Control Laboratory

Chapter 2 Mathematical Models of Systems

逸夫楼B315教室

二教419教室

Mathematical Models-Contents

Important in Modeling Chapter

微分方程 - differential equations

线性化 - linearization

传递函数 - transfer function

- Block diagram

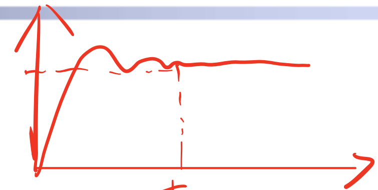
信号流图 - signal flow graph

- Matlab

Mathematical Models–Contents

2.1 Introduction

◻ Mathematical models



Concepts: t_s 调节时间

Descriptions of the behavior of a system using mathematics.

Approach:

1. Define the system and its components.
2. Formulate the mathematical model and fundamental necessary assumptions based on basic principles.
3. Obtain the differential equations representing the mathematical model.
4. Solve the equations for the desired output variables.
5. Examine the solutions and the assumptions.
6. If necessary, reanalyze or redesign the system.

2.2 Differential Equations

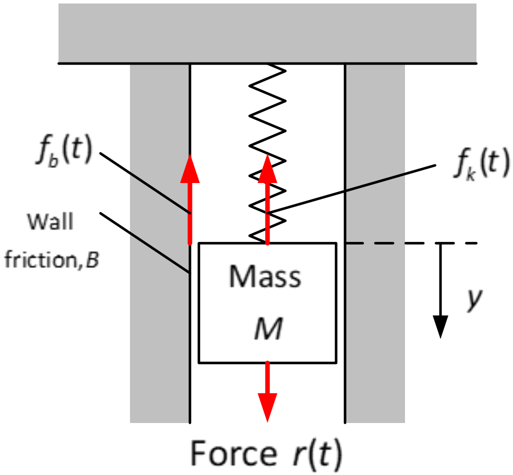
Concepts:

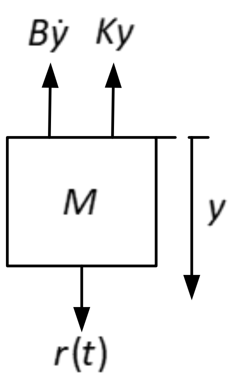
- A mathematical model to describe motion state and dynamic performance of the control system.

Step:

1. Determine the input and output according to the working principle of the components and its role in the control system.
2. Analysis the physical or chemical law the components, write corresponding differential equations.
3. Eliminate intermediate variables, obtain the differential equations to give the relationship between the output and input.

2.2 Differential Equations





$$r(t) - f_b(t) - f_k(t) = M\frac{d^2y(t)}{dt^2}$$

$$f_b(t) = B\frac{dy(t)}{dt}, \quad f_k(t) = Ky(t)$$

the ideal spring

B—— the friction constant;

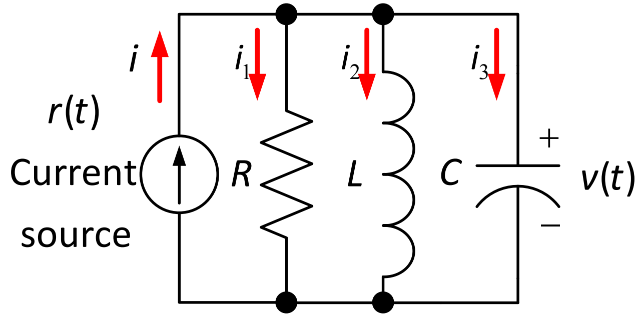
K—— the spring constant.

$$M\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + Ky(t) = r(t)$$

$$f_b(t) = B\frac{dy(t)}{dt} = B\dot{y} \qquad f_k(t) = Ky(t)$$

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = r(t)$$

2.2 Differential Equations

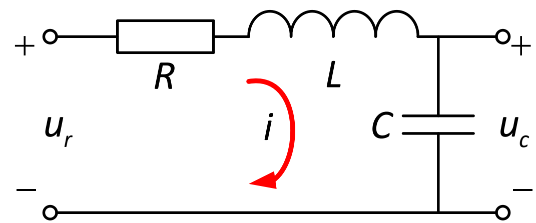


$$i = i_1 + i_2 + i_3$$

$$i_1 = \frac{v(t)}{R}, i_2 = \frac{1}{L}\int_0^t v(t)dt, i_3 = C\frac{dv(t)}{dt}$$

where i_1、i_2、i_3 represent the initial current through R、L、C.

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(t)dt + C\frac{dv(t)}{dt} = r(t)$$



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = u_r(t)$$

$$u_c(t) = \frac{1}{C}\int idt$$

Eliminate intermediate variables *i*.

$$LC\frac{d^2u_c(t)}{dt^2} + RC\frac{du_c(t)}{dt} + u_c(t) = u_r(t)$$

2.3 Linear Approximations

- A system is defined as linear in terms of the system excitation and response.
- if excitation is $x(t)$ and response is $y(t)$, we get necessary conditions:

线性两大特征

  - **Superposition(叠加性):** input $x_1(t)+x_2(t)$ → output $y_1(t)+y_2(t)$
  - **Homogeneity(齐次性):** input $ax(t)$ → output $ay(t)$

- A linear system satisfies properties of superposition and homogeneity.

Example: $y=x^2$,

$y=mx+b$

$y=mx$

Judge whether it is a linear system?

2.3 Linear Approximations

- All systems ultimately become **nonlinear** as the variables are increased without limit.
- A great majority of physical systems are linear within some range of the variables.
- One can often linearize nonlinear elements assuming **small-signal conditions**.

Example: $y(t)=g(x(t))$, $y(t)$ is a function of $x(t)$, and is continuous and differentiable. The normal operating point is designated by x_0. Taylor series expansion about the operating point:

$$y = g(x) = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x0} \frac{(x - x_0)}{1!} + \left.\frac{d^2g}{dx^2}\right|_{x=x0} \frac{(x - x_0)^2}{2!} + \cdots$$

2.3 Linear Approximations

Let the slope(斜率) at the operating point

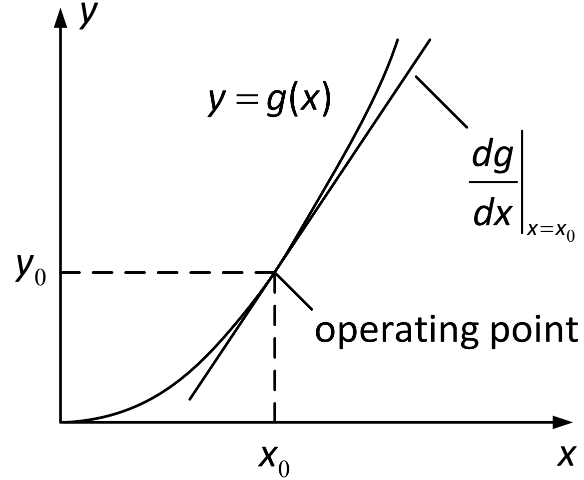
$$m = \left.\frac{dg}{dx}\right|_{x=x_0}$$

A reasonable approximation over a small range of $(x\text{-}x_0)$,

$$y = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} (x - x_0) = y_0 + m(x - x_0)$$

$$y - y_0 = m(x - x_0)$$

$$\Delta y = m\Delta x$$



Note: **1.**The non-linear function curve is continuous over the range of interest.

2.The assumption of small signals is applicable to the specific problem.

2.4 Laplace transformation

- **Purpose**
  - To solve algebraic equations easily for more difficult differential equations.
  - Signals that are physically realizable always have the Laplace transform.

2.4 Laplace transformation

The **Laplace transformation** for a function of time, $f(t)$, is

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt = L\{f(t)\}$$

The inverse Laplace transform is written as

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds = L^{-1}[F(s)]$$

Alternatively, the Laplace variable s can be considered to be the differential operator so that

$$\boxed{s \equiv \frac{d}{dt} \qquad \frac{1}{s} \equiv \int_{0^-}^{t} dt}$$

2.4 Laplace transformation

Important Laplace Transform Pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
$\frac{1}{2}t^2$	$\frac{1}{s^3}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$

ramp 斜坡

加速度

2.4 Laplace transformation

Important Theorems

- Superposition and homogeneity

$$L[f_1(t)] = F_1(s)$$
$$L[f_2(t)] = F_2(s)$$
$$\Rightarrow L[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)$$

- $$L[f'(t)] = s \cdot F(s) - f(0)$$

When $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$ 假设初值

$$L[f'(t)] = s \cdot F(s), L[f''(t)] = s^2 \cdot F(s) \cdots L[f^{(n)}(t)] = s^n \cdot F(s)$$

- When the initial conditions are equal to zeros.

$$L[\int f(t)dt] = \frac{1}{s} F(s)$$

2.4 Laplace transformation

Important Theorems

- Delay function 延迟函数

$$\mathcal{L}[f(t-\tau)] = e^{-\tau s}F(s)$$

- $$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

- Initial value theorem 初值

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\cdot F(s)$$

- Final value theorem 终值定理 → error 误差

$$\lim_{t\to\infty} f(t) = f(\infty) = \lim_{s\to 0} s\cdot F(s)$$

tf：

2.5 Transfer Function

Concept: The transfer function is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.

In general, linear differential equation can be written as

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_1 \frac{dr(t)}{dt} + b_0 r(t)$$

where $y(t)$ is the output and $r(t)$ is the input.

All initial conditions assumed to be zero, 线性系统 叠加定理

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) Y(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0) R(s)$$

$n \geq m$：能量耗散系统，可实现

2.5 Transfer Function

The transfer function is

$$G(s)=\frac{Y(s)}{R(s)}=\frac{b_m s^m+b_{m\text{-}1}s^{m\text{-}1}+\cdots+b_1 s+b_0}{a_n s^n+a_{n\text{-}1}s^{n\text{-}1}+\cdots+a_1 s+a_0}$$

where, n≥m

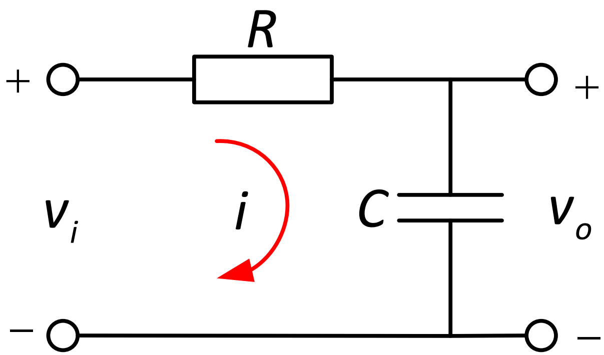
Note

分母

- The order of the denominator is higher than the order of the numerator. 分子
- $G(s)$ is only determined by the structure and the parameters of the system. 不能动
- $G(s)$ may be only for a linear (constant parameter) systems.
- An input-output description of the behavior of a system (not concerning the internal structure of the system and its behavior).

2.5 Transfer Function

Examples (RC network)



Differential Equations

$$v_i = Ri + \frac{1}{C}\int idt, v_o = \frac{1}{C}\int idt$$

Laplace transformation

$$V_i(s) = RI(s) + \frac{1}{Cs}I(s), V_o(s) = \frac{1}{Cs}I(s)$$

Transfer Function

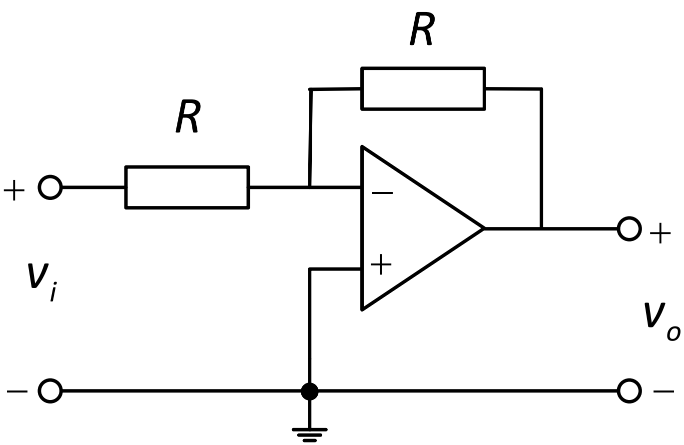
S的阶数

$$G(s) = \frac{V_o(s)}{V_i(s)} == \frac{1}{1+RCs} = \frac{1}{1+\tau s} = \frac{1/\tau}{s+1/\tau}$$

where $\tau = RC$.

2.5 Transfer Function

Examples (op-amp circuit) 运算放大器（课后题）



Differential Equations

$$v_i = iR, v_o = -iR$$

Laplace transformation

$$V_i(s) = I(s)R, V_o(s) = -I(s)R$$

Transfer Function

$$G(s) = \frac{V_o(s)}{V_i(s)} = -1$$

2.5 Poles and Zeros

$$F(s) = k\frac{\prod_{j=1}^{m}(s-z_j)}{\prod_{i=1}^{n}(s-p_i)} = \frac{B(s)}{A(s)}$$

1. Partial fraction expansion

- The roots of $A(s)$=0 are unrepeated

$$F(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n}$$

both sides of the equation is multiplied by (s-p_1) at the same time

$$(s-p_1)\cdot F(s) = C_1 + \frac{C_2(s-p_1)}{s-p_2} + \ldots + \frac{C_n(s-p_1)}{s-p_n}$$

Let $s = p_1, C_1 = (s-p_1)F(s)\big|_{s=p_1}, \cdots C_i = (s-p_i)F(s)\big|_{s=p_i}$

inverse Laplace transfer

$$f(t) = C_1e^{p_1t} + C_2e^{p_2t} + \ldots + C_ne^{p_nt} = \sum_{i=1}^{n} C_ie^{p_it}$$

2.5 Poles and Zeros

- The roots of $A(s)=0$ are repeated

Assuming there are m repeated roots p_1 , and $p_{m+1}, \ldots, p_n$ are unrepeated.

$$F(s)=\frac{C_m}{(s-p_1)^m}+\frac{C_{m-1}}{(s-p_1)^{m-1}}+\ldots+\frac{C_1}{s-p_1}+\ldots+\frac{C_{m+1}}{s-p_{m+1}}+\ldots+\frac{C_n}{s-p_n}$$

Equation $C_i=(s-p_i)F(s)\big|_{s=p_i}$ may be used to calculate $C_{m+1},\cdots,C_n$.

$$C_m=(s-p_1)^m F(s)\Big|_{s=p_1},$$

$$C_{m-1}=\frac{d}{ds}[(s-p_1)^m F(s)]\Big|_{s=p_1},\cdots$$

$$C_1=\frac{1}{(m-1)!}\frac{d^{(m-1)}}{ds^{(m-1)}}[(s-p_1)^m F(s)]\Big|_{s=p_1}$$

inverse Laplace transfer

$$f(t)=[\frac{C_m}{(m-1)!}t^{m-1}+\frac{C_{m-1}}{(m-2)!}t^{m-2}+\ldots$$

$$+C_1]e^{p_1 t}+\sum_{i=m+1}^{n} C_i e^{p_i t}$$

2.5 Poles and Zeros

Examples

$$F(s) = \frac{1}{(s+2)^3(s+3)}$$ Obtain an expression for $f(t)$.

Answer

$$f(t) = [\frac{1}{2}t^2 - t + 1]e^{-2t} - e^{-3t}$$

2.5 Poles and Zeros

2. Characteristic equation

Transfer function is $G(s)=\frac{p(s)}{q(s)}$ .The denominator polynomial $q(s)$, $q(s)=0$ is called the characteristic equation.

characteristic equation：

$$\boldsymbol{q(s)=0}$$

Poles: The roots of the characteristic equation $q(s)=0$.

Zeros: The roots of the numerator polynomic $p(s)=0$.

2.5 Poles and Zeros

Example1

$$G(s)=\frac{Y(s)}{R(s)}=\frac{2}{(s+1)(s+2)}$$

$r(t)$=1(t), obtain $y(t)$.

answer

$$r(t)=1(t)\xrightarrow{\text{Laplace transform}}R(s)=\frac{1}{s}$$

$$Y(s)=G(s)R(s)=\frac{2}{s(s+1)(s+2)}$$

$$=\frac{1}{s}-\frac{2}{s+1}+\frac{1}{s+2}$$

Inverse Laplace transfer

$$y(t)=1-2e^{-t}+e^{-2t}$$

Example2

$$G(s)=\frac{Y(s)}{R(s)}=\frac{0.5(s+4)}{(s+1)(s+2)}$$

$r(t)$=1(t), obtain $y(t)$.

answer

$$r(t)=1(t)\xrightarrow{\text{Laplace transform}}R(s)=\frac{1}{s}$$

$$Y(s)=G(s)R(s)=\frac{0.5(s+4)}{s(s+1)(s+2)}$$

$$=\frac{1}{s}-\frac{1.5}{s+1}+\frac{0.5}{s+2}$$

Inverse Laplace transfer

$$y(t)=1-1.5e^{-t}+0.5e^{-2t}$$

2.5 Poles and Zeros

Note:

- **The poles** determine **the character of the time response** and **the stability of the system**.
- **The zeros** do not affect the free modes.
- The weight of the modes may be affected by the zeros, especially when the zeros are located near the poles.

2.6 Basic terms

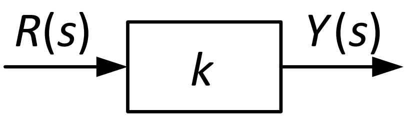
1.Proportional element

Model: $y(t)=k\cdot r(t)$, where k is a constant.

Transfer Function:

$$G(s)=\frac{Y(s)}{R(s)}=k$$

Character:

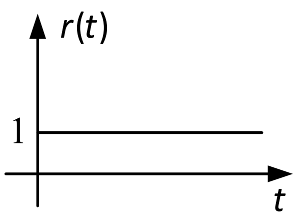


Note:

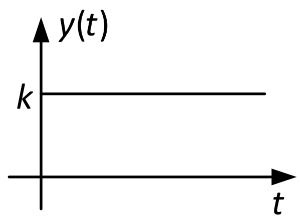
① The output is proportional to the input.

② fast response.

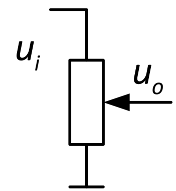
Input:

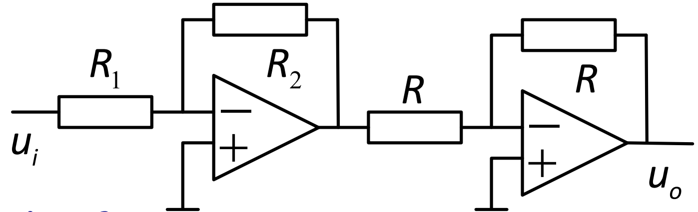


Output:



Physical devices





2.6 Basic terms

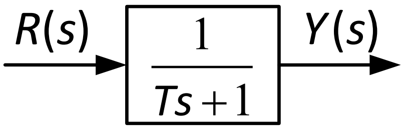
2.Inertial element

Model: $T\dfrac{dy(t)}{dt}+y(t)=r(t)$

Transfer Function:

$$G(s)=\frac{Y(s)}{R(s)}=\frac{1}{Ts+1}$$

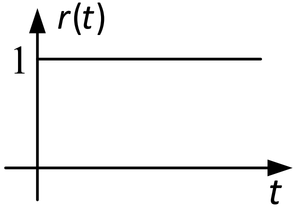
Character:



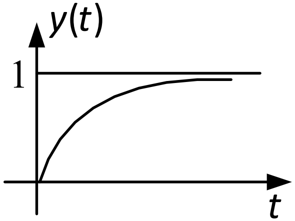
Note:

T: time constant.

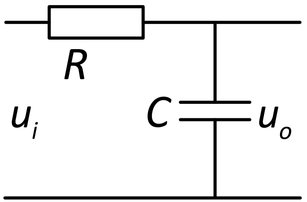
Input:

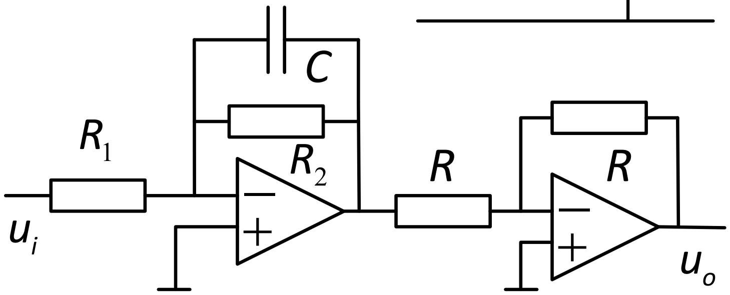


Output:



Physical devices





2.6 Basic terms

3.Integral element

Model: $T\dfrac{dy(t)}{dt}=r(t)$

$$y(t)=\frac{1}{T}\int r(t)dt$$

Transfer Function:

$$G(s)=\frac{Y(s)}{R(s)}=\frac{1}{Ts}$$

Character:

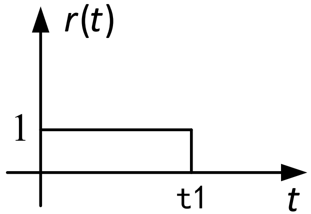
$$R(s)\longrightarrow\boxed{\frac{1}{Ts}}\longrightarrow Y(s)$$

Note:

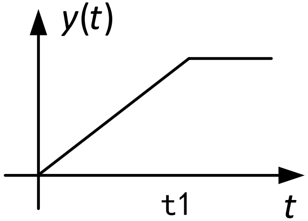
①T: time constant

②Memory facility

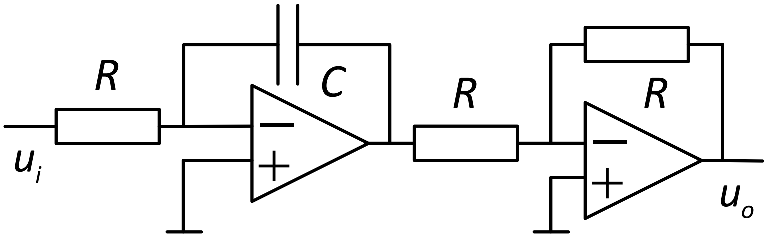
Input:



Output:



Physical devices



2.6 Basic terms

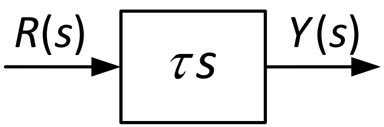
4.Differential element

Model: $y(t) = \tau \dfrac{dr(t)}{dt}$

Transfer Function:

$$G(s) = \frac{Y(s)}{R(s)} = \tau s$$

Character:

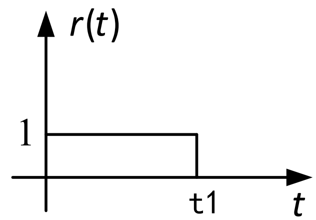


Note:

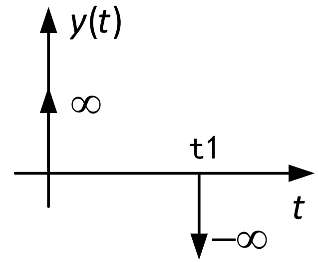
① τ : time constant

②y(t) shows the trend of r(t)

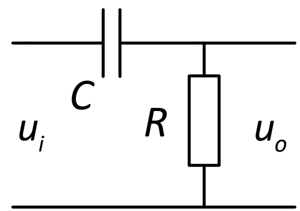
Input:

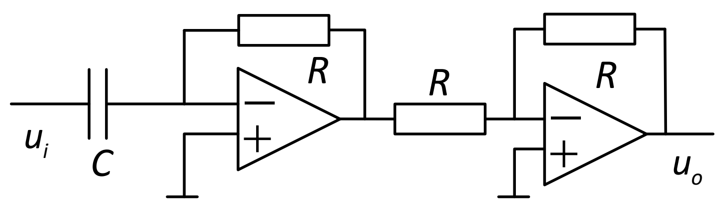


Output:



Physical devices





2.6 Basic terms

5.Oscillating element

Model:

$$T^2\frac{d^2y(t)}{dt^2}+2\xi T\frac{dy(t)}{dt}+y(t)=r(t)$$

Transfer Function:

$$G(s)=\frac{Y(s)}{R(s)}=\frac{1}{T^2s^2+2\xi Ts+1}$$

$$=\frac{\omega_n{}^2}{s^2+2\xi\omega_n s+\omega_n{}^2}$$

Character:

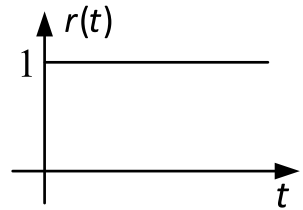
$$R(s)\longrightarrow\boxed{\frac{\omega_n{}^2}{s^2+2\xi\omega_n s+\omega_n{}^2}}\longrightarrow Y(s)$$

Note: $\omega_n=\frac{1}{T}$,Natural frequency

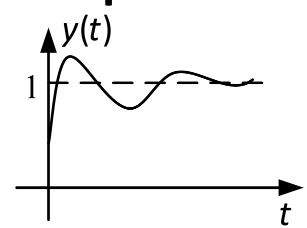
ξ ,Damping ratio

$$s_{1,2}=\begin{cases}-\xi\omega_n\pm\omega_n\sqrt{\xi^2-1} & \xi\geq 1\\ -\xi\omega_n\pm j\omega_n\sqrt{\xi^2-1} & 0<\xi<1\end{cases}$$

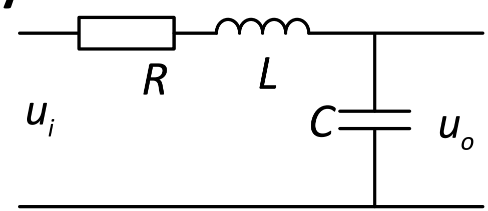
Input:



Output:



Physical devices



2.6 Basic terms

6.Dalay element

Model:

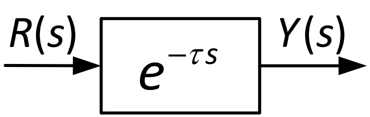
$$r(t) = f(t)$$

$$y(t) = f(t-\tau)$$

Transfer Function:

$$G(s) = \frac{Y(s)}{R(s)} = e^{-\tau s}$$

Character:



Output: (unit step Input)