Control Laboratory

Non-linear control system

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Review

- What is the nonlinearity, nonlinear control system?
- The typical nonlinearity.
- The characteristics of the nonlinear control system.
- The analysis methods of the nonlinear control system:

$$\text{Classical methods:} \begin{cases} \text{describing function} \\ \text{phase plane} \end{cases}$$

10.1 Describing function of nonlinear system

Four items:

1. What is the describing function? } ← (modeling)
2. How to get the describing function? }
3. How to analyze a nonlinear system by describing function? ← (analysis and design)
4. Attentions and development

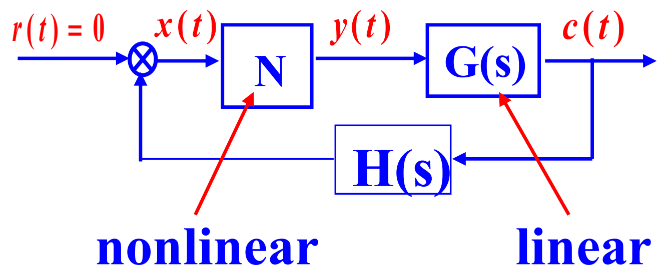
1 What is the describing function?

(Put forwarded by P.J.Daniel, In 1940)

□ 1. Basic idea

For the nonlinear system

Describing function



$x(t) = A\sin wt$

- **Assumptions**
  - No input, r(t) = 0
  - Linear part acts as a low-pass filter, that is, higher order harmonic components are damped
  - Nonlinearity does not generate sub-harmonics
  - Nonlinearity is symmetric
  - Nonlinearity does not depend on frequency
  - Assume that at point A, e(t) = A sin(wt)

10.2 What is the describing function?

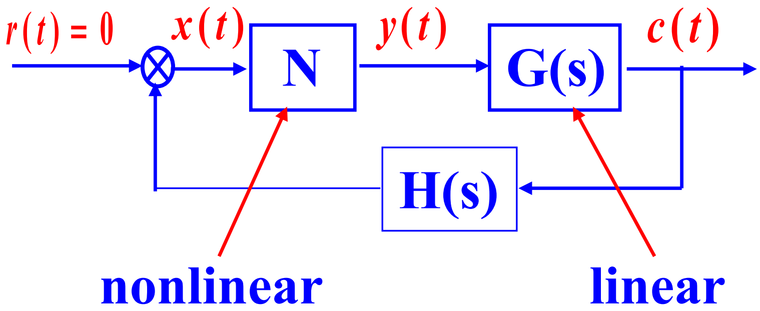


Fig.1 Typical structure of the nonlinear systems

$$x(t) = A\sin wt \Longrightarrow$$ **a sinusoidal input,**

$y(t)$, maybe it is not a sinusoidal but a periodic function, can be expressed as a Fourier series:

$$y(t) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\omega t + B_n \sin n\omega t)$$

$$= A_0 + \sum_{n=1}^{\infty} Y_n \sin(n\omega t + \varphi_n)$$

10.2 What is the describina function?

$$y(t) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\omega t + B_n \sin n\omega t)$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} y(t)d(\omega t)$$

$$= A_0 + \sum_{n=1}^{\infty} Y_n \sin(n\omega t + \varphi_n)$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos n\omega t d(\omega t)$$

$$Y_n = \sqrt{A_n^2 + B_n^2}, \quad \varphi_n = arctg\frac{A_n}{B_n}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin n\omega t d(\omega t)$$

Discuss:

i) For the symmetry nonlinearity: $A_0 = 0$, **and**

ii) the high-order harmonic of *y*(*t*) are neglected, then:

$y(t) \approx Y_1 \sin(\omega t + \varphi_1) \Longrightarrow$ **output frequency is equal to input frequency approximately.**

10.2 What is the describing function?

It means:

We can describe the nonlinear components by frequency response.

Definition of the describing function

The describing function $N(A)$ of the nonlinear element is: the complex ratio of the fundamental component of the output $y(t)$ and the sinusoidal input $x(t)$, that is:

For $x(t) = A\sin wt$

$$y(t) \approx A_1 \cos wt + B_1 \sin wt = Y_1 \sin(wt + \varphi_1) \Longrightarrow N(A) = \frac{Y_1 e^{j\varphi_1}}{A}$$

Here:

10.2 What is the describing function?

$$Y_1 = \sqrt{A_1^2 + B_1^2}$$

$$\varphi_1 = arctg\frac{A_1}{B_1}$$

$$\leftarrow \begin{cases} A_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} y(t)\cos\omega t\, d(\omega t) \\ B_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} y(t)\sin\omega t\, d(\omega t) \end{cases}$$

- **Because the describing function actually is the linearized "frequency response" → "harmonic linearization".**
- ***The nonlinear systems is analyzed did by Frequency Response.***

10.2 How to get the describing function?

1. Steps

(1) Input a sinusoid signal $x(t)$ to the nonlinear elements:

$$x(t) = A\sin wt$$

10.2 How to get the describing function?

(2) Solve $y(t)$ and obtain the fundamental component of $y(t)$.

(3) Calculate describing function $N(A)$

$$\left.\begin{aligned} A_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\cos\omega t\, d(\omega t) \\ B_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t\, d(\omega t) \end{aligned}\right\} \left.\begin{aligned} Y_1 &= \sqrt{A_1^2 + B_1^2} \\ \varphi_1 &= arctg\frac{A_1}{B_1} \end{aligned}\right\} y(t) \approx Y_1 \sin(\omega t + \varphi_1)$$

$$\Downarrow$$

$$N(A) = \frac{Y_1}{A} e^{j\varphi_1}$$

A 正弦信号的幅值

10.2 How to get the describing function?

Example 1

The mathematical description of a nonlinear device is:

$$y = \frac{1}{2}x + \frac{1}{4}x^3$$

Determine the describing function of the device.

Solution

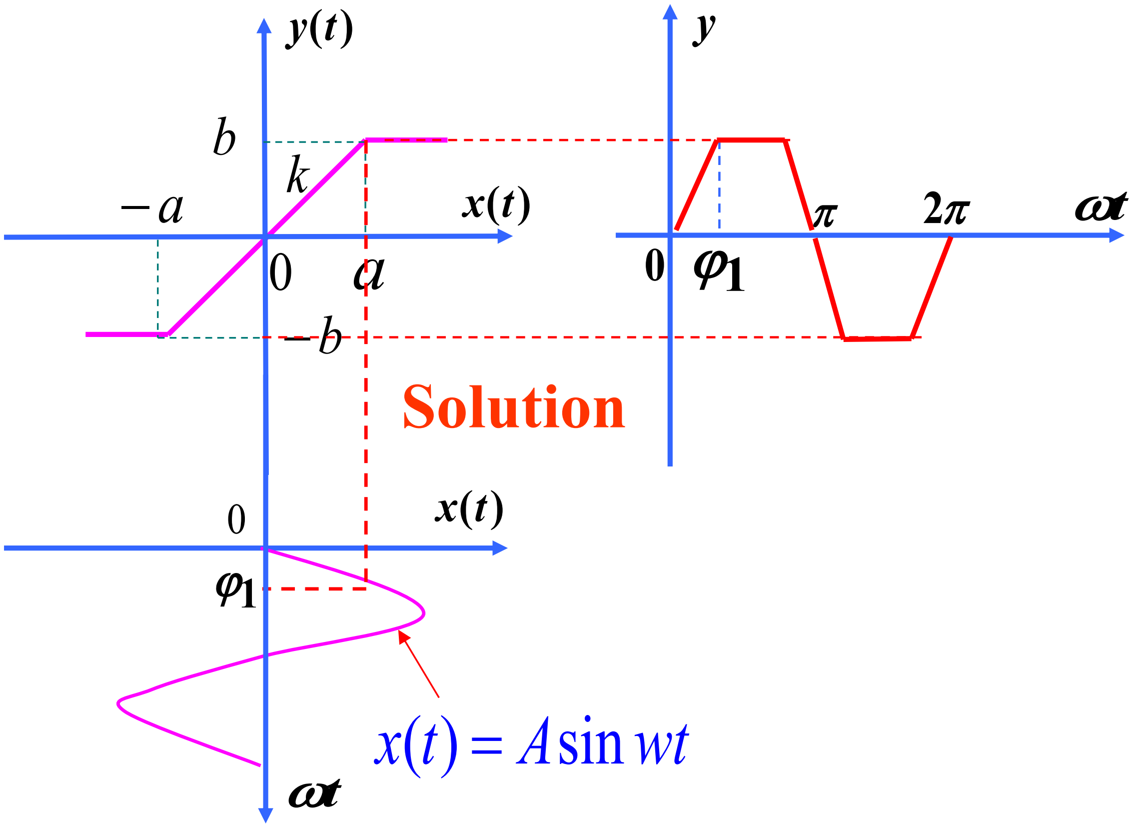
$$y(t) = \frac{1}{2}x + \frac{1}{4}x^3 \Big|_{x = A\sin wt}$$

$$= (\frac{1}{2}A + \frac{3}{16}A^3)\sin wt - \frac{1}{16}A^3 \sin 3wt$$

$$y_1(t) = (\frac{1}{2}A + \frac{3}{16}A^3)\sin wt \longrightarrow N(A) = \frac{\dot{y}_1}{\dot{x}} = \frac{1}{2} + \frac{3}{16}A^2$$

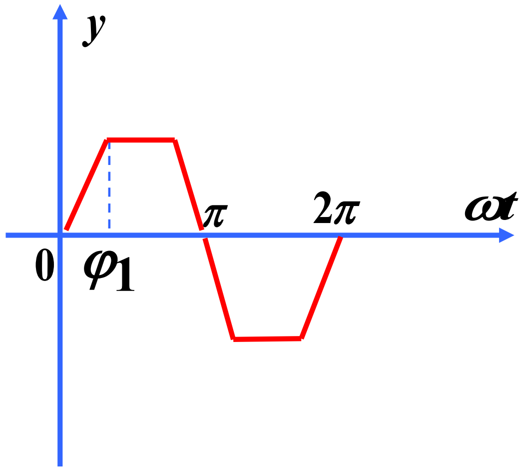
10.2 How to get the describing function?

Example 2 **Determine the describing function of the saturation nonlinearity.**



Solution

10.2 How to get the describing function?



$$y(t)=\begin{cases} kA\sin wt & 0\le wt\le \varphi_1 \\ kA & \varphi_1 < wt \le \dfrac{\pi}{2} \end{cases}$$

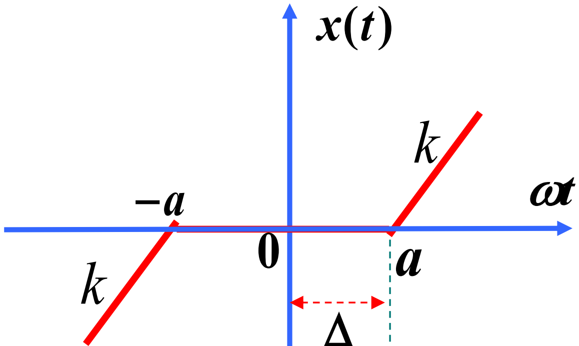
$$A_1 = 0$$

$$B_1=\frac{1}{\pi}\int_0^{2\pi} y(t)\sin wt\,d(wt)=\frac{2kA}{\pi}\left[\arcsin\frac{a}{A}+\frac{a}{A}\sqrt{1-(\frac{a}{A})^2}\right]$$

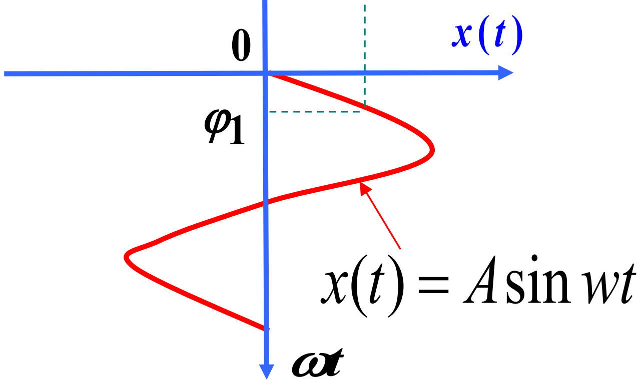
$$N(A)=\frac{B_1}{A}=\frac{2k}{\pi}\left[\arcsin\frac{a}{A}+\frac{a}{A}\sqrt{1-(\frac{a}{A})^2}\right] \qquad A>a$$

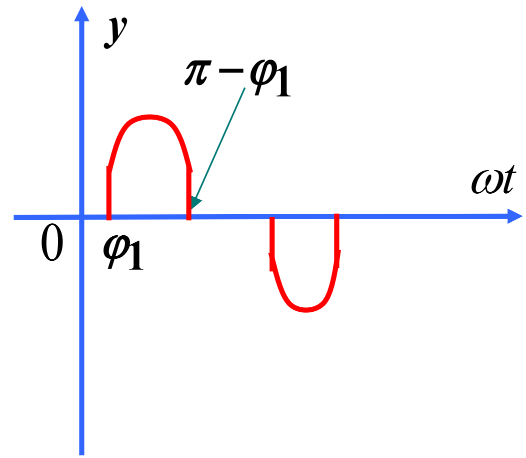
10.2 How to get the describing function?

Example 3 **Determine the describing function of the dead zone nonlinearity.**



solution

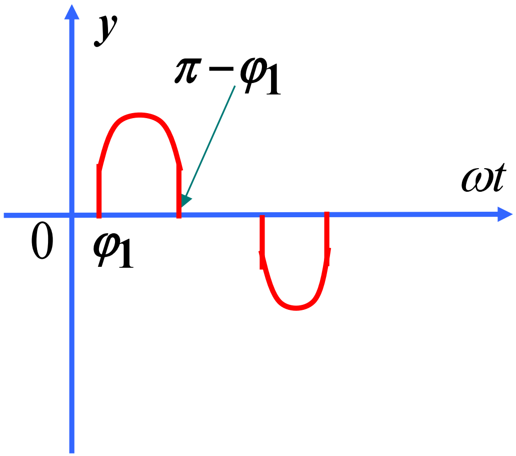




$$\varphi_1 = \arcsin\frac{\Delta}{A}$$

10.2 How to get the describing function?

$$y(t) = \begin{cases} 0 & 0 \le wt \le \varphi_1 \\ k(A\sin wt - \Delta) & \varphi_1 < wt \le \dfrac{\pi}{2} \end{cases}$$



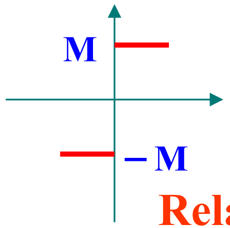
$$A_1 = 0$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin wt d(wt) = \frac{2kA}{\pi}\left[\frac{\pi}{2} - \arcsin\frac{\Delta}{A} - \frac{\Delta}{A}\sqrt{1-(\frac{\Delta}{A})^2}\right] \quad A \ge \Delta$$

$$N(A) = \frac{B_1}{A} = \frac{2k}{\pi}\left[\frac{\pi}{2} - \arcsin\frac{\Delta}{A} - \frac{\Delta}{A}\sqrt{1-(\frac{\Delta}{A})^2}\right] \quad A \ge \Delta$$

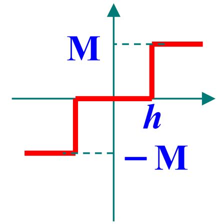
10.2 How to get the describing function?

3. The describing function of some typical nonlinearity

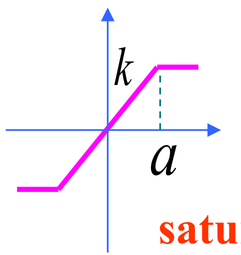


Relay

$$N(A)=\frac{4M}{\pi A}$$

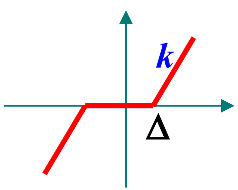


$$N(A)=\frac{4M}{\pi A}\sqrt{1-\left(\frac{h}{A}\right)^2}$$



saturation

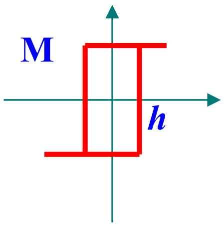
$$N(A)=\frac{2k}{\pi}\left[\arcsin\frac{a}{A}+\frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right] \qquad A>a$$



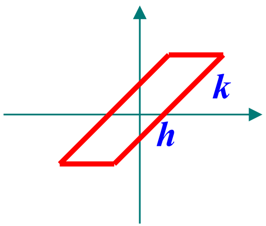
dead zone

$$N(A)=\frac{2k}{\pi}\left[\frac{\pi}{2}-\arcsin\frac{\Delta}{A}-\frac{\Delta}{A}\sqrt{1-\left(\frac{\Delta}{A}\right)^2}\right] \qquad A\geq\Delta$$

10.2 describing function of typical nonlinearity



$$N(A)=\frac{4M}{\pi A}\sqrt{1-(\frac{h}{A})^2}-j\frac{4M}{\pi A^2}$$



backlash
hysteresis

$$N(A)=\frac{k}{\pi}\left[\frac{\pi}{2}+\arcsin(1-\frac{2h}{A})+2(1-\frac{2h}{A})\sqrt{\frac{h}{A}(1-\frac{h}{A})}\right]+j\frac{4kh}{\pi A}(\frac{h}{A}-1)$$

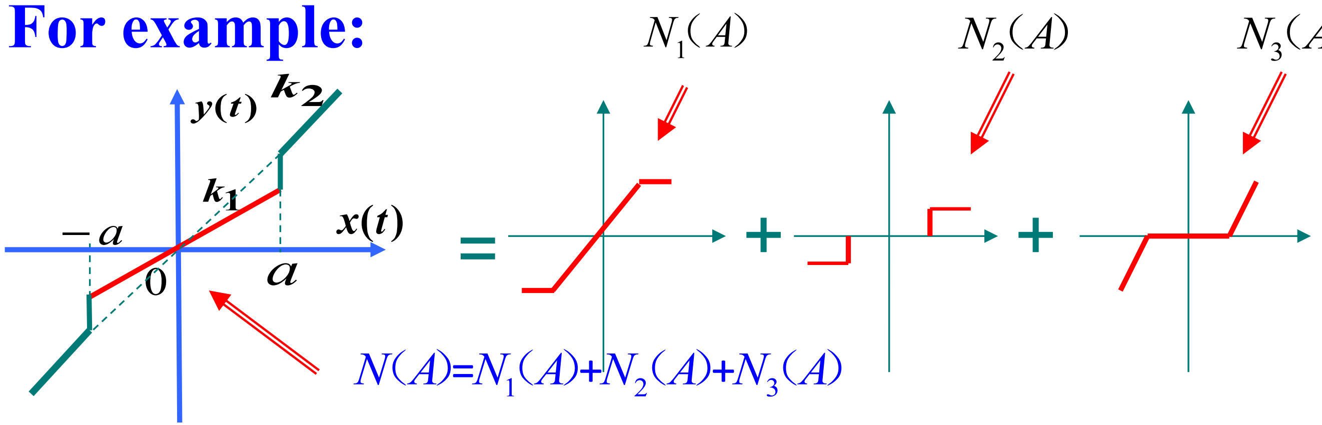
4. characters

(1) For the "single value" nonlinearity, the describing function is a "real number".

such as the dead zone, saturation and the ideal relay nonlinearity etc.

(2) The describing function satisfy the superposition principle (nonlinearity not).

For example:



$N(A) = N_1(A) + N_2(A) + N_3(A)$

10.2 Stability of the nonlinear system

1. Review of Nyquist criterion

For the linear system:

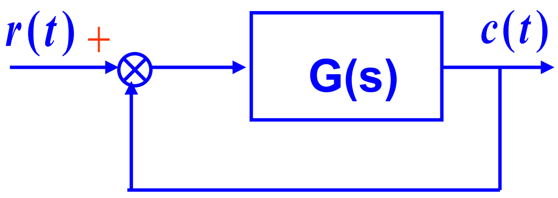


Fig.1

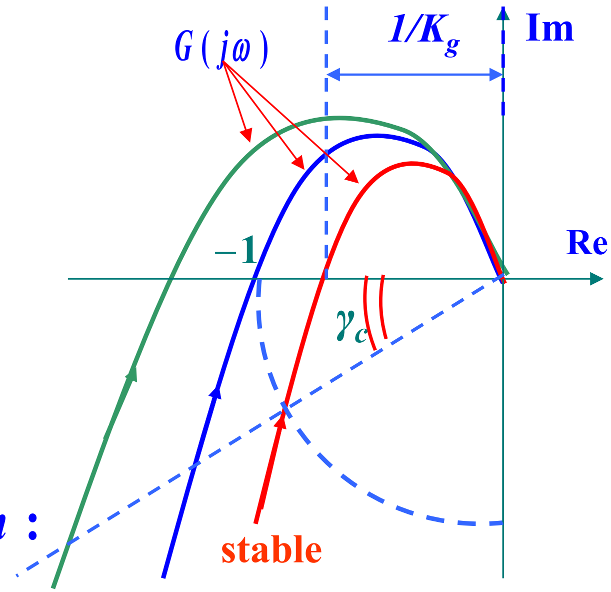


Fig. 2

The characteristic equation of the system :

$$1 + G(j\omega) = 0$$

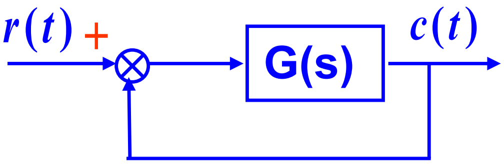
$$\Rightarrow G(j\omega) = -1 + j0$$

If $G(s)$ is a minimum phase transfer function, the necessary and sufficient condition of the stable system is :

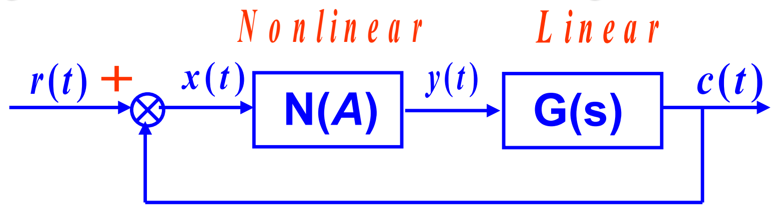
$G(j\omega)$ *does not circle the point* $(-1, j\omega)$

10.2 Stability of the nonlinear system

2. Compare the nonlinear system with the linear system



Linear system



nonlinear system

Transfer function of the system:

$$T(s)=\frac{C(s)}{R(s)}\Rightarrow T(jw)=\frac{C(jw)}{R(jw)}=\frac{G(jw)}{1+G(jw)}$$

$$T(jw)=\frac{C(jw)}{R(jw)}=\frac{N(A)G(jw)}{1+N(A)G(jw)}$$

Characteristic equation:

$$1+G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-1$$

$$1+N(A)G(jw)=0$$

$$G(jw)=-\frac{1}{N(A)}$$

In the $G(j\omega)$ plane

A point

A curve

Because Describing function N(A) is a linearized frequency response, we can expand Nyquist Criterion to the nonlinear system.

10.2 Stability of the nonlinear system

(For example the minimum phase system)

compare with linear system

(1)G(jw) don't circle the $-\dfrac{1}{N(A)}$ curve, the nonlinear system is stable

(2)G(jw) circle the $-\dfrac{1}{N(A)}$ curve, the nonlinear ststem is unstable.

(3)G(jw) intesect with the $-\dfrac{1}{N(A)}$ curve, There is a self-oscillation in the nonlinear system.

(1) $G(j\omega)$ don' t circle the point $(-1, j\omega)$, the system is stable;

(2) $G(j\omega)$ circle the point $(-1, j\omega)$, the system is unstable;

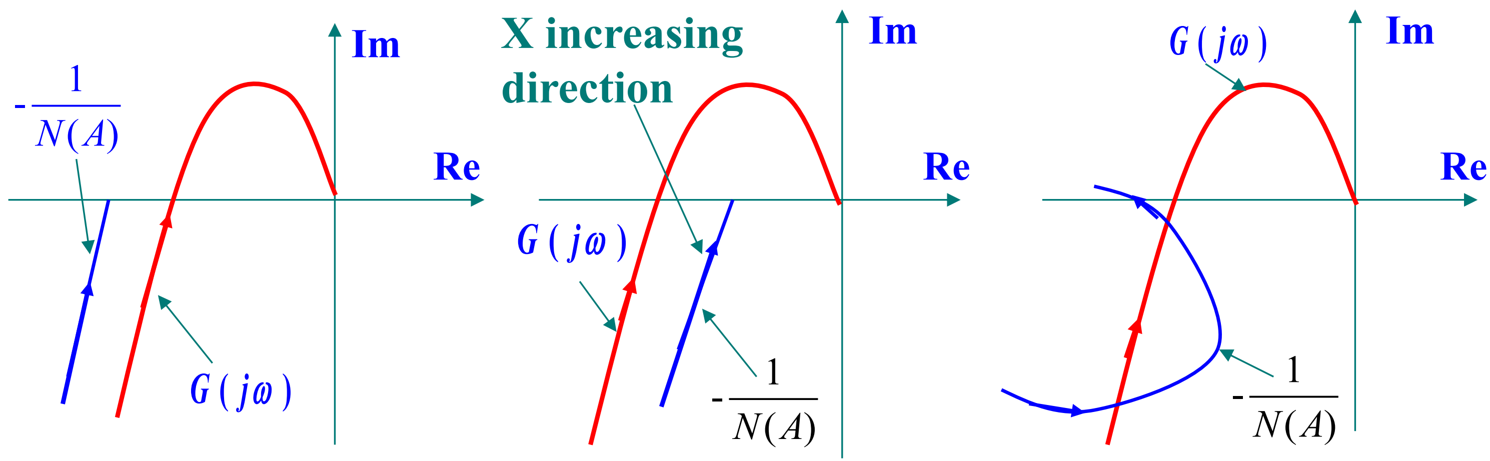
(3) $G(j\omega)$ intersect with the point $(-1, j\omega)$, the system is in the critical stability.

10.2 Stability of the nonlinear system

(For example the minimum phase system)

$-\dfrac{1}{N(A)}$ **:Negative inverse Describing function**

Graphical explanation is shown as following:



$G(j\omega)$ do not circle －1/N(A) (stable)	$G(j\omega)$ circle －1/N(A) (unstable)	$G(j\omega)$ Intersect with －1/N(A) (self-oscillation)

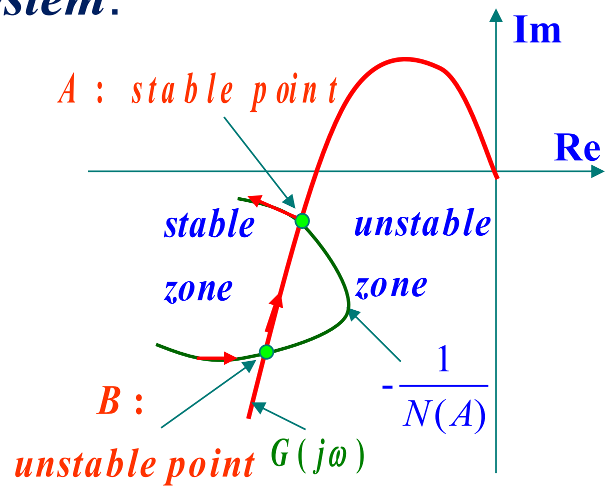
10.2 Stability of the nonlinear system

4. Self-oscillation of the nonlinear system

A special motion of the nonlinear system:

System will be at a continuous oscillation, which has a constant amplitude and frequency, when the system come under a light disturbance.

Correponding to the intersection point of G(jw) with $-\dfrac{1}{N(A)}$



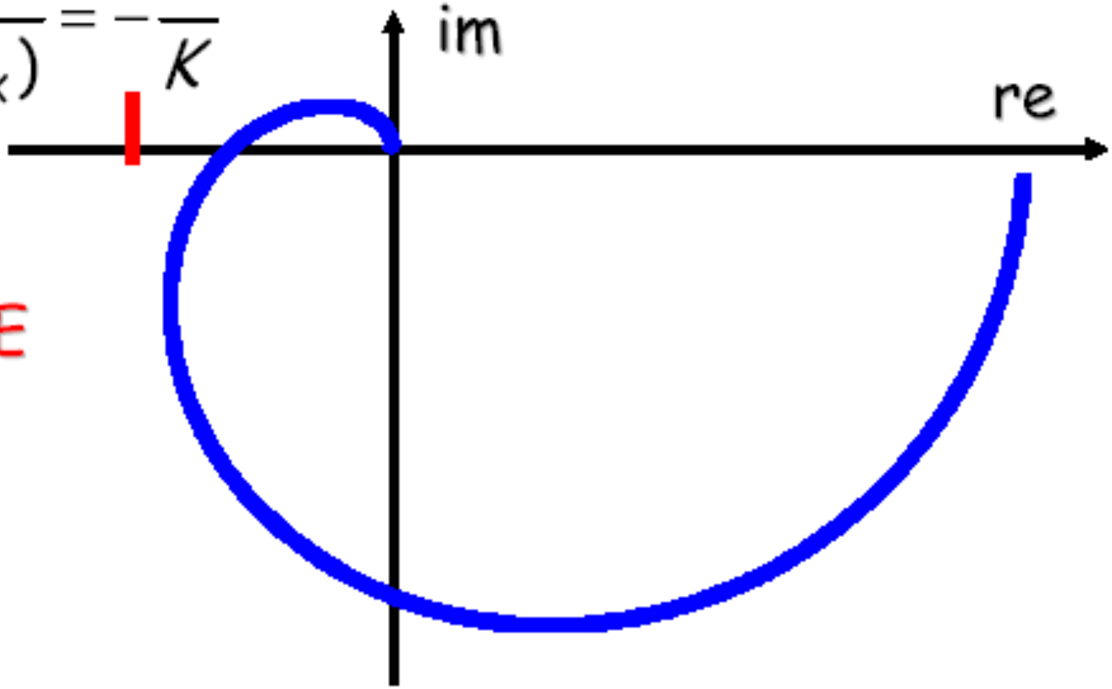
Self-oscillation

B: *unstable self-oscillation point*→*–1/N(A)*enter unstable zone from stable zone.

A: *stable self-oscillation point*→*–1/N(A)* enter stable zone from unstable zone.

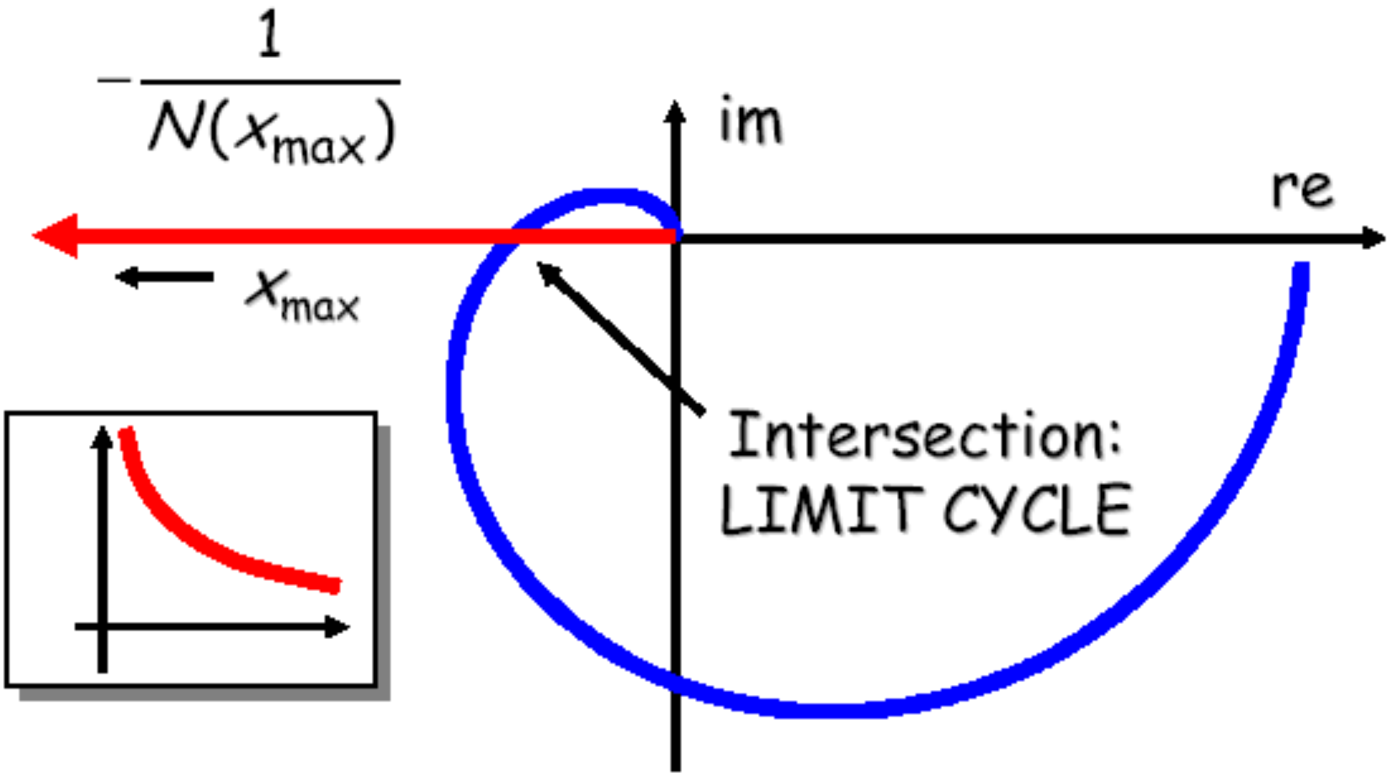
Example (gain)

$$-\frac{1}{N(x_{max})} = -\frac{1}{K}$$

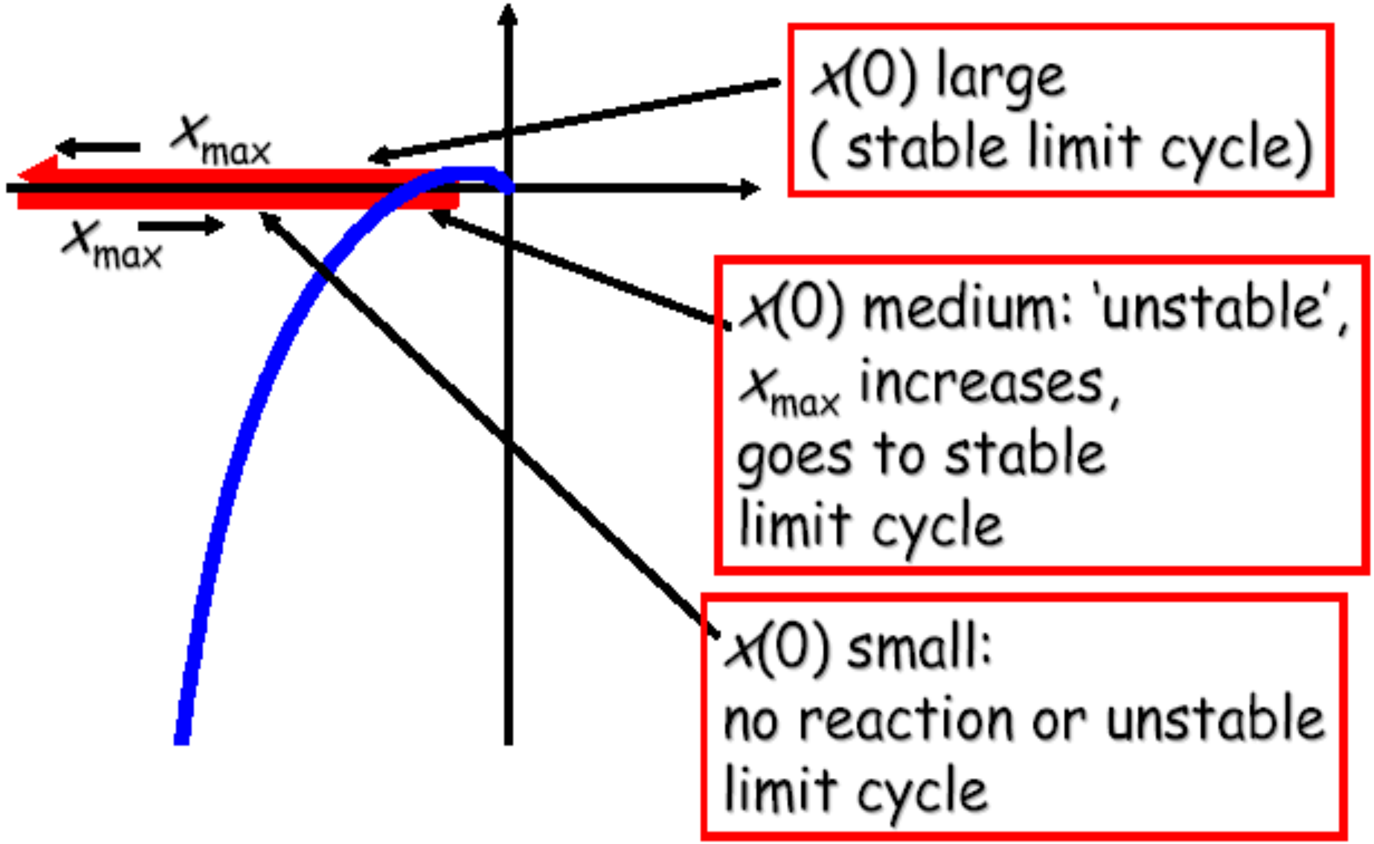


STABLE

Example (relay)

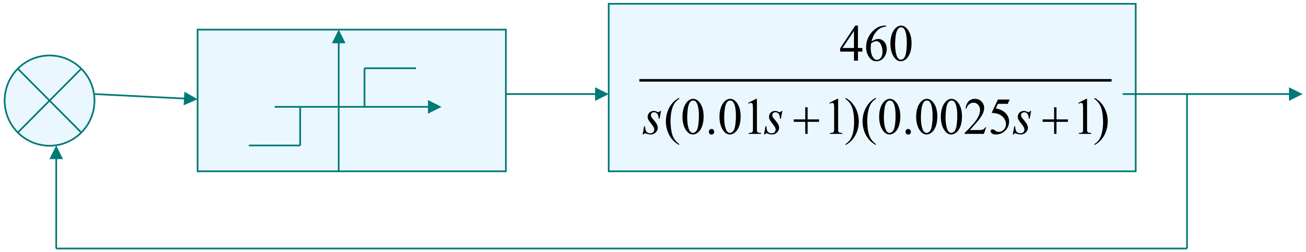


Example (relay +dead zone)



Describing function-Example

E.g the system is shown as follows



the non-linearity is

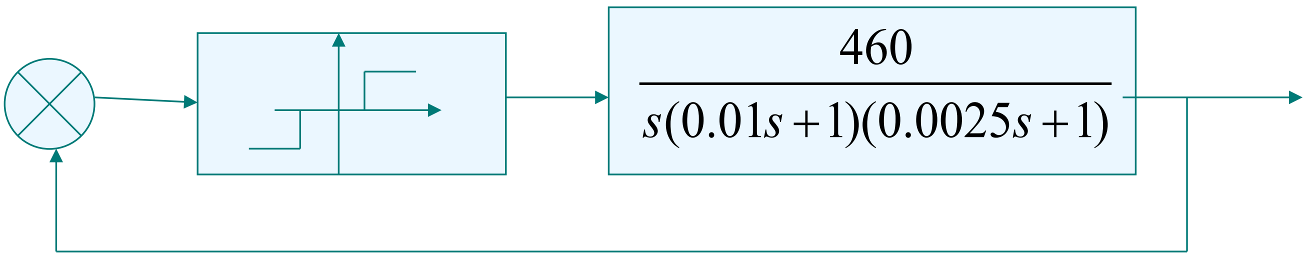
$$N(A) = \frac{4M}{\pi A}\sqrt{1-(\frac{h}{A})^2} \qquad A \geq h\,,\ M=1$$

$$\frac{-1}{N(A)} = \frac{-\pi A}{4M\sqrt{1-(\frac{h}{A})^2}} \qquad A \geq h$$

Let $\frac{h}{A} = u$ $\quad \frac{-1}{N(A)} = \frac{-\pi h}{4Mu\sqrt{1-u^2}} \qquad A \geq h$

Describing function-Example

E.g the system is shown as follows



the non-linearity is

$$N(A)=\frac{4M}{\pi A}\sqrt{1-\left(\frac{h}{A}\right)^2} \qquad A\geq h,\ M=1$$

$$\frac{-1}{N(A)}=\frac{-\pi A}{4M\sqrt{1-\left(\frac{h}{A}\right)^2}} \qquad A\geq h$$

Let $\frac{h}{A}=u$

$$\frac{-1}{N(A)}=\frac{-\pi h}{4Mu\sqrt{1-u^2}} \qquad A\geq h$$

Describing function-Example

E.g the system is shown as follows
the non-linearity is

$$G(s)=\frac{460}{s(0.01s+1)(0.0025s+1)}$$

$$G(jw)=\frac{460}{jw(j0.01w+1)(j0.0025w+1)}$$

$$|G(jw)|=\frac{460}{w\sqrt{1+\frac{w^2}{100^2}}\sqrt{1+\frac{w^2}{400^2}}}$$

$$\angle G(jw)=-90-\arctan 0.01w-\arctan 0.0025w$$

$$\angle G(jw)=-180^\circ$$

$$\arctan 0.01w+\arctan 0.0025w=90^\circ \quad 0.01w=\frac{1}{0.0025w} \quad w=200$$

Describing function-Example

E.g the system is shown as follows
the non-linearity is

$$G(jw)=\frac{460}{jw(j0.01w+1)(j0.0025w+1)}$$

$$M=2,\quad h=1$$

$$\frac{-1}{N(A)}=\frac{-\pi h}{4Mu\sqrt{1-u^2}}=\frac{-\pi h}{4Mu\sqrt{1-u^2}}\qquad A\geq h$$

$$\frac{-1}{N(A)}=-\left|G(jw)\right|_{w=200}=\frac{23}{25}$$

$$\frac{-1}{N(A)}=\frac{-\pi h}{4Mu\sqrt{1-u^2}}=\frac{-\pi A}{8\sqrt{1-\frac{1}{A^2}}}=\frac{-23}{25}$$

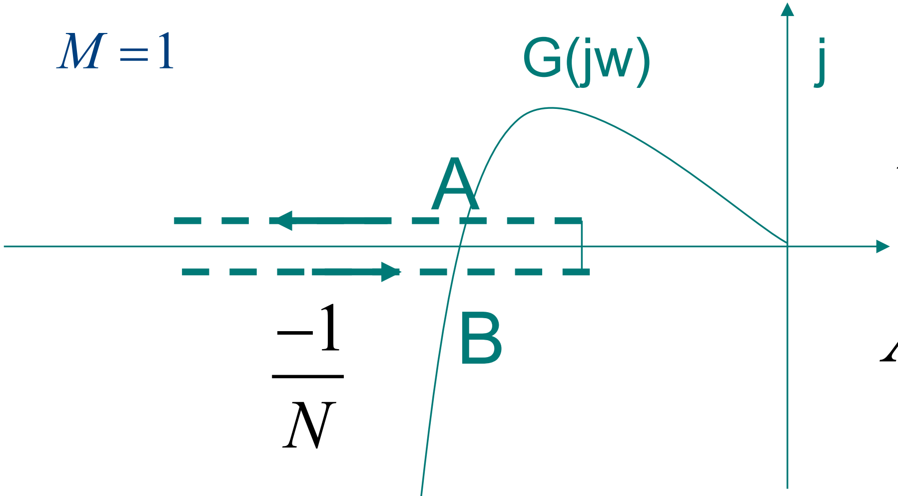
$$A_1=2.04(self\text{-}oscillation)\qquad A_2=1.15$$

Describing function-Example

$$\frac{-1}{N(A)} = \frac{-\pi h}{4Mu\sqrt{1-u^2}} \qquad A \geq h$$

point A has a stable periodic oscillation.

$M = 1$



$B: 1.15\sin 200t$

$A: 2.04\sin 200t$

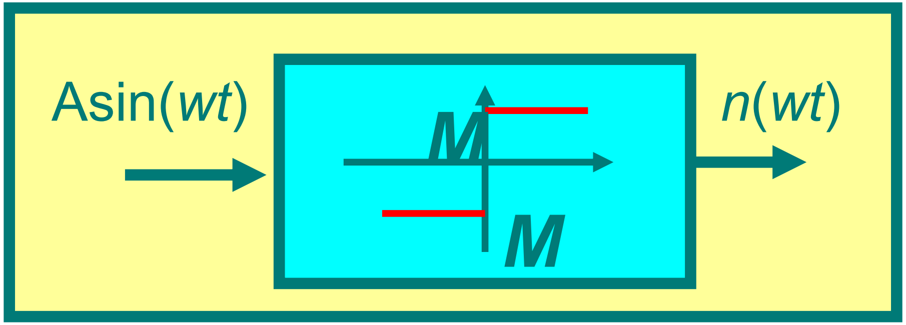
$u=\frac{h}{A}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\frac{-1}{N(A)}$	-7.9	-4.2	-2.7	-2.1	-1.8	-1.6	-1.5	-1.6	-2	$-\infty$

Describing function-Example

- Linear part

$$G(jw) = \frac{e^{-jw}}{1 + jw}$$

- Nonlinear part - relay



Example

- Nonlinear part- replace it with describing function

$$\boxed{N(\mathrm{A}) = \frac{4M}{\pi \mathrm{A}}}$$

- Criterion

$$\boxed{G = -\frac{1}{N} \Rightarrow \frac{e^{-jw}}{1+jw} = -\frac{\pi \mathrm{A}}{4}}$$

Describing function-Example

$$G = -\frac{1}{N} \Rightarrow \frac{e^{-jw}}{1+jw} = -\frac{\pi A}{4}$$

- Two equations to solve unknowns *A* and *w*

$$\frac{1}{\sqrt{1+w^2}} \angle -w - \tan^{-1} w = -\frac{\pi A}{4}$$

$$\frac{1}{\sqrt{1+w^2}} = \frac{\pi A}{4}$$

$$\angle -w - \tan^{-1} w = -\pi$$

The solution is

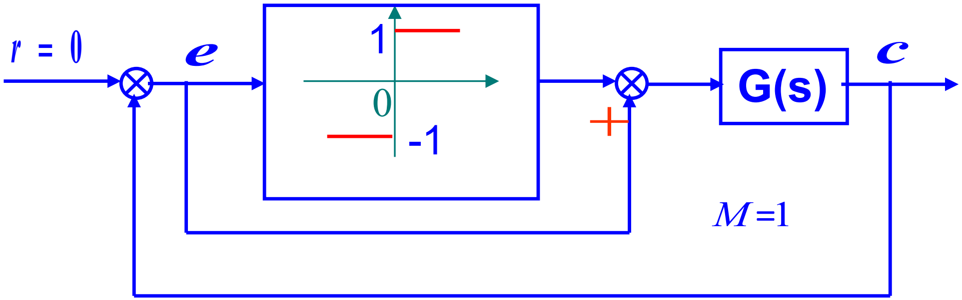
$w = 2.1$

$A = 0.57$

Example:

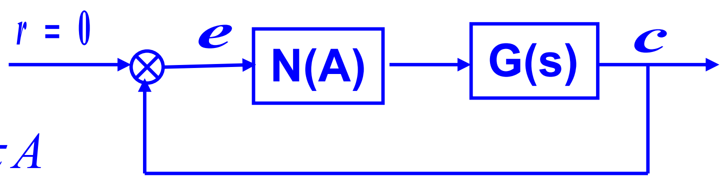
The nonlinear system is in Fig. 1. The describing function of Relay nonlinearity is $\frac{4M}{\pi A}$, $G(s)=\dfrac{K}{s(5s+1)(10s+1)}$

1. Determin the stability of the nonlinear system
2. Determin K and oscillation frequency when the amplitude of the self-oscillation is $A=\dfrac{1}{\pi}$



Solution:

The system is equivalent to



$$N(A)=1+\frac{4M}{\pi A}\Rightarrow -\frac{1}{N(A)}=-\frac{\pi A}{\pi A+4M}$$

Graphical explanation is shown as Fig.1

$$G(jw)=\frac{k}{jw(j5w+1)(j10w+1)}$$

$$-\frac{1}{N(A)}=-\frac{\pi A}{\pi A+4}\bigg|_{A=\frac{1}{\pi}}=-\begin{cases}0 & A=0\\ 1 & A=\infty\end{cases}$$

$$|G(jw)|=\frac{k}{w\sqrt{1+25w^2}\sqrt{1+100w^2}}$$

$$\angle G(jw)=-90-\arctan 5w-\arctan 10w$$

$$\angle G(jw)=-90-\arctan 5w-\arctan 10w=-180$$

$$\arctan 5w+\arctan 10w=90\Rightarrow 5w=1/10w$$

$$|G(jw)|=\frac{k}{w\sqrt{1+25w^2}\sqrt{1+100w^2}}\bigg|_{w=0.14}=\frac{10k}{3}$$

$$\omega=\sqrt{\frac{1}{50}}\approx 0.14$$

$$G(j\omega)=-\frac{10}{3}K$$

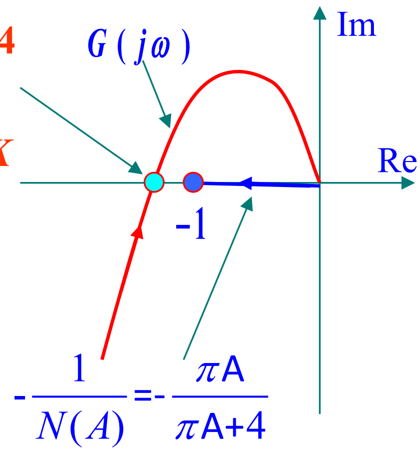


Fig.1

(1) Stability analysis $\begin{cases}K>\dfrac{3}{10}, & \text{unstable.}\\ K\le\dfrac{3}{10}, & \text{self-oscillation}\end{cases}$

(2) $$-\frac{1}{N(A)}=-\frac{\pi A}{\pi A+4}\bigg|_{A=\frac{1}{\pi}}=-\frac{1}{5}\qquad |G(jw)|=-\frac{1}{N(A)}\Rightarrow\frac{10k}{3}=\frac{1}{5}\qquad k=0.06$$

Example (relay +dead zone)

Eg1, A control system shows as follows, where a=3,b=1,

1.Determine if these system have self-excitation?

If it has, solve amplitude and frequency.

2.In case system doesn't have self-excitation, determine the parameters of the relay.

$$N(A) = \frac{4a}{\pi A}\sqrt{1 - (\frac{b}{A})^2}, \quad A \geq b$$

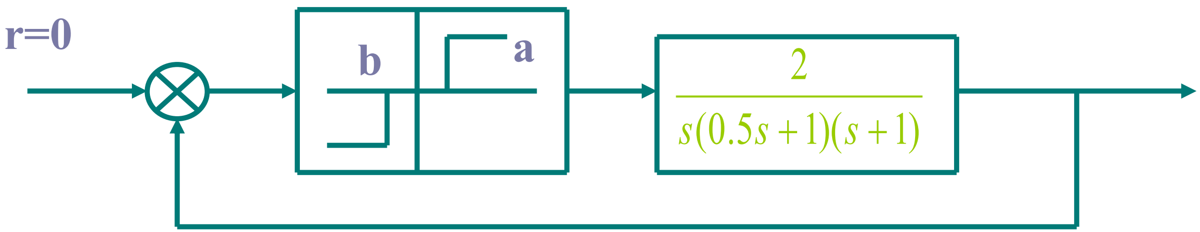


Fig. System with relay

Example (relay +dead zone)

Solve : we know the description function

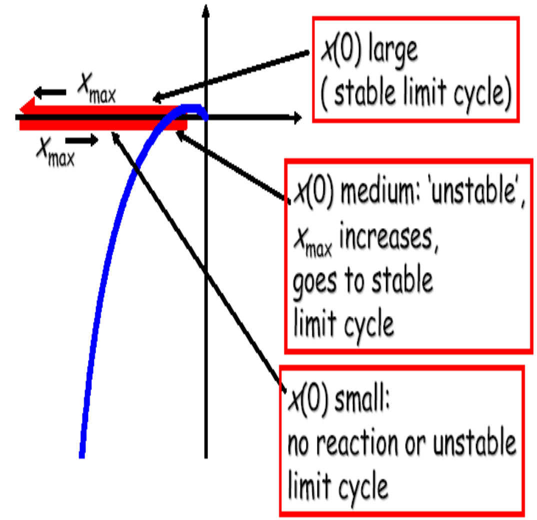
$$-\frac{1}{N(A)} = -\frac{\pi A}{4a\sqrt{1-\left(\frac{b}{A}\right)^2}}, \quad A \ge b \qquad -\frac{1}{N(A)} = \begin{cases} -\infty & A=b \\ -\infty & A \to \infty \end{cases}$$

Let $\boxed{\frac{b}{A} = u}$

$$-\frac{1}{N(A)} = -\frac{\pi a}{4u\sqrt{1-u^2}}, \quad u \le 1$$

$$\frac{d\frac{1}{N(A)}}{dA} = 0 \quad \Rightarrow \sqrt{1-u^2} = u \quad u \le 1$$

$$\Rightarrow u_{\max} = \frac{\sqrt{2}}{2} \Rightarrow A_{\max} = \sqrt{2}b$$



$$h \le A < A_{\max} \quad \Rightarrow \frac{d\left[-\frac{1}{N(A)}\right]}{dA} > 0$$

$$A \ge A_{\max} \quad \Rightarrow \frac{d\left[-\frac{1}{N(A)}\right]}{dA} < 0$$

$$-\frac{1}{N(A)} = -\frac{\pi b}{2a} = -\frac{\pi}{6} \approx -0.5236$$

Example (relay +dead zone)

$$|G(j\omega)| = \frac{2}{\mathrm{w}\sqrt{0.25w^2+1}\sqrt{w^2+1}}$$

$$\mathrm{Im}G(j\omega)=0 \Rightarrow \frac{2(1-0.5\omega^2)}{\omega(0.25\omega^4+1.25\omega^2+1)}=0$$

$$\angle G(j\omega) = \text{-}90 \text{ - } \arctan 0.5w \text{ - } \arctan w$$

Phase angle is zero

$$\angle G(j\omega) = \text{-}90 \text{ - } \arctan 0.5w \text{ - } \arctan w = \text{-}180 \quad \Rightarrow \omega = \sqrt{2}$$

Take $\omega = \sqrt{2}$ into the real part

$$\mathrm{Re}\, G(j\omega)\big|_{\omega=\sqrt{2}} = -\frac{1}{1.5} \approx -0.66$$

$$\mathrm{Re}\, G(j\omega)\big|_{\omega=\sqrt{2}} = \left|-\frac{1}{N(A)}\right| \Rightarrow \quad \frac{\pi A}{12\sqrt{1-\left(\frac{1}{A}\right)^2}} = \frac{1}{1.5}$$

Example (relay +dead zone)

We have $A1=1.11, A2=2.3$

So amplitude and frequency of self-exciting is 2.3 and $\omega=\sqrt{2}$ respectively.

2.If there isn't self-oscillation in the system, let

$$-\frac{1}{N(A)}\bigg|_{A=\sqrt{2}a} \le -\frac{1}{1.5} \quad \Rightarrow \frac{\pi b}{2a} > \frac{1}{1.5}$$

We have $\dfrac{a}{b} < 2.36$

10.2 Attentions and development

1.Attentions

- Using the describing function to analyze the nonlinear system, Linear parts of the system must be provided with a good charact-eristic of the low-pass filter→so that the harmonics produced by the nonlinear element can be neglected.
- Generally, the describing function method can only be used for analyzing the stability and self-oscillation of the nonlinear systems, not the stead-state error and transient specifications.

2. Development

Modern analysis and design method of the nonlinear systems: Computer simulation and intelligent design.

Purpose

- Describing function method for nonlinear systems is prediction of limit cycles.
- It is also used to design controllers that eliminate limit cycles and as a controller design tool.
- to predict sub harmonics, jump phenomena and response of nonlinear systems to sinusoidal inputs.