Note Title 04/11/2022

INTEGRÁLY

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

PZ1

a) $\int 3x^2 + 2x - 4\, dx = 3\frac{x^{2+1}}{2+1} + 2\cdot\frac{x^{1+1}}{1+1} - 4\cdot\frac{x^{0+1}}{0+1} + c =$

$= \frac{3x^3}{3} + \frac{2x^2}{2} - 4x + c = x^3 + x^2 - 4x + c$

b) $\int \frac{x^3}{3} - \frac{x}{5}\, dx = \frac{1}{3}\frac{x^4}{4} - \frac{1}{5}\frac{x^2}{2} + c = \frac{x^4}{12} - \frac{x^2}{10} + c$

c) $\int \sqrt{x^3} - \frac{1}{\sqrt{x}}\, dx = \int x^{\frac{3}{2}} - x^{-\frac{1}{2}} dx = \frac{x^{\frac{3}{2}+\frac{2}{2}}}{\frac{5}{2}} - \frac{x^{-\frac{1}{2}+\frac{2}{2}}}{\frac{1}{2}} + c$

$= \frac{2x^{\frac{5}{2}}}{5} - 2x^{\frac{1}{2}} + c = \frac{2}{5}\sqrt{x^5} - 2\sqrt{x} + c$

d) $\int \frac{\sqrt{x^4 + 2 + x^{-4}}}{x^3} dx = \int \frac{\sqrt{x^4 + 2 + \frac{1}{x^4}}}{x^3} dx =$

$= \int \frac{\sqrt{\frac{x^8 + 2x^4 + 1}{x^4}}}{\frac{x^3}{1}} dx = \int \frac{\sqrt{x^8 + 2x^4 + 1}}{x^2 \cdot x^3} dx =$

$x^4 = t \qquad x^8 + 2x^4 + 1 = t^2 + 2t + 1 = (t+1)^2 = (x^4+1)^2$

$= \int \frac{\sqrt{(x^4+1)^2}}{x^5} dx = \int \frac{x^4+1}{x^5} dx = \int \frac{1}{x} + \frac{1}{x^5} dx$

$= \ln x + \frac{x^{-4}}{-4} + c = \ln x - \frac{1}{4x^4} + c$

w) $\int 5\cos x - 2x^5 + \frac{3}{1+x^2}\,dx = 5\cdot\sin x - \cancel{2}\,1\,\frac{x^6}{\cancel{6}\,3} +$

$+ 3\cdot \operatorname{arctg} x + C$

$\cos^2 x + \sin^2 x = 1$

m) $\int \frac{1+\cos^2 x}{1+\cos(2x)}\,dx = \int \frac{1+\cos^2 x}{\underbrace{1+\cos^2 x - \sin^2 x}_{\cos^2 x}}\,dx =$

$= \int \frac{1+\cos^2 x}{2\cos^2 x}\,dx = \int \frac{1}{2\cos^2 x} + \frac{1}{2}\,dx = \frac{1}{2}\operatorname{tg} x + \frac{1}{2}x + C$

n) $\int \frac{1}{\cos 2x + \sin^2 x}\,dx = \int \frac{1}{\cos^2 x - \sin^2 x + \sin^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx =$

$= \operatorname{tg} x + C$

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q) $\int \frac{1+2x^2}{x^2(1+x^2)}\,dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}\,dx =$

$$\frac{1+2x^2}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$\frac{1+2x^2}{x^2(1+x^2)} = \frac{Ax(1+x^2) + B(1+x^2) + (Cx+D)x^2}{x^2(1+x^2)} \quad / \cdot x^2(1+x^2)$$

$$1+2x^2 = Ax + Ax^3 + B + Bx^2 + Cx^3 + Dx^2$$

$$0 = A + C \Rightarrow C = 0$$

$$2 = B + D \Rightarrow 2 = 1 + D \Rightarrow D = 1$$

$$0 = A$$

$$1 = B$$

$$= \int \frac{0}{x} + \frac{1}{x^2} + \frac{0\cdot x + 1}{1+x^2}\,dx = \frac{x^{-1}}{-1} + \operatorname{arctg} x + c$$

SUBSTITUČNÁ METÓDA :

7.2 a) $\int \frac{1}{3+4x^2}\,dx = \int \frac{1}{3+(2x)^2}\,dx = \left| \begin{array}{l} 2x = t \\ 2dx = 1dt \end{array} \right| =$

$$= \int \frac{1}{2} \frac{\overbrace{2\,dx}^{dt}}{3+(\underbrace{2x}_{t})^2} = \frac{1}{2}\int \frac{dt}{3+t^2} \boxed{=}$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a}$$

$$\boxed{=} = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x}{\sqrt{3}}\right) + c$$

b) $\int \frac{x}{3+4x^2}\,dx = \left| \begin{array}{l} 3+4x^2 = t \\ 0+8x\,dx = dt \end{array} \right| = \frac{1}{8}\int \frac{\overbrace{8x\,dx}^{dt}}{\underbrace{3+4x^2}_{t}} =$

$$= \frac{1}{8}\int \frac{1}{t}\,dt = \frac{1}{8}\ln|t| = \frac{1}{8}\ln|3+4x^2| + c$$

d) $\int \underbrace{e^x \operatorname{tg} e^x\,dx}_{dt} = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right| = \int \operatorname{tg} t\,dt = -\int \frac{\overbrace{-\sin t\,dt}^{dw}}{\underbrace{\cos t}_{w}}$

$$\left|\begin{matrix} \cos t = w \\ -\sin t\,dt = dw \end{matrix}\right| = -\int \frac{dw}{w} = -\ln|w| = -\ln|\cos t| =$$

$$= -\ln|\cos e^x| + c$$

e) $$\int \frac{3x+2}{x^2+4x+5}\,dx = \int \frac{\frac{3}{2}\cdot\frac{2}{3}\cdot(3x+2)}{x^2+4x+5}\,dx =$$

$$\frac{3}{2}\int \frac{\frac{2}{3}\cdot 3x + \frac{2}{3}\cdot 2}{x^2+4x+5}\,dx = \frac{3}{2}\int \frac{2x + \frac{4}{3} + 4 - 4}{x^2+4x+5}\,dx =$$

$$= \frac{3}{2}\Bigg(\underbrace{\int \frac{2x+4}{x^2+4x+5}\,dx}_{\text{I.}} + \underbrace{\int \frac{-4+\frac{4}{3}}{x^2+4x+5}\,dx}_{\text{II.}} \circledequal$$

I. $$\int \frac{2x+4}{x^2+4x+5}\,dx \boxed{=} \int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$$

$$(x^2+4x+5)' = 2x+4$$

$$\boxed{=} \ln|x^2+4x+5|$$

II. $$x^2+4x+5 = x^2+4x+4+1 = (x+2)^2+1$$

$$\int \frac{-\frac{12+4}{3}}{(x+2)^2+1}\,dx = \frac{-8}{3}\int \frac{1}{(x+2)^2+1}\,dx \left|\begin{matrix} x+2 = t \\ dx = dt \end{matrix}\right| =$$

$$= -\frac{8}{3}\int \frac{1}{t^2+1}\,dt = -\frac{8}{3}\operatorname{arctg} t = -\frac{8}{3}\operatorname{arctg}(x+2) + c$$

$$\circledequal \frac{3}{2}\left(\ln|x^2+4x+5| - \frac{8}{3}\operatorname{arctg}(x+2)\right) + c$$

METÓDA PER-PARTES

$$\int u' \cdot v = u \cdot v - \int u v'$$

$$\int u \cdot v' = u v - \int u' v$$

PR 3

a) $\int \underbrace{x \cdot \sin x}_{u \cdot v'} \, dx \quad \left| \begin{array}{ll} u = x & u' = 1 \\ v' = \sin x & v = \int \sin x \, dx \\ & = -\cos x \end{array} \right| =$

$$= \underbrace{x}_{u} \cdot \underbrace{(-\cos x)}_{v} - \int \underbrace{1}_{u'} \cdot \underbrace{(-\cos x)}_{v} \, dx =$$

$$= -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c$$

d) $\int \ln x \, dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v} \, dx = \left| \begin{array}{ll} u' = 1 & u = x \\ v = \ln x & v' = \frac{1}{x} \end{array} \right|$

$$= \underbrace{x}_{u} \cdot \underbrace{\ln x}_{v} - \int \underbrace{x}_{u} \cdot \underbrace{\frac{1}{x}}_{v'} \, dx = x \ln x - \int 1 \, dx =$$

$$= x \ln x - x + c$$

f) $\int \overbrace{\underbrace{e^x}_{u} \underbrace{\sin x}_{v'}}^{IN} \, dx \quad \left| \begin{array}{ll} u = e^x & u' = e^x \\ v' = \sin x & v = -\cos x \end{array} \right| =$

$$= -e^x \cos x - \int e^x (-\cos x) \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$\left| \begin{array}{ll} u = e^x & u' = e^x \\ v' = \cos x & v = \sin x \end{array} \right| = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x \, dx}_{IN}$$

$$\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx \quad \Big/ + \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = -e^x(\cos x - \sin x) + c \quad /:2$$

$$\int e^x \sin x\,dx = -\frac{e^x}{2}(\cos x - \sin x) + c$$

g) $\displaystyle\int \frac{x}{\cos^2 x}\,dx = \int x\cdot\frac{1}{\cos^2 x}\,dx = \left| \begin{array}{ll} u = x & u' = 1 \\ v' = \frac{1}{\cos^2 x} & v = \operatorname{tg} x \end{array} \right| =$

$$= x\cdot\operatorname{tg} x - \int 1\cdot\operatorname{tg} x\,dx = x\operatorname{tg} x + \int \overbrace{\frac{-\sin x}{\cos x}\,dx}^{dt} \quad \left| \begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \end{array} \right|$$

$$= x\operatorname{tg} x + \int \frac{1}{t}\,dt = x\operatorname{tg} x + \ln|t| = x\operatorname{tg} x + \ln|\cos x| + c$$

PR: $\displaystyle\int e^{\cos^2 x}\,\overbrace{\sin 2x\,dx}^{dt} \quad \left| \begin{array}{l} \cos^2 x = t \\ \underbrace{2\cos x\cdot\sin x}_{\sin 2x}\,dx = dt \end{array} \right| = \int e^t\,dt = e^t =$

$$= e^{\cos^2 x} + c$$

PR: $\displaystyle\int \frac{\cos(\ln x)}{x}\,\overbrace{dx}^{dt} \quad \left| \begin{array}{l} \ln x = t \\ \frac{1}{x}\,dx = dt \end{array} \right| = \int \cos t\,dt =$

$$= \sin t = \sin(\ln x) + c$$

PR. $\displaystyle\int x\,a^x\,dx \quad \left| \begin{array}{ll} u = x & u' = 1 \\ v' = a^x & v = \int a^x\,dx = \frac{a^x}{\ln a} \end{array} \right| = \frac{x\cdot a^x}{\ln a} - \int \frac{a^x}{\ln a}\,dx =$

$$= \frac{xa^x}{\ln a} - \frac{1}{\ln a} \cdot \frac{a^x}{\ln a} = \frac{xa^x}{\ln a} - \frac{a^x}{\ln^2 a} + c$$