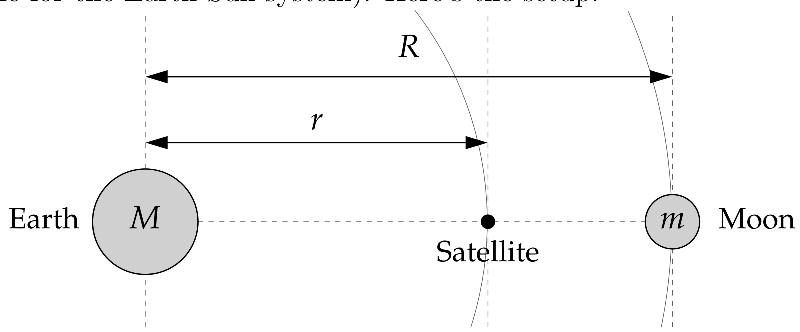
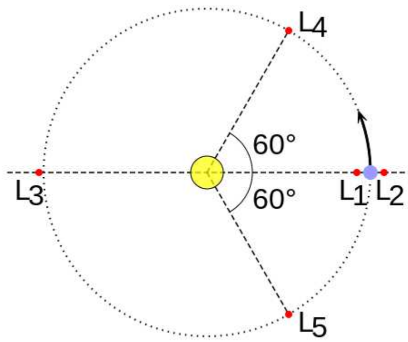
UNIVERSITY OF SUSSEX
Scientific Computing
Tutor: Dr. Ilian Iliev, Office: Pev III 4C5

Assignment 2
Deadline: 12pm on Thursday, November 20th, 2014.
Penalties will be imposed for submissions beyond this date.
Final submission date: Friday, November 21nd, 2014
No submissions will be accepted beyond this date.

1. **The Lagrange point:** There is a magical point between the Earth and the Moon, called the L_1 Lagrange point, at which a satellite will orbit the Earth in perfect synchrony with the Moon, staying always in between the two. This works because the inward pull of the Earth and the outward pull of the Moon combine to create exactly the needed centripetal force that keeps the satellite in its orbit (this could also be done for the Earth-Sun system). Here's the setup:



There are also 4 more Lagrange points - L_2 to L_5, similarly quite useful for parking satellites into orbit. E.g. the Planck and Hershel satellites were send to L_2 for Earth-Sun, in order to always be on Earth's shadow, see figure.



(a) Assuming circular orbits, and assuming that the Earth is much more massive than either the Moon or the satellite, show that the distance r from the center of the Earth to the L_1 point satisfies

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r,$$

where M and m are the Earth and Moon masses, G is Newton's gravitational constant, and ω is the angular velocity of both the Moon and the satellite.

(b) The equation above is a fifth-order polynomial equation in r (also called a quintic equation). Such equations cannot be solved exactly in closed form, but it's straightforward to solve them numerically. Write a program that uses either Newton's method, the Ridder method or the internal `fsolve` routine to solve for the distance r from the Earth to the L_1 point. Compute a solution accurate to at least four significant figures.

The values of the various parameters are:

$$\begin{aligned} G &= 6.674 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}, \\ M &= 5.974 \times 10^{24}\,\mathrm{kg}, \\ m &= 7.348 \times 10^{22}\,\mathrm{kg}, \\ R &= 3.844 \times 10^{8}\,\mathrm{m}, \\ \omega &= 2.662 \times 10^{-6}\,\mathrm{s^{-1}}. \end{aligned}$$

You will also need to choose a suitable starting value for r, or two starting values if you use the secant method. [30]

2. (a) Given the values of $f(x)$ at the points x, $x-h_1$, and $x+h_2$, where $h_1 \neq h_2$, derive the most accurate possible finite difference approximations for $f'(x)$ and $f''(x)$. What is the order of the truncation error in each case? Assume that h_1 and h_2 are of the same order.

(b) Estimate $f'(1)$ and $f''(1)$ from the following data:

x	0.97	1.00	1.05
$f(x)$	0.85040	0.84147	0.82612

[20]

3. (a) Derive central difference approximations for $f'(x)$ accurate to $O(h^4)$ by applying the Richardson extrapolation, starting with the central difference approximation of $O(h^2)$ (given in the lecture notes).

(b) Use the approximations in (a) to estimate the derivative of $f(x) = tanh(x)$ at $x = 1$ using $h = 0.5$ and 0.1. What are the errors for each of the two approximations with respect to the exact answer? Do the errors behave as expected? Why do you think that is? [20]

4. **The Stefan–Boltzmann constant:** The Planck theory of thermal radiation (you will study this in detail later) tells us that in the (angular) frequency interval ω to $\omega + d\omega$, a perfect black body of unit area radiates electromagnetically an amount of thermal energy per second equal to $I(\omega)d\omega$, where

$$I(\omega) = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{(e^{\hbar\omega/k_B T} - 1)} \tag{1}$$

Here $\hbar$ is Planck's constant divided by 2π, c is the speed of light, and k_B is Boltzmann's constant.

(a) Show that the total energy per unit area radiated by a black body is

$$W = \frac{k_B^4 T^4}{4\pi^2 c^2 \hbar^3} \frac{x^3}{(e^x - 1)} \tag{2}$$

(b) Write a program to evaluate the integral in this expression from zero to infinity using a method of your choice (e.g. internal `scipy.integrate.quad`, or the provided Romberg code). Explain what method you used, and how accurate you think (or know) your answer is.

(c) Even before Planck gave his theory of thermal radiation around the turn of the 20th century, it was known that the total energy W given off by a black body per unit area per second followed Stefan's law: $W = \sigma T^4$, where σ is the Stefan–Boltzmann constant. Use your value for the integral above to compute a value for the Stefan–Boltzmann constant (in SI units). Check your result against the known value, which you can find in books or on-line. You should get good agreement. What is the relative error? [30]