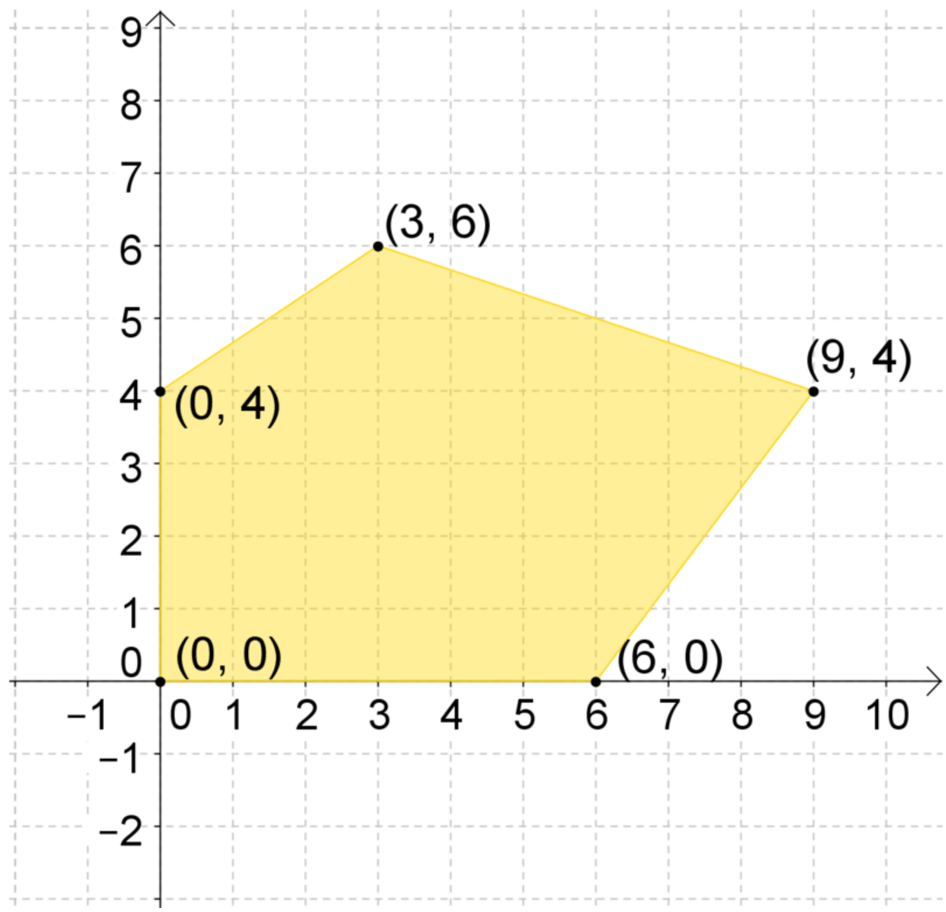
5.7 Applications of Systems of Inequalities

Here you'll learn how to use systems of linear inequalities to solve real-world problems.

The following diagram shows a feasible region solution for a system of linear inequalities.



If $z = 2x + 3y$, at what point on the graph is the value of z the largest?

Applications of Systems of Inequalities

A system of linear inequalities is often used to determine the maximum or minimum values of a situation with multiple constraints. For example, you might be determining how many of a product should be produced to maximize a profit.

In order to solve this type of problem using linear inequalities, follow these steps:

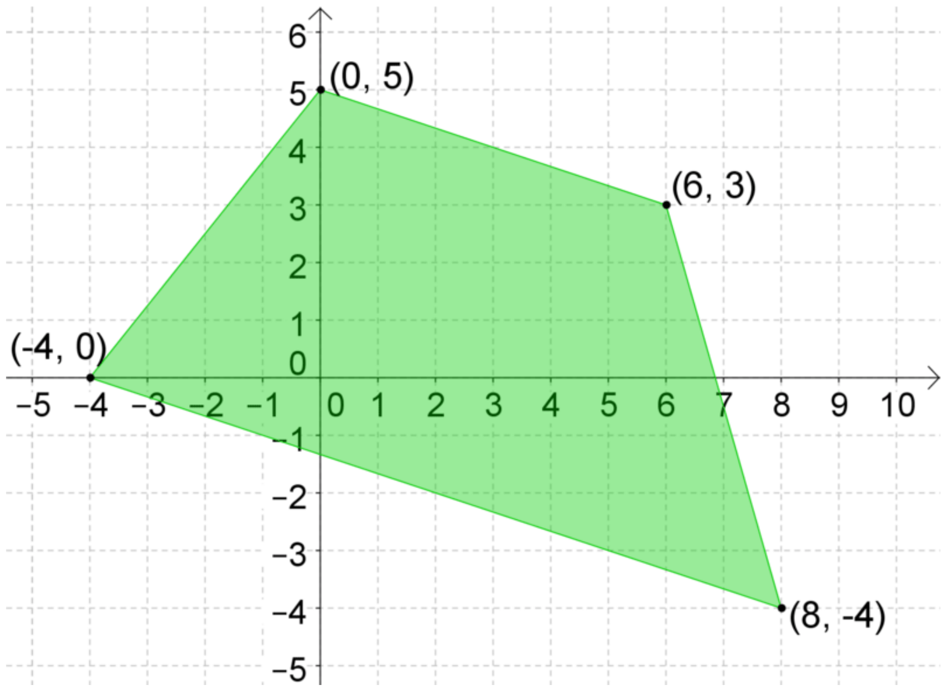
- Step 1: Make a table to organize the given information.
- Step 2: List the constraints of the situation. Write an inequality for each constraint.
- Step 3: Write an equation for the quantity you are trying to maximize (like profit) or minimize (like cost).
- Step 4: Graph the constraints as a system of inequalities.
- Step 5: Find the exact coordinates for each vertex from the graph or algebraically.

- Step 6: Use the Vertex Theorem. Test all vertices of the feasible region in the equation and see which point is the maximum or minimum.

Look at the examples below to see how this process works. Example A allows you to see Steps 5 and 6. Examples B and C allow you to see all the steps in the process.

Evaluate the expression

Evaluate the expression $z = 3x + 4y$ for the given feasible region to determine the point at which z has a maximum value and the point at which z has a minimum value.



$(-4, 0)$	$z = 3x + 4y \rightarrow z = 3(-4) + 4(0) \rightarrow z = -12 + 0 \rightarrow z = -12$ Therefore $3x + 4y = -12$
$(0, 5)$	$z = 3x + 4y \rightarrow z = 3(0) + 4(5) \rightarrow z = 0 + 20 \rightarrow z = 20$ Therefore $3x + 4y = 20$
$(6, 3)$	$z = 3x + 4y \rightarrow z = 3(6) + 4(3) \rightarrow z = 18 + 12 \rightarrow z = 30$ Therefore $3x + 4y = 30$
$(8, -4)$	$z = 3x + 4y \rightarrow z = 3(8) + 4(-4) \rightarrow z = 24 - 16 \rightarrow z = 8$ Therefore $3x + 4y = 8$

The maximum value of z occurred at the vertex (6, 3). The minimum value of z occurred at the vertex (-4, 0).

Note: Using the vertices of the feasible region to determine the maximum or the minimum value is a branch of mathematics known as ***linear programming***. Linear programming is a technique used by businesses to solve problems. The types of problems that usually employ linear programming are those where the profit is to be maximized and those where the expenses are to be minimized. However, linear programming can also be used to solve other types of problems. The solution provides the business with a program to follow to obtain the best results for the company.

Write out the inequalities

A company that produces flags makes two flags for Nova Scotia-the traditional blue flag and the green flag for Cape Breton. To produce each flag, two types of material, nylon and cotton, are used. The company has 450 units of nylon in stock and 300 units of cotton. The traditional blue flag requires 6 units of nylon and 3 units of cotton. The Cape Breton flag requires 5 units of nylon and 5 units of cotton. Each blue flag that is made realizes a profit of $12 for the company, whereas each Cape Breton flag realizes a profit of $15. For the nylon and cotton that the company currently has in stock, how many of each flag should the company make to maximize their profit?

Let 'x' represent the number of blue flags. Let 'y' represent the number of green flags.

Step 1: Transfer the information presented in the problem to a table.

TABLE 5.1:

	Units Required per Blue Flag	**Units Required Per Green Flag**	**Units Available**
Nylon	**6**	**5**	**450**
Cotton	**3**	**5**	**300**
Profit(per flag)	**$12**	**$15**	

The information presented in the problem identifies the restrictions or conditions on the production of the flags. These restrictions are known as ***constraints*** and are written as inequalities to represent the information presented in the problem.

Step 2: From the information (now in the table), list the constraints.

- The number of blue flags that are produced must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{x \geq 0}$$

.

- The number of green flags that are produced must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{y \geq 0}$$

.

- The total number of units of nylon required to make both types of flags cannot exceed 450. Therefore, the constraint is

$$\boxed{6x + 5y \leq 450}$$

.

- The total number of units of cotton required to make both types of flags cannot exceed 300. Therefore, the constraint is

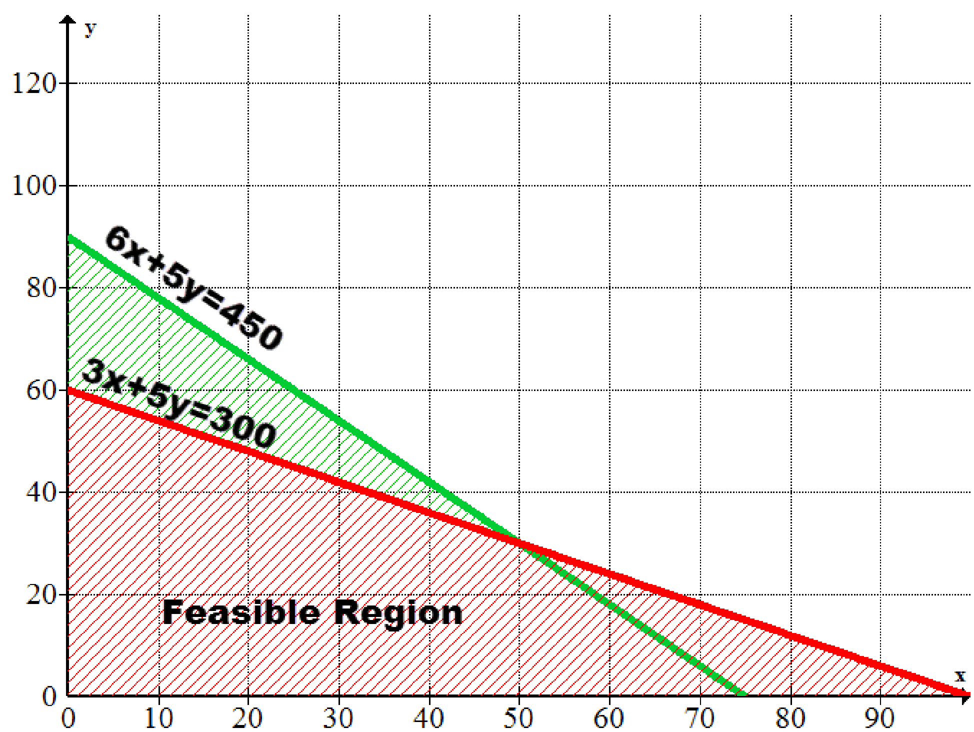
$$\boxed{3x+5y \leq 300}$$

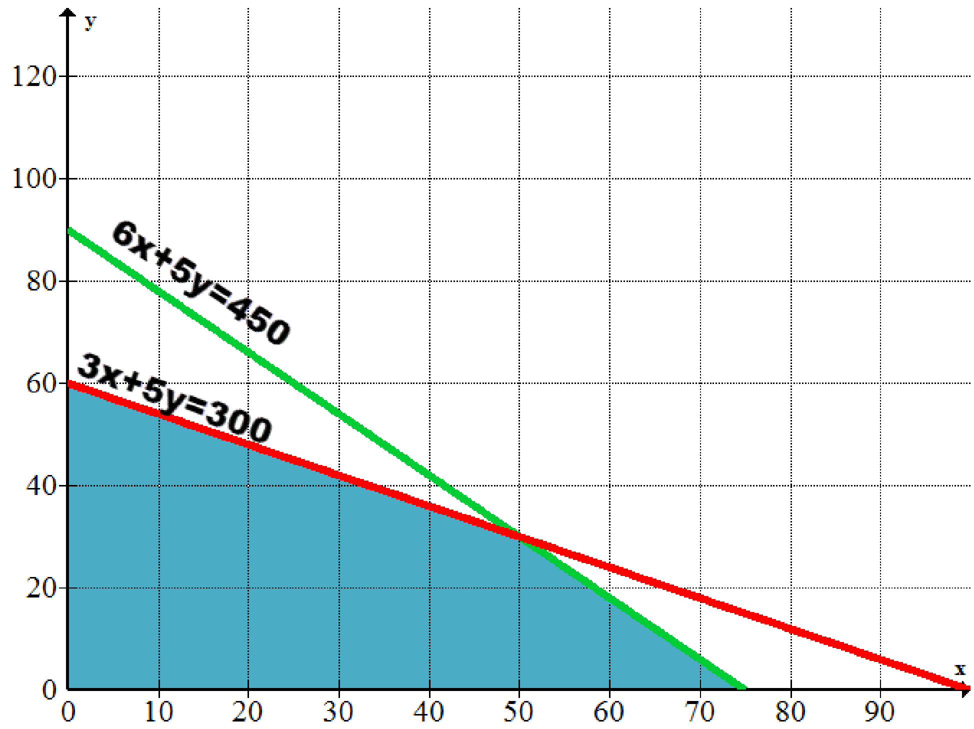
.

Step 3: Write an equation to identify the profit.

$$\boxed{P = 12x+15y}$$

Step 4: Graph the listed constraints to identify the feasible region.





The feasible region is the area shaded in teal blue.

Step 5: Algebraically, determine the exact point of intersection between the constraints. Also, the x-intercept of the feasible region must be calculated. Write the constraints as linear equations and solve the system by elimination.

$$\begin{array}{llll}
6x+5y=450 \rightarrow & 6x+5y=450 \rightarrow & 6x+\cancel{5y}=450 & 6x+5y=450 \\
3x+5y=300 & -1(3x+5y=300) \rightarrow & \underline{-3x-\cancel{5y}=-300} & 6(50)+5y=450 \\
 & & 3x=150 \rightarrow & 300+5y=450 \\
 & & \frac{\cancel{3}x}{\cancel{3}}=\frac{\overset{50}{\cancel{150}}}{\cancel{3}} & 300-300+5y=450-300 \\
 & & x=50 & 5y=150 \\
 & & & \frac{\cancel{5}y}{\cancel{5}}=\frac{\overset{30}{\cancel{150}}}{\cancel{5}} \\
 & & & y=30
\end{array}$$

$$\boxed{l_1 \cap l_2 @ (50,30)}$$

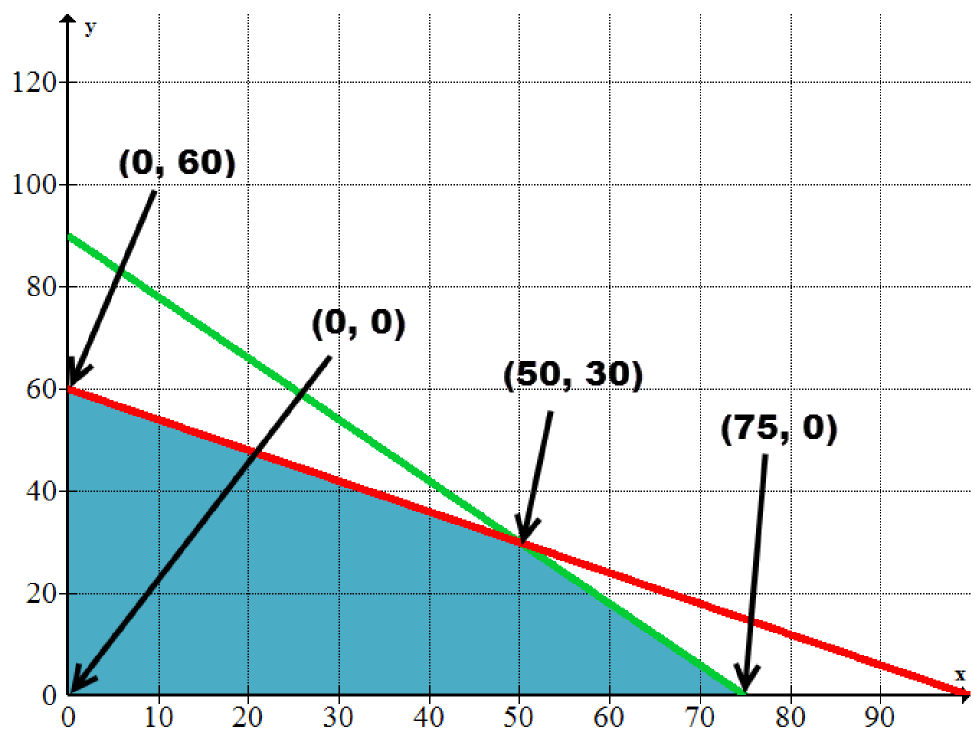
The x-intercept for the inequality $6x+5y \leq 450$ must be calculated. Write the inequality as a linear equation. Set 'y' equal to zero and solve the equation for 'x'.

$$\begin{aligned}6x+5y&=450\\6x+5(0)&=450\\6x&=450\\\frac{\cancel{6}x}{\cancel{6}}&=\frac{\overset{75}{\cancel{450}}}{\cancel{6}}\\x&=75\end{aligned}$$

The x-intercept of the feasible region is (75, 0).

The y-intercept is (0, 60). This point was plotted when the inequalities were put into slope-intercept form for graphing.

The following graph shows the vertices of the polygon that encloses the feasible region.



Step 6: Calculate the profit, using the profit equation, for each vertex of the feasible region:

$(0,0)$ $P = 12x + 15y \rightarrow P = 12(0) + 15(0) \rightarrow P = 0 + 0 \rightarrow P = 0$
Therefore $12x + 15y = \$0$

$(0,60)$ $P = 12x + 15y \rightarrow P = 12(0) + 15(60) \rightarrow P = 0 + 900 \rightarrow P = 900$
Therefore $12x + 15y = \$900$

$(50,30)$ $P = 12x + 15y \rightarrow P = 12(50) + 15(30) \rightarrow P = 600 + 450 \rightarrow P = 1050$
Therefore $12x + 15y = \$1050$

$(75,0)$ $P = 12x + 15y \rightarrow P = 12(75) + 15(0) \rightarrow P = 900 + 0 \rightarrow P = 900$
Therefore $12x + 15y = \$900$

The maximum profit occurred at the vertex (50, 30). This means, with the supplies in stock, the company should make 50 blue flags and 30 green flags to maximize their profit.

Solve the inequalities

A local smelting company is able to provide its customers with iron, lead and copper by melting down either of two ores, A or B. The ores arrive at the company in railroad cars. Each railroad car of ore A contains 3 tons of iron, 3 tons of lead and I ton of copper. Each railroad car of ore B contains 1 ton of iron, 4 tons of lead and 3 tons of copper. The smelting receives an order for 7 tons of iron, 19 tons of lead and 8 tons of copper. The cost to purchase and process a carload of ore A is \$7000 while the cost for ore B is \$6000. If the company wants to fill the order at a minimum cost, how many carloads of each ore must be bought?

Let 'x' represent the number of carloads of ore A to purchase. Let 'y' represent the number of carloads of ore B to purchase.

Step 1: Transfer the information presented in the problem to a table.

TABLE 5.2:

	One Carload of ore A	One Carload of ore B	Number of tons to fill the order
Tons of Iron	**3**	**1**	**7**
Tons of Lead	**3**	**4**	**19**
Tons of Copper	**1**	**3**	**8**

Step 2: From the information, list the constraints.

- The number of carloads of ore A that must be bought is either zero or greater than zero. Therefore, the constraint is

$$\boxed{x \geq 0}$$

.

- The number of carloads of ore B that must be bought is either zero or greater than zero. Therefore, the constraint is

$$\boxed{y \geq 0}$$

.

- The total number of tons of iron from ore A and ore B must be greater than or equal to the 7 tons needed to fill the order. Therefore, the constraint is

$$\boxed{3x+y \geq 7}$$

.

- The total number of tons of lead from ore A and ore B must be greater than or equal to the 20 tons needed to fill the order. Therefore, the constraint is

$$\boxed{3x+4y \geq 19}$$

.

- The total number of tons of copper from ore A and ore B must be greater than or equal to the 8 tons needed to fill the order. Therefore, the constraint is

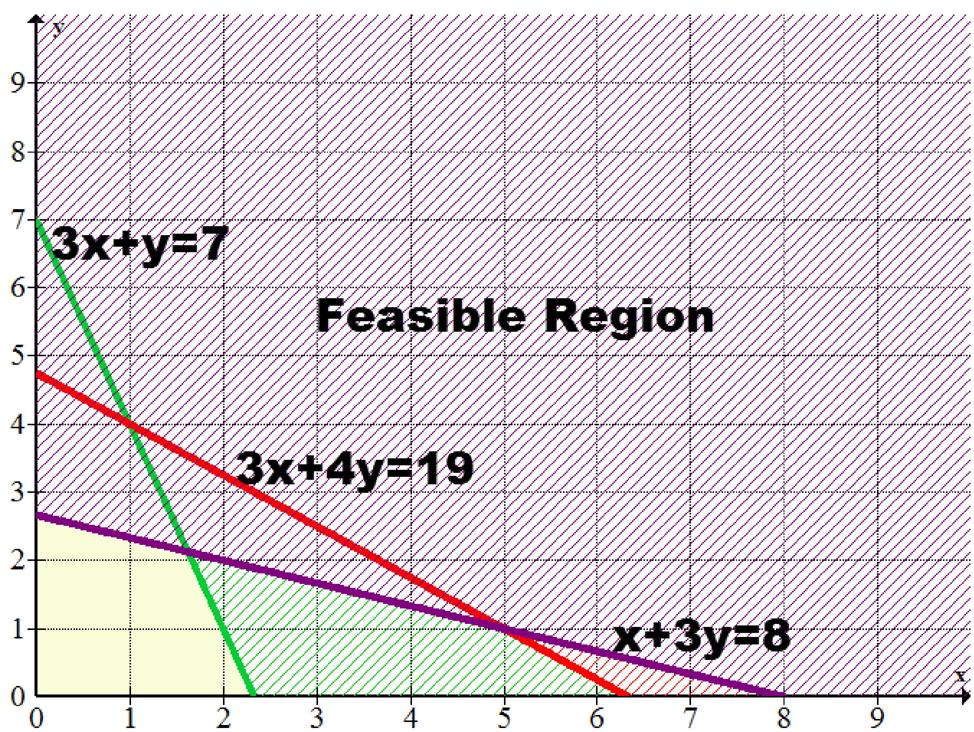
$$\boxed{x+3y \geq 8}$$

.

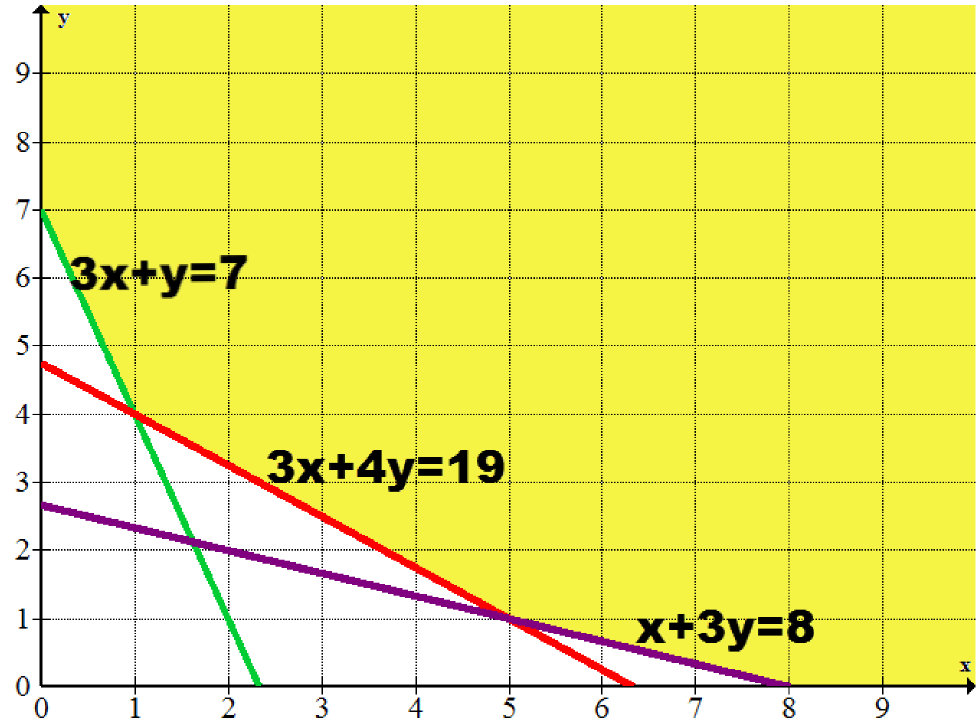
Step 3: Write an equation to represent the cost in dollars of x carloads of ore A and y carloads of ore B.

$$\boxed{c = 7000x + 6000y}$$

Step 4: Graph the listed constraints to identify the feasible region.



The feasible region shows that there are an infinite number of ways to fill the order. The feasible region is the large shaded area that is sitting above the graphed lines.



Step 5: Algebraically, determine the exact point of intersection between the constraints. Also, the x-intercept of the feasible region must be calculated. Write the constraints as linear equations and solve the system by elimination.

$$\begin{array}{lllll}
3x+y=7 \rightarrow & -1(3x+y=7) \rightarrow & -\cancel{3x}-y=-7 & & 3x+4y=19 \\
3x+4y=19 & 3x+4y=19 \rightarrow & \underline{\cancel{3x}+4y=19} & & 3x+4(4)=19 \\
& & 3y=12 & \rightarrow & 3x+16=19 \\
& & \frac{\cancel{3}y}{\cancel{3}}=\frac{\overset{4}{\cancel{12}}}{\cancel{3}} & & 3x+16-16=19-16 \\
& & y=4 & & 3x=3 \\
& & & & \frac{\cancel{3}x}{\cancel{3}}=\frac{\overset{1}{\cancel{3}}}{\cancel{3}} \\
& & & & x=1
\end{array}$$

$$\boxed{l_1 \cap l_2 @ (1,4)}$$

$$\begin{aligned} 3x+4y&=19 \quad\rightarrow & 3x+4y&=19 \quad\rightarrow & \cancel{3x}+4y&=19 & 3x+4y&=19\\ x+3y&=8 & -3(x+3y&=8) \quad\rightarrow & \underline{-\cancel{3x}-9y}&\underline{=-24} & 3x+4(1)&=19\\ &&&& -5y&=-5 \quad\rightarrow & 3x+4&=19\\ &&&& \frac{\cancel{-5}y}{\cancel{-5}}&=\frac{\overset{1}{\cancel{-5}}}{\cancel{-5}} & 3x+4-4&=19-4\\ &&&& y&=1 & 3x&=15\\ &&&&&& \frac{\cancel{3}x}{\cancel{3}}&=\frac{\overset{5}{\cancel{15}}}{\cancel{3}}\\ &&&&&& x&=5 \end{aligned}$$

$$\boxed{l_2 \cap l_3 @ (5,1)}$$

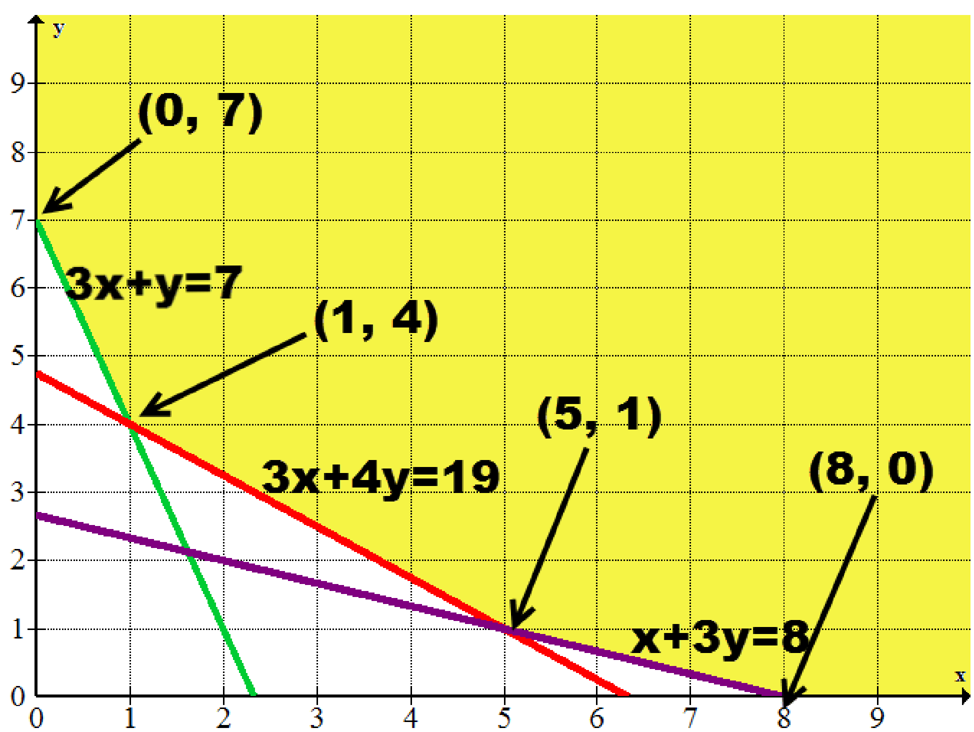
The x-intercept for the inequality $x+3y \geq 8$ must be calculated. Write the inequality as a linear equation. Set 'y' equal to zero and solve the equation for 'x'.

$$\begin{aligned} x+3y&=8\\ x+3(0)&=8\\ x&=8 \end{aligned}$$

The x-intercept of the feasible region is (8, 0).

The y-intercept is (0, 7). This point was plotted when the inequalities were put into slope-intercept form for graphing.

The following graph shows the vertices of the region borders the feasible region.



Step 6: Calculate the cost, using the cost equation, for each vertex of the feasible region:

$(0,7)$ $\quad c = 7000x + 6000y \rightarrow c = 7000(0) + 6000(7) \rightarrow c = 0 + 42,000 \rightarrow P = 42,000$
Therefore $7000x + 6000y = \$42,000$

$(1,4)$ $\quad c = 7000x + 6000y \rightarrow c = 7000(1) + 6000(4) \rightarrow c = 7000 + 24,000 \rightarrow P = 31,000$
Therefore $7000x + 6000y = \$31,000$

$(5,1)$ $\quad c = 7000x + 6000y \rightarrow c = 7000(5) + 6000(1) \rightarrow c = 35,000 + 6000 \rightarrow P = 41,000$
Therefore $7000x + 6000y = \$41,000$

$(8,0)$ $\quad c = 7000x + 6000y \rightarrow c = 7000(8) + 6000(0) \rightarrow c = 56,000 + 0 \rightarrow P = 56,000$
Therefore $7000x + 6000y = \$56,000$

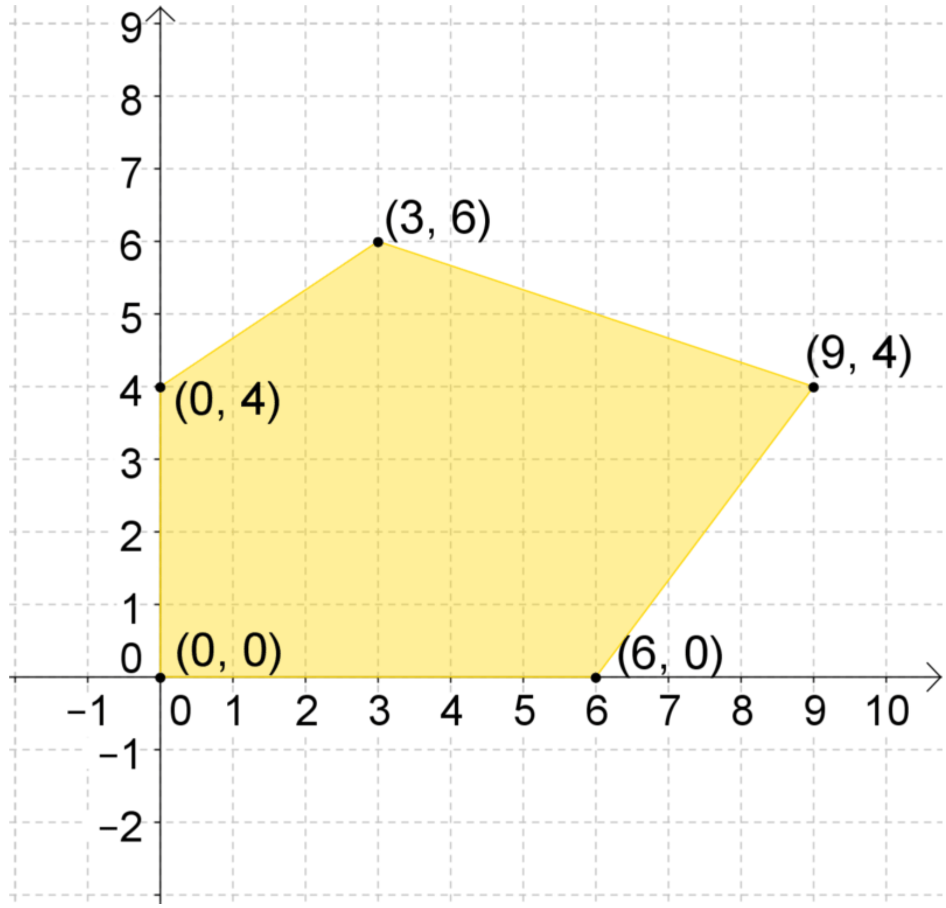
The minimum cost is located at the vertex (1, 4). Therefore the company should buy one carload of ore A and four carloads of ore B.

Examples

Example 1

Earlier, you were asked at what point on the graph is the value of z the largest.

The following diagram shows a feasible region that is within a polygonal region.

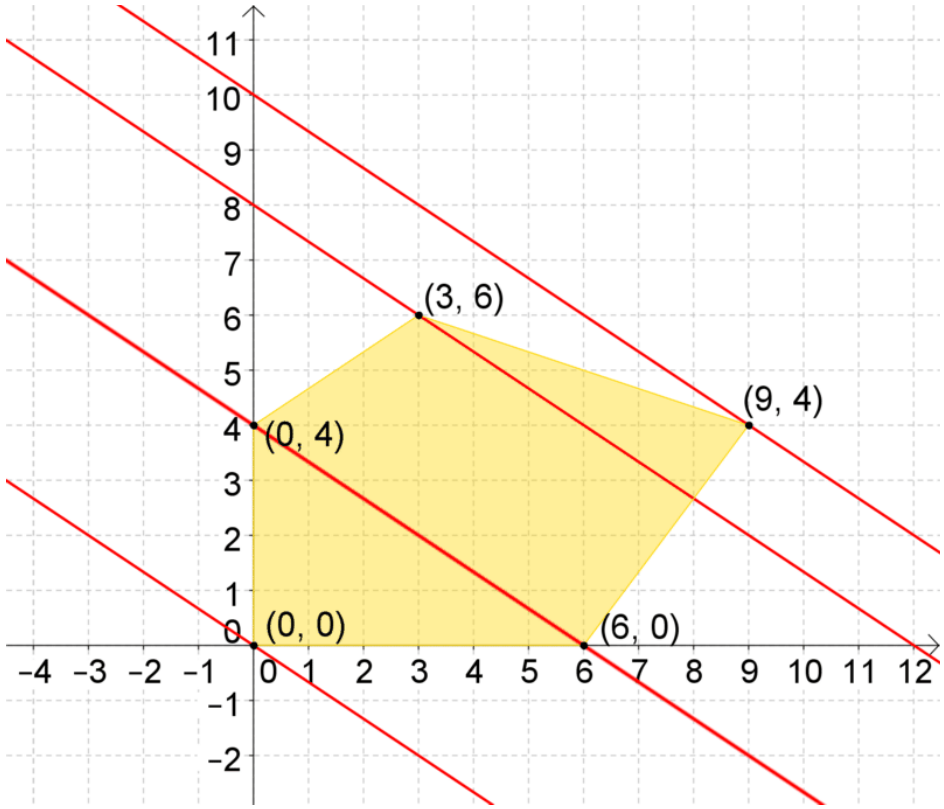


The linear function $z = 2x + 3y$ will now be evaluated for each of the vertices of the polygon.

To evaluate the value of 'z' substitute the coordinates of the point into the expression for 'x' and 'y'.

$(0,0)$	$z = 2x+3y \rightarrow z = 2(0)+3(0) \rightarrow z = 0+0 \rightarrow z = 0$ Therefore $2x+3y = 0$
$(0,4)$	$z = 2x+3y \rightarrow z = 2(0)+3(4) \rightarrow z = 0+12 \rightarrow z = 12$ Therefore $2x+3y = 12$
$(6,0)$	$z = 2x+3y \rightarrow z = 2(6)+3(0) \rightarrow z = 12+0 \rightarrow z = 12$ Therefore $2x+3y = 12$
$(3,6)$	$z = 2x+3y \rightarrow z = 2(3)+3(6) \rightarrow z = 6+18 \rightarrow z = 24$ Therefore $2x+3y = 24$
$(9,4)$	$z = 2x+3y \rightarrow z = 2(9)+3(4) \rightarrow z = 18+12 \rightarrow z = 30$ Therefore $2x+3y = 30$

The value of $z = 2x+3y$, for each of the vertices, remains constant along any line with a slope of $-\frac{2}{3}$. This is obvious on the following graph.



As the line moved away from the origin, the value of $z = 2x + 3y$ increased. The maximum value for the shaded region occurred at the vertex (9, 4) while the minimum value occurred at the vertex (0, 0). These statements confirm the vertex theorem for a feasible region:

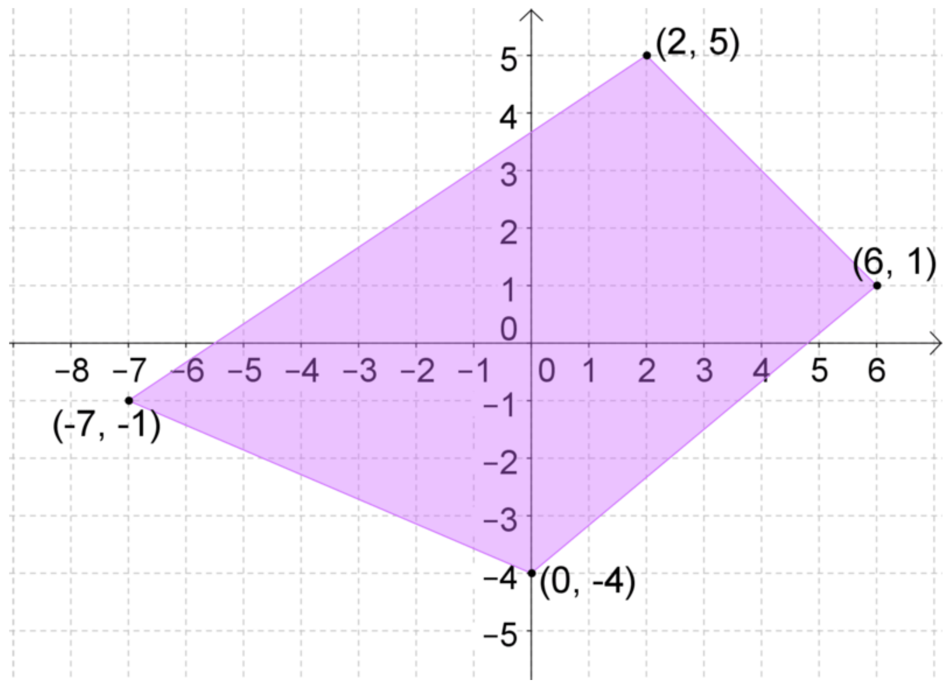
If a linear expression

$$\boxed{z = ax + by + c}$$

is to be evaluated for all points of a convex, polygonal region, then the maximum value of z, if one exists, will occur at one of the vertices of the feasible region. Also, the minimum value of z, if one exists, will occur at one of the vertices of the feasible region.

Example 2

For the following graphed region and the expression $z = 5x + 7y - 1$, find a point where 'z' has a maximum value and a point where 'z' has a minimum value.



The vertices of the polygonal region are (-7, -1); (2, 5); (6, 1); and (0, -4).

$(-7,-1)$ $\quad z=5x+7y-1 \rightarrow z=5(-7)+7(-1)-1 \rightarrow z=-35-7-1 \rightarrow z=-43$
Therefore $5x+7y-1=-43$

$(2,5)$ $\quad z=5x+7y-1 \rightarrow z=5(2)+7(5)-1 \rightarrow z=10+35-1 \rightarrow z=44$
Therefore $5x+7y-1=44$

$(6,1)$ $\quad z=5x+7y-1 \rightarrow z=5(6)+7(1)-1 \rightarrow z=30+7-1 \rightarrow z=36$
Therefore $5x+7y-1=36$

$(0,-4)$ $\quad z=5x+7y-1 \rightarrow z=5(0)+7(-4)-1 \rightarrow z=0-28-1 \rightarrow z=-29$
Therefore $5x+7y-1=-29$

The maximum value of 'z' occurred at the vertex (2, 5). The minimum value of 'z' occurred at the vertex (-7, -1).

Example 3

The following table shows the time required on three machines for a company to produce Super 1 and Super 2 coffee percolators. The table also shows the amount of time that each machine is available during a one hour period. The company is trying to determine how many of each must be made to maximize a profit if they make \$30 on each Super 1 model and \$35 on each Super 2 model. List the constraints and write a profit statement to represent the information.

TABLE 5.3:

	Super 1	Super 2	Time Available
Machine A	1 minute	3minutes	24 minutes
Machine B	3 minutes	2minutes	36 minutes
Machine C	3 minutes	4 minutes	44 minutes

Let 'x' represent the number of Super 1 coffee percolators. Let 'y' represent the number of Super 2 coffee percolators.

- The number of Super 1 coffee percolators that are made must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{x \geq 0}$$

.

- The number of Super 2 coffee percolators that are made must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{y \geq 0}$$

.

- The total amount of time that both a Super 1 and a Super 2 model can be processed on Machine A is less than or equal to 24 minutes. Therefore, the constraint is

$$\boxed{x+3y \leq 24}$$

.

- The total amount of time that both a Super 1 and a Super 2 model can be processed on Machine B is less than or equal to 36 minutes. Therefore, the constraint is

$$\boxed{3x+2y \leq 36}$$

.

- The total amount of time that both a Super1 and a Super 2 model can be processed on Machine C is less than or equal to 44 minutes. Therefore, the constraint is

$$\boxed{3x+4y \leq 44}$$

.

- The profit equation is

$$\boxed{P = 30x+35y}$$

Example 4

A local paint company has created two new paint colors. The company has 28 units of yellow tint and 22 units of red tint and intends to mix as many quarts as possible of color X and color Y. Each quart of color X requires 4 units of yellow tint and 1 unit of red tint. Each quart of color Y requires 1 unit of yellow tint and 4 units of red tint. How many quarts of each color can be mixed with the units of tint that the company has available? List the constraints, complete the graph and determine the solution using linear programming.

Table:

TABLE 5.4:

	Color X	Color Y	Units Available
Yellow Tint	**4 units**	**1 unit**	**28**
Red Tint	**1 unit**	**4 units**	**22**

Constraints: Let 'x' represent the number of quarts of Color X paint to be made. Let 'y' represent the number of quarts of Color Y paint to be made.

- The number of quarts of Color X paint that are mixed must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{x \geq 0}$$

.

- The number of quarts of Color Y paint that are mixed must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{y \geq 0}$$

.

- The total amount of yellow tint that is used to mix Color X and Color Y must be less than or equal to 28. Therefore, the constraint is

$$\boxed{4x + y \leq 28}$$

.

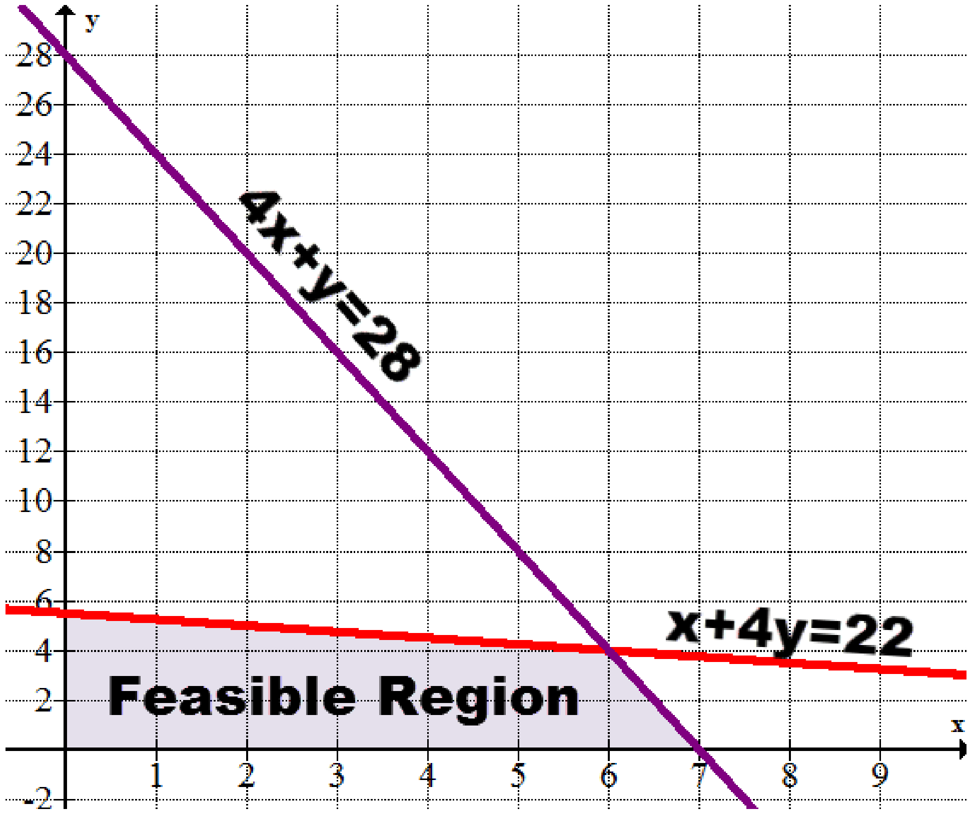
- The total amount of red tint that is used to mix Color X and Color Y must be less than or equal to 22. Therefore, the constraint is

$$\boxed{x + 4y \leq 22}$$

.

Equation: The company wants to mix as many quarts as possible. They want to maximize the value of Q given by $Q = x + y$.

Graph:



Vertices:

$$\begin{array}{llllll}
4x+y=28 & \rightarrow & 4x+y=28 & \rightarrow & \cancel{4x}+y=28 & & 4x+y=28 \\
x+4y=22 & & -4(x+4y=22) & \rightarrow & \underline{-\cancel{4x}-16y=-88} & & 4x+(\color{red}{4})=28 \\
 & & & & -15y=-60 & \rightarrow & 4x=24 \\
 & & & & \frac{\cancel{-15}y}{\cancel{-15}}=\frac{\cancel{-60}^{\color{red}{4}}}{\cancel{-15}} & & \frac{\cancel{4}x}{\cancel{4}}=\frac{\cancel{24}^{\color{red}{6}}}{\cancel{4}} \\
 & & & & y=4 & & x=6
\end{array}$$

$$\boxed{l_1 \cap l_2 @ (6,4)}$$

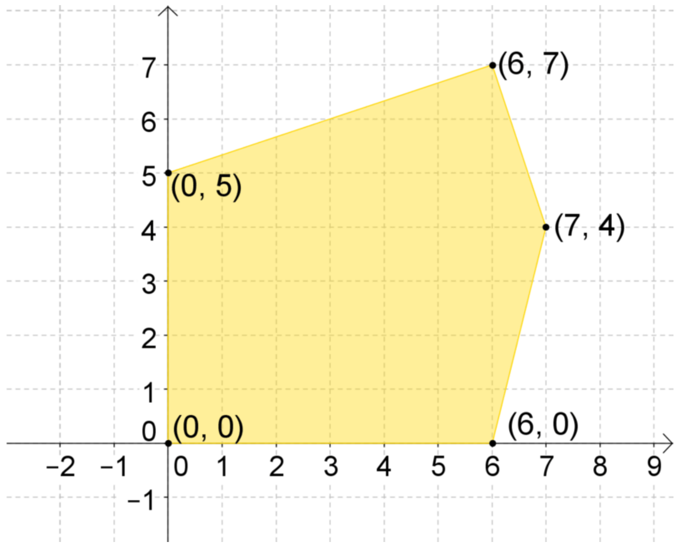
The three points in the feasible region are $(6,4),(7,0),(0,5.5)$. The company wants to maximize $Q=x+y$. The point that produces the maximum value is $(6,4)$.
The company should mix **6** quarts of Color X paint and **4** quarts of Color Y paint.

Review

For each graphed region and corresponding equation, find a point at which 'z' has a maximum value and a point at which 'z' has a minimum value.

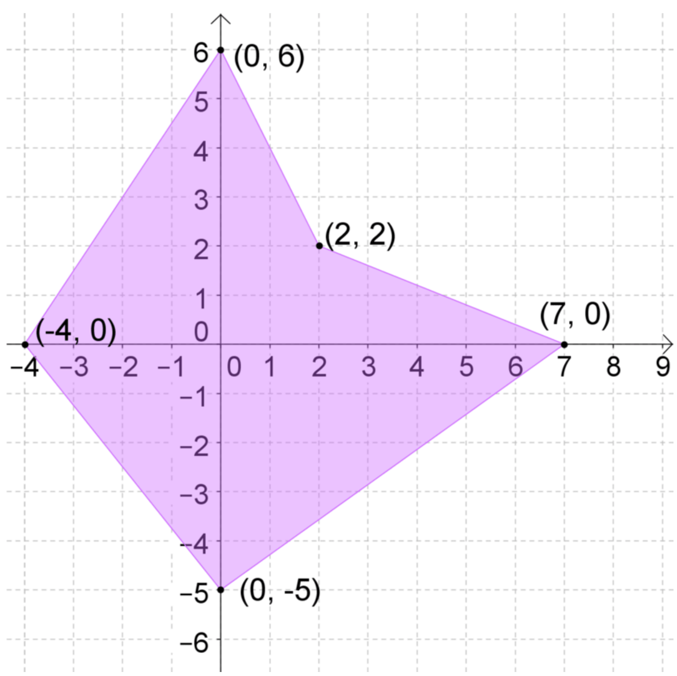
1.

$$\boxed{z=7x-2y}$$



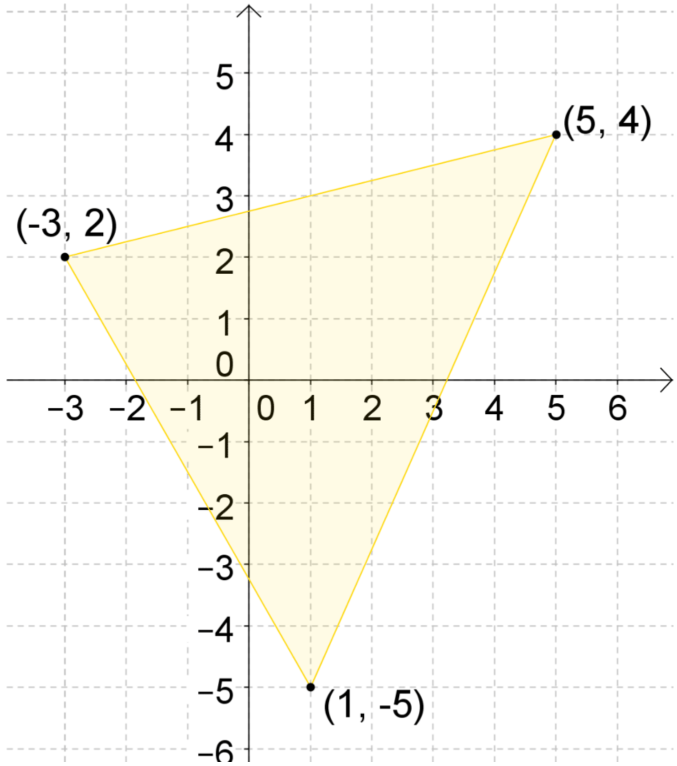
2.

$$\boxed{z = 3y - 4x}$$



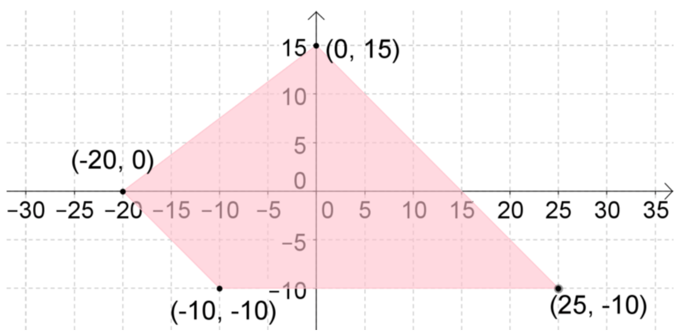
3.

$$\boxed{z = 4x - 2y}$$



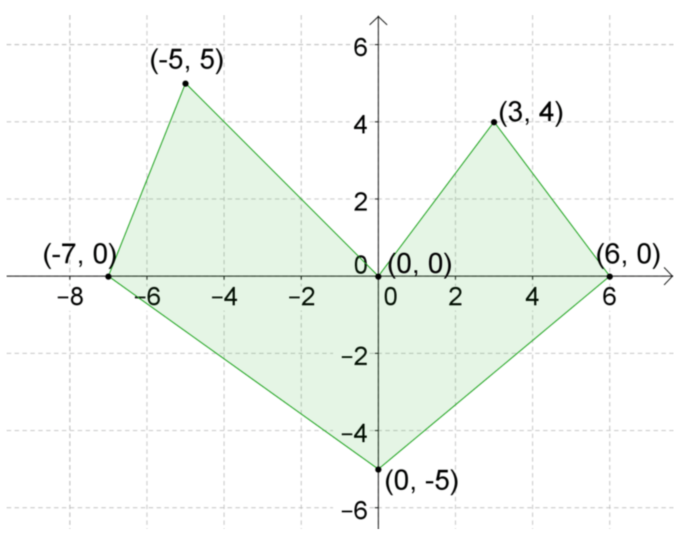
4.

$\boxed{z = 10x + 20y}$



5.

$\boxed{z = 20x - 15y + 4}$



A small manufacturing company makes $125 on each DVD player it produces and $100 profit on each color TV set it makes. Each DVD player and each TV must be processed by a cutting machine (A), a fitting machine (B) and a polishing machine (C). Each DVD player must be processed on Machine A for one hour, on Machine B for one hour and on Machine C for four hours. Each TV set must be processed on Machine A for two hours, on Machine B for one hour and on Machine C for one hour. Machines A, B, and C are available for 16, 9, and 24 hours per day respectively. How many DVD players and TV sets must be made each day to maximize the profit?

6. List the constraints and state the profit equation.
7. Create a graph and identify the feasible region.
8. Determine what the company must do to maximize their profit.

April has a small business during the winter months making hats and scarves. A hat requires 2 hours on Machine A, 4 hours on Machine B and 2 hours on Machine C. A scarf requires 3 hours on Machine A, 3 hours on Machine B and 1 hour on Machine C. Machine A is available 36 hours each week, Machine B is available 42 hours each week and Machine C is available 20 hours each week. The profit on a hat is $7.00 and the profit on a scarf is $4.00. How many of each should be made each week to maximize the profit?

9. List the constraints and state the profit equation.
10. Create a graph and identify the feasible region.
11. Determine what the April must do to maximize her profit.

Beth is knitting mittens and gloves. Each pair must be processed on three machines. Each pair of mittens requires 2 hours on Machine A, 2 hours on Machine B and 4 hours on Machine C. Each pair of gloves requires 4 hours on Machine A, 2 hours on Machine B and 1 hour on Machine C. Machine A, B, and C are available 32, 18 and 24 minutes each day respectively. The profit on a pair of mittens is $8.00 and on a pair of gloves is $10.00. How many pairs of each should be made each day to maximize the profit?

12. List the constraints and state the profit equation.
13. Create a graph and identify the feasible region.
14. Determine what the Beth must do to maximize her profit.

A patient is prescribed a pill that contains vitamins A, B and C. These vitamins are available in two different brands of pills. The first type is called Brand X and the second type is called Brand Y. The following table shows the amount of each vitamin that a Brand X and a Brand Y pill contain. The table also shows the minimum daily requirement needed by the patient. Each Brand X pill costs $0.32 and each Brand Y pill costs $0.29. How many pills of each brand should the patient take each day to minimize the cost?

TABLE 5.5:

	Brand X	**Brand Y**	**Minimum Daily Requirement**
Vitamin A	**2mg**	**1mg**	**5mg**
Vitamin B	**3mg**	**3mg**	**12mg**
Vitamin C	**25mg**	**50mg**	**125mg**

15. List the constraints and state the cost equation.
16. Create a graph and identify the feasible region.
17. Determine what the patient must do to minimize his/her cost.

A local smelting company is able to provide its customers with lead, copper and iron by melting down either of two ores, X or Y. The ores arrive at the company in railroad cars. Each railroad car of ore X contains 5 tons of lead, 1 ton of copper and 1 ton of iron. Each railroad car of ore Y contains 1 ton of lead, 1 ton of copper and 2 tons of iron. The smelting company receives an order and must make at least 20 tons of lead, 12 tons of copper and 20 tons of iron. The cost to purchase and process a carload of ore X is $6000 while the cost for ore Y is $5000. If the company wants to fill the order at a minimum cost, how many carloads of each ore must be bought?

18. List the constraints and state the cost equation.
19. Create a graph and identify the feasible region.
20. Determine what the company must do to minimize their cost.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 5.7.

Summary

You learned that a *system* of equations or inequalities means more than one equation or inequality.

To solve a system of equations you can graph the system and look for the point of intersection or use one of two algebraic methods (substitution or elimination). Sometimes a system of equations has no solution because the two lines are parallel. Other times the system has an infinite number of solutions because the lines coincide.

A linear inequality appears as a region in the Cartesian plane. To solve a system of linear inequalities, graph both and look for where their solution regions overlap. This region is often called the feasible region. Systems of linear inequalities can help you solve problems where you have multiple constraints on different variables and you are trying to figure out how to maximize or minimize something (like profit or cost). The maximum or minimum values will occur at one of the vertices of the feasible region according to the Vertex Theorem. These types of problems are sometimes referred to as linear programming problems.

CHAPTER 6 Exponents and Exponential Functions

Chapter Outline

Introduction

Here you'll learn all about *exponents* in algebra. You will learn the properties of exponents and how to simplify exponential expressions. You will learn how exponents can help you write very large or very small numbers with scientific notation. You will also learn how to solve different types of exponential equations where the variable appears as the exponent or the base. Finally, you will explore different types of exponential functions of the form $y = b^x, y = ab^x, y = ab^{\frac{x}{c}}$, and $y = ab^{\frac{x}{c}} + d$ as well as applications of exponential functions.

6.1 Product Rules for Exponents

Here you'll learn how to multiply two terms with the same base and how to find the power of a product.

Suppose you have the expression:

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot x \cdot x$

How could you write this expression in a more concise way?

Product Rules for Exponents

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn two rules that have to do with exponents and products.

RULE: To multiply two terms with the same base, add the exponents.

$$a^m \times a^n = \underbrace{(a \times a \times \ldots \times a)}_{\substack{\downarrow \\ m \text{ factors}}} \underbrace{(a \times a \times \ldots \times a)}_{\substack{\downarrow \\ n \text{ factors}}}$$

$$a^m \times a^n = \underbrace{(a \times a \times a \ldots \times a)}_{\substack{\downarrow \\ m+n \text{ factors}}}$$

$$a^m \times a^n = a^{m+n}$$

RULE: To raise a product to a power, raise each of the factors to the power.

$$(ab)^n = \underbrace{(ab) \times (ab) \times \ldots \times (ab)}_{\substack{\downarrow \\ n \text{ factors}}}$$

$$(ab)^n = \underbrace{(a \times a \times \ldots \times a)}_{\substack{\downarrow \\ n \text{ factors}}} \times \underbrace{(b \times b \times \ldots \times b)}_{\substack{\downarrow \\ n \text{ factors}}}$$

$$(ab)^n = a^n b^n$$

Simplify

$3^2 \times 3^3$

$3^2 \times 3^3$	The base is 3.
3^{2+3}	Keep the base of 3 and add the exponents.
3^5	This answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated.

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3$$
$$3^5 = 243$$
$$\boxed{3^2 \times 3^3 = 3^5 = 243}$$

Simplify

Simplify $(x^3)(x^6)$.

$(x^3)(x^6)$	The base is x.
x^{3+6}	Keep the base of x and add the exponents.
x^9	The answer is in exponential form.
$\boxed{(x^3)(x^6) = x^9}$	

Simplify

Simplify $y^5 \cdot y^2$.

$y^5 \cdot y^2$	The base is y.
y^{5+2}	Keep the base of y and add the exponents.
y^7	The answer is in exponential form.
$\boxed{y^5 \cdot y^2 = y^7}$	

Simplify

Simplify $5x^2y^3 \cdot 3xy^2$.

$5x^2y^3 \cdot 3xy^2$	The bases are x and y.
$15(x^2y^3)(xy^2)$	Multiply the coefficients - $5 \times 3 = 15$. Keep the base of x and y and add the exponents of the same base. If a base does not have a written exponent, it is understood as 1.
$15x^{2+1}y^{3+2}$	
$15x^3y^5$	The answer is in exponential form.
$\boxed{5x^2y^3 \cdot 3xy^2 = 15x^3y^5}$	

Examples

Example 1

Earlier, you were asked to write the expression in a more concise way.

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot x \cdot x$ can be rewritten as $x^9y^5x^4$. Then, you can use the rules of exponents to simplify the expression to $x^{13}y^5$. This is certainly much quicker to write!

Example 2

Simplify the following expression.

$(-3x)^2$

$9x^2$. Here are the steps:

$$\begin{aligned}(-3x)^2 &= (-3)^2 \cdot (x)^2 \\ &= 9x^2\end{aligned}$$

Example 3

Simplify the following expression.

$(5xy)^3$

$125x^3y^3$. Here are the steps:

$$\begin{aligned}(5x^2y^4)^3 &= (5)^3 \cdot (x)^3 \cdot (y)^3 \\ &= 125x^3y^3\end{aligned}$$

Example 4

$(2^3 \cdot 3^2)^2$

5184. Here are the steps:

$$
\begin{aligned}
(2^3 \cdot 3^2)^2 &= (8 \cdot 9)^2 \\
&= (72)^2 \\
&= 5184
\end{aligned}
$$

OR

$$
\begin{aligned}
(2^3 \cdot 3^2)^2 &= (8 \cdot 9)^2 \\
&= 8^2 \cdot 9^2 \\
&= 64 \cdot 81 \\
&= 5184
\end{aligned}
$$

Review

Simplify each of the following expressions, if possible.

1. $4^2 \times 4^4$
2. $x^4 \cdot x^{12}$
3. $(3x^2y^4)(9xy^5z)$
4. $(2xy)^2(4x^2y^3)$
5. $(3x)^5(2x)^2(3x^4)$
6. $x^3y^2z \cdot 4xy^2z^7$
7. $x^2y^3 + xy^2$
8. $(0.1xy)^4$
9. $(xyz)^6$
10. $2x^4(x^2 - y^2)$
11. $3x^5 - x^2$
12. $3x^8(x^2 - y^4)$

Expand and then simplify each of the following expressions.

13. $(x^5)^3$
14. $(x^6)^8$
15. $(x^a)^b$ *Hint: Look for a pattern in the previous two problems.*

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.1.

6.2 Quotient Rules for Exponents

Here you'll learn how to divide two terms with the same base and find the power of a quotient.

Suppose you have the expression:

$\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}$

How could you write this expression in a more concise way?

Quotient Rules for Exponents

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn two rules that have to do with exponents and quotients.

RULE: To divide two powers with the same base, subtract the exponents.

$$\frac{a^m}{a^n} = \frac{\overbrace{(a \times a \times \ldots \times a)}^{m \text{ factors}}}{\underbrace{(a \times a \times \ldots \times a)}_{n \text{ factors}}} \quad m > n; a \neq 0$$

$$\frac{a^m}{a^n} = \underbrace{(a \times a \times \ldots \times a)}_{m-n \text{ factors}}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

RULE: To raise a quotient to a power, raise both the numerator and the denominator to the power.

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \times \frac{a}{b} \times \ldots \times \frac{a}{b}}_{\substack{\downarrow \\ n \text{ factors}}}$$

$$\left(\frac{a}{b}\right)^n = \frac{\overbrace{(a \times a \times \ldots \times a)}^{\substack{n \text{ factors} \\ \uparrow}}}{\underbrace{(b \times b \times \ldots \times b)}_{\substack{\downarrow \\ n \text{ factors}}}}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\ (b \neq 0)$$

Simplify

Simplify $2^7 \div 2^3$.

$2^7 \div 2^3$	The base is 2.
2^{7-3}	Keep the base of 2 and subtract the exponents.
2^4	The answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated.

$$2^4 = 2 \times 2 \times 2 \times 2$$
$$2^4 = 16$$
$$\boxed{2^7 \div 2^3 = 2^4 = 16}$$

Simplify

Simplify $\frac{x^8}{x^2}$.

$\frac{x^8}{x^2}$	The base is x.
x^{8-2}	Keep the base of x and subtract the exponents.
x^6	The answer is in exponential form.

$$\boxed{\frac{x^8}{x^2} = x^6}$$

Simplify

Simplify $\frac{16x^5y^5}{4x^2y^3}$.

$\frac{16x^5y^5}{4x^2y^3}$ The bases are x and y.

$4\left(\frac{x^5y^5}{x^2y^3}\right)$ Divide the coefficients - $16 \div 4 = 4$. Keep the base of x and y and subtract the exponents of the same base.

$4x^{5-2}y^{5-3}$

$4x^3y^2$

Examples

Example 1

Earlier, you were asked to simplify an expression.

$\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}$ can be rewritten as $\frac{x^9y^5}{x^6y^3}$ and then simplified to x^3y^2.

Example 2

Simplify the following expression.

$\left(\frac{2}{3}\right)^2$

$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$

Example 3

Simplify the following expression.

$\left(\frac{x}{6}\right)^3$

$\left(\frac{x}{6}\right)^3 = \frac{x^3}{6^3} = \frac{x^3}{216}$

Example 4

Simplify the following expression.

$\left(\frac{3x}{4y}\right)^2$

$\left(\frac{3x}{4y}\right)^2 = \frac{3^2x^2}{4^2y^2} = \frac{9x^2}{16y^2}$

Review

Simplify each of the following expressions, if possible.

1. $\left(\frac{2}{5}\right)^6$
2. $\left(\frac{4}{7}\right)^3$
3. $\left(\frac{x}{y}\right)^4$

4. $\frac{20x^4y^5}{5x^2y^4}$
5. $\frac{42x^2y^8z^2}{6xy^4z}$
6. $\left(\frac{3x}{4y}\right)^3$
7. $\frac{72x^2y^4}{8x^2y^3}$
8. $\left(\frac{x}{4}\right)^5$
9. $\frac{24x^{14}y^8}{3x^5y^7}$
10. $\frac{72x^3y^9}{24xy^6}$
11. $\left(\frac{7}{y}\right)^3$
12. $\frac{20x^{12}}{-5x^8}$

13. Simplify using the laws of exponents: $\frac{2^3}{2^5}$
14. Evaluate the numerator and denominator separately and then simplify the fraction: $\frac{2^3}{2^5}$
15. Use your result from the previous problem to determine the value of a: $\frac{2^3}{2^5} = \frac{1}{2^a}$
16. Use your results from the previous three problems to help you evaluate 2^{-4}.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.2.

6.3 Power Rule for Exponents

Here you'll learn how to find the power of a power.

Can you simplify an expression where an exponent has an exponent? For example, how would you simplify $[(2^3)^2]^4$?

Power Rule for Exponents

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn a rule that has to do with raising a power to another power.

RULE: To raise a power to a new power, multiply the exponents.

$$(a^m)^n = (\underbrace{a \times a \times \ldots \times a}_{m \text{ factors}})^n$$

$$(a^m)^n = \underbrace{\underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \times \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}}}_{n \textit{ times}}$$

$$(a^m)^n = \underbrace{a \times a \times a \ldots \times a}_{mn \text{ factors}}$$

$$(a^m)^n = a^{mn}$$

Evaluate the expression

Evaluate $(2^3)^2$.

$(2^3)^2 = 2^6 = 64$.

Simplify the expression

Simplify $(x^7)^4$.

$(x^7)^4 = x^{28}$

Evaluate the expression

Evaluate $(3^2)^3$.

$(3^2)^3 = 3^6 = 729$.

Simplify the expression

Simplify $(x^2y^4)^2 \cdot (xy^4)^3$.

$(x^2y^4)^2 \cdot (xy^4)^3 = x^4y^8 \cdot x^3y^{12} = x^7y^{20}$.

Examples

Example 1

Earlier, you were asked if you can simplify an expression where an exponent has an exponent.

$[(2^3)^2]^4 = [2^6]^4 = 2^{24}$. Notice that the power rule applies even when a number has been raised to more than one power. The overall exponent is 24 which is $3 \cdot 2 \cdot 4$.

You know you can rewrite 2^4 as $2 \times 2 \times 2 \times 2$ and then calculate in order to find that

$$\boxed{2^4 = 16}$$

. This concept can also be reversed. To write 32 as a power of 2, $32 = 2 \times 2 \times 2 \times 2 \times 2$. There are 5 twos; therefore,

$$\boxed{32 = 2^5}$$

. Use this idea to complete the following problems.

Example 2

Write 81 as a power of 3.

$81 = 3 \times 3 = 9 \times 3 = 27 \times 3 = 81$

There are 4 threes. Therefore

$$\boxed{81 = 3^4}$$

Example 3

Write $(9)^3$ as a power of 3.

$9 = 3 \times 3 = 9$

There are 2 threes. Therefore

$$\boxed{9 = 3^2}$$

.

$(3^2)^3$ Apply the law of exponents for power to a power-multiply the exponents.
$3^{2\times 3} = 3^6$
Therefore

$$\boxed{(9)^3 = 3^6}$$

Example 4

Write $(4^3)^2$ as a power of 2.

$4 = 2 \times 2 = 4$

There are 2 twos. Therefore

$$\boxed{4 = 2^2}$$

$\left((2^2)^3\right)^2$ Apply the law of exponents for power to a power-multiply the exponents.

$$\boxed{2^{2\times 3} = 2^6}$$

$(2^6)^2$ Apply the law of exponents for power to a power-multiply the exponents.

$$\boxed{2^{6\times 2} = 2^{12}}$$

Therefore

$$\boxed{(4^3)^2 = 2^{12}}$$

Review

Simplify each of the following expressions.

1. $\left(\frac{x^4}{y^3}\right)^5$
2. $\frac{(5x^2y^4)^5}{(5xy^2)^3}$
3. $\frac{x^8y^9}{(x^2y)^3}$
4. $(x^2y^4)^3$
5. $(3x^2)^2 \cdot (4xy^4)^2$
6. $(2x^3y^5)(5x^2y)^3$
7. $(x^4y^6z^2)^2(3xyz)^3$
8. $\left(\frac{x^2}{2y^3}\right)^4$
9. $\frac{(4xy^3)^4}{(2xy^2)^3}$
10. True or false: $(x^2 + y^3)^2 = x^4 + y^6$
11. True or false: $(x^2y^3)^2 = x^4y^6$
12. Write 64 as a power of 4.
13. Write $(16)^3$ as a power of 2.
14. Write $(9^4)^2$ as a power of 3.
15. Write $(81)^2$ as a power of 3.
16. Write $(25^3)^4$ as a power of 5.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.3.

6.4 Zero and Negative Exponents

Here you'll learn how to work with zero and negative exponents.

How can you use the quotient rules for exponents to understand the meaning of a zero or negative exponent?

Zero and Negative Exponents

Zero Exponent

Recall that

$$\boxed{\frac{a^m}{a^n} = a^{m-n}}$$

. If $m = n$, then the following would be true:

$$\frac{a^m}{a^n} = a^{m-n} = a^0$$
$$\frac{3^3}{3^3} = 3^{3-3} = 3^0$$

However, any quantity divided by itself is equal to one. Therefore, $\frac{3^3}{3^3} = 1$ which means $3^0 = 1$. This is true in general:

$$\boxed{a^0 = 1 \text{ if } a \neq 0.}$$

Note that if $a = 0$, 0^0 is not defined.

Negative Exponents

$4^2 \times 4^{-2} = 4^{2+(-2)} = 4^0 = 1$

Therefore:

$$4^2 \times 4^{-2} = 1$$

$$\frac{4^2 \times 4^{-2}}{4^2} = \frac{1}{4^2} \qquad \text{Divide both sides by } 4^2.$$

$$\frac{\cancel{4^2} \times 4^{-2}}{\cancel{4^2}} = \frac{1}{4^2} \qquad \text{Simplify the equation.}$$

$$\boxed{4^{-2} = \frac{1}{4^2}}$$

This is true in general and creates the following laws for negative exponents:

- $$\boxed{a^{-m} = \frac{1}{a^m}}$$
- $$\boxed{\frac{1}{a^{-m}} = a^m}$$

These laws for negative exponents can be expressed in many ways:

- If a term has a negative exponent, write it as 1 over the term with a positive exponent. For example: $a^{-m} = \frac{1}{a^m}$ and $\frac{1}{a^{-m}} = a^m$
- If a term has a negative exponent, write the reciprocal with a positive exponent. For example: $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$ and $a^{-m} = \frac{a^{-m}}{1} = \frac{1}{a^m}$
- If the term is a factor in the numerator with a negative exponent, write it in the denominator with a positive exponent. For example: $3x^{-3}y = \frac{3y}{x^3}$ and $a^{-m}b^n = \frac{1}{a^m}(b^n) = \frac{b^n}{a^m}$
- If the term is a factor in the denominator with a negative exponent, write it in the numerator with a positive exponent. For example: $\frac{2x^3}{x^{-2}} = 2x^3(x^2)$ and $\frac{b^n}{a^{-m}} = b^n\left(\frac{a^m}{1}\right) = b^n a^m$

These ways for understanding negative exponents provide shortcuts for arriving at solutions without doing tedious calculations. The results will be the same.

Evaluate the exponents

Evaluate the following using the laws of exponents.

$\left(\frac{3}{4}\right)^{-2}$

There are two methods that can be used to evaluate the expression.

Method 1: Apply the negative exponent rule

$$\boxed{a^{-m} = \frac{1}{a^m}}$$

$\left(\frac{3}{4}\right)^{-2} = \frac{1}{\left(\frac{3}{4}\right)^2}$ Write the expression with a positive exponent by applying $\boxed{a^{-m} = \frac{1}{a^m}}$.

$\frac{1}{\left(\frac{3}{4}\right)^2} = \frac{1}{\frac{3^2}{4^2}}$ Apply the law of exponents for raising a quotient to a power. $\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$

$\frac{1}{\frac{3^2}{4^2}} = \frac{1}{\frac{9}{16}}$ Evaluate the powers.

$\frac{1}{\frac{9}{16}} = 1 \div \frac{9}{16}$ Divide

$1 \div \frac{9}{16} = 1 \times \frac{16}{9} = \frac{16}{9}$

$$\boxed{\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}}$$

Method 2: Apply the shortcut and write the reciprocal with a positive exponent.

$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^{2}$ Write the reciprocal with a positive exponent.

$\left(\frac{4}{3}\right)^{2} = \frac{4^2}{3^2}$ Apply the law of exponents for raising a quotient to a power. $\boxed{\left(\frac{a}{b}\right)^{n} = \frac{a^n}{b^n}}$

$\frac{4^2}{3^2} = \frac{16}{9}$ Simplify.

$$\boxed{\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}}$$

Applying the shortcut facilitates the process for obtaining the solution.

State the following using only positive exponents: (If possible, use shortcuts)

i) y^{-6}

y^{-6} Write the expression with a positive exponent by applying $\boxed{a^{-m} = \frac{1}{a^m}}$.

$$\boxed{y^{-6} = \frac{1}{y^6}}$$

ii) $\left(\frac{a}{b}\right)^{-3}$

$\left(\frac{a}{b}\right)^{-3}$ Write the reciprocal with a positive exponent.

$\left(\frac{a}{b}\right)^{-3} = \left(\frac{b}{a}\right)^{3}$ Apply the law of exponents for raising a quotient to a power. $\boxed{\left(\frac{a}{b}\right)^{n} = \frac{a^n}{b^n}}$

$\left(\frac{b}{a}\right)^{3} = \frac{b^3}{a^3}$

$$\boxed{\left(\frac{a}{b}\right)^{-3} = \frac{b^3}{a^3}}$$

iii) $\frac{x^5}{y^{-4}}$

$\frac{x^5}{y^{-4}}$ Apply the negative exponent rule. $\boxed{\frac{1}{a^{-m}} = a^m}$

$\frac{x^5}{y^{-4}} = x^5\left(\frac{y^4}{1}\right)$ Simplify.

$$\boxed{\frac{x^5}{y^{-4}} = x^5y^4}$$

iv) $a^2 \times a^{-5}$

$a^2 \times a^{-5}$ Apply the product rule for exponents $\boxed{a^m \times a^n = a^{m+n}}$.

$a^2 \times a^{-5} = a^{2+(-5)}$ Simplify.

$a^{2+(-5)} = a^{-3}$ Write the expression with a positive exponent by applying $\boxed{a^{-m} = \frac{1}{a^m}}$.

$a^{-3} = \frac{1}{a^3}$

$$\boxed{a^2 \times a^{-5} = \frac{1}{a^3}}$$

Evaluate

Evaluate the following: $\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}}$

There are two methods that can be used to evaluate the problem.

Method 1: Work with the terms in the problem in exponential form.

Numerator:

$7^{-2} = \frac{1}{7^2}$ and $7^{-1} = \frac{1}{7}$ Apply the definition $a^{-m} = \frac{1}{a^m}$

$\frac{1}{7^2} + \frac{1}{7}$ A common denominator is needed to add the fractions.

$\frac{1}{7^2} + \frac{1}{7}\left(\frac{7}{7}\right)$ Multiply $\frac{1}{7}$ by $\frac{7}{7}$ to obtain the common denominator of 7^2

$\frac{1}{7^2} + \frac{7}{7^2} = \frac{1+7}{7^2} = \frac{8}{7^2}$ Add the fractions.

Denominator:

$7^{-3} = \frac{1}{7^3}$ and $7^{-4} = \frac{1}{7^4}$ Apply the definition $a^{-m} = \frac{1}{a^m}$

$\frac{1}{7^3} + \frac{1}{7^4}$ A common denominator is needed to add the fractions.

$\left(\frac{7}{7}\right)\frac{1}{7^3} + \frac{1}{7^4}$ Multiply $\frac{1}{7^3}$ by $\frac{7}{7}$ to obtain the common denominator of 7^4

$\frac{7}{7^4} + \frac{1}{7^4} = \frac{1+7}{7^4} = \frac{8}{7^4}$ Add the fractions.

Numerator and Denominator:

$\frac{8}{7^2} \div \frac{8}{7^4}$ Divide the numerator by the denominator.

$\frac{8}{7^2} \times \frac{7^4}{8}$ Multiply by the reciprocal.

$\frac{\cancel{8}}{7^2} \times \frac{7^4}{\cancel{8}} = \frac{7^4}{7^2} = 7^2 = 49$ Simplify.

$$\boxed{\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}} = 49}$$

Method 2: Multiply the numerator and the denominator by 7^4. This will change all negative exponents to positive exponents. Apply the product rule for exponents and work with the terms in exponential form.

$\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}}$

$\left(\frac{7^4}{7^4}\right)\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}}$ Apply the distributive property with the product rule for exponents.

$\frac{7^2+7^3}{7^1+7^0}$ Evaluate the numerator and the denominator.

$\frac{49+343}{7+1} = \frac{392}{8} = 49$

$$\boxed{\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}} = 49}$$

Whichever method is used, the result is the same.

Examples

Example 1

Earlier, you were asked how can you use the quotient rules for exponents to understand the meaning of a zero or negative exponent?

By the quotient rule for exponents, $\frac{x^m}{x^m} = x^{m-m} = x^0$. Since anything divided by itself is equal to 1 (besides 0), $\frac{x^m}{x^m} = 1$. Therefore, $x^0 = 1$ as long as $x \neq 0$.

Also by the quotient rule for exponents, $\frac{x^2}{x^5} = x^{2-5} = x^{-3}$. If you were to expand and reduce the original expression you would have $\frac{x^2}{x^5} = \frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^3}$. Therefore, $x^{-3} = \frac{1}{x^3}$. This generalizes to $x^{-a} = \frac{1}{x^a}$.

Example 2

Use the laws of exponents to simplify the following: $(-3x^2)^3(9x^4y)^{-2}$

$(-3x^2)^3(9x^4y)^{-2}$ Apply the laws of exponents $\boxed{(a^m)^n = a^{mn}}$ and $\boxed{a^{-m} = \frac{1}{a^m}}$

$(-3x^2)^3(9x^4y)^{-2} = (-3^3x^6)\left(\frac{1}{(9x^4y)^2}\right)$ Simplify and apply $\boxed{(ab)^n = a^nb^n}$

$(-3^3x^6)\left(\frac{1}{(9x^4y)^2}\right) = -27x^6\left(\frac{1}{(9^2x^8y^2)}\right)$ Simplify.

$-27x^6\left(\frac{1}{(9^2x^8y^2)}\right) = \frac{-27x^6}{81x^8y^2}$ Simplify and apply the quotient rule for exponents $\boxed{\frac{a^m}{a^n} = a^{m-n}}$.

$\frac{-27x^6}{81x^8y^2} = -\frac{1x^{-2}}{3y^2}$ Apply the negative exponent rule $\boxed{a^{-m} = \frac{1}{a^m}}$

$$\boxed{(-3x^2)^3(9x^4y)^{-2} = -\frac{1}{3x^2y^2}}$$

Example 3

Rewrite the following using only positive exponents. $(x^2y^{-1})^2$

$$\begin{aligned}(x^2y^{-1})^2 &= x^4y^{-2}\\ &= \frac{x^4}{y^2}\end{aligned}$$

Example 4

Use the laws of exponents to evaluate the following: $[5^{-4} \times (25)^3]^2$

Try to do this one by applying the laws of exponents.

$[5^{-4} \times (25)^3]^2$

$[5^{-4} \times (25)^3]^2 = [5^{-4} \times (5^2)^3]^2$

$[5^{-4} \times (5^2)^3]^2 = [5^{-4} \times 5^6]^2$

$[5^{-4} \times 5^6]^2 = (5^2)^2$

$(5^2)^2 = 5^4$

$5^4 = 625$

$$\boxed{[5^{-4} \times (25)^3]^2 = 5^4 = 625}$$

Review

Evaluate each of the following expressions:

1. $-\left(\frac{2}{3}\right)^0$
2. $\left(-\frac{2}{5}\right)^{-2}$
3. $(-3)^{-3}$

4. $6 \times \left(\frac{1}{2}\right)^{-2}$
5. $7^{-4} \times 7^{4}$

Rewrite the following using positive exponents only. Simplify where possible.

6. $(4wx^{-2}y^{3}z^{-4})^{3}$
7. $\frac{a^{2}b^{3}c^{-2}}{d^{-2}bc^{-6}}$
8. $x^{-2}(x^{2}-1)$
9. $m^{4}(m^{2}+m-5m^{-2})$
10. $\frac{x^{-2}y^{-2}}{x^{-1}y^{-1}}$
11. $\left(\frac{x^{-2}}{y^{4}}\right)^{3}\left(\frac{y^{-4}}{x^{6}}\right)^{-7}$
12. $\frac{(x^{-2}y^{4})^{2}}{(x^{5}y^{-3})^{4}}$
13. $\frac{(3xy^{2})^{3}}{(3x^{2}y)^{4}}$
14. $\left(\frac{x^{2}y^{-25}z^{5}}{-12.4x^{3}y}\right)^{0}$
15. $\left(\frac{x^{-2}}{y^{3}}\right)^{5}\left(\frac{y^{-2}}{x^{4}}\right)^{-3}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.4.

6.5 Fractional Exponents

Here you'll learn how to work with exponents that are fractions.

If an exponent usually tells you the number of times to multiply the base by itself, what does it mean if the exponent is a fraction? How can you think about and calculate $4^{\frac{3}{2}}$?

Fractional Exponents

A fraction exponent is related to a root. Raising a number to the power of $\frac{1}{2}$ is the same as taking the square root of the number. If you have $a^{\frac{m}{n}}$, you can think about this expression in multiple ways:

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} \quad \text{or} \quad a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad \text{or} \quad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

All of these ideas can be summarized as the following rule for fractional exponents:

$$\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \; m,n \in N}$$

Simplify the exponent

Simplify the following:

$(125)^{-\frac{2}{3}}$

$(125)^{-\frac{2}{3}}$ Apply the law of exponents for negative exponents $\boxed{a^{-m} = \frac{1}{a^m}}$.

$\frac{1}{125^{\frac{2}{3}}}$ Apply the law of exponents for rational exponents $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \; m,n \in N.}$

$\frac{1}{\left(\sqrt[3]{125}\right)^2}$

The cube root of 125 is '**5**'.

$\frac{1}{5^2}$

Evaluate the denominator.

$$\frac{1}{25}$$

$$\boxed{(125)^{-\frac{2}{3}} = \frac{1}{25}}$$

Simplify the exponent

Simplify the following:

$(2a^2b^4)^{\frac{3}{2}}$

$(2a^2b^4)^{\frac{3}{2}}$ Apply the law of exponents for raising a power to a power $\boxed{(a^m)^n = a^{mn}}$.

$(2a^2b^4)^{\frac{3}{2}} = 2^{1\times\frac{3}{2}}(a^2)^{\frac{3}{2}}(b^4)^{\frac{3}{2}}$ Simplify the expression.

$2^{1\times\frac{3}{2}}(a^2)^{\frac{3}{2}}(b^4)^{\frac{3}{2}} = 2^{\frac{3}{2}}(a)^{2\times\frac{3}{2}}(b)^{4\times\frac{3}{2}}$ Simplify. Apply the rule for rational exponents $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \; m,n \in N}$.

$2^{\frac{3}{2}}(a)^{\cancel{2}\times\frac{3}{\cancel{2}}}(b)^{\overset{2}{\cancel{4}}\times\frac{3}{\cancel{2}}} = \sqrt{2^3}(a)^3(b)^6$ Simplify

$\sqrt{2^3}(a)^3(b)^6 = \sqrt{8}a^3b^6$

$\sqrt{8}a^3b^6 = 2\sqrt{2}a^3b^6$

$\boxed{(2a^2b^4)^{\frac{3}{2}} = 2\sqrt{2}a^3b^6}$

State the following using radicals:

i) $2^{\frac{3}{8}}$

$2^{\frac{3}{8}} = \sqrt[8]{2^3} = \sqrt[8]{8}$

ii) $7^{-\frac{1}{5}}$

$7^{-\frac{1}{5}} = \frac{1}{\sqrt[5]{7}}$

iii) $3^{\frac{3}{4}}$

$3^{\frac{3}{4}} = \sqrt[4]{3^3} = \sqrt[4]{27}$

State the following using exponents:

i) $\sqrt[3]{7^2}$

$\sqrt[3]{7^2} = 7^{\frac{2}{3}}$

ii) $\frac{1}{\left(\sqrt[4]{5}\right)^3}$

$$\frac{1}{\left(\sqrt[4]{5}\right)^3}$$

$$\frac{1}{\left(\sqrt[4]{5}\right)^3} = \frac{1}{5^{\frac{3}{4}}}$$

$$\frac{1}{5^{\frac{3}{4}}} = 5^{-\frac{3}{4}}$$

iii) $\left(\sqrt[5]{a}\right)^2$

$\left(\sqrt[5]{a}\right)^2 = a^{\frac{2}{5}}$

Examples

Example 1

Earlier, you were asked what does it mean if the exponent is a fraction.

$4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8$

Example 2

Use the laws of exponents to evaluate the following: $9^{\frac{3}{2}} \div 36^{-\frac{1}{2}}$

$9^{\frac{3}{2}} \div 36^{-\frac{1}{2}}$

$$9^{\frac{3}{2}} \div 36^{-\frac{1}{2}} = \left(\sqrt{9}\right)^3 \div \frac{1}{36^{\frac{1}{2}}}$$

$$\left(\sqrt{9}\right)^3 \div \frac{1}{36^{\frac{1}{2}}} = (3)^3 \div \frac{1}{\sqrt{36}}$$ Simplify

$$(3)^3 \div \frac{1}{\sqrt{36}} = 27 \div \frac{1}{6}$$ Perform the indicated operation of division.

$$27 \div \frac{1}{6} = 27 \times \frac{6}{1} = 162$$

$$\boxed{9^{\frac{3}{2}} \div 36^{-\frac{1}{2}} = 162}$$

Example 3

Simplify the following using the laws of exponents. $(20a^2b^3c^{-1})^{\frac{3}{2}}$

$$(20a^2b^3c^{-1})^{\frac{3}{2}}$$ Apply the law of exponents $\boxed{(ab)^n = a^nb^n}$.

$$(20a^2b^3c^{-1})^{\frac{3}{2}} = 20^{1\times\frac{3}{2}}(a)^{2\times\frac{3}{2}}(b)^{3\times\frac{3}{2}}(c)^{-1\times\frac{3}{2}}$$ Simplify the exponents.

$$20^{1\times\frac{3}{2}}(a)^{2\times\frac{3}{2}}(b)^{3\times\frac{3}{2}}(c)^{-1\times\frac{3}{2}} = 20^{\frac{3}{2}}(a)^3(b)^{\frac{9}{2}}(c)^{-\frac{3}{2}}$$

$$20^{\frac{3}{2}}(a)^3(b)^{\frac{9}{2}}(c)^{-\frac{3}{2}} = \left(\sqrt{20}\right)^3(a)^3\sqrt{b^9}\left(\frac{1}{c^{\frac{3}{2}}}\right)$$ Simplify

$$20^{\frac{3}{2}}(a)^3(b)^{\frac{9}{2}}(c)^{-\frac{3}{2}} = \left(2\sqrt{5}\right)^3(a^3)\left(b^4\sqrt{b}\right)\left(\frac{1}{\sqrt{c^3}}\right)$$ Simplify

$$\left(2\sqrt{5}\right)^3(a^3)\left(b^4\sqrt{b}\right)\left(\frac{1}{\sqrt{c^3}}\right) = 8\sqrt{125}a^3b^4\sqrt{b}\frac{1}{c\sqrt{c}}$$

$$8\sqrt{125}a^3b^4\sqrt{b}\frac{1}{c\sqrt{c}} = 40\sqrt{5}a^3b^4\sqrt{b}\left(c\sqrt{c}\right)^{-1}$$ Simplify

$$\boxed{(20a^2b^3c^{-1})^{\frac{3}{2}} = 40\sqrt{5}a^3b^4\sqrt{b}\left(c\sqrt{c}\right)^{-1}}$$

Example 4

Use the laws of exponents to evaluate the following: $\frac{64^{\frac{2}{3}}}{216^{-\frac{1}{3}}}$

$\frac{64^{\frac{2}{3}}}{216^{-\frac{1}{3}}}$.

Numerator	**Denominator**
$64^{\frac{2}{3}}$	$216^{-\frac{1}{3}}$
$64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2$	$216^{-\frac{1}{3}} = \frac{1}{216^{\frac{1}{3}}}$
$\left(\sqrt[3]{64}\right)^2 = (4)^2$	$216^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{216}}$
$(4)^2 = 16$	$\frac{1}{\sqrt[3]{216}} = \frac{1}{6}$

Numerator divided by denominator:

$$16 \div \frac{1}{6}$$
$$16 \times \frac{6}{1} = 96$$
$$\boxed{\frac{64^{\frac{2}{3}}}{216^{-\frac{1}{3}}} = 96}$$

Review

Express each of the following as a radical and if possible, simplify.

1. $x^{\frac{1}{2}}$
2. $5^{\frac{3}{4}}$
3. $2^{\frac{3}{2}}$
4. $2^{-\frac{1}{2}}$
5. $9^{-\frac{1}{5}}$

Express each of the following using exponents:

6. $\sqrt{26}$
7. $\sqrt[3]{5^2}$
8. $(\sqrt[6]{a})^5$
9. $\sqrt[4]{m}$
10. $\left(\sqrt[3]{7}\right)^2$

Evaluate each of the following using the laws of exponents:

11. $3^{\frac{2}{5}} \times 3^{\frac{3}{5}}$
12. $(6^{0.4})^5$
13. $2^{\frac{1}{7}} \times 4^{\frac{3}{7}}$
14. $\left(\frac{64}{125}\right)^{-\frac{1}{2}}$
15. $(81^{-1})^{-\frac{1}{4}}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.5.

6.6 Exponential Expressions

Here you'll learn how to use all of the laws of exponents to simplify and evaluate exponential expressions.

Can you simplify the following expression so that it has only positive exponents?

$$\frac{8x^3y^{-2}}{(-4a^2b^4)^{-2}}$$

Exponential Expressions

The following table summarizes all of the rules for exponents.

Laws of Exponents

If $a \in R, a \geq 0$ and $m, n \in Q$, then

1. $a^m \times a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$ (if $m > n, a \neq 0$)
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $(b \neq 0)$
6. $a^0 = 1$ $(a \neq 0)$
7. $a^{-m} = \frac{1}{a^m}$
8. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Evaluate the expression

Evaluate $81^{-\frac{1}{4}}$.

First, rewrite with a positive exponent:

$81^{-\frac{1}{4}} = \frac{1}{81^{\frac{1}{4}}} = \left(\frac{1}{81}\right)^{\frac{1}{4}}$.

Next, evaluate the fractional exponent:

$\left(\frac{1}{81}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{1}{81}} = \frac{1}{3}$

Simplify the expression

Simplify $(4x^3y)(3x^5y^2)^4$.

$$\begin{aligned}(4x^3y)(3x^5y^2)^4 &= (4x^3y)(81x^{20}y^8)\\ &= 324x^{23}y^9\end{aligned}$$

Simplify the expression

Simplify $\left(\frac{x^{-2}y}{x^4y^3}\right)^{-2}$.

$$\begin{aligned}\left(\frac{x^{-2}y}{x^4y^3}\right)^{-2} &= \left(\frac{x^4y^3}{x^{-2}y}\right)^2 \\ &= (x^6y^2)^2 \\ &= x^{12}y^4\end{aligned}$$

Examples

Example 1

Earlier, you were asked to simplify an expression.

$$\begin{aligned}\frac{8x^3y^{-2}}{(-4x^2y^4)^{-2}} &= (8x^3y^{-2})(-4x^2y^4)^2 \\ &= (8x^3y^{-2})(16x^4y^8) \\ &= 8\cdot 16\cdot x^3\cdot x^4\cdot y^{-2}\cdot y^8 \\ &= 128x^7y^6\end{aligned}$$

Example 2

Use the laws of exponents to simplify the following:

$(-2x)^5(2x^2)$

$(-2x)^5(2x^2) = (-32x^5)(2x^2) = -64x^7$

Example 3

Use the laws of exponents to simplify the following:

$(16x^{10})\left(\frac{3}{4}x^5\right)$

$(16x^{10})\left(\frac{3}{4}x^5\right) = 12x^{15}$

Example 4

Use the laws of exponents to simplify the following:

$\frac{(x^{15})(x^{24})(x^{25})}{(x^7)^8}$

$\frac{(x^{15})(x^{24})(x^{25})}{(x^7)^8} = \frac{x^{64}}{x^{56}} = x^8$

Review

Simplify each expression.

1. $(x^{10})(x^{10})$
2. $(7x^3)(3x^7)$
3. $(x^3y^2)(xy^3)(x^5y)$
4. $\frac{(x^3)(x^2)}{(x^4)}$
5. $\frac{x^2}{x^{-3}}$
6. $\frac{x^6y^8}{x^4y^{-2}}$
7. $(2x^{12})^3$
8. $(x^5y^{10})^7$
9. $\left(\frac{2x^{10}}{3y^{20}}\right)^3$

Express each of the following as a power of 3. Do not evaluate.

10. $(3^3)^5$
11. $(3^9)(3^3)$
12. $(9)(3^7)$
13. 9^4
14. $(9)(27^2)$

Apply the laws of exponents to evaluate each of the following without using a calculator.

15. $(2^3)(2^2)$
16. $6^6 \div 6^5$
17. $-(3^2)^3$
18. $(1^2)^3 + (1^3)^2$
19. $\left(\frac{1}{3}\right)^6 \div \left(\frac{1}{3}\right)^8$

Use the laws of exponents to simplify each of the following.

20. $(4x)^2$
21. $(-3x)^3$
22. $(x^3)^4$
23. $(3x)(x^7)$
24. $(5x)(4x^4)$
25. $(-3x^2)(-6x^3)$
26. $(10x^8) \div (2x^4)$

Simplify each of the following using the laws of exponents.

27. $5^{\frac{1}{2}} \times 5^{\frac{1}{3}}$
28. $(d^4e^8f^{12})^{\frac{1}{4}}$
29. $\sqrt[4]{\frac{y^{\frac{1}{2}}\sqrt{xy}}{x^{\frac{2}{3}}}}$
30. $(32a^{20}b^{-15})^{\frac{1}{5}}$
31. $(729x^{12}y^{-6})^{\frac{2}{3}}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.6.

6.7 Scientific Notation

Here you'll learn about scientific notation.

Very large and very small quantities and measures are often used to provide information in magazines, textbooks, television, newspapers and on the Internet. Some examples are:

- The distance between the sun and Neptune is 4,500,000,000 km.
- The diameter of an electron is approximately 0.00000000000022 inches.

Scientific notation is a convenient way to represent such numbers. How could you write the numbers above using scientific notation?

Scientific Notation

To represent a number in scientific notation means to express the number as a product of two factors: a number between 1 and 10 (including 1) and a power of 10. A positive real number 'x' is said to be written in **scientific notation** if it is expressed as

$$x = a \times 10^n$$

where

$$1 \le a < 10 \text{ and } n \in Z.$$

In other words, a number in scientific notation is a single nonzero digit followed by a decimal point and other digits, all multiplied by a power of 10.

When working with numbers written in scientific notation, you can use the following rules. These rules are proved by example in Example B and Example C.

$$\boxed{(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n}$$

$$\boxed{(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n}$$

$$\boxed{(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})}$$

$$\boxed{(A \times 10^m) \div (B \times 10^n) = (A \div B) \times (10^{m-n})}$$

Write the following numbers using scientific notation:

i) 2,679,000

$$2,679,000 = 2.679 \times 1,000,000$$
$$2.679 \times 1,000,000 = 2.679 \times 10^{6}$$

The exponent, $n = 6$, represents the decimal point that is 6 places to the right of the **standard position of the decimal point.**

ii) 0.00005728

$$0.00005728 = 5.728 \times 0.00001$$
$$5.728 \times 0.00001 = 5.728 \times \frac{1}{100,000}$$
$$5.728 \times \frac{1}{100,000} = 5.728 \times \frac{1}{10^{5}}$$
$$5.728 \times \frac{1}{100,000} = 5.728 \times 10^{-5}$$

The exponent, $n = -5$, represents the decimal point that is 5 places to the left of the **standard position of the decimal point.**

One advantage of scientific notation is that calculations with large or small numbers can be done by **applying the laws of exponents.**

Complete the following table.

TABLE 6.1:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$1.3 \times 10^5 + 2.5 \times 10^5$			
$3.7 \times 10^{-2} + 5.1 \times 10^{-2}$			
$4.6 \times 10^4 - 2.2 \times 10^4$			
$7.9 \times 10^{-2} - 5.4 \times 10^{-2}$			

TABLE 6.2:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$1.3 \times 10^5 + 2.5 \times 10^5$	$130,000 + 250,000$	380,000	3.8×10^5
$3.7 \times 10^{-2} + 5.1 \times 10^{-2}$	$0.037 + 0.051$	0.088	8.8×10^{-2}

TABLE 6.2: (continued)

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$4.6 \times 10^4 - 2.2 \times 10^4$	$46,000 - 22,000$	24,000	2.4×10^4
$7.9 \times 10^{-2} - 5.4 \times 10^{-2}$	$0.079 - 0.054$	0.025	2.5×10^{-2}

Note that the numbers in the last column have the same power of 10 as those in the first column.

Complete the following table.

TABLE 6.3:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$(3.6 \times 10^2) \times (1.4 \times 10^3)$			
$(2.5 \times 10^3) \times (1.1 \times 10^{-6})$			
$(4.4 \times 10^4) \div (2.2 \times 10^2)$			
$(6.8 \times 10^{-4}) \div (3.2 \times 10^{-2})$			

TABLE 6.4:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$(3.6 \times 10^2) \times (1.4 \times 10^3)$	360×1400	504,000	5.04×10^5
$(2.5 \times 10^3) \times (1.1 \times 10^{-6})$	2500×0.0000011	0.00275	2.75×10^{-3}
$(4.4 \times 10^4) \div (2.2 \times 10^2)$	$44,000 \div 220$	200	2.0×10^2
$(6.8 \times 10^{-4}) \div (3.2 \times 10^{-2})$	$0.00068 \div 0.032$	0.02125	2.125×10^{-2}

Note that for multiplication, the power of 10 is the result of adding the exponents of the powers in the first column. For division, the power of 10 is the result of subtracting the exponents of the powers in the first column.

Calculate each of the following:

i) $4.6 \times 10^4 + 5.3 \times 10^5$

Before the rule

$$\boxed{(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n}$$

can be used, one of the numbers must be rewritten so that the powers of 10 are the same.

Rewrite 4.6×10^4

$4.6 \times 10^4 = (0.46 \times 10^1) \times 10^4$ The power 10^1 indicates the number of places to the right that the decimal point must be moved to return 0.46 to the original number of 4.6.

$(0.46 \times 10^1) \times 10^4 = 0.46 \times 10^5$ Add the exponents of the power.

Rewrite the question and substitute 4.6×10^4 with 0.46×10^5.

$0.46 \times 10^5 + 5.3 \times 10^5$

Apply the rule

$$\boxed{(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n}$$

.

$$(0.46 \times 10^5) + (5.3 \times 10^5) = (0.46 + 5.3) \times 10^5$$
$$(0.46 + 5.3) \times 10^5 = 5.76 \times 10^5$$
$$\boxed{4.6 \times 10^4 + 5.3 \times 10^5 = 5.76 \times 10^5}$$

ii) $4.7 \times 10^{-3} - 2.4 \times 10^{-4}$

Before the rule

$$\boxed{(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n}$$

can be used, one of the numbers must be rewritten so that the powers of 10 are the same.

Rewrite 4.7×10^{-3}

$4.7 \times 10^{-3} = (47 \times 10^{-1}) \times 10^{-3}$ The power 10^{-1} indicates the number of places to the left that the decimal point must be moved to return 47 to the original number of 4.7.

$(47 \times 10^{-1}) \times 10^{-3} = 47 \times 10^{-4}$ Add the exponents of the power.

Rewrite the question and substitute 4.7×10^{-3} with 47×10^{-4}.

$47 \times 10^{-4} - 2.4 \times 10^{-4}$

Apply the rule

$$\boxed{(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n}$$

.

$$(47 \times 10^{-4}) - (2.4 \times 10^{-4}) = (47 - 2.4) \times 10^{-4}$$
$$(47 \times 10^{-4}) - (2.4 \times 10^{-4}) = 44.6 \times 10^{-4}$$

The answer must be written in scientific notation.

$$44.6 \times 10^{-4} = (4.46 \times 10^1) \times 10^{-4} \qquad \text{Apply the law of exponents } - \text{ add the exponents of the power.}$$
$$4.46 \times 10 \times 10^{-4} = 4.46 \times 10^{-3}$$
$$\boxed{4.7 \times 10^{-3} - 2.4 \times 10^{-4} = 4.46 \times 10^{-3}}$$

iii) $(7.3 \times 10^5) \times (6.8 \times 10^4)$

$(7.3 \times 10^5) \times (6.8 \times 10^4)$

$7.3 \times 10^5 \times 6.8 \times 10^4$ Apply the rule $\boxed{(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})}$.

$(7.3 \times 10^5) \times (6.8 \times 10^4) = (7.3 \times 6.8) \times (10^{5+4})$

$(7.3 \times 6.8) \times (10^{5+4}) = (49.64) \times (10^9)$

$(49.64) \times (10^9) = 49.64 \times 10^9$ Write the answer in scientific notation.

$49.64 \times 10^9 = (4.964 \times 10^1) \times 10^9$ Apply the law of exponents – add the exponents of the power.

$49.64 \times 10^9 = 4.964 \times 10^{10}$

$\boxed{(7.3 \times 10^5) \times (6.8 \times 10^4) = 4.964 \times 10^{10}}$

iv) $(4.8 \times 10^9) \div (5.79 \times 10^7)$

$(4.8 \times 10^9) \div (5.79 \times 10^7)$

$(4.8 \times 10^9) \div (5.79 \times 10^7)$ Apply the rule $\boxed{(A \times 10^m) \div (B \times 10^n) = (A \div B) \times (10^{m-n})}$.

$(4.8 \times 10^9) \div (5.79 \times 10^7) = (4.8 \div 5.79) \times 10^{9-7}$ Apply the law of exponents – subtract the exponents of the power.

$(4.8 \div 5.79) \times 10^{9-7} = (0.829) \times 10^2$ Write the answer in scientific notation.

$(0.829) \times 10^2 = (8.29 \times 10^{-1}) \times 10^2$ Apply the law of exponents – add the exponents of the power.

$\boxed{(8.29 \times 10^{-1}) \times 10^2 = 8.29 \times 10^1}$

Examples

Example 1

Earlier, you were asked to write the distance between the sun and Neptune and the diameter of an electron in scientific notation.

The distance between the sun and Neptune would be written as 4.5×10^9 *km* and the diameter of an electron would be written as 2.2×10^{-13} *in*.

Example 2

Express the following product in scientific notation: $(4 \times 10^{12})(9.2 \times 10^7)$

Apply the rule

$$\boxed{(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})}$$

$$(4 \times 10^{12}) \times (9.2 \times 10^7) = (4 \times 9.2) \times (10^{12+7})$$
$$(4 \times 9.2) \times (10^{12+7}) = 36.8 \times 10^{19}$$

Express the answer in scientific notation.

$$36.8 \times 10^{19} = (3.68 \times 10^{1}) \times 10^{19}$$
$$(3.68 \times 10^{1}) \times 10^{19} = 3.68 \times 10^{20}$$
$$\boxed{(4 \times 10^{12})(9.2 \times 10^{7}) = 3.68 \times 10^{20}}$$

Example 3

Express the following quotient in scientific notation: $\frac{6,400,000}{0.008}$

Begin by expressing the numerator and the denominator in scientific notation.

$\frac{6.4 \times 10^{6}}{8.0 \times 10^{-3}}$

Apply the rule

$$\boxed{(A \times 10^{m}) \div (B \times 10^{n}) = (A \div B) \times (10^{m+n})}$$

.

$(6.4 \times 10^{6}) \div (8.0 \times 10^{-3}) = (6.4 \div 8.0) \times (10^{6--3})$ Apply the law of exponents − subtract the exponents of the powers.

$(6.4 \div 8.0) \times (10^{6--3}) = (0.8) \times (10^{9})$

$(0.8) \times (10^{9}) = 0.8 \times 10^{9}$ Express the answer in scientific notation.

$0.8 \times 10^{9} = (8.0 \times 10^{-1}) \times 10^{9}$

$0.8 \times 10^{9} = 8.0 \times 10^{-1} \times 10^{9}$ Apply the law of exponents − add the exponents of the powers.

$8.0 \times 10^{-1} \times 10^{9} = 8.0 \times 10^{8}$

$\boxed{\frac{6,400,000}{0.008} = 8.0 \times 10^{8}}$ Express the answer in scientific notation.

Example 4

If $a = 0.000415, b = 521$, and $c = 71,640$, find an approximate value for $\frac{ab}{c}$. Express the answer in scientific notation.

Express all values in scientific notation.

$$0.000415 = 4.15 \times 10^{-4}$$
$$521 = 5.21 \times 10^{2}$$
$$71,640 = 7.1640 \times 10^{4}$$

Use the values in scientific notation to determine an approximate value for $\frac{ab}{c}$.

$\frac{ab}{c} = \frac{(4.15 \times 10^{-4})(5.21 \times 10^{2})}{7.1640 \times 10^{4}}$

In the numerator, apply the rule

$$\boxed{(A \times 10^{m}) \times (B \times 10^{n}) = (A \times B) \times (10^{m+n})}$$

$$\frac{(4.15 \times 10^{-4})(5.21 \times 10^{2})}{7.1640 \times 10^{4}} = \frac{(4.15 \times 5.21) \times (10^{-4} \times 10^{2})}{7.1640 \times 10^{4}}$$
$$\frac{(4.15 \times 5.21) \times (10^{-4} \times 10^{2})}{7.1640 \times 10^{4}} = \frac{21.6215 \times 10^{-2}}{7.1640 \times 10^{4}}$$
Apply the rule $\boxed{(A \times 10^{m}) \div (B \times 10^{n}) = (A \div B) \times (10^{m-n})}$.
$$\frac{21.6215 \times 10^{-2}}{7.1640 \times 10^{4}} = (21.6215 \div 7.1640) \times (10^{-2-4})$$
$$\boxed{(21.6215 \div 7.1640) \times (10^{-2} \times 10^{4}) = 3.018 \times 10^{-6}}$$

Review

Express each of the following in scientific notation:

1. 42,000
2. 0.00087
3. 150.64
4. 56,789
5. 0.00947

Express each of the following in standard form:

6. 4.26×10^{5}
7. 8×10^{4}
8. 5.967×10^{10}
9. 1.482×10^{-6}
10. 7.64×10^{-3}

Perform the indicated operations and express the answer in scientific notation

11. $8.9 \times 10^{4} + 4.3 \times 10^{5}$
12. $8.7 \times 10^{-4} - 6.5 \times 10^{-5}$
13. $(5.3 \times 10^{6}) \times (7.9 \times 10^{5})$
14. $(3.9 \times 10^{8}) \div (2.8 \times 10^{6})$

For the given values, perform the indicated operations for $\frac{ab}{c}$ and express the answer in scientific notation and standard form.

15.

$$a = 76.1$$
$$b = 818,000,000$$
$$c = 0.000016$$

16.

$$
\begin{aligned}
a &= 9.13 \times 10^{9} \\
b &= 5.45 \times 10^{-23} \\
c &= 1.62
\end{aligned}
$$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.7.

6.8 Exponential Equations

Here you'll learn how to solve basic equations that contain exponents.

The following exponential equation is one in which the variable appears in the exponent.

$9^{x+1} = \sqrt{27}$

How can you solve this type of equation where you can't isolate the variable?

Exponential Equations

When an equation has exponents, sometimes the variable will be in the exponent and sometimes it won't. There are different strategies for solving each type of equation.

- ***When the variable is in the exponent:*** Rewrite each side of the equation so that the bases of the exponent are the same. Then, create a new equation where you set the exponents equal to each other and solve (see Example A).
- ***When the variable is not in the exponent:*** Manipulate the equation so the exponent is no longer there (see Example B). Or, rewrite each side of the equation so that both sides have the same exponent. Then, create a new equation where you set the bases equal to each other and solve (see Example C).

Solve the following exponential equation:

$25^{x-3} = \left(\frac{1}{5}\right)^{3x+18}$

The variable appears in the exponent. Write both sides of the equation as a power of 5.

$(5^2)^{x-3} = (5^{-1})^{3x+18}$

Apply the law of exponents for raising a power to a power

$$\boxed{(a^m)^n = a^{mn}}$$

.

$5^{2(x-3)} = 5^{-1(3x+18)}$ Simplify the exponents.

$5^{2x-6} = 5^{-3x-18}$ The bases are the same so the exponents are equal quantities.

$2x-6 = -3x-18$ Set the exponents equal to each other and solve the equation.

$$2x-6+6 = -3x-18+6$$
$$2x = -3x-12$$
$$2x+3x = -3x+3x-12$$
$$5x = -12$$
$$\frac{5x}{5} = \frac{-12}{5}$$
$$\frac{\cancel{5}x}{\cancel{5}} = \frac{-12}{5}$$
$$\boxed{x = \frac{-12}{5}}$$

Solve the following exponential equation:

$4(x-2)^{\frac{1}{2}} = 16$

The variable appears in the base.

$4(x-2)^{\frac{1}{2}} = 16$ Divide both sides of the equation by 4.

$$\frac{4(x-2)^{\frac{1}{2}}}{4} = \frac{16}{4}$$
$$\frac{\cancel{4}(x-2)^{\frac{1}{2}}}{\cancel{4}} = \frac{\overset{4}{\cancel{16}}}{\cancel{4}}$$

$(x-2)^{\frac{1}{2}} = 4$ Multiply the exponents on each side of the equation by the reciprocal of $\frac{1}{2}$.

$\left[(x-2)^{\frac{1}{2}}\right]^2$ Apply the law of exponents $\boxed{(a^m)^n = a^{mn}}$.

$(x-2)^{\frac{1}{2}\times 2} = (4)^{1\times 2}$ Simplify the exponents.

$(x-2)^1 = 4^2$

$x-2 = 16$ Solve the equation.

$$x-2+2 = 16+2$$
$$x = 18$$
$$\boxed{x = 18}$$

Solve the following exponential equation:

$(2x-4)^{\frac{2}{3}} = \sqrt[3]{9}$

The variable appears in the base.

$(2x-4)^{\frac{2}{3}} = \sqrt[3]{9}$ Apply $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \quad m,n \in N}$ to the right side of the equation.

$(2x-4)^{\frac{2}{3}} = (9)^{\frac{1}{3}}$ Write 9 as a power of 3.

$(2x-4)^{\frac{2}{3}} = (3^2)^{\frac{1}{3}}$ Apply the law of exponents $\boxed{(a^m)^n = a^{mn}}$ to the right side of the equation.

$(2x-4)^{\frac{2}{3}} = (3)^{2\times\frac{1}{3}}$ Simplify the exponents.

$(2x-4)^{\frac{2}{3}} = (3)^{\frac{2}{3}}$ The exponents are equal so the bases are equal quantities.

$2x-4 = 3$ Solve the equation.

$$2x-4+4 = 3+4$$
$$2x = 7$$
$$\frac{2x}{2} = \frac{7}{2}$$
$$\frac{\cancel{2}x}{\cancel{2}} = \frac{7}{2}$$
$$\boxed{x = \frac{7}{2}}$$

Examples

Example 1

Earlier, you were asked how can you solve this type of equation where you cannot isolate the variable.

$9^{x+1} = \sqrt{27}$

To begin, write each side of the equation with a common base. Both 9 and 27 can be written as a power of '3'. Therefore, $(3^2)^{x+1} = \sqrt{3^3}$.

Apply

$$\boxed{(a^m)^n = a^{mn}}$$

to the left side of the equation. $3^{2x+2} = \sqrt{3^3}$

Express the right side of the equation in exponential form and apply

$$\boxed{(a^m)^n = a^{mn}}$$

.

$$3^{2x+2} = (3^3)^{\frac{1}{2}}$$
$$3^{2x+2} = 3^{\frac{3}{2}}$$

Now that the bases are the same, then the exponents are equal quantities.

$2x+2 = \frac{3}{2}$

Solve the equation.

$2(2x+2) = 2\left(\frac{3}{2}\right)$ Multiply both sides of the equation by '2'. Simplify and solve.

$$4x+4 = \cancel{2}\left(\frac{3}{\cancel{2}}\right)$$
$$4x+4 = 3$$
$$4x+4-4 = 3-4$$
$$\frac{\cancel{4}x}{\cancel{4}} = \frac{-1}{4}$$
$$\boxed{x = -\frac{1}{4}}$$

Example 2

Use the laws of exponents to solve the following exponential equation: $27^{1-x} = \left(\frac{1}{9}\right)^{2-x}$

$27^{1-x} = \left(\frac{1}{9}\right)^{2-x}$	The variable appears in the exponent.
$(3^3)^{1-x} = (3^{-2})^{2-x}$	Write each side of the equation as a power of 3.
$(3^3)^{1-x} = (3^{-2})^{2-x}$	Apply the law of exponents $\boxed{(a^m)^n = a^{mn}}$.
$(3)^{3(1-x)} = (3)^{-2(2-x)}$	Simplify the exponents.
$3^{3-3x} = 3^{-4+2x}$	The bases are the same so the exponents are equal quantities.
$3-3x = -4+2x$	Solve the equation.
$3-3-3x = -4-3+2x$	
$-3x = -7+2x$	
$-3x-2x = -7x+2x-2x$	
$-5x = -7$	
$\frac{-5x}{5} = \frac{-7}{5}$	
$\frac{\cancel{-5}x}{\cancel{-5}} = \frac{7}{5}$	
$\boxed{x = \frac{7}{5}}$	

Example 3

Use the laws of exponents to solve the following exponential equation: $(x-3)^{\frac{1}{2}} = (25)^{\frac{1}{4}}$

$(x-3)^{\frac{1}{2}} = (25)^{\frac{1}{4}}$	The variable appears in the base.
$(x-3)^{\frac{1}{2}} = (5^2)^{\frac{1}{4}}$	Write 25 as a power of 2.
$(x-3)^{\frac{1}{2}} = (5^2)^{\frac{1}{4}}$	Apply the law of exponents $\boxed{(a^m)^n = a^{mn}}$ to the right side of the equation.
$(x-3)^{\frac{1}{2}} = (5)^{2\times\frac{1}{4}}$	Simplify the exponents.
$(x-3)^{\frac{1}{2}} = (5)^{\frac{2}{4}}$	
$(x-3)^{\frac{1}{2}} = (5)^{\frac{1}{2}}$	The exponents are equal so the bases are equal quantities.
$x-3 = 5$	Solve the equation.
$x-3+3 = 5+3$	
$x = 8$	
$\boxed{x = 8}$	

Example 4

Use the laws of exponents to solve $\frac{(8^{x-4})(2^x)(4^{2x+3})}{32^x} = 16$.

$$\frac{(8^{x-4})(2^x)(4^{2x+3})}{32^x}=16$$ The variable appears in the exponent.

$$\frac{[(2^3)^{x-4}](2^x)[(2^2)^{2x+3}]}{(2^5)^x}=2^4$$ Write all bases as a power of 2. Write 16 as a power of 2.

$$\frac{[(2^3)^{x-4}](2^x)[(2^2)^{2x+3}]}{(2^5)^x}=2^4$$ Apply the law of exponents $\boxed{(a^m)^n=a^{mn}}$.

$$\frac{[(2)^{3(x-4)}](2^x)[(2)^{2(2x+3)}]}{(2)^{5(x)}}=2^4$$

$$\frac{[(2)^{3x-12}](2^x)[(2)^{4x+6}]}{2^{5x}}=2^4$$ Simplify the exponents.

$$\frac{[2^{3x-12}](2^x)[2^{4x+6}]}{2^{5x}}=2^4$$ Apply the law of exponents $\boxed{a^m\times a^n=a^{m+n}}$.

$$\frac{[2^{3x+x+4x-12+6}]}{2^{5x}}=2^4$$ Simplify the exponents.

$$\frac{2^{8x-6}}{2^{5x}}=2^4$$ Apply the laws of exponents $\boxed{\frac{a^m}{a^n}=a^{m-n}}$.

$$2^{8x-6-5x}=2^4$$ Simplify the exponents.

$$2^{3x-6}=2^4$$ The bases are the same so the exponents are equal quantities.

$$3x-6=4$$ Solve the equation.

$$3x-6+6=4+6$$

$$3x=10$$

$$\frac{3x}{3}=\frac{10}{3}$$

$$\frac{\cancel{3}x}{\cancel{3}}=\frac{10}{3}$$

$$\boxed{x=\frac{10}{3}}$$

Review

Use the laws of exponents to solve the following exponential equations:

1. $2^{3x-1}=\sqrt[3]{16}$
2. $36^{x-2}=\left(\frac{1}{6}\right)^{2x+5}$
3. $6(x-4)^{\frac{1}{3}}=18$
4. $(3x-2)^{\frac{2}{5}}=4$
5. $36^{x+1}=\sqrt{6}$
6. $3^{5x-1}=\sqrt[3]{9}$
7. $9^{2x-1}=\left(\sqrt[4]{27}\right)^x$
8. $(3x-2)^{\frac{3}{2}}=8$
9. $(x+1)^{-\frac{5}{2}}=32$
10. $\left(\sqrt{3}\right)^{4x}=27^{x-3}$
11. $4^{3x-1}=\sqrt[3]{32}$
12. $(x+2)^{\frac{2}{3}}=(27)^{\frac{2}{9}}$

13. $(2^{x-3})(8^{x}) = 32$
14. $(x-2)^{\frac{1}{2}} = 9^{\frac{1}{4}}$
15. $8^{x+12} = \left(\frac{1}{16}\right)^{2x-7}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.8.

6.9 Exponential Functions

Here you'll learn to sketch and recognize basic exponential functions. You will also learn a real world application of exponential functions.

Roberta invested \$600 into a mutual fund that paid 4% interest each year compounded annually.

i) Complete a table showing the value of the mutual fund for the first five years.

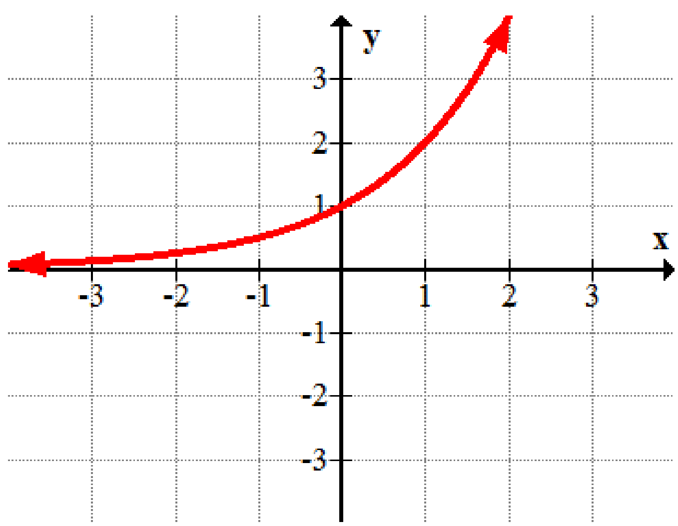
ii) Write an exponential function of the form $y = a \cdot b^x$ to describe the value of the mutual fund.

iii) Use the exponential function to determine the value of the mutual fund in 15 years.

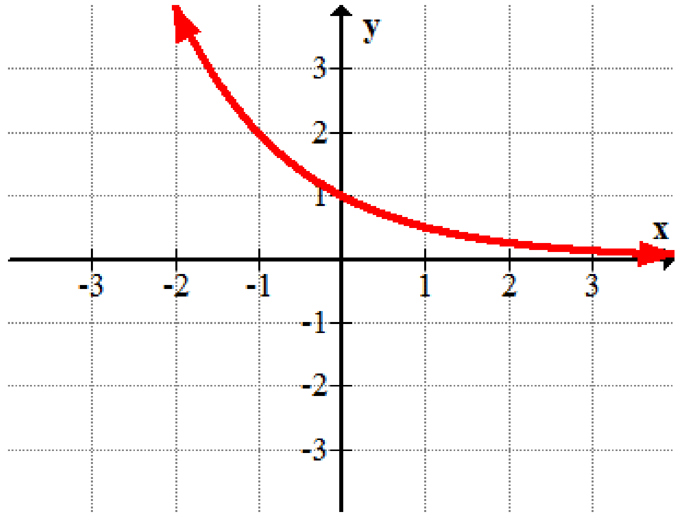
Exponential Functions

An exponential function is a function with a variable in the exponent. Two examples of exponential functions are shown below:

$\boxed{y = 2^x}$



$\boxed{y = \left(\frac{1}{2}\right)^x}$



Here are some facts to notice about the functions and their graphs:

- The graph of $y = 2^x$ is an increasing curve. It shows growth.
- Each y-value of $y = 2^x$ is 2 times the previous y-value (for the integer values of x). For example, the points on the graph go from (0, 1) to (1, 2) to (2, 4). The next point would be (3, 8). The y-values keep being multiplied by 2.
- The graph of $y = \left(\frac{1}{2}\right)^x$ is a decreasing curve. It shows decay.
- Each y-value of $y = \left(\frac{1}{2}\right)^x$ is $\frac{1}{2}$ the value of the previous y-value (for the integer values of x). For example, the points on the graph go from (0, 1) to $\left(1, \frac{1}{2}\right)$ to $\left(2, \frac{1}{4}\right)$. The y-values keep being multiplied by $\frac{1}{2}$.
- Both graphs have a y-intercept of 1. This is because anything to the zero power is equal to 1.
- The domain of each function is $D = \{x|x \in R\}$.
- The range for each function is $R = \{y|y > 0,\ y \in R\}$.

Based on the above observations, you can deduce that an exponential function of the form $y = ab^x$ where $b > 0$ has the following properties:

Properties of an Exponential Function of the form

- 'b' is the value of the common ratio. Within the function, as the x-value increases by 1, the y-value is multiplied by the common ratio.
- If $b > 1$ then the curve will represent exponential growth.
- If $0 < b < 1$ then the curve will represent exponential decay.
- Every exponential function of the form $y = ab^x$ will pass through the point $(0, a)$. a will always be the y-intercept of the function, or its value at time 0.
- Every exponential function of the form $y = ab^x$ will have the domain and range:

$$D = \{x|x \in R\}\ and\ R = \{y|y > 0,\ y \in R\}$$

For the following tables of values that represent exponential functions, determine the common ratio:

i)

x	0	1	2	3	4	$\cdots$
y	1	2	4	8	16	$\cdots$

The common ratio is a constant that is determined by $r = \frac{t_{n+1}}{t_n}$.

$$r = \frac{t_{n+1}}{t_n} = \frac{2}{1} = 2$$
$$r = \frac{t_{n+1}}{t_n} = \frac{4}{2} = 2$$
$$r = \frac{t_{n+1}}{t_n} = \frac{8}{4} = 2$$
$$r = \frac{t_{n+1}}{t_n} = \frac{16}{8} = 2$$
$$\boxed{\text{The common ratio is 2.}}$$

ii)

x	0	1	2	3	4	$\cdots$
y	100	50	25	12.5	6.25	$\cdots$

The common ratio is a constant that is determined by $r = \frac{t_{n+1}}{t_n}$.

$$r = \frac{t_{n+1}}{t_n}$$
$$r = \frac{50}{100} = \frac{1}{2}$$
$$r = \frac{25}{50} = \frac{1}{2}$$
$$r = \frac{12.5}{25} = \frac{1}{2}$$
$$r = \frac{6.25}{12} = \frac{1}{2}$$
$$\boxed{\text{The common ratio is } \frac{1}{2}.}$$

Solve the expression

Using the exponential function

$$\boxed{f(x) = 3^x}$$

, determine the value of each of the following:

i) $f(2)$

ii) $f(3)$

iii) $f(0)$

iv) $f(4)$

v) $f(-2)$

Solutions: $f(x) = 3^x$ is another way to express $y = 3^x$. To determine the value of the function for the given values, replace the exponent with that value and evaluate the expression.

i)

$$f(x) = 3^x$$
$$f(2) = 3^2$$
$$f(2) = 9$$
$$\boxed{f(2) = 9}$$

ii)

$$f(x) = 3^x$$
$$f(3) = 3^3$$
$$f(3) = 27$$
$$\boxed{f(3) = 27}$$

iii)

$$f(x) = 3^x$$
$$f(0) = 3^0$$
$$f(0) = 1$$
$$\boxed{f(0) = 1}$$

iv)

$$f(x) = 3^x$$
$$f(4) = 3^4$$
$$f(4) = 81$$
$$\boxed{f(4) = 81}$$

v)

$$f(x) = 3^x$$
$$f(-2) = 3^{-2}$$
$$f(-2) = \frac{1}{3^2}$$
$$\boxed{f(-2) = \frac{1}{9}}$$

Solve the following problem on exponential function

On January 1, Juan invested \$1.00 at his bank at a rate of 10% interest compounded daily.

i) Create a table of values for the first 8 days of the investment.

ii) What is the common ratio?

iii) Determine the equation of the function that would best represent Juan's investment.

iv) How much money will Juan have in his account on January 31?

v) If Juan had originally invested \$100 instead of \$1.00 at 10%, what exponential equation would describe the investment. How much money would he have in his account on January 31?

i)

$$\begin{array}{llll} 1.00(.10) = 0.10 & 1.10(.10) = 0.11 & 1.21(.10) = 0.12 & 1.33(.10) = 0.13 \\ 1.00 + 0.10 = 1.10 & 1.10 + 0.11 = 1.21 & 1.21 + 0.12 = 1.33 & 1.33 + 0.13 = 1.46 \\ & & & \\ 1.46(.10) = 0.15 & 1.61(.10) = 0.16 & 1.77(.10) = 0.18 & 1.95(.10) = 0.20 \\ 1.46 + 0.15 = 1.61 & 1.61 + 0.16 = 1.77 & 1.77 + 0.18 = 1.95 & 1.95 + 0.20 = 2.15 \end{array}$$

# of days	0	1	2	3	4	5	6	7	8
Money (\$)	1	1.10	1.21	1.33	1.46	1.61	1.77	1.95	2.15

ii) The common ratio is a constant that is determined by $\frac{t_{n+1}}{t_n}$. Therefore, the common ratio for this problem is $r = \frac{t_{n+1}}{t_n} \rightarrow \frac{1.10}{1} = 1.10 \rightarrow \frac{1.21}{1.10} = 1.10 \rightarrow \frac{1.33}{1.21} = 1.10$.

The common ratio is

$$\boxed{1.10}$$

.

iii) The equation of the function to model Juan's investment is $y = 1.10^x$

iv)

$$y = 1.10^x \rightarrow y = 1.10^{31} \rightarrow \boxed{y = \$19.19}$$

.

On January 31, Juan will have \$19.19 in his account.

v)

$$\begin{aligned} & y = 100(1.10)^x \\ & y = 100(1.10)^{31} \rightarrow y = \$1919.43 \rightarrow \boxed{y = \$1919.43}. \end{aligned}$$

On January 31, Juan would have \$1919.43 in his account if he had invested \$100 instead of \$1.00.

Examples

Example 1

Earlier, you were given a problem about Roberta.

Roberta invested \$600 into a mutual fund that paid 4% interest each year compounded annually.

i)

$$\begin{array}{lll} 600(.04) = 24 & 624(.04) = 24.96 & 648.96(.04) = 25.96 \\ 600 + 24 = 624 & 624 + 24.96 = 648.96 & 648.96 + 25.96 = 674.92 \end{array}$$

$$\begin{array}{ll} 674.92(.04) = 27.00 & 701.92(.04) = 28.08 \\ 674.92 + 27.00 = 701.92 & 701.92 + 28.08 = 730.00 \end{array}$$

Time $(years)$	0	1	2	3	4	5
Value (\$)	600	624	648.96	674.92	701.92	730.00

ii) The initial value is \$600. The common ratio is 1.04 which represents the initial investment and the interest rate of 4%. The exponent is the time in years. The exponential function is $y = 600(1.04)^x$ or $v = 600(1.04)^t$.

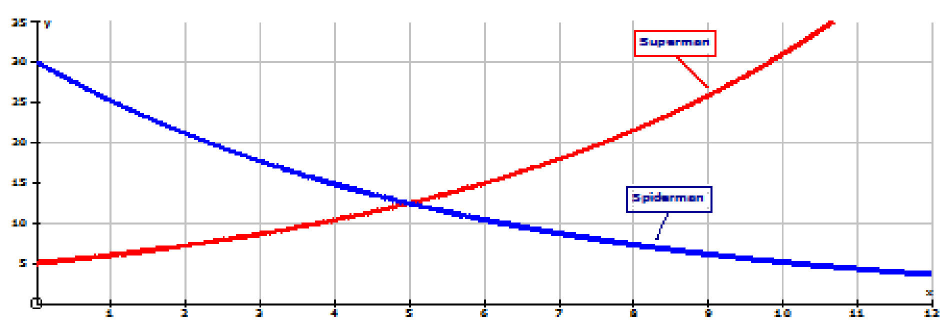
iii)

$$v = 600(1.04)^t$$
$$v = 600(1.04)^{15}$$
$$v = \$1080.57$$
$$\boxed{v = \$1080.57}$$

The value of the mutual fund in fifteen years will be \$1080.57.

Example 2

The graph below shows the change in value of two comic books purchases in the year 2000. Both comics were expected to be good investments, but one of them did not perform as expected. Use the graphs to answer the questions.



a) What was the purchase price of each comic book?
b) Which comic book shows exponential growth?
c) Which comic book shows exponential decay?
d) In what year were both comic books equal in value?
e) State the domain and range for each comic.

a) The purchase price of each comic book is the y-intercept. The y-intercept is the initial value of the books. The Spiderman comic book cost \$30.00 and the Superman comic book cost \$5.00.

b) The Superman comic book shows exponential growth.
c) The Spiderman comic book shows exponential decay.
d) In 2005 both comic books were equal in value. The graphs intersect at approximately (5, \$12.50), where 5 represents five years after the books were purchased.
e) The domain and range for each comic is $D = \{x|x \in R\}$ and $R = \{y|y > 0,\ y \in R\}$

Example 3

Paulette bought a Bobby Orr rookie card for \$300. The value of the card appreciates (increases) by 30% each year.

a) Complete a table of values to show the first five years of the investment.
b) Determine the common ratio for the successive terms.
c) Determine the equation of the exponential function that models this investment.

$$300(.30) = 90 \qquad 390(.30) = 117 \qquad 507(.30) = 152.10$$
$$300 + 90 = 390 \qquad 390 + 117 = 507 \qquad 507 + 152.10 = 659.10$$

$$659.10(.30) = 197.73 \qquad 856.83(.30) = 257.05$$
$$659.10 + 197.73 = 856.83 \qquad 856.83 + 257.05 = 1113.88$$

a)

Time (*years*)	0	1	2	3	4	5
Value (\$)	300	390	507	659.10	856.83	1113.88

b)

$$r = \frac{t_{n+1}}{t_n}$$
$$r = \frac{390}{300} = 1.3$$
$$r = \frac{507}{390} = 1.3$$
$$r = \frac{659.10}{507} = 1.3$$
$$r = \frac{856.83}{659.10} = 1.3$$
$$r = \frac{1113.88}{856.83} = 1.3$$

c) The exponential function that would model Paulette's investment is

$$\boxed{y = 300(1.3)^x \ or \ v = 300(1.3)^t}$$

Example 4

Due to the closure of the pulp and paper mill, the population of the small town is decreasing at a rate of 12% annually. If there are now 2400 people living in the town, what will the town's projected population be in eight years?

The town's population is decreasing by 12% annually. The simplest way to use this in an exponential function is to use the percent of the population that still exists each year - 88%.

Therefore, the exponential function would consist of the present population (a), the common ratio is 0.88 (b) and the time in years would be the exponent (x). The function is $p = 2400(0.88)^t$

The population in eight years would be

$$p = 2400(0.88)^t$$
$$p = 2400(0.88)^8$$
$$p = 863.123$$
$$p \approx 863 \ people$$

Review

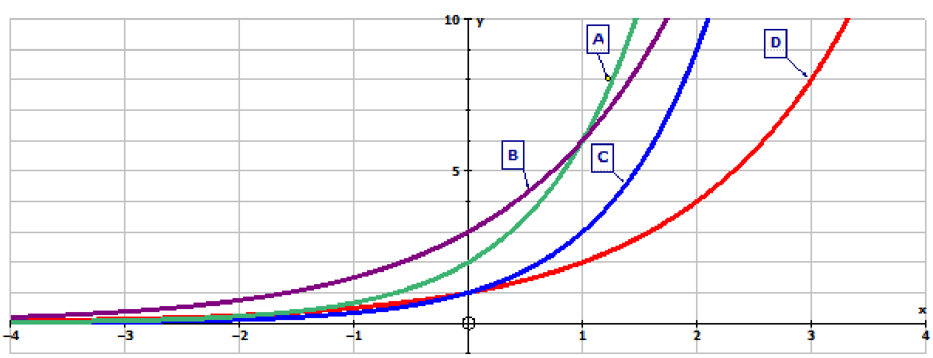
Brandon bought a car for $13,000. The value of the car depreciates by 20% each year.

1. Complete a table of values to show the car's values for the first five years.
2. Determine the exponential function that would model the depreciation of Brandon's car.

For each of the following exponential functions, identify the common ratio and the y-intercept, and tell if the function represents a growth or decay curve.

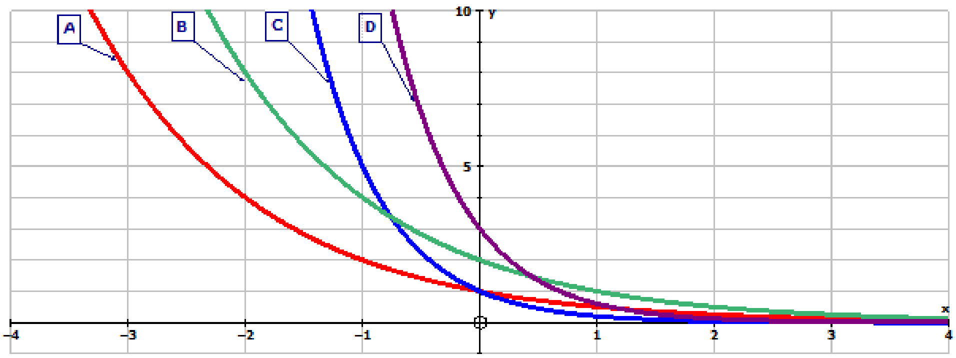
3. $y = 4(5)^x$
4. $y = 13(2.3)^x$
5. $y = 0.85(0.16)^x$
6. $y = 1.6(0.5)^x$
7. $y = 0.4(2.1)^x$

Match each graph below with its corresponding equation:



8. $y = 2^x$
9. $y = 3^x$
10. $y = 2(3)^x$
11. $y = 3(2)^x$
12. Do these graphs represent growth or decay?

Match each graph below with its corresponding equation:



13. $y = 0.5^x$
14. $y = 0.2^x$
15. $y = 2(0.5)^x$
16. $y = 3(0.2)^x$
17. Do these graphs represent growth or decay?

18. Jolene purchased a summer home for \$120,000 in 2002. If the property has consistently increased in value by 11% each year, what will be the value of her summer home in 2012?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.9.

6.10 Advanced Exponential Functions

Here you'll learn more about exponential functions that are of the form $y = a(b)^{\left(\frac{x}{c}\right)}$ and $y = a(b)^{\left(\frac{x}{c}\right)} + d$.

Andrea invested money into a mutual fund. Every three months she received a bank statement indicating the growth of her investment. Andrea recorded the following data from her bank statements:

Time $(months)$	0	3	6	9	12	15
Value ($)	1200	1224	1248.48	1273.45	1298.92	1324.90

i) How much was the initial investment?

ii) What was the rate of interest the bank applied to her investment? Express this as a percent.

iii) What is the exponential function that best models the value of Andrea's investment?

iv) What will be the mutual fund's value in two years?

v) Can you use technology to determine the length of time it will take for the mutual fund to double in value?

Advanced Exponential Functions

You have learned about exponential functions of the form $y = ab^x$ in which the x-value increased by increments of one. However, data is often collected using other intervals such as every three months, every two hours, every fifteen minutes, and so forth. You still want to obtain the exponential function to represent such a situation, but the function must be modified to accommodate this interval change.

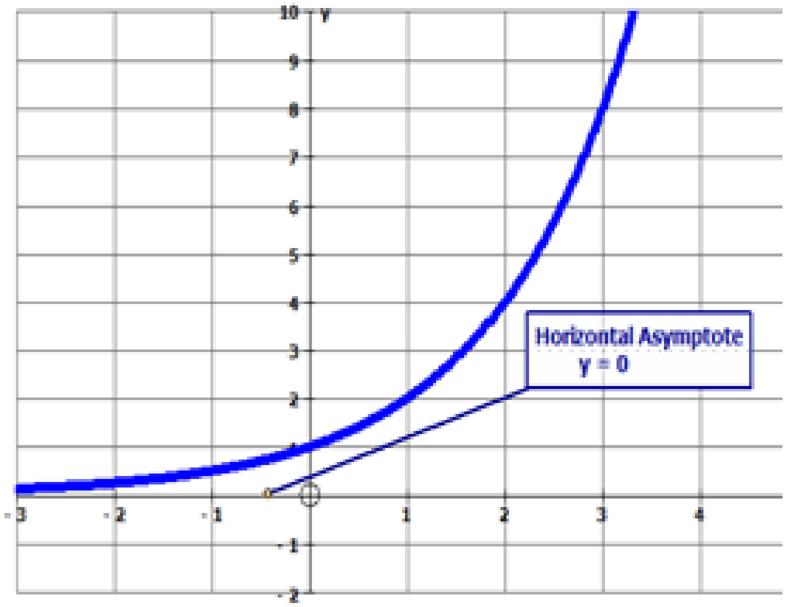
The exponential function will now be expressed in the form $y = a(b)^{\frac{x}{c}}$ where:

- 'a' represents the initial value (y-intercept)
- 'b' represents the common ratio (the rate of growth or decay)
- 'c' represents the increment in value of x.

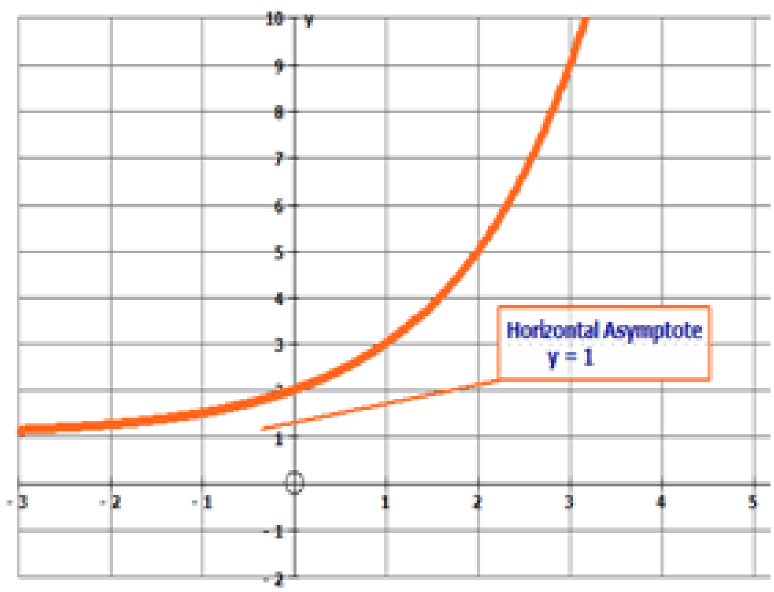
With exponential functions of the form $y = ab^x$, the **horizontal asymptote** is the x-axis. A horizontal asymptote is a horizontal line that a function keeps getting closer and closer to. Look at the graph of the function $y = 2^x$. The growth curve gets closer and closer to the x-axis, so this is the horizontal asymptote. The equation of the horizontal asymptote is

$$\boxed{y = 0}$$

.



Now look at the graph of the function $y = 2^x + 1$. The growth curve gets closer and closer to the line $y = 1$, so this is the horizontal asymptote.



For exponential functions of the form $y = a(b)^{\frac{x}{c}} + d$:

- '$a+d$' represents the initial value (y-intercept)
- 'b' represents the common ratio (the rate of growth or decay)
- 'c' represents the increment in value of x.
- '$y = d$' represents the horizontal asymptote.

Rewrite the exponential form

Write an exponential function of the form $y = a(b)^{\frac{x}{c}}$ to describe the table of values.

X	0	2	4	6	8	10
Y	3	6	12	24	48	96

The initial value 'a' is 3. The common ratio is:

$$\boxed{r=\frac{t_{n+1}}{t_n}=\frac{6}{3}=2} \qquad \boxed{r=\frac{t_{n+1}}{t_n}=\frac{12}{6}=2} \qquad \boxed{r=\frac{t_{n+1}}{t_n}=\frac{24}{12}=2}$$

$$\boxed{r=\frac{t_{n+1}}{t_n}=\frac{48}{24}=2} \qquad \boxed{r=\frac{t_{n+1}}{t_n}=\frac{96}{48}=2}$$

The increment in the x-value is 2. The exponential function to describe the table of values is

$$\boxed{y=3(2)^{\frac{x}{2}}}$$

.

Rewrite in exponential form

A radioactive isotope decays exponentially over time according to the equation $A(t)=42\left(\frac{1}{2}\right)^{\frac{t}{20}}$, where $A(t)$ is the amount of the isotope present at time t, in days.

i) What is the half-life of the isotope?

ii) Explain in words what the equation represents.

iii) How much isotope will remain after 35 days?

i) Half life is the length of time it takes for only half of the original amount of isotope to be present. Since $b=\frac{1}{2}$, the half life of the isotope is the increment in the x-value which is 'c'. The half life is 20 days.

ii) The equation represents the amount (A) of a decaying isotope that remains at any time x, in days. There are 42 units initially present and its half-life is 20 days.

iii)

$$A(t)=42\left(\frac{1}{2}\right)^{\frac{t}{20}}$$

$$A(t)=42\left(\frac{1}{2}\right)^{\frac{35}{20}}=12.5\ units$$

Write in exponential form

Brian bought a cup of coffee from the cafeteria and placed it on the ground while recording its change in temperature (the temperature that day was $0^\circ C$). The temperature of the coffee was recorded every 3 minutes, and the results are shown in the following table:

Time (min)	0	3	6	9	12	15
Temp $(0^\circ C)$	90.0	81.0	72.9	65.6	59.0	53.1

i) Write an exponential function to describe the temperature of the coffee after 't' minutes.

ii) Determine the temperature of the coffee after 30 minutes.

iii) Determine the temperature of the coffee after I hour.

i) The initial value 'a' is 90.

The common ratio is

$$\boxed{r = \frac{81}{90} = 0.9}$$

$$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{72.9}{81} = 0.9}$$

$$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{65.6}{72.9} = 0.899 = 0.9}$$

$$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{59.0}{65.6} = 0.899 = 0.9}$$

$$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{53.1}{59} = 0.9}$$

The increment in the x-value is 3.

The exponential function to describe the table of values is

$$\boxed{y = 90(0.9)^{\frac{x}{3}}}$$

.

ii) Replace 'x' in the exponent with '30' and calculate the value of the function. $y = 90(0.9)^{\frac{30}{3}} = 31.4°C$

iii) One hour equals 60 minutes. Replace 'x' in the exponent with '60' and calculate the value of the function. $y = 90(0.9)^{\frac{60}{3}} = 10.9°C$

Solve the following problem

In the example above, the outdoor temperature was $0°C$. This temperature was chosen to make the value of the horizontal asymptote zero. More importantly, this enabled us to determine the value of the common ratio.

Now you will investigate the same problem again with the outdoor temperature now being $15°C$. What effect will this have on the value of the common ratio? How can the value of 'b' be determined so that an exponential function can be created to model the data?

Brian bought a cup of coffee from the cafeteria and placed it on the ground while recording its change in temperature (the temperature that day was $15°C$). The temperature of the coffee was recorded every 3 minutes, and the results are shown in the following table:

Time (min)	0	3	6	9	12	15
Temp $(0°C)$	105.0	96.0	87.9	80.6	74.0	68.1

i) Determine the rate at which the coffee is cooling.

ii) Write an exponential function to describe the temperature of the coffee after 't' minutes.

iii) Determine the temperature of the coffee after I hour.

i) You can first try to determine the common ratio by using the values in the given table.

$\boxed{r = \frac{96}{105} = 0.914}$ $\boxed{r = \frac{t_{n+1}}{t_n} = \frac{87.9}{96} = 0.916}$ $\boxed{r = \frac{t_{n+1}}{t_n} = \frac{80.6}{87.9} = 0.917}$

$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{74.0}{80.6} = 0.918}$ $\boxed{r = \frac{t_{n+1}}{t_n} = \frac{68.1}{74.0} = 0.920}$

Notice that this doesn't work! The common ratio must be consistent. To eliminate the difficulty with identifying the common ratio, simply subtract the value of the horizontal asymptote (15) from each of the y-values. Then, look for the common ratio.

Time(min)	0	3	6	9	12	15
Temp(^{0} C)	105.0	96.0	87.9	80.6	74.0	68.1
Temp - 15	90.0	81.0	72.9	65.6	59.0	53.1

0.900 0.900 0.899 0.899 0.900

The common ratio is 0.9. Therefore, the temperature of the coffee is decreasing by 10% every 3 minutes.

ii) The value of the 'a' is the initial temperature less the outdoor temperature. $a = 105.0°C - 15.0°C = 90°C$

The initial value 'a' is 90. The common ratio 'b' is 0.9. The increment in the x-values is 3. Therefore, $c = 3$. The coolest temperature that will be reached by the cooling coffee is that of the outdoor temperature. Therefore $d = 15$. The exponential function that models the problem is

$$\boxed{y = 90(0.9)^{\frac{x}{3}} + 15}$$

iii) One hour equals 60 minutes. Replace 'x' in the exponent with '60' and calculate the value of the function.

$$\boxed{y = 90(0.9)^{\frac{60}{3}} + 15 = 25.9°C}$$

Examples

Example 1

Earlier, you were given a problem about Andrea.

Andrea invested money into a mutual fund. Every three months she received a bank statement indicating the growth of her investment. Andrea recorded the following data from her bank statements:

Time $(months)$	0	3	6	9	12	15
Value ($)	1200	1224	1248.48	1273.45	1298.92	1324.90

i) The initial investment was $1200.

ii) The rate of interest that the bank applied to her investment was:

$$r = \frac{t_{n+1}}{t_n} \qquad r = \frac{t_{n+1}}{t_n} \qquad r = \frac{t_{n+1}}{t_n} \qquad r = \frac{t_{n+1}}{t_n} \qquad r = \frac{t_{n+1}}{t_n}$$

$$r = \frac{1224}{1200} \qquad r = \frac{1248.48}{1224} \qquad r = \frac{1273.45}{1248.48} \qquad r = \frac{1298.92}{1273.45} \qquad r = \frac{1324.90}{1298.92}$$

$$\boxed{r = 1.02} \qquad \boxed{r = 1.02} \qquad \boxed{r = 1.02} \qquad \boxed{r = 1.02} \qquad \boxed{r = 1.02}$$

The common ratio is 1.02. Therefore the interest rate was 2%.

iii) The increment of the x-value is 3. Therefore 'c' $= 3$. The exponential function to model Andrea's investment is

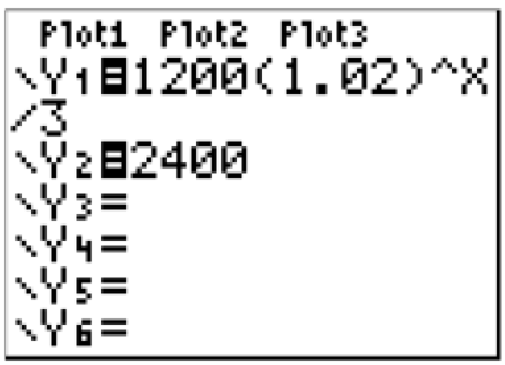
$$\boxed{y = 1200(1.02)^{\frac{x}{3}}}$$

.

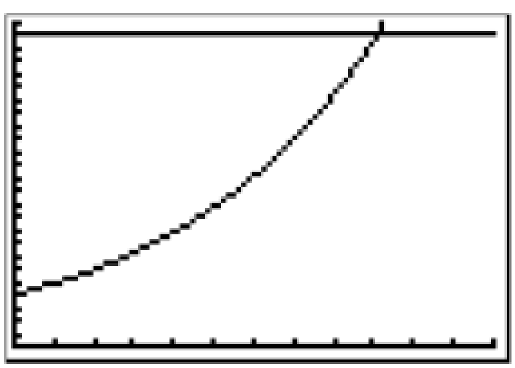
iv) The value of 'c' is 3. This represents the every 3 months that the bank applies the interest to the investment. Therefore, the value of 'x' must be in months. There are 24 months in two years. The value of Andrea's investment in two years time will be:

$$\boxed{y = 1200(1.02)^{\frac{24}{3}} = \$1405.99}$$

v) Press $y =$ and enter the equations $y = 1200(1.02)^{\frac{x}{3}}$ and $y = 2400$ as shown below.

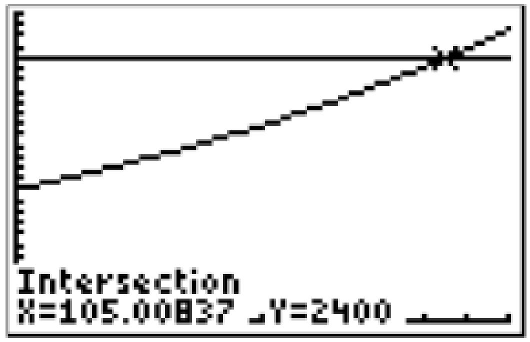


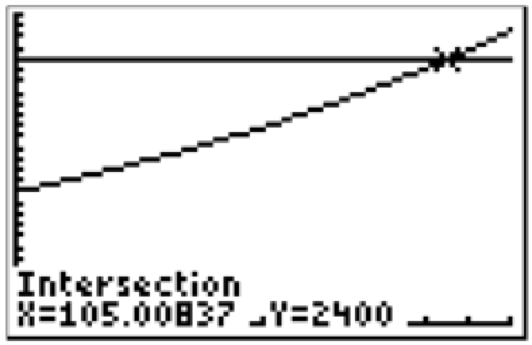
Graph the equations:



Determine the intersection point of the growth curve and the horizontal line by using

$\boxed{2^{nd}\ \text{TRACE}}$ $\boxed{5}$ $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$





The intersection point is $(105.00837, 2400)$. It will take 105 months (8 years and 9 months) for Andrea's investment to double in value.

Example 2

A bucket of tar is heated outside to repair a roof that is leaking. Once the leak has been repaired, the remaining tar is left in the bucket to cool down to use again at a later time. Its change in temperature, T, measured in $^\circ C$, with respect to time, t, in minutes, can be modeled with the function

$$\boxed{T = 95(0.72)^{\frac{t}{15}} + 30}$$

.

i) What is the initial temperature of the tar?
ii) What is the outdoor temperature?
iii) At what rate is the temperature of the tar changing over time?
iv) Determine the temperature of the tar after 30 minutes.
v) Determine the temperature of the tar after 2 hours.
vi) Draw a sketch to show the change in temperature of the tar over time.

i) The initial temperature is the temperature 'a' plus the outdoor temperature of $30^\circ C$.

The exponential function is of the form $y = a(b)^{\frac{x}{c}} + d$. The initial temperature is

$$\boxed{a + d = 95^\circ C + 30^\circ C = 125^\circ C}$$

.

ii) The outdoor temperature is the horizontal asymptote ('d') of the function. The outdoor temperature is

$$\boxed{30^\circ C}$$

.

iii) The common ratio 'b' represents the rate expressed as a decimal that the tar is maintaining. Therefore the rate at which it is cooling down is

$$100 - 0.72 = 0.28$$
$$0.28 \times 100\% = 28\%$$

. The temperature of the tar is dropping 28% every 15 minutes.

iv) The temperature of the tar after 30 minutes is determined by replacing 't' in the exponent with 30 and then performing the calculation.

$$T = 95(0.72)^{\frac{t}{15}} + 30$$
$$T = 95(0.72)^{\frac{30}{15}} + 30$$
$$T = 79.2^\circ C$$

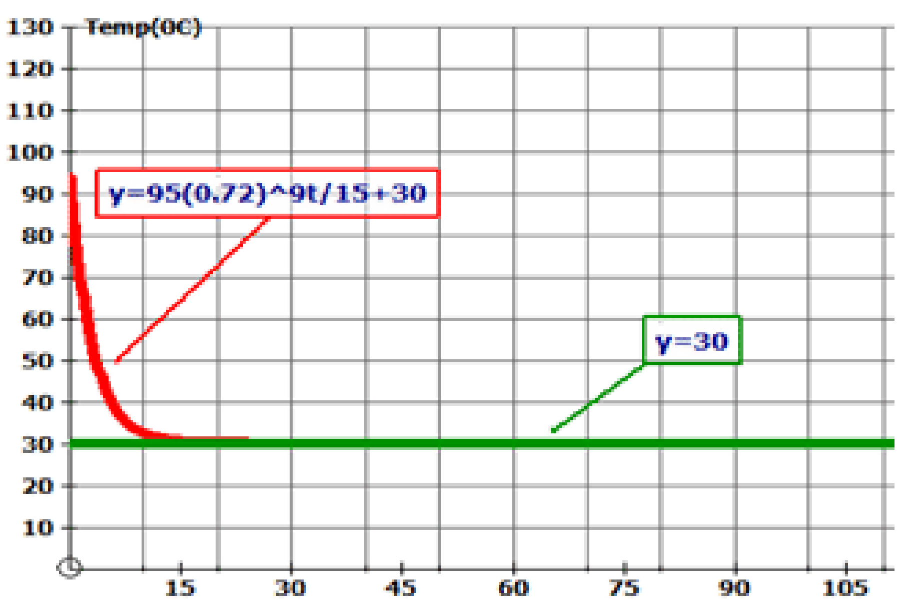
v) There are 60 minutes in one hour and 120 minutes in two hours. The temperature of the tar after 2 hours is determined by replacing 't' in the exponent with 120 and then performing the calculation.

$$T = 95(0.72)^{\frac{t}{15}} + 30$$
$$T = 95(0.72)^{\frac{120}{15}} + 30$$
$$T = 36.9^\circ C$$

vi) A sketch to represent

$$\boxed{T = 95(0.72)^{\frac{t}{15}} + 30}$$

is shown below:



Example 3

Roberto's car radiator is always overheating. He decided to monitor its change in temperature. The outdoor temperature was $73.4°F$, so he did not mind doing the task at hand. Roberto recorded the temperature that was determined every five minutes. The following table is the recorded temperatures.

Time (min)	0	5	10	15	20	25
Temp $(0°F)$	224.6	197.42	175.1	156.7	141.8	129.4

i) Determine the rate at which the radiator is cooling.
ii) Determine the value of 'a' in $y = a(b)^{\frac{x}{c}} + d$ for this problem.
iii) What is the equation of the horizontal asymptote?
iv) Write an exponential function to indicate the temperature of the radiator after 't' minutes.
v) What was the temperature of the radiator after one hour?
vi) Draw a sketch of the radiator's change in temperature over time.

i) The rate by which the radiator is cooling must be determined by first subtracting the outdoor temperature of $73.4°F$ from each of the temperature values.

Time (min)	0	5	10	15	20	25
Temp$(°F)$	224.6	197.4	175.1	156.7	141.8	129.4
Temp $-$ 73.4	151.2	124.0	101.7	83.3	68.4	56.0

Now, determine the common ratio using the final row of the table.

$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{124.0}{151.2} = 0.820}$ $\boxed{r = \frac{t_{n+1}}{t_n} = \frac{101.7}{124.0} = 0.820}$ $\boxed{r = \frac{t_{n+1}}{t_n} = \frac{83.3}{101.7} = 0.819}$

$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{68.4}{83.3} = 0.821}$ $\boxed{r = \frac{t_{n+1}}{t_n} = \frac{56.0}{68.4} = 0.818}$

Therefore the common ratio is 0.82. The rate at which the radiator is cooling is

$$\boxed{100\% - 82\% = 18\%}$$

every five minutes.
ii) The value of 'a' in $y = a(b)^{\frac{x}{c}} + d$ is the initial temperature given in the table less the outdoor temperature.

$$\boxed{a = 224.6°F - 73.4°F = 151.2°F}$$

iii) The equation of the horizontal asymptote is $y =$ the outdoor temperature or

$$\boxed{y = 73.4}$$

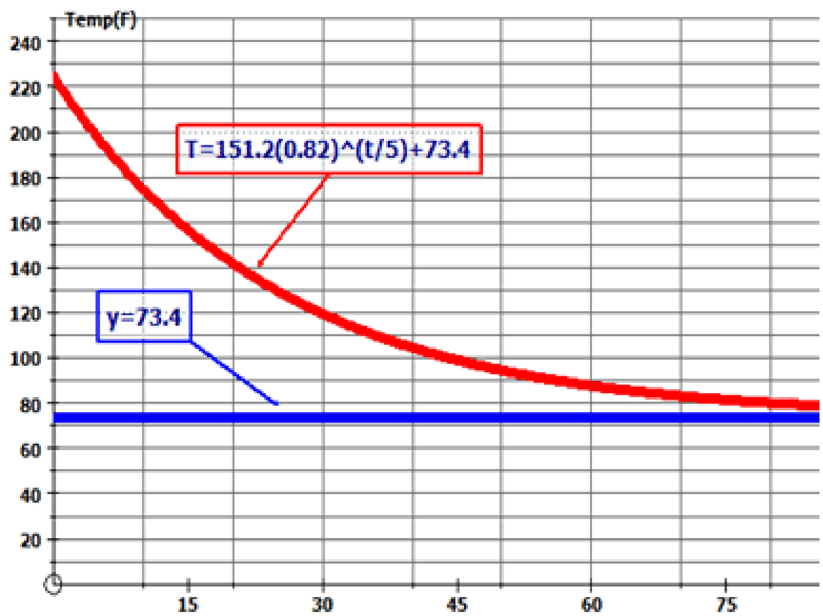
iv) The exponential function that models the change in temperature of the radiator is:

$$\begin{aligned}
a &= 151.2 \\
b &= 0.82 \\
c &= 5 \\
d &= 73.4 \\
T &= 151.2(0.82)^{\frac{t}{5}} + 73.4
\end{aligned}$$

v) There are 60 minutes in one hour. Therefore

$$\boxed{T = 151.2(0.82)^{\frac{60}{5}} + 73.4 = 87.4^\circ F}$$

vi)



A deadly bacteria is threatening the small town of Norman. The bacteria are doubling every 3 hours. If there were initially 250 spores, how many will be present in 12 hours?

Write the exponential function to model the number of spores produced over time.

$$\begin{aligned}
a &= 250 \\
b &= 3 \\
x &= 12 \\
y &= a(b)^x \\
y &= 250(3)^{12}
\end{aligned}$$

Use the exponential function to determine the number of spores present after 12 hours.

$$\boxed{y = 250(3)^{12} = 132,860,250 \; spores}$$

Review

Write an exponential function in the form $y = a(b)^{\frac{x}{c}}$ to model each table of values.

1.

X	0	5	10	15	20	25
Y	8	4	2	1	0.5	0.25

2.

X	0	3	6	9	12	15
Y	2	10	50	250	1250	6250

3.

X	-4	0	4	8	12	16
Y	6	2.4	0.96	0.384	0.1536	0.06144

4.

X	-0.6	-0.3	0	0.3	0.6	0.9
Y	10	12	14.4	17.28	20.736	24.8832

5.

X	0	0.1	0.2	0.3	0.4	0.5
Y	5	15	45	135	405	1215

A radioactive isotope decays at the rate indicated by the exponential function $A(t) = 800\left(\frac{1}{2}\right)^{\frac{t}{1500}}$, where 't' is the time in years and $A(t)$ is the amount of the isotope, in grams, remaining.

6. What is the initial mass of the isotope?
7. How long will it take for the isotope to be reduced to half of its original amount?
8. What will the mass of the isotope be after 4500 years?

For each of the following exponential functions state the equation of the horizontal asymptote, the y-intercept, the range, and whether it is a growth or decay function.

9. $y = 2^x + 5$
10. $y = 2(3)^x$
11. $y = 6\left(\frac{1}{3}\right)^x + 5$
12. $y = 4(0.4)^x + 1.8$
13. $y = 12(1.25)^x$

A hot cup of coffee cools exponentially with time as it sits on the teacher's desk. Its change in temperature, T, measured in $^\circ C$, with respect to time, t, in minutes, is modeled with the following function:

$$T = 82(0.6)^{\frac{t}{12}} + 20$$

14. What is the initial temperature of the coffee?
15. What is the room temperature of the classroom?
16. At what rate is the temperature of the coffee changing over time?
17. What is the coffee's temperature after 30 minutes?
18. What is the temperature of the coffee after 1 hour?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.10.

Summary

You learned that in an expression like 2^x, the "2" is the base and the "x" is the exponent. You learned the following laws of exponents that helped you to simplify expressions with exponents:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$ (if $m > n, a \neq 0$)
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $(b \neq 0)$
- $a^0 = 1$ $(a \neq 0)$
- $a^{-m} = \frac{1}{a^m}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

You learned that scientific notation is a way to express large or small numbers in the form

$$x = a \times 10^n$$

where

$$1 \leq a < 10 \text{ and } n \in Z.$$

You learned that to solve exponential equations with variables in the exponent you should try to rewrite the equations so the bases are the same. Then, set the exponents equal to each other and solve. If the equation has a variable in the base you can try to get rid of the exponent or, make the exponents on each side of the equation the same and then set the bases equal to each other and solve.

Finally, you learned all about exponential functions. You learned that for exponential functions of the form $y = ab^{\frac{x}{c}} + d$, if $0 < b < 1$ then the function is decreasing and represents exponential decay. If $b > 1$ then the function is increasing and represents exponential growth. Exponential functions are used in many real-life situations such as with the decay of radioactive isotopes and with interest that compounds.

CHAPTER 7 Polynomials

Chapter Outline

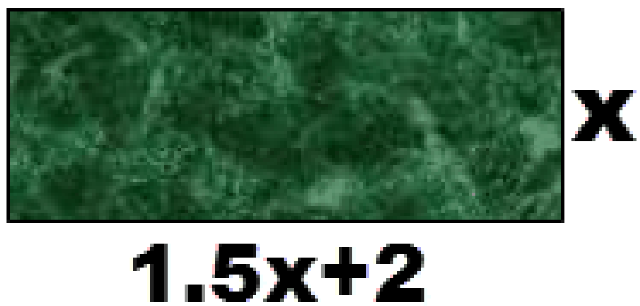
Introduction

Here you'll learn all about polynomials. You'll start by learning how to add, subtract, and multiply polynomials. Then you will learn how to factor polynomials, which can be thought of as the opposite of multiplying. Next you'll learn how to divide polynomials and how this connects to factoring. Finally, you'll learn how to use your graphing calculator to graph polynomials in order to determine the factors and roots of polynomials.

7.1 Addition and Subtraction of Polynomials

Here you'll learn how to add and subtract polynomials.

You are going to build a rectangular garden in your back yard. The garden is 2 m more than 1.5 times as long as it is wide. Write an expression to show the area of the garden.



Adding and Subtracting Polynomials

The word polynomial comes from the Greek word *poly* meaning "many". Polynomials are made up of one or more terms and each term must have an exponent that is 0 or a whole number. This means that $3x^2+2x+1$ is a polynomial, but $3x^{0.5}+2x^{-2}+1$ is not a polynomial. Some common polynomials have special names based on how many terms they have:

- A monomial is a polynomial with just one term. Examples of monomials are $3x$, $2x^2$ and 7.
- A binomial is a polynomial with two terms. Examples of binomials are $2x+1$, $3x^2-5x$ and $x-5$.
- A trinomial is a polynomial with three terms. An example of a trinomial is $2x^2+3x-4$.

To add and subtract polynomials you will go through two steps.

1. Use the distributive property to remove parentheses. Remember that when there is no number in front of the parentheses, it is like there is a 1 in front of the parentheses. Pay attention to whether or not the sign in front of the parentheses is $+$ or $-$, because this will tell you if the number you need to distribute is $+1$ or -1.
2. Combine similar terms. This means, combine the x^2 terms with the x^2 terms, the x terms with the x terms, etc.

Add the polynomials

Find the sum: $(3x^2+2x-7)+(5x^2-3x+3)$.

First you want to remove the parentheses. Because this is an addition problem, it is like there is a $+1$ in front of each set of parentheses. When you distribute a $+1$, none of the terms will change.

$1(3x^2+2x-7)+1(5x^2-3x+3)=3x^2+2x-7+5x^2-3x+3$

Next, combine the similar terms. Sometimes it can help to first reorder the expression to put the similar terms next to one another. Remember to keep the signs with the correct terms. For example, in this problem the 7 is negative and the 3x is negative.

$$\begin{aligned} 3x^2+2x-7+5x^2-3x+3 &= 3x^2+5x^2+2x-3x-7+3 \\ &= 8x^2-x-4 \end{aligned}$$

This is your final answer.

Subtract the polynomials

Find the difference: $(5x^2+8x+6)-(4x^2+5x+4)$.

First you want to remove the parentheses. Because this is a subtraction problem, it is like there is a -1 in front of the second set of parentheses. When you distribute a -1, each term inside that set of parentheses will change its sign.

$1(5x^2+8x+6)-1(4x^2+5x+4)=5x^2+8x+6-4x^2-5x-4$

Next, combine the similar terms. Remember to keep the signs with the correct terms.

$$\begin{aligned} 5x^2+8x+6-4x^2-5x-4 &= 5x^2-4x^2+8x-5x+6-4 \\ &= x^2+3x+2 \end{aligned}$$

This is your final answer.

Subtract the polynomials

Find the difference: $(3x^3+6x^2-7x+5)-(4x^2+3x-8)$

First you want to remove the parentheses. Because this is a subtraction problem, it is like there is a -1 in front of the second set of parentheses. When you distribute a -1, each term inside that set of parentheses will change its sign..

$1(3x^3+6x^2-7x+5)-1(4x^2+3x-8)=3x^3+6x^2-7x+5-4x^2-3x+8$

Next, combine the similar terms. Remember to keep the signs with the correct terms.

$$\begin{aligned} 3x^3+6x^2-7x+5-4x^2-3x+8 &= 3x^3+6x^2-4x^2-7x-3x+5+8 \\ &= 3x^3+2x^2-10x+13 \end{aligned}$$

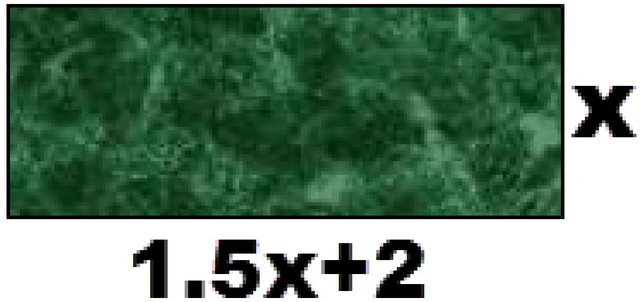
This is your final answer.

Examples

Example 1

Earlier, you were asked to write an expression to show the area of the garden.

Remember that the area of a rectangle is length times width.



$$Area = l \times w$$
$$Area = (1.5x+2)x$$
$$Area = 1.5x^2 + 2x$$

Example 2

Find the sum: $(2x^2+4x+3)+(x^2-3x-2)$.

$(2x^2+4x+3)+(x^2-3x-2) = 2x^2+4x+3+x^2-3x-2 = 3x^2+x+1$

Example 3

Find the difference: $(5x^2-9x+7)-(3x^2-5x+6)$.

$(5x^2-9x+7)-(3x^2-5x+6) = 5x^2-9x+7-3x^2+5x-6 = 2x^2-4x+1$

Example 4

Find the sum: $(8x^3+5x^2-4x+2)+(4x^3+7x-5)$.

$(8x^3+5x^2-4x+2)+(4x^3+7x-5) = 8x^3+5x^2-4x+2+4x^3+7x-5 = 12x^3+5x^2+3x-3$

Review

For each problem, find the sum or difference.

1. $(x^2+4x+5)+(2x^2+3x+7)$
2. $(2r^2+6r+7)-(3r^2+5r+8)$
3. $(3t^2-2t+4)+(2t^2+5t-3)$
4. $(4s^2-2s-3)-(5s^2+7s-6)$
5. $(5y^2+7y-3)+(-2y^2-5y+6)$
6. $(6x^2+36x+13)-(4x^2+13x+33)$
7. $(12a^2+13a+7)+(9a^2+15a+8)$
8. $(9y^2-17y-12)+(5y^2+12y+4)$
9. $(11b^2+7b-12)-(15b^2-19b-21)$
10. $(25x^2+17x-23)-(-14x^3-14x-11)$
11. $(-3y^2+10y-5)-(5y^2+5y+8)$

12. $(-7x^2-5x+11)+(5x^2+4x-9)$
13. $(9a^3-2a^2+7)+(3a^2+8a-4)$
14. $(3x^2-2x+4)-(x^2+x-6)$
15. $(4s^3+4s^2-5s-2)-(-2s^2-5s+6)$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.1.

7.2 Multiplication of Polynomials

Here you will learn how to multiply polynomials using the distributive property.

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack measures the glass and finds the height to width ratio is 4:3. Write the expression to determine the area of the picture frame.

Multiplying Polynomials

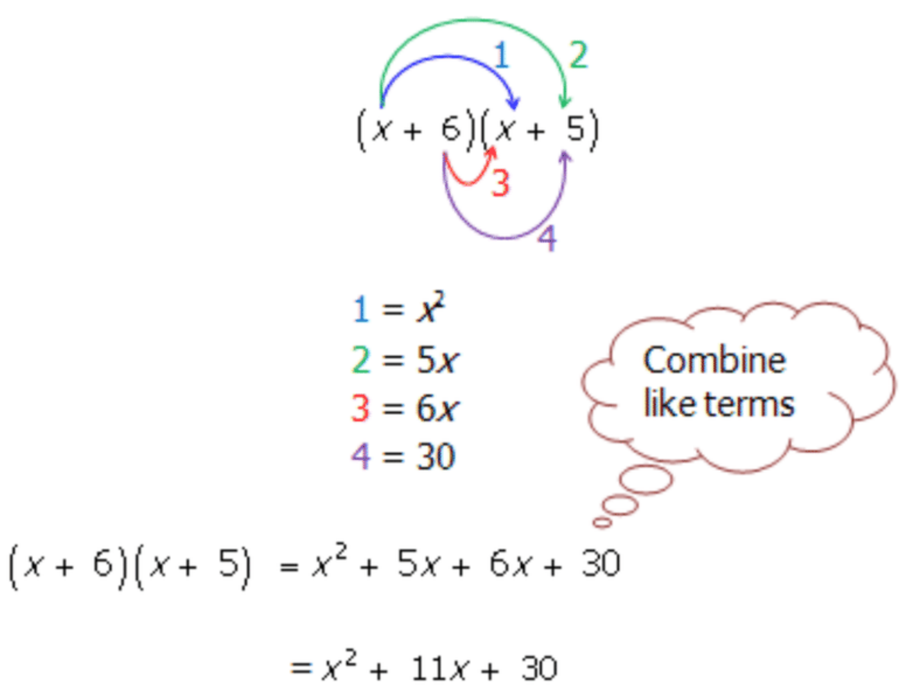
To multiply polynomials you will need to use the distributive property. Recall that the distributive property says that if you start with an expression like $3(5x+2)$, you can simplify it by multiplying both terms inside the parentheses by 3 to get a final answer of $15x+6$.

When multiplying polynomials, you will need to use the distributive property more than once for each problem.

Multiply the polynomials

Find the product: $(x+6)(x+5)$

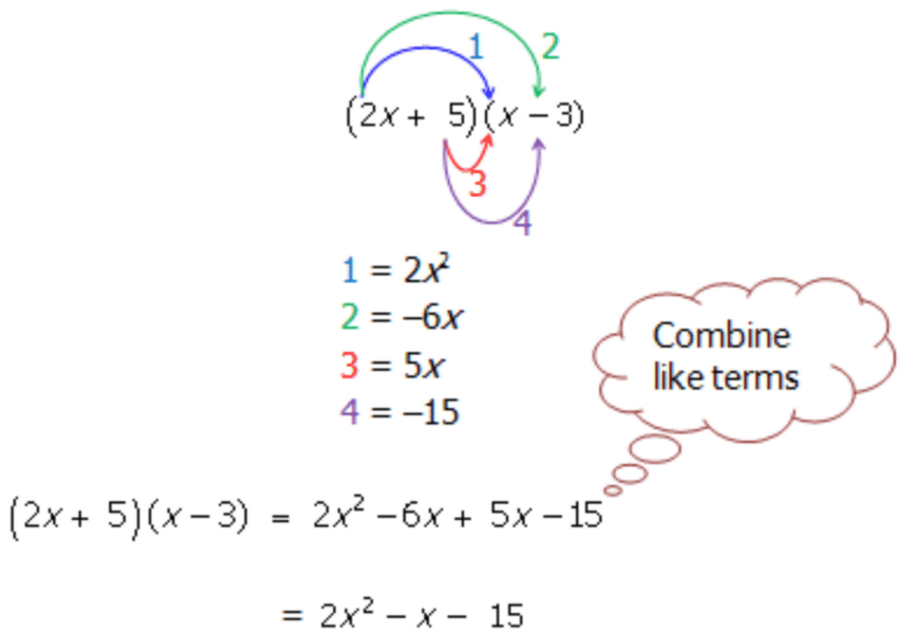
To answer this question you will use the distributive property. The distributive property would tell you to multiply x in the first set of parentheses by everything inside the second set of parentheses , then multiply 6 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



Multiply the ploynomials

Find the product: $(2x+5)(x-3)$

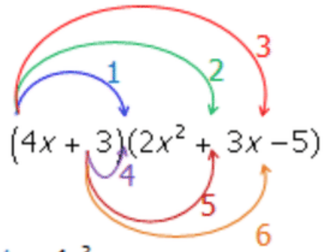
Again, use the distributive property. The distributive property tells you to multiply $2x$ in the first set of parentheses by everything inside the second set of parentheses , then multiply 5 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



Multiply the polynomials

Find the product: $(4x+3)(2x^2+3x-5)$

Even though at first this question may seem different, you can still use the distributive property to find the product. The distributive property tells you to multiply $4x$ in the first set of parentheses by everything inside the second set of parentheses, then multiply 3 in the first set of parentheses by everything in the second set of parentheses. Here is what that looks like:



$1 = 4x^3$
$2 = 12x^2$
$3 = -20x$
$4 = 6x^2$
$5 = 9x$
$6 = -15$

Combine like terms

$$(4x+3)(2x^2+3x-5) = 4x^3 + 12x^2 - 20x + 6x^2 + 9x - 15$$

$$= 4x^3 + 18x^2 - 11x - 15$$

Examples

Example 1

Earlier, you were given a problem about Jack.

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack measures the glass and finds the height to width ratio is 4:3. Write the expression to determine the area of the picture frame.

What is known?

- The width is 5 inches longer than the glass
- The height is 7 inches longer than the glass
- The glass has a height to width ratio of 4:3

The equations:

- The height of the picture frame is $4x+7$
- The width of the picture frame is $3x+5$

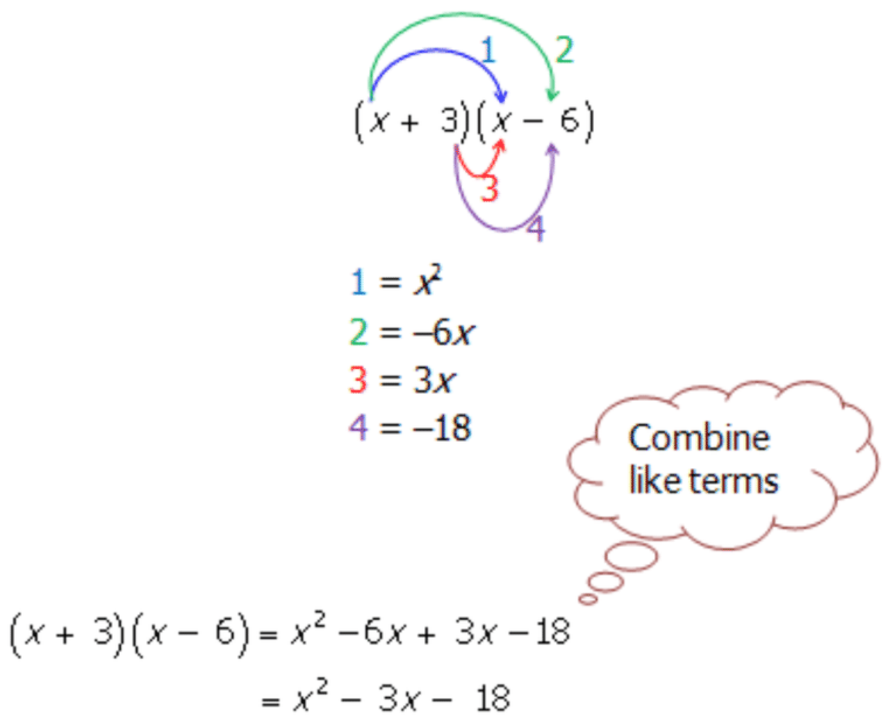
The formula:

$$\text{Area} = w \times h$$
$$\text{Area} = (3x+5)(4x+7)$$
$$\text{Area} = 12x^2 + 21x + 20x + 35$$
$$\text{Area} = 12x^2 + 41x + 35$$

Example 2

Find the product: $(x+3)(x-6)$

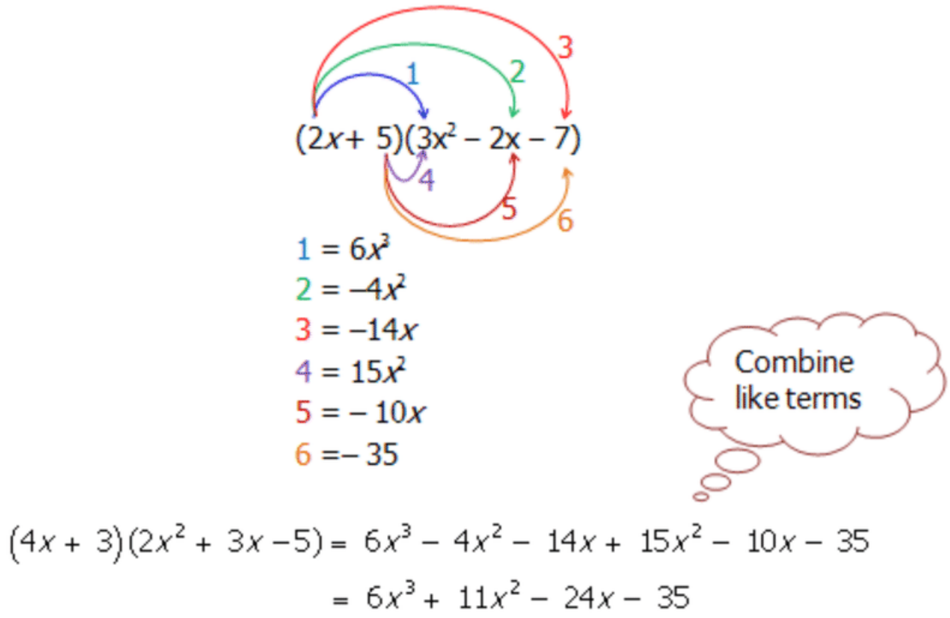
$(x+3)(x-6)$



Example 3

Find the product: $(2x+5)(3x^2-2x-7)$

$(2x+5)(3x^2-2x-7)$



Example 4

An average football field has the dimensions of 160 ft by 360 ft. If the expressions to find these dimensions were $(3x+7)$ and $(7x+3)$, what value of x would give the dimensions of the football field?

$\text{Area} = l \times w$

$$\begin{aligned}
\text{Area} &= 360 \times 160 \\
(7x+3) &= 360 \\
7x &= 360-3 \\
7x &= 357 \\
x &= 51
\end{aligned}$$

$$\begin{aligned}
(3x+7) &= 160 \\
3x &= 160-7 \\
3x &= 153 \\
x &= 51
\end{aligned}$$

The value of x that satisfies these expressions is 51.

Review

Use the distributive property to find the product of each of the following polynomials:

1. $(x+4)(x+6)$
2. $(x+3)(x+5)$
3. $(x+7)(x-8)$
4. $(x-9)(x-5)$
5. $(x-4)(x-7)$

6. $(x+3)(x^2+x+5)$
7. $(x+7)(x^2-3x+6)$
8. $(2x+5)(x^2-8x+3)$
9. $(2x-3)(3x^2+7x+6)$
10. $(5x-4)(4x^2-8x+5)$
11. $9a^2(6a^3+3a+7)$
12. $-4s^2(3s^3+7s^2+11)$
13. $(x+5)(5x^3+2x^2+3x+9)$
14. $(t-3)(6t^3+11t^2+22)$
15. $(2g-5)(3g^3+9g^2+7g+12)$

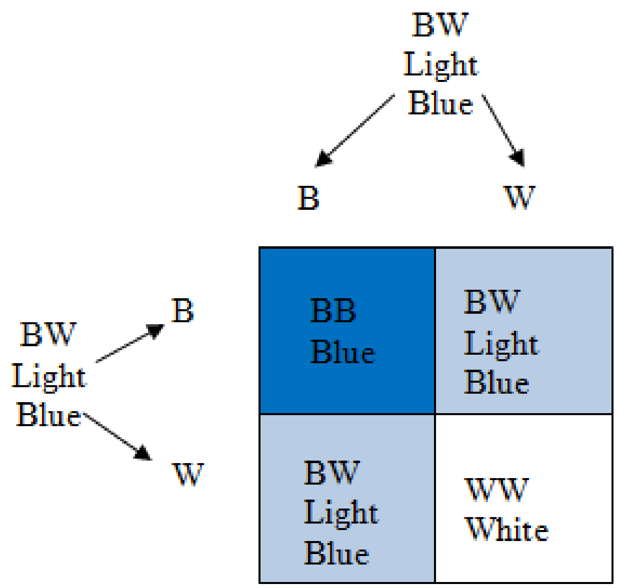
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.2.

7.3 Special Products of Polynomials

Here you will learn about special cases of binomial multiplication.

A flower is homozygous blue (RR) and another flower is homozygous white (rr). Use a Punnett square to show that a mixture of the two can produce a white flower.



Special Products of Polynomials

There are two special cases of multiplying binomials. If you can learn to recognize them, you can multiply these binomials more quickly.

Here are the two special products that you should learn to recognize:

Special Case 1 (Binomial Squared): $(x \pm y)^2 = x^2 \pm 2xy + y^2$

- Example: $(x+5)^2 = x^2 + 10x + 25$
- Example: $(2x-8)^2 = 4x^2 - 32x + 64$

Special Case 2 (Difference of Perfect Squares): $(x+y)(x-y) = x^2 - y^2$

- Example: $(5x+10)(5x-10) = 25x^2 - 100$
- Example: $(2x-4)(2x+4) = 4x^2 - 16$

Keep in mind that you can always use the distributive property to do the multiplications if you don't notice that the problem is a special case.

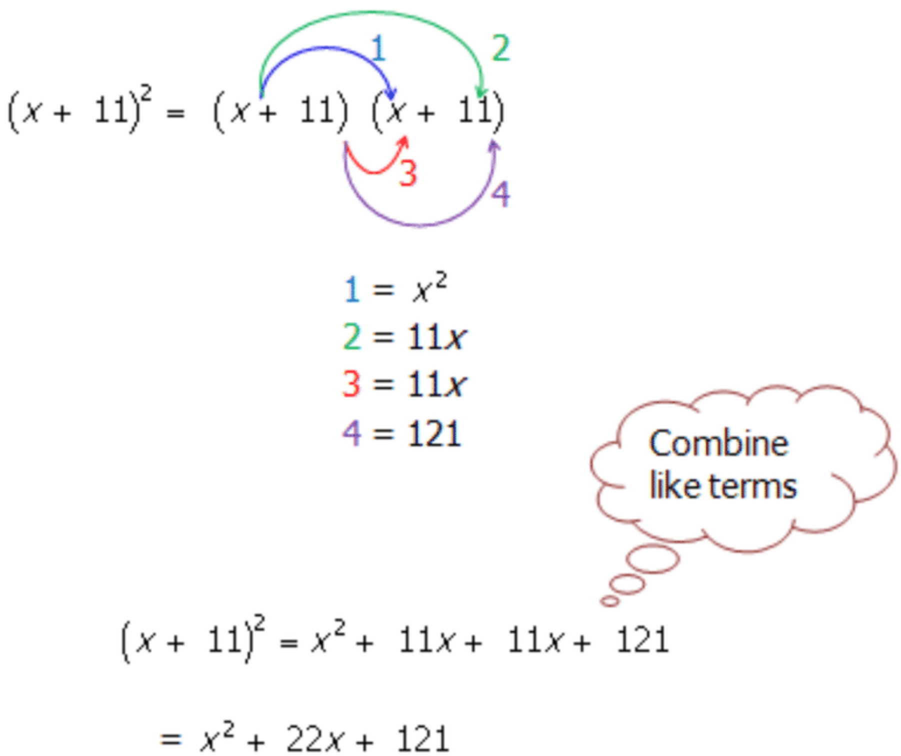
Multiply the polynomials

Find the product: $(x+11)^2$

These is an example of Special Case 1. You can use that pattern to quickly multiply.

$$(x+11)^2 = x^2 + 2 \cdot x \cdot 11 + 11^2$$
$$= x^2 + 22x + 121$$

You can verify that this is the correct answer by using the distributive property:



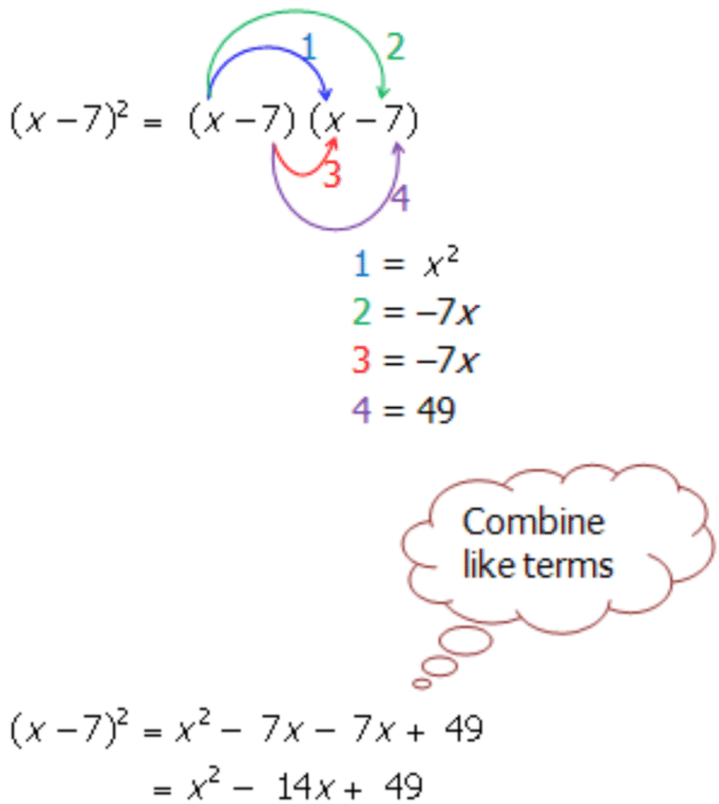
Multiply the polynomials

Find the product: $(x-7)^2$

These is another example of Special Case 1. You can use that pattern to quickly multiply.

$$(x-7)^2 = x^2 - 2 \cdot x \cdot 7 + 7^2$$
$$= x^2 - 14x + 49$$

You can verify that this is the correct answer by using the distributive property:



Multiply the polynomials

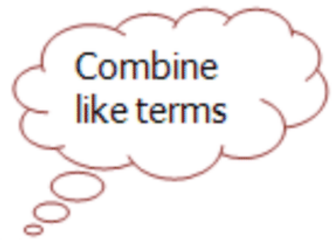
Find the product: $(x+9)(x-9)$

These is an example of Special Case 2. You can use that pattern to quickly multiply.

$$\begin{aligned}(x+9)(x-9) &= x^2-9^2\\ &= x^2-81\end{aligned}$$

You can verify that this is the correct answer by using the distributive property:

$$1 = x^2$$
$$2 = -9x$$
$$3 = 9x$$
$$4 = -81$$

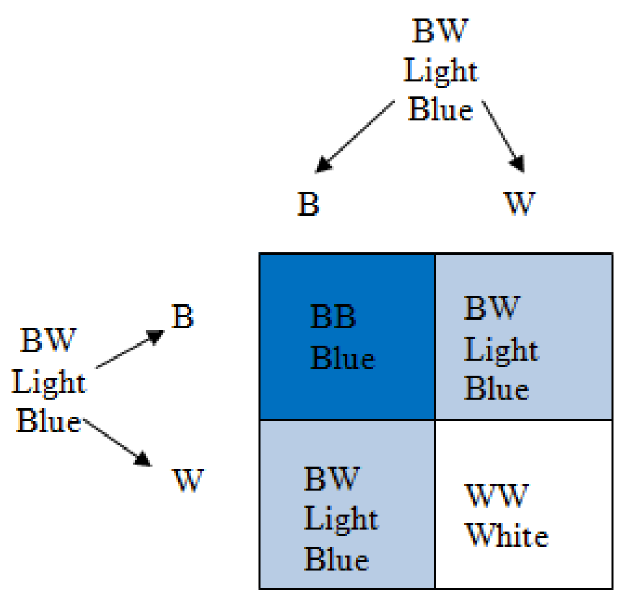


$$\begin{aligned}(x+9)(x-9) &= x^2 + 9x - 9x - 81\\ &= x^2 - 81\end{aligned}$$

Examples

Example 1

Earlier, you were asked to use a Punnett square to show that a mixture of the two flowers can produce a white flower.



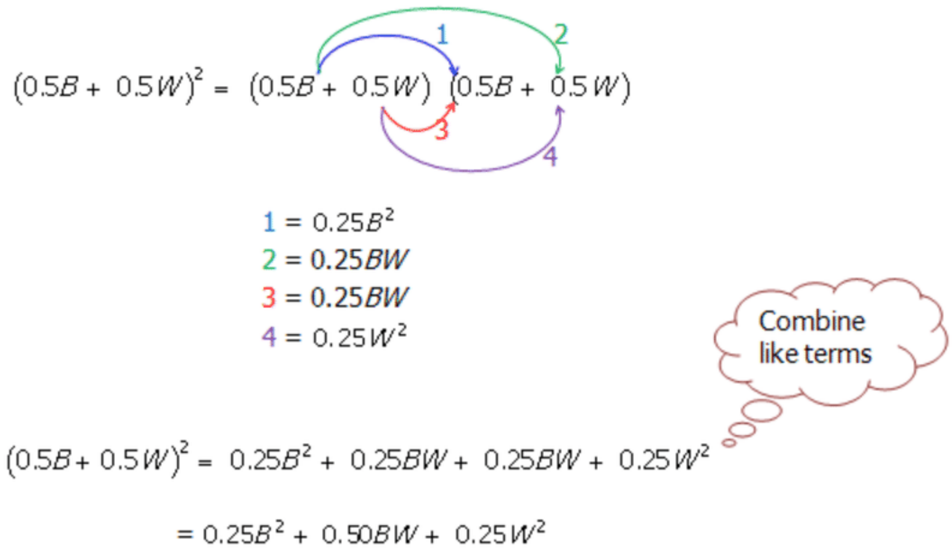
Each flower will have one-half of the blue genes and one-half of the white genes. Therefore the equation formed will be:

$0.5B+0.5W$

The offspring will have the genetic makeup (the mixture produced) using the equation:

$(0.5B+0.5W)^2$

Notice that this is an example of Special Case 1. You can expand the offspring genetic makeup equation to find out the percentage of offspring (or flowers) that will be blue, white, or light blue.



Therefore 25% of the offspring flowers will be blue, 50% will be light blue, and 25% will be white.

Example 2

Expand the following binomial: $(x+4)^2$.

$(x+4)^2 = x^2+4x+4x+16 = x^2+8x+16$.

Example 3

Expand the following binomial: $(5x-3)^2$.

$(5x-3)^2 = 25x^2-15x-15x+9 = 25x^2-30x+9$

Example 4

Determine whether or not each of the following is a difference of two perfect squares:

a) a^2-16

Yes, $a^2-16=(a+4)(a-4)$

b) $9b^2-49$

Yes, $9b^2-49=(3b+7)(3b-7)$

c) c^2-60

No, 60 is not a perfect square.

Review

Expand the following binomials:

1. $(t+12)^2$
2. $(w+15)^2$
3. $(2e+7)^2$
4. $(3z+2)^2$
5. $(7m+6)^2$
6. $(g-6)^2$
7. $(d-15)^2$
8. $(4x-3)^2$
9. $(2p-5)^2$
10. $(6t-7)^2$

Find the product of the following binomials:

11. $(x+13)(x-13)$
12. $(x+6)(x-6)$
13. $(2x+5)(2x-5)$
14. $(3x+4)(3x-4)$
15. $(6x+7)(6x-7)$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.3.

7.4 Monomial Factors of Polynomials

Here you will learn to find a common factor in a polynomial and factor it out of the polynomial.

Can you write the following polynomial as a product of a monomial and a polynomial?

$12x^4+6x^3+3x^2$

Monomial Factors of Polynomials

In the past you have studied common factors of two numbers. Consider the numbers 25 and 35. A common factor of 25 and 35 is 5 because 5 goes into both 25 and 35 evenly.

This idea can be extended to polynomials. A common factor of a polynomial is a number and/or variable that are a factor in all terms of the polynomial. The Greatest Common Factor (or GCF) is the largest monomial that is a factor of each of the terms of the polynomial.

To factor a polynomial means to write the polynomial as a product of other polynomials. One way to factor a polynomial is:

1. Look for the greatest common factor.
2. Write the polynomial as a product of the **greatest common factor** and **the polynomial that results when you divide all the terms of the original polynomial by the greatest common factor.**

One way to think about this type of factoring is that you are essentially doing the distributive property in reverse.

Factor

Factor the following binomial: $5a+15$

Step 1: Identify the GCF. Looking at each of the numbers, you can see that 5 and 15 can both be divided by 5. The GCF for this binomial is 5.

Step 2: Divide the GCF out of each term of the binomial:

$5a+15=5(a+3)$

Factor

Factor the following polynomial: $4x^2+8x-2$

Step 1: Identify the GCF. Looking at each of the numbers, you can see that 4, 8 and 2 can all be divided by 2. The GCF for this polynomial is 2.

Step 2: Divide the GCF out of each term of the polynomial:

$4x^2+8x-2=2(2x^2+4x-1)$

Factor

Factor the following polynomial: $3x^5-9x^3-6x^2$

Step 1: Identify the GCF. Looking at each of the terms, you can see that 3, 9 and 6 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$3x^5 - 9x^3 - 6x^2 = 3x^2(x^3 - 3x - 2)$

Examples

Example 1

Earlier, you were asked if you can write the following polynomial as a product of a monomial and a polynomial.

To write as a product you want to try to factor the polynomial: $12x^4 + 6x^3 + 3x^2$.

Step 1: Identify the GCF of the polynomial. Looking at each of the numbers, you can see that 12, 6, and 3 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$12x^4 + 6x^3 + 3x^2 = 3x^2(4x^2 + 2x + 1)$$

Example 2

Find the common factors of the following: $a^2(b+7) - 6(b+7)$

Step 1: Identify the GCF

This problem is a little different in that if you look at the expression you notice that $(b+7)$ is common in both terms. Therefore $(b+7)$ is the common factor. The GCF for this expression is $(b+7)$.

Step 2: Divide the GCF out of each term of the expression:

$$a^2(b+7) - 6(b+7) = (a^2 - 6)(b+7)$$

Example 3

Factor the following polynomial: $5k^6 + 15k^4 + 10k^3 + 25k^2$

Step 1: Identify the GCF. Looking at each of the numbers, you can see that 5, 15, 10, and 25 can all be divided by 5. Also notice that each of the terms has an k^2 in common. The GCF for this polynomial is $5k^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$5k^6 + 15k^4 + 10k^3 + 25k^2 = 5k^2(k^4 + 3k^2 + 2k + 5)$$

Example 4

Factor the following polynomial: $27x^3y + 18x^2y^2 + 9xy^3$

Step 1: Identify the GCF. Looking at each of the numbers, you can see that 27, 18 and 9 can all be divided by 9. Also notice that each of the terms has an xy in common. The GCF for this polynomial is $9xy$.

Step 2: Divide the GCF out of each term of the polynomial:
$27x^3y+18x^2y^2+9xy^3=9xy(3x^2+2xy+y^2)$

Review

Factor the following polynomials by looking for a common factor:

1. $7x^2+14$
2. $9c^2+3$
3. $8a^2+4a$
4. $16x^2+24y^2$
5. $2x^2-12x+8$
6. $32w^2x+16xy+8x^2$
7. $12abc+6bcd+24acd$
8. $15x^2y-10x^2y^2+25x^2y$
9. $12a^2b-18ab^2-24a^2b^2$
10. $4s^3t^2-16s^2t^3+12st^2-24st^3$

Find the common factors of the following expressions and then factor:

11. $2x(x-5)+7(x-5)$
12. $4x(x-3)+5(x-3)$
13. $3x^2(e+4)-5(e+4)$
14. $8x^2(c-3)-7(c-3)$
15. $ax(x-b)+c(x-b)$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.4.

7.5 Factorization of Quadratic Expressions

Here you'll learn how to factor quadratic expressions.

Jack wants to construct a border around two sides of his garden. The garden measures 5 yards by 18 yards. He has enough stone to build a border with a total area of 30 square yards. The border will be twice as wide on the shorter end. What are the dimensions of the border?

Factorization of Quadratic Expressions

To factor a polynomial means to write the polynomial as a product of other polynomials. Here, you'll focus on factoring quadratic expressions. Quadratic expressions are polynomials of degree 2, of the form ax^2+bx+c. Consider the steps for finding the product of the following binomials:

$$\begin{aligned}(2x+3)(3x-5) &= 6x^2-10x+9x-15\\ &= 6x^2-x-15\end{aligned}$$

When factoring a quadratic expression, your job will be to take an expression like $6x^2-x-15$ and write it as $(2x+3)(3x-5)$. You can think of factoring as the reverse of multiplying. Notice that when factored, the $6x^2$ factors to $2x$ and $3x$. The -15 factors to -5 and 3. You can say then, in general, that with the trinomial ax^2+bx+c, you have to factor both "a" and "c".

- $$ax^2+bx+c = (dx+e)(fx+g) \text{ where } a = d \times f \text{ and } c = e \times g$$
- The middle term (b) is $$b = dg+ef$$

Here you will work through a number of examples to develop mastery at factoring trinomials using a box method.

Factor:

$2x^2+11x+15$

First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the 'a' value is 2 and the 'c' value is 15. Start by making a box and placing these values in the box as shown.

2	
	15

The product of 2 and 15 is 30. To continue filling in the box, you need to find two numbers that multiply to 30, but add up to +11 (the value of b in the original equation). The two numbers that work are 5 and 6: $5+6=11$ and $5 \cdot 6 = 30$. Put 5 and 6 in the box.

2	6
5	15

Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 2 and 6, has a GCF of 2. The second row, 5 and 15, has a GCF of 5.

2	6	2
5	15	5

GCF for horizontal rows

The first column, 2 and 5, has a GCF of 1. The second column, 6 and 15, has a GCF of 3.

2	6	2
5	15	5
1	3	

GCF for vertical rows

Notice that the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(1x+3)$ and $(2x+5)$. You can verify that those binomials multiply to create the original trinomial: $(x+3)(2x+5) = 2x^2+5x+6x+15 = 2x^2+11x+15$.

The factored form of $2x^2+11x+15$ is $(x+3)(2x+5)$.

Factor:

$3x^2-8x-3$

First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the 'a' value is 3 and the 'c' value is -3. Start by making a box and placing these values in the box as shown.

3	
	–3

The product of 3 and -3 is -9. To continue filling in the box, you need to find two numbers that multiply to -9, but add up to -8 (the value of b in the original equation). The two numbers that work are -9 and 1. $-9+1=-8$ and $-9 \cdot 1=-9$. Put -9 and 1 in the box.

3	1
–9	–3

Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 3 and 1, has a GCF of 1. The second row, -9 and -3, has a GCF of -3.

3	1	1
–9	–3	–3

GCF for horizontal rows

The first column, 3 and -9, has a GCF of 3. The second column, 1 and -3, has a GCF of 1.

3	1	1
–9	–3	–3
3	1	

GCF for vertical rows

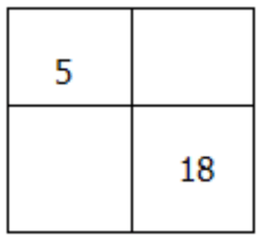
Notice that the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(3x+1)$ and $(1x-3)$. You can verify that those binomials multiply to create the original trinomial: $(3x+1)(x-3) = 3x^2 - 9x + 1x - 3 = 3x^2 - 8x - 3$.

The factored form of $3x^2 - 8x - 3$ is $(3x+1)(x-3)$.

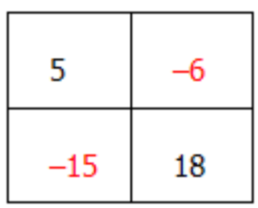
Factor:

$5w^2 - 21w + 18$

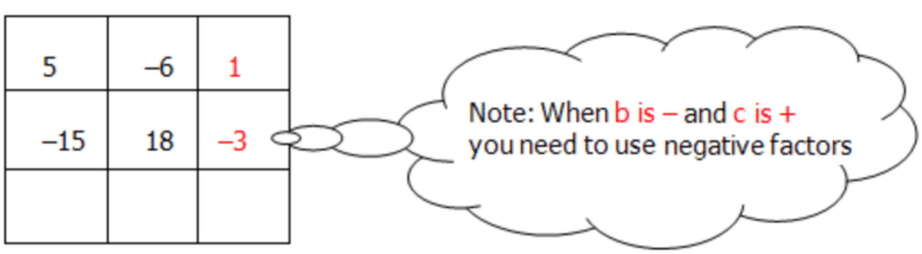
First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the 'a' value is 5 and the 'c' value is 18. Start by making a box and placing these values in the box as shown.



The product of 5 and 18 is 90. To continue filling in the box, you need to find two numbers that multiply to 90, but add up to -21 (the value of b is the original equation). The two numbers that work are -6 and -15. $-6 + (-15) = -21$ and $-6 \cdot -15 = 90$. Put -6 and -15 in the box.

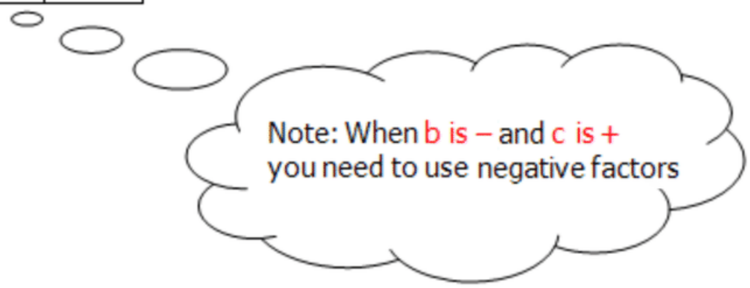


Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 5 and -6, has a GCF of 1. The second row, -15 and 18, has a GCF of 3.



The first column, 5 and -15, has a GCF of 5. The second column, -6 and 18, has a GCF of 6.

5	−6	1
−15	18	−3
5	−6	



Notice that you need to make two of the GCFs negative in order to make the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(5w-6)$ and $(w-3)$. You can verify that those binomials multiply to create the original trinomial: $(5w-6)(w-3) = 5w^2 - 15w - 6w + 18 = 5w^2 - 21w + 18$.

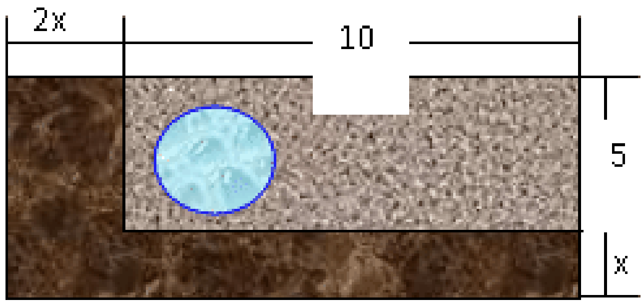
The factored form of $5w^2 - 21w + 18$ is $(5w-6)(w-3)$.

Examples

Example 1

Earlier, you were given a story about Jack.

Jack wants to construct a border around two sides of his garden. The garden measures 5 yards by 18 yards. He has enough stone to build a border with a total area of 30 square yards. The border will be twice as wide on the shorter end. What are the dimensions of the border?



$$\begin{aligned}
\text{Area of Garden} &= 18 \times 5 = 90\ yd^2 \\
\text{Area of border} &= 30\ yd^2 \\
\text{Area of Garden} + \text{border} &= (18+2x)(5+x) \\
\text{Area of border} &= (\text{Area of garden} + \text{border}) - \text{Area of garden} \\
30 &= (18+2x)(5+x) - 90 \\
30 &= 90 + 18x + 10x + 2x^2 - 90 \\
30 &= 28x + 2x^2 \\
0 &= 2x^2 + 28x - 30
\end{aligned}$$

This trinomial has a common factor of 2. First, factor out this common factor:

$2x^2 + 28x - 30 = 2(x^2 + 14x - 15)$

Now, you can use the box method to factor the remaining trinomial. After using the box method, your result should be:

$2(x^2 + 14x - 15) = 2(x+15)(x-1)$

To find the dimensions of the border you need to solve a quadratic equation. This is explored in further detail in another concept:

$$\begin{array}{cc}
\multicolumn{2}{c}{2(x+15)(x-1) = 0} \\
\swarrow & \searrow \\
x+15 = 0 & x-1 = 0 \\
x = -15 & x = 1
\end{array}$$

Since x cannot be negative, x must equal 1.

Width of Border: $2x = 2(1) = 2\ yd$

Length of Border: $x = 1\ yd$

Example 2

Factor the following trinomial: $8c^2 - 2c - 3$

Use the box method and you find that $8c^2 - 2c - 3 = (2c+1)(4c-3)$

Example 3

Factor the following trinomial: $3m^2 + 3m - 60$

First you can factor out the 3 from the polynomial. Then, use the box method. The final answer is $3m^2 + 3m - 60 = 3(m-4)(m+5)$.

Example 4

Factor the following trinomial: $5e^3 + 30e^2 + 40e$

First you can factor out the $5e$ from the polynomial. Then, use the box method. The final answer is $5e^3 + 30e^2 + 40e = 5e(e+2)(e+4)$.

Review

Factor the following trinomials.

1. x^2+5x+4
2. $x^2+12x+20$
3. $a^2+13a+12$
4. $z^2+7z+10$
5. $w^2+8w+15$
6. $x^2-7x+10$
7. $x^2-10x+24$
8. m^2-4m+3
9. s^2-6s+7
10. $y^2-8y+12$
11. x^2-x-12
12. x^2+x-12
13. $x^2-5x-14$
14. $x^2-7x-44$
15. y^2+y-20
16. $3x^2+5x+2$
17. $5x^2+9x-2$
18. $4x^2+x-3$
19. $2x^2+7x+3$
20. $2y^2-15y-8$
21. $2x^2-5x-12$
22. $2x^2+11x+12$
23. $6w^2-7w-20$
24. $12w^2+13w-35$
25. $3w^2+16w+21$
26. $16a^2-18a-9$
27. $36a^2-7a-15$
28. $15a^2+26a+8$
29. $20m^2+11m-4$
30. $3p^2+17p-20$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.5.

7.6 Special Cases of Quadratic Factorization

Here you'll learn to recognize two special kinds of quadratics and how to factor them quickly.

A box is to be designed for packaging with a side length represented by the quadratic $9b^2 - 64$. If this is the most economical box, what are the dimensions?

Quadratic Factorization

When factoring quadratics, there are special cases that can be factored more quickly. There are two special quadratics that you should learn to recognize:

Special Case 1 (Perfect Square Trinomial): $x^2 \pm 2xy + y^2 = (x \pm y)^2$

- Example: $x^2 + 10x + 25 = (x+5)^2$
- Example: $4x^2 - 32x + 64 = (2x-8)^2$

Special Case 2 (Difference of Perfect Squares): $x^2 - y^2 = (x+y)(x-y)$

- Example: $25x^2 - 100 = (5x+10)(5x-10)$
- Example: $4x^2 - 25 = (2x-5)(2x+5)$

Keep in mind that you can always use the box method to do the factoring if you don't notice the problem as a special case.

Factor

$2x^2 + 28x + 98$.

First, notice that there is a common factor of 2. Factor out the common factor:

$2x^2 + 28x + 98 = 2(x^2 + 14x + 49)$

Next, notice that the first and last terms are both perfect squares ($x^2 = x \cdot x$ and $49 = 7 \cdot 7$, and the middle term is 2 times the product of the roots of the other terms ($14x = 2 \cdot x \cdot 7$). This means $x^2 + 14x + 49$ is a perfect square trinomial (Special Case 1). Using the pattern:

$x^2 + 14x + 49 = (x+7)^2$

Therefore, the complete factorization is $2x^2 + 28x + 98 = 2(x+7)^2$.

Factor

$8a^2 - 24a + 18$.

First, notice that there is a common factor of 2. Factor out the common factor:

$8a^2 - 24a + 18 = 2(4a^2 - 12a + 9)$

Next, notice that the first and last terms are both perfect squares and the middle term is 2 times the product of the roots of the other terms ($12a = 2 \cdot 2a \cdot 3$). This means $4a^2 - 12a + 9$ is a perfect square trinomial (Special Case 1). Because the middle term is negative, there will be a negative in the binomial. Using the pattern:

$4a^2 - 12a + 9 = (2a - 3)^2$

Therefore, the complete factorization is $8a^2 - 24a + 18 = 2(2a - 3)^2$.

Factor

$x^2 - 16$.

Notice that there are no common factors. The typical middle term of the quadratic is missing and each of the terms present are perfect squares and being subtracted. This means $x^2 - 16$ is a difference of perfect squares (Special Case 2). Using the pattern:

$x^2 - 16 = (x - 4)(x + 4)$

Note that it would also be correct to say $x^2 - 16 = (x + 4)(x - 4)$. It does not matter whether you put the + version of the binomial first or the - version of the binomial first.

Examples

Example 1

Earlier, you were given a problem about a box.

A box is to be designed for packaging with a side length represented by the quadratic $9b^2 - 64$. If this is the most economical box, what are the dimensions?

First: factor the quadratic to find the value for b.

$9b^2 - 64$

This is a difference of perfect squares (Special Case 2). Use that pattern:

$9b^2 - 64 = (3b - 8)(3b + 8)$

To finish this problem we need to **solve** a quadratic equation. This idea is explored in further detail in another concept.

$$
\begin{array}{cc}
\multicolumn{2}{c}{9b^2 - 64 = (3b + 8)(3b - 8)} \\
\swarrow & \searrow \\
3b + 8 = 0 & 3b - 8 = 0 \\
3b = -8 & 3b = 8 \\
b = \dfrac{-8}{3} & b = \dfrac{8}{3}
\end{array}
$$

The most economical box is a cube. Therefore the dimensions are $\frac{8}{3} \times \frac{8}{3} \times \frac{8}{3}$

Example 2

Factor completely $s^2 - 18s + 81$

This is Special Case 1. $s^2-18s+81=(s-9)^2$

Example 3

Factor completely $50-98x^2$

First factor out the common factor of 2. Then, it is Special Case 2. $50-98x^2=2(5-7x)(5+7x)$

Example 4

Factor completely $4x^2+48x+144$

First factor out the common factor of 4. Then, it is Special Case 1. $4x^2+48x+144=4(x+6)^2$

Review

Factor each of the following:

1. $s^2+18s+81$
2. $x^2+12x+36$
3. $y^2-14y+49$
4. $4a^2+20a+25$
5. $9s^2-48s+64$
6. s^2-81
7. x^2-49
8. $4t^2-25$
9. $25w^2-36$
10. $64-81a^2$
11. $y^2-22y+121$
12. $16t^2-49$
13. $9a^2+30a+25$
14. $100-25b^2$
15. $4s^2-28s+49$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.6.

7.7 Zero Product Property for Quadratic Equations

Here you'll learn how to solve a quadratic equation by factoring and using the zero product property.

The area of a particular rectangle was found to be $A(w) = w^2 - 8w - 58$. Determine the dimensions of the rectangle if the area was known to be 7 units.

Zero Product Property

Recall that when solving an equation, you are trying to determine the values of the variable that make the equation true. For the equation $2x^2 + 10x + 8 = 0$, $x = -1$ and $x = -4$ are both solutions. You can check this:

- $2(-1)^2 + 10(-1) + 8 = 2(1) - 10 + 8 = 0$
- $2(-4)^2 + 10(-4) + 8 = 2(16) - 40 + 8 = 32 - 40 + 8 = 0$

Here you will focus on solving quadratic equations. One of the methods for quadratic equations utilizes your factoring skills and a property called the **zero product property**.

If $a \cdot b = 0$, what can you say about a or b? What you should realize is that either a or b have to be equal to 0, because that is the only way that their product will be 0. If both a and b were non-zero, then their product would have to be non-zero. This is the idea of the zero product property. The zero product property states that if the product of two quantities is zero, then one or both of the quantities must be zero.

The zero product property has to do with products being equal to zero. When you factor, you turn a quadratic expression into a product. If you have a quadratic expression equal to zero, you can factor it and then use the zero product property to solve. So, if you were given the equation $2x^2 + 5x - 3 = 0$, first you would want to turn the quadratic expression into a product by factoring it:

$2x^2 + 5x - 3 = (x + 3)(2x - 1)$

You can rewrite the equation you are trying to solve as $(x + 3)(2x - 1) = 0$.

Now, you have the product of two binomials equal to zero. This means at least one of those binomials must be equal to zero. So, you have two mini-equations that you can solve to find the values of x that cause each binomial to be equal to zero.

- $x + 3 = 0$, which means $x = -3$ OR
- $2x - 1 = 0$, which means $x = \frac{1}{2}$

The two solutions to the equation $2x^2 + 5x - 3 = 0$ are $x = -3$ and $x = \frac{1}{2}$.

Keep in mind that you can only use the zero product property if your equation is set equal to zero! If you have an equation not set equal to zero, first rewrite it so that it is set equal to zero. Then factor and use the zero product property.

Solve for

$x^2 + 5x + 6 = 0$.

First, change x^2+5x+6 into a product so that you can use the zero product property. Change the expression into a product by factoring:

$x^2+5x+6=(x+3)(x+2)$

Next, rewrite the equation you are trying to solve:

$x^2+5x+6=0$ becomes $(x+3)(x+2)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $x+3=0$, which means that $x=-3$
- $x+2=0$, which means that $x=-2$

The solutions are $x=-3$ or $x=-2$.

Solve for

$6x^2+x-15=0$.

In order to solve for x you need to factor the polynomial.

First, change $6x^2+x-15$ into a product so that you can use the zero product property. Change the expression into a product by factoring:

$6x^2+x-15=(3x+5)(2x-3)$

Next, rewrite the equation you are trying to solve:

$6x^2+x-15=0$ becomes $(3x+5)(2x-3)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $3x+5=0$, which means that $x=-\frac{5}{3}$
- $2x-3=0$, which means that $x=\frac{3}{2}$

The solutions are $x=-\frac{5}{3}$ or $x=\frac{3}{2}$.

Solve for

$x^2+2x-35=0$.

First, change $x^2+2x-35$ into a product so that you can use the zero product property. Change the expression into a product by factoring:

$x^2+2x-35=(x+7)(x-5)$

Next, rewrite the equation you are trying to solve:

$x^2+2x-35=0$ becomes $(x+7)(x-5)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $x+7=0$, which means that $x=-7$
- $x-5=0$, which means that $x=5$

The solutions are $x=-7$ or $x=5$.

Examples

Example 1

Earlier, you were given a problem about a rectangle.

The area of a particular rectangle was found to be $A(w)=w^2-8w-58$. Determine the dimensions of the rectangle if the area was known to be 7 units.

In other words, you are being asked to solve the problem:

$$\begin{aligned}&w^2-8w-58=7\\&OR\\&w^2-8w-65=0\end{aligned}$$

You can solve this problem by factoring and using the zero product property.

$w^2-8w-65=0$ becomes $(w+5)(w-13)=0$

$$\begin{array}{ccc} & (w+5)(w-13)=0 & \\ \swarrow & & \searrow \\ w+5=0 & & w-13=0 \\ w=-5 & or & w=13 \end{array}$$

Since you are asked for dimensions, a width of -5 units does not make sense. Therefore for the rectangle, the width would be 13 units.

Example 2

Solve for the variable in the polynomial: $x^2+4x-21=0$

$x^2+4x-21=(x-3)(x+7)$

$$\begin{array}{ccc} & (x-3)(x+7)=0 & \\ \swarrow & & \searrow \\ (x-3)=0 & & (x+7)=0 \\ x=3 & & x=-7 \end{array}$$

Example 3

Solve for the variable in the polynomial: $20m^2+11m-4=0$

$20m^2+11m-4=(4m-1)(5m+4)$

$$
\begin{array}{cc}
\multicolumn{2}{c}{(4m-1)(5m+4)=0} \\
\swarrow & \searrow \\
4m-1=0 & 5m+4=0 \\
4m=1 & 5m=-4 \\
m=\dfrac{1}{4} & m=\dfrac{-4}{5}
\end{array}
$$

Example 4

Solve for the variable in the polynomial: $2e^2+7e+6=0$

$2e^2+7e+6=(2e+3)(e+2)$

$$
\begin{array}{cc}
\multicolumn{2}{c}{(2e+3)(e+2)=0} \\
\swarrow & \searrow \\
2e+3=0 & e+2=0 \\
2e=-3 & e=-2 \\
e=\dfrac{-3}{2} &
\end{array}
$$

Review

Solve for the variable in each of the following equations.

1. $(x+1)(x-3)=0$
2. $(a+3)(a+5)=0$
3. $(x-5)(x+4)=0$
4. $(2t-4)(t+3)=0$
5. $(x-8)(3x-7)=0$
6. $x^2+x-12=0$
7. $b^2+2b-24=0$
8. $t^2+3t-18=0$
9. $w^2+3w-108=0$
10. $e^2-2e-99=0$
11. $6x^2-x-2=0$
12. $2d^2+14d-16=0$
13. $3s^2+20s+12=0$
14. $18x^2+12x+2=0$
15. $3j^2-17j+10=0$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.7.

7.8 Applications of Quadratic Equations

Here you'll learn how to apply your knowledge of factoring quadratic expressions to solve real world application problems.

Two cars leave an intersection at the same time. One car travels north and the other car travels west. When the car traveling north had gone 24 miles, the distance between the cars was four miles more than three times the distance traveled by the car heading west. Find the distance between the cars at that time.

Quadratic Equations

Quadratic functions can be used to help solve many different real world problems. Here are two hints for solving quadratic word problems:

1. It is often helpful to start by drawing a picture in order to visualize what you are asked to solve.
2. Once you have solved the problem, it is important to make sure that your answers are realistic given the context of the problem. For example, if you are solving for the age of a person and one of your answers is a negative number, that answer does not make sense in the context of the problem and is not actually a solution.

Write as a quadratic function

The number of softball games that must be scheduled in a league with n teams is given by $G(n) = \frac{n^2-n}{2}$. Each team can only play every other team exactly once. A league schedules 21 games. How many softball teams are in the league?

You are given the function $G(n) = \frac{n^2-n}{2}$ and you are asked to find n when $G(n) = 21$. This means, you have to solve the equation:

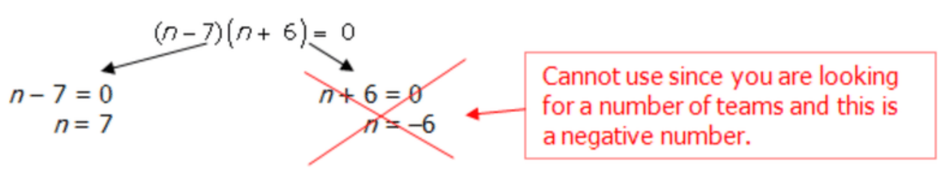
$21 = \frac{n^2-n}{2}$

Start by setting the equation equal to zero:

$$\begin{aligned} 42 &= n^2 - n \\ n^2 - n - 42 &= 0 \end{aligned}$$

Now solve for n to find the number of teams (n) in the league. Start by factoring the left side of the equation and rewriting the equation:

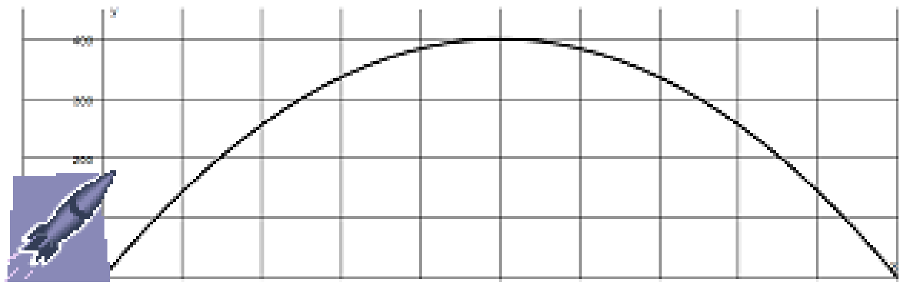
$n^2 - n - 42 = 0$ becomes $(n-7)(n+6) = 0$



There are 7 teams in the softball league.

Write as a quadratic function

When a home-made rocket is launched from the ground, it goes up and falls in the pattern of a parabola. The height, in feet, of a home-made rocket is given by the equation $h(t) = 160t - 16t^2$ where t is the time in seconds. How long will it take for the rocket to return to the ground?



The formula for the path of the rocket is $h(t) = 160t - 16t^2$. You are asked to find t when $h(t) = 0$, or when the rocket hits the ground and no longer has height. Start by factoring:

$160t - 16t^2 = 0$ becomes $16t(10 - t) = 0$

This means $16t = 0$ (so $t = 0$) or $10 - t = 0$ (so $t = 10$). $t = 0$ represents the rocket being on the ground when it starts, so it is not the answer you are looking for. $t = 10$ represents the rocket landing back on the ground.

The rocket will hit the ground after 10 seconds.

Solve for t

Using the information in the previous problem, what is the height of the rocket after 2 seconds?

To solve this problem, you need to replace t with 2 in the quadratic function.

$$\begin{aligned}
h(t) &= 160t - 16t^2 \\
h(2) &= 160(2) - 16(2)^2 \\
h(2) &= 320 - 64 \\
h(2) &= 256.
\end{aligned}$$

Therefore, after 2 seconds, the height of the rocket is 256 feet.

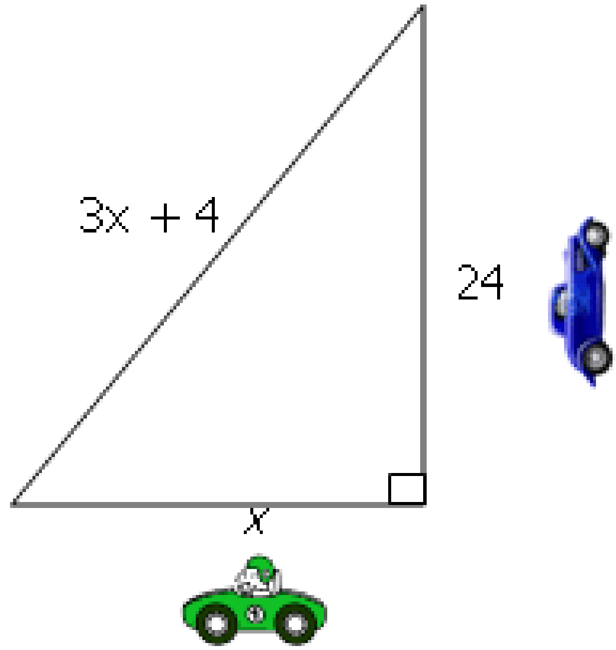
Examples

Example 1

Earlier, you were given a problem about two cars.

Two cars leave an intersection at the same time. One car travels north and the other car travels west. When the car traveling north had gone 24 miles the distance between the cars was four miles more than three times the distance traveled by the car heading west. Find the distance between the cars at that time.

First draw a diagram. Since the cars are traveling north and west from the same starting position, the triangle made to connect the distance between them is a right triangle. Since you have a right triangle, you can use the Pythagorean Theorem to set up an equation relating the lengths of the sides of the triangle.

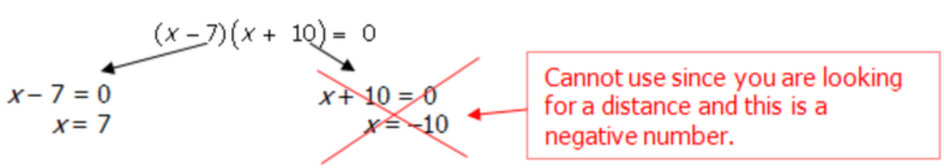


The Pythagorean Theorem is a geometry theorem that says that for all right triangles, $a^2+b^2=c^2$ where a and b are legs of the triangle and c is the longest side of the triangle, the hypotenuse. The equation for this problem is:

$$\begin{aligned}
x^2+24^2 &= (3x+4)^2\\
x^2+576 &= (3x+4)(3x+4)\\
x^2+576 &= 9x^2+24x+16
\end{aligned}$$

Now set the equation equal to zero and factor the quadratic expression so that you can use the zero product property.

$$\begin{aligned}
x^2+576 &= 9x^2+24x+16\\
0 &= 8x^2+24x-560\\
0 &= 8(x^2+3x-70)\\
0 &= 8(x-7)(x+10)
\end{aligned}$$

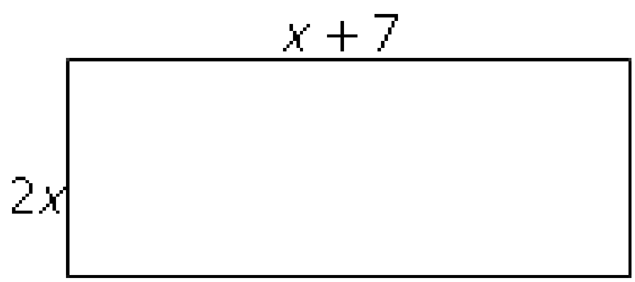


So you now know that $x=7$. Since the distance between the cars is represented by the expression $3x+4$, the actual distance between the two cars after the car going north has traveled 24 miles is:

$$\begin{aligned}
3x+4 &= 3(7)+4\\
&= 21+4\\
&= 25 \ miles
\end{aligned}$$

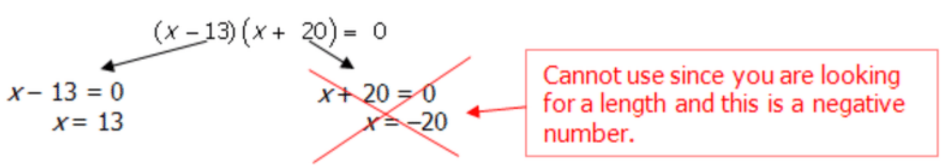
Example 2

A rectangle is known to have an area of 520 square inches. The lengths of the sides are shown in the diagram below. Solve for both the length and the width.



The rectangle has an area of 520 square inches and you know that the area of a rectangle has the formula: $A = l \times w$. Therefore:

$$\begin{aligned}
520 &= (x+7)(2x)\\
520 &= 2x^2 + 14x\\
0 &= 2x^2 + 14x - 520\\
0 &= 2(x^2 + 7x - 260)\\
0 &= 2(x-13)(x+20)
\end{aligned}$$

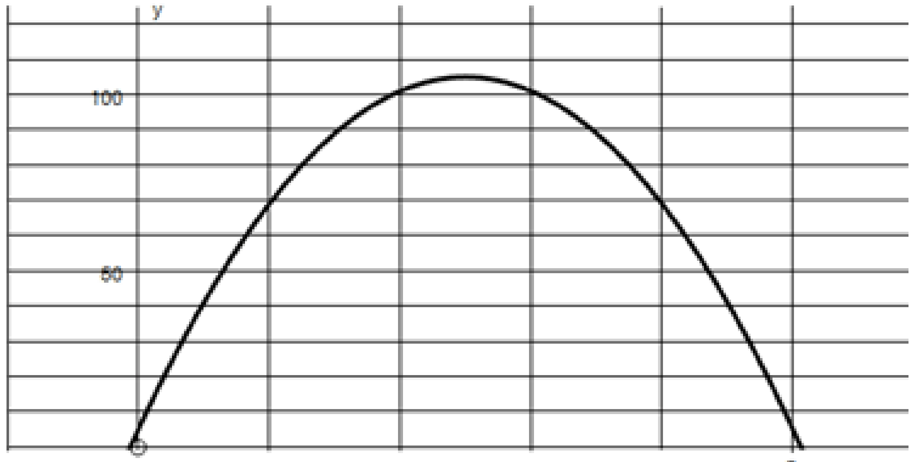


Therefore the value of x is 13. This means that the width is $2x$ or $2(13) = 26$ *inches*. The length is $x+7 = 13+7 = 20$ *inches*.

Example 3

The height of a ball in feet can be found by the quadratic function $h(t) = -16t^2 + 80t + 5$ where t is the time in seconds that the ball is in the air. Determine the time(s) at which the ball is 69 feet high.

The equation for the ball being thrown is $h(t) = -16t^2 + 80t + 5$. If you drew the path of the thrown ball, you would see something like that shown below.



You are asked to find the time(s) when the ball hits a height of 69 feet. In other words, solve for:

$$69 = -16t^2 + 80t + 5$$

To solve for t, you have to factor the quadratic and then solve for the value(s) of t.

$$\begin{aligned} 0 &= -16t^2 + 80t - 64 \\ 0 &= -16(t^2 - 5t + 4) \\ 0 &= -16(t-1)(t-4) \end{aligned}$$

$$\begin{array}{cc} \swarrow & \searrow \\ t-1=0 & t-4=0 \\ t=1 & t=4 \end{array}$$

Since both values are positive, you can conclude that there are two times when the ball hits a height of 69 feet. These times are at 1 second and at 4 seconds.

Example 4

A manufacturer measures the number of cell phones sold using the binomial $0.015c + 2.81$. She also measures the wholesale price on these phones using the binomial $0.011c + 3.52$. Calculate her revenue if she sells 100,000 cell phones.

The number of cell phones sold is the binomial $0.015c + 2.81$. The wholesale price on these phones is the binomial $0.011c + 3.52$. The revenue she takes in is the wholesale price times the number that she sells. Therefore:

$R(c) = (0.015c + 2.81)(0.011c + 3.52)$
First, let's expand the expression for R to get the quadratic expression. Therefore:

$$\begin{aligned} R(c) &= (0.015c + 2.81)(0.011c + 3.52) \\ R(c) &= 0.000165c^2 + 0.08371c + 9.8912 \end{aligned}$$

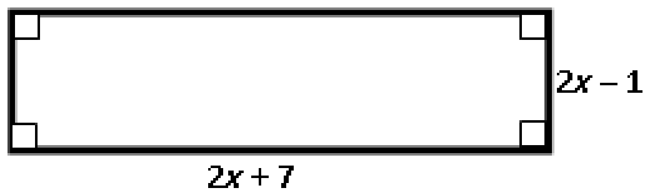
The question then asks if she sold 100,000 cell phones, what would her revenue be. Therefore what is $R(c)$ when $c = 100,000$.

$$R(c) = 0.000165c^2 + 0.08371c + 9.8912$$
$$R(c) = 0.000165(100,000)^2 + 0.08371(100,000) + 9.8912$$
$$R(c) = 1,658,380.89$$

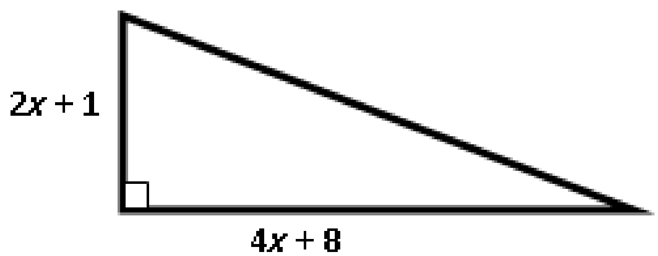
Therefore she would make $1,658,380.89 in revenue.

Review

1. A rectangle is known to have an area of 234 square feet. The length of the rectangle is given by $x+3$ and the width of the rectangle is given by $x+8$. What is the value of x?
2. Solve for x in the rectangle below given that the area is 9 units.



3. Solve for x in the triangle below given that the area is 10 units.



A pool is treated with a chemical to reduce the amount of algae. The amount of algae in the pool t days after the treatment can be approximated by the function $A(t) = 40t^2 - 300t + 500$.

4. How many days after treatment will the pool have the no algae?
5. How much algae is in the pool before treatments are started?
6. How much less algae is in the pool after 1 day?

A football is kicked into the air. The height of the football in meters can be found by the quadratic function $h(t) = -5t^2 + 25t$ where t is the time in seconds since the ball has been kicked.

7. How high is the ball after 3 seconds? At what other time is the ball the same height?
8. When will the ball be 20 meters above the ground?
9. After how many seconds will the ball hit the ground?

A ball is thrown into the air. The height of the ball in meters can be found by the quadratic function $h(t) = -5t^2 + 30t$ where t is the time in seconds since the ball has been thrown.

10. How high is the ball after 3 seconds?
11. When will the ball be 25 meters above the ground?
12. After how many seconds will the ball hit the ground?

Kim is drafting the windows for a new building. Their shape can be modeled by the function $h(w) = -w^2 + 4$, where h is the height and w is the width of points on the window frame, measured in meters.

13. Find the width of each window at its base.
14. Find the width of each window when the height is 3 meters.
15. What is the height of the window when the width is 1 meter?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.8.

7.9 Complete Factorization of Polynomials

Here you will learn how to factor a polynomial completely by first looking for common factors and then factoring the resulting expression.

Can you factor the following polynomial completely?

$8x^3 + 24x^2 - 32x$

Factorizing Polynomials

A cubic polynomial is a polynomial of degree equal to 3. Examples of cubics are:

- $9x^3 + 10x - 5$
- $8x^3 + 2x^2 - 5x - 7$

Recall that to factor a polynomial means to rewrite the polynomial as a product of other polynomials . You will not be able to factor all cubics at this point, but you will be able to factor some using your knowledge of common factors and factoring quadratics. In order to attempt to factor a cubic, you should:

1. Check to see if the cubic has any common factors. If it does, factor them out.
2. Check to see if the resulting expression can be factored, especially if the resulting expression is a quadratic. To factor the quadratic expression you could use the box method, or any method you prefer.

Anytime you are asked to **factor completely**, you should make sure that none of the pieces (factors) of your final answer can be factored any further. If you follow the steps above of first checking for common factors and then checking to see if the resulting expressions can be factored, you can be confident that you have factored completely.

Factor

Factor the following polynomial completely: $3x^3 - 15x$.

Look for the common factors in each of the terms. The common factor is $3x$. Therefore:

$3x^3 - 15x = 3x(x^2 - 5)$

The resulting quadratic, $x^2 - 5$, cannot be factored any further (it is NOT a difference of perfect squares). Your answer is $3x(x^2 - 5)$.

Factor

Factor the following polynomial completely: $2a^3 + 16a^2 + 30a$.

Look for the common factors in each of the terms. The common factor is $2a$. Therefore:

$2a^3 + 16a^2 + 30a = 2a(a^2 + 8a + 15)$

The resulting quadratic, $a^2 + 8a + 15$ can be factored further into $(a+5)(a+3)$. Your final answer is $2a(a+5)(a+3)$.

Factor

Factor the following polynomial completely: $6s^3+36s^2-18s-42$.

Look for the common factors in each of the terms. The common factor is 6. Therefore:

$6s^3+36s^2-18s-42=6(s^3+6s^2-3s-7)$

The resulting expression is a cubic, and you don't know techniques for factoring cubics without common factors at this point. Therefore, your final answer is $6(s^3+6s^2-3s-7)$.

Note: It turns out that the resulting cubic cannot be factored, even with more advanced techniques. Remember that not all expressions can be factored. In fact, in general most expressions ***cannot*** *be factored.*

Examples

Example 1

Earlier, you were asked to factor a polynomial completely.

Factor the following polynomial completely: $8x^3+24x^2-32x$.

Look for the common factors in each of the terms. The common factor is $8x$. Therefore:

$8x^3+24x^2+32x=8x(x^2+3x-4)$

The resulting quadratic can be factored further into $(x+4)(x-1)$. Your final answer is $8x(x+4)(x-1)$.

Example 2

Factor the following polynomial completely.

$9w^3+12w$.

The common factor is $3w$. Therefore, $9w^3+12w=3w(3w^2+4)$. The resulting quadratic cannot be factored any further, so your answer is $3w(3w^2+4)$.

Example 3

Factor the following polynomial completely.

y^3+4y^2+4y.

The common factor is y. Therefore, $y^3+4y^2+4y=y(y^2+4y+4)$. The resulting quadratic can be factored into $(y+2)(y+2)$ or $(y+2)^2$. Your answer is $y(y+2)^2$.

Example 4

Factor the following polynomial completely.

$2t^3-10t^2+8t$.

The common factor is $2t$. Therefore, $2t^3-10t^2+8t=2t(t^2-5t+4)$. The resulting quadratic can be factored into $(t-4)(t-1)$. Your answer is $2t(t-4)(t-1)$.

Review

Factor each of the following polynomials completely.

1. $6x^3 - 12$
2. $4x^3 - 12x^2$
3. $8y^3 + 32y$
4. $15a^3 + 30a^2$
5. $21q^3 + 63q$
6. $4x^3 - 12x^2 - 8$
7. $12e^3 + 6e^2 - 6e$
8. $15s^3 - 30s + 45$
9. $22r^3 + 66r^2 + 44r$
10. $32d^3 - 16d^2 + 12d$
11. $5x^3 + 15x^2 + 25x - 30$
12. $3y^3 - 18y^2 + 27y$
13. $12s^3 - 24s^2 + 36s - 48$
14. $8x^3 + 24x^2 - 80x$
15. $5x^3 - 25x^2 - 70x$

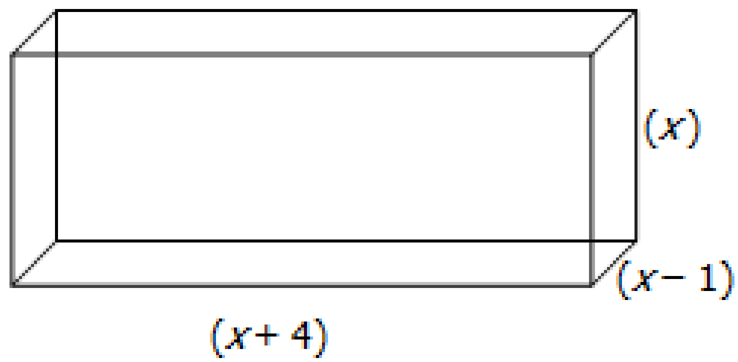
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.9.

7.10 Factorization by Grouping

Here you will learn to factor polynomials by grouping.

A tank is bought at the pet store and is known to have a volume of 12 cubic feet. The dimensions are shown in the diagram below. If your new pet requires the tank to be at least 3 feet high, did you buy a big enough tank?



Factoring by Grouping

Recall that to factor means to rewrite an expression as a product. In general, quadratic expressions are the easiest to factor and cubic expressions are much more difficult to factor.

One method that can be used to factor *some* cubics is the factoring by grouping method. To factor cubic polynomials by grouping there are four steps:

- **Step 1:** Separate the terms into two groups.
- **Step 2:** Factor out the common terms in each of the two groups.
- **Step 3:** Factor out the common binomial.
- **Step 4:** If possible, factor the remaining quadratic expression.

Take a look at the examples to see what factoring by grouping looks like.

Factor the following polynomial by grouping:

$w^3-2w^2-9w+18$.

Step 1: Separate the terms into two groups. Notice the sign change on the second group because of the negative sign.

$w^3-2w^2-9w+18=(w^3-2w^2)-(9w-18)$

Step 2: Factor out the common terms in each of the sets of parentheses.

$(w^3-2w^2)-(9w-18)=w^2(w-2)-9(w-2)$

Step 3: Factor out the common binomial $(w-2)$.

$w^2(w-2)-9(w-2)=(w-2)(w^2-9)$

Step 4: Factor the remaining quadratic expression (w^2-9).

$(w-2)(w^2-9)=(w-2)(w+3)(w-3)$

Therefore, your answer is: $w^3-2w^2-9w+18=(w-2)(w+3)(w-3)$

Factor the following polynomial by grouping:

$2s^3-8s^2+3s-12$.

Step 1: Separate the terms into two groups.

$2s^3-8s^2+3s-12=(2s^3-8s^2)+(3s-12)$

Step 2: Factor out the common terms in each of the sets of parentheses.

$(2s^3-8s^2)+(3s-12)=2s^2(s-4)+3(s-4)$

Step 3: Factor out the common binomial $(s-4)$.

$2s^2(s-4)+3(s-4)=(s-4)(2s^2+3)$

Step 4: Check to see if the remaining quadratic can be factored. In this case, the expression $(2s^3+3)$ cannot be factored.

Therefore, your final answer is $2s^3-8s^2+3s-12=(s-4)(2s^2+3)$

Factor the following polynomial by grouping:

$y^3+5y^2-4y-20$.

Step 1: Separate the terms into two groups. Notice the sign change on the second group because of the negative sign.

$y^3+5y^2-4y-20=(y^3+5y^2)-(4y+20)$

Step 2: Factor out the common terms in each of the sets of parentheses.

$(y^3+5y^2)-(4y+20)=y^2(y+5)-4(y+5)$

Step 3: Factor out the common binomial $(y+5)$.

$y^2(y+5)-4(y+5)=(y+5)(y^2-4)$

Step 4: Factor the remaining quadratic expression (y^2-4).

$(y+5)(y^2-4)=(y+5)(y+2)(y-2)$

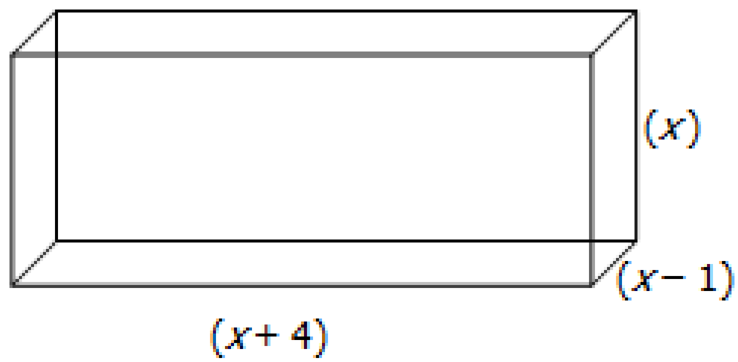
Therefore, your answer is $y^3+5y^2-4y-20=(y+5)(y+2)(y-2)$.

Examples

Example 1

Earlier, you were given a problem about a tank.

A tank is bought at the pet store and is known to have a volume of 12 cubic feet. The dimensions are shown in the diagram below. If your new pet requires the tank to be at least 3 feet high, did you buy a big enough tank?



To solve this problem, you need to calculate the volume of the tank.

$$\begin{aligned}
V &= l \times w \times h \\
12 &= (x+4) \times (x-1) \times (x) \\
12 &= (x^2+3x-4) \times (x) \\
12 &= x^3+3x^2-4x \\
0 &= x^3+3x^2-4x-12
\end{aligned}$$

Now you start to solve by factoring by grouping.

$0 = (x^3+3x^2)-(4x+12)$

Factor out the common terms in each of the sets of parentheses.

$0 = x^2(x+3)-4(x+3)$

Factor out the group of terms $(x+3)$ from the expression.

$0 = (x+3)(x^2-4)$

Completely factor the remaining quadratic expression.

$0 = (x+3)(x-2)(x+2)$

Now solve for the variable x.

$$\begin{array}{ccc}
\multicolumn{3}{c}{0 = (x+3)\ (x-2)\ (x+2)} \\
\swarrow & \downarrow & \searrow \\
x+3=0 & x-2=0 & x+2=0 \\
x=-3 & x=2 & x=-2
\end{array}$$

Since you are looking for a length, only $x = 2$ is a good solution (you can't have a negative length!). But since you need a tank 3 feet high and this one is only 2 feet high, you need to go back to the pet shop and buy a bigger one.

Example 2

Factor the following polynomial by grouping: $y^3 - 4y^2 - 4y + 16$.

Here are the steps:

$$\begin{aligned} y^3 - 4y^2 - 4y + 16 &= y^2(y-4) - 4(y-4) \\ &= (y^2-4)(y-4) \\ &= (y-2)(y+2)(y-4) \end{aligned}$$

Example 3

Factor the following polynomial by grouping: $3x^3 - 4x^2 - 3x + 4$.

Here are the steps:

$$\begin{aligned} 3x^3 - 4x^2 - 3x + 4 &= x^2(3x-4) - 1(3x-4) \\ &= (x^2-1)(3x-4) \\ &= (x-1)(x+1)(3x-4) \end{aligned}$$

Example 4

Factor the following polynomial by grouping: $e^3 + 3e^2 - 4e - 12$.

Here are the steps:

$$\begin{aligned} e^3 + 3e^2 - 4e - 12 &= e^2(e+3) - 4(e+3) \\ &= (e^2-4)(e+3) \\ &= (e+2)(e-2)(e+3) \end{aligned}$$

Review

Factor the following cubic polynomials by grouping.

1. $x^3 - 3x^2 - 36x + 108$
2. $e^3 - 3e^2 - 81e + 243$
3. $x^3 - 10x^2 - 49x + 490$
4. $y^3 - 7y^2 - 5y + 35$
5. $x^3 + 9x^2 + 3x + 27$
6. $3x^3 + x^2 - 3x - 1$
7. $5s^3 - 6s^2 - 45s + 54$
8. $4a^3 - 7a^2 + 4a - 7$
9. $5y^3 + 15y^2 - 45y - 135$
10. $3x^3 + 15x^2 - 12x - 60$
11. $2e^3 + 14e^2 + 7e + 49$
12. $2k^3 + 16k^2 + 38k + 24$

13. $-6x^3 + 3x^2 + 54x - 27$
14. $-5m^3 - 6m^2 + 20m + 24$
15. $-2x^3 - 8x^2 + 14x + 56$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.10.

7.11 Factorization of Special Cubics

Here you'll learn to factor the sum and difference of perfect cubes.

Factor the following cubic polynomial: $375x^3+648$.

Factorization of Special Cubics

While many cubics cannot easily be factored, there are two special cases that can be factored quickly. These special cases are the sum of perfect cubes and the difference of perfect cubes.

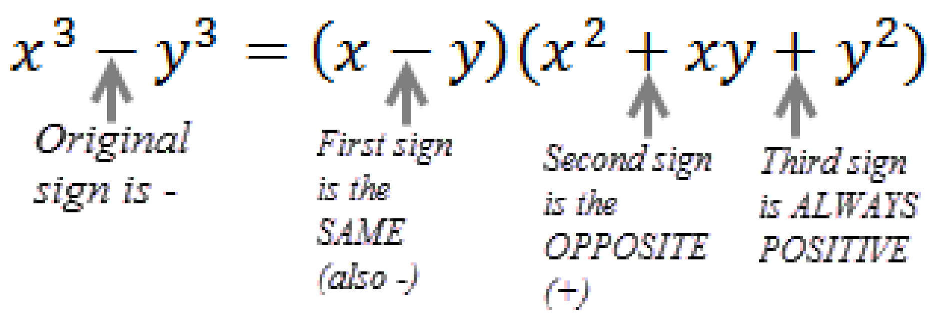
- Factoring the sum of two cubes follows this pattern:

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

- Factoring the difference of two cubes follows this pattern:

$$x^3-y^3=(x-y)(x^2+xy+y^2)$$

The acronym SOAP can be used to help you remember the positive and negative signs when factoring the sum and difference of cubes. SOAP stands for "**S**ame", "**O**pposite", "**A**lways **P**ositive". "Same" refers to the first sign in the factored form of the cubic being the same as the sign in the original cubic. "Opposite" refers to the second sign in the factored cubic being the opposite of the sign in the original cubic. "Always Positive" refers to the last sign in the factored form of the cubic being always positive. See below:



Factor:

x^3+27.

This is the sum of two cubes and uses the factoring pattern: $x^3+y^3=(x+y)(x^2-xy+y^2)$.

$x^3+3^3=(x+3)(x^2-3x+9)$.

Factor:

x^3-343.

This is the difference of two cubes and uses the factoring pattern: $x^3 - y^3 = (x-y)(x^2+xy+y^2)$.

$x^3 - 7^3 = (x-7)(x^2+7x+49)$.

Factor:

$64x^3 - 1$.

This is the difference of two cubes and uses the factoring pattern: $x^3 - y^3 = (x-y)(x^2+xy+y^2)$.

$(4x)^3 - 1^3 = (4x-1)(16x^2+4x+1)$.

Examples

Example 1

Earlier, you were asked to factor the following cubic polynomial: $375x^3+648$.

First you need to recognize that there is a common factor of 3. $375x^3+648 = 3(125x^3+216)$

Notice that the result is the sum of two cubes. Therefore, the factoring pattern is $x^3+y^3 = (x+y)(x^2-xy+y^2)$.

$375x^3+648 = 3(5x+6)(25x^2-30x+36)$

Example 2

Factor the following cubic.

x^3+512

$x^3+8^3 = (x+8)(x^2-8x+64)$.

Example 3

Factor the following cubic.

$8x^3+125$

$(2x)^3+5^3 = (2x+5)(4x^2-10x+25)$.

Example 4

x^3-216

$x^3-6^3 = (x-6)(x^2+6x+36)$.

Review

Factor each of the following cubics.

1. x^3+h^3
2. a^3+125
3. $8x^3+64$

4. $x^3 + 1728$
5. $2x^3 + 6750$
6. $h^3 - 64$
7. $s^3 - 216$
8. $p^3 - 512$
9. $4e^3 - 32$
10. $2w^3 - 250$
11. $x^3 + 8$
12. $y^3 - 1$
13. $125e^3 - 8$
14. $64a^3 + 2197$
15. $54z^3 + 3456$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.11.

7.12 Division of a Polynomial by a Monomial

Here you'll learn how to divide a polynomial by a monomial.

Can you divide the polynomial by the monomial? How does this relate to factoring?

$4e^4 + 6e^3 - 10e^2 \div 2e$

Division of a Polynomial by a Monomial

Recall that a monomial is an algebraic expression that has only one term. So, for example, x, 8, -2, or $3ac$ are all monomials because they have only one term. The term can be a number, a variable, or a combination of a number and a variable. A polynomial is an algebraic expression that has more than one term.

When dividing polynomials by monomials, it is often easiest to separately divide each term in the polynomial by the monomial. When simplifying each mini-division problem, don't forget to use exponent rules for the variables. For example,

$$\frac{8x^5}{2x^3} = 4x^2$$

.

Remember that a fraction is just a division problem!

Divide the polynomial

What is $(14s^2 - 21s + 42) \div (7)$?

This is the same as $\frac{14s^2-21s+42}{7}$. Divide each term of the polynomial numerator by the monomial denominator and simplify.

- $\frac{14s^4}{7} = 2s^4$
- $\frac{-21s}{7} = -3s$
- $\frac{42}{7} = 6$

Therefore, $(14s^2 - 21s + 42) \div (7) = 2s^4 - 3s + 6$.

Divide the polynomial

What is $\frac{3w^3-18w^2-24w}{6w}$?

Divide each term of the polynomial numerator by the monomial denominator and simplify. Remember to use exponent rules when dividing the variables.

- $\frac{3w^3}{6w} = \frac{w^2}{2}$
- $\frac{-18w^2}{6w} = -3w$
- $\frac{-24w}{6w} = -4$

Therefore, $\frac{3w^3-18w^2-24w}{6w} = \frac{w^2}{2} - 3w - 4$.

Divide the polynomial

What is $(-27a^4b^5+81a^3b^4-18a^2b^3)\div(-9a^2b)$?

This is the same as $\frac{-27a^4b^5+81a^3b^4-18a^2b^3}{-9a^2b}$. Divide each term of the polynomial numerator by the monomial denominator and simplify. Remember to use exponent rules when dividing the variables.

- $\frac{-27a^4b^5}{-9a^2b}=3a^2b^4$
- $\frac{81a^3b^4}{-9a^2b}=-9ab^3$
- $\frac{-18a^2b^3}{-9a^2b}=2b^2$

Therefore, $(-27a^4b^5+81a^3b^4-18a^2b^3)\div(-9a^2b)=3a^2b^4-9ab^3+2b^2$.

Examples

Example 1

Earlier, you were asked can you divide the polynomial by the monomial. How does this relate to factoring?

$4e^4+6e^3-10e^2\div 2e$

This process is the same as factoring out a $2e$ from the expression $4e^4+6e^3-10e^2$.

- $\frac{4e^4}{2e}=2e^3$
- $\frac{6e^3}{2e}=3e^2$
- $\frac{-10e^2}{2e}=-5e$

Therefore, $4e^4+6e^3-10e^2\div 2e=2e^3+3e^2-5e$.

Example 2

Complete the following division problem.

$(3a^5-5a^4+17a^3-9a^2)\div(a)$

$(3a^5-5a^4+17a^3-9a^2)\div(a)=3a^4-5a^3+17a^2-9a$

Example 3

$(-40n^3-32n^7+88n^{11}+8n^2)\div(8n^2)$

$(-40n^3-32n^7+88n^{11}+8n^2)\div(8n^2)=-5n-4n^5+11n^9+1$

Example 4

$\frac{16m^6-12m^4+4m^2}{4m^2}$

$\frac{(16m^6-12m^4+4m^2)}{(4m^2)}=4m^4-3m^2+1$

Review

Complete the following division problems.

1. $(6a^3+30a^2+24a)\div 6$
2. $(15b^3+20b^2+5b)\div 5$
3. $(12c^4+18c^2+6c)\div 6c$
4. $(60d^{12}+90d^{11}+30d^8)\div 30d$
5. $(33e^7+99e^3+22e^2)\div 11e$
6. $(-8a^4+8a^2)\div(-4a)$
7. $(-3b^4+6b^3-30b^2+15b)\div(-3b)$
8. $(-40c^{12}-20c^{11}-25c^9-30c^3)\div 5c^2$
9. $(32d^{11}+16d^7+24d^4-64d^2)\div 8d^2$
10. $(14e^{12}-18e^{11}-12e^{10}-18e^7)\div -2e^5$
11. $(18a^{10}-9a^8+72a^7+9a^5+3a^2)\div 3a^2$
12. $(-24b^9+42b^7+42b^6)\div -6b^3$
13. $(24c^{12}-42c^7-18c^6)\div -2c^5$
14. $(14d^{12}+21d^9+42d^7)\div -7d^4$
15. $(-40e^{12}+30e^{10}-10e^4+30e^3+80e)\div -10e^2$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.12.

7.13 Long Division and Synthetic Division

Here you will learn to divide polynomials using polynomial long division and synthetic division.

Can you divide the following polynomials?

$$\frac{x^2-5x+6}{x-2}$$

Long Division and Synthetic Division

Polynomial Long Division

Whenever you want to divide a polynomial by a polynomial, you can use a process called polynomial long division. This process is similar to long division for regular numbers. Look at the example below:

$$\frac{(x^2+3x+2)}{(x+1)}$$

This is the same as the division problem below:

$$x+1\overline{\big)\,x^2+3x+2}$$

Step 1: Divide the first term in the numerator (x^2) by the first term in the denominator (x). Put this result above the division bar in your answer. In this case, $\frac{x^2}{x}=x$.

$$\begin{array}{r} x \\ x+1\overline{\big)\,x^2+3x+2} \end{array}$$

Step 2: Multiply the denominator $(x+1)$ by the result from Step 1 (x), and put the new result below your numerator. Then, ***subtract*** to get your new polynomial. *This is the same process as in regular number long division!*

$$\begin{array}{r} x \\ x+1\overline{\big)\,x^2+3x+2} \\ \underline{x^2+x\downarrow} \\ 2x+2 \end{array}$$

Step 3: Divide the first term in the new polynomial ($2x$) by the first term in the denominator (x). Put this result above the division bar in your answer. Multiply, subtract, and repeat this process until you cannot repeat it anymore.

$$\begin{array}{r|l} & x\ +2 \\ \hline x+1 & x^2+3x+2 \\ & \underline{x^2+x} \downarrow \\ & 2x+2 \\ & \underline{2x+2} \\ & 0 \end{array}$$

Therefore: $\frac{(x^2+3x+2)}{(x+1)} = (x+2)$

Synthetic Division

Synthetic division is another method of dividing polynomials. It is a shorthand of long division that only works when you are dividing by a polynomial of degree 1. Usually the divisor is in the form $(x \pm a)$. In synthetic division, unlike long division, you are only concerned with the coefficients in the polynomials. Consider the same example as above:

$$x+1\overline{\big)\,x^2+3x+2}$$

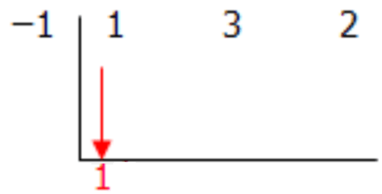
Step 1: Write the coefficients in an upside down division sign.

$$\begin{array}{|ccc} 1 & 3 & 2 \\ & & \\ \hline \end{array}$$

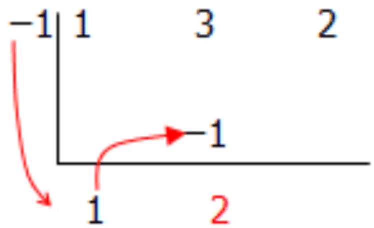
Step 2: Put the opposite of the number from the divisor to the left of the division symbol. In this case, the divisor is $x+1$, so you will use a -1.

$$\begin{array}{r|ccc} -1 & 1 & 3 & 2 \\ & & & \\ \cline{2-4} \end{array}$$

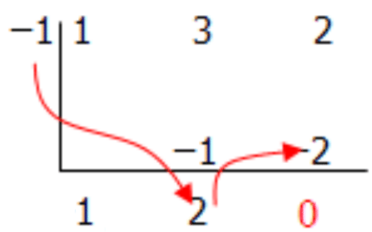
Step 3: Take your leading coefficient and bring it down below the division symbol.



Step 4: Multiply this number by the number to the left of the division symbol and place it in the next column. Add the two numbers together and place this new number below the division sign.



Step 5: Multiply this second number by the number to the left of the division symbol and place it into the third column. Add the two numbers together and place this new number below the division sign.



The numbers below the division sign represent your coefficients. Therefore, $\frac{(x^2+3x+2)}{(x+1)} = (x+2)$.

Use long division to divide:

Step 1: Divide the first term in the numerator by the first term in the denominator, put this in your answer. Therefore $\frac{x^2}{x} = x$.

$$\begin{array}{r}
x \\
(x-1)|\overline{x^2+6x-7}
\end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r}
x \\
(x-1)|\overline{x^2+6x-7} \\
\underline{x^2-x } \\
7x
\end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r|l} & \overline{x+7} \\ (x-1) & x^2+6x-7 \\ & \underline{x^2-x \quad \downarrow} \\ & \quad 7x-7 \\ & \underline{\quad 7x-7} \\ & \qquad 0 \end{array}$$

Therefore: $\frac{x^2+6x-7}{x-1} = (x+7)$

Use long division to divide:

Step 1: Divide the first term in the numerator by the first term in the denominator; put this in your answer. Therefore $\frac{2x^2}{2x} = x$.

$$\begin{array}{r|l} & \overline{x} \\ (2x+5) & 2x^2+7x+5 \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r|l} & \overline{x} \\ (2x+5) & 2x^2+7x+5 \\ & \underline{2x^2+5x} \\ & \qquad 2x \end{array}$$

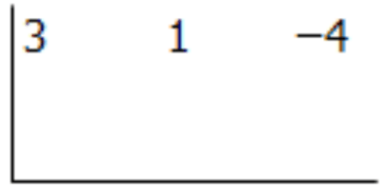
Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r|l} & \overline{x+1} \\ (2x+5) & 2x^2+7x+5 \\ & \underline{2x^2+5x \quad \downarrow} \\ & \quad 2x+5 \\ & \underline{\quad 2x+5} \\ & \qquad 0 \end{array}$$

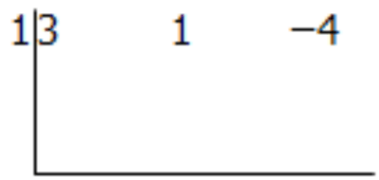
Therefore: $\frac{2x^2+7x+5}{2x+5} = (x+1)$

Use synthetic division to divide:

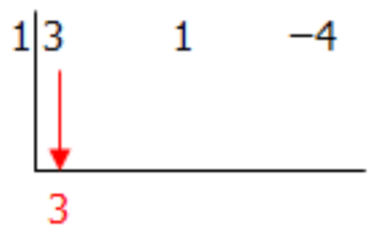
Step 1: Write the coefficients in an upside down division sign.



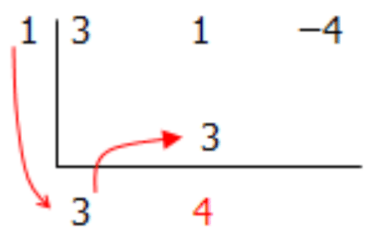
Step 2: Put the opposite of the number from the divisor to the left of the division symbol.



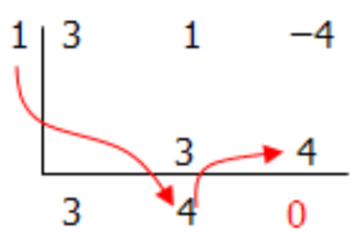
Step 3: Take your leading coefficient and bring it down below the division symbol.



Step 4: Multiply this number by the number to the left of the division symbol and place it in the next column. Add the two numbers together and place this new number below the division sign.



Step 5: Multiply this second number by the number to the left of the division symbol and place it into the third column. Add the two numbers together and place this new number below the division sign.



Therefore: $\frac{3x^2+x-4}{x-1} = (3x+4)$

Examples

Example 1

Earlier, you were asked to divide polynomials.

You can divide using long division or synthetic division.

$$\frac{x^2-5x+6}{x-2}$$

Long Division:

Step 1: Divide the first term in the numerator by the first term in the denominator, put this in your answer. Therefore $\frac{x^2}{x} = x$.

$$(x-2)\overline{|x^2-5x+6}^{\,x}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r} x \\ (x-2)|\overline{x^2-5x+6} \\ \underline{x^2-2x} \\ 3x \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r} x-3 \\ (x-2)|\overline{x^2-5x+6} \\ \underline{x^2-2x\ \downarrow} \\ -3x+6 \\ \underline{-3x+6} \\ 0 \end{array}$$

Therefore: $\frac{x^2-5x+6}{x-2} = (x-3)$

Example 2

Use long division to divide $5x^2+6x+1$ by $x+1$.

$\frac{5x^2+6x+1}{x+1} = (5x+1)$

Example 3

Use synthetic division to divide $3x^2-2x-1$ by $x-1$.

$\frac{3x^2-2x-1}{x-1} = (3x+1)$

Example 4

Use synthetic division to divide $3x^3+11x^2+4x-4$ by $x+1$.

$\frac{3x^3+11x^2+4x-4}{x+1}=(3x^2+8x-4)$

Review

Use long division to divide each of the following:

1. $(x^2+7x+12)\div(x+3)$
2. $(x^2+4x+3)\div(x+3)$
3. $(a^2-4a-45)\div(a-9)$
4. $(3x^2+5x-2)\div(3x-1)$
5. $(2x^2-5x+2)\div(2x-1)$

Use synthetic division to divide each of the following:

6. $(b^2-5b+6)\div(b-3)$
7. $(x^2-6x+8)\div(x-4)$
8. $(a^2-1)\div(a+1)$
9. $(c^2-9)\div(c-3)$
10. $(5r^2+2r-3)\div(r+1)$

Divide each of the following:

11. $\frac{2x^3-7x^2-14x-5}{x-5}$
12. $\frac{9x^4-15x^3+12x^2-11x-15}{3x^3+4x+3}$
13. $\frac{6x^4+4x^3+9x^2+2x+3}{2x^2+1}$
14. $\frac{x^4+4x^3+3x^2+x+1}{x+1}$
15. $\frac{2x^3+7x^2-27x+18}{x+6}$
16. $\frac{8x^3-2x^2+7x+5}{2x+1}$
17. $\frac{3x^3-15x^2+4x-20}{x-5}$
18. $\frac{9x^3+26x^2-48x+5}{x^2+3x-5}$
19. $\frac{-x^3+13x+12}{x+3}$
20. $\frac{x^3-2x^2-5x+10}{x-2}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.13.

7.14 The Factor Theorem

Here you will learn how to factor a polynomial using the factor theorem.

A rectangular shaped container is built in such a way that its volume can be represented by the polynomial $V(w) = w^3 + 7w^2 + 16w + 12$, where w is the width of the container.

a) Factor the polynomial.

b) If $w = 2\ ft$, what are the dimensions of the container?

The Factor Theorem

You know techniques for factoring quadratics and special cases of cubics, but what about other cubics or higher degree polynomials? With the factor theorem, you can attempt to factor these types of polynomials. The factor theorem states that if $(x-a)$ is a factor of $p(x)$, then $p(a) = 0$. To use the factor theorem:

1. Guess factors of the given polynomial $p(x)$. Factors should be of the form $(x-a)$ where a is a factor of the constant term of the polynomial divided by a factor of the first coefficient of the polynomial.
2. Test potential factors by checking $p(a)$. If $p(a) = 0$, then $x-a$ is a factor of the polynomial.
3. Divide the polynomial by one of its factors.
4. Repeat Steps 2 and 3 until the result is a quadratic expression that you can factor using other methods.

Use the factor theorem

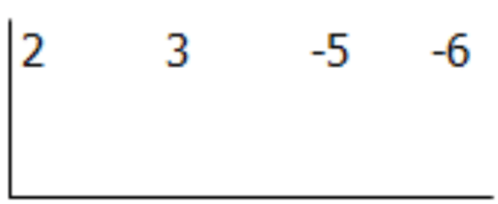
Use the factor theorem to determine if $x+1$ is a factor of $p(x) = 2x^3 + 3x^2 - 5x - 6$. If so, find the other factors.

If $x+1$ is a factor, then $p(-1) = 0$. Test this:

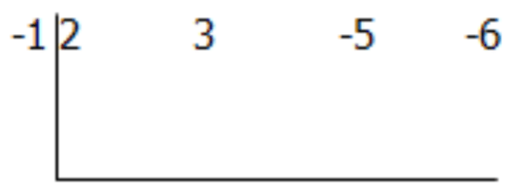
$$\begin{aligned} p(x) &= 2x^3 + 3x^2 - 5x - 6 \\ x = -1 : p(-1) &= 2(-1)^3 + 3(-1)^2 - 5(-1) - 6 \\ p(-1) &= -2 + 3 + 5 - 6 \\ p(-1) &= 0 \quad \text{(IS a factor)} \end{aligned}$$

Now that you have one of the factors, use division to find the others.

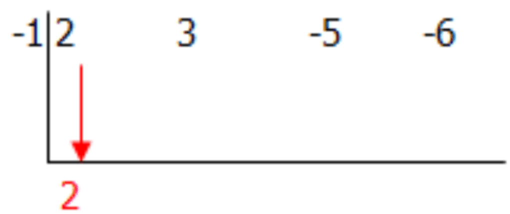
Step 1:



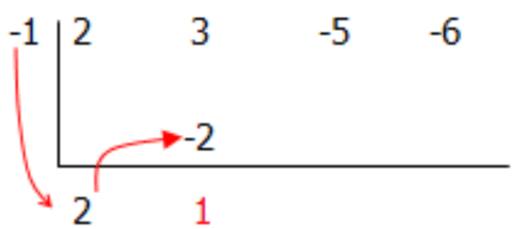
Step 2:



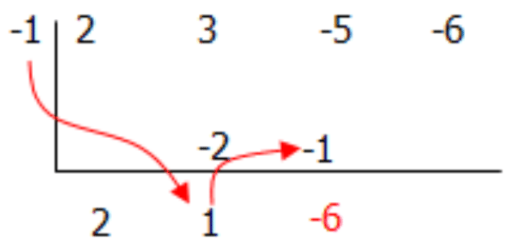
Step 3:



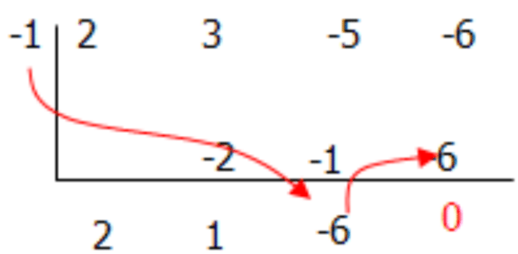
Step 4:



Step 5:



Step 6:



So:

$$\begin{aligned}
p(x) &= 2x^3+3x^2-5x-6\\
p(x) &= (x+1)(2x^2+x-6)\\
p(x) &= (x+1)(2x-3)(x+2)
\end{aligned}$$

Use the factor theorem

Use the factor theorem to determine if $x+3$ is a factor of $s(x)=5x^2-13x-84$. If so, find the other factors.

If $x+3$ is a factor, then $p(-3)=0$. Test this:

$$s(x)=5x^2-13x-84$$

$$\begin{aligned}
x=-3: s(-3) &= 5(-3)^2-13(-3)-84\\
s(-3) &= 45+39-84\\
s(-3) &= 0 \quad (\text{IS a factor})
\end{aligned}$$

Now that you have one of the factors, use division to find the other factor.

Step 1: Divide the first term in the numerator by the first term in the denominator; put this in your answer. Therefore $\frac{5x^2}{x}=5x$.

$$\begin{array}{r|l} & \color{red}{5x} \\ \hline (x+3) & 5x^2-13x-84 \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r|l} & \color{red}{5x} \\ \hline (x+3) & 5x^2-13x-84 \\ & \underline{5x^2+15x} \\ & \quad -28x \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r|l} & 5x\color{red}{-28} \\ \hline (x+3) & 5x^2-13x-84 \\ & \underline{5x^2+13x \quad \downarrow} \\ & \quad -28x-84 \\ & \underline{\quad -28x-84} \\ & \qquad 0 \end{array}$$

Therefore: $\frac{(5x^2-13x-84)}{(x+3)} = (5x-28)$

So:

$$s(x) = 5x^3 - 13x - 84$$
$$s(x) = (x+3)(5x-28)$$

Factor

$f(t) = t^3 - 8t^2 + 17t - 10.$

In order to begin to find the factors, look at the number -10 and find the factors of this number. The factors of -10 are -1, 1, -2, 2, -5, 5, -10, 10. Next, start testing the factors to see if you get a remainder of zero.

$$f(t) = t^3 - 8t^2 + 17t - 10$$
$$t = -1 : f(-1) = (-1)^3 - 8(-1)^2 + 17(-1) - 10$$
$$f(-1) = -36 \quad \text{(NOT a factor)}$$

$$t = 1 : f(1) = (1)^3 - 8(1)^2 + 17(1) - 10$$
$$f(1) = 0 \quad \text{(IS a factor)}$$

Now that you have one of the factors, use division to find the others.

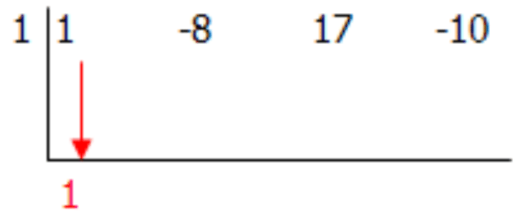
Step 1:

$$\begin{array}{|cccc} 1 & -8 & 17 & -10 \\ & & & \\ \hline \end{array}$$

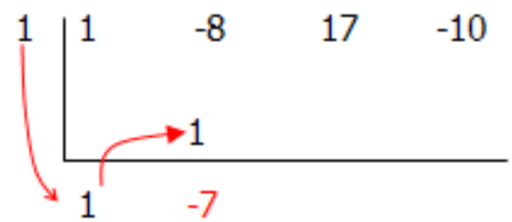
Step 2:

$$\begin{array}{r|cccc} 1 & 1 & -8 & 17 & -10 \\ & & & & \\ \hline \end{array}$$

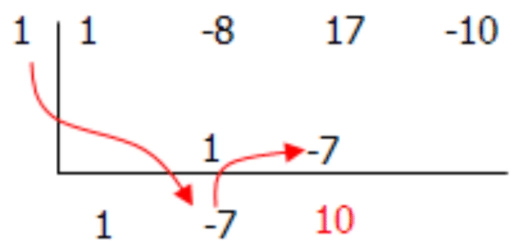
Step 3:



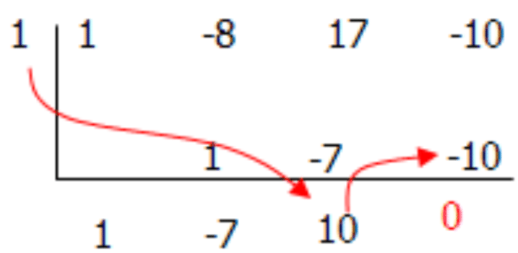
Step 4:



Step 5:



Step 6:



So :

$$f(t) = t^3 - 8t^2 + 17t - 10$$
$$f(t) = (t-1)(t^2 - 7t + 10)$$
$$f(t) = (t-1)(t-5)(t-2)$$

Therefore: $f(t) = (t-1)(t-5)(t-2)$

Examples

Example 1

Earlier, you were given a problem about a container.

A rectangular shaped container is built in such a way that its volume can be represented by the polynomial $V(w) = w^3 + 7w^2 + 16w + 12$, where w is the width of the container.

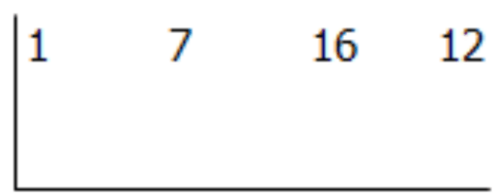
a) In order to begin to find the factors, look at the number 12 and find the factors of this number. The factors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$, and ± 12. Next, start testing the factors to see if you get a remainder of zero.

$$\begin{aligned}
V(w) &= w^3 + 7w^2 + 16w + 12 \\
w = -1 : V(-1) &= (-1)^3 + 7(-1)^2 + 16(-1) + 12 \\
V(-1) &= 2 \quad (\text{NOT a factor})
\end{aligned}$$

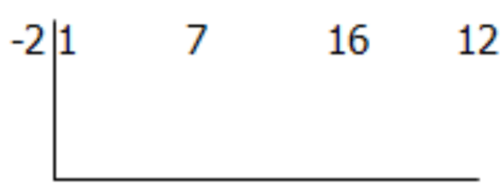
$$\begin{aligned}
w = -2 : V(-2) &= (-2)^3 + 7(-2)^2 + 16(-2) + 12 \\
V(-2) &= 0 \quad (\text{IS a factor})
\end{aligned}$$

Now that you have one of the factors, use division to find the others.

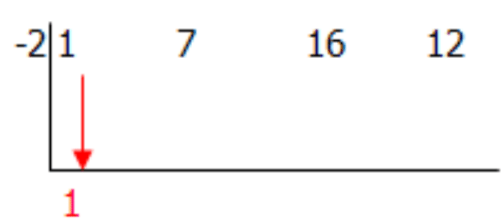
Step 1:



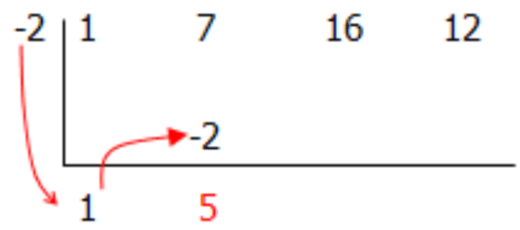
Step 2:



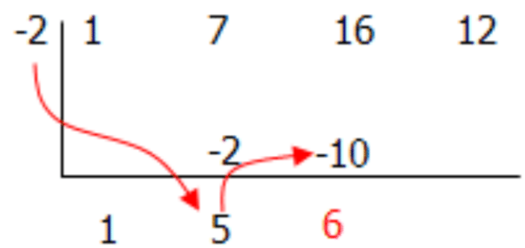
Step 3:



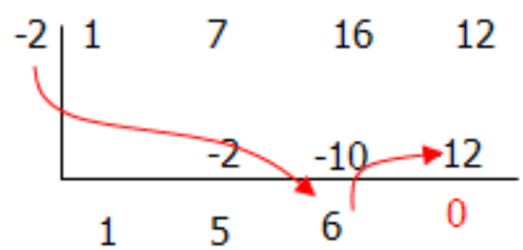
Step 4:



Step 5:



Step 6:



So:

$$
\begin{aligned}
V(w) &= w^3 + 7w^2 + 16w + 12 \\
V(w) &= (w+2)(w^2 + 5w + 6) \\
V(w) &= (w+2)(w+2)(w+3)
\end{aligned}
$$

b) If $w = 2$, what are the dimensions of the container?

$$
\begin{aligned}
(w+2) &= 2+2 = 4 \\
(w+2) &= 2+2 = 4 \\
(w+3) &= 2+3 = 5
\end{aligned}
$$

Therefore. the dimensions of the container are $4\ ft \times 4\ ft \times 5\ ft$.

Example 2

Determine if $e+3$ is a factor of $2e^3-e^2+e-1$.

$e=-3:2(-3)^3-(-3)^2+(-3)-1=-67$. Therefore $(e+3)$ is not a factor of $2e^3-e^2+e-1$.

Example 3

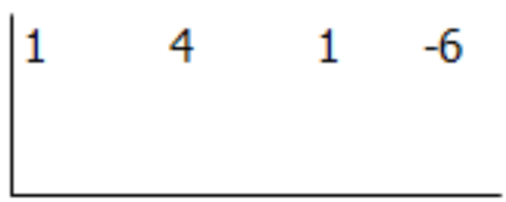
Factor: x^3+4x^2+x-6.

In order to begin to find the factors, look at the number -6 and find the factors of this number. The factors of -6 are $\pm 1, \pm 2, \pm 3$, and ± 6. Next, start testing the factors to see if you get a remainder of zero.

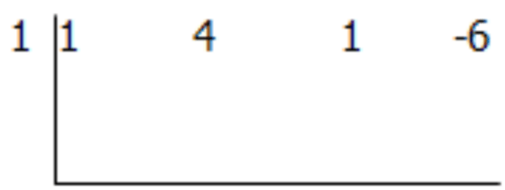
$$\begin{aligned}
&x^3+4x^2+x-6\\
&x=-1:(-1)^3+4(-1)^2+(-1)-6=-4 \quad \text{(NOT a factor)}\\
&x=1:(1)^3+4(1)^2+(1)-6=0 \quad \text{(IS a factor)}
\end{aligned}$$

Now that you have one of the factors, use division to find the others.

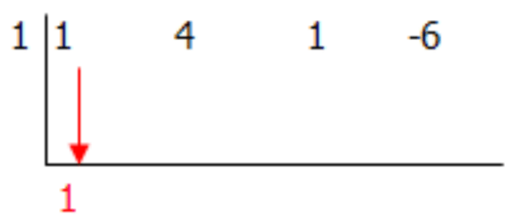
Step 1:



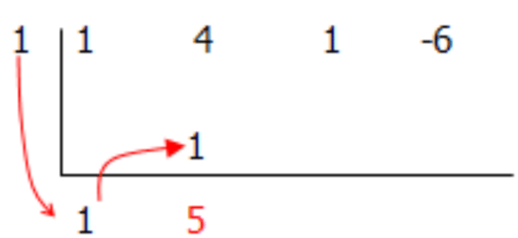
Step 2:



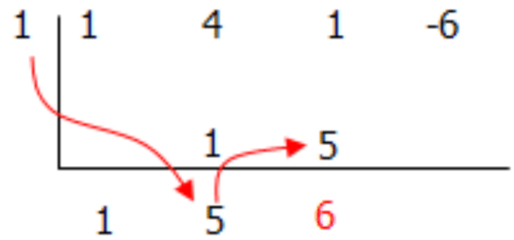
Step 3:



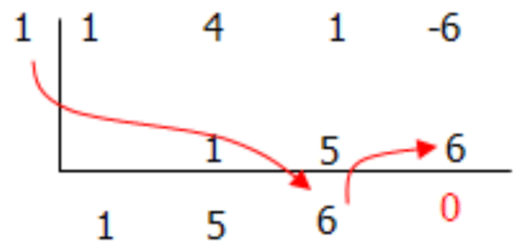
Step 4:



Step 5:



Step 6:



Therefore: $x^3+4x^2+x-6=(x-1)(x^2+5x+6)=(x-1)(x+2)(x+3)$

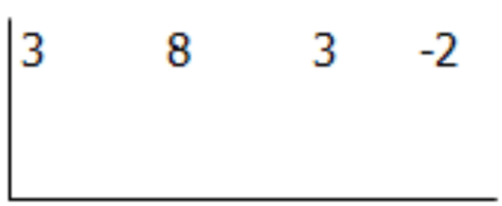
Example 4

A tennis court is being built where the volume is represented by the polynomial $p(L)=3L^3+8L^2+3L-2$, where L represents the length of the court. Determine if $L+1$ is a factor and if so, find the other factors. If $L=5ft$, what are the dimensions of the court.

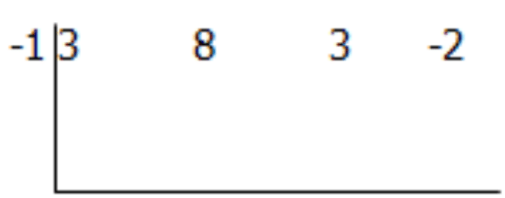
Start by testing the factor $L+1$ to see if you get a remainder of zero.

$$
\begin{aligned}
&p(L)=3L^3+8L^2+3L-2L=-1:\\
&p(L)=3(-1)^3+8(-1)^2+3(-1)-2\\
&p(1)=0 \quad \text{(IS a factor)}
\end{aligned}
$$

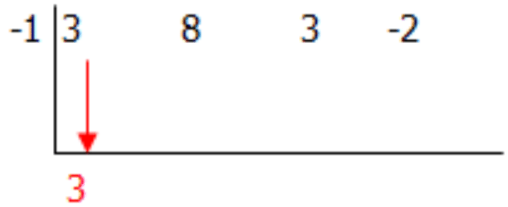
Now that you have one of the factors, use division to find the others.
Step 1:



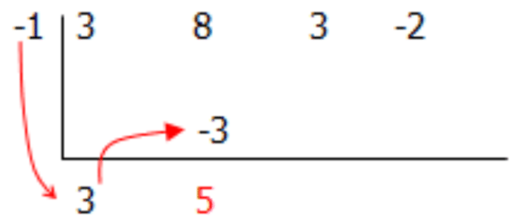
Step 1:



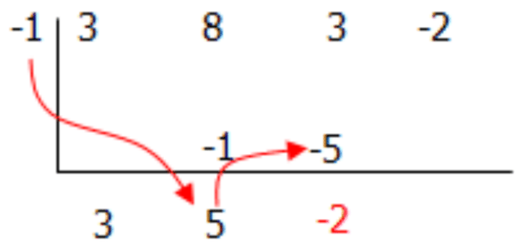
Step 2:



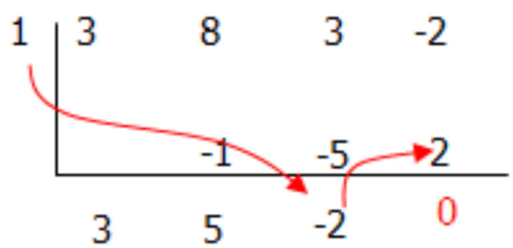
Step 3:



Step 4:



Step 5:



So:

$$p(L) = 3L^3 + 8L^2 + 3L - 2$$
$$p(L) = (L+1)(3L^2 + 5L - 2)$$
$$p(L) = (L+1)(3L-1)(L+2)$$

If $L = 5\ ft$, what are the dimensions of the container?

$$(L+1) = 5+1 = 6$$
$$(3L-1) = 3(5) - 1 = 14$$
$$(L+2) = 5+2 = 7$$

Therefore the dimensions of the container are $6\ ft \times 14\ ft \times 7\ ft$.

Review

Determine if $a-4$ is a factor of each of the following.

1. a^3-5a^2+3a+4
2. $3a^2-7a-20$
3. $-a^4+3a^3+5a^2-16$
4. $a^4-2a^3-8a^2+3a-4$
5. $2a^4-5a^3-7a^2-21a+4$

Factor each of the following:

6. x^3+2x^2+2x+1
7. x^3+x^2-x-1
8. $2x^3-5x^2+2x+1$
9. $2b^3+4b^2-3b-6$
10. $3c^3-4c^2-c+2$
11. $2x^3-13x^2+17x+12$
12. x^3+2x^2-x-2
13. $3x^3+2x^2-53x+60$
14. $x^3-7x^2+7x+15$
15. $x^4+4x^3-7x^2-34x-24$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.14.

7.15 Graphs of Polynomial Functions

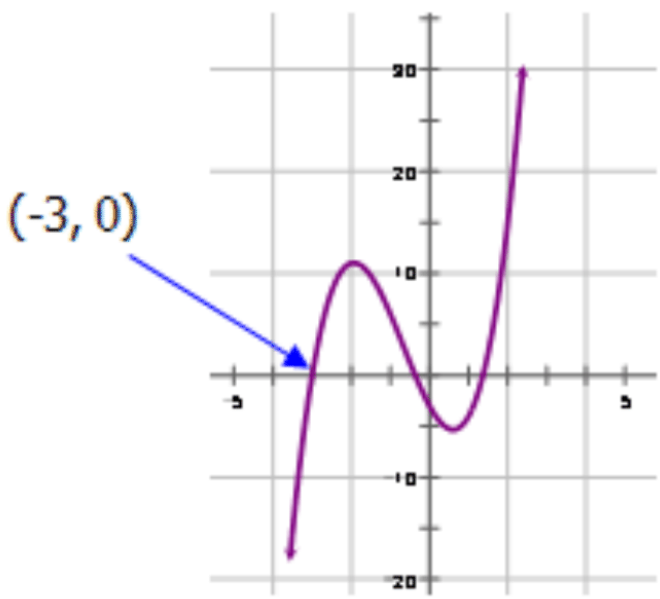
Here you will look at graphs of polynomial functions and identify real roots of polynomial functions from their graphs.

Use your graphing calculator to graph the functions below. What are the real roots of the functions?

1. $f(x) = x^3 - 6x^2 + 11x - 6$
2. $g(x) = 2x^4 - 4x^3 - 3x^2 + 12x - 8$

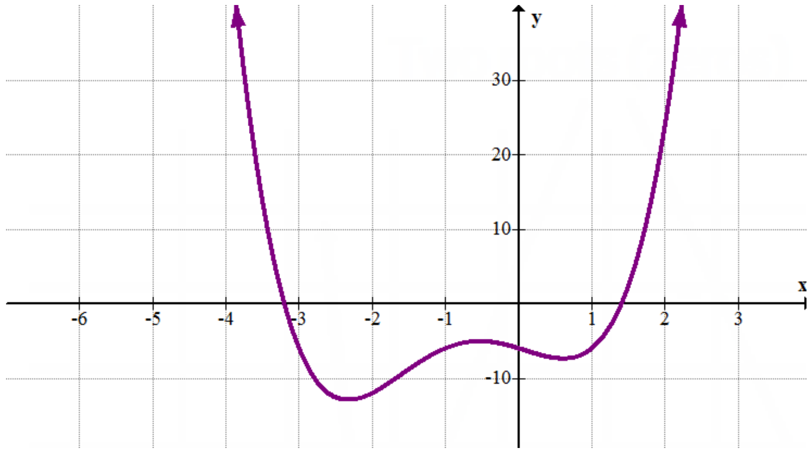
Graphing Polynomial Functions

Recall that $x - a$ is a factor of polynomial $p(x)$ if $p(a) = 0$. This means that on a graph, factors will appears as x-intercepts of a polynomial because they will occur at points with a y-coordinate equal to zero. For the function graphed below, you can see one of the x-intercepts is the point (-3, 0).

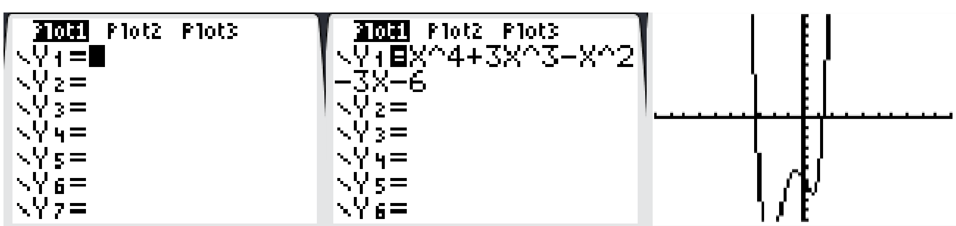


This means that one of the factors of the polynomial must be $(x+3)$. For the polynomial above, -3 is not only known as an x-intercept. It is also known as a **real root** of the polynomial. *Whenever a root (x-intercept) of a polynomial is an integer, it corresponds to a factor of the function.*

Cubic polynomials are degree three and are of the form $y = ax^3 + bx^2 + cx + d$. Graphs of cubics are like the graph above where overall one end of the graph points up and one end of the graph points down. Quartic polynomials are degree four and are of the form $y = ax^4 + bx^3 + cx^2 + dx + e$. Graphs of quartics are like the graph below where overall both ends of the graph point either up or down.



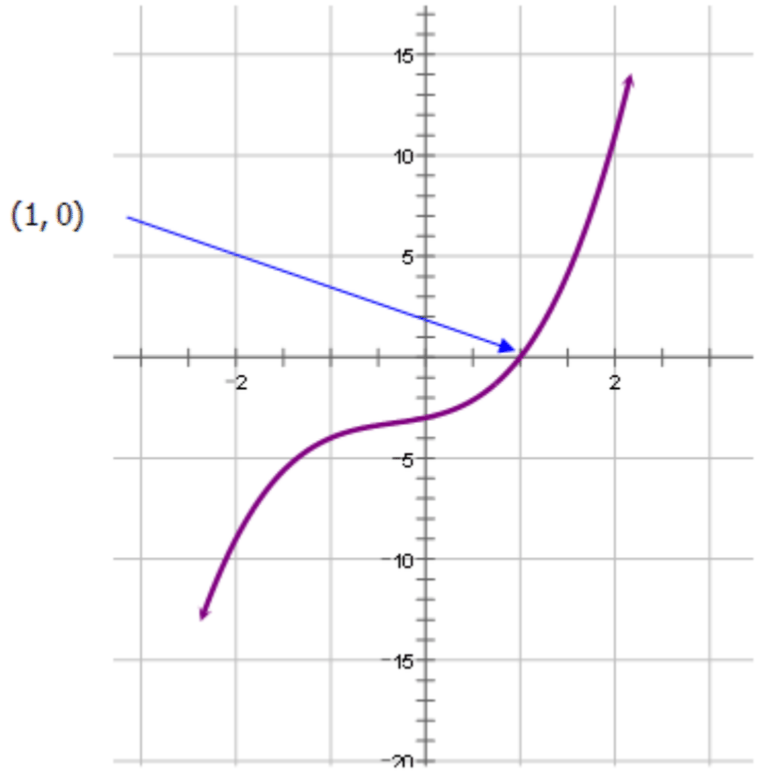
You can use a graphing calculator to graph cubics and quartics. To graph with your graphing calculator, push [Y=], enter your polynomial, push [GRAPH] to see the graph. Then, look at the graph for information about the factors of the polynomial. You can push [TABLE] ([2nd], [GRAPH]) to see the points on the graph more clearly.



Graph the function

Graph the function $f(x) = x^3 + x^2 + x - 3$ to determine the number of real roots (x-intercepts).

Once you graph the function, this is what you should see:

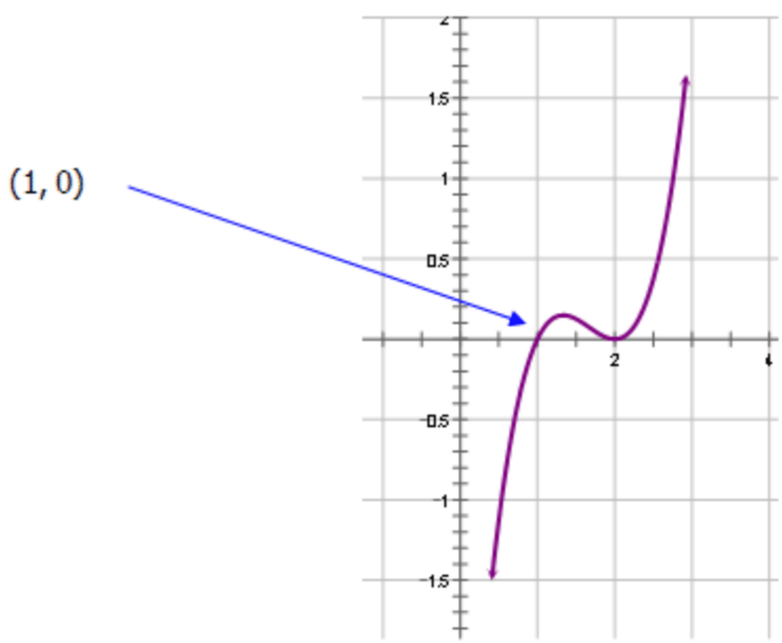


The polynomial $f(x) = x^3 + x^2 + x - 3$ has only one real root (x-intercept) at (1, 0).

Graph the function

Graph the function $g(x) = x^3 - 5x^2 + 8x - 4$ to determine if $x - 1$ is a factor of the polynomial.

Once you graph the function, this is what you should see:

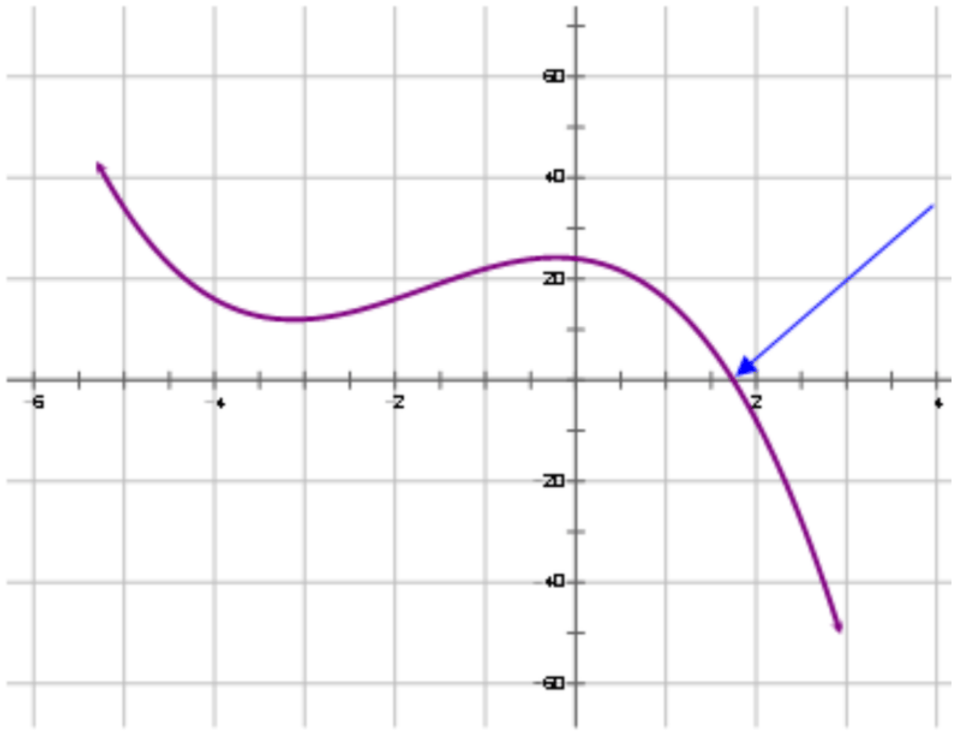


Since (1, 0) is an x-intercept of the polynomial $g(x) = x^3 - 5x^2 + 8x - 4$, $(x - 1)$ is a factor of this cubic.

Find the number of real roots

How many real roots (x-intercepts) are there for the polynomial $h(x) = -x^3 - 5x^2 - 2x + 24$?

Once you graph the function, this is what you should see:



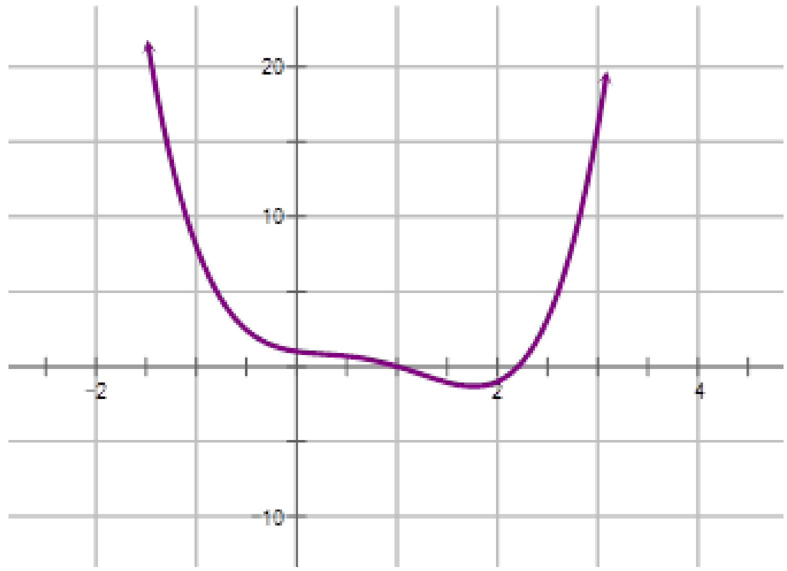
There is one x-intercept so there is one real root.

Find the number of real roots

Find the real root(s) for the following quartic.

$$k(x) = x^4 - 3x^3 + 2x^2 - x + 1$$

This is the graph of the quartic:



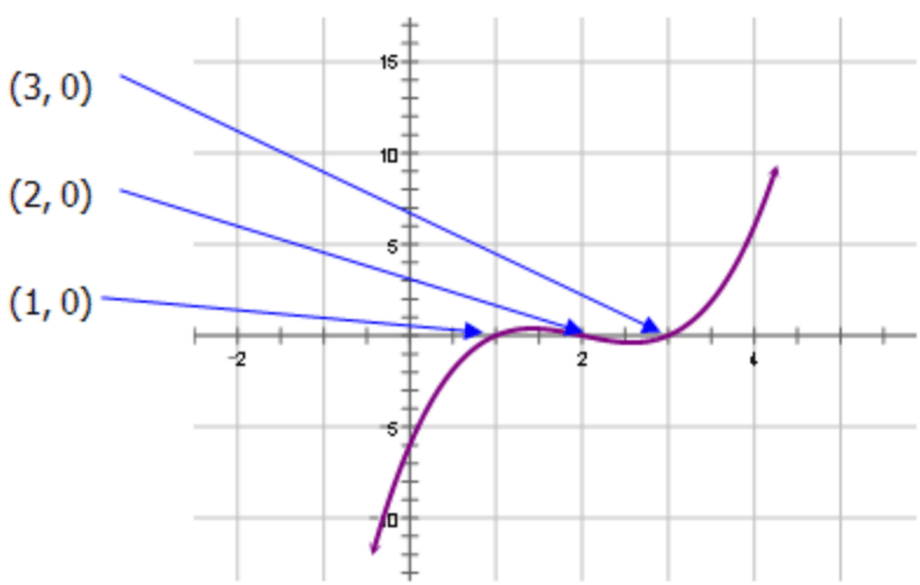
There are two real roots for this quartic. One is (1, 0) and the other occurs around (2.25,0).

Examples

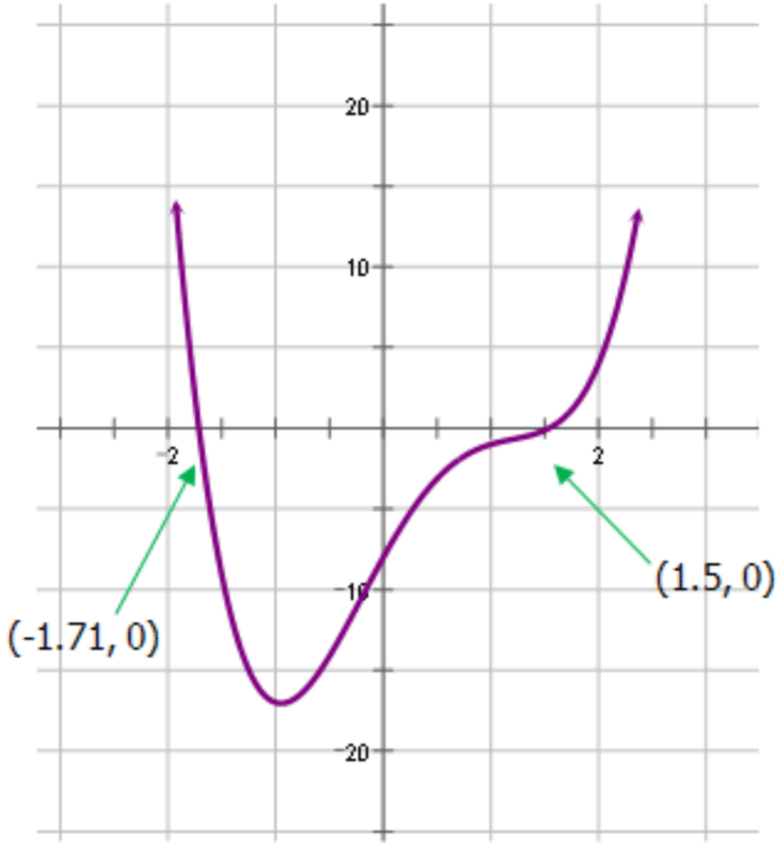
Example 1

Earlier, you were asked to find the real roots of two functions.

Here is the graph of the function $f(x) = x^3 - 6x^2 + 11x - 6$:

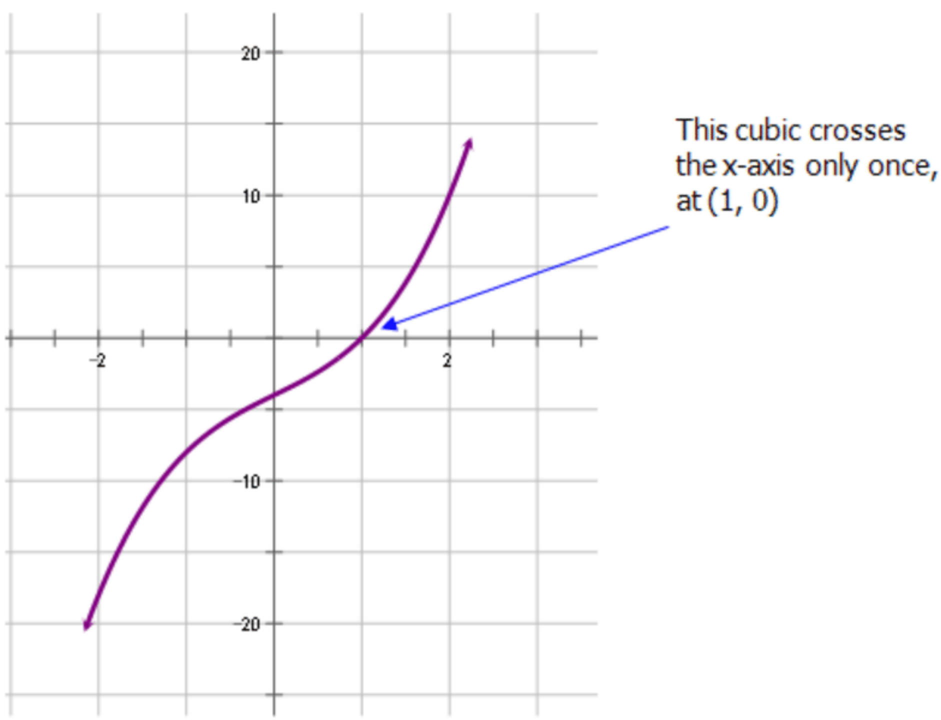


Here is the graph of the function $g(x) = 2x^4 - 4x^3 - 3x^2 + 12x - 8$. It has two real roots as indicated.



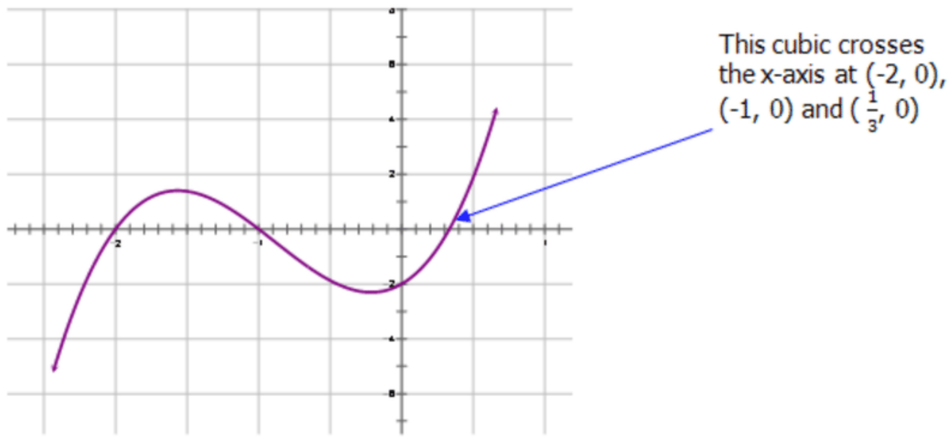
Example 2

Find the real roots for the cubic $y = x^3 + 3x - 4$ using a graph.



Example 3

Graph the function $g(x) = 3x^3 + 8x^2 + 3x - 2$ and determine the number of real roots. Is $(x-2)$ one of the factors of this polynomial?

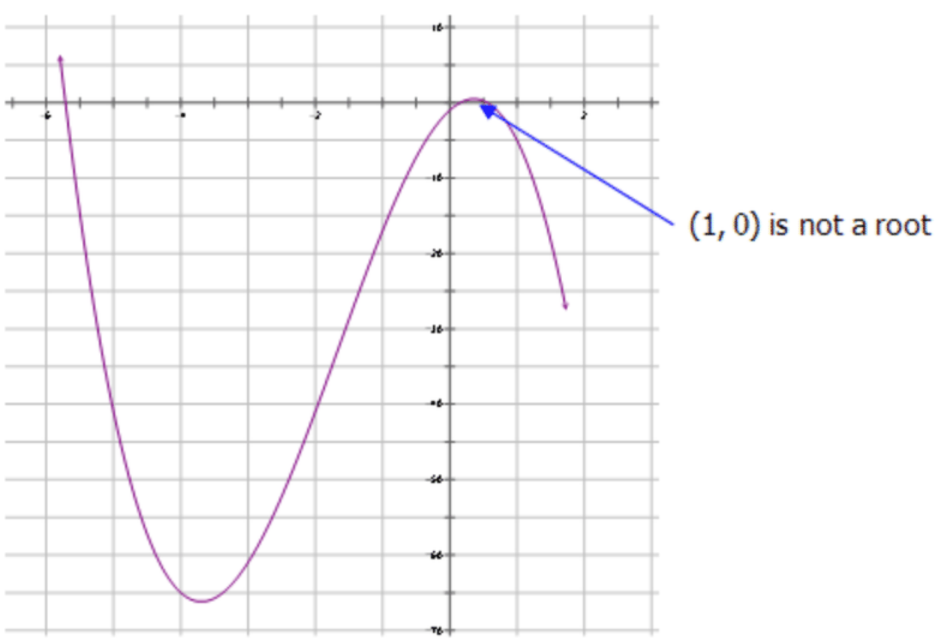


Since one of these root values is (-2, 0), the factor for the polynomial would be $(x+2)$ and not $(x-2)$.

Example 4

Graph the function $m(x) = -2x^3 + 10x^2 + 8x - 1$ to determine if $x - 1$ is a factor of the polynomial.

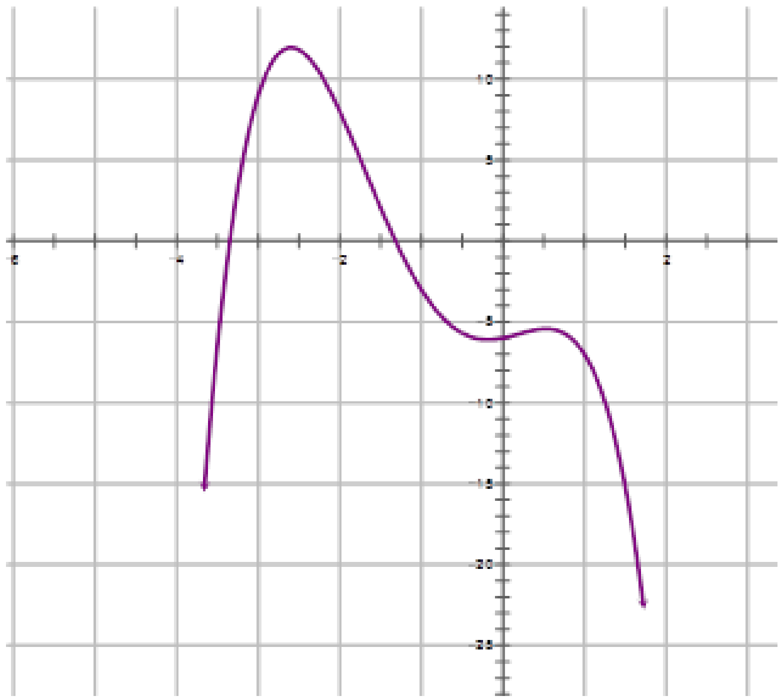
If $(x-1)$ was one of the factors then one of the roots would have to be (1, 0). This is not the case.



Example 5

Describe the graph of the following quartic: $j(x) = -x^4 - 3x^3 + 2x^2 + x - 6$.

The graph has an M shape. It looks like an M because of the -1 coefficient before x^4. There are two real roots located at (-3.35, 0) and (-1.32, 0).



Review

Find the real roots for the following cubic polynomials using a graph.

1. $y = x^3 - 2x^2 - 9x + 18$
2. $y = x^3 + 5x^2 - 4x - 20$
3. $y = 3x^3 - 6x^2 + 12x - 5$
4. $y = 2x^3 - 8x^2 + 3x - 12$
5. $y = -2x^3 - 3x^2 - 5x + 10$

Graph the functions below and determine the number of real roots. Give at least one factor of each polynomial from the graphed solution.

6. $y = x^3 - 3x^2 - 2x + 6$
7. $y = x^3 + x^2 - 3x - 3$
8. $y = x^3 + 2x^2 - 16x - 32$
9. $y = 2x^3 + 13x^2 + 9x + 6$
10. $y = 2x^3 + 15x^2 + 4x - 21$

Graph the functions below to determine if $x - 1$ is a factor of the polynomial.

11. $y = x^3 - 2x^2 + 3x - 6$
12. $y = x^3 + 3x^2 - 2x - 2$
13. $y = 3x^3 + 8x^2 - 5x - 6$
14. $y = x^3 + x^2 - 10x + 8$
15. $y = 2x^3 - x^2 - 3x + 2$

Indicate the real root(s) on the following quartic graphs:

16. $y = x^4 - 3x^3 - 6x^2 - 3$
17. $y = x^4 - 8x^2 - 8$
18. $y = 2x^4 + 2x^3 + x^2 - x - 8$
19. $y = x^4 - 6x^2 - x + 3$
20. $y = x^4 + x^3 - 7x^2 - x + 6$

Describe the following graphs:

21. $y = x^4 - 5x^2 + 2x + 2$
22. $y = x^4 + 3x^3 - x - 3$
23. $y = -x^4 + x^3 + 4x^2 - x + 6$
24. $y = -x^4 - 5x^3 - 5x^2 + 5x + 6$
25. $y = -2x^4 - 4x^3 - 5x^2 - 4x - 4$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.15.

Summary

You learned that adding, subtracting, and multiplying polynomials all rely on the distributive property. You also learned that factoring is the reverse of multiplying because when you factor a polynomial you are trying to rewrite the polynomial as a product of other polynomials. You learned how to factor completely by first looking for common factors and then using other methods to factor the remaining expression. You learned special cases of factoring to watch out for including the difference of perfect squares, perfect square trinomials, and the sum and difference of cubes. You learned how factoring can allow you to solve a quadratic equation with the help of the zero product property.

You learned how to divide polynomials and how dividing polynomials can connect to factoring. You learned about the factor theorem and that polynomial long division can actually help you to factor higher degree polynomials. Finally, you learned that factors of polynomials connect to their x-intercepts, and that you can use a graphing calculator to graph polynomial functions in order to find out more information about the factors of the polynomial.

CHAPTER 8 Rational Expressions and Rational Functions

Chapter Outline

Introduction

Here you'll learn all about rational expressions. You'll start by learning how to apply your factoring skills to simplifying rational expressions. You'll then learn how operations with fractions generalize to operations with rational expressions. Finally, you will learn about rational functions. You will learn what the graphs of rational functions look like and how to find the asymptotes for rational functions.

8.1 Rational Expression Simplification

Here you'll learn how to simplify rational expressions.

How could you use factoring to help simplify the following rational expression?

$\frac{3x^2-27}{x^2+7x+12}$

Rational Expressions

A rational number is any number of the form $\frac{a}{b}$, where $b \neq 0$. A ***rational expression*** is any algebraic expression of the form $\frac{a(x)}{b(x)}$, where $b \neq 0$. An example of a rational expression is: $\frac{4x^2+20x+24}{2x^2+8x+8}$.

Consider that any number or expression divided by itself is equal to 1. For example, $\frac{2}{2} = 1$ and $\frac{(x+2)}{(x+2)} = 1$. This fact allows you to simplify rational expressions that are in factored form by looking for "1's". Consider the following rational expression:

$\frac{4x^2+20x+24}{2x^2+8x+8}$

Factor both the numerator and denominator completely:

$\frac{4(x+2)(x+3)}{2(x+2)(x+2)}$

Notice that there is one factor of $x+2$ in both the numerator and denominator. These factors divide to make 1, so they "cancel out" (the second factor of $(x+2)$ in the denominator will remain there).

$\frac{4\cancel{(x+2)}(x+3)}{2\cancel{(x+2)}(x+2)}$

Also, the $\frac{4}{2}$ reduces to just 2. The simplified expression is:

$\frac{2(x+3)}{x+2}$

Keep in mind that you cannot "cancel out" common factors until both the numerator and denominator have been factored.

A rational expression is like any other fraction in that it is said to be undefined if the denominator is equal to zero. Values of the variable that cause the denominator of a rational expression to be zero are referred to as ***restrictions*** and must be excluded from the set of possible values for the variable. For the original expression above, the restriction is $x \neq -2$ because if $x = -2$ then the denominator would be equal to zero. Note that to determine the restrictions you must look at the **original** expression before any common factors have been cancelled.

Simplify the following and state any restrictions on the denominator.

$\frac{x-2}{x^2-10x+16}$

To begin, factor both the numerator and the denominator:

$\frac{x-2}{(x-8)(x-2)}$

Cancel out the common factor of $x-2$ to create the simplified expression:

$\frac{\cancel{(x-2)}}{(x-8)\cancel{(x-2)}}$

$\frac{1}{x-8}$

The restrictions are $x \neq 2$ and $x \neq 8$ because both of those values for x would have made the denominator of the original expression equal to zero.

Simplify the following and state any restrictions on the denominator.

$\frac{x^2+7x+12}{x^2-16}$

To begin, factor both the numerator and the denominator:

$\frac{(x+4)(x+3)}{(x-4)(x+4)}$

Cancel out the common factor of $x+4$ to create the simplified expression:

$\frac{\cancel{(x+4)}(x+3)}{(x-4)\cancel{(x+4)}}$

$\frac{x+3}{x-4}$

The restrictions are $x \neq 4$ and $x \neq -4$ because both of those values for x would have made the denominator of the original expression equal to zero.

Simplify the following and state any restrictions on the denominator.

$\frac{3x^2-7x-6}{4x^2-13x+3}$

To begin, factor both the numerator and the denominator:

$\frac{(x-3)(3x+2)}{(x-3)(4x-1)}$

Cancel out the common factor of $x-3$ to create the simplified expression:

$\frac{\cancel{(x-3)}(3x+2)}{\cancel{(x-3)}(4x-1)}$

$\frac{3x+2}{4x-1}$

The restrictions are $x \neq 3$ and $x \neq \frac{1}{4}$ because both of those values for x would have made the denominator of the original expression equal to zero.

Examples

Example 1

Earlier, you were asked how could you use factoring to help simplify the following rational expression.

$$\begin{aligned}
&\frac{3x^2-27}{x^2+7x+12}\\
=&\frac{3(x^2-9)}{(x+3)(x+4)}\\
=&\frac{3(x+3)(x-3)}{(x+3)(x+4)}\\
=&\frac{3\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x+4)}\\
=&\frac{3(x-3)}{x+4}
\end{aligned}$$

where $x \neq -3$ and $x \neq -4$

Example 2

Simplify the following and state the restrictions.

$\frac{m^2-9m+18}{4m^2-24m}$

$\frac{m^2-9m+18}{4m^2-24m} = \frac{(m-6)(m-3)}{(4m)(m-6)} = \frac{(m-3)}{4m}, m \neq 0; m \neq 6$

Example 3

Simplify the following and state the restrictions.

$\frac{2x^2-8}{4x+8}$

$\frac{2x^2-8}{4x+8} = \frac{(2)(x-2)(x+2)}{(4)(x+2)} = \frac{(x-2)}{2}, x \neq -2$

Example 4

$\frac{c^2+4c-5}{c^2-2c-35}$

$\frac{c^2+4c-5}{c^2-2c-35} = \frac{(c+5)(c-1)}{(c-7)(c+5)} = \frac{(c-1)}{(c-7)}, c \neq -5; c \neq 7$

Review

For each of the following rational expressions, state the restrictions.

1. $\frac{7}{x+4}$
2. $\frac{-3}{x-5}$
3. $\frac{5x+1}{5x-1}$
4. $\frac{6}{4x-3}$
5. $\frac{(x+1)}{x^2-4}$
6. $\frac{x-8}{x^2+3x+2}$
7. $\frac{x+6}{x^2-5x-24}$
8. $\frac{5x+2}{2x^2+5x+2}$

Simplify each of the following rational expressions and state the restrictions.

9. $\frac{4}{4x+12}$
10. $\frac{4c^2}{8c^2-4c}$
11. $\frac{10x+5}{2x+1}$
12. $\frac{x-4}{x^2-16}$
13. $\frac{y+1}{y^2+5y+4}$
14. $\frac{c+2}{c^2-5c-14}$
15. $\frac{(b-3)^2}{b^2-6b+9}$
16. $\frac{3n^2-27}{6n+18}$
17. $\frac{6k^2+7k-20}{12k^2-19k+4}$
18. $\frac{4x^2-4x-3}{2x^2+3x-9}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 8.1.

8.2 Rational Expression Multiplication and Division

Here you'll learn how to multiply and divide rational expressions.

How can you use your knowledge of multiplying fractions to multiply the following rational expressions?

$\frac{10y+20}{5y-15} \cdot \frac{y-3}{y^2+10y+16}$

Multiplying and Dividing Rational Expressions

Rational expressions are examples of fractions, so you multiply and divide rational expressions in the same ways that you multiply and divide fractions. As with fractions, after multiplying or dividing you will want to simplify your result. You will also want to state any restrictions that would cause the denominator of either original rational expression to be equal to zero. To multiply:

$\frac{x^2+2x}{x+3} \cdot \frac{x^2+4x+3}{x}$

First, factor all expressions that can be factored:

$\frac{x(x+2)}{x+3} \cdot \frac{(x+3)(x+1)}{x}$

Next, multiply the numerators and multiply the denominators to create one big rational expression. Leave in factored form:

$\frac{x(x+2)(x+3)(x+1)}{x(x+3)}$

Simplify:

$\frac{\cancel{x}(x+2)\cancel{(x+3)}(x+1)}{\cancel{x}\cancel{(x+3)}}$

$=(x+2)(x+1)$

$=x^2+3x+2$

Finally, state the restrictions based on the original rational expressions:

$x \neq -3$ and $x \neq 0$.

To divide rational expressions, recall that dividing one fraction by another is the same as multiplying the first fraction by the reciprocal of the second fraction. For example, $\frac{x^2+2x}{x+3} \div \frac{x}{x^2+4x+3}$ is equivalent to, and can be rewritten as, $\frac{x^2+2x}{x+3} \cdot \frac{x^2+4x+3}{x}$, which can then be solved using the same steps as above.

Multiply the following rational expressions and state the restrictions.

$\frac{4x-8}{x^2-7x+10} \cdot \frac{x^2-3x-10}{x^2-4}$

Begin by factoring the numerator and denominator of each expression:

$\frac{4(x-2)}{(x-5)(x-2)} \cdot \frac{(x-5)(x+2)}{(x-2)(x+2)}$

Next, multiply the numerators and the denominators to create one rational expression:

$\frac{4(x-2)(x-5)(x+2)}{(x-5)(x-2)(x-2)(x+2)}$

Simplify by removing common factors from the numerator and denominator that divide to make 1:

$\frac{4\cancel{(x-2)}\cancel{(x-5)}\cancel{(x+2)}}{\cancel{(x-5)}\cancel{(x-2)}\cancel{(x+2)}(x-2)}$

The final answer is: $\frac{4}{(x-2)}$ with restrictions: $x \neq 5$, $x \neq 2$, and $x \neq -2$.

Divide the following rational expressions and state the restrictions.

$\frac{m^2-4}{m^2+9m+14} \div \frac{3m^2-6m}{m^2-49}$

To divide rational expressions, multiply by the reciprocal of the divisor. Then, follow the process for multiplying rational expressions.

$\frac{m^2-4}{m^2+9m+14} \cdot \frac{m^2-49}{3m^2-6m}$

Begin by factoring the numerator and denominator of each expression.

$\frac{(m-2)(m+2)}{(m+7)(m+2)} \cdot \frac{(m-7)(m+7)}{3m(m-2)}$

Next, multiply the numerators and the denominators to create one rational expression:

$\frac{(m-2)(m+2)(m-7)(m+7)}{3m(m+7)(m+2)(m-2)}$

Simplify by removing common factors from the numerator and denominator that divide to make 1:

$\frac{\cancel{(m+2)}\cancel{(m-2)}\cancel{(m+7)}(m-7)}{3m\cancel{(m-2)}\cancel{(m+7)}\cancel{(m+2)}}$

The final answer is: $\frac{(m-7)}{3m}$ with restrictions: $x \neq -7$, $x \neq 2$, $x \neq 0$, $x \neq 7$ and $x \neq -2$. Note that when dividing rational expressions, for restrictions you must consider ALL factors that ever appear in a denominator. This means that both the numerator and denominator of the second rational expression must be considered for restrictions.

Simplify the following rational expressions and state the restrictions.

$\frac{12x^2+13x-35}{5x^2-21x+18} \div \frac{3x^2+16x+21}{5x^2+9x-18}$

To divide rational expressions, multiply by the reciprocal of the divisor. Then, follow the process for multiplying rational expressions.

$\frac{12x^2+13x-35}{5x^2-21x+18} \times \frac{5x^2+9x-18}{3x^2+16x+21}$

Begin by factoring the numerator and denominator of each expression.

$\frac{(4x-5)(3x+7)}{(5x-6)(x-3)} \times \frac{(5x-6)(x+3)}{(3x+7)(x+3)}$

Next, multiply the numerators and the denominators to create one rational expression:

$\frac{(4x-5)(3x+7)(5x-6)(x+3)}{(5x-6)(x-3)(3x+7)(x+3)}$

Simplify by removing common factors from the numerator and denominator that divide to make 1:

$\frac{(4x-5)\cancel{(3x+7)}\cancel{(5x-6)}\cancel{(x+3)}}{\cancel{(5x-6)}(x-3)\cancel{(3x+7)}\cancel{(x+3)}}$

The final answer is: $\frac{(4x-5)}{(x-3)}$ with restrictions: $x \neq \frac{6}{5}$, $x \neq 3$, $x \neq -\frac{7}{3}$, and $x \neq -3$.

Examples

Example 1

Earlier, you were asked how can you use your knowledge of multiplying fractions to multiply the following rational expressions.

$\frac{10y+20}{5y-15} \cdot \frac{y-3}{y^2+10y+16}$

Begin by factoring the numerator and denominator of each expression.

$\frac{10(y+2)}{5(y-3)} \cdot \frac{y-3}{(y+8)(y+2)}$

Express the factored rational expressions as a single rational expression.

$\frac{10(y+2)(y-3)}{5(y-3)(y+8)(y+2)}$

Cancel the common factors that exist in the numerator and the denominator.

$\frac{\cancel{10}^{2}\cancel{(y+2)}\cancel{(y-3)}}{\cancel{5}\cancel{(y-3)}(y+8)\cancel{(y+2)}}$

The result of cancelling the common factors is the answer. Don't forget to include the restrictions.

$$\boxed{\frac{2}{y+8}; y \neq 3; y \neq -8; y \neq -2}$$

Example 2

Multiply or divide each of the following and state the restrictions.

$\frac{x+7}{x^2-5x-36} \div \frac{x^2-2x-63}{x+4} \cdot \frac{x^2-15x+54}{x^2-36}$

Write the term after the division sign as a reciprocal and multiply.

$\frac{x+7}{x^2-5x-36} \cdot \frac{x+4}{x^2-2x-63} \cdot \frac{x^2-15x+54}{x^2-36}$

Factor the numerator and denominator of each expression.

$\frac{x+7}{(x-9)(x+4)} \cdot \frac{x+4}{(x-9)(x+7)} \cdot \frac{(x-9)(x-6)}{(x+6)(x-6)}$

Express the factored rational expressions as a single rational expression.

$\frac{(x+7)(x+4)(x-9)(x-6)}{(x-9)(x+4)(x-9)(x+7)(x+6)(x-6)}$

Cancel the common factors that exist in the numerator and the denominator.

$\frac{\overbrace{\cancel{(x+7)}\cancel{(x+4)}\cancel{(x-9)}\cancel{(x-6)}}^{1}}{\cancel{(x-9)}\cancel{(x+4)}(x-9)\cancel{(x+7)}(x+6)\cancel{(x-6)}}$

The result of cancelling the common factors is the answer.

$$\boxed{= \frac{1}{(x-9)(x-6)}; x \neq 9; x \neq -4; x \neq -7; x \neq -6; x \neq 6;}$$

Example 3

$\frac{y^2-25}{y^2-6y} \cdot \frac{y^2-12y+36}{y^2+2y-15} \div \frac{y^2-11y+30}{y^2+4y-21}$

Write the term after the division sign as a reciprocal and multiply.

$\frac{y^2-25}{y^2-6y} \cdot \frac{y^2-12y+36}{y^2+2y-15} \cdot \frac{y^2+4y-21}{y^2-11y+30}$

Factor the numerator and denominator of each expression.

$\frac{(y+5)(y-5)}{y(y-6)} \cdot \frac{(y-6)(y-6)}{(y+5)(y-3)} \cdot \frac{(y+7)(y-3)}{(y-6)(y-5)}$

Express the factored rational expressions as a single rational expression.

$\frac{(y+5)(y-5)(y-6)(y-6)(y+7)(y-3)}{y(y-6)(y+5)(y-3)(y-6)(y-5)}$

Cancel the common factors that exist in the numerator and the denominator.

$\frac{\cancel{(y+5)}\cancel{(y-5)}\cancel{(y-6)}\cancel{(y-6)}(y+7)\cancel{(y-3)}}{y\cancel{(y-6)}\cancel{(y+5)}\cancel{(y-3)}\cancel{(y-6)}\cancel{(y-5)}}$

The result of cancelling the common factors is the answer.

$$\boxed{=\frac{y+7}{y}; y \neq 0; y \neq 6; y \neq -5; y \neq 3; y \neq 5; y \neq -7}$$

Example 4

$\frac{2x^2+7x-4}{6x^2+x-2} \cdot \frac{15x^2+7x-2}{5x^2+19x-4}$

Factor the numerator and denominator of each expression.

$\frac{(2x-1)(x+4)}{(2x-1)(3x+2)} \cdot \frac{(5x-1)(3x+2)}{(5x-1)(x+4)}$

Express the factored rational expressions as a single rational expression.

$\frac{(2x-1)(x+4)(5x-1)(3x+2)}{(2x-1)(3x+2)(5x-1)(x+4)}$

Cancel the common factors that exist in the numerator and the denominator.

$$\frac{\overbrace{\cancel{(2x-1)}\cancel{(x+4)}\cancel{(5x-1)}\cancel{(3x+2)}}^{1}}{\cancel{(2x-1)}\cancel{(3x+2)}\cancel{(5x-1)}\cancel{(x+4)}}$$

The result of cancelling the common factors is the answer.

$$\boxed{=1; x \neq \frac{1}{2}; x \neq -\frac{2}{3}; x \neq \frac{1}{5}; x \neq -4}$$

Review

Multiply or divide each of the following and state the restrictions for each.

1. $\frac{3x+9}{6x} \cdot \frac{x^2}{x^2-9}$
2. $\frac{c^2+5c+6}{c-1} \cdot \frac{c^2-1}{c+3}$
3. $\frac{a^2+3a}{3a-9} \cdot \frac{a^2-a-6}{2a^2+6a}$
4. $\frac{y-3}{y+3} \cdot \frac{y^2-9}{y+3} \cdot \frac{y^2+6y+9}{y^2-6y+9}$
5. $\frac{m^2-4m-5}{m^2-5m} \cdot \frac{m^2-6m+5}{m^2-1} \cdot \frac{m}{m-5}$
6. $\frac{x^2-x-20}{x^2-25} \div \frac{3x+12}{x+5}$
7. $\frac{d^2-9}{3-3d} \div \frac{d^2+5d+6}{d^2+3d-4}$

8. $\frac{4x^2-20x}{3x+6} \div \frac{x-5}{x^2-x-6}$
9. $\frac{4n^2-9}{2n^3+2n^2-4n} \div \frac{2n^2-n-3}{3n^2-6n+3}$
10. $\frac{e^2+10e+21}{2e^2+7e-15} \div \frac{e^2+8e+15}{e^2+10e+25}$
11. $\frac{x^2+2x-15}{x^2-6x+8} \cdot \frac{x^2+2x-8}{x^2-6x+9} \cdot \frac{x^2-7x+12}{x^2-x-30}$
12. $\frac{2x^2+5x-3}{4x^2-12x+5} \div \frac{3x^2+13x+12}{6x^2-7x-20}$
13. $\frac{5m^2-20}{m^2+14m+33} \cdot \frac{m^2+10m-11}{m^2-8m+12} \cdot \frac{m^2-3m-18}{m^2+m-2}$
14. $\frac{2y^2+5y-12}{y^2+9y+14} \div \frac{6y^2-7y-3}{3y^2+25y+8} \cdot \frac{y^2+3y-28}{y^2-16}$
15. $\frac{x^2-49}{x^2+3x-88} \cdot \frac{x^2+6x-55}{x^2-11x+28} \div \frac{x^2+2x-35}{x^2-12x+32}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 8.2.

8.3 Rational Expression Addition and Subtraction

Here you'll learn how to add and subtract rational expressions.

Can you use your knowledge of rational expressions and adding fractions to add the following rational expressions?

$\frac{3x}{x^2+6x-16}+\frac{2x}{x-2}$

Adding and Subtracting Rational Expressions

Rational expressions are examples of fractions, so you add and subtract rational expressions in the same way that you add and subtract fractions. As with fractions, you will need a common denominator, ideally the lowest common denominator (LCD), in order to add or subtract the expressions.

For example, to add:

$\frac{x^2+2x}{x+3}+\frac{x}{x^2+4x+3}$

First, factor the denominators to get:

$\frac{x^2+2x}{x+3}+\frac{x}{(x+3)(x+1)}$.

Next, find the lowest common denominator (LCD). This will be the product of each unique factor in the denominators. In this case, the LCD is $(x+3)(x+1)$. Multiply the numerator and denominator of each fraction by the factors necessary to create the common denominator. In this case, you only need to multiply the fraction on the left by $\left(\frac{x+1}{x+1}\right)$. The expression becomes:

$\frac{x^2+2x}{x+3}\left(\frac{x+1}{x+1}\right)+\frac{x}{(x+3)(x+1)}$

$=\frac{(x^2+2x)(x+1)}{(x+3)(x+1)}+\frac{x}{(x+3)(x+1)}$

Now add the numerators and write as one rational expression:

$=\frac{(x^2+2x)(x+1)+x}{(x+3)(x+1)}$

Simplify the numerator by multiplying, combining like terms, and factoring if possible (the denominator is left in factored form):

$\frac{x^3+3x^2+3x}{(x+3)(x+1)}$

$=\frac{x(x^2+3x+3)}{(x+3)(x+1)}$

The rational expression cannot be simplified any further so this is your answer. The restrictions are $x\neq -3$ and $x\neq -1$ because those values would cause one or both of the original denominators to be equal to zero.

Identify the lowest common denominator (LCD) in factored form.

$\frac{2x-3}{x^2-7x+10}-\frac{x-5}{x^2-2x-15}$

To determine the LCD, begin by factoring the denominators.

$\frac{2x-3}{x^2-7x+10}-\frac{x-5}{x^2-2x-15}=\frac{2x-3}{(x-5)(x-2)}-\frac{x-5}{(x-5)(x+3)}$

The LCD is

$$\boxed{(x-5)(x-2)(x+3)}$$

$\frac{2x+1}{x^2+6x+9}+\frac{3x-2}{x^2+x-6}$ii)
$\frac{2x+1}{x^2+6x+9}+\frac{3x-2}{x^2+x-6}=\frac{2x+1}{(x+3)(x+3)}+\frac{3x-2}{(x+3)(x-2)}$
The LCD is

$$\boxed{(x+3)(x+3)(x-2)}$$

Add the following rational expressions and state the restrictions.

$\frac{3x+1}{x^2+8x+16}+\frac{2x-3}{x^2+x-12}$
Begin by determining the LCD. Factor the denominators of each expression.
$\frac{3x+1}{x^2+8x+16}+\frac{2x-3}{x^2+x-12}=\frac{3x+1}{(x+4)(x+4)}+\frac{2x-3}{(x+4)(x-3)}$
The LCD is

$$\boxed{(x+4)(x+4)(x-3)}$$

Multiply the numerators and denominators of each expression by the necessary factors to create the LCD.
$\frac{3x+1}{(x+4)(x+4)}\left(\frac{x-3}{x-3}\right)+\frac{2x-3}{(x+4)(x-3)}\left(\frac{x+4}{x+4}\right)$
Multiply the numerators. Keep the denominators in factored form.
$\frac{3x^2-8x-3}{(x+4)(x+4)(x-3)}+\frac{2x^2+5x-12}{(x+4)(x+4)(x-3)}$
Write the two expressions as one rational expression.
$\frac{3x^2-8x-3+2x^2+5x-12}{(x+4)(x+4)(x-3)}$
Simplify the numerator by combining like terms.
$\frac{5x^2-3x-15}{(x+4)(x+4)(x-3)}$
The numerator cannot be factored so the expression cannot be further simplified. The answer in lowest terms is:

$$\boxed{\frac{5x^2-3x-15}{(x+4)(x+4)(x-3)};x\neq-4;x\neq3}$$

Subtract the following rational expressions and state the restrictions.

$\frac{x}{x^2-9x+18}-\frac{x-2}{x^2-10x+24}$
Begin by determining the LCD. Factor the denominators of each expression.
$\frac{x}{(x-6)(x-3)}-\frac{x-2}{(x-6)(x-4)}$
The LCD is

$$\boxed{(x-6)(x-3)(x-4)}$$

Multiply the numerators and denominators of each expression to get the LCD.

$\frac{x}{(x-6)(x-3)}\left(\frac{x-4}{x-4}\right)-\frac{x-2}{(x-6)(x-4)}\left(\frac{x-3}{x-3}\right)$

Multiply the numerators.

$\frac{x^2-4x}{(x-6)(x-3)(x-4)}-\frac{x^2-5x+6}{(x-6)(x-3)(x-4)}$

Write the expressions as one rational expression.

$\frac{x^2-4x-(x^2-5x+6)}{(x-6)(x-3)(x-4)}$

$=\frac{x^2-4x-x^2+5x-6}{(x-6)(x-3)(x-4)}$

Simplify the numerator by combining like terms.

$\frac{x-6}{(x-6)(x-3)(x-4)}$

The term $(x-6)$ is common to both the numerator and the denominator. This term can be "cancelled." The solution is:

$$\boxed{\frac{1}{(x-3)(x-4)};x\neq 3;x\neq 4;x\neq 6}$$

Examples

Example 1

Earlier, you were asked can you use your knowledge of rational expressions and adding fractions to add the following expressions.

$\frac{3x}{x^2+6x-16}+\frac{2x}{x-2}$

Factor the denominator of the first fraction and rewrite the problem:

$\frac{3x}{(x+8)(x-2)}+\frac{2x}{x-2}$

The LCD is $(x+8)(x-2)$.

$\frac{3x}{(x+8)(x-2)}+\frac{2x}{x-2}\left(\frac{x+8}{x+8}\right)$

Multiply the numerators.

$\frac{3x}{(x+8)(x-2)}+\frac{2x^2+16x}{(x-2)(x+8)}$

Write the two expressions a one rational expression.

$\frac{3x+2x^2+16x}{(x+8)(x-2)}$

Simplify the numerator by combining like terms. Your final answer is:

$$\boxed{\frac{2x^2+19x}{(x+8)(x-2)};x\neq -8;x\neq 2}$$

Example 2

Add or subtract the following and state the restrictions.

$\frac{2x}{x^2-4}-\frac{1}{x-2}$

$$\begin{aligned}
\frac{2x}{x^2-4}-\frac{1}{x-2} &= \frac{2x}{(x-2)(x+2)}-\frac{x+2}{(x-2)(x+2)} \\
&= \frac{2x-(x+2)}{(x-2)(x+2)} \\
&= \frac{(x-2)}{(x-2)(x+2)} \\
&= \frac{1}{(x+2)};x \neq -2;x \neq 2
\end{aligned}$$

Example 3

Add or subtract the following and state the restrictions.

$\frac{-2}{3y^2+5y+2}+\frac{3}{y^2-7y-8}$

$$\begin{aligned}
\frac{-2}{3y^2+5y+2}+\frac{3}{y^2-7y-8} &= \frac{-2}{(3y+2)(y+1)}+\frac{3}{(y-8)(y+1)} \\
&= \frac{-2(y-8)}{(3y+2)(y+1)(y-8)}+\frac{3(3y+2)}{(3y+2)(y-8)(y+1)} \\
&= \frac{-2y+16+9y+6}{(3y+2)(y-8)(y+1)} \\
&= \frac{7y+22}{(3y+2)(y-8)(y+1)};y \neq -\frac{2}{3};y \neq 8;y \neq -1
\end{aligned}$$

Example 4

$\frac{3m-1}{9m^3-36m^2}+\frac{2m+1}{2m^2-5m-12}$

$$\begin{aligned}
\frac{3m-1}{9m^3-36m^2}+\frac{2m+1}{2m^2-5m-12} &= \frac{3m-1}{9m^2(m-4)}+\frac{2m+1}{(2m+3)(m-4)} \\
&= \frac{(3m-1)(2m+3)}{9m^2(m-4)(2m+3)}+\frac{9m^2(2m+1)}{9m^2(2m+3)(m-4)} \\
&= \frac{6m^2-2m+9m-3+18m^3+9m^2}{9m^2(m-4)(2m+3)} \\
&= \frac{18m^3+15m^2+7m-3}{9m^2(m-4)(2m+3)};m \neq -\frac{3}{2};m \neq 4;m \neq 0
\end{aligned}$$

Review

For each of the following rational expressions, determine the LCD.

1. $\frac{2a-3}{4}+\frac{3a-1}{5}-\frac{a-5}{2}$
2. $\frac{5}{3x^2}-\frac{1}{2x}+\frac{3}{5x^3}$
3. $\frac{x}{a^2b}-\frac{y}{ab^2}+\frac{z}{3a^3b^2}$
4. $\frac{2w}{w^2-6w+5}-\frac{3w}{w^2-11w+30}$

5. $\frac{1}{y^2+5y} - \frac{2}{y^2+12y+35} - \frac{3}{y^3+7y^2}$

For each of the following rational expressions, state the restrictions.

6. $\frac{3}{x^2-5x+4} + \frac{4}{x^2-16}$
7. $\frac{5}{a^2+a} - \frac{2}{a^2+3a+2}$
8. $\frac{6}{m^2-5m} + \frac{7}{m^2-4m-5}$
9. $\frac{3n}{n^2+2n-3} - \frac{4n}{n^2+n-6}$
10. $\frac{6}{y^2-4} + \frac{4}{y^2+4y+4}$

Add or subtract each of the following rational expressions and state the restrictions.

11. $\frac{2a-3}{4} + \frac{3a-1}{5} - \frac{a-5}{2}$
12. $\frac{5}{3x^2} - \frac{1}{2x} + \frac{3}{5x^3}$
13. $\frac{x}{a^2b} - \frac{y}{ab^2} + \frac{z}{3a^3b^2}$
14. $\frac{2w}{w^2-6w+5} - \frac{3w}{w^2-11w+30}$
15. $\frac{1}{y^2+5y} - \frac{2}{y^2+12y+35} - \frac{3}{y^3+7y^2}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 8.3.

8.4 Graphs of Rational Functions

Here you'll investigate the graphs of rational functions using a graphing calculator.

Can you use your graphing calculator to help you sketch the graph of the rational function? Does this function have any zeros or asymptotes?

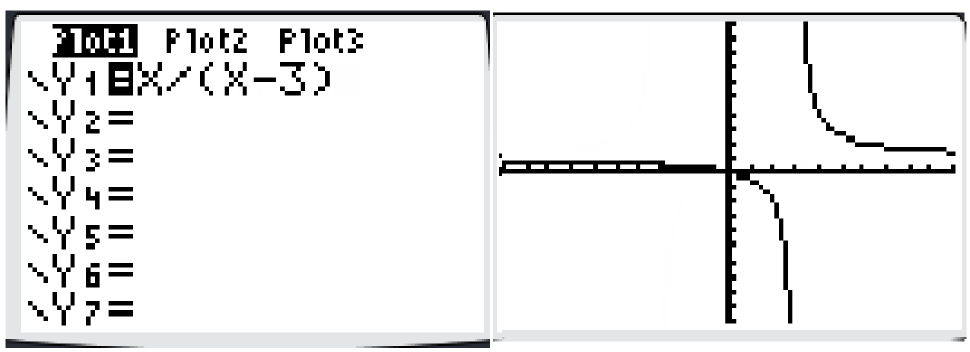
$y = \frac{4}{x^2+1}$

Graphing Rational Functions

A rational function is the quotient of two polynomial functions. In general,

$f(x) = \frac{A(x)}{B(x)}$

where A and B are polynomials and $B \neq 0$. You can use your graphing calculator to graph a rational function and look for important features. Consider the function $y = \frac{x}{x-3}$.



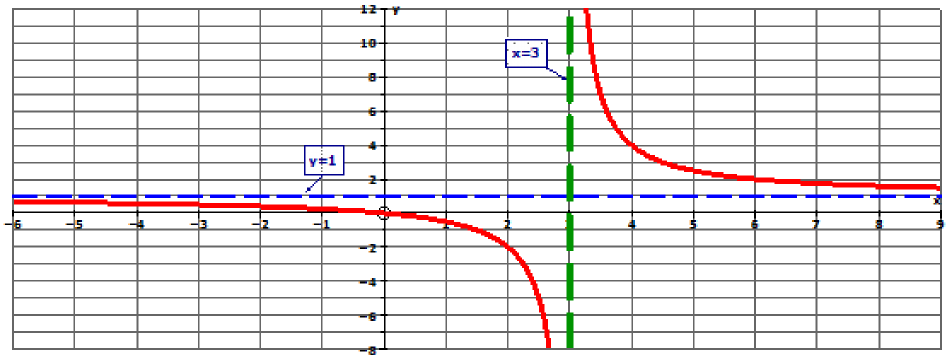
Notice that the function has two pieces. In between those two pieces are the asymptotes.

- A vertical asymptote occurs at the x value(s) that cause the denominator of the function to be equal to zero (which is undefined). This function has a vertical asymptote at $x = 3$.
- A horizontal asymptote occurs at the y value(s) that cause the denominator of the function to be zero if the function is rewritten and solved for x instead of y. This is what it looks like to solve for x:

$$\begin{aligned}
&y = \frac{x}{x-3} \\
&(x-3)(y) = (x-3)\left(\frac{x}{x-3}\right) \\
&xy - 3y = \cancel{(x-3)}\left(\frac{x}{\cancel{x-3}}\right) \\
&xy - 3y = x \\
&xy - x = 3y \\
&x(y-1) = 3y \\
&\boxed{x = \frac{3y}{y-1}}
\end{aligned}$$

Thus, $y \neq 1$ and this function has a horizontal asymptote at $y = 1$.

The image below shows the graph with the asymptotes drawn in and labelled. For rational functions, the asymptotes represent the lines that the function will approach but never touch.



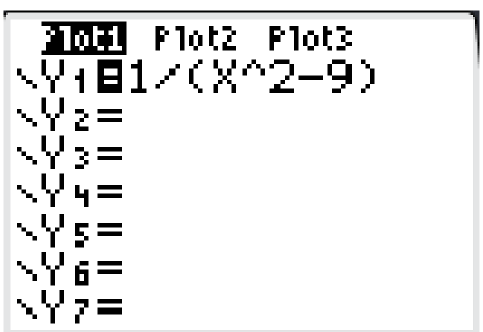
It is also important to note the x-intercepts (zeros) of the function. The zeros of the function will be the values for x that cause the numerator, but not also the denominator, to be equal to zero.

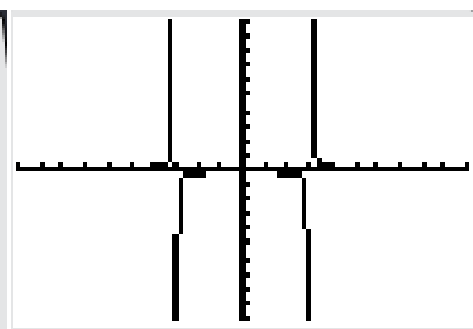
Graph the rational function

Use technology to sketch the graph of the rational function. Find all zeros and asymptotes and label those on your sketch.

$y = \frac{1}{x^2-9} = \frac{1}{(x+3)(x-3)}$

Here is the sketch from the calculator:

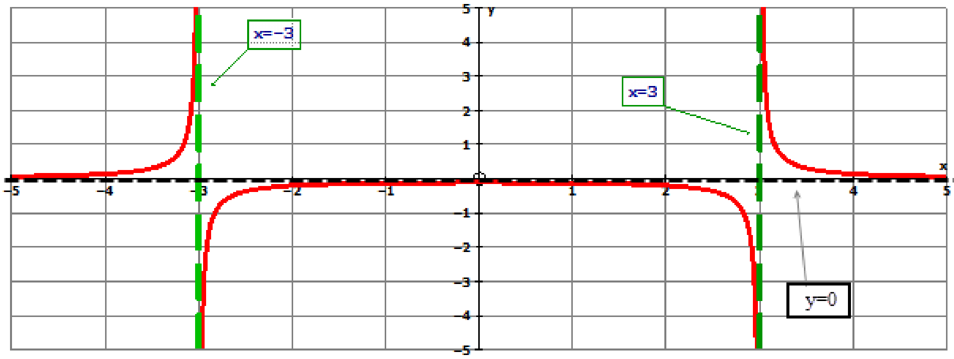




It can sometimes be hard to interpret what you see on the graphing calculator screen. Use algebra to find the asymptotes and zeros and sketch those first.

- There are no zeros for the numerator. Therefore, there are no x-intercepts for the function.
- The zeros of the denominator are 3 and -3. This means that there are two vertical asymptotes. One vertical asymptote is the line $x = 3$ and the other is the line $x = -3$.
- Another way to determine a horizontal asymptote besides solving the equation for x is to look at the degrees of the numerators and denominators. The degree is the highest exponent. The degree of the numerator is 0 and the degree of the denominator is 2. In general, if the degree of the numerator is less than the degree of the denominator, there will be a horizontal asymptote at $y = 0$.

Once you have sketched the asymptotes, use the table and/or graph from the graphing calculator to decide what the rest of the graph looks like. Here is the graph with the asymptotes labelled.

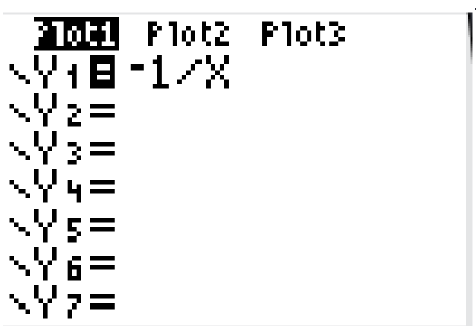


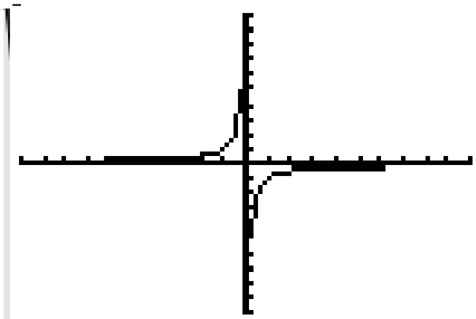
Graph the rational function

Use technology to sketch the graph of the rational function. Find all zeros and asymptotes and label those on your sketch.

$y = -\frac{1}{x}$

Here is the sketch from the calculator:



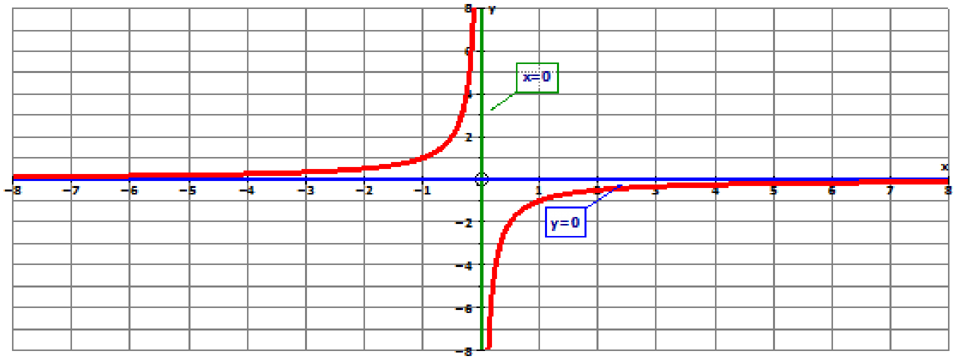


Use algebra to find the asymptotes and zeros and sketch those first.

- There are no zeros for the numerator. Therefore, there are no x-intercepts for the function.
- The zero of the denominator is 0. This means that there is one vertical asymptote, the line $x = 0$.
- The horizontal asymptote is $y = 0$. You can use algebra to solve the equation for x and look for the values of y that will cause the denominator to be equal to zero:

$$
\begin{aligned}
y &= -\frac{1}{x} \\
xy &= -1 \\
\frac{xy}{y} &= \frac{-1}{y} \\
\Aboxed{x &= -\frac{1}{y}}
\end{aligned}
$$

Once you have sketched the asymptotes, use the table and/or graph from the graphing calculator to decide what the rest of the graph looks like. Here is the graph with the asymptotes labelled.

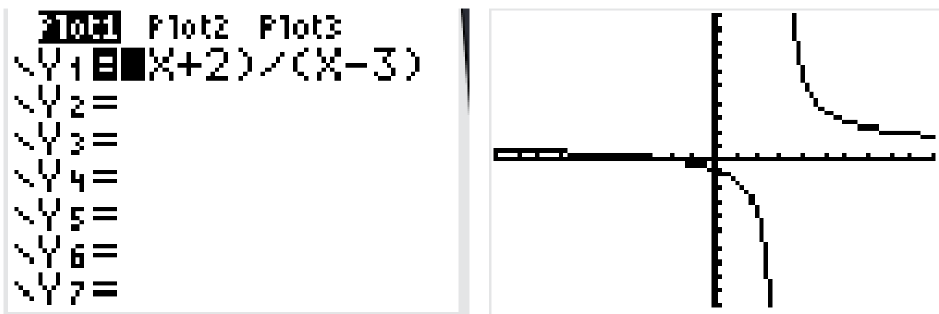


Graph the rational function

Use technology to sketch the graph of the rational function. Find all zeros and asymptotes and label those on your sketch.

$y = \frac{x+2}{x-3}$

Here is the sketch from the calculator:

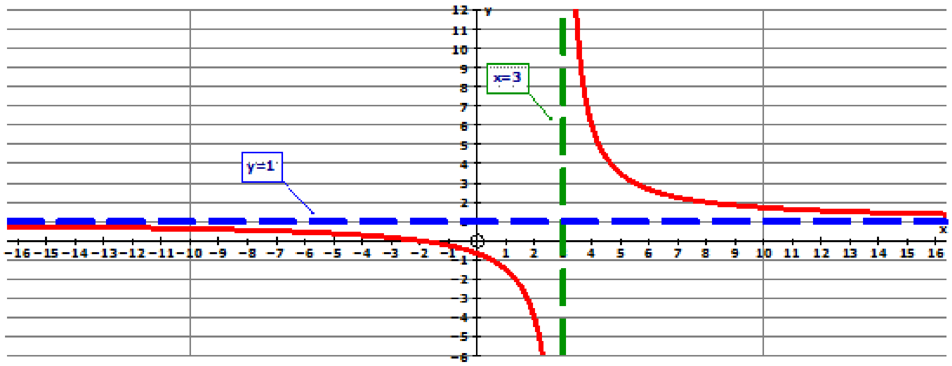


Use algebra to find the asymptotes and zeros and sketch those first.

- The zero of the numerator is -2 so the zero (x-intercept) of the function is (-2, 0).
- The zero of the denominator is 3. This means that there is one vertical asymptote, the line $x = 3$.
- The horizontal asymptote is $y = 1$. You can use algebra to solve the equation for x and look for the values of y that will cause the denominator to be equal to zero:

$$\begin{aligned}
&y = \frac{x+2}{x-3} \\
&(x-3)(y) = \left(\frac{x+2}{x-3}\right)(x-3) \\
&xy - 3y = x + 2 \\
&xy - x = 2 + 3y \\
&x(y-1) = 2 + 3y \\
&\boxed{x = \frac{3y+2}{y-1}}
\end{aligned}$$

Once you have sketched the asymptotes, use the table and/or graph from the graphing calculator to decide what the rest of the graph looks like. Here is the graph with the asymptotes labelled.



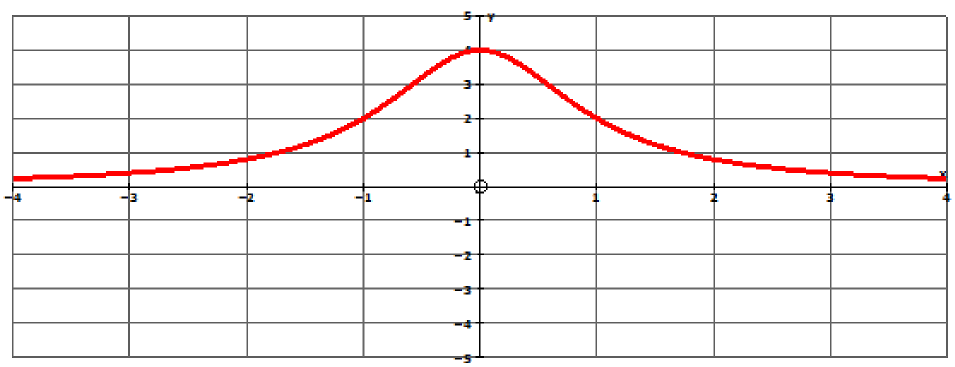
Examples

Example 1

Earlier, you were asked does this function have any zeros or asymptotes.

$y = \frac{4}{x^2+1}$

Here is a sketch of the function:

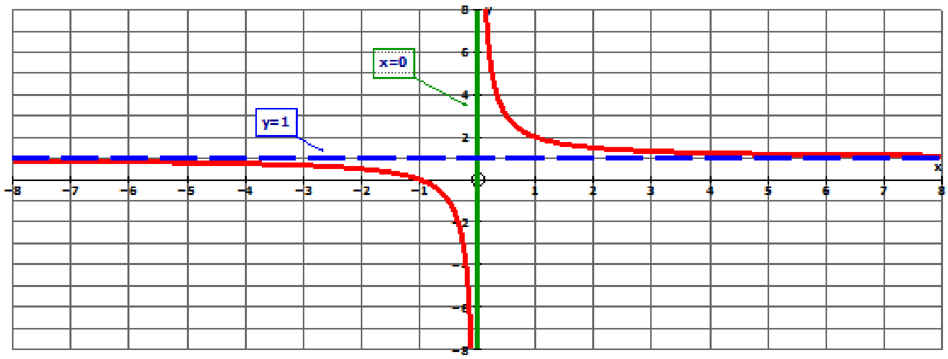


- There are no zeros for this function since there are no zeros for the numerator. The graph does not cross the x-axis.
- There are no zeros for the denominator. Therefore, there are no vertical asymptotes.
- Because the degree of the numerator is less than the degree of the denominator, there is a horizontal asymptote at $y = 0$.

Example 2

Sketch the graph of the rational function: $y = \frac{x+1}{x}$

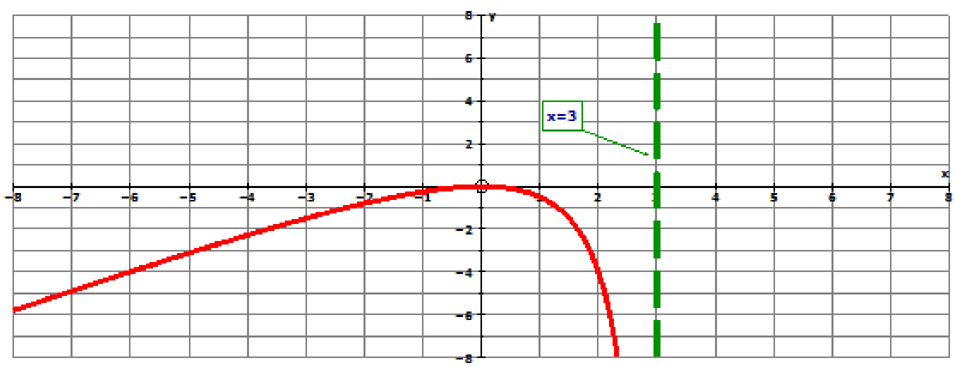
The zero of the denominator is 0. This means that there will be a vertical asymptote at $x = 0$ (the y-axis). Because the degree of the denominator and numerator are the same, you can solve the equation for x and get $x = \frac{1}{y-1}$. The zero of the denominator is now 1. This means there is a horizontal asymptote at $y = 1$. The numerator has a zero at -1, so there is an x-intercept at -1.



Example 3

Sketch the graph of the rational function: $y = \frac{x^2}{x-3}$

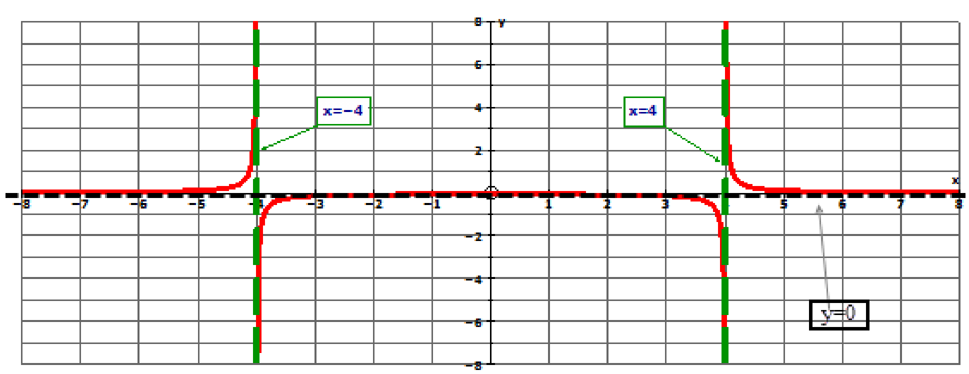
The zero of the denominator is 3. This means that there will be a vertical asymptote at $x = 3$. Because the degree of the numerator is greater than the degree of the denominator, there are no horizontal asymptotes. The zero of the numerator is 0, so there is an x-intercept at 0.



Example 4

Find the asymptotes of the function: $y = \frac{1}{x^2-16}$

The zeros of the denominator are 4 and -4. This means that there will be two vertical asymptotes. One vertical asymptote will be the line $x = 4$ and the other will be the line $x = -4$. Because the degree of the numerator is less than the degree of the denominator, there is a horizontal asymptote at $y = 0$. Though you did not need to sketch the graph, here is the graph of the function:



Review

Sketch the graph of each of the following rational functions.

1. $y = \frac{2}{x+3}$
2. $y = \frac{x}{x-1}$
3. $y = \frac{1}{x^2-4}$
4. $y = \frac{x+2}{x}$
5. $y = \frac{1}{x^2+2}$
6. $y = \frac{x}{x+2}$
7. $y = \frac{1}{x^2-x-12}$
8. $y = \frac{x-1}{x+3}$
9. $y = \frac{x-1}{x+4}$
10. $y = \frac{5}{x^2+1}$

Without graphing the following rational functions, state what you know about their asymptotes and zeros.

11. $y = \frac{1}{x^2-x-2}$
12. $y = -\frac{2}{x-4}$
13. $y = -\frac{2}{x^2+1}$
14. $y = \frac{6}{x^2+1}$
15. $y = \frac{x-1}{x+3}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 8.4.

Summary

You learned that operations with rational expressions rely on factoring and operations with fractions. To multiply rational expressions, multiply across and simplify. To divide rational expressions, change the problem to a multiplication problem by multiplying the first fraction by the reciprocal of the second fraction. To add or subtract, find the lowest common denominator in order to combine the expressions.

Rational functions can be graphed on the graphing calculator as an aid for making a sketch. You can algebraically find both the vertical and horizontal asymptotes of a rational function. To find the vertical asymptotes, consider the values of x that cause the denominator to be equal to zero and thus the function to be undefined. One method for finding horizontal asymptotes is to solve the equation of the function for x, and then look for the values of y that cause the denominator to be equal to zero. In the case where the degree of the numerator is less than the degree of the denominator, the horizontal asymptote will automatically be at $y = 0$. Rational functions will be explored in further detail in future courses like Algebra II and PreCalculus.

CHAPTER 9 Quadratic Equations and Quadratic Functions

Chapter Outline

Introduction

Here you'll learn more about quadratic equations and quadratic functions. You will learn three new methods for solving quadratic equations and discover the connections between a quadratic equation and its corresponding quadratic function. You will discover a new set of numbers called complex numbers and see how complex numbers are related to quadratic functions with no x-intercepts. Finally, you will learn how to solve radical equations.

9.1 Graphs to Solve Quadratic Equations

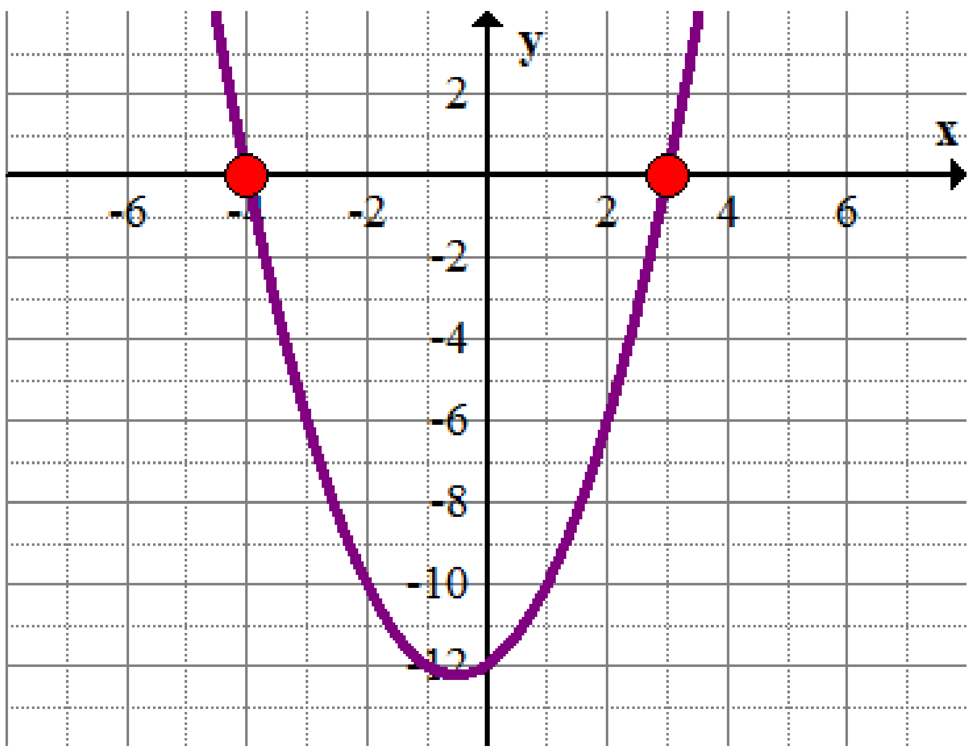
Here you will learn how to solve a quadratic equation by graphing.

One way to solve the equation $x^2-2x-3=0$ is to use factoring and the zero product property. How could you use a graph to solve $x^2-2x-3=0$?

Graphing to Solve Quadratic Equations

Recall that a quadratic equation is a degree 2 equation that can be written in the form $ax^2+bx+c=0$. Every quadratic equation has a corresponding quadratic function that you get by changing the "0" to a "y". Standard form for a quadratic function is $y=ax^2+bx+c$. Quadratic functions can be graphed by hand, or with a graphing calculator.

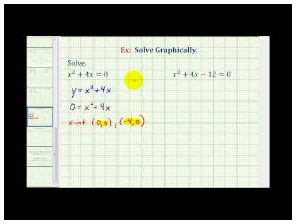
How do the solutions to the equation $x^2+x-12=0$ show up on the graph of $y=x^2+x-12$? On the graph you are looking for the points that have a y-coordinate that is equal to 0. Therefore, the solutions to the equation will show up as the x-intercepts on the graph of the function. These are also known as the ***roots*** or ***zeros*** of the function. Here is the graph of $y=x^2+x-12$:



You can see the x-intercepts are at $(-4,0)$ and $(3,0)$. This means that the solutions to the equation $x^2+x-12=0$ are $x=-4$ and $x=3$. You can verify these solutions by substituting them back into the equation:

- $(-4)^2+(-4)-12=16-4-12=0$
- $(3)^2+(3)-12=9+3-12=0$

Graphing is a great way to solve quadratic equations. Keep in mind that you can also solve many quadratic equations by factoring or using other algebraic methods such as the quadratic formula or completing the square.



MEDIA

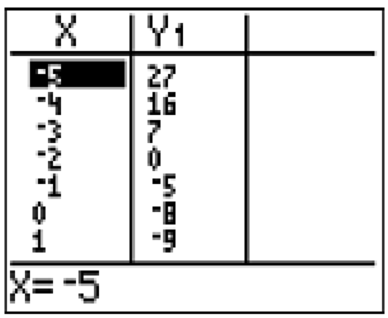
Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/181667

Find the x intercept

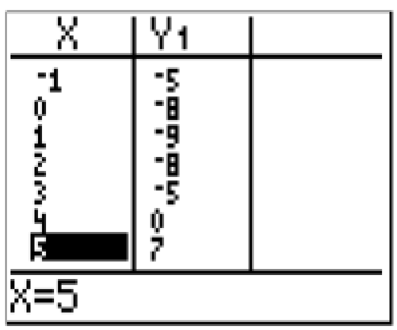
Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2-2x-8=0$

The corresponding function is $y=x^2-2x-8$. Use your graphing calculator to make a table and a graph for this function.



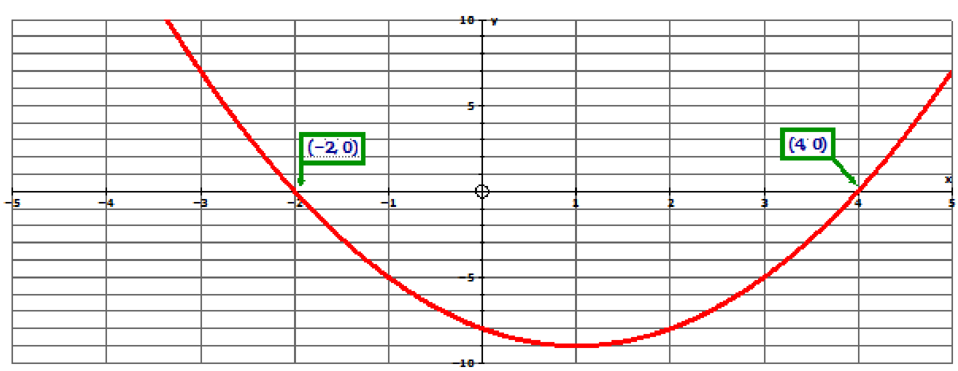
X	Y1
-5	27
-4	16
-3	7
-2	0
-1	-5
0	-8
1	-9

X= -5



X	Y1
-1	-5
0	-8
1	-9
2	-8
3	-5
4	0
5	7

X=5

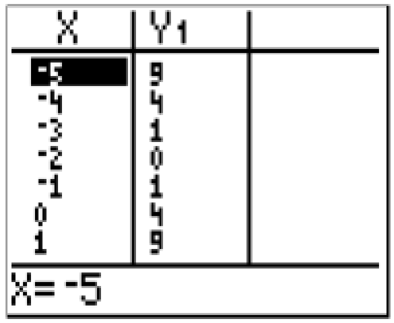


The x-intercepts are (-2, 0) and (4, 0). The x-intercepts are the values for 'x' that result in $y=0$ and are therefore the solutions to your equation. The solutions for the quadratic are $x=-2$ and $x=4$.

Find the x intercept

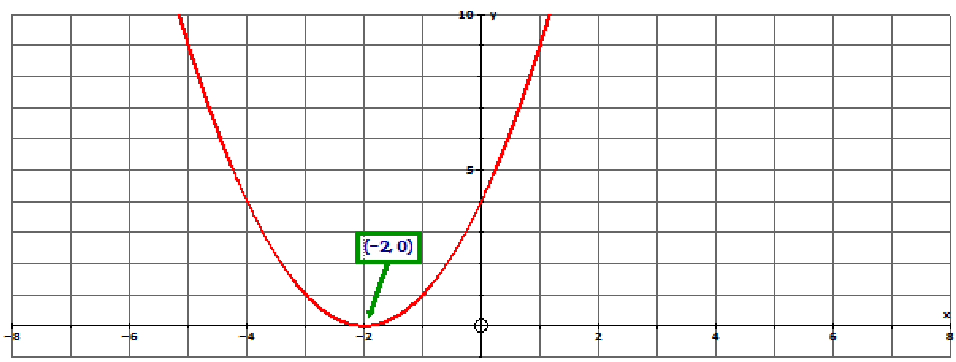
Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2+4x+4=0$

The corresponding function is $y = x^2+4x+4$. Use your graphing calculator to make a table and a graph for this function.



X	Y1	
-5	9	
-4	4	
-3	1	
-2	0	
-1	1	
0	4	
1	9	

X= -5

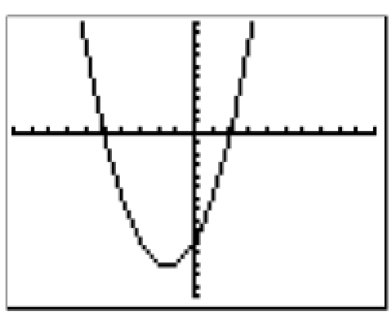


The only x-intercept is (-2, 0). There is only one solution to the equation: $x = -2$. Keep in mind that quadratic equations can have 0, 1, or 2 real solutions. If you were to factor the quadratic x^2+4x+4, you would get $(x+2)(x+2)$–two of the same factors. The root of -2 for this function is said to have a ***multiplicity*** of 2, because 2 factors produce the same solution. You will learn more about ***multiplicity*** when you study polynomials in future courses.

Find the x intercept

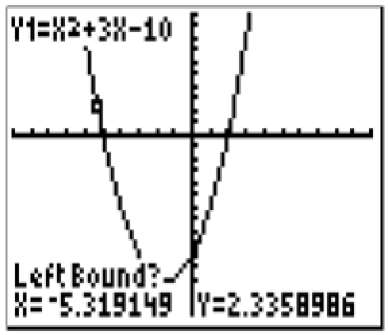
Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2+3x=10$

First rewrite the equation so it is set equal to zero to get $x^2+3x-10=0$. Now, the corresponding function is $y = x^2+3x-10$. Use your graphing calculator to make a graph for this function. You will see that there are two x-intercepts.



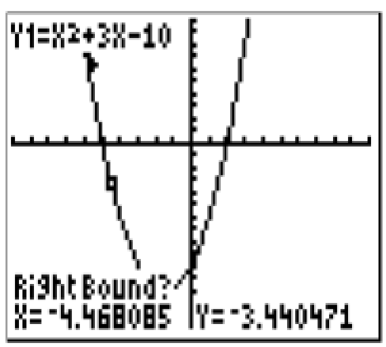
For this example you will see how the calculator can calculate the zeros of a function on a graph. *This technique is particularly useful when the intercepts are not at whole numbers.* Have the calculator find the x-intercept on the left first. Press

The calculator will display "Left Bound?" Use the arrow to position the cursor so that it is to the left and above the x-axis.



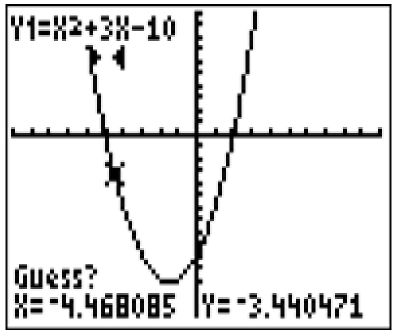
When the cursor has been positioned, press

The calculator will now display "Right Bound?" Use the arrows to position the cursor so that it is to the right and below the x-axis.

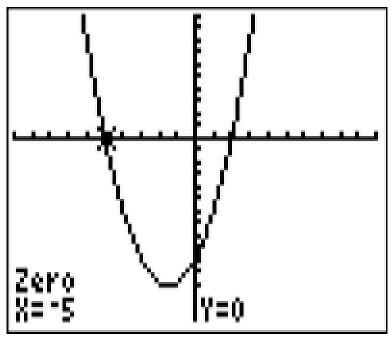


When the cursor has been positioned, press

The calculator will now display "Guess?"

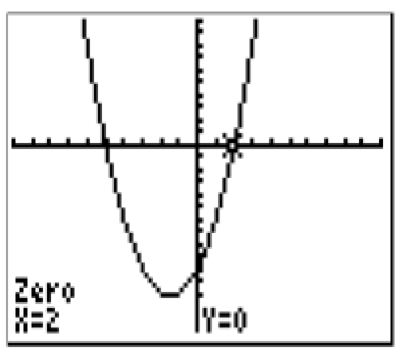


Press



At the bottom of the screen you can see it says "Zero" and the x and y coordinates. You are interested in the x-coordinate because that is one of the solutions to the original equation. The x-intercept is (-5, 0) which means that one of the solutions is $x = -5$.

Repeat this same process to determine the value of the x-intercept on the right.



The x-intercept is (2, 0) which means that the second solution is $x = 2$.

Examples

Example 1

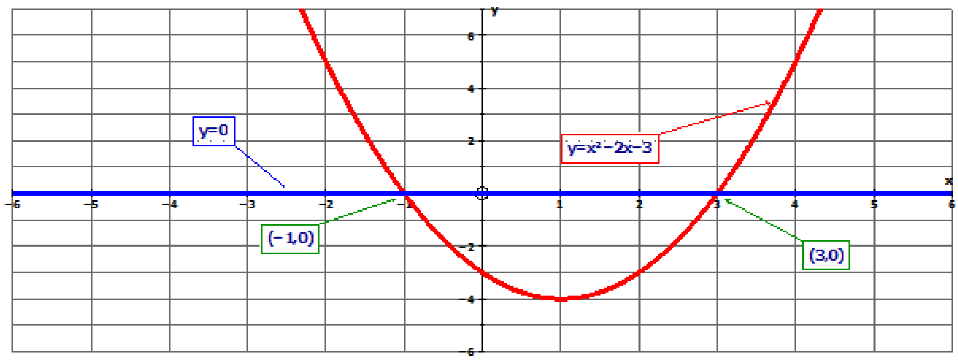
Earlier, you were asked how could you use a graph to solve $x^2 - 2x - 3 = 0$.

To solve the equation $x^2-2x-3=0$ using a graph, use a calculator to graph the corresponding function $y=x^2-2x-3$. Then, look for the values on the graph where $y=0$, which will be the x-intercepts.

Another way to think about this problem is to solve the system:

$$\begin{Bmatrix} y=x^2-2x-3 \\ y=0 \end{Bmatrix}$$

You are looking for where the parabola $y=x^2-2x-3$ intersects with the line $y=0$.



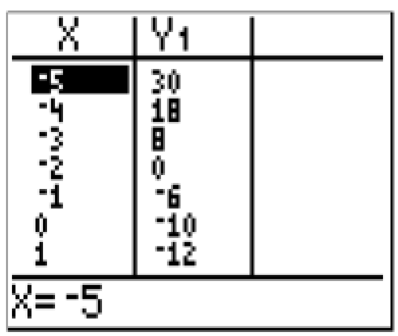
The points of intersection are (-1, 0) and (3, 0). The solutions to the original equation are $x=-1$ and $x=3$.

Example 2

Solve the quadratic equation using a graph.

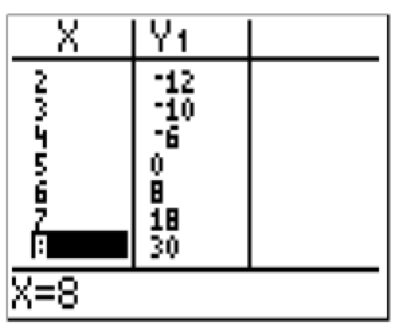
$$x^2-3x-10=0$$

To begin, create a table of values for the corresponding function $y=x^2-3x-10$ by using your graphing calculator:



X	Y1
-5	30
-4	18
-3	8
-2	0
-1	-6
0	-10
1	-12

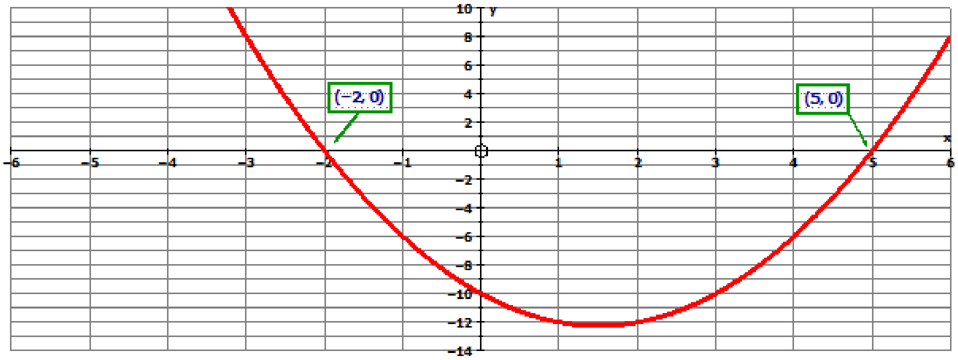
X= -5



X	Y1
2	-12
3	-10
4	-6
5	0
6	8
7	18
8	30

X=8

From the table, the x-intercepts are (-2, 0) and (5, 0). The x-intercepts are the values for 'x' that result in $y = 0$ and are therefore the solutions to the equation.

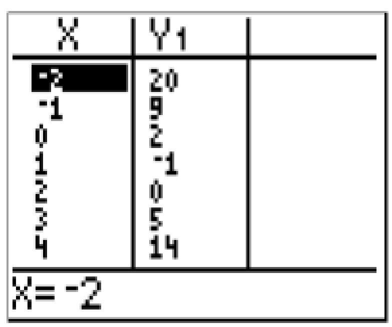


The solutions to the equation are are $x = -2$ and $x = 5$.

Example 3

$$2x^2 - 5x + 2 = 0$$

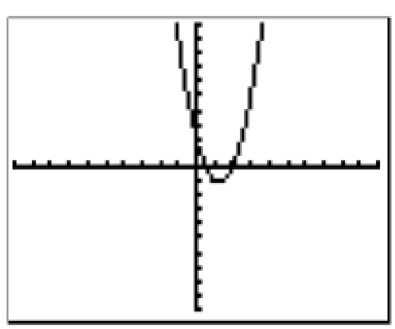
To begin, create a table of values for the corresponding function $y = 2x^2 - 5x + 2$ by using your graphing calculator:



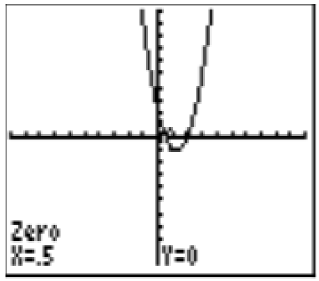
X	Y1
-2	20
-1	9
0	2
1	-1
2	0
3	5
4	14

X= -2

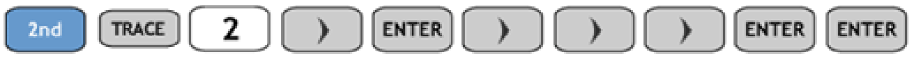
Press

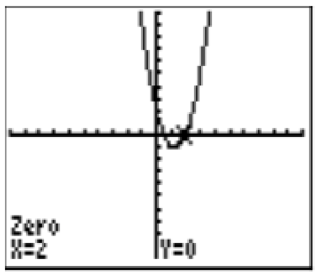


Press the following keys to determine the x-intercept to the left:



Press the following keys to determine the x-intercept to the right:

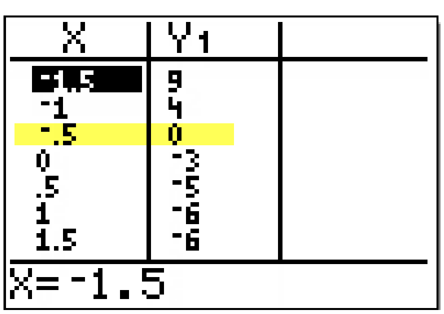


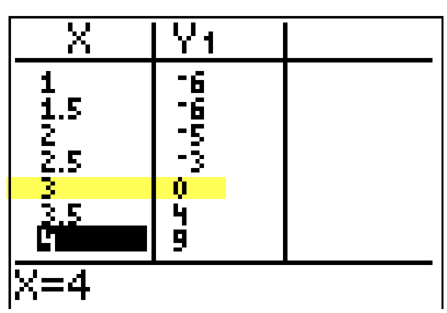


The x-intercepts of the function are (0.5, 0) and (2, 0). The solutions to the equation are, therefore, $x = 0.5$ and $x = 2$.

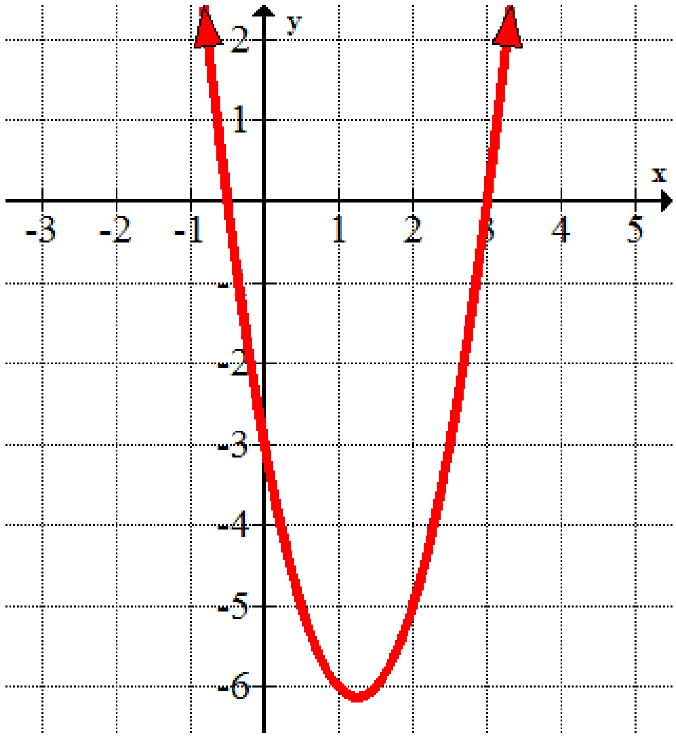
Example 4

First rewrite the equation so it is set equal to zero: $2x^2 - 5x - 3 = 0$. Next, create a table of values for the corresponding function $y = 2x^2 - 5x - 3$ by using your graphing calculator:





Now sketch the graph of the function.



The zeros of the function are (-0.5, 0) and (3, 0). Therefore, the solutions to the equation are $x = -0.5$ and $x = 3$.

Review

Use your graphing calculator to solve each of the following quadratic equations by graphing:

1. $2x^2 + 9x - 18 = 0$
2. $3x^2 + 8x - 3 = 0$
3. $-5x^2 + 13x + 6 = 0$
4. $2x^2 - 11x + 5 = 0$
5. $3x^2 + 8x - 3 = 0$
6. $x^2 - x - 20 = 0$
7. $2x^2 - 7x + 5 = 0$
8. $3x^2 + 7x = -2$
9. $2x^2 - 15 = -x$
10. $3x^2 - 10x = 8$
11. How could you use the graphs of a system of equations to solve $3x^2 - 10x = 8$?
12. What's the difference between a quadratic equation and a quadratic function?
13. Will a quadratic equation always have 2 solutions? Explain.
14. The quadratic equation$x^2 + 4 = 0$ has no real solutions. How does the graph of $y = x^2 + 4$ verify this fact?
15. When does it make sense to use the graphing method for solving a quadratic equation?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.1.

9.2 Completing the Square

Here you will learn how to solve a quadratic equation by completing the square.

How can you use square roots to solve the quadratic equation $(x-3)^2 = 15$?

The equation $x^2-6x-6=0$ is equivalent to $(x-3)^2 = 15$. How can you algebraically change $x^2-6x-6=0$ into $(x-3)^2 = 15$ in order to solve it?

Completing the Square

Recall that some quadratics are known as perfect square trinomials because they can be represented as a binomial squared. For example:

- $x^2+10x+25$ can be written as $(x+5)^2$
- x^2+6x+9 can be written as $(x+3)^2$
- x^2+2x+1 can be written as $(x+1)^2$

You can solve a quadratic such as $(x+5)^2 = 0$ by taking the square root of both sides as shown:

$$\begin{gathered}(x+5)^2 = 0 \\ \sqrt{(x+5)^2} = \pm\sqrt{0} \\ x+5 = +0 \text{ OR } x+5 = -0 \\ x = -5\end{gathered}$$

You can also solve such a quadratic even if it is not set equal to zero. For example, you can solve $(x+5)^2 = 9$ in a similar way:

$$\begin{gathered}(x+5)^2 = 9 \\ \sqrt{(x+5)^2} = \pm\sqrt{9} \\ x+5 = +3 \text{ OR } x+5 = -3 \\ x = -2 \quad x = -8\end{gathered}$$

Completing the square is a technique for solving quadratic equations that turns a given equation into a perfect square trinomial set equal to a number so that it can be solved using the method above. Consider the equation:

$x^2-12x+20=0$

Step 1: Move 20 to the right side of the equation.

$$x^2-12x=-20$$

Step 2: *Complete the square* on the left side of the equation. This means, figure out what could be added to the left side to turn that equation into a perfect square trinomial. One method for figuring this out is to take the value of 'b' (which is -12), divide it by 2, and square the result:

$$\begin{aligned}
&b = -12\\
&\frac{b}{2} = \frac{-12}{2} = -6\\
&\left(\frac{b}{2}\right)^2 = (-6)^2 = 36\\
&x^2 - 12x + 36 = -20 + 36
\end{aligned}$$

Remember that what is added to one side of the equation must be added to the other side of the equation. Once you simplify you get:

$x^2 - 12x + 36 = 16$

Step 3: Rewrite the left side of the equation as a binomial squared.

$$(x-6)^2 = 16$$

Step 4: Take the square root of both sides of the equation.

$$\begin{aligned}
\sqrt{(x-6)^2} &= \sqrt{16}\\
x - 6 &= \pm 4
\end{aligned}$$

Step 5: Set the left side of the equation equal to each of the roots on the right side and solve each linear equation.

$x - 6 = 4$ and $x - 6 = -4$

$x - 6 + 6 = 4 + 6$ and $x - 6 + 6 = -4 + 6$

$x = 10$ or $x = 2$

The solutions to the equation are:

$$\boxed{x = 10 \text{ or } x = 2}$$

Complete the square

What would you need to add to the following expression to turn it into a perfect square trinomial?

$x^2 + 16x$

To determine the value to add, you can divide 16 by 2 and square the result.

$$\begin{aligned}
&b = 16\\
&\frac{b}{2} = \frac{16}{2} = 8\\
&\left(\frac{b}{2}\right)^2 = (8)^2 = 64
\end{aligned}$$

The expression is now $x^2 + 16x + 64$, which can be rewritten as $(x+8)^2$.

Complete the Square

Solve the following quadratic equation by completing the square: $x^2+2x-35=0$

Step 1: Move -35 to the right side of the equation.

$$x^2+2x=35$$

Step 2: Complete the square on the left side of the equation. The value of b is 2.

$$\begin{aligned}
&b=2\\
&\frac{b}{2}=\frac{2}{2}=1\\
&\left(\frac{b}{2}\right)^2=(1)^2=1\\
&x^2+2x+1=35+1
\end{aligned}$$

$x^2+2x+1=36$

Step 3: Rewrite the left side of the equation as a binomial squared.

$$(x+1)^2=36$$

Step 4: Take the square root of both sides of the equation.

$$\begin{aligned}
\sqrt{(x+1)^2}&=\sqrt{36}\\
x+1&=\pm 6
\end{aligned}$$

Step 5: Set the left side of the equation equal to each of the roots on the right side and solve each linear equation.

$x+1=6$ and $x+1=-6$

$x+1-1=6-1$ and $x+1-1=-6-1$

$x=5$ or $x=-7$

The solutions to the equation are:

$$\boxed{x=5 \text{ } or \text{ } x=-7}$$

Complete the square

Solve the following quadratic equation by completing the square:

$$\boxed{2x^2-5x+2=0}$$

In this quadratic equation, the value of 'a' is not one. The first step will be to factor out the value of a.

Step 1: Factor out the '2'. Then, because the equation is set equal to zero, divide both sides by 2 in order to have a simpler equation.

$$2(x^2-\frac{5}{2}x+1)=0$$
$$x^2-\frac{5}{2}x+1=0$$

Step 2: Move 1 to the right side of the equation.

$$x^2-\frac{5}{2}x=-1$$

Step 3: Complete the square on the left side of the equation.

$$b=-\frac{5}{2}$$
$$\frac{b}{2}=-\frac{5}{2}\div 2=-\frac{5}{2}\left(\frac{1}{2}\right)=-\frac{5}{4}$$
$$\left(\frac{b}{2}\right)^2=\left(-\frac{5}{4}\right)^2=\frac{25}{16}$$
$$x^2-\frac{5}{2}x+\frac{25}{16}=-1+\frac{25}{16}$$

$x^2-\frac{5}{2}x+\frac{25}{16}=\frac{9}{16}$

Step 4: Rewrite the perfect square trinomial on the left side of the equation as a binomial squared.

$$\left(x-\frac{5}{4}\right)^2=\frac{9}{16}$$

Step 4: Take the square root of both sides of the equation.

$$\sqrt{\left(x-\frac{5}{4}\right)^2}=\sqrt{\frac{9}{16}}$$
$$x-\frac{5}{4}=\pm\frac{3}{4}$$

Step 5: Set the left side of the equation equal to each of the roots on the right side and solve each linear equation.

$x-\frac{5}{4}=\frac{3}{4}$ and $x-\frac{5}{4}=-\frac{3}{4}$

$x-\frac{5}{4}+\frac{5}{4}=\frac{3}{4}+\frac{5}{4}$ and $x-\frac{5}{4}+\frac{5}{4}=-\frac{3}{4}+\frac{5}{4}$

$x=\frac{8}{4}$ or $x=\frac{2}{4}$

$x=2$ and $x=\frac{1}{2}$

The solutions to the equation are:

$$\boxed{x = 2 \ or \ x = \frac{1}{2}}$$

Examples

Example 1

Earlier, you were asked to solve $x-3=\sqrt{15}$ by completing the square.

To solve $(x-3)^2=15$, take the square root of both sides. You will get that $x-3=\sqrt{15}$ or $x-3=-\sqrt{15}$. The two solutions are therefore $x=3+\sqrt{15}, 3-\sqrt{15}$.

You can turn $x^2-6x-6=0$ into $(x-3)^2=15$ by completing the square.

Example 2

What would you need to add to the following expression to turn it into a perfect square trinomial?

$$x^2-7x$$

Divide b by $\frac{1}{2}$ and square the result.

$(\frac{-7}{2})^2=\frac{49}{4}$

Example 3

Solve the following quadratic equation by completing the square.

$$x^2-4x+1=0$$

$x^2-4x+1=0$

$$\begin{aligned}
x^2-4x &= -1\\
x^2-4x+4 &= -1+4\\
x^2-4x+4 &= 3\\
(x-2)^2 &= 3\\
\sqrt{(x-2)^2} &= \sqrt{3}\\
x-2 &= \pm\sqrt{3}
\end{aligned}$$

The solutions to the equation are:

$$\boxed{x = 2+\sqrt{3} \ or \ x = 2-\sqrt{3}}$$

$$\boxed{x = 2 \pm \sqrt{3}}$$

Example 4

Solve the following quadratic equation by completing the square.

$$7x^2 - 2x - 2 = 0$$

$7x^2 - 2x - 2 = 0$

$$\begin{aligned}
7(x^2 - \frac{2}{7}x - \frac{2}{7}) &= 0 \\
x^2 - \frac{2}{7}x - \frac{2}{7} &= 0 \\
x^2 - \frac{2}{7}x &= \frac{2}{7} \\
x^2 - \frac{2}{7}x + \frac{1}{49} &= \frac{2}{7} + \frac{1}{49} \\
\left(x - \frac{1}{7}\right)^2 &= \frac{15}{49}
\end{aligned}$$

The exact solutions to the equation are:

$$\boxed{x = \frac{\sqrt{15}+1}{7} \ or \ x = \frac{-\sqrt{15}+1}{7}}$$

The approximate solutions (which have been rounded) are:

$$\boxed{x = 0.70 \ or \ x = -0.41}$$

Review

State the value of m that makes each trinomial a perfect square:

1. $x^2 - 10x + m$
2. $x^2 - 22x + m$
3. $x^2 + \frac{1}{2}x + m$
4. $x^2 + 9x + m$
5. $x^2 + x + m$

Solve each of the following quadratic equations by completing the square:

6. $x^2 + 18x = 85$
7. $x^2 - \frac{2}{3}x = 1$

8. $x^2 - 7x = 3$
9. $x^2 + \frac{1}{5}x = 2$
10. $x^2 - \frac{2}{3}x = 5$

Solve each of the following quadratic equations by completing the square:

11. $x^2 - 2x - 8 = 0$
12. $2x^2 + 8x + 5 = 0$
13. $3x^2 - 6x - 2 = 0$
14. $2x^2 - 3x - 5 = 0$
15. $3x^2 + 4x - 2 = 0$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.2.

9.3 The Quadratic Formula

Here you will learn the quadratic formula and how to use it.

Solve the following quadratic equation algebraically:

$$3x^2-5x+1=0$$

The Quadratic Formula

You can use the method of completing the square to solve the general quadratic equation $ax^2+bx+c=0$. The result will be a formula that you can use to solve any quadratic equation given the values for a, b, and c. The following is a derivation of the quadratic formula:

Step 1: Divide the general equation by a. Then, move the third term on the left side to the right side of the equation.

$$\begin{aligned} ax^2+bx+c&=0\\ x^2+\frac{b}{a}x+\frac{c}{a}&=0\\ x^2+\frac{b}{a}x&=-\frac{c}{a} \end{aligned}$$

Step 2: Complete the square. Note that your "b" value in this case is actually $\frac{b}{a}$.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

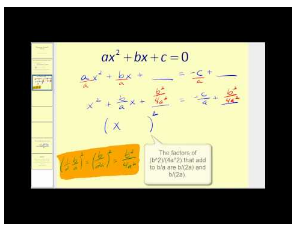
Step 3: Simplify.

$$\begin{aligned} x^2+\frac{b}{a}x+\frac{b^2}{4a^2}&=\frac{b^2}{4a^2}-\frac{c}{a}\left(\frac{4a}{4a}\right)\\ x^2+\frac{b}{a}x+\frac{b^2}{4a^2}&=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}\\ x^2+\frac{b}{a}x+\frac{b^2}{4a^2}&=\frac{b^2-4ac}{4a^2} \end{aligned}$$

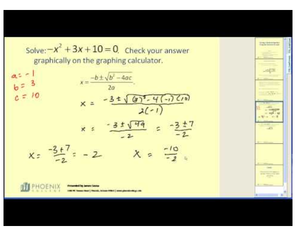
Step 4: Rewrite the left side of the equation as a binominal squared. Then, take the square root of both sides and solve for x.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$
$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\sqrt{\frac{b^2-4ac}{4a^2}}$$
$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$
$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This is known as the quadratic formula. You can use the quadratic formula to solve ANY quadratic equation. All you need to know are the values of a, b, and c. Keep in mind that while the factoring method for solving a quadratic equation will only sometimes work, the quadratic formula will ALWAYS work. You should memorize the quadratic formula because you will use it in algebra and future math courses.



MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/181665



MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/181666

QUADRATIC FORMULA: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Use the quadratic formula

Find the exact solutions of the following quadratic equation using the quadratic formula:

$$5x^2+2x-2=0$$

For this quadratic equation,

$$a=5, b=2, c=-2$$

. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
x &= \frac{-(2) \pm \sqrt{(2)^2-4(5)(-2)}}{2(5)} \\
x &= \frac{-2 \pm \sqrt{4+40}}{10} \\
x &= \frac{-2 \pm \sqrt{44}}{10} \\
x &= \frac{-2 \pm 2\sqrt{11}}{10} \\
x &= \frac{-1 \pm \sqrt{11}}{5}
\end{aligned}$$

The exact solutions to the quadratic equation are $\frac{-1+\sqrt{11}}{5}$ *or* $\frac{-1-\sqrt{11}}{5}$.

Use the quadratic formula

Use the quadratic formula to determine the approximate solutions of the equation:

$2x^2 - 3x = 3$

Start by rewriting the equation in standard form so that it is set equal to zero. $2x^2 - 3x = 3$ becomes $2x^2 - 3x - 3 = 0$. For this quadratic equation, $a = 2, b = -3, c = -3$. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
m &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
m &= \frac{-(-3) \pm \sqrt{(-3)^2-4(2)(-3)}}{2(2)} \\
x &= \frac{3 \pm \sqrt{9+24}}{4} \\
x &= \frac{3 \pm \sqrt{33}}{4} \qquad \sqrt{33} = 5.74 \\
x &= \frac{3 \pm 5.74}{4} \\
x &= \frac{3+5.74}{4} \text{ or } x = \frac{3-5.74}{4} \\
x &= \frac{8.74}{4} \text{ or } x = \frac{-2.74}{4} \\
x &= 2.2 \text{ or } x = -0.7
\end{aligned}$$

The approximate solutions to the quadratic equation to the nearest tenth are $x = 2.2$ or $x = -0.7$.

Use the quadratic formula

Solve the following equation using the quadratic formula:

$$\boxed{\frac{2}{y} - \frac{3}{y+1} = 1}$$

While this does not look like a quadratic equation (it is actually a rational equation because it contains rational expressions), you can rewrite it as a quadratic equation by multiplying by $(y)(y+1)$ to get rid of the fractions. *Note that y and $y+1$ are the denominators you want to eliminate. This is why you want to multiply by* $(y)(y+1)$. After multiplying, simplify and put the equation in standard quadratic form set equal to 0.

$$\begin{aligned}
&\frac{2}{y} - \frac{3}{y+1} = 1\\
&\frac{2}{y}(y)(y+1) - \frac{3}{y+1}(y)(y+1) = 1(y)(y+1)\\
&\frac{2}{\cancel{y}}(\cancel{y})(y+1) - \frac{3}{\cancel{y+1}}(y)(\cancel{y+1}) = 1(y)(y+1)\\
&2(y+1) - 3(y) = 1(y^2+y)\\
&2y+2-3y = y^2+y\\
&2-y = y^2+y\\
&y^2+2y-2 = 0
\end{aligned}$$

For this quadratic equation, $a = 1, b = 2, c = -2$. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
y &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\
y &= \frac{-(2) \pm \sqrt{(2)^2-4(1)(-2)}}{2(1)}\\
y &= \frac{-2 \pm \sqrt{4+8}}{2}\\
y &= \frac{-2 \pm \sqrt{12}}{2}\\
y &= \frac{-2 \pm 2\sqrt{3}}{2}\\
y &= -1 \pm \sqrt{3}
\end{aligned}$$

The exact solutions to the equation are $-1+\sqrt{3}$ or $-1-\sqrt{3}$. Note that neither of these solutions will cause the original equation to have a zero in the denominator, so they both work.

Examples

Example 1

Earlier, you were asked to solve $3x^2-5x+1 = 0$ algebraically.

To solve the equation $3x^2-5x+1 = 0$ algebraically, you can use the quadratic formula. For this quadratic equation, $a = 3, b = -5, c = 1$. Substitute these values into the quadratic formula and simplify.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\
x &= \frac{-(-5) \pm \sqrt{(-5)^2-4(3)(1)}}{2(3)}\\
x &= \frac{5 \pm \sqrt{25-12}}{6}\\
x &= \frac{5 \pm \sqrt{13}}{6}
\end{aligned}
$$

The solutions are $x = \frac{5 \pm \sqrt{13}}{6}$.

Example 2

For the following equation, rewrite as a quadratic equation and state the values for a, b and c:

$$\frac{2}{x-1}+\frac{3}{x+2}=1$$

Multiply by $(x-1)(x+2)$ to clear the fractions.

$$
\begin{aligned}
&\frac{2}{x-1}+\frac{3}{x+2}=1\\
&\frac{2}{x-1}(x-1)(x+2)+\frac{3}{x+2}(x-1)(x+2)=1(x-1)(x+2)\\
&\frac{2}{\cancel{x-1}}(\cancel{x-1})(x+2)+\frac{3}{\cancel{x+2}}(x-1)(\cancel{x+2})=1(x^2+2x-1x-2)\\
&2(x+2)+3(x-1)=1(x^2+x-2)\\
&2x+4+3x-3=x^2+x-2\\
&x^2+x-2=5x+1\\
&x^2-4x-3=0
\end{aligned}
$$

For this equation, $a = 1, b = -4, c = -3$.

Example 3

Solve the following quadratic equation using the quadratic formula:

$$6x^2-8x=0$$

This equation does not have a 'c' term. The value of 'c' is 0. For this equation, $a = 6, b = -8, c = 0$.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(6)(0)}}{2(6)}$$

$$\begin{aligned}
x &= \frac{8 \pm \sqrt{64-0}}{12}\\
x &= \frac{8 \pm \sqrt{64}}{12}\\
x &= \frac{-8 \pm 8}{12}\\
x &= \frac{8+8}{12} \ or \ x = \frac{8-8}{12}\\
&\boxed{x = \frac{4}{3} \ or \ x = 0}
\end{aligned}$$

Example 4

Find the approximate solutions to the following equation:

$$\frac{x+3}{2x-1} = \frac{2x+3}{x+5}$$

Multiply by $(2x-1)(x+5)$ to clear the fractions.

$$\begin{aligned}
&\frac{x+3}{2x-1} = \frac{2x+3}{x+5}\\
&\frac{x+3}{2x-1}(2x-1)(x+5) = \frac{2x+3}{x+5}(2x-1)(x+5)\\
&\frac{x+3}{\cancel{2x-1}}(\cancel{2x-1})(x+5) = \frac{2x+3}{\cancel{x+5}}(2x-1)(\cancel{x+5})\\
&(x+3)(x+5) = (2x+3)(2x-1)
\end{aligned}$$

$$\begin{aligned}
x^2+5x+3x+15 &= 4x^2-2x+6x-3\\
x^2+8x+15 &= 4x^2+4x-3\\
&-3x^2+4x+18 = 0
\end{aligned}$$

For this equation, $a = -3, b = 4, c = 18$.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(-3)(18)}}{2(-3)}$$

$$
\begin{aligned}
x &= \frac{-4 \pm \sqrt{16+216}}{-6} \\
x &= \frac{-4 \pm \sqrt{232}}{-6} \\
x &= 3.21 \textit{ and } x = -1.87
\end{aligned}
$$

The solutions to the quadratic equation to the nearest tenth are $x = 3.2$ or $x = -1.9$.

Review

State the value of a, b and c for each of the following quadratic equations.

1. $2x^2 + 7x - 1 = 0$
2. $3x^2 + 2x = 7$
3. $9x^2 - 7 = 4x$
4. $2x^2 - 7 = 0$
5. $4 - 2x^2 = 11x$

Determine the exact roots of the following quadratic equations using the quadratic formula.

6. $2x^2 = 8x - 7$
7. $6y = 2 - y^2$
8. $1 = 8x + 3x^2$
9. $2(n-2)(n+1) - (n+3) = 0$
10. $\frac{2e}{e+1} - \frac{3}{e-1} = \frac{4}{e^2-1}$
11. $x^2 - 2x - 5 = 0$
12. $\frac{m}{4} - \frac{m^2}{2} = -1$
13. $\frac{3}{y} - \frac{4}{y+2} = 2$
14. $\frac{1}{2}x^2 - \frac{x}{4} - 1 = 0$
15. $3x^2 + 8x = 1$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.3.

9.4 Applications of Quadratic Functions

Here you will consider real-world applications of quadratic functions.

A toy rocket is fired into the air from the top of a barn. Its height (h) above the ground in yards after t seconds is given by the function $h(t) = -5t^2 + 10t + 20$.

a) What was the maximum height of the rocket?

b) How long was the rocket in the air before hitting the ground?

c) At what time(s) will the rocket be at a height of 22 yd?

Applications of Quadratic Functions

There are many real-world situations that deal with quadratics and parabolas. Throwing a ball, shooting a cannon, diving from a platform and hitting a golf ball are all examples of situations that can be modeled by quadratic functions.

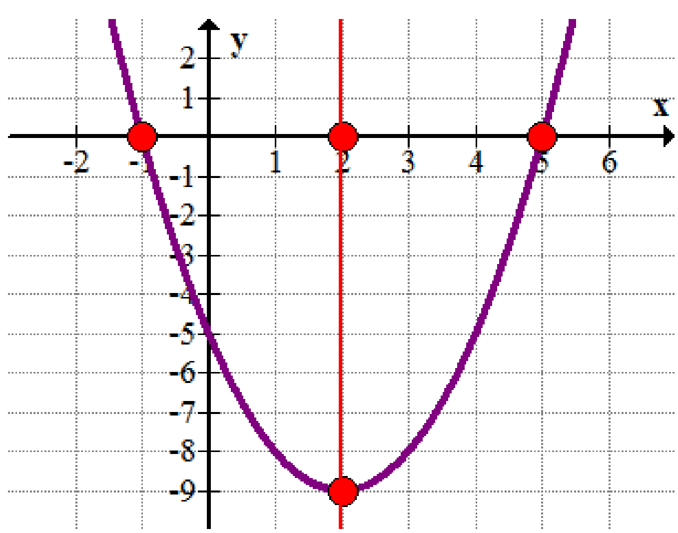
In many of these situations you will want to know the highest or lowest point of the parabola, which is known as the vertex. For example, consider that when you throw a football, the path it takes through the air is a parabola. Natural questions to ask are:

- "When does the football reach its maximum height?"
- "How high does the football get?"

If you know the equation for the function that models the situation, you can find the vertex. If the function is $f(x) = ax^2 + bx + c$, the x-coordinate of the vertex will be $-\frac{b}{2a}$. The y-coordinate of the vertex can be found by substituting the x-coordinate into the function. In the case of the football:

- The x-coordinate of the vertex will give you the time when the football is at its maximum height.
- The y-coordinate will give you the maximum height.

One way to understand where the $-\frac{b}{2a}$ comes from is to consider where the vertex is on a parabola.

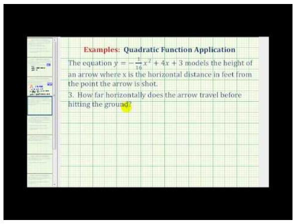


Due to the symmetry of parabolas, the x-coordinate of the vertex is directly between the two x-intercepts. The two x-intercepts are, according to the quadratic formula:

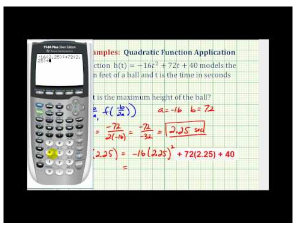
$$x = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$

So, $x = -\frac{b}{2a}$ is in the middle. One x-intercept is $\frac{\sqrt{b^2-4ac}}{2a}$ to the right and the other x-intercept is $\frac{\sqrt{b^2-4ac}}{2a}$ to the left.



MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/181668



MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/181669

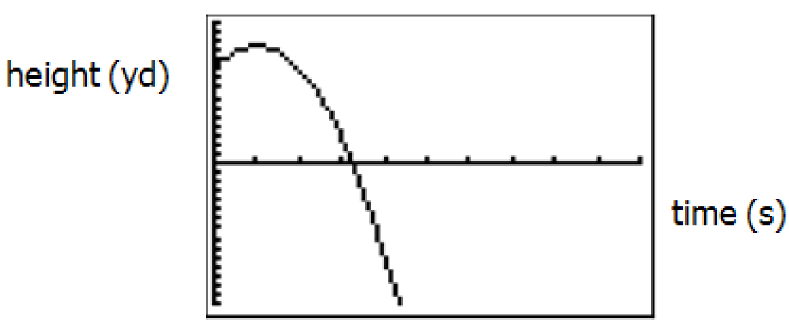
Solve the problem using quadratic function

A toy rocket is fired into the air from the top of a barn. Its height (h) above the ground in yards after t seconds is given by the function $h(t) = -5t^2 + 10t + 20$.

a) What was the initial height of the rocket?

b) When did the rocket reach its maximum height?

Sketch a graph of the function. Your graphing calculator can be used to produce the graph.



a) The initial height of the rocket is the height from which it was fired. The time is zero.

$$h(t) = -5t^2 + 10t + 20$$
$$h(t) = -5(0)^2 + 10(0) + 20$$
$$\boxed{h(t) = 20\ yd}$$

The initial height of the toy rocket is 20 yards. This is the y-intercept of the graph. The y-intercept of a quadratic function written in general form is the value of 'c'.

b) The time at which the rocket reaches its maximum height is the x-coordinate of the vertex.

$$t = -\frac{b}{2a}$$
$$t = -\frac{10}{2(-5)}$$
$$\boxed{t = 1\ sec}$$

It takes the toy rocket 1 second to reach its maximum height.

Solve using quadratic function

The sum of a number and its square is 272. Find the number.

Let n represent the number. Write an equation to represent the problem.

$n^2 + n = 272$

You can solve this equation using a few different methods. Here, solve by completing the square.

$$n^2 + n + \frac{1}{4} = 272 + \frac{1}{4}$$

$$n^2 + n + \frac{1}{4} = \frac{1089}{4}$$

$$\left(n + \frac{1}{2}\right)^2 = \frac{1089}{4}$$

$$\sqrt{\left(n + \frac{1}{2}\right)^2} = \sqrt{\frac{1089}{4}}$$
$$n + \frac{1}{2} = \pm\frac{33}{2}$$

$$n = \frac{32}{2}\ or\ n = -\frac{34}{2}$$
$$n = 16\ or\ n = -17$$

These are both solutions to the problem. There are no restrictions listed in the problem regarding the solution.

Solve the quadratic function

The product of two consecutive positive odd integers is 195. Find the integers.

Let n represent the first positive odd integer. Let $n+2$ represent the second positive odd integer. Write an equation to represent the problem.

$$n(n+2) = 195$$
$$n^2 + 2n = 195$$

You can solve this equation with a few different methods. Here, use the quadratic formula.

$$n^2 + 2n - 195 = 0$$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$n = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-195)}}{2(1)}$$

$$n = \frac{-2 \pm \sqrt{784}}{2}$$
$$n = \frac{-2 \pm 28}{2}$$

$$n = \frac{-2+28}{2} \; or \; n = \frac{-2-28}{2}$$

$$n = 13 \; or \; n = -15$$

There was a restriction on the solution presented in the problem. The solution must be an odd positive integer. Therefore, 13 is the solution you can use. The two positive odd integers are 13 and 15.

Examples

Example 1

Earlier, you were given a problem about a toy rocket.

a) The maximum height was reached by the rocket at one second as you found in Example A.

$$\begin{aligned}
h(t) &= -5t^2 + 10t + 20 \\
h(t) &= -5(1)^2 + 10(1) + 20 \\
&\boxed{h(t) = 25\ yd}
\end{aligned}$$

The maximum height reached by the rocket was 25 yd.

b) When the rocket hits the ground, its height will be zero.

$$\begin{aligned}
h(t) &= -5t^2 + 10t + 20 \\
0 &= -5t^2 + 10t + 20
\end{aligned}$$

Use the quadratic formula to solve for 't'. You have $a = -5, b = 10, c = 20$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(10) \pm \sqrt{(10)^2 - 4(-5)(20)}}{2(-5)}$$

$$\begin{aligned}
t &= \frac{-10 \pm 10\sqrt{5}}{-10} \\
t &= 1 \pm \sqrt{5}
\end{aligned}$$

$$t = 1 + \sqrt{5}\ or\ t = 1 - \sqrt{5}$$

$$t = 3.24\ s\ or\ t = -1.24\ s$$

$$\boxed{t = 3.24\ s}$$

Accept this solution

$$\boxed{t = -1.24\ s}$$

Reject this solution. Time cannot be a negative quantity.

The toy rocket stayed in the air for approximately 3.24 seconds.

c) The rocket reached a maximum height of 25 yd at a time of 1 second. The rocket must reach a height of 22yd before and after one second. Remember the old saying: "What goes up must come down."

Use the quadratic formula to determine these times.

$$h(t) = -5t^2 + 10t + 20$$
$$22 = -5t^2 + 10t + 20$$

$$0 = -5t^2 + 10t - 2$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(10) \pm \sqrt{(10)^2 - 4(-5)(-2)}}{2(-5)}$$

$$t = \frac{5 + \sqrt{15}}{5} \quad or\ t = \frac{5 - \sqrt{15}}{5}$$

$$t = 1.77\ s\ or\ t = 0.23\ s$$

$$\boxed{t = 1.77\ s}$$

Accept this solution

$$\boxed{t = 0.23\ s}$$

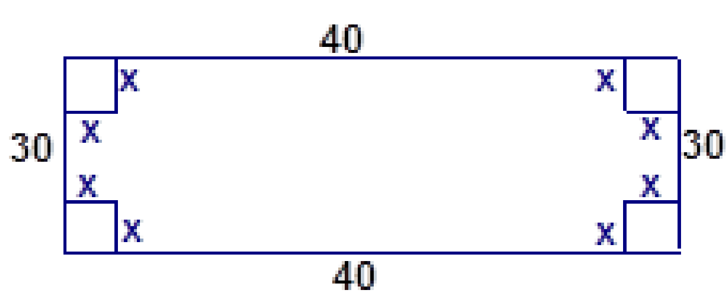
Accept this solution.

The rocket reached a height of 22 yd at 0.23 seconds on its way up and again at 1.77 seconds on its way down.

Example 2

A rectangular piece of cardboard measuring 40 in. by 30 in. is to be made into an open box with a base (bottom) of 900 in^2 by cutting equal squares from the four corners and then bending up the sides. Find, to the nearest tenth of an inch, the length of the side of the square that must be cut from each corner.

Sketch a diagram to represent the problem.



Let the variable x represent the side length of the square.

- $L = 40 - 2x$
- $W = 30 - 2x$

The area of a rectangle is the product of its length and its width. The area of the base of the rectangle must be 900 in^2, after the squares have been removed.

$$\begin{aligned} L \cdot W &= Area \\ (40-2x)(30-2x) &= 900 \end{aligned}$$

$$\begin{aligned} 1200 - 80x - 60x + 4x^2 &= 900 \\ 4x^2 - 140x + 1200 &= 900 \end{aligned}$$

You can now solve using the quadratic formula.

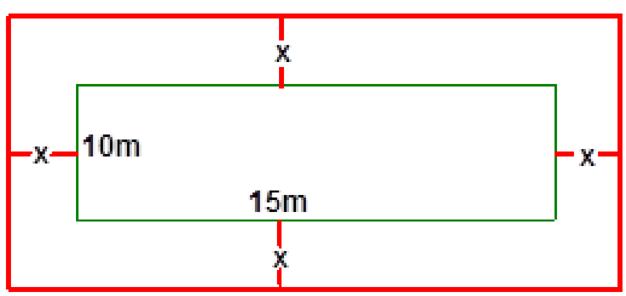
$$x = 32.7 \text{ or } x = 2.3$$

The solution of 32.7 in must be rejected since it would cause the length and the width of the rectangle to result in negative values. The length of the side of the square that was cut from the cardboard was 2.3 inches.

Example 3

The local park has a rectangular flower bed that measures 10 feet by 15 feet. The caretaker plans on doubling its area by adding a strip of uniform width around the flower bed. Determine the width of the strip.

Sketch a diagram to represent the problem.



Let the variable x, represent the side length of the uniform strip.

$$\begin{aligned} L &= 15 + 2x & L \cdot W &= Area \\ W &= 10 + 2x & (15)(10) &= 150\ ft^2 \end{aligned}$$

The area of a rectangle is the product of its length and its width. The area of the original flower bed is 150 ft^2. The new flower bed must be twice this area which is 300 ft^2.

$$\begin{aligned} L \cdot W &= Area \\ (15+2x)(10+2x) &= 300 \end{aligned}$$

$$\begin{aligned} 150+30x+20x+4x^2 &= 300 \\ 4x^2+50x+150 &= 300 \end{aligned}$$

You can now solve using the quadratic formula.

$$x = 2.5 \ ft \ and \ x = -15 \ ft$$

$$\boxed{x = 2.5 \ ft}$$

Accept this solution.

$$\boxed{x = -15 \ ft}$$

Reject this solution. The width of the strip cannot be a negative value.

The width of the strip that is to be added to the flower bed is 2.5 feet.

Example 4

$h(t) = -4.9t^2 + 8t + 5$ represents Jeremiah's height (h) in meters above the water t seconds after he leaves the diving board.

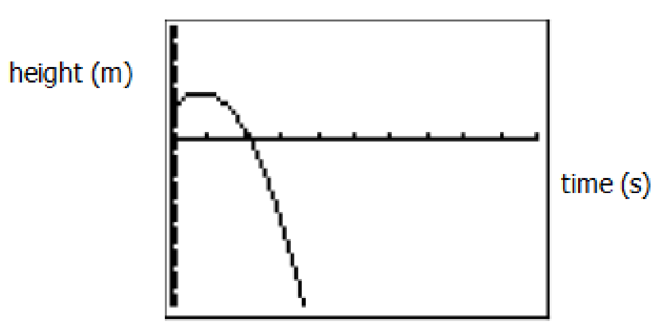
i) What is the initial height of the diving board?

ii) At what time did Jeremiah reach his maximum height?

iii) What was Jeremiah's maximum height?

iv) How long was Jeremiah in the air?

Sketch a graph of the function.



i) The initial height of the diving board is when the time is zero.

$$\begin{aligned}
&h(t) = -4.9t^2 + 8t + 5\\
&h(t) = -4.9(0)^2 + 8(0) + 5\\
&h(t) = 0 = 0 + 5\\
&\boxed{h(t) = 5\ m}
\end{aligned}$$

The initial height of the diving board is 5 m.

ii) The time at which Jeremiah reaches his maximum height is the x-coordinate of the vertex.

$$\begin{aligned}
&t = -\frac{b}{2a}\\
&a = -4.9\\
&b = 8\\
&t = -\frac{8}{2(-4.9)}\\
&t = \frac{-8}{-9.8}\\
&\boxed{t = 0.82\ sec}
\end{aligned}$$

It took Jeremiah 0.82 seconds to reach his maximum height.

iii) The maximum height was reached Jeremiah at 0.82 seconds.

$$\begin{aligned}
&h(t) = -4.9t^2 + 8t + 5\\
&h(t) = -4.9(0.82)^2 + 8(0.82) + 5\\
&h(t) = -3.29 + 6.56 + 5\\
&\boxed{h(t) = 8.27\ m}
\end{aligned}$$

The maximum height reached by Jeremiah was 8.27 m.

iv) When Jeremiah hits the water, his height will be zero.

$$\begin{aligned}
h(t) &= -4.9t^2 + 8t + 5\\
0 &= -4.9t^2 + 8t + 5
\end{aligned}$$

Use the quadratic formula to solve for 't'.

$$t = -0.48\ s\ or\ t = 2.12\ s$$

$$\boxed{t = 2.12\ s}$$

Accept this solution

$$\boxed{t = -0.48\ s}$$

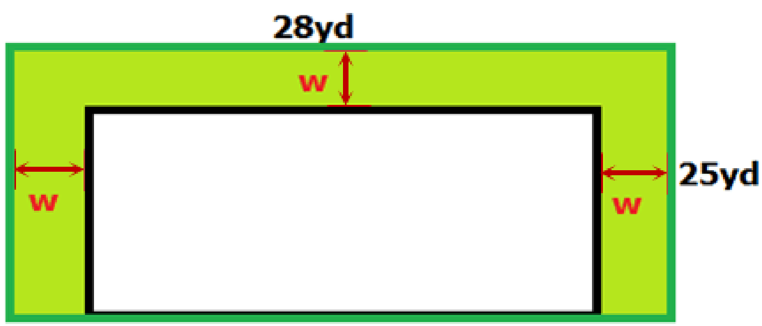
Reject this solution. Time cannot be a negative quantity.

Jeremiah stayed in the air for approximately 2.12 seconds.

Review

Solve the following problems using your knowledge of quadratic functions.

1. The product of two consecutive even integers is 224. Find the integers.
2. The hypotenuse of a right triangle is 26 inches. The sum of the legs is 34 inches. Find the length of the legs of the triangle.
3. The product of two consecutive integers is 812. What are the integers?
4. The width of a rectangle is 3 inches longer than the length. The area of the rectangle is 674.7904 square inches. What are the dimensions of the rectangle?
5. The product of two consecutive odd integers is 3135. What are the integers?
6. Josie wants to landscape her rectangular back garden by planting shrubs and flowers along a border of uniform width as shown in the diagram. Determine the width of the border if the outside fence has dimensions of 28 yd by 25 yd and the remaining garden is to be $\frac{3}{4}$ of the original size.



7. Gregory ran the 1800 yard race last year but knows that if he could run 0.5 yd/s faster, he could reduce his time by 30 seconds. What was Gregory's time when he ran the race last year?

During a high school baseball tournament, Lexie hits a pitch and the baseball stays in the air for 4.42 seconds. The function describes the height over time, where h is its height, in yards, and t is the time, in seconds, from the instant the ball is hit.

$h = -5t^2 + 22t + 0.5$

8. Algebraically determine the maximum height the ball reaches.
9. When will the ball reach its maximum height?
10. How long will the ball be at a height of less than 20 meters while it is in the air?

A rock is thrown off a 75 meter high cliff into some water. The height of the rock relative to the cliff after t seconds is given by $h(t) = -5t^2 + 20t$.

11. Where will the rock be after five seconds?

12. How long before the rock reaches its maximum height?
13. When will the rock hit the water?

You jump off the end of a ski jump. Your height in meters relative to the height of the ski jump after t seconds is given by $h(t) = -5t^2 + 12t$.

14. How high will you be after 2 seconds? At this point are you going up or coming down?
15. If you spend 6.1 seconds in the air, how far below the end of the ski jump do you land? (What is the vertical distance?)

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.4.

9.5 Roots to Determine a Quadratic Function

Here you will learn to find the equation of a quadratic function given its roots.

What quadratic function has roots of 2 and 7? Does more than one function have these roots?

Roots to Determine a Quadratic Function

If $x=2$ and $x=5$ are roots of a quadratic function, then $(x-2)$ and $(x-5)$ must have been factors of the quadratic equation. Therefore, a quadratic function with roots 2 and 5 is:

$$\begin{aligned} y &= (x-2)(x-5) \\ y &= x^2-7x+10 \end{aligned}$$

Note that there are many other functions with roots of 2 and 5. These functions will all be multiples of the function above. The function below would also work:

$$\begin{aligned} y &= 2(x-2)(x-5) \\ y &= 2x^2-14x+20 \end{aligned}$$

If you don't know the solutions of a quadratic equation, but you know the sum and the product of the solutions, you can find the equation. If r_1 and r_2 are the solutions to the quadratic equation $ax^2+bx+c=0$, then

$r_1+r_2=-\frac{b}{a} \text{ and } r_1 \times r_2 = \frac{c}{a}$.

The quadratic can be rewritten as:

$x^2-(\text{sum of the solutions})x+(\text{product of the solutions})=0$

$x^2-\left(-\frac{b}{a}\right)x+\left(\frac{c}{a}\right)=0$

Where did these sum and products come from? Consider the sum of the two solutions obtained from the quadratic formula:

$$\begin{aligned} r_1+r_2 &= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)+\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) \\ r_1+r_2 &= -\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a} \\ r_1+r_2 &= -\frac{b}{2a}+\cancel{\frac{\sqrt{b^2-4ac}}{2a}}-\frac{b}{2a}-\cancel{\frac{\sqrt{b^2-4ac}}{2a}} \\ r_1+r_2 &= -\frac{b}{2a}-\frac{b}{2a} \\ r_1+r_2 &= -\frac{2b}{2a} \\ &\boxed{r_1+r_2=-\frac{b}{a}} \end{aligned}$$

Now consider the product of the two solutions obtained from the quadratic formula:

$$r_1 \times r_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) \times \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$
$$r_1 \times r_2 = \left(-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right) \times \left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)$$

$$r_1 \times r_2 = \frac{b^2}{4a^2}-\frac{b^2-4ac}{4a^2}$$
$$r_1 \times r_2 = \frac{b^2-b^2+4ac}{4a^2}$$
$$r_1 \times r_2 = \frac{4ac}{4a^2}$$
$$\boxed{r_1 \times r_2 = \frac{c}{a}}$$

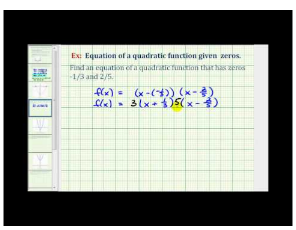
You can use these ideas to determine a quadratic equation with solutions $2+\sqrt{3}$ and $2-\sqrt{3}$.

- The sum of the solutions is: $2+\sqrt{3}+2-\sqrt{3}=4$.
- The product of the solutions: $(2+\sqrt{3})(2-\sqrt{3})=4-3=1$.

Therefore, the equation is:

$x^2-(\text{sum of the solutions})x+(\text{product of the solutions})=0$

$x^2-4x+1=0$



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/181671

Without solving, determine the sum and the product of the solutions for the following quadratic equations:

i) $x^2+3x+2=0$

ii) $3m^2+4m-3=0$

Remember that the sum of the solutions is $-\frac{b}{a}$ and the product of the solutions is $\frac{c}{a}$.

i) For $x^2+3x+2=0$, $a=1, b=3, c=2$. Therefore, the sum of the solutions is: $-\frac{3}{1}=-3$. The product of the solutions is $\frac{2}{1}=2$.

ii) For $3m^2+4m-3=0$, $a=3, b=4, c=-3$. Therefore, the sum of the solutions is: $-\frac{4}{3}$. The product of the solutions is $\frac{-3}{3}=-1$.

Find a quadratic function with the roots:

The solutions to the quadratic equation are:

$x = 3 + \sqrt{5}$ and $x = 3 - \sqrt{5}$

The factors are $(x - (3 + \sqrt{5}))$ and $(x - (3 - \sqrt{5}))$. One possible function in factored form is:

$$y = \left(x - 3 - \sqrt{5}\right)\left(x - 3 + \sqrt{5}\right)$$

Multiply and simplify:

$$y = x^2 - 3x + \sqrt{5}x - 3x + 9 - 3\sqrt{5} - \sqrt{5}x + 3\sqrt{5} - \sqrt{25}$$

$$\begin{aligned} y &= x^2 - 3x + \cancel{\sqrt{5}x} - 3x + 9 - \cancel{3\sqrt{5}} - \cancel{\sqrt{5}x} + \cancel{3\sqrt{5}} - 5 \\ y &= x^2 - 3x - 3x + 9 - 5 \\ &\boxed{y = x^2 - 6x + 4} \end{aligned}$$

Keep in mind that any multiple of the right side of the above function would also have the given roots.

Using the solutions indicated below; determine the quadratic equation by using the sum and the product of the solutions:

$3 + 2\sqrt{2}$ and $3 - 2\sqrt{2}$

The sum of the solutions is $3 + 2\sqrt{2} + 3 - 2\sqrt{2} = 6$.

The product of the solutions is $(3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 9 - 8 = 1$.

$x^2 - (\text{sum of the solutions})x + (\text{product of the solutions}) = 0$

$x^2 - 6x + 1 = 0$

Examples

Example 1

Earlier, you were asked what quadratic function has roots of 2 and 7. There are multiple functions with these roots. The basic example is $y = (x-2)(x-7)$ which is $y = x^2 - 9x + 14$. However any function of the form $y = a(x-2)(x-7)$ with $a \neq 0$ would work.

Example 2

By solving the given equation, find an equation whose solutions are each one less than the solutions to:

$$y^2 - 3y - 6 = 0$$

Determine the solutions of the quadratic equation with the quadratic formula. You should get that the solutions to the quadratic equation are:

$$y = \frac{3+\sqrt{33}}{2} \ or \ y = \frac{3-\sqrt{33}}{2}$$

The solutions of the new equation must be one less than each of the above solutions.

$$\begin{aligned}
y &= \frac{3}{2} + \frac{\sqrt{33}}{2} \ or \ y = \frac{3}{2} - \frac{\sqrt{33}}{2} \\
y &= \frac{3}{2} + \frac{\sqrt{33}}{2} - 1 \ or \ y = \frac{3}{2} - \frac{\sqrt{33}}{2} - 1 \\
y &= \frac{3}{2} + \frac{\sqrt{33}}{2} - \frac{2}{2} \ or \ y = \frac{3}{2} - \frac{\sqrt{33}}{2} - \frac{2}{2} \\
y &= \frac{1}{2} + \frac{\sqrt{33}}{2} \ or \ y = \frac{1}{2} - \frac{\sqrt{33}}{2}
\end{aligned}$$

The solutions of the new equation are:

$$y = \frac{1+\sqrt{33}}{2} \ or \ y = \frac{1-\sqrt{33}}{2}$$

The sum of the solutions is 1. The product of the solutions is -8. One possible quadratic equation is $y^2 - 1y - 8 = 0$.

Example 3

Without solving the given equation, find an equation whose solutions are the reciprocals of the solutions to:

$$2x^2 - 3x + 5 = 0$$

The sum of the solutions is $\frac{3}{2}$. The product of the solutions is $\frac{5}{2}$. The solutions of the new equation must be the reciprocals of the solutions of the original equation. Therefore, the sum of the solutions of the new equation will be:

$$R_1 + R_2 = \frac{1}{r_1} + \frac{1}{r_2}$$
$$R_1 + R_2 = \frac{1}{r_1}\left(\frac{r_2}{r_2}\right) + \frac{1}{r_2}\left(\frac{r_1}{r_1}\right)$$
$$R_1 + R_2 = \frac{r_2}{r_1 r_2} + \frac{r_1}{r_1 r_2}$$
$$R_1 + R_2 = \frac{r_2 + r_1}{r_1 r_2}$$
$$R_1 + R_2 = \frac{\frac{3}{2}}{\frac{5}{2}}$$
$$R_1 + R_2 = \frac{3}{2} \times \frac{2}{5}$$
$$\boxed{R_1 + R_2 = \frac{3}{5}}$$

The product of the solutions of the new equation will be:

$$R_1 \times R_2 = \frac{1}{r_1} \times \frac{1}{r_2}$$
$$R_1 \times R_2 = \frac{1}{r_1 r_2}$$
$$R_1 \times R_2 = \frac{1}{\frac{5}{2}}$$
$$R_1 \times R_2 = 1\left(\frac{2}{5}\right)$$
$$\boxed{R_1 \times R_2 = \frac{2}{5}}$$

The new equation is:

$$x^2 - (r_1 + r_2)x + (r_1 \times r_2) = 0$$
$$x^2 - \left(\frac{3}{5}\right)x + \left(\frac{2}{5}\right) = 0$$
$$5\left(x^2 - \left(\frac{3}{5}\right)x + \left(\frac{2}{5}\right) = 0\right)$$
$$\boxed{5x^2 - 3x + 2 = 0}$$

Example 4

Without solving the given equation, find an equation whose solutions are the negatives of the solutions to:

$$m^2 - 4m + 9 = 0$$

The sum of the solutions is 4 and the product of the solutions is 9. The solutions to the new equation must be negatives of the solutions of the original equation. Therefore, the sum and the product of the new solutions are:

$$\boxed{R_1+R_2=-r_1+(-r_2)=-(r_1+r_2)=-4} \textit{ and } \boxed{R_1\times R_2=(-r_1)\times(-r_2)=r_1\times r_2=9}$$

The new equation is:

$$m^2-(r_1+r_2)m+(r_1\times r_2)=0$$
$$m^2-(-4)m+(+9)=0$$
$$\boxed{m^2+4m+9=0}$$

Review

Without solving, determine the sum and the product of the roots of the following quadratic equations.

1. $2y^2-8y+3=0$
2. $3e^2-6e=4$
3. $0=14-12x+18x^2$
4. $5x^2+6=7x$
5. $2(2x-1)(x+5)=x^2+4$

For the following sums and products of the solutions, state one possible quadratic equation:

6. sum: 4; product: 3
7. sum: 0; product: -16
8. sum: -9; product: -7
9. sum: -6; product: -5
10. sum: $-\frac{2}{3}$; product: $\frac{5}{3}$

For the given roots, determine the factors of the quadratic function:

11. $-\frac{3}{2}$ and 5
12. $\frac{1}{4}$ and $\frac{3}{2}$
13. -5 and 3
14. $-\frac{5}{2}$ and $-\frac{4}{3}$
15. $\pm\frac{5}{2}$

For the given roots, determine a potential quadratic function:

16. -2 and -4
17. -3 and $-\frac{1}{3}$
18. $2+\sqrt{3}$ and $2-\sqrt{3}$
19. $\pm 2\sqrt{5}$
20. $-3\pm\sqrt{7}$

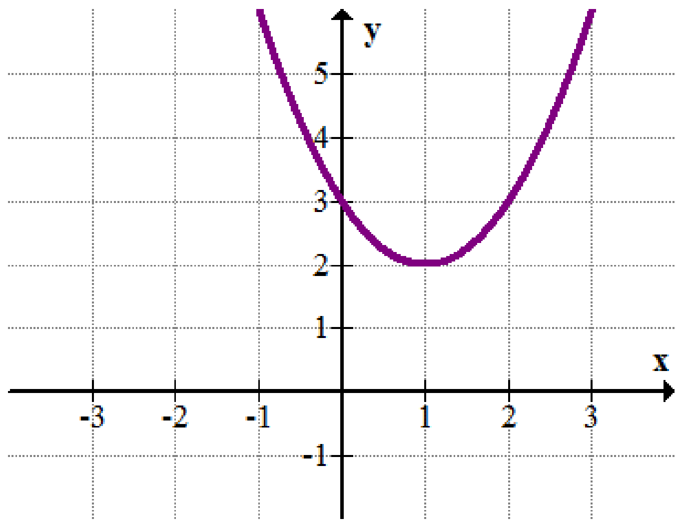
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.5.

9.6 Imaginary Numbers

Here you will learn about imaginary numbers.

The solutions to a quadratic equation show up as the x-intercepts of the corresponding quadratic function. The parabola $y = x^2 - 2x + 3$ is shown below.



What does this graph tell you about the solutions to $x^2 - 2x + 3 = 0$?

Imaginary Numbers

When humans first created the concept of numbers, they only had the counting numbers (whole numbers) $\{1, 2, 3, \ldots\}$ because numbers were meant to count physical objects. It took a long time (more than 700 years!) before even the concept of 0 was invented in the year 500 AD in India. Since then, humans have slowly been adding to our number system so that it can work for us. Fractions, decimals, and negative numbers have become an important part of our world. For a long time, it was accepted that the square root of negative numbers did not exist. In order for a solution to exist to the equation $x^2 = -1$, mathematicians invented a solution. This solution is called the imaginary number and is noted by the letter i:

$\sqrt{-1} = i$ and $i^2 = -1$

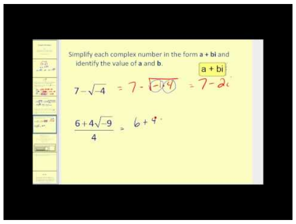
Imaginary numbers were not commonly accepted in mathematics until the 1700s, but since then they have become an important part of our number system and are especially important in physics. They are called imaginary because they cannot be found on a traditional number line of real numbers. The square root of any negative number can be written in terms of the imaginary number i:

- $\sqrt{-4} = \sqrt{4 \cdot -1} = \sqrt{4}\sqrt{-1} = 2i$
- $\sqrt{-5} = \sqrt{5 \cdot -1} = \sqrt{5}\sqrt{-1} = \sqrt{5}i$
- $\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16}\sqrt{-1} = 4i$

You can perform addition, subtraction, and multiplication with imaginary numbers just like regular numbers (you can also do division, but that is a bit more complicated and won't be considered here). When performing multiplication, remember that $i^2 = -1$. You should always express answers in such a way that the i does not have an exponent.

- $2i+1-3i+4$ simplifies to $-i+5$ or $5-i$
- $3i \cdot 2i$ can be multiplied to $6i^2 = 6(-1) = -6$

Numbers that are a combination of imaginary and real numbers, such as $5-i$, are called ***complex numbers***. You will study complex numbers in much more depth in future courses.



MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/181678

Express as an imaginary number

Express $\sqrt{-49}$ as a simplified imaginary number.

$\sqrt{-49} = \sqrt{49 \cdot -1} = \sqrt{49}\sqrt{-1} = 7i$

Express as an imaginary number

Express $\sqrt{-40}$ as a simplified imaginary number.

$\sqrt{-40} = \sqrt{4 \cdot 10 \cdot -1} = \sqrt{4}\sqrt{10}\sqrt{-1} = 2\sqrt{10}i$

Simplify

Simplify the following expression: $(4+3i)+(6-5i)$

Simplify by combining the real numbers with the real numbers and the imaginary numbers with the imaginary numbers:

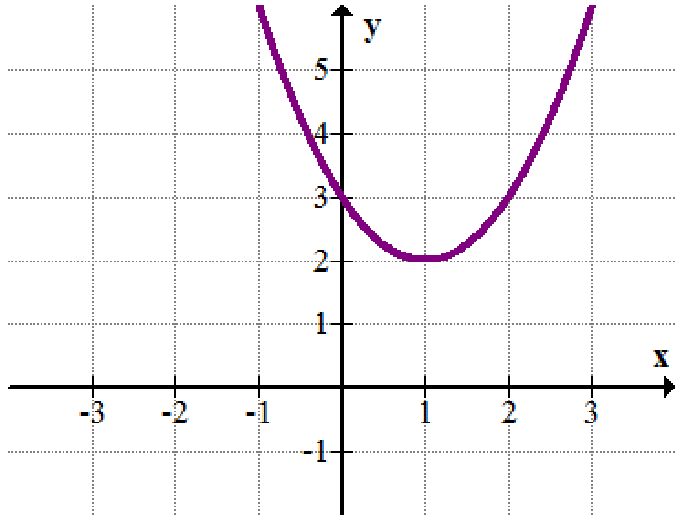
$(4+3i)+(6-5i) = 10-2i$

Examples

Example 1

Earlier, you were given a problem on a parabola.

The parabola $y = x^2 - 2x + 3$ has no x-intercepts, as shown below.



This means that the solutions to the equation $x^2 - 2x + 3 = 0$ are not real numbers. The solutions are complex numbers. You can still find the solutions using the quadratic formula, but your result will be 2 complex number solutions.

Example 2

Simplify the following:

$(5-3i)-(2+4i)$

$(5-3i)-(2+4i) = 5-3i-2-4i = 3-7i$

Example 3

Simplify the following:

$3i(4i^2-5i+3)$

$$\begin{aligned} 3i(4i^2-5i+3) &= 3i(4\cdot -1-5i+3) \\ &= 3i(-4-5i+3) \\ &= 3i(-1-5i) \\ &= -3i-15i^2 \\ &= -3i-15(-1) \\ &= -3i+15 \end{aligned}$$

Example 4

$(7+2i)(3-i)$

$$\begin{aligned}(7+2i)(3-i) =& 21-7i+6i-2i^2\\ &= 21-i-2i^2\\ &= 21-i-2(-1)\\ &= 21-i+2\\ &= 23-i\end{aligned}$$

Example 5

$\sqrt{-12}$

$\sqrt{-12} = \sqrt{4\cdot 3\cdot -1} = 2i\sqrt{3}$

Review

Express each as a simplified imaginary number.

1. $\sqrt{-300}$
2. $\sqrt{-32}$
3. $4\sqrt{-18}$
4. $\sqrt{-75}$
5. $\sqrt{-98}$

Simplify each of the following:

6. $(8+5i)-(12+8i)$
7. $(7+3i)(4-5i)$
8. $(2+i)(4-i)$
9. $3(5i-4)-2(6i-7)$
10. $5i(3i-2i^2+4)$
11. $(3+4i)+(11+6i)$
12. $(5+2i)(1-5i)$
13. $(1+i)(1-i)$
14. $2(6i-3)-4(2i+6)$
15. i^3
16. i^4
17. i^6
18. $5i(3i-2i^2+4)$

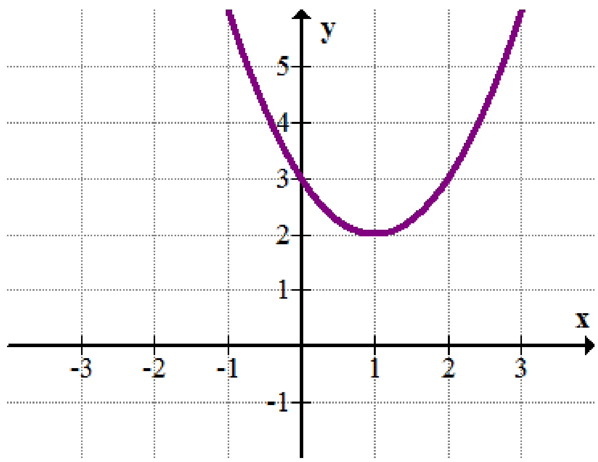
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.6.

9.7 Complex Roots of Quadratic Functions

Here you'll learn how to find complex roots of a quadratic function and what it means when a function has complex roots.

The quadratic function $y = x^2 - 2x + 3$ (shown below) does not intersect the x-axis and therefore has no real roots. What are the complex roots of the function?



Complex Roots of Quadratic Functions

Recall that the imaginary number, i, is a number whose square is -1:

$i^2 = -1$ and $i = \sqrt{-1}$

The sum of a real number and an imaginary number is called a ***complex number***. Examples of complex numbers are $5 + 4i$ and $3 - 2i$. All complex numbers can be written in the form $a + bi$ where a and b are real numbers. Two important points:

- The set of real numbers is a subset of the set of complex numbers where $b = 0$. Examples of real numbers are $2, 7, \frac{1}{2}, -4.2$.
- The set of imaginary numbers is a subset of the set of complex numbers where $a = 0$. Examples of imaginary numbers are $i, -4i, \sqrt{2}i$.

This means that the set of complex numbers includes real numbers, imaginary numbers, and combinations of real and imaginary numbers.

When a quadratic function does not intersect the x-axis, it has complex roots. When solving for the roots of a function algebraically using the quadratic formula, you will end up with a negative under the square root symbol. With your knowledge of complex numbers, you can still state the complex roots of a function just like you would state the real roots of a function.

Solve the following quadratic equation for

$$\boxed{m^2-2m+5=0}$$

You can use the quadratic formula to solve. For this quadratic equation, $a=1, b=-2, c=5$.

$$\begin{aligned}
m &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
m &= \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(5)}}{2(1)} \\
m &= \frac{2 \pm \sqrt{4-20}}{2} \\
m &= \frac{2 \pm \sqrt{-16}}{2} \qquad \sqrt{-16} = \sqrt{16} \times i = 4i \\
m &= \frac{2 \pm 4i}{2} \\
m &= 1 \pm 2i \\
m &= 1+2i \text{ or } m = 1-2i
\end{aligned}$$

There are no real solutions to the equation. The solutions to the quadratic equation are $1+2i$ *and* $1-2i$.

Solve the following equation by rewriting it as a quadratic and using the quadratic formula.

$\frac{3}{e+3} - \frac{2}{e+2} = 1$

To rewrite as a quadratic equation, multiply each term by $(e+3)(e+2)$.

$$\begin{aligned}
&\frac{3}{e+3}(e+3)(e+2) - \frac{2}{e+2}(e+3)(e+2) = 1(e+3)(e+2) \\
&3(e+2)-2(e+3) = (e+3)(e+2)
\end{aligned}$$

Expand and simplify.

$$\begin{aligned}
&3e+6-2e-6 = e^2+2e+3e+6 \\
&e^2+4e+6=0
\end{aligned}$$

Solve using the quadratic formula. For this quadratic equation, $a=1, b=4, c=6$.

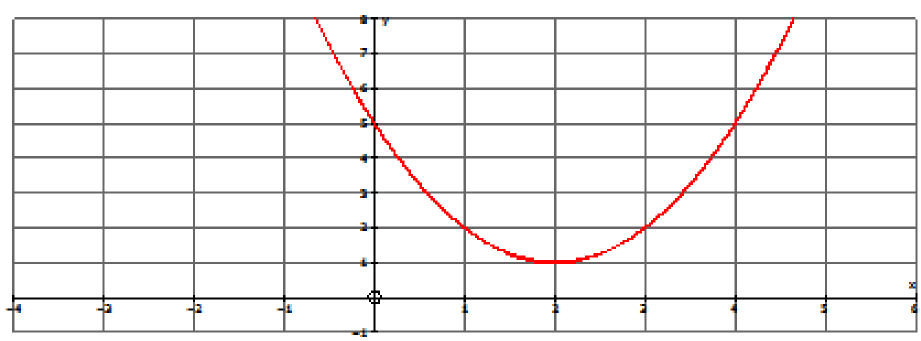
$$e = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$e = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(6)}}{2(1)}$$
$$e = \frac{-4 \pm \sqrt{16 - 24}}{2}$$
$$e = \frac{-4 \pm \sqrt{-8}}{2} \qquad \sqrt{-8} = \sqrt{8} \times i = \sqrt{4 \cdot 2} \times i = 2i\sqrt{2}$$
$$e = \frac{-4 \pm 2i\sqrt{2}}{2}$$
$$e = -2 \pm i\sqrt{2}$$
$$e = -2 + i\sqrt{2} \text{ or } e = -2 - i\sqrt{2}$$

There are no real solutions to the equation. The solutions to the equation are $-2 + i\sqrt{2}$ *and* $-2 - i\sqrt{2}$

Sketch the graph of the following quadratic function. What are the roots of the function?

$$y = x^2 - 4x + 5$$

Use your calculator or a table to make a sketch of the function. You should get the following:



As you can see, the quadratic function has no x-intercepts; therefore, the function has no real roots. To find the roots (which will be complex), you must use the quadratic formula.

For this quadratic function, $a = 1, b = -4, c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$
$$x = \frac{4 \pm \sqrt{16 - 20}}{2}$$
$$x = \frac{4 \pm \sqrt{-4}}{2} \qquad \sqrt{-4} = \sqrt{4} \times i = 2i$$
$$x = \frac{4 \pm 2i}{2}$$
$$x = 2 \pm i$$
$$x = 2 + i \ or \ x = 2 - i$$

The complex roots of the quadratic function are $2 + i$ *and* $2 - i$.

Examples

Example 1

Earlier, you were asked what are the complex roots of $y = x^2 - 2x + 3$.

To find the complex roots of the function $y = x^2 - 2x + 3$, you must use the quadratic formula.

For this quadratic function, $a = 1, b = -2, c = 3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{4 - 12}}{2}$$
$$x = \frac{2 \pm \sqrt{-8}}{2} \qquad \sqrt{-8} = \sqrt{8} \times i = 2\sqrt{2}i$$
$$x = \frac{2 \pm 2\sqrt{2}i}{2}$$
$$x = 1 \pm \sqrt{2}i$$

Example 2

Solve the following quadratic equation. Express all solutions in simplest radical form.

$$2n^2 + n = -4$$

$$2n^2 + n = -4$$

Set the equation equal to zero.

$$2n^2+n+4=0$$

Solve using the quadratic formula.

$$\begin{aligned}
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
n &= \frac{-(1)\pm\sqrt{(1)^2-4(2)(4)}}{2(2)}\\
n &= \frac{-1\pm\sqrt{1-32}}{4}\\
n &= \frac{-1\pm\sqrt{-31}}{4}\\
n &= \frac{-1\pm i\sqrt{31}}{4}
\end{aligned}$$

Example 3

Solve the following quadratic equation. Express all solutions in simplest radical form.

$$m^2+(m+1)^2+(m+2)^2=-1$$

$$m^2+(m+1)^2+(m+2)^2=-1$$

Expand and simplify.

$$\begin{aligned}
&m^2+(m+1)(m+1)+(m+2)(m+2)=-1\\
&m^2+m^2+m+m+1+m^2+2m+2m+4=-1\\
&3m^2+6m+5=-1
\end{aligned}$$

Write the equation in general form.

$$3m^2+6m+6=0$$

Divide by 3 to simplify the equation.

$$\begin{aligned}
&\frac{3m^2}{3}+\frac{6m}{3}+\frac{6}{3}=\frac{0}{3}\\
&m^2+2m+2=0
\end{aligned}$$

Solve using the quadratic formula:

$$
\begin{aligned}
m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
m &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)} \\
m &= \frac{-2 \pm \sqrt{4-8}}{2} \\
m &= \frac{-2 \pm \sqrt{-4}}{2} \\
m &= \frac{-2 \pm 2i}{2} \\
m &= -1 \pm i
\end{aligned}
$$

Example 4

Is it possible for a quadratic function to have exactly one complex root?

No, even in higher degree polynomials, complex roots will always come in pairs. Consider when you use the quadratic formula– if you have a negative under the square root symbol, both the + version and the - version of the two answers will end up being complex.

Review

1. If a quadratic function has 2 x-intercepts, how many complex roots does it have? Explain.
2. If a quadratic function has no x-intercepts, how many complex roots does it have? Explain.
3. If a quadratic function has 1 x-intercept, how many complex roots does it have? Explain.
4. If you want to know whether a function has complex roots, which part of the quadratic formula is it important to focus on?
5. You solve a quadratic equation and get 2 complex solutions. How can you check your solutions?
6. In general, you can attempt to solve a quadratic equation by graphing, factoring, completing the square, or using the quadratic formula. If a quadratic equation has complex solutions, what methods do you have for solving the equation?

Solve the following quadratic equations. Express all solutions in simplest radical form.

7. $x^2 + x + 1 = 0$
8. $5y^2 - 8y = -6$
9. $2m^2 - 12m + 19 = 0$
10. $-3x^2 - 2x = 2$
11. $2x^2 + 4x = -11$
12. $-x^2 + x - 23 = 0$
13. $-3x^2 + 2x = 14$
14. $x^2 + 5 = -x$
15. $\frac{1}{2}d^2 + 4d = -12$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.7.

9.8 The Discriminant

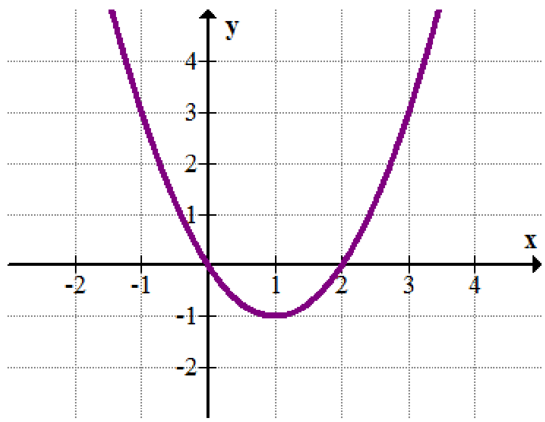
Here you'll learn what the discriminant is and how to use it to help you to describe the roots and graph of a quadratic function.

Suppose you want to know whether the function $y = x^2 - 5x + 12$ has real roots. What part of the quadratic formula would you need to test in order to determine if the roots of the function are real or complex?

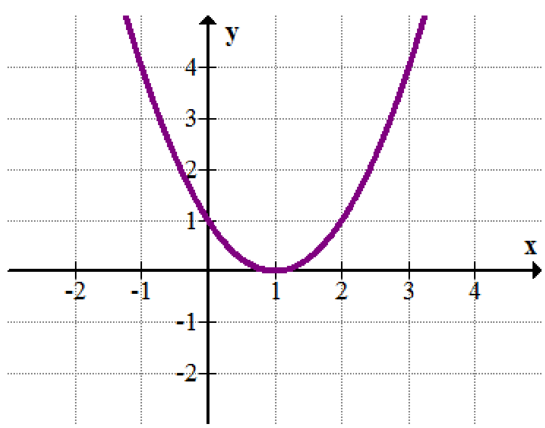
The Discriminant

Consider the potential roots of a quadratic function. There are three options:

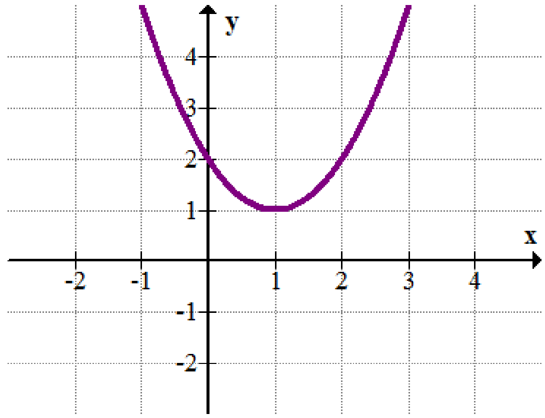
1. The function has 2 distinct real roots and 2 x-intercepts.



2. The function has 1 real root (of multiplicity 2) and 1 x-intercept.



3. The function has 0 real roots, 2 complex roots, and 0 x-intercepts.



The quadratic formula states that the two roots of a quadratic function $y = ax^2 + bx + c$ are:

$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Under the radical sign is the expression $b^2 - 4ac$. This expression is known as the ***discriminant*** of the quadratic function and is important especially because when it is negative the roots of the function are complex. There are only three possible outcomes for the value of the discriminant. These outcomes are:

1. The value of the discriminant is positive ($b^2 - 4ac > 0$). This means the function has 2 distinct real roots and 2 x-intercepts.
2. The value of the discriminant is zero ($b^2 - 4ac = 0$). This means the function has 1 real root (of multiplicity 2) and 1 x-intercept.
3. The value of the discriminant is negative ($b^2 - 4ac < 0$). This means the function has 0 real roots, 2 complex roots, and 0 x-intercepts.

This discriminant is helpful when you are only looking to describe the graph of a quadratic function or its roots, but don't need to know its exact roots. If you need to know its exact roots, you will still have to use the complete quadratic formula.

If the discriminant of a quadratic function has the value shown below, determine if the function will have two distinct real roots, 1 real root of multiplicity 2, or two distinct complex roots.

a) 7

$b^2 - 4ac > 0$ so the quadratic function will have two distinct real roots.

b) -3

$b^2 - 4ac < 0$ so the quadratic function will have two distinct complex roots.

c) $\frac{1}{2}$

$b^2 - 4ac > 0$ so the quadratic function will have two distinct real roots.

d) 0

$b^2 - 4ac = 0$ so the quadratic function will have 1 real root of multiplicity 2.

Given the function

a) 5

$b^2 - 4ac > 0$ so the parabola will cross the x-axis twice.

b) -6

$b^2 - 4ac < 0$ so the parabola will not cross the x-axis.

c) 0

$b^2 - 4ac = 0$ so the parabola will cross the x-axis once.

d) 0.2

$b^2 - 4ac > 0$ so the parabola will cross the x-axis twice.

For each of the following quadratic equations, determine the value of the discriminant and use that value to describe the nature of the roots.

Let D represent the discriminant.

a) $4x^2 - 4x + 1 = 0$

The quadratic equation will have 1 real solution of multiplicity 2.

$$\begin{aligned}
&D = b^2 - 4ac\\
&D = (-4)^2 - 4(4)(1)\\
&D = 16 - 16\\
&\boxed{D = 0}\\
&b^2 - 4ac = 0
\end{aligned}$$

b) $x^2 + 3x + 9 = 0$

The quadratic equation will have two distinct complex solutions and no real solutions.

$$\begin{aligned}
&D = b^2 - 4ac\\
&D = (3)^2 - 4(1)(9)\\
&D = 9 - 36\\
&\boxed{D = -27}\\
&b^2 - 4ac < 0
\end{aligned}$$

c) $2x^2 + 3x - 4 = 0$

The quadratic equation will have two distinct real solutions.

$$\begin{aligned}
&D = b^2 - 4ac\\
&D = (3)^2 - 4(2)(-4)\\
&D = 9 + 32\\
&\boxed{D = 41}\\
&b^2 - 4ac > 0
\end{aligned}$$

d) $9x^2+12x+4=0$

The quadratic equation will have 1 real root of multiplicity 2.

$$
\begin{aligned}
&D=b^2-4ac\\
&D=(12)^2-4(9)(4)\\
&D=144-144\\
&\boxed{D=0}\\
&b^2-4ac=0
\end{aligned}
$$

Examples

Example 1

Earlier, you were asked what part of the quadratic formula would you need to test in order to determine if the roots of the function are real or complex.

In order to determine if the roots of $y=x^2-5x+12$ are real or complex, you must find the discriminant, b^2-4ac. For this function:

$$
\begin{aligned}
b^2-4ac&=(-5)^2-4(1)(12)\\
&=25-48\\
&=-23
\end{aligned}
$$

Because the discriminant is negative, this function has two distinct complex roots.

Example 2

Given the following quadratic equation, find the value of 'm' such that the equation will have 1 real solution of multiplicity 2.

$mx^2+(m+8)x+9=0$

Begin by determining the value of the discriminant.

$$b^2-4ac=(m+8)^2-4(m)(9)$$

Expand and Simplify

$$
\begin{aligned}
b^2-4ac&=(m+8)(m+8)-4(m)(9)\\
b^2-4ac&=m^2+8m+8m+64-36m\\
b^2-4ac&=m^2-20m+64
\end{aligned}
$$

If the equation has 1 real solution of multiplicity 2, the value of the discriminant must equal zero.

$$b^2 - 4ac = 0$$
$$\therefore m^2 - 20m + 64 = 0$$

Factor the quadratic equation and solve for the variable 'm'.

$$(m-16)(m-4) = 0$$
$$m - 16 = 0 \text{ or } m - 4 = 0$$
$$m = 16 \text{ or } m = 4$$

The values of 'm' that would produce 1 solution of multiplicity 2 for the quadratic equation are $m = 16$ *or* $m = 4$.

Example 3

Given the following quadratic equation, determine the nature of the solutions:

$4(y^2 - 5y + 5) = -5$

Write the quadratic equation in general form.

$$4(y^2 - 5y + 5) = -5$$

Apply the distributive property.

$$4y^2 - 20y + 20 = -5$$

Set the equation equal to zero.

$$4y^2 - 20y + 20 + 5 = -5 + 5$$
$$4y^2 - 20y + 25 = 0$$

Determine the value of the discriminant for this quadratic equation.

$$D = b^2 - 4ac$$
$$D = (-20)^2 - 4(4)(25)$$
$$D = 400 - 400$$
$$\boxed{D = 0}$$
$$b^2 - 4ac = 0$$

The quadratic equation will have 1 real solution of multiplicity 2.

Example 4

Given the following quadratic equation, find the value of 'm' such that the equation will have two distinct complex solutions.

$(m+1)e^2-2e-3=0$

Begin by determining the value of the discriminant.

$$b^2-4ac=(-2)^2-4(m+1)(-3)$$

Expand and Simplify

$$b^2-4ac=12m+16$$

If the equation has two distinct complex solutions the value of the discriminant must be less than zero.

$$\begin{aligned}&b^2-4ac<0\\ &\therefore 12m+16<0\end{aligned}$$

Solve the inequality.

$$\begin{aligned}&12m+16-16<0-16\\ &12m<-16\\ &\frac{12m}{12}<\frac{-16}{12}\\ &\frac{\cancel{12}m}{\cancel{12}}<-\frac{16}{12}\\ &\boxed{m<-\frac{4}{3}}\end{aligned}$$

The value of 'm' that would produce two distinct complex solutions for the quadratic equation is $m<-\frac{4}{3}$.

Review

If the discriminant of a quadratic equation has the value shown below, describe the nature of the solutions.

1. -14
2. 11
3. 0
4. -0.25
5. 124

State the nature of the solutions for each of the following quadratic equations.

6. $2x^2+7x-1=0$

7. $3x^2+2x=-7$
8. $-9x^2-7=4x$
9. $x^2-8x+16=0$
10. $4+2x^2=11x$

Determine the value(s) of 'm' that will produce the indicated solution for each of the following:

11. $y^2+(m+2)y+2m=0$; 1 real solution of multiplicity 2
12. $g^2+(m-1)g+1=0$; 2 real solutions
13. $3mx^2-3x+1=0$; 2 complex solutions
14. $x^2+4mx+1=0$; 1 real solution of multiplicity 2
15. $p^2+mp+16=0$; 2 real solutions

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.8.

9.9 Radical Equations

Here you will learn how to solve a radical equation.

Can you solve the following equation?

$x+\sqrt{x-2}=4$

Radical Equations

A radical equation is an equation with a variable under a radical sign. The following are all examples of radical equations:

- $\sqrt{x}=5$
- $\sqrt{x-4}+5=0$
- $x+\sqrt{x-2}=4$
- $\sqrt{4x+5}-\sqrt{2x-6}=3$

Just like multiplication and division or addition and subtraction are inverse operations (they "undo" each other), squaring and finding a square root are inverse operations:

- $(\sqrt{5})^2=5$
- $\sqrt{(x+2)^2}=x+2$

Therefore, to eliminate a square root in an equation, isolate the square root part of the equation and then square both sides of the equation. If there are multiple square roots, you might have to go through this process multiple times. For example, consider the following equation:

$\sqrt{x-1}-5=0$

Step 1: Isolate the radical:

$$\sqrt{x-1}=5$$

Step 2: Square both sides of the equation.

$$\left(\sqrt{x-1}\right)^2=(5)^2$$
$$x-1=25$$

Step 3: Solve the resulting equation.

$$x=26$$

Just like any equation, you can check your answer(s) by substituting them back into the original equation. Unlike other equations, ***it is very important to check your answers to radical equations.*** Due to the fact that the process of squaring produces positive numbers, sometimes you will end up with a solution to a radical equation that does not actually work in the original equation. These solutions are called "extraneous" and are not actually solutions. Therefore, ***you must always check your answers to radical equations***. Check the solution to the above equation by verifying that the left side (L.S.) of the equation equals the right side (R.S.) of the equation:

$$\begin{aligned}
&\sqrt{x-1}-5=0 \\
&L.S. = \sqrt{x-1}-5 \qquad R.S. = 0 \\
&L.S. = \sqrt{26-1}-5 \\
&L.S. = \sqrt{25}-5 \\
&L.S. = 5-5 \\
&L.S. = 0 \\
&L.S. = R.S.
\end{aligned}$$

Therefore, the solution of 26 is correct.

Solve the following radical equation and verify the solution(s):

$2\sqrt{x-1}-1=9$

Begin by isolating the radical:

$$\begin{aligned}
2\sqrt{x-1}-1+1 &= 9+1 \\
2\sqrt{x-1} &= 10 \\
\frac{2\sqrt{x-1}}{2} &= \frac{10}{2} \\
\sqrt{x-1} &= 5
\end{aligned}$$

Now square both sides of the equation:

$$\begin{aligned}
\left(\sqrt{x-1}\right)^2 &= (5)^2 \\
x-1 &= 25
\end{aligned}$$

Solve the equation:

$$x = 26$$

Finally, verify the result by substituting the value of 26 for 'x' into the original equation. If 26 is a solution to the equation, the left side ($L.S.$) will equal the right side ($R.S.$).

$$\begin{aligned}
&2\sqrt{x-1}-1=9\\
&L.S.=2\sqrt{26-1}-1 \quad R.S.=9\\
&L.S.=2\sqrt{25}-1\\
&L.S.=2(5)-1\\
&L.S.=10-1\\
&L.S.=9\\
&L.S.=R.S.
\end{aligned}$$

Therefore, the solution of 26 is correct.

Solve the following radical equation and verify the solution(s):

$\sqrt{4x+5}-\sqrt{2x-6}=3$

When a radical equation has more than one radical, begin by writing the equation with one radical on each side of the equation. You will ultimately have to square the equation more than once throughout the solution process.

$$\sqrt{4x+5}=3+\sqrt{2x-6}$$

Square both sides of the equation:

$$\left(\sqrt{4x+5}\right)^2=\left(3+\sqrt{2x-6}\right)^2$$

Expand and simplify:

$$\begin{aligned}
&\left(\sqrt{4x+5}\right)^2=\left(3+\sqrt{2x-6}\right)\left(3+\sqrt{2x-6}\right)\\
&4x+5=9+3\sqrt{2x-6}+3\sqrt{2x-6}+2x-6\\
&4x+5=3+6\sqrt{2x-6}+2x
\end{aligned}$$

Isolate the radical:

$$2x+2=6\sqrt{2x-6}$$

Simplify the equation:

$$\begin{aligned}
&\frac{2x}{2}+\frac{2}{2}=\frac{6\sqrt{2x-6}}{2}\\
&x+1=3\sqrt{2x-6}
\end{aligned}$$

Square both sides of the equation:

$$(x+1)^2 = \left(3\sqrt{2x-6}\right)^2$$
$$x^2+x+x+1 = 9(2x-6)$$
$$x^2+2x+1 = 18x-54$$

The equation is quadratic. Write the equation in standard form:

$$x^2-16x+55=0$$

Solve the equation by factoring:

$$(x-11)(x-5)=0$$
$$x = 11 \text{ or } x = 5$$

Verify the results by first substituting the value of 11 for 'x' into the original equation and then substituting the value of 5 for 'x' into the original equation:

Verify $x=11$.
$$\sqrt{4x+5}-\sqrt{2x-6}=3$$
$$L.S. = \sqrt{4x+5}-\sqrt{2x-6} \quad R.S. = 3$$
$$L.S. = \sqrt{4(11)+5}-\sqrt{2(11)-6}$$
$$L.S. = \sqrt{49}-\sqrt{16}$$
$$L.S. = 7-4$$
$$L.S. = 3$$
$$R.S. = 3$$
$$L.S. = R.S.$$

Verify $x=5$.
$$\sqrt{4x+5}-\sqrt{2x-6}=3$$
$$L.S. = \sqrt{4x+5}-\sqrt{2x-6} \quad R.S. = 3$$
$$L.S. = \sqrt{4(5)+5}-\sqrt{2(5)-6}$$
$$L.S. = \sqrt{25}-\sqrt{4}$$
$$L.S. = 5-2$$
$$L.S. = 3$$
$$R.S. = 3$$
$$L.S. = R.S.$$

Therefore, the solutions of 11 and 5 are both correct.

Solve the following radical equation and verify the solution(s):

$\sqrt{5x+6}-4=0$

Begin by isolating the radical:

$$\sqrt{5x+6}=4$$

Square both sides of the equation:

$$\left(\sqrt{5x+6}\right)^{2} = (4)^{2}$$
$$5x+6 = 16$$

Solve the equation:

$$5x = 10$$
$$\frac{5x}{5} = \frac{10}{5}$$
$$x = 2$$

Verify the result by substituting the value of 2 for 'x' into the original equation:

$$\begin{aligned}
&\text{Verify } x = 2.\\
&\sqrt{5x+6}-4 = 0\\
&L.S. = \sqrt{5x+6}-4 \quad R.S. = 0\\
&L.S. = \sqrt{5(2)+6}-4\\
&L.S. = \sqrt{10+6}-4\\
&L.S. = \sqrt{16}-4\\
&L.S. = 4-4\\
&L.S. = 0\\
&R.S. = 0\\
&L.S. = R.S.
\end{aligned}$$

Therefore, the solution of 2 is correct.

Examples

Example 1

Earlier, you were asked if you could solve the following equation.

$x+\sqrt{x-2} = 4$

Begin by isolating the radical:

$$\sqrt{x-2} = 4-x$$

Square both sides of the equation:

$\left(\sqrt{x-2}\right)^{2} = (4-x)^{2}$

Expand and simplify:

$$x-2=16-8x+x^2$$

The equation is a quadratic. Write the equation in standard form:

$$x^2-9x+18=0$$

Solve the equation:

$$\begin{aligned}&(x-6)(x-3)=0\\&x-6=0 \text{ or } x-3=0\\&x=6 \text{ or } x=3\end{aligned}$$

Verify the results by first substituting the value of 6 for 'x' into the original equation and then substituting the value of 3 for 'x' into the original equation.

Verify $x=6$.	Verify $x=3$.
$x+\sqrt{x-2}=4$	$x+\sqrt{x-2}=4$
$L.S.=6+\sqrt{6-2} \quad R.S.=4$	$L.S.=3+\sqrt{3-2} \quad R.S.=4$
$L.S.=6+\sqrt{4}$	$L.S.=3+\sqrt{1}$
$L.S.=6+2$	$L.S.=3+1=4$
$L.S.=8$	$L.S.=4$
$R.S.=4$	$R.S.=4$
$L.S.\neq R.S.$	$L.S.=R.S.$

The value $x=6$ did not satisfy the original equation and is not a solution to the radical equation. It is called an *extraneous solution*. $x=3$ is a solution to the equation.

Example 2

Verify whether or not $x=3$ and $x=-5$ are solutions to the radical equation:

$2\sqrt{x+6}+\sqrt{2x+10}=2$

Verify the results by first substituting the value of 3 for 'x' into the original equation and then substituting the value of -5 for 'x' into the original equation.

Verify $x = 3$.

$2\sqrt{x+6} + \sqrt{2x+10} = 2$

$L.S. = 2\sqrt{x+6} + \sqrt{2x+10} \quad R.S. = 2$

$L.S. = 2\sqrt{(3)+6} + \sqrt{2(3)+10}$

$L.S. = 2\sqrt{9} + \sqrt{16}$

$L.S. = 2(3) + 4$

$L.S. = 6 + 4$

$L.S. = 10$

$R.S. = 2$

$L.S. \neq R.S.$

Verify $x = -5$.

$2\sqrt{x+6} + \sqrt{2x+10} = 2$

$L.S. = 2\sqrt{x+6} + \sqrt{2x+10} \quad R.S. = 2$

$L.S. = 2\sqrt{(-5)+6} + \sqrt{2(-5)+10}$

$L.S. = 2(1) + \sqrt{0}$

$L.S. = 2(1) + 0$

$L.S. = 2 + 0$

$L.S. = 2$

$R.S. = 2$

$L.S. = R.S.$

The value $x = 3$ did not satisfy the original equation and is not a solution to the radical equation. It is an extraneous solution. $x = -5$ is a solution.

Example 3

Solve the following radical equation and verify the solution(s)to the equation.

$x - \sqrt{x-1} = 7$

$-\sqrt{x-1} = 7 - x$

$$x - 1 = 49 - 14x + x^2$$

$$x^2 - 15x + 50 = 0$$

$$(x-10)(x-5) = 0$$
$$x = 10 \text{ or } x = 5$$

Verify $x = 10$.

$x - \sqrt{x-1} = 7$

$L.S. = x - \sqrt{x-1} \quad R.S. = 7$

$L.S. = (10) - \sqrt{(10)-1}$

$L.S. = 10 - \sqrt{9}$

$L.S. = 10 - 3$

$L.S. = 7$

$R.S. = 7$

$L.S. = R.S.$

Verify $x = 5$.

$x - \sqrt{x-1} = 7$

$L.S. = x - \sqrt{x-1} \quad R.S. = 7$

$L.S. = (5) - \sqrt{(5)-1}$

$L.S. = 5 - \sqrt{4}$

$L.S. = 5 - 2$

$L.S. = 3$

$R.S. = 7$

$L.S. \neq R.S.$

The value $x = 5$ did not satisfy the original equation and is not a solution to the radical equation. It is an extraneous solution. $x = 10$ is a solution.

Example 4

Solve the following radical equation and verify the solution(s) to the equation.

$\sqrt{x+7} - \sqrt{x} = 1$

$\sqrt{x+7} - \sqrt{x} = 1$

$$\sqrt{x+7} = 1 + \sqrt{x}$$

$$\left(\sqrt{x+7}\right)^2 = \left(1 + \sqrt{x}\right)^2$$

$$x+7 = 1 + 2\sqrt{x} + x$$

$$6 = 2\sqrt{x}$$

$$\begin{aligned}(6)^2 &= \left(2\sqrt{x}\right)^2 \\ 36 &= 4x\end{aligned}$$

$$\begin{aligned}\frac{36}{4} &= \frac{4x}{4} \\ 9 &= x\end{aligned}$$

Verify $x = 9$.

$\sqrt{x+7} - \sqrt{x} = 1$

$L.S. = \sqrt{x+7} - \sqrt{x} \quad R.S. = 1$

$L.S. = \sqrt{(9)+7} - \sqrt{9}$

$L.S. = \sqrt{16} - \sqrt{9}$

$L.S. = 4 - 3$

$L.S. = 1$

$L.S. = 1$

$R.S. = 1$

$L.S. = R.S.$

$x = 9$ is a solution to the equation.

Review

1. Is $x = 7$ a solution to $\sqrt{x+2} = -3$?
2. Is $x = 1$ a solution to $\sqrt{x^2+4x+4} - \sqrt{x^2+3x} = 1$?
3. Is $x = -3$ a solution to $\sqrt{x^2+4x+4} - \sqrt{x^2+3x} = 1$?
4. Is $x = 12$ a solution to $\sqrt{x+4} + 8 = x$?
5. Is $x = 5$ a solution to $\sqrt{x+4} + 8 = x$?
6. Is $x = 8$ a solution to $\sqrt{x+1} = 1 + \sqrt{x-4}$?
7. Is $x = 3$ a solution to $\sqrt{x+1} + \frac{2}{\sqrt{x+1}} = \sqrt{x+6}$?

Solve the following radical equations and verify the solution(s).

8. $x = 3 + \sqrt{x-1}$
9. $\sqrt{x+1} = 1 + \sqrt{x-4}$
10. $\sqrt{x} - \sqrt{x-16} = 2$
11. $\sqrt{3x-2} - 1 = \sqrt{2x-3}$
12. $5\sqrt{x-6} = x$
13. $x = 5 + \sqrt{x-4}$
14. $\sqrt{x+2} = 5 - \sqrt{x-3}$
15. $\sqrt{x} + \sqrt{x-9} = 5$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.9.

Summary

You learned that all quadratic equations have a corresponding quadratic function. Real solutions to quadratic equations are the x-intercepts of the quadratic function. If a quadratic equation has only complex solutions, the quadratic function will not have x-intercepts.

You also learned that there are four methods for solving quadratic equations:

1. Factoring and the zero product property (learned previously)
2. Graphing and looking for x-intercepts
3. Completing the square
4. The quadratic formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

The advantage of the quadratic formula is that it will always work to give you solutions, even if the solutions are not real numbers.

If you want to determine whether the roots of a given quadratic function are real or complex, but you don't need to know specifically what the roots are, you can use the discriminant. The discriminant is the part of the quadratic formula under the square root symbol ($b^2 - 4ac$). If the discriminant is negative, the roots will be complex. If the discriminant is equal to zero, there will only be one root (of multiplicity 2). If the discriminant is positive, the roots will be real.

You also learned that radical equations are equations with variables under square roots. Radical equations can be solved by isolating the square root and squaring both sides. Sometimes radical equations will produce extraneous solutions, which are not really solutions, so it is important to always check your answers to radical equations.

CHAPTER 10 Geometric Transformations

Chapter Outline

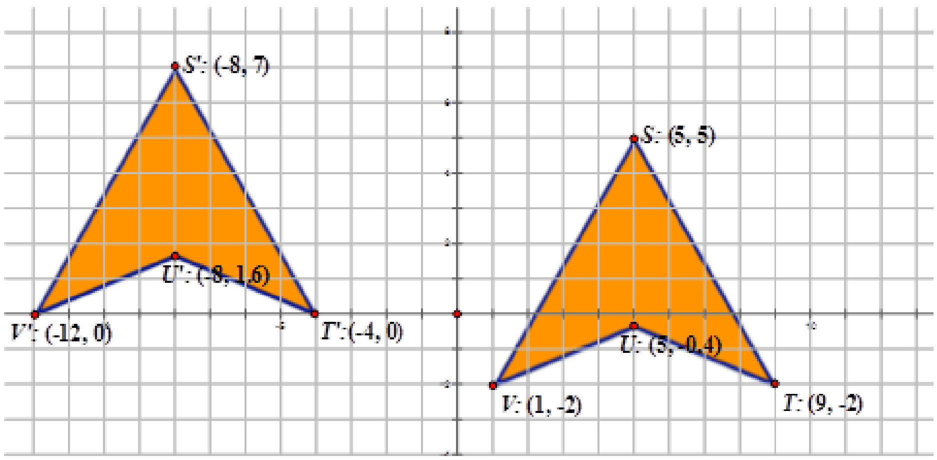
Introduction

Here you'll learn all about geometric transformations. You will learn about reflections, rotations, translations and dilations. You will also learn the distance formula and the midpoint formulas and see how algebra and geometry are connected.

10.1 Translations

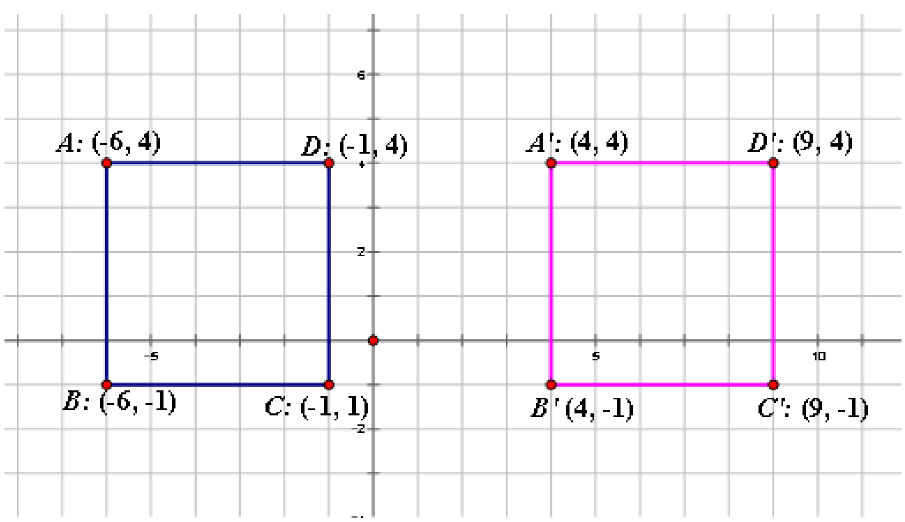
Here you will learn to describe translations.

Karen looked at the image below and stated that the image was translated thirteen units backwards. Is she correct? Explain.



Translations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A translation is a type of transformation that moves each point in a figure the same distance in the same direction. Translations are often referred to as slides. If you look at the picture below, you can see that the square $ABCD$ is moved 10 units to the right. All points of the square have been moved 10 units to the right to make the translated image $(A'B'C'D')$. The original square $(ABCD)$ is called a **preimage**. The final square is called the **image.**

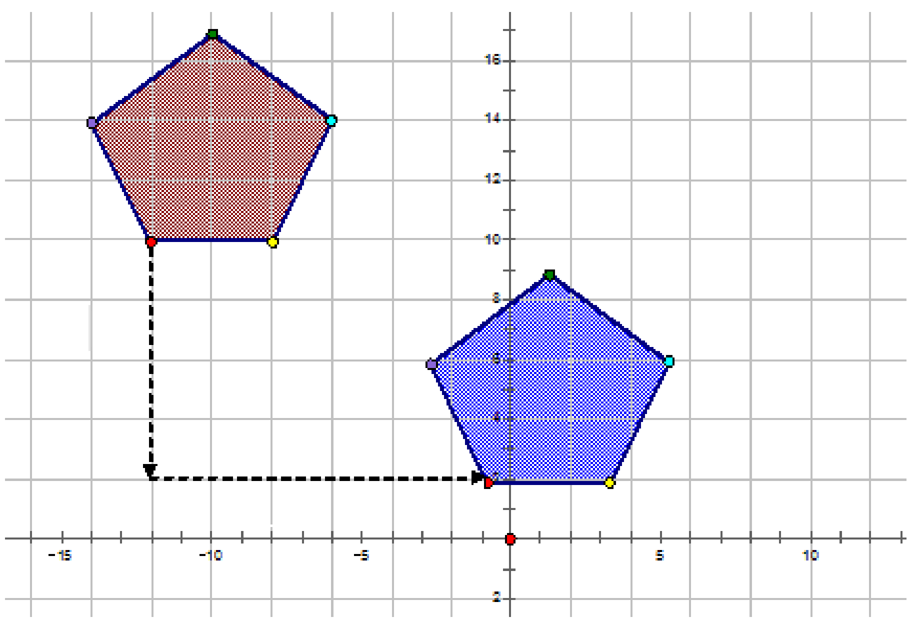


MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65238

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65239

Describe the translation

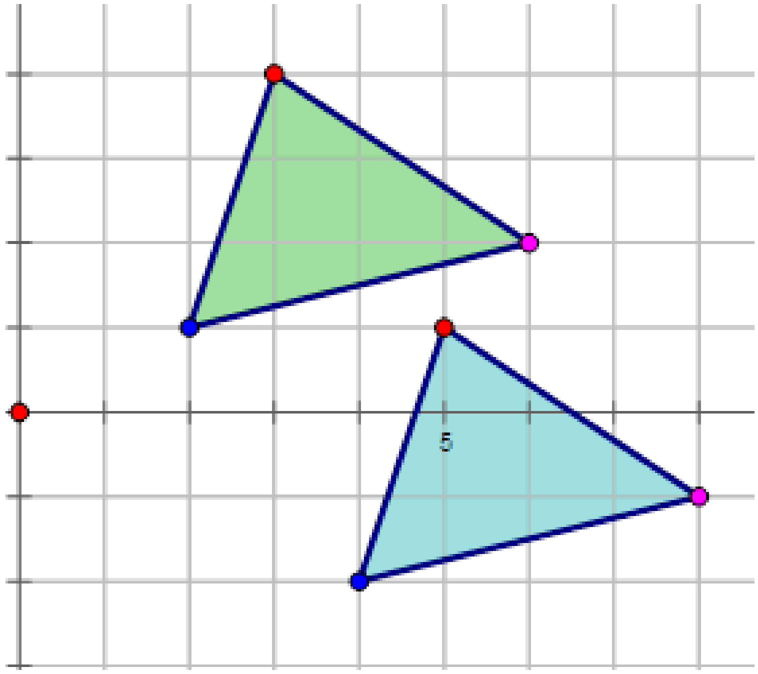
Describe the translation of the purple pentagon in the diagram below.



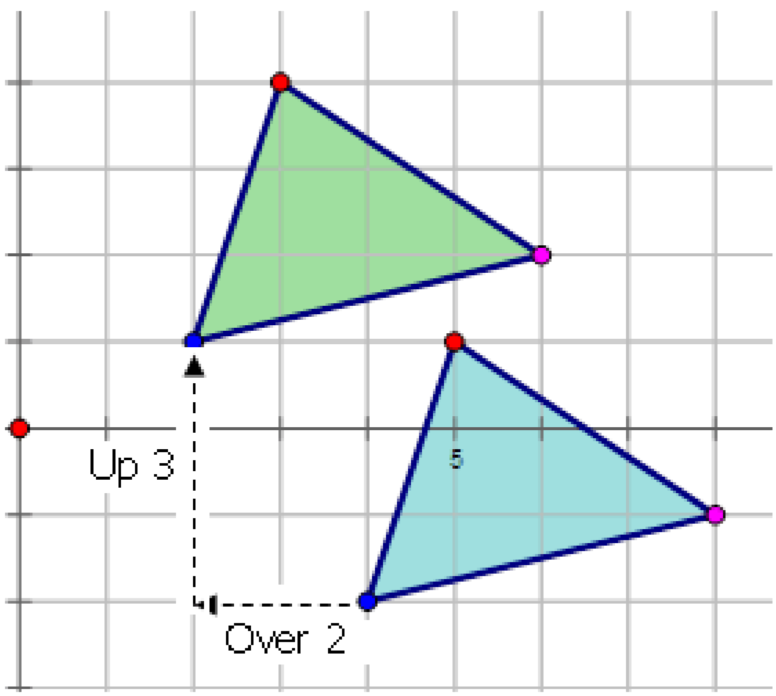
The pentagon is translated down 8 and over 11 to the right.

Describe the translation

Describe the translation of the light blue triangle in the diagram to the right.

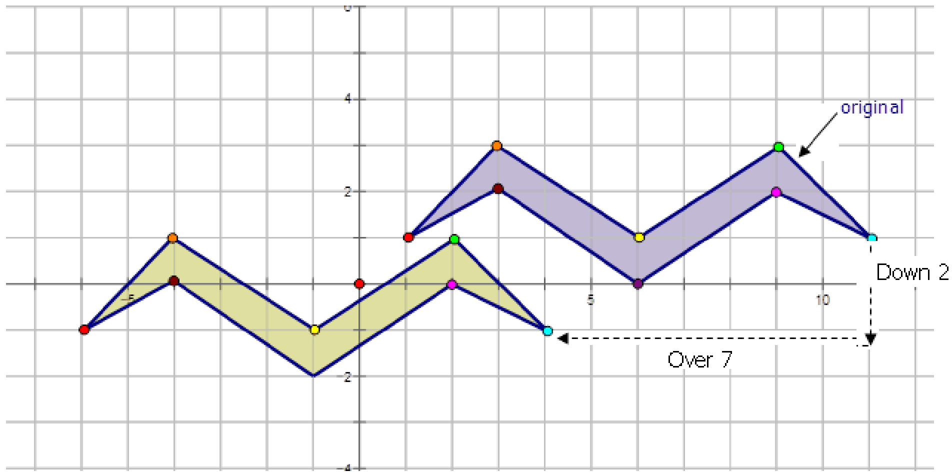


The blue triangle moves up 3 units and over 2 units to the left to make the green triangle image.



Describe the translation

Describe the translation in the diagram below.



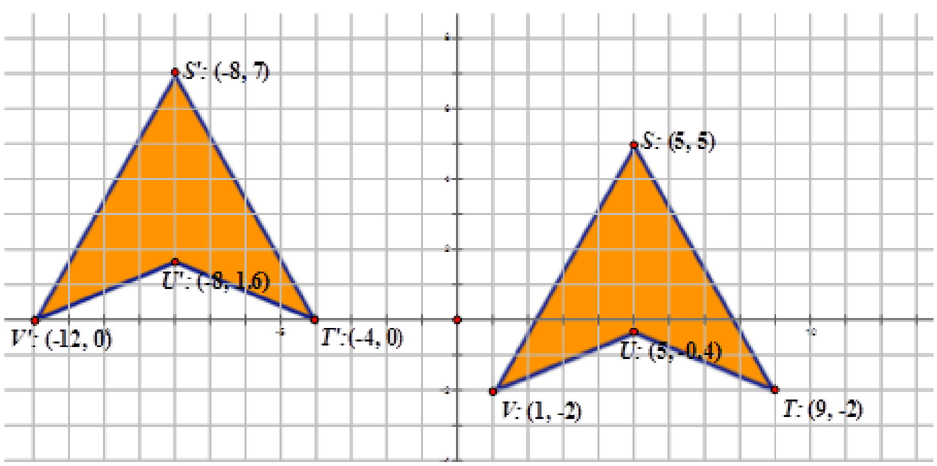
The original shape is translated down 2 and over 7 to the left.

Examples

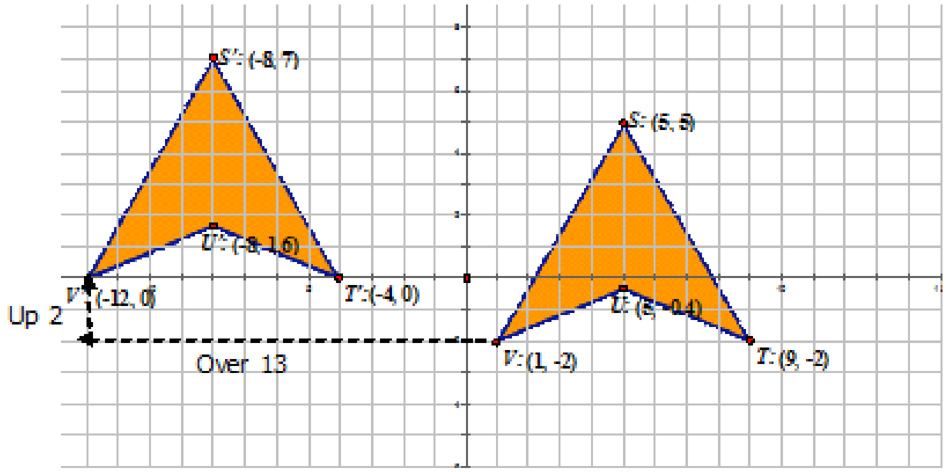
Example 1

Earlier, you were given a problem about Karen.

Karen looked at the image below and stated that the image was translated thirteen units backwards. Is she correct? Explain.

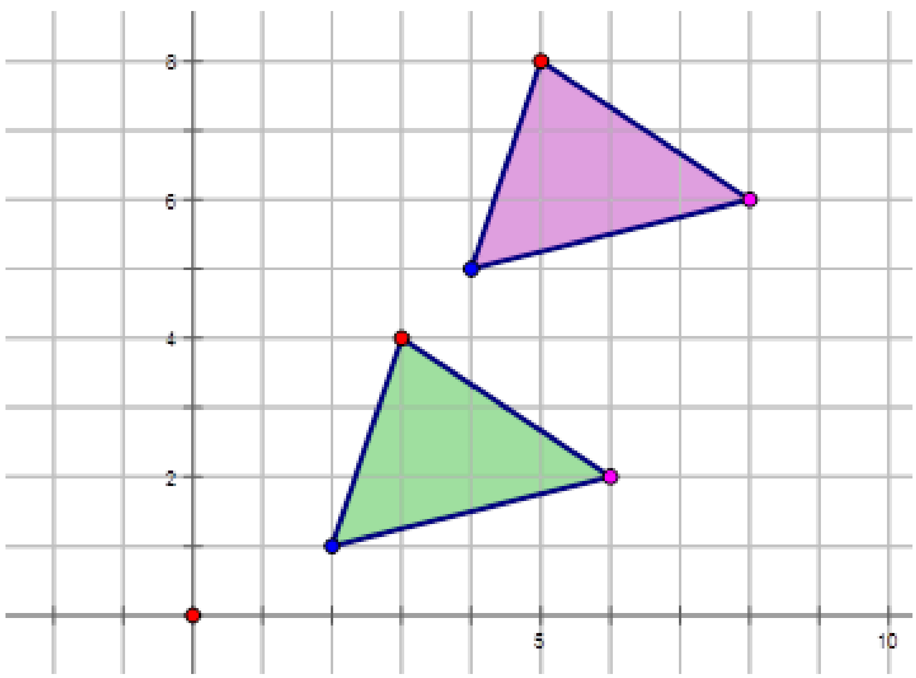


Karen is somewhat correct in that the translation is moving to the left (backwards). The proper way to describe the translation is to say that the image $STUV$ has moved 13 units to the left and 2 units up.

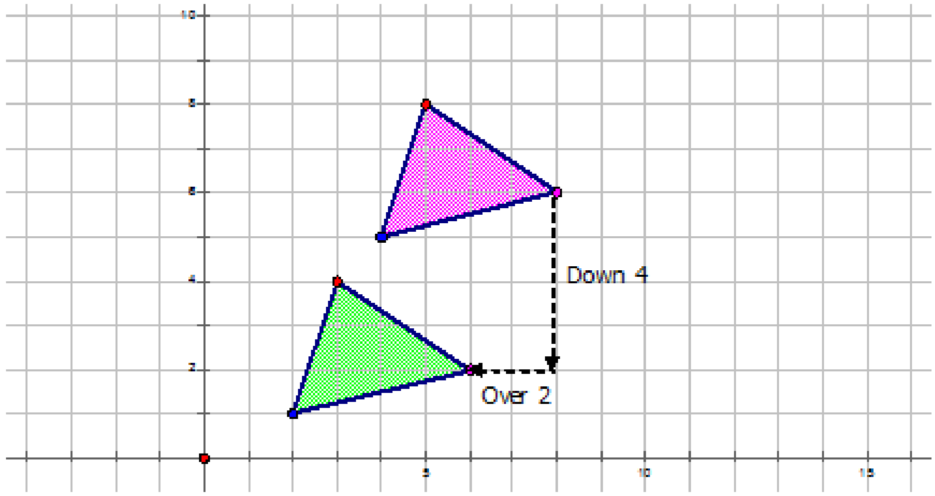


Example 2

Describe the translation of the pink triangle in the diagram below.

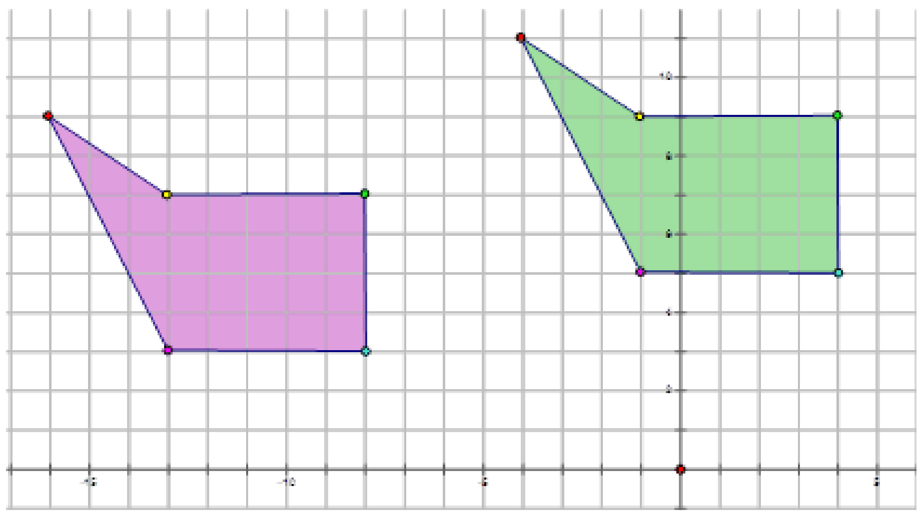


The pink triangle is translated down 4 and over 2 to the left.

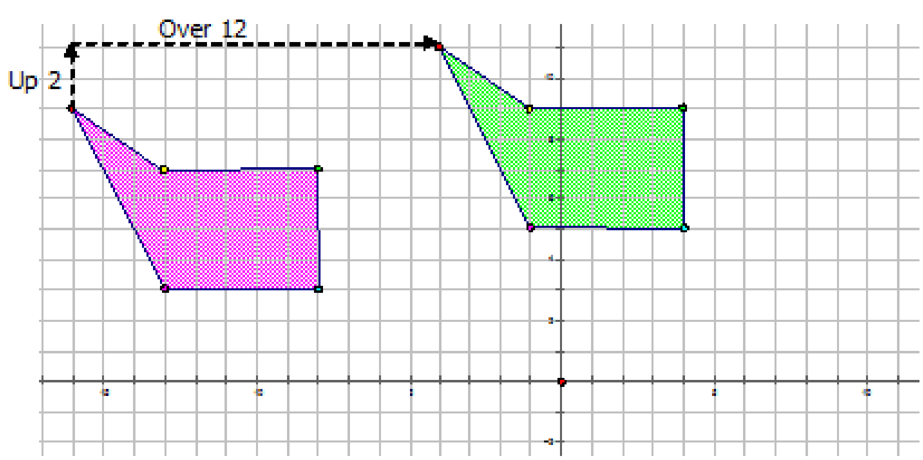


Example 3

Describe the translation of the purple polygon in the diagram below.

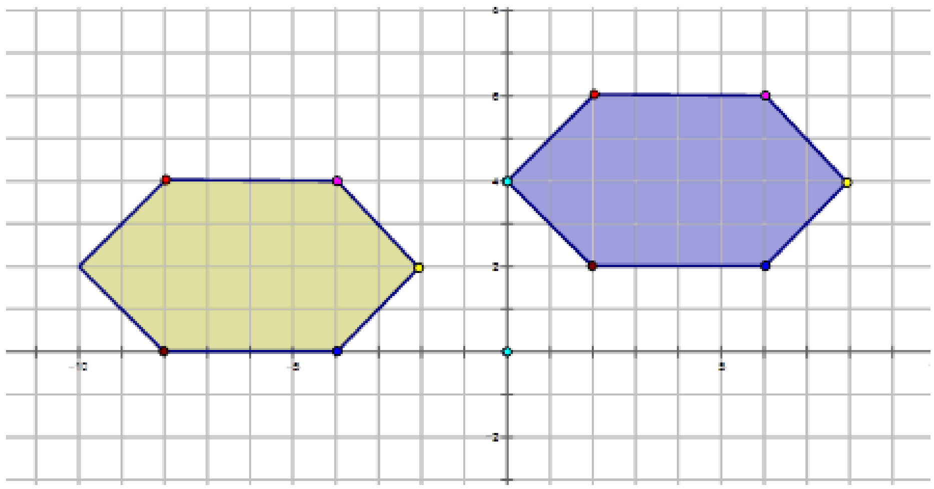


The purple polygon is translated up 2 and over 12 to the right.

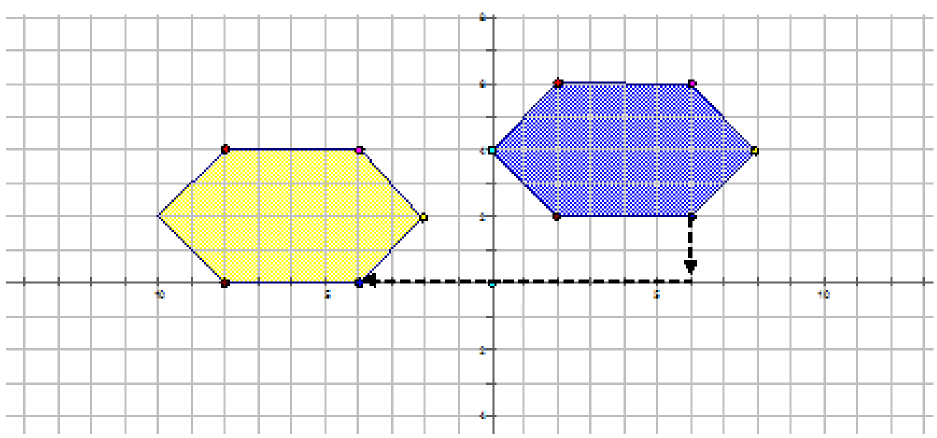


Example 4

Describe the translation of the blue hexagon in the diagram below.

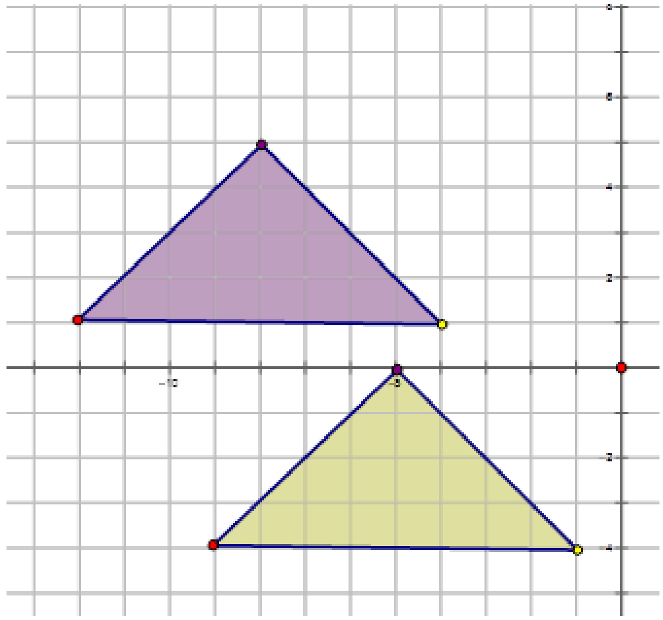


The blue hexagon is translated down 2 and over 10 to the left.

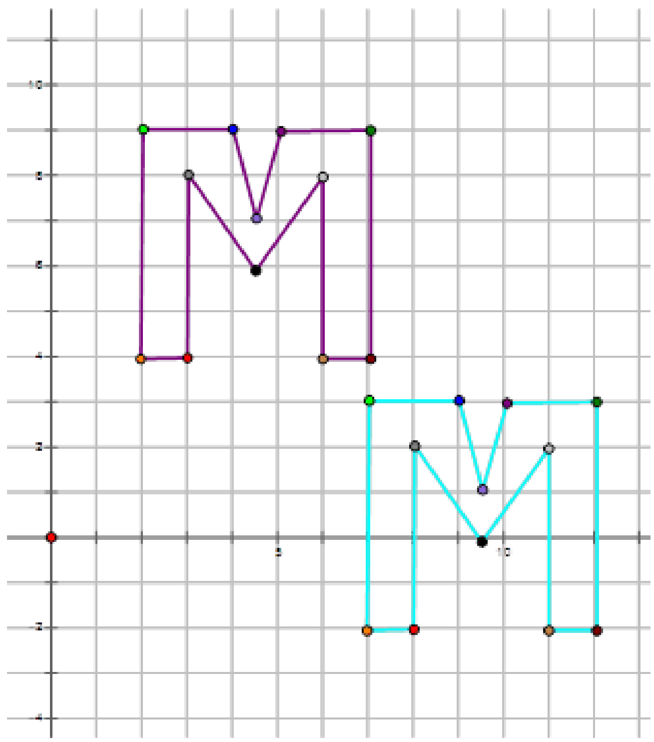


Review

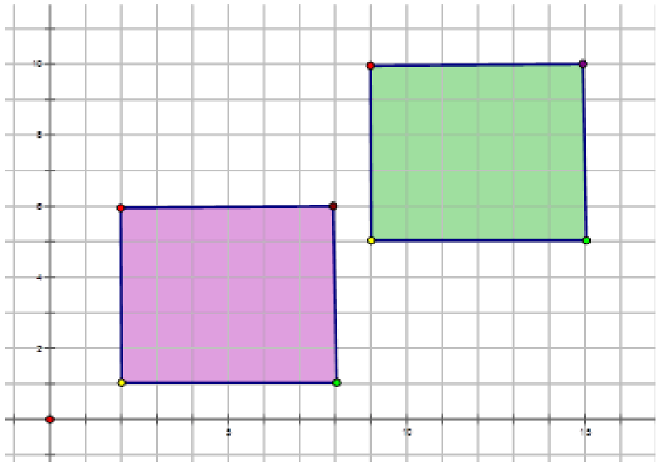
Describe the translation of the purple original figures in the diagrams:

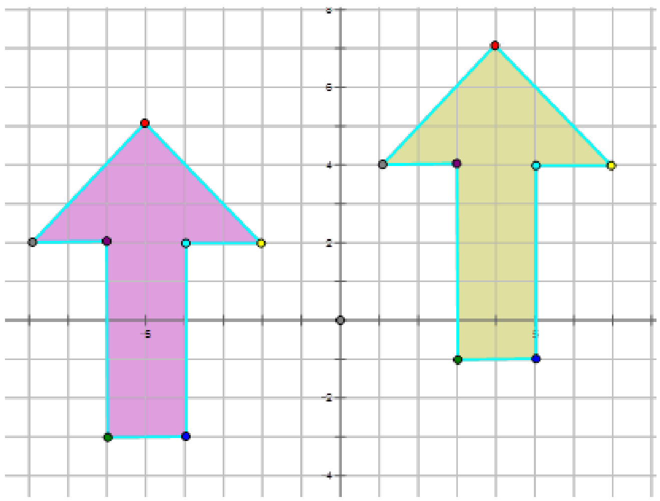


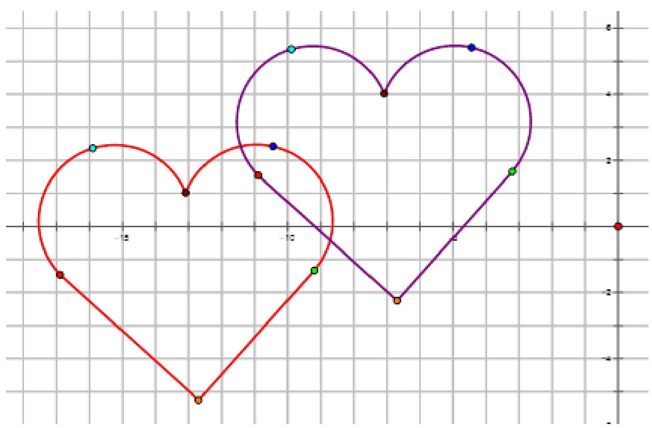
1.



2.

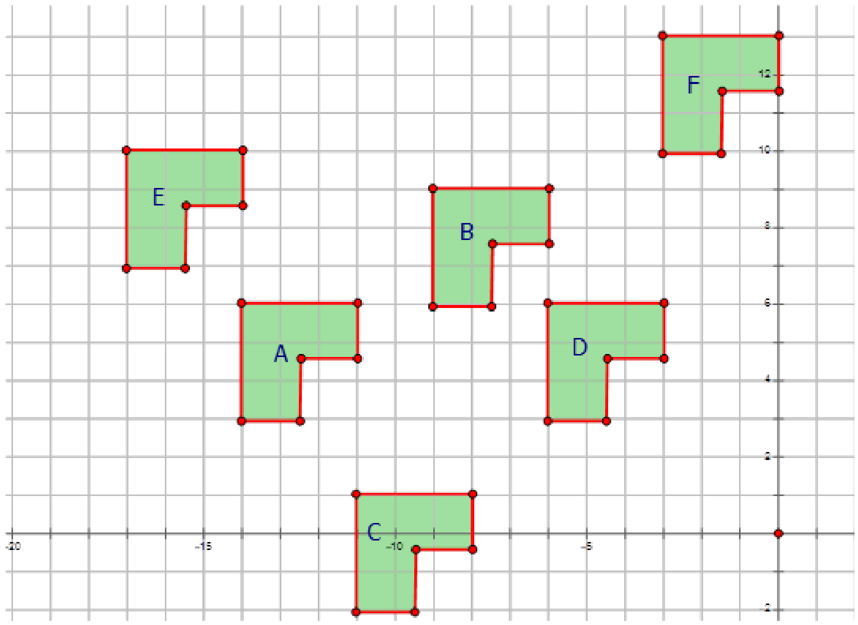
3. 

4. 

5. 

Use the diagram below to describe the following translations:

6. A onto B
7. A onto C
8. A onto D
9. A onto E
10. A onto F



On a piece of graph paper, plot the points $A(2,3), B(6,3)$ and $C(6,1)$ to form $\triangle ABC$.

11. Translate the triangle 3 units to the right and 2 units down. Label this $\triangle A'B'C'$.
12. Translate $\triangle A'B'C'$ 3 units to the left and 4 units down. Label this $\triangle A''B''C''$.
13. Describe the translation necessary to bring $\triangle A''B''C''$ to $\triangle ABC$.

On a piece of graph paper, plot the points $D(1,5), E(2,3)$ and $F(1,0)$ to form $\triangle ABC$.

14. Translate the triangle 2 units to the left and 4 units down. Label this $\triangle D'E'F'$.
15. Translate $\triangle D'E'F'$ 5 units to the right and 2 units up. Label this $\triangle D''E''F''$.
16. Describe the translation necessary to bring $\triangle D''E''F''$ to $\triangle DEF$.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.1.

10.2 Graphs of Translations

Here you will learn how to graph a translation given a description of the translation.

Triangle ABC has coordinates $A(1,1), B(8,1)$ and $C(5,8)$. Draw the triangle on the Cartesian plane. Translate the triangle up 4 units and over 2 units to the right. State the coordinates of the resulting image.

Graphs of Translations

In geometry, a transformation is an operation that moves, flips, or changes a shape (called the preimage) to create a new shape (called the image). A translation is a type of transformation that moves each point in a figure the same distance in the same direction. Translations are often referred to as slides. When you perform a translation on a shape, the coordinates of that shape will change:

- translating up means you will add the translated unit to the y coordinate of the (x,y) points in the preimage
- translating down means you will subtract the translated unit from the y coordinate of the (x,y) points in the preimage
- translating right means you will add the translated unit to the x coordinate of the (x,y) points in the preimage
- translating left means you will subtract the translated unit from the x coordinate of the (x,y) points in the preimage

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65248

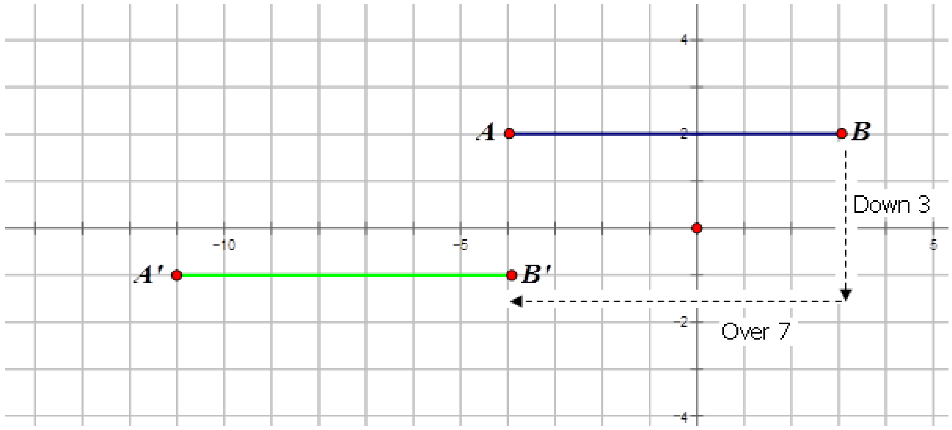
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65249

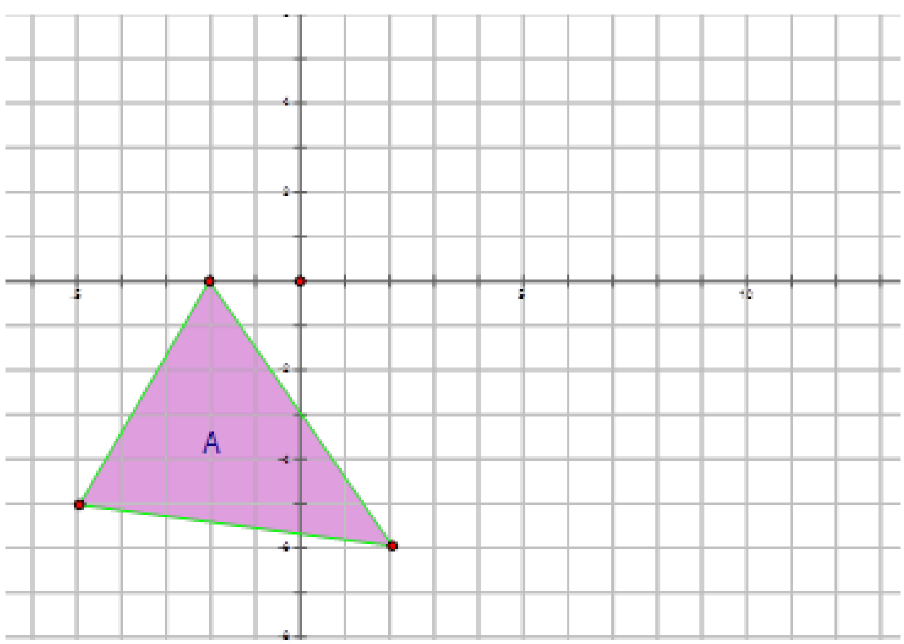
Graph the translation

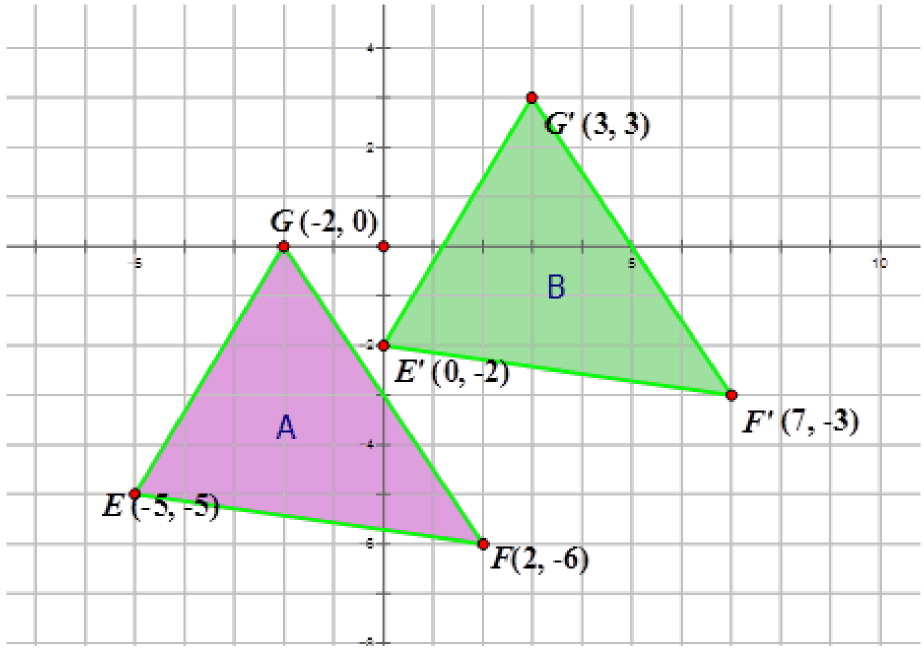
Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has been translated 3 units down and 7 units to the left. Draw the preimage and image and properly label each.



Graph the translation

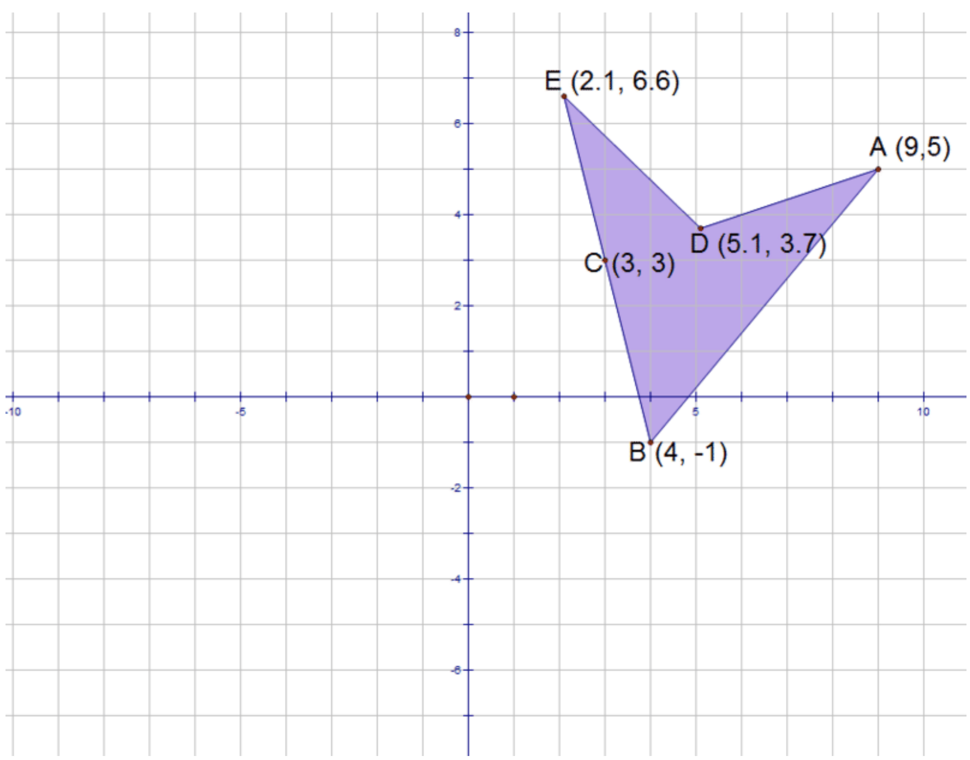
Triangle A is translated 3 units up and 5 units to the right to make triangle B. Find the coordinates of triangle B. On the diagram, draw and label triangle B.

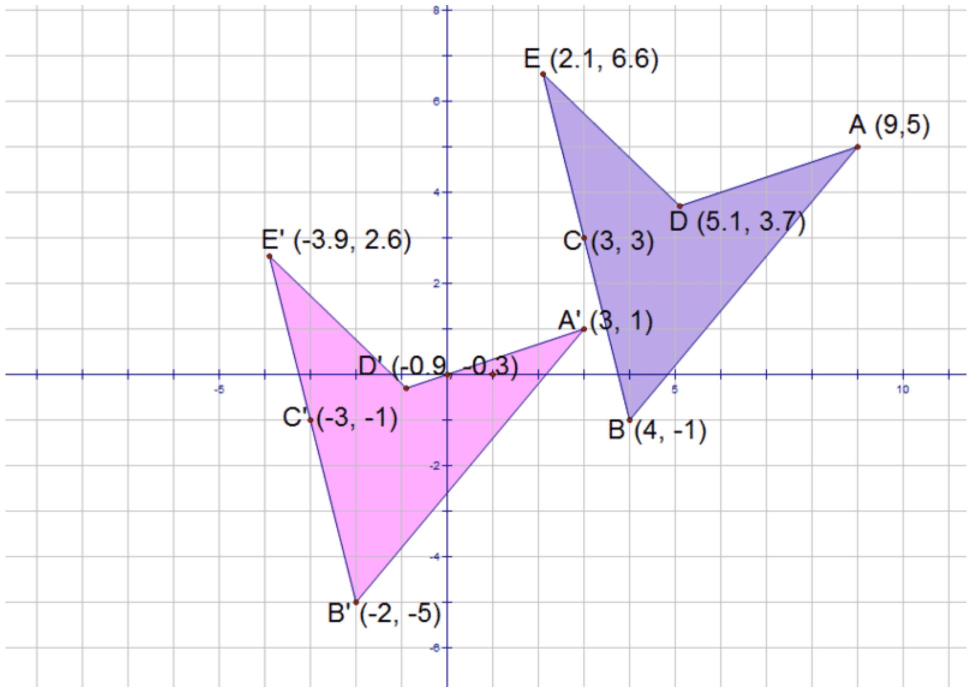




Graph the translation

The following figure is translated 4 units down and 6 units to the left to make a translated image. Find the coordinates of the translated image. On the diagram, draw and label the image.

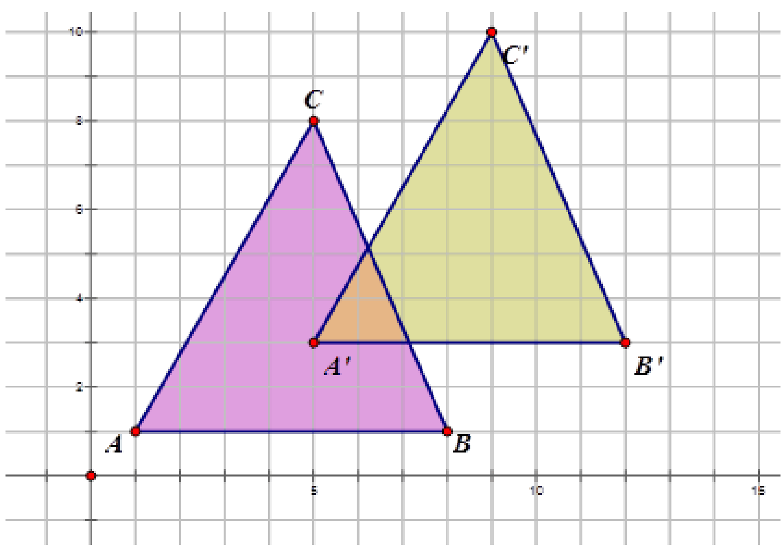




Examples

Example 1

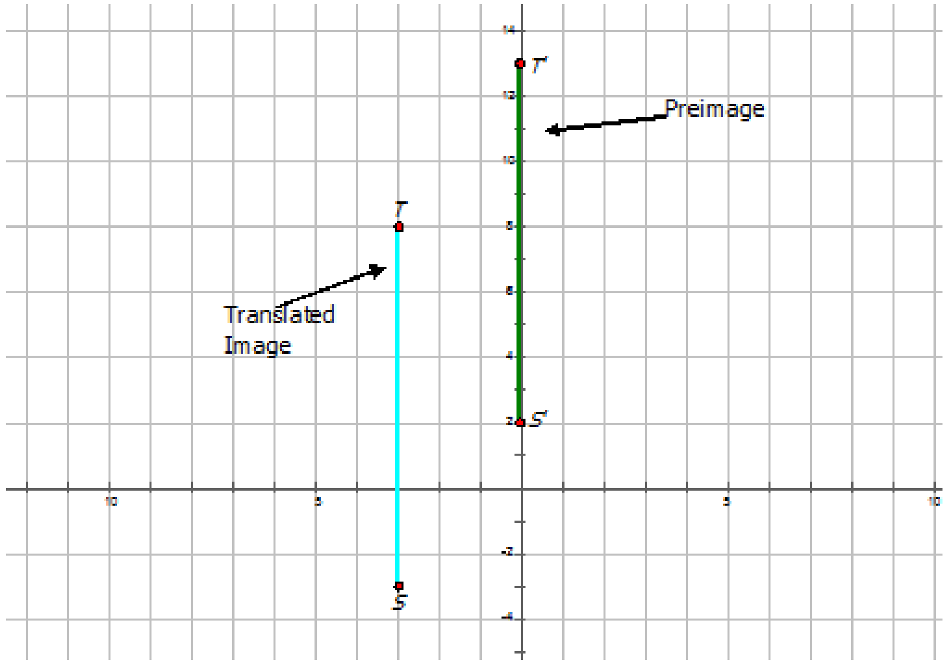
Earlier, you were asked to state the coordinates of the resulting image.



The coordinates of the new image are $A'(5,3), B'(12,3)$ and $C'(9,10)$.

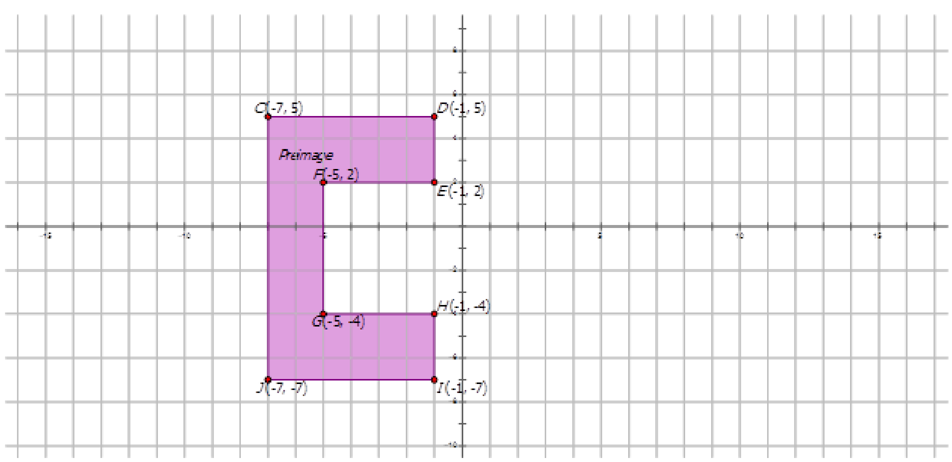
Example 2

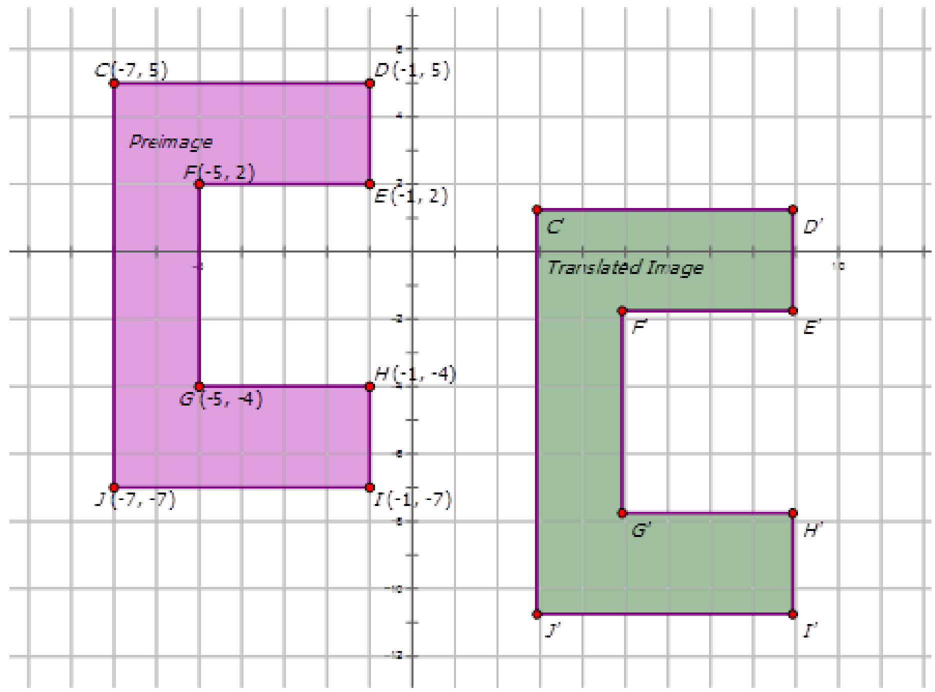
Line $\overline{ST}$ drawn from (-3, -3) to (-3, 8) has been translated 4 units up and 3 units to the right. Draw the preimage and image and properly label each.



Example 3

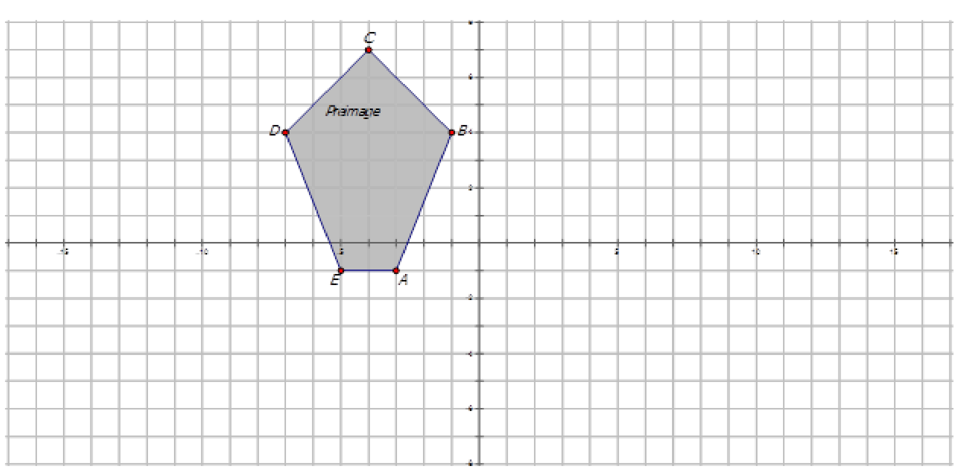
The polygon below has been translated 3 units down and 10 units to the right. Draw the translated image and properly label each.

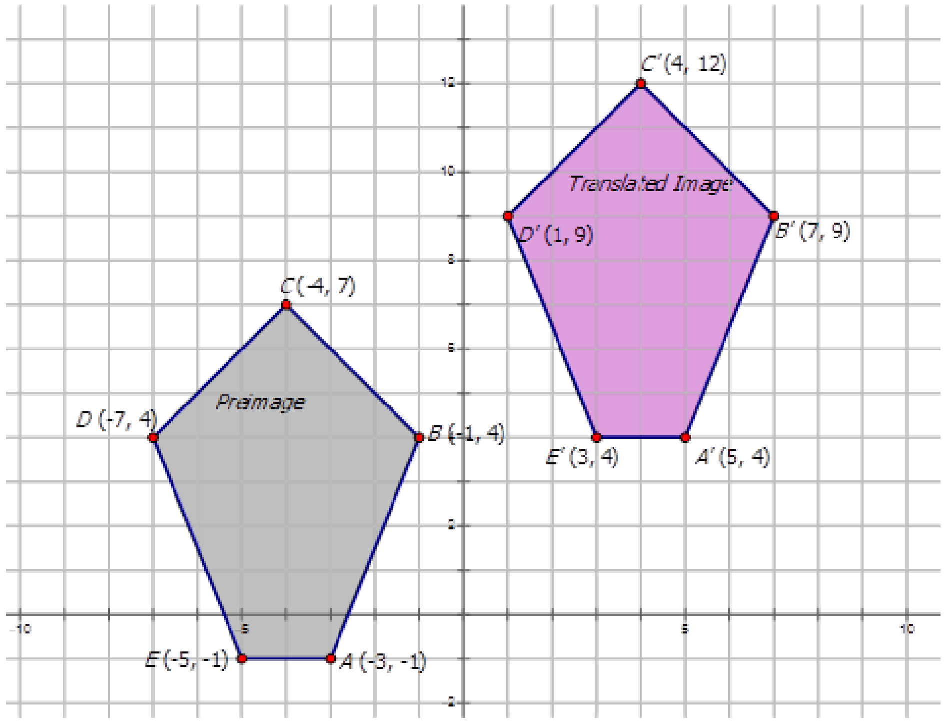




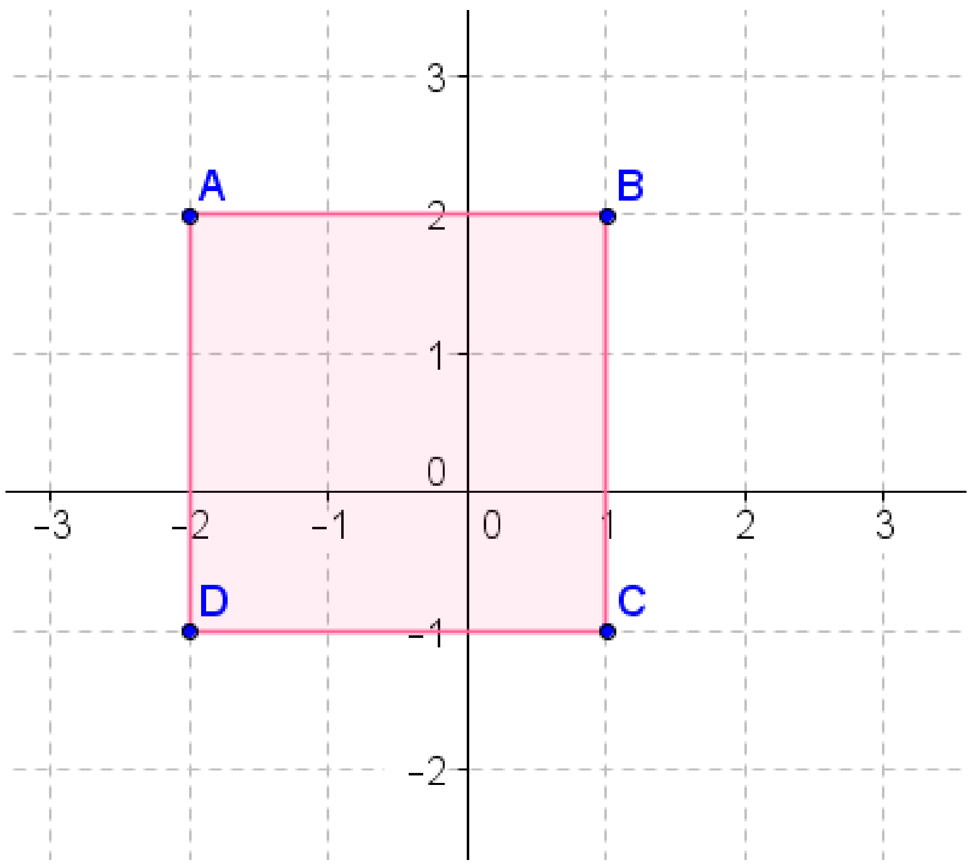
Example 4

The purple pentagon is translated 5 units up and 8 units to the right to make the translated pentagon. Find the coordinates of the purple pentagon. On the diagram, draw and label the translated pentagon.

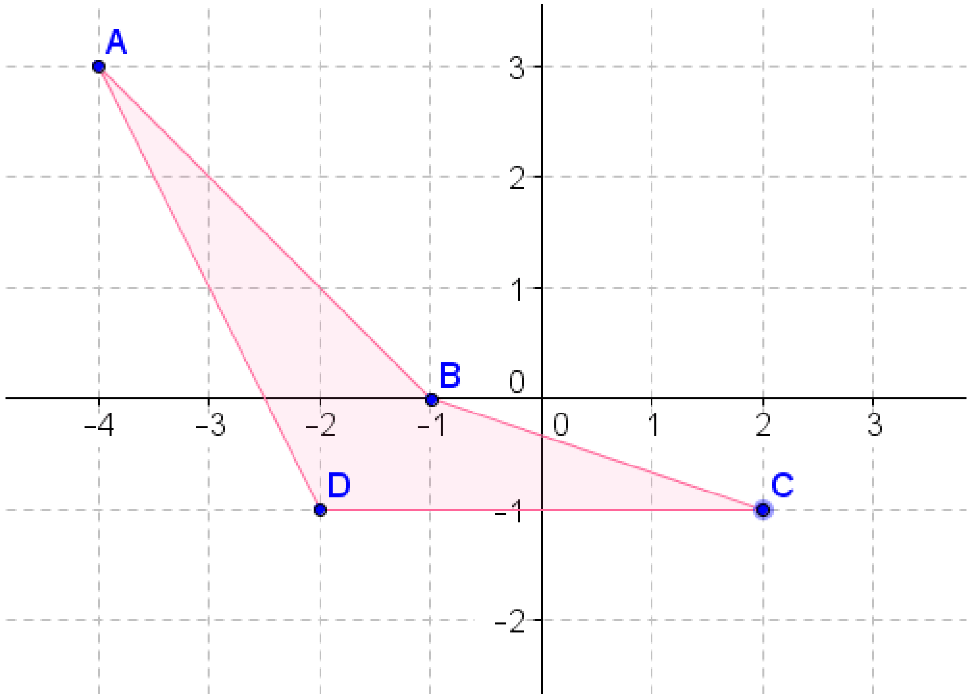




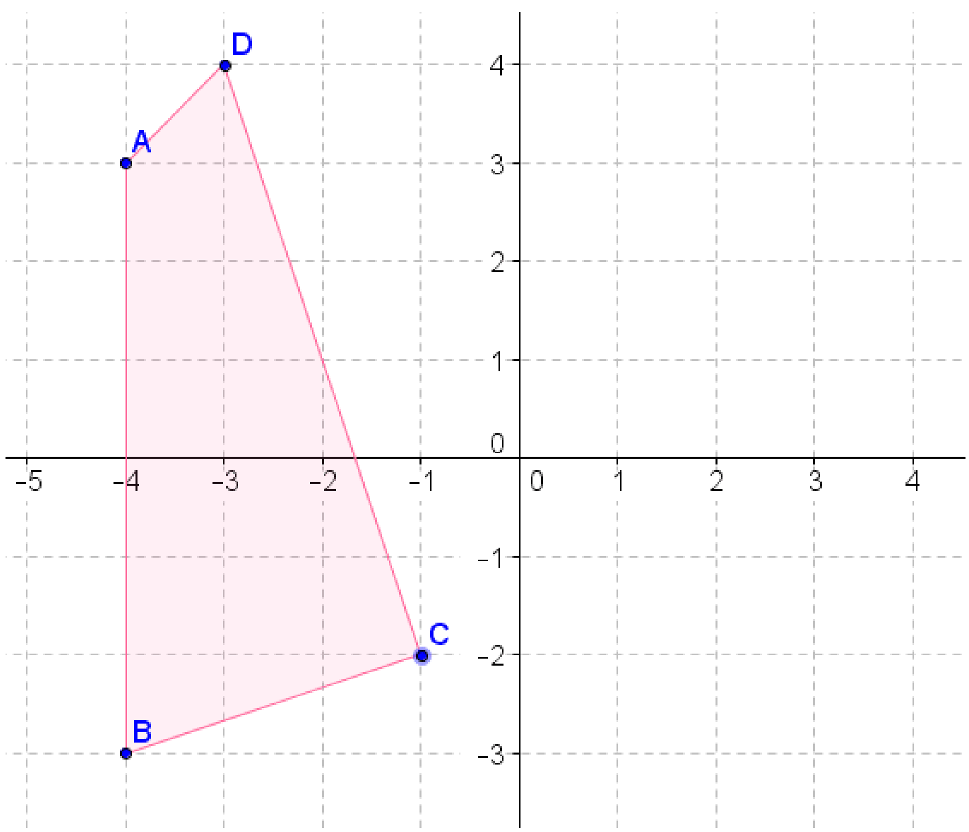
Review



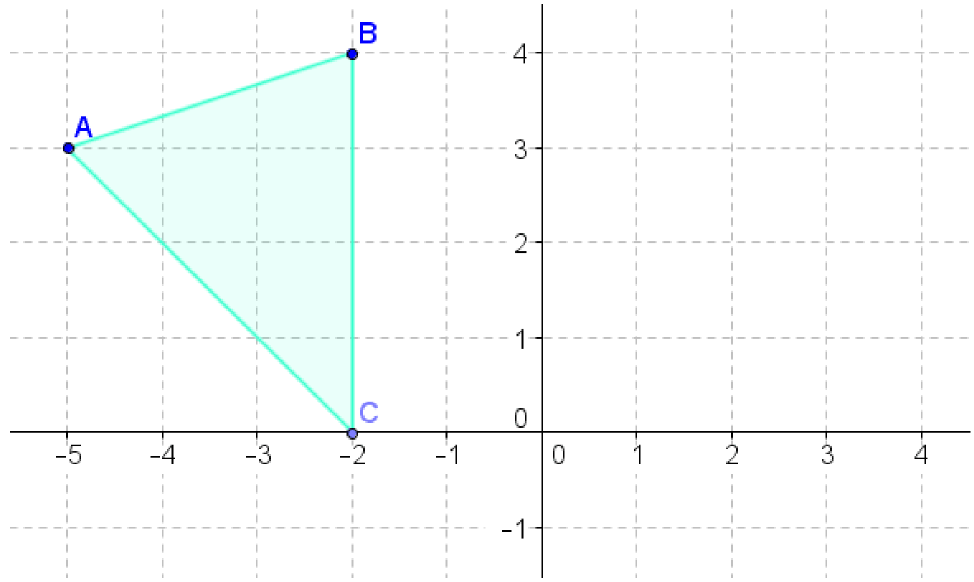
1. Translate the above figure 3 units to the right and 4 units down.
2. Translate the above figure 2 units to the left and 2 units up.



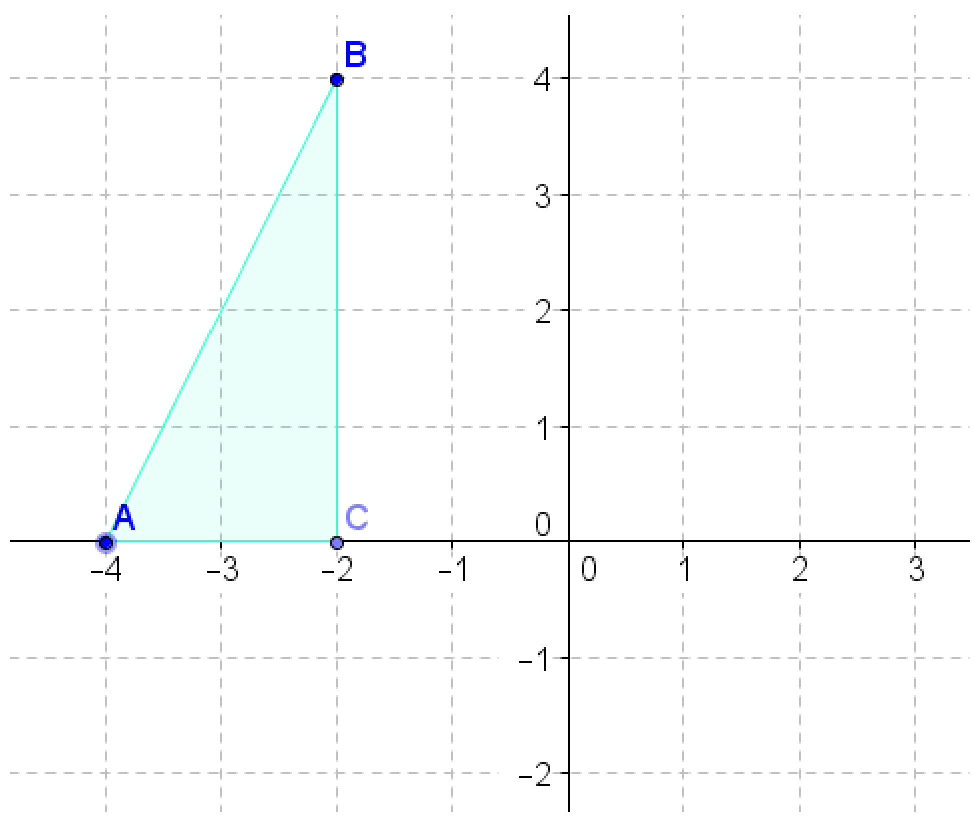
3. Translate the above figure 3 units to the right and 2 units down.
4. Translate the above figure 5 units to the left and 1 unit up.



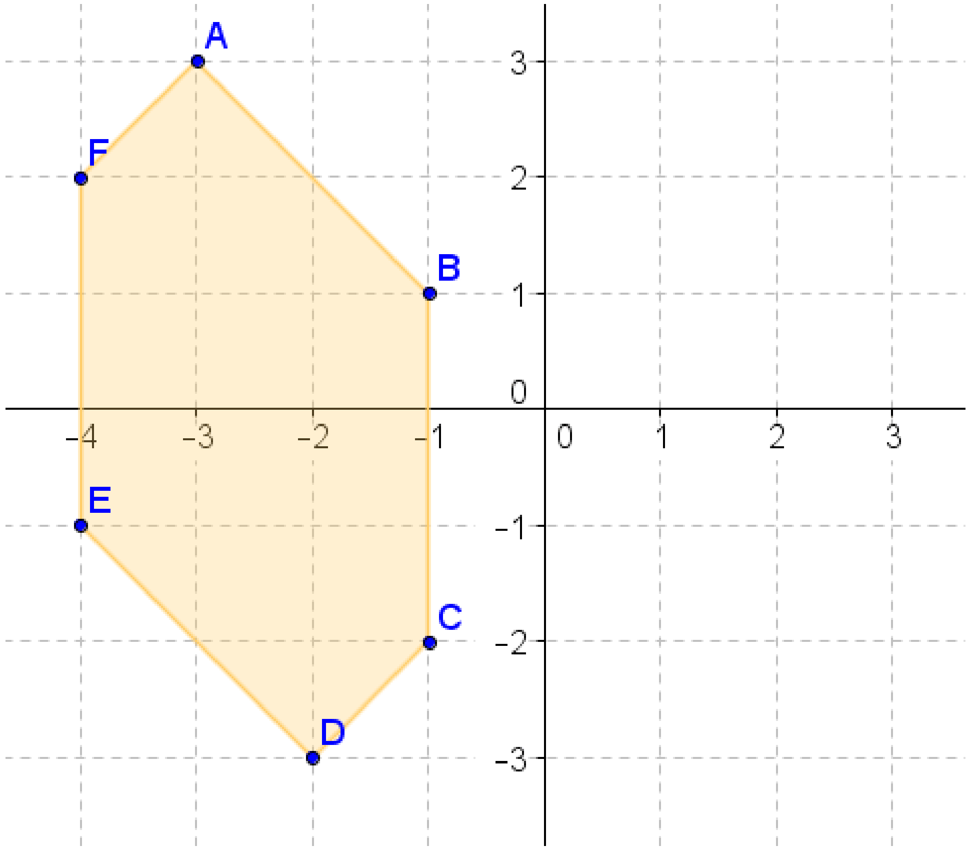
5. Translate the above figure 2 units to the right and 3 units down.
6. Translate the above figure 4 units to the left and 1 unit up.



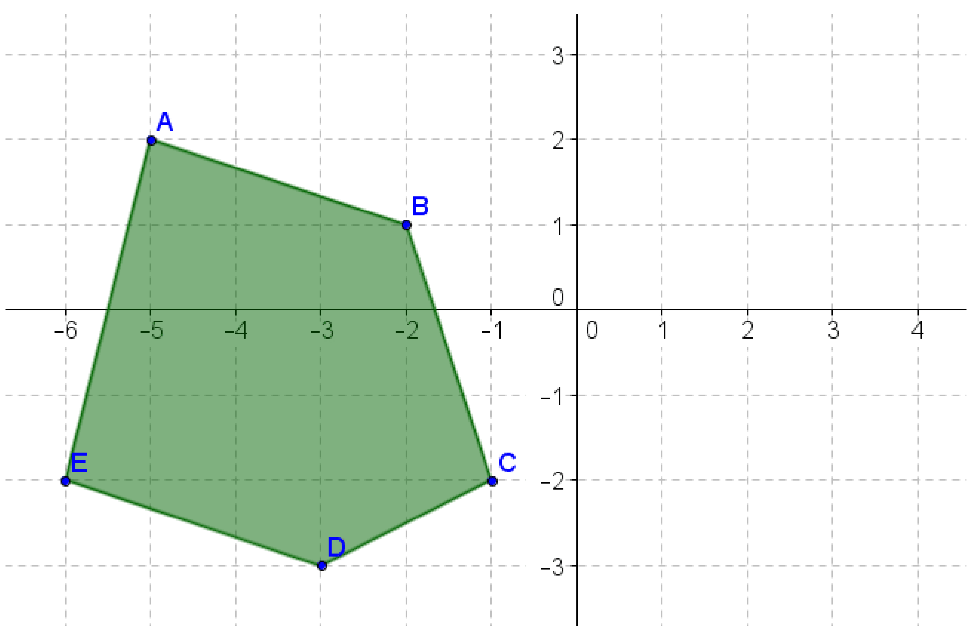
7. Translate the above figure 6 units to the right and 2 units down.
8. Translate the above figure 4 units to the left and 6 units up.



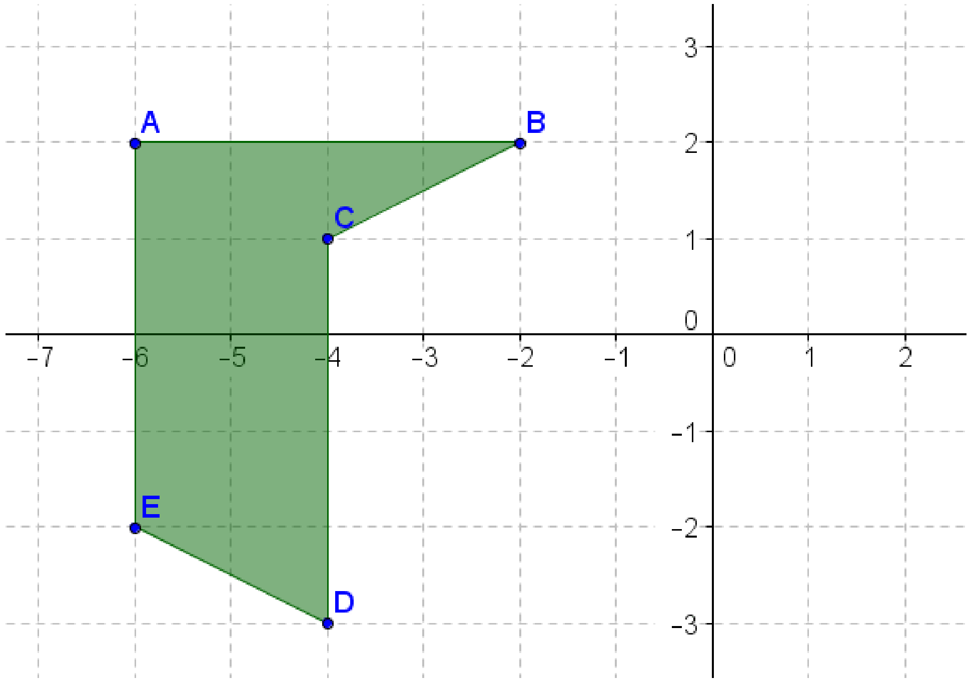
9. Translate the above figure 2 units to the right and 3 units down.
10. Translate the above figure 5 units to the left and 5 units up.



11. Translate the above figure 3 units to the right and 6 units down.
12. Translate the above figure 2 units to the left and 2 units up.



13. Translate the above figure 3 units to the right and 3 units down.
14. Translate the above figure 5 units to the left and 2 units up.



15. Translate the above figure 7 units to the right and 4 units down.
16. Translate the above figure 1 unit to the left and 2 units up.

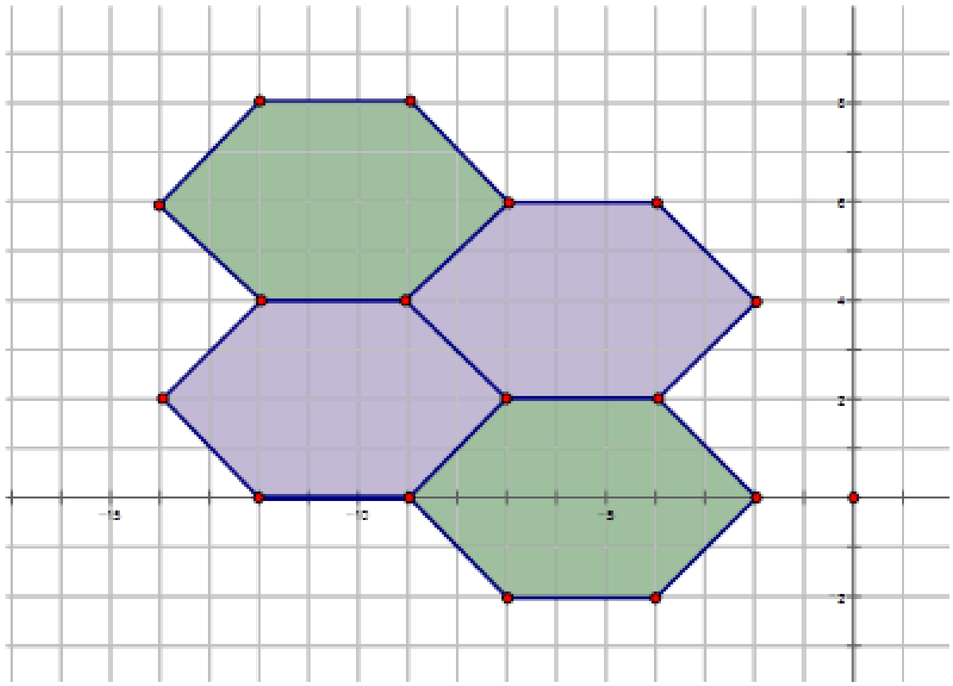
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.2.

10.3 Rules for Translations

Here you will learn the different notation used for translations.

The figure below shows a pattern of a floor tile. Write the mapping rule for the translation of the two blue floor tiles.



Rules for Translation

In geometry, a transformation is an operation that moves, flips, or changes a shape (called the preimage) to create a new shape (called the image). A translation is a type of transformation that moves each point in a figure the same distance in the same direction. Translations are often referred to as slides. You can describe a translation using words like "moved up 3 and over 5 to the left" or with notation. There are two types of notation to know.

1. One notation looks like $T_{(3,\ 5)}$. This notation tells you to add 3 to the x values and add 5 to the y values.
2. The second notation is a mapping rule of the form $(x,y) \rightarrow (x-7, y+5)$. This notation tells you that the x and y coordinates are translated to $x-7$ and $y+5$.

The mapping rule notation is the most common.

Multimedia

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65236

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65237

Write the mapping rule

Sarah describes a translation as point P moving from $P(-2,2)$ to $P'(1,-1)$. Write the mapping rule to describe this translation for Sarah.

In general ,$P(x,y) \rightarrow P'(x+a, y+b)$.

In this case, $P(-2,2) \rightarrow P'(-2+a, 2+b)$ or $P(-2,2) \rightarrow P'(1,-1)$

Therefore:

$$\begin{aligned} -2+a &= 1 \qquad \textit{and} \qquad & 2+b &= -1 \\ a &= 3 & b &= -3 \end{aligned}$$

The rule is:

$$(x,y) \rightarrow (x+3, y-3)$$

Write the mapping rule

Mikah describes a translation as point D in a diagram moving from $D(1,-5)$ to $D'(-3,1)$. Write the mapping rule to describe this translation for Mikah.

In general ,$P(x,y) \rightarrow P'(x+a, y+b)$.

In this case, $D(1,-5) \rightarrow D'(1+a, -5+b)$ or $D(1,-5) \rightarrow D'(-3,1)$

Therefore:

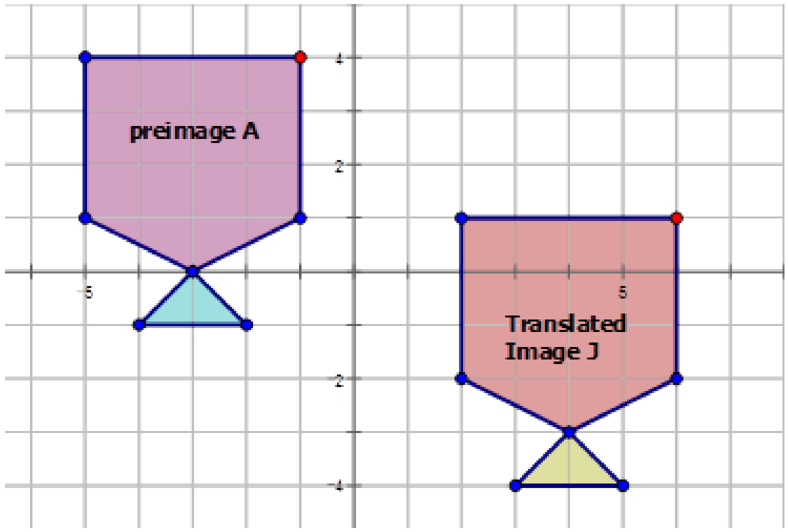
$$\begin{aligned} 1+a &= -3 \qquad \textit{and} \qquad & -5+b &= 1 \\ a &= -4 & b &= 6 \end{aligned}$$

The rule is:

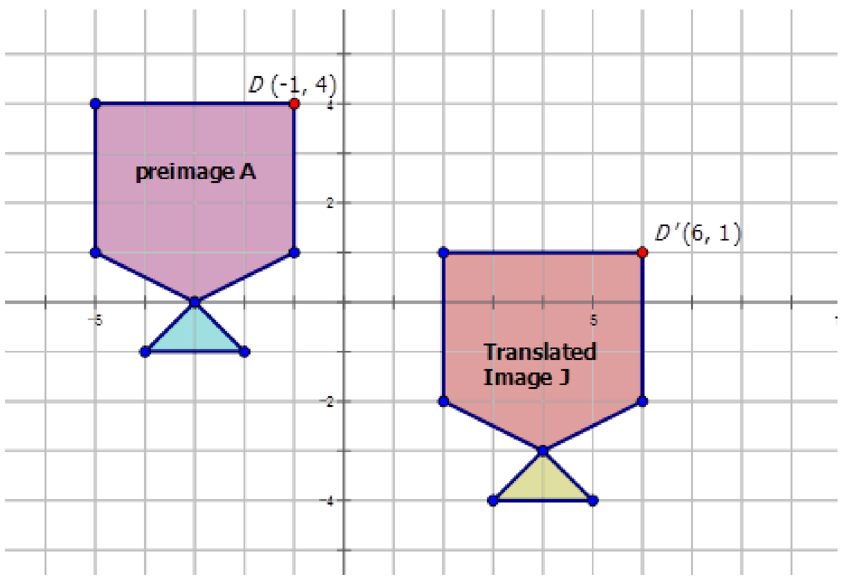
$$(x,y) \rightarrow (x-4, y+6)$$

Write the mapping rule

Write the mapping rule that represents the translation of the preimage A to the translated image J in the diagram below.



First, pick a point in the diagram to use to see how it is translated.



$D:(-1,4)\ D':(6,1)$

$D(x,y) \rightarrow D'(x+a,y+b)$

So: $D(-1,4) \rightarrow D'(-1+a,4+b)$ or $D(-1,4) \rightarrow D'(6,1)$

Therefore:

$$\begin{aligned} -1+a &= 6 \qquad \textit{and} \qquad & 4+b &= 1 \\ a &= 7 & b &= -3 \end{aligned}$$

The rule is:

$$(x,y) \rightarrow (x+7,y-3)$$

Examples

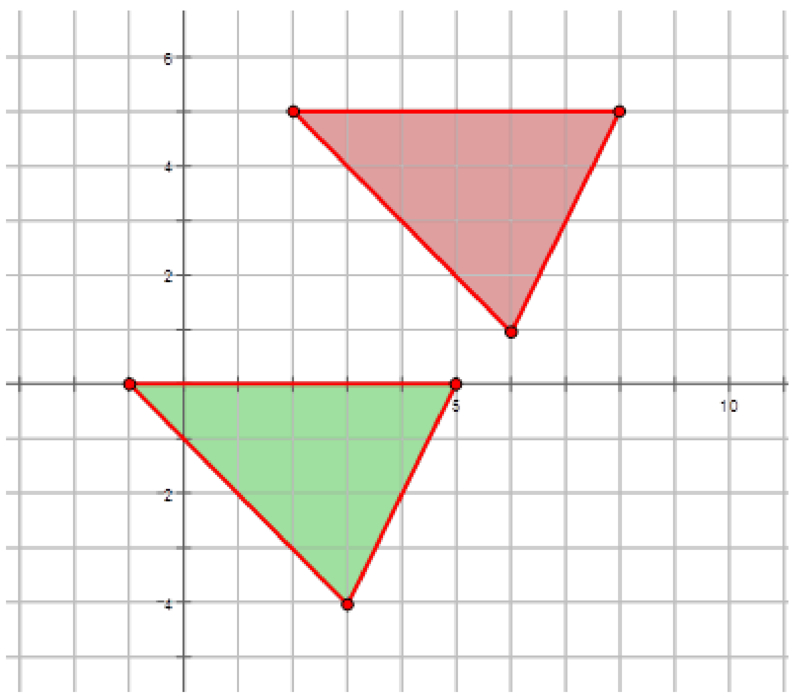
Example 1

Jack describes a translation as point J moving from $J(-2,6)$ to $J'(4,9)$. Write the mapping rule to describe this translation for Jack.

$$(x,y) \rightarrow (x+6,y+3)$$

Example 2

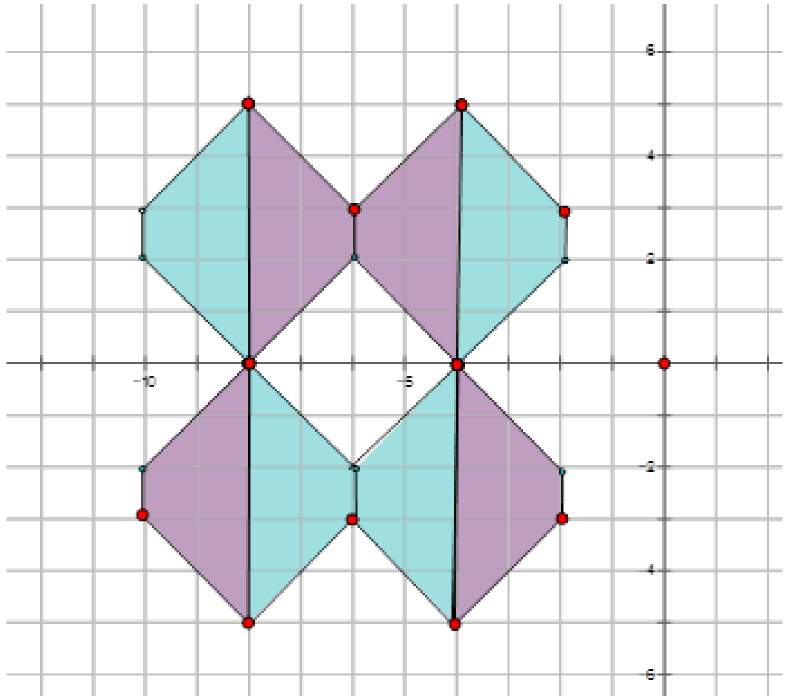
Write the mapping rule that represents the translation of the red triangle to the translated green triangle in the diagram below.



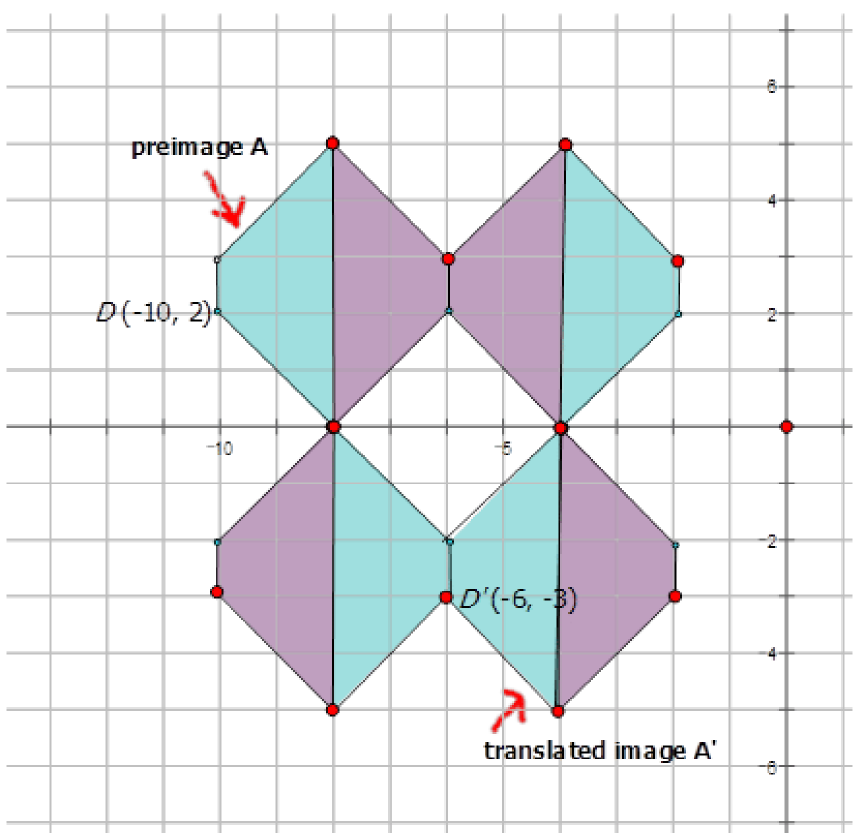
$$(x,y) \rightarrow (x-3,y-5)$$

Example 3

The following pattern is part of wallpaper found in a hotel lobby. Write the mapping rule that represents the translation of one blue trapezoid to a translated blue trapezoid shown in the diagram below.



If you look closely at the diagram below, there two pairs of trapezoids that are translations of each other. Therefore you can choose one blue trapezoid that is a translation of the other and pick a point to find out how much the shape has moved to get to the translated position.



For those two trapezoids:

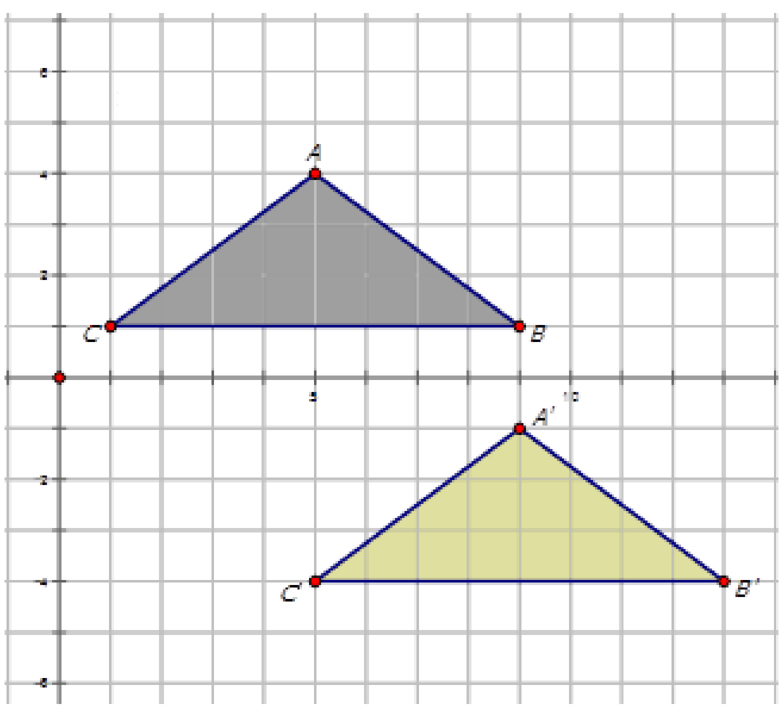
$$(x,y) \rightarrow (x+4,y-5)$$

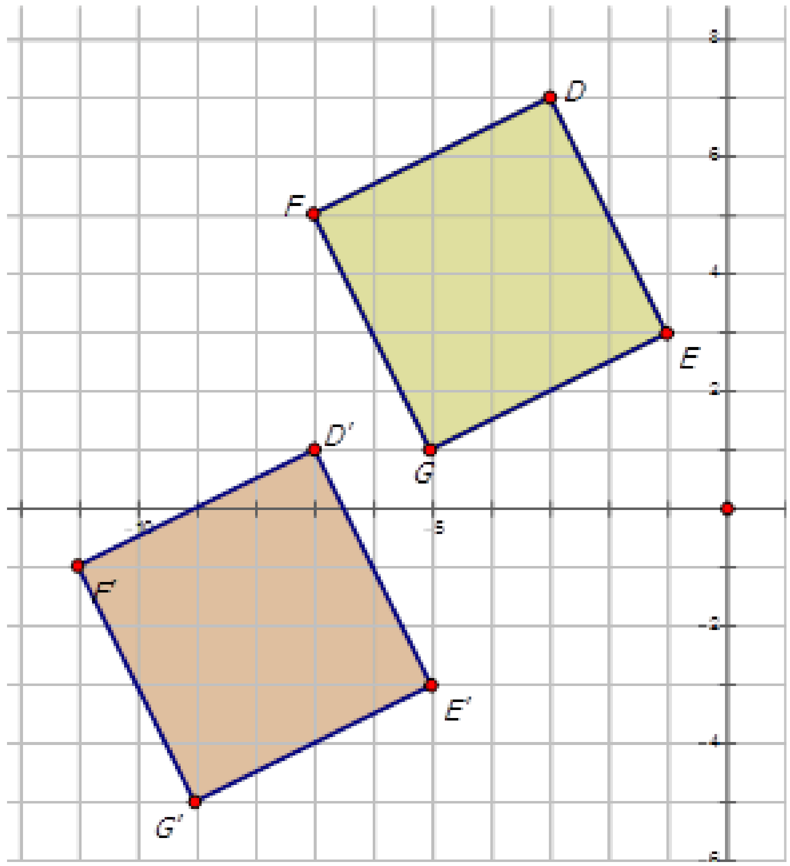
Review

Write the mapping rule to describe the movement of the points in each of the translations below.

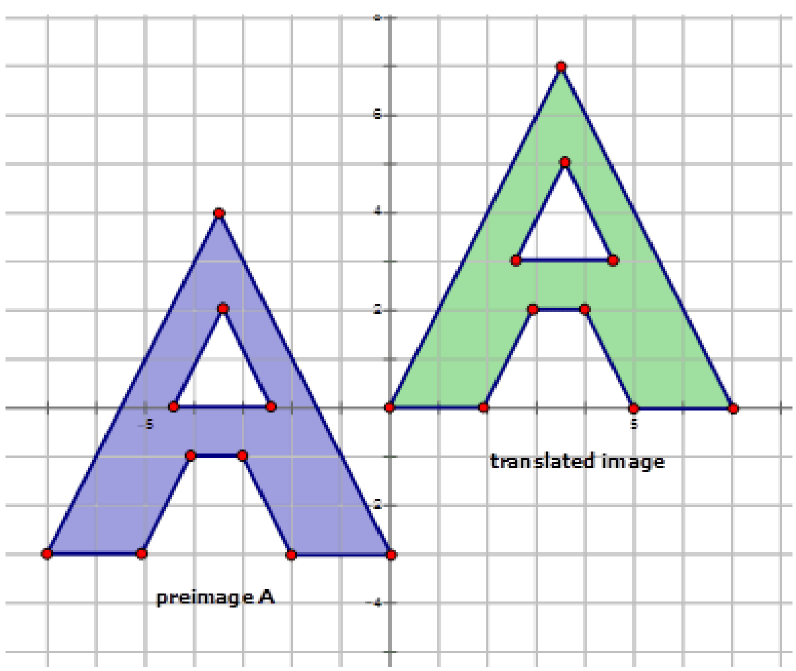
1. $S(1,5) \rightarrow S'(2,7)$
2. $W(-5,-1) \rightarrow W'(-3,1)$
3. $Q(2,-5) \rightarrow Q'(-6,3)$
4. $M(4,3) \rightarrow M'(-2,9)$
5. $B(-4,-2) \rightarrow B'(2,-2)$
6. $A(2,4) \rightarrow A'(2,6)$
7. $C(-5,-3) \rightarrow C'(-3,4)$
8. $D(4,-1) \rightarrow D'(-4,2)$
9. $Z(7,2) \rightarrow Z'(-3,6)$
10. $L(-3,-2) \rightarrow L'(4,-1)$

Write the mapping rule that represents the translation of the preimage to the image for each diagram below.

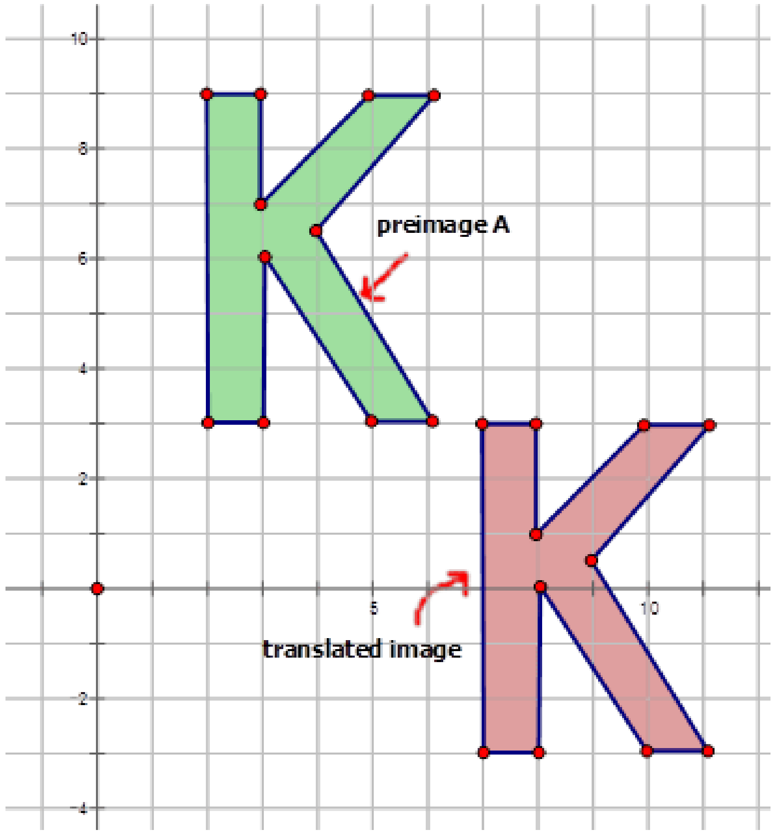
11. 

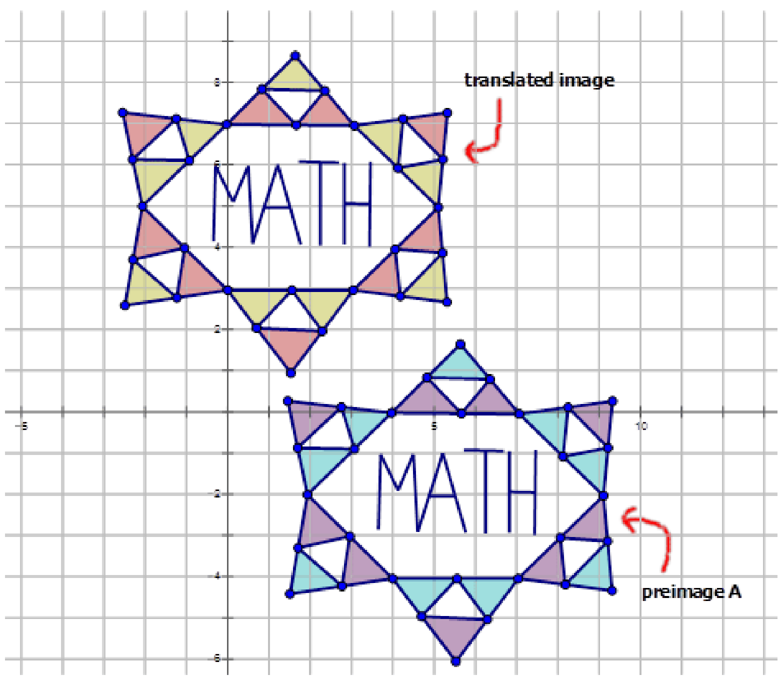


12.



13.

14. 

15. 

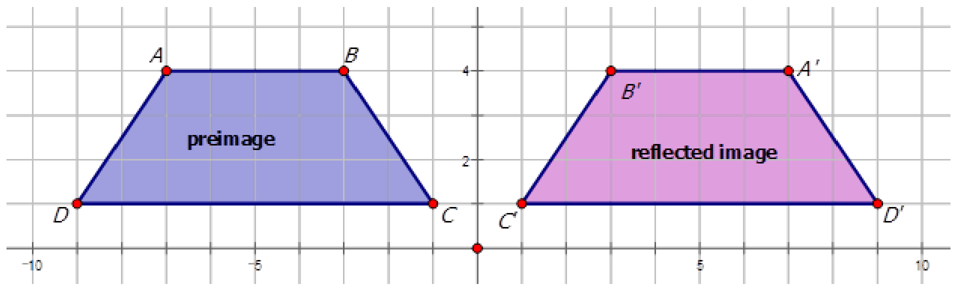
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.3.

10.4 Reflections

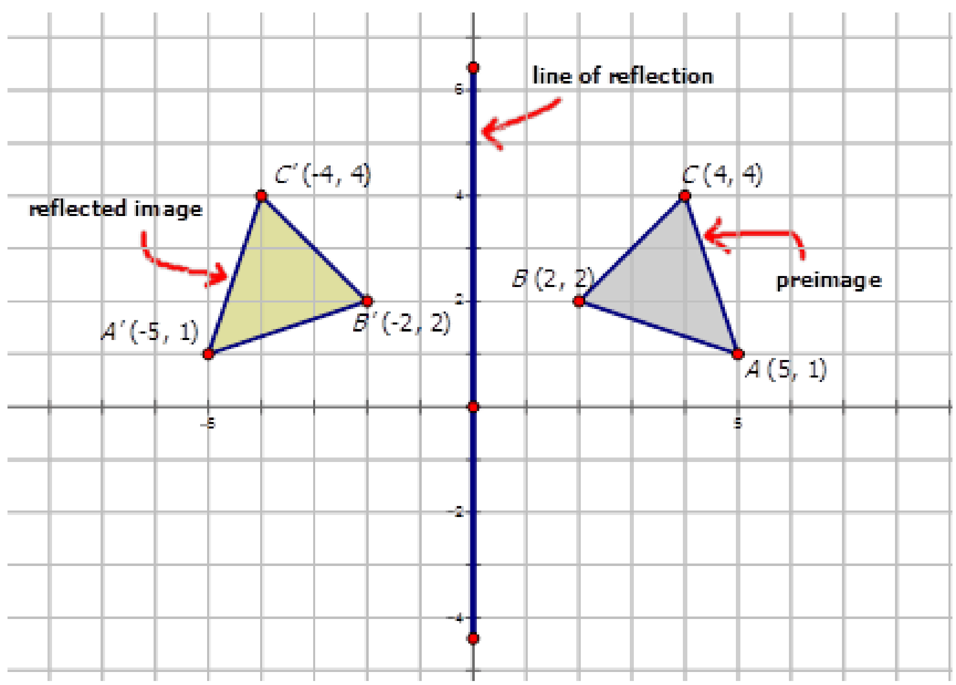
Here you'll learn about geometric reflections.

Scott looked at the image below and stated that the image was reflected about the y-axis. Is he correct? Explain.



Reflection

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A reflection is an example of a transformation that takes a shape (called the preimage) and flips it across a line (called the line of reflection) to create a new shape (called the image).



You can reflect a shape across any line, but the most common reflections are the following:

- reflections across the x-axis: y values are multiplied by -1.
- reflections across the y-axis: x values are multiplied by -1.
- reflections across the line $y = x$: x and y values switch places.
- reflections across the line $y = -x$. x and y values switch places and are multiplied by -1.

MEDIA

Click image to the left or use the URL below.

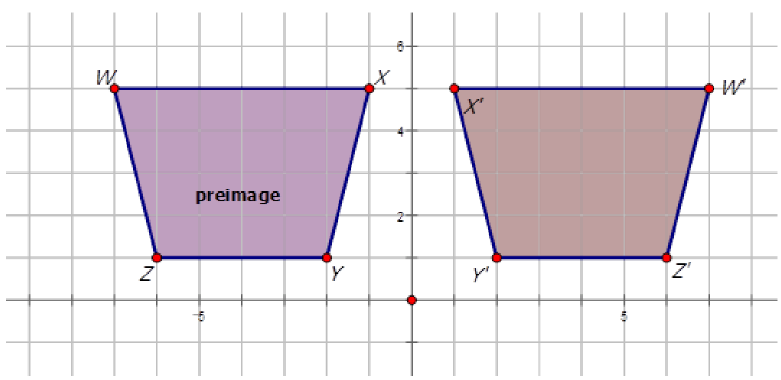
URL: https://www.ck12.org/flx/render/embeddedobject/65240

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65241

Describe the reflection shown in the diagram below.



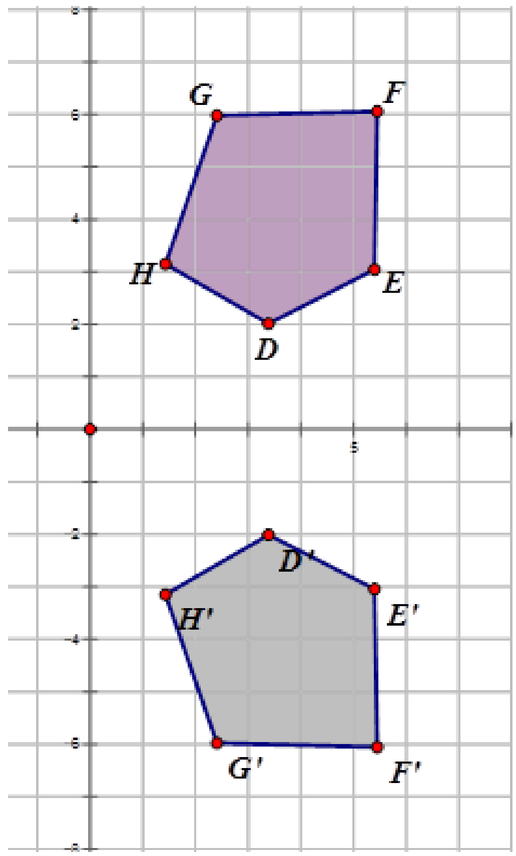
The shape is reflected across the y-axis. Let's examine the points of the shapes.

TABLE 10.1:

Points on $WXYZ$	$W(-7,5)$	$X(-1,5)$	$Y(-2,1)$	$Z(-6,1)$
Points on $W'X'Y'Z'$	$W'(7,8)$	$X'(1,5)$	$Y'(2,1)$	$Z'(6,1)$

In the table above, all of the x-coordinates are multiplied by -1. Whenever a shape is reflected across the y-axis, it's x-coordinates will be multiplied by -1.

Describe the reflection of the purple pentagon in the diagram below.



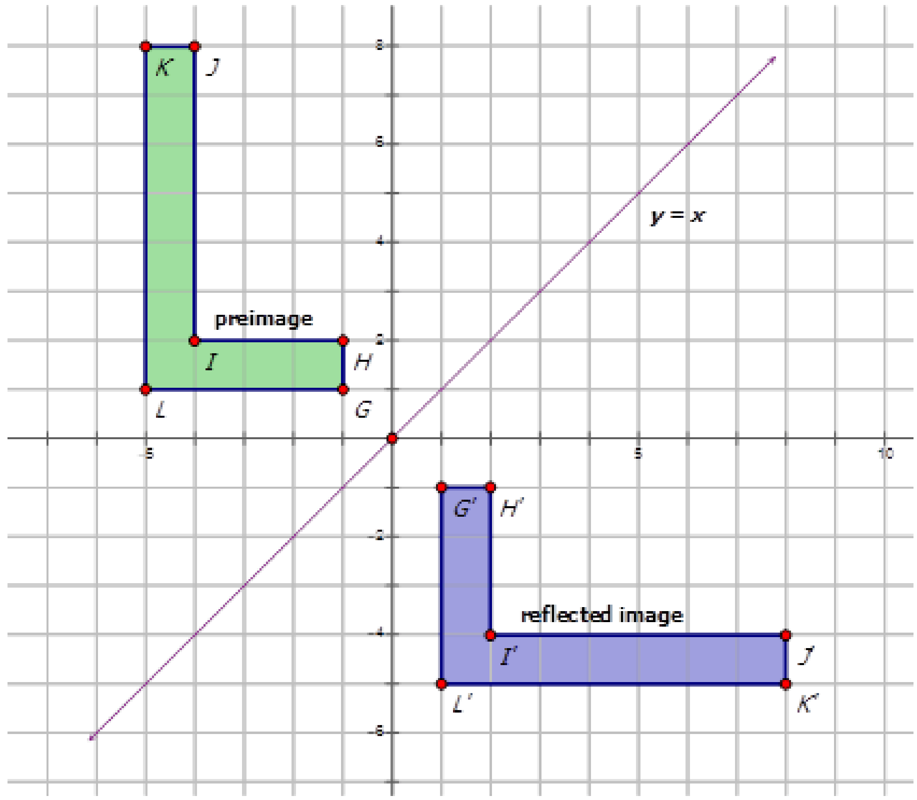
The pentagon is reflected across the x-axis. Let's examine the points of the pentagon.

TABLE 10.2:

Points on $DEFGH$	$D(3.5,2)$	$E(5.4,3)$	$F(5.5,6)$	$G(2.3,6)$	$H(1.4,3.2)$
Points on $D'E'F'G'H'$	$D'(3.5,-2)$	$E'(5.4,-3)$	$F'(5.5,-6)$	$G'(2.3,-6)$	$H'(1.4,-3.2)$

In the table above, all of the x-coordinates are the same but the y-coordinates are multiplied by -1. This is what will happen anytime a shape is reflected across the x-axis.

Describe the reflection in the diagram below.



The shape is reflected across the line $y = x$. Let's examine the points of the preimage and the reflected image.

TABLE 10.3:

Points on $GHIJKL$	$G(-1,1)$	$H(-1,2)$	$I(-4,2)$	$J(-4,8)$	$K(-5,8)$	$L(-5,1)$
Points on $G'H'I'J'K'L'$	$G'(1,-1)$	$H'(2,-1)$	$I'(2,-4)$	$J'(8,-4)$	$K'(8,-5)$	$L'(1,-5)$

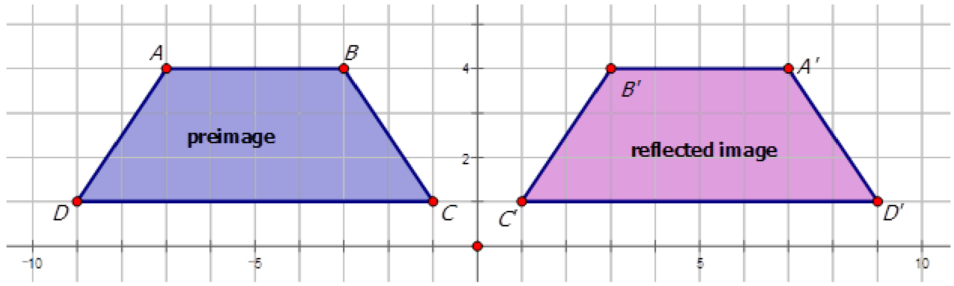
Notice that all of the points on the preimage reverse order (or interchange) to form the corresponding points on the reflected image. So for example the point G on the preimage is at (-1, 1) but the corresponding point G' on the reflected image is at (1, -1). The x values and the y values change places anytime a shape is reflected across the line $y = x$.

Examples

Example 1

Earlier, you were given a problem about Scott.

Scott looked at the image below and stated that the image was reflected across the y-axis. Is he correct? Explain.



Scott is correct in that the preimage is reflected about the y-axis to form the translated image. You can tell this because all points are equidistant from the line of reflection. Let's examine the points of the trapezoid and see.

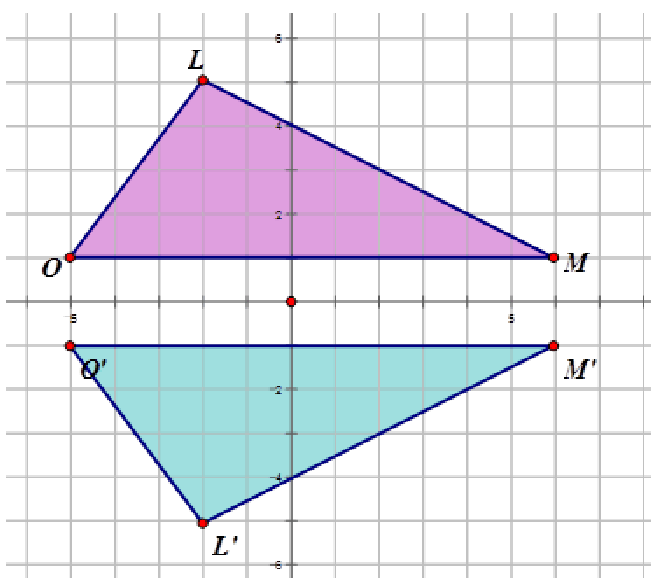
TABLE 10.4:

Point for $ABCD$	**Point for** $A'B'C'D'$
$A(-7,4)$	$A'(7,4)$
$B(-3,4)$	$B'(3,4)$
$C(-1,1)$	$C'(1,1)$
$D(-9,1)$	$D'(9,1)$

All of the y-coordinates for the reflected image are the same as their corresponding points in the preimage. However, the x-coordinates have been multiplied by -1.

Example 2

Describe the reflection of the pink triangle in the diagram below.



Examine the points of the preimage and the reflected image.

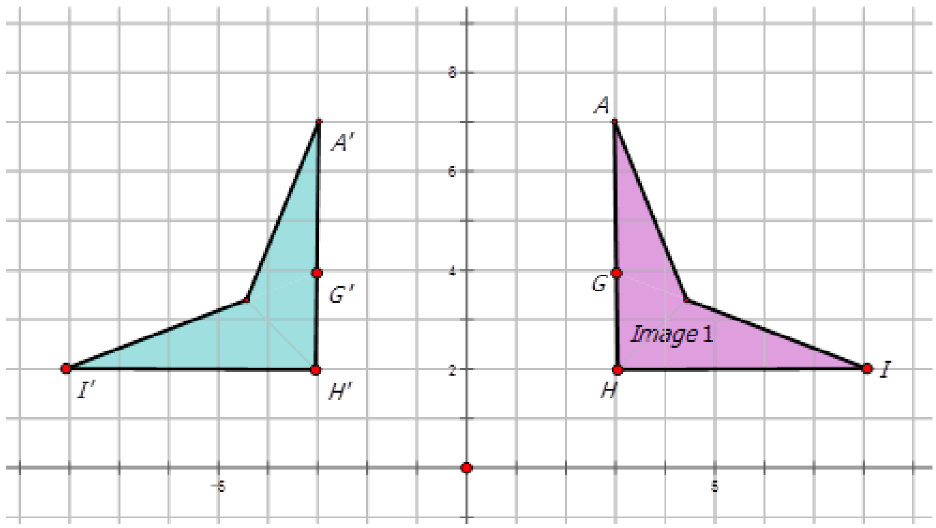
TABLE 10.5:

Points on LMO	$L(-2,5)$	$M(6,1)$	$O(-5,1)$
Points on $L'M'O'$	$L'(-2,-5)$	$M'(6,-1)$	$O'(-5,-1)$

Notice that all of the y-coordinates of the preimage (purple triangle) are multiplied by -1 to make the reflected image. The line of reflection is the x-axis.

Example 3

Describe the reflection of the purple polygon in the diagram below.



Examine the points of the preimage and the reflected image.

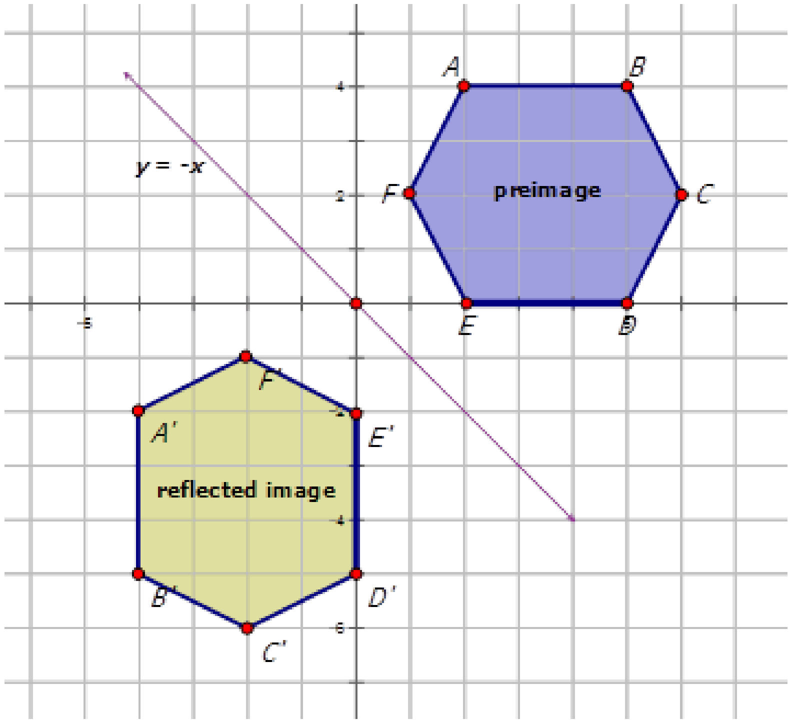
TABLE 10.6:

Points on $AGHI$	$A(3,7)$	$G(3,4)$	$H(3,2)$	$I(8,2)$
Points on $A'G'H'I'$	$A'(-3,7)$	$G'(-3,4)$	$H'(-3,2)$	$I'(-8,2)$

Notice that all of the x-coordinates of the preimage (image 1) is multiplied by -1 to make the reflected image. The line of reflection is the y-axis.

Example 4

Describe the reflection of the blue hexagon in the diagram below.



Examine the points of the preimage and the reflected image.

TABLE 10.7:

Points on $ABCDEF$	$A(2,4)$	$B(5,4)$	$C(6,2)$	$D(5,0)$	$E(2,0)$	$F(1,2)$
Points on $A'B'C'D'E'F'$	$A'(-4,-2)$	$B'(-4,-5)$	$C'(-2,-6)$	$D'(0,-5)$	$E'(0,-2)$	$F'(-2,-1)$

Notice that both the x-coordinates and the y-coordinates of the preimage (image 1) change places to form the reflected image. As well the points are multiplied by -1. The line of reflection is the line $y = -x$.

Review

If the following points were reflected across the x-axis, what would be the coordinates of the reflected points? Show these reflections on a graph.

1. (3, 1)
2. (4, -2)
3. (-5, 3)
4. (-6, 4)

If the following points were reflected across the y-axis, what would be the coordinates of the reflected points? Show these reflections on a graph.

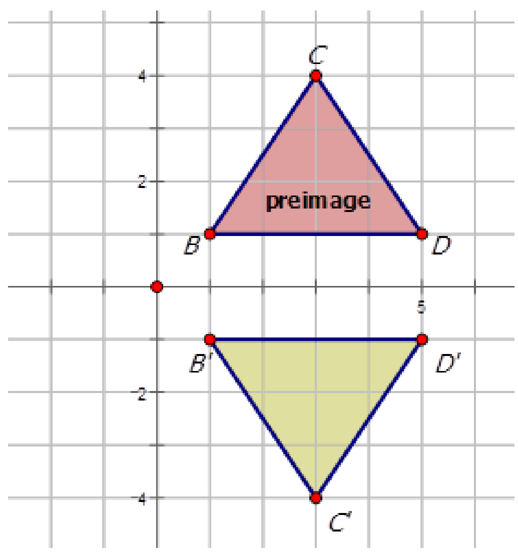
5. (-4, 3)
6. (5, -4)
7. (-5, -4)

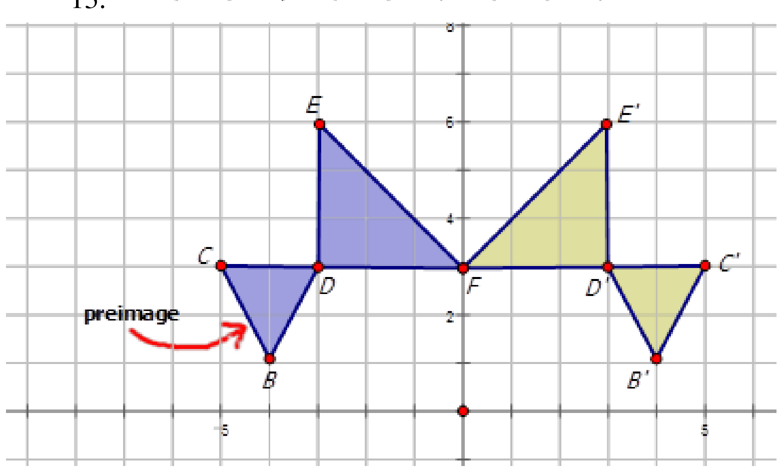
8. (3, 3)

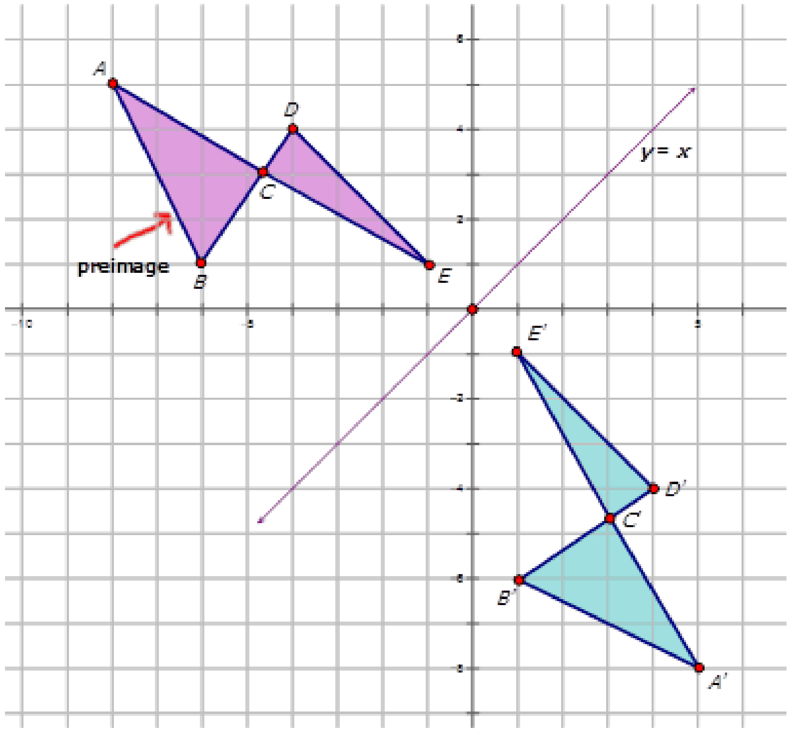
If the following points were reflected about the line $y = x$, what would be the coordinates of the reflected points? Show these reflections on a graph.

9. (3, 1)
10. (4, -2)
11. (-5, 3)
12. (-6, 4)

Describe the following reflections:

13. 

14. 

15. 

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.4.

10.5 Graphs of Reflections

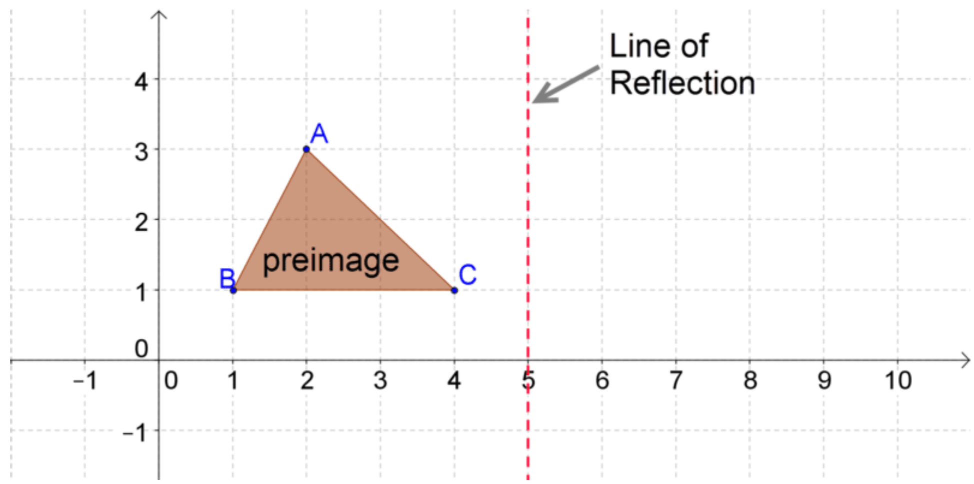
Here you will learn how to reflect an image on a coordinate grid.

Triangle A has coordinates $E(-5,-5)$, $F(2,-6)$ and $G(-2,0)$. Draw the triangle on the Cartesian plane. Reflect the image across the y-axis. State the coordinates of the resulting image.

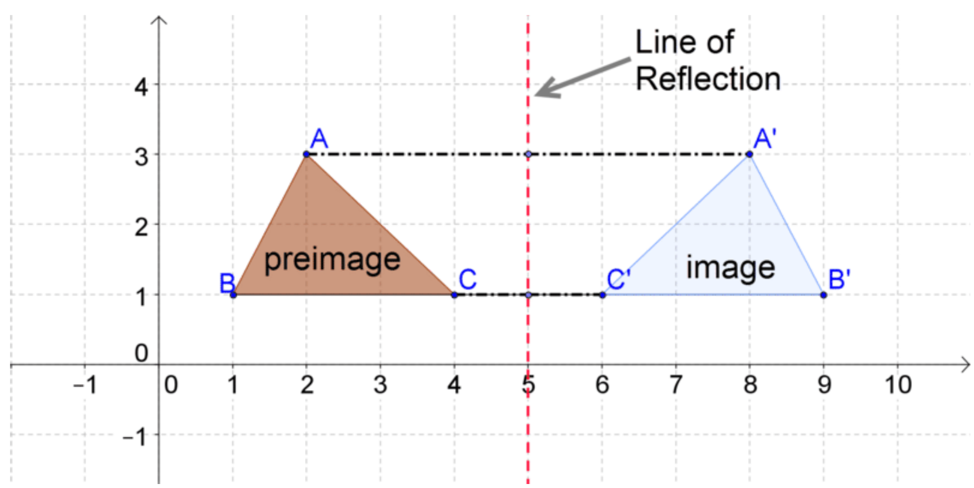
Graphs of Reflections

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A reflection is an example of a transformation that takes a shape (called the preimage) and flips it across a line (called the line of reflection) to create a new shape (called the image).

To graph a reflection, you can visualize what would happen if you flipped the shape across the line.



Each point on the preimage will be the same distance from the line of reflection as it's corresponding point in the image. For example, for the pair of triangles below, both A and A' are 3 units away from the line of reflection.



For common reflections, you can also remember what happens to their coordinates:

- reflections across the x-axis: y values are multiplied by -1.
- reflections across the y-axis: x values are multiplied by -1.
- reflections across the line $y = x$: x and y values switch places.
- reflections across the line $y = -x$. x and y values switch places and are multiplied by -1.

Knowing the rules above will allow you to recognize reflections even when a graph is not available.

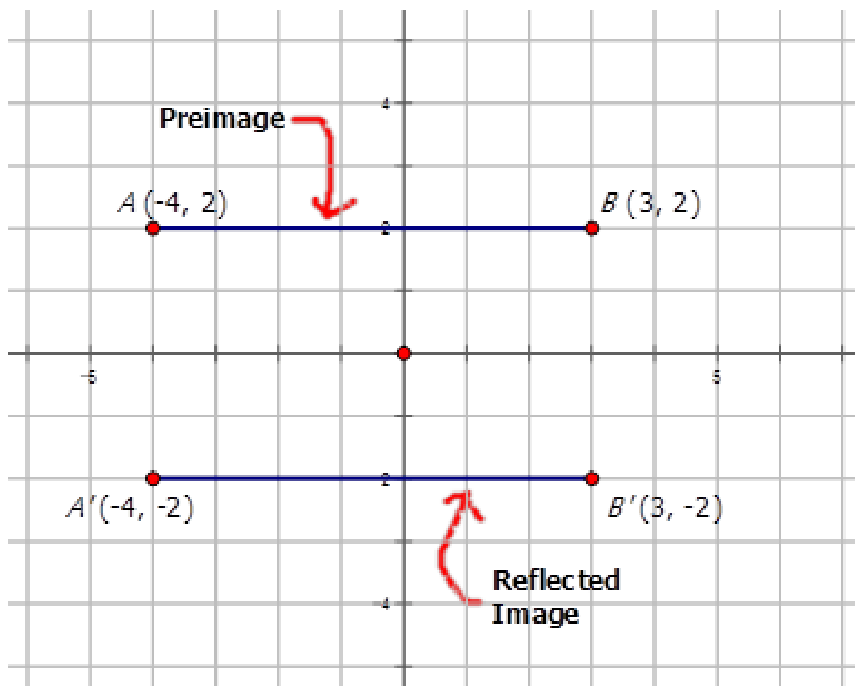
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65256

Draw the preimage and image and properly label each.

Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has been reflected across the x-axis.



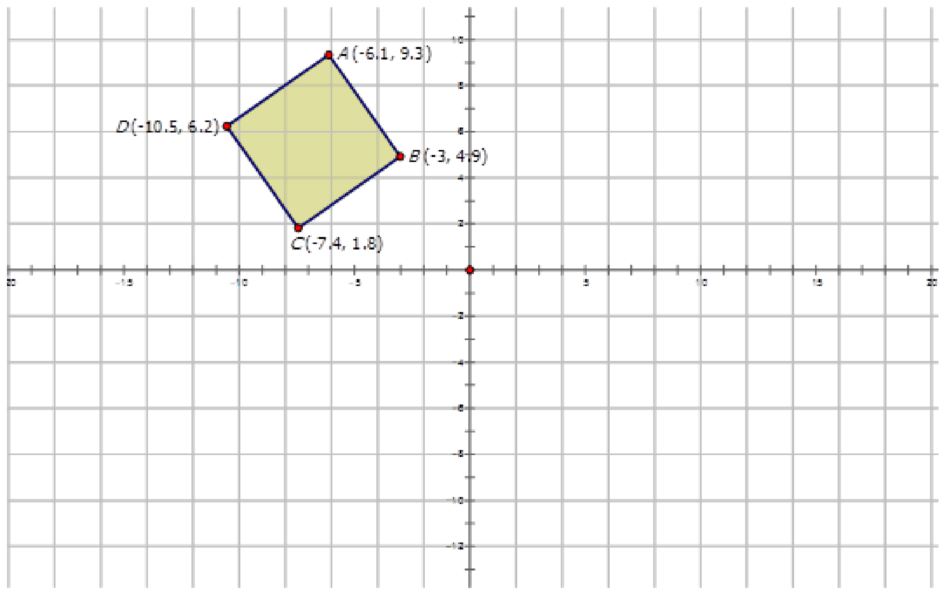
MEDIA

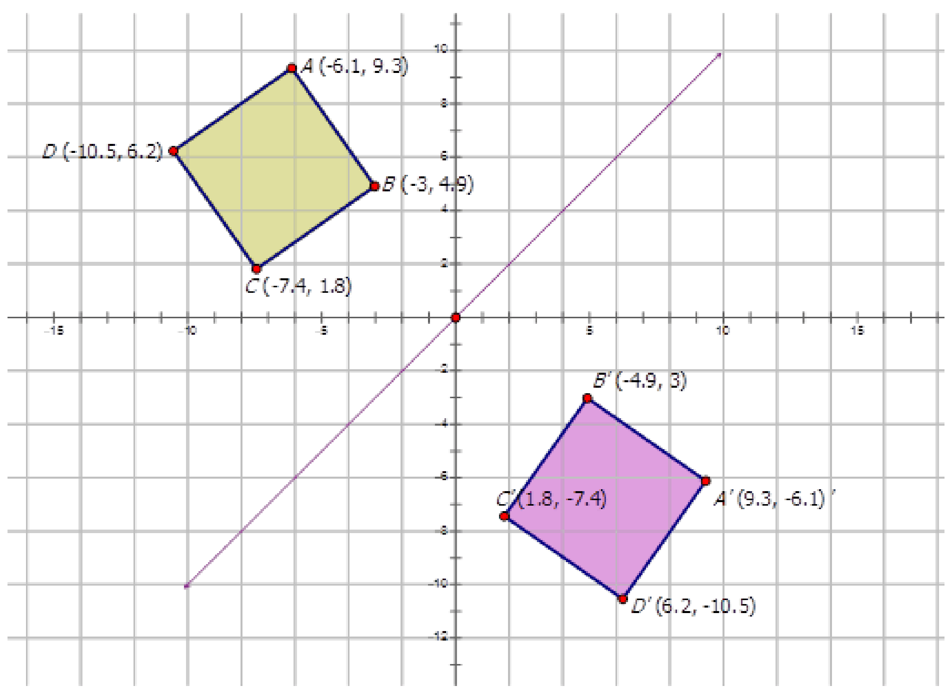
Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65257

Find the coordinates of the reflected.

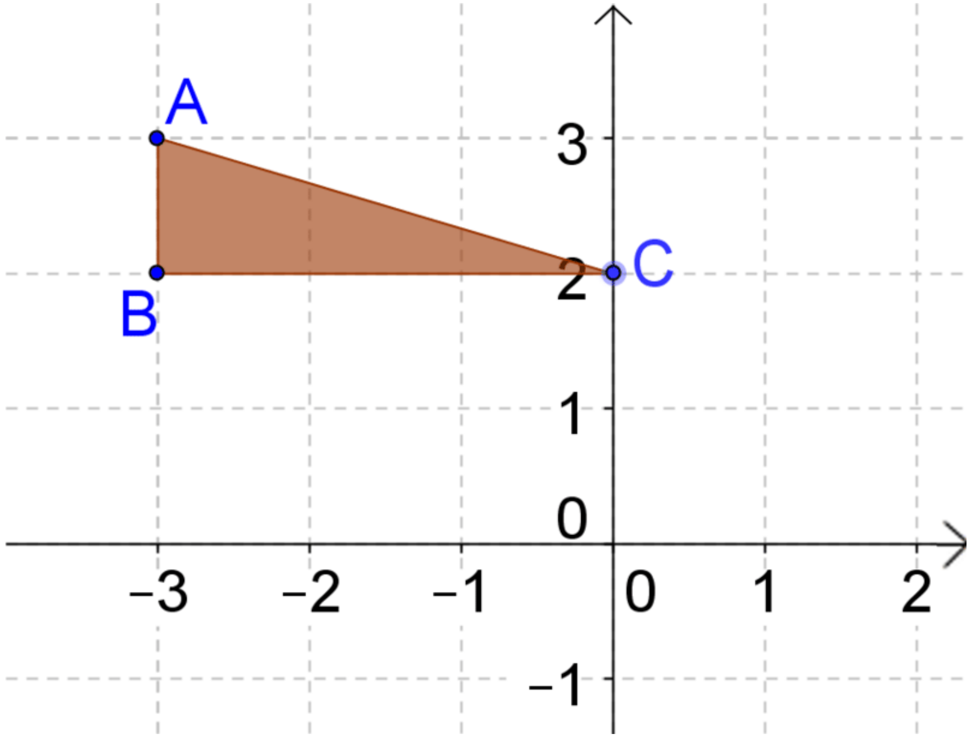
The diamond $ABCD$ is reflected across the line $y = x$ to form the image $A'B'C'D'$. Find the coordinates of the reflected image. On the diagram, draw and label the reflected image.

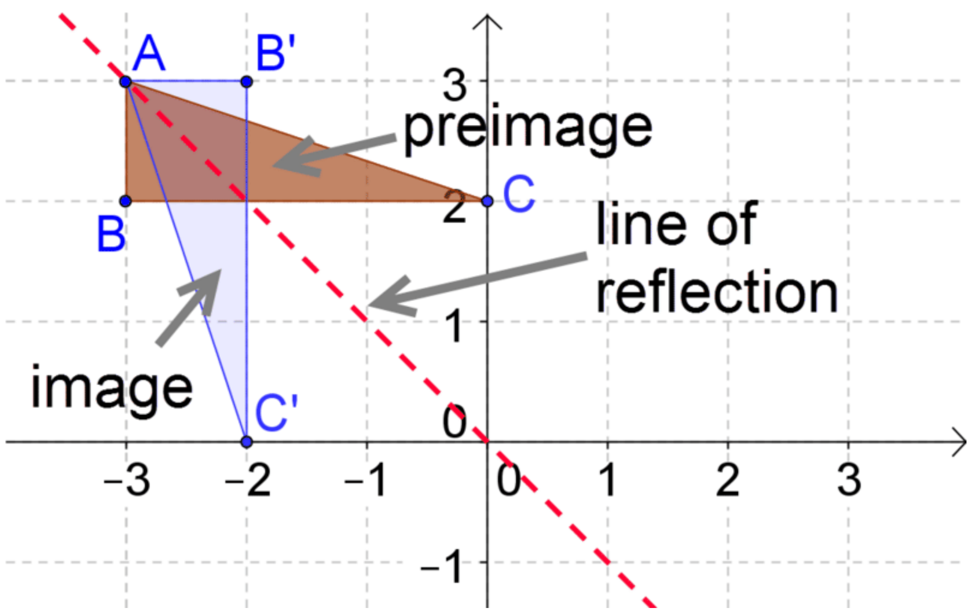




Draw and label the reflected image.

Triangle ABC is reflected across the line $y = -x$ to form the image $A'B'C'$. Draw and label the reflected image.

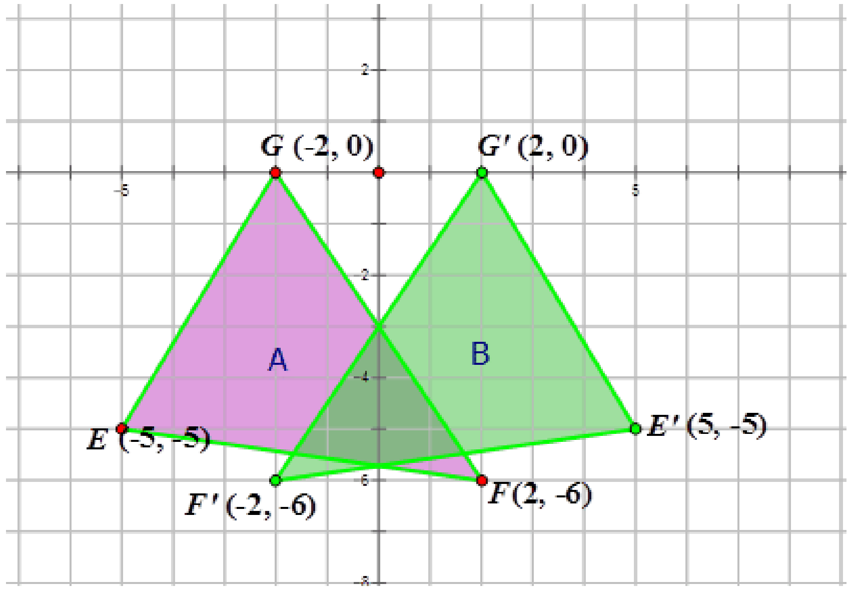




Examples

Example 1

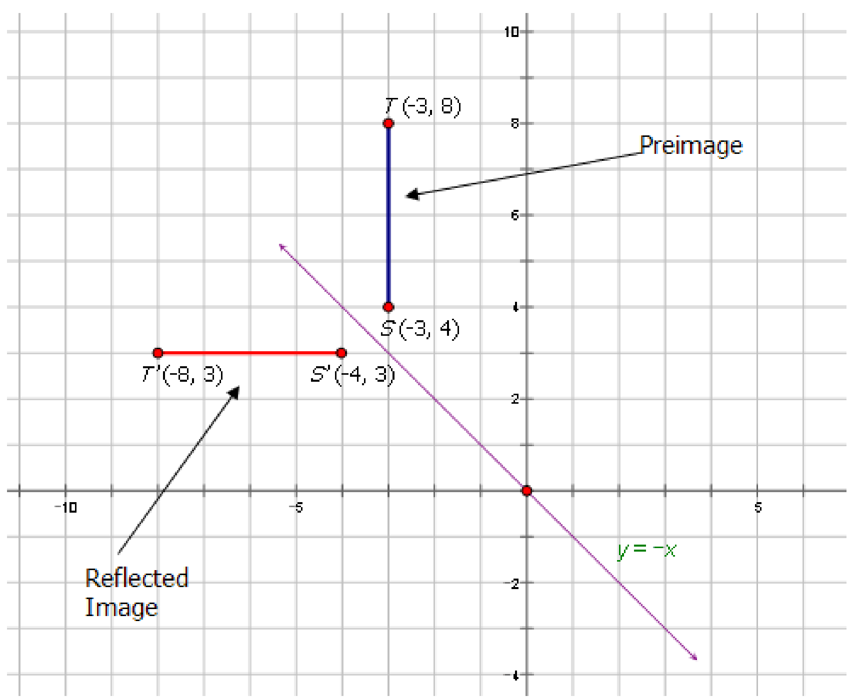
Earlier, you were asked to state the coordinates of the resulting image.



The coordinates of the new image (B) are $E'(5,-5)$, $F'(2,-6)$ and $G'(2,0)$.

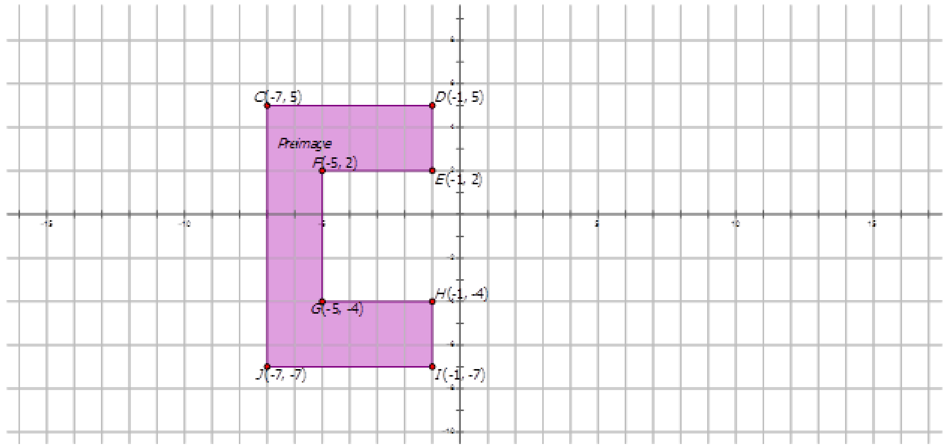
Example 2

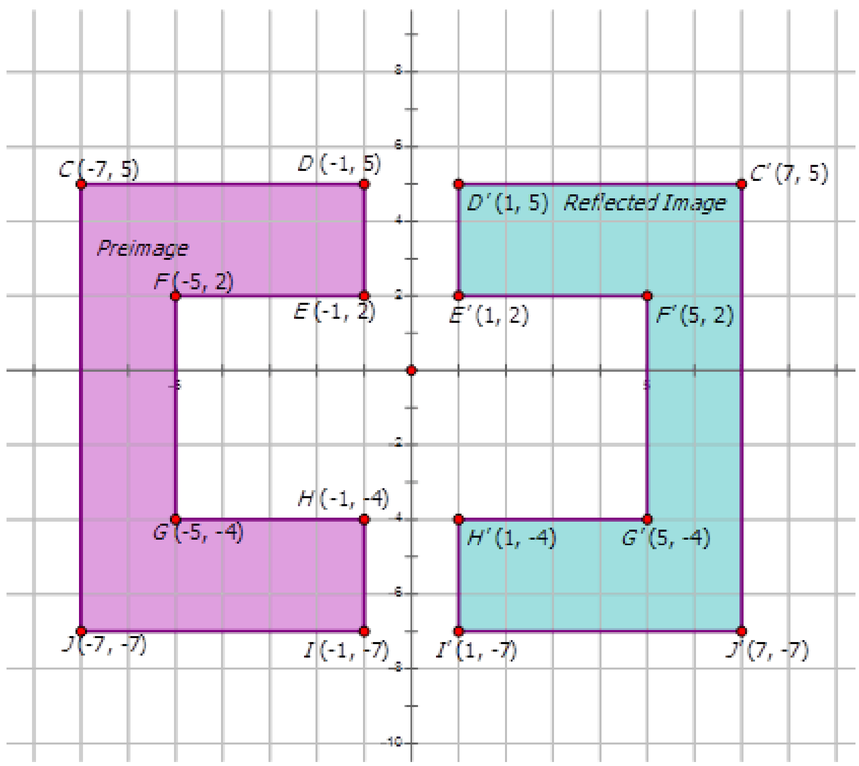
Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has been reflected across the line $y = -x$. Draw the preimage and image and properly label each.



Example 3

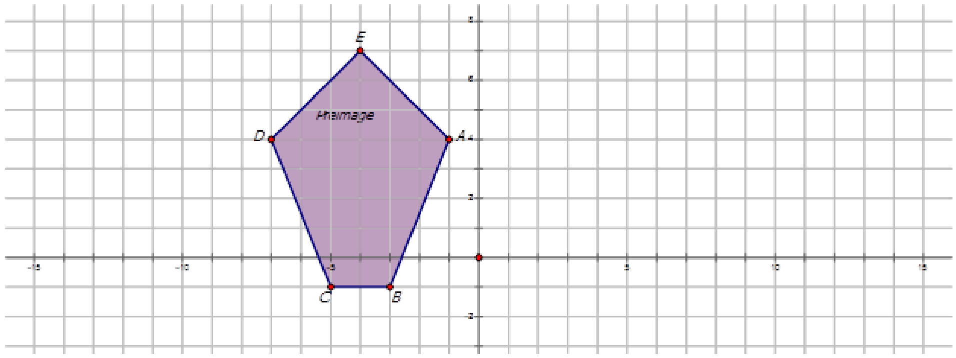
The polygon below has been reflected across the y-axis. Draw the reflected image and properly label each.

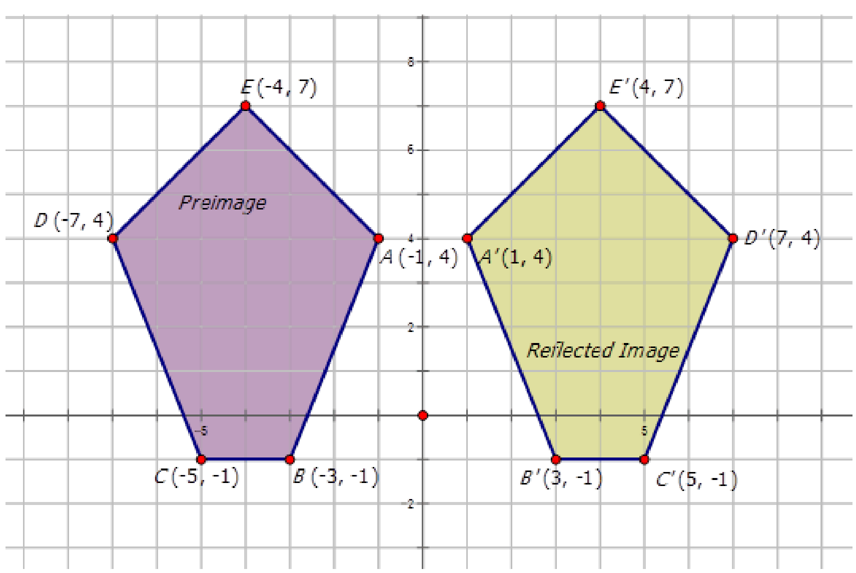




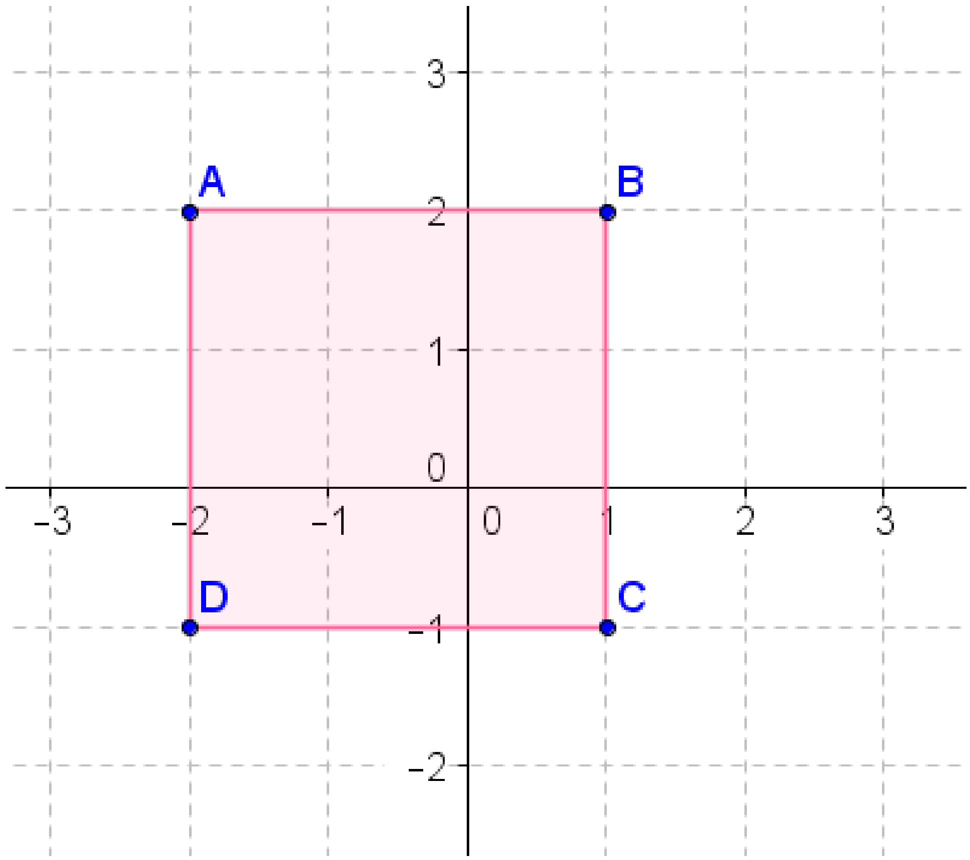
Example 4

The purple pentagon is reflected across the $y-axis$ to make the new image. Find the coordinates of the purple pentagon. On the diagram, draw and label the reflected pentagon.

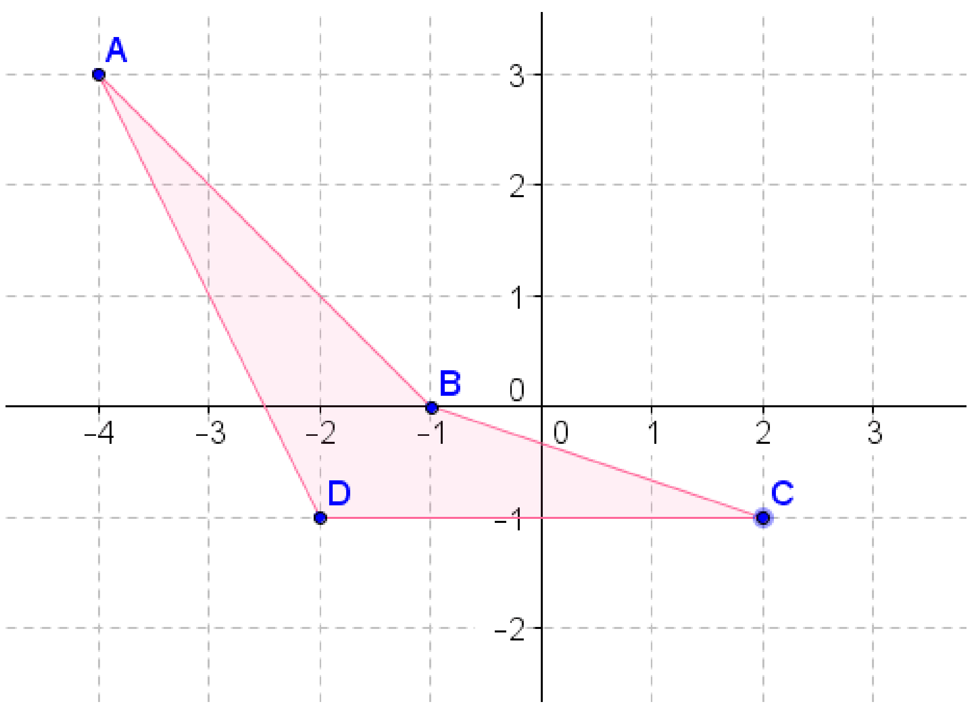

E
Preimage
D
A
C
B


E (-4, 7)
E′(4, 7)
Preimage
D (-7, 4)
A (-1, 4)
A′(1, 4)
D′(7, 4)
Reflected Image
C (-5, -1)
B (-3, -1)
B′(3, -1)
C′(5, -1)

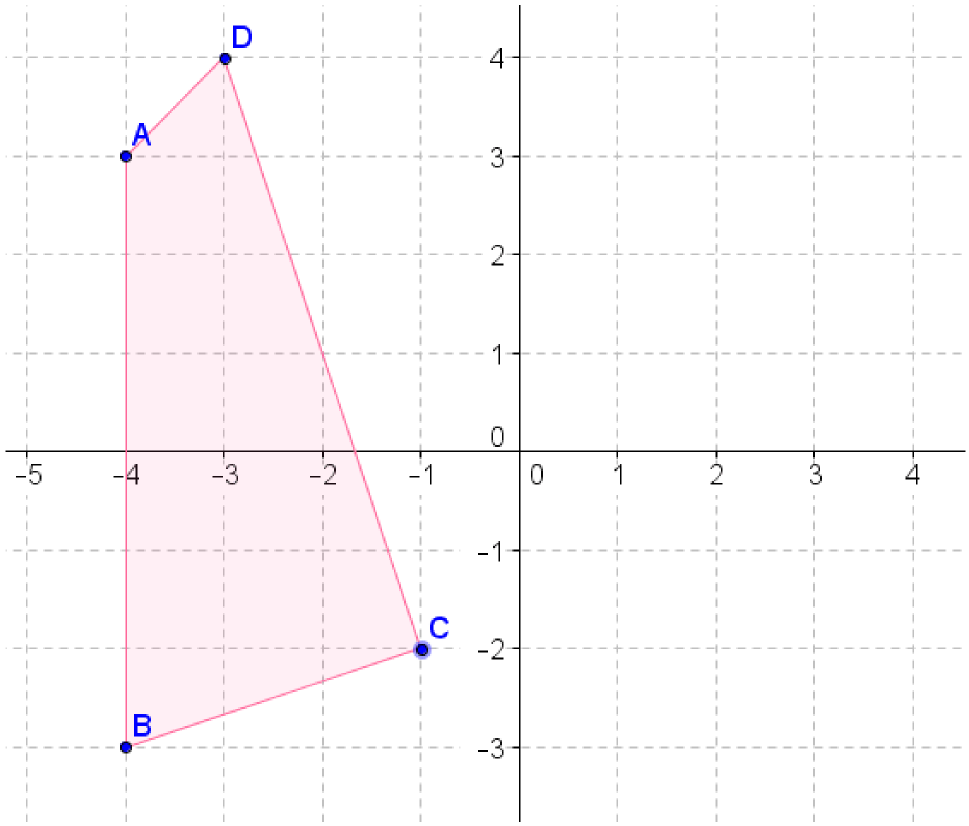
Review



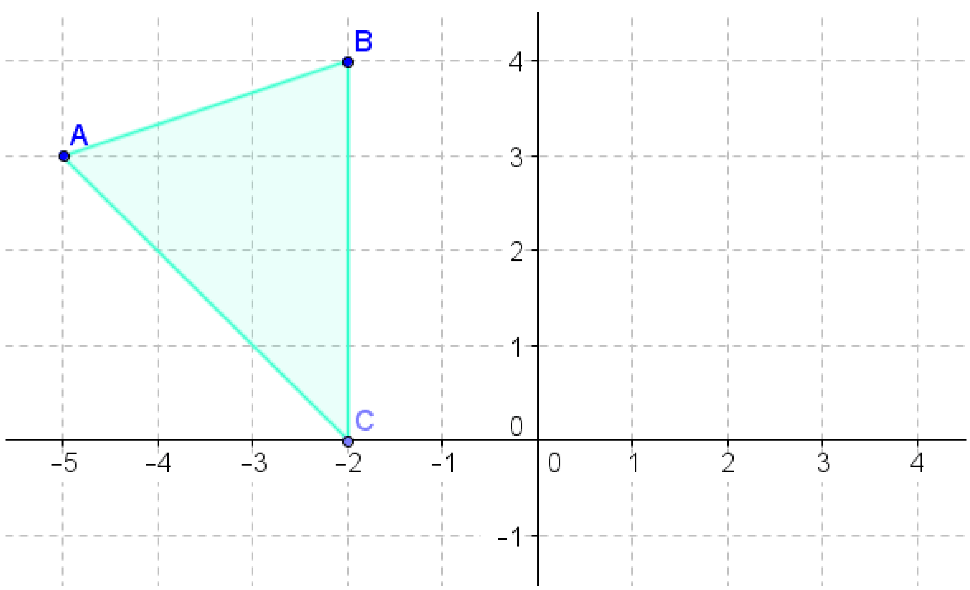
1. Reflect the above figure across the x-axis.
2. Reflect the above figure across the y-axis.
3. Reflect the above figure across the line $y = x$.



4. Reflect the above figure across the x-axis.
5. Reflect the above figure across the y-axis.
6. Reflect the above figure across the line $y = x$.

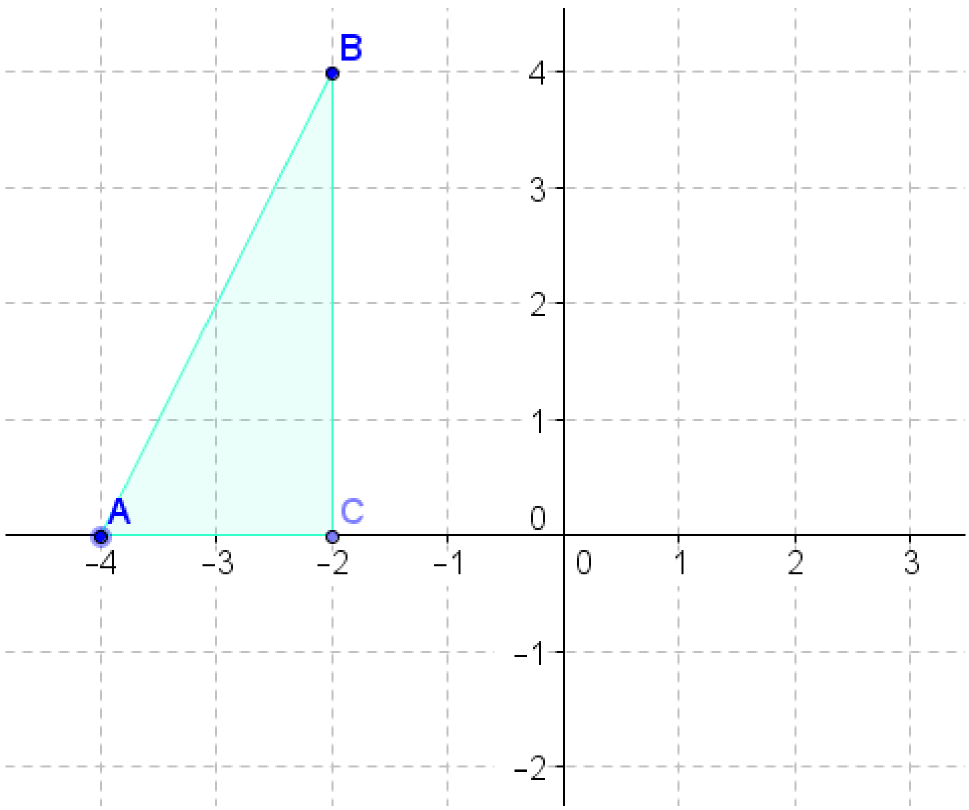


7. Reflect the above figure across the x-axis.
8. Reflect the above figure across the y-axis.
9. Reflect the above figure across the line $y = x$.

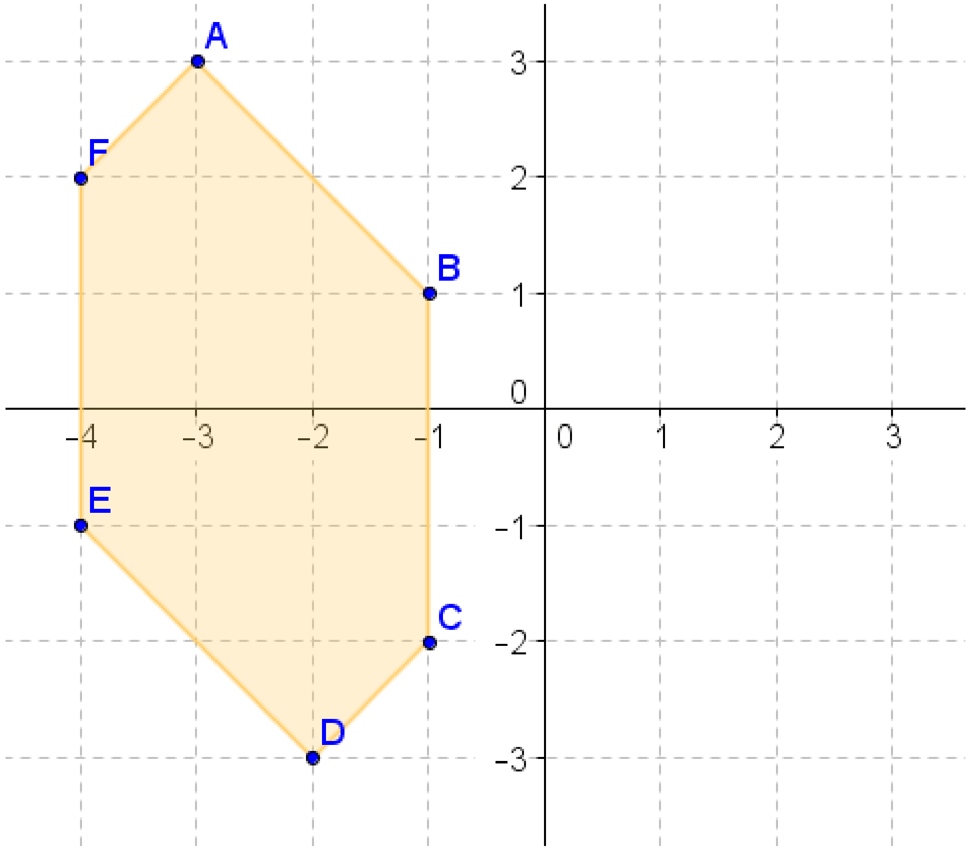


10. Reflect the above figure across the x-axis.

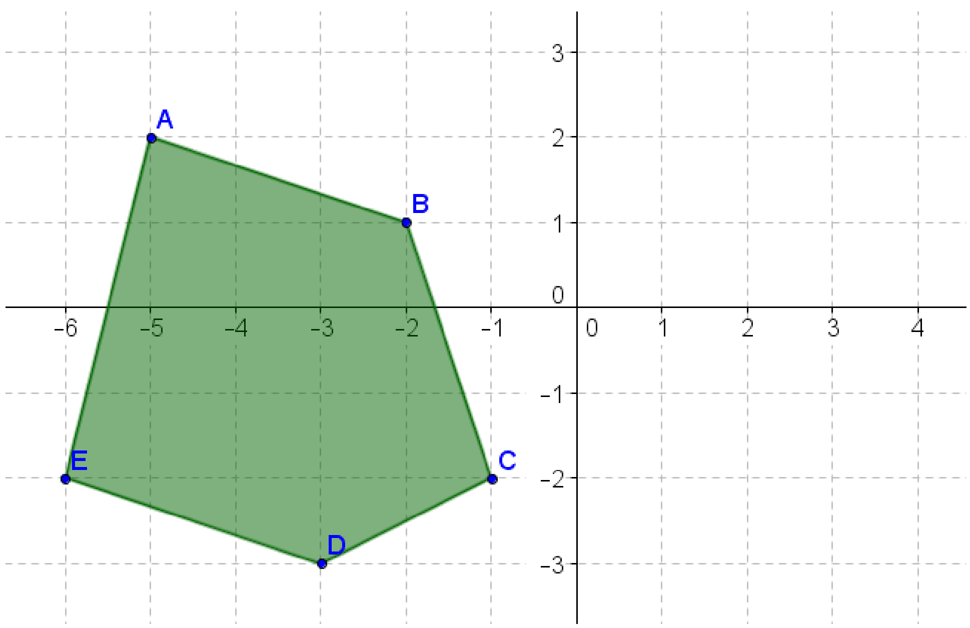
11. Reflect the above figure across the y-axis.
12. Reflect the above figure across the line $y = x$.



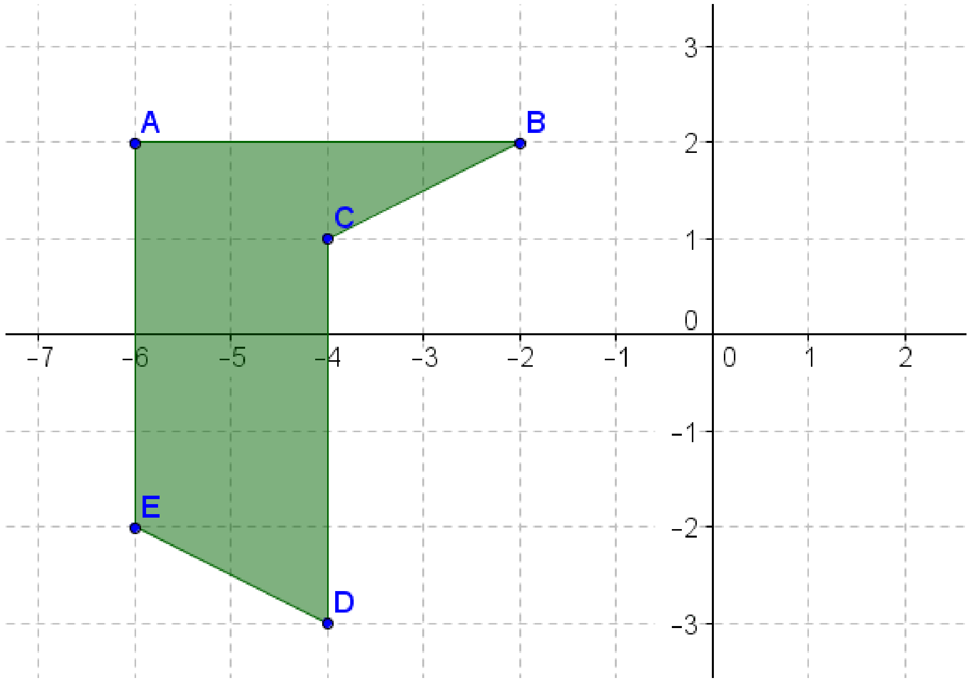
13. Reflect the above figure across the x-axis.
14. Reflect the above figure across the y-axis.
15. Reflect the above figure across the line $y = x$.



16. Reflect the above figure across the x-axis.
17. Reflect the above figure across the y-axis.
18. Reflect the above figure across the line $y = x$.



19. Reflect the above figure across the x-axis.
20. Reflect the above figure across the y-axis.
21. Reflect the above figure across the line $y = x$.



22. Reflect the above figure across the x-axis.
23. Reflect the above figure across the y-axis.
24. Reflect the above figure across the line $y = x$.

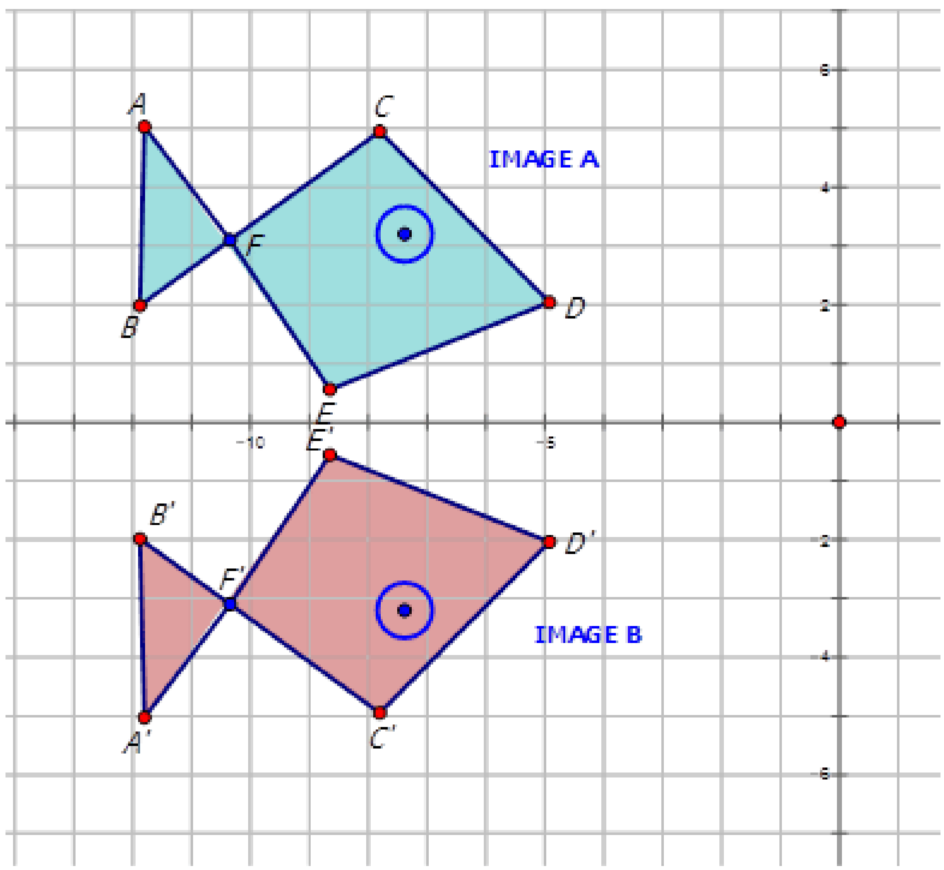
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.5.

10.6 Rules for Reflections

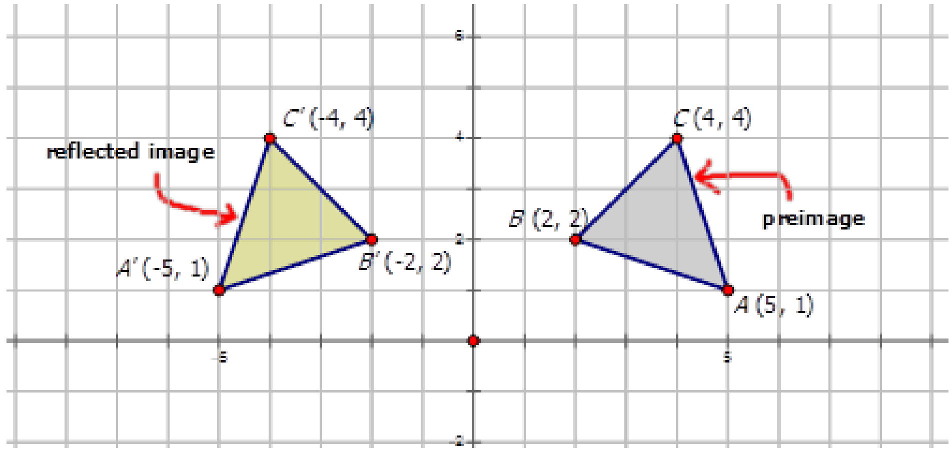
Here you will learn notation for describing a reflection with a rule.

The figure below shows a pattern of two fish. Write the mapping rule for the reflection of Image A to Image B.



Rules for Reflections

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A reflection is an example of a transformation that takes a shape (called the preimage) and flips it across a line (called the line of reflection) to create a new shape (called the image). By examining the coordinates of the reflected image, you can determine the line of reflection. The most common lines of reflection are the x-axis, the y-axis, or the lines $y = x$ or $y = -x$.



The preimage has been reflected across he y-axis. This means, all of the x-coordinates have been multiplied by -1. You can describe the reflection in words, or with the following notation:

$r_{y-axis}(x,y) \rightarrow (-x,y)$

Notice that the notation tells you exactly how each (x,y) point changes as a result of the transformation.

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65250

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65251

Find the image of the point (3, 2) that has undergone a reflection across

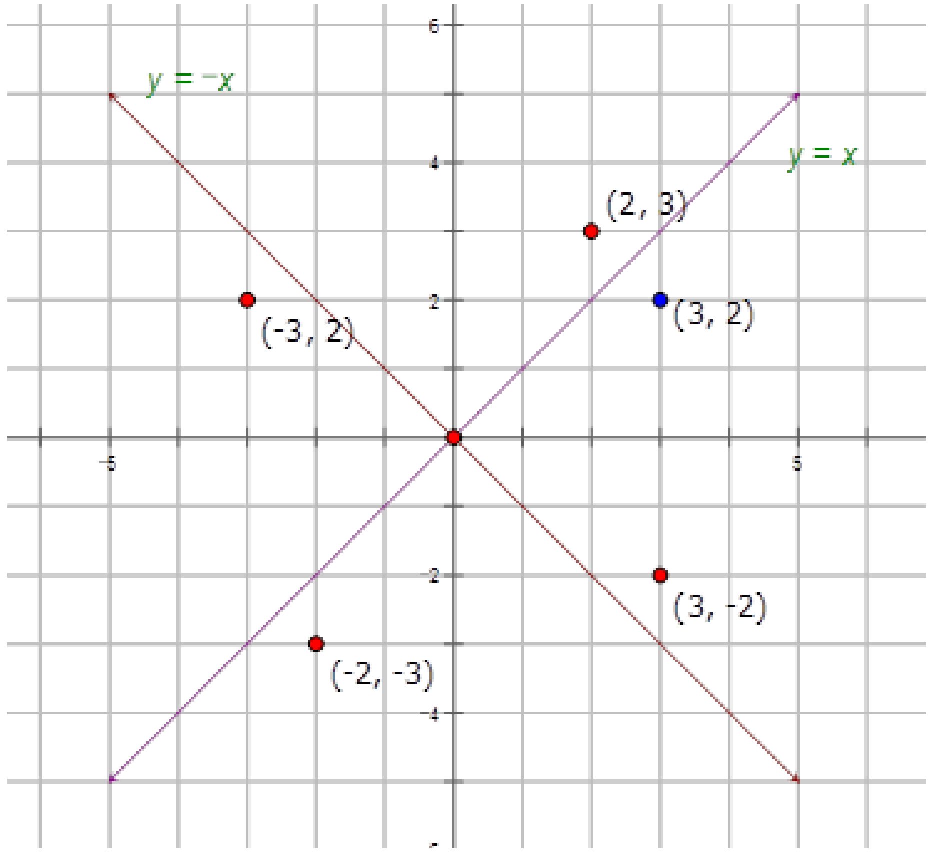
a) the x-axis,

b) the y-axis,

c) the line $y = x$, and

d) the line $y = -x$.

Write the notation to describe the reflection.



a) Reflection across the x-axis: $r_{x-axis}(3,2) \rightarrow (3,-2)$

b) Reflection across the y-axis: $r_{y-axis}(3,2) \rightarrow (3,-2)$

c) Reflection across the line $y = x$: $r_{y=x}(3,2) \rightarrow (2,3)$

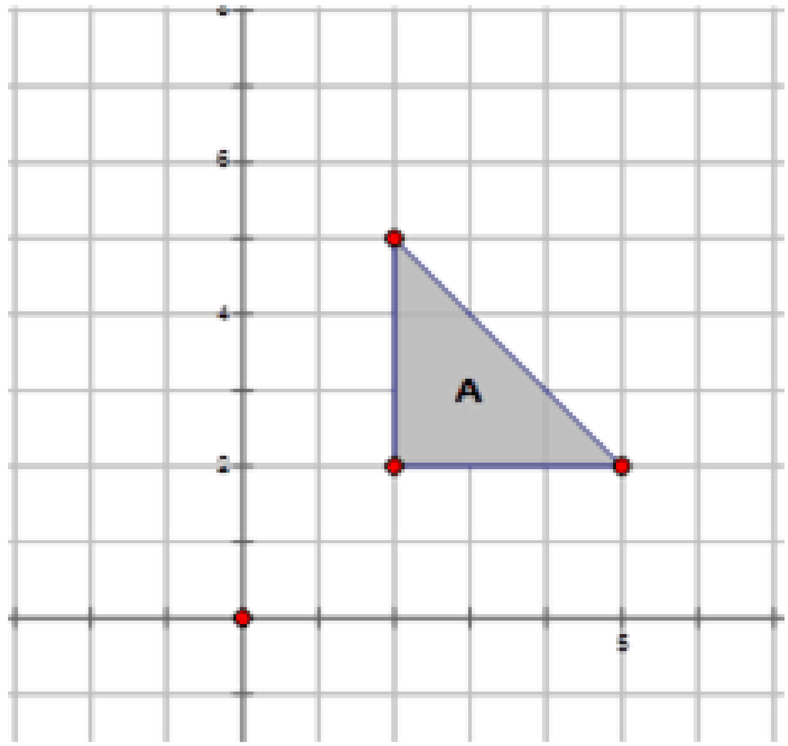
d) Reflection across the line $y = -x$: $r_{y=-x}(3,2) \rightarrow (-2,-3)$

Reflect Image A in the diagram below:

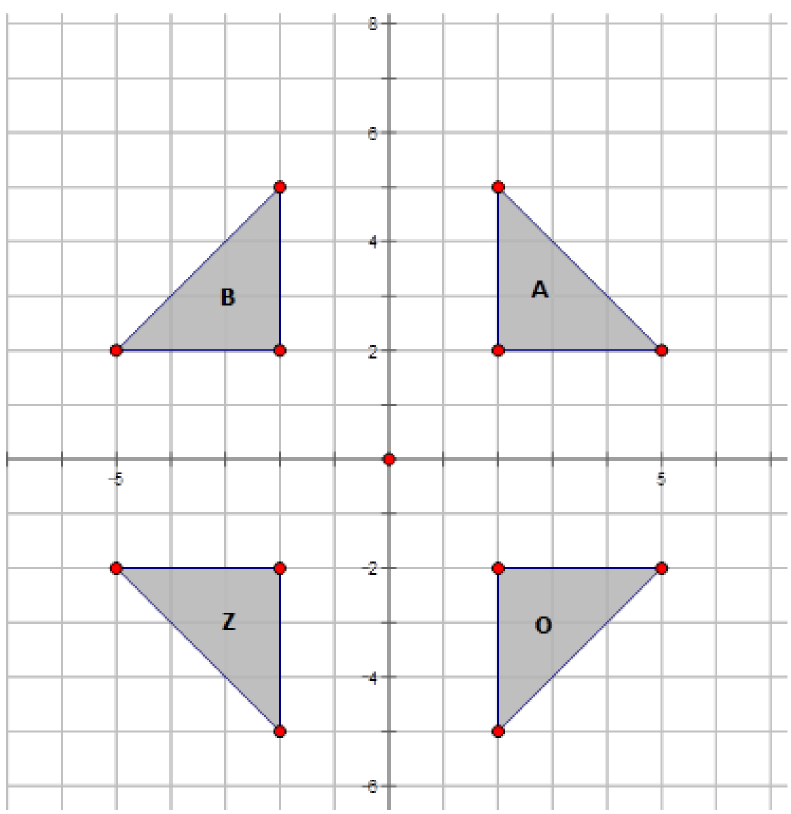
a) Across the y-axis and label it B.

b) Across the x-axis and label it O.

c) Across the line $y = -x$ and label it Z.



Write notation for each to indicate the type of reflection.

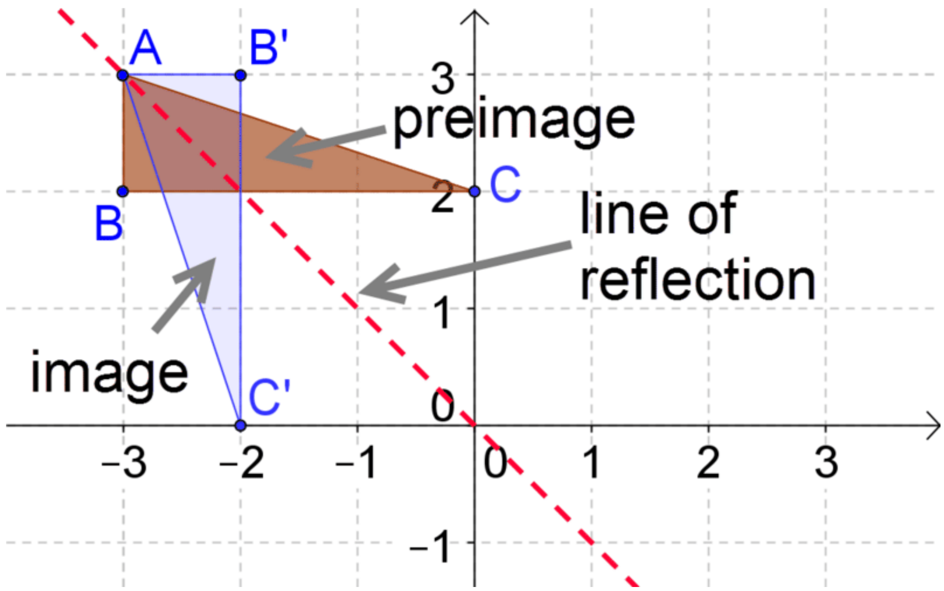


a) Reflection across the y-axis: $r_{y-axis}A \rightarrow B = r_{y-axis}(x,y) \rightarrow (-x,y)$

b) Reflection across the x-axis: $r_{x-axis}A \rightarrow O = r_{x-axis}(x,y) \rightarrow (x,-y)$

c) Reflection across the $y = -x$: $r_{y=-x}A \rightarrow Z = r_{y=-x}(x,y) \rightarrow (-y,-x)$

Write the notation that represents the reflection of the preimage to the image in the diagram below.



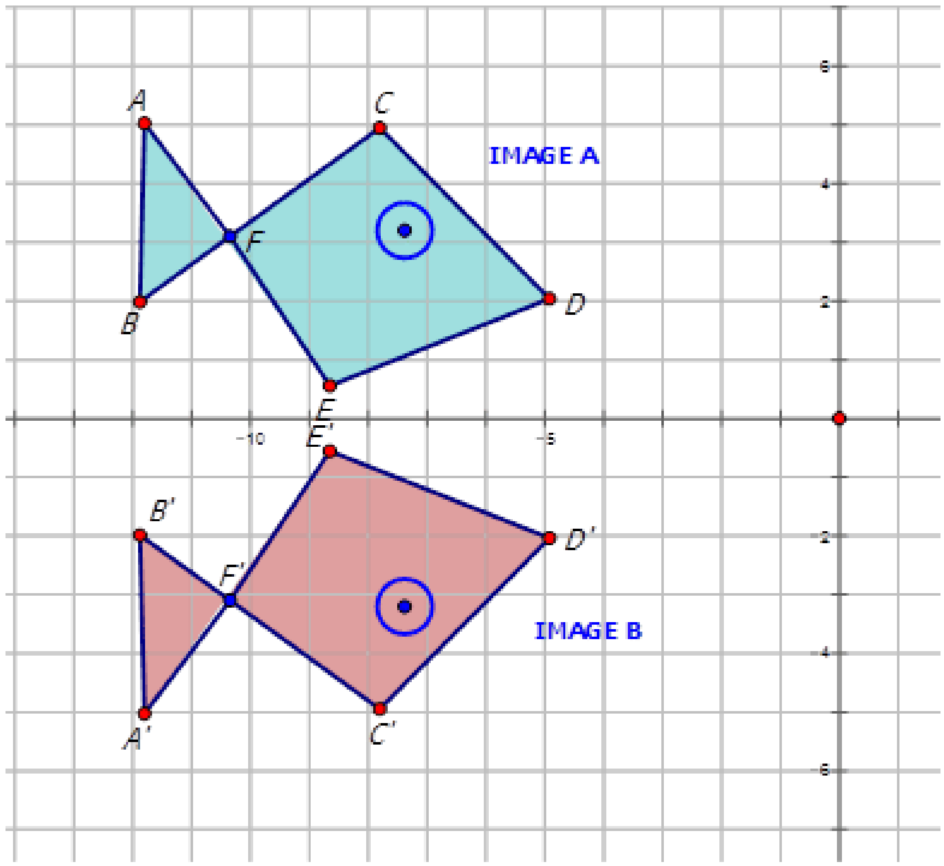
This is a reflection across the line $y- = -x$. The notation is $r_{y=-x}(x,y) \rightarrow (-y,-x)$.

Examples

Example 1

Earlier, you were given a problem about a figure.

The figure below shows a pattern of two fish. Write the mapping rule for the reflection of Image A to Image B.



To answer this question, look at the coordinate points for Image A and Image B.

TABLE 10.8:

Image A	$A(-11.8,5)$	$B(-11.8,2)$	$C(-7.8,5)$	$D(-4.9,2)$	$E(-8.7,0.5)$	$F(-10.4,3.1)$
Image B	$A'(-11.8,-5)$	$B'(-11.8,-2)$	$C'(-7.8,-5)$	$D'(-4.9,-2)$	$E'(-8.7,-0.5)$	$F'(-10.4,-3.1)$

Notice that all of the y-coordinates have changed sign. Therefore Image A has reflected across the x-axis. To write a rule for this reflection you would write: $r_{x-axis}(x,y) \rightarrow (x,-y)$.

Example 2

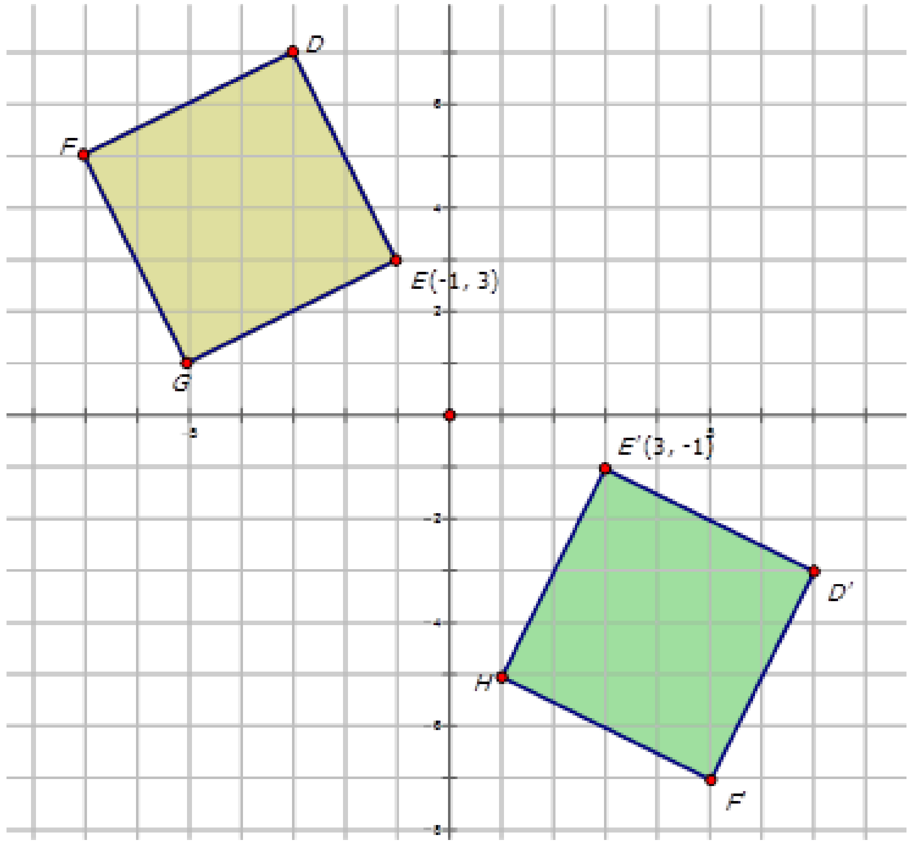
Thomas describes a reflection as point J moving from $J(-2,6)$ to $J'(-2,-6)$. Write the notation to describe this reflection for Thomas.

$J:(-2,6) \quad J':(-2,-6)$

Since the y-coordinate is multiplied by -1 and the x-coordinate remains the same, this is a reflection in the x-axis. The notation is: $r_{x-axis}J \rightarrow J' = r_{x-axis}(-2,6) \rightarrow (-2,6)$

Example 3

Write the notation that represents the reflection of the yellow diamond to the reflected green diamond in the diagram below.



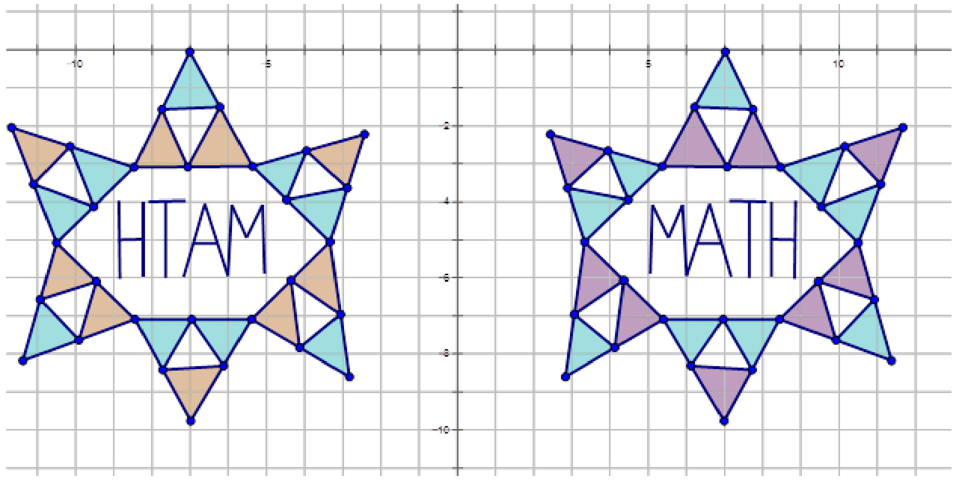
In order to write the notation to describe the reflection, choose one point on the preimage (the yellow diamond) and then the reflected point on the green diamond to see how the point has moved. Notice that point E is shown in the diagram:

$E(-1,3) \rightarrow E'(3,-1)$

Since both x- and y-coordinates are reversed numbers, the reflection is in the line $y = x$. The notation for this reflection would be: $r_{y=x}(x,y) \rightarrow (y,x)$.

Example 4

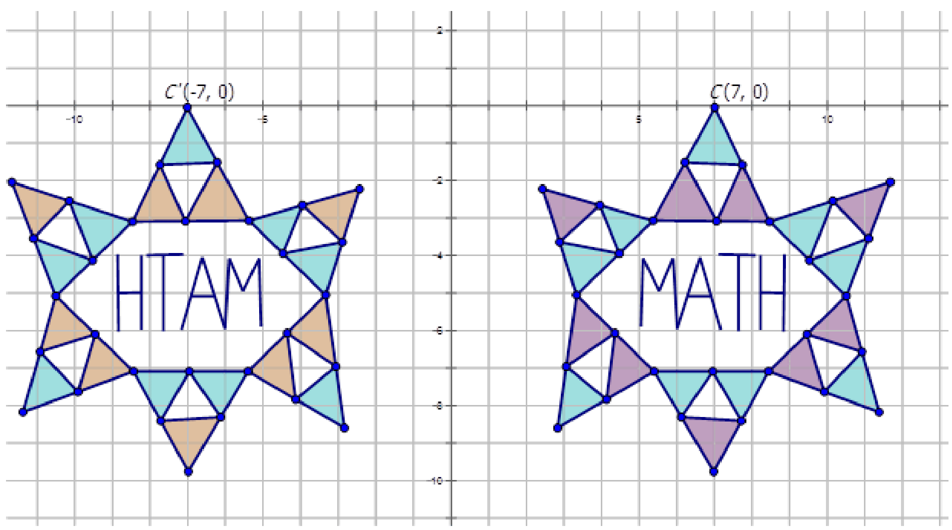
Karen was playing around with a drawing program on her computer. She created the following diagrams and then wanted to determine the transformations. Write the notation rule that represents the transformation of the purple and blue diagram to the orange and blue diagram.



In order to write the notation to describe the transformation, choose one point on the preimage (purple and blue diagram) and then the transformed point on the orange and blue diagram to see how the point has moved. Notice that point A is shown in the diagram:

$C(7,0) \rightarrow C'(-7,0)$

Since both x-coordinates only are multiplied by -1, the transformation is a reflection is in y-axis. The notation for this reflection would be: $r_{y-axis}(x,y) \rightarrow (-x,y)$.



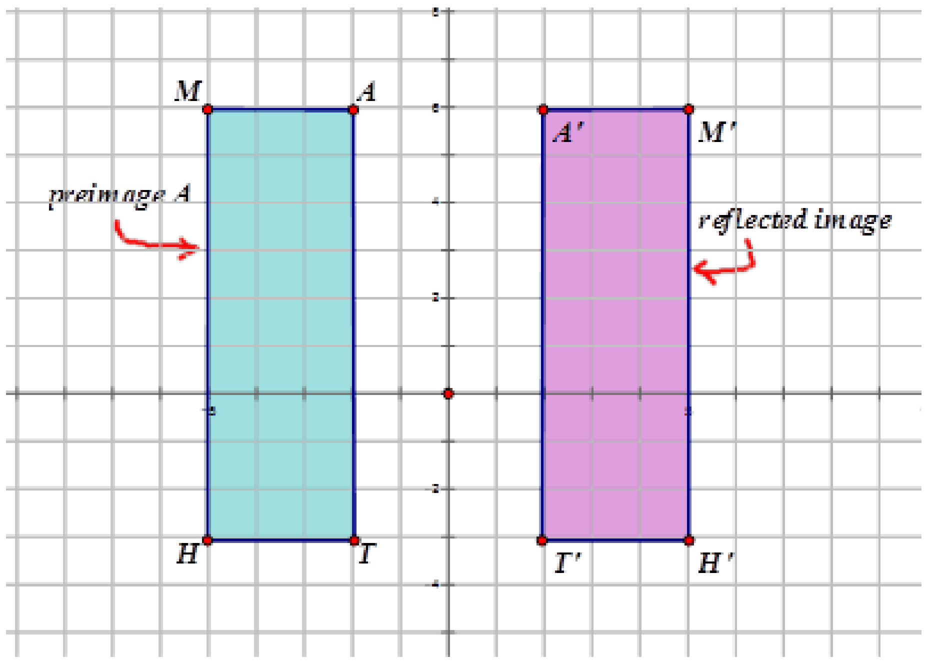
Review

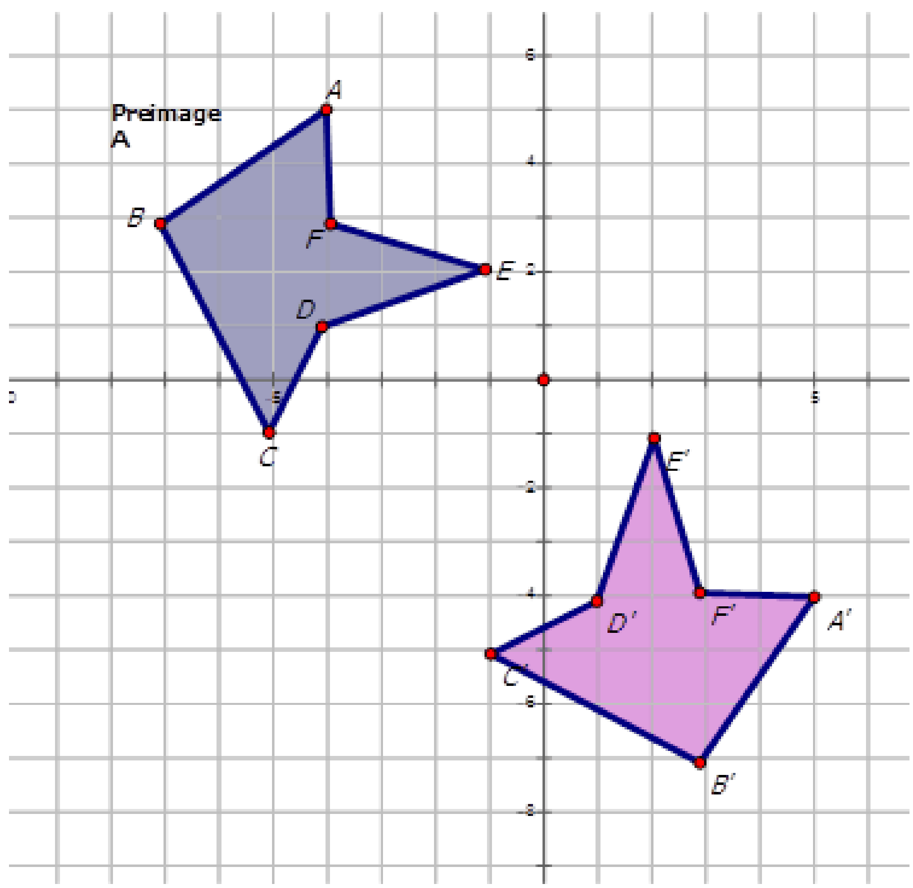
Write the notation to describe the movement of the points in each of the reflections below.

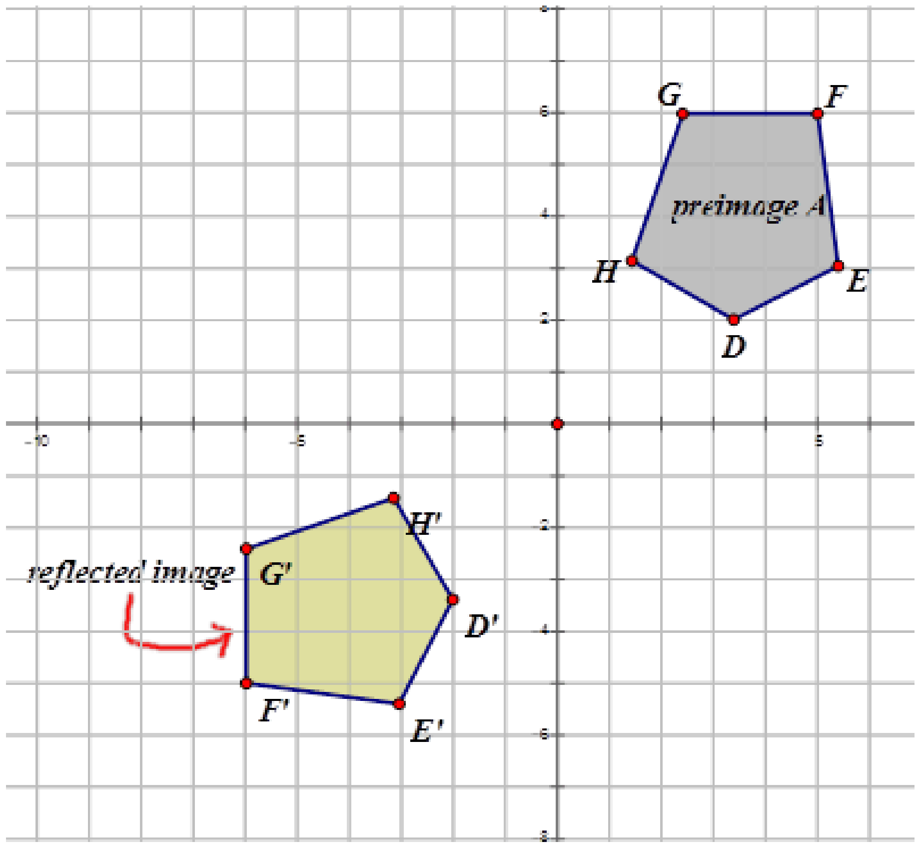
1. $S(1,5) \rightarrow S'(-1,5)$
2. $W(-5,-1) \rightarrow W'(5,-1)$
3. $Q(2,-5) \rightarrow Q'(2,5)$
4. $M(4,3) \rightarrow M'(-3,-4)$
5. $B(-4,-2) \rightarrow B'(-2,-4)$
6. $A(3,5) \rightarrow A'(-3,5)$
7. $C(1,2) \rightarrow C'(2,1)$
8. $D(2,-5) \rightarrow D'(5,-2)$

9. $E(3,1) \rightarrow E'(-3,1)$
10. $F(-4,2) \rightarrow F'(-4,-2)$
11. $G(1,3) \rightarrow G'(1,-3)$

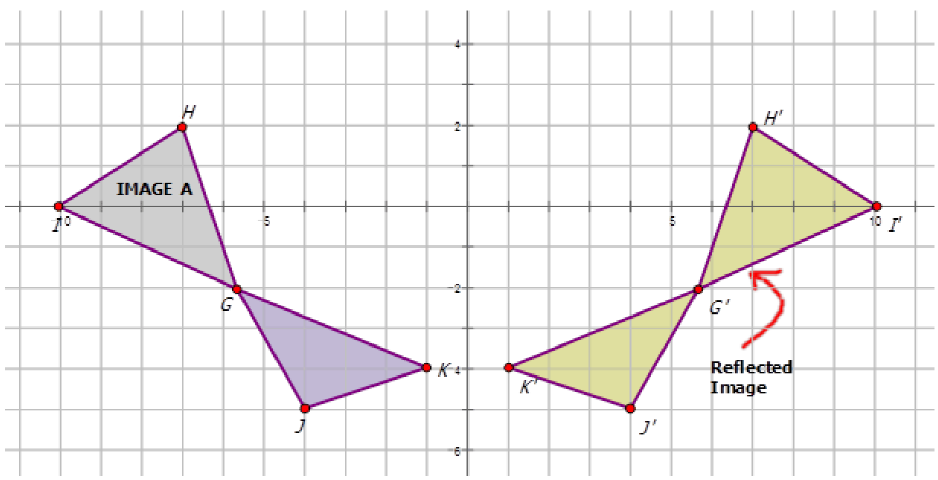
Write the notation that represents the reflection of the preimage image for each diagram below.

12. 

13. 



14.



15.

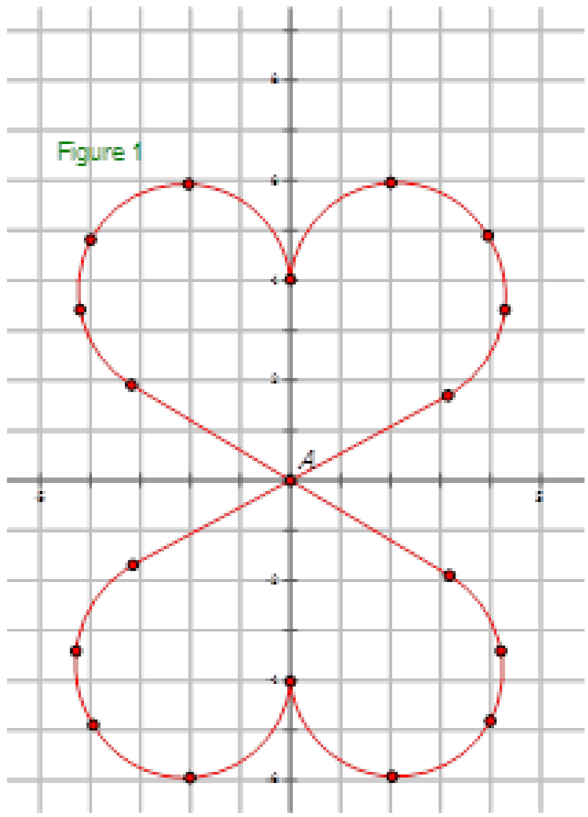
Answers for Review Problems

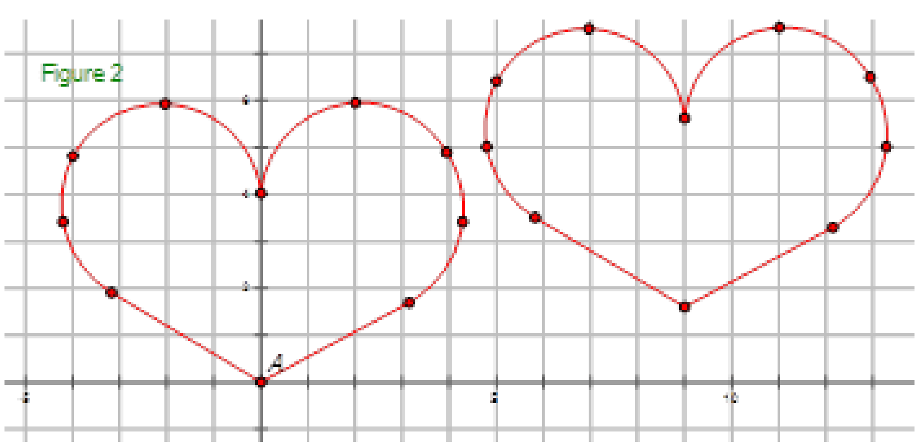
To see the Review answers, open this PDF file and look for section 10.6.

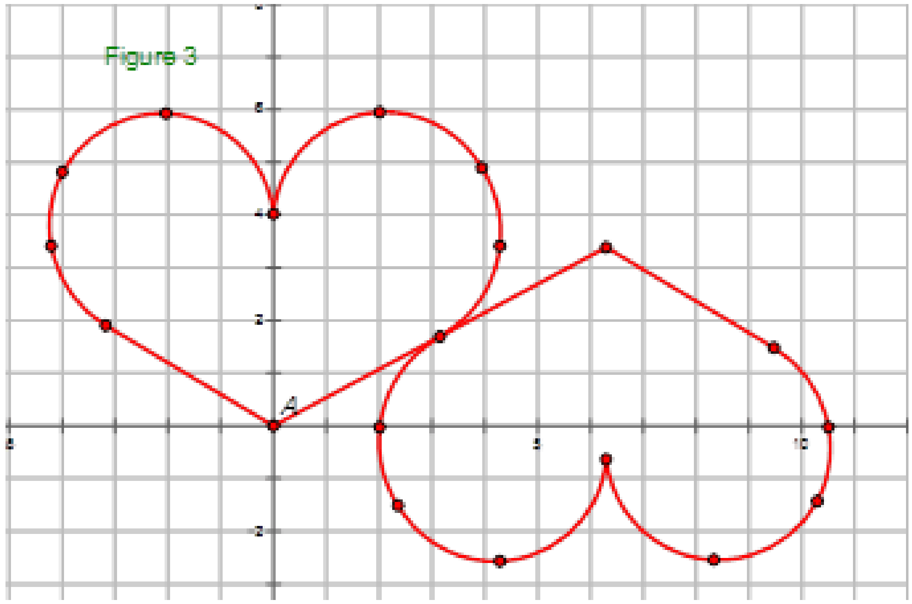
10.7 Rotations

Here you will learn about geometric rotations.

Which one of the following figures represents a rotation? Explain.

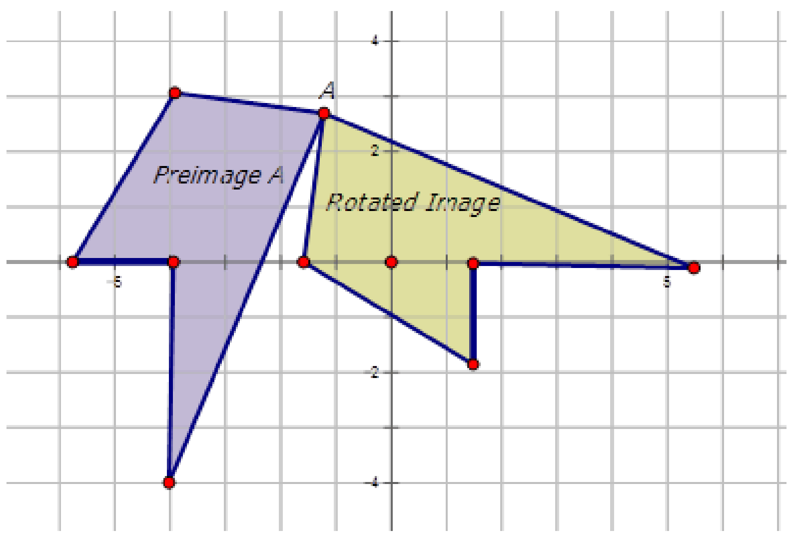






Rotations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A rotation is an example of a transformation where a figure is rotated about a specific point (called the center of rotation), a certain number of degrees. The figure below shows that the Preimage A has been rotated 90° about point A to form the rotated image. Point A is the center of rotation.



In order to describe a rotation, you need to state how many degrees the preimage rotated, the center of rotation, and the direction of the rotation (clockwise or counterclockwise). The most common center of rotation is the origin. The table below shows what happens to points when they have undergone a rotation about the origin. The angles are given as counterclockwise.

TABLE 10.9:

Center of Rotation	Angle of Rotation	Preimage (Point P)	Rotated Image (Point P')
(0, 0)	90° (or -270°)	(x, y)	$(-y, x)$

TABLE 10.9: (continued)

Center of Rotation	Angle of Rotation	Preimage (Point P)	Rotated Image (Point P')
(0, 0)	180° (or −180°)	(x, y)	$(-x, -y)$
(0, 0)	270° (or −90°)	(x, y)	$(y, -x)$

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65242

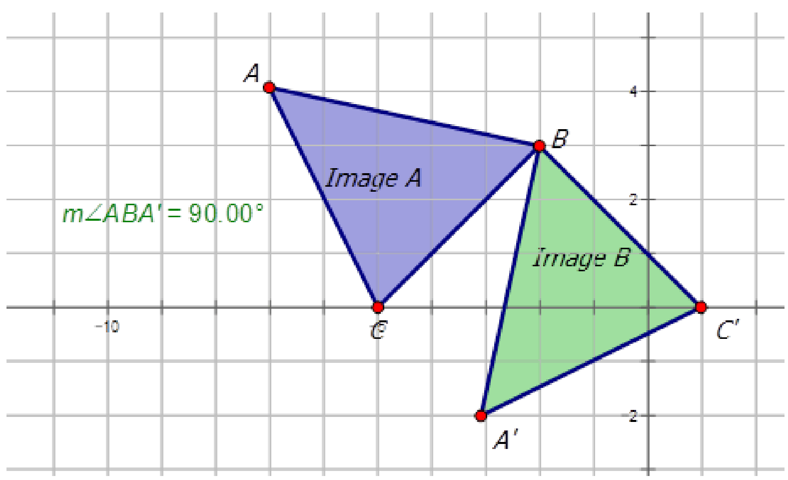
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65243

Describe the rotation

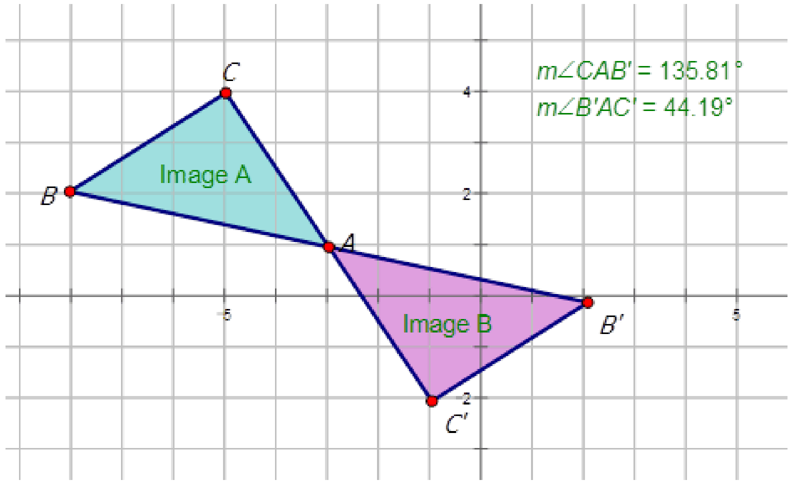
Describe the rotation of the blue triangle in the diagram below.



Looking at the angle measures, $\angle ABA' = 90°$. Therefore the preimage, Image A, has been rotated 90° counterclockwise about the point B.

Describe the rotation

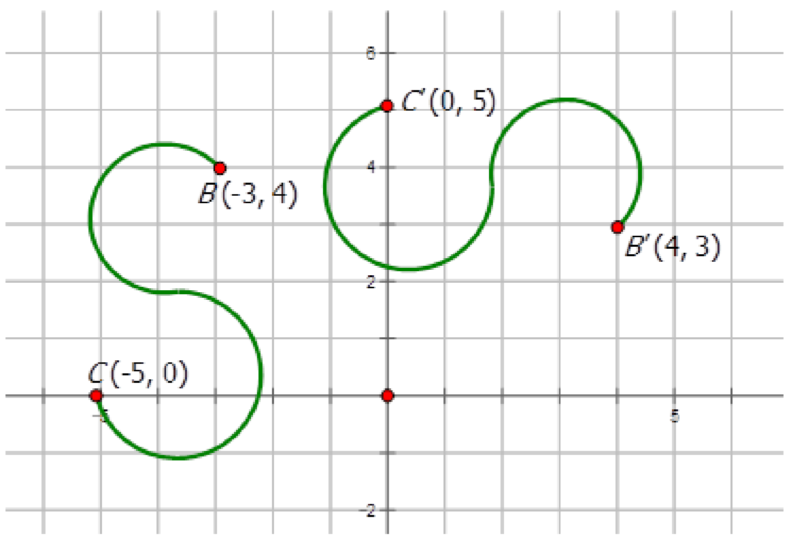
Describe the rotation of the triangles in the diagram below.



Looking at the angle measures, $\angle CAB' + \angle B'AC' = 180°$. The triangle ABC has been rotated 180°CCW about the center of rotation Point A.

Describe the rotation

Describe the rotation in the diagram below.



To describe the rotation in this diagram, look at the points indicated on the S shape.

- Points BC: $B(-3,4)C(-5,0)$
- Points $B'C'$: $B'(4,3)C'(0,5)$

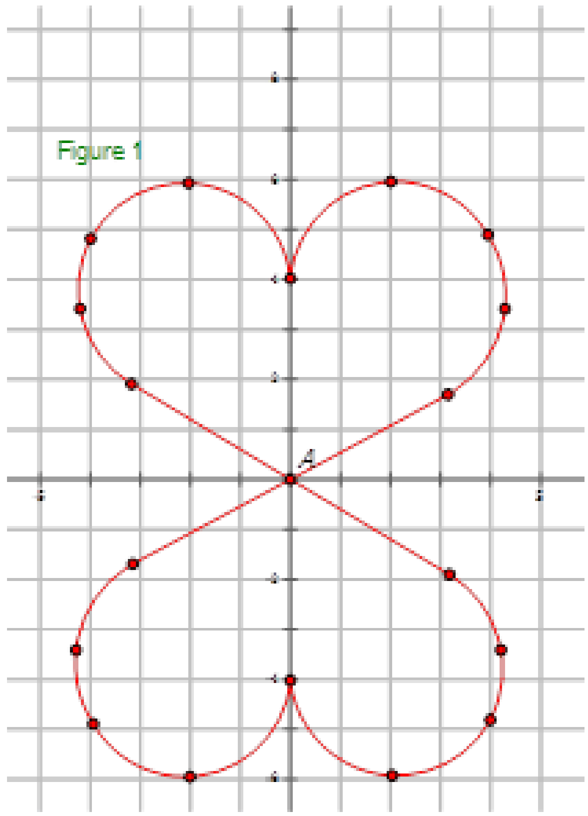
These points represent a rotation of 90° clockwise about the origin. Each coordinate point (x,y) has become the point $(y,-x)$.

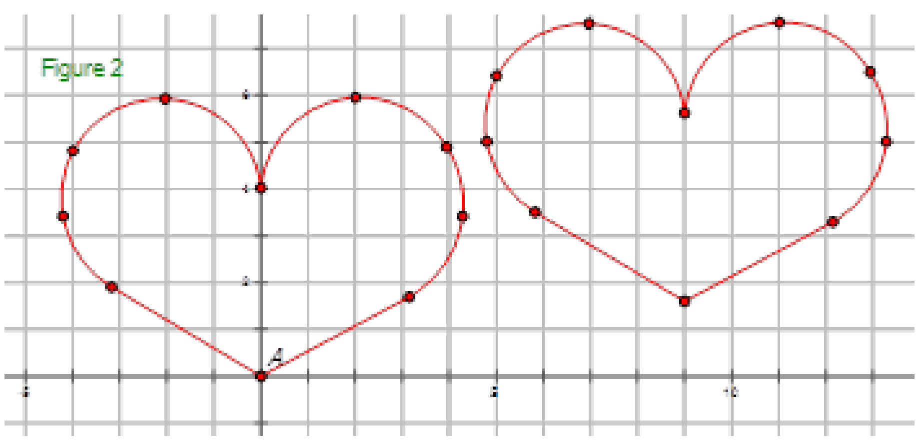
Examples

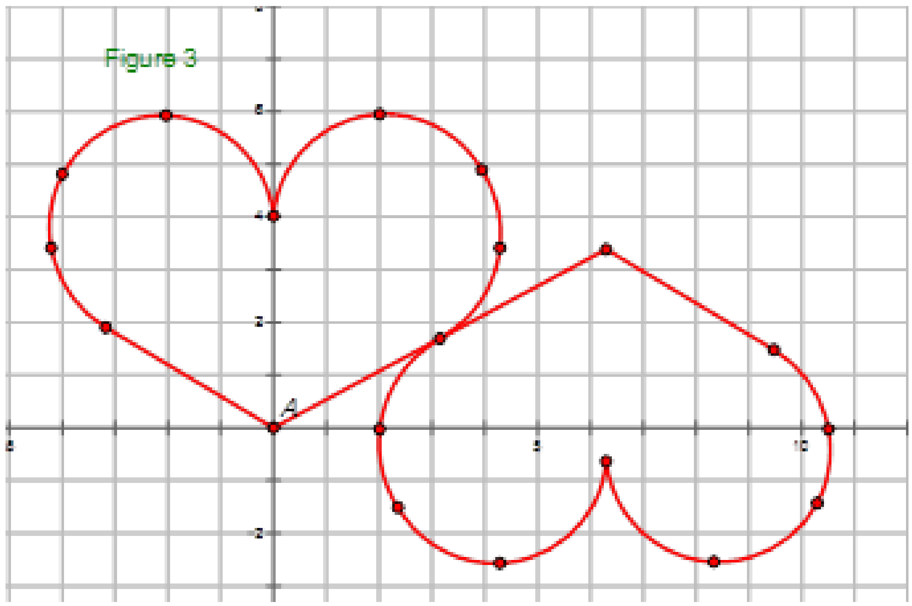
Example 1

Earlier, you were given a problem about two figures.

Which one of the following figures represents a rotation? Explain.



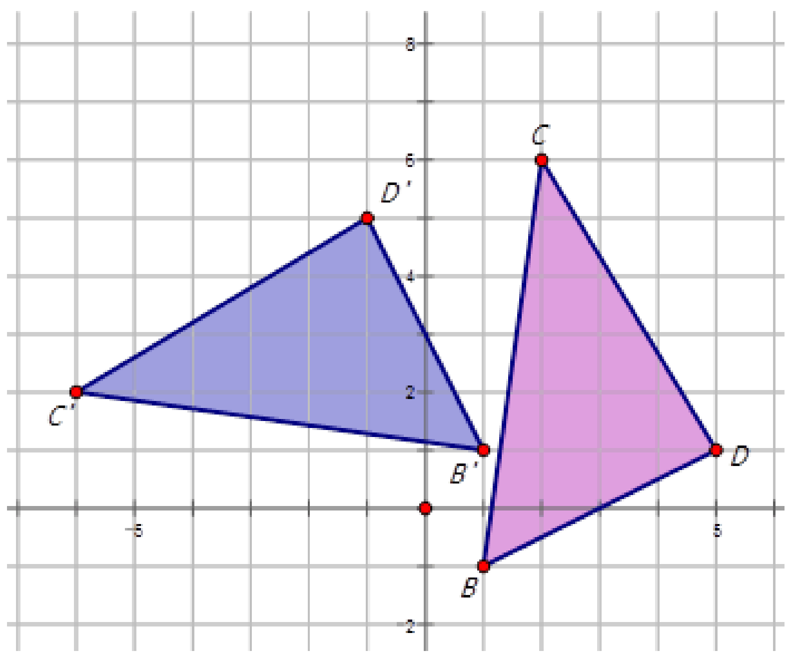




You know that a rotation is a transformation that turns a figure about a fixed point. This fixed point is the turn center or the center of rotation. In the figures above, Figure 1 and Figure 3 involve turning the heart about a fixed point. Figure 1 rotates the heart about the point A. Figure 3 rotates the heart about the point directly to the right of A. Figure 2 does a translation, not a rotation.

Example 2

Describe the rotation of the pink triangle in the diagram below.



Examine the points of the preimage and the rotated image (the blue triangle).

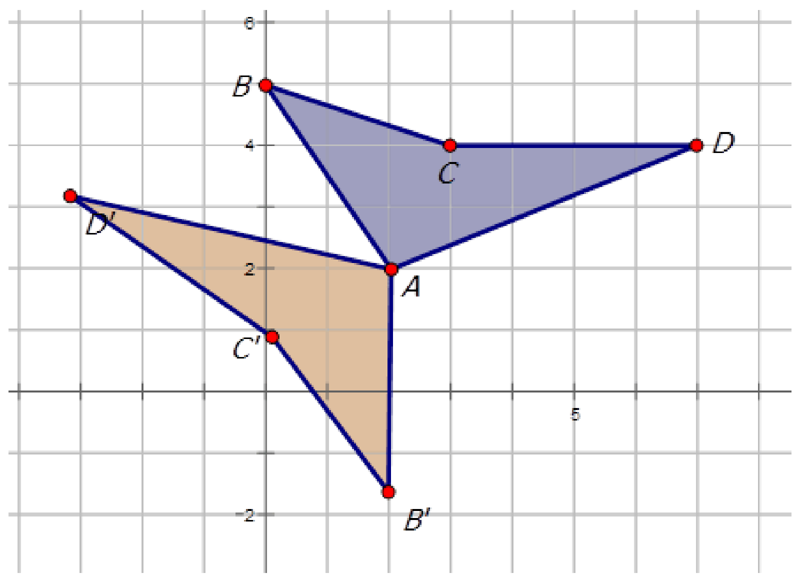
TABLE 10.10:

Points on BCD	$B(1,-1)$	$C(2,6)$	$D(5,1)$
Points on $B'C'D'$	$B'(1,1)$	$C'(-6,2)$	$D'(-1,5)$

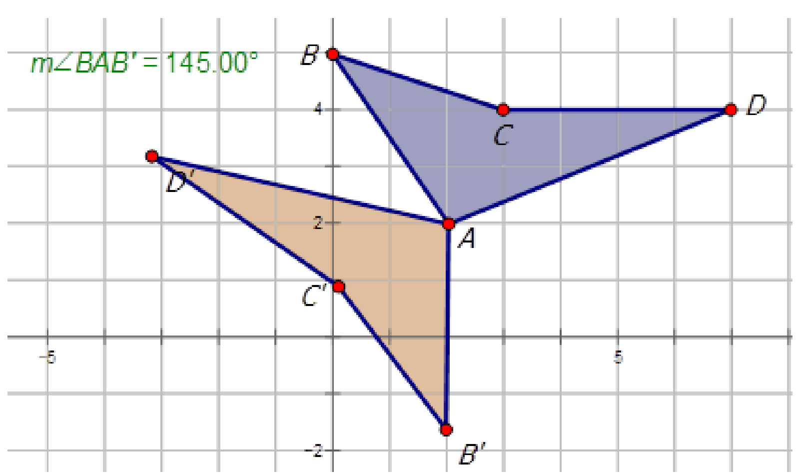
These points represent a rotation of 90°CW about the origin. Each coordinate point (x,y) has become the point $(-y,x)$.

Example 3

Describe the rotation of the blue polygon in the diagram below.



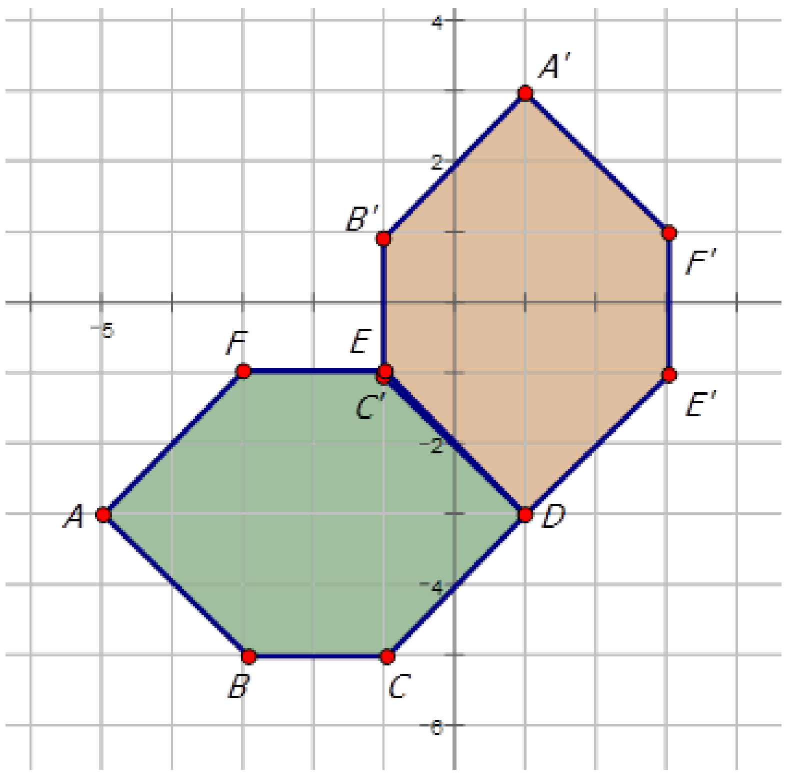
For this image, look at the rotation. It is not rotated about the origin but rather about the point A. We can measure the angle of rotation:



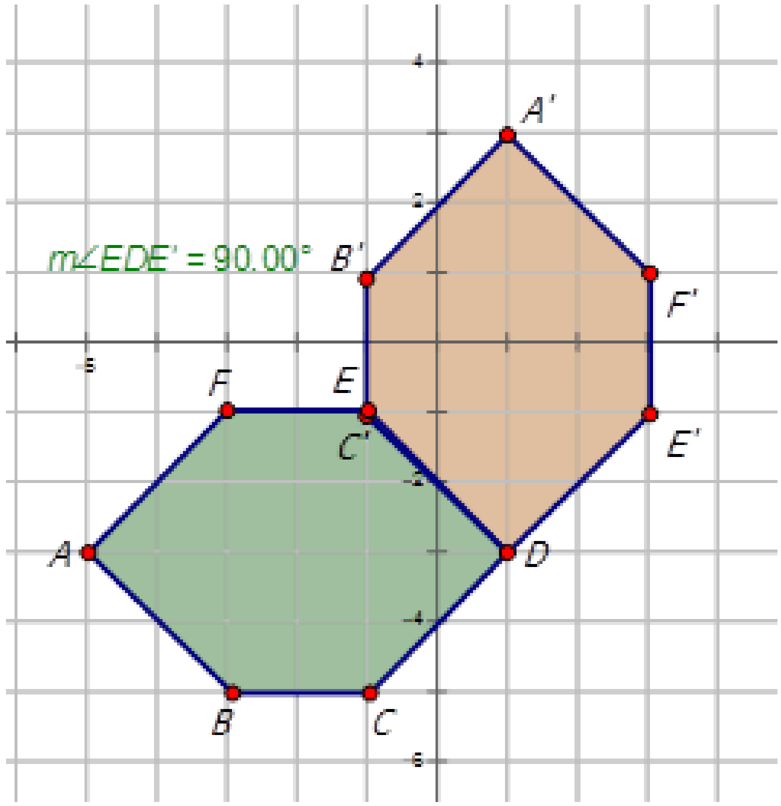
The blue polygon is being rotated about the point A 145° clockwise. You would say that the blue polygon is rotated 145°CW to form the orange polygon.

Example 4

Describe the rotation of the green hexagon in the diagram below.



For this image, look at the rotation. It is not rotated about the origin but rather about the point A. We can measure the angle of rotation:



The green polygon is being rotated about the point D 90° clockwise. You would say that the green hexagon is rotated 90°CW to form the orange hexagon.

Review

If the following points were rotated about the origin with a 180°CCW rotation, what would be the coordinates of the rotated points?

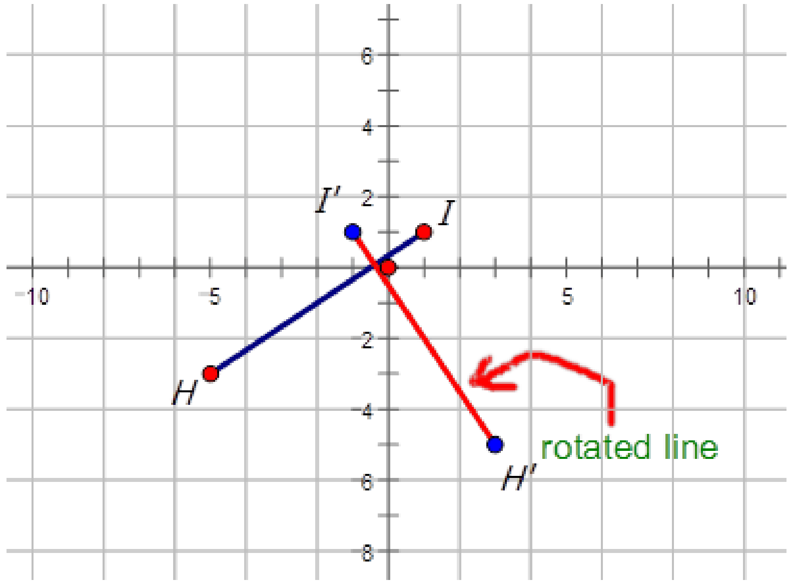
1. (3, 1)
2. (4, -2)
3. (-5, 3)
4. (-6, 4)
5. (-3, -3)

If the following points were rotated about the origin with a 90°CW rotation, what would be the coordinates of the rotated points?

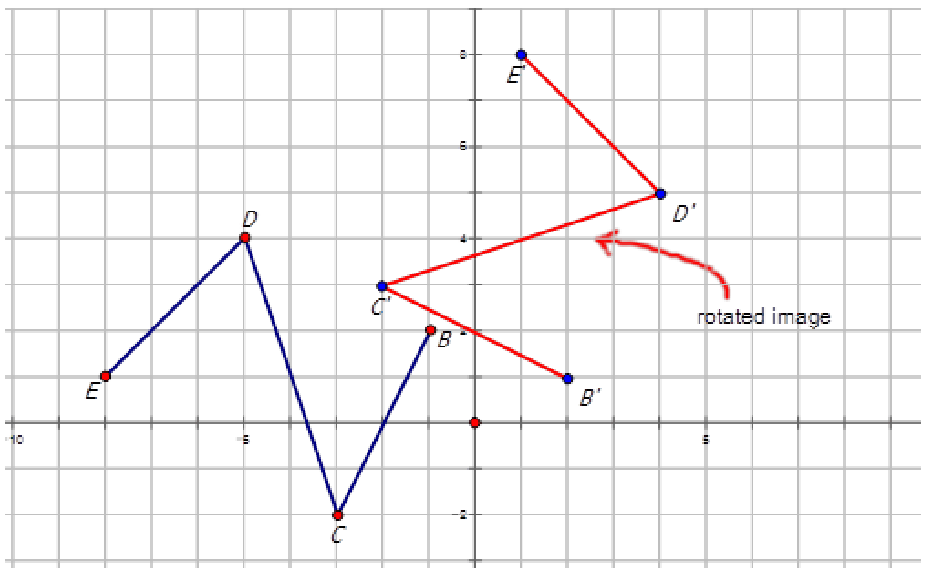
6. (-4, 3)
7. (5, -4)
8. (-5, -4)
9. (3, 3)
10. (-8, -9)

Describe the following rotations:

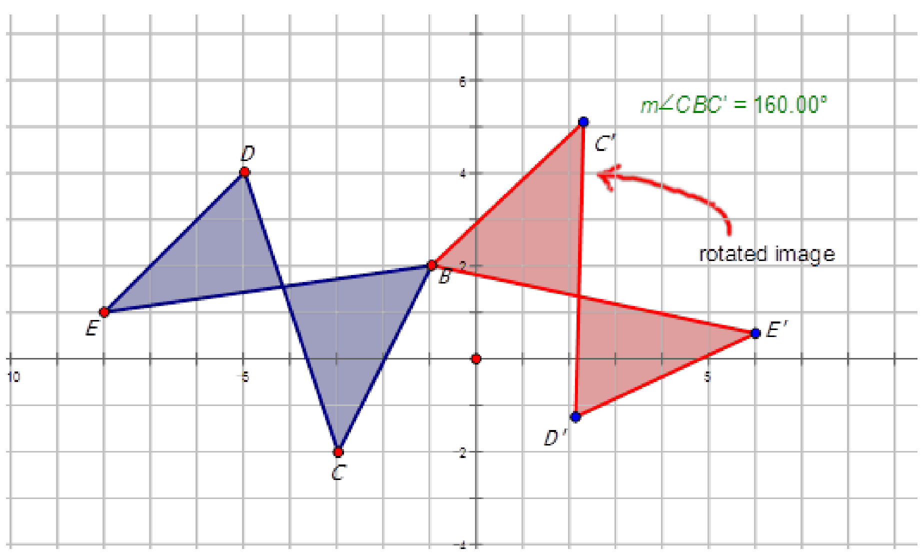
11.

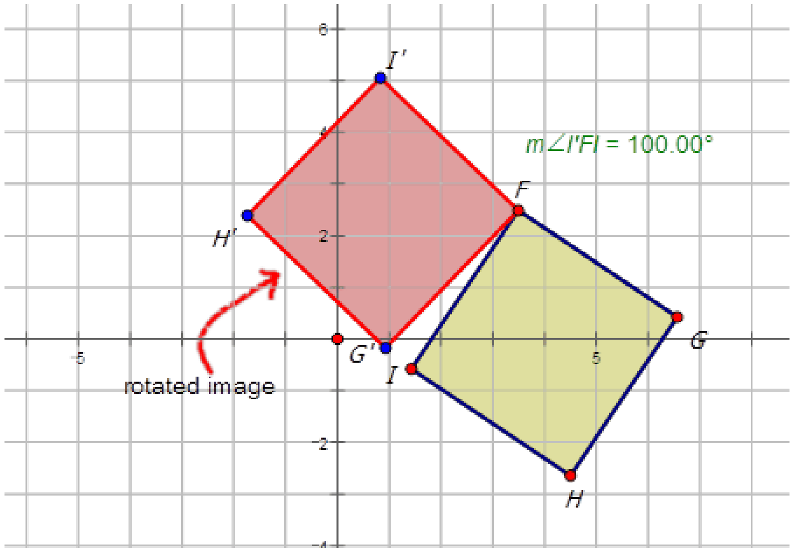

I'
I
H
H'
rotated line

12.


E'
D'
D
C'
B
B'
E
C
rotated image

13.


m∠CBC' = 160.00°
C'
D
rotated image
B
E'
E
D'
C

14. 

15. Why is it not necessary to specify the direction when rotating 180°?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.7.

10.8 Graphs of Rotations

Here you will learn how to graph a rotation.

Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane. Rotate the image 110° counterclockwise about the point X. Show the resulting image.

Graphs of Rotations

In geometry, a **transformation** is an operation that moves, flips, or changes a shape to create a new shape. A rotation is an example of a transformation where a figure is rotated about a specific point (called the center of rotation), a certain number of degrees.

For now, in order to graph a rotation in general you will use geometry software. This will allow you to rotate any figure any number of degrees about any point. There are a few common rotations that are good to know how to do without geometry software, shown in the table below.

TABLE 10.11:

Center of Rotation	**Angle of Rotation**	**Preimage (Point P)**	**Rotated Image (Point P')**
(0, 0)	90°(or -270°)	(x,y)	$(-y,x)$
(0, 0)	180°(or -180°)	(x,y)	$(-x,-y)$
(0, 0)	270°(or -90°)	(x,y)	$(y,-x)$

Multimedia

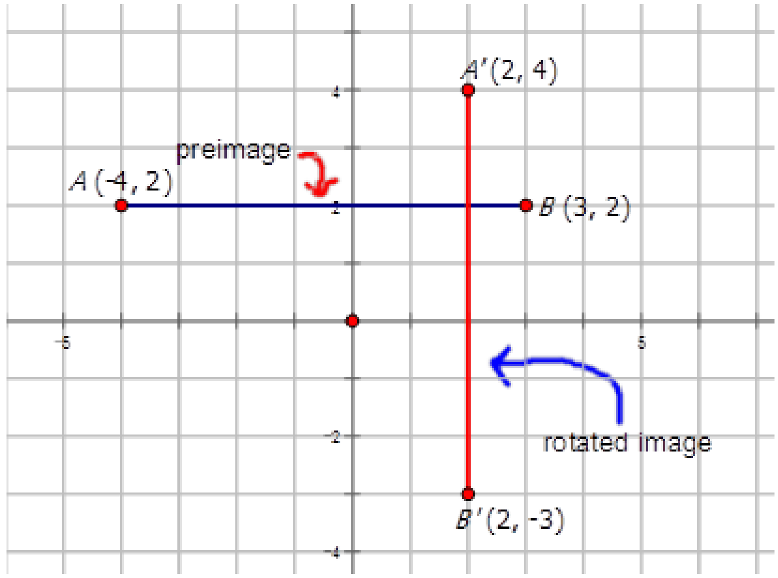
MEDIA

Click image to the left or use the URL below.

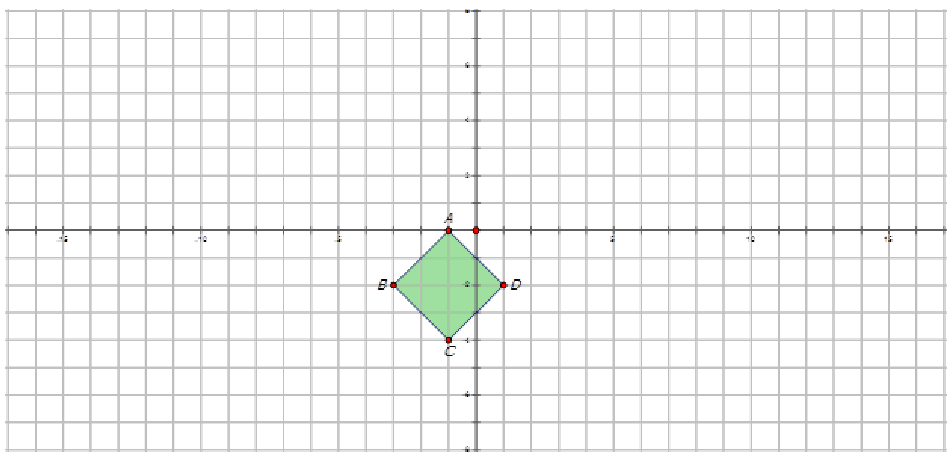
URL: https://www.ck12.org/flx/render/embeddedobject/65244

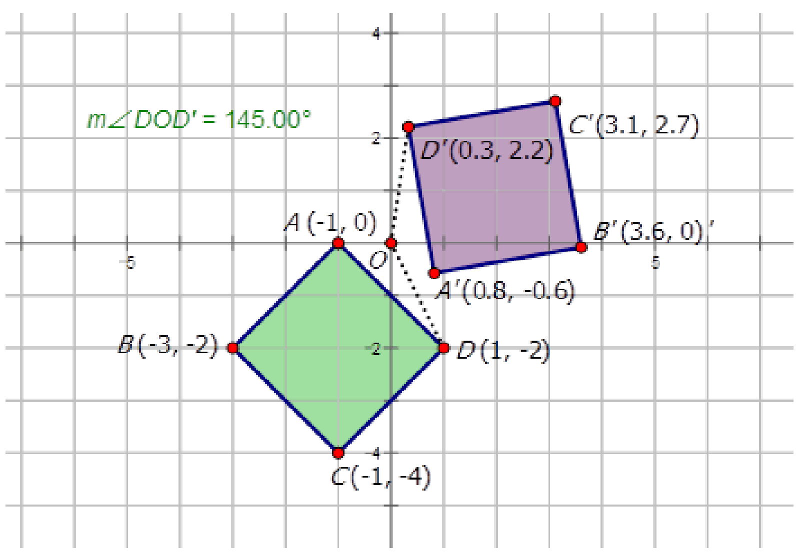
Draw the preimage and image and properly label each for the following transformation:

Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has been rotated about the origin at an angle of 90°CW.



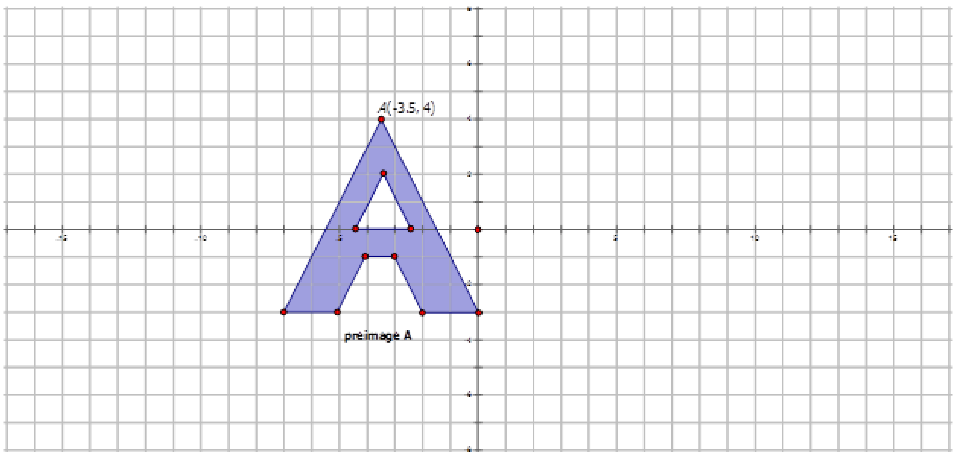
On the diagram below, draw and label the rotated image for the rotation: The diamond

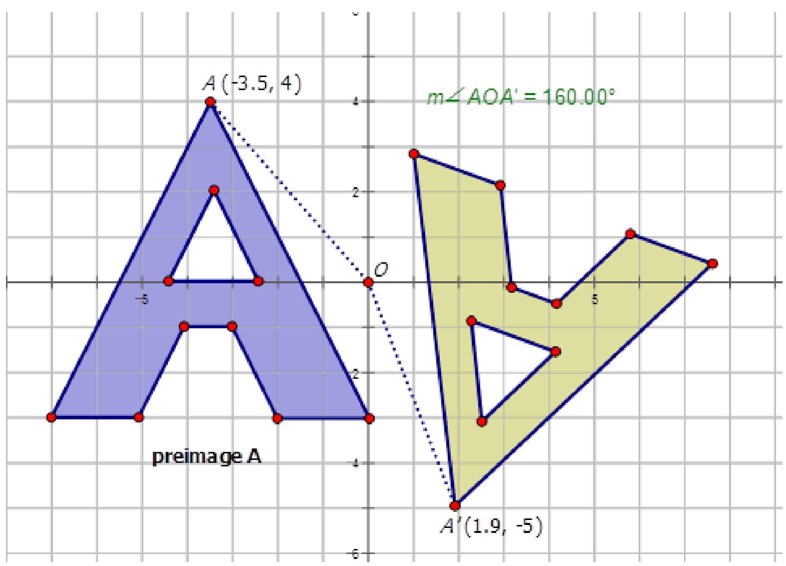




Notice the direction is counter-clockwise.

On the diagram below, draw and label the image for the rotation: The following figure is rotated about the origin





Notice the direction of the rotation is counter-clockwise, therefore the angle of rotation is 160°.

MEDIA

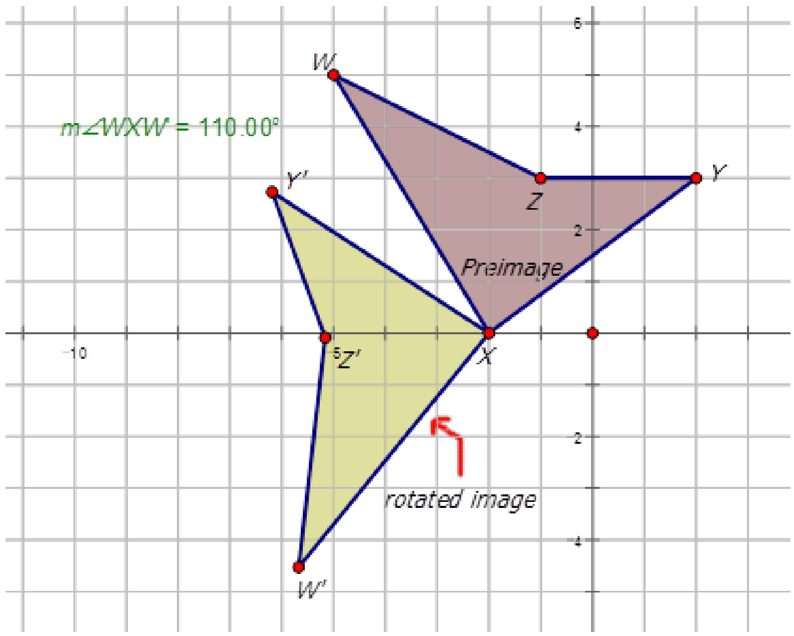
Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65245

Examples

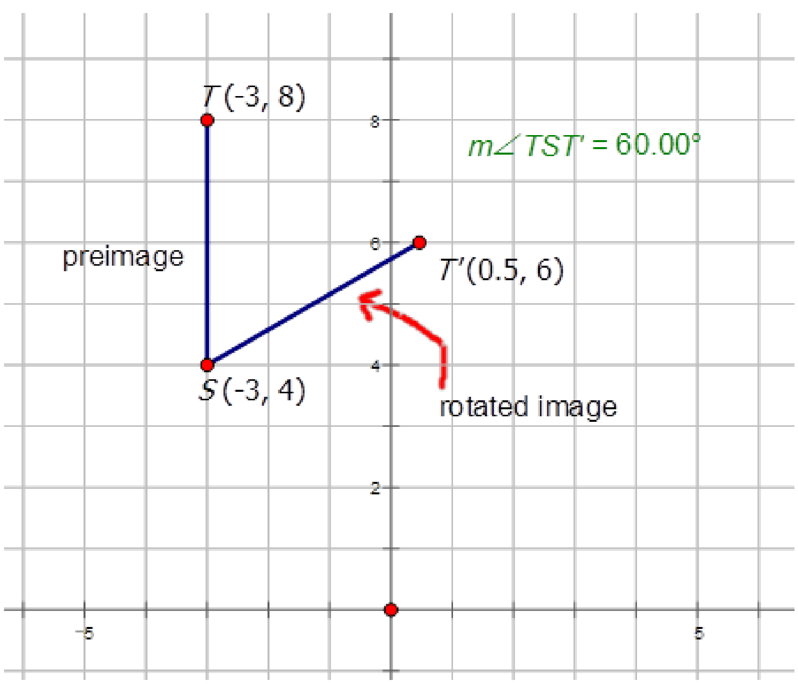
Example 1

Earlier, you were asked about a quadrilateral. Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane. Rotate the image 110° counterclockwise about the point X. Show the resulting image.



Example 2

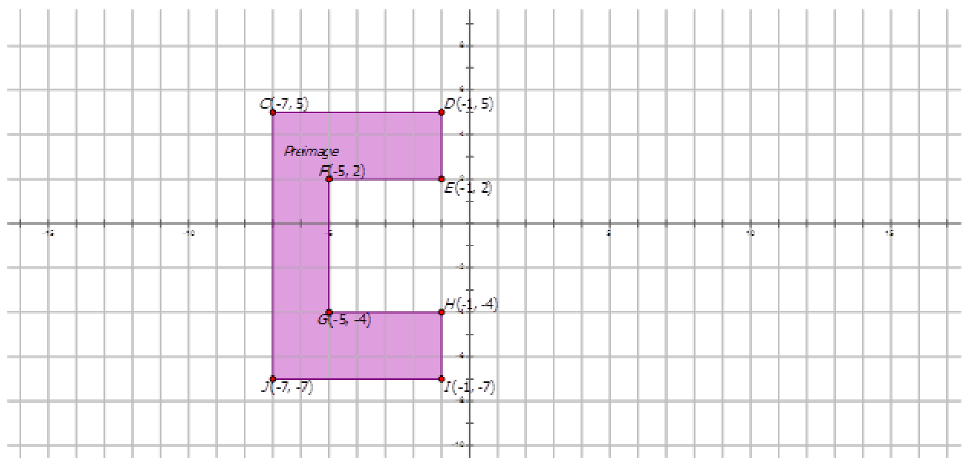
Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has been rotated 60°CW about the point S. Draw the preimage and image and properly label each.



Notice the direction of the angle is clockwise, therefore the angle measure is 60°CW or −60°.

Example 3

The polygon below has been rotated 155°CCW about the origin. Draw the rotated image and properly label each.



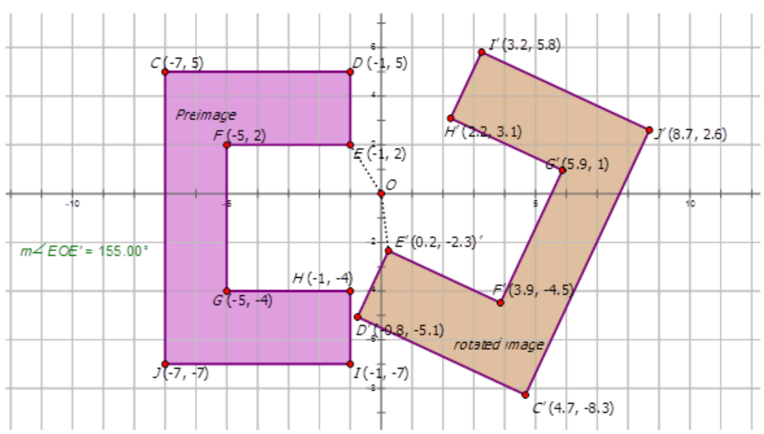
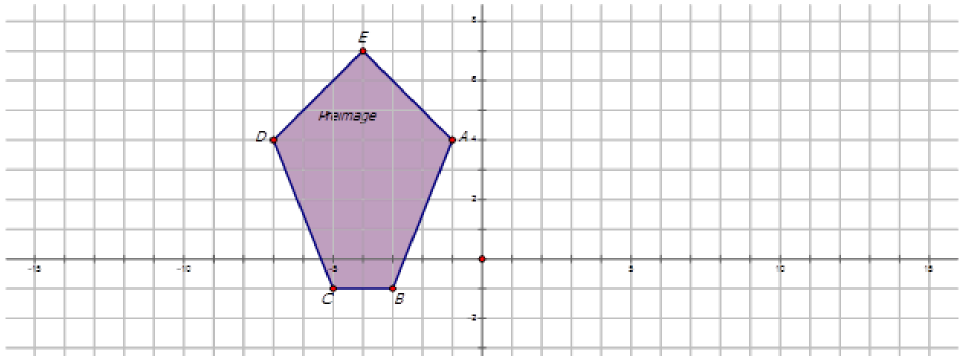


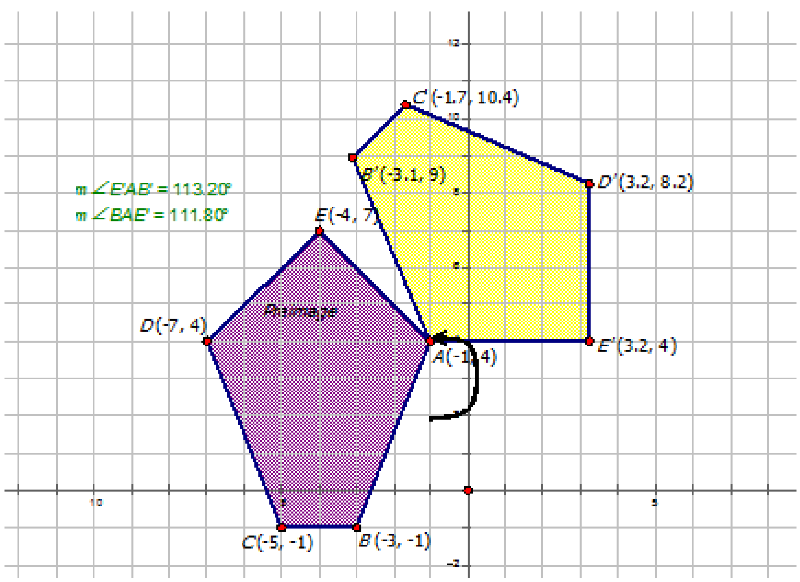
FIGURE 10.1

Notice the direction of the angle is counter-clockwise, therefore the angle measure is 155°CCW or 155°.

Example 4

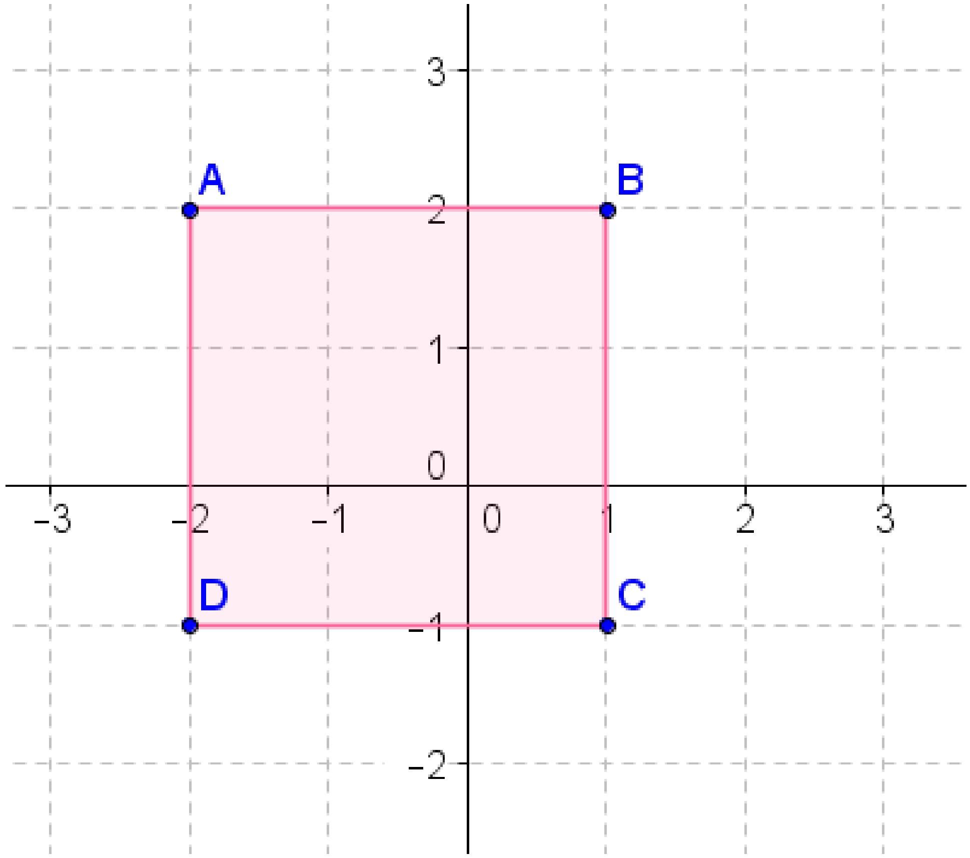
The purple pentagon is rotated about the point A 225°. Find the coordinates of the purple pentagon. On the diagram, draw and label the rotated pentagon.



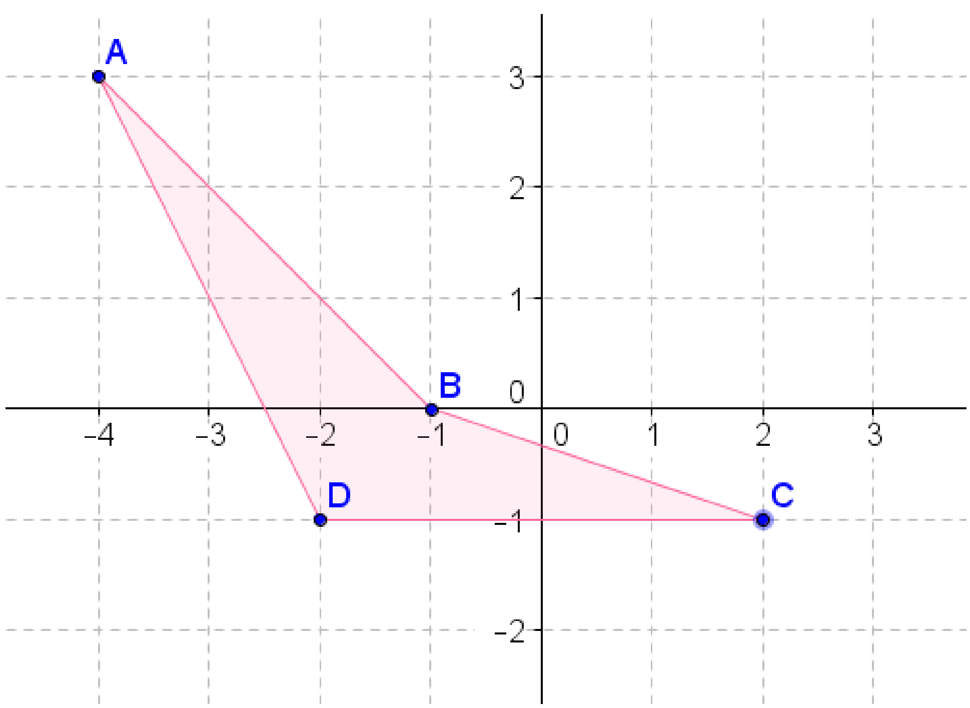


The measure of $\angle BAB' = m\angle BAE' + m\angle E'AB'$. Therefore $\angle BAB' = 111.80° + 113.20°$ or $225°$. Notice the direction of the angle is counter-clockwise, therefore the angle measure is $225°$CCW or $225°$.

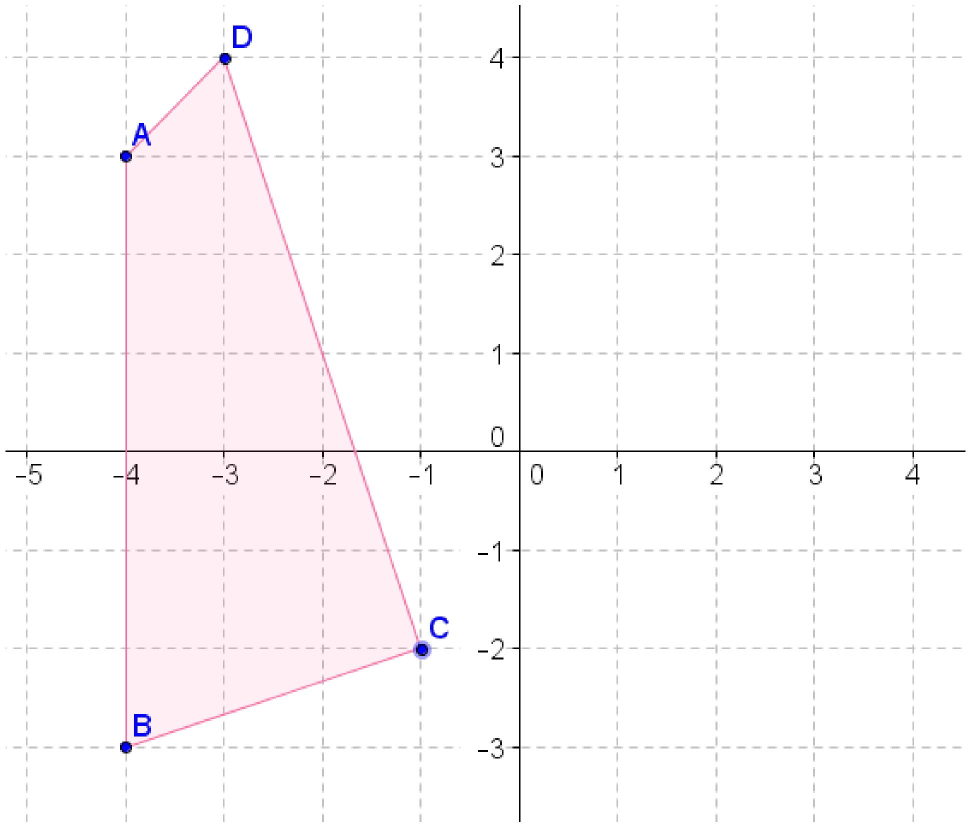
Review



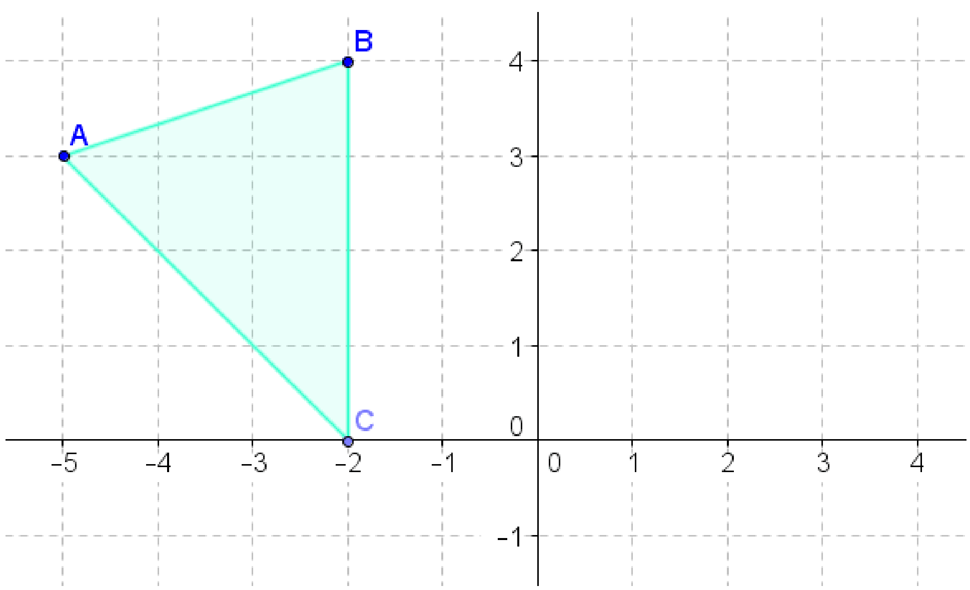
1. Rotate the above figure 90° clockwise about the origin.
2. Rotate the above figure 270° clockwise about the origin.
3. Rotate the above figure 180° about the origin.



4. Rotate the above figure 90° counterclockwise about the origin.
5. Rotate the above figure 270° counterclockwise about the origin.
6. Rotate the above figure 180° about the origin.

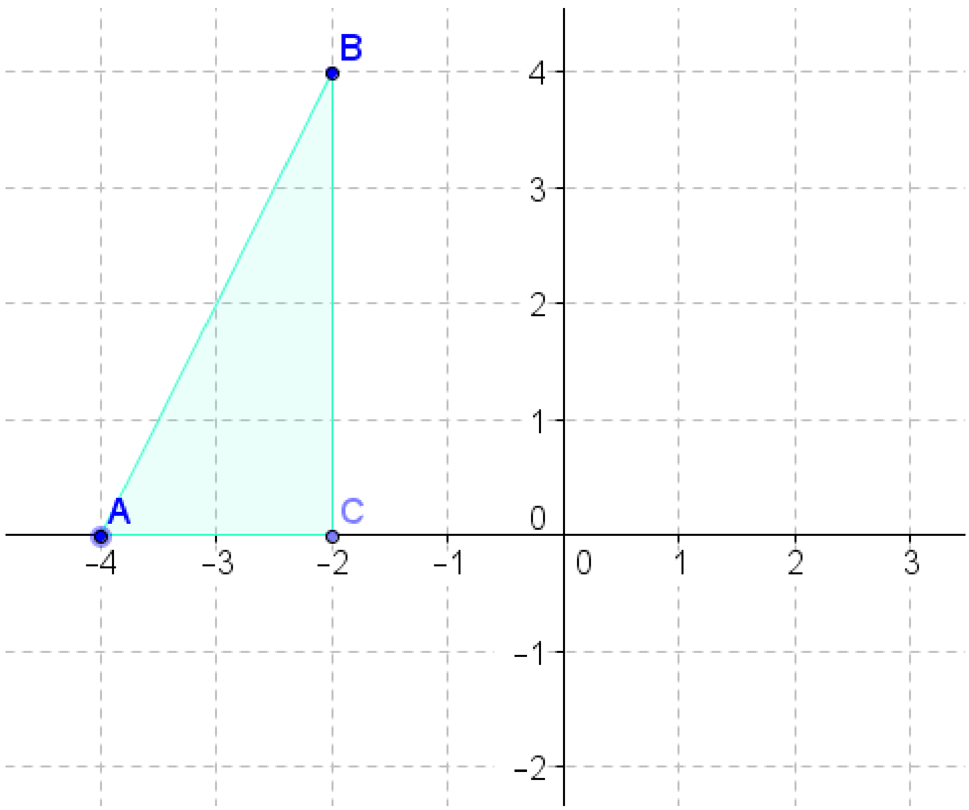


7. Rotate the above figure 90° clockwise about the origin.
8. Rotate the above figure 270° clockwise about the origin.
9. Rotate the above figure 180° about the origin.

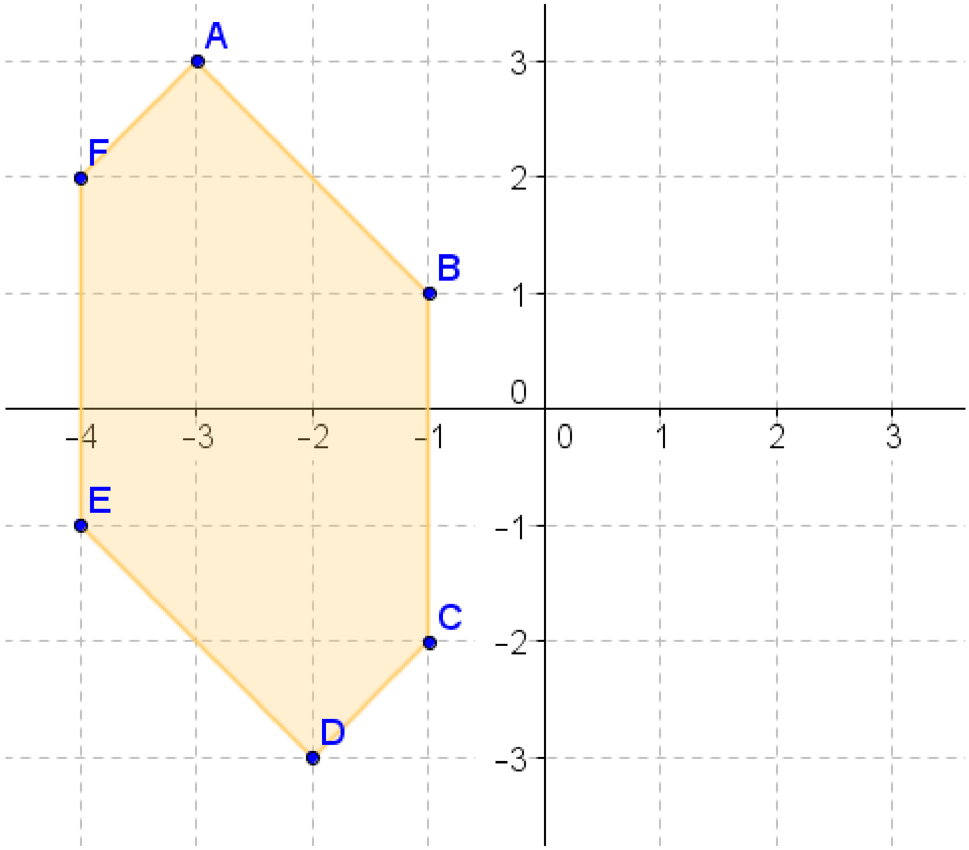


10. Rotate the above figure 90° counterclockwise about the origin.

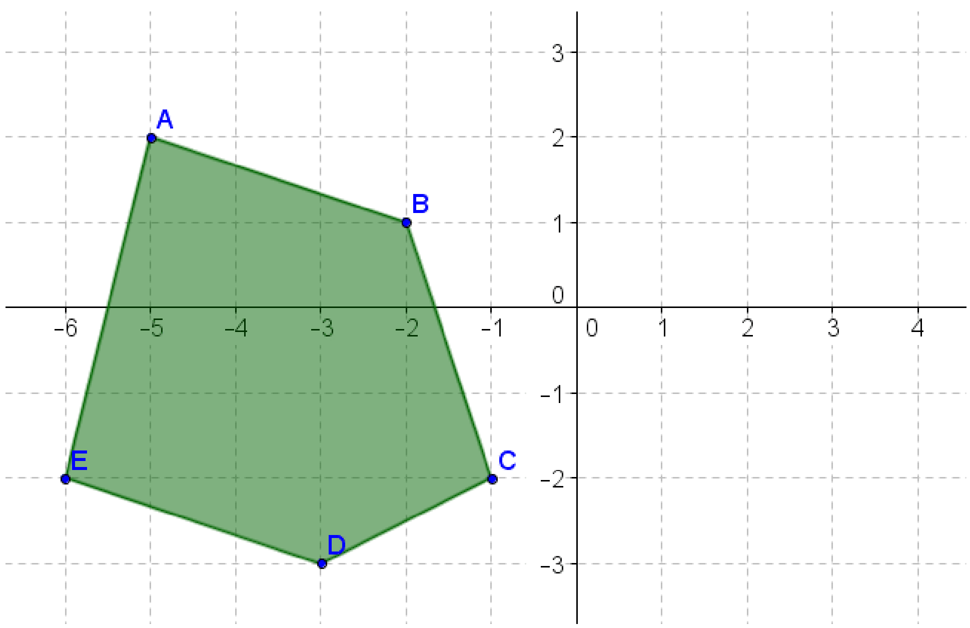
11. Rotate the above figure 270° counterclockwise about the origin.
12. Rotate the above figure 180° about the origin.



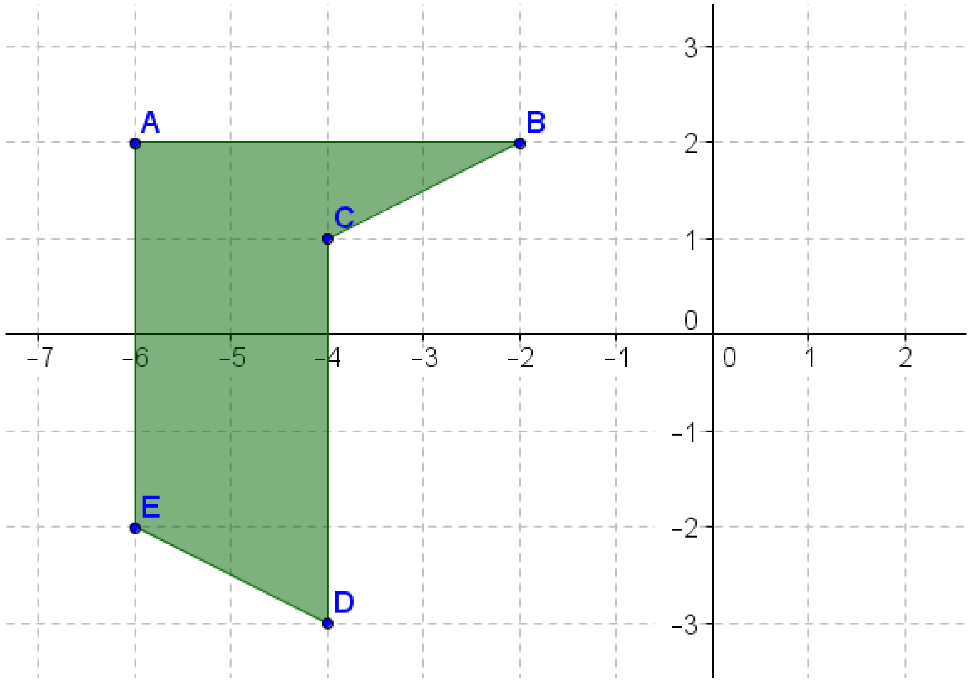
13. Rotate the above figure 90° clockwise about the origin.
14. Rotate the above figure 270° clockwise about the origin.
15. Rotate the above figure 180° about the origin.



16. Rotate the above figure 90° counterclockwise about the origin.
17. Rotate the above figure 270° counterclockwise about the origin.
18. Rotate the above figure 180° about the origin.



19. Rotate the above figure 90° clockwise about the origin.
20. Rotate the above figure 270° clockwise about the origin.
21. Rotate the above figure 180° about the origin.



22. Rotate the above figure 90° counterclockwise about the origin.
23. Rotate the above figure 270° counterclockwise about the origin.
24. Rotate the above figure 180° about the origin.

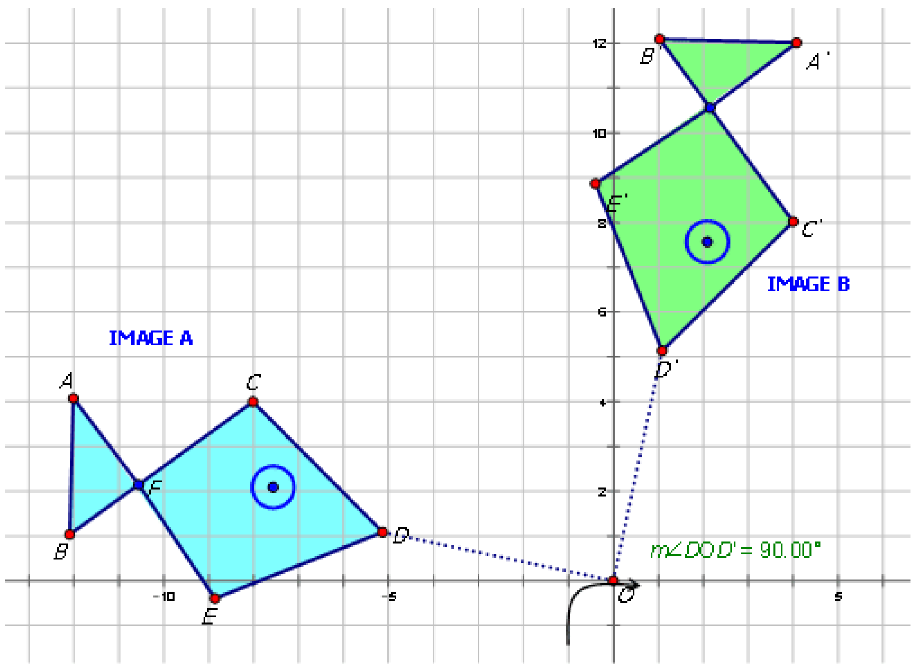
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.8.

10.9 Rules for Rotations

Here you will learn the notation used for rotations.

The figure below shows a pattern of two fish. Write the mapping rule for the rotation of Image A to Image B.



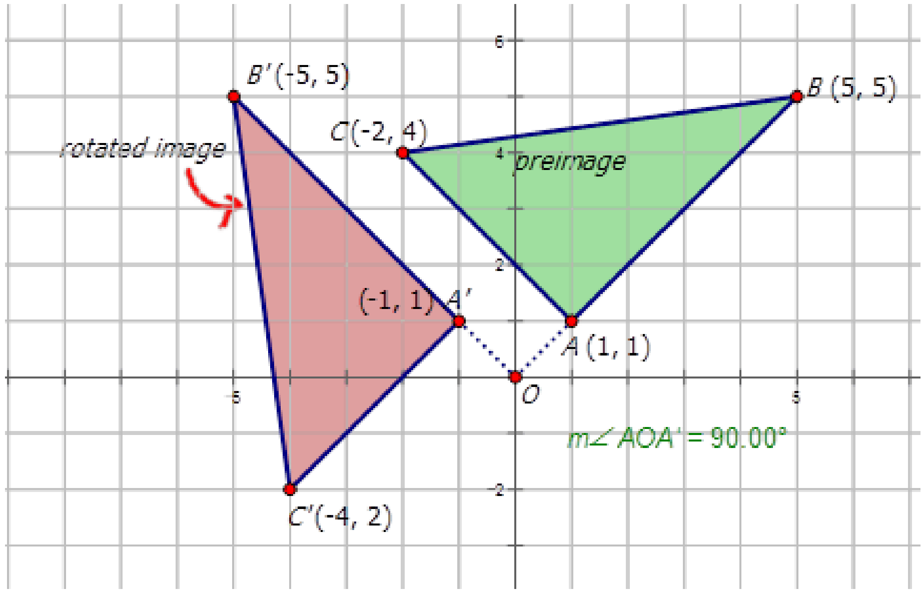
Rules for Rotations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A rotation is an example of a transformation where a figure is rotated about a specific point (called the center of rotation), a certain number of degrees. Common rotations about the origin are shown below:

TABLE 10.12:

Center of Rotation	Angle of Rotation	Preimage (Point P)	Rotated Image (Point P')	Notation (Point P')
(0, 0)	90°(or −270°)	(x,y)	$(-y,x)$	$(x,y) \rightarrow (-y,x)$
(0, 0)	180°(or −180°)	(x,y)	$(-x,-y)$	$(x,y) \rightarrow (-x,-y)$
(0, 0)	270°(or −90°)	(x,y)	$(y,-x)$	$(x,y) \rightarrow (y,-x)$

You can describe rotations in words, or with notation. Consider the image below:



Notice that the preimage is rotated about the origin 90°CCW. If you were to describe the rotated image using notation, you would write the following:

$$R_{90^\circ}(x,y) = (-y,x)$$

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65258

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65259

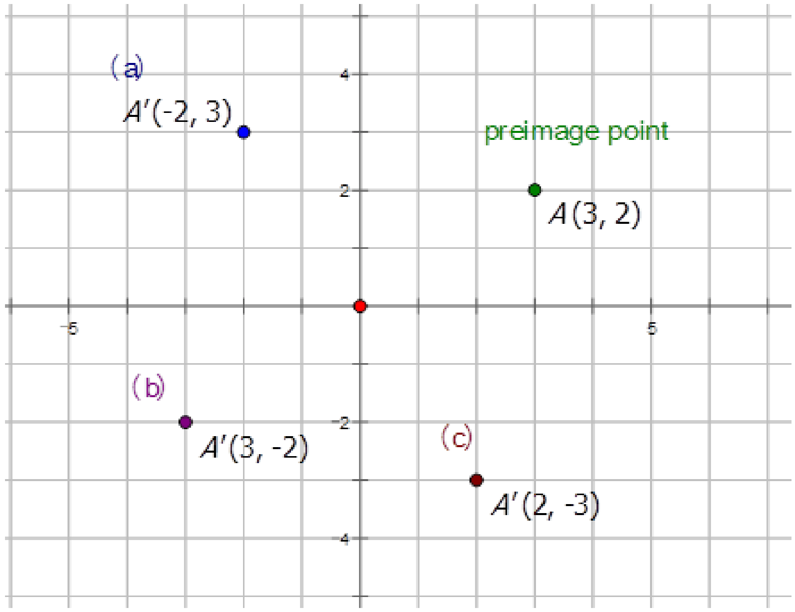
Find an image of the point (3, 2) that has undergone a clockwise rotation:

a) about the origin at 90°,

b) about the origin at 180°, and

c) about the origin at 270°.

Write the notation to describe the rotation.



a) Rotation about the origin at 90° : $R_{90^\circ}(x,y) = (-y,x)$

b) Rotation about the origin at 180° : $R_{180^\circ}(x,y) = (-x,-y)$

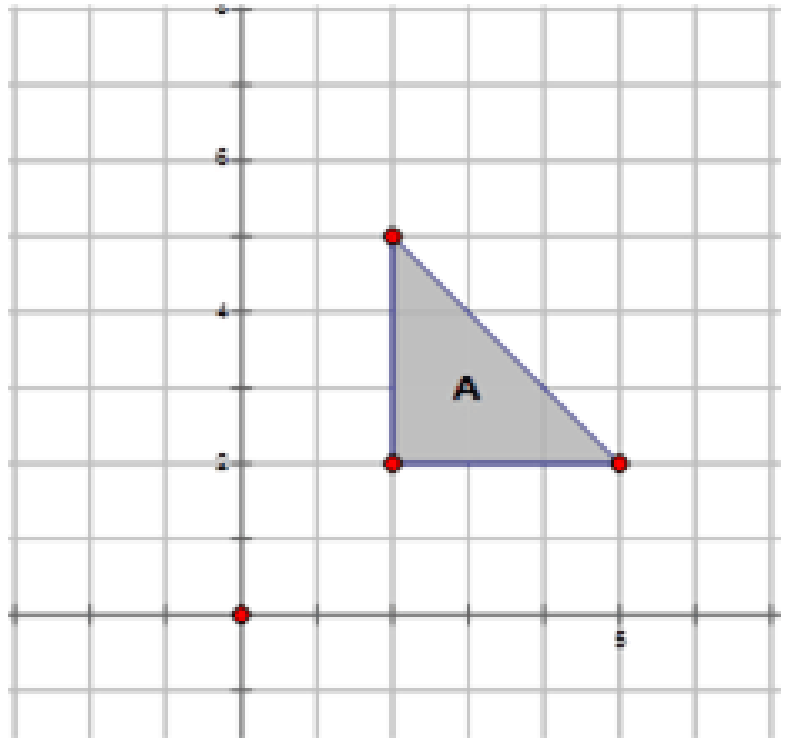
c) Rotation about the origin at 270° : $R_{270^\circ}(x,y) = (y,-x)$

Rotate Image A in the diagram below:

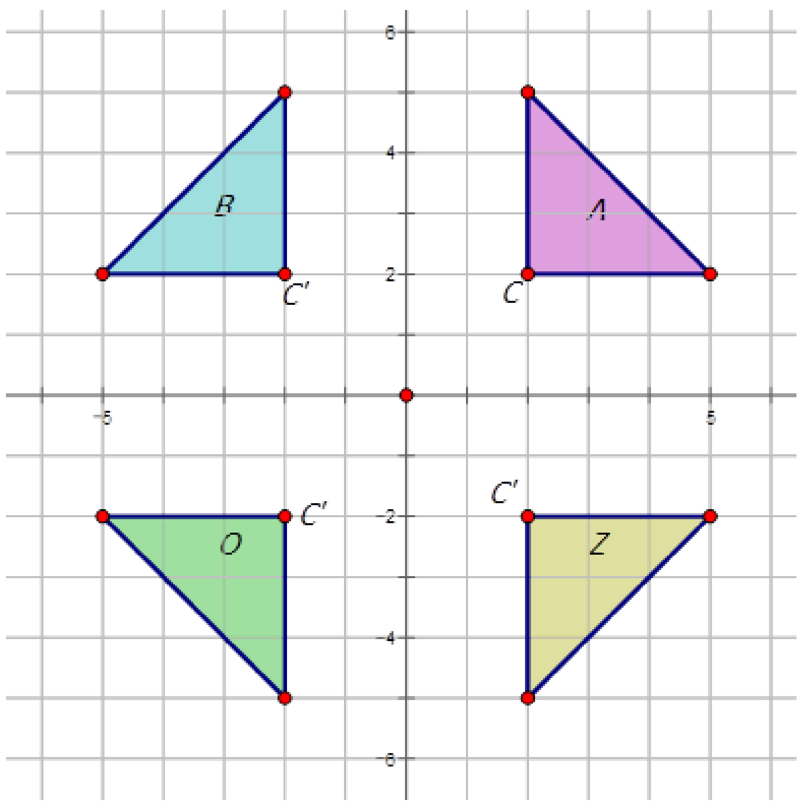
a) about the origin at 90°, and label it B.

b) about the origin at 180°, and label it O.

c) about the origin at 270°, and label it Z.



Write notation for each to indicate the type of rotation.

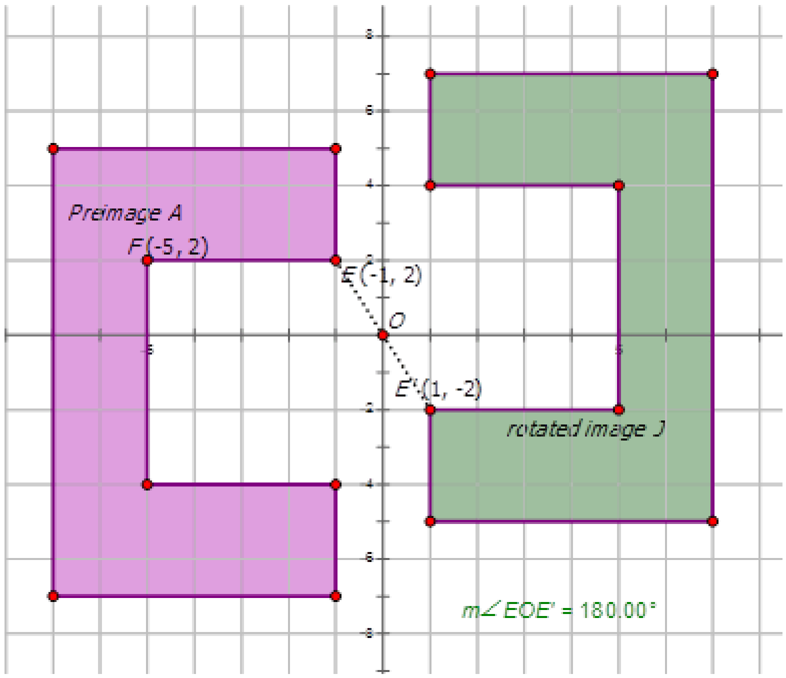


a) Rotation about the origin at 90°: $R_{90^\circ}A \rightarrow B = R_{90^\circ}(x,y) \rightarrow (-y,x)$

b) Rotation about the origin at 180°: $R_{180^\circ}A \rightarrow O = R_{180^\circ}(x,y) \rightarrow (-x,-y)$

c) Rotation about the origin at 270°: $R_{270^\circ}A \rightarrow Z = R_{270^\circ}(x,y) \rightarrow (y,-x)$

Write the notation that represents the rotation of the preimage A to the rotated image J in the diagram below.



First, pick a point in the diagram to use to see how it is rotated.

$$E:(-1,2) \quad E':(1,-2)$$

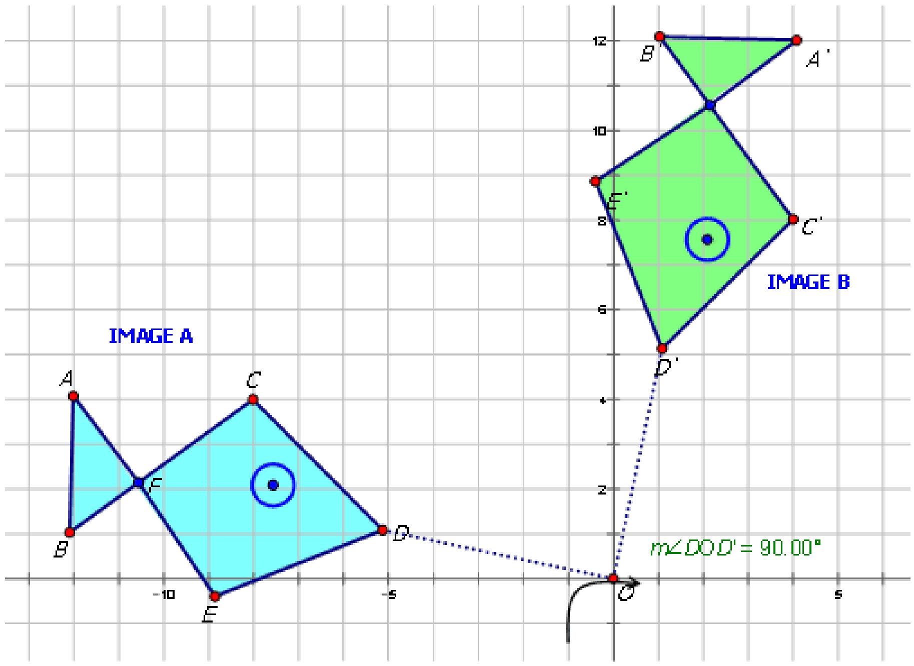
Notice how both the x- and y-coordinates are multiplied by -1. This indicates that the preimage A is reflected about the origin by 180°CCW to form the rotated image J. Therefore the notation is $R_{180^\circ}A \rightarrow J = R_{180^\circ}(x,y) \rightarrow (-x,-y)$.

Examples

Example 1

Earlier, you were given a problem about a figure.

The figure below shows a pattern of two fish. Write the mapping rule for the rotation of Image A to Image B.

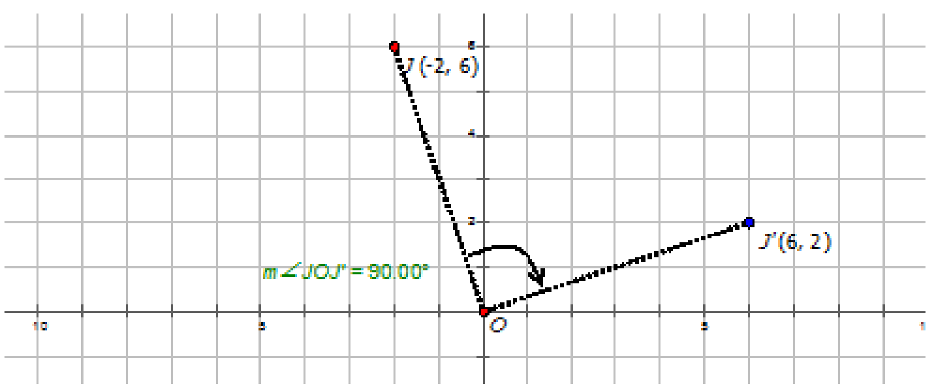


Notice that the angle measure is 90° and the direction is clockwise. Therefore the Image A has been rotated $-90°$ to form Image B. To write a rule for this rotation you would write: $R_{270°}(x,y) = (-y,x)$.

Example 2

Thomas describes a rotation as point J moving from $J(-2,6)$ to $J'(6,2)$. Write the notation to describe this rotation for Thomas.

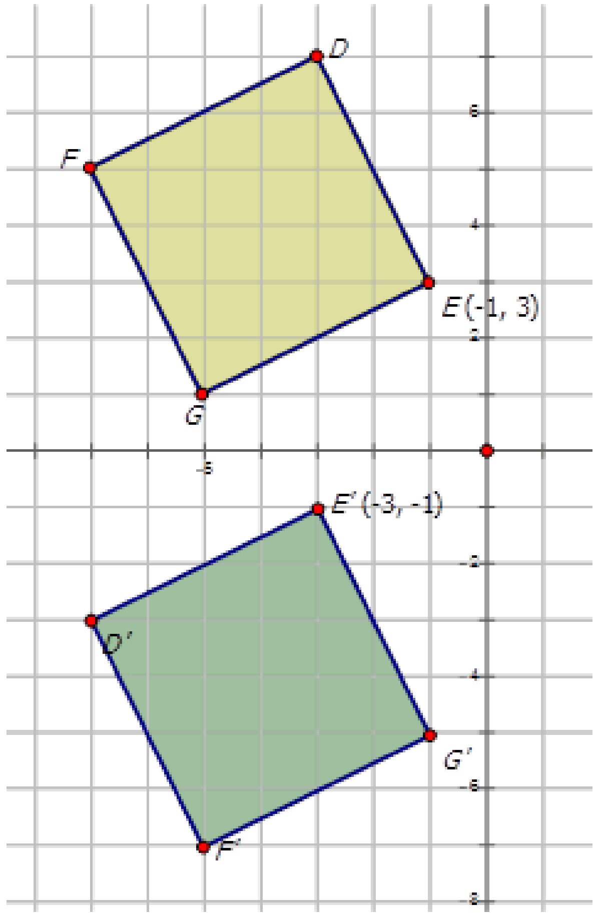
$J:(-2,6) \quad J':(6,2)$



Since the x-coordinate is multiplied by -1, the y-coordinate remains the same, and finally the x- and y-coordinates change places, this is a rotation about the origin by 270° or $-90°$. The notation is: $R_{270°}J \rightarrow J' = R_{270°}(x,y) \rightarrow (y,-x)$

Example 3

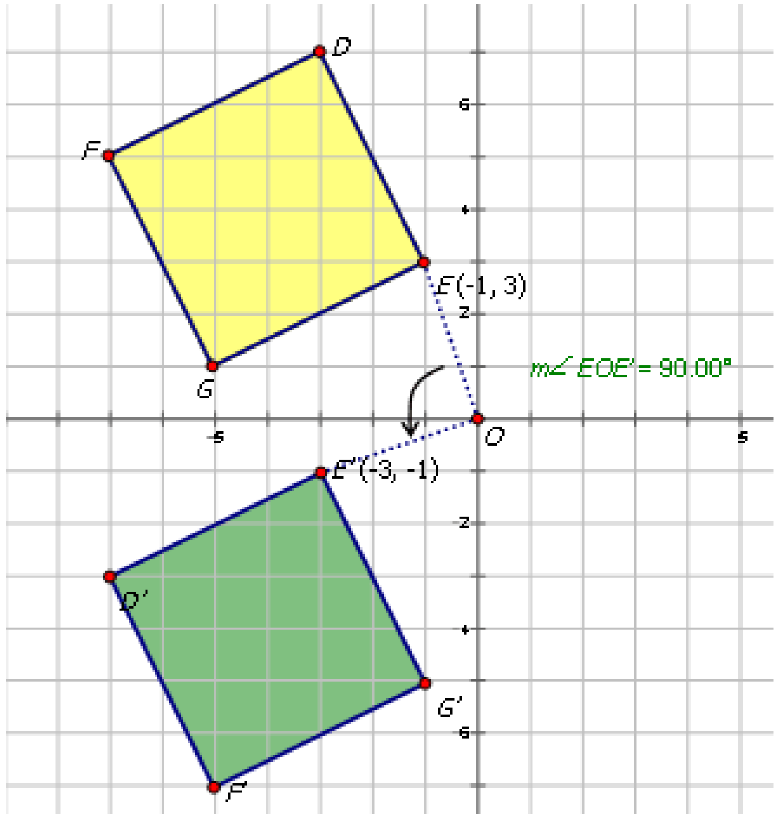
Write the notation that represents the rotation of the yellow diamond to the rotated green diamond in the diagram below.



In order to write the notation to describe the rotation, choose one point on the preimage (the yellow diamond) and then the rotated point on the green diamond to see how the point has moved. Notice that point E is shown in the diagram:

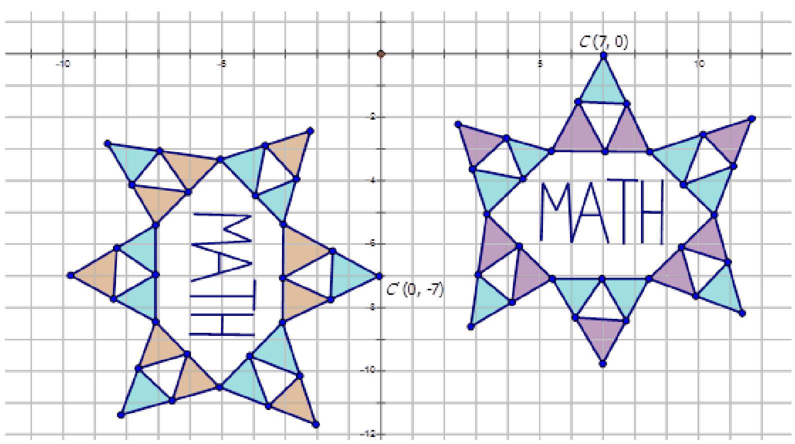
$$E(-1,3) \rightarrow E'(-3,-1)$$

Since both x- and y-coordinates are reversed places and the y-coordinate has been multiplied by -1, the rotation is about the origin 90°. The notation for this rotation would be: $R_{90^\circ}(x,y) \rightarrow (-y,x)$.



Example 4

Karen was playing around with a drawing program on her computer. She created the following diagrams and then wanted to determine the transformations. Write the notation rule that represents the transformation of the purple and blue diagram to the orange and blue diagram.



In order to write the notation to describe the transformation, choose one point on the preimage (purple and blue diagram) and then the transformed point on the orange and blue diagram to see how the point has moved. Notice that point C is shown in the diagram:

$$C(7,0) \rightarrow C'(0,-7)$$

Since the x-coordinates only are multiplied by -1, and then x- and y-coordinates change places, the transformation is a rotation is about the origin by 270°. The notation for this rotation would be: $R_{270^\circ}(x,y) \rightarrow (y,-x)$.

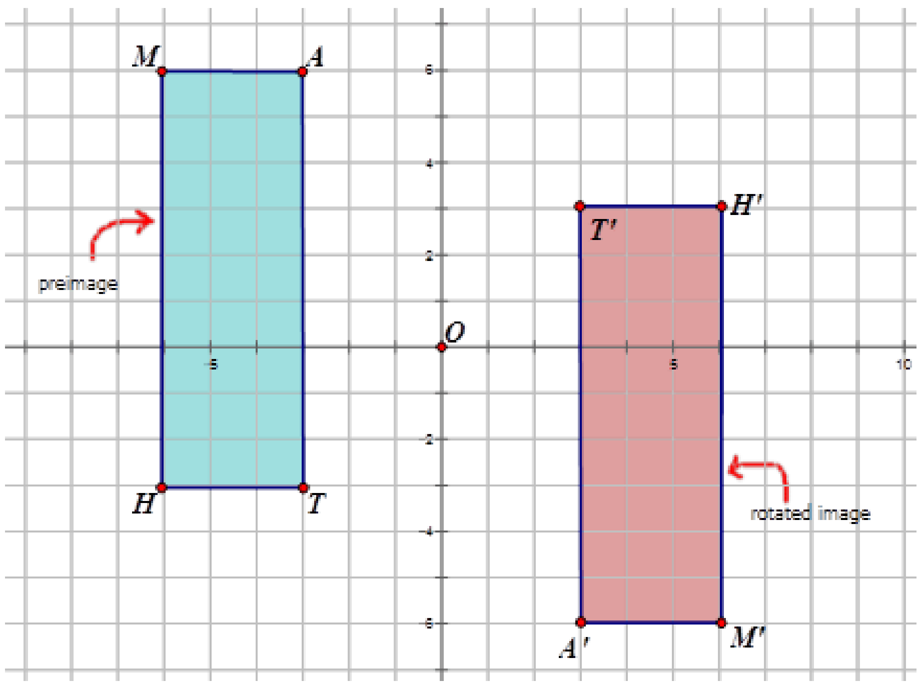
Review

Complete the following table:

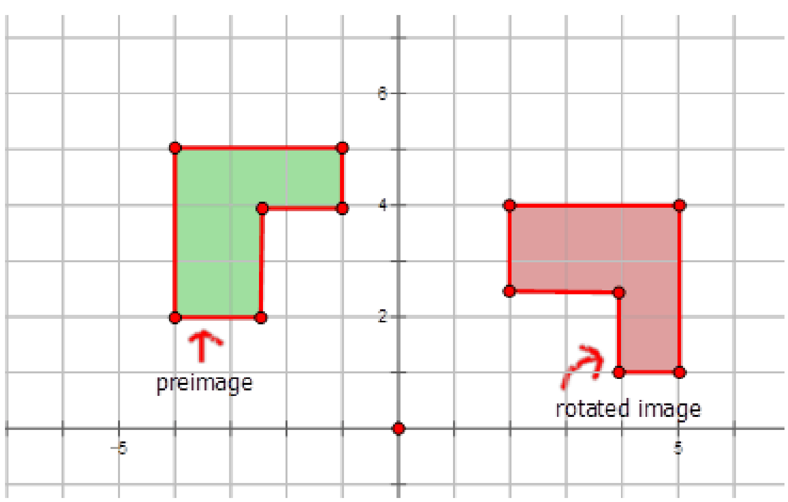
TABLE 10.13:

Starting Point	90° **Rotation**	180° **Rotation**	270° **Rotation**	360° **Rotation**
1. (1, 4)				
2. (4, 2)				
3. (2, 0)				
4. (-1, 2)				
5. (-2, -3)				

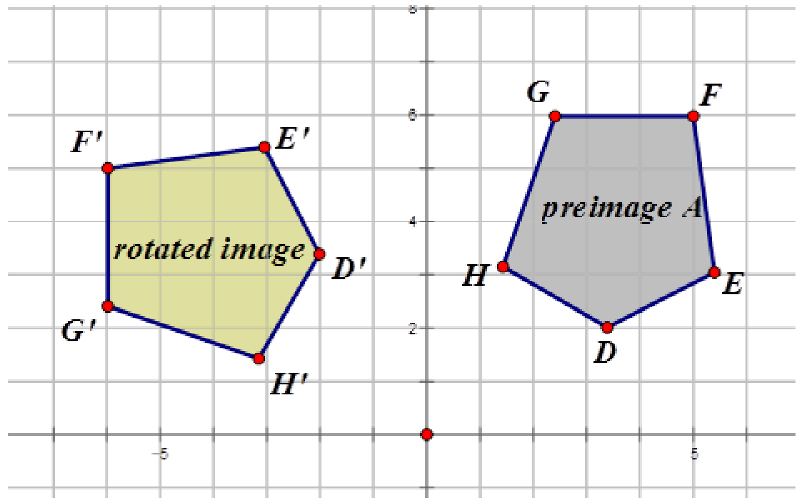
Write the notation that represents the rotation of the preimage to the image for each diagram below.

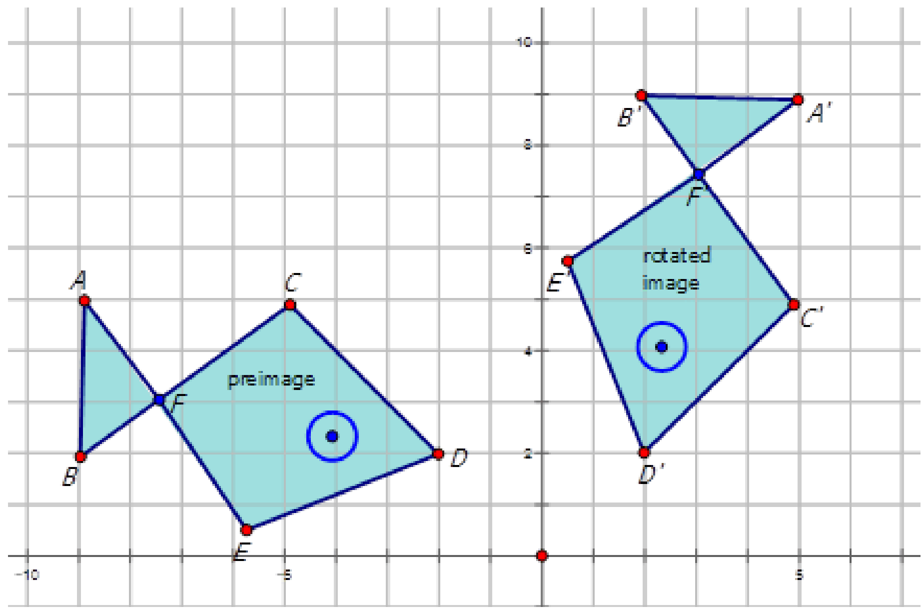


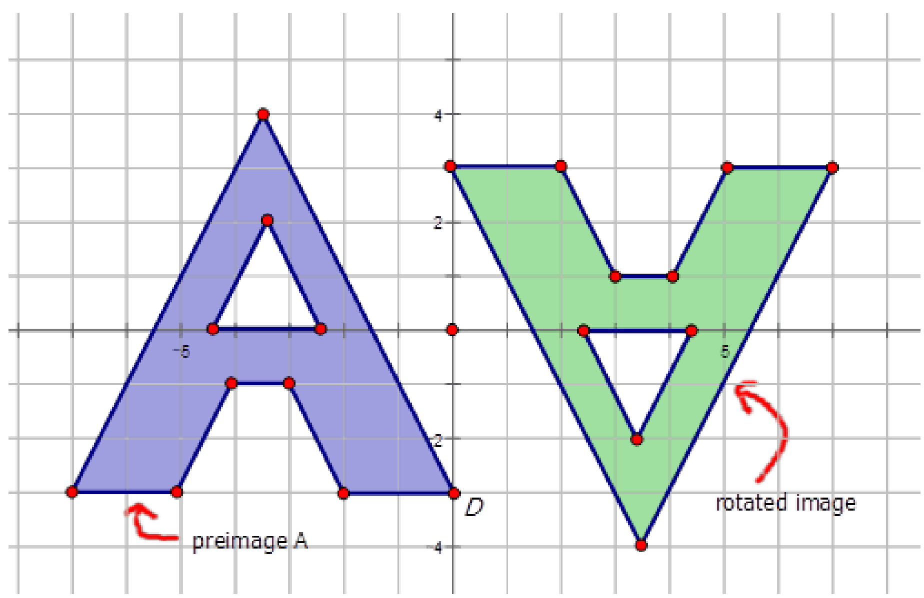
6.



7.

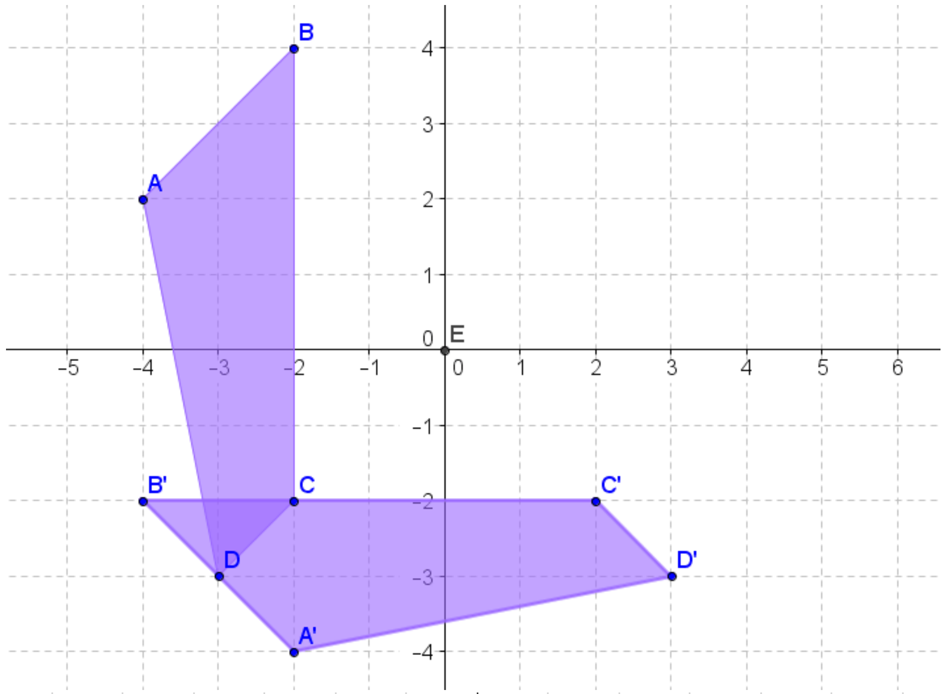
8. 

9. 

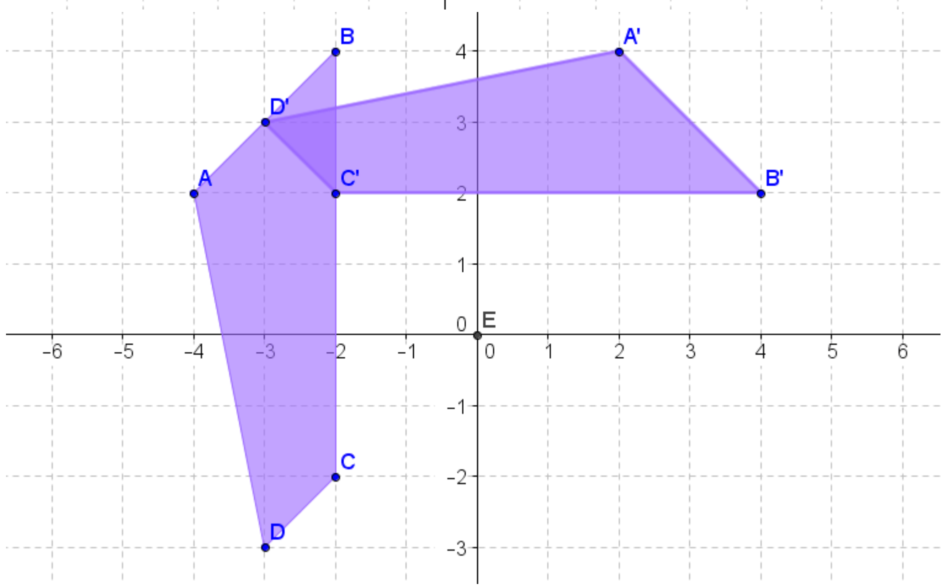
10. 

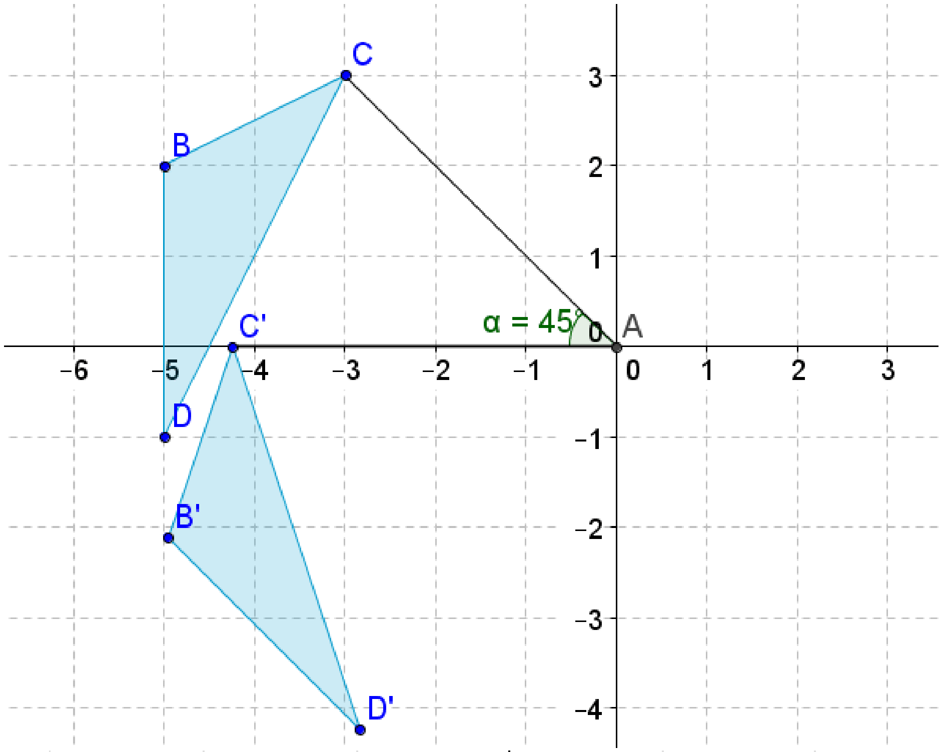
Write the notation that represents the rotation of the preimage to the image for each diagram below.

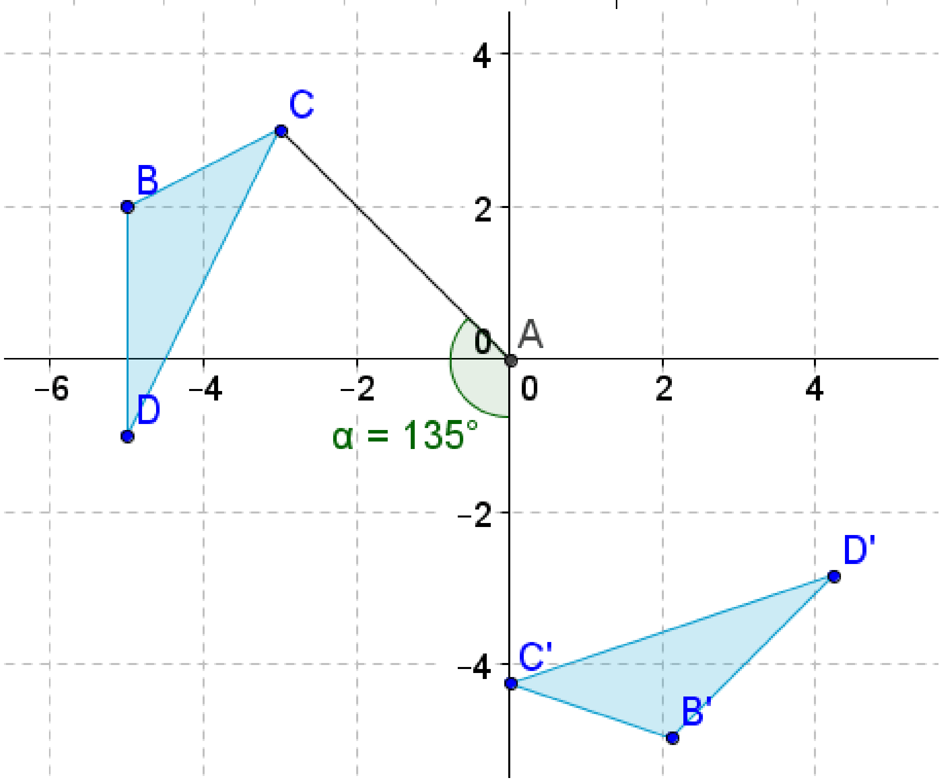
11.

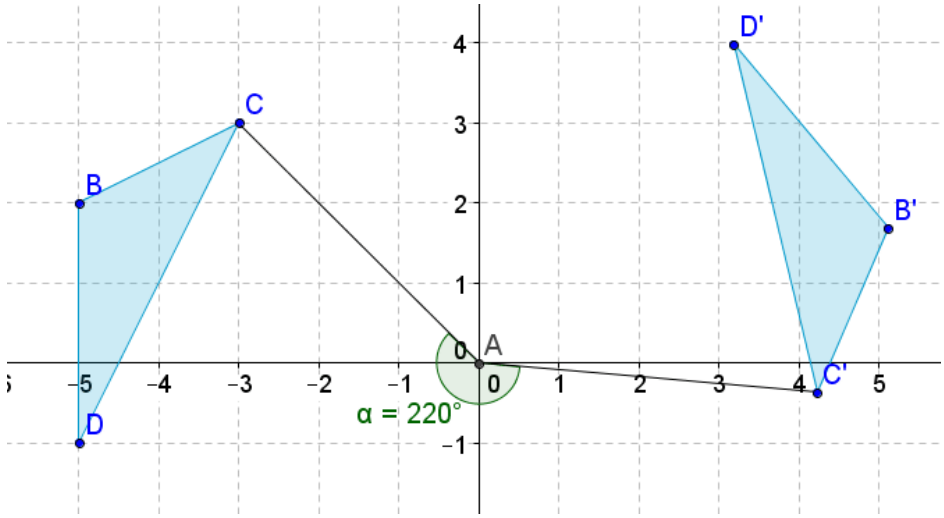


12.



13. 
C
B
C'
α = 45°
A
D
B'
D'
3
2
1
0
−1
−2
−3
−4
−6
−5
−4
−3
−2
−1
0
1
2
3

14. 
C
B
A
D
α = 135°
D'
C'
B'
4
2
0
−2
−4
−6
−4
−2
0
2
4

15. 

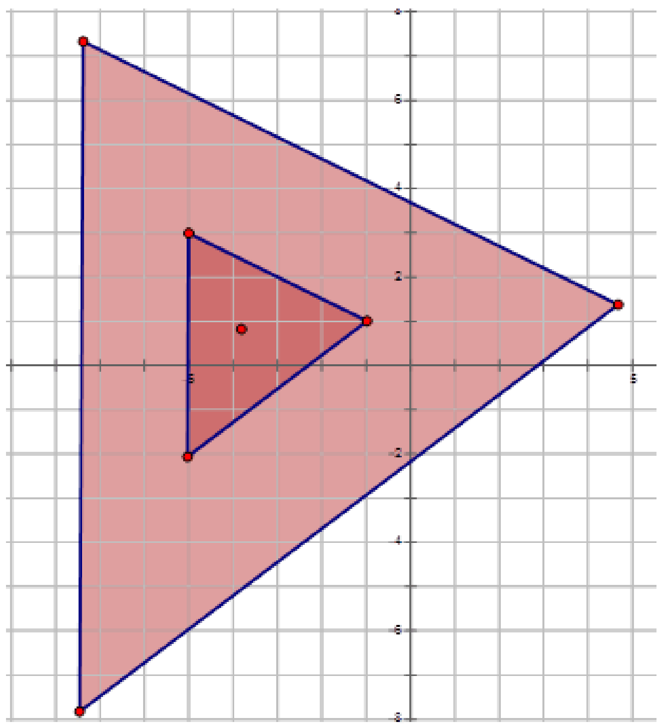
Answers for Review Problems

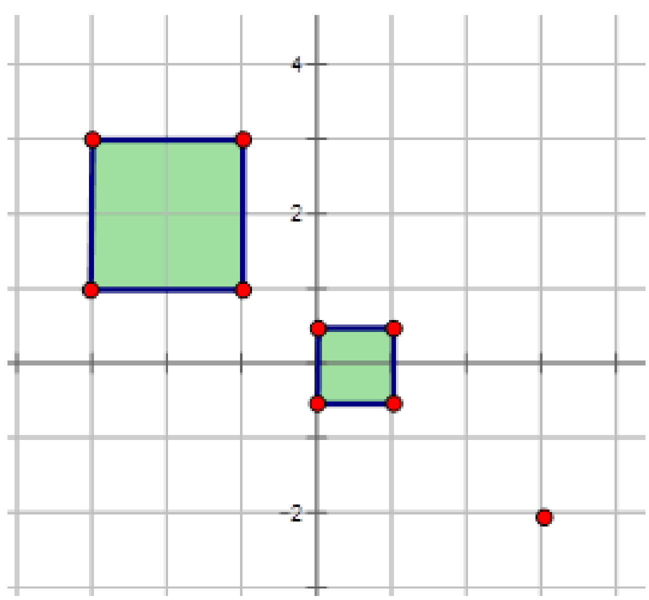
To see the Review answers, open this PDF file and look for section 10.9.

10.10 Dilations

Here you will learn about geometric dilations.

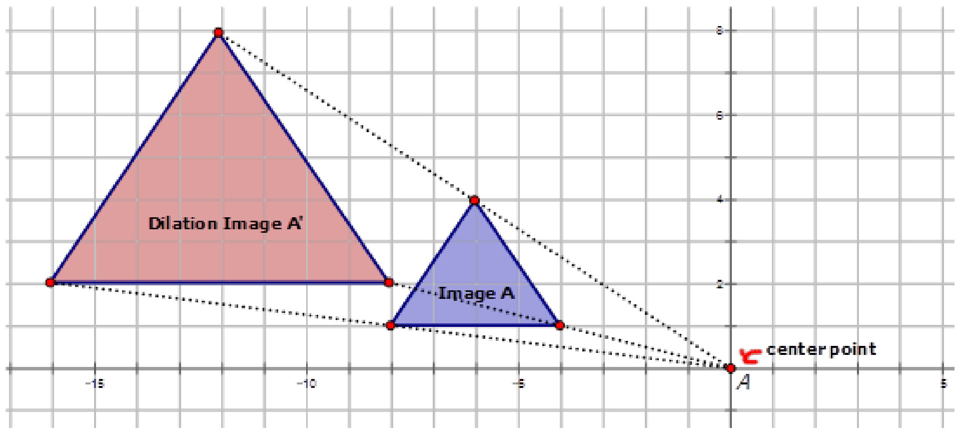
Which one of the following figures represents a dilation? Explain.





Dilations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A dilation is a type of transformation that enlarges or reduces a figure (called the preimage) to create a new figure (called the image). The scale factor, r, determines how much bigger or smaller the dilation image will be compared to the preimage. The figure below shows that the image A' is a dilation by a scale factor of 2.



Dilations also need a center point. The center point is the center of the dilation. You use the center point to measure the distances to the preimage and the dilation image. It is these distances that determine the scale factor.

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65264

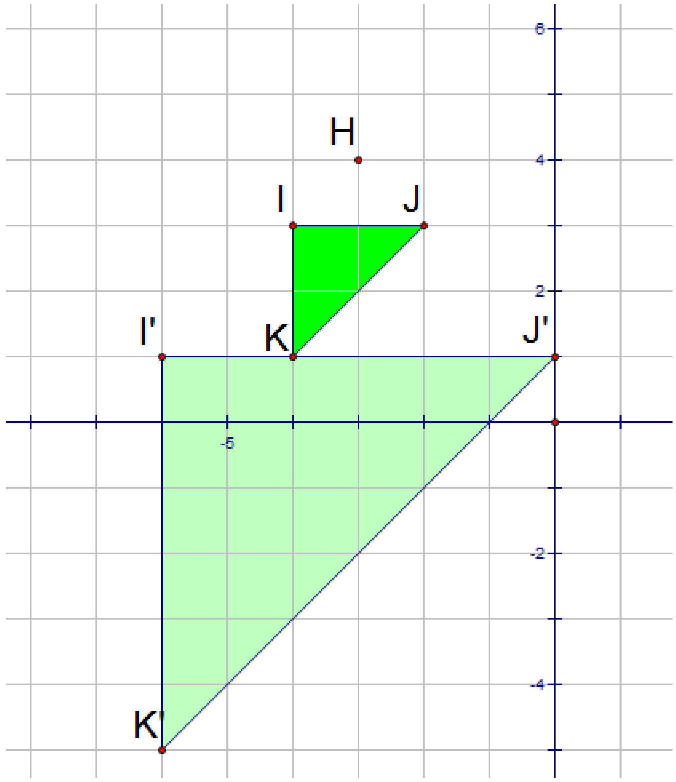
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65265

Describe the dilation in the diagram below.

The center of dilation is point H.



Compare the lengths of corresponding sides to determine the scale factor. $\overline{IJ}$ is 2 units long and $\overline{I'J'}$ is 6 units long. $\frac{6}{2} = 3$, so the scale factor is 3. Therefore, the center point H is used to dilate $\triangle IJK$ to $\triangle I'J'K'$ by a factor of 3.

Using the measurement below and the scale factor, determine the measure of the dilated image.

$$m\overline{AB} = 15\ cm$$
$$r = \frac{1}{3}$$

You need to multiply the scale factor by the measure of AB in order to find the measurement of the dilated image $A'B'$.

$m\overline{A'B'} = (r)m\overline{AB}$

$$m\overline{A'B'} = \frac{1}{3}(15)$$
$$m\overline{A'B'} = 5\ cm$$

Using the measurement below and the scale factor, determine the measure of the preimage.

$$m\overline{H'I'} = 24\ cm$$
$$r = 2$$

Here, you need to divide the scale factor by the measurement of $H'I'$ in order to find the measurement of the preimage HI.

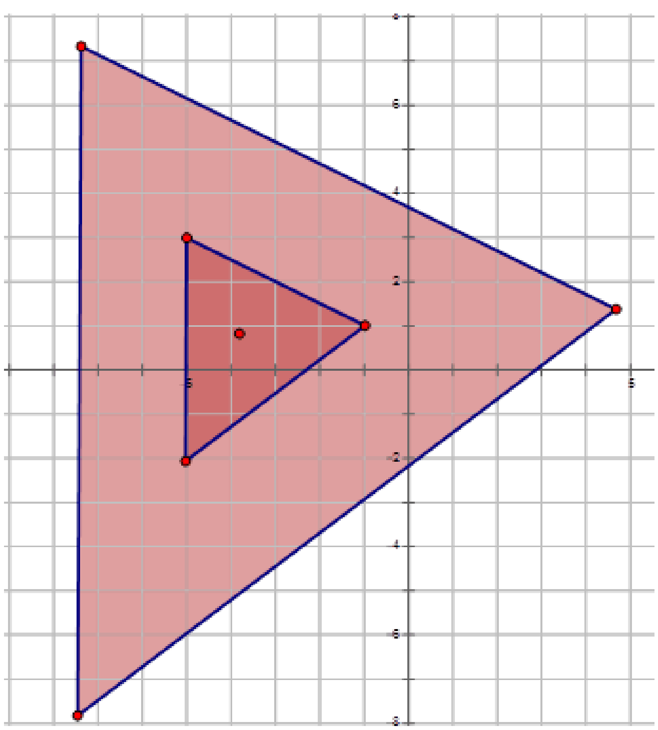
$m\overline{H'I'} = (r)m\overline{HI}$

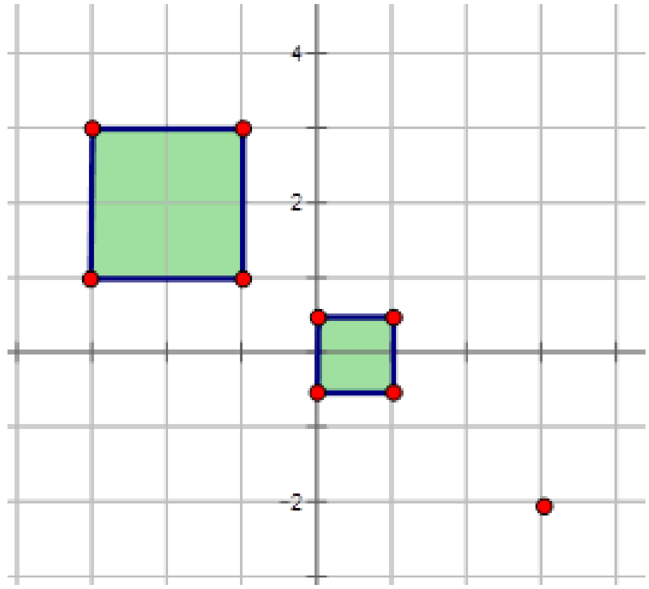
$$24 = 2m\overline{HI}$$
$$m\overline{HI} = \frac{24}{2}$$
$$m\overline{HI} = 12\ cm$$

Examples

Example 1

Earlier, you were asked which one of the following figures represents a dilation? Explain.





You know that a dilation is a transformation that produces an image of the same shape but larger or smaller. Both of the figures above represent objects that involve dilations. In the figure with the triangles, the scale factor is 3.

The second figure with the squares also represents a dilation. In this figure, the center point $(3,-2)$ is used to dilate the small square by a factor of 2.

Example 2

Using the measurement below and the scale factor, determine the measure of the preimage.

$$m\overline{T'U'} = 12\ cm$$
$$r = 4\ cm$$

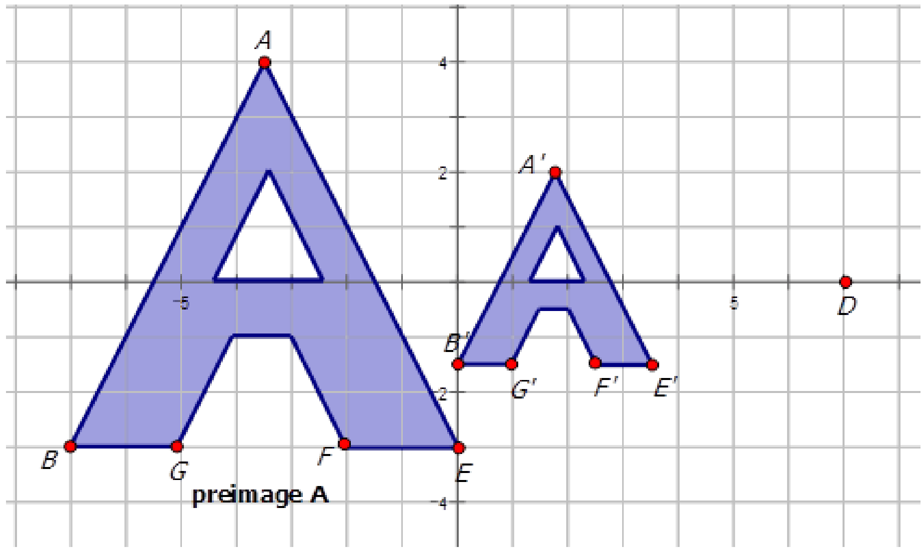
Here, you need to divide the scale factor by the measurement of $H'I'$ in order to find the measurement of the preimage HI.

$m\overline{T'U'} = |r|m\overline{TU}$

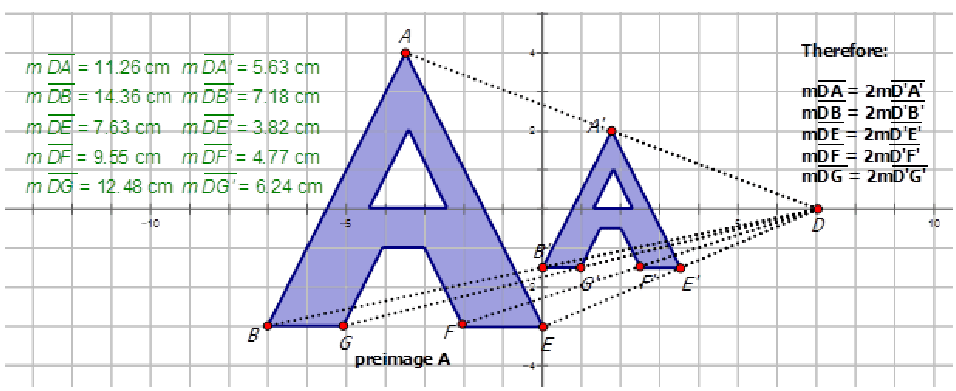
$$12 = 4m\overline{TU}$$
$$m\overline{TU} = \frac{12}{4}$$
$$m\overline{TU} = 3\ cm$$

Example 3

Describe the dilation in the diagram below.



Look at the diagram below:

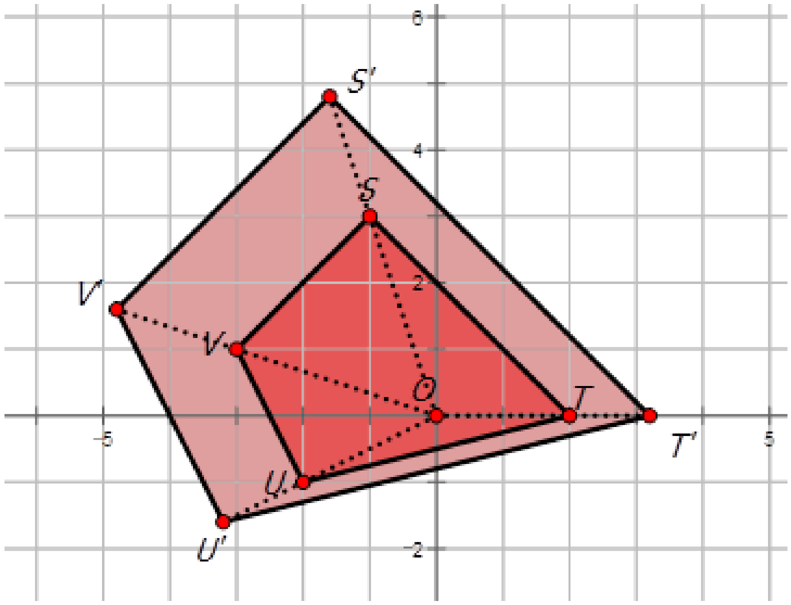


In the figure, the center point D is used to dilate the A by a factor of $\frac{1}{2}$.

Example 4

Quadrilateral $STUV$ has vertices $S(-1,3), T(2,0), U(-2,-1)$, and $V(-3,1)$. The quadrilateral undergoes a dilation about the origin with a scale factor of $\frac{8}{5}$. Sketch the preimage and the dilation image.

Look at the diagram below:



Review

Find the measure of the dilation image given the following information:

1.

$$m\overline{AB} = 12\ cm$$
$$r = 2$$

2.

$$m\overline{CD} = 25\ cm$$
$$r = \frac{1}{5}$$

3.

$$m\overline{EF} = 18\ cm$$
$$r = \frac{2}{3}$$

4.

$$m\overline{GH} = 18\ cm$$
$$r = 3$$

5.

$$m\overline{IJ} = 100\ cm$$
$$r = \frac{1}{10}$$

Find the measure of the preimage given the following information:

6.

$$m\overline{K'L'} = 48\ cm$$
$$r = 4$$

7.

$$m\overline{M'N'} = 32\ cm$$
$$r = 4$$

8.

$$m\overline{O'P'} = 36\ cm$$
$$r = 6$$

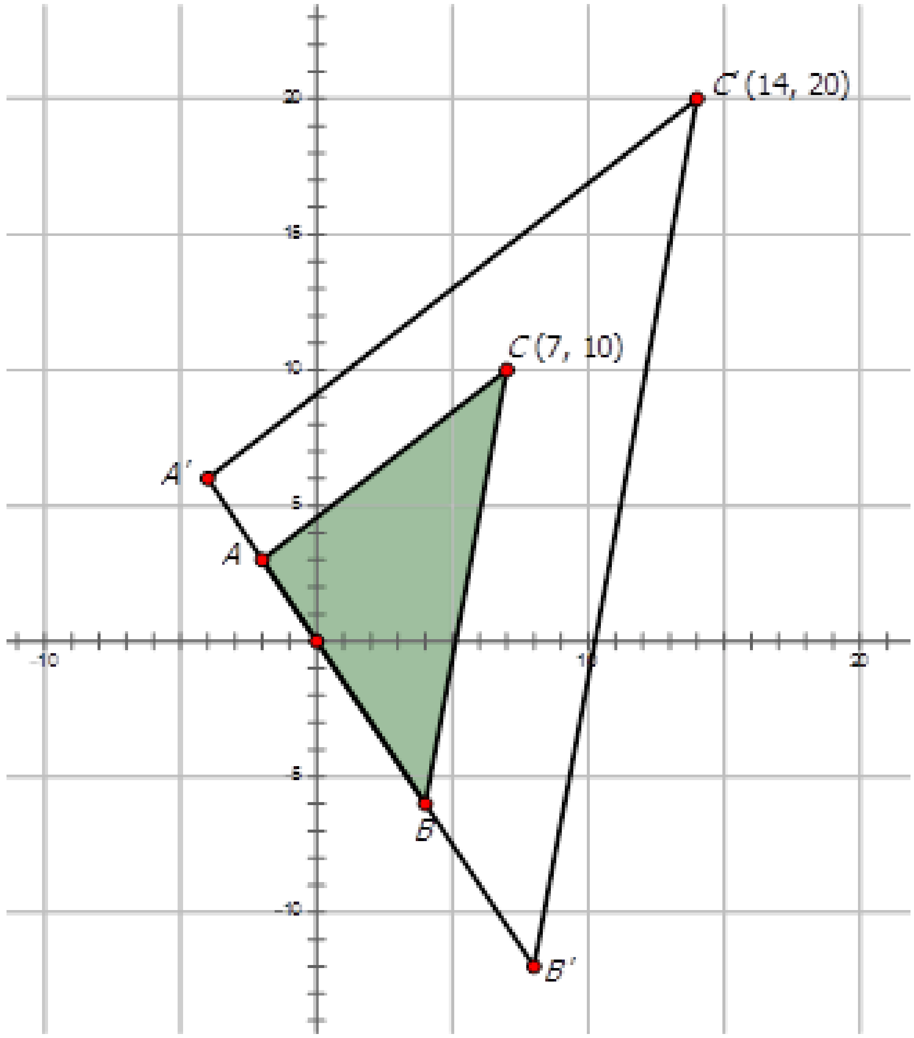
9.

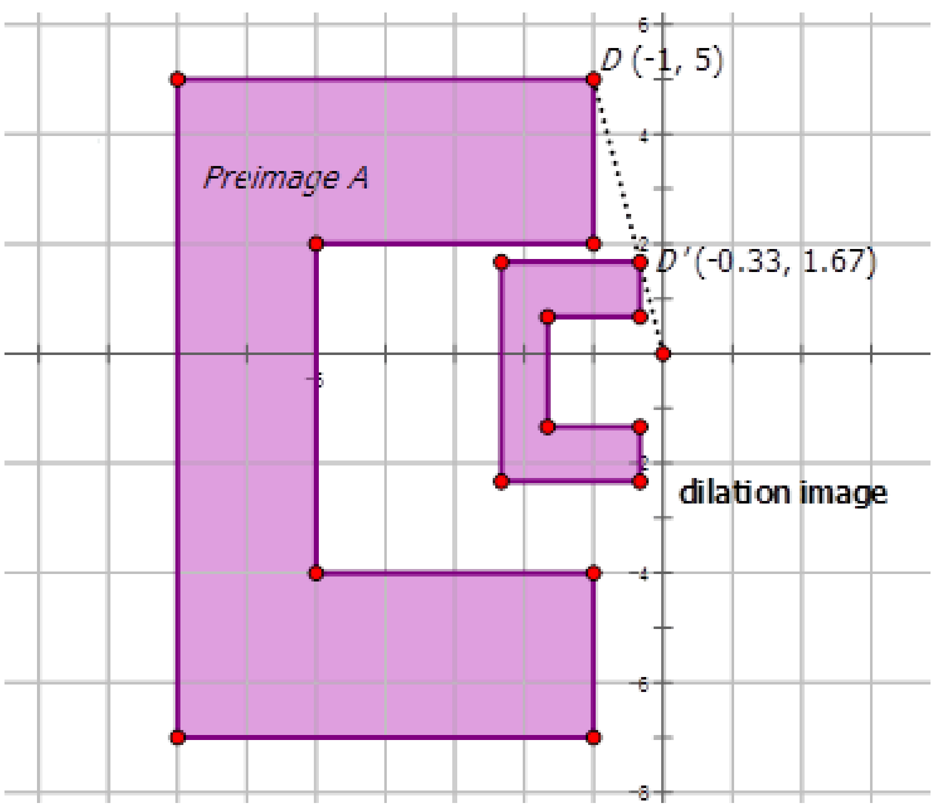
$$m\overline{Q'R'} = 20\ cm$$
$$r = \frac{1}{4}$$

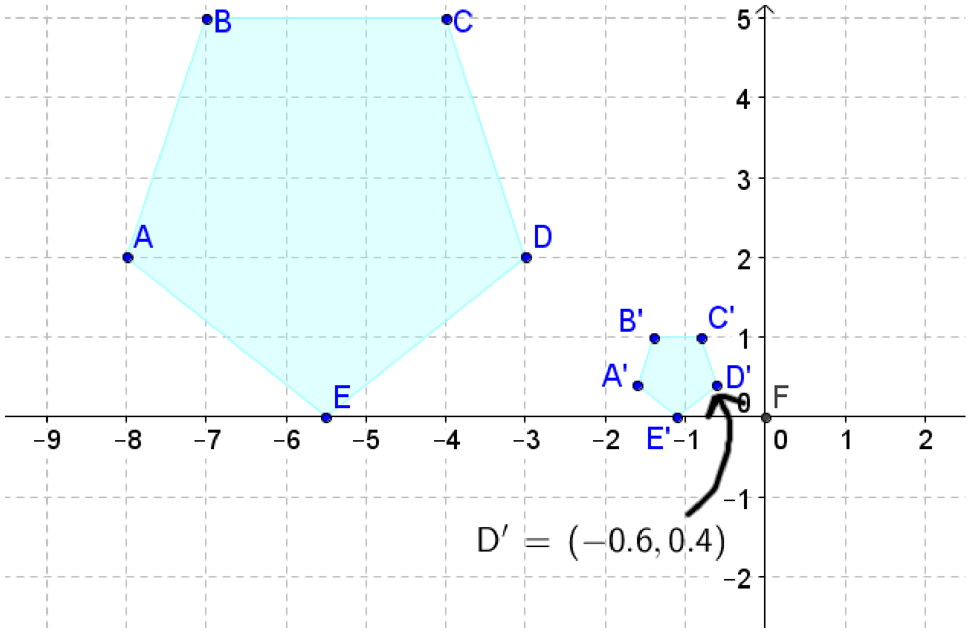
10.

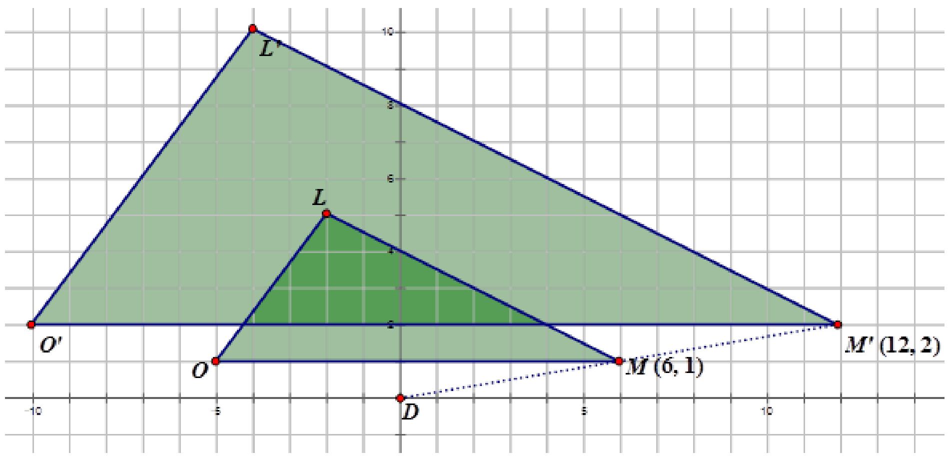
$$m\overline{S'T'} = 40\ cm$$
$$r = \frac{4}{5}$$

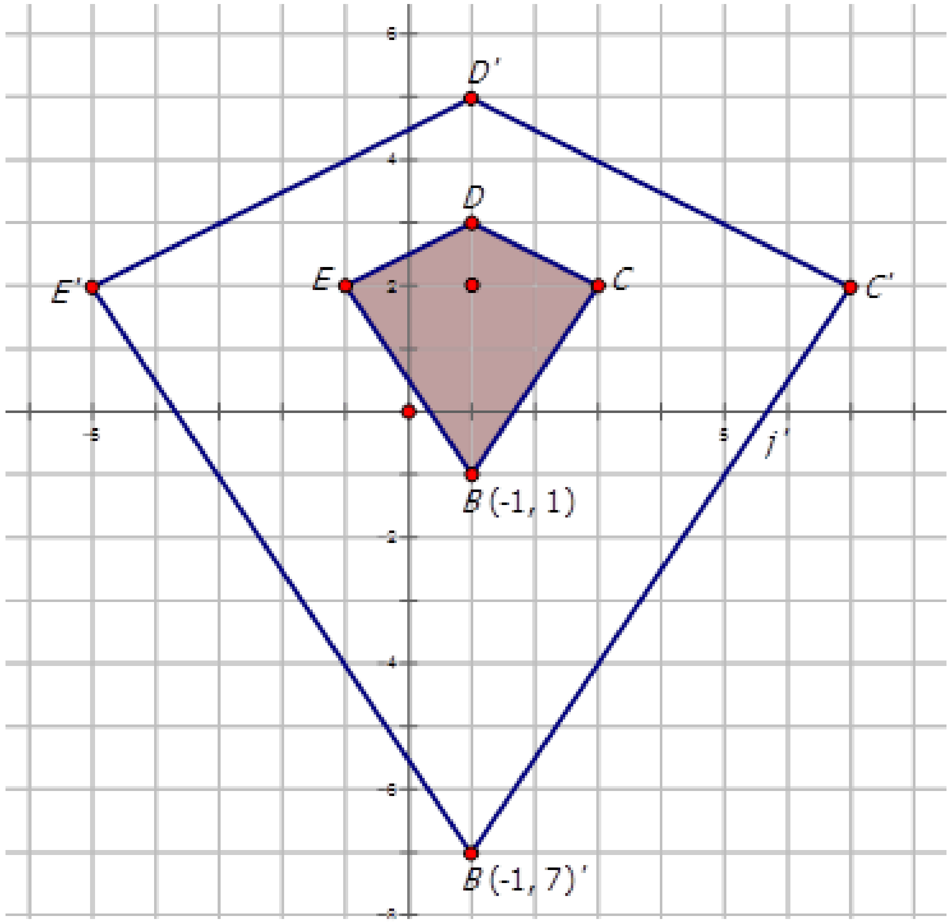
Describe the following dilations:

11. 

12. 

13. 

14. 

15. 

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.10.

10.11 Graphs of Dilations

Here you will learn how to graph a dilation given a description of the dilation.

Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane.

The quadrilateral undergoes a dilation centered at the origin of scale factor $\frac{1}{3}$. Show the resulting image.

Graphs of Dilations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A dilation is a type of transformation that enlarges or reduces a figure (called the preimage) to create a new figure (called the image). The scale factor, r, determines how much bigger or smaller the dilation image will be compared to the preimage.

In order to graph a dilation, use the center of dilation and the scale factor. Find the distance between a point on the preimage and the center of dilation. Multiply this length by the scale factor. The corresponding point on the image will be this distance away from the center of dilation in the same direction as the original point.

If you compare the length of a side on the preimage to the length of the corresponding side on the image, the length of the side on the image will be the length of the side on the preimage multiplied by the scale factor.

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65260

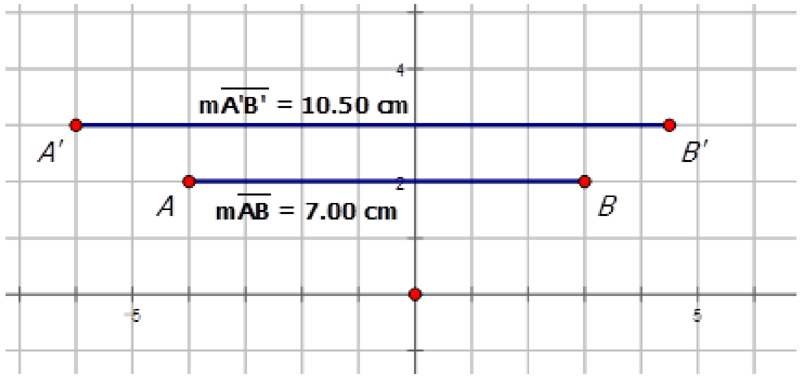
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65261

Draw the dilation image

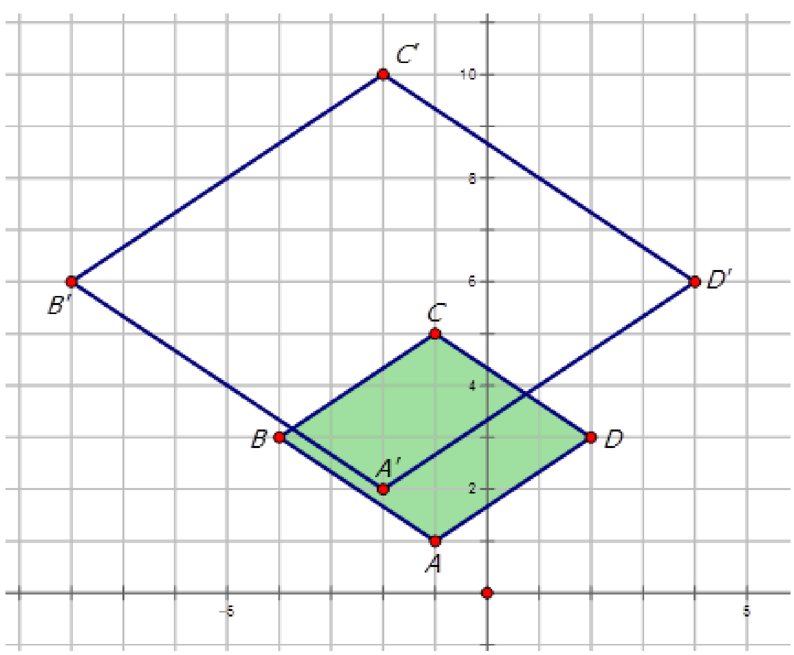
Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has undergone a dilation about the origin to produce $A'(-6,3)$ and $B'(4.5,3)$. Draw the preimage and dilation image and determine the scale factor.

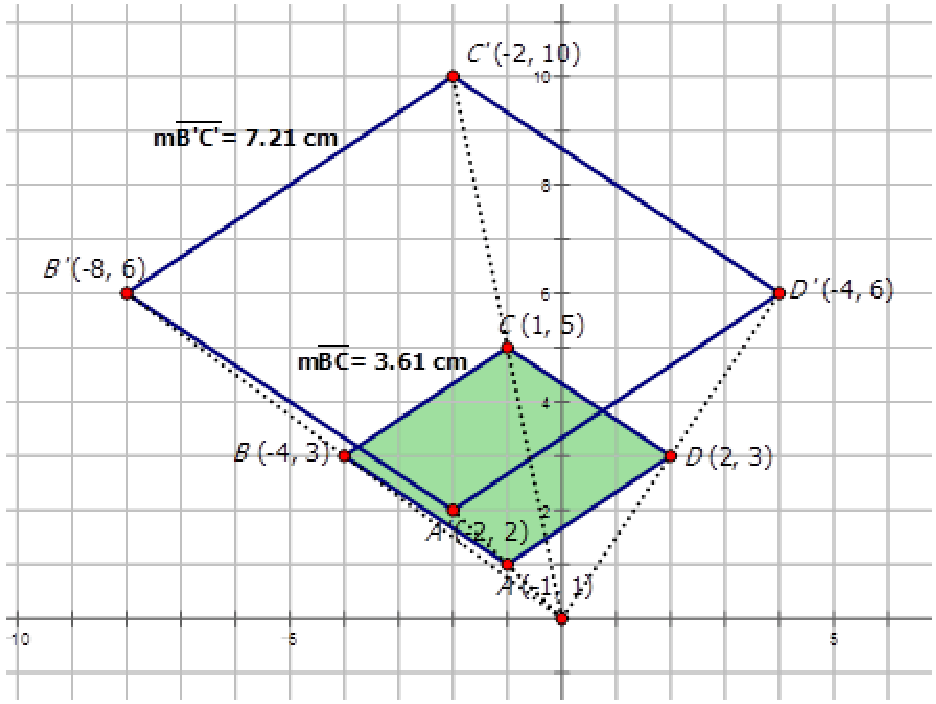


$$\begin{aligned}
scale\ factor &= \frac{dilation\ image\ length}{preimage\ length} \\
scale\ factor &= \frac{10.5}{7.0} \\
scale\ factor &= \frac{3}{2}
\end{aligned}$$

Determine the scale factor

The diamond $ABCD$ undergoes a dilation about the origin to form the image $A'B'C'D'$. Find the coordinates of the dilation image. Using the diagram, determine the scale factor.

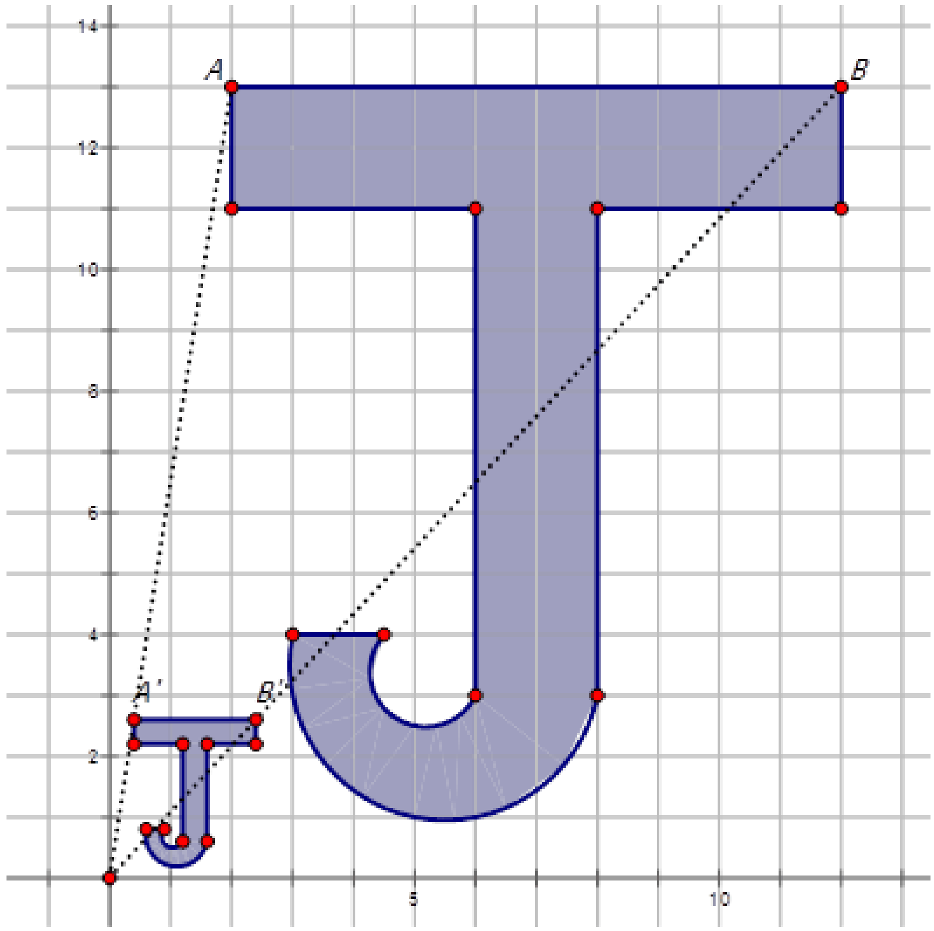


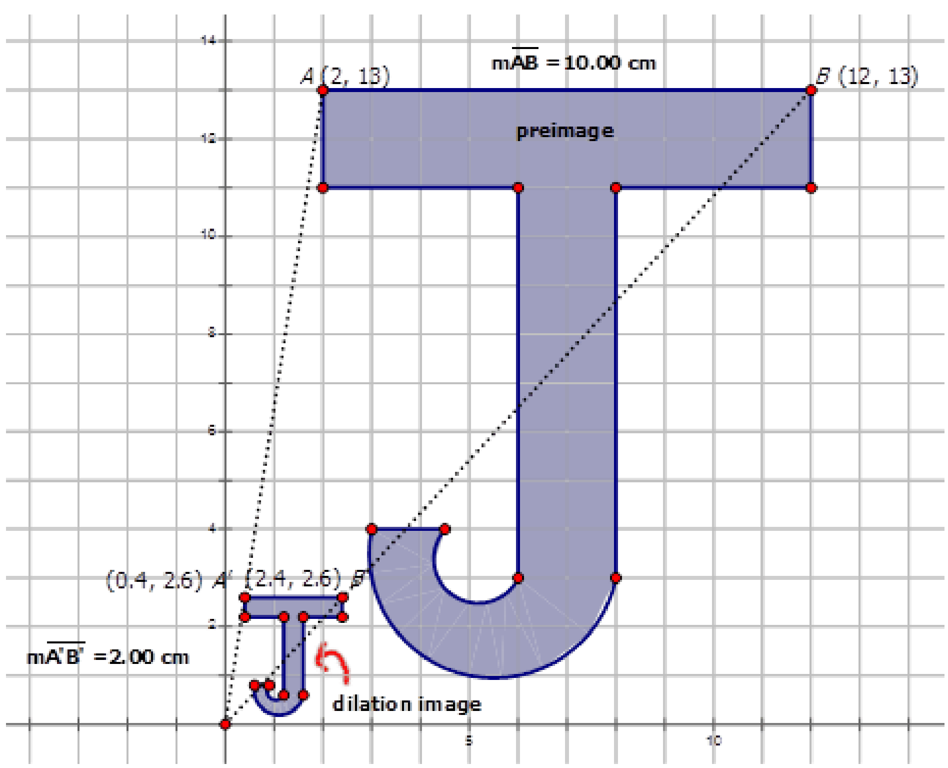


$$scale\ factor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scale factor = \frac{7.21}{3.61}$$
$$scale factor = 2$$

Determine the scale factor

The diagram below undergoes a dilation about the origin to form the dilation image. Find the coordinates of A and B and A' and B' of the dilation image. Using the diagram, determine the scale factor.


A
B
A'
B'

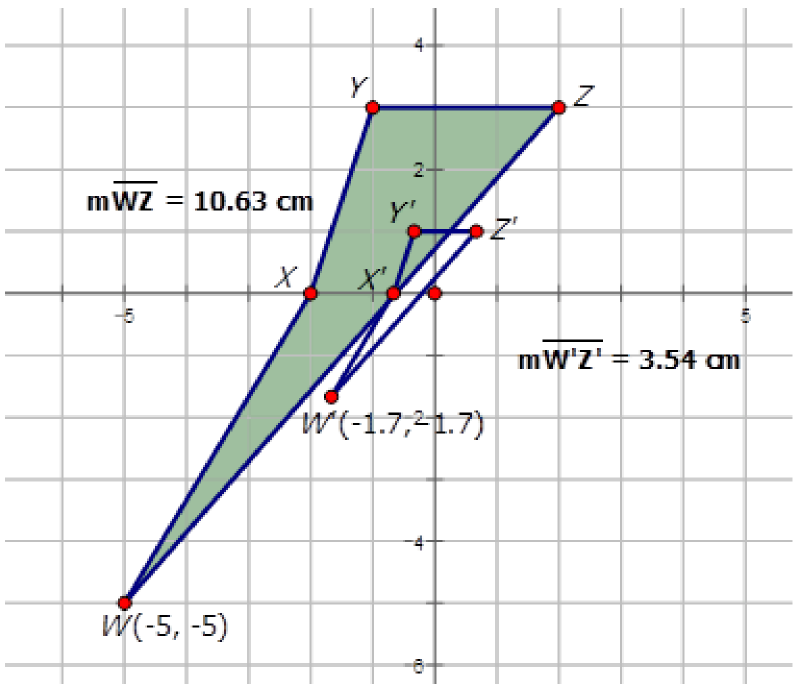

mAB = 10.00 cm
A (2, 13)
B (12, 13)
preimage
(0.4, 2.6) A'
(2.4, 2.6) B'
mA'B' = 2.00 cm
dilation image

$$scalefactor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scale\ factor = \frac{2.00}{10.00}$$
$$scale\ factor = \frac{1}{5}$$

Examples

Example 1

Earlier, you were asked to show the resulting image of the dilation.

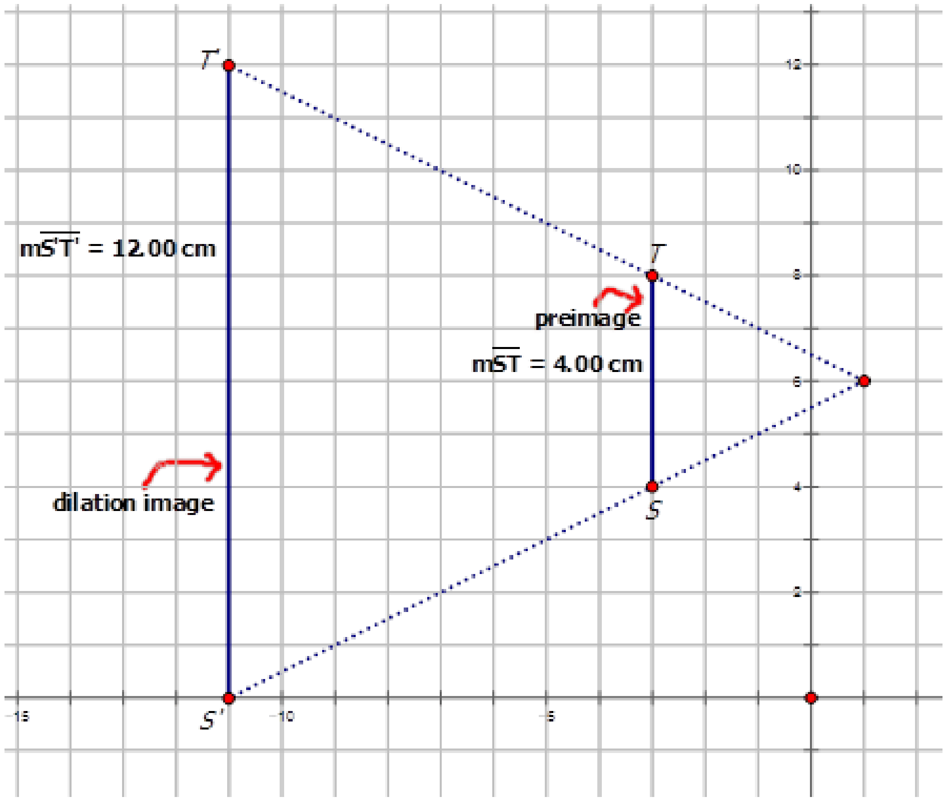


Test to see if the dilation is correct by determining the scale factor.

$$scale\ factor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scale\ factor = \frac{10.63}{3.54}$$
$$scale\ factor = 3$$

Example 2

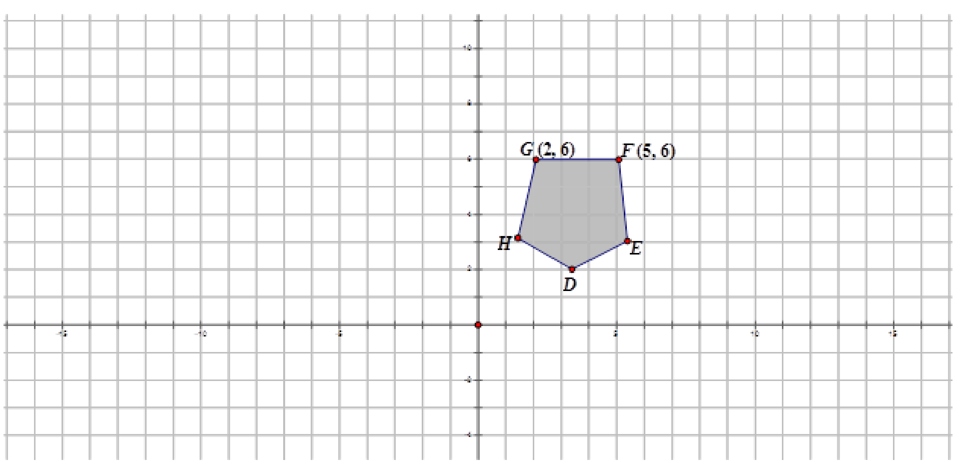
Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has undergone a dilation of scale factor 3 about the point $A(1,6)$. Draw the preimage and image and properly label each.

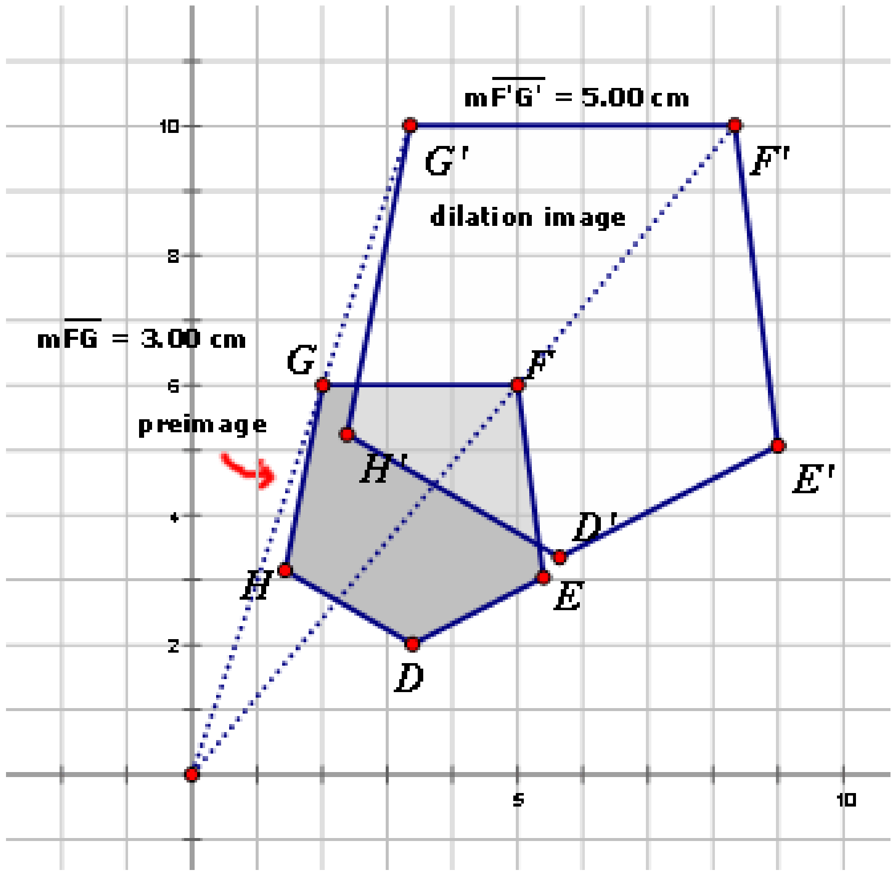


$$scale\ factor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scale\ factor = \frac{12.00}{4.00}$$
$$scale\ factor = 3$$

Example 3

The polygon below has undergone a dilation about the origin with a scale factor of $\frac{5}{3}$. Draw the dilation image and properly label each.

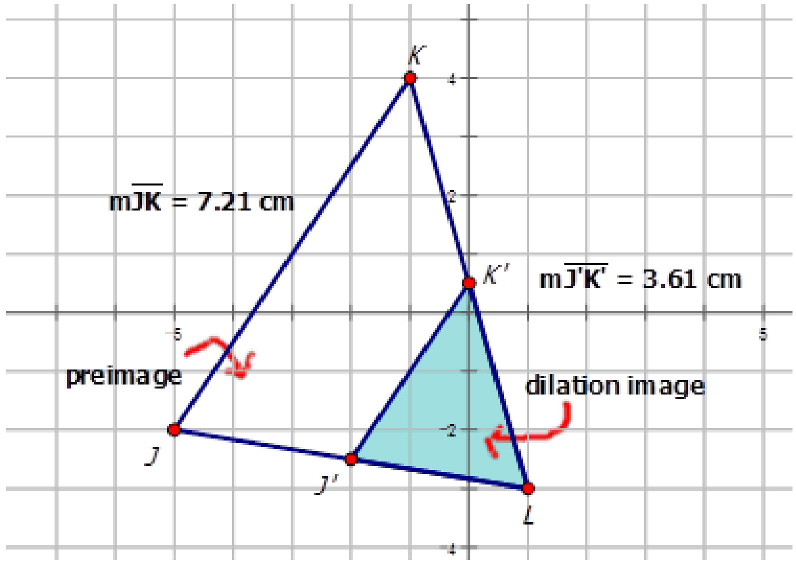




$$scale factor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scale\ factor = \frac{5.00}{3.00}$$
$$scale\ factor = \frac{5}{3}$$

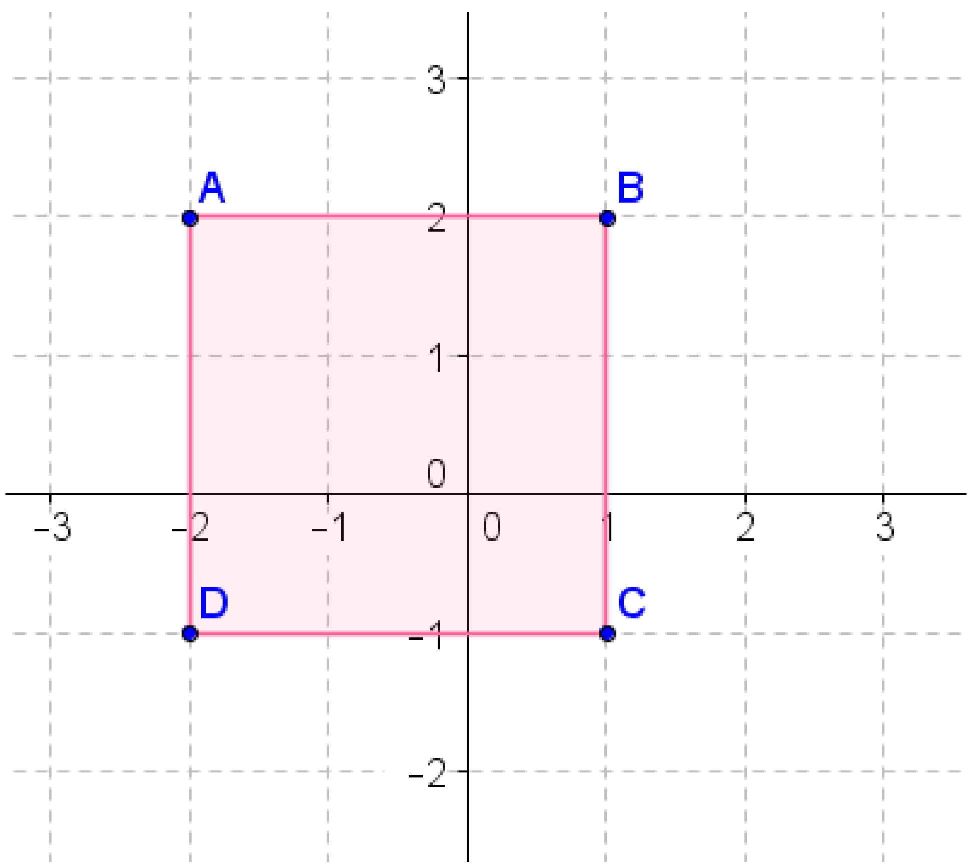
Example 4

The triangle with vertices $J(-5,-2), K(-1,4)$ and $L(1,-3)$ has undergone a dilation of scale factor $\frac{1}{2}$. about the center point L. Draw and label the dilation image and the preimage then check the scale factor.

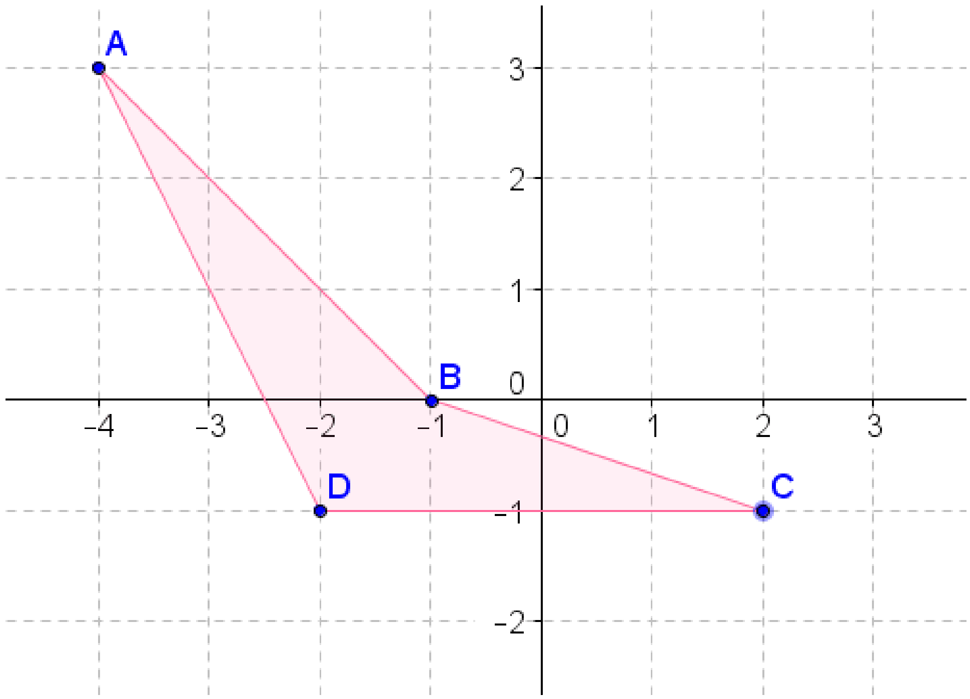


$$scale\ factor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scale\ factor = \frac{7.21}{3.61}$$
$$scale\ factor = \frac{1}{2}$$

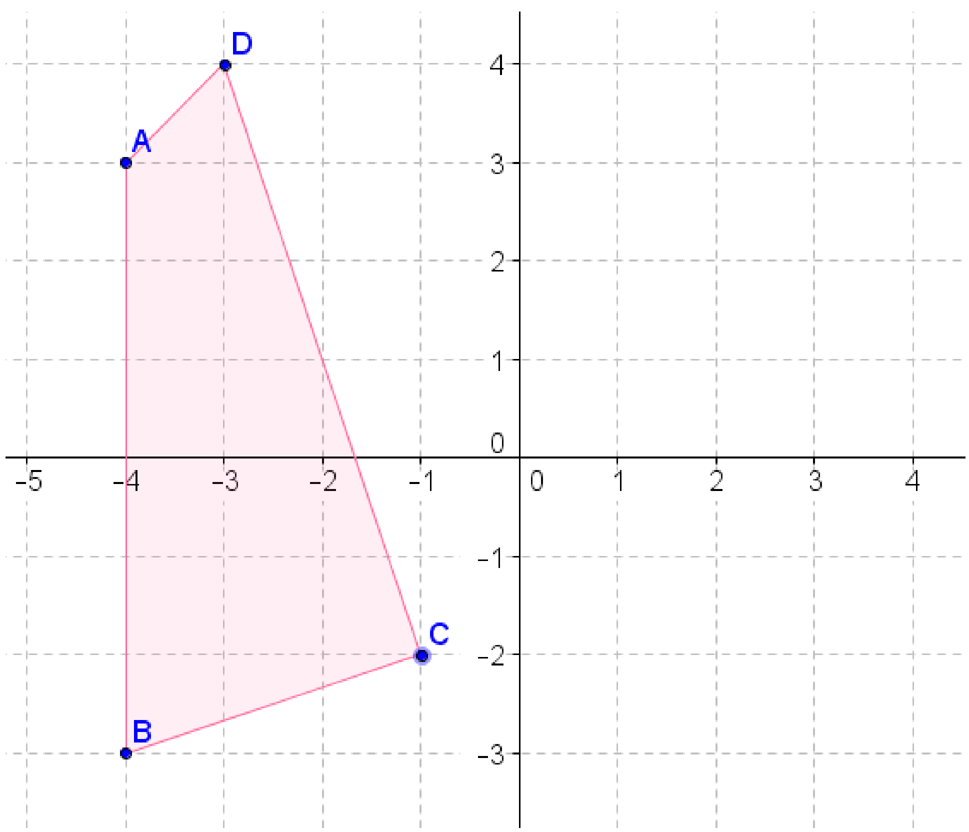
Review



1. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
2. Dilate the above figure by a factor of 2 about point D.

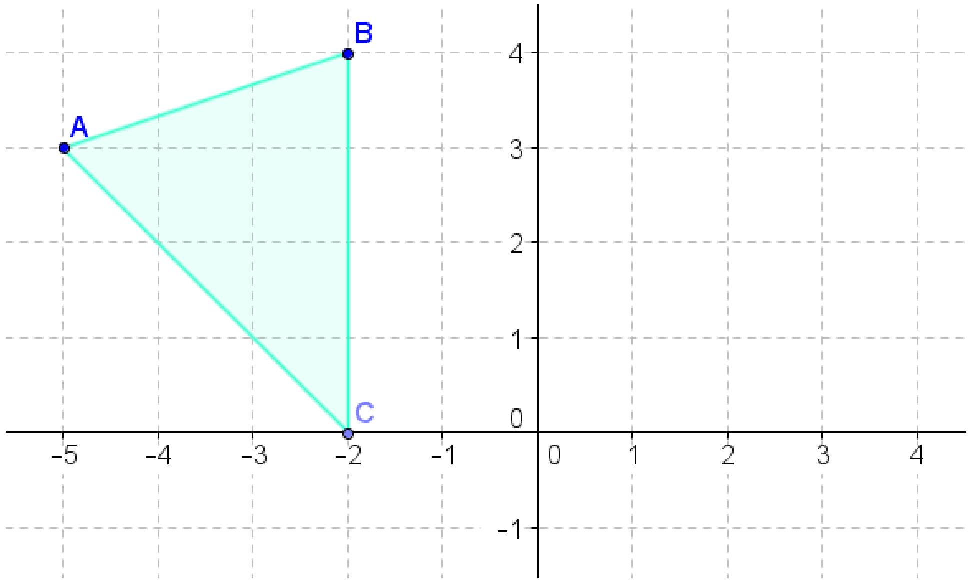


3. Dilate the above figure by a factor of 3 about the origin.
4. Dilate the above figure by a factor of $\frac{1}{2}$ about point C.

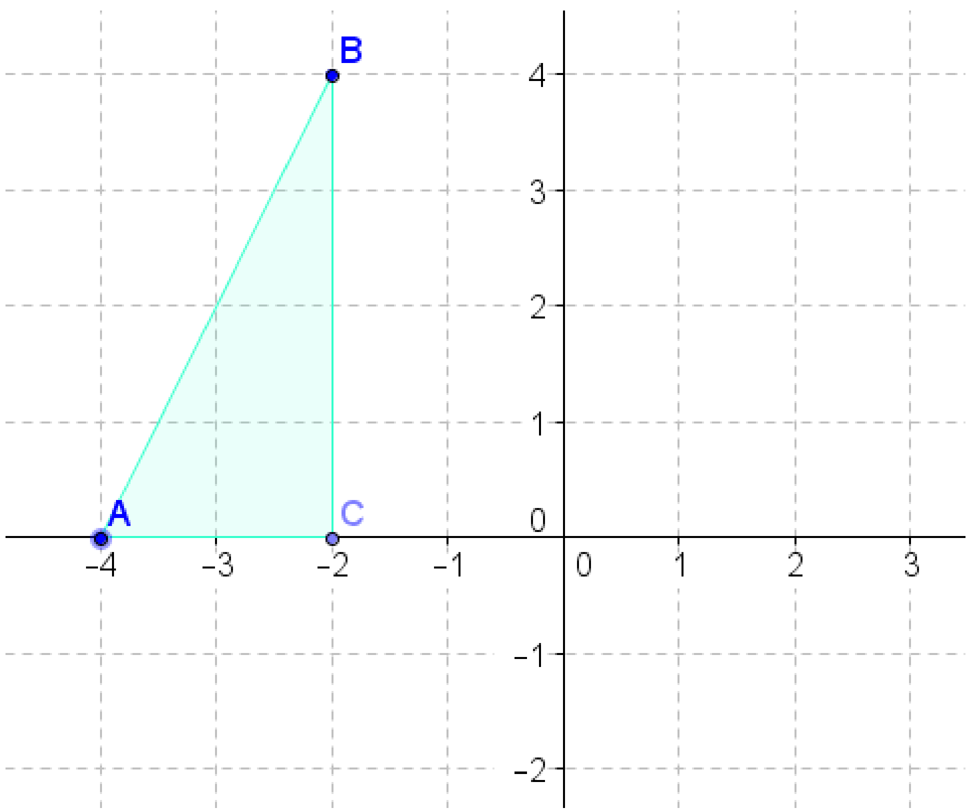


5. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.

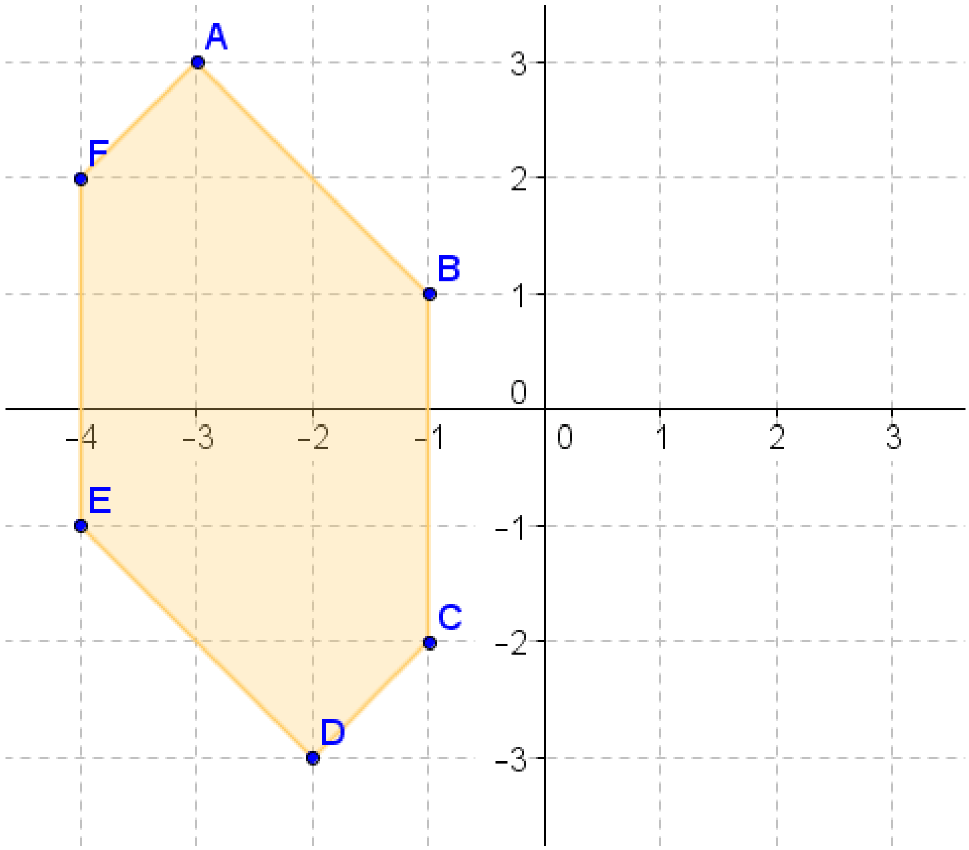
6. Dilate the above figure by a factor of $\frac{1}{2}$ about point C.



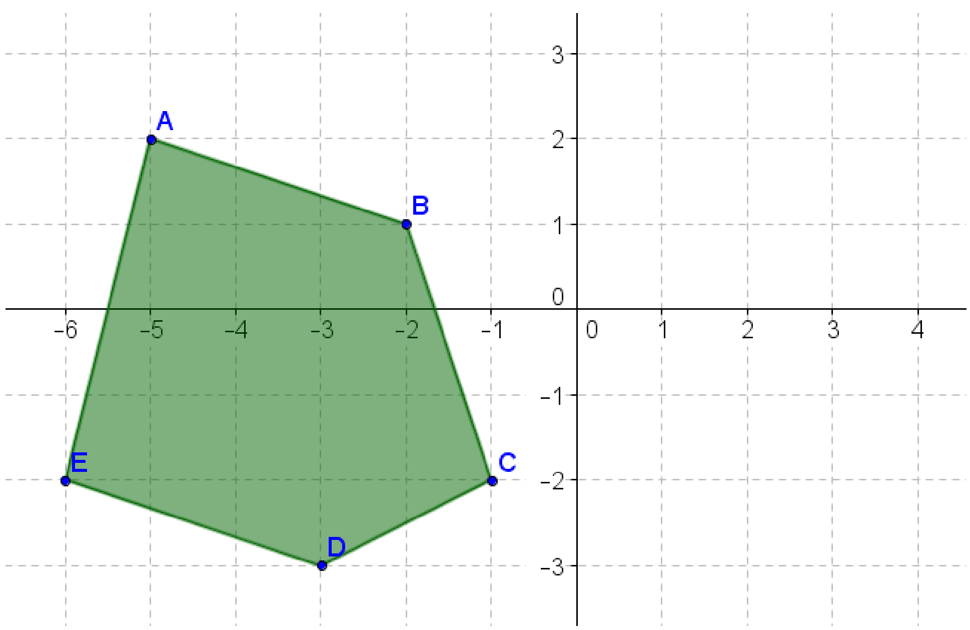
7. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
8. Dilate the above figure by a factor of $\frac{1}{4}$ about point C.



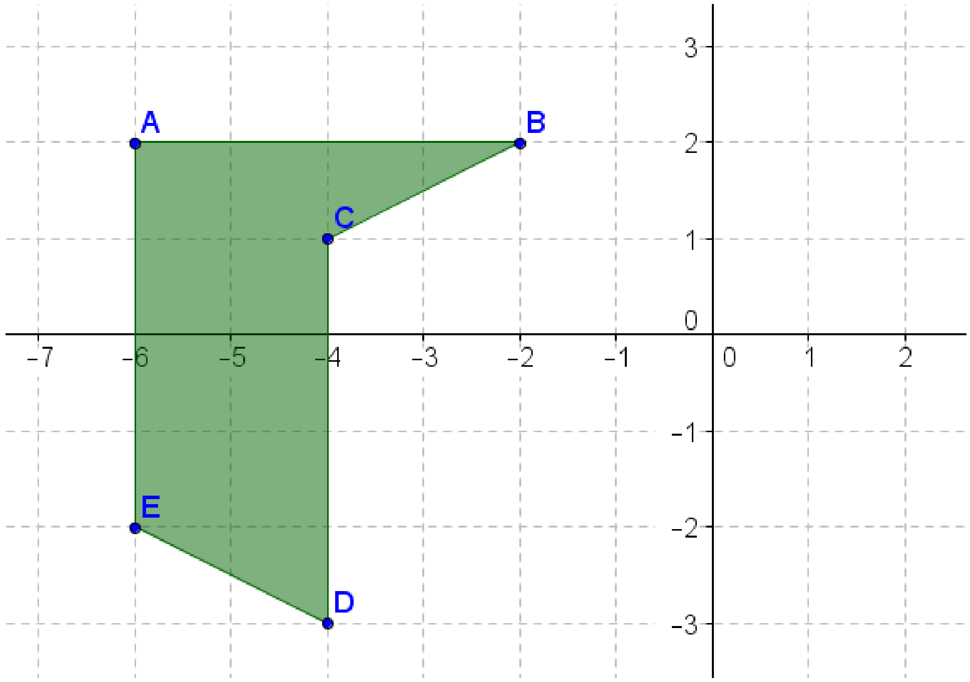
9. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
10. Dilate the above figure by a factor of 2 about point A.



11. Dilate the above figure by a factor of 2 about the origin.
12. Dilate the above figure by a factor of $\frac{1}{2}$ about point D.



13. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
14. Dilate the above figure by a factor of 3 about point D.



15. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
16. Dilate the above figure by a factor of $\frac{1}{2}$ about point C.

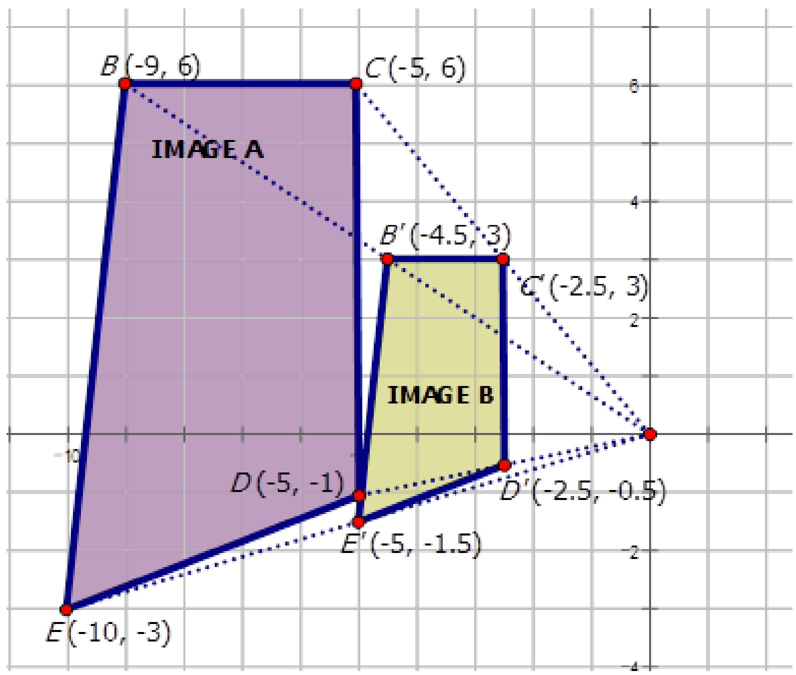
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.11.

10.12 Rules for Dilations

Here you will learn the notation for describing a dilation.

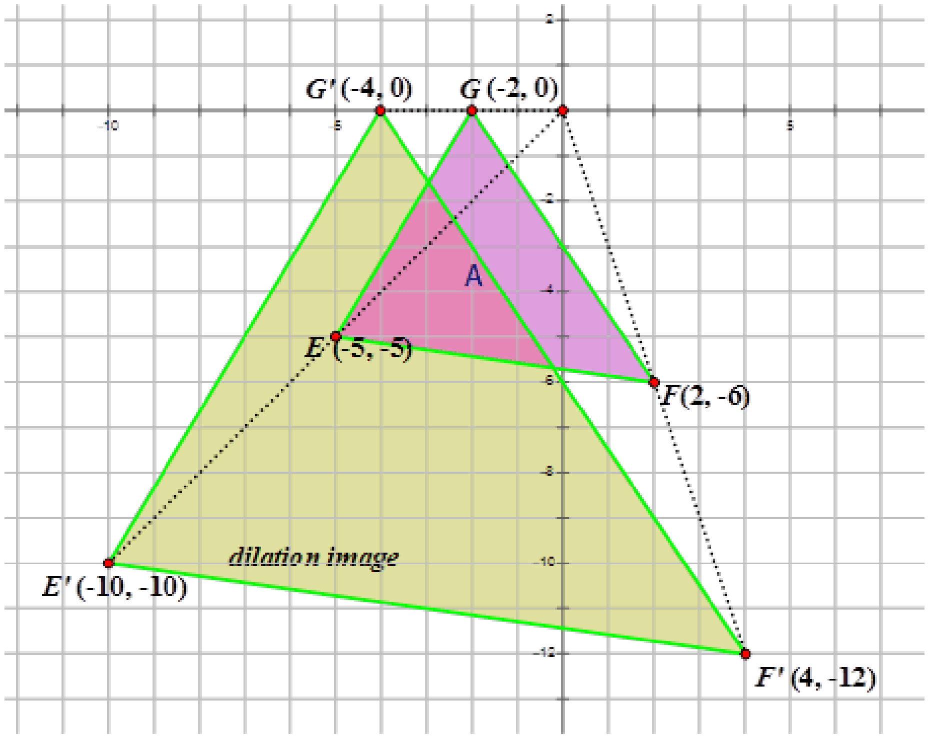
The figure below shows a dilation of two trapezoids. Write the mapping rule for the dilation of Image A to Image B.



Rules for Dilations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A dilation is a type of transformation that enlarges or reduces a figure (called the preimage) to create a new figure (called the image). The scale factor, r, determines how much bigger or smaller the dilation image will be compared to the preimage.

Look at the diagram below:



The Image A has undergone a dilation about the origin with a scale factor of 2. Notice that the points in the dilation image are all double the coordinate points in the preimage. A dilation with a scale factor k about the origin can be described using the following notation:

$$D_k(x,y) = (kx,ky)$$

k will always be a value that is greater than 0.

TABLE 10.14:

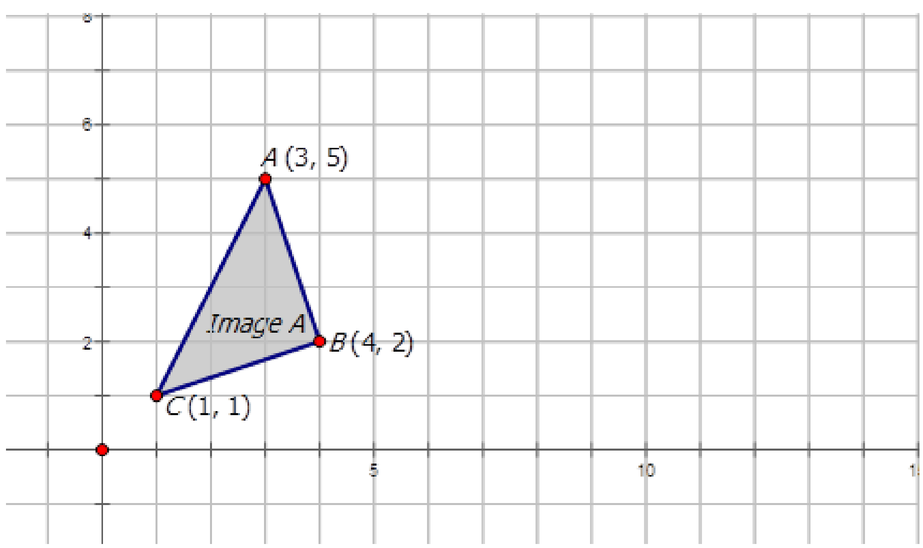
Scale Factor, k	**Size change for preimage**
$k > 1$	Dilation image is larger than preimage
$0 < k < 1$	Dilation image is smaller than preimage
$k = 1$	Dilation image is the same size as the preimage

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65234

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65235

Draw the dilation image

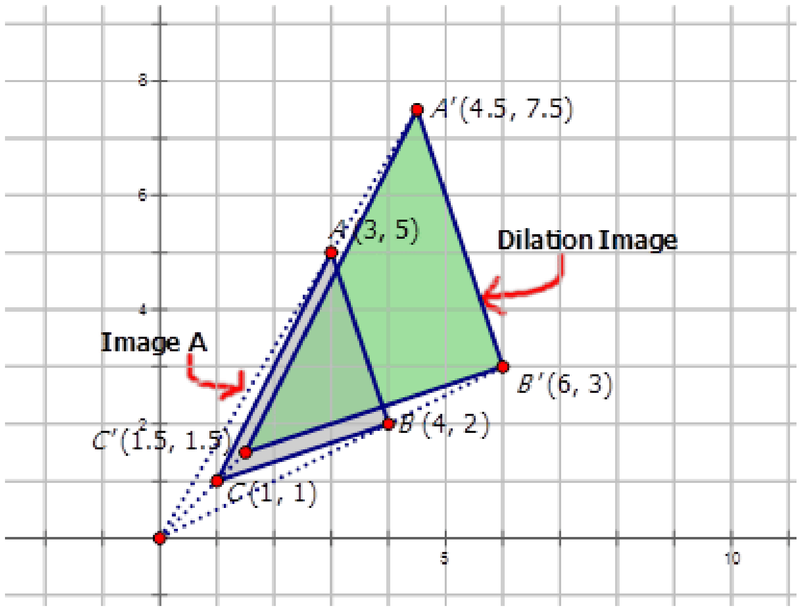
The mapping rule for the dilation applied to the triangle below is $(x,y) \rightarrow (1.5x, 1.5y)$. Draw the dilation image.



With a scale factor of 1.5, each coordinate point will be multiplied by 1.5.

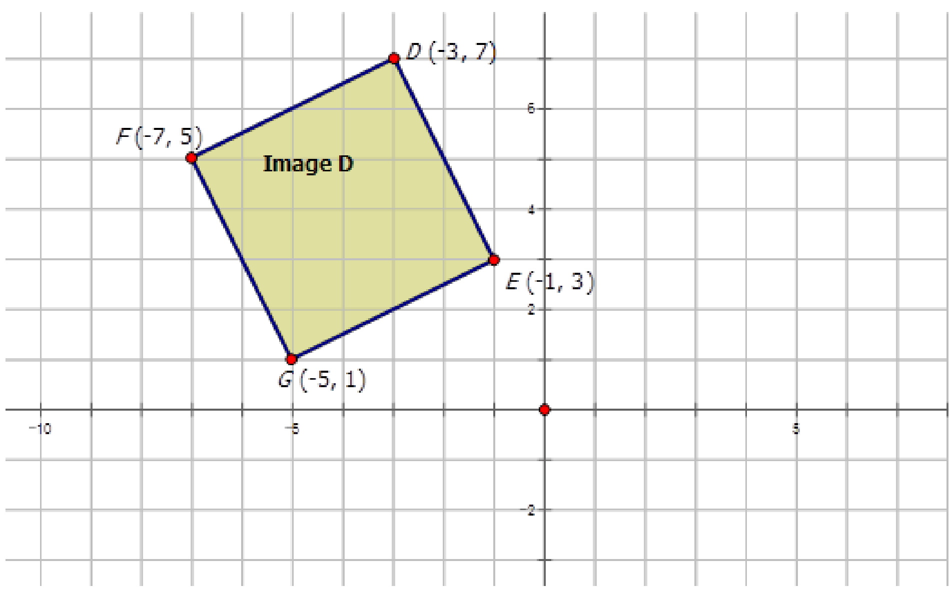
Image A	$A(3,5)$	$B(4,2)$	$C(1,1)$
Dilation Image	$A'(4.5,7.5)$	$B'(6,3)$	$C'(1.5,1.5)$

The dilation image looks like the following:



Draw the dilation image

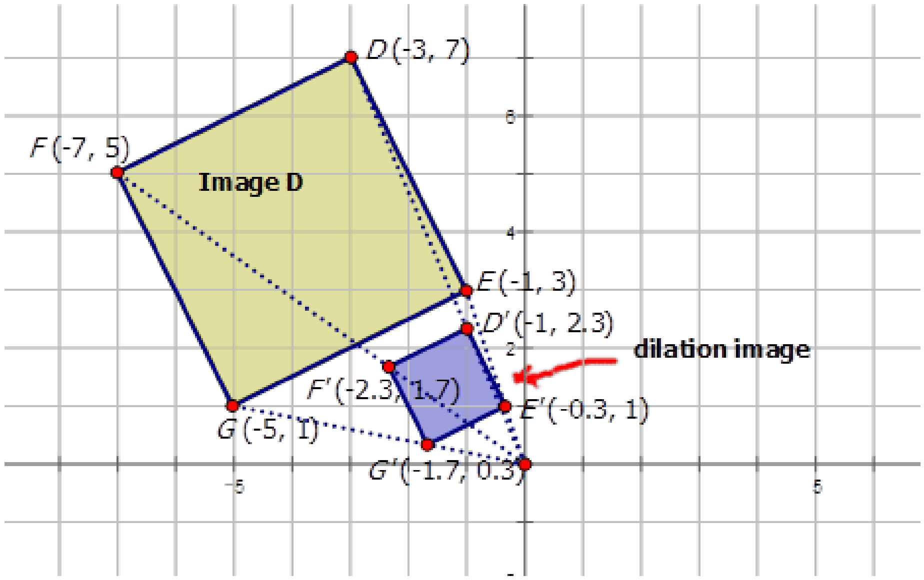
The mapping rule for the dilation applied to the diagram below is $(x,y) \rightarrow \left(\frac{1}{3}x, \frac{1}{3}y\right)$. Draw the dilation image.



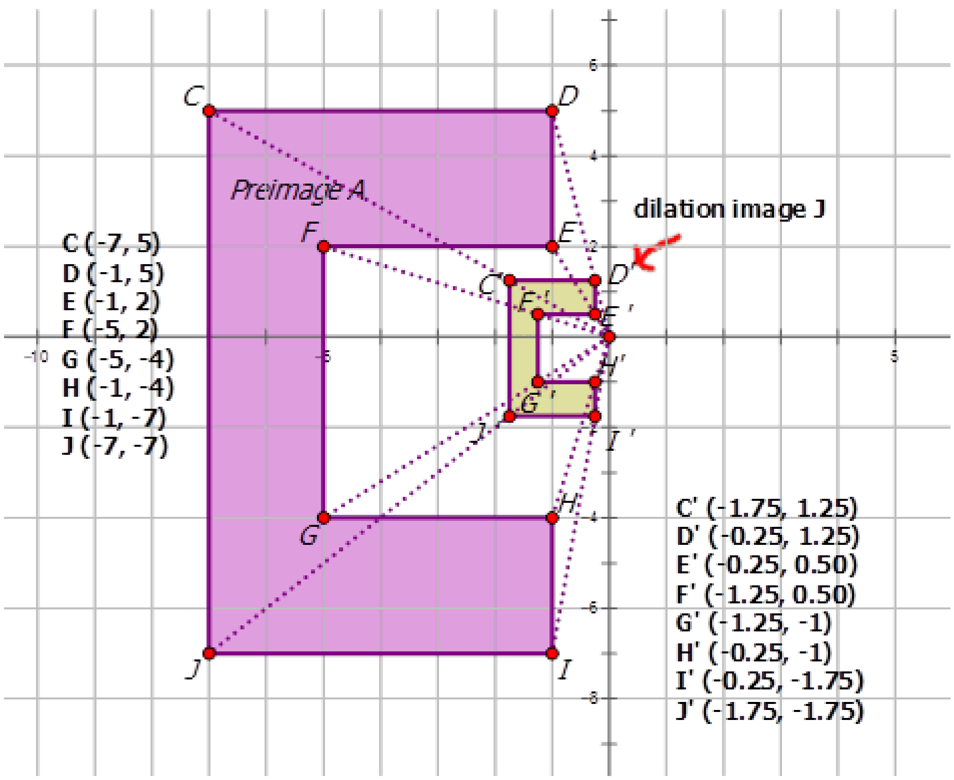
With a scale factor of $\frac{1}{3}$, each coordinate point will be multiplied by $\frac{1}{3}$.

Image *D*	$D(-3,7)$	$E(-1,3)$	$F(-7,5)$	$G(-5,1)$
Dilation Image	$D'(-1,2.3)$	$E'(-0.3,1)$	$F'(-2.3,1.7)$	$G'(-1.7,0.3)$

The dilation image looks like the following:



Write the notation that represents the dilation of the preimage A to the dilation image J in the diagram below.



First, pick a point in the diagram to use to see how it has been affected by the dilation.

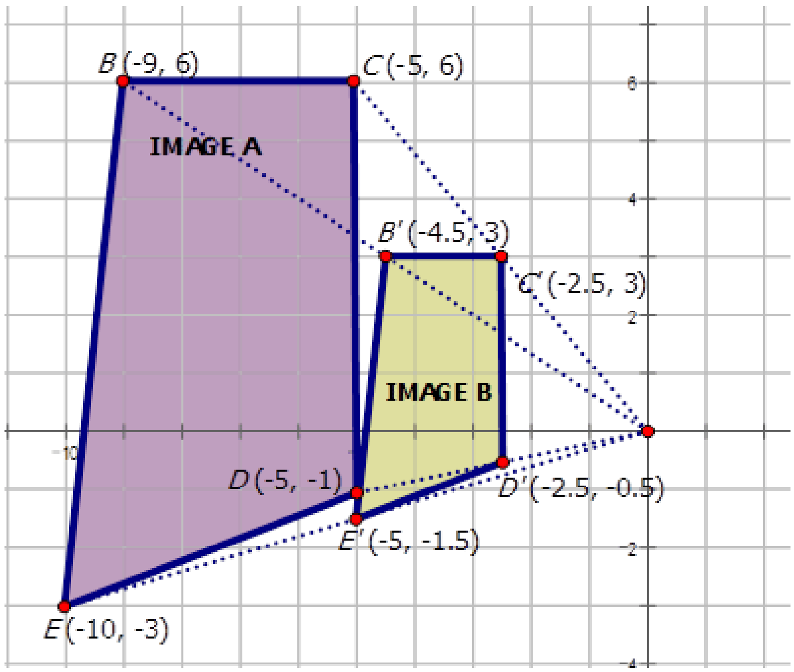
$C : (-7, 5) \quad C' : (-1.75, 1.25)$

Notice how both the x- and y-coordinates are multiplied by $\frac{1}{4}$. This indicates that the preimage A undergoes a dilation about the origin by a scale factor of $\frac{1}{4}$ to form the dilation image J. Therefore the mapping notation is $(x, y) \rightarrow \left(\frac{1}{4}x, \frac{1}{4}y\right)$.

Examples

Example 1

Earlier, you were asked to write the mapping rule for the dilation of Image A to Image B.



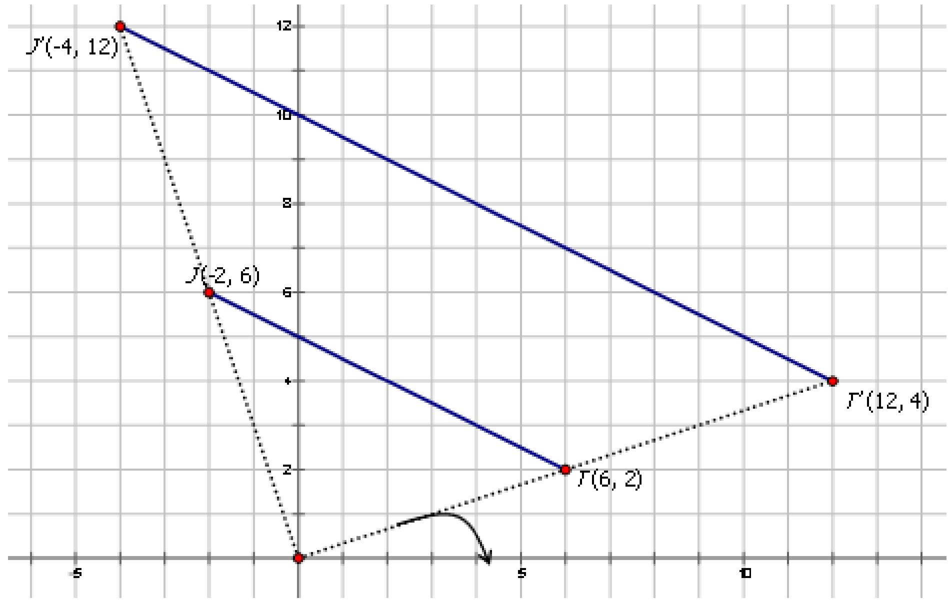
Look at the points in each image:

Image A	$B(-9,6)$	$C(-5,6)$	$D(-5,-1)$	$E(-10,-3)$
Image B	$B'(-4.5,3)$	$C'(-2.5,3)$	$D'(-2.5,-0.5)$	$E'(-5,-1.5)$

Notice that the coordinate points in Image B (the dilation image) are $\frac{1}{2}$ that found in Image A. Therefore the Image A undergoes a dilation about the origin of scale factor $\frac{1}{2}$. To write a mapping rule for this dilation you would write: $(x,y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$.

Example 2

Thomas describes a dilation of point JT with vertices $J(-2,6)$ to $T(6,2)$ to point $J'T'$ with vertices $J'(-4,12)$ and $T'(12,4)$. Write the notation to describe this dilation for Thomas.



Since the x- and y-coordinates are each multiplied by 2, the *scale factor* is 2. The mapping notation is: $(x,y) \rightarrow (2x,2y)$

Example 3

Given the points $A(12,8)$ and $B(8,4)$ on a line undergoing a dilation to produce $A'(6,4)$ and $B'(4,2)$, write the notation that represents the dilation.

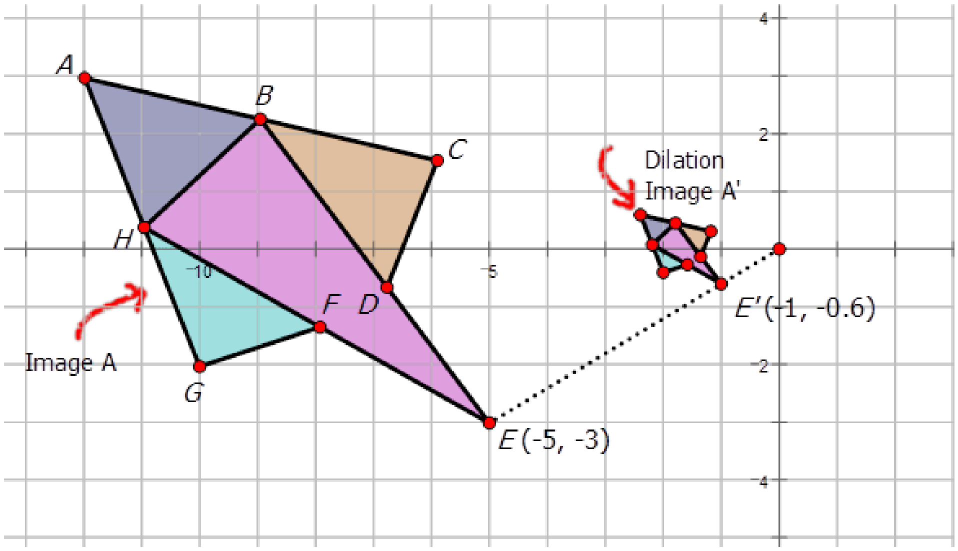
In order to write the notation to describe the dilation, choose one point on the preimage and then the corresponding point on the dilation image to see how the point has moved. Notice that point EA is:

$$A(12,8) \rightarrow A'(6,4)$$

Since both x- and y-coordinates are multiplied by $\frac{1}{2}$, the dilation is about the origin has a scale factor of $\frac{1}{2}$. The notation for this dilation would be: $(x,y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$.

Example 4

Janet was playing around with a drawing program on her computer. She created the following diagrams and then wanted to determine the transformations. Write the notation rule that represents the transformation of the purple and blue diagram to the orange and blue diagram.



In order to write the notation to describe the dilation, choose one point on the preimage A and then the corresponding point on the dilation image A' to see how the point has changed. Notice that point E is shown in the diagram:

$$E(-5,-3) \rightarrow E'(-1,-0.6)$$

Since both x- and y-coordinates are multiplied by $\frac{1}{5}$, the dilation is about the origin has a scale factor of $\frac{1}{5}$. The notation for this dilation would be: $(x,y) \rightarrow \left(\frac{1}{5}x, \frac{1}{5}y\right)$.

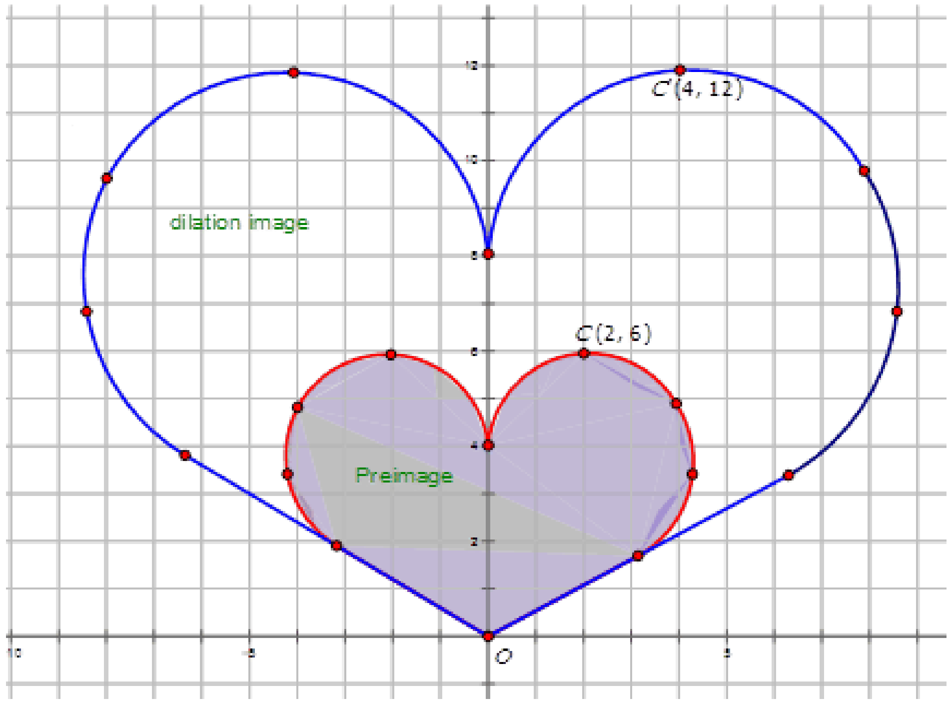
Review

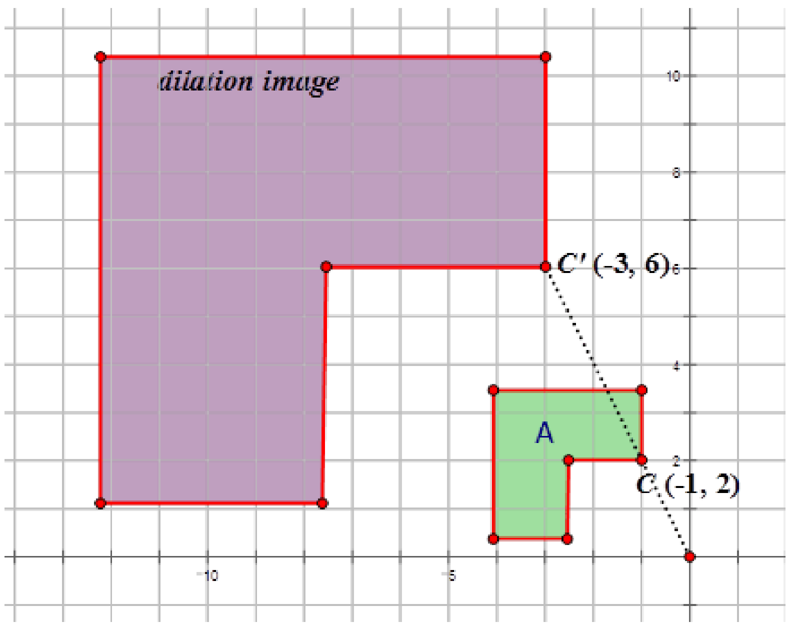
Complete the following table. Assume that the center of dilation is the origin.

TABLE 10.15:

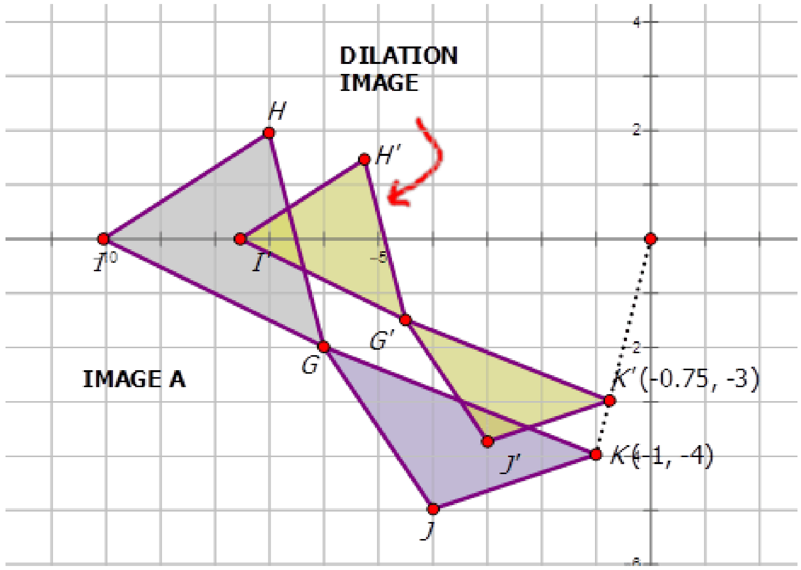
Starting Point	D_2	D_5	$D_{\frac{1}{2}}$	$D_{\frac{3}{4}}$
1. (1, 4)				
2. (4, 2)				
3. (2, 0)				
4. (-1, 2)				
5. (-2, -3)				
6. (9, 4)				
7. (-1, 3)				
8. (-5, 2)				
9. (2, 6)				
10. (-5, 7)				

Write the notation that represents the dilation of the preimage to the image for each diagram below.

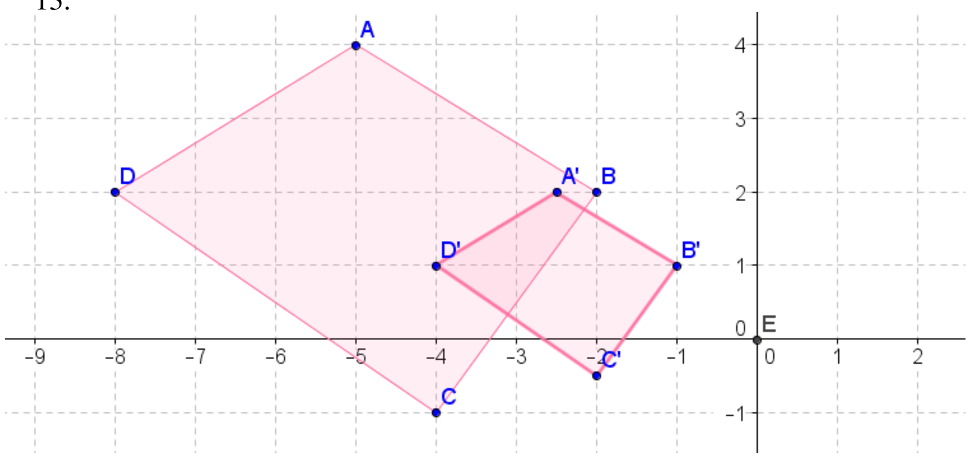
11. 

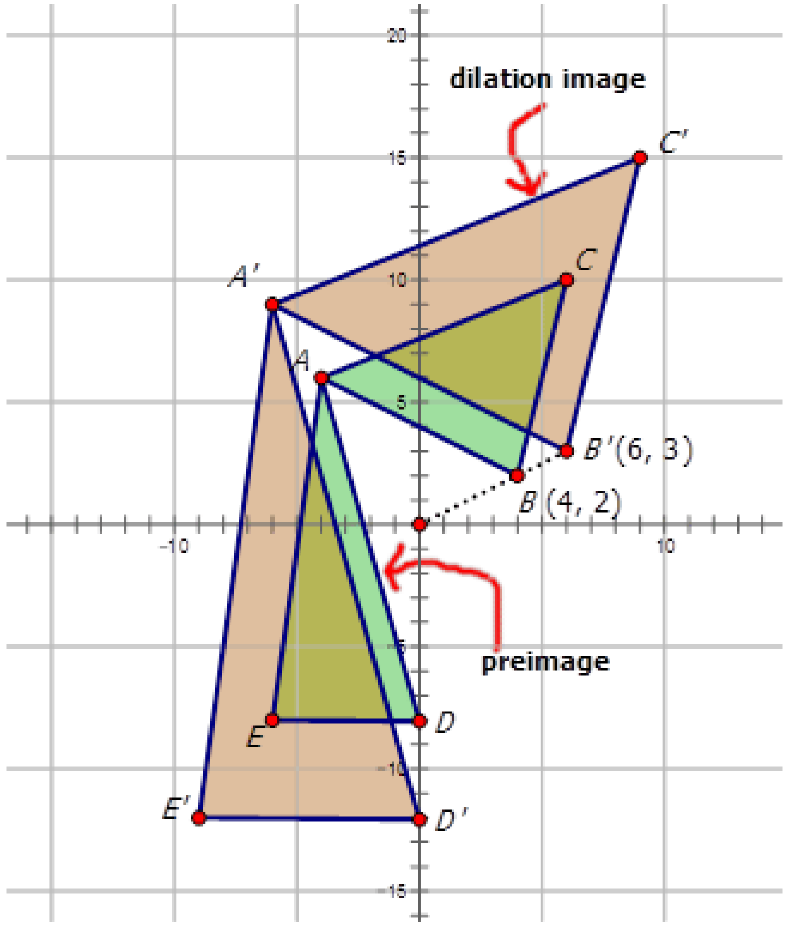
12. 

13.



14.



15. 

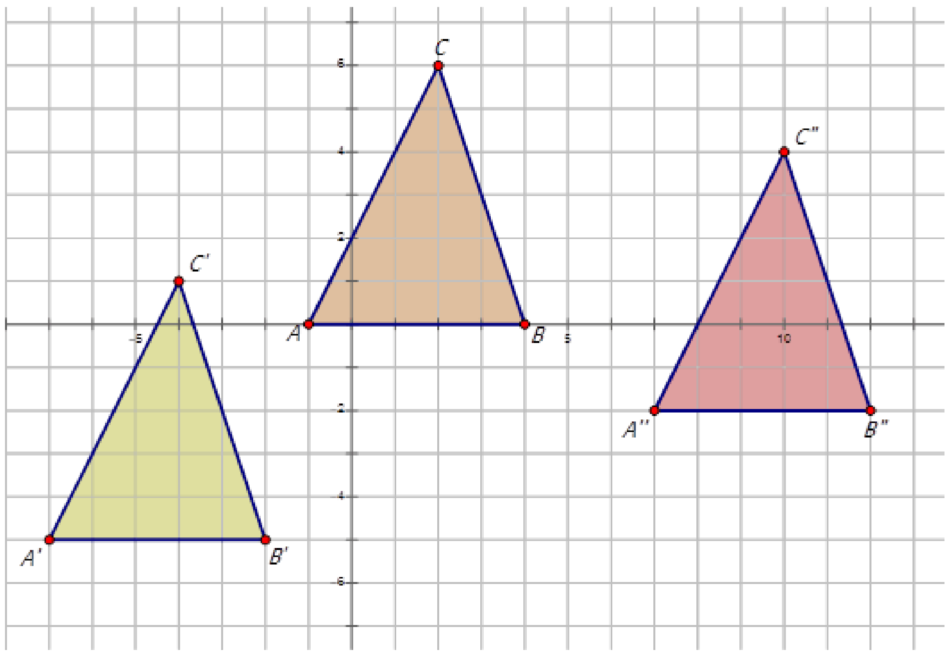
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.12.

10.13 Composite Transformations

Here you will learn about composite transformations.

Look at the following diagram. It involves two translations. Identify the two translations of triangle ABC.



Composite Transformations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A composite transformation is when two or more transformations are performed on a figure (called the preimage) to produce a new figure (called the image).

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65246

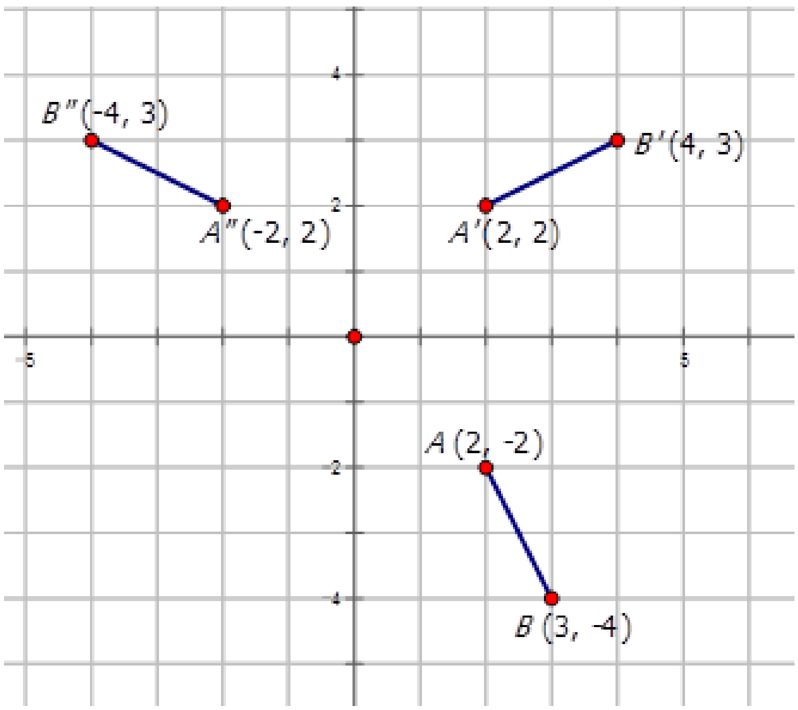
MEDIA

Click image to the left or use the URL below.

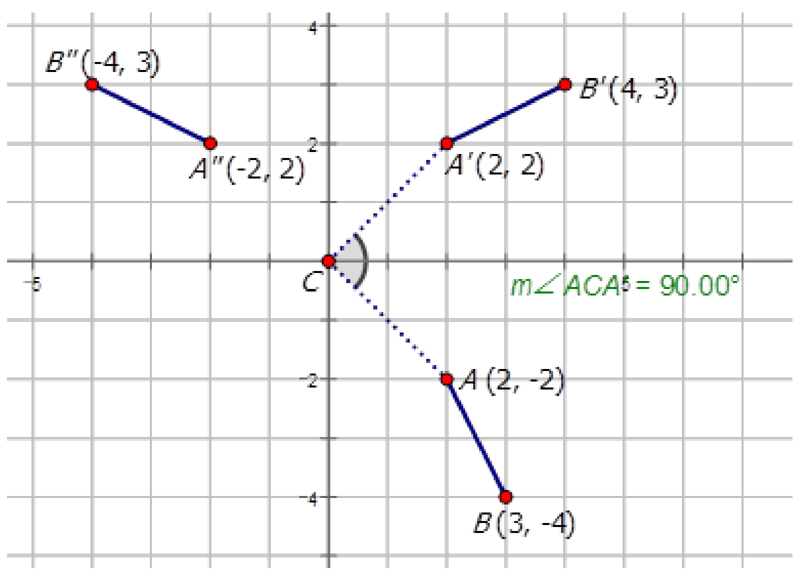
URL: https://www.ck12.org/flx/render/embeddedobject/65247

Describe the transformations in the diagram below.

The transformations involve a reflection and a rotation.

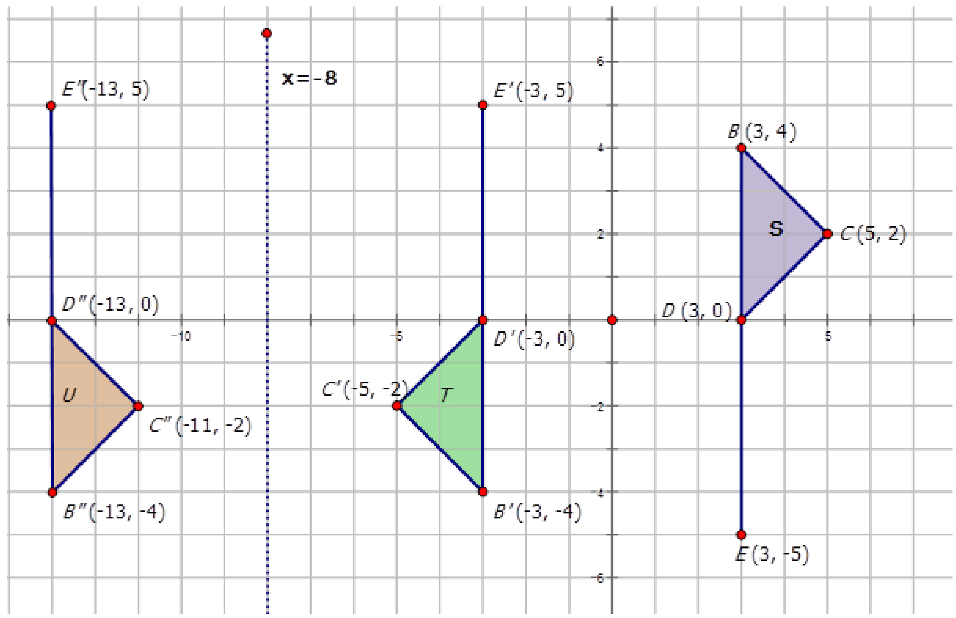


First line AB is rotated about the origin by 90°CCW.



Then the line $A'B'$ is reflected about the y-axis to produce line $A''B''$.

Describe the transformations in the diagram below.



The flag in diagram S is rotated about the origin 180° to produce flag T. You know this because if you look at one point you notice that both x- and y-coordinate points is multiplied by -1 which is consistent with a 180° rotation about the origin. Flag T is then reflected about the line $x = -8$ to produce Flag U.

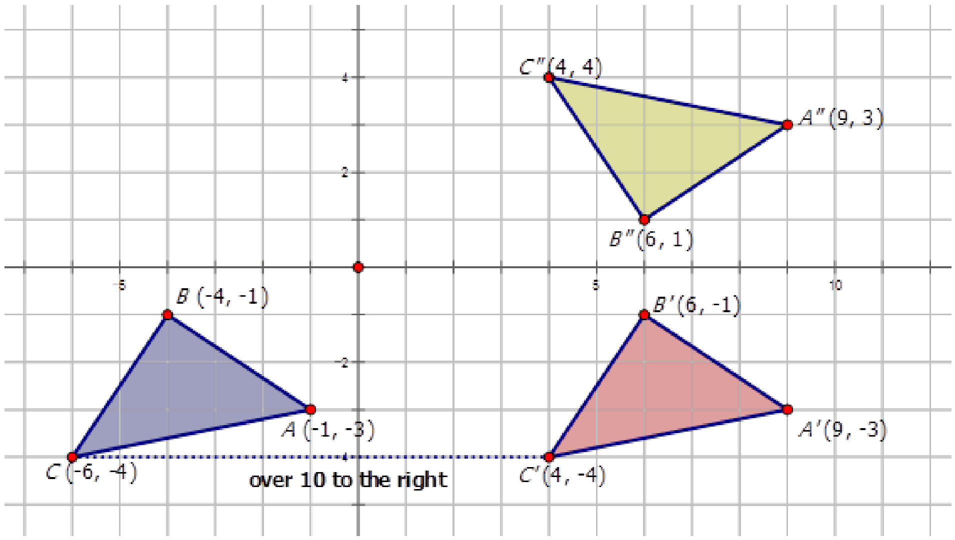
Draw the diagram

Triangle ABC where the vertices of ΔABC are $A(-1,-3)$, $B(-4,-1)$, and $C(-6,-4)$ undergoes a composition of transformations described as:

a) a translation 10 units to the right, then

b) a reflection in the x-axis.

Draw the diagram to represent this composition of transformations. What are the vertices of the triangle after both transformations are applied?

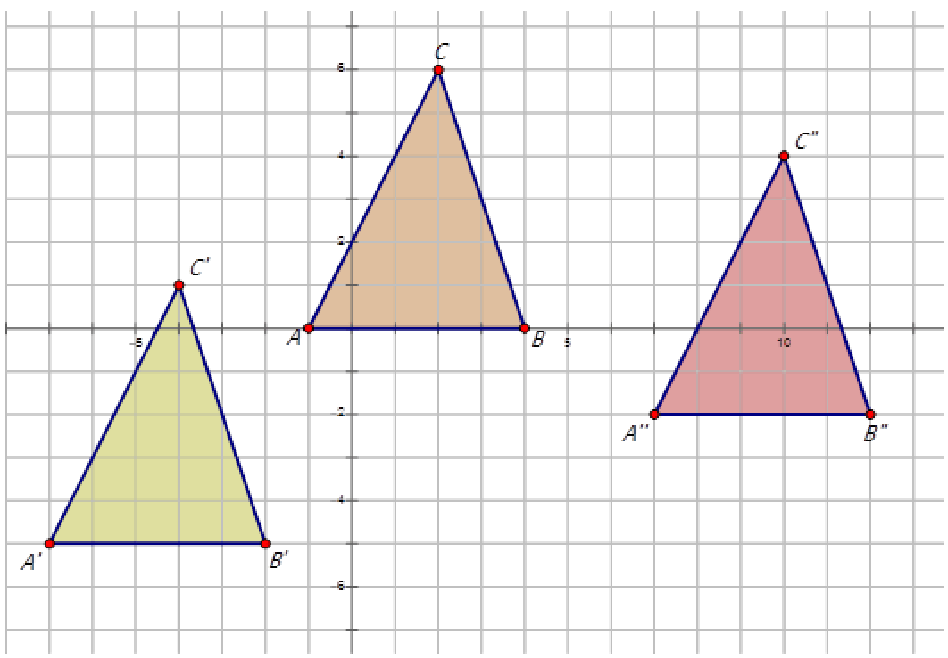


Triangle $A''B''C''$ is the final triangle after all transformations are applied. It has vertices of $A''(9,3)$, $B''(6,1)$, and $C''(4,4)$.

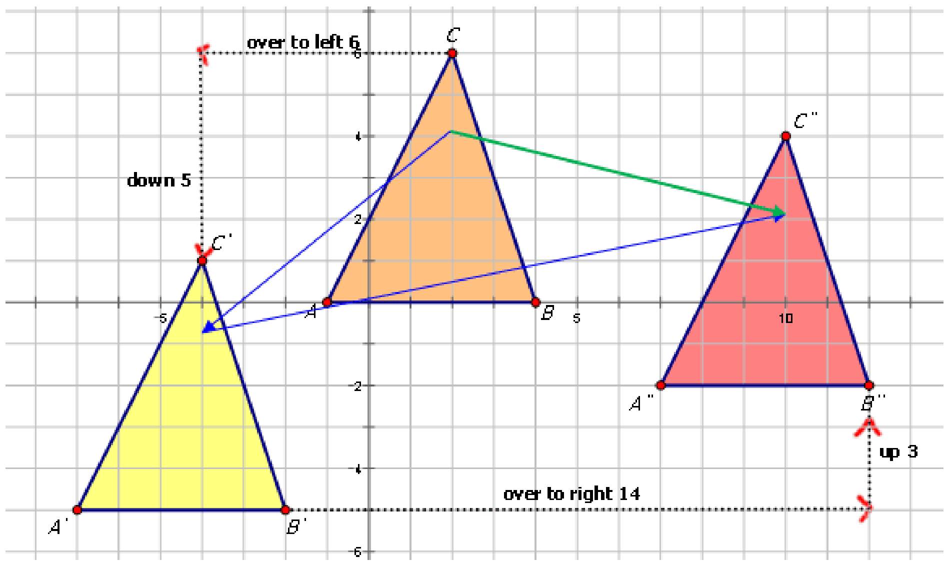
Examples

Example 1

Earlier, you were asked to identify the two translations of triangle ABC.



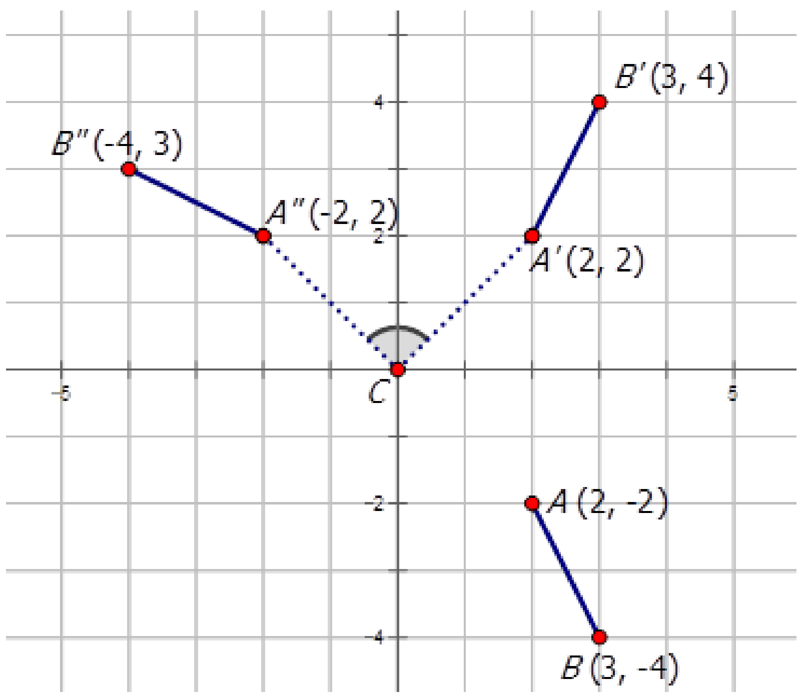
ΔABC moves over 6 to the left and down 5 to produce $\Delta A'B'C'$. Then $\Delta A'B'C'$ moves over 14 to the right and up 3 to produce $\Delta A''B''C''$. These translations are represented by the blue arrows in the diagram.



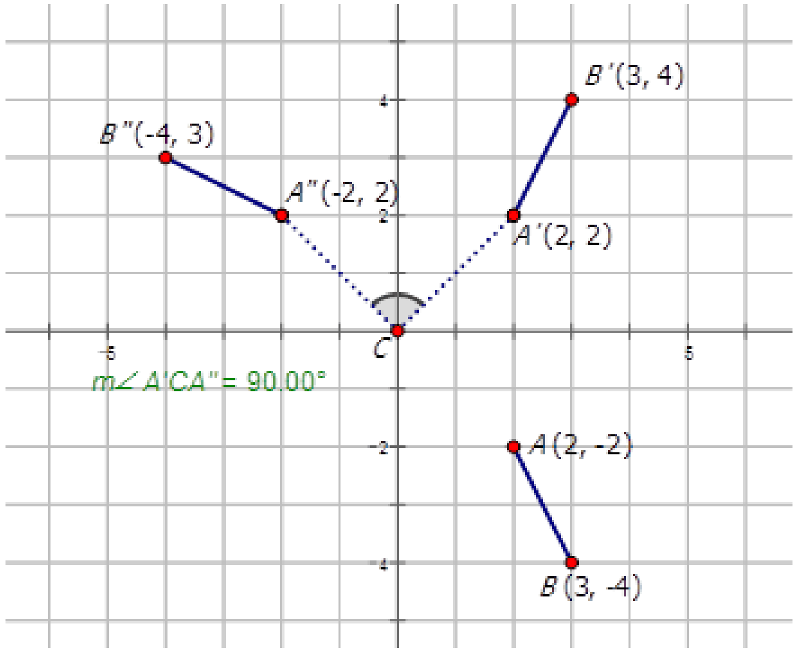
All together ΔABC moves over 8 to the right and down 2 to produce $\Delta A''B''C''$. The total translations for this movement are seen by the green arrow in the diagram above.

Example 2

Describe the transformations in the diagram below. The transformations involve a rotation and a reflection.



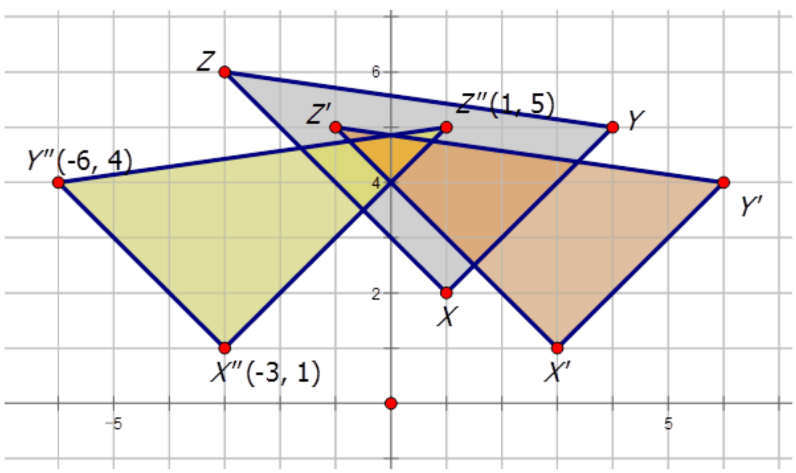
The transformations involve a reflection and a rotation. First line AB is reflected about the y-axis to produce line $A'B'$.



Then the line $A'B'$ is rotated about the origin by 90°CCW to produce line $A''B''$.

Example 3

Triangle XYZ has coordinates $X(1,2)$, $Y(-3,6)$ and $Z(4,5)$.The triangle undergoes a rotation of 2 units to the right and 1 unit down to form triangle $X'Y'Z'$. Triangle $X'Y'Z'$ is then reflected about the y-axis to form triangle $X''Y''Z''$. Draw the diagram of this composite transformation and determine the vertices for triangle $X''Y''Z''$.



Example 4

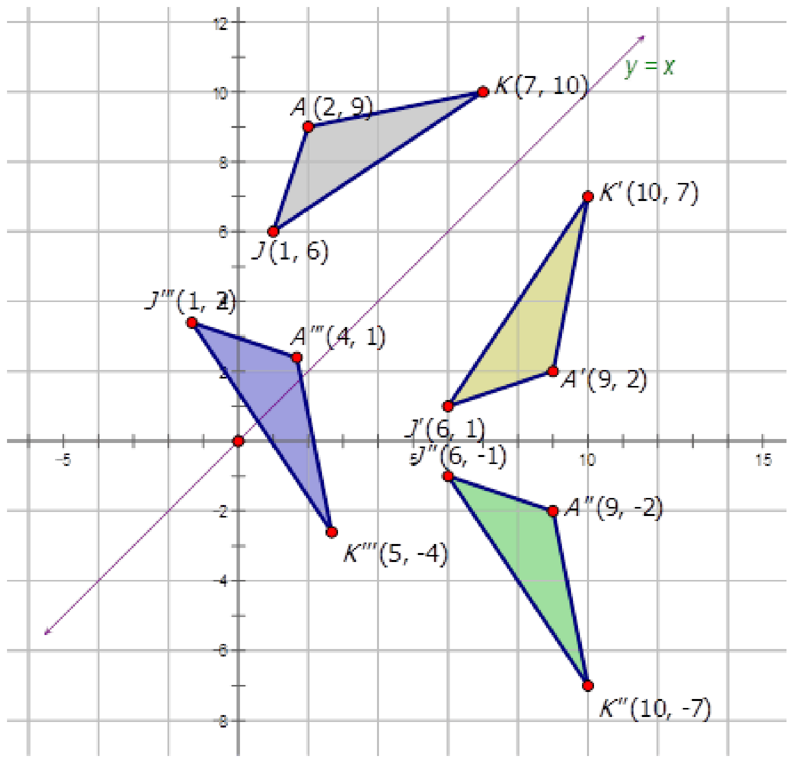
The coordinates of the vertices of ΔJAK are $J(1,6)$, $B(2,9)$, and $C(7,10)$.

a) Draw and label ΔJAK.

b) ΔJAK is reflected over the line $y = x$. Graph and state the coordinates of $\Delta J'A'K'$.

c) $\Delta J'A'K'$ is then reflected about the x-axis. Graph and state the coordinates of $\Delta J''A''K''$.

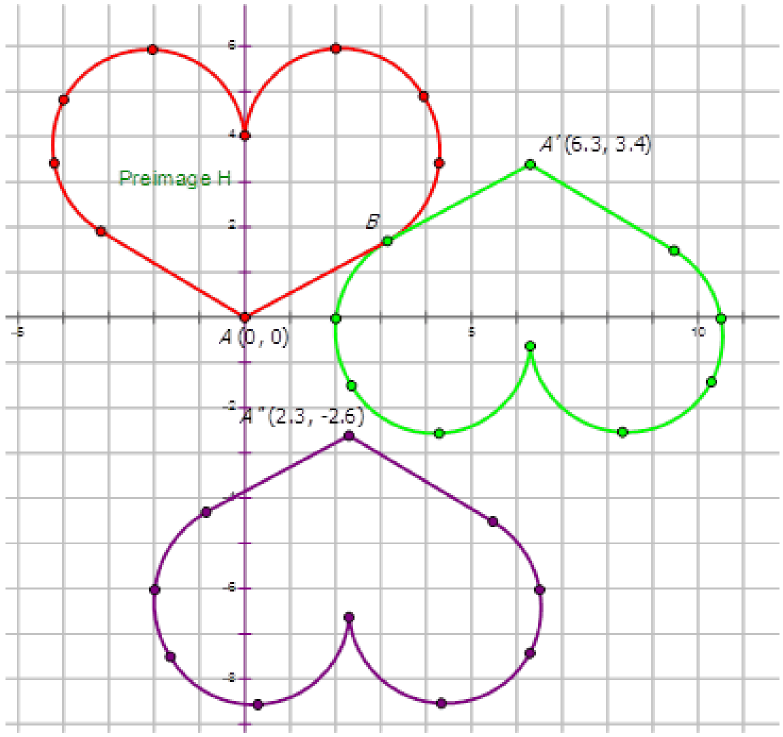
d) $\Delta J''A''K''$ undergoes a translation of 5 units to the left and 3 units up. Graph and state the coordinates of $\Delta J'''A'''K'''$.

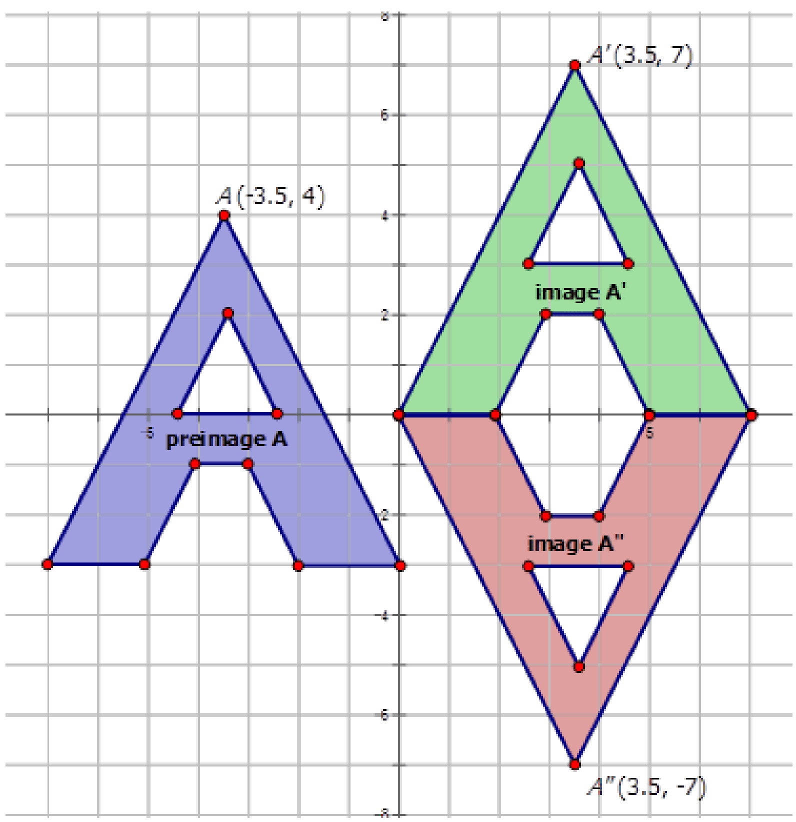


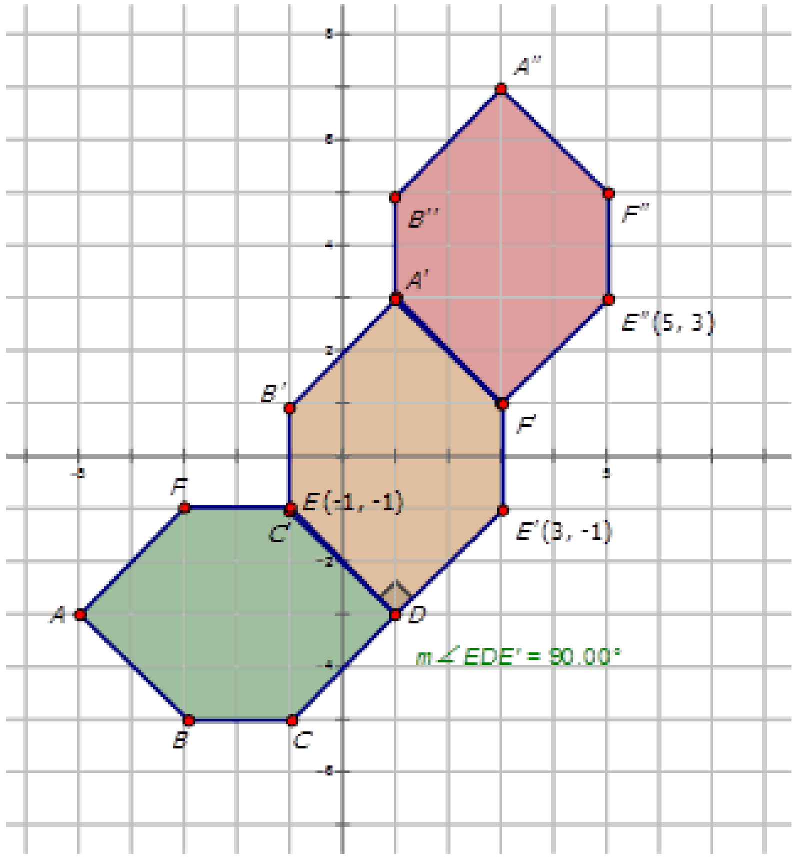
Review

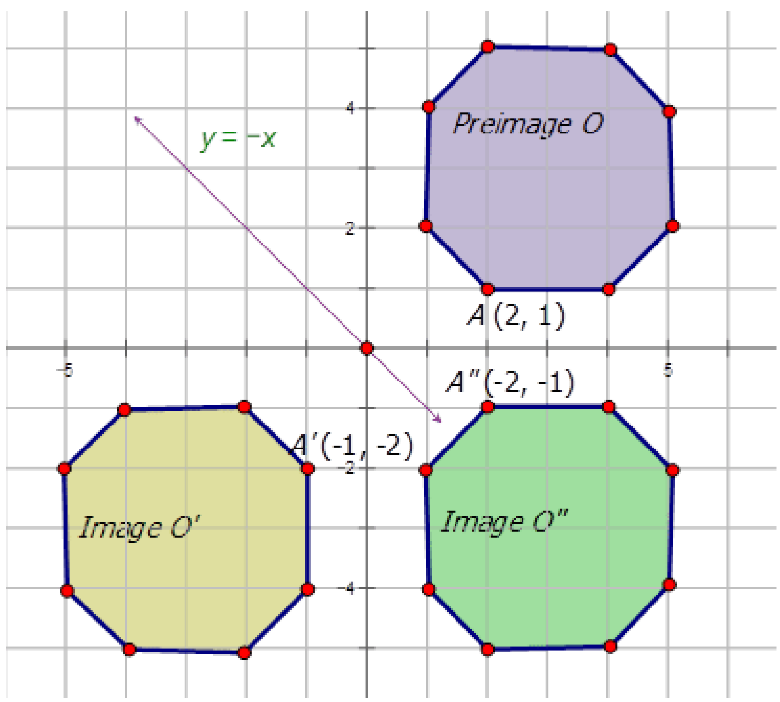
1. A point X has coordinates (-1, -8). The point is reflected across the y-axis to form X'. X' is translated over 4 to the right and up 6 to form X''. What are the coordinates of X' and X''?
2. A point A has coordinates (2, -3). The point is translated over 3 to the left and up 5 to form A'. A' is reflected across the x-axis to form A''. What are the coordinates of A' and A''?
3. A point P has coordinates (5, -6). The point is reflected across the line $y = -x$ to form P'. P' is rotated about the origin 90°CW to form P''. What are the coordinates of P' and P''?
4. Line JT has coordinates $J(-2,-5)$ and $T(2,3)$. The segment is rotated about the origin 180° to form $J'T'$. $J'T'$ is translated over 6 to the right and down 3 to form $J''T''$. What are the coordinates of $J'T'$ and $J''T''$?
5. Line SK has coordinates $S(-1,-8)$ and $K(1,2)$. The segment is translated over 3 to the right and up 3 to form $S'K'$. $S'K'$ is rotated about the origin 90°CCW to form $S''K''$. What are the coordinates of $S'K'$ and $S''K''$?
6. A point K has coordinates (-1, 4). The point is reflected across the line $y = x$ to form K'. K' is rotated about the origin 270°CW to form K''. What are the coordinates of K' and K''?

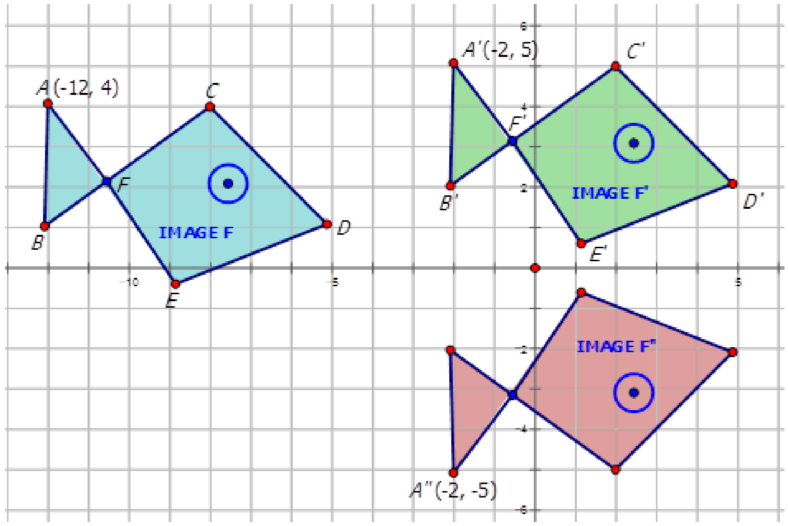
Describe the following composite transformations:

7. 
A'(6.3, 3.4)
Preimage H
B
A (0, 0)
A"(2.3, -2.6)

8. 
A'(3.5, 7)
A(-3.5, 4)
image A'
preimage A
image A"
A"(3.5, -7)

9. 
A''
B''
F''
A'
E''(5, 3)
B'
F'
F
E(-1, -1)
C'
E'(3, -1)
A
D
m∠EDE' = 90.00°
B
C

10. 
Preimage O
y = −x
A (2, 1)
A''(-2, -1)
A'(-1, -2)
Image O'
Image O''

11. 

12. Explore what happens when you reflect a shape twice, over a pair of parallel lines. What one transformation could have been performed to achieve the same result?
13. Explore what happens when you reflect a shape twice, over a pair of intersecting lines. What one transformation could have been performed to achieve the same result?
14. Explore what happens when you reflect a shape over the x-axis and then the y-axis. What one transformation could have been performed to achieve the same result?
15. A composition of a reflection and a translation is often called a glide reflection. Make up an example of a glide reflection. Why do you think it's called a **glide** reflection?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.13.

10.14 Order of Composite Transformations

Here you will investigate whether or not the order that transformations are performed matters when doing a composite transformation.

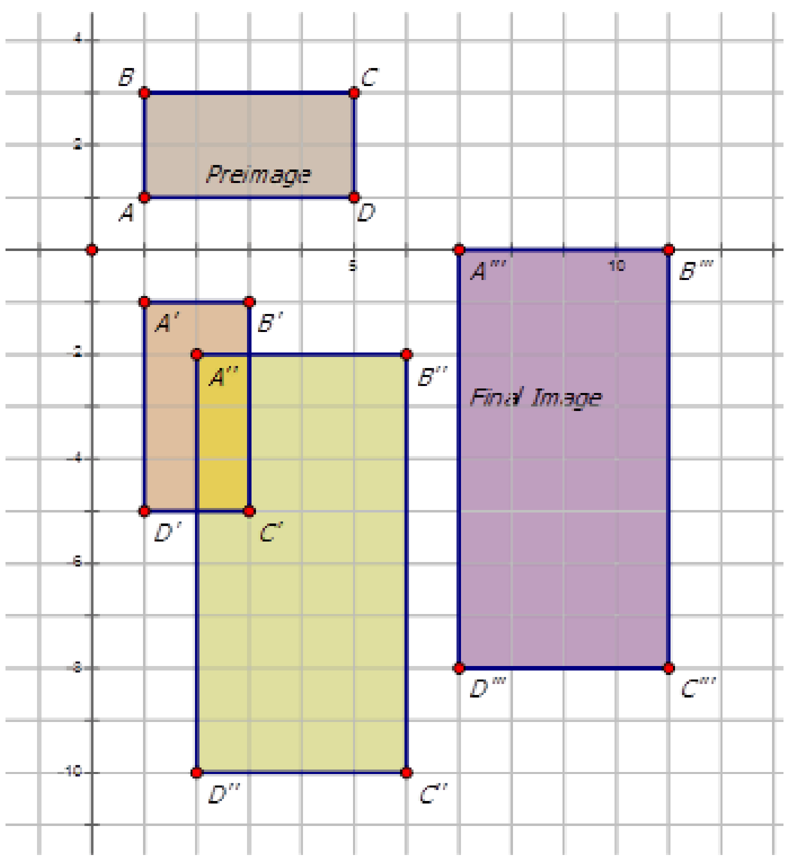
Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane.

The quadrilateral undergoes a dilation centered at the origin of scale factor $\frac{1}{3}$ and then is translated 4 units to the right and 5 units down. Show the resulting image.

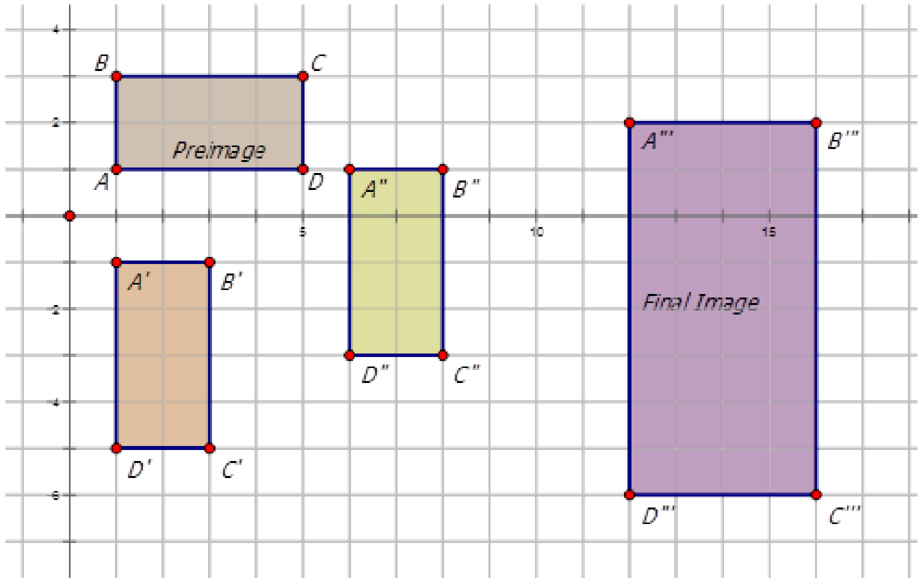
Order of Composite Transformations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A composite transformation is when two or more transformations are performed on a figure (called the preimage) to produce a new figure (called the image).

Imagine if you rotate, then dilate, and then translate a rectangle of vertices $A(1,1), B(1,3), C(5,3)$, and $D(5,1)$. You would end up with a diagram similar to that found below:



If you take the same preimage and rotate, translate it, and finally dilate it, you could end up with the following diagram:



Therefore, the order is important when performing a composite transformation. Remember that the composite transformation involves a series of one or more transformations in which each transformation after the first is performed on the image that was transformed. Only the first transformation will be performed on the initial preimage.

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65254

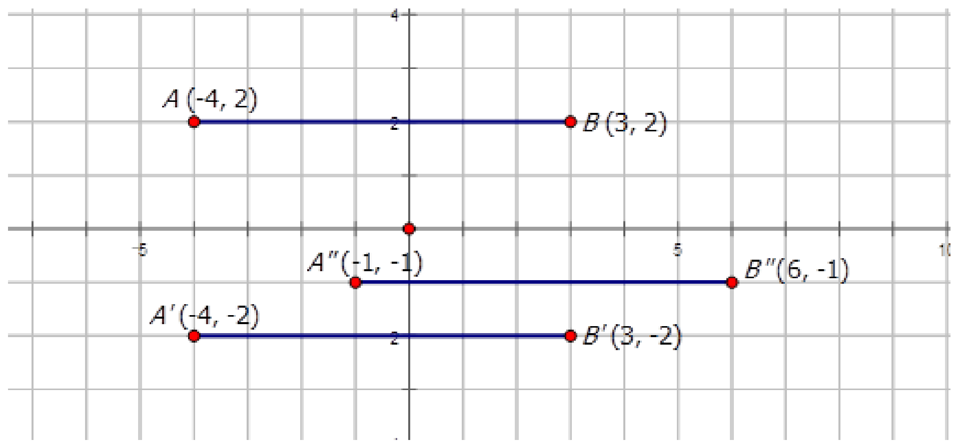
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65255

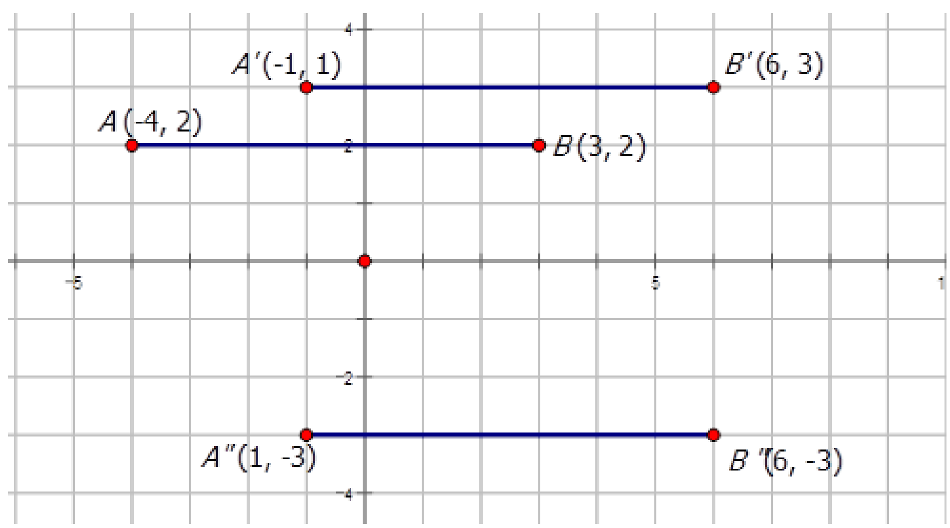
Draw a diagram

Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has undergone a reflection across the x-axis. It then undergoes a translation up one unit and over 3 units to the right to produce $A''B''$. Draw a diagram to represent this composite transformation and indicate the vertices for each transformation.



Solve using order of composite transformations

For the composite transformation in the previous problem, suppose the preimage *AB* undergoes a translation up one unit and over 3 units to the right and then undergoes a reflection across the *x*-axis. Does the order matter?

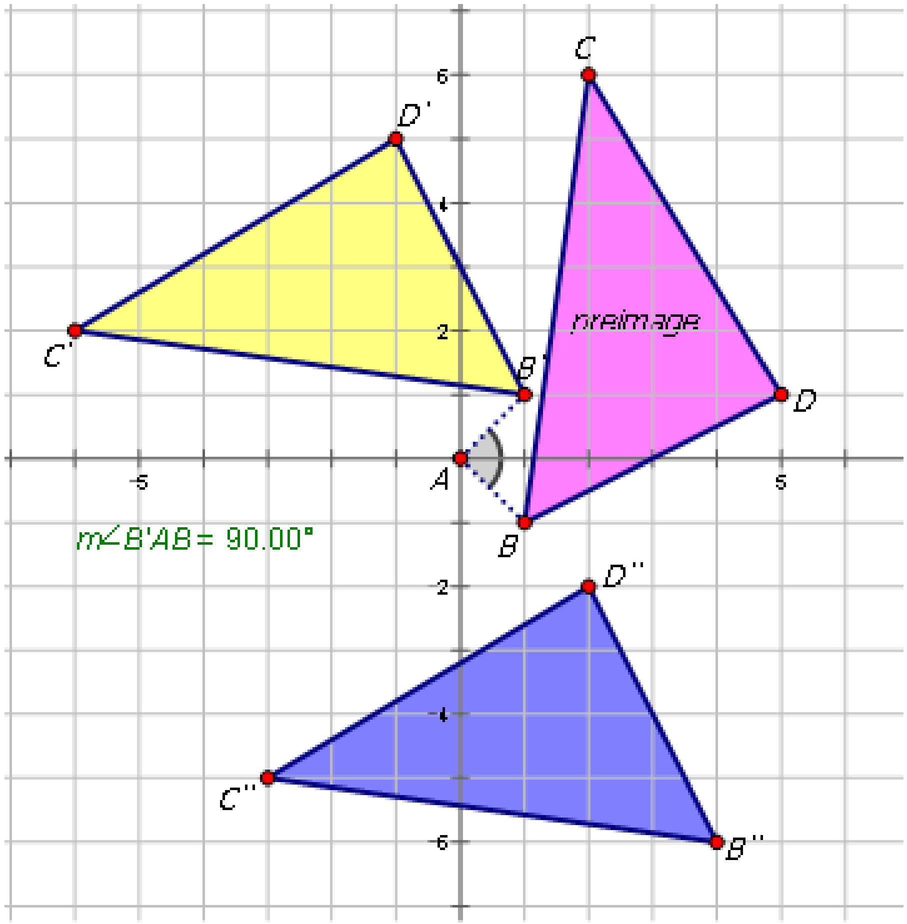


For this example $A''B''$ is not the same as $A''B''$ from the previous example (example A). Therefore order does matter.

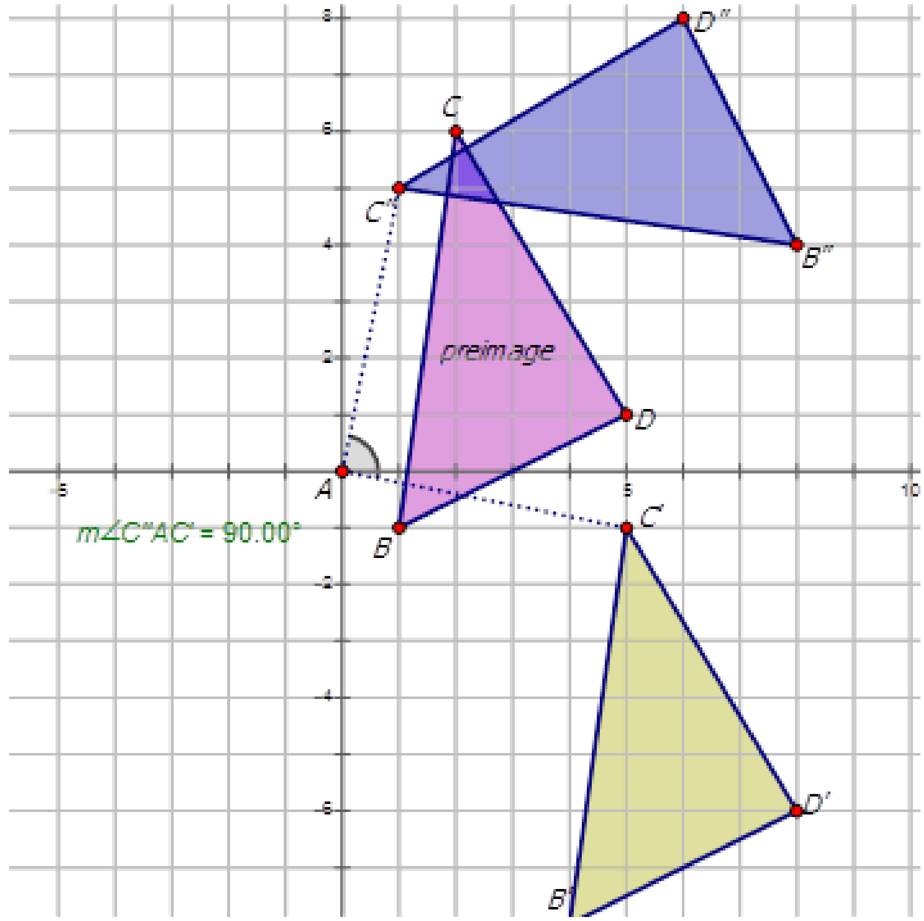
Solve using order of transformation

Triangle *BCD* is rotated 90°CCW about the origin. The resulting figure is then translated over 3 to the right and down 7. Does order matter?

Order: Rotation then Translation



Order: Translation then Rotation



The blue triangle represents the final image after the composite transformation. In this example, order does matter as the blue triangles do not end up in the same locations.

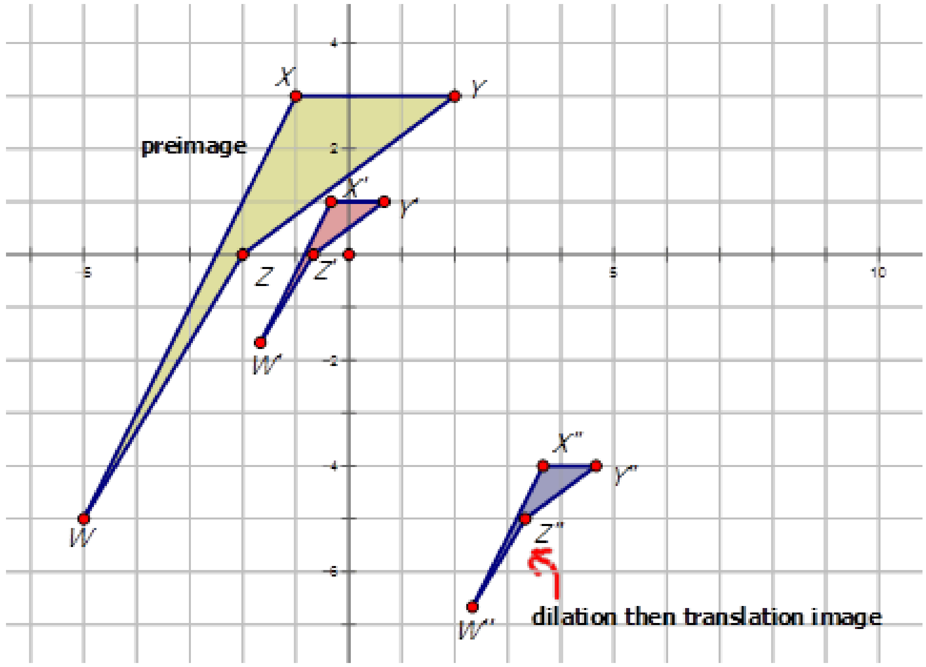
Examples

Example 1

Earlier, you were given a problem about a quadrilateral.

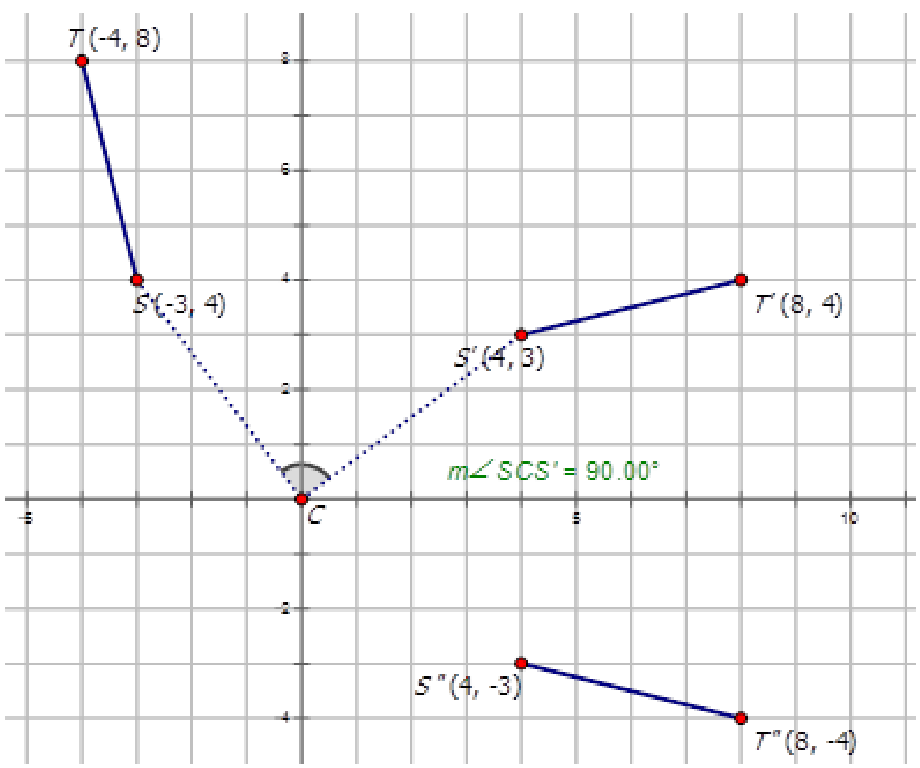
Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane.

The quadrilateral undergoes a dilation centered at the origin of scale factor $\frac{1}{3}$ and then is translated 4 units to the right and 5 units down. Show the resulting image.



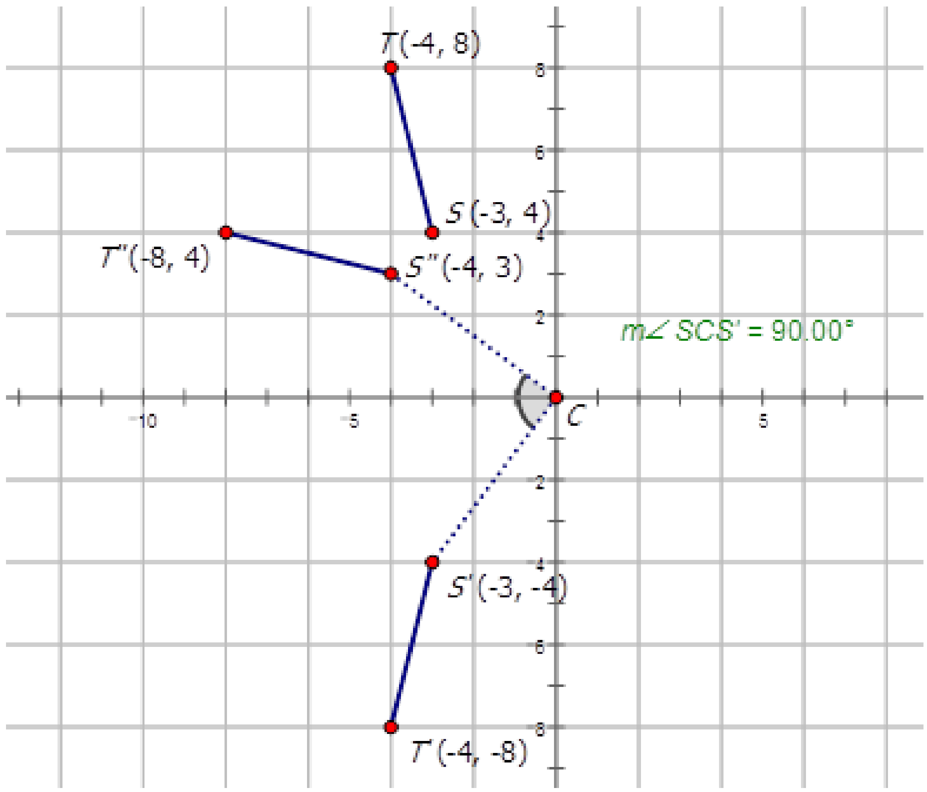
Example 2

Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has undergone a rotation about the origin at 90°CW and then a reflection in the x-axis. Draw a diagram with labeled vertices to represent this composite transformation.



Example 3

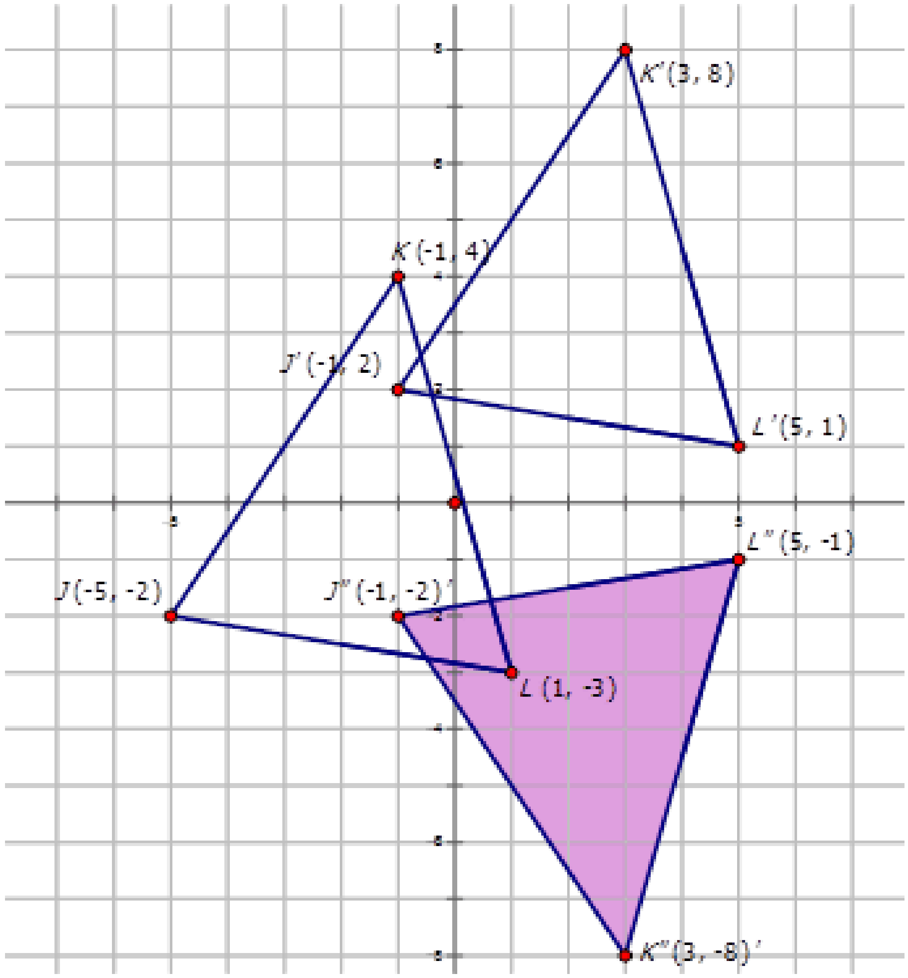
Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has undergone a reflection in the x-axis and then a rotation about the origin at 90°CW. Draw a diagram with labeled vertices to represent this composite transformation. Is the graph the same as the diagram in #1?



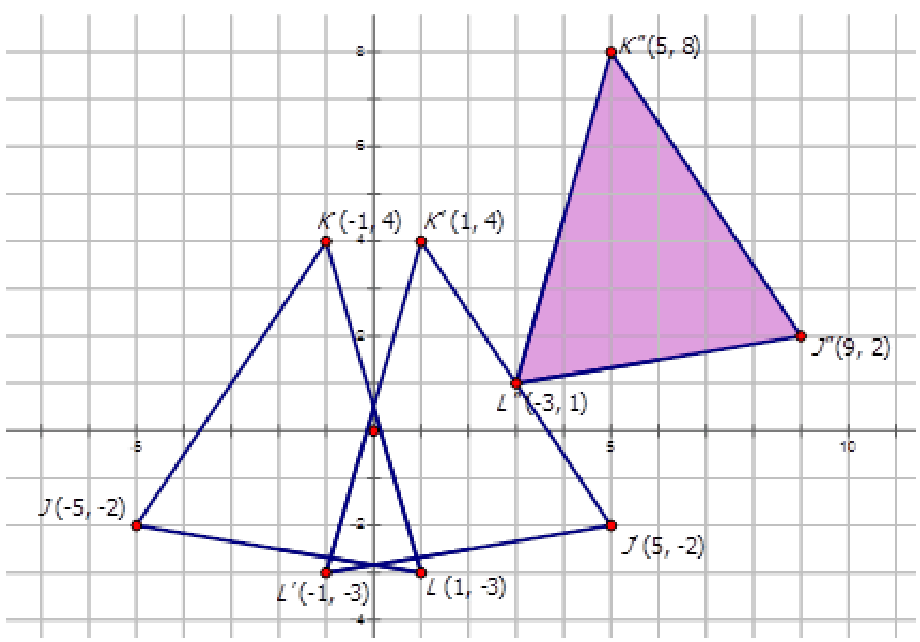
If you compare the graph above to that found in Question 1, you see that the final transformation image $S''T''$ has different coordinates than the image $S''T''$ in question 2. Therefore order does matter.

Example 4

The triangle with vertices $J(-5,-2), K(-1,4)$ and $L(1,-3)$ has undergone a transformation of up 4 and over to the right 4 and then a reflection in the x-axis. Draw and label the composite transformation. Does order matter?



Order: Translation then Reflection

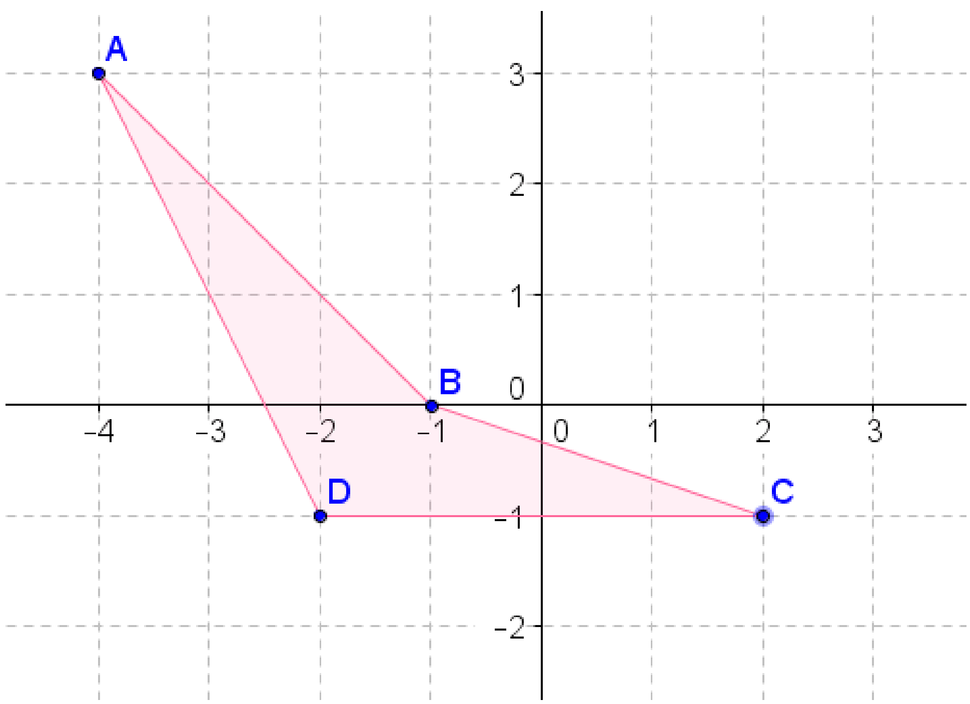


Order: Reflection then Transformation

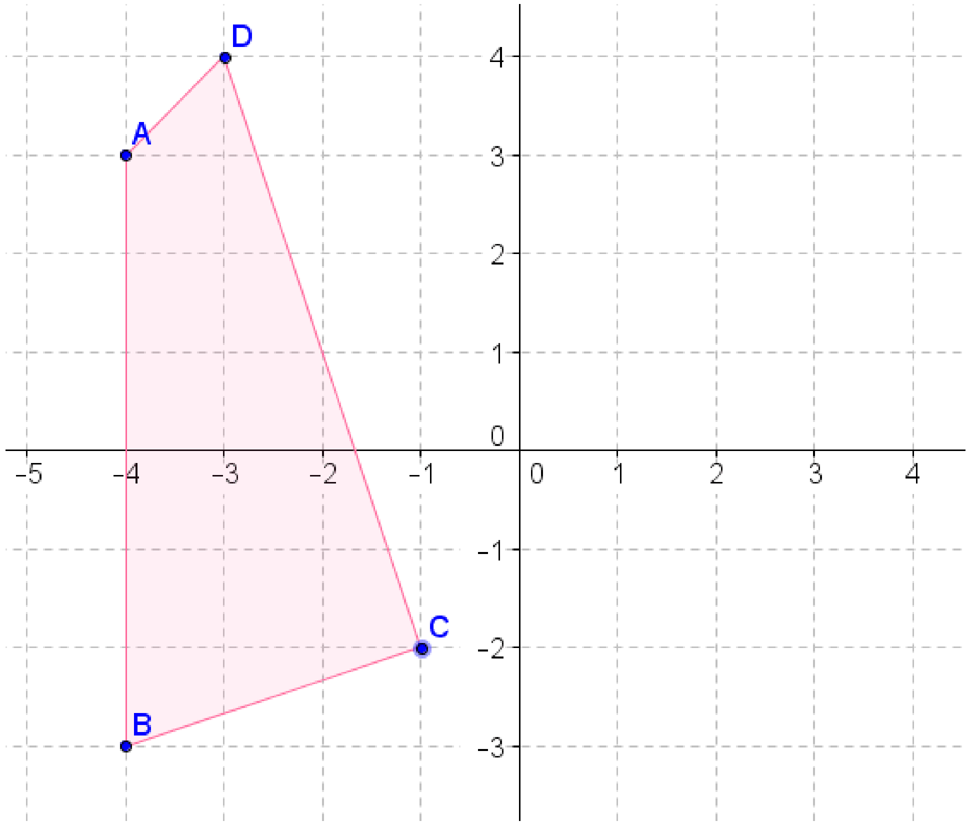
In this problem, order did matter. The final image after the composite transformation changed when the order changed.

Review

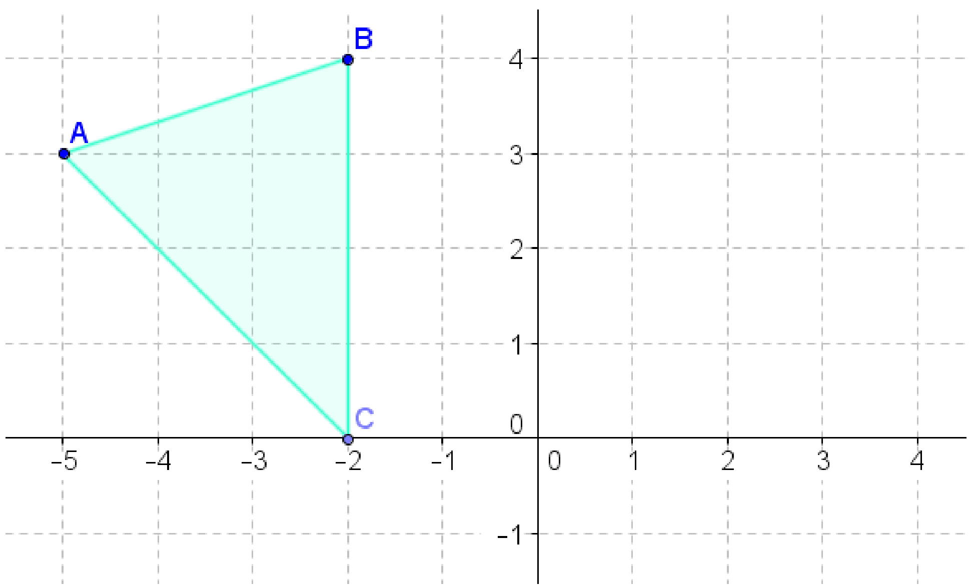
1. Reflect the above figure across the x-axis and then rotate it 90° CW about the origin.
2. Translate the above figure 2 units to the left and 2 units up and then reflect it across the line $y = x$



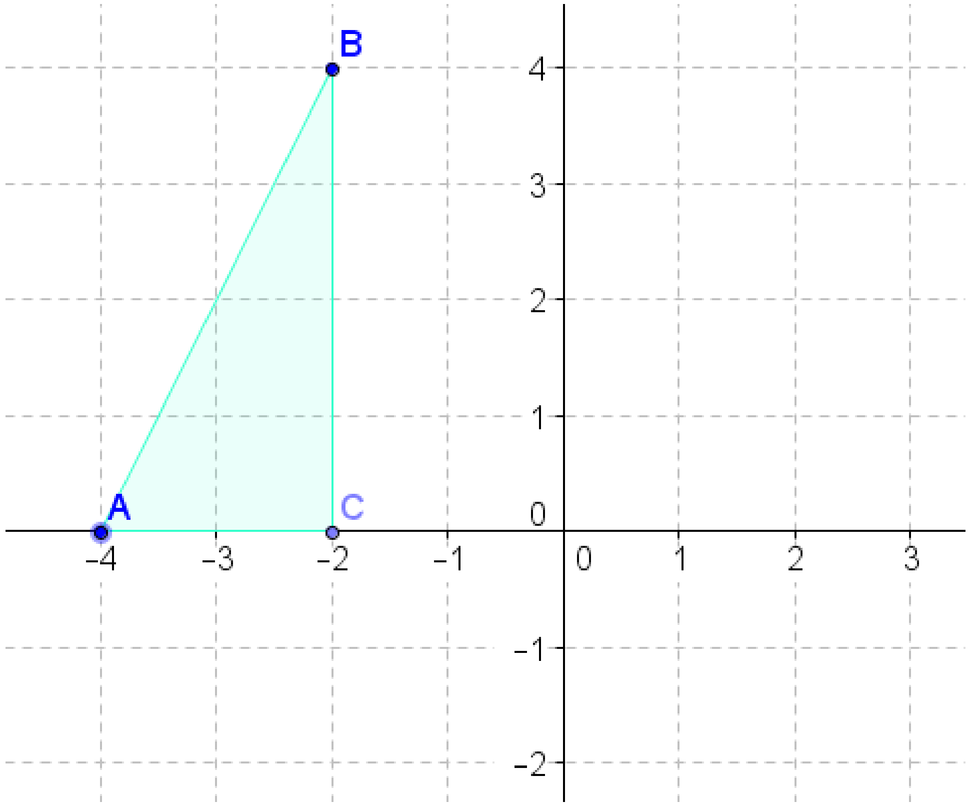
3. Reflect the above figure across the y-axis and then reflect it across the x-axis.
4. What single transformation would have produced the same result as in #3?



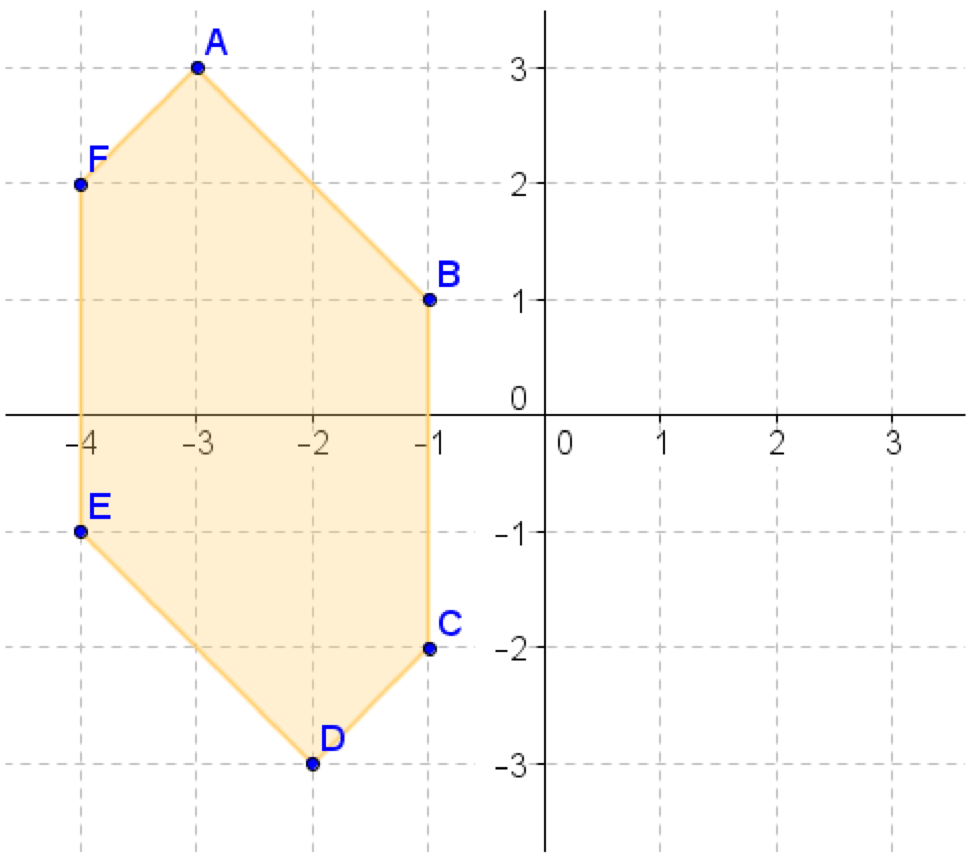
5. Reflect the above figure across the line $x = 2$ and then across the line $x = 8$.
6. What single transformation would have produced the same result a in #5?



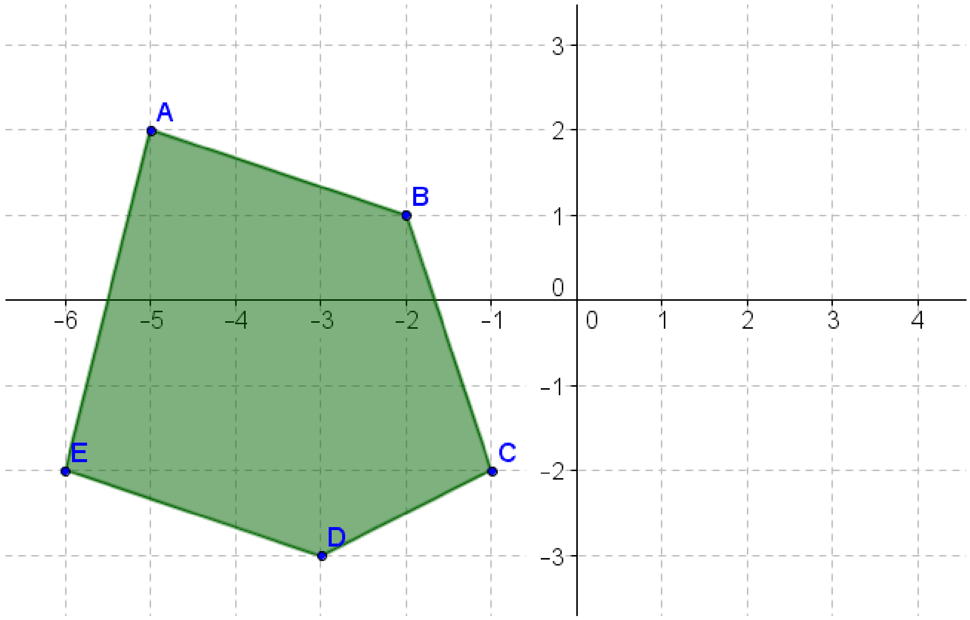
7. Reflect the above figure across the line y=x and then across the x-axis.
8. What single transformation would have produced the same result a in #7?



9. Translate the above figure 2 units to the right and 3 units down and then reflect it across the y-axis.
10. Rotate the above figure 270° CCW about the origin and then translate it over 1 unit to the right and down 1 unit.



11. Reflect the above figure across the line $y = -x$ and then translate it 2 units to the left and 3 units down.
12. Translate the above figure 2 units to the left and 3 units down and then reflect it across the line $y = -x$.



13. Translate the above figure 3 units to the right and 4 units down and then rotate it about the origin 90° CW.
14. Rotate the above figure about the origin 90° CW and then translate it 3 units to the right and 4 units down.
15. How did your result to #13 compare to your result to #14?

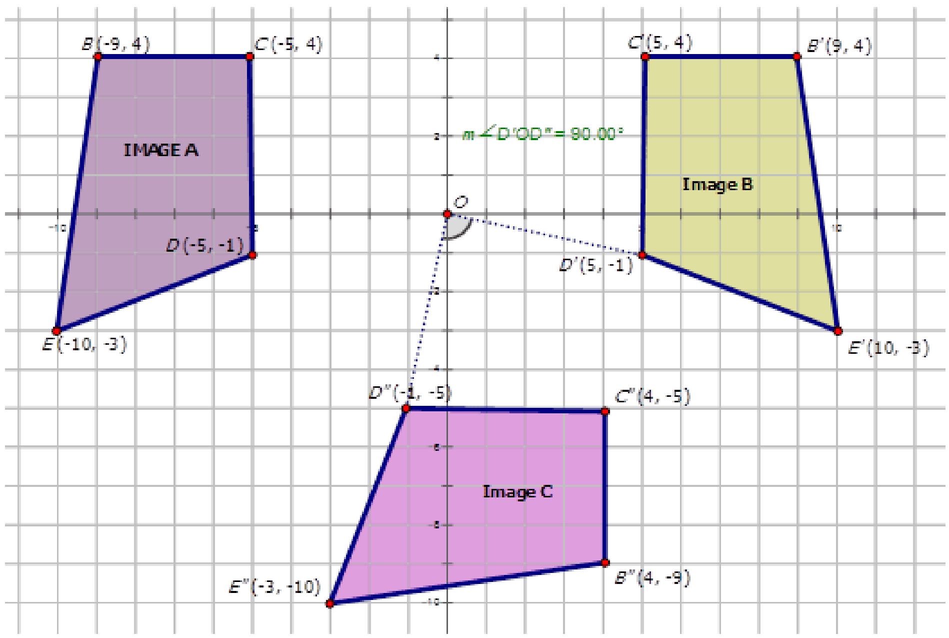
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.14.

10.15 Notation for Composite Transformations

Here you will learn notation for describing a composite transformation.

The figure below shows a composite transformation of a trapezoid. Write the mapping rule for the composite transformation.



Notation for Composite Transformations

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A composite transformation is when two or more transformations are performed on a figure (called the preimage) to produce a new figure (called the image). The order of transformations performed in a composite transformation matters.

To describe a composite transformation using notation, state each of the transformations that make up the composite transformation and link them with the symbol $\circ$. The transformations are performed in order from right to left. Recall the following notation for translations, reflections, and rotations:

- Translation: $T_{a,b} : (x,y) \rightarrow (x+a, y+b)$ is a translation of a units to the right and b units up.
- Reflection: $r_{y-axis}(x,y) \rightarrow (-x,y)$.
- Rotation: $R_{90^\circ}(x,y) = (-y,x)$

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65262

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65263

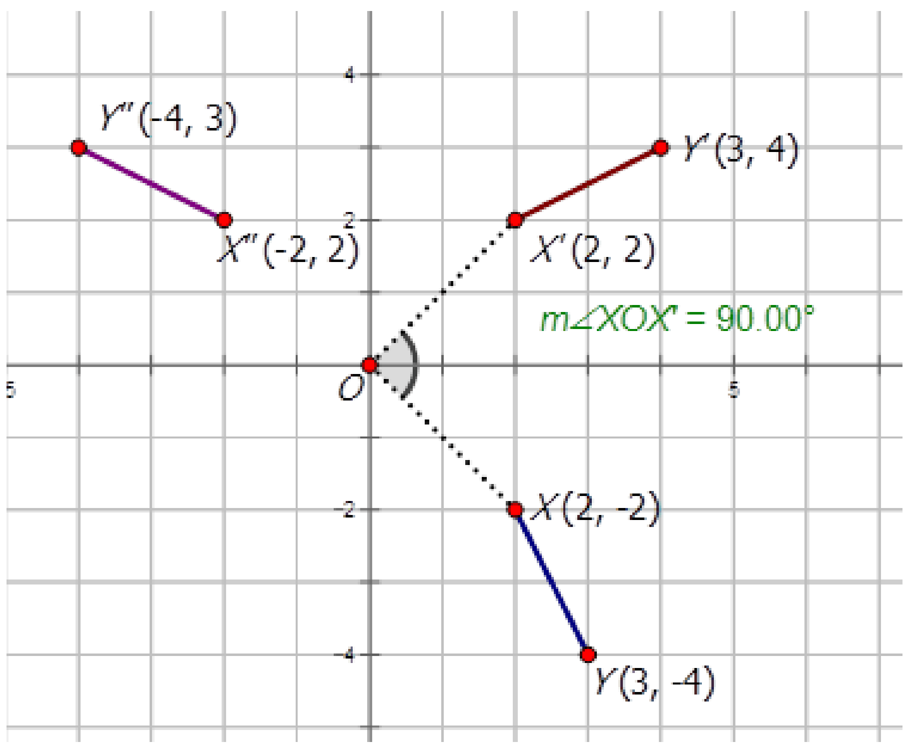
Graph the line

Graph the line XY given that $X(2,-2)$ and $Y(3,-4)$. Also graph the composite image that satisfies the rule

$$r_{y-axis} \circ R_{90^\circ}$$

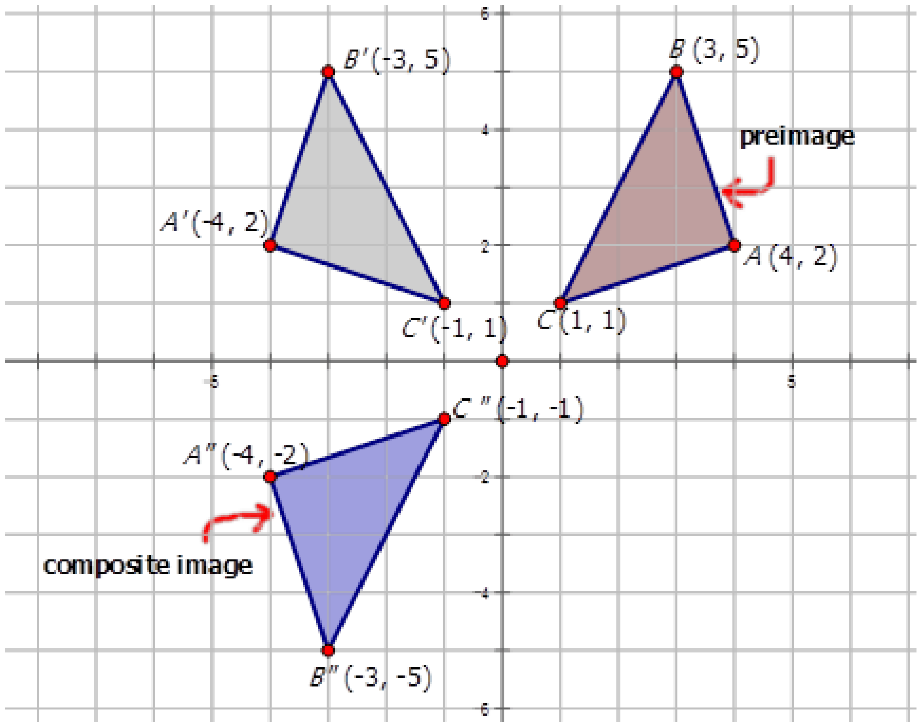
.

The first translation is a 90°CCW turn about the origin to produce $X'Y'$. The second translation is a reflection about the y-axis to produce $X''Y''$.



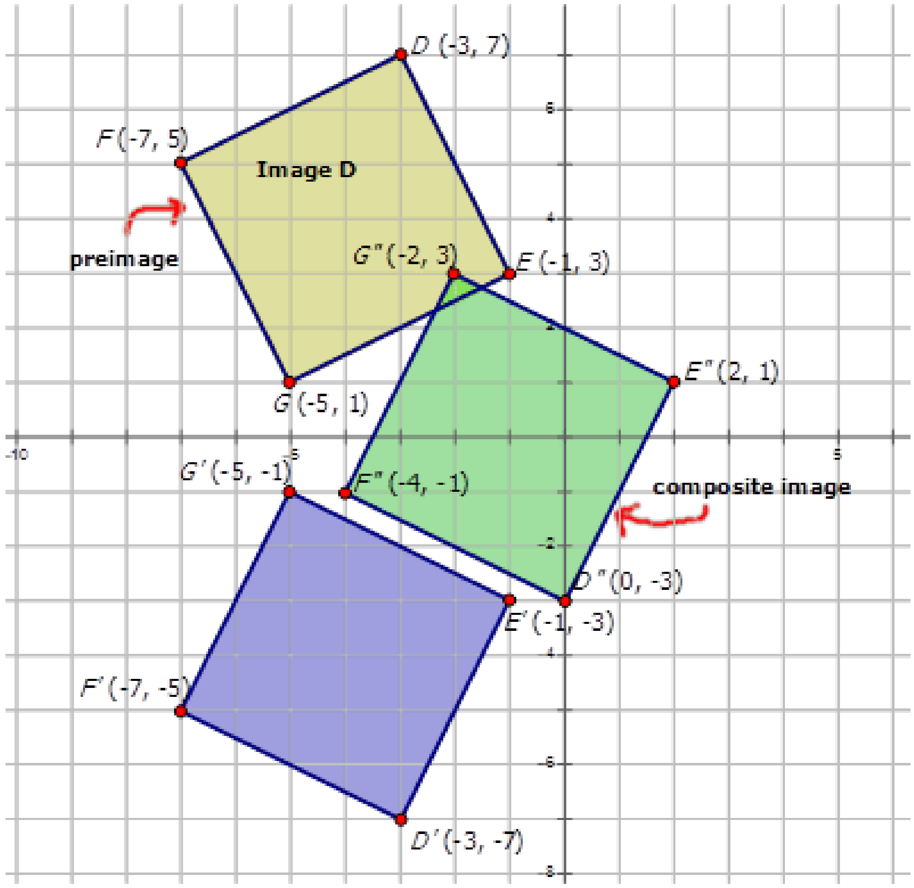
Draw the composite image

Image A with vertices $A(3,5), B(4,2)$ and $C(1,1)$ undergoes a composite transformation with mapping rule $r_{x-axis} \circ r_{y-axis}$. Draw the preimage and the composite image and show the vertices of the composite image.



Draw the composite image

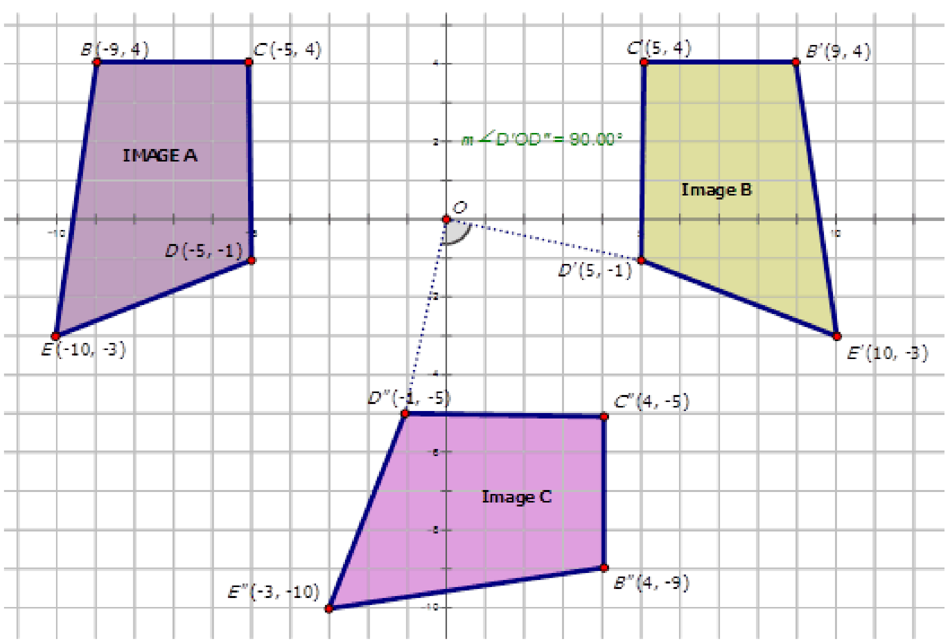
Image D with vertices $D(-3,7), E(-1,3), F(-7,5)$ and $G(-5,1)$ undergoes a composite transformation with mapping rule $T_{3,4} \circ r_{x-axis}$. Draw the preimage and the composite image and show the vertices of the composite image.



Examples

Example 1

Earlier, you were asked to write the mapping rule for the composite transformation.



The transformation from Image A to Image B is a reflection across the y-axis. The notation for this is r_{y-axis}. The transformation for image B to form image C is a rotation about the origin of 90°CW. The notation for this transformation is R_{270°. Therefore, the notation to describe the transformation of Image A to Image C is

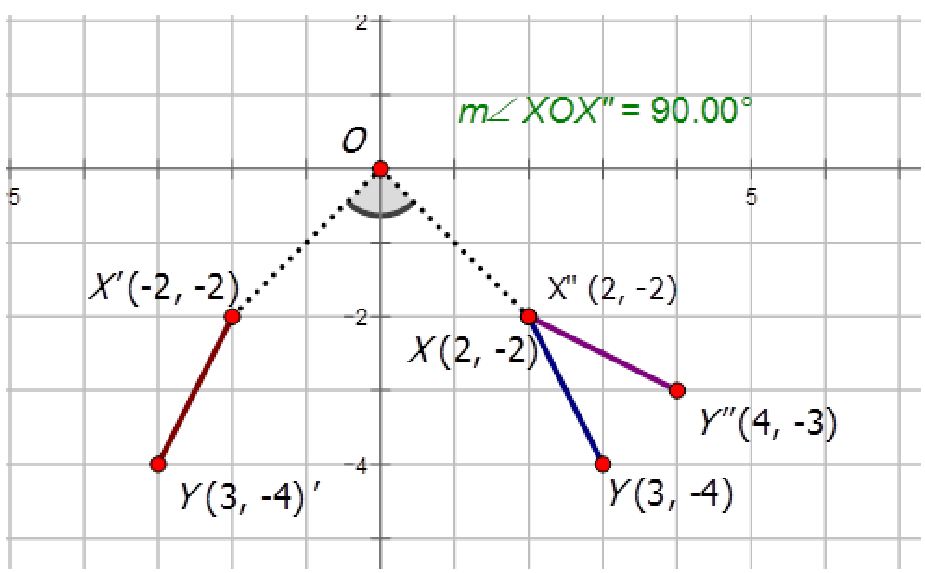
$$R_{270^\circ} \circ r_{y-axis}$$

Example 2

Graph the line XY given that $X(2,-2)$ and $Y(3,-4)$. Also graph the composite image that satisfies the rule

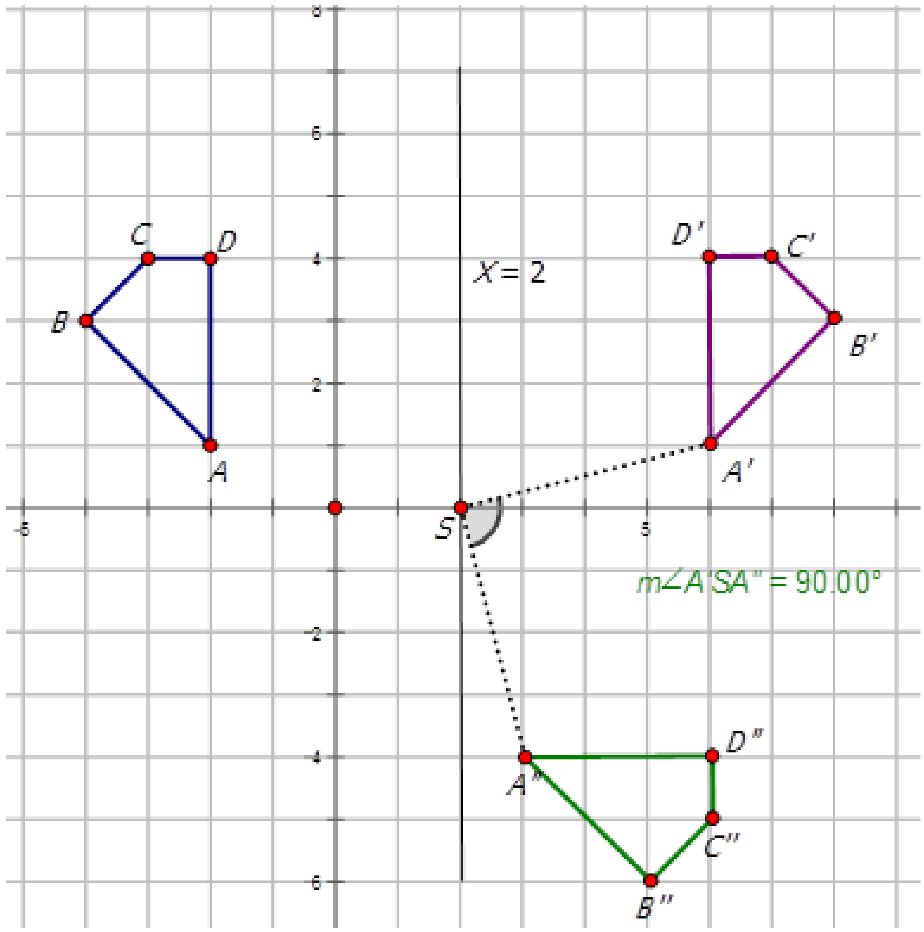
$$R_{90^\circ} \circ r_{y-axis}$$

The first transformation is a reflection about the y-axis to produce $X'Y'$. The second transformation is a 90°CCW turn about the origin to produce $X''Y''$.



Example 3

Describe the composite transformations in the diagram below and write the notation to represent the transformation of figure $ABCD$ to $A''B''C''D''$.

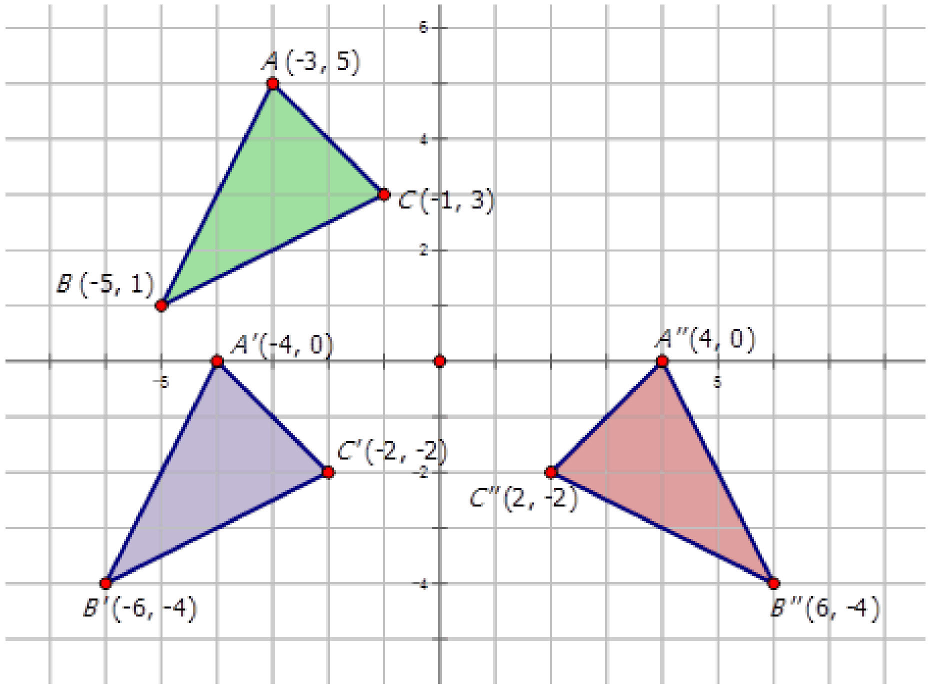


There are two transformations shown in the diagram. The first transformation is a reflection about the line $X = 2$ to produce $A'B'C'D'$. The second transformation is a 90°CW (or 270°CCW) rotation about the point (2, 0) to produce the figure $A''B''C''D''$. Notation for this composite transformation is:

$$R_{270^\circ} \circ r_{x=2}$$

Example 4

Describe the composite transformations in the diagram below and write the notation to represent the transformation of figure ABC to $A''B''C''$.



There are two transformations shown in the diagram. The first transformation is a translation of 1 unit to the left and 5 units down to produce $A'B'C'$. The second reflection in the y-axis to produce the figure $A''B''C''$. Notation for this composite transformation is:

$$r_{y-axis} \circ T_{-1,-5}$$

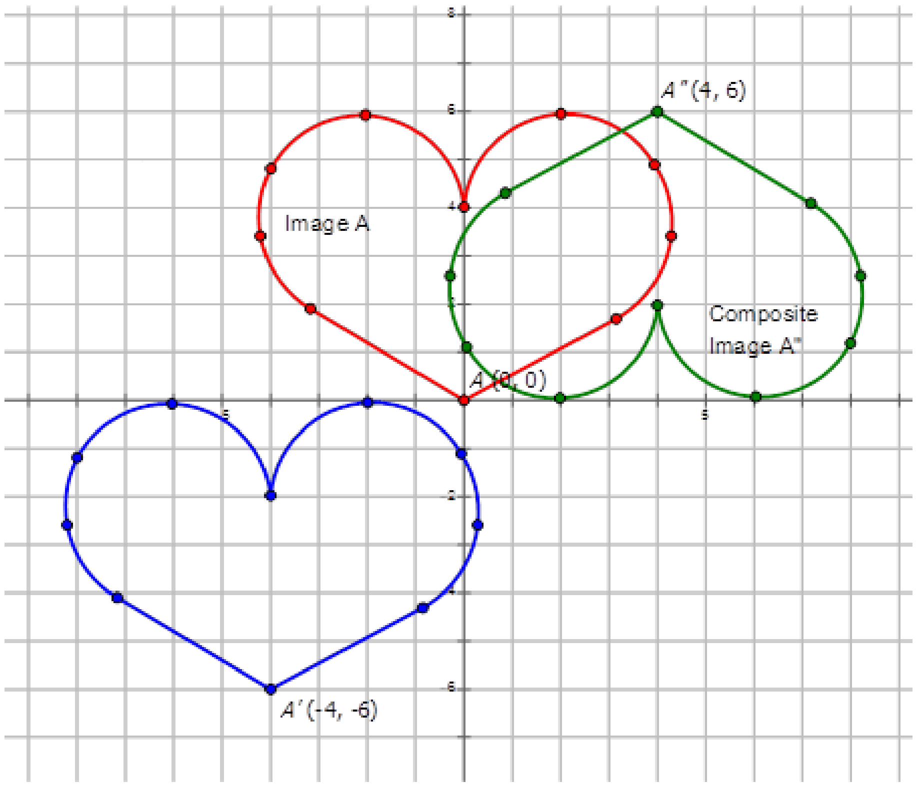
Review

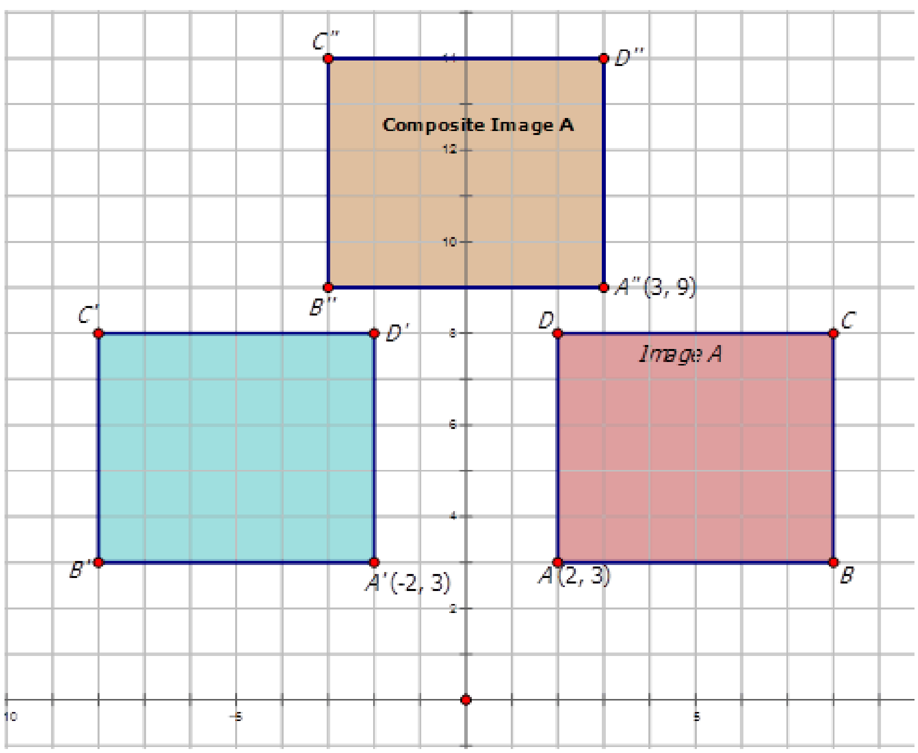
Complete the following table:

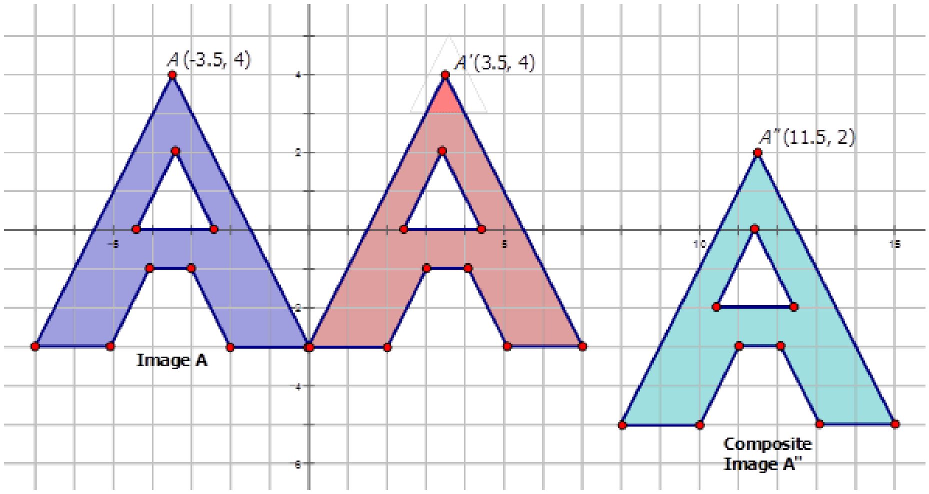
TABLE 10.16:

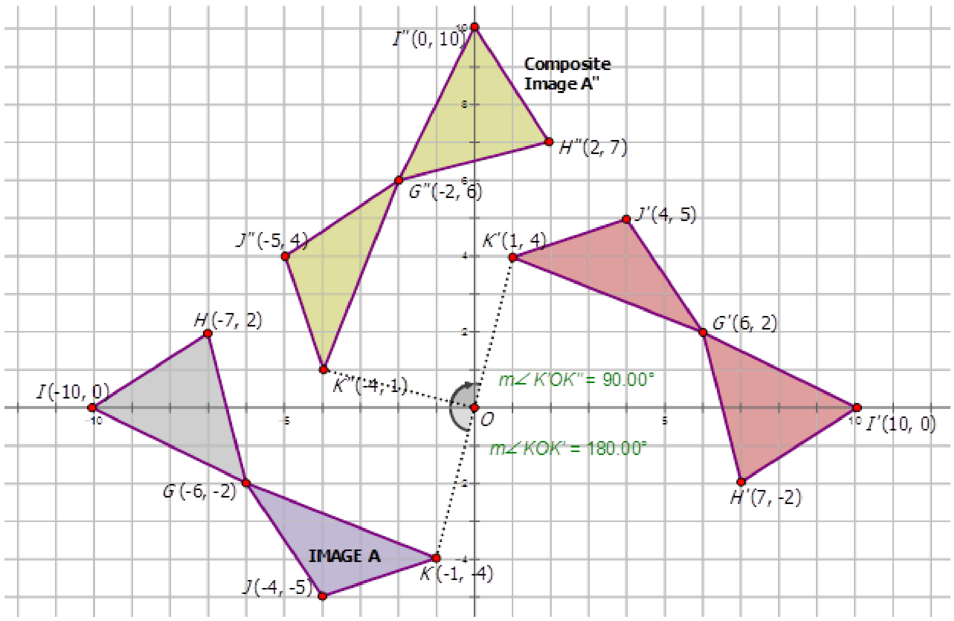
Starting Point	$T_{3,-4} \circ R_{90^\circ}$	$r_{x-axis} \circ r_{y-axis}$	$T_{1,6} \circ r_{x-axis}$	$r_{y-axis} \circ R_{180^\circ}$
1. (1, 4)				
2. (4, 2)				
3. (2, 0)				
4. (-1, 2)				
5. (-2, -3)				
6. (4, -1)				
7. (3, -2)				
8. (5, 4)				
9. (-3, 7)				
10. (0, 0)				

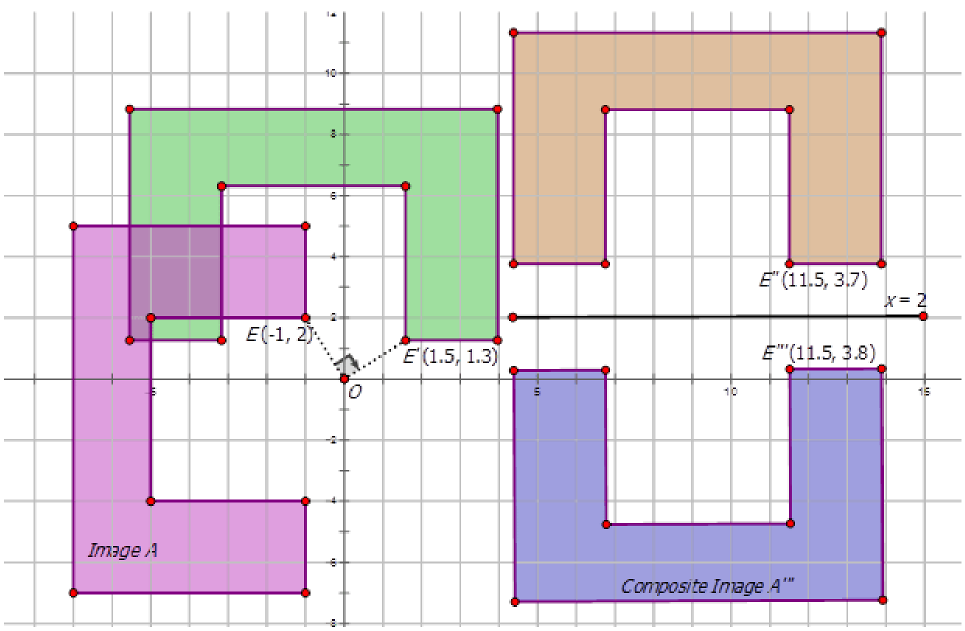
Write the notation that represents the composite transformation of the preimage A to the composite images in the diagrams below.

11. 
A"(4, 6)
Image A
Composite
Image A"
A(0, 0)
A'(-4, -6)

12. 
C"
D"
Composite Image A
A"(3, 9)
B"
C'
D'
D
C
Image A
B'
A'(-2, 3)
A(2, 3)
B

13. 
A(-3.5, 4)
A'(3.5, 4)
A"(11.5, 2)
Image A
Composite
Image A"

14. 
I"(0, 10)
Composite
Image A"
H"(2, 7)
G"(-2, 6)
J'(4, 5)
J"(-5, 4)
K'(1, 4)
H(-7, 2)
G'(6, 2)
I(-10, 0)
K"(-4, 1)
m∠K'OK" = 90.00°
O
I'(10, 0)
m∠KOK' = 180.00°
G(-6, -2)
H'(7, -2)
IMAGE A
K(-1, -4)
J(-4, -5)

15. 
E"(11.5, 3.7)
x = 2
E(-1, 2)
E'(1.5, 1.3)
E'"(11.5, 3.8)
O
Image A
Composite Image A'"

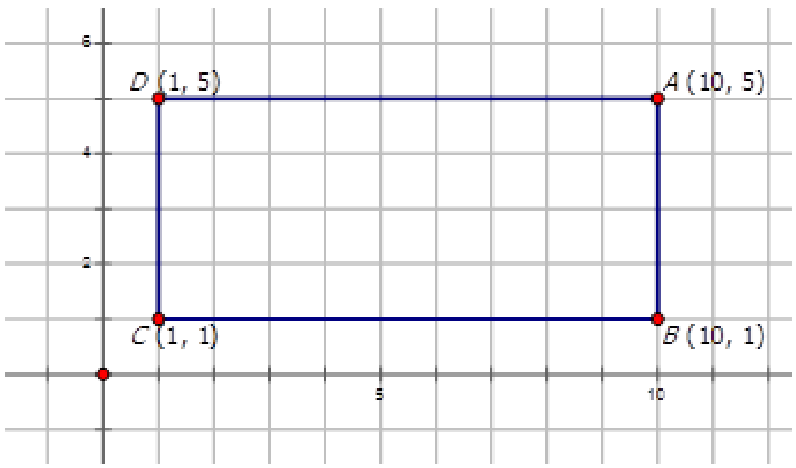
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.15.

10.16 The Midpoint Formula

Here you will learn how to find the midpoint of a line segment.

Find the midpoints for the diagram below and then draw the lines of reflection.

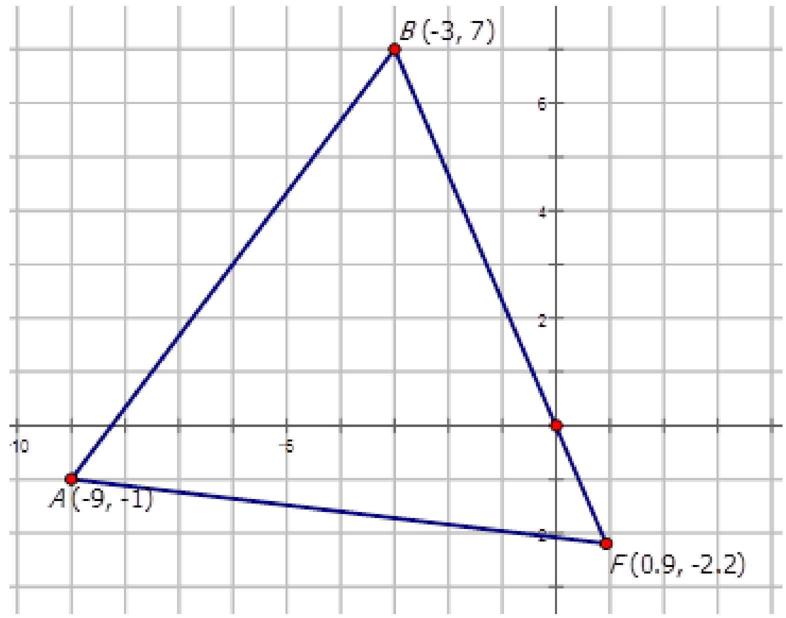


The Midpoint Formula

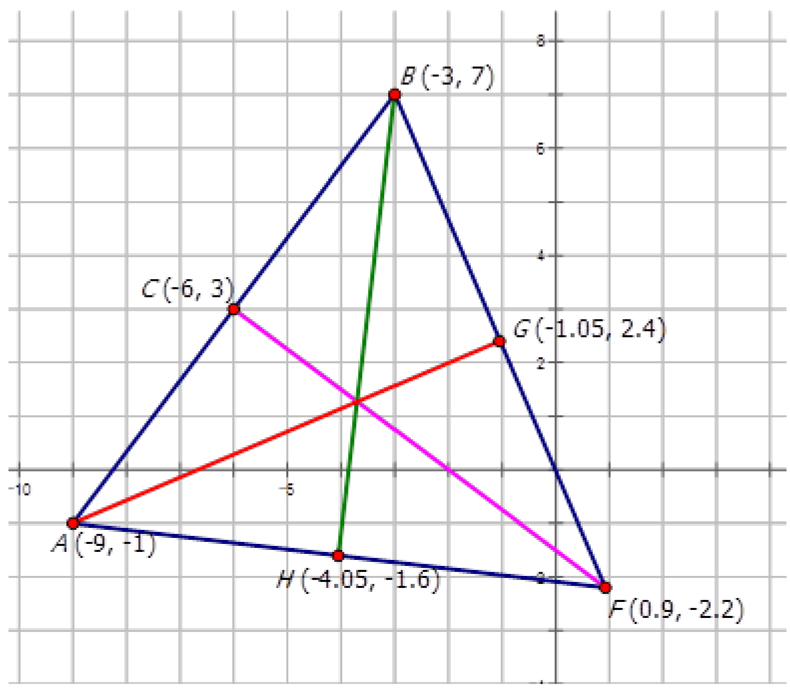
The midpoint of a line segment is the point exactly in the middle of the two endpoints. In order to calculate the coordinates of the midpoint, find the average of the two endpoints:

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Sometimes midpoints can help you to find lines of reflection (lines of symmetry) in shapes. Look at the equilateral triangle in the diagram below.



In an equilateral triangle there are three lines of symmetry. The lines of symmetry connect each vertex to the midpoint on the opposite side.



C is the mid-point of AB, G is the midpoint of BF, and H is the midpoint of AF. The lines AG, FC, and BH are all lines of symmetry or lines of reflection.

Keep in mind that not all midpoints will create lines of symmetry!

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65252

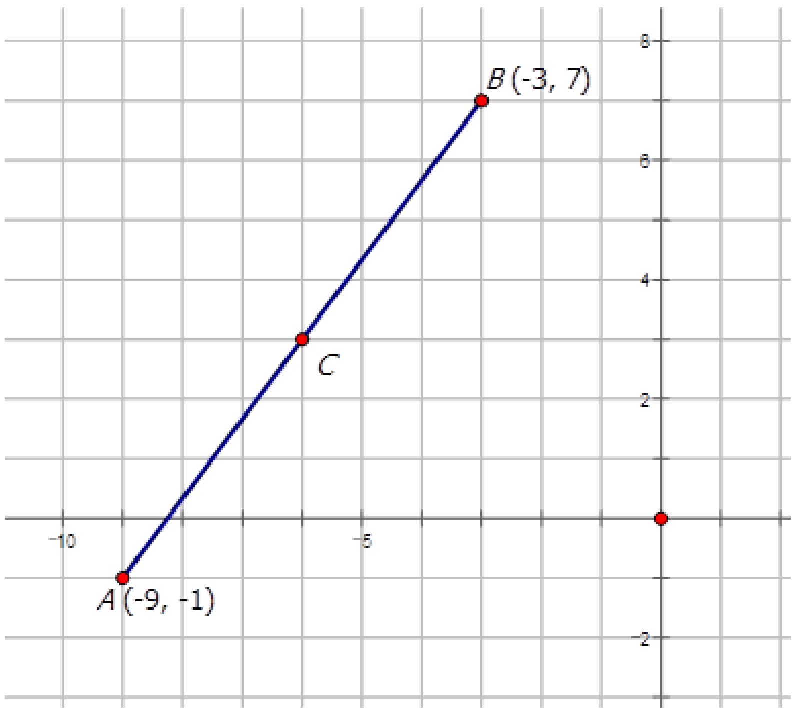
MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65253

Solve using the midpoint formula

In the diagram below, C is the midpoint between $A(-9,-1)$ and $B(-3,7)$. Find the coordinates of C.



$$M_{AB} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{AB} = \left(\frac{-9+-3}{2}, \frac{-1+7}{2}\right)$$
$$M_{AB} = \left(\frac{-12}{2}, \frac{6}{2}\right)$$
$$M_{AB} = (-6,3)$$

Solve using the midpoint formula

Find the coordinates of point T on the line ST knowing that S has coordinates (-3, 8) and the midpoint is (12, 1).

Look at the midpoint formula:

$$M = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$

For this problem, if you let point T have the coordinates x_1 and y_1, then you need to find x_1 and y_1 using the midpoint formula.

$$\begin{aligned} M_{ST} &= \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right) \\ (12,1) &= \left(\frac{-3+x_1}{2}, \frac{8+y_1}{2}\right) \end{aligned}$$

Next you need to separate the x-coordinate formula and the y-coordinate formula to solve for your unknowns.

$$12 = \frac{-3+x_1}{2} \quad 1 = \frac{8+y_1}{2}$$

Now multiply each of the equations by 2 in order to get rid of the fraction.

$$24 = -3+x_1 \quad 2 = 8+y_1$$

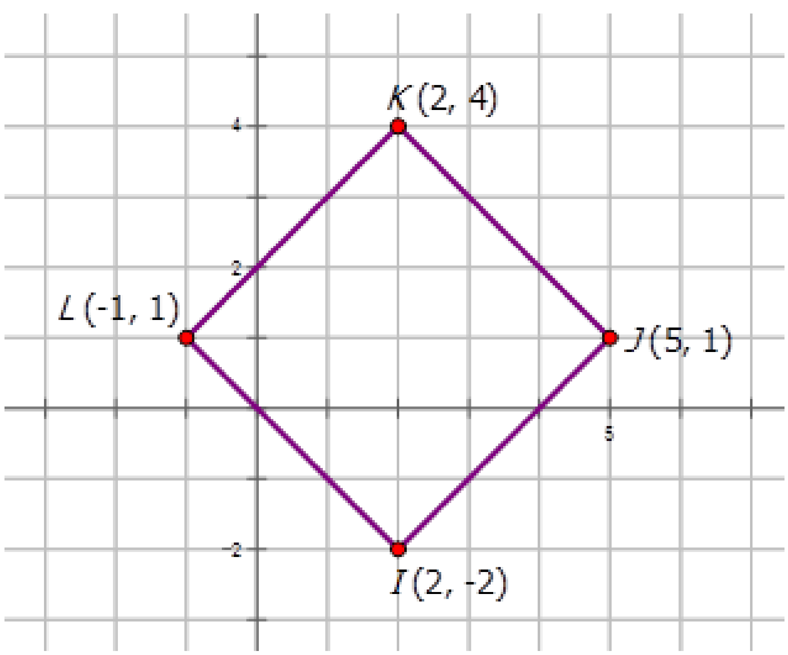
Now you can solve for x_1 and y_1.

$$27 = x_1 \quad -6 = y_1$$

Therefore the point T in the line ST has coordinates (27, -6).

Solve using the midpoint formula

Find the midpoints for the diagram below in order to draw the lines of reflection (or the line of symmetry).

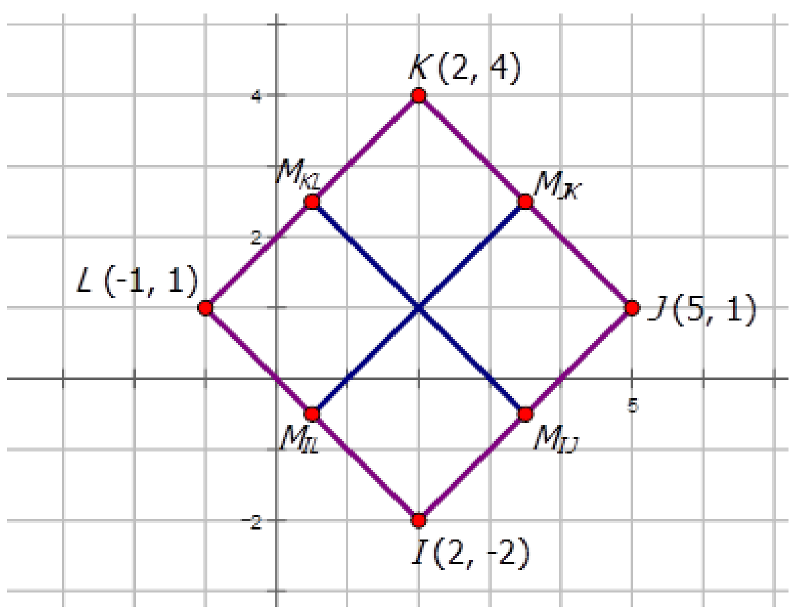


$$M_{IL} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{IL} = \left(\frac{2+-1}{2}, \frac{-2+1}{2}\right)$$
$$M_{IL} = \left(\frac{1}{2}, \frac{-1}{2}\right)$$

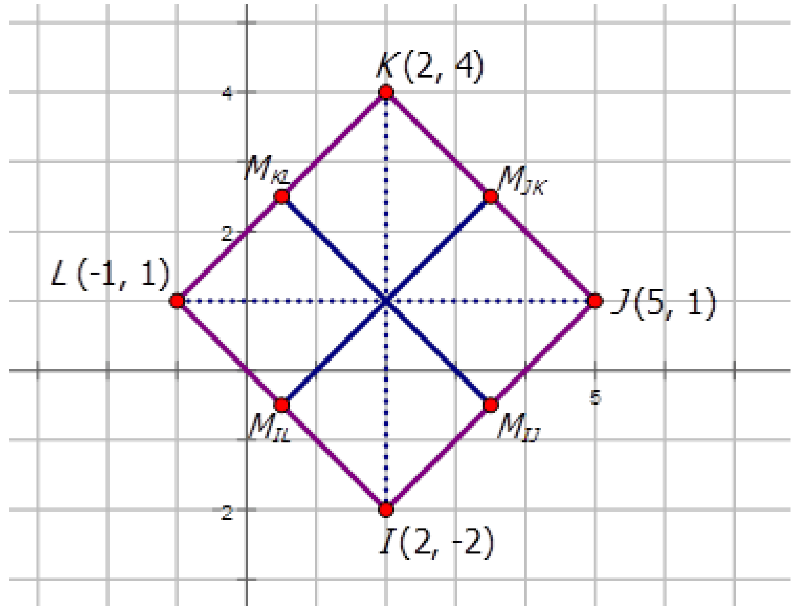
$$M_{IJ} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{IJ} = \left(\frac{2+5}{2}, \frac{-2+1}{2}\right)$$
$$M_{IJ} = \left(\frac{7}{2}, \frac{-1}{2}\right)$$
$$M_{IJ} = (3.5, -0.5)$$

$$M_{JK} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{JK} = \left(\frac{5+2}{2}, \frac{4+1}{2}\right)$$
$$M_{JK} = \left(\frac{7}{2}, \frac{5}{2}\right)$$
$$M_{JK} = (3.5, 2.5)$$

$$M_{KL} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{KL} = \left(\frac{2+-1}{2}, \frac{1+4}{2}\right)$$
$$M_{KL} = \left(\frac{1}{2}, \frac{5}{2}\right)$$
$$M_{KL} = (0.5, 2.5)$$



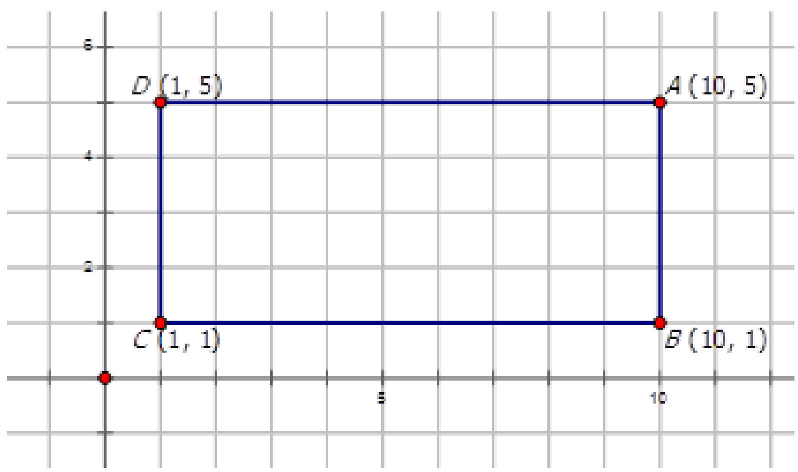
As seen in the graph above, a square has two lines of symmetry drawn from the mid-points of the opposite sides. A square actually has two more lines of symmetry that are the diagonals of the square.



Examples

Example 1

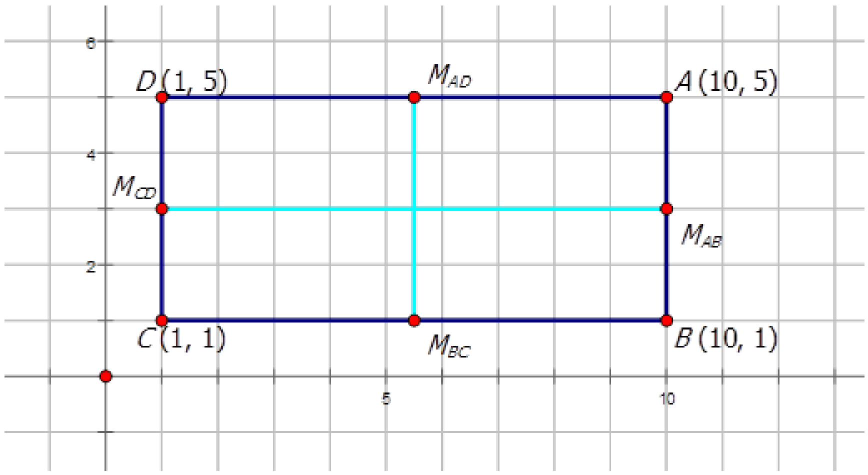
Earlier, you were asked to find the midpoints for the diagram below in order to draw the lines of reflection.



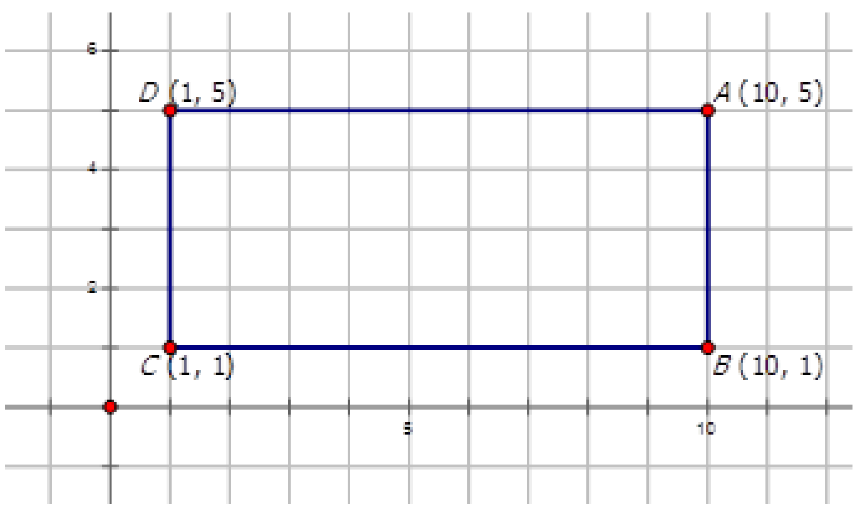
$$M_{AB} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{AB} = \left(\frac{10+10}{2}, \frac{1+5}{2}\right)$$
$$M_{AB} = \left(\frac{20}{2}, \frac{6}{2}\right)$$
$$M_{AB} = (10,3)$$

$$M_{AD} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{AD} = \left(\frac{10+1}{2}, \frac{5+5}{2}\right)$$
$$M_{AD} = \left(\frac{11}{2}, \frac{10}{2}\right)$$
$$M_{AD} = (5.5,5)$$

$$M_{BC} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right) \qquad M_{CD} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{BC} = \left(\frac{10+1}{2}, \frac{1+1}{2}\right) \qquad M_{CD} = \left(\frac{1+1}{2}, \frac{1+5}{2}\right)$$
$$M_{BC} = \left(\frac{11}{2}, \frac{2}{2}\right) \qquad M_{CD} = \left(\frac{2}{2}, \frac{6}{2}\right)$$
$$M_{BC} = (5.5, 1) \qquad M_{CD} = (1, 3)$$

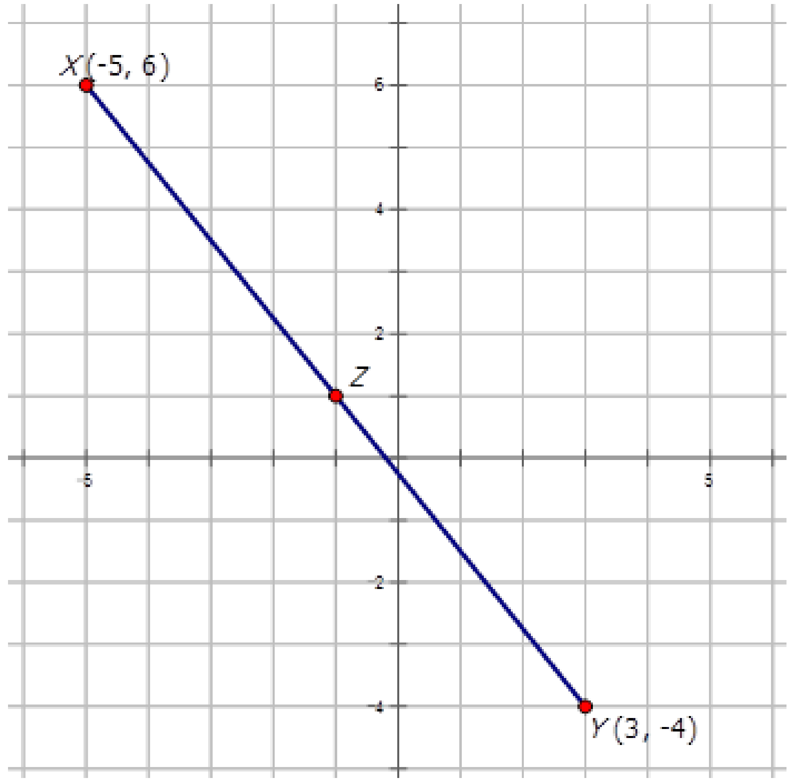


As seen in the graph above, a rectangle has two lines of symmetry.



Example 2

In the diagram below, Z is the midpoint between $X(-5, 6)$ and $Y(3, -4)$. Find the coordinates of Z.



$$M_{XY} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{XY} = \left(\frac{-5+3}{2}, \frac{-4+6}{2}\right)$$
$$M_{XY} = \left(\frac{-2}{2}, \frac{2}{2}\right)$$
$$M_{XY} = (-1, 1)$$

Example 3

Find the coordinates of point K on the line JK knowing that J has coordinates (-2, 5) and the midpoint is (10, 1).

Let point K have the coordinates x_1 and y_1, then find x_1 and y_1 using the midpoint formula.

$$M_{JK} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$(10, 1) = \left(\frac{-2+x_1}{2}, \frac{5+y_1}{2}\right)$$

Next you need to separate the x-coordinate formula and the y-coordinate formula to solve for your unknowns.

$$10 = \frac{-2+x_1}{2} \quad 1 = \frac{5+y_1}{2}$$

Now multiply each of the equations by 2 in order to get rid of the fraction.

$$20 = -2 + x_1 \quad 2 = 5 + y_1$$

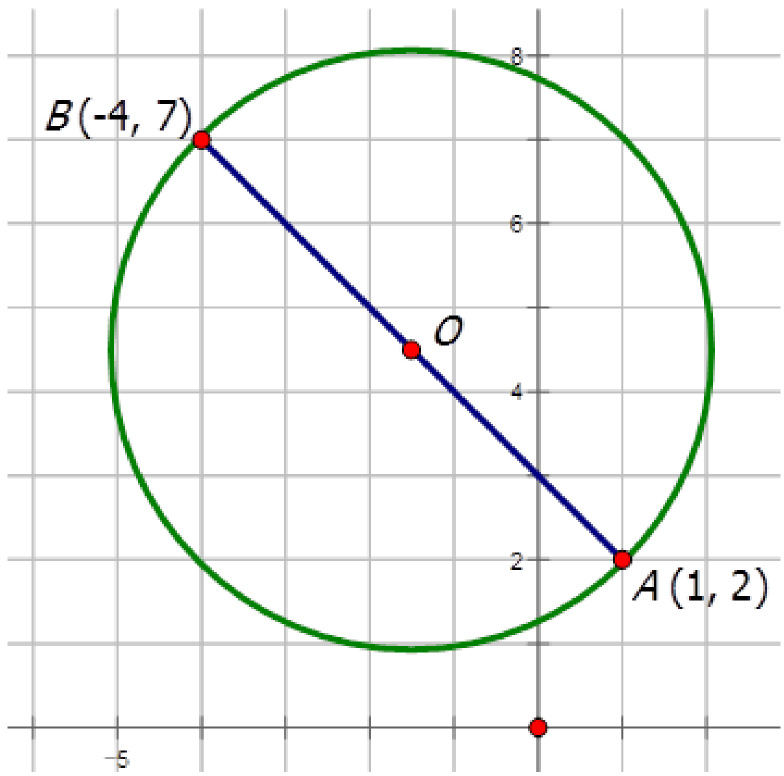
Now you can solve for x_1 and y_1.

$$22 = x_1 \quad -3 = y_1$$

Therefore the point K in the line JK has coordinates (22, -3).

Example 4

A diameter is drawn in the circle as shown in the diagram below. What are the coordinates for the center of the circle, O?



$$M_{AB} = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$
$$M_{AB} = \left(\frac{1 + -4}{2}, \frac{2 + 7}{2}\right)$$
$$M_{AB} = \left(\frac{-3}{2}, \frac{9}{2}\right)$$
$$M_{AB} = (-1.5, 4.5)$$

Review

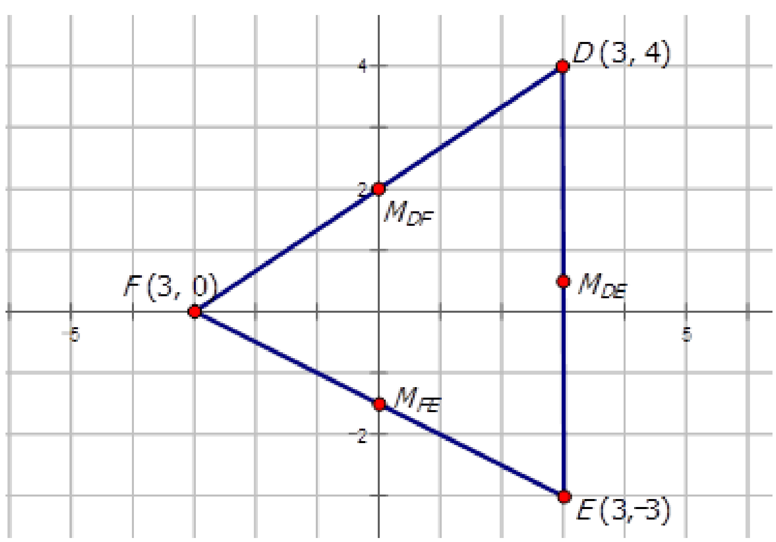
Find the mid-point for each line below given the endpoints:

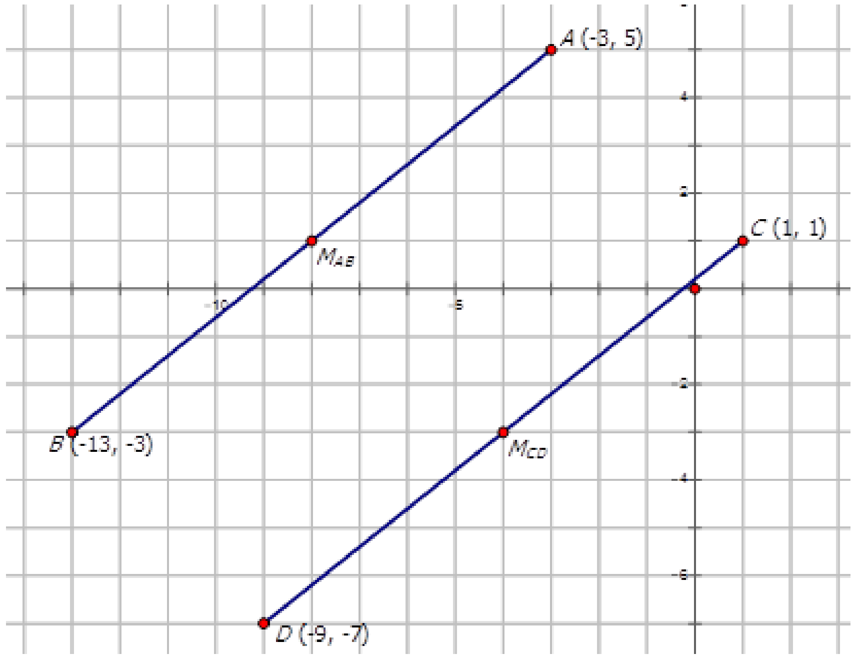
1. Line AB given $A(5,7)$ and $B(3,9)$.
2. Line BC given $B(3,8)$ and $C(5,2)$.
3. Line CD given $C(4,6)$ and $D(3,5)$.
4. Line DE given $D(9,11)$ and $E(2,2)$.
5. Line EF given $E(1,1)$ and $F(8,7)$.
6. Line FG given $F(1,8)$ and $G(1,4)$.

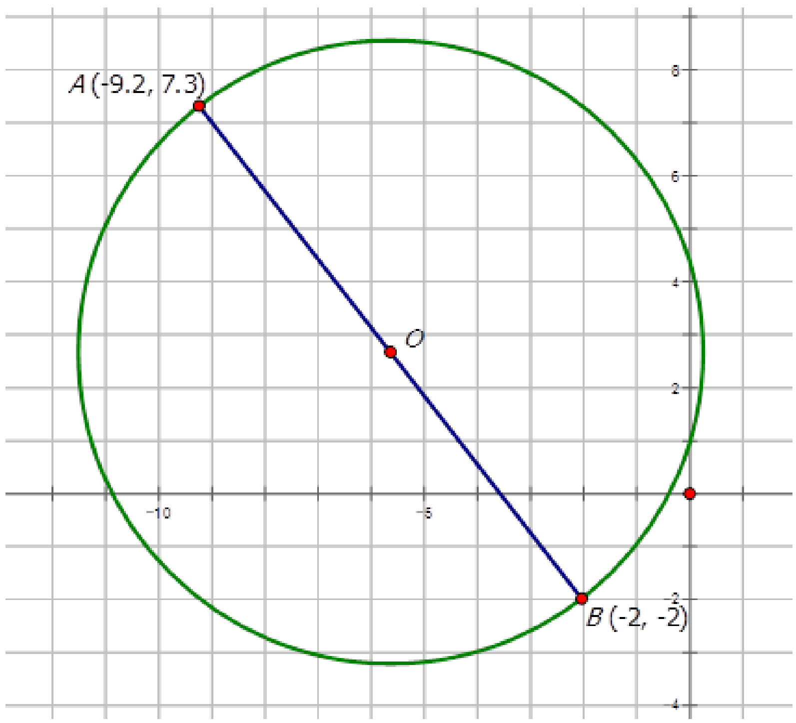
For the following lines, one endpoint is given and then the mid-point. Find the other endpoint.

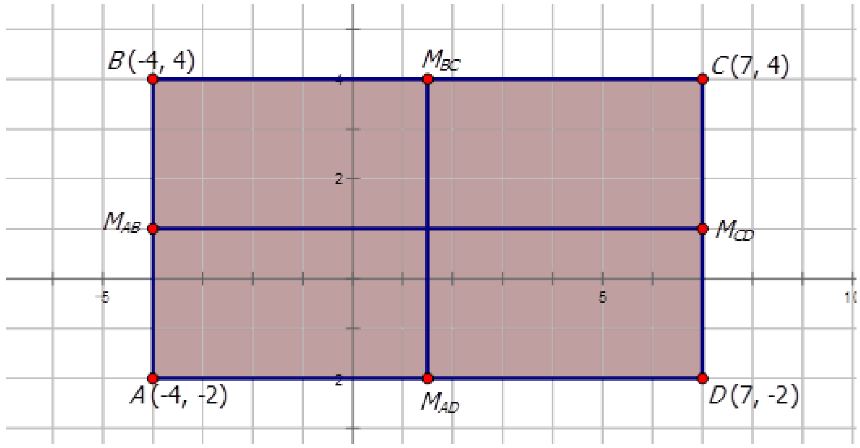
7. Line AB given $A(3,-5)$ and $M_{AB}(7,7)$.
8. Line BC given $B(2,4)$ and $M_{BC}(4,9)$.
9. Line CD given $C(-2,6)$ and $M_{CD}(1,1)$.
10. Line DE given $D(2,9)$ and $M_{DE}(8,2)$.
11. Line EF given $E(-6,-5)$ and $M_{EF}(-2,6)$.

For each of the diagrams below, find the midpoints.

12. 

13. 

14. 

15. 

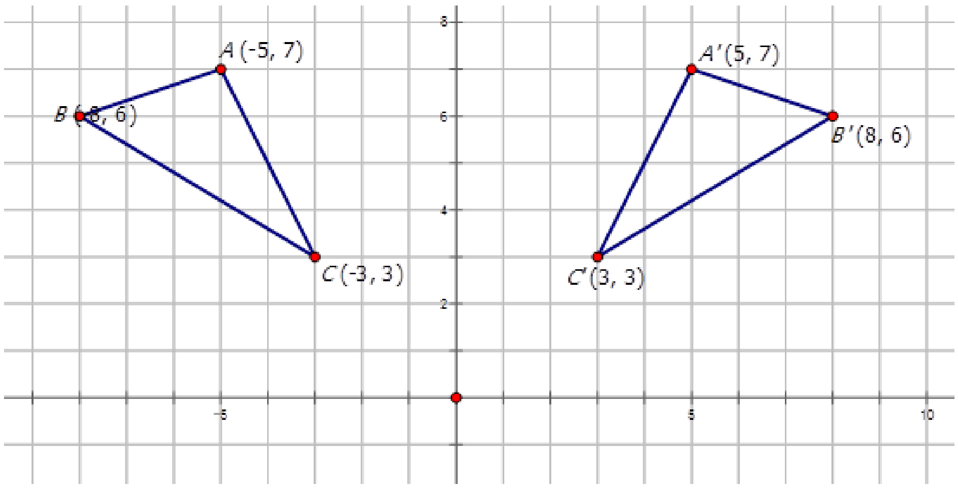
Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.16.

10.17 The Distance Formula

Here you will learn the distance formula and how to use it to determine whether or not two line segments are congruent.

Triangle ABC has vertices $A(-5,7), B(-8,6)$ and $C(-3,3)$. The triangle is reflected about the y-axis to form triangle $A'B'C'$. Assuming that $\angle A = \angle A', \angle B = \angle B', \textit{and } \angle C = \angle C'$, prove the two triangles are congruent.



The Distance Formula

Two shapes are **congruent** if they are exactly the same shape and exactly the same size. In congruent shapes, all corresponding sides will be the same length and all corresponding angles will be the same measure. Translations, reflections, and rotations all create congruent shapes.

If you want to determine whether two segments are the same length, you could try to use a ruler. Unfortunately, it's hard to be very precise with a ruler. You could also use geometry software, but that is not always available. If the segments are on the coordinate plane and you know their endpoints, you can use the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance formula helps justify congruence by proving that the sides of the preimage have the same length as the sides of the transformed image. The distance formula is derived using the Pythagorean Theorem, which you will learn more about in geometry.

MEDIA

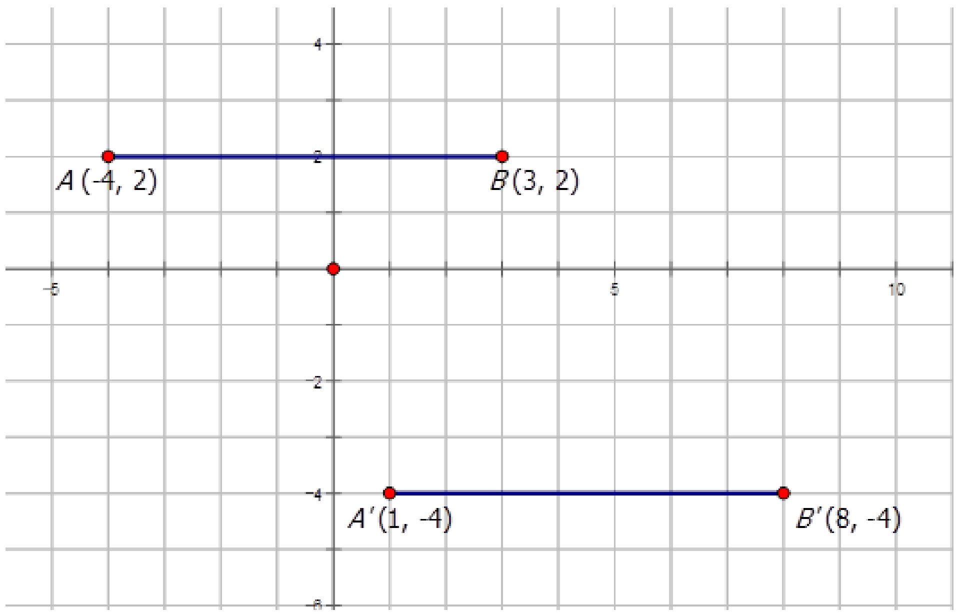
Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/65232

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/65233

Solve using the distance formula

Line segment AB is translated 5 units to the right and 6 units down to produce line $A'B'$. The diagram below shows the endpoints of lines AB and $A'B'$. Prove the two line segments are congruent.

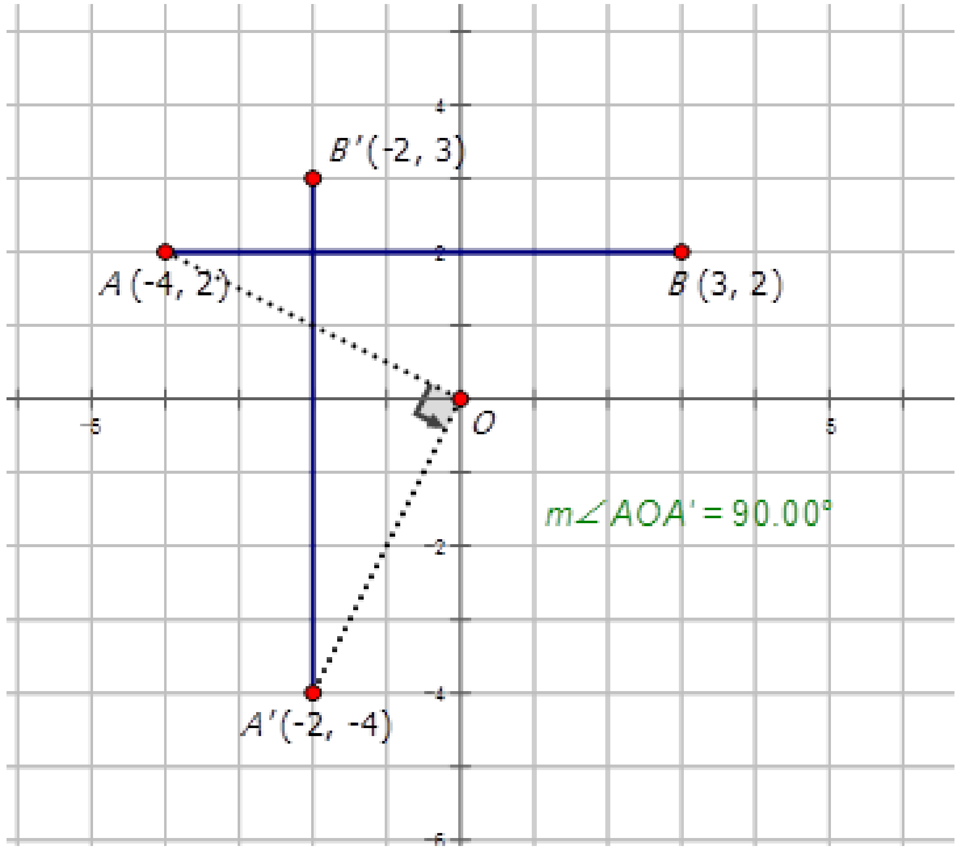


$$\begin{aligned}
d_{AB} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{AB} &= \sqrt{(-4-3)^2+(2-2)^2} \\
d_{AB} &= \sqrt{(-7)^2+(0)^2} \\
d_{AB} &= \sqrt{49+0} \\
d_{AB} &= \sqrt{49} \\
d_{AB} &= 7\ cm
\end{aligned}$$

$$\begin{aligned}
d_{A'B'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{A'B'} &= \sqrt{(1-8)^2+(-4-(-4))^2} \\
d_{A'B'} &= \sqrt{(-7)^2+(0)^2} \\
d_{A'B'} &= \sqrt{49+0} \\
d_{A'B'} &= \sqrt{49} \\
d_{A'B'} &= 7\ cm
\end{aligned}$$

Solve using the distance formula

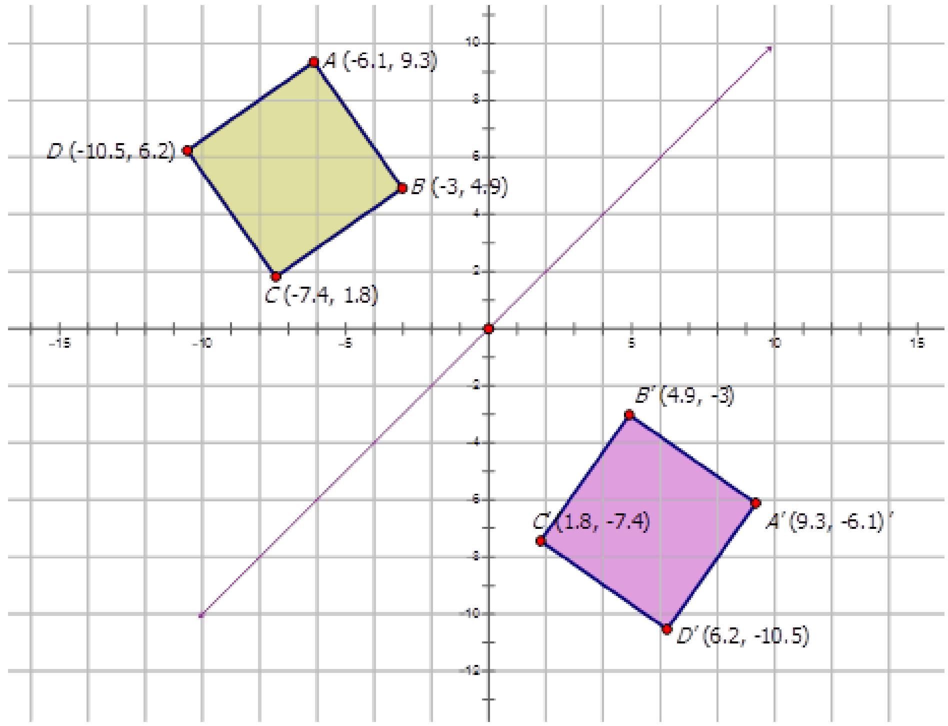
Line segment AB has been rotated about the origin 90°CCW to produce $A'B'$. The diagram below shows the lines AB and $A'B'$. Prove the two line segments are congruent.



$$
\begin{aligned}
d_{AB} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{AB} &= \sqrt{(-4-3)^2+(2-2)^2} \\
d_{AB} &= \sqrt{(-7)^2+(0)^2} \\
d_{AB} &= \sqrt{49+0} \\
d_{AB} &= \sqrt{49} \\
d_{AB} &= 7\ cm
\end{aligned}
\qquad
\begin{aligned}
d_{A'B'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{A'B'} &= \sqrt{(-2-(-2))^2+(-4-3)^2} \\
d_{A'B'} &= \sqrt{(0)^2+(-7)^2} \\
d_{A'B'} &= \sqrt{0+49} \\
d_{A'B'} &= \sqrt{49} \\
d_{A'B'} &= 7\ cm
\end{aligned}
$$

Solve using the distance formula

The square $ABCD$ has been reflected about the line $y = x$ to produce $A'B'C'D'$ as shown in the diagram below. Prove the two are congruent.



Since the figures are squares, you can conclude that all angles are the same and equal to 90°. You can also conclude that for each square, all the sides are the same length. Therefore, all you need to verify is that $m\overline{AB} = m\overline{A'B'}$.

$$\begin{aligned} d_{AB} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ d_{AB} &= \sqrt{(-6.1-(-3))^2+(9.3-4.9)^2} \\ d_{AB} &= \sqrt{(-3.1)^2+(4.4)^2} \\ d_{AB} &= \sqrt{9.61+19.36} \\ d_{AB} &= \sqrt{28.97} \\ d_{AB} &= 5.38\ cm \end{aligned}$$

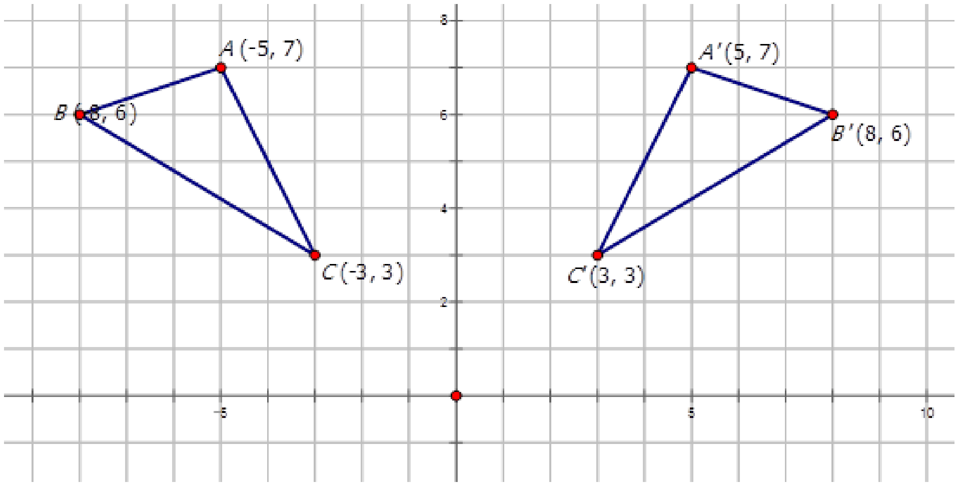
$$\begin{aligned} d_{A'B'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ d_{A'B'} &= \sqrt{(9.3-4.9)^2+(-6.1-(-3))^2} \\ d_{A'B'} &= \sqrt{(4.4)^2+(-3.1)^2} \\ d_{A'B'} &= \sqrt{19.36+9.61} \\ d_{A'B'} &= \sqrt{28.97} \\ d_{A'B'} &= 5.38\ cm \end{aligned}$$

Since $m\overline{AB} = m\overline{A'B'}$ and both shapes are squares, all 8 sides must be the same length. Therefore, the two squares are congruent.

Examples

Example 1

Earlier, you were asked to prove that these two triangles are congruent.



To prove congruence, prove that $m\overline{AB} = m\overline{A'B'}, m\overline{AC} = m\overline{A'C'}$, *and* $m\overline{BC} = m\overline{B'C'}$.

$$
\begin{aligned}
d_{AB} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d_{AB} &= \sqrt{(-5 - (-8))^2 + (7 - 6)^2} \\
d_{AB} &= \sqrt{(3)^2 + (1)^2} \\
d_{AB} &= \sqrt{9 + 1} \\
d_{AB} &= \sqrt{10} \\
d_{AB} &= 3.16 \ cm
\end{aligned}
\qquad
\begin{aligned}
d_{A'B'} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d_{A'B'} &= \sqrt{(5 - 8)^2 + (7 - 6)^2} \\
d_{A'B'} &= \sqrt{(-3)^2 + (1)^2} \\
d_{A'B'} &= \sqrt{9 + 1} \\
d_{A'B'} &= \sqrt{10} \\
d_{A'B'} &= 3.16 \ cm
\end{aligned}
$$

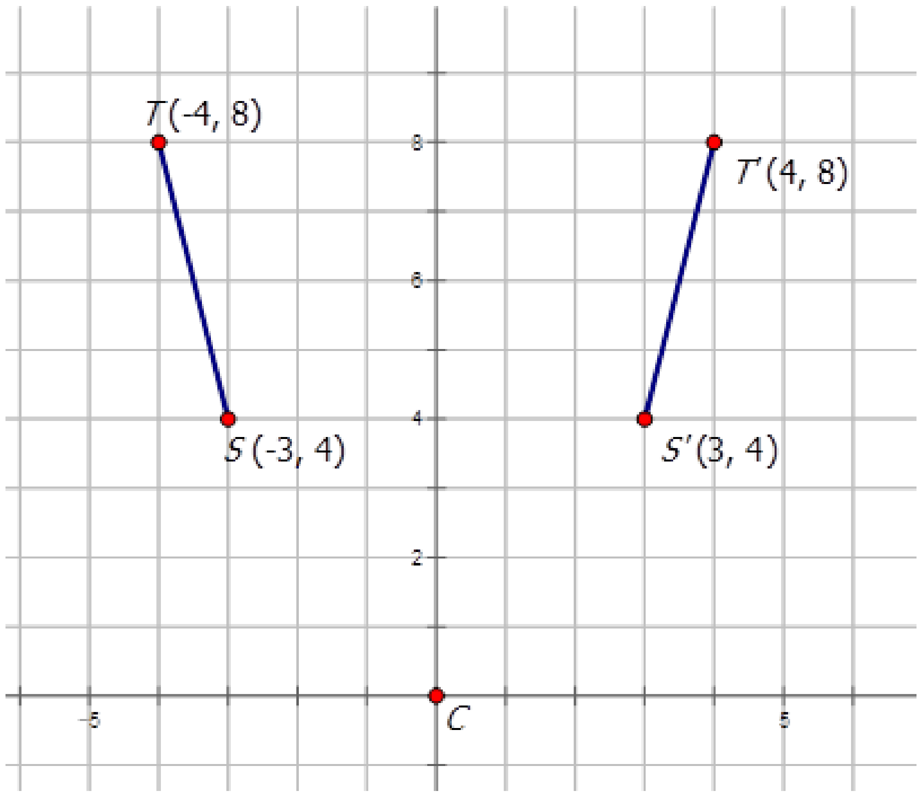
$$
\begin{aligned}
d_{AC} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d_{AC} &= \sqrt{(-5 - (-3))^2 + (7 - 3)^2} \\
d_{AC} &= \sqrt{(-2)^2 + (4)^2} \\
d_{AC} &= \sqrt{4 + 16} \\
d_{AC} &= \sqrt{20} \\
d_{AC} &= 4.47 \ cm
\end{aligned}
\qquad
\begin{aligned}
d_{A'C'} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d_{A'C'} &= \sqrt{(5 - 3)^2 + (7 - 3)^2} \\
d_{A'C'} &= \sqrt{(2)^2 + (4)^2} \\
d_{A'C'} &= \sqrt{4 + 16} \\
d_{A'C'} &= \sqrt{20} \\
d_{A'C'} &= 4.72 \ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{BC} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d_{BC} &= \sqrt{(-8 - (-3))^2 + (6 - 3)^2} \\
d_{BC} &= \sqrt{(-5)^2 + (3)^2} \\
d_{BC} &= \sqrt{25 + 9} \\
d_{BC} &= \sqrt{34} \\
d_{BC} &= 5.83 \ cm
\end{aligned}
\qquad
\begin{aligned}
d_{A'C'} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d_{A'C'} &= \sqrt{(8 - 3)^2 + (6 - 3)^2} \\
d_{A'C'} &= \sqrt{(5)^2 + (3)^2} \\
d_{A'C'} &= \sqrt{25 + 9} \\
d_{A'C'} &= \sqrt{34} \\
d_{A'C'} &= 5.83 \ cm
\end{aligned}
$$

It is given that $\angle A = \angle A', \angle B = \angle B'$, *and* $\angle C = \angle C'$, and the distance formula proved that $m\overline{AB} = m\overline{A'B'}, m\overline{AC} = m\overline{A'C'}$, *and* $m\overline{BC} = m\overline{B'C'}$. Therefore the two triangles are congruent.

Example 2

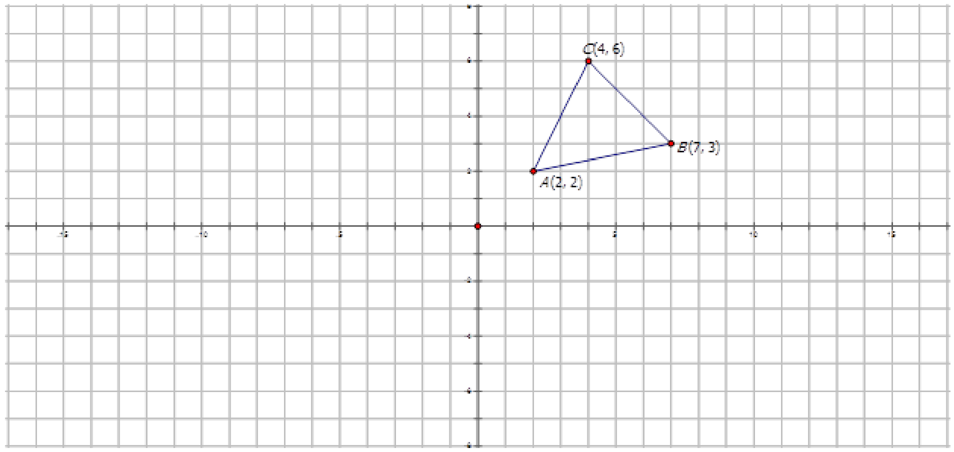
Line segment $\overline{ST}$ drawn from $S(-3,4)$ to $T(-3,8)$ has undergone a reflection in the y-axis to produce Line $S'T'$ drawn from $S'(3,4)$ to $T'(4,8)$. Draw the preimage and image and prove the two lines are congruent.



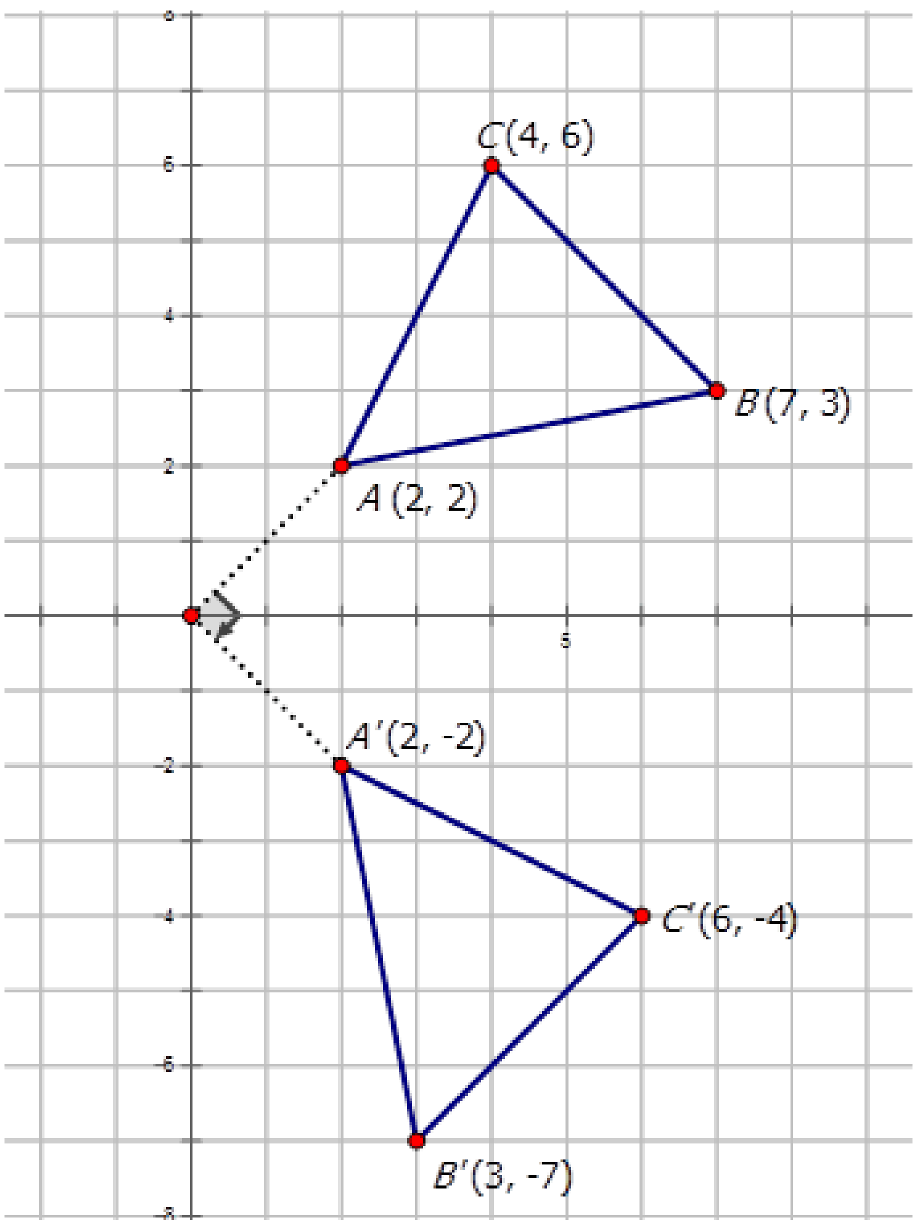
$$
\begin{aligned}
d_{ST} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{ST} &= \sqrt{(-3-(-4))^2+(4-8)^2} \\
d_{ST} &= \sqrt{(1)^2+(-4)^2} \\
d_{ST} &= \sqrt{1+16} \\
d_{ST} &= \sqrt{17} \\
d_{ST} &= 4.12\ cm
\end{aligned}
\qquad
\begin{aligned}
d_{S'T'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{S'T'} &= \sqrt{(3-4)^2+(4-8)^2} \\
d_{S'T'} &= \sqrt{(-1)^2+(-4)^2} \\
d_{S'T'} &= \sqrt{1+16} \\
d_{S'T'} &= \sqrt{17} \\
d_{S'T'} &= 4.12\ cm
\end{aligned}
$$

Example 3

The triangle below has undergone a rotation of 90°CW about the origin. Given that all of the angles are equal, draw the transformed image and prove the two figures are congruent.



C(4, 6)
B(7, 3)
A(2, 2)



C(4, 6)
B(7, 3)
A (2, 2)
A′(2, -2)
C′(6, -4)
B′(3, -7)

$$d_{AB} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{AB} = \sqrt{(2-7)^2+(2-3)^2}$$
$$d_{AB} = \sqrt{(-5)^2+(-1)^2}$$
$$d_{AB} = \sqrt{25+1}$$
$$d_{AB} = \sqrt{26}$$
$$d_{AB} = 5.10\ cm$$

$$d_{A'B'} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{A'B'} = \sqrt{(3-2)^2+(-7-(-2))^2}$$
$$d_{A'B'} = \sqrt{(1)^2+(-5)^2}$$
$$d_{A'B'} = \sqrt{1+25}$$
$$d_{A'B'} = \sqrt{26}$$
$$d_{A'B'} = 5.10\ cm$$

$$d_{AC} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{AC} = \sqrt{(2-4)^2+(2-6)^2}$$
$$d_{AC} = \sqrt{(-2)^2+(-4)^2}$$
$$d_{AC} = \sqrt{4+16}$$
$$d_{AC} = \sqrt{20}$$
$$d_{AC} = 4.47\ cm$$

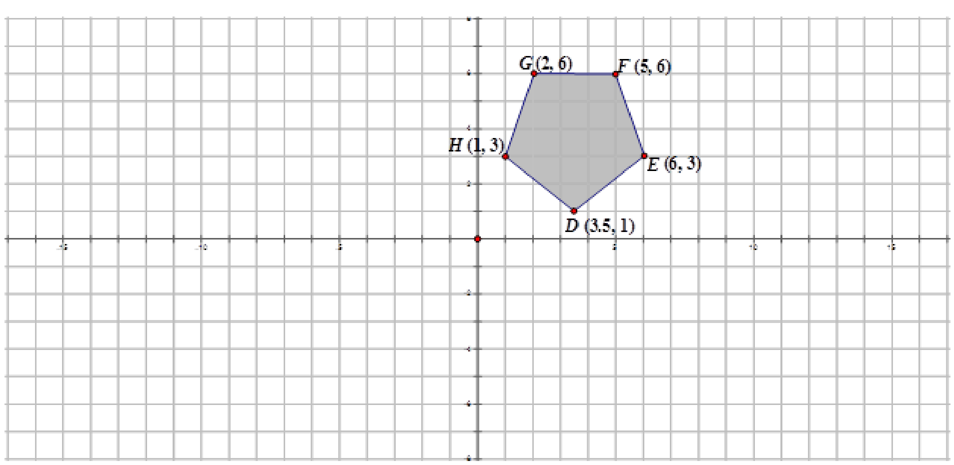
$$d_{A'C'} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{A'C'} = \sqrt{(2-6)^2+(-2-(-4))^2}$$
$$d_{A'C'} = \sqrt{(-4)^2+(2)^2}$$
$$d_{A'C'} = \sqrt{16+4}$$
$$d_{A'C'} = \sqrt{20}$$
$$d_{A'C'} = 4.72\ cm$$

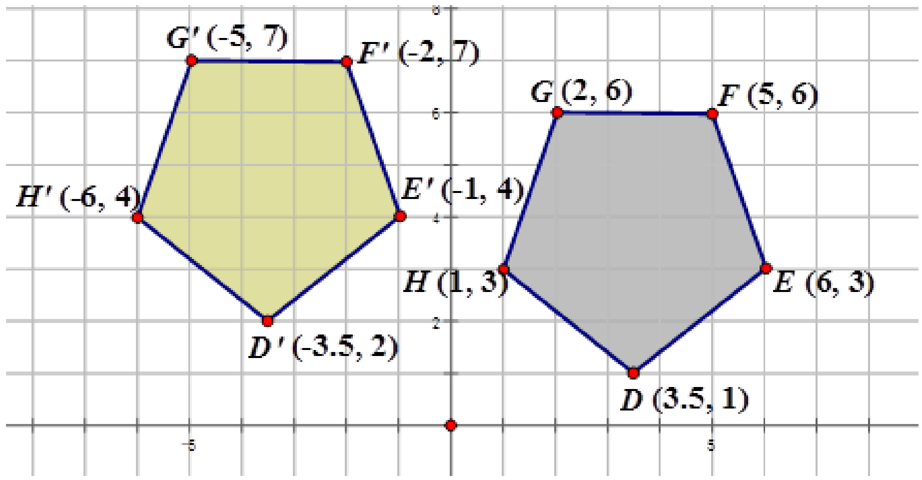
$$d_{BC} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{BC} = \sqrt{(7-4)^2+(3-6)^2}$$
$$d_{BC} = \sqrt{(3)^2+(-3)^2}$$
$$d_{BC} = \sqrt{9+9}$$
$$d_{BC} = \sqrt{18}$$
$$d_{BC} = 4.24\ cm$$

$$d_{B'C'} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{B'C'} = \sqrt{(3-6)^2+(-7-(-4))^2}$$
$$d_{B'C'} = \sqrt{(-3)^2+(-3)^2}$$
$$d_{B'C'} = \sqrt{9+9}$$
$$d_{B'C'} = \sqrt{18}$$
$$d_{B'C'} = 4.24\ cm$$

Example 4

The polygon below has undergone a translation of 7 units to the left and 1 unit up. Given that all of the angles are equal, draw the transformed image and prove the two figures are congruent.





$$\begin{aligned}
d_{DE} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{DE} &= \sqrt{(3.5-6)^2+(1-3)^2} \\
d_{DE} &= \sqrt{(-2.5)^2+(-2)^2} \\
d_{DE} &= \sqrt{6.25+4} \\
d_{DE} &= \sqrt{10.25} \\
d_{DE} &= 3.20\ cm
\end{aligned}$$

$$\begin{aligned}
d_{D'E'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{D'E'} &= \sqrt{(-3.5-(-1))^2+(2-4)^2} \\
d_{D'E'} &= \sqrt{(-2.5)^2+(-2)^2} \\
d_{D'E'} &= \sqrt{6.25+4} \\
d_{D'E'} &= \sqrt{10.25} \\
d_{D'E'} &= 3.20\ cm
\end{aligned}$$

$$\begin{aligned}
d_{EF} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{EF} &= \sqrt{(6-5)^2+(3-6)^2} \\
d_{EF} &= \sqrt{(1)^2+(-3)^2} \\
d_{EF} &= \sqrt{1+9} \\
d_{EF} &= \sqrt{10} \\
d_{EF} &= 3.16\ cm
\end{aligned}$$

$$\begin{aligned}
d_{E'F'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{E'F'} &= \sqrt{(-1-(-2))^2+(4-7)^2} \\
d_{E'F'} &= \sqrt{(1)^2+(-3)^2} \\
d_{E'F'} &= \sqrt{1+9} \\
d_{E'F'} &= \sqrt{10} \\
d_{E'F'} &= 3.16\ cm
\end{aligned}$$

$$\begin{aligned}
d_{FG} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{FG} &= \sqrt{(5-2)^2+(6-6)^2} \\
d_{FG} &= \sqrt{(3)^2+(0)^2} \\
d_{FG} &= \sqrt{9+0} \\
d_{FG} &= \sqrt{9} \\
d_{FG} &= 3.00\ cm
\end{aligned}$$

$$\begin{aligned}
d_{F'G'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{F'G'} &= \sqrt{(-2-(-5))^2+(7-7)^2} \\
d_{F'G'} &= \sqrt{(3)^2+(0)^2} \\
d_{F'G'} &= \sqrt{9+0} \\
d_{F'G'} &= \sqrt{9} \\
d_{F'G'} &= 3.00\ cm
\end{aligned}$$

$$d_{GH} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{GH} = \sqrt{(2-1)^2+(6-3)^2}$$
$$d_{GH} = \sqrt{(1)^2+(3)^2}$$
$$d_{GH} = \sqrt{1+9}$$
$$d_{GH} = \sqrt{10}$$
$$d_{GH} = 3.16\ cm$$

$$d_{G'H'} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{G'H'} = \sqrt{(-5-(-6))^2+(7-4)^2}$$
$$d_{G'H'} = \sqrt{(1)^2+(3)^2}$$
$$d_{G'H'} = \sqrt{1+9}$$
$$d_{G'H'} = \sqrt{10}$$
$$d_{G'H'} = 3.16\ cm$$

$$d_{HD} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{HD} = \sqrt{(1-3.5)^2+(3-1)^2}$$
$$d_{HD} = \sqrt{(-2.5)^2+(2)^2}$$
$$d_{HD} = \sqrt{6.25+4}$$
$$d_{HD} = \sqrt{10.25}$$
$$d_{HD} = 3.20\ cm$$

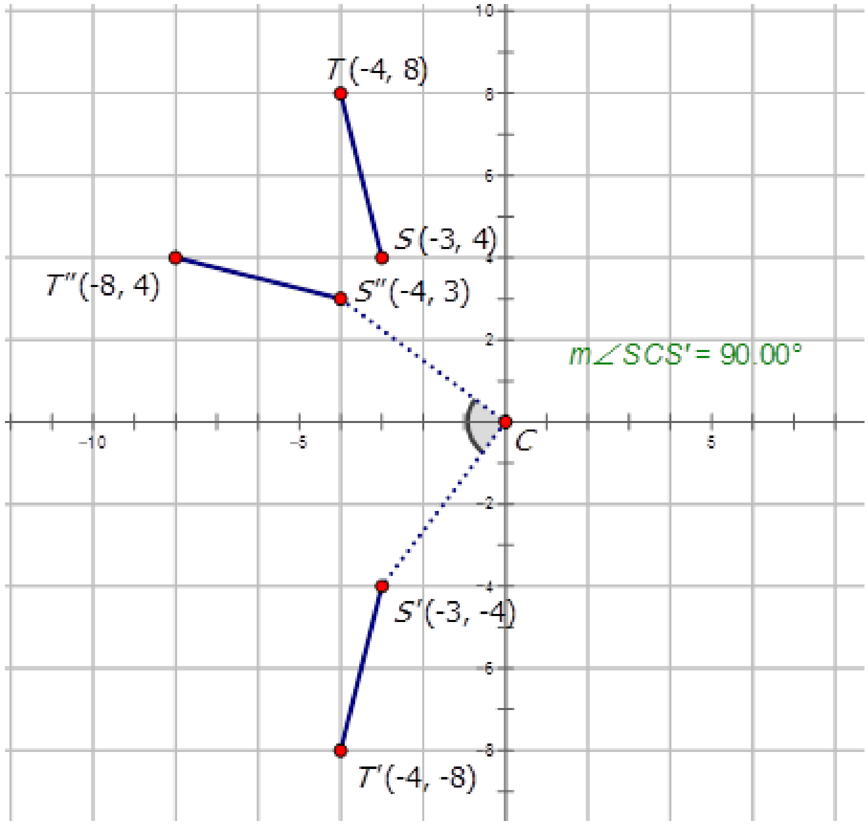
$$d_{H'D'} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{H'D'} = \sqrt{(-6-(-3.5))^2+(4-2)^2}$$
$$d_{H'D'} = \sqrt{(-2.5)^2+(2)^2}$$
$$d_{H'D'} = \sqrt{6.25+4}$$
$$d_{H'D'} = \sqrt{10.25}$$
$$d_{H'D'} = 3.20\ cm$$

Review

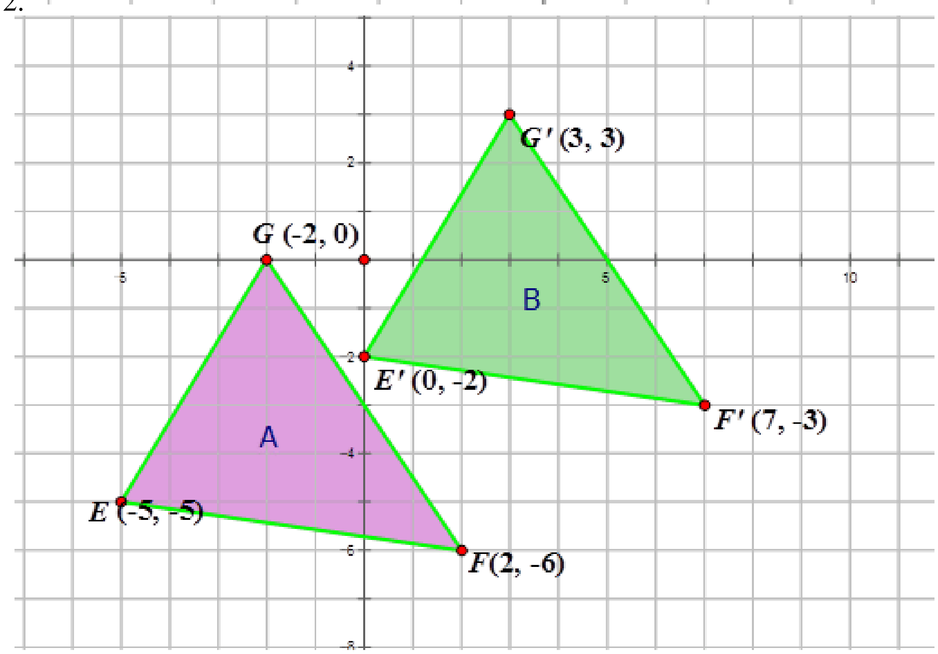
Find the length of each line segment below given its endpoints. Leave all answers in simplest radical form.

1. Line segment AB given $A(5,7)$ and $B(3,9)$.
2. Line segment BC given $B(3,8)$ and $C(5,2)$.
3. Line segment CD given $C(4,6)$ and $D(3,5)$.
4. Line segment DE given $D(9,11)$ and $E(2,2)$.
5. Line segment EF given $E(1,1)$ and $F(8,7)$.
6. Line segment FG given $F(3,6)$ and $G(2,4)$.
7. Line segment GH given $G(-2,4)$ and $H(6,-1)$.
8. Line segment HI given $H(1,-5)$ and $I(3,3)$.
9. Line segment IJ given $I(3.4,7)$ and $J(1,6)$.
10. Line segment JK given $J(6,-3)$ and $K(-2,4)$.
11. Line segment KL given $K(-3,-3)$ and $L(2,-1)$.

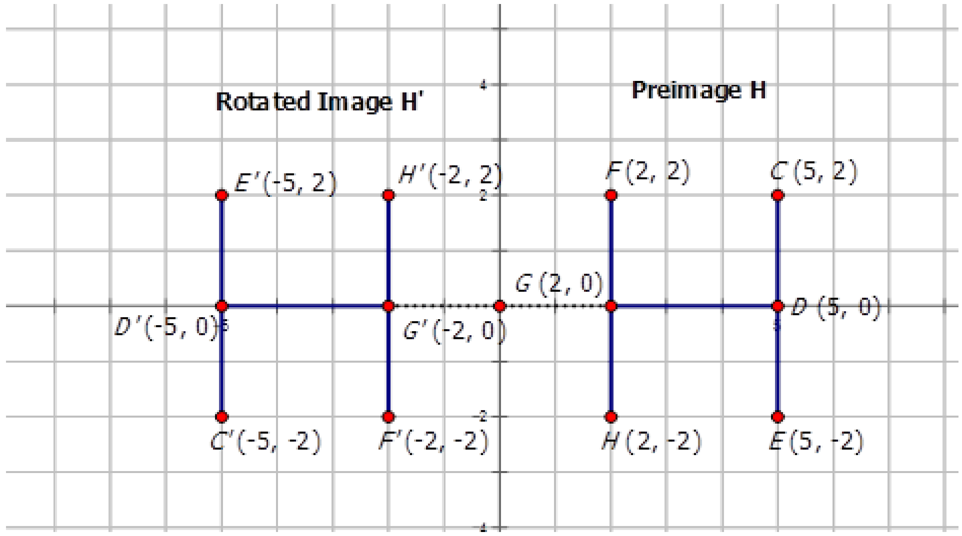
For each of the diagrams below, assume the corresponding angles are congruent. Find the lengths of the line segments to prove congruence.

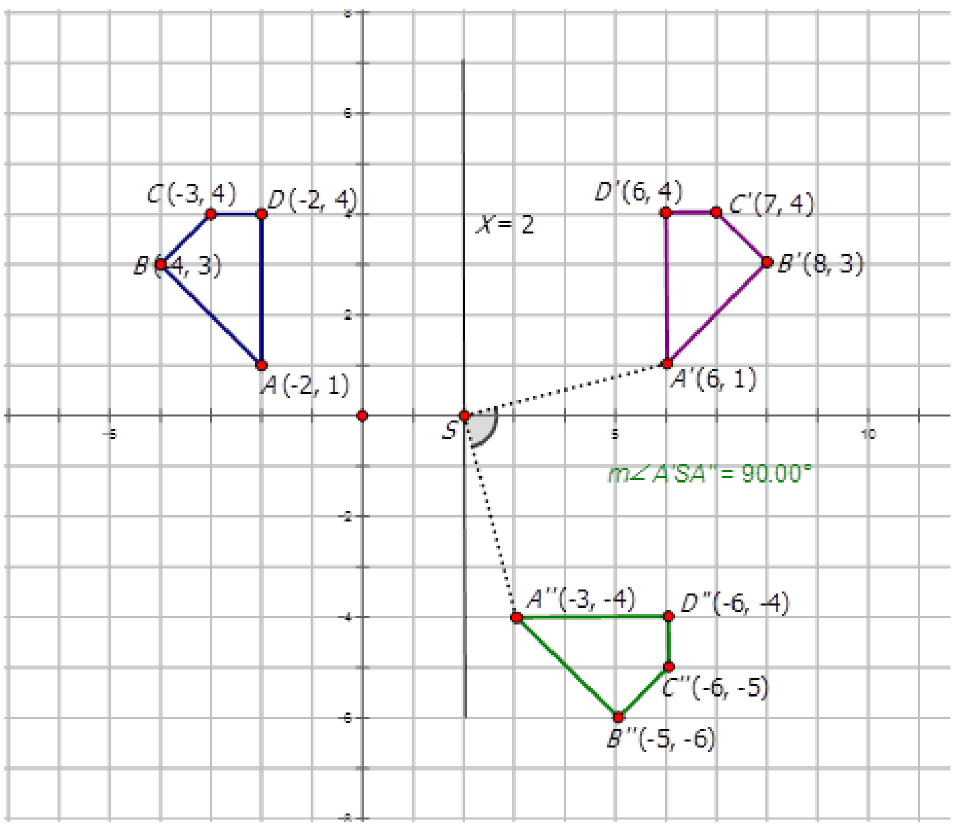


12.



13.

14. 

15. 

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.17.

Summary

You learned that there are four geometric transformations. Translations, reflections, and rotations all produce congruent shapes. Congruent shapes are exactly the same shape and size. Translations are slides, reflections are flips, and rotations are turns.

The fourth geometric transformation is the dilation. A dilation produces a shape that is an enlargement or reduction of the preimage.

You also learned that two or more transformations can be performed in sequence. The result is called a composite transformation.

Finally, you learned the midpoint formula and the distance formula. The midpoint of a line segment is the point exactly in the middle of the two endpoints. Sometimes midpoints can help you to find lines of reflection (also known as lines of symmetry). The distance formula helps you to calculate the length of line segments. The distance formula is useful for determining whether or not the corresponding sides of shapes are the same length. This can help you to determine whether one shape has been transformed to create another shape.