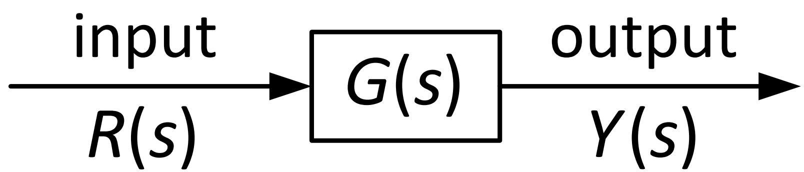
Control Laboratory

Chapter 5 The Performance of Feedback System

Contents

5.1 Introduction

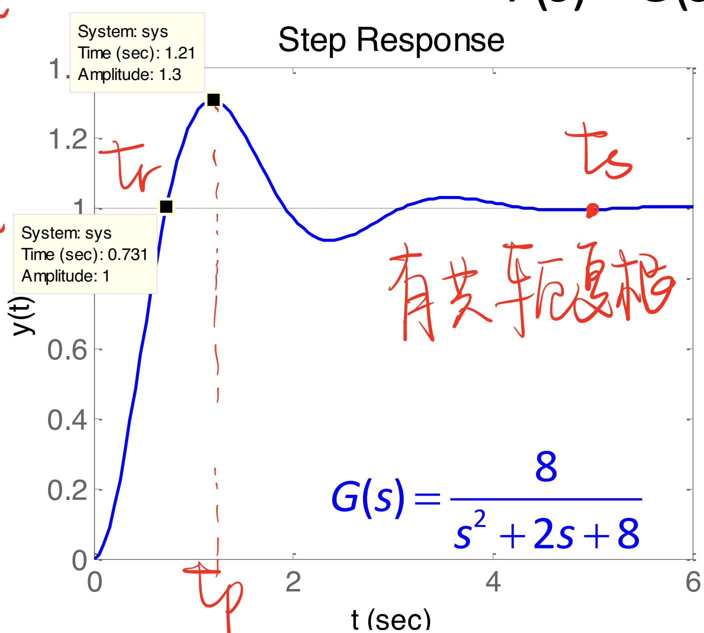
影响曲线的系数



$$Y(s)=G(s)R(s)$$

极点

曲线大概怎样



Character:
stability, transient, steady-state

Parameters:

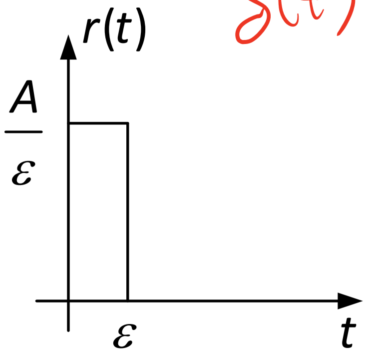
t_r, t_p, t_s, $P.O.$, e_{ss}

泰勒展开 任一函数展成

5.2 Test Input Signal

Impulse

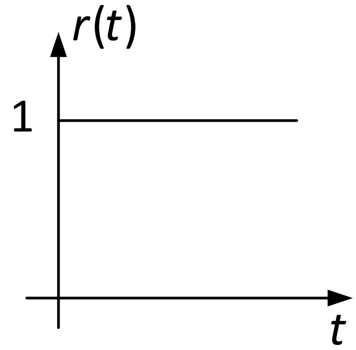
δ(t)



$$r(t)=\begin{cases}\dfrac{A}{\varepsilon} & 0\le t\le\varepsilon\\ 0 & \text{otherwise}\end{cases}$$

$$R(s)=A$$

Step

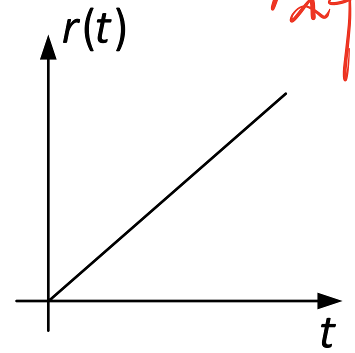


$$r(t)=\begin{cases}A & t\ge 0\\ 0 & t<0\end{cases}$$

$$R(s)=\frac{A}{s}$$

Ramp

斜坡



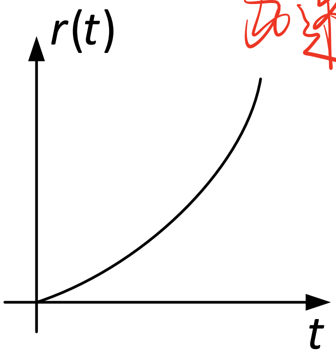
$$r(t)=\begin{cases}At & t\ge 0\\ 0 & t<0\end{cases}$$

$$R(s)=\frac{A}{s^2}$$

5.2 Test Input Signal

Acceleration (Parabolic)

加速度

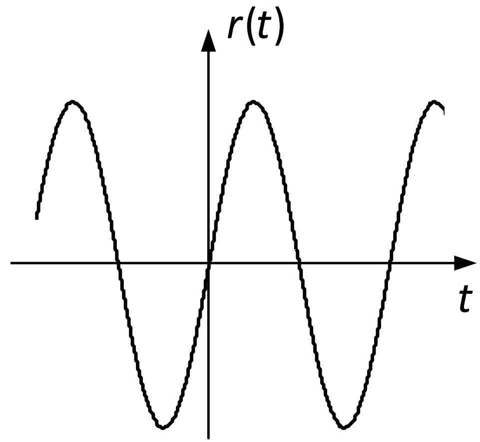


$$r(t)=\begin{cases} At^2 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$R(s)=\frac{2A}{s^3}$$

Sine & cosine

周期信号



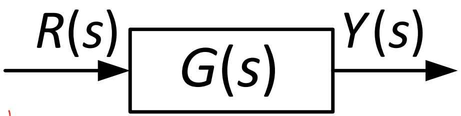
$$r(t)=A\sin(\omega t) \qquad R(s)=\frac{A\omega}{s^2+\omega^2}$$

$$r(t)=A\cos(\omega t) \qquad R(s)=\frac{As}{s^2+\omega^2}$$

5.2 Test Input Signal

Example:

系统的增益 $\frac{9}{10}$ 的倒 决定放大 缩小



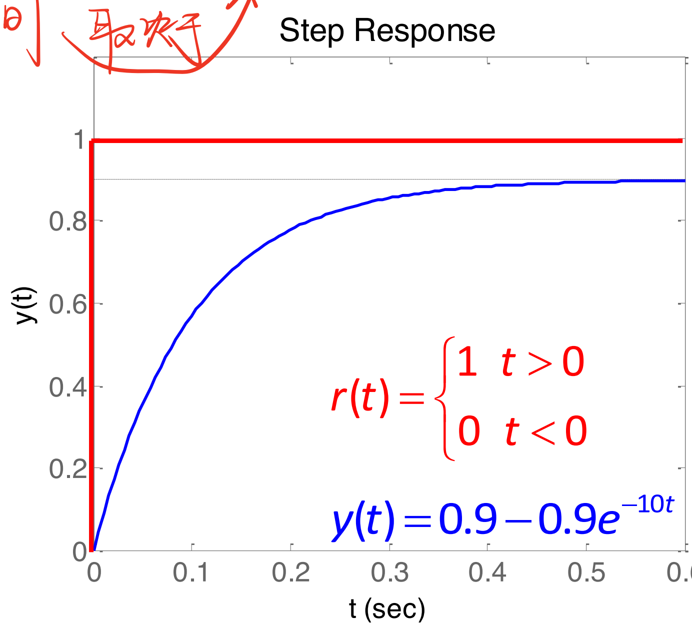
$$G(s)=\frac{9}{s+10},R(s)=\frac{1}{s},y(t)=?$$

t_r = 稳态的 0.9 对应的时间 取决于

$$Y(s)=G(s)\cdot R(s)=\frac{9}{s+10}\cdot\frac{1}{s}$$

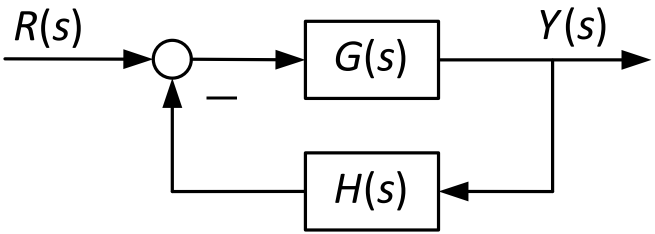
$$=\frac{0.9}{s}-\frac{0.9}{s+10}$$

$$y(t)=0.9-0.9e^{-10t}$$



5.3 2nd-order System Performance

$$G(s)=\frac{k}{s(Ts+1)},R(s)=\frac{1}{s},\mathrm{H}(s)=1,y(t)=?$$



$$G_{closed-loop}(s)=\frac{G(s)}{1+G(s)H(s)}=\frac{k/T}{s^2+(1/T)s+k/T}=\frac{\omega_n^2}{s^2+2\omega_n\zeta s+\omega_n^2}$$

$$\text{Let }\begin{cases}\omega_n{}^2=k/T\\ 2\zeta\omega_n=1/T\end{cases}\Rightarrow\begin{cases}\omega_n=\sqrt{k/T}\ \rightarrow\text{natural frequency}\\ \zeta=\dfrac{1}{2\sqrt{kT}}\ \rightarrow\text{damping ratio}\end{cases}$$

阻尼比

5.3 2nd-order System Performance

$$Y(s)=\frac{\omega_n^2}{s^2+2\omega_n\zeta s+\omega_n^2}\cdot R(s)$$ CE: $q(s)=s^2+2\omega_n\zeta s+\omega_n^2=0$

$$s_{1,2}=\frac{-2\zeta\omega_n\pm\sqrt{4\zeta^2\omega_n^2-4\omega_n^2}}{2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$ （ω_n>0， $\zeta\geq 0$）

过阻尼

① **overdamped** $\zeta>1$

临界阻尼 ② **critical damping** $\zeta=1$

欠阻尼 ③ **underdamped** $0<\zeta<1$

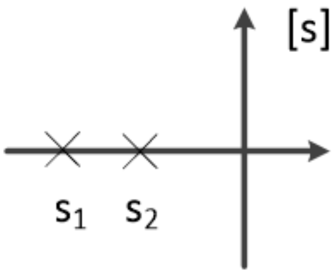
无阻尼 ④ **undamped** $\zeta=0$

5.3 2nd-order System Performance

CE: $s^2 + 2\omega_n\zeta s + \omega_n^2 = 0, s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

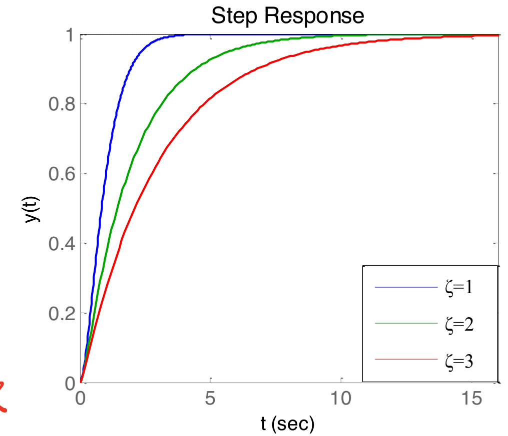
① **overdamped** $\zeta > 1$

s_1, s_2 are negative real roots



$$Y(s) = \frac{\omega_n^2}{s(s\text{-}s_1)(s\text{-}s_2)}$$

There is no oscillating term



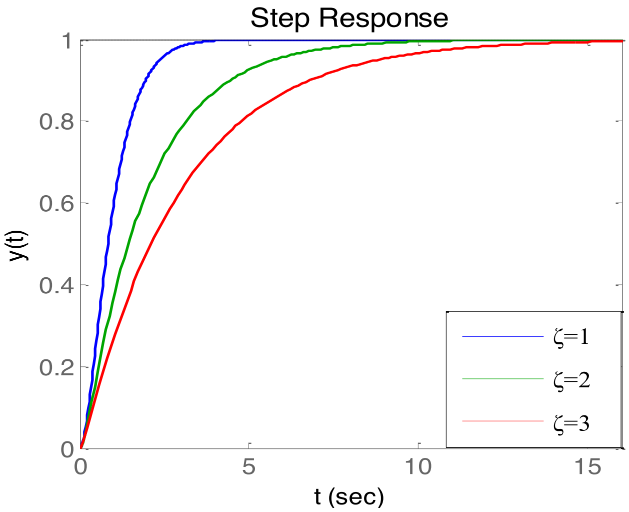
$$y(t) = \frac{\omega_n^2}{s_1 s_2} + \frac{\omega_n^2}{s_1^2 - s_1 s_2} e^{s_1 t} + \frac{\omega_n^2}{s_2^2 - s_1 s_2} e^{s_2 t}$$

阻尼越大，曲线越慢，启动慢

ξ

5.3 2nd-order System Performance

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n \zeta s + \omega_n^2}, \omega_n = 2, \zeta = 1,2,3 \quad R(s) = \frac{1}{s}$$



$$\frac{Y(s)}{R(s)} = \frac{4}{s^2 + 4s + 4}$$

$$\frac{Y(s)}{R(s)} = \frac{4}{s^2 + 8s + 4}$$

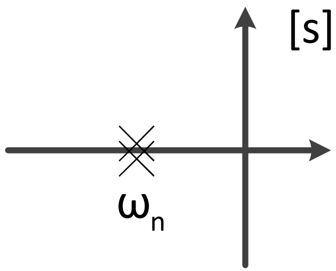
$$\frac{Y(s)}{R(s)} = \frac{4}{s^2 + 12s + 4}$$

```
>> num=[4]; den=[1 4 4]; step(num,den);hold on;
>> num=[4]; den=[1 8 4]; step(num,den);hold on;
>> num=[4]; den=[1 12 4]; step(num,den);hold on;
```

5.3 2nd-order System Performance

② **critical damping** $\zeta = 1$

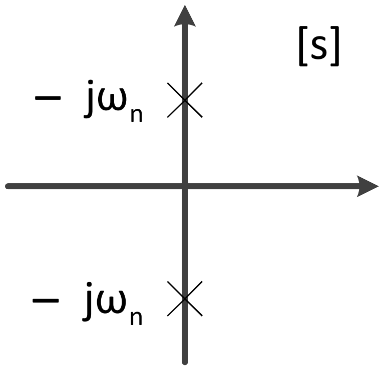
s_1, s_2 are repeated real roots, s_1, s_2= $-\omega_n$.



$$Y(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{1}{s} + \frac{-1}{s+\omega_n} + \frac{-\omega_n}{(s+\omega_n)^2}$$

$$y(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

③ **undamped** $\zeta = 0$



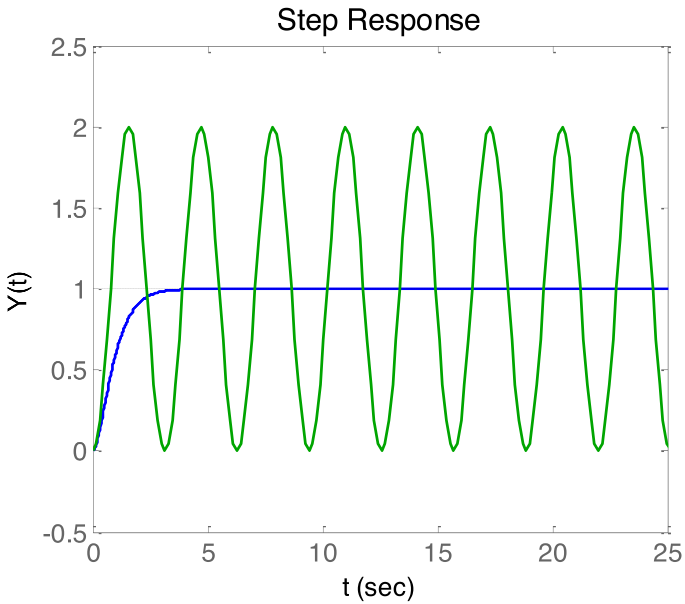
$$s_1, s_2 = \pm j\omega_n$$

$$Y(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

$$y(t) = 1 - \cos\omega_n t$$

5.3 2nd-order System Performance

$$\frac{Y(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\omega_n\zeta s+\omega_n^2},\omega_n=2,\zeta=0,1,\text{step response}$$



$$\frac{Y(s)}{R(s)}=\frac{4}{s^2+4s+1}$$

$$\frac{Y(s)}{R(s)}=\frac{4}{s^2+4}$$

```
>> num=[4];den=[1 4 4];step(num,den);hold on;
>> num=[4];den=[1 0 4];step(num,den);hold on;
```

5.3 2nd-order System Performance

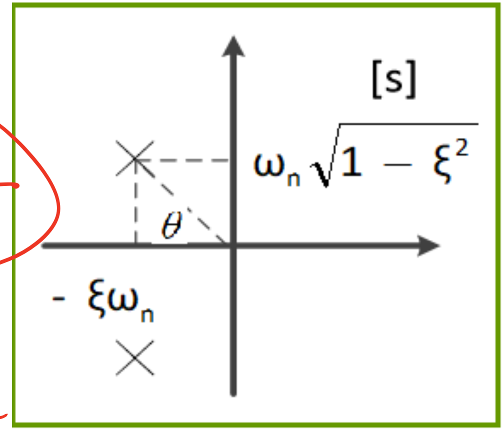
④ **underdamped** $0<\zeta<1$

t_r

$$\frac{\pi-\arccos\zeta}{\omega_n\sqrt{1-\zeta^2}}$$

s_1, s_2 are complex roots

$$Y(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n+\omega_n{}^2)}$$



$$y(t)=1-\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

$\omega_d t+\theta=k\pi$

$t=\frac{k\pi-\theta}{\omega_d}$

$$\sigma^2+\omega_d^2=\omega_n^2$$

$\omega_d=\omega_n\sqrt{1-\zeta^2}$: Damped oscillation frequency

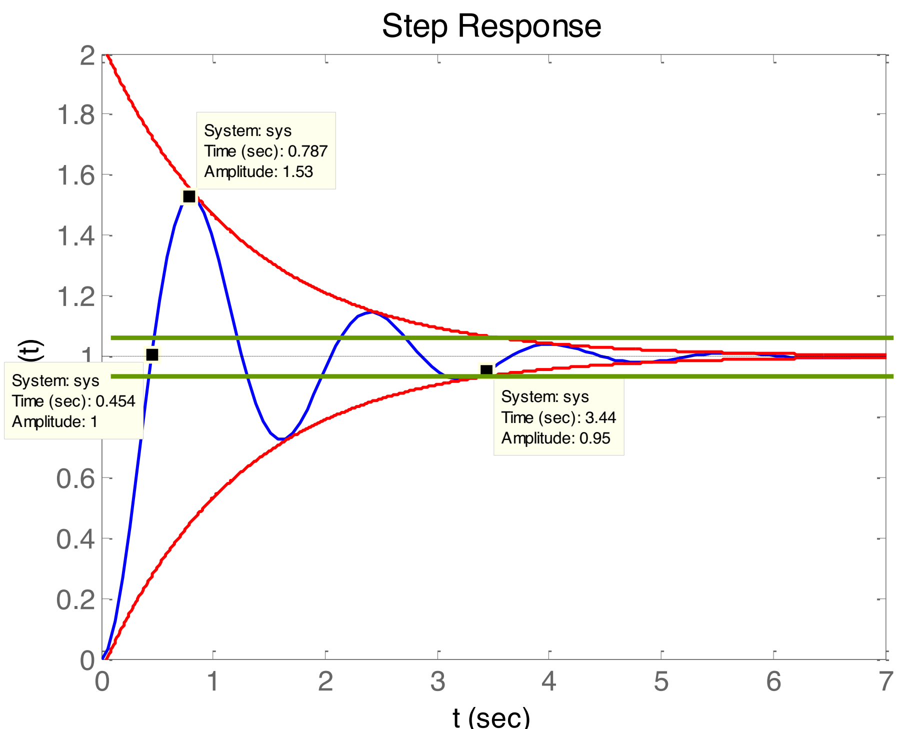
$\sigma=\zeta\omega_n$: Attenuation coefficient

$$\theta=\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$=\arccos\zeta$$

5.3 2nd-order System Performance

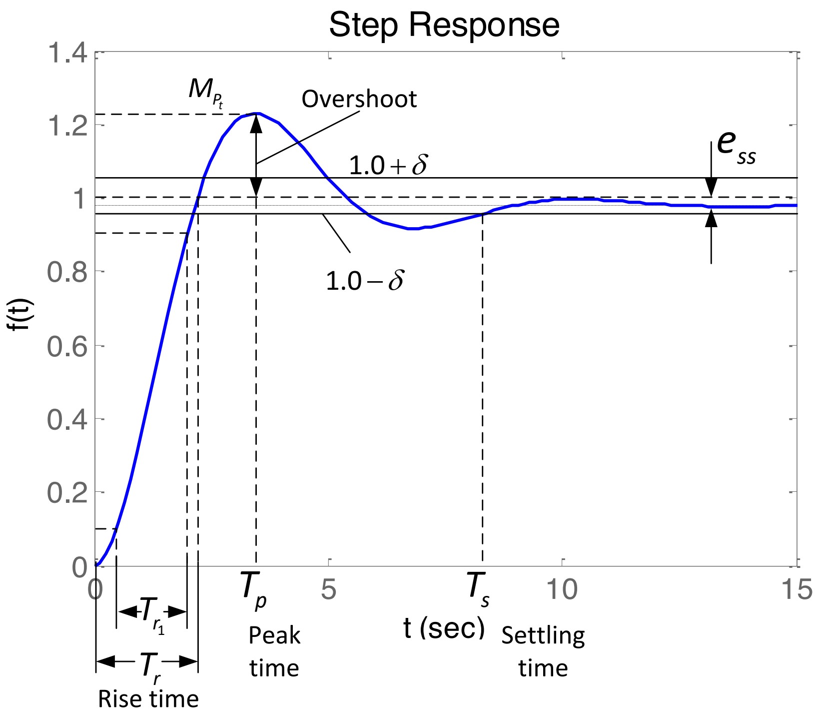
$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n \zeta s + \omega_n^2}, \omega_n = 4, \zeta = 0.2$$



5.3 2nd-order System Performance

The parameters: T_p, $P.O.$, T_r, T_s.

For a 2nd-order system with a unit step input $r(t)$=1:



5.3 2nd-order System Performance

① **T_p (peak time):** the time it takes to rise from 0 to the peak value of the time response

$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$\frac{dy(t)}{dt} = 0$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_p \propto \frac{1}{\omega_n}$$

$$M_{pt} = y(T_p) = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

5.3 2nd-order System Performance

② ***P.O.* (percentage overshoot)**

$$P.O. = \frac{M_{pt} - fv}{fv} \times 100\%$$

Mpt: the peak value of the output response

fv: final value

For the typical second order system (unit step input), *fv*=1

$$P.O. = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$\zeta \uparrow \Rightarrow P.O. \downarrow$$

5.3 2nd-order System Performance

$$y(t)=1-\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

③ T_r (rising time)

The time it takes to rise from 0 to 100% of the magnitude of the step input

$$y(T_r)=1 \Rightarrow \sin(\omega_n\sqrt{1-\zeta^2}T_r+\arccos\zeta)=0$$

$$\omega_n\sqrt{1-\zeta^2}T_r+\arccos\zeta=k\pi \Rightarrow T_r=\frac{\pi-\arccos\zeta}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_r \propto \frac{1}{\omega_n}$$

5.3 2nd-order System Performance

④ **T_s (settling time):**

The time required for the system to settle within a certain percentage of the input amplitude.

$$|y(T_s)-1|<\Delta, \quad \Delta=2\%\,\&\,5\%$$

3-4

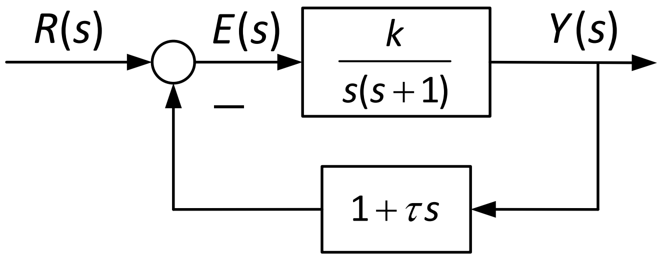
$$T_s=\frac{4}{\zeta\omega_n}, \quad \Delta=2\%$$

$$T_s=\frac{3}{\zeta\omega_n}, \quad \Delta=5\%$$

$$T_s \propto \frac{1}{\omega_n}$$

5.3 2nd-order System Performance

Example 1



The response to a unit step input is specified as $P.O.$=20%, T_p=1s.

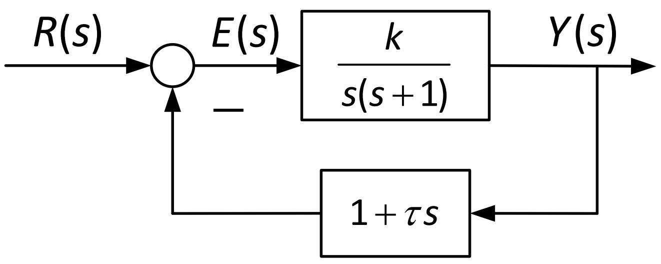
Determine k, τ, T_r, and T_s.

5.3 2nd-order Sys Performance

Example 1

$P.O.=20\%$, $t_p=1s$

k, τ to a unit step input



$$T(s)=\frac{\frac{k}{s(s+1)}}{1+\frac{k}{s(s+1)}(1+\tau s)}=\frac{k}{s^2+(1+k\tau)s+k}=\frac{\omega_n^2}{s^2+2\omega_n\zeta s+\omega_n^2}$$

where $k=\omega_n^2, 1+k\tau=2\omega_n\zeta \Rightarrow k=\omega_n^2, \tau=\dfrac{2\omega_n\zeta-1}{k}$

$$P.O.=\frac{M_{pt}-fv}{fv}\times 100\%=e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\times 100\%=20\%\Rightarrow \zeta=0.45$$

$$T_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}=1s\Rightarrow \omega_n=3.5 \qquad \Rightarrow k=12.25, \tau=3.52$$

5.3 2nd-order System Performance

Example 2

Discuss how *T* and *k* influence the system performance

$$\frac{Y(s)}{R(s)} = \frac{k/T}{s^2 + (1/T)s + k/T} = \frac{\omega_n^2}{s^2 + 2\omega_n \zeta s + \omega_n^2},\ 1/T = 2\omega_n \zeta, k/T = \omega_n^2$$

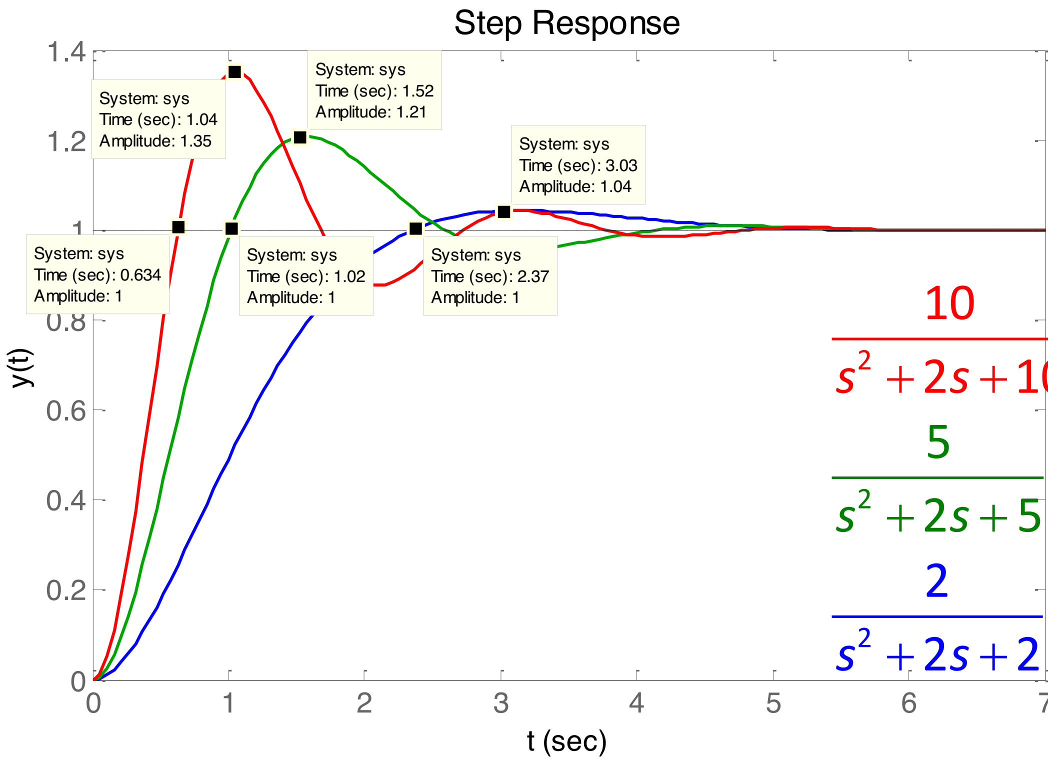
$$\boxed{1)\ for\ \ a\ \ given\ \ T,\ \ k\uparrow \Leftrightarrow \zeta \downarrow \omega_n \uparrow}$$

无阻尼、没有超调

$$P.O. = 100 e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-\pi}{\sqrt{\frac{1}{\zeta^2}-1}}} \uparrow \qquad T_s = \frac{3\sim 4}{\zeta\omega_n} \rightarrow$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \downarrow \qquad T_r = \frac{\pi - \arccos\zeta}{\omega_n\sqrt{1-\zeta^2}} \downarrow$$

5.3 2nd-order System Performance



$$\frac{10}{s^2+2s+10}, T=0.5, k=5$$

$$\frac{5}{s^2+2s+5}, T=0.5, k=2.5$$

$$\frac{2}{s^2+2s+2}, T=0.5, K=1$$

```
>> num=[2];den=[1 2 2];step(num,den);hold on;
>> num=[5];den=[1 2 5];step(num,den);hold on;
>> num=[10];den=[1 2 10];step(num,den);hold on;
```

5.3 2nd-order System Performance

$$\frac{Y(s)}{R(s)}=\frac{k/T}{s^2+(1/T)s+k/T}=\frac{\omega_n^2}{s^2+2\omega_n\zeta s+\omega_n^2},\ 1/T=2\omega_n\zeta,k/T=\omega_n^2$$

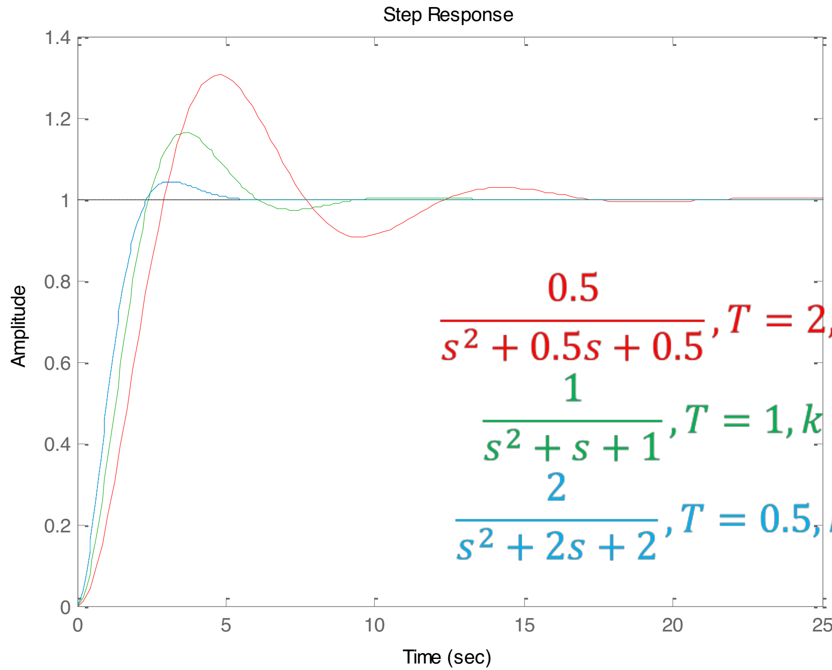
$$\boxed{2)\,for\ \ a\ \ given\ \ k,T\uparrow\Leftrightarrow\omega_n\downarrow,\zeta\downarrow}$$

tradeoff

$$P.O.=100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}=e^{\frac{-\pi}{\sqrt{\frac{1}{\zeta^2}-1}}}\uparrow \qquad T_s=\frac{3.5\sim 4}{\zeta\omega_n}\uparrow$$

$$T_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\uparrow \qquad T_r=\frac{\pi-\arccos\zeta}{\omega_n\sqrt{1-\zeta^2}}\uparrow$$

5.3 2nd-order System Performance



$$\frac{0.5}{s^2+0.5s+0.5}, T=2, k=1$$

$$\frac{1}{s^2+s+1}, T=1, k=1$$

$$\frac{2}{s^2+2s+2}, T=0.5, k=1$$

```
>> num=[1];den=[1 0.4 1];step(num,den);hold on;
>> num=[1];den=[1 0.8 1];step(num,den);hold on;
>> num=[1];den=[1 1.4 1];step(num,den);hold on;
>> num=[1];den=[1 1.8 1];step(num,den);hold on;
```

5.3 2nd-order System Performance

$$\left.\begin{matrix} T_r, T_p, T_s \\ P.O. \end{matrix}\right\rangle \begin{matrix} T: consistant \\ K: \ contradictory \end{matrix}$$

Sometimes, we can not meet all specifications simultaneously, a necessary compromise is needed.

5.4 Effect of a third pole and zero

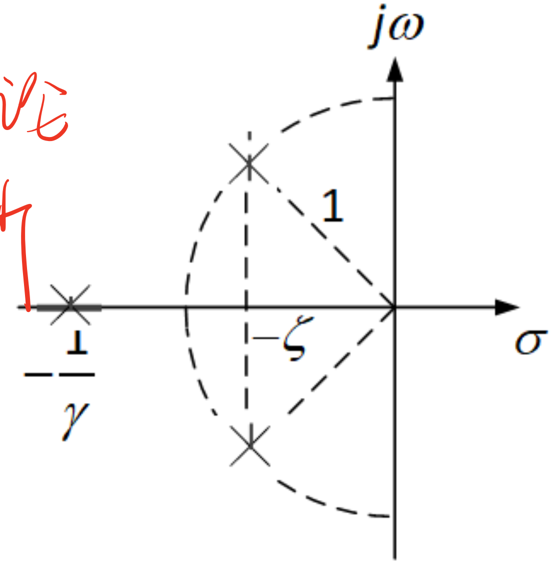
1. A third pole effect

$$T(s)=\frac{Y(s)}{R(s)}=\frac{1}{(s^2+2\zeta s+1)(\gamma s+1)},0<\zeta<1,R(s)=\frac{1}{s},y(t)=?$$

$$\text{Let } \zeta=0.5,T(s)=\frac{1}{(s^2+s+1)(\gamma s+1)}$$

γ	$\frac{1}{\gamma}$	$P.O.$	t_s
2.25	0.444	0	9.63
1.5	0.666	3.9	6.3
0.9	1.111	12.3	8.81
0.4	2.50	18.6	8.67
0.05	20.0	20.5	8.37
0	∞	20.5	8.24

理论分析

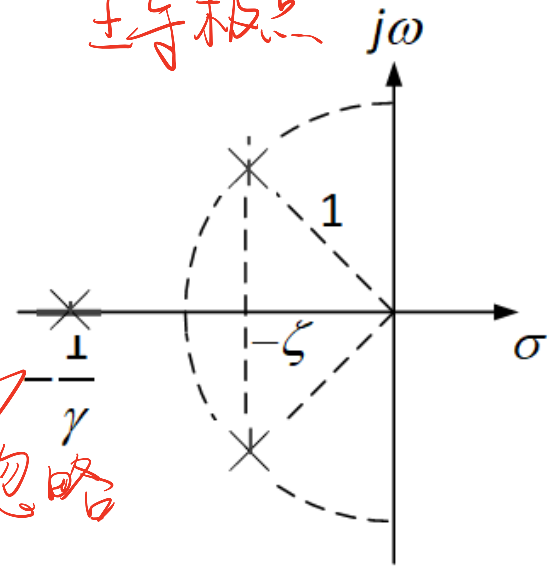


5.4 Effect of a third pole and zero

1. A third pole effect

确定 通过实验 主导极点

It was ascertained experimentally that the response of the third-order system can be approximated by the dominant roots of the second-order system when



$$\left|\frac{1}{\gamma}\right| \geq (10 \text{ or } 5)\,|\zeta\omega_n|$$

工程

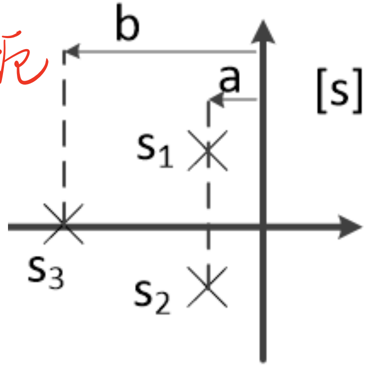
i.e. the real part of the dominant roots is less than one tenth(fifth) of the real part of the third root.

5.4 Effect of a third pole and zero

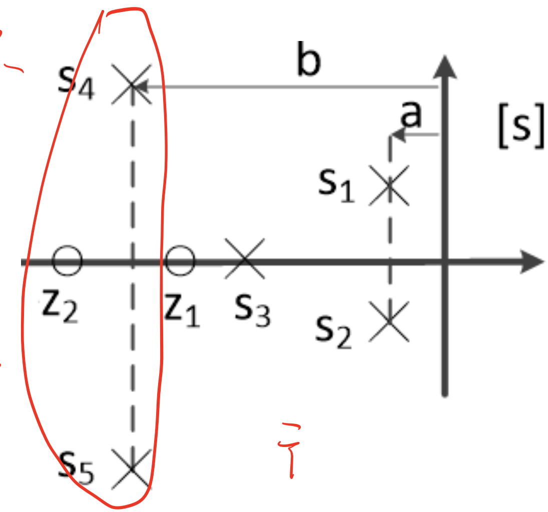
2. Dominant poles

主导极点

一定指共轭的复根



忽略



- For a stable higher-order system, the poles would be the dominant poles when the location of the close-loop is

①The closest poles to the imaginary axis are the conjugate complex poles

②Other poles are far away from Im axis and $b \geq 5a$

- The transient response can be determined by dominant poles.

5.4 Effect of a third pole and zero

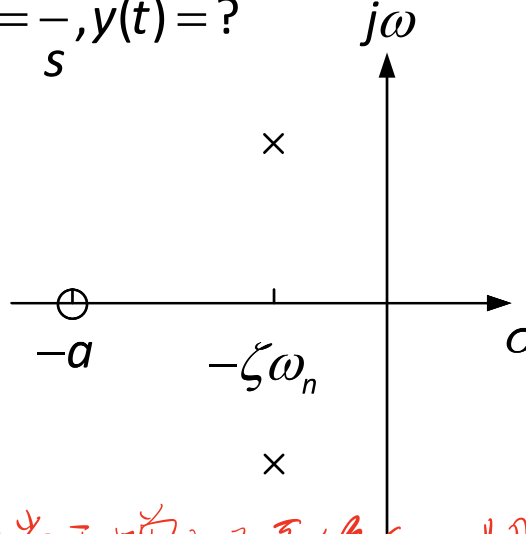
3. Effects of a zero

$\frac{a}{\zeta\omega_n} \uparrow \Rightarrow P.O. \downarrow, t_s \downarrow, t_p \uparrow$

超调时间

$$T(s)=\frac{(\omega_n^2/a)(s+a)}{(s^2+2\zeta\omega_n s+\omega_n^2)}, 0<\zeta<1, R(s)=\frac{1}{s}, y(t)=?$$

$\frac{a}{\zeta\omega_n}$	$P.O.$	t_s	t_p
5	23.1	8	3
2	39.7	7.6	2.2
1	89.9	10.1	1.8
0.5	210.0	10.3	1.5



相当于增加了系统的阻尼

Avoid selecting zeros near the imaginary axis or the dominant poles.

5.5 s-plane root location & the transient response

Closed-loop TF:

$$\frac{Y(s)}{R(s)}=\frac{\sum P_k\Delta_k}{\Delta}=\frac{G(s)}{1+G(s)}=\frac{\text{zeros}}{\text{poles}}$$

$$\text{CE: } \Delta=0\Rightarrow 1+G(s)=0$$

$$R(s)=\frac{1}{s}\Rightarrow Y(s)=\frac{\text{zeros}}{\text{poles}}\cdot R(s)$$

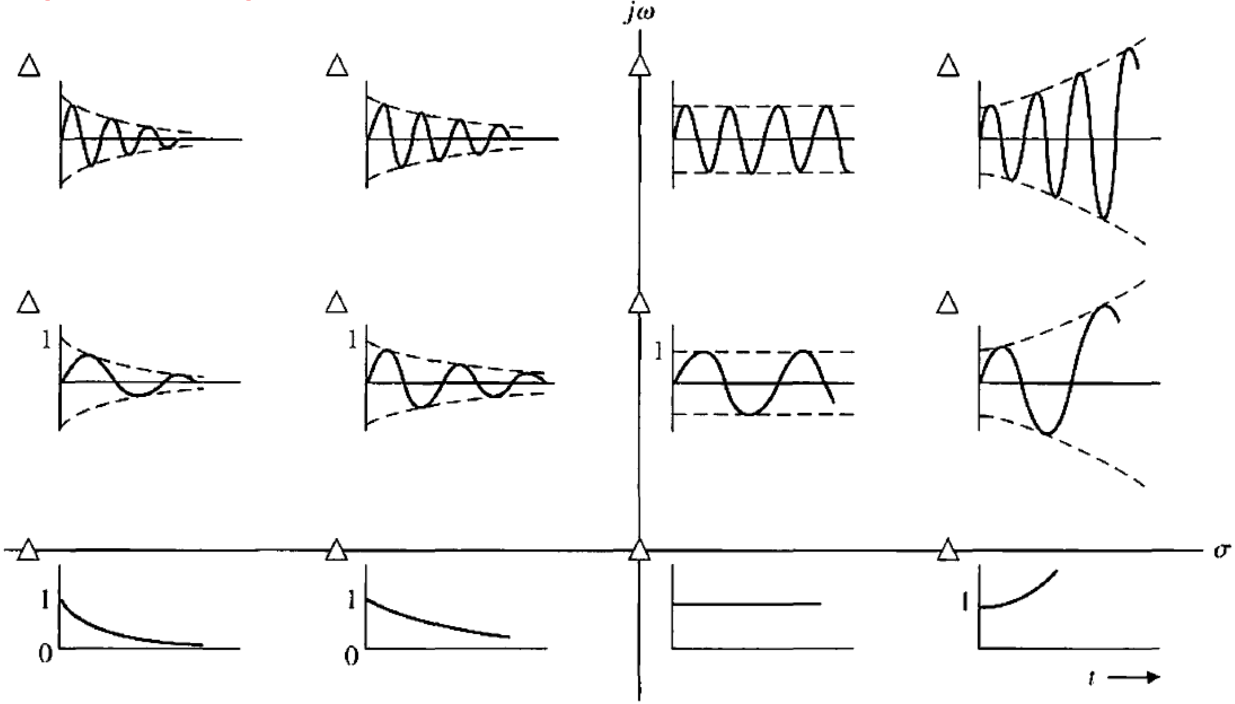
Under none repeated roots assumptions:

$$Y(s)=\frac{1}{s}+\sum_{i=1}^{M}\frac{A_i}{s+\sigma_i}+\sum_{k=1}^{N}\frac{B_k s+C_k}{s^2+2\alpha_k s+(\alpha_k{}^2+\omega_k{}^2)}$$

$$y(t)=1+\sum_{i=1}^{M}A_i e^{-\sigma_i t}+\sum_{k=1}^{N}D_k e^{-\alpha_k t}\sin(\omega_k t+\theta_k)$$

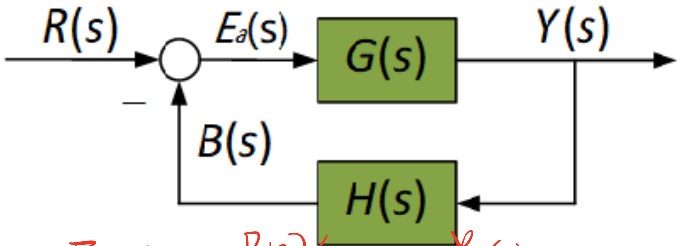
5.5 s-plane root location & the transient response

The impulse response for various root locations:



5.6 the Steady-state Error

稳态误差



$E(s) = R(s)/H(s) - Y(s)$

$$E(s) = R(s) - Y(s) = R(s) - T(s)R(s)$$

$$= (1 - \frac{G(s)}{1+G(s)H(s)})R(s)$$

$$= \frac{1+G(s)H(s)-G(s)}{1+G(s)H(s)}R(s)$$

$$E_a(s) = R(s) - B(s)$$

$$= R(s) - H(s)Y(s)$$

$$= \frac{1}{1+G(s)H(s)}R(s)$$

$$E(s) = \frac{1}{1+G(s)H(s)}R(s) = E_a(s)$$

两种形式相同

$H(s)=1$

$$e_{ss} = e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+G(s)}R(s)$$

5.6 the Steady-state Error

$$e_{ss} = e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+G(s)}R(s)$$

The general open-loop transfer function is

$$G(s) = \frac{K\prod_{j=1}^{M}(s+z_j)}{s^N\prod_{i=1}^{Q}(s+p_i)}$$

N=0	N=1	N=2
Type 0	Type Ⅰ	Type Ⅱ

N:The type number of the system

Three steady-state error constants:

① position error constant

$$K_p = \lim_{s\to 0} G(s)$$ 位置误差系数

② velocity error constant

$$K_v = \lim_{s\to 0} sG(s)$$ 速度

③ acceleration error constant

$$K_a = \lim_{s\to 0} s^2 G(s)$$ 加速度

5.6 the Steady-state Error

① For step input, $R(s) = \dfrac{A}{s}$

$$e_{ss} = e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+G(s)}R(s)$$

$$= \lim_{s\to 0} s\frac{1}{1+G(s)}\frac{A}{s} = \frac{A}{1+\lim\limits_{s\to 0} G(s)}$$

$$= \frac{A}{1+K_p} \qquad \text{(type 0)}$$

$$= \frac{A}{1+\infty} = 0 \ \ \text{(type I)}$$

$$= \frac{A}{1+\infty} = 0 \ \ \text{(type II)}$$

5.6 the Steady-state Error

② For ramp input, $R(s)=\dfrac{A}{s^2}$

$$e_{ss}=e(\infty)=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\lim_{s\to 0}s\frac{1}{1+G(s)}R(s)$$

$$=\lim_{s\to 0}s\frac{1}{1+G(s)}\frac{A}{s^2}=\lim_{s\to 0}\frac{A}{s+sG(s)}=\frac{A}{\lim\limits_{s\to 0}sG(s)}$$

$$=\frac{A}{0}=\infty \qquad \text{(type 0)}$$

$$=\frac{A}{K_v} \qquad \text{(type I)}$$

$$=\frac{A}{\infty}=0 \qquad \text{(type II)}$$

5.5 the Steady-state Error

③ For acceleration input, $R(s)=\dfrac{A}{s^3}$

$$e_{ss}=e(\infty)=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\lim_{s\to 0}s\frac{1}{1+G(s)}R(s)$$

$$=\lim_{s\to 0}s\frac{1}{1+G(s)}\frac{A}{s^3}=\frac{A}{\lim\limits_{s\to 0}s^2G(s)}$$

$$=\frac{A}{0}=\infty \qquad \text{(type 0)}$$

$$=\frac{\mathrm{A}}{0}=\infty \qquad \text{(type I)}$$

$$=\frac{A}{K_a} \qquad \text{(type II)}$$

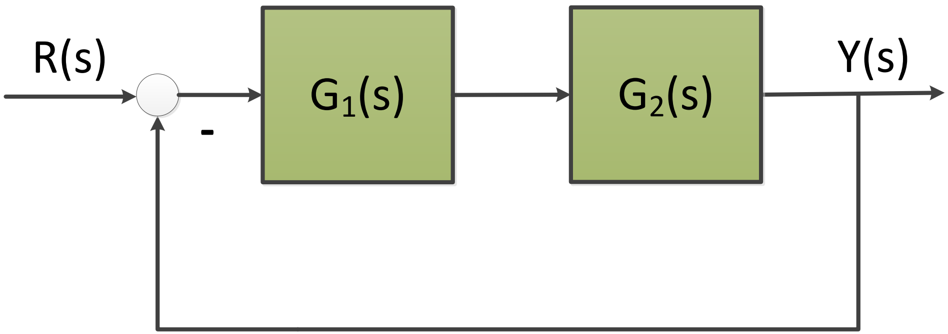
5.6 the Steady-state Error

The value of e_{ss} depends on the type of $R(s)$ and $G(s)$

	Type 0	Type 1	Type 2
Step input	$\frac{A}{1+K_p}$	0	0
Ramp input	∞	$\frac{A}{K_v}$	0
Acceleration input	∞	∞	$\frac{A}{K_a}$

设置系统

Example(5.6)



$$G_1(s) = k_1 + \frac{k_2}{s}$$

$$G_2(s) = \frac{k}{\tau s + 1}$$

Determine e_{ss}

5.7 Summary

Concept

- Test input signals: step, ramp, parabolic, impulse
- Performance of the second-order system with unity step input
  rise time, peak time, settling time, percent overshoot

忽略高阶极点

- Dominant poles, effects of the zero 降阶
- The relationship between the s-plane root location and the transient response
- The steady-state error
  Kp, ka, kv

Technique

- Calculate Tr , Tp , Ts, P.O. 百分超调量
- Calculate steady-state error

End