Note Title 11/10/2022

ODVODENIE UNIVERZÁLNEJ SUBSTITÚCIE

a) $\boxed{t = tg\frac{x}{2}}$

$$arctg\, t = \frac{x}{2} \Rightarrow \boxed{x = 2 arctg\, t}$$

$$\boxed{dx = \frac{2}{1+t^2}\, dt}$$

$\sin(2x) = 2\sin x \cdot \cos x$ } GONIOMETRICKÉ VZŤAHY
$\cos(2x) = \cos^2 x - \sin^2 x$ } PRE 2-NÁSOBNÉ UHLY

$$\sin\left(\cancel{2}\frac{x}{\cancel{2}}\right) = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \cdot \frac{\frac{1}{\cos^2\frac{x}{2}}}{\frac{1}{\cos^2\frac{x}{2}}} =$$

$$= \frac{\underbrace{\frac{2\sin\frac{x}{2}\cdot\cancel{\cos\frac{x}{2}}}{\cos^{\cancel{2}}\frac{x}{2}}}_{tg\frac{x}{2}}}{\underbrace{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}_{tg^2\frac{x}{2}} + \underbrace{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}}_{=1}} = \frac{2t}{t^2+1}$$

$$\cos\left(\cancel{2}\cdot\frac{x}{\cancel{2}}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \cdot \frac{\frac{1}{\cos^2\frac{x}{2}}}{\frac{1}{\cos^2\frac{x}{2}}} =$$

$$= \frac{\underbrace{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}}_{1} - \underbrace{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}_{tg^2\frac{x}{2}}}{\underbrace{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}_{tg^2\frac{x}{2}} + \underbrace{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}}_{1}} = \frac{1-t^2}{t^2+1}$$

b) $\boxed{t = tg\, x}$

$\boxed{arctg\, t = x}$

$$\Downarrow$$

$$\boxed{dx = \frac{1}{1+t^2}\, dt}$$

$$\sin^2 x = \frac{\sin^2 x}{1} = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} \cdot \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} = \frac{\overbrace{\frac{\sin^2 x}{\cos^2 x}}^{tg^2 x}}{\underbrace{\frac{\sin^2 x}{\cos^2 x}}_{} + \underbrace{\frac{\cos^2 x}{\cos^2 x}}_{1}} = \frac{t^2}{t^2+1}$$

$$\cos^2 x = \frac{\cos^2 x}{1} = \frac{\cos^2 x}{\sin^2 x + \cos^2 x} \cdot \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} = \frac{\overbrace{\frac{\cos^2 x}{\cos^2 x}}^{1}}{\underbrace{\frac{\sin^2 x}{\cos^2 x}}_{tg^2 x} + \underbrace{\frac{\cos^2 x}{\cos^2 x}}_{1}} = \frac{1}{t^2+1}$$

PR 1

a) $$\int \frac{1-\sin x}{1+\cos x}\,dx \quad \Big| \; t = tg\frac{x}{2} \; \Big| = \int \frac{1-\overbrace{\frac{2t}{t^2+1}}^{\sin x}}{1+\underbrace{\frac{1-t^2}{t^2+1}}_{\cos x}} \cdot \overbrace{\frac{2}{1+t^2}\,dt}^{dx} =$$

$$= \int \frac{\frac{t^2+1-2t}{t^2+1}}{\frac{\cancel{t^2}+1+1-\cancel{t^2}}{t^2+1}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{(t^2+1-2t)\cdot\cancel{(t^2+1)}}{\cancel{(t^2+1)}\cdot\cancel{2}} \cdot \frac{\cancel{2}}{(1+t^2)}\,dt =$$

$$= \int \frac{t^2+1-2t}{1+t^2}\,dt = \int 1 - \frac{2t}{1+t^2}\,dt = t - \ln|1+t^2| + C \;;\; t = tg\frac{x}{2}$$

b) $$\int \frac{tg x}{1+tg x}\,dx \quad \Big| \; t = tg x \; \Big| = \int \frac{\overset{tg x}{t}}{1+\underset{tg x}{t}} \cdot \overbrace{\frac{dt}{1+t^2}}^{dx} = \int \frac{t}{(1+t)(1+t^2)}\,dt$$

$$\frac{t}{(1+t)(1+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2} = \frac{A + At^2 + Bt + C + Bt^2 + Ct}{(1+t)(1+t^2)}$$

$$0 \cdot t^2 + 0 + 1 \cdot t = (A+B)t^2 + (B+C)t + A + C$$

$$0 = A + B \Rightarrow A = -B \qquad \Rightarrow B = \frac{1}{2}$$

$$1 = B + C \qquad 1 = B + C$$

$$0 = A + C \qquad = 0 = -B + C \quad (+) = 1 = 2C \Rightarrow C = \frac{1}{2}$$

$$A = -C \Rightarrow A = -\frac{1}{2}$$

$$= \int \frac{-\frac{1}{2}}{1+t} + \frac{\frac{1}{2}t + \frac{1}{2}}{1+t^2}\,dt = -\frac{1}{2}\ln|1+t| + \frac{1}{2}\int \frac{t+1}{t^2+1}\,dt$$

$$= -\frac{1}{2}\ln|1+t| + \frac{1}{2\cdot 2}\int \frac{2t}{t^2+1}\,dt + \frac{1}{2}\int \frac{1}{t^2+1}\,dt =$$

$$= -\frac{1}{2}\ln|1+t| + \frac{1}{4}\ln(t^2+1) + \frac{1}{2}\,\mathrm{arctg}\, t + C \;;\; t = tg x$$

ⓒ $\int \frac{1}{\cos x}\,dx = \left| t = \mathrm{tg}\frac{x}{2} \right| = \int \frac{1}{\frac{1-t^2}{\cancel{t^2+1}}} \cdot \frac{2}{\cancel{1+t^2}}\,dt =$

$= \int \frac{2}{1-t^2}\,dt = \int \frac{2}{(1-t)(1+t)}\,dt$

$$\frac{2}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t} = \frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$$

$$2 = A(1+t) + B(1-t)$$

① volím $\boxed{t=-1}$ $\Rightarrow 2 = A(1-1) + B(1+1) = 2B \Rightarrow \boxed{B=1}$

volím $\boxed{t=1}$ $\Rightarrow 2 = A(1+1) + B(1-1) = 2A \Rightarrow \boxed{A=1}$

②
$$2 = A + At + B - Bt$$
$$2 = (A-B)t + A + B$$
$$0 = A - B \Rightarrow A = B \Rightarrow \boxed{A=1}$$
$$2 = A + B \Rightarrow 2 = 2B \Rightarrow \boxed{B=1}$$

$= \int \frac{1}{1-t} + \frac{1}{1+t}\,dt = -\ln|1-t| + \ln|1+t| + C; \quad t = \mathrm{tg}\frac{x}{2}$

ⓓ $\int \frac{1}{\sin x - \cos x}\,dx = \left| t = \mathrm{tg}\frac{x}{2} \right| = \int \frac{1}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$

$= \int \frac{1}{\frac{2t-1+t^2}{\cancel{1+t^2}}} \cdot \frac{2}{\cancel{1+t^2}}\,dt = \int \frac{2}{t^2+2t-1}\,dt = \int \frac{2}{(t+1-\sqrt{2})(t+1+\sqrt{2})}\,dt$

$$D = b^2 - 4ac = 4 - 4.(-1) = 8 > 0$$

$$t_{1,2} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

$$\frac{A}{t+1-\sqrt{2}} + \frac{B}{t+1+\sqrt{2}} = \frac{At + A + \sqrt{2}A + Bt + B - \sqrt{2}B}{(t+1-\sqrt{2})(t+1+\sqrt{2})}$$

$$2 = (A+B).t + A + \sqrt{2}A + B - \sqrt{2}B$$

$$0 = A + B \Rightarrow A = -B$$
$$2 = A + \sqrt{2}A + B - \sqrt{2}B = -\cancel{B} - \sqrt{2}B + \cancel{B} - \sqrt{2}B$$

$$2 = -2\sqrt{2}B$$

$$1 = -\sqrt{2}B \Rightarrow \underline{B} = -\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \underline{-\frac{\sqrt{2}}{2}} \qquad \underline{A = \frac{\sqrt{2}}{2}}$$

$$= \int \frac{\frac{\sqrt{2}}{2}}{t+1-\sqrt{2}} + \frac{-\frac{\sqrt{2}}{2}}{t+1+\sqrt{2}}\,dt = \underline{\frac{\sqrt{2}}{2}\ln|t+1-\sqrt{2}| - \frac{\sqrt{2}}{2}\ln|t+1+\sqrt{2}| + C}$$

$$;\ \underline{t = \operatorname{tg}\frac{x}{2}}$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta) + \sin(\alpha+\beta)\right)$$

a) $\int \sin^3 x\cos^2 x\,dx = \int \sin x\cdot\sin^2 x\cos^2 x\,dx = \int \sin x\cdot(1-\cos^2 x)\cos^2 x\,dx$

$$\left|\begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array}\right| = \int \cancel{\sin x}\cdot(1-t^2)\cdot t^2\cdot\frac{dt}{-\cancel{\sin x}} = \int (t^2-1)t^2\,dt$$

$$= \int t^4 - t^2\,dt = \frac{t^5}{5} - \frac{t^3}{3} + C;\ t = \cos x$$

b) $\int \frac{1}{\cos x}\,dx = \int \frac{1}{\cos x}\cdot\frac{\cos x}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$ $\quad dt$

$$= \left|\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array}\right| = \int \frac{dt}{1-t^2} \cdots\cdots = \underbrace{\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|}_{\text{VZOREC + DOKAZALI SMO SI TO}}$$

c) $\int \sin(3x)\cdot\cos(2x)\,dx$ $\qquad$ $\alpha = 3x$, $\beta = 2x$

$$\left|\begin{array}{l} \alpha+\beta = 3x+2x = 5x \\ \alpha-\beta = 3x-2x = x \end{array}\right|$$

$$= \int \frac{1}{2}(\sin x + \sin 5x)\,dx = -\frac{1}{2}\cos x - \frac{1}{2}\frac{\cos(5x)}{5} + C$$

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a) $$\int \frac{\sqrt[3]{x}}{x+\sqrt[6]{x^5}}\,dx = \left|\begin{array}{l} t = \sqrt[6]{x} \Rightarrow x = t^6 \\ dx = 6t^5\,dt \\ \sqrt[3]{x} = t^2 \\ \sqrt[6]{x^5} = t^5 \end{array}\right| =$$

$$= \int \frac{t^2}{t^6+t^5}\cdot 6t^5\,dt = \int \frac{6t^2\cdot\cancel{t^5}}{\cancel{t^5}(t+1)}\,dt = 6\int \frac{t^2}{t+1}\,dt =$$

$$= 6\int \frac{t^2-1+1}{t+1}\,dt = 6\int \frac{t^2-1}{t+1} + \frac{1}{t+1}\,dt = 6\int \frac{(t-1)\cancel{(t+1)}}{\cancel{t+1}} + \frac{1}{t+1}\,dt$$

$$= 6\left[\frac{t^2}{2} - t + \ln|t+1|\right] + C\;;\quad t = \sqrt[6]{x}$$

c) $$\int \frac{\sqrt{1+x}}{x}\,dx = \left|\begin{array}{l} t = \sqrt{1+x} \\ t^2 = 1+x \Rightarrow x = t^2-1 \\ dx = 2t\,dt \end{array}\right| = \int \frac{t}{t^2-1}\,2t\,dt =$$

$$= \int \frac{2t^2}{t^2-1}\,dt = 2\int \frac{t^2-1+1}{t^2-1}\,dt = 2\int 1 + \frac{1}{t^2-1}\,dt =$$

$$= 2t + 2\int \frac{1}{(t-1)(t+1)}\,dt =$$

PR7

a) $$\int e^{x+e^x}\,dx = \int e^x e^{e^x}\,dx = \left|\begin{array}{l} t = e^x \\ dt = e^x\,dx \end{array}\right| = \int e^t\,dt =$$

$$= e^t + C = e^{e^x} + C$$

b) $$\int x\sqrt[3]{x+4}\,dx = \left|\begin{array}{l} t = \sqrt[3]{x+4} \\ t^3 = x+4 \Rightarrow x = t^3-4 \\ dx = 3t^2\,dt \end{array}\right| =$$

$$= \int (t^3-4)\,t \cdot 3t^2\,dt = \int 3t^6 - 12t^3\,dt = \frac{3t^7}{7} - \frac{12t^4}{4} + C =$$

$$= \frac{3}{7}t^7 - 3t^4 + C\;;\quad t = \sqrt[3]{x+4}$$

c) $$\int x^2 \operatorname{arctg}\frac{1}{x}\,dx = \left|\begin{array}{ll} u = \operatorname{arctg}\frac{1}{x} & u' = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(\frac{-1}{x^2}\right) \\ v' = x^2 & v = \frac{x^3}{3} \end{array}\right|$$

$$\frac{1}{1+\frac{1}{x^2}}\cdot\left(\frac{-1}{x^2}\right) = \frac{1}{\frac{x^2+1}{\cancel{x^2}}}\left(\frac{-1}{\cancel{x^2}}\right) = \frac{-1}{x^2+1}$$

$$= \frac{x^3}{3}\operatorname{arctg}\frac{1}{x} - \int -\frac{1}{x^2+1}\cdot\left(\frac{x^3}{3}\right)dx = \frac{x^3}{3}\operatorname{arctg}\frac{1}{x} + \frac{1}{3}\int\frac{x^3}{x^2+1}\,dx$$

$$x^3 : (x^2+1) = x - \frac{x}{x^2+1}$$
$$\frac{-(x^3+x)}{-x}$$

$$= \circledast + \frac{1}{3}\int x - \frac{x}{x^2+1}\,dx =$$

$$= \circledast + \frac{1}{3}\frac{x^2}{2} - \frac{1}{3}\cdot\frac{1}{2}\ln|x^2+1| + C = \underline{\underline{\frac{x^3}{3}\operatorname{arctg}\left(\frac{1}{x}\right) + \frac{x^2}{6} - \frac{1}{6}\ln|x^2+1| + C}}$$

ⓐ $$\int \frac{8x - \arcsin^2 x}{\sqrt{1-x^2}}\,dx = \int \underbrace{\frac{8x}{\sqrt{1-x^2}}}_{I_1} - \underbrace{\frac{\arcsin^2 x}{\sqrt{1-x^2}}}_{I_2}\,dx$$

I_1: $$\int \frac{8x}{\sqrt{1-x^2}}\,dx = \left| \begin{array}{l} t = 1-x^2 \\ dt = -2x\,dx \\ dx = \frac{dt}{-2x} \end{array} \right| = \int \frac{\overset{-4}{\cancel{8}}\cancel{x}}{\sqrt{t}} \cdot \frac{dt}{\cancel{-2}\cancel{x}} = -4\int t^{-\frac{1}{2}}\,dt$$

$$= -4\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \underline{\underline{-8\sqrt{1-x^2} + C}}$$

I_2: $$\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx = \left| \begin{array}{l} t = \arcsin x \\ dt = \frac{1}{\sqrt{1-x^2}}\,dx \end{array} \right| = \int t^2\,dt = \frac{t^3}{3} + C =$$
$$= \frac{\arcsin^3 x}{3} + C$$

$$= \underline{\underline{-8\sqrt{1-x^2} - \frac{\arcsin^3 x}{3} + C}}$$