

Plug-and-play Sanitization for TFHE

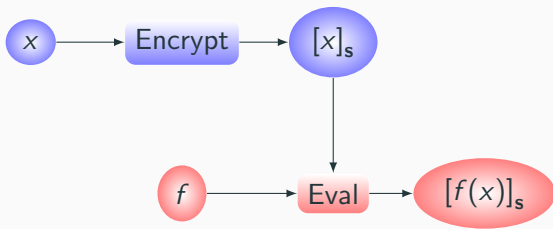
Florian Bourse and Malika Izabachène

March 24, 2024: FHE.org Toronto 2024 Conference

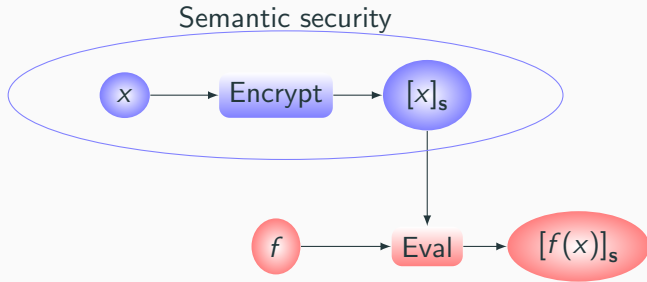
Motivation :

Circuit privacy and sanitization

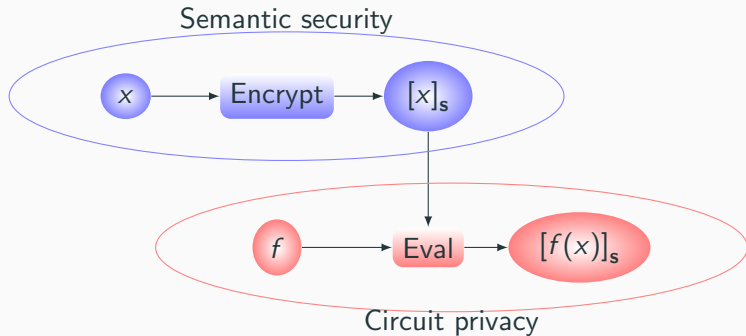
Motivation : Typical FHE application



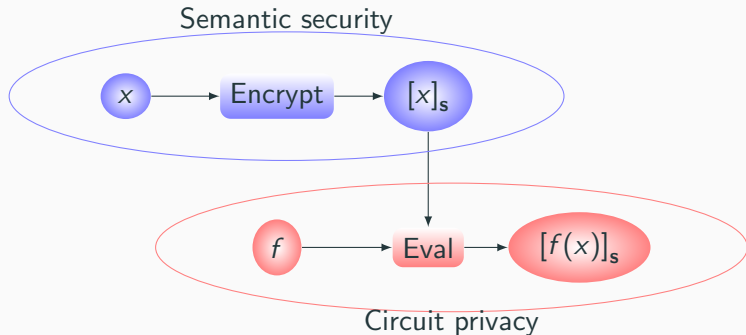
Motivation : Typical FHE application



Motivation : Typical FHE application



Motivation : Typical FHE application



Computational vs statistical.
Indistinguishability-based vs simulation-based.

Warm-up example : Circuit privacy

LWE in dimension $n = 1$, modulo $q = 1024$, secret key $\mathbf{s} = (0)$:

Warm-up example : Circuit privacy

LWE in dimension $n = 1$, modulo $q = 1024$, secret key $\mathbf{s} = (0)$:

$$C_0 = (42, 0) \quad C_1 = (13, 37)$$

Warm-up example : Circuit privacy

LWE in dimension $n = 1$, modulo $q = 1024$, secret key $\mathbf{s} = (0)$:

$$C_0 = (42, 0) \quad C_1 = (13, 37)$$

$$f_\beta(x_0, x_1) = \text{mux}(\beta, x_0, x_1) = x_\beta$$

Warm-up example : Circuit privacy

LWE in dimension $n = 1$, modulo $q = 1024$, secret key $\mathbf{s} = (0)$:

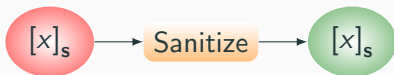
$$C_0 = (42, 0) \quad C_1 = (13, 37)$$

$$f_\beta(x_0, x_1) = \text{mux}(\beta, x_0, x_1) = x_\beta$$

$\text{Eval}(f_\beta, C_0, C_1) = (42, 0)$, can you find β ?

Sanitization \Rightarrow Circuit privacy

Sanitization :



Sanitization \Rightarrow Circuit privacy

Sanitization :

$\text{Sanitize}(C_0) \approx \text{Sanitize}(C_1)$ if $\text{Decrypt}(s, C_0) = \text{Decrypt}(s, C_1)$



Independent of original ciphertext

Sanitization \Rightarrow Circuit privacy

Circuit Privacy for a class of function \mathcal{F} :

$$O_0 = \text{Eval}(f_1, C_1, \dots, C_k) \approx \text{Eval}(f_2, C_1, \dots, C_k) = O_1$$

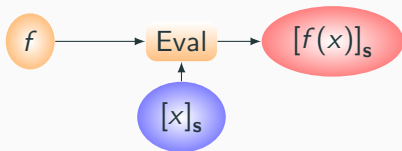
if $\text{Decrypt}(\mathbf{s}, O_0) = \text{Decrypt}(\mathbf{s}, O_1)$

Sanitization \Rightarrow Circuit privacy

Circuit Privacy for a class of function \mathcal{F} :

$$O_0 = \text{Eval}(f_1, C_1, \dots, C_k) \approx \text{Eval}(f_2, C_1, \dots, C_k) = O_1$$

if $\text{Decrypt}(\mathbf{s}, O_0) = \text{Decrypt}(\mathbf{s}, O_1)$

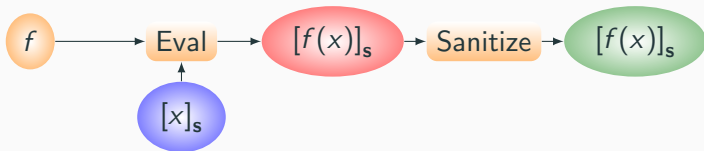


Sanitization \Rightarrow Circuit privacy

Circuit Privacy for a class of function \mathcal{F} :

$$O_0 = \text{Eval}(f_1, C_1, \dots, C_k) \approx \text{Eval}(f_2, C_1, \dots, C_k) = O_1$$

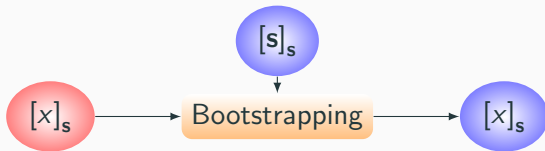
if $\text{Decrypt}(\mathbf{s}, O_0) = \text{Decrypt}(\mathbf{s}, O_1)$



Circuit privacy for bootstrapping \Rightarrow Sanitization

Bootstrapping :

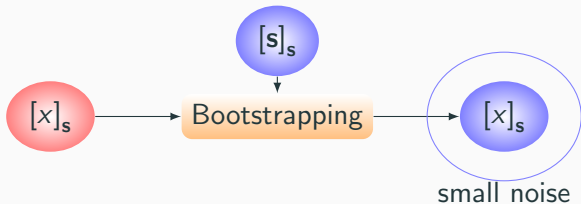
$$\text{Bootstrap}(C) = \text{Eval}(\text{Decrypt}(\cdot, C), [s]_s)$$



Circuit privacy for bootstrapping \Rightarrow Sanitization

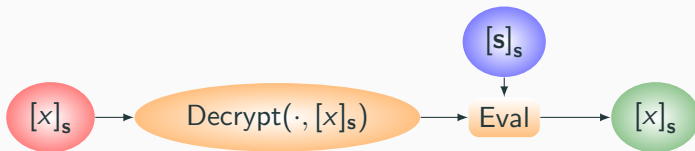
Bootstrapping :

$$\text{Bootstrap}(C) = \text{Eval}(\text{Decrypt}(\cdot, C), [s]_s)$$



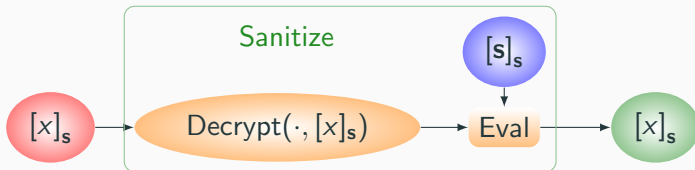
Circuit privacy for bootstrapping \Rightarrow Sanitization

With circuit privacy for the class of functions $(\text{Decrypt}(\cdot, C), [s]_s)_C$:



Circuit privacy for bootstrapping \Rightarrow Sanitization

With circuit privacy for the class of functions $(\text{Decrypt}(\cdot, C), [s]_s)_C$:



Randomization techniques

A common misconception : Circuit privacy is easy [Gen09], noise flooding

The idea is simple: to construct a random encryption ψ' of π from a particular encryption ψ of π , we simply add an encryption of 0 that has a **much larger random “error”** vector than ψ – **super-polynomially larger**, so that the new error vector statistically obliterates all information about ψ 's error vector.

A common misconception : Circuit privacy is easy [Gen09], noise flooding

The idea is simple: to construct a random encryption ψ' of π from a particular encryption ψ of π , we simply add an encryption of 0 that has a **much larger random “error”** vector than ψ – **super-polynomially larger**, so that the new error vector statistically obliterates all information about ψ 's error vector.

But : $(0, e)$ is such an encryption of 0 :

$$C_0 + (0, 200) = (\underline{42}, 200) \quad C_1 + (0, 200) = (\underline{13}, 237)$$

A common misconception : Circuit privacy is easy [Gen09], noise flooding

The idea is simple: to construct a random encryption ψ' of π from a particular encryption ψ of π , we simply add an encryption of 0 that has a **much larger random “error”** vector than ψ – **super-polynomially larger**, so that the new error vector statistically obliterates all information about ψ 's error vector.

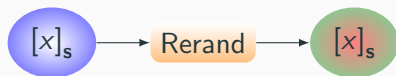
But : $(0, e)$ is such an encryption of 0 :

$$C_0 + (0, 200) = (\underline{42}, 200) \quad C_1 + (0, 200) = (\underline{13}, 237)$$

Requires additional properties !

Soak-and-Spin paradigm [DS16]

We assume that one may build an efficient randomized algorithm Rerand such that :

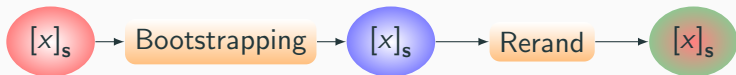


$$\Delta(\text{Rerand}(C_0), \text{Rerand}(C_1)) \leq \delta < 1$$

$$\text{if } \text{Decrypt}(\mathbf{s}, C_0) = \text{Decrypt}(\mathbf{s}, C_1)$$

Soak-and-Spin paradigm [DS16]

We assume that one may build an efficient randomized algorithm Rerand such that :



$$\Delta(\text{Rerand}(C_0), \text{Rerand}(C_1)) \leq \delta < 1$$

$$\text{if } \text{Decrypt}(\mathbf{s}, C_0) = \text{Decrypt}(\mathbf{s}, C_1)$$

Repeat for sanitization.

How to choose your Rerand

Previous work:

- Public key: $O(n \log q)$ LWE encryptions of 0.
- Rerand: random subset sum of those (+ noise).
- Statistical security.

How to choose your Rerand

Previous work:

- Public key: $O(n \log q)$ LWE encryptions of 0.
- Rerand: random subset sum of those (+ noise).
- Statistical security.

This work:

- Public key: 1 RLWE encryption of 0: $(a, as + e)$.
- Rerand: 1 RLWE encryption (+ noise):

$$(ra + e', r(as + e) + e'').$$

- Security under decisional RLWE assumption.

How to choose your Rerand

Previous work:

- Public key: $O(n \log q)$ LWE encryptions of 0.
- Rerand: random subset sum of those (+ noise).
- Statistical security.

This work:

- Public key: 1 RLWE encryption of 0: $(a, as + e)$.
- Rerand: 1 RLWE encryption (+ noise):

$$(ra + e', r(as + e) + e'').$$

- Security under decisional RLWE assumption.
- Ciphertext is well distributed even if the secret key is leaked.

Circuit private MUX operation [BdMW16]

GSW ciphertexts:

$$[\beta]_{\mathbf{s}} = (a, as + e) + \beta \mathbf{G}, \quad \beta \in \{0, 1\}$$

Circuit privacy for $(\text{mux}(\cdot, C_0, C_1))_{C_0, C_1}$:

Circuit private MUX operation [BdMW16]

GSW ciphertexts:

$$[\beta]_{\mathbf{s}} = (a, as + e) + \beta \mathbf{G}, \quad \beta \in \{0, 1\}$$

Circuit privacy for $(\text{mux}(\cdot, C_0, C_1))_{C_0, C_1}$:

$$C_0 + \mathbf{G}_{\text{rand}}^{-1} (C_1 - C_0) \cdot [\beta]_{\mathbf{s}} + \text{Rerand} \approx C_{\beta} + \mathcal{D}$$

where $\mathbf{G}_{\text{rand}}^{-1}(\cdot)$ is a Gaussian, **Rerand** is just Gaussian noise and \mathcal{D} is independent of C_0, C_1

Circuit private mux operation [BdMW16]

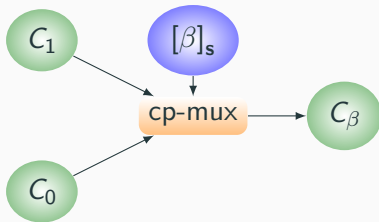
GSW ciphertexts:

$$[\beta]_s = (a, as + e) + \beta \mathbf{G}, \quad \beta \in \{0, 1\}$$

Circuit privacy for $(\text{mux}(\cdot, C_0, C_1))_{C_0, C_1}$:

$$C_0 + \mathbf{G}_{\text{rand}}^{-1} (C_1 - C_0) \cdot [\beta]_s + \text{Rerand} \approx C_\beta + \mathcal{D}$$

where $\mathbf{G}_{\text{rand}}^{-1}(\cdot)$ is a Gaussian, **Rerand** is just Gaussian noise and \mathcal{D} is independent of C_0, C_1



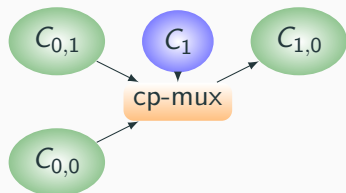
Circuit private BlindRotate

Application to branching programs :

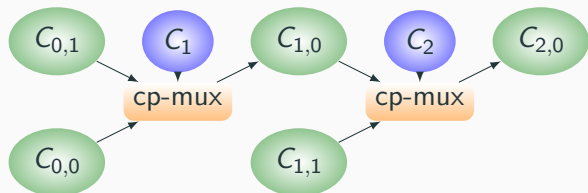
$C_{0,1}$

$C_{0,0}$

Application to branching programs :

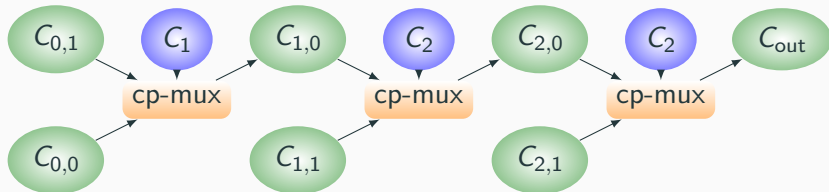


Application to branching programs :



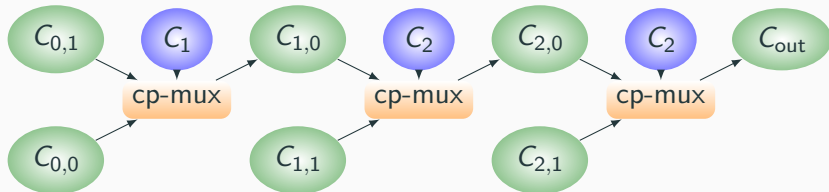
Circuit privacy for branching program evaluation [BdMW16]

Application to branching programs :



Circuit privacy for branching program evaluation [BdMW16]

Application to branching programs :



Proof by induction, adding Rerand noise at each step :
Lemma requires fresh Gaussian samples,
but each step depends on the previous one.

TFHE BlindRotate algorithm

Building block of the Bootstrapping,
computes $[X^{b-sa}]_s$ from (a, b) and $[s]_s$.

TFHE BlindRotate algorithm

Building block of the Bootstrapping,
computes $[X^{b-sa}]_s$ from (a, b) and $[s]_s$.

BlindRotate

```
ACC = (0, ..., 0, TESTVECTOR · X-b)
```

```
for  $i = 1$  to  $n - 1$ 
```

```
    ACC = cp-mux( $[s_i]_s$ , ACC, ACC × (Xai - 1))
```

```
Return ACC
```

TESTVECTOR allows for functional bootstrapping.

TFHE BlindRotate algorithm

Building block of the Bootstrapping,
computes $[X^{b-sa}]_s$ from (\mathbf{a}, b) and $[s]_s$.

BlindRotate

```
ACC = (0, ..., 0, TESTVECTOR · X-b)
```

```
for  $i = 1$  to  $n - 1$ 
```

```
    ACC = cp-mux( $[s]_s$ , ACC, ACC × (Xai - 1))
```

```
Return ACC
```

TESTVECTOR allows for functional bootstrapping.
RGSW ciphertexts, over $\mathcal{R} = \mathbb{Z}/q\mathbb{Z}[X]/(X^N + 1)$,
 q and N powers-of-two.

$$\mathbf{e}\mathbf{x} + \mathbf{y} \approx \mathbf{e}'$$

holds over rings for any \mathbf{v} , $\mathbf{x} = \mathbf{G}_{rand}^{-1}(\mathbf{v})$.

$$\mathbf{e}\mathbf{x} + y \approx e'$$

holds over rings for any \mathbf{v} , $\mathbf{x} = \mathbf{G}_{rand}^{-1}(\mathbf{v})$.

$$C_0 + \mathbf{G}_{rand}^{-1}(C_1 - C_0) \cdot [\beta]_s + \mathbf{Rerand} \approx C_\beta + \mathcal{D}$$

where $\mathbf{G}_{rand}^{-1}(\cdot)$ is a Gaussian, \mathbf{Rerand} is just Gaussian noise and \mathcal{D} is independent of C_0, C_1 .

$$\mathbf{e}\mathbf{x} + y \approx e'$$

holds over rings for any \mathbf{v} , $\mathbf{x} = \mathbf{G}_{rand}^{-1}(\mathbf{v})$.

$$C_0 + \mathbf{G}_{rand}^{-1}(C_1 - C_0) \cdot [\beta]_s + \mathbf{Rerand} \approx C_\beta + \mathcal{D}$$

where $\mathbf{G}_{rand}^{-1}(\cdot)$ is a Gaussian, \mathbf{Rerand} is just Gaussian noise and \mathcal{D} is independent of C_0, C_1 .

Holds, but only for impractical parameters (Leftover Hash Lemma).

Lower bound for required parameters

Leftover Hash Lemma : If $\mathbf{a}, \mathbf{r}, u$ uniform

$$\mathbf{a}, \mathbf{ar} \approx \mathbf{a}, u$$

Lower bound for required parameters

Leftover Hash Lemma : If $\mathbf{a}, \mathbf{r}, u$ uniform

$$\mathbf{a}, \mathbf{a}\mathbf{r} \approx \mathbf{a}, u$$

However : in \mathcal{R} : $\frac{1}{2}$ invertible, $\frac{1}{2}$ not invertible

Lower bound for required parameters

Leftover Hash Lemma : If $\mathbf{a}, \mathbf{r}, u$ uniform

$$\mathbf{a}, \mathbf{ar} \approx \mathbf{a}, u$$

However : in \mathcal{R} : $\frac{1}{2}$ invertible, $\frac{1}{2}$ not invertible

$$\Delta(\mathbf{ar}, u) \geq \frac{1}{2^\ell}$$

with ℓ length of \mathbf{a} .

Exemple in dimension 1 :

a	0	1	1	0	0	1	1	0
r	0	0	1	1	1	1	0	0
u	0	0	0	0	1	1	1	1
ar	0	0	1	0	0	1	0	0

Our version of cp-mux

Public key : $pk = (a, as + e)$

$$\text{cp-mux}(C, C_0, C_1) = C_0 + \mathbf{G}_{\text{rand}}^{-1} (C_1 - C_0) \cdot C + \text{Rerand}$$

with $\text{Rerand} = (ra + e', r(as + e) + e'')$

Our version of cp-mux

Public key : $pk = (a, as + e)$

$$\text{cp-mux}(C, C_0, C_1) = C_0 + \mathbf{G}_{\text{rand}}^{-1} (C_1 - C_0) \cdot C + \text{Rerand}$$

with $\text{Rerand} = (ra + e', r(as + e) + e'')$

$$\text{cp-mux}([\beta]_s, C_0, C_1) \approx C_\beta + \mathcal{D}$$

Our version of cp-mux

Public key : $pk = (a, as + e)$

$$\text{cp-mux}(C, C_0, C_1) = C_0 + \mathbf{G}_{\text{rand}}^{-1} (C_1 - C_0) \cdot C + \text{Rerand}$$

with $\text{Rerand} = (ra + e', r(as + e) + e'')$

$$\begin{aligned} \text{cp-mux}([\beta]_s, C_0, C_1) &\approx C_\beta + \mathcal{D} \\ &\approx \text{cp-mux}([\beta]_s, C_\beta, C_\beta) \end{aligned}$$

Our version of cp-mux

Public key : $pk = (a, as + e)$

$$\text{cp-mux}(C, C_0, C_1) = C_0 + \mathbf{G}_{\text{rand}}^{-1} (C_1 - C_0) \cdot C + \text{Rerand}$$

with $\text{Rerand} = (ra + e', r(as + e) + e'')$

$$\begin{aligned} \text{cp-mux}([\beta]_s, C_0, C_1) &\approx C_\beta + \mathcal{D} \\ &\approx \text{cp-mux}([\beta]_s, C_\beta, C_\beta) \end{aligned}$$

$$C_0 + \mathbf{G}_{\text{rand}}^{-1} (C_1 - C_0) \cdot [\beta]_s + \text{Rerand} \approx C_\beta + \mathbf{G}_{\text{rand}}^{-1} (\mathbf{0}) \cdot [\beta]_s + \text{Rerand}$$

Circuit private BlindRotate

New proof technique : reuse the **Rerand**

Unfolding the loop :

$$\begin{aligned} & X^{b-sa} + \\ & + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_1(X^{a_1} - 1)) \cdot [s_1]_s \\ & + \dots \\ & + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_{n-1}(X^{a_{n-1}} - 1)) \cdot [s_{n-1}]_s \\ & + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_n(X^{a_n} - 1)) \cdot [s_n]_s \\ & + \text{Rerand} \end{aligned}$$

Circuit private BlindRotate

New proof technique : reuse the **Rerand**

Unfolding the loop :

$$\begin{aligned} & X^{b-sa} + \\ & + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_1(X^{a_1} - 1)) \cdot [s_1]_s \\ & + \dots \\ & + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_{n-1}(X^{a_{n-1}} - 1)) \cdot [s_{n-1}]_s \\ \text{Rerand} & + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_n(X^{a_n} - 1)) \cdot [s_n]_s \end{aligned}$$

Circuit private BlindRotate

New proof technique : reuse the **Rerand**

Unfolding the loop :

$$\begin{aligned} & X^{b-sa} + \\ & + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_1(X^{a_1} - 1)) \cdot [s_1]_s \\ & + \dots \\ & + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_{n-1}(X^{a_{n-1}} - 1)) \cdot [s_{n-1}]_s \\ \text{Rerand} & + \mathbf{G}_{\text{rand}}^{-1} (\mathbf{0}) \cdot [s_n]_s \end{aligned}$$

Circuit private BlindRotate

New proof technique : reuse the **Rerand**

Unfolding the loop :

$$\begin{aligned} & X^{b-sa} + \\ & \quad + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_1(X^{a_1} - 1)) \cdot [s_1]_s \\ & \quad + \dots \\ \text{Rerand} & \quad + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_{n-1}(X^{a_{n-1}} - 1)) \cdot [s_{n-1}]_s \\ & \quad + \mathbf{G}_{\text{rand}}^{-1} (\mathbf{0}) \cdot [s_n]_s \end{aligned}$$

New proof technique : reuse the **Rerand**

Unfolding the loop :

$$\begin{aligned} & X^{b-sa} + \\ & \quad + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_1(X^{a_1} - 1)) \cdot [s_1]_s \\ & \quad + \dots \\ \text{Rerand} & + \mathbf{G}_{\text{rand}}^{-1} (\mathbf{0}) \cdot [s_{n-1}]_s \\ & + \mathbf{G}_{\text{rand}}^{-1} (\mathbf{0}) \cdot [s_n]_s \end{aligned}$$

New proof technique : reuse the Rerand

Unfolding the loop :

$$\begin{aligned} & X^{b-sa} + \\ & \text{Rerand} + \mathbf{G}_{\text{rand}}^{-1} (\text{ACC}_1(X^{a_1} - 1)) \cdot [s_1]_s \\ & + \dots \\ & + \mathbf{G}_{\text{rand}}^{-1} (\mathbf{0}) \cdot [s_{n-1}]_s \\ & + \mathbf{G}_{\text{rand}}^{-1} (\mathbf{0}) \cdot [s_n]_s \end{aligned}$$

Circuit private BlindRotate

New proof technique : reuse the Rerand

Unfolding the loop :

$$\begin{aligned} & X^{b-sa} + \\ & \text{Rerand} + \mathbf{G}_{\text{rand}}^{-1}(\mathbf{0}) \cdot [s_1]_s \\ & + \dots \\ & + \mathbf{G}_{\text{rand}}^{-1}(\mathbf{0}) \cdot [s_{n-1}]_s \\ & + \mathbf{G}_{\text{rand}}^{-1}(\mathbf{0}) \cdot [s_n]_s \end{aligned}$$

Circuit private BlindRotate

New proof technique : reuse the **Rerand**

Unfolding the loop :

$$\begin{aligned} & X^{b-sa} + \mathbf{Rerand} \\ & + \mathbf{G}_{\mathbf{rand}}^{-1}(\mathbf{0}) \cdot [s_1]_s \\ & + \dots \\ & + \mathbf{G}_{\mathbf{rand}}^{-1}(\mathbf{0}) \cdot [s_{n-1}]_s \\ & + \mathbf{G}_{\mathbf{rand}}^{-1}(\mathbf{0}) \cdot [s_n]_s \end{aligned}$$

Building Sanitize

Putting together Sanitize

TFHE Bootstrapping :

ModSwitch

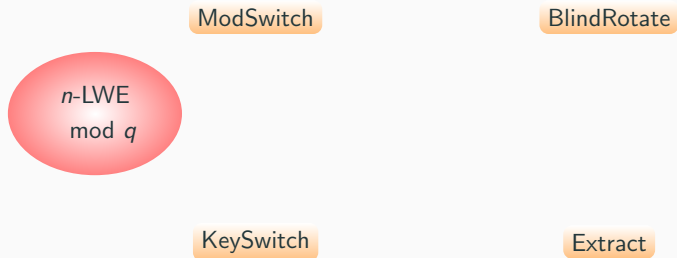
BlindRotate

KeySwitch

Extract

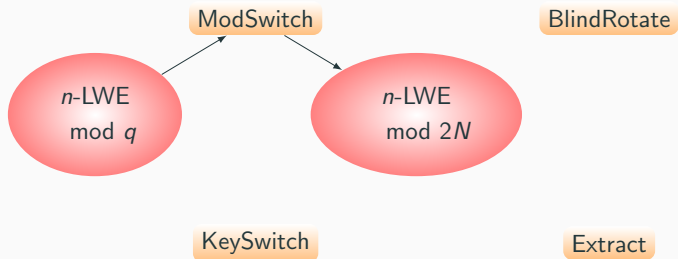
Putting together Sanitize

TFHE Bootstrapping :



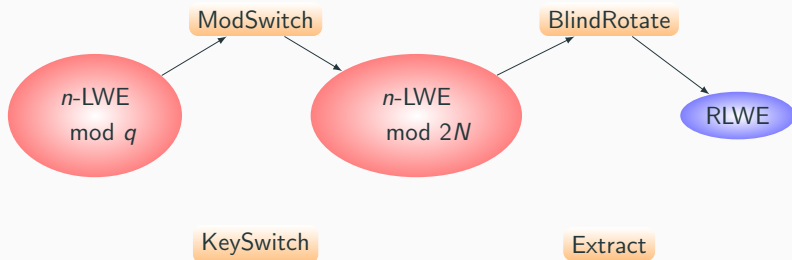
Putting together Sanitize

TFHE Bootstrapping :



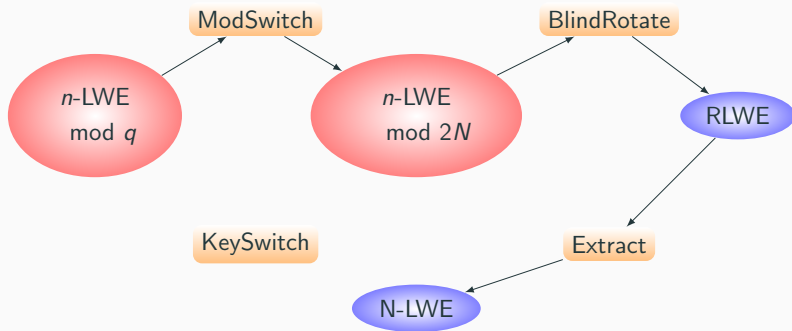
Putting together Sanitize

TFHE Bootstrapping :



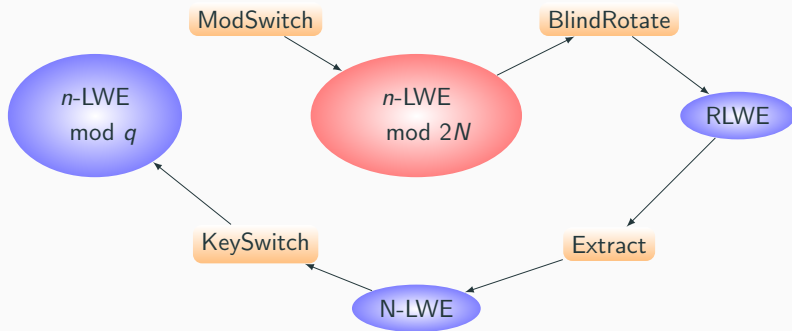
Putting together Sanitize

TFHE Bootstrapping :



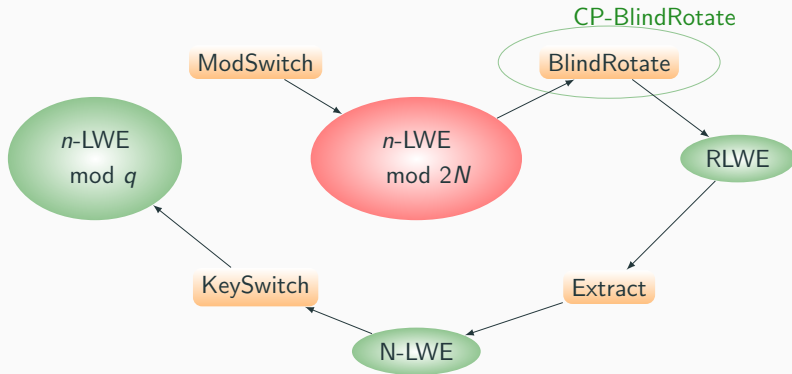
Putting together Sanitize

TFHE Bootstrapping :



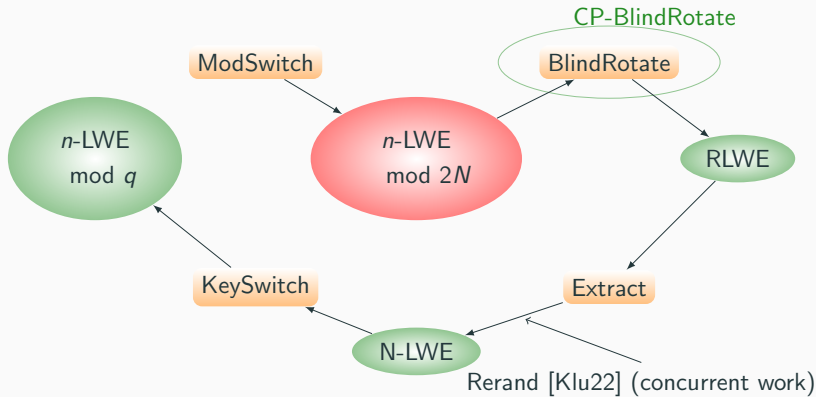
Putting together Sanitize

TFHE Bootstrapping :



Putting together Sanitize

TFHE Bootstrapping :



Empirical results and conclusion

Performance

LWE: $q = 2^{35}$, $\vartheta_{lwe} = 2^{-27.2}$, $B_{ks} = 4$, $\ell_{ks} = 7$

RLWE: $N = 1024$, $\vartheta_{rlwe} = 2^{-67.6}$, $\vartheta_{pk} = 2^{-65.6}$, $2^{-49.5}$

RGSW: $B_G = 32$, $\ell_G = 7$, $\vartheta_G = 2^{-39.4}$

Bootstrap	Sanitize w/ pre-computation	Sanitize w/o pre-computation	Rerand
-			-
0.42 s	0.52 s	7.5 s	1 ms

100 bits of security

Conclusion

- [BdMW16] over rings :
multi-hop circuit privacy for branching programs for RGSW
without circular security

Conclusion

- [BdMW16] over rings :
multi-hop circuit privacy for branching programs for RGSW
without circular security
- Better induction proof :
reduces parameters and improve efficiency for [BdMW16],
even on GSW

Conclusion

- [BdMW16] over rings :
multi-hop circuit privacy for branching programs for RGSW
without circular security
- Better induction proof :
reduces parameters and improve efficiency for [BdMW16],
even on GSW
- Alternative Rerand implementation :
greatly reduces the size of public key, applicable to RLWE and
LWE

Conclusion

- [BdMW16] over rings :
multi-hop circuit privacy for branching programs for RGSW
without circular security
- Better induction proof :
reduces parameters and improve efficiency for [BdMW16],
even on GSW
- Alternative Rerand implementation :
greatly reduces the size of public key, applicable to RLWE and
LWE
- Sanitization with one round of bootstrapping :
works with power-of-two modulus, adaptable to other modulus

Conclusion

- [BdMW16] over rings :
multi-hop circuit privacy for branching programs for RGSW
without circular security
- Better induction proof :
reduces parameters and improve efficiency for [BdMW16],
even on GSW
- Alternative Rerand implementation :
greatly reduces the size of public key, applicable to RLWE and
LWE
- Sanitization with one round of bootstrapping :
works with power-of-two modulus, adaptable to other modulus

Thanks for listening !