Note Title 12/5/2022

PR10

$$1{,}212121\ldots = 1 + 0{,}21 + 0{,}0021 + 0{,}000021 + \ldots$$

$$= 1 + \frac{21}{\underbrace{100}_{10^2}} + \frac{21}{\underbrace{10000}_{10^4}} + \frac{21}{\underbrace{1000000}_{10^6}}$$

$$= 1 + \frac{21}{10^2}\left(\underbrace{1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots}\right) \qquad q = \frac{1}{100}$$

$$s_n = \frac{1}{1-\frac{1}{100}} = \frac{100}{99}$$

$$\Rightarrow 1 + \frac{21}{\cancel{100}} \cdot \frac{\cancel{100}}{99} = 1 + \frac{21}{99} = \frac{120}{99} = \underline{\underline{\frac{40}{33}}}$$

PR11

$$\log x + \log\sqrt{x} + \log\sqrt[4]{x} + \log\sqrt[8]{x} + \cdots = 2$$

$$\log x + \log x^{\frac{1}{2}} + \log x^{\frac{1}{4}} + \log x^{\frac{1}{8}} + \cdots = 2$$

$$\log x + \tfrac{1}{2}\log x + \tfrac{1}{4}\log x + \tfrac{1}{8}\log x + \cdots = 2$$

$$\log x \left(\underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}\right) = 2$$

$$q = \frac{1}{2} < 1 \Rightarrow s_n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \underline{\underline{2}}$$

$$\log x \cdot \cancel{2} = \cancel{2}$$

$$\log x = 1 \Rightarrow \underline{\underline{x = 10}}$$

CAUCHYHO ODMOCNINOVÉ KRITÉRIUM

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad ; \quad \begin{array}{ll} L < 1 & (K) \\ L > 1 & (D) \\ L = 1 & ? \end{array}$$

PR1 (a) $\displaystyle\sum_{n=1}^{\infty} (\operatorname{arctg}(n+1))^n$

$$\lim_{n\to\infty} \sqrt[n]{|(\operatorname{arctg}(n+1))^n|} = \lim_{n\to\infty} \operatorname{arctg}(n+1) = \frac{\pi}{2} > 1$$

NA ZÁKLADE CAUCHYHO ODM. KRITÉRIA RAD $\displaystyle\sum_{n=1}^{\infty}(\operatorname{arctg}(n+1))^n$ DIVERGUJE.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln^n(n+1)}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{\ln^n(n+1)}\right|} = \lim_{n\to\infty} \frac{1}{\underbrace{\ln(n+1)}_{\infty}} = 0 < 1 \quad (K)$$

ⓒ $\sum_{n=2}^{\infty} \left(\frac{1}{\ln n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{\ln n}\right)^n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1 \quad \text{Ⓚ}$$

ⓓ $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1 \quad \text{Ⓚ}$$

ⓔ $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^1 \cdot 3^{3n}}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3 \cdot 27^n}} = \lim_{n\to\infty} \frac{n}{\underbrace{3^{\frac{1}{n}}}_{1} \cdot 27} = \infty > 1 \quad \text{Ⓓ}$$

ⓕ $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ $\quad \lim_{n\to\infty} \sqrt[n]{\frac{2^n \cdot 2}{n^n}} = \lim_{n\to\infty} \frac{2 \cdot \overbrace{2^{\frac{1}{n}}}^{1}}{n} = 0 < 1 \quad \text{Ⓚ}$

ⓖ $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{n+1-1}{n+1}\right)^n =$$

$$= \lim_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^{n+1-1} = \underbrace{\lim_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^{n+1}}_{e^{-1}} \cdot \underbrace{\lim_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^{-1}}_{1}$$

$$= e^{-1} = \frac{1}{e} < 1 \quad \text{Ⓚ}$$

ⓗ $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ $\quad \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{n^{\frac{2}{n}}}{2} = \frac{1}{2} < 1 \quad \text{Ⓚ}$

$$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1}$$

POUŽÍVAŤ AKO FAKT

D'ALAMBERTOVO PODIELOVÉ KRITÉRIUM

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

$L < 1$ Ⓚ

$L > 1$ Ⓓ

$L = 1$?

ⓐ $\sum_{n=1}^{\infty} \frac{n+1}{2^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{n+2}{2^{n+1}}}{\frac{n+1}{2^n}} \right| = \lim_{n\to\infty} \frac{n+2}{\cancel{2^n}\cdot 2} \cdot \frac{\cancel{2^n}}{n+1} = \frac{1}{2} \lim_{n\to\infty} \frac{n+2}{n+1} = \frac{1}{2} < 1$$

Ⓚ

ⓑ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1$$

NEVIEME ROZHODNÚŤ O KON. RADU

ⓒ $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} = \lim_{n\to\infty} \frac{(n+1)!\, 2^n}{2^{n+1}\cdot n!} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}\,\cancel{2^n}}{\cancel{2^n}\cdot 2 \cdot \cancel{n!}} = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$$

Ⓓ

ⓓ $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)!}$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+2)!}}{\frac{2^n}{(n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{2^n}\cdot 2 \cdot \cancel{(n+1)!}}{(n+2)\cancel{(n+1)!}\, \cancel{2^n}} \right| = 0 < 1$$

Ⓚ

ⓔ $\sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)\cdot \cancel{\left(\frac{2}{3}\right)^n} \cdot \frac{2}{3}}{n \cdot \cancel{\left(\frac{2}{3}\right)^n}} \right| = \frac{2}{3} \lim_{n\to\infty} \frac{n+1}{n} =$$

$$= \frac{2}{3} < 1$$

Ⓚ

ⓕ $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 2)^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{(\ln 2)^{n+1}}}{\frac{n^3}{(\ln 2)^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^3 \cdot \cancel{(\ln 2)^n}}{\cancel{(\ln 2)^n}\cdot \ln 2 \cdot n^3} =$$

$$= \frac{1}{\ln 2} \lim_{n\to\infty} \frac{n^3+3n^2+3n+1}{n^3} = \frac{1}{\ln 2} > 1$$

$\ln 2 < 1$

Ⓓ

ⓗ $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n \cdot \cancel{(n+1)}\,\cancel{n!}}{\cancel{(n+1)}\,\cancel{n!}\; n^n} =$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

Ⓓ

POROVNÁVACIE KRITÉRIUM

ZOVŠEOBECNENÝ HARMONICKÝ RAD $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p>1$ RAD KONVERGUJE

PR3 ⓐ $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ $\frac{\sqrt{n}}{\ln n} > \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}$ $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ JE ZOVŠEOBECNENÝ HARMONICKÝ RAD S $p=\frac{1}{2}<1$ A DIVERGUJE

$\Rightarrow$ AJ RAD $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ DIV.

ⓑ $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+3}}$ $\frac{1}{n\sqrt{n^2+3}} < \frac{1}{n\sqrt{n^2}} = \frac{1}{n^2}$ $p=2>1 \Rightarrow$ RAD $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (K) $\Rightarrow$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+3}}$ (K)

ⓒ $\sum_{n=1}^{\infty} \frac{\sin^2 n}{3^n}$ $\frac{\overset{\langle 0,1\rangle}{\sin^2 n}}{3^n} < \frac{1}{3^n}$ $\to$ GR $q=\frac{1}{3}<1$ KONVERGENTNÝ

$\Rightarrow$ AJ RAD $\sum_{n=1}^{\infty} \frac{\sin^2 n}{3^n}$ (K)

CAUCHYHO INTEGRÁLNE KRITÉRIUM

FUNKCIA $f(x)$: SPOJITÁ, NEZÁPORNÁ, NERASTÚCA

$$\int_k^{\infty} f(x)\,dx = \text{ČÍSLO (K)} \quad \Big/ \quad \pm\infty \text{ (D)}$$

ⓐ $\sum_{n=1}^{\infty} \frac{1}{n}$ $f(x)=\frac{1}{x}$
- SPOJITÁ
- NEZÁPORNÁ
- KLESAJÚCA

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \left[\ln|x|\right]_1^b =$$

$$= \lim_{b\to\infty} \left[\underbrace{\ln b}_{\ln\infty=\infty} - \ln 1\right] = \infty - 1 = \infty$$ $\Rightarrow$ RAD $\sum_{n=1}^{\infty} \frac{1}{n}$ (D)

ⓑ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $f(x)=\frac{1}{x^2}$
- SPOJITÁ
- NEZÁPORNÁ
- KLESAJÚCA

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b \to \infty} \left[-\frac{1}{b} + \frac{1}{1} \right] = 1 \quad \Rightarrow \quad \text{RAD } \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (K)$$

(c) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ $\qquad f(x) = \frac{1}{1+x^2}$

$$\int_1^{\infty} \frac{1}{1+x^2} dx = \lim_{b \to \infty} \int_1^b \frac{1}{1+x^2} dx = \lim_{b \to \infty} \left[\operatorname{arctg} x \right]_1^b =$$

$$= \lim_{b \to \infty} \left[\operatorname{arctg} b - \operatorname{arctg} 1 \right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \ldots \text{RAD } (K)$$

(d) $\sum_{n=2}^{\infty} \frac{2}{n \ln n}$ $\qquad f(x) = \frac{2}{x \ln x}$

$$\int_2^{\infty} \frac{2}{x \ln x} dx = 2 \int_2^{\infty} \frac{1}{x \ln x} dx = 2 \lim_{b \to \infty} \int_2^{b} \frac{1}{x \ln x} dx =$$

$$= \left| \begin{array}{ll} t = \ln x & t_1 = \ln b \\ dt = \frac{1}{x} dx & t_2 = \ln 2 \end{array} \right| = 2 \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{t} dt =$$

$$= 2 \lim_{b \to \infty} \left[\ln t \right]_{\ln 2}^{\ln b} = 2 \lim_{b \to \infty} \left[\ln|\ln b| - \ln|\ln 2| \right] =$$

$$= \infty \qquad \text{RAD DIV.}$$