# Designs for practical SHE schemes based on RLWR

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(with thanks to A. Costache and R. Player for their slides!)

# What is RLWR?

# To start: what is RLWE?

Let N be a power of 2, and q a modulus.

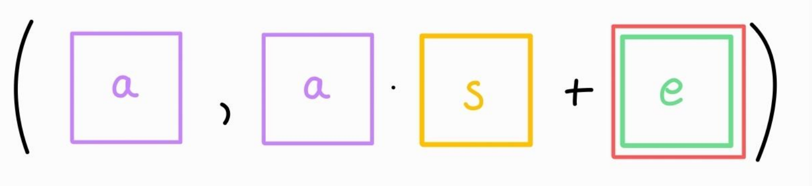
Consider also the ring $R_q = \mathbb{Z}_q[X]/(X^N+1)$

$$\left( \boxed{a} , \boxed{a} \cdot \boxed{s} + \boxed{e} \right) \in R_q \times R_q$$

Decision problem: is (a,b) formed like this, or uniformly random?
Search problem: what is the secret s?

# How is RLWR different?

- No Gaussian errors
- Achieve this through rounding operation on a.s.



# What is RLWR?

Rounding operation

$$\lfloor x \rceil_{q,p} := \left\lfloor \frac{p}{q} x \right\rceil$$

Let N be a power of 2, and q a modulus.

Consider also the ring $R_q = \mathbb{Z}_q[X]/(X^N + 1)$

$$\left( \boxed{a} \, , \left\lfloor \boxed{a} \cdot \boxed{s} \right\rceil_{q,p} \right) \in R_q \times R_p$$

Decision problem: is (a,b) formed like this, or uniformly random?

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# Motivation for considering RLWR

- Avoids need for Gaussian sampling
  - Side channels
  - Cost
- Easy to implement rounding
- Potential for improved bandwidth
- LWR is used by several NIST candidates (Lizard, Round5, Saber)

# How might we make SHE schemes from LWR?

# Initial attempt: BFV analogue [CS17]

Method: Swap out everything from BFV for rounded versions

uniform random bits

$$(\mathsf{ct}_0, \mathsf{ct}_1) = \left( \sum_{k=1}^{\ell} \boxed{r_k} \cdot \underline{v_k}, \boxed{\Delta_p} \cdot \underline{m} + \sum_{k=1}^{\ell} \boxed{r_k} \cdot \underline{w_k} \right) \in R_q \times R_p$$

Public key Pairs:

$(v_k, w_k) \in R_q \times R_p$

$=$

$(v_k, \lceil v_k s \rfloor_{q,p})$

scale parameter

$\Delta_p = \lfloor \frac{p}{t} \rceil$

message

$m \in R_t$

Costache, A., & Smart, N. P. (2017). Homomorphic encryption without gaussian noise. *Cryptology ePrint Archive*. https://eprint.iacr.org/2017/163.pdf

# There’s a problem…

When we multiply two ciphertexts,ct=($ct_0$,$ct_1$) and ct’=($ct_0$’,$ct_1$’), we do a tensor product, and it is not clear what ring each component should be in

$\boxed{R_q}$

$\boxed{R_p}$

$$\left(\boxed{\mathtt{ct}_0}\boxed{\mathtt{ct}_0'}, \boxed{\mathtt{ct}_0}\boxed{\mathtt{ct}_1'} + \boxed{\mathtt{ct}_0'}\boxed{\mathtt{ct}_1}, \boxed{\mathtt{ct}_1}\boxed{\mathtt{ct}_1'}\right) = (c_2, c_1, c_0)$$

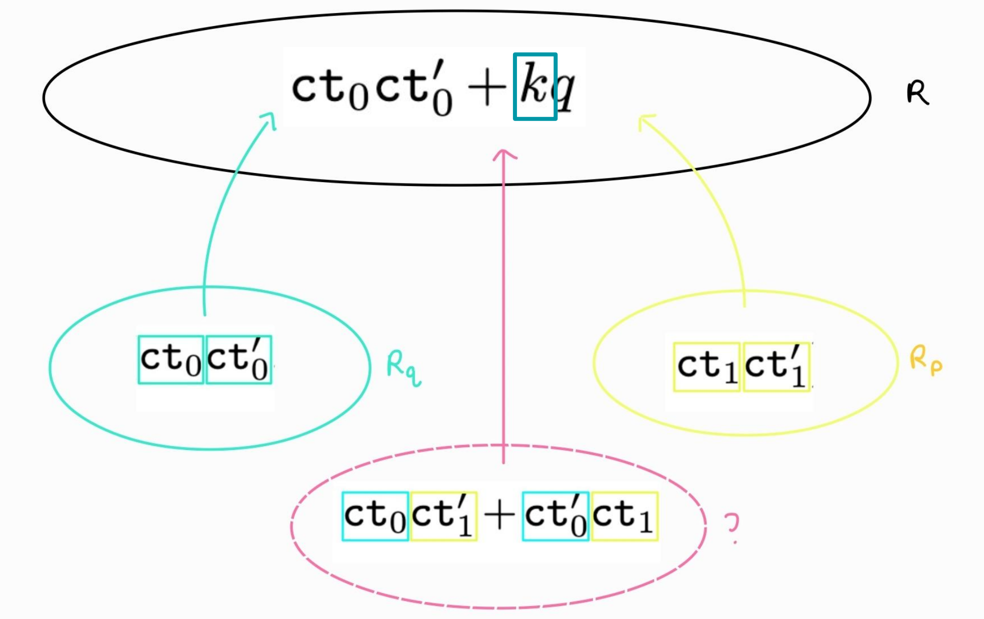
# When we do noise analysis…

We normally lift to R

Luo et al suggest we consider $c_2$ in the ring $R_q$

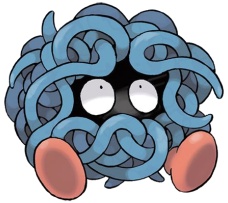
Creates unmanageable noise growth, due to integer polynomial $\boxed{k}$, if we consider in the wrong modulus

When we're done with analysis, where do we map it to?




Luo, F., Wang, F., Wang, K., Li, J., & Chen, K. (2018). LWR-based fully homomorphic encryption, revisited. *Security and Communication Networks*, *2018*.

# The Tangled Modulus problem



# Where should the elements live after multiplication?

In which ring should we consider each element, in order to avoid unmanageable noise?

$$\left(\boxed{\mathtt{ct}_0}\boxed{\mathtt{ct}'_0}, \boxed{\mathtt{ct}_0}\boxed{\mathtt{ct}'_1} + \boxed{\mathtt{ct}'_0}\boxed{\mathtt{ct}_1}, \boxed{\mathtt{ct}_1}\boxed{\mathtt{ct}'_1}\right) = (c_2, c_1, c_0)$$

We define the invariant noise as $N_{\text{mult}}$ in the following equation:

$$\frac{t}{p}\left[\left(\frac{p}{q}\right)^2 c_2 s^2 - \frac{p}{q} c_1 s + c_0\right] = m_{\text{mult}} + N_{\text{mult}} + tG_{\text{mult}}$$

# Possible solutions

$\boxed{\text{Solution 1: Carefully choose modulus and do noise analysis to ensure manageable noise (our work, LPR-style)}}$

Solution 2: Modify the evaluation key to avoid the problem (our work, Regev-style)

Solution 3: Base the RLWR scheme on a different scheme to BFV (LWWC18)

# RLWE LPR variant of BFV interlude

# LPR-style encryption scheme

Key Generation:

$$\mathtt{sk} = s \in R_q$$
$$\mathtt{pk} = (a, as + e) \in R_q \times R_q$$

Encryption:

$$\mathtt{ct} = (c_0, c_1) \in R_q \times R_q, \text{ where}$$

$$c_0 = \mathtt{pk}_0 u + e_0$$
$$c_1 = \mathtt{pk}_1 u + e_1 + \lfloor q/t \rfloor \cdot \mathtt{m} \qquad \mathtt{m} \in R_t$$

Decryption of ct:

$$\mathtt{m}' = \left[ \left\lfloor \frac{t}{q} \left[ (-c_0 s) + c_1 \right]_q \right\rceil \right]_t$$



Lyubashevsky, V., Peikert, C., & Regev, O. (2013). On ideal lattices and learning with errors over rings. *Journal of the ACM (JACM)*, *60*(6), 1-35.

# LPR-style encryption: multiplication

Multiplication of ct=($ct_0$,$ct_1$) and ct'=($ct_0$',$ct_1$'):

$$\mathtt{ct}_{\mathsf{mult}} := \left( \left\lfloor \frac{t}{q} c_0 c_0' \right\rceil,\ \left\lfloor \frac{t}{q} (c_0 c_1' + c_1 c_0') \right\rceil,\ \left\lfloor \frac{t}{q} c_1 c_1' \right\rceil \right) = (d_0, d_1, d_2)$$



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# LPR-style RLWR scheme

# Our LPR-style SHE scheme from RLWR

Key Generation:

$$\mathtt{sk} = s \in R_r$$
$$\mathtt{pk} = (a, b = \lfloor a \cdot s \rceil_{r,q}) \in R_r \times R_q$$

Encryption: $\mathtt{ct} = (c_0, c_1) \in R_q \times R_p$, where

$$c_0 = \lfloor \mathrm{pk}_0 u \rceil_{r,q}$$
$$c_1 = \lfloor \mathrm{pk}_1 u \rceil_{q,p} + \lfloor p/t \rfloor \cdot \mathtt{m}$$

Decryption:

$$\mathtt{m}' = \left[ \left\lfloor \frac{t}{p} \left( -\frac{p}{q} c_0 s + c_1 \right) \right\rceil \right]_t$$

# Our LPR scheme mult from RLWR

Here's where the elements should live

This combination avoids the unmanageable noise growth

$$\mathsf{ct}_{\mathsf{mult}} := \left( \left[ \left\lfloor \frac{t}{p} c_0 c_0' \right\rceil \right]_{q^2/p}, \left[ \left\lfloor \frac{t}{p} (c_0 c_1' + c_1 c_0') \right\rceil \right]_q, \left[ \left\lfloor \frac{t}{p} c_1 c_1' \right\rceil \right]_p \right)$$

**LPR-style encryption: multiplication**

Multiplication of ct=($c_0$,$c_1$) and ct'=($c_0$',$c_1$'):

$$\mathsf{ct}_{\mathsf{mult}} := \left( \left\lfloor \frac{t}{q} c_0 c_0' \right\rceil, \left\lfloor \frac{t}{q} (c_0 c_1' + c_1 c_0') \right\rceil, \left\lfloor \frac{t}{q} c_1 c_1' \right\rceil \right) = (d_0, d_1, d_2)$$

# Security proof?

# Security proof for regular LPR (from RLWE)

Hyb1 Honestly generated $\mathtt{pk}$ and $\mathtt{ct}$ (Real IND-CPA game)



Lyubashevsky, V., Peikert, C., & Regev, O. (2013). On ideal lattices and learning with errors over rings. *Journal of the ACM (JACM)*, *60*(6), 1-35.

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$\downarrow$ Decision Ring-LWE

Hyb2 Uniform $\mathtt{pk} = (a, b)$, honestly generated $\mathtt{ct} = (au + e_1, bu + e_2)$



Lyubashevsky, V., Peikert, C., & Regev, O. (2013). On ideal lattices and learning with errors over rings. *Journal of the ACM (JACM)*, *60*(6), 1-35.

# Security proof for regular LPR (from RLWE)

**Hyb1** Honestly generated $\texttt{pk}$ and $\texttt{ct}$ (Real IND-CPA game)

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**Hyb2** Uniform $\texttt{pk} = (a, b)$, honestly generated $\texttt{ct} = (au + e_1, bu + e_2)$

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**Hyb3** Uniform $\texttt{pk}$, uniform $\texttt{ct}$ ($\mathcal{A}$ has no information)



Lyubashevsky, V., Peikert, C., & Regev, O. (2013). On ideal lattices and learning with errors over rings. *Journal of the ACM (JACM)*, *60*(6), 1-35.

# Our RLWR LPR-style security proof

Hyb1 Honestly generated $\mathtt{pk} \in R_r \times R_q$, $\mathtt{ct} \in R_q \times R_p$ (IND-CPA)

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# Our RLWR LPR-style security proof

Hyb1 Honestly generated $\texttt{pk} \in R_r \times R_q$, $\texttt{ct} \in R_q \times R_p$ (IND-CPA)

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$\downarrow$ 3-moduli Ring-LWR

Hyb3 Uniform $\texttt{pk}$, uniform $\texttt{ct}$ ($\mathcal{A}$ has no information)

# 3-moduli RLWR

Let $a \leftarrow R_r$ and $b \leftarrow R_q$ be chosen uniformly, and let $u$ be a uniformly chosen polynomial with ternary coefficients. The (decisional) 3-moduli Ring-LWR problem with parameters $n$, $r$, $q$, $p$ asks to distinguish between $(a, b, \lfloor au \rceil_{r,q}, \lfloor bu \rceil_{q,p})$ and uniform tuples $(a, b, c, d) \in R_r \times R_q \times R_q \times R_p$.

# Reduction from Ring-LWR to 3-moduli Ring-LWR

Let $p$, $q$, $r$ be integers such that $q|r$ and $pr = q^2$. If there is an efficient algorithm for 3-moduli Ring-LWR with parameters $n$, $r$, $q$, $p$, then there is an efficient algorithm for Ring-LWR with parameters $n$, $r$, $q$.

# Regev-style scheme

# Can we reduce restrictions on p and q?

LPR-style scheme works well for powers of 2 → natural for rounding

LPR-style scheme requires: $q|r$ and $pr = q^2$

We might want to explore alternative schemes to avoid this restriction

Our solution:

- Regev-style scheme
  - Generalise the relinearisation technique to remove the condition $q|r$
  - Provable security for prime p and q with no other restrictions

# Results

# Comparing to BFV asymptotically

| | LPR-style scheme | Regev-style scheme | Prior work [LWWC18] | BFV |
|---|---|---|---|---|
| $\texttt{pk}$ size | $n \log(rq)$ | $\ell n \log(pq)$ | $(\ell + 1) n \log(p)$ | $2n \log(q)$ |
| $\texttt{ct}$ size | $n \log(pq)$ | $n \log(pq)$ | $(\ell + 1) n \log(p)$ | $2n \log(q)$ |
| Security | RLWR | RLWR | RLWR | RLWE |

# Methodology for concrete comparison with BFV

Goal: compare ciphertext sizes between our schemes and BFV, using our proof-of-concept Python implementation

Find parameter set with minimal ciphertext size such that we have:

1) 128 bits security
2) Correct decryption

Tree-shaped arithmetic circuit with depth L

Each level is 8 additions and 1 multiplication

# What is the ciphertext size?

Logarithm to base 2 of the minimal ciphertext size in kilobytes that supports L levels for the specified circuit with plaintext modulus t = 3

Calculated using proof of concept Python implementation

| Scheme | Level $L$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 |
| BFV | 4.17 | 6.21 | 7.86 | 8.27 | 8.58 | 9.92 | 10.14 | 10.34 | 10.50 | 10.65 | 11.86 | 11.99 | 12.10 | 12.21 | 12.31 |
| **LPR-like** | 3.91 | 6.09 | 7.79 | 8.22 | 8.55 | 9.89 | 10.11 | 10.31 | 10.48 | 10.64 | 11.85 | 11.97 | 12.09 | 12.19 | 12.30 |
| **Regev($\ell = 3$)** | 4.09 | 6.17 | 7.83 | 8.25 | 8.57 | 9.91 | 10.13 | 10.33 | 10.50 | 10.65 | 11.85 | 11.98 | 12.10 | 12.20 | 12.30 |

# What is the ciphertext size?

Logarithm to base 2 of the minimal ciphertext size in kilobytes that supports L levels for the specified circuit with plaintext modulus $t = 2^8$

Calculated using proof of concept Python implementation

| Scheme | Level $L$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 |
| BFV | 4.58 | 6.63 | 8.27 | 8.68 | 10.06 | 10.32 | 10.54 | 10.73 | 11.95 | 12.11 | 12.24 | 12.37 | 12.48 | 12.59 | 12.69 |
| **LPR-like** | 4.43 | 6.55 | 8.22 | 8.64 | 10.03 | 10.29 | 10.52 | 10.77 | 11.94 | 12.09 | 12.23 | 12.36 | 12.47 | 12.58 | 12.68 |
| **Regev($\ell = 3$)** | 4.52 | 6.60 | 8.25 | 8.67 | 10.04 | 10.31 | 10.53 | 10.72 | 11.95 | 12.10 | 12.24 | 12.36 | 12.48 | 12.59 | 12.69 |

# Parameter selection methodology

- We would like to choose parameters for concrete security using Lattice Estimator
- But Lattice Estimator is built for LWE instances
- Solution: interpret our LWR instances as LWE instances
- Model the implied LWE error distribution as a Gaussian with standard deviation $\sigma = \sqrt{\left((q/p)^2 - 1\right)/12}$ following 'Estimate all the schemes'
- Take p and q powers of 2 for performance
- We choose q/p = 16 to make $\sigma$ = 4.61, which is close to the standard choice for FHE of $\sigma$ = 3.2

Albrecht, M. R., Curtis, B. R., Deo, A., Davidson, A., Player, R., Postlethwaite, E. W., ... & Wunderer, T. (2018). Estimate all the {LWE, NTRU} schemes!. In *Security and Cryptography for Networks: 11th International Conference, SCN 2018*

# Summary

# What did we do?

Designed two SHE schemes based on the RLWR problem, Regev-style and LPR-style

- BFV-like scheme is possible from RLWR
- Comparable parameters to BFV
- Improved ciphertext sizes
- Security analysis

# Next steps…

- Library integration?
- RNS variant?
- Building other things from RLWR?
- [your cool idea here!]

# Thank you!

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Paper coming soon to an eprint near you…

Designed two SHE schemes based on the RLWR problem, Regev-style and LPR-style

- BFV-like scheme is possible from RLWR
- Comparable parameters to BFV
- Improved ciphertext sizes
- Thorough security analysis

# Parameter sizes for LPR-style RLWR scheme

Assuming uniform ternary secret, targeting 128 bits security

| $n$ | $r$ | $q$ | $p$ |
|---|---|---|---|
| $2^{15}$ | $2^{856}$ | $2^{852}$ | $2^{848}$ |
| $2^{14}$ | $2^{425}$ | $2^{421}$ | $2^{417}$ |
| $2^{13}$ | $2^{211}$ | $2^{207}$ | $2^{203}$ |
| $2^{12}$ | $2^{105}$ | $2^{101}$ | $2^{97}$ |
| $2^{11}$ | $2^{52}$ | $2^{48}$ | $2^{44}$ |
| $2^{10}$ | $2^{26}$ | $2^{22}$ | $2^{18}$ |

# Comparing to BFV asymptotically

| | Regev-type scheme | LPR-type scheme | LWWC [54] | BFV [37] |
|---|---|---|---|---|
| Size of pk | $\ell n \log(PQ)$ | $n \log(rq)$ | $(\ell' + 1)n \log(q')$ | $2n \log(q'')$ |
| Size of ct | $n \log(PQ)$ | $n \log(pq)$ | $(\ell' + 1)n \log(p')$ | $2n \log(q'')$ |
| Security | Ring-LWR$_{n,Q,P}$ | Ring-LWR$_{n,r,q}$ | Ring-LWR$_{n,q',p'}$ | Ring-LWE$_{n,\chi,q''}$ |
| Constraints | $P$ and $Q$ prime | $q \mid r$ and $pr = q^2$ | $p' \mid q'$ | N/A |

**Table 1.** A comparison of our Regev-type scheme and our LPR-type scheme with the prior schemes LWWC [54] and BFV [37]. The parameter constraints specified are required for provable security. We may assume $\log(q'') = \log(Q) = \log(r)$, while $q' > p'$ and $p'$ is a polynomial factor larger than $q''$. If the encryption randomness in the Regev-type scheme is sampled from the set of scalars $\{-B/2, \ldots B/2\}$ then $\ell \geq 1/\log(B+1)(n\log(PQ) + 2\lambda - 2)$, while $\ell' > \log(q') + 2\lambda$.