Control Laboratory

Chapter 9 Stability in the Frequency Domain

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Contents

9.1 Introduction

Stability: bound input →bound output

① Closed-loop poles

② Routh-Hurwitz criterion

③ Root locus

④ Nyquist stability criterion

Advantages: a. open-loop $G(jw)$

b. $G(jw)$ can be obtain through experiment

c. relative stability

9.2 Mapping Contours

Cauthy's theorem (principle of the argument)

If a contour Γ_s in the s-plane encircles Z zeros and P poles of $F(s)$ and does not pass through any poles or zeros of $F(s)$ and the traversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the $F(s)$ plane encircles the origin of the $F(s)$ plane $N = Z - P$ times in the clockwise direction.

9.2 Mapping Contours

Cauthy's theorem (principle of the argument)

Open-loop TF $GH(s) = \dfrac{A(s)}{B(s)}$

Closed-loop TF $F(s) = 1 + GH(s) = 1 + \dfrac{A(s)}{B(s)} = \dfrac{A(s) + B(s)}{B(s)}$

$A(s) + B(s) = 0$ Closed-loop poles = zeros of $F(s)$

$B(s) = 0$ Open-loop poles = zeros of $F(s)$

$F(s)$ Encircle the origin of the $F(s)$-plane.

$GH(s) = F(s) - 1$ Encircle the point $(-1, j0)$ of the $F(s)$-plane.

9.3 The Nyquist Criterion

Z=P+N

Z :the number of closed-loop in the right-hand S-plane

P :the number of open-loop in the right-hand S-plane

N : Γ_{GH} encircle the $(-1, j0)$ point N times in the clockwise direction.

9.3 The Nyquist Criterion

Z=P+N

- **A feedback system is stable if and only if the contour Γ_L in the $L(s)$-plane does not encircle the $(-1, 0)$ point when the number of poles of $L(s)$ in the right-hand s -plane is zero $(P = 0)$.**
- **A feedback control system is stable if and only if, for the contour Γ_L , the number of counter-clockwise encirclements of the $(-1, 0)$ point is equal to the number of poles of $L(s)$ with positive real parts.**

9.3 The Nyquist Criterion

①System with no pole at the origin

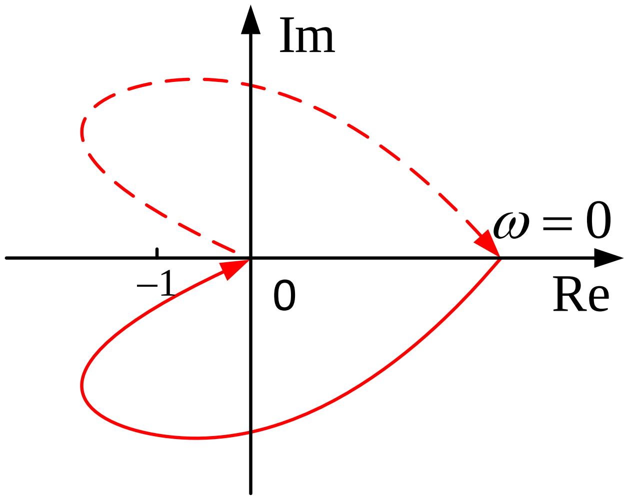
$$GH(s) = \frac{k(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)}$$

Example 1

$$G(s) = \frac{k}{(T_1 s+1)(T_2 s+1)}$$

$$G(j\omega) = \frac{k}{(j\omega T_1+1)(j\omega T_2+1)}$$

$$|G(j\omega)| = \frac{k}{\sqrt{(\omega^2 T_1^{\ 2}+1)}\sqrt{(\omega^2 T_2^{\ 2}+1)}}$$



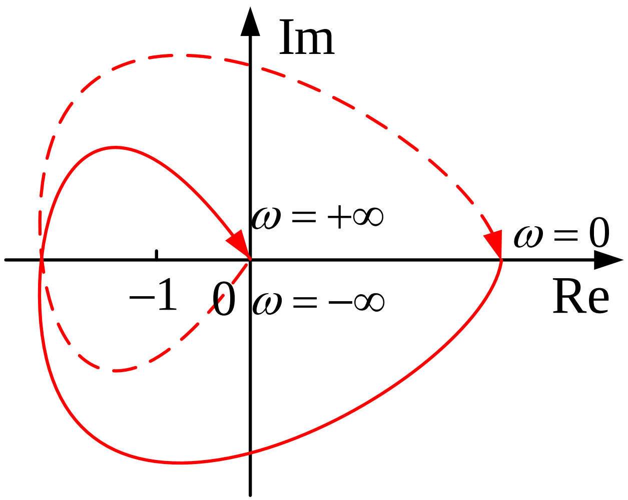
$$P=0, N=0, Z=P+N=0$$

The system is stable.

9.3 The Nyquist Criterion

①System with no pole at the origin

$$GH(s)=\frac{k(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)}$$



Example 2

$$G(s)=\frac{k}{(T_1s+1)(T_2s+1)(T_3s+1)}$$

$$G(j\omega)=\frac{k}{(j\omega T_1+1)(j\omega T_2+1)(j\omega T_3+1)}$$

$$|G(j\omega)|=\frac{k}{\sqrt{(\omega^2T_1^{\ 2}+1)(\omega^2T_2^{\ 2}+1)(\omega^2T_3^{\ 2}+1)}}$$

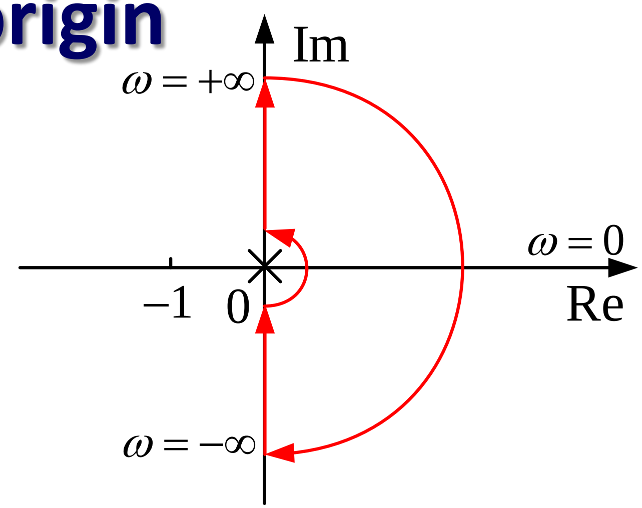
$$P=0, N=2, Z=P+N=2$$

The system is unstable.

9.3 The Nyquist Criterion

②System with poles at the origin

$$GH(s)=\frac{k(s+z_1)\cdots(s+z_m)}{s^r(s+p_1)\cdots(s+p_n)}$$



$$G(s)=\frac{k}{s(s+1)}$$

$$G(j\omega)=\frac{k}{j\omega(j\omega+1)} \quad |G(j\omega)|=\frac{k}{\omega\sqrt{(\omega^2+1)}}$$

$$P=0, N=2, Z=P+N=2$$

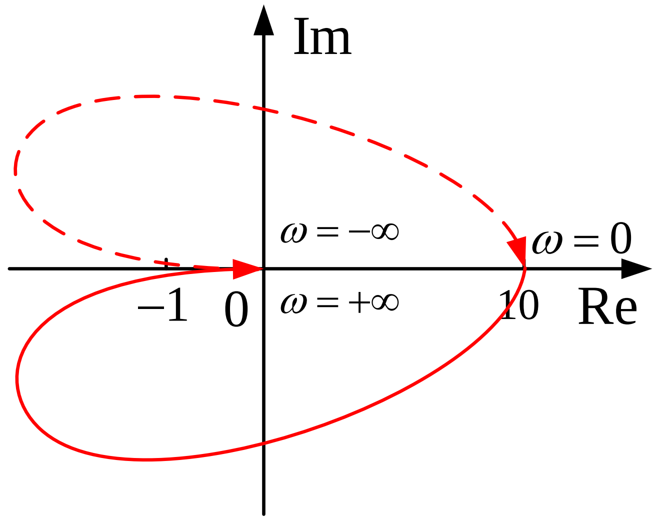
Note: If $GH(s)$ contains the integral element $\frac{1}{s^r}$, the contour Γ_{GH} ranges from an angle of $\frac{r\pi}{2}$ at $\omega=0^-$ to $-\frac{r\pi}{2}$ at $\omega=0^+$ and passes through a $r\pi$ circle whose radius is infinite.

9.3 The Nyquist Criterion

②System with poles at the origin

Example 1

$$G(s)=\frac{10}{(0.2s+1)(0.02s+1)}$$



$$G(j\omega)=\frac{10}{(0.2j\omega+1)(0.02j\omega+1)}$$

$$\left|G(j\omega)\right|=\frac{10}{\sqrt{(0.04\omega^2+1)(0.0004\omega^2+1)}}$$

$$P=0, N=0, Z=P+N=0$$

The system is stable.

9.3 The Nyquist Criterion

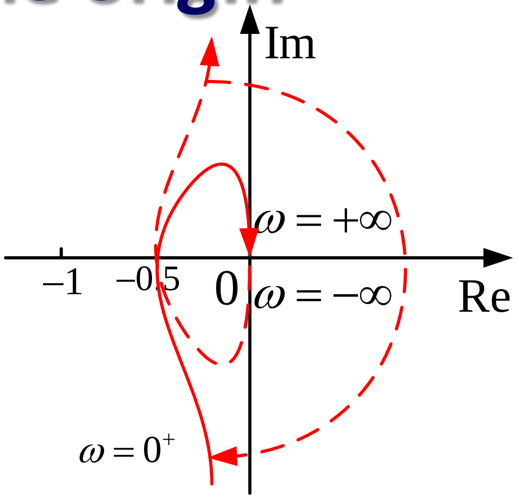
②System with poles at the origin

Example 2

$$G(s)=\frac{10}{s(0.1s+1)^2}$$

$$G(j\omega)=\frac{10}{j\omega(0.1j\omega+1)^2}$$

$$\left|G(j\omega)\right|=\frac{10}{\omega(0.01\omega^2+1)}$$



$$P=0, N=0, Z=P+N=0$$

The system is stable.

9.3 The Nyquist Criterion

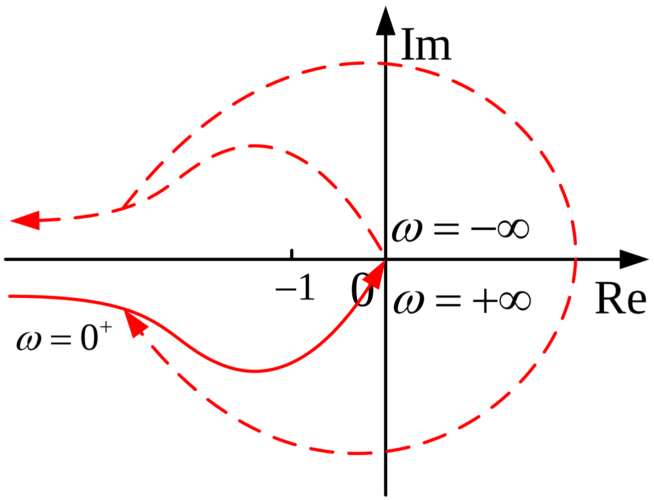
②System with poles at the origin

Example 3

$$G(s)=\frac{k(0.1s+1)}{s^2(0.01s+1)}$$

$$G(j\omega)=\frac{-k(0.1j\omega+1)}{\omega^2(0.01j\omega+1)}$$

$$|G(j\omega)|=\frac{k\sqrt{(0.01\omega^2+1)}}{\omega^2\sqrt{(0.0001\omega^2+1)}}$$



$$P=0, N=0, Z=P+N=0$$

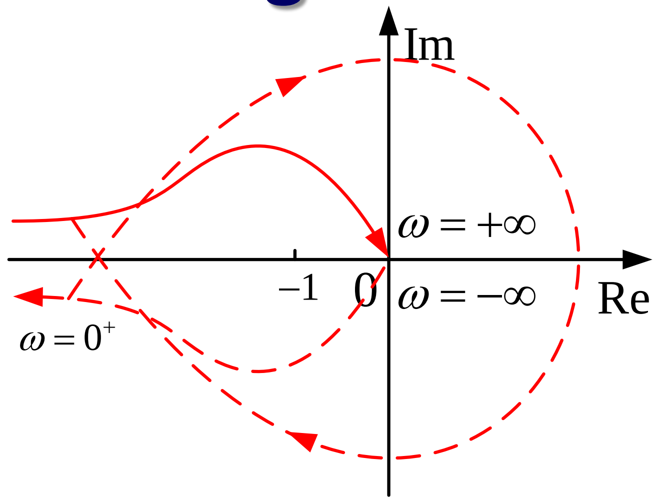
The system is stable.

9.3 The Nyquist Criterion

②System with poles at the origin

Example 4

$$G(s)=\frac{k}{s^2(Ts+1)}$$



$$G(j\omega)=\frac{-k(0.1j\omega+1)}{\omega^2(0.01j\omega+1)}$$

$$P=0, N=2, Z=P+N=2$$

$$|G(j\omega)|=\frac{k\sqrt{(0.01\omega^2+1)}}{\omega^2\sqrt{(0.0001\omega^2+1)}}$$

Two roots lie in the right-halt plane .
The system is unstable.

9.3 The Nyquist Criterion

③Non-minimum phase

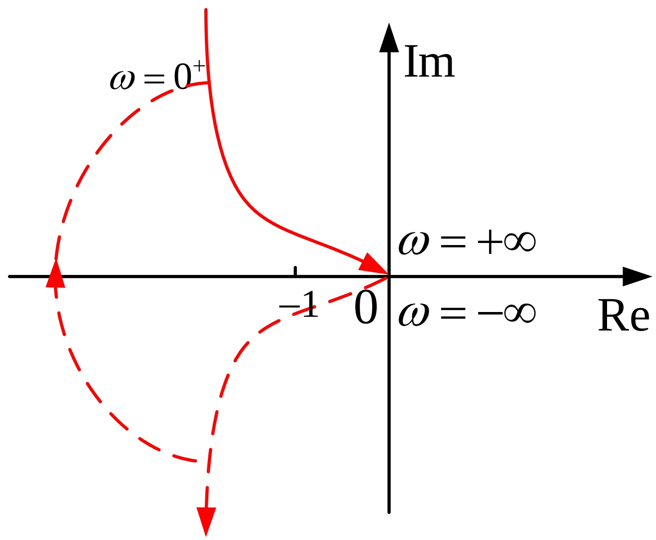
Example 5

$$G(s)=\frac{k}{s(s-1)}$$

$$G(j\omega)=\frac{k}{j\omega(j\omega-1)}$$

$$|G(j\omega)|=\frac{k}{\omega\sqrt{\omega^2+1}}$$

$$\phi(\omega)=-90^{\circ}-(\pi-\arctan\omega)$$
$$=-270^{\circ}+\arctan\omega$$



$$P=1, N=1, Z=P+N=2$$

The system is unstable.

9.3 The Nyquist Criterion

③Non-minimum phase

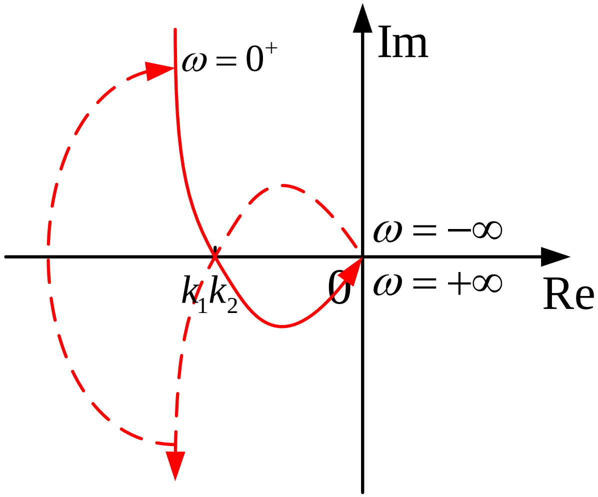
Example 6

$$G(s)=\frac{k_1(1+k_2 s)}{s(s-1)}$$

$$G(j\omega)=\frac{k_1(1+k_2 j\omega)}{j\omega(j\omega-1)} \quad |G(j\omega)|=\frac{k_1\sqrt{k_2{}^2\omega^2+1}}{\omega\sqrt{\omega^2+1}}$$

$$\phi(\omega)=-90^{\mathrm{o}}+\arctan k_2\omega-(\pi-\arctan\omega)$$

$$=-270^{\mathrm{o}}+\arctan k_2\omega+\arctan\omega$$



ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	$-270°$
$\frac{1}{\sqrt{k_2}}$	k_1k_2	$-180°$
∞	0	$-90°$

$|k_1k_2|>1, P=1, N=-1, Z=P+N=0,$ stable

$|k_1k_2|<1, P=1, N=1, Z=P+N=2,$ unstable

$|k_1k_2|=1,$ marginally stable

9.3 The Nyquist Criterion

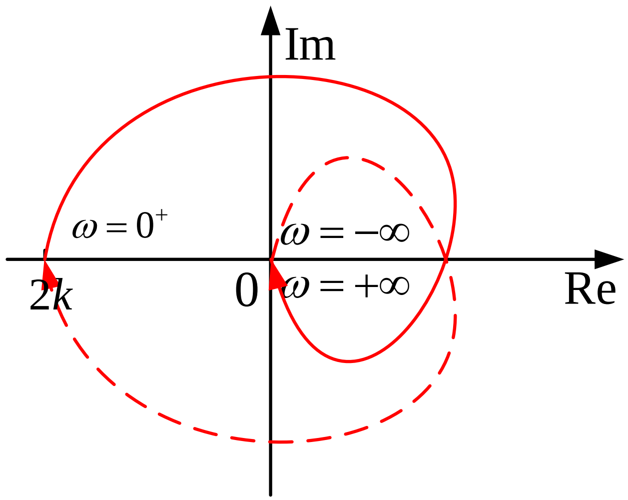
③Non-minimum phase

Example 7

$$G(s)=\frac{k(s-2)}{(s+1)^2}$$

$$G(j\omega)=\frac{k(j\omega-2)}{(j\omega+1)^2} \qquad |G(j\omega)|=\frac{k\sqrt{\omega^2+4}}{\omega^2+1}$$

$$\phi(\omega)=\pi-\arctan\frac{\omega}{2}-2\arctan\omega$$



ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	$2k$	π
∞	0	-90°

$2k>1, P=0, N=1, Z=P+N=1,$ unstable

$2k<1, P=0, N=0, Z=P+N=0,$ stable

$2k=1,$ marginally stable

9.4 Relative Stability

1. Gain margin

The gain margin (h)is the reciprocal of the gain $|GH(j\omega)|$ at the frequency at which the phase angle reaches -180° .

$$h=\frac{1}{|GH(j\omega_x)|}$$ **,when** $\phi(\omega_x)=-180^{o}$

Logarithmic measure

$$h(\mathrm{dB})=20\lg h=-20\lg h$$

In general,

$$h(\mathrm{dB})>6\,\mathrm{dB}$$

9.4 Relative Stability

2. Phase margin

The phase margin (γ)

$$\gamma(\omega_c) = 180^{\circ} + \phi(\omega_c) \qquad \text{,when } |GH(j\omega)| = 1.$$

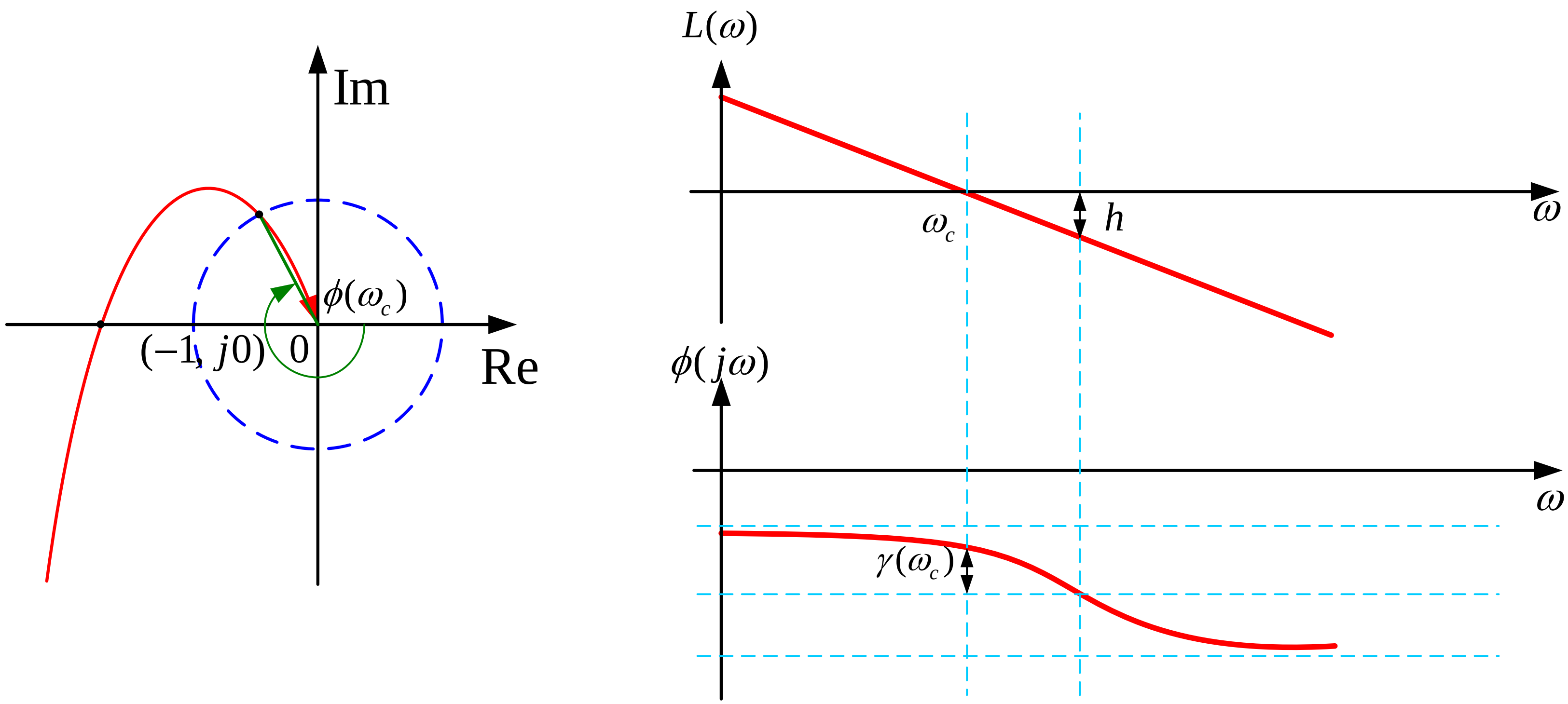
In general, $\gamma(\omega_c) \in (30^{\circ}, 60^{\circ})$**.**

Logarithmic measure when $20\lg|GH(j\omega)| = 0$**.**

Thus,

$$\begin{cases} \text{stable} & h > 1, \gamma > 0 \\ \text{marginaly} & h = 1, \gamma = 0 \\ \text{unstable} & h < 1, \gamma < 0 \end{cases}$$

9.4 Relative Stability



9.4 Relative Stability

Example 1

$$GH(s) = \frac{1}{s(s+1)(0.2s+1)}$$

$$\Rightarrow \begin{cases} L(\omega) = 20\,\mathbf{lg}\,1 - 20\,\mathbf{lg}\,\omega - 20\,\mathbf{lg}(\omega+1) - 20\,\mathbf{lg}(0.2\omega+1) \\ \phi(\omega) = -90^{\circ} - \mathbf{tan}^{-1}\,\omega - \mathbf{tan}^{-1}(\omega+1) - \mathbf{tan}^{-1}(0.2\omega+1) \end{cases}$$

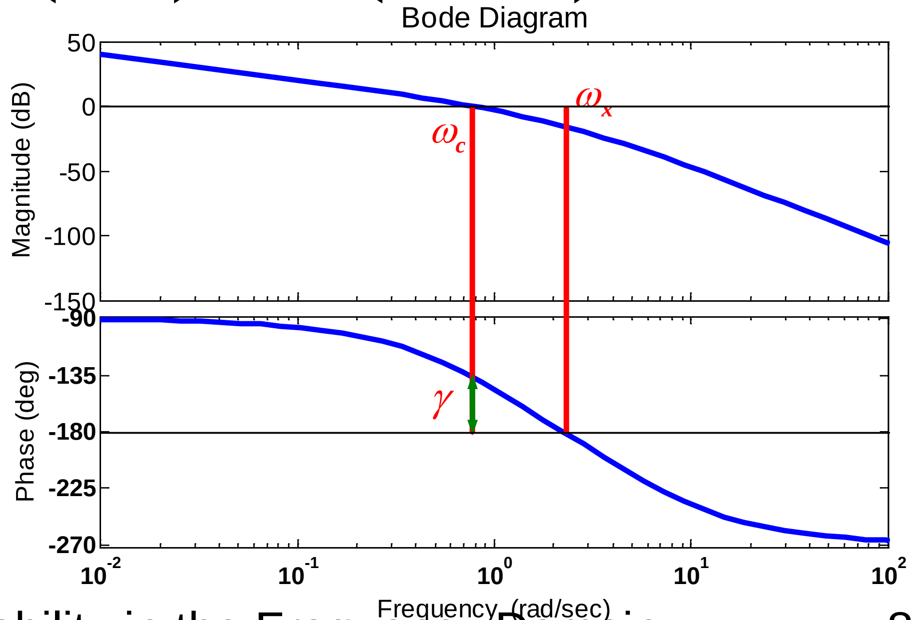
$$L(\omega_c) = 0 \Rightarrow \omega_c = 1$$

$$\gamma = 180^{\circ} + \phi(\omega_c) = 43.2^{\circ}$$

$$\phi(\omega_x) = 180^{\circ} \Rightarrow \omega_x = \sqrt{5}$$

$$h = \frac{1}{|GH(j\omega)|} = 6$$

$$h(\mathbf{dB}) = 20\,\mathbf{lg}\,6 = 15.6\,\mathbf{dB}$$



9.4 Relative Stability

Example 2

$$GH(s)=\frac{10(s+1)}{s(s-1)}$$

$$\Rightarrow\begin{cases} L(\omega)=20\,\mathbf{lg}\,10+20\,\mathbf{lg}(\omega+1)-20\,\mathbf{lg}\,\omega \\ \phi(\omega)=\mathbf{tan}^{-1}\omega-90^{\circ}-(\pi-\mathbf{tan}^{-1}\omega) \\ \qquad\;=-270^{\circ}+2\,\mathbf{tan}^{-1}\omega \end{cases}$$

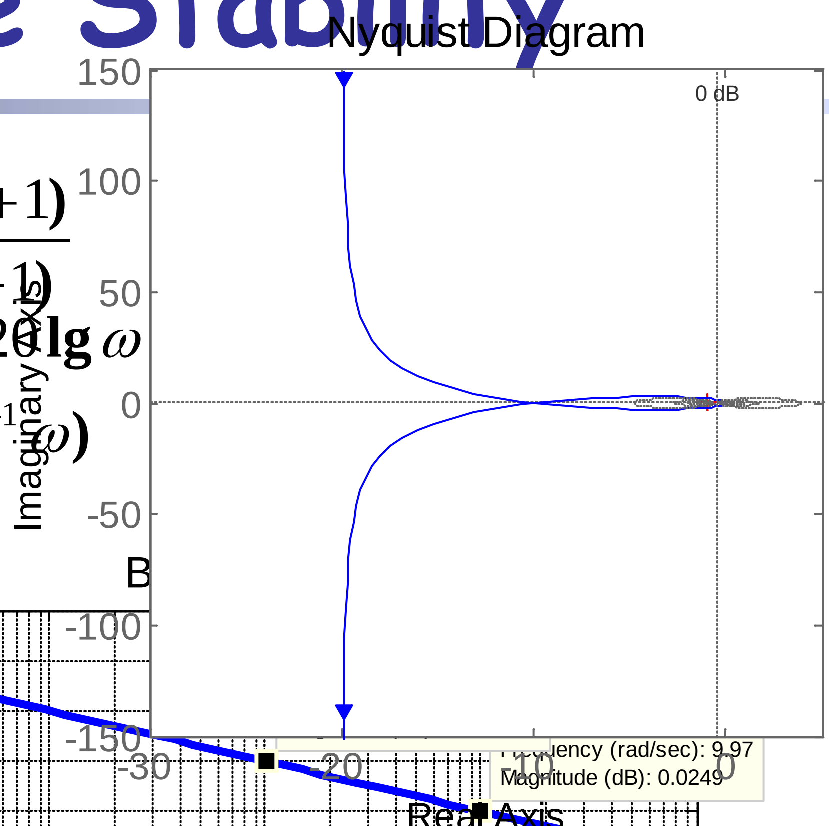
$$L(\omega_c)=0\Rightarrow\omega_c=10$$

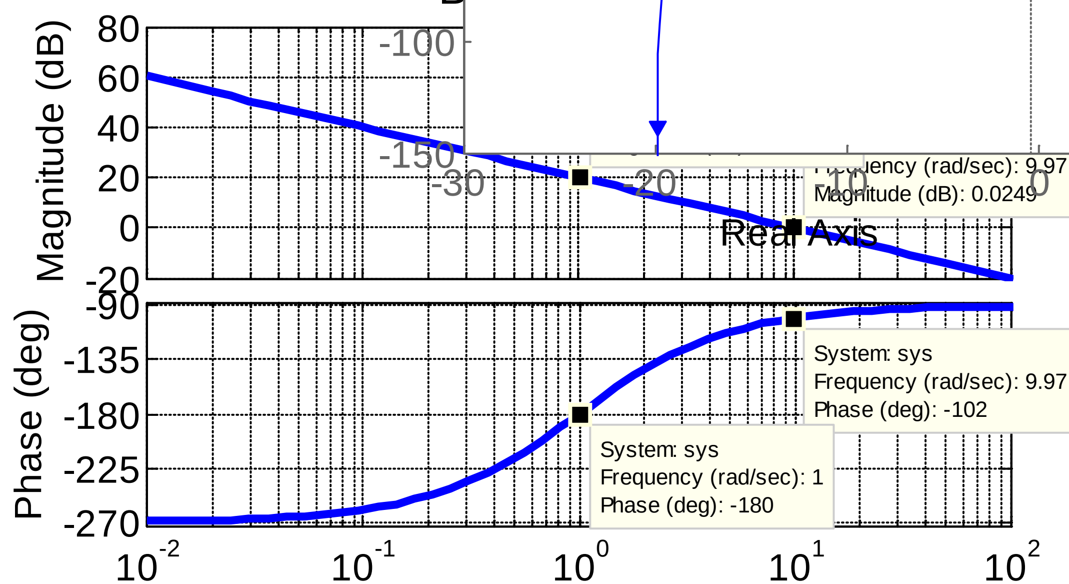
$$\gamma=180^{\circ}+\phi(\omega_c)=78.6^{\circ}$$

$$\phi(\omega_x)=180^{\circ}\Rightarrow\omega_x=1$$

$$h=\frac{1}{|GH(j\omega)|}=10$$

$$h(\mathbf{dB})=20\,\mathbf{lg}\,10=20\,\mathbf{dB}$$





9.4 Relative Stability

The relationship between the slope of $L(\omega)$ and γ

i. **Middle frequency**

   In general, the slope of L(w) in the middle frequency segment is equal to 20dB/dec.

ii. **Low frequency**

   When the slope of low frequency is bigger than middle frequency, $\gamma(\omega_c)$ decreases.

   When w1<<wc, the effect is small.

iii. **High frequency**

   When the slope of high frequency is bigger than middle frequency, $\gamma(\omega_c)$ increases.

   When w1<<wc, the effect is small.

9.4 Relative Stability

The relationship between the slope of $L(\omega)$ and γ

Summary: to ensure enough $\gamma(\omega_c)$

① The slop of $L(\omega)$ in the middle frequency segment should be equal to -20dB/dec.

② Relatively wide middle frequency segment

$$\omega_2 \geq 4\omega_1$$

9.4 Relative Stability

The relationship between k and γ

$$G(s)=\frac{k(\tau s+1)}{s^2(Ts+1)}$$

$$k\uparrow\Rightarrow\omega_c\uparrow,\omega_c\to\omega_2,\gamma\downarrow$$

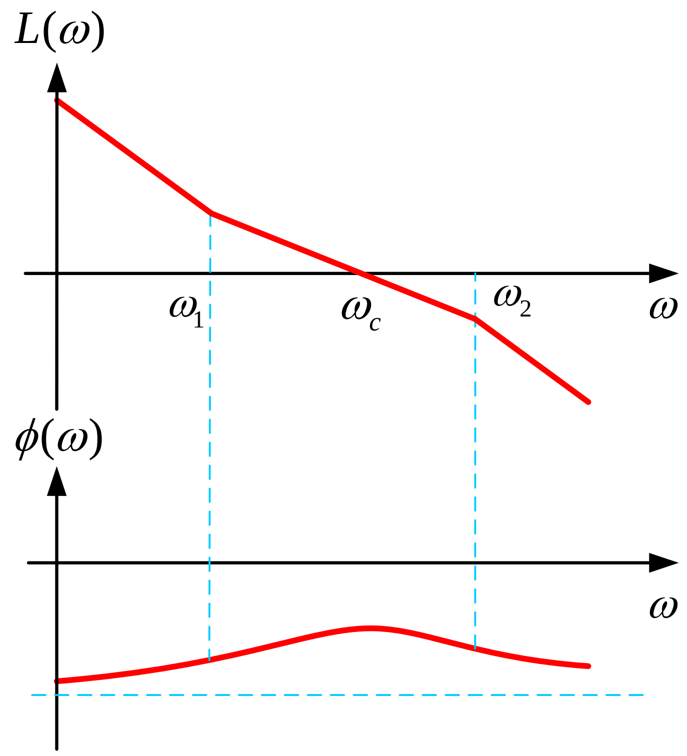
$$k\downarrow\Rightarrow\omega_c\downarrow,\omega_c\to\omega_1,\gamma\downarrow$$

$$\lg\omega_c=\frac{1}{2}(\lg\omega_1+\lg\omega_2)\Rightarrow\omega_c=\sqrt{\omega_1\omega_2}$$

Let $\omega_2=n\omega_1$, $\omega_c=\sqrt{n}\omega_1$

$$L(\omega_c)=20\lg k-20\lg\omega_c{}^2+20\lg\frac{\omega_c}{\omega_1}=0$$

$$k\frac{\omega_c}{\omega_1}=\omega_c{}^2,k=\omega_1\omega_c=\sqrt{n}\omega_1{}^2=\frac{\sqrt{n}}{\tau^2}$$



We get the max of γ when $k=\frac{\sqrt{n}}{\tau^2}$

9.4 Relative Stability

The relationship between γ、ω_c and the transient response performance.

$$G(s)=\frac{\omega_n{}^2}{s(s+2\zeta\omega_n)} \quad \Rightarrow G(j\omega)=\frac{\omega_n{}^2}{j\omega(j\omega+2\zeta\omega_n)},\left|G(j\omega)\right|=\frac{\omega_n{}^2}{\omega\sqrt{\omega^2+4\zeta^2\omega_n{}^2}}$$

$$\left|G(j\omega)\right|=1\Rightarrow \omega_c=\sqrt{\sqrt{4\zeta^2+1}-2\zeta^2}\cdot\omega_n$$

$$\gamma(\omega_c)=180^{\circ}+\phi(\omega_c)=\tan^{-1}\frac{\omega_c}{2\zeta\omega_n}$$

$$Ts=\frac{4}{\zeta\omega_n}=\frac{4}{\zeta\omega_c}\sqrt{\sqrt{4\zeta^2+1}-2\zeta^2}$$

$$=\tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{4\zeta^2+1}-2\zeta^2}}$$

The system response more quickly when ω_c is bigger.

$$\gamma\xleftrightarrow{\text{only related}}\zeta\xleftrightarrow{\text{only related}}P.O.$$

$$\gamma\uparrow\Rightarrow P.O.\downarrow$$

9.5 Summary

- **Nyquist's criterion**

  determine the stability of a feedback control system in the frequency domain.

- **Gain margin and phase margin**

  relative stability measures