Control Laboratory

Chapter 6 The Stability of Linear Feedback Systems

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6.1 Stability

<u>Concept</u>

A stable system is a dynamic system with a bounded response to a bounded input.

- **Absolute stability**

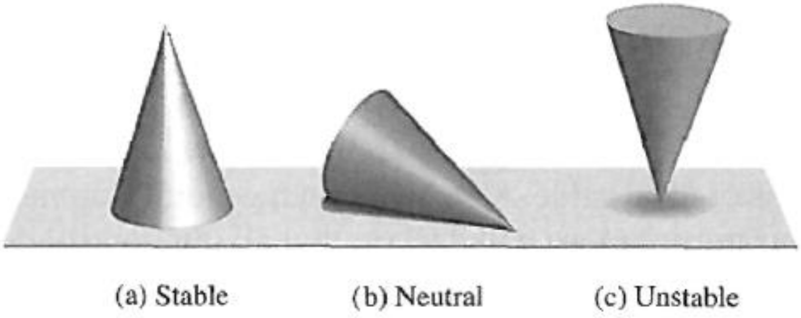
A system possessing absolute stability is called a stable system.

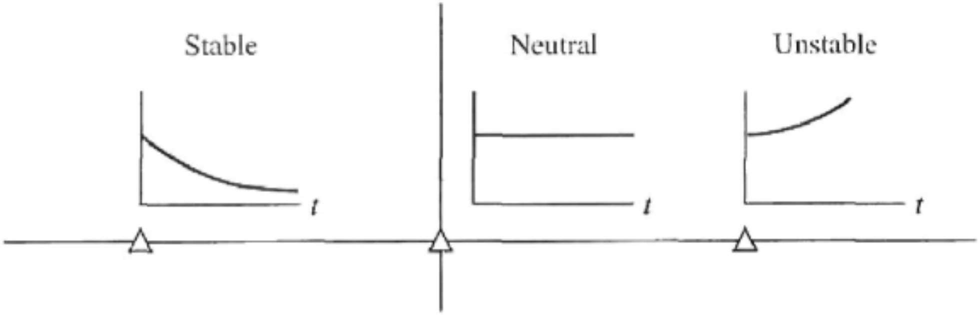
- **Relative stability**

Give that a closed-loop system is stable, we can further characterize the degree of stability, i.e., relative stability.

6.1 Stability

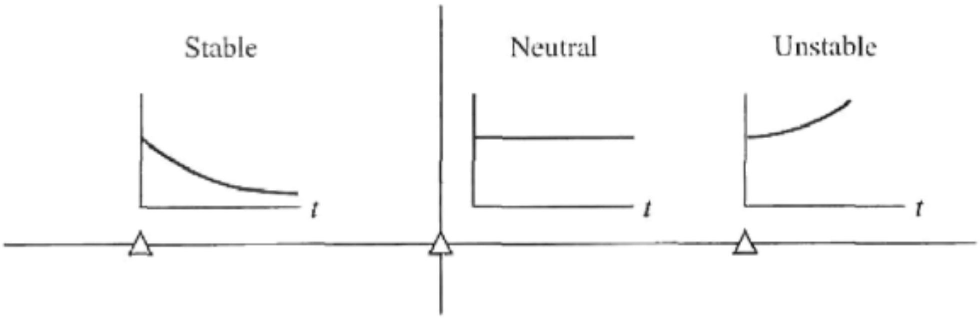
Example





6.1 Stability

- **A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.**



6.1 Stability

- **Stable:**

  All the poles of the transfer function in the left-hand s-plane.

- **Marginally Stable:**

  Some roots on the imaginary axis ($j\omega$ -axis)with all other roots in the left half-plane.

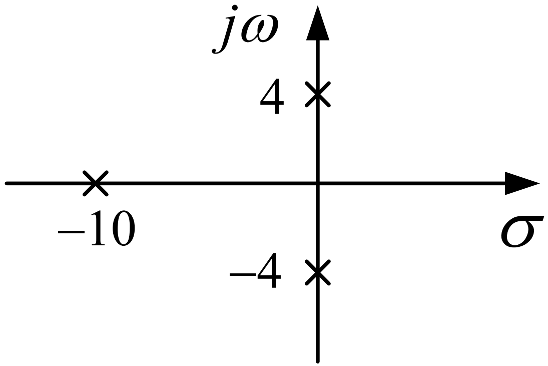
- **Unstable:**

  At least one root in the right half of the s-plane or repeated $j\omega$ roots

6.1 Stability

Example

$$q(s) = (s+10)(s^2+16) = 0$$



$s_1 = -10$

$s_{2,3} = \pm j4$

The system is marginally stable

6.2 Routh Criterion

The characteristic equation in the Laplace variable is written as

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

The necessary condition for the stability is that :

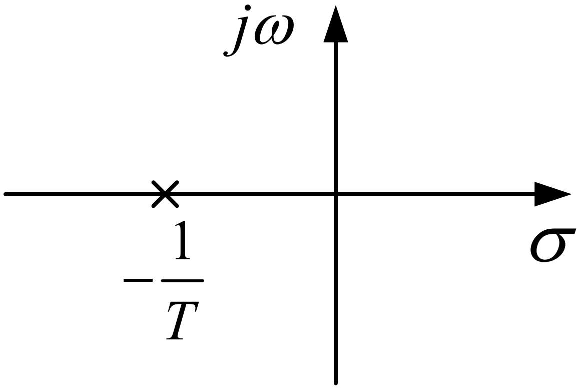
all the coefficients of the polynomial are nonzero and have the same sign.

6.2 Routh Criterion

Example

The first-order closed-loop characteristic equation

$$q(s) = Ts + 1 = 0 \qquad T > 0$$



$$s = -\frac{1}{T}$$

$$\begin{cases} a_0 = 1 > 0 \\ a_1 = T > 0 \end{cases}$$

The system is stable

6.2 Routh Criterion

Example

The second-order closed-loop characteristic equation:

$$q(s) = s^2 + as + b = 0 \qquad a > 0 \qquad b > 0$$

$$s_{1,2} = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}$$

$$a^2 > 4b, \frac{1}{2}\sqrt{a^2 - 4b} < \frac{1}{2}a, s < 0$$

$a^2 < 4b$, the roots have negative real part

The system is stable

6.2 Routh Criterion

Note

- For the first-order or the second-order system, a necessary and sufficient condition for the stability is that the characteristic equation possesses all positive coefficients.
- For the higher-order system, **the condition is necessary but not sufficient.**

6.2 Routh Criterion

For linear systems

CE: $q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$

Routh array

The first column

$$
\begin{array}{ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
s^0 & h_{n-1} & & &
\end{array}
$$

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n\text{-}1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

6.2 Routh Criterion

Routh Criterion:

- If there is changes in sign of the first column of the Routh-array, the system is unstable.
- The number of roots of q(s) with positive real parts is equal to the number of changes in sign of the first column of the Routh array.
- The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability.

Note:

- No change in sign in the first column

  → The system is stable.

- The number of changes in sign

  = the number of positive roots.

6.2 Routh Criterion

Example:

$$q(s) = a_2 s^2 + a_1 s + a_0 = 0$$

The Routh array is

$$a_2 > 0 \qquad a_1 > 0$$

$$\begin{array}{lll} s^2 & a_2 & a_0 \\ s^1 & a_1 & \\ s^0 & b_0 & \end{array}$$

$$b_0 = \frac{a_0 a_1 - a_2 \cdot 0}{a_1} = a_0 > 0$$

the system is stable.

- **Therefore, the requirement for a stable second-order system is that all the coefficients exist and positive.**

6.2 Routh Criterion

Case 1: No element in the first column is zero.

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

The Routh array is

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$\dfrac{a_2 a_1 - a_3 a_0}{a_2}$	
s^0	a_0	

If $\begin{cases} a_0, a_1, a_2, a_3 > 0 \\ a_2 a_1 > a_3 a_0 \end{cases}$

the system is stable.

6.2 Routh Criterion

Example:

$$q(s) = s^3 + s^2 + 2s + 24 = 0$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & \\ s^0 & 24 & \end{array}$$

- The system is unstable.
- The system has two roots with positive real parts(lie in the right half-plane).

6.2 Routh Criterion

Case 2: There is a zero in the first column.

If only one element in the array is zero, it may be replaced with a small positive number ε .

$$q(s) = s^4 + s^3 + s^2 + s + k = 0$$

Exercise: Do it yourself

6.2 Routh Criterion

Examle: There is a zero in the first column.

$$q(s) = s^4 + s^3 + s^2 + s + k = 0$$

The Routh array is

$$\begin{array}{cccc} s^4 & 1 & 1 & k \\ s^3 & 1 & 1 & \\ s^2 & 0 \to \varepsilon & k & \\ s^1 & \dfrac{\varepsilon - k}{\varepsilon} & & \\ s^0 & k & & \end{array}$$

When $k > 0$, $\dfrac{\varepsilon - k}{\varepsilon} < 0$

The system is unstable for all k.

6.2 Routh Criterion

Case 3: All the elements in one row are zero.

occurs when factors such as $(s+\sigma)(s-\sigma)$ **or** $(s+j\omega)(s-j\omega)$

Establish auxiliary polynomial using the row preceding the row of zeros

$$q(s)=s^3+2s^2+4s+k=0$$

s^3	1	4
s^2	2	k
s^1	$\frac{8-k}{2}$	
s^0	k	

auxiliary polynomial $u(s)=2s^2+k$

When $0<k<8$, the system is stable.

When $k=8$, the system is marginally stable

$$u(s)=2s^2+8=2(s+2j)(s-2j)$$

$$(s=-2,\pm 2j)$$

6.2 Routh Criterion

Case 4: **There is a zero in the row, which means there are the repeated roots on the $j\omega$-axis.**

$$q(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1 = 0$$

The Routh array is

$$\begin{array}{llll}
s^5 & 1 & 2 & 1 \\
s^4 & 1 & 2 & 1 \\
s^3 & \boxed{4 \to \varepsilon \quad 4 \to \varepsilon} & & \\
s^2 & 1 & 1 & \\
s^1 & \varepsilon & 0 & \\
s^0 & 1 & &
\end{array}$$

Establish auxiliary polynomial using the row preceding the row of zeros

$$\frac{du(s)}{ds} = 4s^3 + 4s^1 \quad u(s) = s^4 + 2s^2 + 1$$

$$= (s^2 + 1)^2$$

$$= (s + j)^2 (s - j)^2$$

repeated roots on the $j\omega$ -axis

The system is unstable.

6.2 Routh Criterion

Case 4: Repeated roots on the $j\omega$-axis.

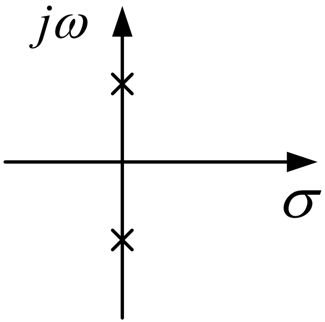
- **If there are the $j\omega$-axis roots of the characteristic equation, it is called marginally stable, since it has an undamped sinusoidal mode.**
- **If the $j\omega$-axis roots are repeated, the system response will be unstable with a form**

  $t\sin(\omega \mathrm{t}+\phi)$

- **The Routh-Hurwitz criteria will not reveal this form of instability .**

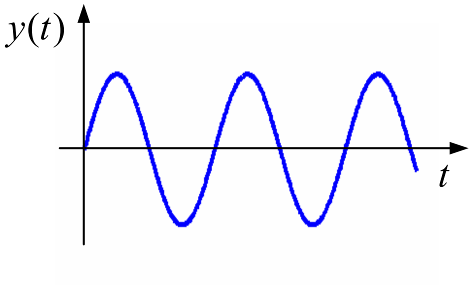
6.2 Routh Criterion

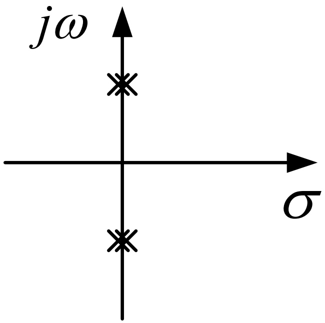
Case 4: Repeated roots on the $j\omega$-axis.



simple roots on the $j\omega$-axis

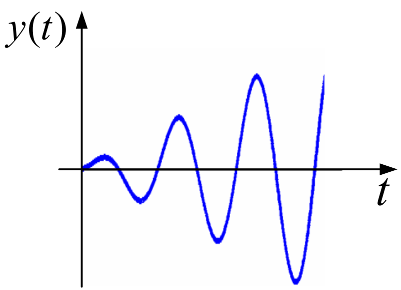
Marginally stable





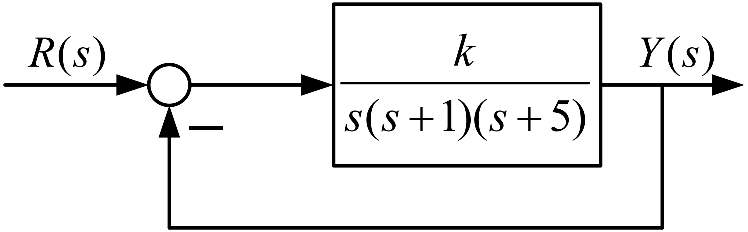
repeated roots on the $j\omega$-axis

Unstable



6.2 Routh Criterion

Example



Characteristic equation: $q(s) = 1 + G(s) = 0$

$$q(s) = s^3 + 6s^2 + 5s^1 + k = 0$$

The Routh array is

$$\begin{array}{ccc}
s^3 & 1 & 5 \\
s^2 & 6 & k \\
s^1 & \frac{30-k}{6} & \\
s^0 & k &
\end{array}$$

$$\begin{cases} \frac{30-k}{6} > 0 \\ k > 0 \end{cases} \Rightarrow 0 < k < 30$$

When $0 < k < 30$, the system is stable.

6.2 Routh Criterion

Example

$$G(s) = \frac{k}{s^2(0.5s+1)}, k > 0$$

Characteristic equation:

$$q(s) = 0.5s^3 + s^2 + k = 0$$

The necessary condition is not satisfied for the missing of s^1 item.

The Routh array is

s^3	0.5	0
s^2	1	k
s^1	$-0.5k$	
s^0	k	

- For $-0.5k < 0$, the system is unstable.
- Two roots lie in the right-half plane

6.2 Routh Criterion

Example

$$G_{\text{open}}(s) = \frac{k(\tau s+1)}{s^2(0.5s+1)}, k>0, \tau>0$$

$$q(s) = 0.5s^3 + s^2 + k\tau s + k = 0$$

The Routh array is

$$\begin{array}{lcc} s^3 & 0.5 & k\tau \\ s^2 & 1 & k \\ s^1 & k\tau - 0.5k & \\ s^0 & k & \end{array}$$

$$\begin{cases} k\tau - 0.5k > 0 \\ k > 0 \end{cases} \Rightarrow \begin{cases} \tau > 0.5 \\ k > 0 \end{cases}$$

When $k > 0, \tau > 0.5$, the system is stable.

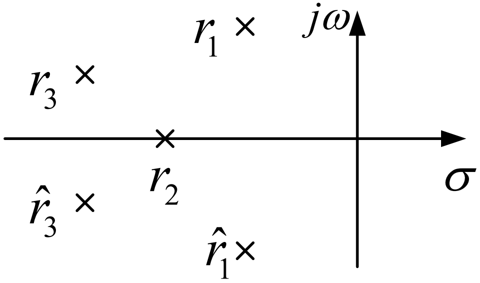
In general, two integral elements should be accompanied by one zero.

6.3 Relative Stability

The relative stability is dictated by the location of the roots of the characteristic equation.

Root r_2 is relatively more stable than the roots r_1 , $\hat{r}_1$.

A shift of the vertical axis in the s-plane to $-\sigma_1$ will result in the roots r_1 , $\hat{r}_1$ appearing on the shifted axis.



- The correct magnitude to shift the vertical axis must be obtained on a trial-and-error basis.

6.3 Relative Stability

Example: Axis shift

$$q(s) = s^3 + 4s^2 + 6s + 4$$

Let $s_n = s + 1$ $\qquad q(s_n) = (s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4$

The Routh array is $\qquad = s_n^{\,3} + s_n^{\,2} + s_n + 1$

$s_n^{\,3}$	1	1
$s_n^{\,2}$	1	1
$s_n^{\,1}$	ε	
$s_n^{\,0}$	1	

$$U(s_n) = s_n^{\,2} + 1 = (s_n + j)(s_n - j)$$
$$= (s + 1 + j)(s + 1 - j)$$

Marginally stable. There are roots on the shifted imaginary axis.

For higher-order system, axis shift method may determine the real part of dominant roots without solving the higher-order polynomial $q(s)$.

6.4 Summary

Concept

- Stability in terms of a bounded system response
- The necessary and sufficient criterion for the stability.
  - All the poles have negative real parts.
  - The Routh-Hurwitz criterion.
- Relative stability

Technique

- The Routh-Hurwitz criterion