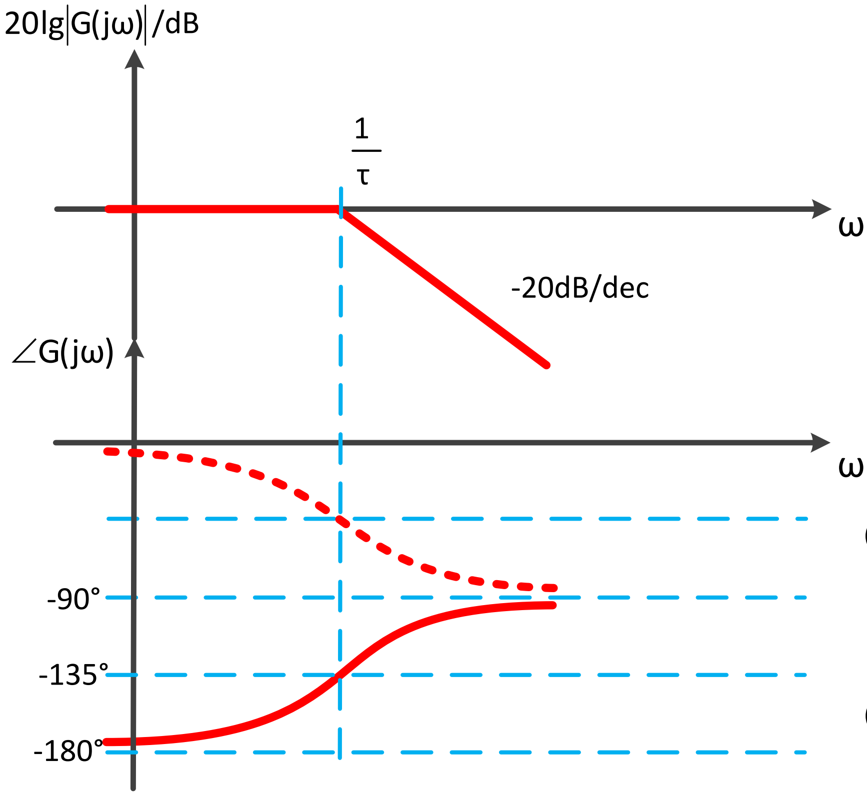
$$G(s)=\frac{1}{\tau s-1}\overset{s=j\omega}{\Rightarrow}G(j\omega)=\frac{1}{\tau\omega j-1},\left|G(j\omega)\right|=\frac{1}{\sqrt{\tau^2\omega^2+1}}$$

$$\angle G(j\omega)=-\arctan\frac{\tau\omega}{-1}=-\pi+\arctan\tau\omega$$

ω	$\left\|G(j\omega)\right\|$	$\angle G(j\omega)$
0	1	-180°
$\frac{1}{\tau}$	$\frac{1}{\sqrt{2}}$	-135°
∞	0	-90°



8.2 Frequency Response Plots



$$20\lg|G(j\omega)| = \begin{cases} 0, & \omega << \frac{1}{\tau} \\ -20\lg|\tau\omega|, & \omega >> \frac{1}{\tau} \end{cases}$$

$$G(s) = \frac{1}{\tau s + 1}$$ Minimum phase

$$G(s) = \frac{1}{\tau s - 1}$$ Non-minimum phase

The range of phase shift of a minimum phase transfer function is the minimum corresponding to a given amplitude curve.

8.3 Examples of Bode Diagram

Example $G(s)=\dfrac{100(0.1s+1)}{s^2(0.01s+1)}$ First we divide it into 4 typical elements

$$G_1(s)=100, G_1(j\omega)=100, |G_1|=100, \angle G_1=0$$

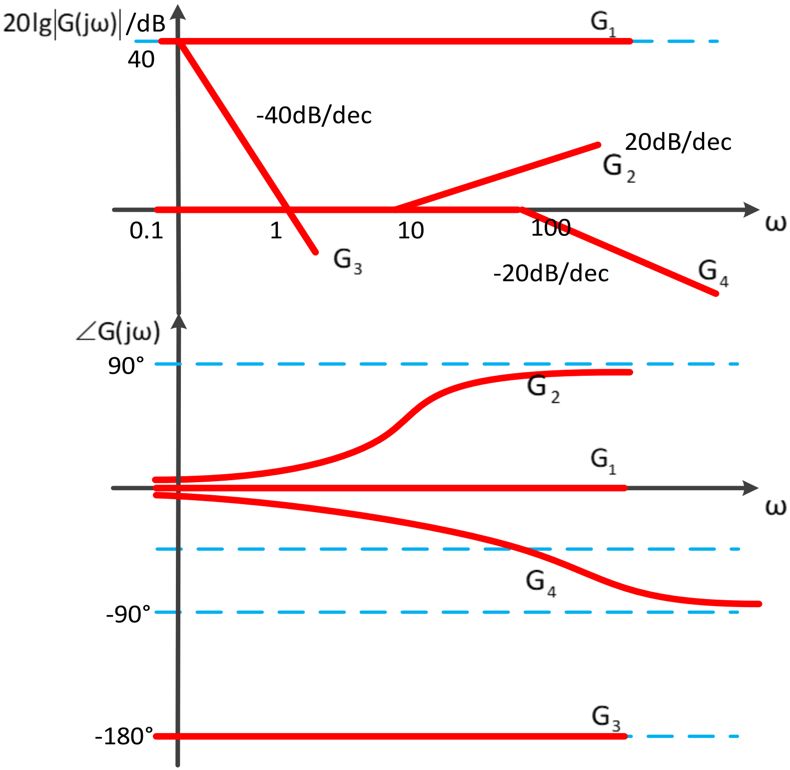
$$G_2(s)=0.1s+1, G_2(j\omega)=0.1j\omega+1, |G_2|=\sqrt{1+0.1^2\omega^2},\quad \angle G_2=tg^{-1}0.1\omega$$

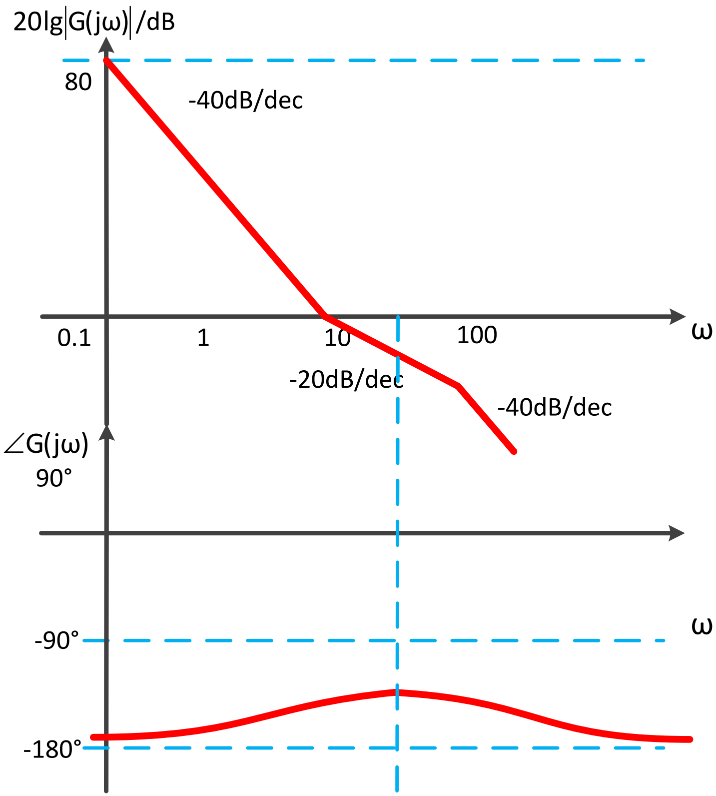
$$G_3(s)=\frac{1}{s^2}, G_3(j\omega)=\frac{1}{-\omega^2}, |G_3|=\frac{1}{\omega^2}, G_3=-180°$$

$$G_4(s)=\frac{1}{0.01s+1}, G_4(j\omega)=\frac{1}{0.01j\omega+1},$$

$$|G_4|=\frac{1}{\sqrt{1+0.01^2\omega^2}}, \angle G_4=-tg^{-1}0.01\omega$$

8.3 Examples of Bode Diagram

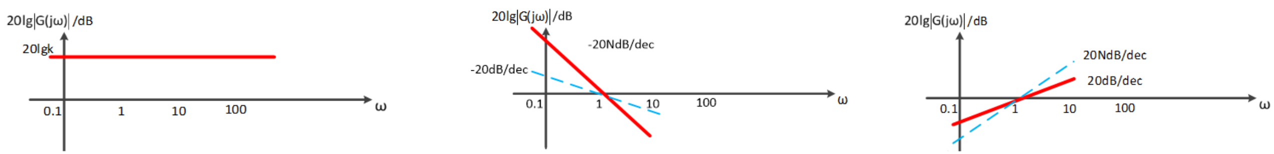




8.3 Examples of Bode Diagram

Steps:

①Locate break frequencies on the horizontal axis;
②In low frequency domain, the magnitude curve is determined by proportional element, integral element or differential element.



③Based on the low frequency domain, the slope changes at every break frequencies.

Inertial element(-20dB/dec); first order differential element(+20dB/dec)
oscillating element(-40dB/dec); second order differential element(+40dB/dec)

crossover frequency ω_c

when $\omega = \omega_c, 20\lg|G(j\omega_c)| = 0, |G(j\omega_c)| = 1$

穿越频率

8.3 Examples of Bode Diagram

常数项置一

化为标准型

Examples:

$$G(s)=\frac{100}{(s+1)(s+10)}$$

$$G(s)=\frac{10(s+10)}{s(0.01s+1)}$$

8.4 Examples of Polar Plot

$$G(s)\big|_{s=j\omega} = G(j\omega) = R(\omega) + jX(\omega) = |G(j\omega)|e^{\angle G(j\omega)} \qquad \omega: 0 \to \infty$$

① $\omega \to 0$

② $\omega \to \infty$

③ Analyze the tendency at the middle frequency segments

④ Key point (where $G(j\omega)$ intersects the negative real axis)

Example:

$$G(s) = \frac{k(\tau s + 1)}{s^2(Ts+1)}$$

$$G(j\omega) = \frac{k(\tau j\omega + 1)}{-\omega^2(Tj\omega + 1)}, \quad |G(j\omega)| = \frac{k\sqrt{\tau^2\omega^2 + 1}}{\omega^2\sqrt{T^2\omega^2 + 1}},$$

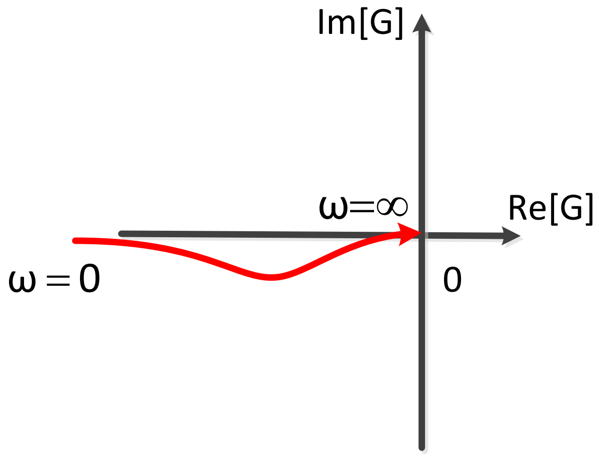
$$\angle G(j\omega) = \arctan \tau\omega - 180^\circ - \arctan T\omega$$

8.4 Examples of Polar Plot

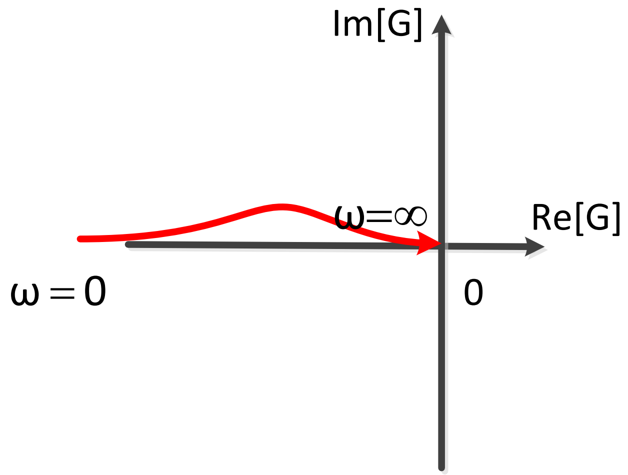
ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	-180°
∞	0	-180°

Then find the tendency at the middle frequency segments

① $\tau > T,\ \arctan\omega\tau > \arctan T\omega$



② $\tau < T,\ \arctan\omega\tau < \arctan T\omega$



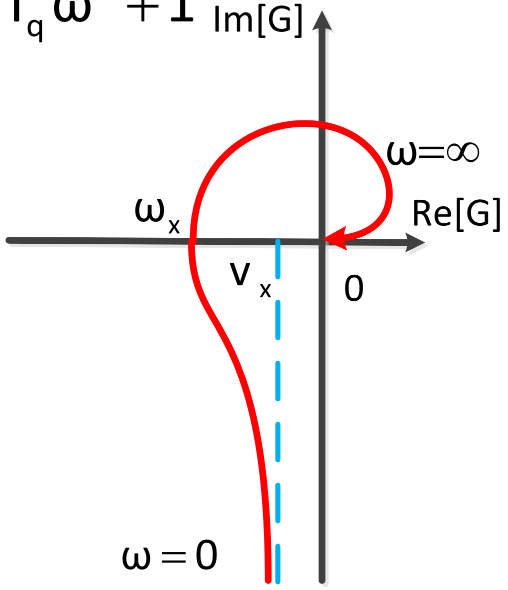
8.4 Examples of Polar Plot

Example:

$$G(j\omega)=\frac{k}{j\omega(T_m j\omega+1)(T_c j\omega+1)(T_q j\omega+1)}$$

$$\left|G(j\omega)\right|=\frac{k}{\omega\sqrt{T_m^2\omega^2+1}\sqrt{T_c^2\omega^2+1}\sqrt{T_q^2\omega^2+1}}$$

$$\angle G(j\omega)=0-90^\circ-\arctan T_m\omega$$
$$-\arctan T_c\omega-\arctan T_q\omega$$



ω	$\left\|G(j\omega)\right\|$	$\angle G(j\omega)$
0	∞	-90°
∞	0	-360°

Then determine v_x and ω_x

8.4 Examples of Polar Plot

$$G(j\omega)=\frac{-k[\omega(T_m+T_c+T_q)-\omega^2T_mT_cT_q]}{\omega(T_m^2\omega^2+1)(T_c^2\omega^2+1)(T_q^2\omega^2+1)}+j\frac{-k[1-\omega^2(T_mT_c+T_cT_q+T_mT_q)]}{\omega(T_m^2\omega^2+1)(T_c^2\omega^2+1)(T_q^2\omega^2+1)}$$

① $\omega\to 0, v_x=\lim_{\omega\to 0}\mathrm{Re}[G(j\omega)]=-k(T_m+T_c+T_q)$

② use $\angle G(j\omega)=-180^\circ$ or $\mathrm{Im}\left[G(j\omega)\right]=0$

we can get $\omega_x=\dfrac{1}{\sqrt{T_mT_c+T_cT_q+T_mT_q}}$

also we have $\left|G(j\omega_x)\right|=\omega_x$

ω_x is the most important Nyquist characteristic point

8.4 Examples of Polar Plot

Tips: For a minimum phase system, the start phase is determined by the integral element

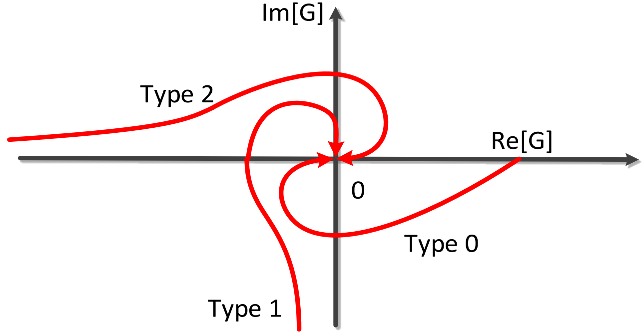
① $\omega = 0$ the start phase $\angle G(j0) = N \times (-90^\circ)$

② $\omega = \infty$ each non-zero pole adds the phase angle -90^0

each non-zero zero adds the phase angle $+90^0$

$$n>m,\ \omega=\infty,\ |G(j\omega)| = 0 \text{ and } \angle G(j\omega) = (n-m)\times(-90^\circ)$$

Example: $G(s) = \dfrac{10}{s^N(0.1s+1)^2}$



8.4 Examples of Polar Plot

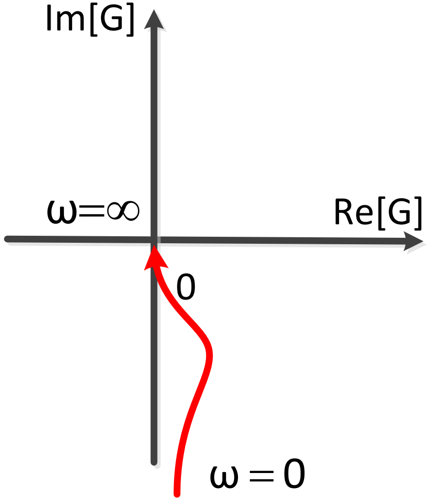
Example:

$$G(s)=\frac{k(T_1 s+1)}{s(T_2 s+1)}(T_1>T_2),\quad G(j\omega)=\frac{k(T_1 j\omega+1)}{j\omega(T_2 j\omega+1)}$$

$$|G(j\omega)|=\frac{k\sqrt{T_1^2\omega^2+1}}{\omega\sqrt{T_2^2\omega^2+1}}$$

$$\angle G(j\omega)=-90^\circ+tg^{-1}T_1\omega-tg^{-1}T_2\omega$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	-90°
∞	0	-90°



8.4 Examples of Polar Plot

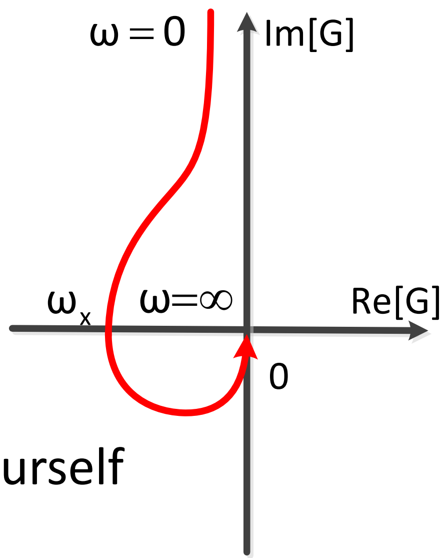
Example:

$$G(s)=\frac{k(s+1)}{s(s-1)},\quad G(j\omega)=\frac{k(j\omega+1)}{j\omega(j\omega-1)},\quad |G(j\omega)|=\frac{k\sqrt{\omega^2+1}}{\omega\sqrt{\omega^2+1}}=\frac{k}{\omega}$$

$$\angle G(j\omega)=-90^\circ+tg^{-1}\omega-(180^\circ-tg^{-1}\omega)$$

$$=-270^\circ+2tg^{-1}\omega$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	-270°
∞	0	-90°



Then determine k of the point ω_x yourself

8.4 Examples of Polar Plot

Stability:

- Time domain: Ⅰ $y(\infty) = \lim_{t \to \infty} y(t)$

  Ⅱ use Laplace transformation and observe the roots of CE

  Ⅲ Routh criterion

  Ⅳ root locus method

- Frequency domain:

  use $G(j\omega)$ to sketch polar plot and then use Nyquist criterion