

Remarks on some Limit Geometric Properties related to an Idempotent and Non-Associative Algebraic Structure

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April 8, 2025

Abstract

This article analyzes the geometric properties of an idempotent, non-associative algebraic structure that extends the Max-Times semiring. This algebraic structure is useful for studying systems of Max-Times and Max-Plus equations, employing an appropriate notion of a non-associative determinant. We consider a connected ultrametric distance and demonstrate that it implies, among other properties, an analogue of the Pythagorean relation. To this end, we introduce a suitable notion of a right angle between two vectors and investigate a trigonometric concept associated with the Chebyshev unit ball. Following this approach, we explore the potential implications of these properties in the complex plane.

We provide an algebraic definition of a line passing through two points, which corresponds to the Painlevé-Peano-Kuratowski limit of a sequence of generalized lines. We establish that this definition leads to distinctive geometric properties; in particular, two distinct parallel lines may share an infinite number of points.

AMS: 06D50, 32F17

Keywords: Generalized mean, convexity, convex hull, ultrametric, semilattice, \mathbb{B} -convexity.

Introduction

It is well known that algebras associated with idempotent semirings are related to particular geometric representations. This article investigates some of these properties through the non-associative symmetrization of Max-Times algebras. The concept of \mathbb{B} -convexity was introduced in [12] as a Painlevé-Kuratowski limit of generalized convexities.¹ This fundamental idea is related to earlier work by Ben-Tal [5] and Avriel [2]. When restricted to the nonnegative orthant, \mathbb{B} -convexity coincides with Max-Times convexity. In this context, an algebraic formulation has been established in the idempotent Max-Times semiring $(\mathbb{R}_+, \max, \times)$ by applying the transformation $+ \rightarrow \max$ in the nonnegative Euclidean orthant. The idempotent Max-Times semiring is homeomorphic to the Max-Plus semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$ via a logarithmic transformation, and \mathbb{B} -convex sets emerge from a broad class of idempotent algebraic structures introduced in [26] and derived from the notion of dequantization (see also [24]).

Hahn-Banach-type separation properties [14] and fixed-point results have been established in [13]. Other idempotent convex structures have been proposed in [1] and [30]. The algebraic formulation of \mathbb{B} -convexity was previously limited to the Max-Times semi-module $(\mathbb{R}_+^n, \vee, \times)$. To address this limitation, an idempotent and non-associative binary operation was introduced in [8] by considering a special class of idempotent and non-associative magmas. In this paper, we introduce an n -ary operation associated with a suitable class of

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¹The letter \mathbb{B} , used to define a corresponding notion of convexity may be confusing in light of the concept of convexity introduced by Anatole Beck in Banach spaces in 1962. It was chosen due to the origins of the names of several members of the mathematics department at the University of Perpignan at the time of the publication of the first paper [12]. Additionally, it can be linked to the work of Aaron Ben-Tal [5], whose paper plays a crucial role in the context of these articles. Another reason for this choice is that the operation \boxplus , pronounced “Boxplus,” bears a graphical resemblance to the Chebyshev unit ball (ℓ_∞) when represented in two dimensions with Cartesian coordinates. This ball plays a central role in the proposed approach and is itself the Hausdorff limit of the sequence of ℓ_p balls as $p \rightarrow \infty$.

regularized semicontinuous operators. It is worth noting that, when involving only two points, the idempotent and non-associative algebraic structures considered are also mentioned by Gaubert [16] and discussed Viro [31], who explored the complex case.

Note that there exist other approaches to extending idempotent algebraic structures. For example, Izhakian and Rowen [19] introduced supertropical algebra, which involves a similar type of construction based on the notion of a valued monoid. Additionally, the concept of hyperfields, introduced in [21] and [25], provides an alternative approach.

Relaxing associativity preserves both symmetry and idempotency. It has been shown that this notion of convexity can be equivalently characterized by the Kuratowski-Painlevé limit of the generalized convex hull of two points as defined in [12]. Some separation results in \mathbb{R}^n have been established in [9], and the limiting structure of polytopes has been analyzed in [10].

More recently, these structures have been examined from an algebraic perspective to compute the limit solutions of systems of Max-Times equations, particularly by introducing a special determinant formulation. These results also have geometric implications for determining the equation of a limit hyperplane passing through n points, with possible extensions to Max-Plus algebra.

This paper continues prior research on this topic from a geometric perspective, introducing appropriate notions of distance. Throughout the paper, an idempotent analogue of subtraction plays a key role.

First, we define a space notion suited to the algebraic structure under consideration and generalize the definition of \mathbb{B} -space introduced in [13]. Second, we construct a sequence of generalized metrics via an appropriate homeomorphic transformation of the vector space structure. We demonstrate that their limit properties yield an ultrametric distance intimately linked to the algebraic structure considered. To this end, we derive a notion of orthogonality from the limit inner product defined in [8]. As a result, we establish geometric properties of a special class of right-angle triples, specifically proving a Pythagorean-type relation within this framework. Using the limiting properties of the generalized inner product, we construct a pseudo-cosine limit function, where the pseudo-cosine is derived from the limiting inner product, and the pseudo-sine stems from the two-dimensional determinant formula established in [11]. Following this approach, we explore potential implications of these properties in the complex plane.

Finally, the paper investigates a notion of line corresponding to the non-associative algebraic structure under study. We show that the Painlevé-Kuratowski limit of a sequence of generalized lines admits an algebraic description extending a previous result in [8] for a line segment. We establish a condition ensuring that two boundary lines are parallel, analogous (in a certain sense) to the criterion for parallel lines in a vector space. The paper concludes with an analysis of the two-dimensional case, in which a special idempotent geometry is examined. Building on [11], we demonstrate how to compute the equation of a two-dimensional line. The final result describes a plane geometry where a unique line passes through two distinct points, and where a line segment can be extended to a half-line. However, two distinct parallel lines may share an infinite number of points.

The paper is structured as follows. Section 1 introduces the framework of the non-associative idempotent algebraic structure defined in [8], along with some associated convexity notions. Section 2 defines field and space concepts related to this algebraic structure, providing illustrative examples. Section 3 derives an ultrametric distance as the limit of a sequence of generalized metrics. In this context, a triangular equality is established for a special class of right-angle triples. Section 4 explores inner product properties established in [9] and introduces a related notion of pseudo-cosine. Section 5 presents an algebraic description of a line. The paper concludes with an analysis of the geometric implications of the considered algebraic structure.

1 Idempotent and Non-Associative Algebraic and Convex Structure

1.1 Isomorphism of vector-Space Structures in Limit

For all $p \in \mathbb{N}$, let us consider a bijection $\varphi_p : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$\varphi_p : \lambda \longrightarrow \lambda^{2p+1} \tag{1}$$

and $\phi_p(x_1, \dots, x_n) = (\varphi_p(x_1), \dots, \varphi_p(x_n))$. This is closely related to the approach proposed by Ben-Tal [5] and Avriel [2]. One can induce a field structure on \mathbb{R} for which φ_p becomes a field isomorphism. For $\lambda, \mu \in \mathbb{R}$, the indexed sum and the indexed product are simply given by

$$\lambda \stackrel{p}{+} \mu = \varphi_p^{-1}(\varphi_p(\lambda) + \varphi_p(\mu)) = (\lambda^{2p+1} + \mu^{2p+1})^{\frac{1}{2p+1}}, \quad (2)$$

and

$$\lambda \stackrel{p}{\cdot} \mu = \varphi_p^{-1}(\varphi_p(\lambda) \cdot \varphi_p(\mu)) = \lambda \cdot \mu. \quad (3)$$

These two operations define a scalar field $(\mathbb{R}, \stackrel{p}{+}, \cdot)$. Given this change of notation via φ_p and ϕ_p , we can define a \mathbb{R} -vector space structure on \mathbb{R}^n by: $\lambda \stackrel{\varphi_p}{\cdot} x = \phi_p^{-1}(\varphi_p(\lambda) \cdot \phi_p(x)) = \lambda \cdot x$ and $x \stackrel{\varphi_p}{+} y = \phi_p^{-1}(\phi_p(x) + \phi_p(y))$. We call these two operations the multiplication by scalar and the indexed sum (indexed by φ_p).

The φ_p -sum denoted $\sum_{j \in [m]}^{\varphi_p}$ of $(x^{(1)}, \dots, x^{(m)}) \in \mathbb{R}^{n \times m}$ is defined by²

$$\sum_{j \in [m]}^{\varphi_p} x^{(j)} = \phi_p^{-1}\left(\sum_{j \in [m]} \phi_p(x^{(j)})\right). \quad (4)$$

For the sake simplicity, throughout the paper we denote for all $x, y \in \mathbb{R}^n$:

$$x \stackrel{p}{+} y = x \stackrel{\varphi_p}{+} y. \quad (5)$$

Recall that Kuratowski-Painlevé lower limit of the sequence of sets $\{A_n\}_{n \in \mathbb{N}}$, denoted $\text{Li}_{n \rightarrow \infty} A_n$, is the set of points x for which there exists a sequence $\{x^{(n)}\}_{n \in \mathbb{N}}$ of points such that $x^{(n)} \in A_n$ for all n and $x = \lim_{n \rightarrow \infty} x^{(n)}$. The Kuratowski-Painlevé upper limit of the sequence of sets $\{A_n\}_{n \in \mathbb{N}}$, denoted $\text{Ls}_{n \rightarrow \infty} A_n$, is the set of points x for which there exists a subsequence $\{x^{(n_k)}\}_{k \in \mathbb{N}}$ of points such that $x^{(n_k)} \in A_{n_k}$ for all k and $x = \lim_{k \rightarrow \infty} x^{(n_k)}$. A sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets of \mathbb{R}^n is said to converge, in the Kuratowski-Painlevé sense, to a set A if $\text{Ls}_{n \rightarrow \infty} A_n = A = \text{Li}_{n \rightarrow \infty} A_n$, in which case we write $A = \text{Lim}_{n \rightarrow \infty} A_n$.

In [8] it was shown that for all $\lambda, \mu \in \mathbb{R}$ we have:

$$\lim_{p \rightarrow +\infty} \lambda \stackrel{p}{+} \mu = \begin{cases} \lambda & \text{if } |\lambda| > |\mu| \\ \frac{1}{2}(\lambda + \mu) & \text{if } |\lambda| = |\mu| \\ \mu & \text{if } |\lambda| < |\mu|. \end{cases} \quad (6)$$

Along this line one can introduce the binary operation \boxplus defined for all $\lambda, \mu \in \mathbb{R}$ by:

$$\lambda \boxplus \mu = \lim_{p \rightarrow +\infty} \lambda \stackrel{p}{+} \mu. \quad (7)$$

Though the operation \boxplus does not satisfy associativity, it can be extended by constructing a non-associative algebraic structure which returns to a given n -tuple a real value. For all $x \in \mathbb{R}^n$ and all subsets I of $[n]$, let us consider the map $\xi_I[x] : \mathbb{R} \rightarrow \mathbb{Z}$ defined for all $\alpha \in \mathbb{R}$ by

$$\xi_I[x](\alpha) = \text{Card}\{i \in I : x_i = \alpha\} - \text{Card}\{i \in I : x_i = -\alpha\}. \quad (8)$$

This map measures the symmetry of the occurrences of a given value α in the coordinates of a vector x .

For all $x \in \mathbb{R}^n$ let $\mathcal{J}_I(x)$ be a subset of I defined by

$$\mathcal{J}_I(x) = \left\{ j \in I : \xi_I[x](x_j) \neq 0 \right\} = I \setminus \{i \in I : \xi_I[x](x_i) = 0\}. \quad (9)$$

$\mathcal{J}_I(x)$ is called **the residual index set** of x . It is obtained by dropping from I all the i 's such that $\text{Card}\{j \in I : x_j = x_i\} = \text{Card}\{j \in I : x_j = -x_i\}$.

For all positive natural numbers n and for all subsets I of $[n]$, let $F_I : \mathbb{R}^n \rightarrow \mathbb{R}$ be the map defined for all $x \in \mathbb{R}^n$ by

²For all positive natural numbers n , $[n] = \{1, \dots, n\}$.

$$F_I(x) = \begin{cases} \max_{i \in \mathcal{J}_I(x)} x_i & \text{if } \mathcal{J}_I(x) \neq \emptyset \quad \text{and} \quad \xi_I[x] \left(\max_{i \in \mathcal{J}_I(x)} |x_i| \right) > 0 \\ \min_{i \in \mathcal{J}_I(x)} x_i & \text{if } \mathcal{J}_I(x) \neq \emptyset \quad \text{and} \quad \xi_I[x] \left(\max_{i \in \mathcal{J}_I(x)} |x_i| \right) < 0 \\ 0 & \text{if } \mathcal{J}_I(x) = \emptyset, \end{cases} \quad (10)$$

where $\xi_I[x]$ is the map defined in (8) and $\mathcal{J}_I(x)$ is the residual index set of x . The operation that takes an n -tuple (x_1, \dots, x_n) from \mathbb{R}^n and returns a single real value $F_I(x_1, \dots, x_n)$ is called an **n -ary extension** of the binary operation \boxplus , for all natural numbers $n \geq 1$ and all $x \in \mathbb{R}^n$, where I is a nonempty subset of $[n]$.

According to [8], for any n -tuple $x = (x_1, \dots, x_n)$, the operation can be defined as:

$$\bigboxplus_{i \in I} x_i = \lim_{p \rightarrow \infty} \sum_{i \in I}^{\varphi_p} x_i = F_I(x). \quad (11)$$

Clearly, this operation encompasses as a special case the binary operation mentioned in [16] and defined in equation (6). For all $(x_1, x_2) \in \mathbb{R}^2$:

$$\bigboxplus_{i \in \{1,2\}} x_i = x_1 \boxplus x_2.$$

For example, if $x = (-3, -2, 3, 3, 1, -3)$, we have $F_{[6]}(-3, -2, 3, 3, 1, -3) = F_{[2]}(-2, 1) = -2 = \bigboxplus_{i \in [6]} x_i$. There are some basic properties that can be inherited from the above algebraic structure: (i) for all $\alpha \in \mathbb{R}$, one has: $\alpha \left(\bigboxplus_{i \in I} x_i \right) = \bigboxplus_{i \in I} (\alpha x_i)$; (ii) suppose that $x \in \epsilon \mathbb{R}_+^n$ where ϵ is +1 or -1. Then $\bigboxplus_{i \in I} x_i = \epsilon \max_{i \in I} \{ \epsilon x_i \}$; (iii) we have $|\bigboxplus_{i \in I} x_i| \leq \bigboxplus_{i \in I} |x_i|$; (iv) for all $x \in \mathbb{R}^n$:

$$\left[x_i \boxplus \left(\bigboxplus_{j \in I \setminus \{i\}} x_j \right) \right] \in \left\{ 0, \bigboxplus_{j \in I} x_j \right\} \quad \text{and} \quad \bigboxplus_{i \in I} x_i = \bigboxplus_{i \in I} \left[x_i \boxplus \left(\bigboxplus_{j \in I \setminus \{i\}} x_j \right) \right]. \quad (12)$$

The algebraic structure $(\mathbb{R}, \boxplus, \cdot)$ can be extended to \mathbb{R}^n . Suppose that $x, y \in \mathbb{R}^n$, and let us denote $x \boxplus y = (x_1 \boxplus y_1, \dots, x_n \boxplus y_n)$. Moreover, let us consider m vectors $x^{(1)}, \dots, x^{(m)} \in \mathbb{R}^n$, and define

$$\bigboxplus_{j \in [m]} x^{(j)} = \left(\bigboxplus_{j \in [m]} x_1^{(j)}, \dots, \bigboxplus_{j \in [m]} x_n^{(j)} \right). \quad (13)$$

The n -ary operation $(x_1, \dots, x_n) \rightarrow \bigboxplus_{i \in [n]} x_i$ is not associative. To simplify the notations of the paper, for all $z \in \{z_{i_1, \dots, i_m} : i_k \in I_k, k \in [m]\}$, where I_1, \dots, I_m are m index subsets of \mathbb{N} , we use the notation:

$$\bigboxplus_{\substack{i_k \in I_k \\ k \in [m]}} z_{i_1, \dots, i_m} = \bigboxplus_{(i_1, \dots, i_m) \in \prod_{k \in [m]} I_k} z_{i_1, \dots, i_m}. \quad (14)$$

Notice that since the operation \boxplus is not associative, it can be ambiguous to apply without using the symbol \bigboxplus indexed on a given finite subset I . In the remainder, for the sake of simplicity, and when it is more convenient, we will adopt the following notational convention. For all $x \in \mathbb{R}^n$:

$$\bigboxplus_{i \in [n]} x_i = x_1 \boxplus \cdots \boxplus x_n. \quad (15)$$

In the sequel, for all $x, y \in \mathbb{R}^n$ we will often use the following notation to define an idempotent analogue of the subtraction operation:

$$x \boxminus y = x \boxplus (-y). \quad (16)$$

1.2 Scalar Products and Determinants in Limit

This section presents the algebraic properties induced by an isomorphism between a scalar field and a set and their implications on the scalar product. If $\langle \cdot, \cdot \rangle$ is the canonical inner product over \mathbb{R}^n , then there exists a symmetric bilinear form $\langle \cdot, \cdot \rangle_{\varphi_p} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by:

$$\langle x, y \rangle_{\varphi_p} = \varphi_p^{-1}(\langle \phi_p(x), \phi_p(y) \rangle) = \left(\sum_{i \in [n]} x_i^{2p+1} y_i^{2p+1} \right)^{\frac{1}{2p+1}}. \quad (17)$$

Now, let us denote $[\langle y, \cdot \rangle_{\varphi_p} \leq \lambda] = \{x \in \mathbb{R}^n : \langle y, x \rangle_{\varphi_p} \leq \lambda\}$. For the sake of simplicity, let us denote $\langle \cdot, \cdot \rangle_p$ this scalar product.

In what follows, we state some results obtained in [8, 9] by taking the limit. We first introduce the operation $\langle \cdot, \cdot \rangle_\infty : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined for all $x, y \in \mathbb{R}^n$ by $\langle x, y \rangle_\infty = \bigoplus_{i \in [n]} x_i y_i$. Let $\|\cdot\|_\infty$ be the Tchebychev norm defined by $\|x\|_\infty = \max_{i \in [n]} |x_i|$. It is established in [8] that for all $x, y \in \mathbb{R}^n$, we have: (i) $\sqrt{\langle x, x \rangle_\infty} = \|x\|_\infty$; (ii) $|\langle x, y \rangle_\infty| \leq \|x\|_\infty \|y\|_\infty$; (iii) for all $\alpha \in \mathbb{R}$, $\alpha \langle x, y \rangle_\infty = \langle \alpha x, y \rangle_\infty = \langle x, \alpha y \rangle_\infty$. By definition, we have for all $x, y \in \mathbb{R}^n$:

$$\langle x, y \rangle_\infty = \lim_{p \rightarrow \infty} \langle x, y \rangle_p. \quad (18)$$

A map $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as $f(x) = \langle a, x \rangle_\infty$ for some $a \in \mathbb{R}^n$ is called an **idempotent symmetric form**. In the following, for all maps $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and all real numbers c , the notation $[f \leq c]$ stands for the set $f^{-1}([\infty, c])$. Similarly, $[f < c]$ stands for $f^{-1}([\infty, c[)$ and $[f \geq c] = [-f \leq -c]$.

For all $u, v \in \mathbb{R}$, let us define the binary operation

$$u \bar{\odot} v = \begin{cases} u & \text{if } |u| > |v| \\ \min\{u, v\} & \text{if } |u| = |v| \\ v & \text{if } |u| < |v|. \end{cases}$$

Note that in [19], a polynomial theory for supertropical algebra was proposed. To achieve this, a similar operation was constructed by extending the tropical semiring with a suitable valuation function. Both proposed constructions follow a similar principle. However, in our approach, this binary operation is derived from the idempotent semiring $(\mathbb{R}_+, \max, \cdot)$ and the absolute value function.

An elementary calculus shows that $u \boxplus v = \frac{1}{2} [u \bar{\odot} v - [(-u) \bar{\odot} (-v)]]$. It was shown in [14], that this operation is associative.

Given m elements u_1, \dots, u_m of \mathbb{R} , not all of which are 0, let I_+ , respectively I_- , be the set of indices for which $0 < u_i$, respectively $u_i < 0$. We can then write $u_1 \bar{\odot} \dots \bar{\odot} u_m = (\bar{\odot}_{i \in I_+} u_i) \bar{\odot} (\bar{\odot}_{i \in I_-} u_i) = (\max_{i \in I_+} u_i) \bar{\odot} (\min_{i \in I_-} u_i)$ from which we have

$$u_1 \bar{\odot} \dots \bar{\odot} u_m = \begin{cases} \max_{i \in I_+} u_i & \text{if } I_- = \emptyset \text{ or } \max_{i \in I_-} |u_i| < \max_{i \in I_+} u_i \\ \min_{i \in I_-} u_i & \text{if } I_+ = \emptyset \text{ or } \max_{i \in I_+} u_i < \max_{i \in I_-} |u_i| \\ \min_{i \in I_-} u_i & \text{if } \max_{i \in I_-} |u_i| = \max_{i \in I_+} u_i. \end{cases} \quad (19)$$

We define an **idempotent lower symmetric form** on \mathbb{R}^n as a map $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for all $(x_1, \dots, x_n) \in \mathbb{R}^n$,

$$g(x_1, \dots, x_n) = \langle a, x \rangle_\infty^- = a_1 x_1 \bar{\odot} \dots \bar{\odot} a_n x_n. \quad (20)$$

It was established in [8] that for all $c \in \mathbb{R}$, the set $g^{-1}([\infty, c]) = \{x \in \mathbb{R}^n \mid g(x) \leq c\}$ is closed. It follows that an idempotent lower symmetric form is lower semi-continuous. It was also shown in [8] that for any $a \in \mathbb{R}^n$, the idempotent lower symmetrical form g defined by $g(x_1, \dots, x_n) = a_1 x_1 \bar{\odot} \dots \bar{\odot} a_n x_n$, is the lower semi-continuous regularization of the map $x \mapsto \langle a, x \rangle_\infty = \bigoplus_{i \in [n]} a_i x_i$.

Similarly, one can introduce a binary operation defined for all $u, v \in \mathbb{R}$ as

$$u \dot{+} v = -[(-u) \bar{\odot} (-v)], \quad (21)$$

which is associative. Along these lines, one can define an **idempotent upper symmetric form** as a map $h : \mathbb{R}^n \rightarrow \mathbb{R}$ such that, for all $(x_1, \dots, x_n) \in \mathbb{R}^n$,

$$h(x_1, \dots, x_n) = \langle a, x \rangle_\infty^+ = a_1 x_1 \dot{+} \dots \dot{+} a_n x_n. \quad (22)$$

For all $x \in \mathbb{R}^n$, we clearly, have the following identities

$$\langle a, x \rangle_\infty^+ = -\langle a, -x \rangle_\infty^- \text{ and } \langle a, x \rangle_\infty^- = -\langle a, -x \rangle_\infty^+. \quad (23)$$

For any $a \in \mathbb{R}^n$, the idempotent upper symmetric form h defined by $h(x_1, \dots, x_n) = a_1 x_1 \dot{+} \dots \dot{+} a_n x_n$, is the upper semi-continuous regularization of the map $x \mapsto \langle a, x \rangle_\infty = \bigoplus_{i \in [n]} a_i x_i$.

These forms can be related to the construction of a ring involving a balance relation symmetrizing the semi-ring $(\mathbb{R}_+, \vee, \cdot)$ (see [11], see also [18] and [28] in a Max-Plus context).

More, recently, the following notion of determinant was introduced in [11]. The set \mathbb{R} , equipped with the operation \boxplus , allows for the definition of a suitable notion of determinant (see [10, 11]).

We denote by $\mathcal{M}_n(\mathbb{R})$ the set of square matrices of order n with real entries. We introduce over $\mathcal{M}_n(\mathbb{R})$ the map $\Phi_p : \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathcal{M}_n(\mathbb{R})$ defined for all matrices $A = (a_{i,j})_{\substack{i=1\dots n \\ j=1\dots n}} \in \mathcal{M}_n(\mathbb{R})$ by:

$$\Phi_p(A) = (\varphi_p(a_{i,j}))_{\substack{i=1\dots n \\ j=1\dots n}} = (a_{i,j}^{2p+1})_{\substack{i=1\dots n \\ j=1\dots n}}. \quad (24)$$

Its reciprocal is the map $\Phi_p^{-1} : \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathcal{M}_n(\mathbb{R})$ defined by:

$$\Phi_p^{-1}(A) = (\varphi_p^{-1}(a_{i,j}))_{\substack{i=1\dots n \\ j=1\dots n}} = (a_{i,j}^{\frac{1}{2p+1}})_{\substack{i=1\dots n \\ j=1\dots n}}. \quad (25)$$

Φ_p is a natural extension of the map ϕ_p from \mathbb{R}^n to $\mathcal{M}_n(\mathbb{R})$. $\Phi_p(A)$ is the $2p+1$ Hadamard power of matrix A . In the following we introduce the matrix product:

$$A \overset{p}{\cdot} x = \phi_p^{-1}(\Phi_p(A) \cdot \phi_p(x)). \quad (26)$$

It is easy to see that the map $x \mapsto A \overset{p}{\cdot} x$ is φ_p -linear, that is linear with respect to the algebraic operation $\overset{p}{\cdot}$ and the usual scalar multiplication. Conversely, if g is a φ_p -linear map then it can be represented by a matrix A such that $g(x) = A \overset{p}{\cdot} x$ for all $x \in \mathbb{R}^n$.

For any $n \times n$ matrix A , let $|A|$ denote its determinant. We now introduce the following definition of the φ_p -determinant:

$$|A|_p = \varphi_p^{-1}|\Phi_p(A)|. \quad (27)$$

Using the Leibnitz formula yields

$$|A|_p = \left(\sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i \in [n]} a_{i,\sigma(i)}^{2p+1} \right)^{\frac{1}{2p+1}}, \quad (28)$$

where \mathfrak{S}_n is the set of all the permutations defined on $[n]$.

Along this line, one can define a determinant in limit as follows:

$$\lim_{p \rightarrow \infty} |A|_p := |A|_\infty = \bigoplus_{\sigma \in \mathfrak{S}_n} (\text{sgn}(\sigma) \prod_{i \in [n]} a_{i,\sigma(i)}). \quad (29)$$

This notion is compared in [11] to that of permanent originally defined in the framework of Max-Plus algebras (see [7] and [6]).

1.3 An Idempotent and Non-associative Convex Structure

In [12], \mathbb{B} -convexity is introduced as a limit of linear convexities. The ϕ_p -convex hull of a finite set $A = \{x^{(1)}, \dots, x^{(m)}\} \subset \mathbb{R}^n$ is defined by:

$$Co^{\phi_p}(A) = \left\{ \sum_{i \in [m]}^{\varphi_p} t_i \overset{\varphi_p}{\cdot} x^{(i)} : \sum_{i \in [m]}^{\varphi_p} t_i = \varphi_p^{-1}(1), \varphi_p(t_i) \geq 0, i \in [m] \right\} \quad (30)$$

which can be rewritten using the fact that $\varphi_p^{-1}(1) = 1$ and φ_p is increasing:

$$Co^{\phi_p}(A) = \left\{ \phi_p^{-1} \left(\sum_{i \in [m]} t_i^{2p+1} \cdot \phi_p(x^{(i)}) \right) : \left(\sum_{i \in [m]} t_i^{2p+1} \right)^{\frac{1}{2p+1}} = 1, t_i \geq 0, i \in [m] \right\}. \quad (31)$$

Moreover, for all $L \subset \mathbb{R}$, we simplify the notations denoting $\text{Co}^p(L) = \text{Co}^{\phi_p}(L)$. The definition of \mathbb{B} -convex sets is based on the definition of the Kuratowski-Painlevé limit of a sequence of ϕ_p -convex sets. For a non-empty finite subset $A \subset \mathbb{R}^n$ we let $\text{Co}^\infty(A) = \lim_{p \rightarrow \infty} \text{Co}^p(A)$. A subset C of \mathbb{R}^n is \mathbb{B} -convex if for all finite subsets A of C , $\text{Co}^\infty(A) \subset C$. The structure of \mathbb{B} -convex sets in \mathbb{R}_+^n will involve the order structure, with respect to the positive cone of \mathbb{R}^n ; we denote by $\bigvee_{i \in [m]} x^{(i)}$ the least upper bound of $x^{(1)}, \dots, x^{(m)} \in \mathbb{R}^n$, that is:

$$\bigvee_{i \in [m]} x^{(i)} = (\max\{x_1^{(1)}, \dots, x_1^{(m)}\}, \dots, \max\{x_n^{(1)}, \dots, x_n^{(m)}\})$$

When $A = \{x^{(1)}, \dots, x^{(m)}\} \subset \mathbb{R}_+^n$, it was shown in [12] that:

$$\text{Co}^\infty(A) = \left\{ \bigvee_{i \in [m]} t_i x^{(i)} : t_i \in [0, 1], \max_{i \in [m]} t_i = 1 \right\} = \lim_{p \rightarrow \infty} \text{Co}^p(A).$$

Moreover it is also proved that the sequence is Hausdorff convergent. In addition, if the set A is contained in any n -dimensional orthant, it was established in [8] that $\text{Co}^\infty(A) = \left\{ \bigoplus_{i \in [m]} t_i x^{(i)} : t_i \in [0, 1], \max_{i \in [m]} t_i = 1 \right\}$.

If $L \subset \mathbb{R}_+^n$ is \mathbb{B} -convex, then $L = \lim_{p \rightarrow \infty} \text{Co}^p(L) = \bigcap_{r=0}^\infty \text{Co}^p(L)$. For all $S \subset \mathbb{R}_+^n$, we denote $\text{Co}^\infty(S) = \lim_{p \rightarrow \infty} \text{Co}^p(S)$. Properties of this operation are developed in [12]. In particular, for all $S \subset \mathbb{R}_+^n$, $\text{Co}^\infty(S)$ is \mathbb{B} -convex or equivalently Max-Times convex.

An important limitation of this definition of \mathbb{B} -convexity is that its algebraic form is restricted to the non-negative orthant \mathbb{R}_+^n , over which the idempotent Max-Times semiring is considered. To circumvent this difficulty an extended definition of \mathbb{B} -convexity other \mathbb{R}^n was suggested in [8]. A subset C of \mathbb{R}^n is said to be **idempotent symmetric convex**³ if, for all $x, y \in C$ and for all $t \in [0, 1]$,

$$x \boxplus ty \in C. \quad (32)$$

It was shown in [8] that the sequence of subsets $\{\text{Co}^{(p)}(x, y)\}_{p \in \mathbb{N}}$ of \mathbb{R}^n has a Painlevé-Kuratowski limit denoted $\text{Co}^\infty(x, y)$. These points can be expressed through an explicit algebraic formula in term of the idempotent and non-associative algebraic structure defined on $(\mathbb{R}^n, \boxplus, \cdot)$.

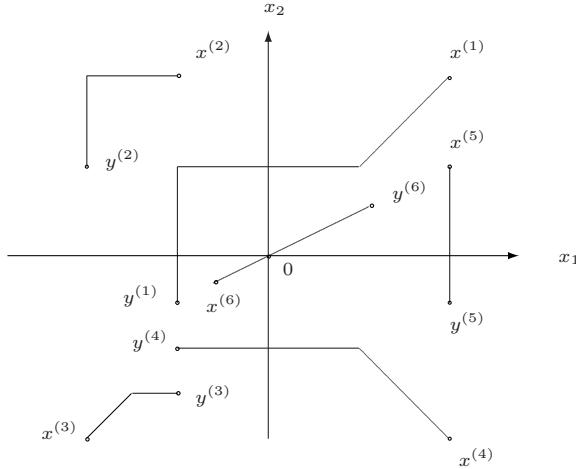


Fig. 1: Examples of sets $\text{Co}^\infty(x, y)$.

Moreover, it was established in [8] that this limit entirely characterizes an idempotent symmetric convex set. A subset C of \mathbb{R}^n is idempotent symmetric convex if and only if for all $x, y \in \mathbb{R}^n$ $\text{Co}^\infty(x, y) \subset C$. More recently, it has been established in [10] that for all $x, y \in \mathbb{R}^n$, $\text{Co}^\infty(x, y)$ is idempotent symmetric convex and thereby $\text{Co}^\infty(x, y)$ is the smallest idempotent symmetric convex set that contains $\{x, y\}$.

³Referred to as $\mathbb{B}^\#$ -convexity in [8], this concept is given a more intuitive terminology here.

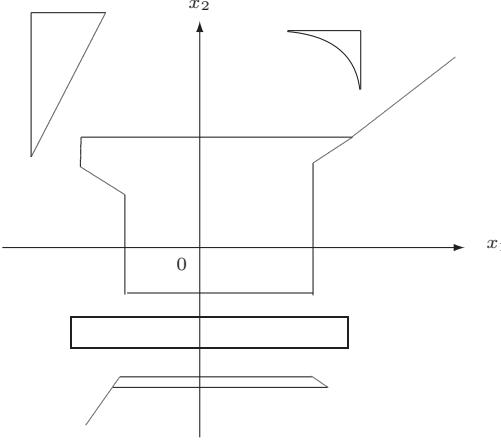


Fig. 2: Examples of idempotent symmetric convex sets

2 Idempotent Symmetric and Ultrametric Spaces

The concept of \mathbb{B} -space, as introduced in [14], is restricted to the non-negative n -dimensional orthant \mathbb{R}_+^n . In this work, we extend it to the entire Euclidean vector space. This new definition will be useful for analyzing the algebraic structure of a line defined on $(\mathbb{R}^n, \boxplus, \cdot)$.

2.1 Idempotent Symmetric Fields and Spaces

A finite-dimensional \mathbb{B} -space (Max-Times space) is, by definition, a subset X of \mathbb{R}_+^n such that $0 \in X$, for all $t \geq 0$ and all $x \in X$, $tx \in X$ and for all $x, y \in X$, $x \vee ty \in X$.

We first define a suitable notion of **idempotent pseudo-field**. By comparing this definition with that of a field, note that we consider a weakened form of associativity and idempotence is added.

Definition 2.1.1 A triple (K, \boxplus_K, \cdot_K) is an idempotent pseudo-field if:

(a) There exists a binary operation $\boxplus_K : K \times K \rightarrow K$ that satisfies the following properties: (i) for all $\lambda \in K$ we have $\lambda \boxplus_K \lambda = \lambda$ (idempotence); (ii) for all $(\lambda, \mu) \in K \times K$ we have $\lambda \boxplus_K \mu = \mu \boxplus_K \lambda$ (commutativity); (iii) there exists a neutral element $0_K \in K$ such that $\lambda \boxplus_K 0_K = 0_K \boxplus_K \lambda = \lambda$ for all $\lambda \in K$; (iv) every $\lambda \in K$ has a symmetric element $-\lambda$ such that $(-\lambda) \boxplus_K \lambda = \lambda \boxplus_K (-\lambda) = 0_X$ (symmetry); (v) for any mutually non-symmetric triple of elements $(\lambda, \mu, \eta) \in K^3$, $(\lambda \boxplus_K \mu) \boxplus_K \eta = \lambda \boxplus_K (\mu \boxplus_K \eta)$ (weakened form of associativity).

(b) There exists a scalar multiplication $\cdot_K : K \times K \rightarrow K$ that satisfies the following properties: (i) for all $\lambda \in K$ and all $(\lambda, \mu) \in K \times K$, we have $\lambda \cdot_K \mu = \mu \cdot_K \lambda$ (commutativity); (ii) for all $\lambda, \mu, \eta \in K \times K \times K$ we have $(\lambda \cdot_K \mu) \cdot_K \eta = \lambda \cdot_K (\mu \cdot_K \eta)$ (associativity); (iii) there is an element $1_K \in K$ such that for all $\lambda \in K$ we have $1_K \cdot_K \lambda = \lambda$; (iv) for all $\lambda \in K \setminus \{0_K\}$ there exists an element λ^{-1} such that $\lambda \cdot_K \lambda^{-1} = 1_K$.

(c) The multiplication \cdot_K is distributive over the operation \boxplus_K : for all $\lambda \in K$ and all $(\mu, \nu) \in K \times K$, we have

$$\lambda \cdot_K (\mu \boxplus_K \nu) = (\lambda \cdot_K \mu) \boxplus_K (\lambda \cdot_K \nu).$$

Example 2.1.2 From [8], the triple $(\mathbb{R}, \boxplus, \cdot)$, where \boxplus is the binary operation defined in (6) is an idempotent pseudo-field.

In the following, when we use the \boxplus symbol without further specification on \mathbb{R}^n this will mean that we are considering the binary operation defined in equation (6) and its related n -ary extension.

Example 2.1.3 We consider the case of the non-associative symmetrisation of the Max-Plus semi-module analysed in [11]. Let $\mathbb{M} = \mathbb{R} \cup \{-\infty\}$ and let us denote $(\mathbb{M}, \oplus, \otimes)$ the Maslov's semi-module where we replace the operations \vee with \oplus and $+$ with \otimes . More precisely one can define on \mathbb{M} the operations \oplus and \otimes respectively as $x \oplus y = \max\{x, y\}$ and $x \otimes y = x + y$, where $-\infty$ is the neutral element of the operation \oplus .

Suppose now that $x \in \mathbb{R}_-$ and let us extend the logarithm function to the whole set of real numbers. This we do by introducing the set

$$\tilde{\mathbb{M}} = \mathbb{M} \cup (\mathbb{R} + i\pi) \quad (33)$$

where i is the complex number such that $i^2 = -1$ and $\mathbb{R} + i\pi = \{x + i\pi : x \in \mathbb{R}\}$. Note that this approach is not an extension to the complex numbers. In fact, it designates a copy of the real numbers. Note also that in [19], the authors propose a closely related construction, though not a symmetric one, by creating a copy of the real numbers. The formalism proposed in this paper is however convenient for introducing the following extended logarithmic function to $\tilde{\mathbb{M}}$ and transferring the algebraic structure of the real set. Let $\psi_{\ln} : \mathbb{R} \rightarrow \tilde{\mathbb{M}}$ be the map defined by:

$$\psi_{\ln}(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ -\infty & \text{if } x = 0 \\ \ln(-x) + i\pi & \text{if } x < 0. \end{cases} \quad (34)$$

The map $x \mapsto \psi_{\ln}(x)$ is an isomorphism from \mathbb{R} to $\tilde{\mathbb{M}}$. Let $\psi_{\exp}(x) : \tilde{\mathbb{M}} \rightarrow \mathbb{R}$ be its inverse. Notice that $\psi_{\ln}(-1) = i\pi$. By definition we have $z \tilde{\boxplus} u = \psi_{\ln}(\psi_{\exp}(z) \boxplus \psi_{\exp}(u))$. Moreover, we have $z \tilde{\otimes} u = \psi_{\ln}(\psi_{\exp}(z) \otimes \psi_{\exp}(u))$. It follows that $(\tilde{\mathbb{M}}, \tilde{\oplus}, \tilde{\otimes})$ is isomorphic to $(\mathbb{R}, \boxplus, \cdot)$ and is also an idempotent pseudo-field with the neutral elements $0_{\tilde{\mathbb{M}}} = -\infty$ and $1_{\tilde{\mathbb{M}}} = 0$.

In Chapter 3 of [22] it was shown that Max-Plus algebra can also be viewed as a limit algebraic structure via the dequantization principle. For all $z \in \tilde{\mathbb{M}}^n$, let us denote:

$$\widetilde{\bigoplus}_{i \in [n]} z_i = \psi_{\ln} \left(\bigoplus_{i \in [n]} \psi_{\exp}(z_i) \right). \quad (35)$$

From [11], it follows that for all $z \in \tilde{\mathbb{M}}^n$, we have

$$\widetilde{\bigoplus}_{i \in [n]} z_i = \lim_{p \rightarrow \infty} \frac{1}{2p+1} \psi_{\ln} \left(\sum_{i \in [n]} \psi_{\exp}((2p+1)z_i) \right).$$

In the following, we give another example arising in a Min-Times context (see [1] for more details.)

Example 2.1.4 Let us define $\mathbb{R}^{-1} := \mathbb{R} \setminus \{0\} \cup \{\infty\}$. Let us consider the binary operation $\boxplus^{-1} : \mathbb{R}^{-1} \times \mathbb{R}^{-1} \rightarrow \mathbb{R}^{-1}$ defined as:

$$x \boxplus^{-1} y = \begin{cases} x & \text{if } |x| < |y| \\ y & \text{if } |x| > |y| \\ +\infty & \text{if } x + y = 0 \\ x & \text{if } x = y. \end{cases} \quad (36)$$

This operation is idempotent, has $+\infty$ as neutral element and is symmetric. Moreover, the standard product of non-zero numbers in \mathbb{R}^{-1} satisfies the desired condition.

Remark 2.1.5 In [31], an idempotent algebraic structure was derived from a suitable dequantization principle defined over the complex scalar field. Let us consider the binary operation $\boxplus_{\mathbb{C}} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ defined as:

$$z \boxplus_{\mathbb{C}} w = \begin{cases} z & \text{if } |z| > |w| \\ w & \text{if } |z| < |w| \\ 0 & \text{if } z + w = 0 \\ \rho \frac{z+w}{|z+w|} & \text{if } |z| = |w| = \rho > 0. \end{cases} \quad (37)$$

It is easy to check that this operation is idempotent, has 0 as neutral element and is symmetric. Moreover, the standard product of complex numbers fulfills all the desired conditions. In addition note that this product is distributive over the operation $\boxplus_{\mathbb{C}}$. In particular, if $|z| = |w| = \rho > 0$ then we have for all $u \in \mathbb{C}$:

$$u \cdot (z \boxplus_{\mathbb{C}} w) = u\rho \frac{z+w}{|z+w|} = \rho \frac{uz+uw}{|z+w|} = \rho|u| \frac{uz+uw}{|uz+uw|} = (uz) \boxplus_{\mathbb{C}} (uw).$$

However, it does not satisfy the weakened form of associativity of an idempotent pseudo-field.

Definition 2.1.6 A triple (X, \boxplus_X, \cdot_K) is an idempotent symmetric space defined over a idempotent pseudo-field K if:

- (a) There exists a binary operation $\boxplus_X : X \times X \rightarrow X$ that satisfies the following properties: (i) for all $x \in X$ we have $x \boxplus_X x = x$; (ii) for all $(x, y) \in X \times X$ we have $x \boxplus_X y = y \boxplus_X x$; (iii) there exists a neutral element $0_X \in X$ such that $x \boxplus_X 0_X = 0_X \boxplus_X x$ for all $x \in X$; (iv) every $x \in X$ has a symmetric element $-x$ such that $(-x) \boxplus_X x = x \boxplus_X (-x) = 0_X$.
- (b) There exists a scalar multiplication $\cdot_K : \mathbb{R} \times X \rightarrow X$ that satisfies the following properties: (i) for all $\lambda \in \mathbb{R}$ and all $(x, y) \in X \times X$, we have $\lambda \cdot_K (x \boxplus_X y) = (\lambda \cdot_K x) \boxplus_X (\lambda \cdot_K y)$; (ii) for all $\lambda, \mu \in \mathbb{R} \times \mathbb{R}$ we have $(\lambda \boxplus_K \mu) \cdot_K x = (\lambda \cdot_K x) \boxplus_X (\mu \cdot_K x)$, (iii) for all $\lambda, \mu \in \mathbb{R} \times \mathbb{R}$ we have $(\lambda \cdot_K \mu) \cdot_K x = \lambda \cdot_K (\mu \cdot_K x)$; (iv) for all $x \in X$ we have $1 \cdot_K x = x$ and $-x = (-1) \cdot_K x$.

Suppose that X is an idempotent symmetric space. A subset Y of X is an **idempotent symmetric subspace** if for all $x, y \in Y$ and all $t \in \mathbb{R}$, $x \boxplus_X ty \in Y$. Notice that, in this definition, associativity has been replaced with idempotence. Clearly, $(\mathbb{R}, \boxplus, \cdot)$ is an idempotent symmetric space.

Lemma 2.1.7 Suppose that (X, \boxplus_X, \cdot_K) is an idempotent symmetric space defined over an idempotent pseudo-field K . Then, for all $x \in X$, one has:

- (a) $\lambda \cdot_K 0_X = 0_X$
- (b) $0_K \cdot_K x = 0_X$
- (c) $((-y) \boxplus_X (-x)) \boxplus_X (x \boxplus_X y) = 0_X$.

Proof: (a) By definition, for all $x \in X$ and all $\lambda \in K$, $\lambda \cdot_K 0_X = \lambda \cdot_K (x \boxplus_X (-x))$ for all $x \in X$. Using axioms (b₁), (b₂) and (b₄) yields $\lambda \cdot_K 0_X = (\lambda \cdot_K x) \boxplus_X (\lambda \cdot_K (-x)) = (\lambda \cdot_K x) \boxplus_X (\lambda \cdot_K (-1) \cdot_K x) = (\lambda \cdot_K x) \boxplus_X ((-1) \cdot_K \lambda \cdot_K x) = (\lambda \cdot_K x) \boxplus_X ((-\lambda \cdot_K x)) = 0_X$. (b) We have for all $x \in X$, $0_K \cdot_K x = (1_K \boxplus (-1_K))x = (1_K \cdot_K x) \boxplus_X ((-1_K) \cdot_K x) = x \boxplus_X (-x) = 0_X$. (c) For all $x, y \in X$, $((-y) \boxplus_X (-x)) \boxplus_X (x \boxplus_X y) = ((-1_K) \cdot_K y \boxplus_X (-1_K) \cdot_K x) \boxplus_X (x \boxplus_X y) = ((-1_K) \cdot (y \boxplus_X x)) \boxplus_X (x \boxplus_X y) = -(x \boxplus_X y) \boxplus_X (x \boxplus_X y) = 0_X$. \square

Proposition 2.1.8 Let X be an idempotent symmetric space. The intersection of two idempotent symmetric subspaces of X is an idempotent symmetric subspace.

Example 2.1.9 $(\mathbb{R}^n, \boxplus, \cdot)$ is an idempotent symmetric space. $Y = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2, x_3 = x_4\}$ is an idempotent symmetric subspace of \mathbb{R}^4 .

In the following, if K is an idempotent pseudo-field, we define the operations for all $x, y \in K^n$ and all $\lambda \in K$ as follows:

$$x \boxplus_K y = (x_1, \dots, x_n) \boxplus_K (y_1, \dots, y_n) = (x_1 \boxplus_K y_1, \dots, x_n \boxplus_K y_n) \quad (38)$$

and

$$\lambda \cdot_K x = \lambda \cdot_K (x_1, \dots, x_n) = (\lambda \cdot_K x_1, \dots, \lambda \cdot_K x_n). \quad (39)$$

The two following examples are based on this construction.

Example 2.1.10 $(\tilde{\mathbb{M}}^n, \tilde{\boxplus}, \tilde{\otimes})$ is an idempotent symmetric space.

Example 2.1.11 $(\mathbb{R}^{-n}, \boxplus^{-1}, \cdot)$ is an idempotent symmetric space, with $\mathbb{R}^{-n} = (\mathbb{R}^{-1})^n$.

In the next example, it is shown that a similar construction can be applied to certain infinite-dimensional spaces.

Example 2.1.12 Suppose that I is a subset of \mathbb{R} and let λ be the Lebesgue measure. Let $L_p(\lambda)$ be the set of the real valued maps $f : I \rightarrow \mathbb{R}$ such that

$$\int_I |f|^p d\lambda < +\infty$$

for $p \in]1, +\infty[$. Let us consider the binary operation defined over $L_p(\lambda)$ by:

$$f \boxplus g : x \mapsto f(x) \boxplus g(x) = \begin{cases} f(x) & \text{if } |f(x)| > |g(y)| \\ \frac{1}{2}(f(x) + f(y)) & \text{if } |f(x)| = |g(y)| \\ f(y) & \text{if } |f(x)| < |g(y)|. \end{cases} \quad (40)$$

Clearly, one has $|f(x) \boxplus g(x)| \leq |f(x)| + |g(x)|$. It follows that

$$\int_I |f \boxplus g|^p d\lambda < \int_I |f|^p d\lambda + \int_I |g|^p d\lambda < \infty.$$

Let \cdot be the standard product of a real valued function by a real scalar, it follows that $(L_p(\lambda), \boxplus, \cdot)$ is an idempotent symmetric space.

Remark 2.1.13 Notice that Examples 2.1.10 and 2.1.11 are constructed from an homeomorphic transformation of $(\mathbb{R}^n, \boxplus, \cdot)$. It is therefore easy to define two convex structures on $(\tilde{\mathbb{M}}^n, \tilde{\boxplus}, \tilde{\otimes})$ and $(\mathbb{R}^{-n}, \boxplus^{-1}, \cdot)$ respectively. Namely a subset M of $\tilde{\mathbb{M}}$ is \mathbb{M} -convex if for all $x, y \in M$ and all $t \in \mathbb{M}$, $x \tilde{\boxplus} ty \in C$. Paralleling [1], a subset C of \mathbb{R}^{-n} is inverse- \mathbb{B} -convex if for all $x, y \in C$ and all $t \in [1, +\infty]$, $x \boxplus^{-1} ty \in C$.

The next result specifically concerns the algebraic structure considered in [8]. However it has some immediate implications for the idempotent symmetric spaces defined in examples 2.1.10 and 2.1.11.

Proposition 2.1.14 For all subsets Y of \mathbb{R}^n , the following claims are equivalent:

(a) Y is an idempotent symmetric subspace of $(\mathbb{R}^n, \boxplus, \cdot)$.

(b) For all $(x_1, \dots, x_m) \in Y^m$ and all $t \in \mathbb{R}^m$, we have $\bigoplus_{i \in [m]} t_i x_i \in Y$.

Proof: Let us prove that (a) implies (b). If Y is an idempotent symmetric subspace, this property is true for $m = 2$. Suppose it is true at rank m and let us prove that it is true at rank $m + 1$. In other words, assume that for all $(x_1, \dots, x_m) \in Y^m$ we have: $\bigoplus_{i \in [m]} t_i x_i \in Y$ for all $t \in \mathbb{R}^m$, we need to prove that if $(x_1, \dots, x_m, x_{m+1}) \in Y^{m+1}$ then for all $t \in \mathbb{R}^{m+1}$ we have $\bigoplus_{i \in [m+1]} t_i x_i \in Y$. To establish this property, we use equation (12) which implies that if $(x_1, \dots, x_m, x_{m+1}) \in Y^{m+1}$ then, for all $t \in \mathbb{R}^{m+1}$, we have

$$\bigoplus_{i \in [m+1]} t_i x_i = \bigoplus_{i \in [m+1]} \left[t_i x_i \boxplus \left(\bigoplus_{j \in [m+1] \setminus \{i\}} t_j x_j \right) \right]. \quad (\star)$$

For all i , the vectors $z_i = t_i x_i \boxplus \left(\bigoplus_{j \in [m+1] \setminus \{i\}} t_j x_j \right)$ are copositive. Thus the associativity holds true for the m -tuple (z_1, \dots, z_m) . However, since we assume that the property holds true at rank m , $z_i \in Y$ for all i . Thus we deduce that $\bigoplus_{i \in [m+1]} t_i x_i \in Y$. Therefore, the first part of the proof is established. Since Y is a subset of \mathbb{R}^n , that is an idempotent symmetric space, the converse inclusion is immediate. \square

The following property is an immediate consequence.

Proposition 2.1.15 Let Y be an idempotent symmetric subspace of $(\mathbb{R}^n, \boxplus, \cdot)$. Then Y is idempotent symmetric convex.

2.2 Generalized Norms and Ultrametric in Limit.

Let $\|\cdot\|$ be the Euclidean norm defined for all $x \in \mathbb{R}^n$ as $\|x\| = (\sum_{i \in [n]} x_i^2)^{\frac{1}{2}}$. Let d be the Euclidean distance defined for all $x, y \in \mathbb{R}^n$ as $d(x, y) = \|x - y\|$. For all positive natural numbers p , we introduce the map $\|\cdot\|_{\varphi_p} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined as

$$\|x\|_{\varphi_p} = \varphi_p^{-1}(\|\phi_p(x)\|). \quad (41)$$

The following properties are immediate: $\|x\|_{\varphi_p} = 0 \iff x = 0$; $\|\alpha x\|_{\varphi_p} = |\alpha| \|x\|_{\varphi_p}$; $\|x + y\|_{\varphi_p} \leq \|x\|_{\varphi_p} + \|y\|_{\varphi_p}$. We deduce the following statements:

Proposition 2.2.1 For all $x, y \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$, we have the the following properties:

- (a) $\|x\|_\infty = \lim_{p \rightarrow \infty} \|x\|_{\varphi_p}$
- (b) $\|x \boxplus y\|_\infty \leq \max\{\|x\|_\infty, \|y\|_\infty\}$.

Proof: (a) For all $x \in \mathbb{R}^n$, we have

$$\|x\|_{\varphi_p} = \left(\sum_{i \in [n]} |x_i|^{2(2p+1)} \right)^{\frac{1}{2(2p+1)}}.$$

Taking the limit yields (a). (b) For all natural numbers p , we have:

$$\|x + y\|_{\varphi_p}^p \leq \|x\|_{\varphi_p}^p + \|y\|_{\varphi_p}^p. \quad (A)$$

Suppose that $\{z^{(p)}\}_{p \in \mathbb{N}}$ is a sequence of \mathbb{R}_+^n which converges to some $z \in \mathbb{R}_+^n$. It follows that we have $\lim_{p \rightarrow \infty} \|z^{(p)}\|_{\varphi_p} = \|z\|_\infty$. For all $x \in \mathbb{R}^n$, let us denote $|x| = (|x_1|, \dots, |x_n|)$. By definition, we have

$$\|x + y\|_{\varphi_p} = \||x + y|\|_{\varphi_p}.$$

Moreover, $\lim_{p \rightarrow \infty} |x + y| = |x \boxplus y|$. It follows that:

$$\lim_{p \rightarrow \infty} \|x + y\|_{\varphi_p} = \|x \boxplus y\|_\infty = \|x \boxplus y\|_\infty.$$

Since $\|x\|_{\varphi_p} \geq 0$ and $\|y\|_{\varphi_p} \geq 0$, taking the limit on both sides in (A) yields the result. \square

Recall that, for all $\alpha, \beta \in \mathbb{R}$, we adopt the notation $\alpha \boxminus \beta = \alpha \boxplus (-\beta)$. Let us consider the map $d_{\boxplus} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined as:

$$d_{\boxplus}(x, y) = \max_{i \in [n]} |x_i \boxplus y_i| \quad (42)$$

In addition, let us consider the map $d_{\varphi_p} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by:

$$d_{\varphi_p}(x, y) = \|x - y\|_{\varphi_p}. \quad (43)$$

This function is a distance relative to the algebraic structure obtained from the binary operation $\overset{p}{+}$. It defines a metric on any φ_p -vector space. Note that d_{φ_p} is also a distance in the usual sense because as $p \geq 1$, we have for all $z \in \mathbb{R}^n$ the relation:

$$d_{\varphi_p}(x, y) \leq d_{\varphi_p}(x, z) \overset{p}{+} d_{\varphi_p}(z, y) \leq d_{\varphi_p}(x, z) + d_{\varphi_p}(z, y). \quad (44)$$

Lemma 2.2.2 For all $x, y \in \mathbb{R}^n$, we have

$$\lim_{p \rightarrow \infty} d_{\varphi_p}(x, y) = d_{\boxplus}(x, y).$$

Proof: For all $x \in \mathbb{R}^n$, we have

$$\|x - y\|_{\varphi_p} = \left(\sum_{i \in [n]} \left(\left| (x_i^{2p+1} - y_i^{2p+1})^{\frac{1}{2p+1}} \right|^2 \right)^{2p+1} \right)^{\frac{1}{2(2p+1)}}.$$

For all $i \in [n]$ and any natural number p , set $z_i^{(p)} = |(x_i^{2p+1} - y_i^{2p+1})^{\frac{1}{2p+1}}|^2$. For all i , we have: $\lim_{p \rightarrow \infty} z_i^{(p)} = |x_i \boxplus y_i|^2$. Since for all p , $z_i^{(p)} \geq 0$, it follows that

$$\lim_{p \rightarrow \infty} \|x - y\|_{\varphi_p} = \left(\max_{i \in [n]} |x_i \boxplus y_i|^2 \right)^{\frac{1}{2}} = d_{\boxplus}(x, y). \quad \square$$

Proposition 2.2.3 *The map $d_{\boxplus} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ defines an ultrametric on \mathbb{R}^n . For all $x, y, z \in \mathbb{R}^n$:*

- (1) $d_{\boxplus}(x, y) = d_{\boxplus}(y, x)$;
- (2) $d_{\boxplus}(x, y) = 0$ if and only if $x = y$;
- (3) $d_{\boxplus}(x, y) \leq \max\{d_{\boxplus}(x, z), d_{\boxplus}(z, y)\}$.

Proof: (1) is immediate. (2) follows from the fact that for all i we have $x_i \boxminus y_i = 0 \iff x_i = -y_i$. (3) For all $x, y, z \in \mathbb{R}^n$ and for all natural numbers p , we have

$$d_{\varphi_p}(x, y) \leq d_{\varphi_p}(x, z) + d_{\varphi_p}(z, y).$$

Since for all $x, y \in \mathbb{R}^n$ we have $d_{\varphi_p}(x, y) \geq 0$ and $\lim_{p \rightarrow \infty} d_{\varphi_p}(x, y) = d_{\boxplus}(x, y)$, we deduce the result. \square

Note that any ultrametric defines a metric since the inequality $d(x, y) \leq \max\{d(x, z), d(y, z)\}$ implies the triangular inequality $d(x, y) \leq d(x, z) + d(y, z)$. In particular for all $x, y, z \in E$, at least one of the three equalities $d(x, y) = d(y, z)$ or $d(x, z) = d(y, z)$ or $d(x, y) = d(z, x)$ holds. From the above definition, one can conclude several typical properties of ultrametrics.

Proposition 2.2.4 *Let Y be an idempotent symmetric subspace of \mathbb{R}^n . (Y, d_{\boxplus}) is an ultrametric space.*

Proof: Since Y is an idempotent symmetric space, the metric d_{\boxplus} is well-defined on Y . From Proposition 2.2.3, the result is immediate. \square

In the following we say that a sequence $\{x^{(k)}\}_{k \in \mathbb{N}}$ is **F-convergent** to $x \in \mathbb{R}^n$ if $\lim_{k \rightarrow \infty} d_{\boxplus}(x^{(k)}, x) = 0$.

Lemma 2.2.5 *Suppose that $\{x^{(k)}\}_{k \in \mathbb{N}}$ is F-convergent to $x \in \mathbb{R}^n$. Let $I(x) = \{i \in [n] : x_i \neq 0\}$. Then, there exists some natural number k_0 such that for all $k \geq k_0$ and for all $i \in I(x)$ we have $x_i^{(k)} = x_i$. Moreover, for all $i \notin I(x)$, $\lim_{k \rightarrow \infty} x_i^{(k)} = 0$.*

Proof: Suppose that $\lim_{k \rightarrow \infty} d_{\boxplus}(x_i^{(k)}, x) = 0$. Assume that there exists an increasing subsequence $\{k_q\}_{q \in \mathbb{N}}$ and some $i_0 \in I(x)$ such that, for all q , $x_i^{(k_q)} \neq x_i$ and let us show a contradiction. In such a case, for all $q \in \mathbb{N}$, we have $|x_{i_0}^{(k_q)} \boxminus x_{i_0}| = \max\{|x_{i_0}^{(k_q)}|, |x_{i_0}|\} \neq 0$. This implies that

$$d_{\boxplus}(x^{(k_q)}, x) = \max_{i \in I(x)} \{|x_i^{(k_q)} \boxminus x_i|\} \geq |x_{i_0}^{(k_q)} \boxminus x_{i_0}| = \max\{|x_{i_0}^{(k_q)}|, |x_{i_0}|\} > 0,$$

that is a contradiction. Hence, there is some k_0 such that for all $k \geq k_0$ and for all $i \in I(x)$ we have $x_i^{(k)} = x_i$. If $i \notin I(x)$ then $|x_i^{(k)} \boxminus x_i| = |x_i^{(k)} \boxminus 0| = |x_i^{(k)}|$ and $\lim_{k \rightarrow \infty} d_{\boxplus}(x^{(k)}, x) = 0$. Therefore, $\lim_{k \rightarrow \infty} x_i^{(k)} = 0$. \square

It is now interesting to see that the idempotent symmetrized space $(\tilde{\mathbb{M}}^n, \tilde{\boxplus}, \tilde{\otimes})$ can also be endowed with a suitable form of ultrametric. Let $d_{\tilde{\boxplus}} : \tilde{\mathbb{M}}^n \times \tilde{\mathbb{M}}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ defined as:

$$d_{\tilde{\boxplus}}(z, w) = \widetilde{\bigoplus}_{i \in [n]} |z_i \tilde{\boxminus} w_i|_{\tilde{\mathbb{M}}}, \quad (45)$$

where for all $\gamma \in \tilde{\mathbb{M}}$, $|\gamma|_{\tilde{\mathbb{M}}} = \psi_{\ln}(|\psi_{\exp}(\gamma)|)$ and, for all i , $z_i \tilde{\boxminus} w_i = \psi_{\ln}(\psi_{\exp}(z_i) \boxminus \psi_{\exp}(w_i))$. It is therefore easy to check that: (i) for all $(z, w) \in \tilde{\mathbb{M}}^n \times \tilde{\mathbb{M}}^n$, $d_{\tilde{\boxplus}}(z, w) = -\infty \iff z = w$; (ii) for all $z, t, w \in \tilde{\mathbb{M}}^n$, $d_{\tilde{\boxplus}}(z, w) \leq \max\{d_{\tilde{\boxplus}}(z, t), d_{\tilde{\boxplus}}(t, w)\}$. Therefore, paralleling Proposition 2.2.4, $(\tilde{\mathbb{M}}, \tilde{\boxplus}, \tilde{\otimes})$ can also be endowed with a topological structure. In particular, we can construct a standard ultrametric by defining for all $z, w \in \tilde{\mathbb{M}}^n$ the distance $\bar{d}_{\tilde{\boxplus}}(z, w) = \psi_{\exp}(d_{\tilde{\boxplus}}(z, w))$.

Proposition 2.2.6 *Let Z be an idempotent symmetric subspace of $\tilde{\mathbb{M}}^n$. Then $(Z, \bar{d}_{\tilde{\boxplus}})$ is an ultrametric space.*

2.3 Geometry of the Ultrametric Ball

In the following, let $B_{\boxplus}(x, r) = \{z \in \mathbb{R}^n : d_{\boxplus}(x, z) < r\}$ and $B_{\boxplus}(x, r] = \{z \in \mathbb{R}^n : d_{\boxplus}(x, z) \leq r\}$ respectively denote the open and closed balls centered at x with radius r .

For all $x \in \mathbb{R}^n$ and all subsets I of $[n]$, let us consider the map $\mathcal{A}_x : I \rightarrow \mathbb{R}_+$ defined for all $i \in I$ by $\mathcal{A}_x(i) = |x_i|$. This map associates to each index i the absolute value of the i -th coordinate of x . For all $\alpha \in \mathbb{R}_+$, $\mathcal{A}_x^{-1}(\alpha) = \{i \in I : |x_i| = \alpha\}$.

For all $x, y \in \mathbb{R}^n$, let $\mathcal{L}(x, y)$ be a subset of I defined by

$$\mathcal{L}(x, y) = \left\{ i \in [n] : x_i \neq y_i \right\}. \quad (46)$$

$\mathcal{L}(x, y)$ is obtained by dropping from $[n]$ all the i 's such that $x_i = y_i$. It follows that $d_{\boxplus}(x, y) = 0$ if $x = y$ and

$$d_{\boxplus}(x, y) = \begin{cases} \max_{i \in \mathcal{L}(x, y)} |x_i| & \text{if } \max_{i \in \mathcal{L}(x, y)} |x_i| \geq \max_{i \in \mathcal{L}(x, y)} |y_i| \\ \max_{i \in \mathcal{L}(x, y)} |y_i| & \text{if } \max_{i \in \mathcal{L}(x, y)} |x_i| \leq \max_{i \in \mathcal{L}(x, y)} |y_i|, \end{cases} \quad (47)$$

if $x \neq y$. In the next statement we describe the geometry of any ball centered at $x \in \mathbb{R}^n$.

Proposition 2.3.1 *Let $x \in \mathbb{R}^n$. Suppose that $\mathcal{A}_x([n]) = \bigcup_{i \in [n]} \mathcal{A}_x(i) = \{\alpha_j\}_{j \in [m]}$ with $\alpha_j < \alpha_{j+1}$ for all $j \in [m-1]$ and $\alpha_j \geq 0$ for all j .*

- (i) *If $x = 0$ and $\mathcal{A}_x([n]) = \{0\}$, then for all $\alpha \geq 0$, $B_{\boxplus}(0, \alpha] = B_{\infty}(0, \alpha]$.*
- (ii) *If either $\alpha = 0$ or $\alpha \in [0, \alpha_1[$ then $B_{\boxplus}(x, \alpha] = \{x\}$.*
- (iii) *If $\alpha \in [\alpha_j, \alpha_{j+1}[$ then $z \in B_{\boxplus}(x, \alpha]$ if and only if:*

$$|z_k| \leq \alpha, \text{ for all } k \in \mathcal{A}_x^{-1}([0, \alpha_j]) \text{ and } z_k = x_k \text{ for all } k \in \mathcal{A}_x^{-1}([\alpha_{j+1}, \alpha_m]).$$

- (iv) *If $\alpha \geq \alpha_m = \|x\|_{\infty}$ then $B_{\boxplus}(x, \alpha] = B_{\infty}(0, \alpha]$.*

Proof: (i) is immediate. (ii) If $\alpha = 0$, then $B_{\boxplus}(x, 0] = \{x\}$. If $\alpha_1 = 0$, the case $\alpha < \alpha_1$ is then excluded. Suppose that $\alpha_1 > 0$. For all $z \neq x$, $\max_{i \in [n]} |x_i \boxplus (-z_i)| \geq \max_{i \in [n]} |x_i| \geq \alpha_1$. Therefore $\alpha < \alpha_1$ implies that $B_{\boxplus}(x, \alpha] = \{x\}$. (iii) Suppose that $\alpha \in [\alpha_1, \alpha_2]$. If there is some $i \in \mathcal{A}_x^{-1}([\alpha_2, \alpha_m])$ such that $z_i \neq x_i$, then $d_{\boxplus}(x, z) \geq \alpha_2 > \alpha_1$. Therefore, if $z \in B_{\boxplus}(x, 0]$ we should have $z_i = x_i$ for all $i \in \mathcal{A}_x^{-1}([\alpha_2, \alpha_m])$. Moreover, for all $i \in \mathcal{A}_x^{-1}(\{\alpha_1\})$ we should have $|z_i| \leq \alpha$. Let us extend this result to some arbitrary $j \leq m-1$. Suppose that $\alpha \in [\alpha_j, \alpha_{j+1}]$. If there is some $i \in \mathcal{A}_x^{-1}([\alpha_{j+1}, \alpha_m])$ such that $z_i \neq x_i$, then $d_{\boxplus}(x, z) \geq \alpha_{j+1} > \alpha_j$. Therefore, if $z \in B_{\boxplus}(x, \alpha]$, we should have $z_i = x_i$ for all $i \in \mathcal{A}_x^{-1}([\alpha_{j+1}, \alpha_m])$. Moreover, for all $i \in \mathcal{A}_x^{-1}([\alpha_1, \alpha_j])$ we should have $|z_i| \leq \alpha$. (iv) Suppose now that $\alpha \geq \|x\|_{\infty}$. By construction $\alpha_m = \|x\|_{\infty}$ and $z \in B_{\boxplus}(x, \alpha]$ if and only if $|z_i| \leq \|x\|_{\infty}$, which ends the proof. \square

Corollary 2.3.2 *For all $\alpha \geq 0$, and all $x \in \mathbb{R}^n$, $B_{\boxplus}(x, \alpha]$ is closed with respect to the norm topology.*

Proof: Let $\mathcal{A}_x([n]) = \{\alpha_j\}_{j \in [m]}$ with $\alpha_j < \alpha_{j+1}$ for all $j \in [m-1]$ and $\alpha_j \geq 0$ for all j . The result is immediate if either $\alpha = 0$ or $\alpha \in [0, \alpha_1[$ since $B_{\boxplus}(x, \alpha] = \{x\}$. From Proposition 2.3.1, if $\alpha \in [\alpha_j, \alpha_{j+1}[$, then

$$B_{\boxplus}(x, \alpha] = \left\{ z \in \mathbb{R}^n : |z_k| \leq \alpha, k \in \mathcal{A}_x^{-1}([0, \alpha_j]) \right\} \cap \left\{ z \in \mathbb{R}^n : z_k = x_k, k \in \mathcal{A}_x^{-1}([\alpha_{j+1}, \alpha_m]) \right\}.$$

Since the intersection of two closed sets is a closed set, we deduce that $B_{\boxplus}(x, \alpha]$ is closed. If $\alpha \geq \|x\|_{\infty}$, the result is immediate. \square

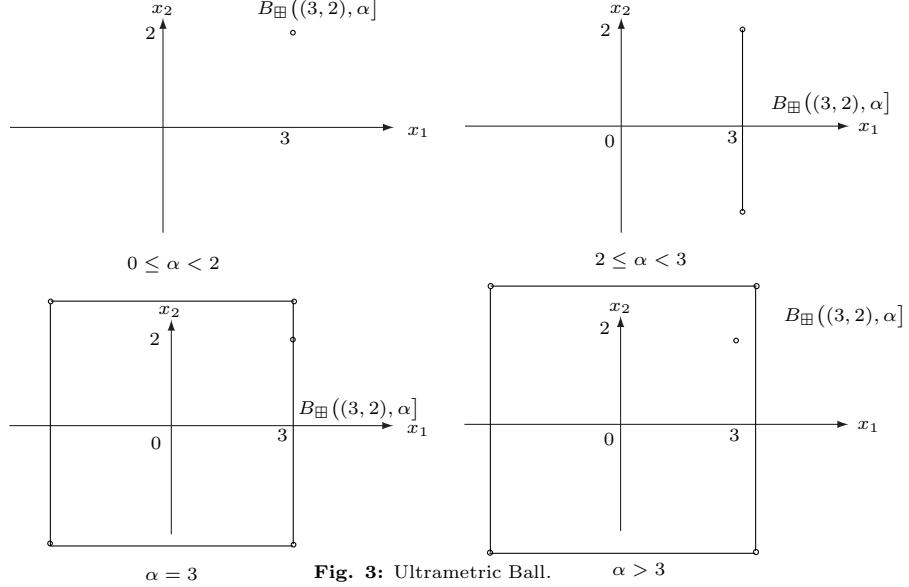


Fig. 3: Ultrametric Ball.

Corollary 2.3.3 For all $\alpha \geq 0$, and all $x \in \mathbb{R}^n$, $B_{\boxplus}(x, \alpha]$ is idempotent symmetric convex.

Proof: From Proposition 2.3.1, $B_{\boxplus}(x, \alpha]$ is the cartesian product between a box and a singleton. Since any box is idempotent symmetric convex, the result immediately follows. \square

In [8] the following formulation of a \mathbb{B} -polytope was established in the case of two points.⁴ For all $x, y \in \mathbb{R}^n$,

$$Co^\infty(x, y) = \left\{ tx \boxplus rx \boxplus sy \boxplus wy : \max\{t, r, s, w\} = 1, t, r, s, w \geq 0 \right\}. \quad (48)$$

In classical convexity, counting the same point twice in a finite set does not alter its convex hull. However, this is not the case here due to the specific properties of the non-associative operation we consider. This formulation is useful to prove the next decomposition result over $(\mathbb{R}^n, \boxplus, \cdot)$.

Proposition 2.3.4 For all $x, y \in \mathbb{R}^n$, and all $z \in Co^\infty(x, y)$ we have:

$$d_{\boxplus}(x, y) = \max\{d_{\boxplus}(x, z), d_{\boxplus}(z, y)\}.$$

Proof: If $x = y$, then the result is immediate. Let us assume that $x \neq y$. Suppose that $z \in Co^\infty(x, y)$, in such a case there are some $r, s, t, w \in [0, 1]$ with $\max\{r, s, t, w\} = 1$ such that

$$z = rx \boxplus sx \boxplus ty \boxplus wy.$$

Suppose that $\alpha = d_{\boxplus}(x, y)$. Let $\mathcal{L}(x, y) = \{i \in [n] : x_i \neq y_i\}$. By construction

$$d_{\boxplus}(x, y) = \max\{|x_i|, |y_i| : i \in \mathcal{L}(x, y)\}.$$

Therefore, for all $i \notin \mathcal{L}(x, y)$, if $|x_i| > d_{\boxplus}(x, y)$ then $x_i = y_i$. Thus, we deduce that $z_i = rx_i \boxplus sx_i \boxplus ty_i \boxplus wy_i = x_i = y_i$. It follows that $x_i \boxminus z_i = y_i \boxminus z_i = 0$. Hence, we have:

$$d_{\boxplus}(x, z) = \max\{|x_i \boxminus z_i| : i \in \mathcal{L}(x, y)\}$$

and

$$d_{\boxplus}(y, z) = \max\{|y_i \boxminus z_i| : i \in \mathcal{L}(x, y)\}.$$

Now, notice that for all i

$$|z_i| = |rx_i \boxplus sx_i \boxplus ty_i \boxplus wy_i| \leq \max\{|x_i|, |y_i|\}$$

⁴We use the convention that for all $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$, $\alpha \boxplus \beta \boxplus \gamma \boxplus \delta = F_{[4]}(\alpha, \beta, \gamma, \delta)$.

Since, for all i , $|x_i \boxplus z_i| \leq \max\{|x_i|, |y_i|\}$ and $|y_i \boxplus z_i| \leq \max\{|x_i|, |y_i|\}$, we deduce that

$$d_{\boxplus}(x, z) \leq \max\{|x_i|, |y_i| : i \in \mathcal{L}(x, y)\} = d_{\boxplus}(x, y)$$

and

$$d_{\boxplus}(y, z) \leq \max\{|x_i|, |y_i| : i \in \mathcal{L}(x, y)\} = d_{\boxplus}(x, y).$$

Thus

$$\max\{d_{\boxplus}(x, z), d_{\boxplus}(z, y)\} \leq d_{\boxplus}(x, y).$$

However, from the triangular inequality, we have

$$d_{\boxplus}(x, y) \leq \max\{d_{\boxplus}(x, z), d_{\boxplus}(z, y)\}.$$

Since the converse inequality holds, this completes the proof. \square

The next result is an immediate consequence.

Proposition 2.3.5 *Let $A = \{x^{(1)}, \dots, x^{(m)}\}$ be a finite subset of \mathbb{R}^n . Suppose moreover that (i) $Co^\infty(x^{(1)}, x^{(m)}) = \bigcup_{i \in [m-1]} Co^\infty(x^{(i)}, x^{(i+1)})$; (ii) for all i $Co^\infty(x^{(i-1)}, x^{(i)}) \cap Co^\infty(x^{(i)}, x^{(i+1)}) = \{x^{(i)}\}$. Then we have:*

$$d_{\boxplus}(x^{(1)}, x^{(m)}) = \max_{i \in [m-1]} d_{\boxplus}(x^{(i)}, x^{(i+1)}).$$

Proof: Since $x^{(2)} \in Co^\infty(x^{(1)}, x^{(m)})$, we have:

$$d_{\boxplus}(x^{(1)}, x^{(m)}) = \max \{d_{\boxplus}(x^{(1)}, x^{(2)}), d_{\boxplus}(x^{(2)}, x^{(m)})\}.$$

From the decomposition property established in [10], we have $Co^\infty(x^{(k)}, x^{(m)}) = Co^\infty(x^{(k)}, x^{(k+1)}) \cup Co^\infty(x^{(k+1)}, x^{(m)})$. It follows that for any $k \in [m-1]$, $x^{(k+1)} \in Co^\infty(x^{(k)}, x^{(m)})$. Hence:

$$d_{\boxplus}(x^{(k)}, x^{(m)}) = \max \{d_{\boxplus}(x^{(k)}, x^{(k+1)}), d_{\boxplus}(x^{(k+1)}, x^{(m)})\}.$$

By recurrence the proof is then immediate. \square

Proposition 2.3.6 *Let $x, y \in \mathbb{R}^n$. If $x = -y$, then for all $z \in Co^\infty(x, y) \setminus \{x, y\}$, $d_{\boxplus}(x, y) = d_{\boxplus}(x, z) = d_{\boxplus}(z, y)$.*

Proof: Suppose that $z \in Co^\infty(x, y)$, in such a case there are some $r, s, t, w \in [0, 1]$ with $\max\{r, s, t, w\} = 1$ such that $z = rx \boxplus sx \boxplus ty \boxplus wy$. Since $z \notin \{x, y\}$ we have $\max\{r, s\} = \max\{t, w\} = 1$, and $\min\{r, s\} = \max\{t, w\} = 1$, otherwise either $z = x$ or $z = y$. If $r = s = t = w = 1$, then $z = 0$ and the result is immediate. Consequently $\max\{r, s\} = \max\{t, w\} = 1$, $\min\{r, s\} < 1$ and $\min\{t, w\} < 1$. Dropping the symmetric terms, it follows that

$$z = rx \boxplus sx \boxplus ty \boxplus wy = \min\{r, s\}x \boxplus \min\{t, w\}y.$$

Since $\min\{r, s\} < 1$ and $\min\{t, w\} < 1$, this implies that $d_{\boxplus}(x, y) = d_{\boxplus}(x, z) = d_{\boxplus}(z, y)$. \square

3 Some Trigonometric Properties in Limit

In this section, we show that certain properties related to the notion of orthogonality can be derived within the framework of the idempotent and symmetric algebraic structure we consider.

3.1 Orthogonality and Pythagorean Properties in Limit

In the following, for all $p \in \mathbb{N}$, we say that $x, y \in \mathbb{R}^n$ are φ_p -orthogonal if:

$$\langle x, y \rangle_p = 0. \quad (49)$$

In addition, we say that $x, y \in \mathbb{R}^n$ are F -orthogonal if:

$$\langle x, y \rangle_\infty = 0. \quad (50)$$

In the sequel, the earlier results are extended to the general case of a triple $[x, y, z]$ of \mathbb{R}^n . Notice that we have for all natural numbers p

$$\langle x - z, y - z \rangle_p = \sum_{i \in [n]}^{q_p} x_i y_i + \sum_{i \in [n]}^{q_p} (-x_i z_i) + \sum_{i \in [n]}^{q_p} (-y_i z_i) + \sum_{i \in [n]}^{q_p} z_i^2. \quad (51)$$

Therefore:

$$\lim_{p \rightarrow \infty} \langle x - z, y - z \rangle_p = F_{[4n]}(x_1 y_1, \dots, x_n y_n, -x_1 z_1, \dots, -x_n z_n, -y_1 z_1, \dots, -y_n z_n, z_1^2, \dots, z_n^2). \quad (52)$$

In the following, we introduce the operation $\langle\langle \cdot, \cdot, \cdot \rangle\rangle_\boxplus : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined as:

$$\langle\langle x, y, z \rangle\rangle_\infty = \lim_{p \rightarrow \infty} \langle x - z, y - z \rangle_p. \quad (53)$$

In the following, for all $x, y, z \in \mathbb{R}^n$, we say that $[x, y, z]$ is F -right-angled in z if

$$\langle\langle x, y, z \rangle\rangle_\infty = 0. \quad (54)$$

In general, a right-angled triangle is not a F -right-angled triple. This is shown in the next example.

Example 3.1.1 Let $x = (1, -2, -1), y = (2, 3, -2)$ and $z = (3, 2, -3)$. We have $x - z = (-2, -4, 2)$ and $y - z = (-1, 1, 1)$. We have $\langle x - z, y - z \rangle = 0$. Therefore $[x, y, z]$ is a right-angled triangle. However $F_{[12]}(2, -6, 2, 3, 2, -3, -6, -6, -6, 9, 4, 9) = 9 \neq 0$. Hence $[x, y, z]$ is not a F -right-angled triple.

The next result was established in [10]. It plays an important role in the remainder. It means that if the limit sum is zero then the ϕ_p -generalized sum is zero for any p .

Lemma 3.1.2 Suppose that there is some $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that $\bigoplus_{i \in [n]} x_i = 0$. Then for all $p \in \mathbb{N}$, $\sum_{i \in [n]}^{q_p} x_i = 0$.

Proposition 3.1.3 Let $x, y, z \in \mathbb{R}^n$, and let $[x, y, z]$ be a triple. The triple $[x, y, z]$ is F -right-angled in z all if and only if for all $p \in \mathbb{N}$, $[x, y, z]$ is φ_p -right-angled in z , i.e. $\langle x - z, y - z \rangle_p = 0$.

Proof: If $[x, y, z]$ is a F -right-angled triple in z then:

$$\langle\langle x, y, z \rangle\rangle_\infty = 0.$$

It follows that

$$\langle x - z, y - z \rangle_p = 0$$

for all $p \in \mathbb{N}$ and the first implication follows.

Conversely if for all p , $[x, y, z]$ is a φ_p -right-angled triple in z , then for all p

$$\langle x - z, y - z \rangle_p = 0.$$

Hence

$$\lim_{p \rightarrow \infty} \langle x - z, y - z \rangle_p = \langle\langle x, y, z \rangle\rangle_\infty = 0,$$

which proves the reciprocal. \square

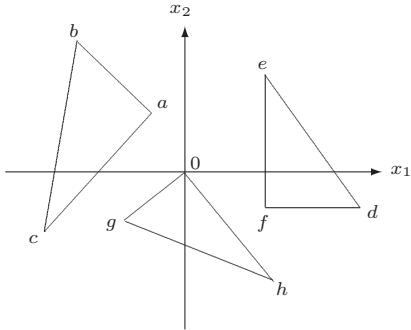


Fig. 4: F -right-angled triples with convex lines.

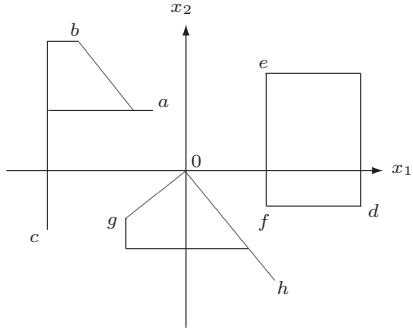


Fig. 5: F -right-angled triples with idempotent symmetric convex lines.

Remark 3.1.4 A F -right-angled triple is a right-angled triangle. However, the converse is not true. In Figure 4, $[a, b, c]$ is right-angled but not F -right-angled. The triples $[d, f, e]$ and $[g, 0, h]$ are F -right-angled.

In Figure 5, the same triples are represented using idempotent symmetric convex lines. Clearly, the right angle is preserved only for the triples $[d, f, e]$ and $[g, 0, h]$.

In the following, for all subsets I of $[n]$ and all $x \in \mathbb{R}^n$, let us denote $x_{[I]} = \sum_{i \in I} x_i e_i$, where $\{e_1, \dots, e_n\}$ is the canonic basis of \mathbb{R}^n .

Proposition 3.1.5 Let $x, y, z \in \mathbb{R}^n$, and let $[x, y, 0]$ is a F -right-angled triple in 0. Then there exists a partition $\{I_1, \dots, I_m, J\}$ of $[n]$ with $m = \lfloor \frac{n}{2} \rfloor$, $\text{Card}(J) \in \{0, 1\}$ and such that for all k :

$$\left[x_{[I_k]}, y_{[I_k]}, 0_{[I_k]} \right]$$

is a right-angled triangle in $0_{[I_k]}$ of \mathbb{R}^{I_k} with $\text{Card}(I_k) = 2$. Moreover, if $J = \{j\}$ is nonempty then $x_j y_j = 0$.

Proof: If $[x, y, 0]$ is a F -right-angled triple in 0 then:

$$\bigoplus_{i \in [n]} x_i y_i = 0.$$

$\bigoplus_{i \in [n]} x_i y_i = 0$ implies that there exists a partition $\{I_1, \dots, I_m, J\}$ of $[n]$ with $m = \lfloor \frac{n}{2} \rfloor$, $\text{Card}(I_k) = 2$ for all k and $\text{Card}(J) \in \{0, 1\}$ such that:

$$\sum_{i \in [I_k]} x_i y_i = \bigoplus_{i \in [I_k]} x_i y_i = 0,$$

and $x_j y_j = 0$ if $J = \{j\}$ is a nonempty set. Consequently

$$\sum_{i \in [I_k]} x_i y_i = 0.$$

Suppose that $I_k = \{i', i''\}$. Then

$$x_{i'} y_{i'} \boxminus x_{i''} y_{i''} = 0 \iff x_{i'} y_{i'} - x_{i''} y_{i''} = 0.$$

It follows that $\langle x_{[I_k]}, y_{[I_k]} \rangle_p = 0$ for all p . \square

Corollary 3.1.6 Suppose that $n = 2$. If $[x, y, 0]$ is a right-angled triangle in z of \mathbb{R}^2 , then $[x, y, 0]$ is F -right-angled in 0 and it is φ_p -right-angled in 0 for all natural numbers p .

The following result is an idempotent and non-associative analogue of the Pythagorean theorem.

Proposition 3.1.7 Let $x, y, z \in \mathbb{R}^n$, and let $[x, y, z]$ is a F -right-angled triple in z . Then for all $p \in \mathbb{N}$:

$$d_{\varphi_p}^2(x, y) = d_{\varphi_p}^2(x, z) + d_{\varphi_p}^2(y, z),$$

and

$$d_{\boxplus}(x, y) = \max\{d_{\boxplus}(x, z), d_{\boxplus}(y, z)\}.$$

Proof: If $[x, y, z]$ is a F -right-angled triple in z then

$$\langle x, y, z \rangle_\infty = 0.$$

Thus, for all p , $\langle x - z, y - z \rangle_p = 0$. It follows that

$$\langle \phi_p(x) - \phi_p(z), \phi_p(y) - \phi_p(z) \rangle = 0.$$

Hence

$$\|\phi_p(x) - \phi_p(y)\|^2 = \|\phi_p(x) - \phi_p(z)\|^2 + \|\phi_p(y) - \phi_p(z)\|^2.$$

We deduce that

$$\left\| \phi_p \left(\phi_p^{-1}(\phi_p(x) - \phi_p(y)) \right) \right\|^2 = \varphi_p \left(\varphi_p^{-1} \left(\|\phi_p \left(\phi_p^{-1}(\phi_p(x) - \phi_p(z)) \right)\|^2 \right) \right) + \varphi_p \left(\varphi_p^{-1} \left(\|\phi_p \left(\phi_p^{-1}(\phi_p(y) - \phi_p(z)) \right)\|^2 \right) \right).$$

Thus:

$$d_{\varphi_p}^2(x, y) = \|x - y\|_{\varphi_p}^2 = \|x - z\|_{\varphi_p}^p + \|y - z\|_{\varphi_p}^p = d_{\varphi_p}^2(x, z) + d_{\varphi_p}^2(y, z).$$

Taking the limit, yields:

$$d_{\boxplus}^2(x, y) = \lim_{p \rightarrow \infty} d_{\varphi_p}^2(x, y) = \lim_{p \rightarrow \infty} d_{\varphi_p}^2(x, z) + d_{\varphi_p}^2(y, z) = \max\{d_{\boxplus}^2(x, z), d_{\boxplus}^2(y, z)\}.$$

The final statement immediately follows. \square

3.2 Cosine, Sine in Limit and some Parametrization of the Unit Square

The purpose of this section is to demonstrate that we can use the limiting properties of standard trigonometric concepts to derive analogues of well-known geometric properties within the framework of the non-associative algebra we are studying. Paralleling the usual definition of a cosine, let us define the φ_p -pseudo-cosine of the angle between two vectors as:

$$\cos_p(x, y) = \frac{\langle x, y \rangle_p}{\|x\|_{\varphi_p} \|y\|_{\varphi_p}}. \quad (55)$$

Taking the limit, we define the pseudo F -cosine as:

$$\cos_\infty(x, y) = \frac{\langle x, y \rangle_\infty}{\|x\|_\infty \|y\|_\infty}. \quad (56)$$

It is now natural to define a sine function in limit. In [11] a φ_p exterior product is defined for all $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ as:

$$(v_1 \stackrel{p}{\wedge} v_2 \stackrel{p}{\wedge} \cdots \stackrel{p}{\wedge} v_n) = |v_1, v_2, \dots, v_n|_p (e_1 \wedge e_2 \wedge \cdots \wedge e_n), \quad (57)$$

where:

$$(v_1 \stackrel{p}{\wedge} v_2 \stackrel{p}{\wedge} \cdots \stackrel{p}{\wedge} v_n) = \varphi_p^{-1}(\phi_p(v_1) \wedge \phi_p(v_2) \wedge \cdots \wedge \phi_p(v_n)) \quad (58)$$

$$= \varphi_p^{-1}(|\phi_p(v_1), \phi_p(v_2), \dots, \phi_p(v_n)|) (e_1 \wedge e_2 \wedge \cdots \wedge e_n). \quad (59)$$

Taking the limit yields:

$$(v_1 \stackrel{\infty}{\wedge} v_2 \stackrel{\infty}{\wedge} \cdots \stackrel{\infty}{\wedge} v_n) = |v_1, v_2, \dots, v_n|_\infty (e_1 \stackrel{\infty}{\wedge} e_2 \stackrel{\infty}{\wedge} \cdots \stackrel{\infty}{\wedge} e_n). \quad (60)$$

In the case of two vectors the exterior product can be identified to the exterior product and the oriented area of the cross product can be interpreted as the positive area of the parallelogram having x and y as sides:

$$x \wedge y = |x, y| = \|x\| \|y\| \sin(x, y). \quad (61)$$

The φ_p -sine function for two vectors $x, y \in \mathbb{R}^2$ is defined as

$$\sin_p(x, y) = \frac{|x, y|_p}{\|x\|_{\varphi_p} \|y\|_{\varphi_p}} = \frac{\varphi_p^{-1}(\varphi_p(x_1)\varphi_p(y_2) - \varphi_p(x_2)\varphi_p(y_1))}{\|x\|_{\varphi_p} \|y\|_{\varphi_p}}. \quad (62)$$

Along this line, let us define

$$\sin_\infty(x, y) = \frac{|x, y|_\infty}{\|x\|_\infty \|y\|_\infty} = \lim_{p \rightarrow \infty} \sin_p(x, y) = \frac{x_1 y_2 \boxminus x_2 y_1}{\max\{|x_1, x_2|\} \max\{|y_1, y_2|\}}. \quad (63)$$

Lemma 3.2.1 For all $x, y \in \mathbb{R}^n$, we have the following properties.

- (a) $\cos_p(x, y) = \varphi_p^{-1}(\cos(\phi_p(x), \phi_p(y)))$ and $\sin_p(x, y) = \varphi_p^{-1}(\sin(\phi_p(x), \phi_p(y)))$.
- (b) $\cos_p(x, y)^2 + \sin_p(x, y)^2 = 1$.
- (c) $\max\{|\cos_\infty(x, y)|, |\sin_\infty(x, y)|\} = 1$.

Proof: (a) The first statement is immediate. To prove the second part note that:

$$\sin_p(x, y) = \frac{|x, y|_p}{\|x\|_{\varphi_p} \|y\|_{\varphi_p}} = \frac{\varphi_p^{-1}(|\phi_p(x), \phi_p(y)|)}{\varphi_p^{-1}(\|\phi_p(x)\|) \varphi_p^{-1}(\|\phi_p(y)\|)}.$$

Therefore

$$\sin_p(x, y) = \varphi_p^{-1}\left(\frac{|\phi_p(x), \phi_p(y)|}{\|\phi_p(x)\| \|\phi_p(y)\|}\right) = \varphi_p^{-1}(\sin(\phi_p(x), \phi_p(y))).$$

(b) We have

$$\begin{aligned} \cos_p(x, y)^2 + \sin_p(x, y)^2 &= \varphi_p\left([\varphi_p^{-1}(\cos(\phi_p(x), \phi_p(y)))]^2\right) + \varphi_p\left([\varphi_p^{-1}(\sin(\phi_p(x), \phi_p(y)))]^2\right) \\ &= \varphi_p\left(\varphi_p^{-1}[(\cos(\phi_p(x), \phi_p(y))]^2\right) + \varphi_p\left(\varphi_p^{-1}[(\sin(\phi_p(x), \phi_p(y))]^2\right) \\ &= (\cos(\phi_p(x), \phi_p(y)))^2 + (\sin(\phi_p(x), \phi_p(y)))^2 = 1. \end{aligned}$$

(c) From (a) and (b), we have

$$\max\{|\cos_\infty(x, y)|^2, |\sin_\infty(x, y)|^2\} = \lim_{p \rightarrow \infty} \cos_p(x, y)^2 + \sin_p(x, y)^2 = 1.$$

Therefore,

$$\max\{|\cos_\infty(x, y)|, |\sin_\infty(x, y)|\} = 1. \quad \square$$

Let $C_\infty(0, 1) = \{(x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} = 1\}$ be the circle defined with respect to the $\|\cdot\|_\infty$ metric in \mathbb{R}^2 . Let us consider the points $a = (1, 0)$, $b = (1, 1)$, $c = (0, 1)$, $d = (-1, 1)$, $e = (-1, 0)$, $f = (-1, -1)$, $g = (0, -1)$, $h = (1, -1)$. By construction:

$$C_\infty(0, 1) = C_\boxplus(0, 1) = [a, b] \cup [b, d] \cup [d, f] \cup [f, h] \cup [h, a]. \quad (64)$$

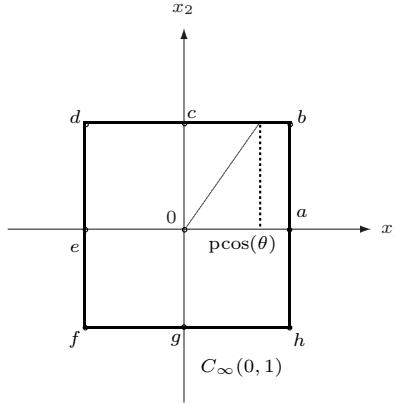


Fig. 6: Pseudo Cosine.

In the following we will identify the angles of the square with the length of the segments of its circumference. Let us consider the map $\alpha : C_\infty(0, 1) \rightarrow [0, 8]$ defined by:

$$\alpha(x_1, x_2) = \begin{cases} x_2 & \text{if } x \in [a, b] \\ 2 - x_1 & \text{if } x \in [b, d] \\ 4 - x_2 & \text{if } x \in [d, f] \\ 6 + x_1 & \text{if } x \in [f, h] \\ 8 + x_2 & \text{if } x \in [h, a] \end{cases}. \quad (65)$$

Clearly, we have $\alpha(a) = 0$, $\alpha(b) = 1$, $\alpha(c) = 2$, $\alpha(d) = 3$, $\alpha(e) = 4$, $\alpha(f) = 5$, $\alpha(g) = 6$, $\alpha(h) = 7$.

It is easy to prove that this map is a bijection from $C_\infty(0, 1)$ to $[0, 8]$. Notice that 8 is the perimeter of $C_\infty(0, 1)$. For all $\theta \in [0, 8]$ we define the pseudo-cosine $\text{pcos} : \mathbb{R} \rightarrow [-1, 1]$ as a periodic function with period 8:

$$\text{pcos}(\theta) = \cos_\infty(\alpha^{-1}(\theta), e_1) \quad \text{mod } 8. \quad (66)$$

It follows that for any integer $k \in \mathbb{Z}$

$$\text{pcos}(\theta + 8k) = \begin{cases} 1 & \text{if } \theta \in [0, 1] \\ 2 - \theta & \text{if } \theta \in [1, 3] \\ -1 & \text{if } \theta \in [3, 5] \\ -6 + \theta & \text{if } \theta \in [5, 7] \\ 1 & \text{if } \theta \in [7, 8] \end{cases}. \quad (67)$$

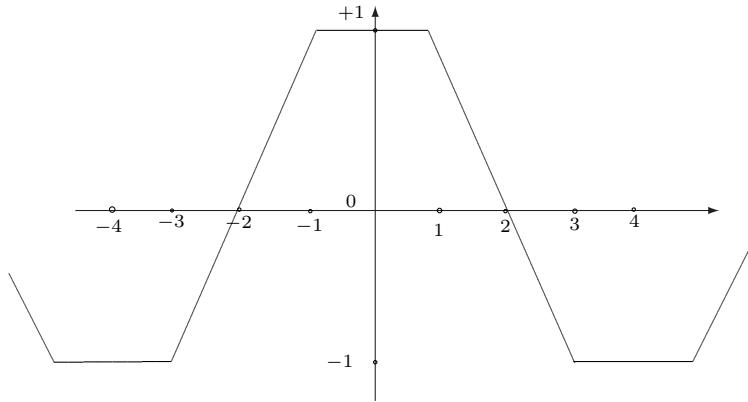


Fig. 7: Pseudo Cosine Function

Parallel to the previous definition, for all $\theta \in [0, 8]$, we define the pseudo-sine function $\text{psin} : \mathbb{R} \rightarrow [-1, 1]$ as a periodic function with period 8:

$$\text{psin}(\theta) = \sin_\infty(\alpha^{-1}(\theta), e_1) \quad \mod 8. \quad (68)$$

It follows that for any integer $z \in \mathbb{Z}$

$$\text{psin}(\theta + 8k) = \begin{cases} \theta & \text{if } \theta \in [0, 1] \\ 1 & \text{if } \theta \in [1, 3] \\ 4 - \theta & \text{if } \theta \in [3, 5] \\ -1 & \text{if } \theta \in [5, 7] \\ 8 - \theta & \text{if } \theta \in [7, 8] \end{cases}. \quad (69)$$

Hence, for all $\theta \in \mathbb{R}$, we have from Lemma 3.2.1

$$\max\{|\text{pcos}(\theta)|, |\text{psin}(\theta)|\} = \max\{|\cos_\infty(\alpha^{-1}(\theta), e_1)|, |\sin_\infty(\alpha^{-1}(\theta), e_1)|\} = 1. \quad (70)$$

3.3 Some Formalism on the Complex Scalar Field

We can continue the previous analogies, this time using complex numbers and studying the relative properties of the unit square. Let us denote $C_{\varphi_p}(0, 1] = \{x \in \mathbb{R}^n : \|x\|_{\varphi_p} = 1\}$. We have the relations $\phi_p(C_{\varphi_p}(0, 1]) = C(0, 1]$. Now, let us consider the subsets of \mathbb{C} defined as $C_{\varphi_p}^\natural(0, 1] = \{z = a + ib \in \mathbb{C} : \|(a, b)\|_{\varphi_p} = 1\}$ and $C_\infty^\natural(0, 1] = \{z = a + ib \in \mathbb{C} : \|(a, b)\|_\infty = 1\}$. Let us consider the bijective map $\varphi^\natural : \mathbb{C} \rightarrow \mathbb{C}$ defined as:

$$\varphi_p^\natural(a + ib) = \varphi_p(a) + i\varphi_p(b). \quad (71)$$

Its reciprocal is the map $\varphi_p^{\natural -1} : \mathbb{C} \rightarrow \mathbb{C}$ defined as $\varphi_p^{\natural -1}(a + ib) = \varphi_p^{-1}(a) + i\varphi_p^{-1}(b)$. The map φ_p^\natural is homogeneous with respect to the real scalar field, i.e. for all $\rho \in \mathbb{R}$, $\varphi_p^\natural(\rho(a + ib)) = \rho(\varphi_p(a) + i\varphi_p(b)) = \rho\varphi_p^\natural(a + ib)$. Therefore, for all $z = \rho e^{i\theta} \in \mathbb{C}$, with $\rho = |z|$, we have

$$\varphi_p^{\natural -1}(z) = \rho(\varphi_p^{-1}(\cos \theta) + i\varphi_p^{-1}(\sin \theta)). \quad (72)$$

For all $z, w \in \mathbb{C}^2$, let us define:

$$z \stackrel{p}{\cdot} w = \varphi_p^{\natural -1}(\varphi_p^\natural(z)\varphi_p^\natural(w)) \quad (73)$$

It follows that if $z = a + ib$ and $w = c + id$, then:

$$z \stackrel{p}{\cdot} w = \varphi_p^{-1}(\varphi_p(a)\varphi_p(b) - \varphi_p(b)\varphi_p(d)) + i\varphi_p^{-1}(\varphi_p(a)\varphi_p(c) + \varphi_p(b)\varphi_p(c)). \quad (74)$$

Equivalently

$$z \stackrel{p}{\cdot} w = (ab \stackrel{p}{\boxminus} bd) + i(ad \stackrel{p}{\boxplus} bc). \quad (75)$$

Along this line, taking the limit with respect to the product topology of \mathbb{R}^2 , we define the product:

$$z \boxtimes w = \lim_{p \rightarrow \infty} z \stackrel{p}{\cdot} w = (ab \boxminus bd) + i(ad \boxplus bc). \quad (76)$$

Note that this product is equivalent to the matrix product defined in [11] for matrices representing complex numbers. We have for all complex numbers of the form $z = a + ib$ and $w = c + id$:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \boxtimes \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac \boxminus bd & -(ad \boxplus bc) \\ ad \boxplus bc & ac \boxminus bd \end{pmatrix}. \quad (77)$$

Moreover, for all $p \in \mathbb{N}$, and all $z = a + ib \in \mathbb{C}$ we define the φ_p -module of z as $|z|_p = \|(a, b)\|_{\varphi_p}$. Similarly, $|z|_\infty = \|(a, b)\|_\infty = \max\{|a|, |b|\}$. In the following, we also extend the operation \boxplus to the complex numbers. For all $z = a + ib$ and $w = c + id$, we have :

$$z \boxplus w = (a \boxplus c) + i(b \boxplus d). \quad (78)$$

We can then establish the following relations.

Lemma 3.3.1 *We have the following properties:*

(a) *For all $z \in \mathbb{C}$, there exists some $\theta \in [0, 8]$ such that*

$$z = \|z\|_\infty(\text{pcos}(\theta) + i \text{psin}(\theta))$$

(b) *If $z = a + ib$ then $\sqrt{z \boxtimes \bar{z}} = |z|_\infty = \max\{|a|, |b|\}$.*

(c) *If $z \neq 0$, then $z \cdot \frac{1}{\|z\|_\infty^2} \bar{z} = 1$.*

(d) *Suppose that $z, w \in C_\infty^\natural(0, 1)$, then $z \boxtimes w \in C_\infty^\natural(0, 1)$.*

(e) *Suppose that $z, w \in \mathbb{C}$, then $|z \boxtimes w|_\infty = |z|_\infty |w|_\infty$.*

Proof: (a) Suppose that $z = a + ib$. If $z = 0$, the result is trivial. If $z \neq 0$ then

$$z = \max\{|a|, |b|\} \frac{a + ib}{\max\{|a|, |b|\}} = |z|_\infty \left(\frac{a}{\max\{|a|, |b|\}} + i \frac{b}{\max\{|a|, |b|\}} \right).$$

However $\left(\frac{a}{\max\{|a|, |b|\}} + i \frac{b}{\max\{|a|, |b|\}} \right) \in C_\infty^\natural(0, 1)$. Hence there is some $\theta \in [0, 8]$ such that

$$\left(\frac{a}{\max\{|a|, |b|\}} + i \frac{b}{\max\{|a|, |b|\}} \right) = \text{pcos}\theta + i \text{psin}\theta.$$

Hence

$$z = \|z\|_\infty(\text{pcos}(\theta) + i \text{psin}(\theta)).$$

(b) We have $z \boxtimes \bar{z} = (a + ib) \boxtimes (a - ib) = (a^2 \boxplus b^2) + i(ab \boxminus ab) = a^2 \boxplus b^2 = \max\{|a|, |b|\}^2$ which yields the result. (c) is immediate from (b).

(d) Suppose that $z = a + ib$ and $w = c + id$. We first assume that $z, w \in C_\infty^\natural(0, 1)$. This implies that $\max\{|a|, |b|\} = \max\{|c|, |d|\} = 1$. By definition we have:

$$z \boxtimes w = (ac \boxminus bd) + i(ad \boxplus bc).$$

We need to prove that $\max\{|ac \boxminus bd|, |ad \boxplus bc|\} = 1$. We consider the following cases:

(i) if $|a| < 1$ and $|c| < 1$ then $|bd| = 1$ and consequently $|ac \boxminus bd| = 1$; (ii) if $|a| < 1$ and $|d| < 1$ then $|bc| = 1$, therefore $|ad \boxplus bc| = 1$; (iii) if $|c| < 1$ and $|b| < 1$ then $|ad| = 1$ and we deduce that $|ad \boxplus bc| = 1$; (iv) if $|b| < 1$ and $|d| < 1$ then $|ac| = 1$, hence we have $|ac \boxminus bd| = 1$.

Suppose now that $|a| = |b| = |c| = |d| = 1$. All we need to show is that we cannot have $ac \boxminus bd = ad \boxplus bc = 0$. In such a case we have $ac = bd$ and $ad = -bc$. However, since $|c| = |d| = 1$, this implies that $d^2 = -c^2$ that is a contradiction. Consequently, we deduce that that if $z, w \in C_\infty^\natural(0, 1)$ then $z \boxtimes w \in C_\infty^\natural(0, 1)$.

(e) Suppose that $z = |z|_\infty(\text{pcos}(\theta) + i \text{psin}(\theta))$ and $w = |w|_\infty(\text{pcos}(\alpha) + i \text{psin}(\alpha))$. We have:

$$z \boxtimes w = |z|_\infty |w|_\infty (\text{pcos}(\theta)\text{pcos}(\alpha) \boxminus \text{psin}(\theta)\text{psin}(\alpha) + i(\text{pcos}(\theta)\text{psin}(\alpha) \boxplus \text{psin}(\theta)\text{pcos}(\alpha))).$$

Hence, since

$$|\text{pcos}(\theta)\text{pcos}(\alpha) \boxminus \text{psin}(\theta)\text{psin}(\alpha) + i(\text{pcos}(\theta)\text{psin}(\alpha) \boxplus \text{psin}(\theta)\text{pcos}(\alpha))|_\infty = 1$$

we deduce the result. \square

4 Algebraic Formulation of the Kuratowski-Peano Limit of a Line

4.1 Algebraic Formulation of a Line

In this section, we demonstrate that the non-associative algebraic structure we consider allows for establishing an algebraic formulation of any limiting line passing through two points. In the standard convex case it is easy to give an algebraic formulation of a line passing through two different points. In such a case one can define the line passing through x and y ($x \neq y$) as:

$$\mathcal{D}(x, y) = \{tx + sy : s + t = 1, s, t \in \mathbb{R}\}. \quad (79)$$

Note that there is no restriction imposing the non-negativity of s and t which limits a line to a line-segment. Paralleling this definition one can define a φ_p -line as:

$$\mathcal{D}^{(p)}(x, y) = \left\{ tx + sy : s + t = 1, s, t \in \mathbb{R} \right\}. \quad (80)$$

It was pointed out in [8] that the subset $\{tx \boxplus sy : \max\{s, t\} = 1, s, t \geq 0\}$ is not path-connected and does not describe, over \mathbb{R}^n , the convex hull in the limit of $\{x, y\}$. To that end it was shown in [8] that $Co^\infty(x, y) = \left\{ tx \boxplus rx \boxplus sy \boxplus wy : \max\{t, r, s, w\} = 1, t, r, s, w \geq 0 \right\}$. In what follows, we take a parallel approach to give the algebraic description of a line. Let us consider the following subset:

$$\mathcal{D}^\infty(x, y) := \left\{ tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R} \right\}. \quad (81)$$

Lemma 4.1.1 *For all $x, y \in \mathbb{R}^n$ with $x \neq y$,*

$$\mathcal{D}^\infty(x, y) \subset \text{Li}_{p \rightarrow \infty} \mathcal{D}^{(p)}(x, y).$$

Proof: Suppose that $z \in \left\{ tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R} \right\}$. By definition, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ with $\alpha_1 \boxplus \alpha_2 \boxplus \beta_1 \boxplus \beta_2 = 1$ and such that

$$z = \alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y.$$

Since $\alpha_1 \boxplus \alpha_2 \boxplus \beta_1 \boxplus \beta_2 = 1$, there exists some $p_0 \in \mathbb{N}$ such that, for all $p \geq p_0$, $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 \neq 0$. For all $p \geq p_0$, define

$$z^{(p)} = \frac{1}{\alpha_1 + \alpha_2 + \beta_1 + \beta_2} (\alpha_1 x + \alpha_2 x + \beta_1 y + \beta_2 y).$$

By construction, for all $p \geq p_0$, $z^{(p)} \in \mathcal{D}^{(p)}(x, y)$. Taking the limit on both sides yields:

$$\begin{aligned} \lim_{p \rightarrow \infty} z^{(p)} &= \frac{1}{\alpha_1 \boxplus \alpha_2 \boxplus \beta_1 \boxplus \beta_2} (\alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y) \\ &= \alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y = z. \end{aligned}$$

Consequently, $z \in \text{Li}_{p \rightarrow +\infty} \mathcal{D}^{(p)}(x, y)$. \square

Lemma 4.1.2 *Let $\{(s^{(p)}, t^{(p)})\}_{p \in \mathbb{N}}$ be sequence of \mathbb{R}^2 . Suppose there is an increasing subsequence $\{p_q\}_{q \in \mathbb{N}}$ of natural numbers such that $s^{(p_q)} + t^{(p_q)} = 1$ and $\lim_{q \rightarrow \infty} (s^{(p_q)}, t^{(p_q)}) = (s, t)$. We have the following properties:*

- (a) *If $\max\{|s|, |t|\} > 1$ then $s = -t$.*
- (b) *If $s = -t$ then $|s| = |t| \geq 1$.*
- (c) *If $\min\{|s|, |t|\} \leq 1$ then $\max\{s, t\} = 1$.*
- (d) *If $s \neq -t$ then $\max\{s, t\} = 1$.*

Proof: (a) In such a case if $|s| > 1$ then $t = \lim_{q \rightarrow \infty} t_{p_q} = \lim_{q \rightarrow \infty} 1 - s_{p_q} = -s$. The proof is symmetrical if $|t| > 1$. (b) Let us prove that in such a case we have $|s| \geq 1$ and $|t| \geq 1$. If $|s| < 1$ then $t = \lim_{q \rightarrow \infty} t_{p_q} = \lim_{q \rightarrow \infty} 1 - s_{p_q} = 1$, which contradicts $s = -t$. Similarly, if $|t| < 1$ we have a contradiction. Therefore $|s| \geq 1$ and $|t| \geq 1$. (c) If $s = 1$, then the conclusion is obvious. If $|s| < 1$, then $t = \lim_{q \rightarrow \infty} t_{p_q} = \lim_{q \rightarrow \infty} 1 - s_{p_q} = 1$ and $\max\{s, t\} = 1$. To conclude if $s = -1$, then $t = \lim_{q \rightarrow \infty} t_{p_q} = \lim_{q \rightarrow \infty} 1 + 1 = 1$, which ends the proof of (c). (d) From (a), $s \neq -t$ implies that $\max\{|s|, |t|\} \leq 1$. Hence $\min\{|s|, |t|\} \leq 1$ and the result is then immediate from (c). \square

The next lemma is a preliminary result to show the upper limit of the sequence $\{\mathcal{D}^{(p)}\}_{p \in \mathbb{N}}$ is identical to \mathcal{D}^∞ .

Lemma 4.1.3 Let $x, y \in \mathbb{R}^n$ with $x \neq y$ and suppose that $z \in \text{Ls}_{p \rightarrow \infty} \mathcal{D}^{(p)}(x, y)$. Then there exists a sequence $\{(s^{(p)}, t^{(p)})\}_{p \in \mathbb{N}} \subset \mathbb{R}^2$ and an increasing subsequence $\{p_q\}_{q \in \mathbb{N}}$ of natural numbers such that $s^{(p_q)} + t^{(p_q)} = 1$, $\lim_{q \rightarrow \infty} (s^{(p_q)}, t^{(p_q)}) = (s, t)$ and

$$z_i = sx_i \boxplus ty_i,$$

for all $i \in [n]$ such that $sx_i \neq -ty_i$.

Proof: Suppose that $z \in \text{Ls}_{p \rightarrow \infty} \mathcal{D}^{(p)}(x, y)$. In such a case there exists an increasing sequence of natural numbers $\{p_k\}_{k \in \mathbb{N}}$ and sequence $\{z^{(p_k)}\}_{k \in \mathbb{N}}$ with $z^{(p_k)} \in \mathcal{D}^{(p_k)}(x, y)$ for all k and $\lim_{q \rightarrow \infty} z^{(p_k)} = z$. This implies that there exists a sequence $\{s^{(p_k)}, t^{(p_k)}\}_{k \in \mathbb{N}} \subset \mathbb{R} \times \mathbb{R}$ such that

$$z^{(p_k)} = t^{(p_k)}x + s^{(p_k)}y, \quad (82)$$

with $s^{(p_k)} + t^{(p_k)} = 1$ for all k . Let us prove that $\{(s^{(p_k)}, t^{(p_k)})\}_{q \in \mathbb{N}}$ is a bounded sequence. Suppose that this is not the case and let us show a contradiction. Suppose for example that $\{s^{(p_k)}\}_{k \in \mathbb{N}}$ is not bounded. An elementary calculus shows that since $t^{(p_k)} = 1 - s^{(p_k)}$. Thus:

$$z^{(p_k)} = x + s^{(p_k)}(y - x). \quad (83)$$

However $\lim_{k \rightarrow \infty} y - x = y \boxminus x$. Since $x \neq y$, $y \boxminus x \neq 0$ and this implies that $\lim_{k \rightarrow \infty} \|z^{(p_k)}\| = \infty$ that is a contradiction. Consequently $\{s^{(p_k)}\}_{k \in \mathbb{N}}$ is bounded. Since $s^{(p_k)} = 1 - t^{(p_k)}$ this also implies that $\{t^{(p_k)}\}_{k \in \mathbb{N}}$ is bounded. Therefore, we deduce that the sequence $\{(s^{(p_k)}, t^{(p_k)})\}_{k \in \mathbb{N}}$ is bounded. Consequently one can extract an increasing subsequence $\{p_{k_m}\}_{m \in \mathbb{N}}$ such that

$$\lim_{m \rightarrow \infty} (s^{(p_{k_m})}, t^{(p_{k_m})}) = (s, p).$$

Let us denote $I = \{i \in [n] : sx_i \boxplus ty_i = 0\}$. For all $i \notin I$, since $sx_i \neq -ty_i$ we have

$$\lim_{m \rightarrow \infty} s^{(p_{k_m})}x_i + t^{(p_{k_m})}y_i = sx_i \boxplus ty_i,$$

which yields the result. \square

Proposition 4.1.4 For all $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$\lim_{p \rightarrow \infty} \mathcal{D}^{(p)}(x, y) = \mathcal{D}^\infty(x, y).$$

Proof: To prove this result, we need to establish that $\text{Ls}_{p \rightarrow \infty} \mathcal{D}^{(p)}(x, y) \subset \{tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R}\}$. Suppose that $z \in \text{Ls}_{p \rightarrow \infty} \mathcal{D}^{(p)}(x, y)$. By hypothesis, there exists an increasing sequence of natural numbers $\{p_q\}_{q \in \mathbb{N}}$ and sequence $\{z^{(p_q)}\}_{q \in \mathbb{N}}$ with $z^{(p_q)} = t^{(p_q)}x + s^{(p_q)}y \in \mathcal{D}^{(p_q)}(x, y)$ for all q such that $\lim_{q \rightarrow \infty} z^{(p_q)} = z$. Moreover, $\lim_{p \rightarrow \infty} (s^{(p_q)}, t^{(p_q)}) = (s, p)$ and

$$\lim_{p \rightarrow \infty} s^{(p_q)}x_i + t^{(p_q)}y_i = sx_i \boxplus ty_i, \quad (1)$$

for all i such that $sx_i \neq -ty_i$. Let $I = \{i \in [n] : sx_i \boxplus ty_i = 0\}$. We consider three cases.

(a) $I = \emptyset$. If $\min\{|s|, |t|\} > 1$ then, from Lemma 4.1.2.(a), $s = -t$. Let us consider the vector

$$sx \boxplus x \boxplus (1 - \epsilon)y \boxplus ty,$$

where $\epsilon > 0$ is sufficiently small. Clearly, we have $s \boxplus 1 \boxplus (1 - \epsilon) \boxplus t = 1 \boxplus (1 - \epsilon) = 1$. Moreover, since $|s| = |t| > 1$, for all $i \in [n]$ we have $sx_i \boxplus ty_i = sx_i \boxplus x_i \boxplus (1 - \epsilon)y_i \boxplus ty_i$. From equation (1)

$$z = sx \boxplus ty = sx \boxplus x \boxplus (1 - \epsilon)y \boxplus ty.$$

Therefore $z \in \mathcal{D}^\infty(x, y)$. If $\min\{|s|, |t|\} \leq 1$ then, from Lemma 4.1.2, $\max\{s, t\} = 1$. Suppose that $t = 1$. Then, $sx \boxplus ty = sx \boxplus 0x \boxplus y \boxplus y$ and $s \boxplus 0 \boxplus 1 \boxplus 1 = 1$. One can proceed symmetrically if $s = 1$.

(b) $I \neq \emptyset$ and for all $i \in I$, $x_i = y_i$. In such a case for all $i \in I$, $z_i^{(p_q)} = x_i = z_i$. For all $i \in I$, $sx_i = -ty_i = -tx_i$ and this implies that $s = -t$. From Lemma 4.1.2.(b) we have $|s| \geq 1$ and $|t| \geq 1$. Let us consider the vector

$$sx \boxplus x \boxplus (1 - \epsilon)y \boxplus ty,$$

where $\epsilon > 0$ is sufficiently small. Clearly, since $s = -t$, we have $s \boxplus 1 \boxplus (1 - \epsilon) \boxplus t = 1 \boxplus (1 - \epsilon) = 1$.

If $i \in I$ then, since $s = -t$ and $x_i = y_i$, one has $sx_i \boxplus x_i \boxplus (1 - \epsilon)y_i \boxplus ty_i = x_i \boxplus (1 - \epsilon)y_i = x_i = y_i = z_i$.

If $i \notin I$ then, since $sx_i \neq -tx_i$, $|s| \geq 1$ and $|t| \geq 1$, we have $sx_i \boxplus x_i \boxplus (1 - \epsilon)y_i \boxplus ty_i = sx_i \boxplus ty_i = z_i$. We conclude that $z \in \mathcal{D}^\infty(x, y)$.

(c) $I \neq \emptyset$ and there is some $i_0 \in I$, such that $x_{i_0} \neq y_{i_0}$. We first prove that in such a case we have $s \neq -t$. Suppose that this is not the case. We have $sx_{i_0} \boxplus ty_{i_0} = sx_{i_0} \boxplus (-s)y_{i_0} = 0$. However this implies $x_{i_0} = y_{i_0}$ that is a contradiction. Moreover, from Lemma 4.1.2.(c) we have $\max\{|s|, |t|\} = 1$ and $\max\{s, t\} = 1$. Assume for example that $t = 1$.

Suppose that $i \in I$. We have $ty_i = y_i = -sx_i$ and therefore

$$s^{(p_q)}x_i \stackrel{p_q}{+} t^{(p_q)}y_i = s^{(p_q)}x_i \stackrel{p_q}{-} st^{(p_q)}x_i = \left(s^{(p_q)} \stackrel{p_q}{-} st^{(p_q)}\right)x_i.$$

By hypothesis $\lim_{k \rightarrow \infty} z_i^{(p_q)} = z_i$. Hence the sequence $\left\{s^{(p_q)} \stackrel{p_q}{-} st^{(p_q)}\right\}_{q \in \mathbb{N}}$ has a limit. Let us denote

$$s' = \lim_{q \rightarrow \infty} s^{(p_q)} \stackrel{p_q}{-} st^{(p_q)}.$$

By hypothesis we have $z_i = s'x_i$. Note that, since

$$|s^{(p_q)} \stackrel{p_q}{-} st^{(p_q)}| \leq |s^{(p_q)}| \stackrel{p_q}{+} |st^{(p_q)}|,$$

we have taking the limit $|s'| \leq \max\{|s|, |t|\} \leq |s|$. We consider two cases.

(i) Suppose that $s' \neq -s$. Let us consider the vector

$$sx \boxplus s'x \boxplus 0y \boxplus y.$$

If $i \in I$ then, since $sx_i = -y_i$, we have $w_i = sx_i \boxplus s'x_i \boxplus 0y_i \boxplus y_i = s'x_i = z_i$.

If $i \notin I$, then since $s \neq -s'$, we have either $s = s'$ or $|s'| < |s|$. Hence, $sx_i \boxplus s'x_i \boxplus 0y_i \boxplus y_i = sx_i \boxplus y_i = z_i$.

It follows that $z = sx \boxplus s'x \boxplus 0y \boxplus y$. Moreover, since $-s \neq t = 1$ and $|s'| \leq |s|$, we have $s \boxplus s' \boxplus 0 \boxplus 1 = 1$. Hence $z \in \mathcal{D}^\infty(x, y)$.

(ii) Suppose now that $s' = -s$. Let us consider the vector

$$sx \boxplus 0x \boxplus y \boxplus y.$$

If $i \in I$ then, since $sx_i = -y_i$, we have $sx_i \boxplus 0x_i \boxplus y_i \boxplus y_i = y_i$. However since $y_i = -sx_i$, we deduce that $y_i = s'x_i = z_i$.

If $i \notin I$ then, $sx_i \neq -y_i$. Thus, we have $sx_i \boxplus 0x_i \boxplus y_i \boxplus y_i = sx_i \boxplus y_i = z_i$. Hence, we conclude that $z = sx \boxplus 0x \boxplus y \boxplus y$. Moreover, since $\max\{|s|, |t|\} = 1$, $s \boxplus 0 \boxplus 1 \boxplus 1 = 1$.

Consequently, we have shown that $z \in \{tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R}\}$. If $s = 1$ the proof is symmetrical. Hence

$$\lim_{p \rightarrow \infty} \mathcal{D}^{(p)}(x, y) \subset \{tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R}\}. \quad \square$$

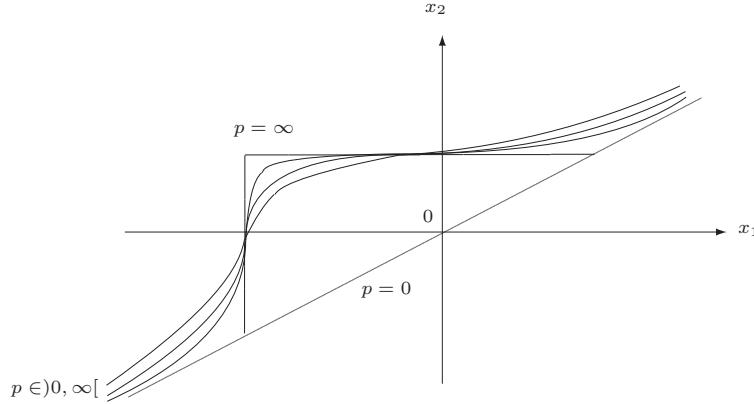


Fig. 8: Limit of a sequence of φ_p -lines.

In the next example, it is shown that the formalism described above allows us to define subsets of \mathbb{R}^n that are unbounded.

Example 4.1.5 Suppose that $n = 3$ and let us consider the points $x = (3, -2, 1)$ and $y = (1, -1, 1)$. Suppose that $t = 1$, $r = \delta = -w$ and $s = 0$ where δ is a real number. Clearly, $t \boxplus r \boxplus s \boxplus w = 1 \boxplus \delta \boxplus (-\delta) \boxplus 0 = 1$. It follows that

$$z_\delta := tx \boxplus rx \boxplus sy \boxplus wy = t \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \boxplus \delta \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \boxplus (-\delta) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \boxplus 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\delta \\ -2\delta \\ 1 \end{pmatrix} \in \mathcal{D}^\infty(x, y).$$

It follows that

$$\lim_{|\delta| \rightarrow \infty} \|z_\delta\| = +\infty.$$

In the following it is shown that $\mathcal{D}^\infty(x, y)$ is path connected.

Proposition 4.1.6 Let $x, y \in \mathbb{R}^n$ and suppose that $x \neq y$. Then $\mathcal{D}^\infty(x, y)$ is path-connected.

Proof: Suppose that $u, v \in \mathcal{D}^\infty(x, y)$ with $u \neq v$. Then, there exists a sequence $\{(u^{(p)}, v^{(p)})\}_{p \in \mathbb{N}}$ such that for all $p \in \mathbb{N}$ $(u^{(p)}, v^{(p)}) \in \mathcal{D}^{(p)}(x, y) \times \mathcal{D}^{(p)}(x, y)$ and $\lim_{p \rightarrow \infty} (u^{(p)}, v^{(p)}) = (u, v)$. Let $\mathcal{I}(u, v) = \{i \in [n] : u_i v_i < 0\}$. Since $\lim_{p \rightarrow \infty} (u^{(p)}, v^{(p)}) = (u, v)$, there is some $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$, $\mathcal{I}(u, v) = \mathcal{I}(u^{(p)}, v^{(p)})$. For all $i \in \mathcal{I}(u, v)$ and all $p \geq p_0$, there is a unique point $\gamma_i^{(p)} \in \text{Co}^p(x, y)$ such that $\{\gamma_i^{(p)}\} = H_i \cap \mathcal{D}^{(p)}(x, y)$ where, for all $i \in [n]$, $H_i = \{x \in \mathbb{R}^n : x_i = 0\}$. For all $i \in \mathcal{I}(u, v)$,

$$\lim_{p \rightarrow \infty} (H_i \cap \mathcal{D}^{(p)}(x, y)) \subset H_i \cap \mathcal{D}^\infty(x, y)$$

that is a nonempty subset of $\mathcal{D}^\infty(x, y)$. Let $\gamma_i \in \lim_{p \rightarrow \infty} (H_i \cap \mathcal{D}^{(p)}(x, y))$. This implies that there is an increasing sequence $\{p_q\}$ such that

$$\lim_{q \rightarrow \infty} \gamma_i^{(p_q)} = \gamma_i \in H_i \cap \mathcal{D}^\infty(x, y).$$

Let us denote $n(u, v) = \text{Card } \mathcal{I}(u, v)$. For all $p \geq p_0$ one can find a sequence $\{i_k\}_{k \in [n(u, v)-1]}$ such that

$$\gamma_{i_k}^{(p_q)} \cdot \gamma_{i_{k+1}}^{(p_q)} \geq 0.$$

It follows that

$$\lim_{q \rightarrow \infty} \text{Co}^{p_q}(\gamma_{i_k}^{(p_q)}, \gamma_{i_{k+1}}^{(p_q)}) = \text{Co}^\infty(\gamma_{i_k}, \gamma_{i_{k+1}}).$$

However,

$$\begin{aligned}
\mathop{\text{Ls}}_{p \rightarrow \infty} Co^p(u^{(p)}, v^{(p)}) &= \bigcup_{k \in [n(u,v)-1]} q \mathop{\text{Ls}}_{q \rightarrow \infty} Co^{p_q}\left(\gamma_{i_k}^{(p_q)}, \gamma_{i_{k+1}}^{(p_q)}\right) \\
&= \bigcup_{k \in [n(u,v)-1]} \mathop{\text{Lim}}_{q \rightarrow \infty} Co^{p_q}\left(\gamma_{i_k}^{(p_q)}, \gamma_{i_{k+1}}^{(p_q)}\right) \\
&= \bigcup_{k \in [n(u,v)-1]} Co^\infty\left(\gamma_{i_k}, \gamma_{i_{k+1}}\right).
\end{aligned}$$

Now, since any set $Co^\infty(\gamma_{i_k}, \gamma_{i_{k+1}})$ is path-connected, it follows that there exists a continuous map $\psi : [0, 1] \rightarrow \mathcal{D}^\infty(x, y)$ such that $\psi(0) = u$ and $\psi(1) = v$. \square

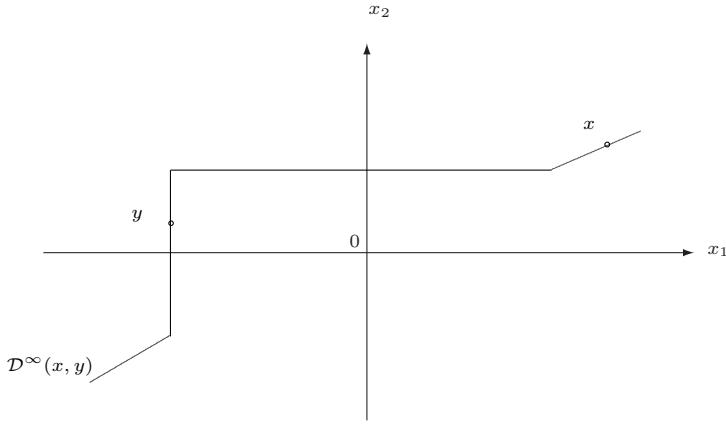


Fig. 9: F -line spanned from x and y .

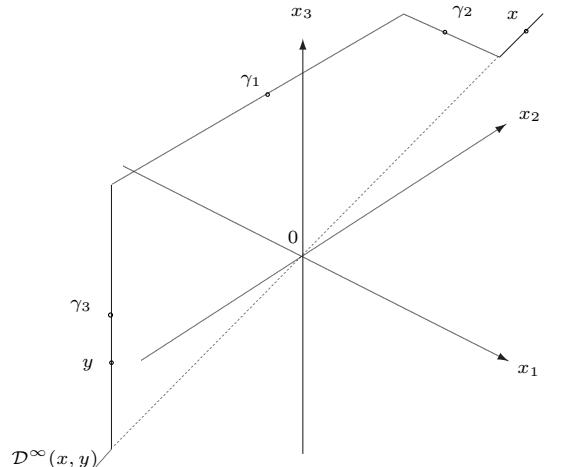


Fig. 10: 3-dimensional F -line.

Figure 10 depicts the case of a 3-dimensional F -line passing through x and y . For $i = 1, 2, 3$, γ_i is the intersection between the F -line and the 2-dimensions plane with equation $x_i = 0$.

Corollary 4.1.7 For all $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$Co^\infty(x, y) \subset \mathcal{D}^\infty(x, y).$$

Proof: From [8]

$$Co^\infty(x, y) := \left\{ tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R}_+ \right\}.$$

The result is then immediate, since $Co^\infty(x, y)$ is the set of all the \mathbb{B} -convex combinations such that $t, r, s, w \geq 0$. \square

Corollary 4.1.7 does not imply that \mathcal{D}^∞ is idempotent symmetric convex. In general, this is not true. This can be seen in Figure 10, where $x \boxminus y$ and $y \boxminus x$ belong to $\mathcal{D}^\infty(x, y)$. However, $(x \boxminus y) \boxplus (y \boxminus x) = 0 \notin \mathcal{D}^\infty(x, y)$.

Proposition 4.1.8 *Let Y be an idempotent symmetric subspace of \mathbb{R}^n . Then, for all $x, y \in \mathbb{R}^n$ with $x \neq y$ we have $\mathcal{D}^\infty(x, y) \subset Y$.*

Proof: We have shown that

$$\mathcal{D}^\infty(x, y) = \left\{ tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R} \right\}.$$

From Proposition 2.1.14 the result is then immediate. \square

Notice that since $(\mathbb{R}^n, \boxplus, \cdot)$ and $(\tilde{\mathbb{M}}^n, \tilde{\boxplus}, \otimes)$ are isomorphic one can deduce the algebraic form of a line passing through z and u . The multiplicative neutral element of \cdot is replaced with $0 = 1_{\tilde{\mathbb{M}}}$ and we have

$$\tilde{\mathcal{D}}^\infty(z, u) = \left\{ tz \tilde{\boxplus} rz \tilde{\boxplus} su \tilde{\boxplus} wu : t \tilde{\boxplus} r \tilde{\boxplus} s \tilde{\boxplus} w = 0, t, r, s, w \in \tilde{\mathbb{M}} \right\}. \quad (84)$$

4.2 Half-Lines and Inclusion Properties

In the next statement, it is shown that, for all $x, y \in \mathbb{R}^n$ with $x \neq y$, the line $\mathcal{D}^\infty(x, y)$ contains two half-lines spanned from two specific points. $\mathcal{D}^\infty(x, y)$ is called the F -line spanned by x and y .

Proposition 4.2.1 *Let $x, y \in \mathbb{R}^n$ and suppose that $x \neq y$. Let $I = \{i \in [n] : x_i \neq y_i\}$ and $J = [n] \setminus I$. We have the following inclusions:*

(a) *If $J = \emptyset$ then*

$$\{t(y \boxminus x) : t \in [1, \infty[\} \subset \mathcal{D}^\infty(x, y) \quad \text{and} \quad \{t(x \boxminus y) : t \in [1, \infty[\} \subset \mathcal{D}^\infty(x, y).$$

(b) *If $J \neq \emptyset$ then*

$$x_{[J]} + \left\{ t(y \boxminus x) : t \in [1, \infty[\right\} \subset \mathcal{D}^\infty(x, y) \quad \text{and} \quad x_{[J]} + \left\{ t(x \boxminus y) : t \in [1, \infty[\right\} \subset \mathcal{D}^\infty(x, y),$$

$$\text{where } x_{[J]} = \sum_{k \in J} x_k e_k = \sum_{k \in J} y_k e_k = y_{[J]}.$$

Proof: (a) We first note that since $J = \emptyset$ and $I = [n]$

$$x \boxminus y = x \boxplus x \boxplus 0y \boxplus (-y) \in \left\{ tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R} \right\}.$$

For all $t \geq 1$, let us consider the vector

$$z = tx \boxplus (-t)y.$$

Since $t \geq 1$, we have

$$z = tx \boxplus x \boxplus 0y \boxplus (-t)y.$$

Moreover $t \boxplus 1 \boxplus 0 \boxplus (-t) = 1$. Clearly $z \in \left\{ tx \boxplus rx \boxplus sy \boxplus wy : t \boxplus r \boxplus s \boxplus w = 1, t, r, s, w \in \mathbb{R} \right\}$, which proves the first part of (a). The proof of the second part is symmetrical swapping x and y . (b) Let us denote $\mathbb{R}_{[I]} = \{\sum_{k \in I} x_k e_k : x \in \mathbb{R}^n\}$. By hypothesis $x, y \in x_{[J]} + \mathbb{R}_{[I]}$. Hence $\mathcal{D}^\infty(x, y) \subset x_{[J]} + \mathbb{R}_{[I]}$. Let $n_I = \text{Card}(I)$ and let $\mathcal{D}_{[I]}^\infty(x, y)$ be the canonical decomposition of $\mathcal{D}^\infty(x, y)$ over $\mathbb{R}_{[I]}$. By definition for all $i \in I$ we have $x_i \neq y_i$. Since $\mathbb{R}_{[I]}$ is isomorphic to its restriction projection onto \mathbb{R}^{n_I} we deduce from (a) that

$$\{t(x_{[I]} \boxminus y_{[I]}) : t \in [1, \infty[\} \subset \mathcal{D}_{[I]}^\infty(x, y).$$

However $\mathcal{D}^\infty(x, y) = x_{[J]} + \mathcal{D}_{[I]}^\infty(x, y)$. Moreover $x_{[J]} \boxminus y_{[J]} = 0$. Hence

$$\begin{aligned} x_{[J]} + \{t(x \boxminus y) : t \in [1, \infty[\} &= x_{[J]} + \{t(x_{[I]} \boxminus y_{[I]}) : t \in [1, \infty[\} \\ &\subset x_{[J]} + \mathcal{D}_{[I]}^\infty(x, y) \\ &\subset \mathcal{D}^\infty(x, y). \end{aligned}$$

The proof of the second part of the statement is symmetrical. \square

Notice that Proposition 4.2.1.(b) can be deduced from Proposition 4.2.1.(a) by considering the canonical isomorphism ψ_I defined from the set $\{z \in \mathbb{R}^n : z_j = 0, j \notin I\}$ to \mathbb{R}^m where $m = \text{Card}(I)$.

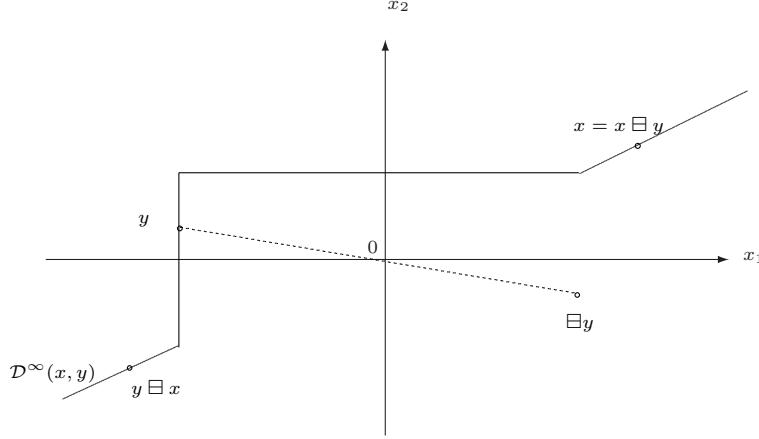


Fig. 11: Inclusion of half-lines.

Let $x, y \in \mathbb{R}^n$. The half lines $\mathcal{D}_+(x, y) = \{t(x \boxminus y) : t \in [1, \infty[\}$ and $\mathcal{D}_-(x, y) = \{t(y \boxminus x) : t \in [1, \infty[\}$ are respectively called **the upper and lower half-lines components** of the F -line $\mathcal{D}^\infty(x, y)$.

Example 4.2.2 Let us consider the case proposed in 4.1.5 with $x = (3, -2, 1)$ and $y = (1, -1, 1)$. In such a case $I = \{1, 2\}$ and $J = \{3\}$. We have:

$$x \boxminus y = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \boxminus \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$$

and

$$x_{\{3\}} = y_{\{3\}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It follows that

$$\mathcal{D}^\infty(x, y) \supset \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left\{ t \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 3t \\ -2t \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} = \mathcal{D}_+(x, y)$$

and

$$\mathcal{D}^\infty(x, y) \supset \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left\{ t \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -3t \\ 2t \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} = \mathcal{D}_-(x, y).$$

4.3 Parallel Lines

From Proposition 4.2.1, any F -lines contains two half-lines components. Let $\mathcal{D}^\infty(x, y)$ and $\mathcal{E}^\infty(u, v)$ be two F -lines respectively spanned from x and y and u and v . We say that $\mathcal{E}^\infty(u, v)$ and $\mathcal{D}^\infty(x, y)$ are F -parallel if $x \boxminus y \propto u \boxminus v$. Equivalently, this means that $x \boxminus y$ and $u \boxminus v$ are collinear, that is there is some $\alpha \in \mathbb{R} \setminus \{0\}$ such that $(x \boxminus y) = \alpha(u \boxminus v)$. From Proposition 4.2.1, there exist two half lines components $\mathcal{E}_-(u, v)$ and $\mathcal{E}_+(u, v)$ of $\mathcal{E}^\infty(u, v)$ and two half lines components $\mathcal{D}_-(x, y)$ and $\mathcal{D}_+(x, y)$ of $\mathcal{D}^\infty(x, y)$ such that $\mathcal{E}_+(u, v) \supset (\subset) \mathcal{D}_+(x, y)$ and $\mathcal{E}_-(u, v) \supset (\subset) \mathcal{D}_-(x, y)$.

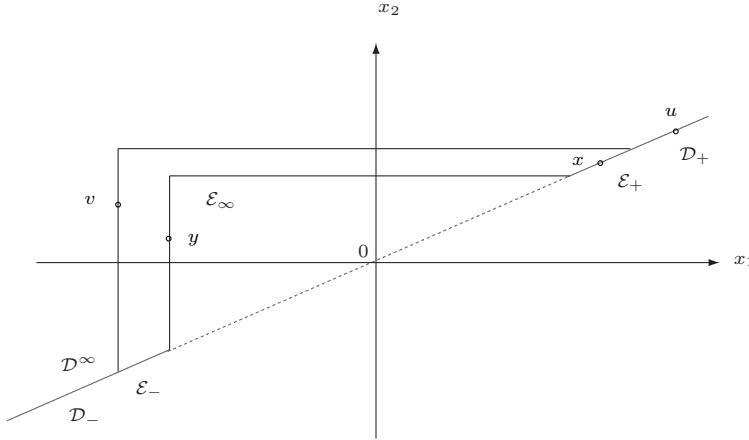


Fig. 12: Parallel F -lines.

Figure 12 depicts the case of two parallel lines. In Figure 13, we consider the case of two secant lines.

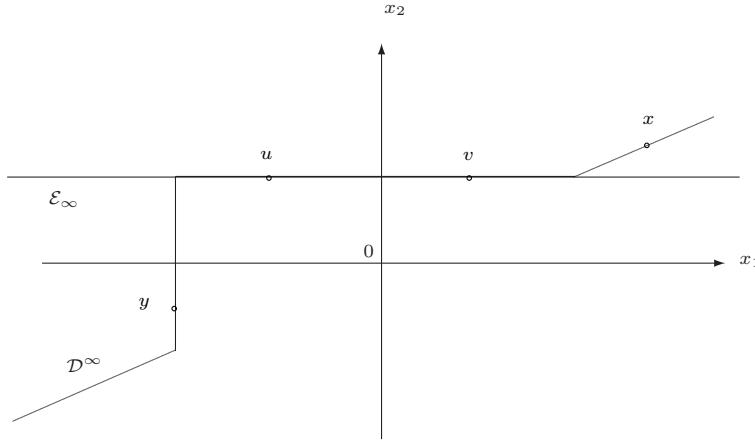


Fig. 13: Secant F -lines.

Traditional axioms of planar Euclidean geometry postulate that: (a₁) a straight line may be drawn between any two points; (a₂) any terminated straight line may be extended indefinitely; (a₃) a circle may be drawn with any given point as center and any given radius, (a₄) all right angles are equal; (v) for any given point not on a given line, there is exactly one line through the point that does not meet the given line.

It is well known that the fifth axiom does not follow from the first four. Clearly, the geometric implications of the algebraic structure proposed in this paper do not conflict with the first 4 axioms. Not surprisingly, this is not the case for the fifth. In particular:

- (i) two distinct parallel lines may have an infinity of common points (Figure 12);
- (ii) there may exist an infinity of F -lines passing by two distinct points x and y (Figure 12);
- (iii) two secant lines may have an infinity of common points (Figure 13);
- (iv) a F -line may not be spanned by two distinct points it contains (Figure 13).

In the following a system of inequations characterizing a F -line that passes from two distinct points is provided. This we do by considering the suitable notion of determinant introduced in [10]. The next intermediary result establishes an algebraic description of the limit of a sequence of φ_p -hyperplanes. The next result was established in [11].

Proposition 4.3.1 *Let V be the $n \times n$ matrix with v_i as i -th column for each i . Let $V_{(i)}$ be the matrix obtained from V by replacing line i with the transpose of the unit vector $\mathbf{1}_n$. Suppose that $|V|_\infty \neq 0$. Let $\{H_p(V)\}_{p \in \mathbb{N}}$ be the sequence of φ_p -hyperplanes passing through each point v_i . Then*

$$\lim_{p \rightarrow \infty} H_p(V) = \left\{ x \in \mathbb{R}^n : \sum_{i \in [n]} |V_{(i)}|_\infty x_i \leq |V|_\infty \leq \sum_{i \in [n]} |V_{(i)}|_\infty x_i \right\}.$$

Given two distinct points $x, y \in \mathbb{R}^2$, any point $z \in \mathcal{D}(x, y)$ satisfies the relation:

$$|z - x, z - y| = \begin{vmatrix} z_1 - x_1 & z_1 - y_1 \\ z_2 - x_2 & z_2 - y_2 \end{vmatrix} = 0. \quad (85)$$

Equivalently, we have

$$(x_2 - y_2)z_1 + (y_1 - x_1)z_2 = |x, y|. \quad (86)$$

The following proposition establishes an analogous result in the case of the F -line \mathcal{D}^∞ . This algebraic characterization is an immediate consequence of Proposition 4.3.1 (see [11]). Let $x, y \in \mathbb{R}^2$ with $x \neq y$. Then:

$$\mathcal{D}^\infty(x, y) = \left\{ z \in \mathbb{R}^2 : (x_2 \boxminus y_2)z_1 \bar{\sim} (y_1 \boxminus x_1)z_2 \leq |x, y|_\infty \leq (x_2 \boxminus y_2)z_1 \dot{+} (y_1 \boxminus x_1)z_2 \right\}. \quad (87)$$

In the following, we show that the equations of two parallel lines in the plane obey algebraic rules analogous to those of the linear case.

Lemma 4.3.2 *Let \mathcal{D}^∞ and \mathcal{E}^∞ be two parallel F -lines of \mathbb{R}^2 . Then there exists $a \in \mathbb{R}^2$, $c, d \in \mathbb{R}$ such that:*

$$\mathcal{D}^\infty = \{z \in \mathbb{R}^2 : \langle a, z \rangle_\infty^- \leq c \leq \langle a, z \rangle_\infty^+ \leq d\}$$

and

$$\mathcal{E}^\infty = \{z \in \mathbb{R}^2 : \langle a, z \rangle_\infty^- \leq d \leq \langle a, z \rangle_\infty^+ \leq c\}.$$

Proof: Suppose now that \mathcal{D}^∞ and \mathcal{E}^∞ are two F -line respectively spanned from x, y and u, v , where $x, y, u, v \in \mathbb{R}^2$. By definition we have:

$$\mathcal{E}^\infty = \left\{ z \in \mathbb{R}^2 : (u_2 \boxminus v_2)z_1 \bar{\sim} (v_1 \boxminus u_1)z_2 \leq |u, v|_\infty \leq (u_2 \boxminus v_2)z_1 \dot{+} (v_1 \boxminus u_1)z_2 \right\}.$$

If \mathcal{D}^∞ and \mathcal{E}^∞ are parallel then $(u \boxminus v) \propto (x \boxminus y)$. Therefore, there is a real number $\alpha \neq 0$ such that $(u \boxminus v) = \alpha(x \boxminus y)$. Hence:

$$\mathcal{E}^\infty = \left\{ z \in \mathbb{R}^2 : \alpha(x_2 \boxminus y_2)z_1 \bar{\sim} \alpha(y_1 \boxminus x_1)z_2 \leq |u, v|_\infty \leq \alpha(x_2 \boxminus y_2)z_1 \dot{+} \alpha(y_1 \boxminus x_1)z_2 \right\}.$$

Suppose that $\alpha > 0$, then:

$$\mathcal{E}^\infty = \left\{ z \in \mathbb{R}^2 : (x_2 \boxminus y_2)z_1 \bar{\sim} (y_1 \boxminus x_1)z_2 \leq \alpha^{-1}|u, v|_\infty \leq (x_2 \boxminus y_2)z_1 \dot{+} (y_1 \boxminus x_1)z_2 \right\}.$$

We deduce the result taking $a = x \boxminus y$, $c = |x, y|_\infty$, and $d = \alpha^{-1}|u, v|_\infty$. If $\alpha < 0$, then:

$$\mathcal{E}^\infty = \left\{ z \in \mathbb{R}^2 : (x_2 \boxminus y_2)z_1 \bar{\sim} (y_1 \boxminus x_1)z_2 \geq \alpha^{-1}|u, v|_\infty \geq (x_2 \boxminus y_2)z_1 \dot{+} (y_1 \boxminus x_1)z_2 \right\}.$$

Since $(x_2 \boxminus y_2)z_1 \bar{\sim} (y_1 \boxminus x_1) = -((y_2 \boxminus x_2)z_1 \dot{+} (x_1 \boxminus y_1))$ and $(x_2 \boxminus y_2)z_1 \dot{+} (y_1 \boxminus x_1) = -((y_2 \boxminus x_2)z_1 \bar{\sim} (x_1 \boxminus y_1))$ we deduce that:

$$\mathcal{E}^\infty = \left\{ z \in \mathbb{R}^2 : (y_2 \boxminus x_2)z_1 \bar{\sim} (x_1 \boxminus y_1)z_2 \leq |\alpha|^{-1}|u, v|_\infty \leq (y_2 \boxminus x_2)z_1 \dot{+} (x_1 \boxminus y_1)z_2 \right\}.$$

Now note that, since $\mathcal{D}^\infty(x, y) = \mathcal{D}^\infty(y, x)$, we also have

$$\mathcal{D}^\infty = \left\{ z \in \mathbb{R}^2 : (y_2 \boxminus x_2)z_1 \boxdot (x_1 \boxminus y_1)z_2 \leq |y, x|_\infty \leq (y_2 \boxminus x_2)z_1 \boxplus (x_1 \boxminus y_1)z_2 \right\}.$$

Hence, taking $a = y \boxminus x$, $c = |y, x|_\infty$ and $d = |\alpha|^{-1}|u, v|_\infty$ yields the result. \square

Example 4.3.3 Suppose that $x = (3, 1)$, $y = (1, -2)$, $u = (-2, 4)$ and $v = (-6, 1)$. We have:

$$x \boxminus y = \begin{pmatrix} 3 \boxminus 1 \\ 1 \boxminus (-2) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad u \boxminus v = \begin{pmatrix} -2 \boxminus (-6) \\ 4 \boxminus 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$$

It follows that $u \boxminus v = 2(x \boxminus y)$. Moreover,

$$|x, y|_\infty = 3 \cdot (-2) \boxminus 1 \cdot 1 = -6 \quad \text{and} \quad |u, v|_\infty = (-2) \cdot 1 \boxminus (-6) \cdot 4 = 24.$$

Therefore:

$$\mathcal{D}^\infty(x, y) = \left\{ z \in \mathbb{R}^2 : (1 \boxminus (-2))z_1 \boxdot (1 \boxminus 3)z_2 \leq -6 \leq (1 \boxminus (-2))z_1 \boxplus (1 \boxminus 3)z_2 \right\}.$$

Hence:

$$\mathcal{D}^\infty(x, y) = \left\{ z \in \mathbb{R}^2 : 2z_1 \boxdot (-3)z_2 \leq -6 \leq 2z_1 \boxplus (-3)z_2 \right\}.$$

In addition we have:

$$\mathcal{D}^\infty(u, v) = \left\{ z \in \mathbb{R}^2 : (4 \boxminus (-1))z_1 \boxdot ((-6) \boxminus (-2))z_2 \leq -6 \leq (4 \boxminus (-1))z_1 \boxplus ((-6) \boxminus (-2))z_2 \right\}.$$

Hence:

$$\mathcal{D}^\infty(u, v) = \left\{ z \in \mathbb{R}^2 : (-4)z_1 \boxdot 6z_2 \leq 24 \leq (-4)z_1 \boxplus 6z_2 \right\}.$$

Equivalently, by changing the sign and simplifying by 2:

$$\mathcal{D}^\infty(u, v) = \left\{ z \in \mathbb{R}^2 : 2z_1 \boxdot (-3)z_2 \leq -12 \leq 2z_1 \boxplus (-3)z_2 \right\}.$$

We can notice that the coefficients multiplying z_1 and z_2 are the same the difference relating only to the constant term.

Conclusion

In this article, we have explored certain geometric properties emerging from the deformation of a vector space structure defined over a scalar field. In particular, we have focused on notions of lines that can be seen as special limits of linear varieties. For future research, it would be valuable to investigate the connections between these notions and those typically derived from Max-Plus algebras, which are traditionally used in the formalism of tropical mathematics, as initiated in [29] and [27].

Compliance with Ethical Standards: Not applicable.

Data Availability Statement (DAS): No new data were created or analyzed in this study. Data sharing is not applicable to this article.

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