Quantitative Risk Management
MATH-471

Working results of Assignment 1

Tianxiao Ma Sciper N° 269118
Mengjie Zhao Sciper N° 256195

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Question 1

1)

According to lecture slides, we know that,

$$L(t, t+\Delta) = -\lambda S(e^X - 1)$$

Let T be a standard Student's t-distributed variable generated by Matlab with freedom in $f = 3, 10, 50$ degrees. However, the mean and variance of $X_{1,t+\Delta}$ is provided, which is $\mu = 0$ and $\sigma = 0.01$, thus, we need to introduce a constant α to rescale T to make it fulfill the problem's requirement, which is,

$$\begin{aligned} X &= \alpha T \\ \Rightarrow Var[X] &= \alpha^2 Var[T] \\ \Rightarrow \alpha &= \sqrt{\frac{Var[X]}{Var[T]}} \\ \Rightarrow \alpha &= \frac{0.01}{\sqrt{\frac{f}{f-2}}} \end{aligned}$$

Hence, by using Matlab, we can find the empirical distribution of losses in different distribution.

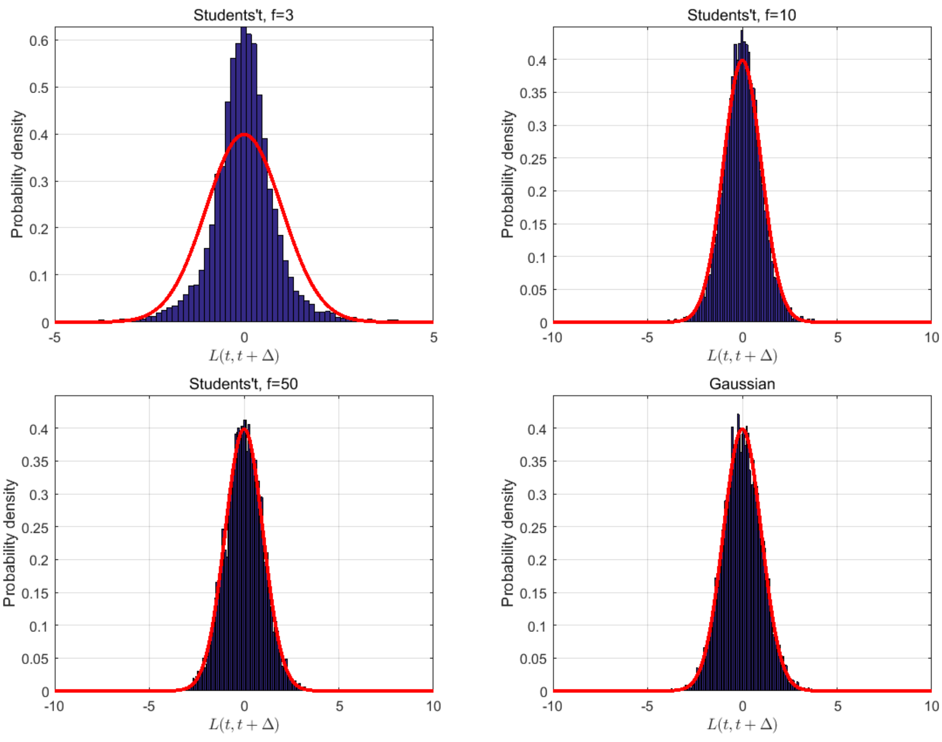

Figure 1: empirical distribution of $L(t, t+\Delta)$

To computed the mean and variance of $L(t, t+\Delta)$, we check $X_{1,t+\Delta}$, which is actually small, thus, we can use the following approximation,

$$L(t, t + \Delta) = L^{\delta}(t, t + \Delta) = -\lambda SX$$

Thus, the mean and variance of $L(t, t + \Delta)$ can be computed easily,

$$\begin{aligned} E[L^{\delta}(t, t + \Delta)] &= -\lambda SE[X] = 0 \\ Var[L^{\delta}(t, t + \Delta)] &= \lambda^2 S^2 Var[X] \\ &= 1^2 \times 100^2 \times 0.01^2 \\ &= 1 \end{aligned}$$

By the way, although the distributions of L above look like normal distribution, none of them is normal distribution because of the primary relation of L and X, $L(t, t + \Delta) = -\lambda S(e^X - 1)$, which means that L is smaller that λS, while normal distribution is unbounded. Therefore, $L(t, t + \Delta)$ is not normal distribution.

Question 2

1)

According to lecture slides, we know that:

$$\begin{aligned} Z_{1,t} &= log(S_t) & & X_{1,t+\Delta} = log(S_{t+\Delta}) - log(S_t) \\ Z_{2,t} &= r_t & \Longrightarrow \quad & X_{2,t+\Delta} = r_{t+\Delta} - r_t \\ Z_{3,t} &= \sigma_t & & X_{3,t+\Delta} = \sigma_{t+\Delta} - \sigma_t \end{aligned}$$

As the risk factor changes $X_{i,t+\Delta}$ is generated according to normal distribution. It is possible that $r_{t+\Delta}$ and $\sigma_{t+\Delta}$ being negative. But we claim that this possibility is extremely small, given the extremely small standard deviation of $X_{i,t+\Delta}$ and small number of samples (10000). For example, we can calculate the probability that $r_{t+\Delta} < 0$ as follow (given $r_t = 0.05$):

$$\Pr(X_{2,t+\Delta} < -0.05) \xrightarrow[\text{standard}]{\text{to}} \Pr(\frac{X_{2,t+\Delta}}{10^{-4}} < \frac{-0.05}{10^{-4}}) = \Phi(-500)$$

which is small enough to be ignored. If we do meet negative values then we set them to their absolute value. Due to the low probability of this event, the whole distribution will not be affected significantly. Following figure displays the empirical distribution:

2)

The linearized loss is given by:

$$L^{\delta}(t, t + \Delta) = -\partial_S C^{BS} S_t X_{1,t+\Delta} - \partial_r C^{BS} X_{2,t+\Delta} - \partial_{\sigma} C^{BS} X_{3,t+\Delta}$$

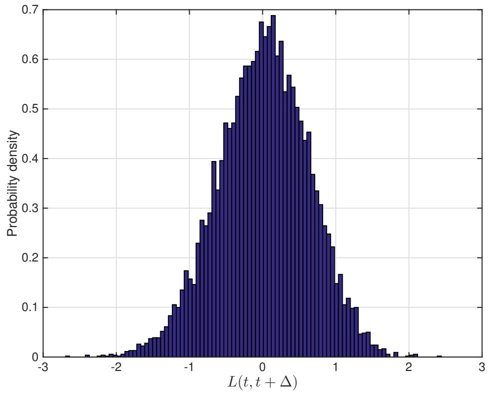

Figure 2: empirical distribution of $L(t, t + \Delta)$

where

$$\partial_S C^{BS} = \Phi(d_1)$$
$$\partial_r C^{BS} = KTe^{-rT}\Phi(d_2)$$
$$\partial_\sigma C^{BS} = S\phi(d_1)\sqrt{T}$$

in which d_1 and d_2 are the same to that defined in lecture slides. Following figure illustrates the empirical distribution of linearized loss: To determine which of the

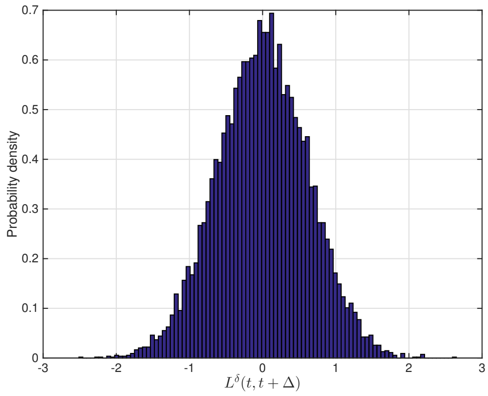

Figure 3: empirical distribution of $L^\delta(t, t + \Delta)$

three risk factors contribute most to $L^\delta(t, t + \Delta)$, we can compare variances of the three terms used to calculate $L^\delta(t, t + \Delta)$:

$$Var_1 = (\partial_S C^{BS})^2 S_t^2 Var[X_{1,t+\Delta}] = 0.4134$$

$$Var_2 = (\partial_r C^{BS})^2 Var[X_{2,t+\Delta}] = 2.8727 \times 10^{-5}$$

$$Var_3 = (\partial_\sigma C^{BS})^2 Var[X_{3,t+\Delta}] = 0.0014$$

Hence we can conclude that the risk factor $Z_{1,t}$, i.e., the stock price contributes most to the linearized loss.

Question 3

We have that $x \sim \mathcal{N}(\mu, \sigma^2)$, then:

$$\mathbb{E}[e^x] = \int_{-\infty}^{+\infty} e^x f(x)dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^x dx$$

We know that:

$$\begin{aligned}
e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^x &= e^{-\frac{(x-\mu)^2}{2\sigma^2}+x} \\
&= e^{-\frac{(x-\mu)^2-2\sigma^2 x}{2\sigma^2}} \\
&= e^{-\frac{(x^2-2x\mu+\mu^2)-2\sigma^2 x}{2\sigma^2}} \\
&= e^{-\frac{(x^2-2x(\mu+\sigma^2)+\mu^2)}{2\sigma^2}} \\
&= e^{-\frac{(x^2-2x(\mu+\sigma^2)+\mu^2+(\mu+\sigma^2)^2-(\mu+\sigma^2)^2)}{2\sigma^2}} \\
&= e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu}{2\sigma^2}} e^{-\frac{(x^2-2x(\mu+\sigma^2)+(\mu+\sigma^2)^2)}{2\sigma^2}} \\
&= e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu}{2\sigma^2}} e^{-\frac{(x-\sigma^2-\mu)^2}{2\sigma^2}}
\end{aligned}$$

Let $X = x - \sigma^2$ then we have:

$$\begin{aligned}
\mathbb{E}[e^x] &= e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu}{2\sigma^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\sigma^2-\mu)^2}{2\sigma^2}} dx \\
&= e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu}{2\sigma^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(X-\mu)^2}{2\sigma^2}} dX \\
&= e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu}{2\sigma^2}} \times 1 = e^{\mu+\frac{1}{2}\sigma^2}
\end{aligned}$$

Appendix

Following are our codes for sovling the questions of this assignment. In addition, for commented and runnable codes for above problems, please refer to:

https://github.com/FINEPFL/Quant-Risk-Manage-MATH-471

```
% This is a script for solving assignment 1 of Quantitative Risk Management
% course at EPFL 2017. If you meet any problem when evalutating or running
% the code, please feel free to drop an email to the authors.
```

```
% *************************************************************************
% Author:
%         Mengjie Zhao, mengjie.zhao@epfl.ch
%         Tianxiao Ma,  tianxiao.ma@epfl.ch
% *************************************************************************
% Here we acknowlege that all following contents are working results of
% ourselves, w.r.t. EPFL honor code.
clear; clc

S_t = 100; lambda = 1; d = 1; mu_X = 0; SD_X = 0.01; f = [3, 10, 50]; N = 10000;

alpha =  SD_X ./ sqrt(f ./ (f-2));
T(:, 1) = trnd (f(1, 1), N, 1);
T(:, 2) = trnd (f(1, 2), N, 1);
T(:, 3) = trnd (f(1, 3), N, 1);
T(:, 4) = randn (N, 1);

X(:, 1) = alpha(:, 1) * T(:, 1);
X(:, 2) = alpha(:, 2) * T(:, 2);
X(:, 3) = alpha(:, 3) * T(:, 3);
X(:, 4) = SD_X * T(:, 4);

L_t = lambda * S_t * (exp(X)-1);

figure(1)
%subplot(2, 2, 1)
[N_1, X] = hist(L_t(:, 1), 150);
bar(X, N_1/(sum(N_1)*diff(X(1:2))), 1);
x = -10:0.1:10;
y = normpdf(x, 0, 1);
hold on
plot(x,y, 'r', 'linewidth', 3);
axis([-5 5 -inf inf]);
hold off
grid on;
xlabel('$L(t,t+\Delta)$', 'interpreter', 'latex', 'fontsize', 15)
ylabel('Probability density')
title('Students''t, f=3');
set(gca, 'fontsize', 12)

figure(2)
%subplot(2, 2, 2)
[N_2, X] = hist(L_t(:, 2), 100);
bar(X, N_2/(sum(N_2)*diff(X(1:2))), 1);
hold on
plot(x,y, 'r', 'linewidth', 3);
hold off
grid on;
xlabel('$L(t,t+\Delta)$', 'interpreter', 'latex', 'fontsize', 15)
ylabel('Probability density')
title('Students''t, f=10');
```

```
set(gca, 'fontsize', 12)

figure(3)
%subplot(2, 2, 3)
[N_3, X] = hist(L_t(:, 3), 100);
bar(X, N_3/(sum(N_3)*diff(X(1:2))), 1);
hold on
plot(x,y, 'r', 'linewidth', 3);
hold off
grid on;
xlabel('$L(t,t+\Delta)$', 'interpreter', 'latex', 'fontsize', 15)
ylabel('Probability density')
title('Students''t, f=50');
set(gca, 'fontsize', 12)

figure(4)
%subplot(2, 2, 4)
[N_4, X] = hist(L_t(:, 4), 100);
bar(X, N_4/(sum(N_4)*diff(X(1:2))), 1);
hold on
plot(x,y, 'r', 'linewidth', 3);
hold off
grid on;
xlabel('$L(t,t+\Delta)$', 'interpreter', 'latex', 'fontsize', 15)
ylabel('Probability density')
title('Gaussian');
set(gca, 'fontsize', 12)
```

```
clearvars; close all; clc

%% Question 2
S_t = 100; r_t = 0.05; sigma_t = 0.2;

u_1_tpd   = 0;
std_1_tpd = 0.01;

u_2_tpd   = 0;
std_2_tpd = 0.0001;

u_3_tpd   = 0;
std_3_tpd = 0.001;

corr_1_3 = -0.5;

% generating data for calculating loss -- L and L_Δ
N = 10000;
MU = [u_1_tpd; u_2_tpd; u_3_tpd]';

% X2_tpd is independent with X1_tpd and X3_tpd, cor(X1_tpd, X3_tpd) = -0.5,
% then we build the correlation matrix, then map it to covariace matrix
% since it is required by the mvnrnd function
cor_mat = [   1    0   -0.5;
              0    1      0;
           -0.5    0      1];
```

```
var_map = [std_1_tpd; std_2_tpd; std_3_tpd] *...
                             [std_1_tpd; std_2_tpd; std_3_tpd]';
cov_mat = cor_mat .* var_map;
R = mvnrnd(MU, cov_mat, N);
corr(R(:, 1), R(:, 3)) % verify the correlation is -0.5
corr(R(:, 1), R(:, 2)) % verify the correlation is 0
corr(R(:, 3), R(:, 2)) % verify the correlation is 0

% as the r_tpd and sigma_tpd could be zero, we force then to its absolute
% value. However, the probability of this event is extremely small.
neg_r_tpd_idx = find(R(:, 2) + r_t < 0);
neg_sigma_tpd_idx = find(R(:, 3) + sigma_t < 0);

% convert the negative values to its absolute value
R(neg_r_tpd_idx, 2) = - r_t - R(neg_r_tpd_idx, 2);
R(neg_sigma_tpd_idx, 3) = - sigma_t - R(neg_sigma_tpd_idx, 3);

% def parameters and helper functions for calculation
t = 0; T = 1; K = 100; Delta = 1/252;
get_d1 = @(t, T, K, S, r, sigma) (log(S./K) + (r + 0.5*sigma.^2) * (T-t))/...
                                                     (sigma * sqrt(T-t));
get_d2 = @(t, T, K, S, r, sigma) (get_d1(t, T, K, S, r, sigma) - ...
                                                      sigma * sqrt(T-t));

get_C_bs = @(t, T, K, S, r, sigma) (S*normcdf(get_d1(t, T, K, S, r, sigma)))...
                 - exp(-r*(T-t)) * K * normcdf(get_d2(t, T, K, S, r, sigma));

d1_t = get_d1(t, T, K, S_t, r_t, sigma_t);
d2_t = get_d2(t, T, K, S_t, r_t, sigma_t);
C_bs_t = get_C_bs(t, T, K, S_t, r_t, sigma_t);

% in matrix R we have X1_tpd, X2_tpd and X3_tpd respectively.
% tpd means t plus delat
t = Delta;
S_tpd = exp(R(:, 1) + log(S_t));
r_tpd = R(:, 2) + r_t;
sigma_tpd = R(:, 3) + sigma_t;

for i=1:N
C_bs_tpd(i) = get_C_bs(t, T, K, S_tpd(i), r_tpd(i), sigma_tpd(i));
end
L_t_tpd = -(C_bs_tpd - C_bs_t);

% get handles from hist to make the hist to pdf and integral to 1
[N, X] = hist(L_t_tpd, 95);
bar(X, N/(sum(N)*diff(X(1:2))), 1)
grid on;
xlabel('$L(t,t+\Delta)$', 'interpreter', 'latex', 'fontsize', 15)
ylabel('Probability density')
set(gca, 'fontsize', 12)
```

```
% calculate Δ, rho and vega
Δ = normcdf(d1_t)
rho = K * T * exp(-r_t * T) * normcdf(d2_t)
vega = S_t * normpdf(d1_t) * sqrt(T)
% calculate contribution to linear loss of each term
var1 = (Δ)^2 * S_t^2 * var(R(:, 1))
var2 = (rho)^2 * var(R(:, 2))
var3 = (vega)^2 * var(R(:, 3))

% calculate linearlized loss
lized_L_t_tpd = -(Δ * S_t * R(:, 1) + rho * R(:, 2) + vega * R(:, 3));
figure(2)
[N_, X_] = hist(lized_L_t_tpd, 95);
bar(X_, N_/(sum(N_)*diff(X_(1:2))), 1)
grid on
xlabel('$L^\delta(t,t+\Delta)$', 'interpreter', 'latex', 'fontsize', 15)
ylabel('Probability density')
set(gca, 'fontsize', 12)
```