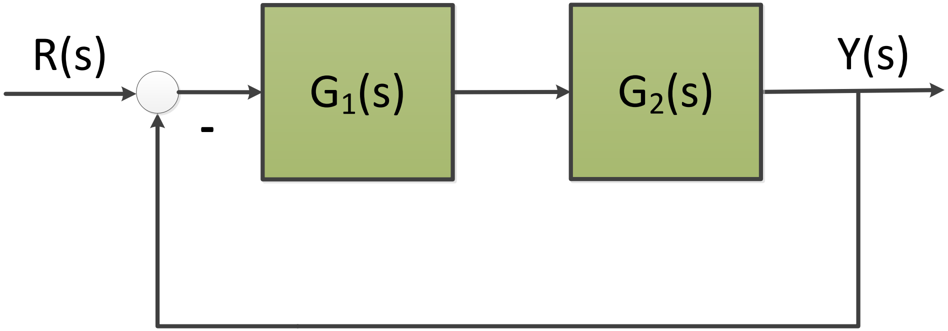
# Chapter 7 The Root Locus Method

# Contents

# 7.1 Introduction




For a first-order system,

$$G_1(s) = \frac{1}{s+1},$$

$$G_2(s) = k$$

open-loop transfer function

$$G(s) = G_1(s)G_2(s) = \frac{k}{s+1}$$

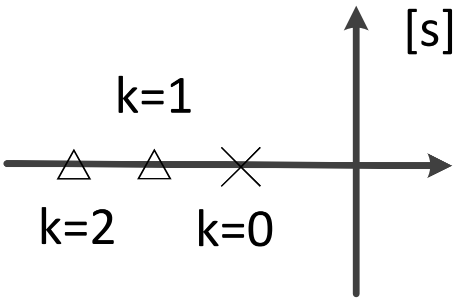
closed-loop transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{k}{s+1+k}$$

# 7.1 Introduction

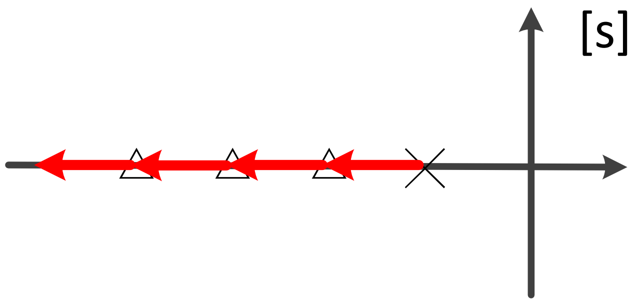
gain : $\dfrac{k}{1+k}$

pole : $s = -(1+k)$

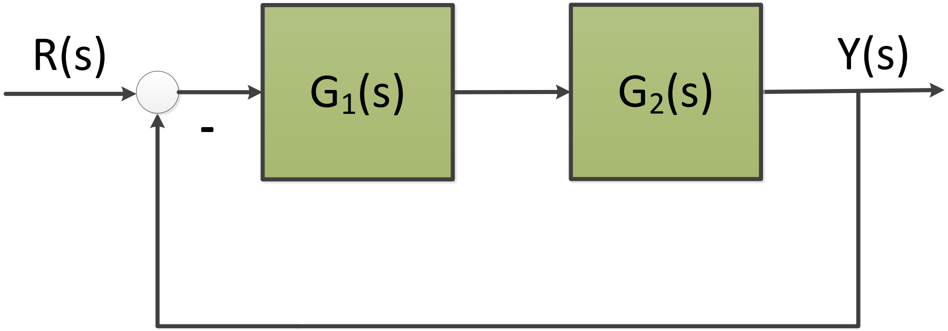



In this feedback system, when k increases,

- The pole goes further to the left,
- Transient response is faster,
- Relative stability is better.



# 7.1 Introduction




For a second-order system

$$G_1(s) = \frac{1}{(s+1)(s+3)},$$

$$G_2(s) = k$$

open-loop transfer function

$$G(s) = G_1G_2 = \frac{k}{(s+1)(s+3)}$$

closed-loop transfer function

$$T(s) = \frac{k}{s^2 + 4s + 3 + k}$$

# 7.1 Introduction

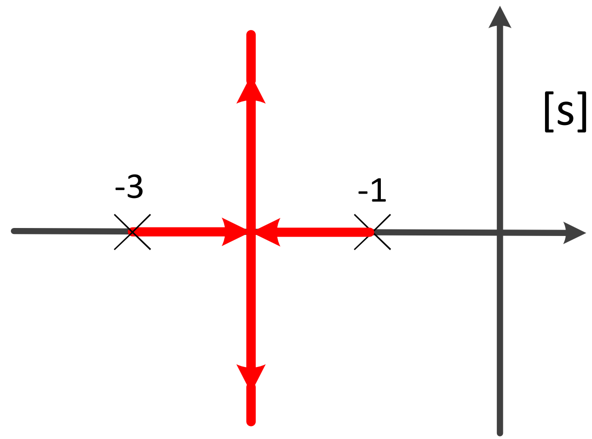
Poles: $$s_{1,2} = \frac{-4 \pm \sqrt{16 - 4(3+k)}}{2} = -2 \pm \sqrt{1-k}$$

k=0: poles at -1 and -3, $\xi > 1$

k<1: two real poles between -1 and -3, $\xi > 1$

k=1: two real poles at -2, $\xi = 1$

k>1: two complex poles with real part -2, $0 < \xi < 1$




Please find the relation between k & $\xi$, $\omega_n$, tp, ts, tr, p.o.

# 7.1 Introduction

**Notes:**

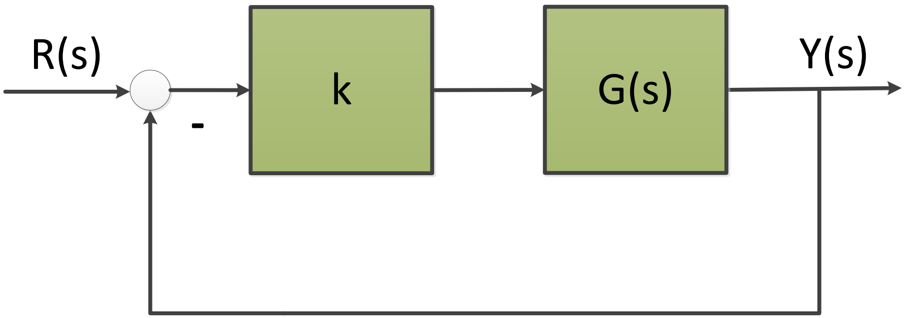
- -1 and -3 are the poles of open-loop transfer function;
- The number of loci equals to the number of system order;
- In this example, k:0→ ∞, the poles are always in the left of s-plane (the system is stable);
- No poles at the original point, so the system is type 0;
- When k is smaller than 1, there is no overshoot;
- A faster response when k increases between 0 and 1.

# 7.1 Introduction

**Introduction:**

- The (relative) stability and the transient performance are related to the location of the roots of CE in the s-plane;
- It is difficult to manually solve the problem of a high order system's (relative) stability and transient performance, we can use MATLAB, or a graphical method: >>rlocus(num,den);
- The root locus gives the location of the poles of the closed system for variations in the open-loop gain of the system;
- To be more specific, the root locus is the path of the roots of the CE traced out in the s-plane as a system parameter (k) is changed from 0 to $\infty$.

# 7.2 Root Locus Concept




As a general feedback system

$$T(s) = \frac{Y(s)}{R(s)} = \frac{kG(s)}{1 + kG(s)}$$

$$\text{CE:}\ \ q(s) = 1 + kG(s) = 1 + k\frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 0$$

$$kG(s) = k\frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = -1$$

根轨迹方程

# 7.2 Root Locus Concept

**The magnitude condition and the phase angle condition** are

$$\begin{cases} |kGH(s)| = 1 \\ \angle kGH(s) = 180^\circ + 2l\pi \quad (l = 0,\ \pm 1,\ \pm 2\ \cdots) \end{cases}$$

用来求K

充分必要条件

The conditions representing in details are as follows

$$\left| k \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} \right| = 1 \qquad \angle \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 180^\circ + 2l\pi$$

$$(l = 0,\ \pm 1,\ \pm 2\ \cdots)$$

# 7.2 Root Locus Concept

**Example:** Given a unit feedback system

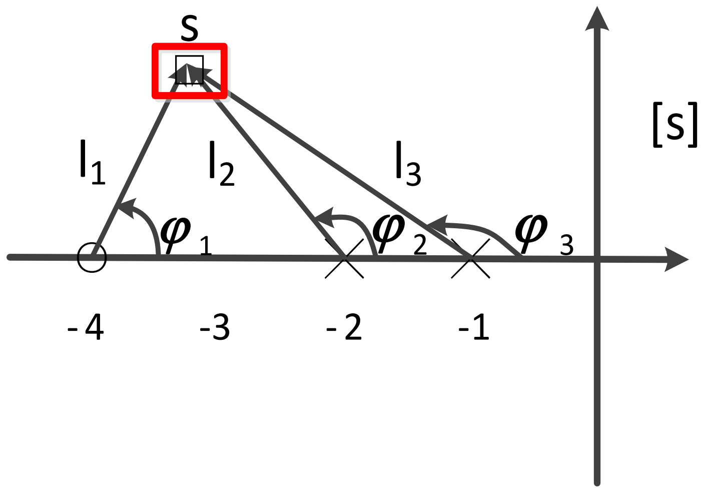
$$G_{open}(s) = \frac{k(s+4)}{(s+1)(s+2)}$$

If a point s in s-plane is on the root locus, it matches the angle condition and the magnitude condition.

$$\angle kG_{open}(s) = \angle(s+4) - \angle(s+1) - \angle(s+2)$$

$$= \phi_1 - \phi_2 - \phi_3 = 180^\circ \pm 2l\pi \ (l = 0, 1, 2 \cdots)$$

$$\left|kG_{open}(s)\right| = \left|\frac{k\,l_1}{l_2\, l_3}\right| = 1$$



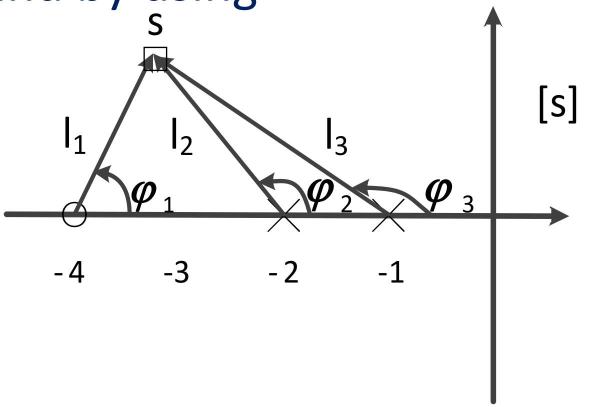
# 7.2 Root Locus Concept

□ **Notes:**

逆时针

- All angles are measured in a counterclockwise direction from a horizontal line.
- The angle requirement is satisfied at any point on the root locus.
- The angle requirement **is the necessary and sufficient condition to determine the closed-loop poles.**
- The gain k at the particular points is found by using

$$\left|kG_{open}(s)\right| = \left|\frac{k\,l_1}{l_2\,l_3}\right| = 1$$



# 7.3 Root Locus Procedure

**Step 1:** Prepare the root locus sketch

(a): Write the characteristic equation so that the parameter of interest, K, appears as a multiplier

$$\text{CE}: \quad q(s) = 1 + G(s)H(s) = 0$$

(b): or in terms of n poles and m zeros

$$1 + k\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$$

# 7.3 Root Locus Procedure

(c): Locate the open-loop poles and zeros in the s-plane.

$$1+G(s)H(s)=\frac{\prod_{j=1}^{n}(s+p_j)+k\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=0$$

where k is from 0 to ∞

起始于开环极点

① $k \to 0$ $\quad \prod_{j=1}^{n}(s+p_j)=0 \quad s=-p_j$ **begin at poles**

开环零点

② $k \to \infty$ $\quad \prod_{i=1}^{m}(s+z_i)=0 \quad s=z_i$ **end at zeros**

**Root loci begin at the open-loop poles and end at the open-loop zeros.**

# 7.3 Root Locus Procedure

(d): Determine the number of separate loci, sL (sL=n). Have n-m branches of the root loci approaching the n-m zeros at infinity.

无穷远处

Eg: open-loop TF: $GH(s) = \dfrac{k}{s(s+2)}$ , n=2, m=0, so there are

(n-m=2) zeros at $\infty$ ;

the root loci of the system begin at s=0 and s=-2,

and end at two $\infty$ zeros.

(e): Root loci are symmetrical with respect to the horizontal real axis.

关于 实轴对称

# 7.3 Root Locus Procedure

**Step 2:Lacate the segments of the real axis that are root loci.**

$$\angle \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = \pi \pm 2l\pi$$

Locus lies to the ~~left~~ of an **odd (奇数)** number of poles and zeros.

右侧

Eg: Sketch the root loci of the system with $GH(s) = \dfrac{2k(s+2)}{s(s+4)}$

# 7.3 Root Locus Procedure

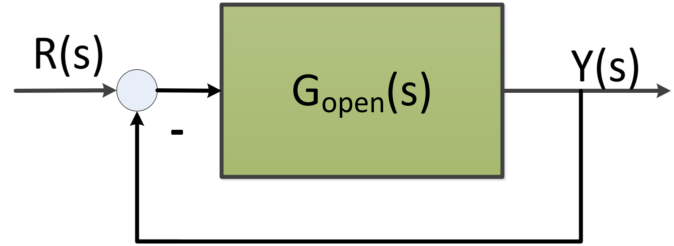
**Step 3: With loci proceed to the zeros at infinity, asymptotes centered at $\sigma_A$ and with angles $\varphi_A$ .**

$$\begin{cases} \sigma_A = \dfrac{\sum \text{poles} - \sum \text{zeros}}{n-m} & \text{(on the real axis)} \\ \varphi_A = \dfrac{2l+1}{n-m}\pi & l = 0, 1, 2 \cdots (n-m-1) \end{cases}$$

The number of asymptotes: n-m

# 7.3 Root Locus Procedure

**Examples**




$$G_{open}(s)=\frac{k(s+1)}{s(s+4)(s^2+2s+4)}$$

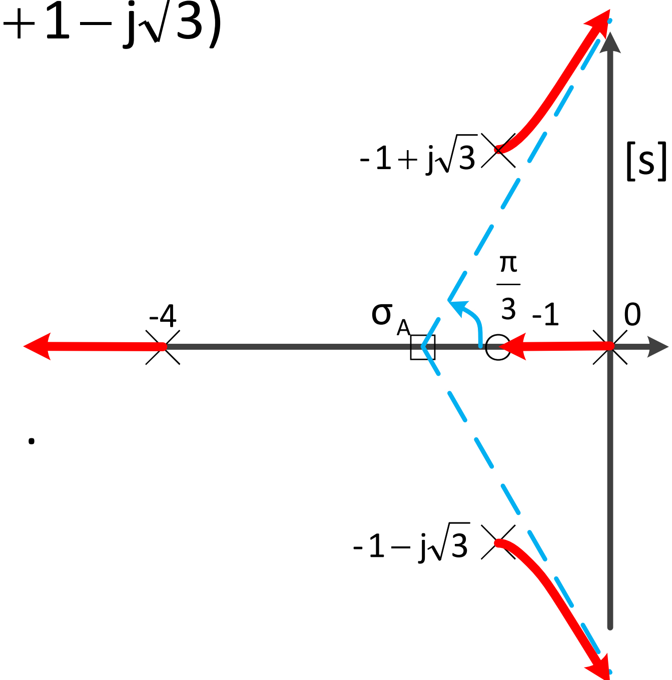
$$q(s)=s(s+4)(s^2+2s+4)$$
$$=s(s+4)(s+1+j\sqrt{3})(s+1-j\sqrt{3})$$

① n=4, m=1;

4 separate loci;

② determine real loci;

③ determine $\sigma_A$ and $\varphi_A$ .




多项式求重根

# 7.3 Root Locus Procedure

$$\sigma_A = \frac{\sum poles - \sum zeros}{n-m} = \frac{-4-2+1}{3} = \frac{-5}{3}$$

$$\varphi_A = \frac{2l+1}{n-m}\pi = \frac{2l+1}{3}\pi = \begin{cases} \frac{1}{3}\pi & l=0 \\ \pi & l=1 \\ \frac{5}{3}\pi & l=2 \end{cases}$$

- Infinity root loci from the complex roots to infinity along the asymptotes
- In this example, the root loci would enter right of the s-plane, then the system is no longer stable.

# 7.3 Root Locus Procedure

**Step 4: Determine the points at which the locus crosses the imaginary axis (if it does so);**

① Routh-Hurwitz criterion (case 3);

② Let s=jw, Re[1+GH(jw)]=0 and Im[1+GH(jw)]=0

Eg: For the closed CE, $s^4+6s^3+12s^2+ks+16s+k=0$.

Use the Routh-Hurwitz criterion, we get the schedule.

$$
\begin{array}{llll}
s^4 & 1 & 12 & k \\
s^3 & 6 & 16+k & 0 \\
s^2 & 12-\dfrac{16+k}{6} & k & 0 \\
s^1 & 16+k-\dfrac{36k}{56-k} & 0 & 0 \\
s^0 & k & 0 & 0
\end{array}
$$

Let $16+k-\dfrac{36k}{56-k}=0$

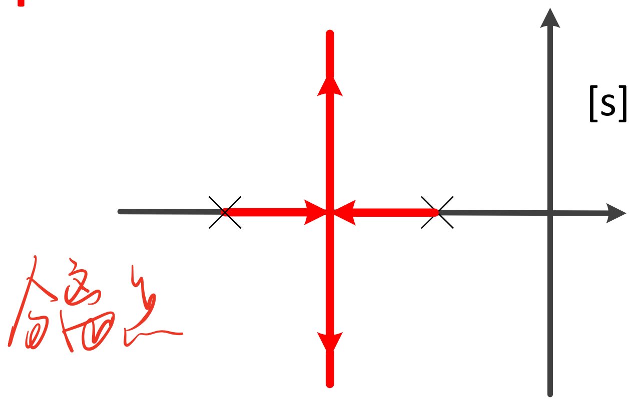
We get0< k<32 for stable system. Then from auxiliary equation, we get $s_{1,2}=\pm 2\sqrt{2}j$

求与虚轴交点

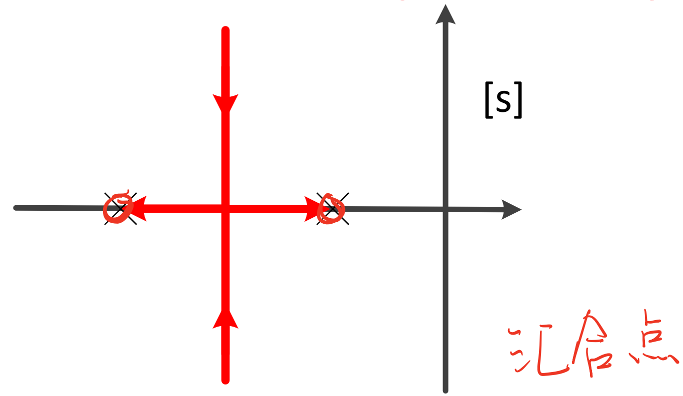
辅助方程

# 7.3 Root Locus Procedure

**Step 5: Determine the break-away point and break-in point (if any)**




break-away point




break-in point

The points are at real axis

- break-away point: When k 0→∞, the two branches break away from the real axis and enter complex plane;
- break-in point: When k 0→∞, the two branches are coming from the complex region and enter real axis.

# 7.3 Root Locus Procedure

① determine the point: When two loci meet at the real axis, the two roots are multiple roots;

$$1+kGH(s)=0 \Rightarrow k=-\frac{\prod_{j=1}^{n}(s+p_j)}{\prod_{i=1}^{m}(s+z_i)} \Rightarrow \frac{d(k)}{ds}=0$$

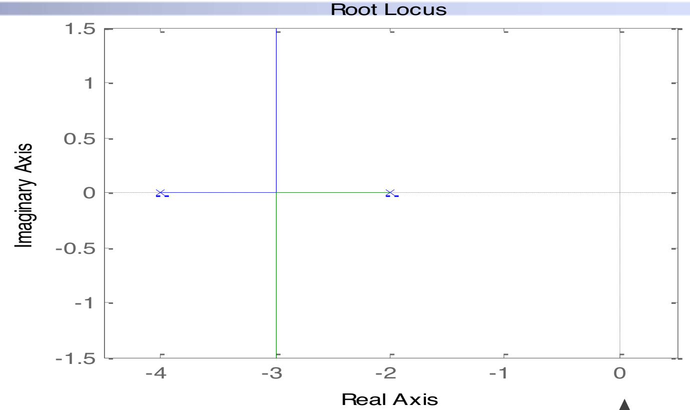
② The angle of the tangents to the loci are

$$\alpha=\frac{\pm(2l+1)}{L}\pi \quad (l=0,1,2\cdots L-1)$$

L is the number of branches reaching or leaving the break-away (break-in)point.

# 7.3 Root Locus Procedure

Example: $G_{open}(s) = \dfrac{k}{(s+2)(s+4)}$



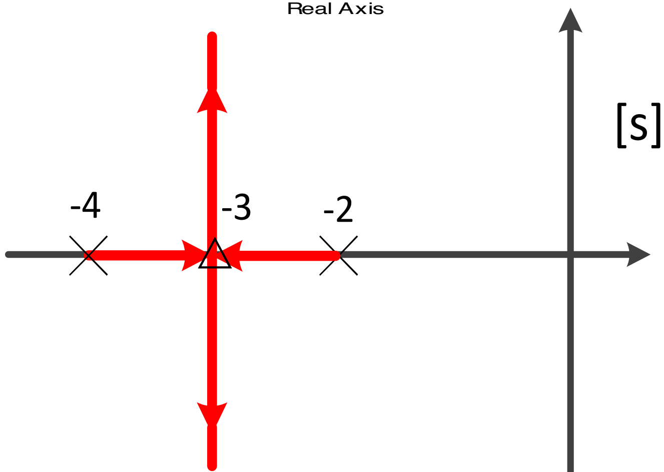

① n=2, m=0; 2 separate loci;

② determine real loci;

③ determine $\sigma_A$ and $\varphi_A$ ;

$$\sigma_A = \frac{\sum p - \sum z}{n-m} = \frac{-6}{2} = -3$$

$$\phi_A = \frac{2l+1}{2}\pi = \begin{cases} \dfrac{1}{2}\pi & l=0 \\ \dfrac{3}{2}\pi & l=1 \end{cases}$$



# 7.3 Root Locus Procedure

④ determine the break-away point and angle;

$$\frac{d(k)}{ds} = \frac{d[-(s^2+6s+8)]}{ds} = 0 \Rightarrow s = -3$$

$$\alpha = \frac{\pm(2l+1)}{2}\pi = \begin{cases} \frac{1}{2}\pi & l = 0 \\ \frac{3}{2}\pi & l = 1 \end{cases}$$

**Examples:**

$$G_{open}(s) = \frac{k}{s(s+1)(s+2)}$$ Do it yourselves

# 7.3 Root Locus Procedure

**Step 6: Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros.**

$$\theta_{pk} = \pi + \sum_{i=1}^{m} \varphi_{z_i p_k} - \sum_{\substack{j=1 \\ j \neq k}}^{n} \theta_{p_j p_k}$$

$$\varphi_{zk} = \pi - \left( \sum_{\substack{i=1 \\ i \neq k}}^{m} \varphi_{z_i z_k} - \sum_{j=1}^{n} \theta_{p_j z_k} \right)$$

**Step 7:Complete the root locus sketch.**

$$\sum_{j=1}^{n} s_j = \sum_{j=1}^{n} p_j \qquad n - m \geq 2$$

Closed-loop poles

open-loop poles

# Example(7.3)

**Example1:** $$GH(s)=\frac{k}{s^4+12s^3+64s^2+128s}$$

**Example2:** $$GH(s)=\frac{k(s+1)}{s(s+0.2)}$$

**Example3:** $$GH(s)=\frac{k(s+1)}{s(s-3)}$$

**Example4:** $$GH(s)=\frac{k}{s(s+1)(s+3)}$$

# Example(7.3)

**Example1:**

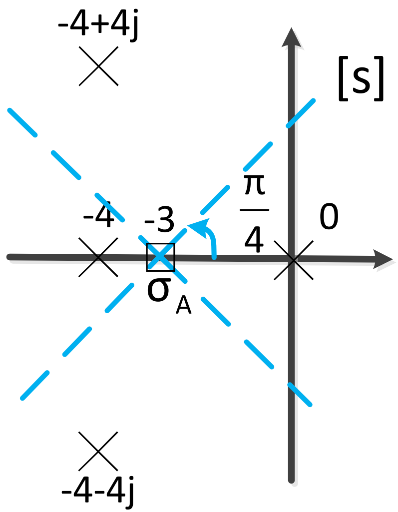
$$G(s)=\frac{k}{s^4+12s^3+64s^2+128s}=\frac{k}{s(s+4)(s+4+4j)(s+4-4j)}$$

① n=4, m=0; 4 separate loci; 4 loci towards infinity;

② determine real loci; ③ determine $\sigma_A$ and $\varphi_A$;

$$\sigma_A=\frac{\sum ps-\sum zs}{n-m}=\frac{-4-4-4}{4}=-3$$

$$\varphi_A=\frac{2l+1}{2}\pi=\begin{cases}\frac{1}{4}\pi \quad l=0; \quad \frac{3}{4}\pi \quad l=1;\\ \frac{5}{4}\pi \quad l=2; \quad \frac{7}{4}\pi \quad l=3.\end{cases}$$



# Example(7.3)

④ marginal points;

$$\begin{array}{llll} s^4 & 1 & 64 & k \\ s^3 & 12 & 128 & 0 \\ s^2 & 53.33 & k & 0 \\ s^1 & 128-\dfrac{12k}{53.33k} & 0 & 0 \\ s^0 & k & 0 & 0 \end{array}$$

$$128-\frac{12k}{53.33}=0$$

$$k=568.89$$

$$53.33s^2+568.89=0$$

$$s^2=10.67$$

$$s_{1,2}=\pm j3.27$$

⑤ break-away point;

$$\frac{d(-k)}{ds}=\frac{d(s^4+12s^3+64s^2+128s)}{ds}=0\Rightarrow s=-1.5$$

$$\alpha=\frac{\pm(2l+1)}{2}\pi=\left\{\frac{1}{2}\pi \quad l=0;\ \ \frac{3}{2}\pi \quad l=1\right.$$

# Example(7.3)

⑥ departure angle;

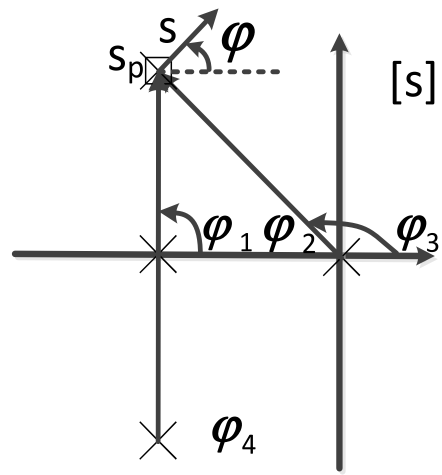
For the pole -4+4j, $-\varphi_1-\varphi_2-\varphi_3-\varphi=180^\circ\pm 2l\pi\ \ (l=0,1,2\cdots)$

$$\varphi=-\varphi_1-\varphi_2-\varphi_3-180^\circ\pm 2l\pi\ \ (l=0,1,2\cdots)$$

$$=-90^\circ-90^\circ-135^\circ-180^\circ\pm 2l\pi\ \ (l=0,1,2\cdots)$$

$$=-135^\circ\pm 2l\pi\ \ (l=0,1,2\cdots)$$

$$=-135^\circ$$




For the pole -4-4j, use the same method, we can get

$$\varphi=135^\circ\pm 2l\pi\ \ (l=0,1,2\cdots)$$

$$=135^\circ$$

⑦ Complete the root locus sketch.

# Example(7.3)

determine $s_1$, $s_2$ and k for given $\xi = 0.707$ ,$\cos^{-1}\xi = 45°$;

we can set $s_1$, $s_2$ as

$$s_1 = s_2 = -\sigma \pm j\sigma$$

use the angle condition

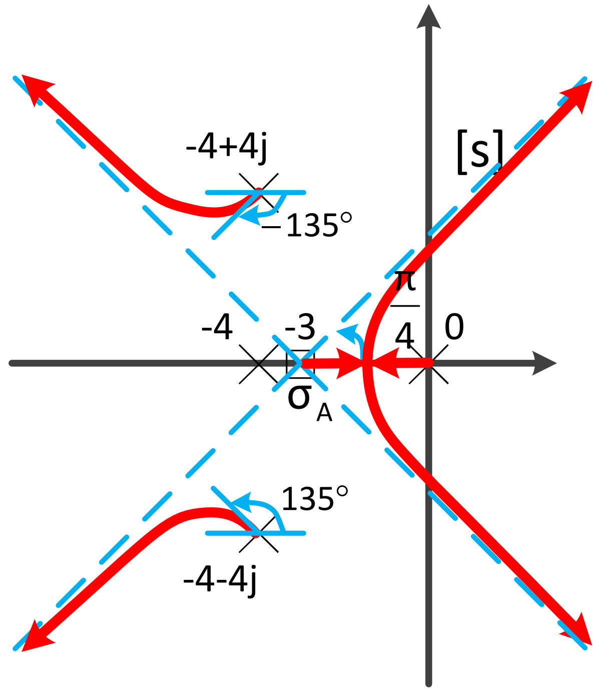
to determine $s_1$, $s_2$ :

$$-(135° - 45° + \tan\frac{4+\sigma}{4-\sigma} + \tan\frac{\sigma}{4-\sigma})$$

$$= 180° + 2l\pi$$

$$\tan\frac{4+\sigma}{4-\sigma} + \tan\frac{\sigma}{4-\sigma} = -270°$$

$$\sigma = \frac{4}{3}$$



# Example(7.3)

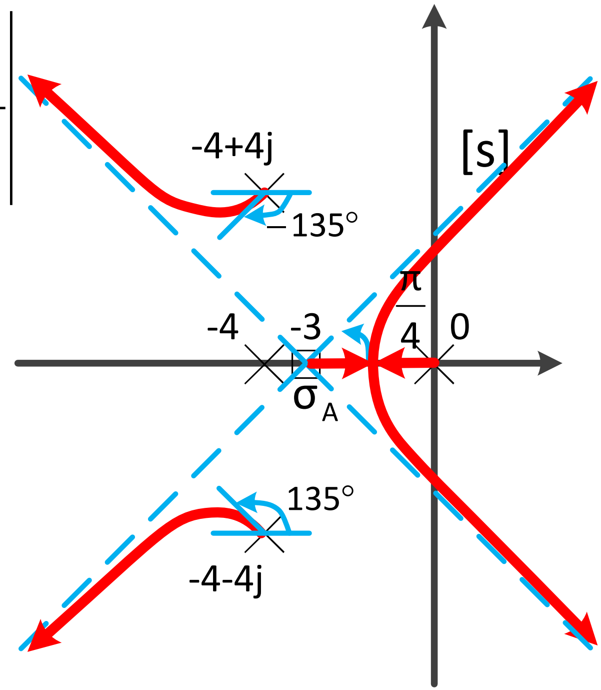
use the angle condition to determine k:

Substitute $s_1$ into k, we get $k = \left| \dfrac{1}{G(s)\big|_{s=s_1}} \right|$

$$= \left| s_1(s_1+4)(s_1+4+4j)(s_1+4-4j) \right|$$

$$= |s_1||s_1+4||s_1+4+4j||s_1+4-4j|$$

(Try to determine k for $s_3$, $s_4$ yourself)



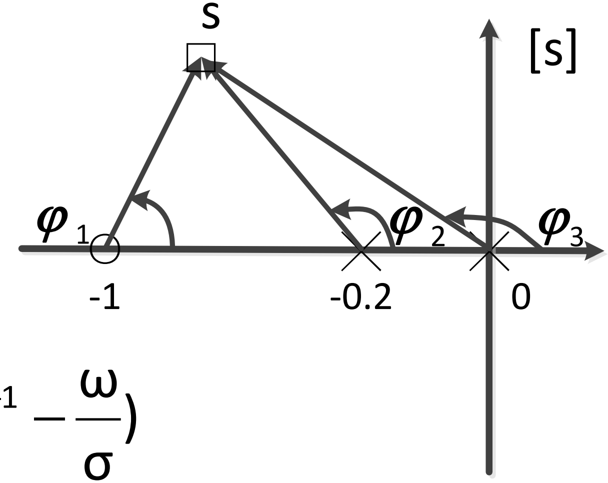
# Example(7.3)

Eg2*: $G(s)=\dfrac{k(s+1)}{s(s+0.2)}$ verify that this system's root loci is a circle

① n=2, m=1; 2 separate loci;

② determine real loci;

③ $\sigma_A=0.8,\ \varphi_A=\pi$




For a point on this root locus s,

use the argument condition:

$$\varphi_1-\varphi_2-\varphi_3=180^\circ\pm 2l\pi$$

set $s=\sigma+j\omega$

$$\tan^{-1}\frac{\omega}{1-\sigma}-(\tan^{-1}-\frac{\omega}{\sigma-0.2})-(\tan^{-1}-\frac{\omega}{\sigma})$$

$$=180^\circ\pm 2l\pi$$

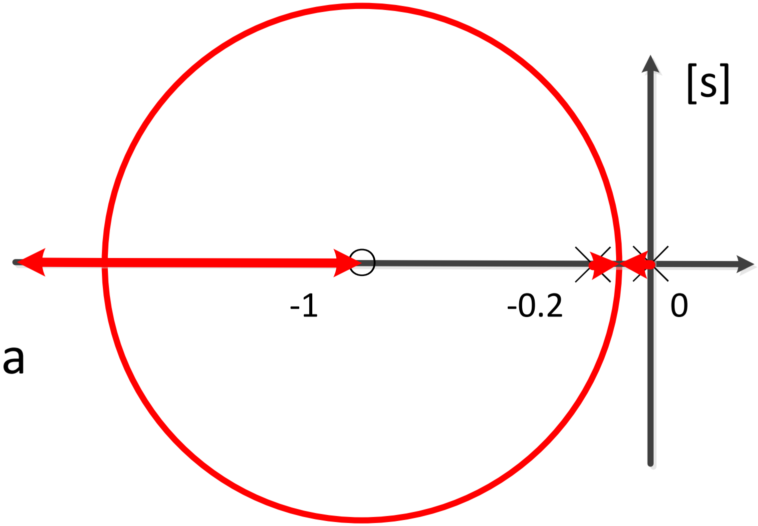
# Example(7.3)

$$\tan^{-1}\frac{\omega}{1-\sigma}+\tan^{-1}\frac{\omega}{\sigma-0.2}+\tan^{-1}\frac{\omega}{\sigma}=180^\circ \pm 2l\pi$$

Use $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

We get $\sigma^2 - 2\sigma + 1 + \omega^2 = 0.8$

$$(\sigma+1)^2+\omega^2=(\sqrt{0.8})^2$$




So the root loci of this system is a circle with the center (-1,0) and Radius $\sqrt{0.8}$

# Example(7.3)

Eg3*: $G(s)=\dfrac{k(s+1)}{s(s-3)}$ determine range of k when system is stable and determine the time domain parameters

① n=2, m=1;   ② 2 separate loci;

③ determine real loci;   ④ n-m=2-1=1, 1 loci towards infinity;

⑤ $\sigma_A=4,\ \varphi_A=\pi$   ⑥ break-away point;

$$\frac{d(-k)}{ds}=\frac{d\left(\dfrac{s(s-3)}{s+1}\right)}{ds}=0\Rightarrow s_1=-1.5,s_2=\text{-}3$$

$$k_1=\left|\frac{s^2-3s}{s+1}\right|_{s_1=1}=1,\ k_2=\left|\frac{s^2-3s}{s+1}\right|_{s_2=\text{-}3}=9$$

$$\alpha_{1,2}=\frac{\pm(2l+1)}{2}\pi=\begin{cases}\dfrac{1}{2}\pi & l=0;\ \dfrac{3}{2}\pi\quad l=1\end{cases}$$

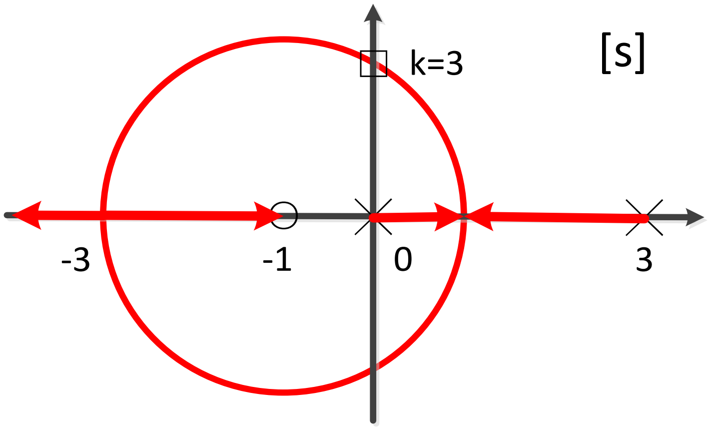
# Example(7.3)

⑦ marginal points;

$$\begin{array}{lcc} s^2 & 1 & k \\ s^1 & k-3 & 0 \\ s^0 & k & 0 \end{array}$$

$k-3=0,\ k=3$

$s^2+3=0,\ s_{1,2}=\pm j\sqrt{3}$




So for a stable system, the range of k is k>3.

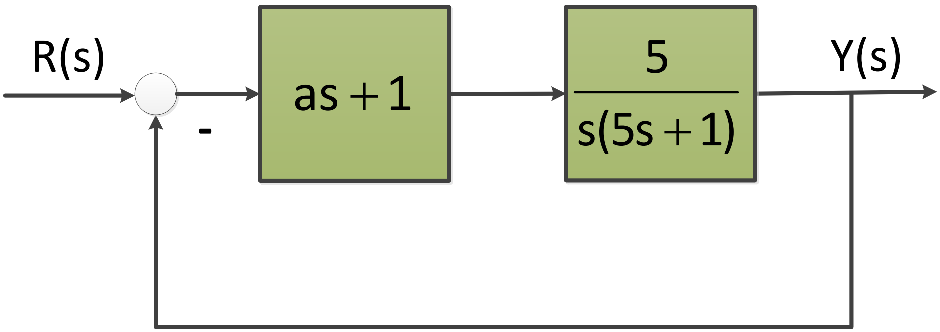
⑧ time domain parameters;

Set k=10, determine y(t), $e_{ss}$, p.o. for unit step input;

$$Y(s)=\frac{10(s+1)}{s^2+7s+10}\cdot\frac{1}{s}=\frac{10(s+1)}{s(s+2)(s+5)}=\frac{1}{s}+\frac{1.67}{s+2}+\frac{-2.67}{s}$$

# 7.4 Parameter Root Loci

When the parameter in the system is no longer k, how to deal with it via root locus method




Exemple:

First we can get the characteristic equation:

$$s(5s+1)+5(as+1)=$$

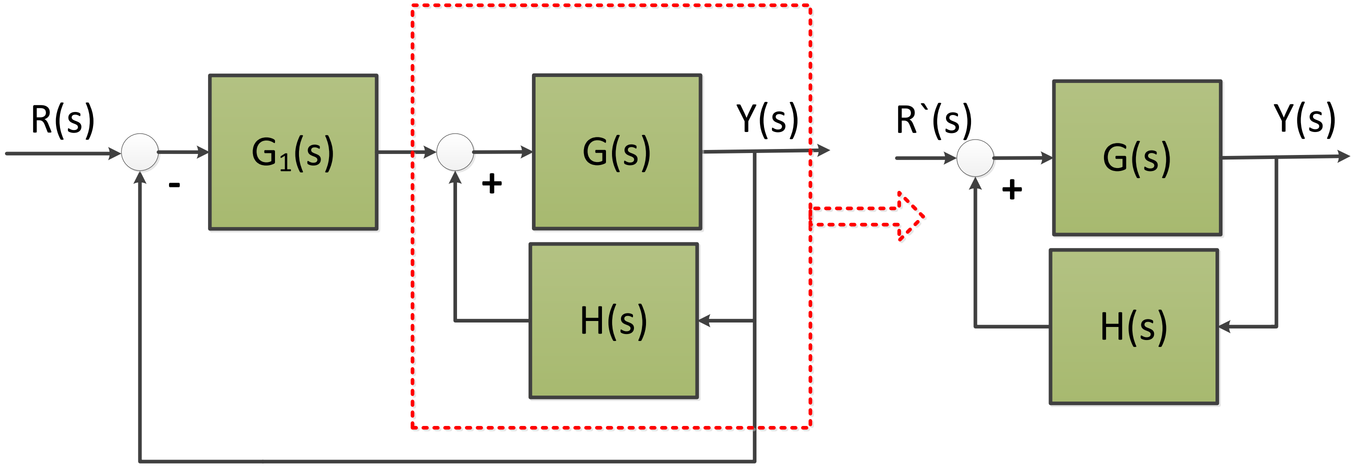
$$5s^2+(5a+1)s+5=0,$$

$$1+\frac{5as}{5s^2+s+5}=0,\ \text{so we can get } G(s)=\frac{5as}{5s^2+s+5}=\frac{as}{s^2+0.2s+1}$$

In this way the parameter root loci system is switched into regular root loci system

# 7.5 Zero-degree Root Locus

For some system, elements as shown in the following figure exist




Then CE would be 1-G(s)H(s)=0. Set H(s)=1 , highlight k, and then we get 1-kG(s)=0, kG(s)=1. The magnitude condition and the phase angle condition would be as follows

# 7.5 Zero-degree Root Locus

$$\begin{cases} |kG(s)| = 1 \\ \angle kG(s) = 0^\circ + 2l\pi \ \ (l = 0,\ \ 1,\ \ 2\ \ \cdots) \end{cases}$$

where the magnitude condition remains and the argle condition changes. Therefore, all the root locus concepts about phase angle condition change.

① real loci. If the total number of real poles and zeros to the right of a search point on the real axis is even, the point lies on the Root Locus.

180 degree: odd; 0 degree: even

② asymptote angle

180 degree $\varphi_A = \dfrac{2l+1}{n-m}\pi \qquad l = 0, 1, 2\cdots(n-m-1)$

0 degree $\varphi_A = \dfrac{2l}{n-m}\pi \qquad l = 0, 1, 2\cdots(n-m-1)$

# 7.5 Zero-degree Root Locus

③ departure angle

180 degree
$$\theta_{pk} = \pi + \left(\sum_{i=1}^{m} \varphi_{z_i p_k} - \sum_{\substack{j=1 \\ j\neq k}}^{n} \theta_{p_j p_k}\right)$$

0 degree
$$\theta_{pk} = 0 + \left(\sum_{i=1}^{m} \varphi_{z_i p_k} - \sum_{\substack{j=1 \\ j\neq k}}^{n} \theta_{p_j p_k}\right)$$

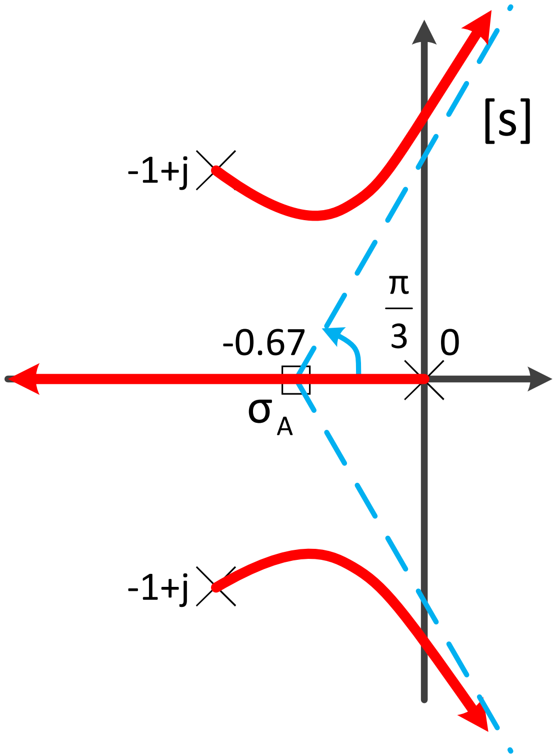
Eg: $G(s) = \dfrac{-k(s+6)(s-6)}{s(s+3)(s+4+4j)(s+4-4j)}$ CE: $1+G(s)=0$

$$1+\frac{-k(s+6)(s-6)}{s(s+3)(s+4+4j)(s+4-4j)} = 0 \qquad \angle kG(s) = 0^\circ + 2l\pi$$

# 7.6 Effect of Additional Zero

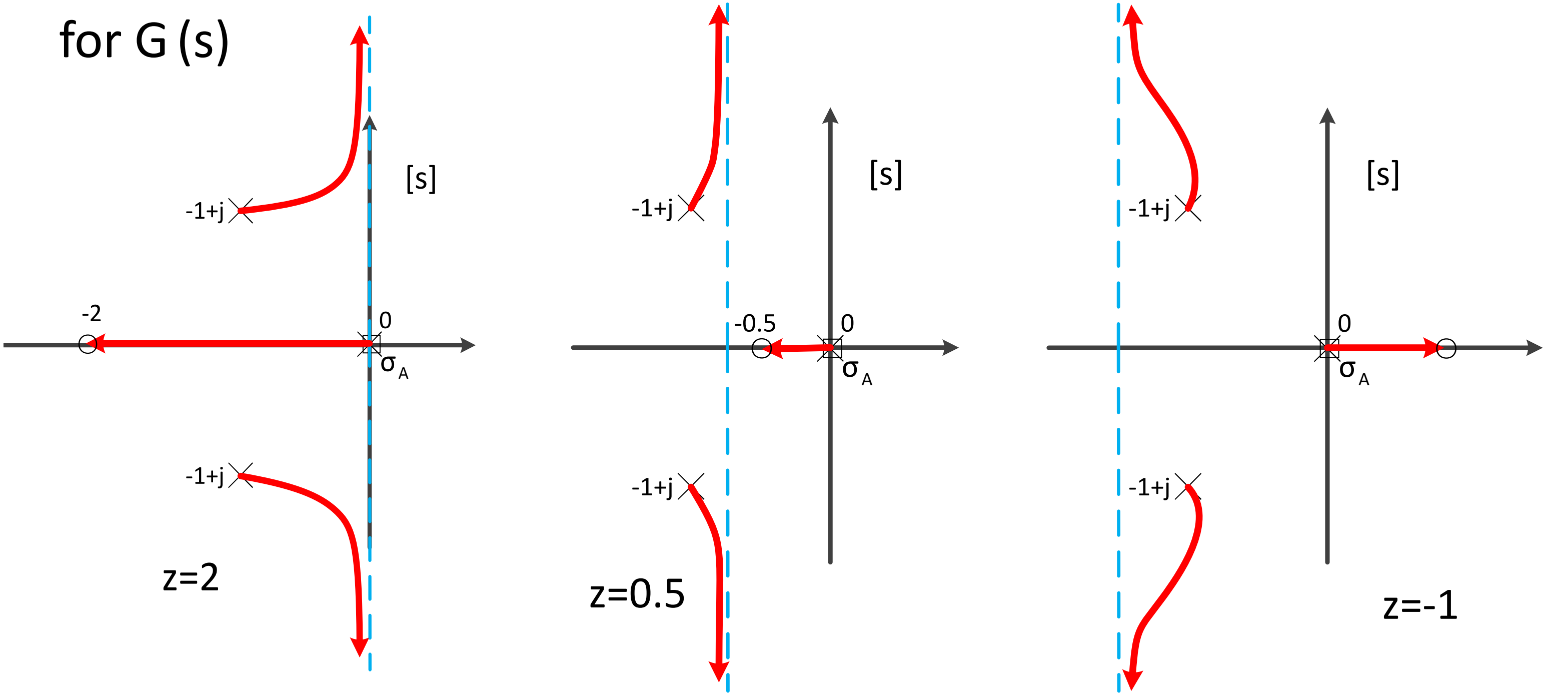
Eg: $G(s)=\dfrac{k(s+z)}{s(s^2+2s+2)}$ $\qquad G_2(s)=\dfrac{k}{s(s^2+2s+2)}$

for $G_2(s)$



# 7.6 Effect of Additional Zero

for G (s)




Along with the value of z decreases, the asymptotes move from the right plane to the left, and the system is more stable

# 7.6 Effect of Additional Zero

Notes:① the additional negative open-loop zero makes the root locus move to the left half of s-plane;

② the stability is increased;

③ if system is stable for any k, we have

$$\sigma_A = \frac{\sum ps - \sum zs}{n-m} \le 0$$

# 7.7 Minimum Phase System

When all the open-loop zeros and poles are located on the left hand of s-plane, the system is called minimum phase system

$$G(s) = \frac{k(s+1)}{s(s+4)}$$

If a zero or a pole is located on the right hand of the s-plane, this system is called non-minimum phase system

$$G_1(s) = \frac{k(s-1)}{s(s+4)} \quad G_2(s) = \frac{k(1-s)}{s(s+4)}$$

$$G_3(s) = \frac{k(s+1)}{s(s-4)} \quad G_4(s) = \frac{k(s+z)}{s(4-s)}$$