MATRICES AND LINEAR ALGEBRA: A QUICK INTRODUCTION

DENNIS BREWER, LUCA CAPOGNA, PHILLIP HARRINGTON, JAMES MEEK, ANDREW RAICH

1. An Introduction to Matrices

1.1. Definition of a matrix.

Definition 1.1. An $m \times n$ **matrix** is a rectangular array of numbers with m rows and n columns.

Example 1.2. A 2×1 matrix is a column vector and a 1×2 matrix is a row vector.

Example 1.3. 2×2 matrices. $A = \begin{pmatrix} 3 & 4 \\ -1 & \pi \end{pmatrix}$ is an example of a 2×2 matrix. We sometimes write $A = (a_{jk})$ where a_{jk} is the element in the jth row and kth column of A. For example $a_{12} = 4$.

Definition 1.4. We say that two matrices A and B are **equal** if $a_{jk} = b_{jk}$ for all j and k.

Example 1.5. 3×3 matrices. $B = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 1 & -3 \\ -1 & 3 & -2 \end{pmatrix}$ is a 3×3 matrix.

Example 1.6. $C = \begin{pmatrix} 4 & 2 \\ -1 & 1 \\ e & 45 \end{pmatrix}$ is a 3×2 matrix. $c_{12} = 2$ while $c_{21} = -1$.

1.2. Matrix Addition.

Matrices of the same size can be added, and the addition is performed component-wise.

Example 1.7. $\begin{pmatrix} 2 & 3 \\ 0 & 1 \\ -3 & 6 \end{pmatrix} + \begin{pmatrix} -2 & 5 \\ 3 & 0 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 8 \\ 3 & 1 \\ -5 & 10 \end{pmatrix}$

Example 1.8. $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 2 & 1 \\ -2 & 1 & 0 \end{pmatrix}$ cannot be added! They are different sizes!

Symbolically, if $A = (a_{jk})$ and $B = (b_{jk})$ are both $m \times n$ matrices, then $A + B = (a_{jk} + b_{jk})$.

1.3. Multiplication of a matrix by a scalar.

If $\alpha \in \mathbb{R}$ is a scalar and $A = (a_{jk})$ is an $m \times n$ matrix, then the matrix $\alpha A = (\alpha a_{jk})$.

Example 1.9. If $A = \begin{pmatrix} 4 & 2 & 3 \\ 3 & -3 & 8 \\ 0 & 0 & 2 \end{pmatrix}$, then $\frac{1}{2}A = \begin{pmatrix} 2 & 1 & \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} & 4 \\ 0 & 0 & 1 \end{pmatrix}$.

1.4. **A special class of matrices.**

Definition 1.10. An $n \times n$ matrix is called a **square matrix**.

Example 1.11. $A = \begin{pmatrix} 3 & 9 & 8 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a 3×3 square matrix. $A = \begin{pmatrix} 3 & 9 & 8 \\ 1 & 3 & 0 \end{pmatrix}$ is a 2×3 matrix, and hence not square.

Definition 1.12. The **identity matrix** is an $n \times n$ matrix with 1's on the diagonal and 0's off of the diagonal and denoted by I or I_n.

Example 1.13. $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A **zero matrix** is any matrix that contains only 0's.

Theorem 1.14. *Let A, B, C be $m \times n$ matrices and $\alpha, \beta \in \mathbb{R}$ be scalars. Then the following algebraic rules hold:*

(i) $A + B = B + A$ Commutativity of Matrix Addition
(ii) $(A + B) + C = A + (B + C)$ Associativity of Matrix Addition
(iii) $\alpha(A + B) = \alpha A + \alpha B$
(iv) $(\alpha + \beta)A = \alpha A + \beta A$
(v) $(\alpha\beta)A = \alpha(\beta A)$

1.5. **Problems.**

1. Give an example of a 4×2 matrix and a 2×4 matrix.
2. If $M = \begin{pmatrix} 4 & 3 & 1 \\ -1 & \pi & 2 \\ -4 & 3 & 1 \end{pmatrix}$, then find m_{23} and $m_{22} + m_{13}$.

Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 3 & 8 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$,

3. What is $b_{11} - 3a_{22} + 2c_{21}$?
4. Compute $A + B$.
5. Compute $3C$.
6. Compute $A + 4B$.
7. Compute $3B - 2C$.
8. Give an example two matrices F and G so that $F + G$ does not exist.

2. Determinants of 2×2 and 3×3 matrices

Definition 2.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A, denoted $\det(A)$ or $|A|$ is the quantity

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant is a fundamental quantity in linear algebra. For example, the system of equations

$$ax + by = \alpha$$
$$cx + dy = \beta$$

has the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} \alpha d - \beta b \\ a\beta - c\alpha \end{pmatrix}$$

if $ad - bc \neq 0$. If $ad - bc = 0$, then the two equations are multiples of one another. From this example, we see that the determinant indicates whether or not a 2×2 system of linear equations has a unique solution.

Example 2.2. $\begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 2 \cdot 0 - 3 \cdot (-1) = 3.$

Example 2.3. $\det \begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix} = (-1)(-6) - 3 \cdot 2 = 0.$

Definition 2.4. If A is a square matrix and $\det(A) \neq 0$, then A is called **nonsingular**. If $\det(A) = 0$, then A is called **singular**.

The matrix in Example 2.2 is nonsingular while the matrix in Example 2.3 is singular.

Definition 2.5. Let $A = (a_{jk})_{1 \leq j,k \leq 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. The **determinant** of A is the quantity

$$\begin{aligned} |A| &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}). \end{aligned}$$

In analog of the 2×2 case, the system of equations

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= \alpha \\ a_{21}x + a_{22}y + a_{23}z &= \beta \\ a_{31}x + a_{32}y + a_{33}z &= \gamma \end{aligned}$$

has a unique solution if and only if $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$ and has no solutions or infinitely many solutions otherwise.

Example 2.6. $\begin{vmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ -1 & 4 & 2 \end{vmatrix} = 3(3 \cdot 2 - 2 \cdot 4) - 2(1 \cdot 2 - 2(-1)) + 0(1 \cdot 4 - 3(-1)) = -6 - 8 = -14.$

Example 2.7. $\begin{vmatrix} i & j & k \\ -1 & 0 & 4 \\ -2 & 1 & 3 \end{vmatrix} = i(0 \cdot 3 - 4 \cdot 1) - j((-1)3 - 4(-2)) + k((-1)1 - 0(-2)) = -4i - 5j - k.$

2.1. Problems.

1. Evaluate $\begin{vmatrix} 5 & -1 \\ 4 & 6 \end{vmatrix}$.

2. Evaluate $\det \begin{pmatrix} -2 & -7 \\ 3 & 8 \end{pmatrix}$.

3. Which of the following is a singular matrix?
   (a)
$$\begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$$
   (b)
$$\begin{pmatrix} 8 & 3 \\ -5 & -2 \end{pmatrix}$$
4. Use determinants to determine whether the system of linear equations
$$\begin{aligned} 2x - 3y &= 2 \\ 3x - 4y &= -1 \end{aligned}$$
   has a unique solution.
5. Evaluate $\begin{vmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ -2 & 2 & 1 \end{vmatrix}$.
6. Evaluate $\det \begin{pmatrix} 2 & 0 & 7 \\ -2 & -1 & -2 \\ 5 & 6 & 1 \end{pmatrix}$.
7. Which of the following is a singular matrix?
   (a)
$$\begin{pmatrix} 4 & 1 & -2 \\ 5 & 3 & -1 \\ 2 & 4 & 1 \end{pmatrix}$$
   (b)
$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 4 & 2 \\ -2 & 2 & 2 \end{pmatrix}$$
8. Use determinants to determine whether the system of linear equations
$$\begin{aligned} 8y + 2z &= 3 \\ 3x + 5y - 3z &= 4 \\ -6x - 2y + 8z &= -1 \end{aligned}$$
   has a unique solution.

3. Matrix Multiplication

3.1. **Matrix Multiplication.** Let A and B be two matrices. The product AB can be defined if the number of columns of A equals the number of rows of B, that is, if A is an $m \times n$ matrix and B is an $n \times k$ matrix, then AB will be an $m \times k$ matrix. If
$$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}, \quad B = (b_{j\ell})_{\substack{1 \le j \le n \\ 1 \le \ell \le k}},$$

then the matrix $C = AB$ is an $m \times k$ matrix and if $C = (c_{j\ell})_{\substack{1 \le i \le n \\ 1 \le \ell \le k}}$, then

$$c_{i\ell} = \sum_{j=1}^{n} a_{ij} b_{j\ell}.$$

Example 3.1. Let $A = \begin{pmatrix} 1 & -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + (-1)4 + 4(-1) \end{pmatrix} = \begin{pmatrix} -6 \end{pmatrix}.$$

On the other hand,

$$BA = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 8 \\ 4 & -4 & 16 \\ -1 & 1 & -4 \end{pmatrix}$$

Matrix multiplication is NOT commutative!

Example 3.2. Let $A = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 3 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \cdot 2 + 2 \cdot 1 + (-1)5 \\ 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

while BA is not defined. Also,

$$AC = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + 2 \cdot 2 + (-1)0 & 4(-1) + 2(-1) + (-1)3 \\ 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 0 & 2(-1) + 1(-1) + 1 \cdot 3 \end{pmatrix} = \begin{pmatrix} 8 & -9 \\ 4 & 0 \end{pmatrix}$$

while

$$CA = \begin{pmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + (-1)2 & 1 \cdot 2 + (-1)1 & 1(-1) + (-1)1 \\ 2 \cdot 4 + (-1)2 & 2 \cdot 2 + (-1)1 & 2(-1) + (-1)1 \\ 0 \cdot 4 + 3 \cdot 2 & 0 \cdot 2 + 3 \cdot 1 & 0(-1) + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -2 \\ 6 & 3 & -3 \\ 6 & 3 & 3 \end{pmatrix}$$

Example 3.3. Multiplying square matrices. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 + 3 \cdot 4 & 2 \cdot 1 + 3(-1) \\ (-1)3 + (-2)4 & (-1)1 + (-2)(-1) \end{pmatrix} = \begin{pmatrix} 18 & -1 \\ -11 & 1 \end{pmatrix}$$

while

$$BA = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 2 + 1(-1) & 3 \cdot 3 + 1(-2) \\ 4 \cdot 2 + (-1)(-1) & 4 \cdot 3 + (-1)(-2) \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 9 & 14 \end{pmatrix}$$

Matrix multiplication is not necessarily commutative even for square matrices!

Example 3.4. If A is any $n \times n$ matrix, then $AI = IA = A$. Multiplication by the identity matrix is the analog of multiplying by 1.

Example 3.5. Multiplicative inverses. Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$. Observe that

$$AB = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

and

$$BA = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Definition 3.6. An $n \times n$ matrix A for which there exists an $n \times n$ matrix B satisfying $AB = BA = I_n$ is called **invertible** and B is denoted by A^{-1}.

The inverse of a nonsingular matrix A is unique. Suppose B and C are inverses of A, i.e., $AB = BA = I$ and $AC = CA = I$. Then

$$C = CI = C(AB) = (CA)B = IB = B.$$

Example 3.7. Two strange examples. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This is an example of a nonzero matrix A so that $A^2 = 0$. A matrix A so that $A^k = 0$ for some positive integer k is called a **nilpotent matrix**.

Let $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B.$$

B is an example of a matrix that is neither 0 nor the identity so that $B^2 = B$. A nonzero matrix so that $B^2 = B$ is called an **idempotent matrix**.

We summarize the arithmetic properties of matrix multiplication with the following theorem.

Theorem 3.8. *Let $\alpha \in \mathbb{R}$ be a matrix and A, B, C be matrices so that the indicated operations are defined.*

(i) $A(B + C) = AB + AC$
(ii) $(A + B)C = AC + BC$
(iii) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

3.2. **Problems.**

1. Let $A = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 3 & 1 \\ 0 & -3 \end{pmatrix}$. Compute AB and BA.

2. Let $A = \begin{pmatrix} 2 & 4 \\ -1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Compute AB. Can you compute BA?

3. Given matrices $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, show that $AB = BA = I$.
4. Given matrices $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, show that $AB = BA = I$.
5. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Show that there cannot be a 2×2 matrix B satisfying $AB = BA = I$.
6. Show that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det A \neq 0$, then $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
7. Let $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$.
   a. Write the vector equation $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ as a system of linear equations and solve for x and y.
   b. Find A^{-1}, apply to $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and solve for the vector $\begin{pmatrix} x \\ y \end{pmatrix}$.
8. Let A be an invertible $n \times n$ matrix. Is it true that $(A^{-1})^{-1} = A$?

4. Linear Independence

A **linear combination** of vectors $\vec{v}_i \in \mathbb{R}^n$ is any vector $\vec{w}$ of the form $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_m\vec{v}_m$ for scalars $\alpha_i \in \mathbb{R}$. The following property assures that no vector in a set can be expressed as a linear combination of the remaining vectors.

Definition 4.1. A set $\{v_1, v_2, ..., v_m\}$ of vectors is called **linearly independent** if the relationship $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_m\vec{v}_m = \vec{0}$ can hold only if all of the scalars α_i are zero. Otherwise, the set is **linearly dependent**.

Example 4.2. The set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \right\}$ is linearly dependent because

$$3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

.

Example 4.3. The set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \right\}$ is linearly independent because

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 + 3\alpha_3 \\ \alpha_2 + 5\alpha_3 \\ 2\alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

can hold only if all of the α_i are 0.

Note that any set of vectors $\{v_1, v_2, ..., v_m, \vec{0}\}$ which includes the zero vector among its members is linearly dependent. This is shown by forming the linear combination $0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_m + 1 \cdot \vec{0} = \vec{0}$.