ON LINEARITY OF HOLOMORPHIC 1-FORMS WITH ZEROS

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Abstract. The goal of this article is two-fold, first we

1. Introduction

In this article we study the set of global holomorphic 1-forms on smooth projective varieties that admit zeros. Interestingly, it has been indicated by plethora of results ([HK05, ?, ?, SS19, ?, PS14] to name just a few) that a lot of the geometry and topology of the variety depends on vanishing of such forms. We first state our result and then put it in the context of the existing vast literature.

Theorem A. Let $f : X \to A$ be a morphism from a smooth projective variety to a simple abelian variety A. Then f is smooth if and only if there is a holomorphic 1-form ω on A such that $f^*\omega$ is nowhere vanishing.

In fact we prove a bit more; When A is not necessarily simple existence of a non-vanishing global holomorphic 1-form is equivalent to very restrictive set of strata that can show up in the Whitney stratification of the morphism f. See Theorem 3.8 for more details.

Remark 1.1 (Previous results).
(1) When X is of general type, it was conjectured in [HK05, ?] and was proved by Popa and Schnell [PS14] every global holomorphic 1- form vanishes on X.
(2) On the opposite extreme for any A, not necessarily simple when f is singular along a divisor of general type in A, Hacon and Kovács [HK05, Proposition 3.5.] show that $f^*\omega$ always admits zero.
(3) The more general version of Theorem A recovers both Item (1) (see Corollary **??**) and Item (2) (see Corollary **??**
(4) A special case of a result of Schreider [SS19, Corollary 1.5] shows for any A, not necessarily simple when X admits a nowhere vanishing holomorphic 1-form of the form $f^*\omega$, then $f(X)$ is fibered by tori. In Theorem 3.8 we show more generally that existence of non-vanishing forms is in fact equivalent to existence of subvarieties $Z \subseteq f(X)$ fibered by tori.

In a different direction we deal with the question of linearity of the set of vanishing 1-forms. This is inspired by the work of [CL73], Carrell and Lieberman, who show that for a smooth complex projective variety X, the set of holomorphic tangent vector fields with zeros is a subvector space of $H^0(X, T_X)$. On the other hand its incarnation for 1-form is far from complete. It is known due to the theory of cohomology jump loci that only the forms satisfying certain resonance property (see Definition **??** below) admitting zeros form a finite union of subvector spaces inside $H^0(X, \Omega^1_X)$. Henceforth we call finite unions of subvector spaces as *linear subvarieties*. In this article we complete the picture. To this end we need the following notations. Let X be a smooth complex projective variety of $\dim X = n$. The zero set of a global holomorphic 1-form ω is given by

$$Z(\omega) := \{x \in X | \omega(T_x X) = 0\}$$

where $T_x X$ is the holomorphic tangent space of X at x. We further denote by

$$V^i(X) := \{\omega \in H^0(X, \Omega^1_X) | \ \operatorname{codim} Z(\omega) \leq i\},$$

when $i = n$ we simply use the notation $V(X)$. Furthermore, for the morphism $f : X \to A$ from X to an abelian variety A, we denote

$$V^i(f) := \{\omega \in f^* H^0(A, \Omega^1_A) | \ \operatorname{codim} Z(\omega) \leq i\},$$

As before when $i = n$ we use the notation $V(f)$.

Theorem B. Let $f : X \to A$ be a morphism from a smooth complex projective variety X to an abelian variety A. Then $V(f)$ is a linear subvariety of $H^0(X, \Omega^1_X)$. In particular, $V(X)$ is a linear subvariety. Moreover, each irreducible components of $V(f)$ are $\mathbb{Q}$-sub Hodge structures of $H^0(X, \Omega^1_X)$. In particular, $V(f)$ and $V(X)$ are defined over $\mathbb{Q}$.

Remark 1.2.
1. When X is of general type, the result of Popa and Schnell (see **??**) it follows that $V(X) = H^0(X, \Omega^1_X)$ which is of course linear.
2. When $\dim X \leq 3$ an immediate corollary of the main theorems in [SS19] and [HS19] is that $V(X)$ is linear.
3. The *holomorphic resonance varieties* are roughly speaking, the "topological parts" $R^i_h(X)$ of $V(X)$. More precisely they are defined as follows
$$R^i_h(X) := \{\omega \in H^0(X, \Omega^1_X) | \ H^i(H^*(X, \mathbb{C}), \wedge\omega) \neq 0\} \subseteq V(X),$$
is linear (see, e.g., [DiPa13]). Note that when $R^i_h(X) = V(X)$, our theorem is a direct consequence of the linearity of the resonance variety. This for instance is the case when $\chi(X) \neq 0$ (see Corollary 2.6).

The equality $R^i_h(X) = V(X)$ in Item (3) may not always be true. An example of Debarre, Jiang and Lahoz [**?**, Example 1.11] shows that the there exists a bi-elliptic surface S admitting a 1-form ω for which $(H^*(X, \mathbb{C}), \wedge\omega)$ is exact, yet ω admits zeros on S.

More generally we expect the following to be true.

Conjecture 1.3. *Let $f : X \to A$ be a morphism from a smooth complex projective variety X to an abelian variety A. Then $V^i(f)$ are linear subvarieties of $H^0(X, \Omega^1_X)$, i.e., $V^i(X)$ is a finite union of linear subspaces of $H^0(X, \Omega^1_X)$ for each nonnegative integer i. In particular $V^i(X)$ are linear.*

To this end, first using a result of Spurr [Sp88], we directly get $V^1(X)$ is linear.

Theorem C. Let $f : X \to A$ be a morphism from a smooth complex projective variety X to an abelian variety A. Then $V^1(f)$ is linear. In particular, $V^1(X)$ is linear.

We also generalize a result of Spurr [Sp88] to quasi-projective varieties.

Theorem D. Let (X, D) be a pair with X a complex smooth projective variety and D a simple normal crossing divisor of X. Let H be a very ample divisor on X. If (X, D) carries a holomorphic log 1-form $0 \neq \omega \in H^0(X, \Omega^1_X(\log D))$ which pullbacks to zero on an effective divisor E with $E^2 \cdot H^{n-2} \geq 0$, then there is a morphism $f : X - D \to C$ to a smooth quasi-projective curve $C = \bar{C} - B$ (where $\bar{C}$ is the completion of C and B can be empty) with

(1) $\omega = f^*\eta$ for some $\eta \in H^0(X, \Omega^1_{\bar{C}}(\log B))$.
(2) E is contained in the fiber of f and $E^2 \cdot H^{n-2} = 0$.

As a corollary we prove the linearity of the set of logarithmic holomorphic 1-forms admitting codimension one zeros.

Corollary 1.4. *Let (X, D) be a pair with X a complex smooth projective variety and D a simple normal crossing divisor of X. Then the set*

$$V^1(X, D) := \{w \in W(U) \mid \operatorname{codim}_X Z(w) \leq i\}$$

is linear.

We show that $V^1(X, D)$ consists of resonant holomorphic logarithmic 1-forms vanishing along a divisor, holomorphic logarthmic 1-forms vanishing along a rigid divisor, and holomorphic logarithmic 1-forms vanishing along some components of the boundary divisor D.

1.1. **Comments about the proof.** For a morphism $f : X \to A$, we consider the following diagram

$$\begin{array}{ccc} X \times f^*H^0(A, \Omega^1_A) & \xrightarrow{df} & T^*X \\ \big\downarrow{\scriptstyle f\times \mathrm{id}} & & \\ A \times H^0(A, \Omega^1_A) \simeq T^*A & & \\ \big\downarrow{\scriptstyle \mathrm{pr}_2} & & \\ H^0(A, \Omega^1_A) & & \end{array}$$

where T^*X denotes the $2n$ dimensional cotangent bundle of X, df is the usual differential, and $\tilde{f} = pr_2 \circ (f \times \mathrm{id})$ is the natural projection. Note that $V(f) \simeq \tilde{f}(df^{-1}(0))$. The key ingredient of Theorem A and B is a result of Migliorini and Shende [MiSh18, Theorem C] which ensures that $(f \times \mathrm{id})(df^{-1}(0))$ is a finite union of closures of conormal bundles $T^*_Z X$ along various subvarieties of $Z \subset A$. Then the linearity follows from the following proposition

Proposition 1.5. *Let A be an abelian variety and X be a proper subvariety of A. Then the following are equivalent*

(1) X is not fibred by tori or of dimension 0.

(2) Any holomorphic 1-form $\omega \in H^0(A, \Omega^1_A)$ restricted to X^{reg}, i.e. $\omega|_{X_{\mathrm{reg}}}$ admits zeros on the smooth locus X_{reg}.

When A is a simple abelian variety this proposition says that any 1-form restricted to X admits zeros on the regular locus of X. A proof of this statement seems to be abundantly found in the literature see for instance [HK05, Proposition 3.1] or [LMW20, Proposition 5.12]. Our contribution is to provide a proof in general (see 3.3).

words about the proof of second part

Notation.

Acknowledgement. Stefan, Jiang, Kovacs, Nero, Sabbah

2. Preliminary

Definition 2.1. *For $\omega \in H^0(X, \Omega^1_X)$, the zero set of ω is the algebraic set of closed point x in X, such that $\omega(x) = 0$. The zero scheme of ω is defined by the ideal sheaf $\mathcal{I}_\omega$, where $\mathcal{I}_\omega$ is the image of the morphism*

$$\mathcal{T}_X \xrightarrow{\langle \omega, \cdot \rangle} \mathcal{O}_X$$

with $\mathcal{T}_X$ being the tangent sheaf of X and $\langle \omega, \cdot \rangle$ being the pairing of tangent field with 1-form ω.

Definition 2.2. *Let X be a smooth projective variety. We call a holomorphic 1-form $\omega \in H^0(X, \Omega^1_X)$ nonresonant if the zero scheme $Z(\omega)$ is nonempty and the complex*

$$\ldots \to H^{i-1}(X, \mathbb{C}) \xrightarrow{\wedge \omega} H^i(X, \mathbb{C}) \xrightarrow{\wedge \omega} H^{i+1}(X, \mathbb{C}) \to \ldots$$

is exact. A holomorphic 1-form $\omega \in H^0(X, \Omega^1_X)$ is called universally nonresonant if the zero scheme $Z(\omega)$ is nonempty and the complex

$$\ldots \to H^{i-1}(X', \mathbb{C}) \xrightarrow{\wedge \tau^* \omega} H^i(X', \mathbb{C}) \xrightarrow{\wedge \tau^* \omega} H^{i+1}(X', \mathbb{C}) \to \ldots$$

is exact for any étale over $\tau : X' \to X$.

Remark 2.3. (1) (Universally) nonresonant holomorphic 1-forms were first studied in [SS19].

(2) In general by [GL87, Proposition 3.4], if a holomorphic ω has zero scheme of codimension greater than or equal to k, then the complex

$$\ldots \to H^i(X,\mathbb{C}) \stackrel{\wedge\omega}{\to} H^i(X,\mathbb{C}) \stackrel{\wedge\omega}{\to} H^i(X,\mathbb{C}) \to \ldots$$

is exact whenever $i<k$.

The set of resonant 1-forms is very well understood (Provide references as much as we can). We recall the following theorem (Due to Voisin?). For $i=1$, this is due to Dimca, Papadima, Suciu. For higher i, this is due to Dimca and Papadima:

Theorem 2.4. [DiPa13, Theorem C, Corollary 1.7] *Let X be a compact Kähler manifold. Each irreducible component of resonance varieties $R^i(X)$ is a linear subspace of $H^1(X,\mathbb{C})$ defined over $\mathbb{Q}$, i.e.,*

$$R^i(X) = (R^i(X)\cap H^1(X,\mathbb{Q}))\otimes \mathbb{C}.$$

Also, each irreducible component of $R^i(X)$ carries $\mathbb{Q}$-Hodge substructures of $H^1(X,\mathbb{C})$.

Corollary 2.5. *Let $f: X\to A$ be a morphism from smooth projective variety X to a simple abelian variety A. Then $R^i(X)\cap f^*H(A,\Omega^1_A)$ is $f^*H(A,\Omega^1_A)$ or 0.*

Proof. $R^i(X)\cap f^*H(A,\Omega^1_A)$ is a $\mathbb{Q}$-Hodge substructure of $H^1(X,\mathbb{C})$ of the simple Hodge structure $f^*H(A,\Omega^1_A)$. □

Corollary 2.6. *For X being a smooth projective variety X such that the topological Euler characteristic class $\chi(X)\neq 0$, $V(X)$ is a linear subvariety.*

3. Linearity of holomorphic 1-forms with zeros

In this section, we consider the holomorphic 1-forms with zero locus of arbitrary dimension.

3.1. **Higher discriminants and singularities of morphisms by Migliorini and Shende.** In this subsection, we recall a very powerful tool by Migliorini and Shende [MiSh18], which will be used in the proof of Theorem B.

Definition 3.1 (Migliorini and Shende)**.** *Let $f: X\to Y$ be a proper morphism between smooth algebraic varieties X and Y. The higher discriminants of f are defined to be*

$$\Delta^i(f) := \{y\in Y \mid \text{no } (i-1)\text{-dimensional subspace of } T_yY \text{ is transversal to } f\},$$

for each nonnegative integer i.

Remark 3.2. (1) In the above definition an i-dimensional subspace W of T_yY is transversal to f means

$$df_x(T_xX) + W = T_yY,$$

for any closed point $x \in f^{-1}(y)$. Thus no i-dimensional subspace of T_yY is transversal to f means the rank of f at some point in $f^{-1}(y)$ is less than $n-i$, where $n = \dim Y$.

(2) By definition, $\Delta^1(f)$ is the usual discriminant, i.e., the locus in Y such that f is not smooth. Also by [Har77, Proposition 10.6], we have the filtration

$$Y = \Delta^0(f) \supset \Delta^1(f) \supset \Delta^2(f) \supset \Delta^3(f) \supset \dots.$$

(3) $\Delta^i(f)$ are closed subvarieties of Y, which measure how singular f can be.

For any morphism $f : X \to Y$, we consider the following diagram

$$T^*X \xleftarrow{df} X \times_Y T^*Y \xrightarrow{\tilde{f}} T^*Y \ ,$$

where T^*X and T^*Y are holomorphic cotangent bundle, df is the usual differential, and $\tilde{f}$ is the natural projection. Now we can formulate the following theorem in [MiSh18, Theorem C].

Theorem 3.3 (Migliorini, Shende). *Let $f : X \to Y$ be a proper morphism between smooth algebraic varieties X and Y. Then*

$$\tilde{f}(df^{-1}(0_X)) = \bigcup_{i \geq 0} \overline{T^*_{D^i(f)_{\mathrm{reg}}} Y},$$

where 0_X is the zero section of the cotangent bundle T^*X, and $\overline{T^*_{D^i(f)_{\mathrm{reg}}} Y}$ is the closure of the conormal bundle of the smooth locus $D^i(f)_{\mathrm{reg}}$ of codimension i components $D^i(f)$ of $\Delta^i(f)$ in Y.

Since the codimension i components $D^i(f)$ of $\Delta^i(f)$ will be used for many times, we give the following definition.

Definition 3.4. *Let $f : X \to Y$ be a proper morphism between smooth algebraic varieties X and Y. The pure i-th discriminants of f are defined to be the union of codimension i components of $\Delta^i(f)$, denoted by $D^i(f)$.*

From now on we will focus on a morphism $f : X \to A$ from smooth projective variety X to abelian variety A. Also, we fix the following notation

$$T^*X \xleftarrow{df} X \times T_eA \xrightarrow{\tilde{f}} T^*A \overset{p}{\twoheadrightarrow} T^*_eA = H^0(A, \Omega^1_A) \ ,$$

where T_eA is the tangent space of A at a chosen origin e. Notice first $p(\tilde{f}(df^{-1}(0_X)))$ is exactly the set of holomorphic 1-forms $\omega \in H^0(A, \Omega^1_A)$ such that $f^*\omega$ has zeros on X.

Corollary 3.5. *Let $f : X \to A$ be a morphism from smooth projective variety X to an abelian variety. Suppose there is a nontrivial holomorphic 1-form $\omega \in H^0(A, \Omega_A^1)$ such that $f^*\omega$ has zeros. Then there exists $i \geq 1$ such that the pure i-th discriminant $D^i(f)$ is nonempty. In other words, f is smooth if and only if $D^i(f) = \emptyset$ for all $i \geq 1$.*

Proof. Assume that for all $i \geq 1$, $D^i(f) = \emptyset$, then

$$\tilde{f}(df^{-1}(0_X)) = \overline{T^*_{D^0(f)_{\mathrm{reg}}} Y} = \overline{T^*_{\Delta^0(f)_{\mathrm{reg}}} Y} = A \times \{0\}.$$

Hence $p(\tilde{f}(df^{-1}(0_X))) = \{0\}$, i.e., the only holomorphic 1-form admitting zeros is the trivial one, which is a contradiction. □

3.2. Proof of Theorem A.

Proof of Theorem A. "⇒" is trivial.

"⇐" Assume f is not smooth, then there is a holomorphic 1-form has zeros. Then there exist $i \geq 1$ such that $D^i(f)$ is nonempty by Corollary 3.5. Since A is simple, $\omega|_{D^i(f)_{reg}}$ has zero for general holomorphic 1-forms $\omega \in H^0(A, \Omega_A^1)$ by [HK05, Proposition 3.1], i.e., $p(\overline{T^*_{D^i(f)_{\mathrm{reg}}} A}) = H^0(A, \Omega_A^1)$, which means every holomorphic 1-form pulled back from A has zero. □

3.3. **Case for general abelian variety.** First we generalize a proposition of Hacon and Kovács [HK05, Proposition 3.1]. Part of the proof of the following proposition is due to [HK05].

Proposition 3.6. *Let A be an abelian variety and X be a proper subvariety of A. Then the following are equivalent*

(1) X is not fibred by tori or of dimension 0.

(2) For general holomorphic 1-forms $\omega \in H^0(A, \Omega_A^1)$, the restricted holomorphic 1-form $\omega|_{X_{\mathrm{reg}}}$ admit zeros on the smooth locus X_{reg}.

Proof. (2)⇒ (1): Suppose X is fibred by tori and $\dim X > 0$, i.e. there exists a subabelian variety $B \subset A$ such that the fibres of the composition $X \hookrightarrow A \twoheadrightarrow A/B$ are B. Taking an étale cover $\tau : A' \to A$, we can assume $A' = B \times C$ and $X' := \tau^{-1}(X) = B \times Y \subset A'$. Hence every 1-form coming from B does not vanish on smooth locus of X', hence on X.

(1)⇒(2): Denote $d = \dim X$, $g = \dim A$. If $d = 0$ it is trivial, so we assume $d > 0$. For any point $x \in X_{\mathrm{reg}}$, we have

$$T_x X_{\mathrm{reg}} \simeq \mathbb{C}^{g-d} \subset T_x A \simeq H^0(A, \Omega_A^1)^\vee \simeq \mathbb{C}^g.$$

Hence we have the following diagram

$$\begin{tikzcd} \mathbb{P}T^*_{X_{\mathrm{reg}}}A \arrow[r, hook, "i"] \arrow[d, "\pi"] & X_{\mathrm{reg}} \times \mathbb{P}H^0(A,\Omega^1_A)^\vee \arrow[r, two heads, "p"] \arrow[dl] & \mathbb{P}H^0(A,\Omega^1_A)^\vee . \\ X_{\mathrm{reg}} & & \end{tikzcd}$$

We then denote S be the image of $\mathbb{P}T^*_{X_{\mathrm{reg}}}A$ in $\mathbb{P}H^0(A,\Omega^1_A)^\vee$ through the map $p\circ i$ in the above diagram. Notice that it suffices to show S is dense in $\mathbb{P}H^0(A,\Omega^1_A)^\vee$.

Now assume S is not dense in $\mathbb{P}H^0(A,\Omega^1_A)^\vee$ and $m := \dim S$. Then for a general point $s \in S$,

$$\dim(p\circ i)^{-1}(s) = g - m - 1 > 0,$$

hence

$$\dim \pi((p\circ i)^{-1}(s)) = g - m - 1 > 0.$$

We denote $Z_s := \pi((p\circ i)^{-1}(s)) \subset X_{\mathrm{reg}}$. For each Z_s, we consider the nontrivial subabelian variety $A_s := \langle Z_s \rangle$ generated by Z_s. Notice first A_s is a proper subabelian variety, because for general $x \in Z_s$, and the line $L_s \subset \mathbb{P}H^0(A,\Omega^1_A)^\vee$ corresponding to s, we have

$$T_x(Z_s) \subset T_xX_{\mathrm{reg}} \subset L_s^\perp.$$

Since A only contains at most countably many subabelian varieties, we have for general $s \in S$, A_s are isomorphic to each other. We denote this subabelian variety to be A_0 with a chosen origin e. Notice first that $T_e(A_0)$ is perpendicular to all L_s, since it is true for general s. Also, notice that since X is not fibred by tori, Z_s is a proper subvariety of the translated tori $A_0 + x$ for general s and any $x \in Z_s$ (Notice that $A_0 + x$ does not depend on the choice of $x \in Z_s$). Thus $\dim A_0 > g - m - 1$, i.e., $\dim T_eA_0 \geq g - m$. However, $\dim \bigcup_{s\in S} L_s = m+1$ ($\dim S = m$). So $T_e(A_0)$ cannot perpendicular to all L_s for $s \in S$. This proves S is dense in $\mathbb{P}H^0(A,\Omega^1_A)^\vee$ and hence the lemma. □

For any subvariety X of an abelian variety A, we consider the following diagram

$$T^*_{X_{\mathrm{reg}}}A \xrightarrow{p} T^*_eA = H^0(A,\Omega^1_A)\ ,$$

where p is the projection given by moving the cotangent vectors to T^*_eA via group action. Notice that $p(T^*_{X_{\mathrm{reg}}}A)$ is exactly the set of holomorphic 1-forms in $H^0(A,\Omega^1_A)$ whose restriction on X_{reg} admitting zeros. Then we have the following corollary of Proposition 3.6.

Corollary 3.7. *Let $i : X \to A$ be a subvariety of an abelian variety A. Then $\overline{p(T^*_{X_{\mathrm{reg}}}A)}$ is linear.*

Proof. By Poincaré complete reducibility theorem, after passing to a étale covering $\tau : A' \to A$ we have a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, hook, "i'"] & X \arrow[d, hook, "i"] \\ A' \arrow[r] & A \end{tikzcd},$$

and we may assume $A' \simeq B \times C$ with B and C being abelian varieties, $X' \simeq Y \times C$ with $Y \subset B$ being a subvariety which is not fibred by tori in B. Then we have

$$T^*_{X'_{\mathrm{reg}}} A' \simeq T^*_{Y_{\mathrm{reg}}} B \times C.$$

As the above, we denote p' be the natural projection

$$T^*_{X'_{\mathrm{reg}}} A' \xrightarrow{p'} T^*_e A' = H^0(A', \Omega'^1_A) \ .$$

Then by Proposition 3.6, we have $\overline{p'(T^*_{X'_{\mathrm{reg}}} A')} = H^0(B, \Omega^1_B)$, which is linear. Hence $\overline{p(T^*_{X_{\mathrm{reg}}} A)}$ is linear, since $X' \to X$ is étale. □

Next we will use Corollary 3.7 and Theorem 3.3 to prove our first main result Theorem B.

Proof of Theorem B. By Theorem 3.3, we have

$$\tilde{f}(df^{-1}(0_X)) = \bigcup_{i \geq 0} \overline{T^*_{D^i(f)_{\mathrm{reg}}} Y},$$

where $\overline{T^*_{D^i(f)_{\mathrm{reg}}} Y}$ is the closure of the conormal bundle of the smooth locus $D^i(f)_{\mathrm{reg}}$ of pure i-th discriminant $D^i(f)$. By Corollary 3.7, we have $p(\overline{T^*_{D^i(f)_{\mathrm{reg}}} Y}) = \overline{p(T^*_{D^i(f)_{\mathrm{reg}}} Y)}$ is linear. Hence

$$p(\tilde{f}(df^{-1}(0_X))) = \bigcup_{i \geq 0} p(\overline{T^*_{D^i(f)_{\mathrm{reg}}} Y})$$

is linear. Also, by the proof of Corollary 3.7, it is clear that each irreducible component of $V(f)$ are $\mathbb{Q}$-sub Hodge structures associated to subtori which are generated by $D^i(f)$. □

Add two corollaries, the above main theorem implies the main theorems of [Papa Schnell], [Hacon Kovacs]

An immediate consequence of the above is the following

Proposition 3.8. *Given a morphism $f\colon X \to A$ from a smooth projective variety to an abelian variety A, there is a holomorphic 1-form $\omega \in H^0(A, \Omega^1_A)$ such that $f^*\omega$ has no zeros if and only if for all $i \geq 1$, the pure i-th discriminant $D^i(f)$ of f are*

(1) either empty, i.e., f is smooth;

(2) or fibered by tori and $\dim D^i(f) > 0$.

Proof. "$\Leftarrow$" By Theorem 3.3, (1) implies that for every holomorphic 1-form $\omega \in H^0(A, \Omega^1_A)$, $f^*\omega$ has no zeros. By Proposition 3.6, (2) implies that there exist a holomorphic 1-form $\omega \in H^0(A, \Omega^1_A)$, $f^*\omega$ has no zeros.

"$\Rightarrow$" If there is i such that the pure i-th discriminant $D^i(f)$ is not fibred by tori and dim ≥ 0, applying Proposition 3.6, we have for general holomorphic 1-form $\omega \in H^0(A, \Omega^1_A)$, $\omega|_{D^i(f)_{\mathrm{reg}}}$ has zeros. Hence

$$p(\overline{T^*_{D^i(f)_{\mathrm{reg}}} Y}) = H^0(A, \Omega^1_A).$$

Hence

$$p(\tilde{f}(df^{-1}(0_X))) = H^0(A, \Omega^1_A),$$

i.e.,

$$V(f) = f^* H^0(A, \Omega^1_A).$$

□

This is a strengthening of a "special case" of Stefan's Theorem 1.4 (2)

Lemma 3.9. *Let A be any abelian variety. Let $Y_1 \times B_1$ and B_2 be two subvarieties of A, where Y_1 is not fibred by tori, B_2 is a simple abelian variety and B_1 is an abelian variety. Suppose $B_2 \subset Y_1 \times B_1$. Then B_2 is a simple factor of B_1 upto isogeny.*

Proof. It suffices to show that the natural map $p : B_2 \to Y_1 \times B_1 \to Y_1$ maps B_2 to a point. Assume not, then we have a positive dimensional abelian variety $C := p(B_2)$ contained in Y_1. Then we consider the inclusion

$$C \subset Y_1 \subset A.$$

Without losing of generality we can assume that C contains a chosen origin 0 by translating $C \subset Y_1$ using group action if necessary. Then we want to find contradiction from

$$0 \in C \subset Y_1 \subset A,$$

where Y_1 is not fibred by tori.

TBA □

With Lemma 3.9 and the above theorem we prove, then we have the following strong result.

Theorem 3.10. *Let X be any smooth complex projective variety. Then X admits a nowhere vanishing holomorphic 1-form ω if and only if X admits a smooth morphism to positive dimensional abelian variety A*

Proof. "$\Leftarrow$" is trivial.

"$\Rightarrow$" Consider the albanese map $a: X \to A_X$ of X. By assumption we have a holomorphic 1-form $\omega \in H^0(A_X, \Omega^1_{A_X})$ such that $a^*\omega$ is nowhere vanishing. By Corollary 3.5, we may assume that not all $D^i(f)$ are empty for $i \geq 1$. Then by Theorem 3.8, we get all the nonempty pure i-th discriminants $D^i(a)$ are fibred by tori and $\dim D^i(a) > 0$. Noticed also, by (Hartshorne generic smooth), we have

$$A_X = D^0(a) \supset D^1(a) \supset D^2(a) \supset D^3(a) \supset \dots.$$

For pure i-th discriminants, does this filtration stil hold? I Need to think about this, which seems not OK! Assume the highest nontrivial discriminant is $D^m(a)$, which is fibred by tori, then we can take simple abelian subvarietyA_0 of A_X so that A_0 is a factor of the torus part of $\Delta^m(a)$. By Lemma 3.9, A_0 is a factor of all the discriminants. Notice that there is a étale covering $\tau : A_X \to A_0 \times B$ with B another abelian variety. Then from the proof of Corollary 3.7, we have all the nontrivial holomorphic 1-forms in $(\tau \circ a)^* H^0(A_0, \Omega^1_{A_0})$ has no zeros on X. But since A_0 is simple, we have $\tau \circ a : X \to A_0$ is smooth by Theorem A. □

4. Linearity of holomorphic 1-forms with codimension one zeros

- Use Normal bundle also to explain nonresonant 1-forms form finite points in $\mathbb{P}H^0(X, \Omega^1_X)$.
- Application in bounding the genus of 1-dimensional locus of nonresonant 1-forms on surfaces.

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