

Arrangements of inflectional lines, hyperosculating conics, and plane cubics

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Abstract

In the present note we study some arrangements of inflectional lines, hyperosculating conics, and a nodal plane cubic that are free. Moreover, we study weak combinatorics of arrangements consisting of lines, conics, and elliptic curves providing natural constraints on them.

Keywords arrangements of rational curves, weak combinatrics, BMY inequalities

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1 Introduction

Our main goal in this short note is to deliver new constructions of free curves using inflectional lines and hyperosculating conics to nodal cubic curves. Our research is strictly motivated by the recent developments, namely by a paper due to Dimca, Ilardi, Pokora and Sticlaru [5], where the authors constructed new examples of arrangements consisting of smooth plane curves and inflectional lines, and by a paper due to Dimca, Ilardi, Malara and Pokora [4], where the authors construct examples of free plane curve using hyperosculating conics to smooth and nodal plane cubic curve. Here we want to make a merger of these two approaches by constructing new examples of free arrangements consisting of inflectional lines and hyperosculating conics to nodal plane cubic curves. Our result, Theorem 3.1, provides a classification of such free arrangements. Moreover, we provide constraints on the weak combinatorics of such curve arrangements under the assumption that we have some naturally selected types of singularities. Our second main result is a Hirzebruch-type inequality for such selected arrangements, see Theorem 4.2.

2 Preliminary definitions

Let $S = \mathbb{C}[x, y, z]$ be the graded polynomial ring in three variables x, y, z with complex coefficients. Let $C : f = 0$ be a reduced curve of degree d in the complex projective plane $\mathbb{P}_{\mathbb{C}}^2$. We denote by $J_f = \langle f_x, f_y, f_z \rangle$ the Jacobian ideal, so the ideal in S generated by the partial derivatives.

Consider the graded S -module of Jacobian syzygies of f , namely

$$AR(f) = \{(a, b, c) \in S^3 : af_x + bf_y + cf_z = 0\}.$$

We define the minimal degree of non-trivial Jacobian syzygies for f as

$$\text{mdr}(f) := \min\{k : AR(f)_k \neq (0)\}.$$

Now we are going to define the freeness of a reduced plane curve $C : f = 0$ in the language of $r := \text{mdr}(f)$ of C , and the total Tjurina number of C which will be denoted by $\tau(C)$.

Definition 2.1. A reduced plane curve $C : f = 0$ is free if and only if $r \leq (d-1)/2$ and

$$\tau(C) = (d-1)^2 - r(d-r-1). \quad (1)$$

For the completeness of the note, we need to recall definitions of inflectional lines and hyperosculating conics. The Hessian of f is defined as

$$H = H(C) = \det \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}. \quad (2)$$

It is known that the intersection $X_C = C \cap H(C)$ consists exactly of the set of inflection points I_C of C union with the set of singular points Y_C of C . Recall that if $p \in C$ is a smooth point of this curve, and $T_p C$ denotes the projective line tangent to C at p , then the *inflection order* of p is by definition

$$\iota_p(C) = (C, T_p C)_p - 2, \quad (3)$$

where $(C, T_p C)_p$ denotes the intersection multiplicity of the curves C and $T_p C$ at the common point p . Moreover, we say that p is an inflection point of C , i.e., $p \in I_C$, if and only if $\iota_p(C) > 0$, and the associated tangent line at p is called as *the inflectional line*.

In 1859, Cayley proved in [1] that for a smooth point on a plane curve of degree $d \geq 3$ there exists a unique osculating conic. It has local intersection multiplicity with the curve at the point of contact at least 5. Then he observed that a plane curve, as for the flexes, has points where the osculating conics has contact ≥ 6 . He called these points sextactic points and the associated conics are called *hyperosculating conics*. It is well-known that if E is an elliptic curve, then E has exactly 27 sextactic points (so exactly 27 hyperosculating conics), and if C is a nodal cubic curve, then it has exactly 3 hyperosculating conics, and these numbers follow form the well-known formula due to Coolidge [2, Chapter VI, Theorem 17].

3 Free arrangements of lines, conics and cubics

Here is our classification result devoted to arrangements of free curves.

Theorem 3.1. 1) If C is an arrangement consisting of a nodal cubic curve, one hyperosculating conic, and one inflectional line, then the arrangement is free with the exponents $(2, 3)$.

2) If C is an arrangement consisting of a nodal cubic curve, one hyperosculating conic, and two distinct inflectional lines, then the arrangement is free with the exponents $(3, 3)$.

Proof. Recall that all nodal cubics are projectively equivalent and we can take

$$E : \quad x^3 + y^3 - xyz = 0$$

as our main object of studies. Following the lines in [7, Appendix], we can determine the three sextactic points of E , namely

$$P_1 = (1 : 1 : 2), \quad P_2 = (\omega : \omega^2 : 2), \quad P_3 = (\omega^2 : \omega : 2),$$

where $\omega = e^{2\pi i/3}$, and the corresponding hyperosculating conics, namely

- $C_1 : 21(x^2 + y^2) - 22xy - 6(x + y)z + z^2 = 0$,
- $C_2 : 21(\omega x^2 + \omega^2 y^2) - 22xy - 6(\omega^2 x + \omega y)z + z^2 = 0$,
- $C_3 : 21(\omega^2 x^2 + \omega y^2) - 22xy - 6(\omega x + \omega^2 y)z + z^2 = 0$.

Recall that at each point P_i , where the curves E and C_i meet, we get A_{11} -singularity. Furthermore, as it was pointed out in [7, Appendix], we have three flexes, namely

$$Q_1 = (1 : -1 : 0), \quad Q_2 = (1 : -\omega : 0), \quad Q_3 = (1 : \omega + 1 : 0),$$

and the corresponding inflectional lines are the following:

- $\ell_1 : 3x + 3y + z = 0$,
- $\ell_2 : 3x + 3\omega^2 y + \omega z = 0$,
- $\ell_3 : 3x + 3\omega y - (\omega + 1)z = 0$.

Observe that each point Q_j , where the curves E and ℓ_j meet, we get A_5 -singularity. Having the collected data at hand, we can proceed with our classification.

First of all, let us consider the arrangement $F_{i,j} = \{E, C_i, \ell_j\}$ with $i, j \in \{1, 2, 3\}$. Observe that $\tau(F_{i,j}) = 19$ and this follows from the fact that our arrangement has three A_1 points, one A_5 , and one A_{11} point as singularities. Using **SINGULAR** we can check that $r := \text{mdr}(F_{i,j}) = 2$ and hence we have

$$r^2 - r(d-1) + (d-1)^2 = 4 - 10 + 25 = 19 = \tau(C_i),$$

so the arrangement C_i is free with the exponents $(d_1, d_2) = (r, d-1-r) = (2, 3)$ for each $i, j \in \{1, 2, 3\}$.

Let us consider now $C_{i,j,k} = \{E, C_i, \ell_j, \ell_k\}$ with $i, j, k \in \{1, 2, 3\}$ and $j \neq k$. Observe that $\tau(C_{i,j,k}) = 27$ and this follows from the fact that we have exactly six A_1 points, two A_5 points, and one A_{11} point as singularities. Moreover, using **SINGULAR** we can check that $\text{mdr}(C_{i,j,k}) = 3$ and hence we have

$$r^2 - r(d-1) + (d-1)^2 = 9 - 18 + 36 = 27 = \tau(C_{i,j,k}),$$

so the arrangement $C_{i,j,k}$ is free with the exponents $(d_1, d_2) = (r, d-1-r) = (3, 3)$ for $i, j, k \in \{1, 2, 3\}$ and each pair of indices $j \neq k$. \square

Using the above considerations, we can find a slightly different example of a reduced free plane curve.

Example 3.2. Let us consider the arrangement $\mathcal{C} = \{E, C_1, \ell\}$, where E and C_1 are the curves indicated in Theorem 3.1, and

$$\ell : \quad x - y = 0.$$

Observe that the line ℓ is passing through the node of E and the sextactic point. Using this observation we can detect all singularities of the curve \mathcal{C} , namely we have one A_1 point, one ordinary triple point D_4 , and one singularity of type D_{14} , and hence $\tau(\mathcal{C}) = 19$. Since $\text{mdr}(\mathcal{C}) = 2$, we have construed an example of a reduced free curve.

Example 3.3. Let us consider the arrangement $\mathcal{C} = \{E, \ell_1, \ell_2, \ell_3\}$, where the curves are indicated in Theorem 3.1. The curve \mathcal{C} is free. Observe that we have four singularities of type A_1 and three singularities of type A_5 and hence $\tau(\mathcal{C}) = 19$. Since $\text{mdr}(\mathcal{C}) = 2$ it allows us to conclude that \mathcal{C} is indeed free.

Remark 3.4. Using Theorem 3.1 we can also construct some examples of **nearly-free** curves. For instance, we can take, the arrangement $\mathcal{C}_1 = \{E, C_1, \ell_1, \ell_2, \ell_3\}$, or $\mathcal{C}_2 = \{E, C_1, C_2, \ell_2\}$ and check that $\tau(\mathcal{C}_1) = \tau(\mathcal{C}_2) = 36$, which allows us to conclude that the constructed arrangements are nearly-free.

4 Combinatorial constraints on arrangements of lines, conics, and cubics with certain simple singularities

In this section we want to focus on combinatorial constraints of arrangements consisting of lines, smooth conics, and elliptic curves in the complex projective plane admitting singularities of type $A_1, A_3, D_4, A_5, A_7, A_{11}, D_{14}$. Our choice of singularities is motivated by arrangements constructed in the previous section in the context of the freeness. First of all, we have the following naive count.

Proposition 4.1. *Let $\mathcal{C} = \{\ell_1, \dots, \ell_d, C_1, \dots, C_k, \mathcal{E}_1, \dots, \mathcal{E}_l\} \subset \mathbb{P}_{\mathbb{C}}^2$ be an arrangement of $d \geq 1$ lines, $k \geq 1$ smooth conics, and $l \geq 1$ elliptic curves. Assume that the arrangement \mathcal{C} admits n_2 singularities of type A_1 , t_3 singularities of type A_3 , n_3 singularities of type D_4 , t_5 singularities of type A_5 , t_7 singularities of type A_7 , t_{11} singularities of type A_{11} , and d_{14} singularities of type D_{14} . Then we have the following count*

$$(d + 2k + 3l)^2 - d - 4k - 9l = d(d - 1) + 4k(k - 1) + 9l(l - 1) + 4dk + 6dl + 12kl = 2n_2 + 4t_3 + 6n_3 + 6t_5 + 8t_7 + 12t_{11} + 16d_{14}. \quad (4)$$

Proof. It follows from Bézout's Theorem and from counting of the intersection indices for singular points that our curve \mathcal{C} admits. \square

Now we can present our main result in this section.

Theorem 4.2. *Let $\mathcal{C} = \{\ell_1, \dots, \ell_d, C_1, \dots, C_k, \mathcal{E}_1, \dots, \mathcal{E}_l\} \subset \mathbb{P}_{\mathbb{C}}^2$ be an arrangement of $d \geq 1$ lines, $k \geq 1$ smooth conics, and $l \geq 1$ elliptic curves. Assume that the arrangement \mathcal{C} admits n_2 singularities of type A_1 , t_3 singularities of type A_3 , n_3 singularities of type D_4 , t_5 singularities of type A_5 , t_7 singularities of type A_7 , t_{11} singularities of type A_{11} , and d_{14} singularities of type D_{14} . Then the following inequality holds:*

$$27l + 8k + n_2 + \frac{3}{4}n_3 \geq d + \frac{5}{2}t_3 + 5t_5 + \frac{29}{4}t_7 + \frac{23}{2}t_{11} + \frac{79}{8}d_{14}. \quad (5)$$

Proof. Let $C = \ell_1 + \dots + \ell_d + C_1 + \dots + C_k + \mathcal{E}_1 + \dots + \mathcal{E}_l$ be the divisor associated with our arrangement \mathcal{C} . In order to show our result we are going to use an orbifold BMY inequality by Langer [6], namely

$$(\star) : \sum_{p \in \text{Sing}(C)} 3 \left(\alpha(\mu_p - 1) + 1 - e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C) \right) \leq (3\alpha - \alpha^2)m^2 - 3\alpha m,$$

where $e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C)$ is the local orbifold Euler number of a given singularity p , μ_p denotes the local Milnor number of p , and $\deg(C) = m$. Using [6, Theorem 8.7, Theorem 9.4.2], we obtain the following:

- if p is singularity of type A_1 , then $e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C) = (1 - \alpha)^2$ for $\alpha \in [0, 1]$,
- if p is singularity of type D_4 , then $e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C) = \frac{(2-3\alpha)^2}{4}$ for $\alpha \in (0, 2/3]$,
- if p is singularity of type A_3 , then $e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C) = \frac{(3-4\alpha)^2}{8}$ for $\alpha \in (1/3, 3/4]$,
- if p is singularity of type A_5 , then $e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C) = \frac{(4-6\alpha)^2}{12}$ for $\alpha \in (1/3, 2/3]$,
- if p is singularity of type A_7 , then $e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C) = \frac{(5-8\alpha)^2}{16}$ for $\alpha \in (3/8, 5/8]$,
- if p is singularity of type A_{11} , then $e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C) = \frac{(7-12\alpha)^2}{24}$ for $\alpha \in (5/12, 7/12]$,
- if p is singularity of type D_{14} , then $e_{\text{orb}}(p; \mathbb{P}_{\mathbb{C}}^2, \alpha C) = \frac{(7-13\alpha)^2}{24}$ for $\alpha \in (5/11, 7/13]$.

Since $m = d + 2k + 3l \geq 6$, we can take for our considerations $\alpha = \frac{1}{2}$, and from now one we are working with the pair $(\mathbb{P}_{\mathbb{C}}^2, \frac{1}{2}C)$. We start with the left-hand side of (\star) , which is equal to

$$\frac{9}{4}n_2 + \frac{45}{8}t_3 + \frac{117}{16}n_3 + \frac{35}{4}t_5 + \frac{189}{4}t_7 + \frac{143}{8}t_{11} + \frac{719}{32}d_{14}.$$

Now we look at the right-hand side of (\star) , namely

$$\frac{5}{4}(d + 2k + 3l)^2 - \frac{3}{2}(d + 2k + 3l).$$

Using the naive combinatorial count presented in Proposition 4.1, we have

$$\frac{5}{4}(d + 4k + 9l + 2n_2 + 4t_3 + 6n_3 + 6t_5 + 8t_7 + 12t_{11} + 16d_{14}) - \frac{3}{2}(d + 2k + 3l).$$

Plugging the collected data above to (\star) we obtain that

$$27l + 8k + n_2 + \frac{3}{4}n_3 \geq d + \frac{5}{2}t_3 + 5t_5 + \frac{29}{4}t_7 + \frac{23}{2}t_{11} + \frac{79}{8}d_{14},$$

which completes the proof. □

Conflict of Interests

I declare that there is no conflict of interest regarding the publication of this paper.

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