Post-Graduation in Data Science for Finance

Fixed Income Securities

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Group Project

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NOVA Information Management School February 2025

1. Government and Corporate Bonds

a) Bootstrap Spot Rates

We started by bootstrapping the spot rate curve for 10 Bonds for the maturities presented in the exercise, ranging from half a year to 5 years. Each bond with a face value of $100 and a semi-annual coupon payment.

The printed spot curve data frame shows each maturity, its discount factor, and the corresponding spot rate (in both decimal and percent). We can observe that the discount factor declines for longer maturity (as should be expected). So, for the bond with maturity at 5y with a discount factor of 0.88 means that $1 payable in 5 years is only worth $0.88 in present value.

	Maturity	DiscountFactor	SpotRateDecimal	SpotRatePercent
1	0.5	0.9776	0.0454	4.54
2	1.0	0.9777	0.0226	2.26
3	1.5	0.9853	0.0099	0.99
4	2.0	0.9706	0.0149	1.49
5	2.5	0.9535	0.0190	1.90
6	3.0	0.9555	0.0152	1.52
7	3.5	0.9491	0.0149	1.49
8	4.0	0.9330	0.0173	1.73
9	4.5	0.9180	0.0190	1.90
10	5.0	0.8862	0.0242	2.42

Table 1 – Bootstrapped Spot Curve and Discount Factors

The plot below of "Bootstrapped Spot Curve (Continuous Compounding)" is the representation of the yield curve. We can clearly observe that the short end of the curve is steepened, While the belly of this curve shows a more flatten curve. At the end low-end of this curve there is a slight steepening happening. Clearly this curve shows that there are no fears of recession, because the short end of the curve is not showing us demand from investors.

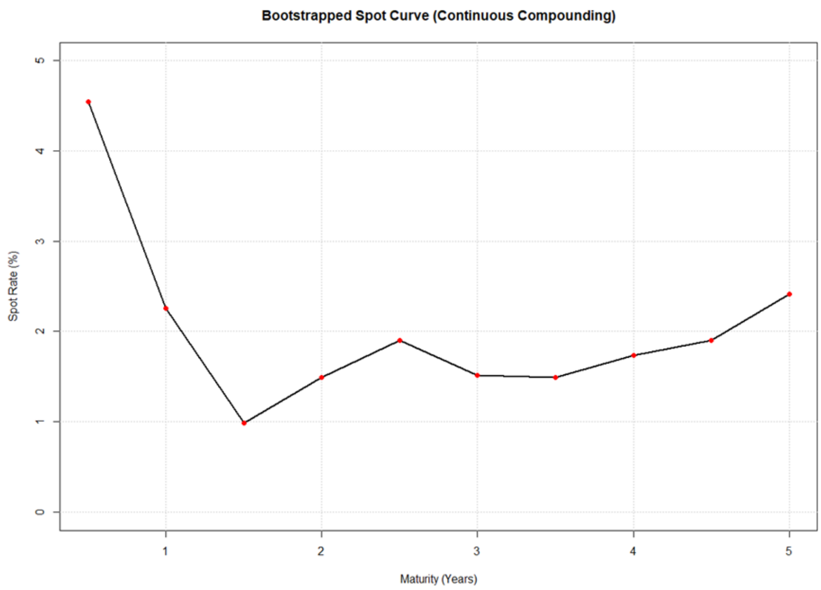

Figure 1 – Bootsrapped Spot Curve

b) Bond Volatility

We used the Macaulay duration and then Modified duration to compute the plot below.

Macaulay duration, D_{mac}, is the weighted average time to cash flow receipt, where weights are the present value of each cash flow as a fraction of total present value.

Modified duration, $D_{\text{mod}} = \frac{D_{\text{mac}}}{1+\frac{\text{YTM}}{2}}$, measures the price sensitivity to a 1% change in yield.

The plot below of "Bond Price Volatility by Maturity," shows an upward trend and the flattening at the low-end of the curve because longer maturities (hence, higher duration) lead to greater price volatility for a given yield volatility.

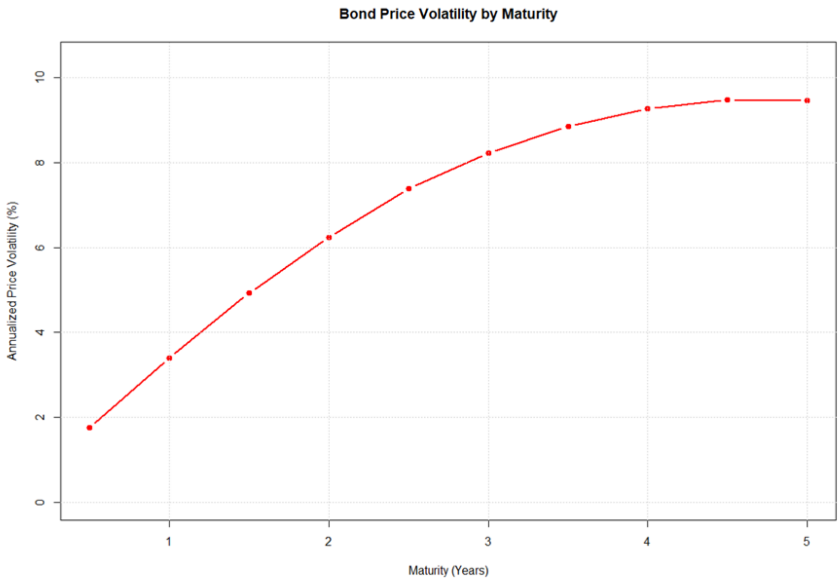

Figure 2 – Bond Price Volatility

As maturity increases, so does duration, and thus the product of (duration × yield volatility) tends to increase. This gives a typical upward-sloping volatility vs. maturity profile. YTM might not be strictly monotonic across maturities (depending on bond prices and coupons), which reflects the shape of the yield curve implied by these market bonds.

c) Corporate Bond

Finally, we price a 3% annual coupon corporate bond (semiannual = 1.5% each half-year) of face value $100,000 by adding a flat 1% credit spread to the risk-free rates bootstrapped in part (a).

The results below show that the corporate bond price is $98,175.02, thus being below par value which means investors can buy it at a discount. We used the Macaulay/Modified Duration to measure the interest-rate risk and the Convexity for second-order interest-rate sensitivity.

A duration of about 4.673 years for a 5-year bond tells us that on average, when considering the present value of each coupon plus principal repayment, the time-weighted location of those cash flows is about 4.673 years.

From an interest-rate risk perspective (Modified Duration):

- If interest rates rise instantaneously by 1%, the bond's price drops by about 4.7%.
- Conversely, if rates fall by 1%, the bond's price would approximately rise by about 4.6%.

By analyzing the results below of the convexity, using the below formula:

$$\frac{\Delta P}{P} \approx -D_{\mathrm{mod}}\,\Delta y + \frac{1}{2}\,\mathrm{Convexity}\,\left(\Delta y\right)^2.$$

We get 1/2 ×22.8×(0.01)^2=0.00114.

A +0.114% price change that either mitigates losses or enhances gains beyond the linear (duration) estimate. So in practical terms, a 22.8 convexity value, in conjunction with our duration of 4.673, indicates that while the bond's first-order sensitivity is about 4.7% price change per 1% yield change, we also have a second-order effect of around 0.5 × 22.8 × (yield change)^2 that makes the bond's price more resilient to large yield increases and more responsive to large yield decreases than a purely linear (duration-only) approximation would suggest.

2. Yield curve

a) NS parameter estimates and mean squared error (MSE)

The Nelson–Siegel model is one of the most widely used parametric frameworks for describing the term structure of interest rates. It has as primary goal to propose a simple yet flexible specification that captures the essential shapes (e.g., upward sloping, downward sloping, humped) observed in empirical yield curves. It's given by the following formula:

$$r(0,t) = \beta_0 + \beta_1 \frac{1 - \exp\left(-\frac{t}{\tau_1}\right)}{\frac{t}{\tau_1}} + \beta_2 \left[\frac{1 - \exp\left(-\frac{t}{\tau_1}\right)}{\frac{t}{\tau_1}} - \exp\left(-\frac{t}{\tau_1}\right) \right]$$

Where:

- $r(0,t)$: spot interest rate for maturity t.
  - $\beta 0$: level parameter, long term interest rate; Shifts the entire yield curve up or down, capturing long-term average rates.
  - $\beta 1$: slope parameter (positive $\beta 1$ < 0; negative $\beta 1$ > 0), short term vs long term spread; Controls the tilt of the curve; for example, a larger positive β1 implies a steeper curve.
  - $\beta 2$: curvature parameter; if positive (negative), concave (convex) yield curve profile. Allows for a hump or inverted hump (valley) around intermediate maturities.
  - $\tau 1$: scale parameter; how quickly the yield curve transitions from its short-end shape (dominated by slope/curvature) toward the long-run level.

Regarding the NS parameters estimates after implementing the NS function and fitting the NS model for each date, we have obtained the respective parameters:[1]

A tibble: 6 × 6

	Date	Beta0	Beta1	Beta2	Lambda	MSE
	<date>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	2005-01-31	5.19	-2.74	-2.49	0.611	0.00364
2	2005-02-28	5.21	-2.72	-2.49	0.649	0.00397
3	2005-03-31	5.09	-2.57	-1.38	0.580	0.00400
4	2005-04-29	4.99	-2.55	-1.62	0.561	0.00434
5	2005-05-31	4.77	-2.33	-1.90	0.578	0.00389
6	2005-06-30	4.64	-2.14	-1.99	0.525	0.00350

Table 2 – First Rows of NS Parameters and MSE

[1] NS parameters estimates can be seen for all dates in the Appendix section

```
      Date                  Beta0             Beta1
 Min.   :2005-01-31   Min.   :1.266   Min.   :-4.782
 1st Qu.:2009-02-27   1st Qu.:2.519   1st Qu.:-2.494
 Median :2013-03-29   Median :3.188   Median :-1.994
 Mean   :2013-03-30   Mean   :3.354   Mean   :-1.896
 3rd Qu.:2017-04-28   3rd Qu.:4.273   3rd Qu.:-1.226
 Max.   :2021-05-31   Max.   :5.206   Max.   : 3.232
     Beta2              Lambda               MSE
 Min.   :-4.569   Min.   :0.02653   Min.   :0.0002166
 1st Qu.:-3.103   1st Qu.:0.35061   1st Qu.:0.0013975
 Median :-2.201   Median :0.49654   Median :0.0031854
 Mean   :-1.993   Mean   :0.56442   Mean   :0.0049473
 3rd Qu.:-1.001   3rd Qu.:0.61891   3rd Qu.:0.0048313
 Max.   : 3.565   Max.   :3.04643   Max.   :0.0368279
```

Figure 3 – Parameter Statistics

In a quick overview of Nelson–Siegel Parameters (2005-2021), the complete set of parameter estimates reported spans a range of market environments and captures key structural shifts in the term structure of interest rates over nearly two decades. A few notable insights emerge from the data collected.

Over the sample, β0 varies between roughly 1.2 and 5.2 percent, reflecting substantial fluctuations in the overall yield-curve *level*. Higher values tend to coincide with the mid-2000s environment, characterized by comparatively robust economic growth and moderately higher policy rates, at that specific time, whereas the lowest levels occur in the aftermath of the global financial crisis and into the 2010s (sovereign debt crisis), when prolonged monetary accommodation make use of downward pressure on longer-term yields.

The slope factor (β1) mostly remains negative—often in the range of −1 to −3, which is indicative of an upward-sloping curve (short-term yields below their long-term counterparts). Occasional positive or near-zero estimates reflect periods of heightened policy tightening or market stress, during which the yield curve has either flattened or temporarily inverted. These instances frequently align with well-documented macroeconomic events, such as the pre-crisis tightening cycle (2006-2007) or the sporadic recession signals in later years.

Regarding the curvature factor (β2), it exhibits both negative and positive extremes (approximately from -4.6 to +3.6), capturing how the mid-maturity segment of the curve can pivot from concave (when negative) to more humped (when positive). In general, negative values of β2 imply a concave shape, with mid-term yields lower than either short or long-term yields. By contrast, large positive swings indicate a pronounced "hump" structure, suggesting that investor demand or risk shifted strongly in the intermediate part of the maturity spectrum. Notably, some of the largest swings occur around episodes of intense market uncertainty, for example, the crisis period of 2008-2009 or sudden policy shifts, where variations in intermediate maturities can be especially pronounced.

In relation to the scale parameter, this factor regulates how quickly short-term forces transition into the longer-term segment of the yield curve. Lower values of λ concentrate curvature effects at longer maturities, whereas higher values shift the curve's "hump" to shorter maturities. It ranges from about 0.03 up to around 3.0, underscoring how the yield curve's "hump" or curvature can shift substantially over time.

Throughout the dataset, the mean squared error (MSE) remains consistently modest (often under 0.01). Although certain months, especially during extreme market dislocations, display slightly elevated errors, the overall pattern affirms that the NS specification reliably explains most of the cross-sectional variation in yields. Meaning, even during times of considerable turbulence or policy experimentation (for example, quantitative easing measures), the three-factor structure plus a scale term generally provides an accurate snapshot of the term structure.

In summary, the time-series progression of these parameters put in perspective shifts in monetary policy, macroeconomic cycles, and investor sentiment over the 2005-2021 period.

Where high β0 values coincide with periods of elevated interest rates, and large swings in β2 or λ\lambda captured curvature adjustments under evolving market conditions.

Meanwhile, low MSE score across most dates underscores the robustness of the Nelson–Siegel approach in modeling and interpreting the yield curve under diverse macro-financial environments.

b) Yield Curve Dynamics

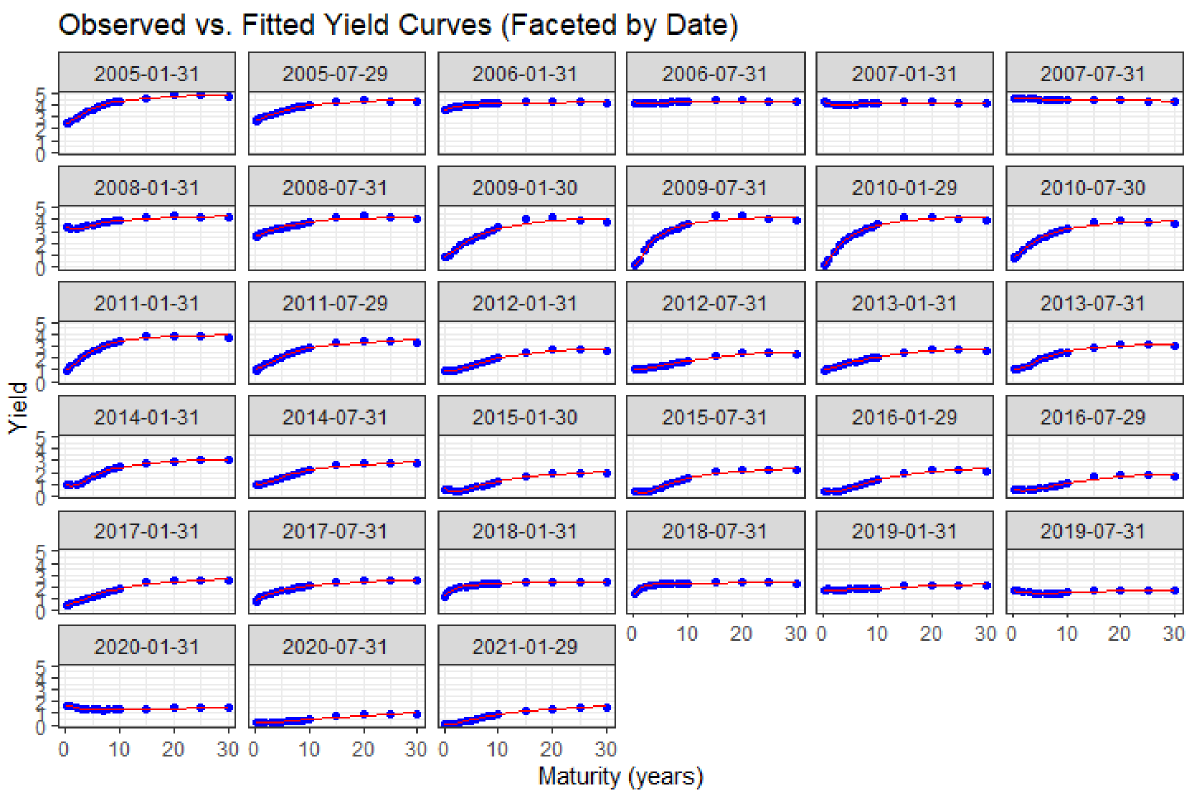

Figure 4 – Observed vs. Fitted Yield Curves for every 6-months

Regarding the Figure XX, we can observed yield curves (represented by the blue dots) for multiple dates from 2005 through 2021, alongside the corresponding Nelson–Siegel (NS) fits (represented by the red lines).

Visually, the Nelson–Siegel model provides a close approximation of the term structure across a wide range of market conditions, as evidenced by minimal vertical discrepancies between the model and the empirical data. The accuracy of these fits corroborates the well-established flexibility of the NS specification for capturing both the level and shape of yields (Nelson & Siegel, 1987; Diebold & Li, 2006), as well as the MSE registered above.

Nelson–Siegel framework is an effective tool for modeling and interpreting yield-curve movements. It captures the broad macroeconomic narratives—such as monetary easing, cyclical recoveries, and secular declines in equilibrium interest rates—through its three principal components.

From a theoretical perspective, variations in the yield curve are commonly interpreted through the lenses of expectations theory and liquidity preference theory (Campbell & Shiller, 1991; Fama, 1984).

2) Under expectations theory, long-term yields primarily reflect investors' anticipations of future short rates, which in turn are influenced by monetary policy and macroeconomic indicators.
2) Liquidity preference theory posits that investors demand a premium for holding longer-dated bonds due to higher interest-rate risk. Consequently, yield curves can exhibit time-varying slopes that are sensitive to central bank policy actions and shifts in investor risk appetite.

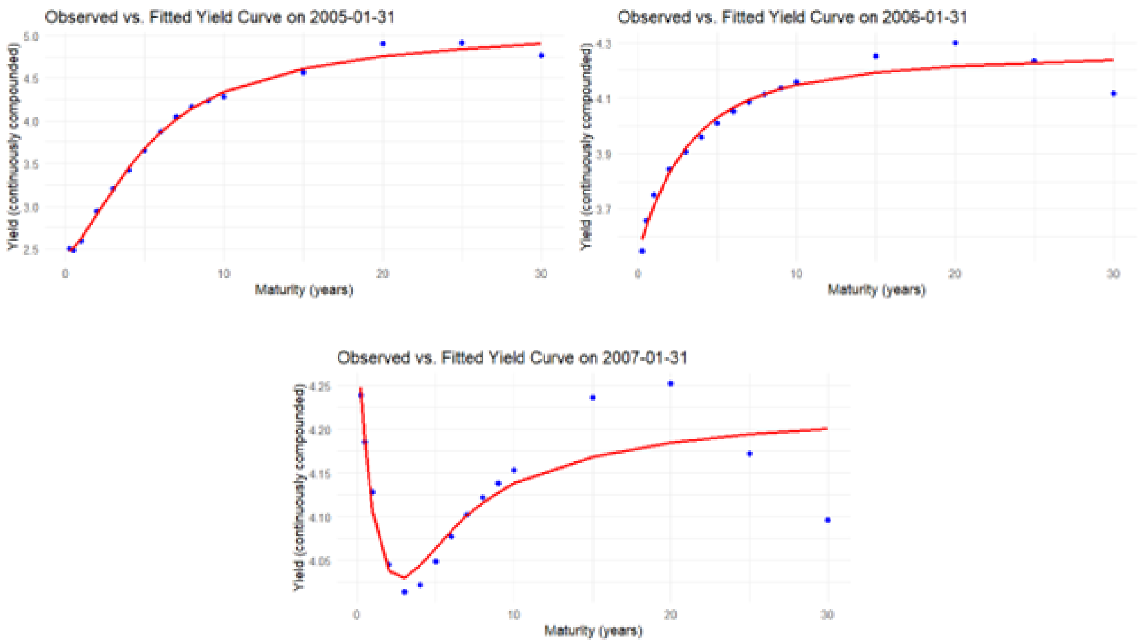

Figure 5 – Early Sample Observed vs. Fitted

In relation to that, in the early part of the sample (2005–2007), yields stand at higher overall levels (around 4%–5% for longer maturities) and show a more pronounced upward slope from short to long maturities. This steep slope aligns with a period of relatively tighter monetary policy and robust economic expansion. Post-2008, the global financial crisis and subsequent recession led to aggressive monetary easing, which compressed short rates toward the zero lower bound while creating a significant drop in the general level of yields.

In many of the 2009–2012 panels, the short end is noticeable low relative to historical norms, while the intermediate-to-long segment remains higher, resulting in a temporarily steep curve. At other times, especially from roughly 2012 onward, the curve flattens in together with forward-guidance policies and persistently low inflation expectations.

In conclusion, the spread between the highest and lowest curves underscores the impact of recessionary conditions (when policy rates are near zero) versus expansions (when central banks allow higher nominal rates). Over the final span (2015 onward), many curves remain at historically low levels, consistent with post-crisis policy frameworks and subdued inflation expectations.

According to the Nelson–Siegel model's decomposition into level (β0), slope (β1), and curvature (β2) factors allow an interpretation of these trends (Litterman & Scheinkman, 1991; Diebold & Li, 2006).

- The level factor typically tracks the general height of the curve across all maturities and often moves downward when monetary authorities lower rates or when long-term inflation expectations abate.
- The slope factor (for example, the difference between short- and long-term rates) reveals whether the term structure is flattening or steepening, frequently aligning with changes in the business cycle and monetary policy.
- The curvature factor influences the shape around intermediate maturities; certain crisis periods induce or reduce mid-curve "humps," reflecting changes in market segmentation or investor risk aversion.

By plotting the fitted NS curves against actual yields over multiple dates (Figure XX), we see these theoretical elements at play. In more volatile times, such as 2009, when the Fed funds rate approached zero, the short end plummets, creating a pronounced upward slope. Later panels, such as 2015–2017, exhibit a somewhat flatter curve at lower yield levels, consistent with ongoing accommodative policy and subdued inflation. As a result, the model continuously adapts its level, slope, and curvature parameters to reflect shifting market expectations.

Furthermore, the consistently small residual errors across all sample dates highlight the robustness of the model in practical yield-curve estimation.

c) Parameter Dynamics

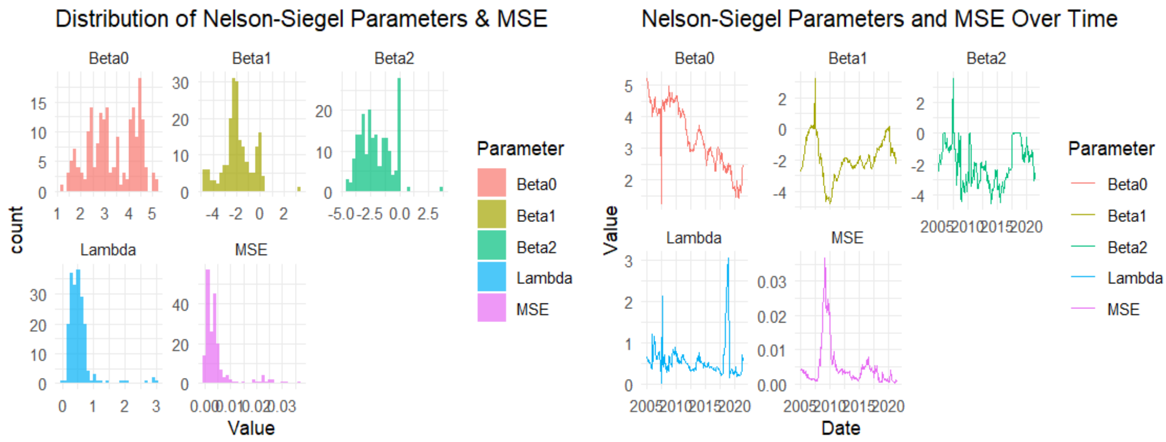

Figure 6 – Distribution and Evolution of NS Parameters and MSE

Figure 6 displays the evolution of the Nelson–Siegel (NS) parameters over time. Each parameter encodes a distinct element of the term structure, and their fluctuations thus provide insight into both market conditions and monetary policy shifts:

The level factor begins in the mid-2000s around 4-5 percent, underscoring the higher yield environment of that era. It then drops following the 2008 global financial crisis, reflecting persistent monetary easing and the subsequent low-rate regime. Although β0 exhibits cyclical fluctuations, it remains relatively subdued for much of the 2010s, aligning with extended periods of unconventional monetary policies and low inflation expectations. A moderate increase appears around 2021, suggesting a partial reversal or expectations of rising yields.

The slope factor typically hovers below zero (-2 to -4), indicating a generally upward-sloping yield curve (short-term rates lower than long-term rates). Occasional surges toward positive territory, most notably around 2006–2007; capture *flattening* or *inverted* segments, often associated with tightening cycles and heightened recession fears. The large downshift after 2008 corresponds to a sharp steepening of the curve as short rates were pushed toward the zero lower bound while long rates remained higher.

The curvature parameter experiences marked spikes and dips, signifying mid-maturity "humps" or "bowing" of the yield curve. Large positive values can signal a pronounced hump, while negative values reflect more concave shapes. Elevated volatility in β2 around 2009–2010, for instance, may be related to sudden shifts in investor demand for intermediate maturities during the crisis period, whereas relatively moderate variations later in the decade suggest more stable curve shapes.

Scale parameter controls how quickly the short-maturity portion of the curve transitions toward the long-term level. Periods with smaller λ values concentrate the slope-to-level transition at longer maturities, whereas larger values push the inflection (hump) closer to the short end. Notably, λ remains stable from 2010 to about 2018 but shows spikes—such as around 2009 and again in 2020–2021—indicating greater impermanence in short-end dynamics. These surges often coincide with heightened policy interventions or sudden adjustments in investor rate expectations.

Overall, the coordinated shifts in β0, β1, and β2 highlight major transitions in level, slope, and curvature of the yield curve, often triggered by macroeconomic and monetary policy changes. On the other hand, the variability in λ captures how swiftly the curve's short-end shape adjusts.

Taken together, these parameters revealed a pronounced downward trend in yields during and after the financial crisis, cycles of steepening and flattening, and evolving mid-curve dynamics -underscoring the flexibility of the Nelson–Siegel model to represent complex term-structure movements across diverse economic conditions.

One additional key metric in these plots is the mean squared error (MSE), which measures the fit of the Nelson-Siegel model to the observed yields at each point in time. Notably, the figure shows spikes in MSE around major economic events - such as the 2008 global financial crisis and the 2020–2021 period - when rapid, atypical yield-curve shifts outpaced the model's standard assumptions. Denote that a rise in MSE alerts analysts that the model's forecasts or risk estimates may be less reliable - especially if the market has transitioned to a new environment or faced an exogenous shock (such as a major policy announcement). In contrast, sustained low MSE during calmer market stretches suggests that Nelson–Siegel remains robust for typical rate environments, effectively capturing level, slope, curvature, and short-end dynamics with a relatively small margin of error.

3. Nelson-Siegel-Svensson Hedging

Interest rate risk is a key concern in fixed-income portfolio management, as shifts in the yield curve directly impact bond valuations. To effectively model and quantify these shifts, the Nelson-Siegel-Svensson (NSS) model provides a flexible framework for fitting and interpreting the term structure of interest rates. The NSS model extends the Nelson-Siegel approach by incorporating an additional curvature term, allowing for greater flexibility in capturing the shape of the yield curve.

The NSS model decomposes the yield curve into the following components:

- Level (β_0): Long-term interest rate expectations, defines the yield curve's overall height.
- Slope (β_1): Short-term movement in rates, representing the difference between short-term and long-term yields.
- Curvature (β_2, β_3): Medium-term fluctuations, allowing the yield curve to exhibit humps or inflection points.
- Decay factors (τ_1, τ_2): Control how quickly the slope and curvature effects diminish over different maturities.

Mathematically, the NSS model defines the yield $y(t)$, at a given maturity t as:

$$y(t) = \beta_0 + \beta_1 \frac{1 - e^{-t/\tau_1}}{t/\tau_1} + \beta_2 \left(\frac{1 - e^{-t/\tau_1}}{t/\tau_1} - e^{-t/\tau_1} \right) + \beta_3 \left(\frac{1 - e^{-t/\tau_2}}{t/\tau_2} - e^{-t/\tau_2} \right)$$

In this report, we examine an asset manager's target portfolio. Using the NSS model, we estimate the level, slope, and curvature durations to quantify the portfolio's exposure to yield curve shifts. These duration measures help assess how different types of yield curve changes—such as parallel shifts, steepening/flattening, and curvature changes—affect the portfolio's value.

To hedge against these risks, the asset manager has selected a hedging portfolio of five Treasury bonds, which will be used to construct a self-financing hedging strategy. The goal is to offset the target portfolio's exposure to level, slope, and curvature risk, ensuring that interest rate movements have a minimized impact on overall portfolio performance.

The NSS parameters on the valuation date (09/02/2022) were provided as follows:

$$\beta_0 = 5.9\%, \qquad \beta_1 = -1.6\%, \qquad \beta_2 = -0.5\%, \qquad \beta_3 = 1.0\%, \qquad \tau_1 = 5, \qquad \tau_2 = 0.5$$

These parameters define the yield curve at the valuation date and serve as the foundation for our duration calculations and hedging strategy formulation.

a) Target Portfolio

The target portfolio consists of 13 fixed-rate Treasury bonds, with varying maturities and coupon rates. The portfolio details are summarized below:

Bond	Maturity	Coupon rate (%)	Quantity
1	01/12/2025	4,00	10 000
2	04/12/2026	7,75	250 000
3	06/12/2027	4,00	50 000
4	10/12/2028	7,00	100 000
5	03/12/2029	5,75	10 000
6	09/12/2030	5,50	200 000
7	06/12/2032	4,00	15 000
8	03/12/2035	4,75	10 000
9	03/12/2030	4,50	30 000

10	04/12/2045	5,00	75 000
11	04/12/2050	4,50	100 000
12	01/12/2051	4,00	10 000
13	07/12/2052	5,00	10 000

Table 3 – Target Portfolio

Each bond's time to maturity was computed as the difference between its maturity date and the valuation date (09/02/2022), converted into years

Bond	Maturity	Maturity in years	Coupon rate (%)	Quantity
1	01/12/2025	3.81	4,00	10 000
2	04/12/2026	4.82	7,75	250 000
3	06/12/2027	5.82	4,00	50 000
4	10/12/2028	6.83	7,00	100 000
5	03/12/2029	7.81	5,75	10 000
6	09/12/2030	8.83	5,50	200 000
7	06/12/2032	10.82	4,00	15 000
8	03/12/2035	13.81	4,75	10 000
9	03/12/2030	8.81	4,50	30 000
10	04/12/2045	23.82	5,00	75 000
11	04/12/2050	28.82	4,50	100 000
12	01/12/2051	29.81	4,00	10 000
13	07/12/2052	30.83	5,00	10 000

Table 4 – Target Portfolio including Time To Maturity

To assess the interest rate risk of the portfolio, we employed two key functions:

1. NSS_spot – Computes continuously compounded spot rates for discounting bon cash flows
2. NSS_Sens – Calculates bond fair value, dollar-durations and parametric durations, measuring exposure to yield curve shifts

Since the bond values depend on discounting, and discounting depends on interest rates, we must first compute NSS spot rates before determining risk measures.

The NSS Sensitivity analysis quantifies how bond values react to changes in the yield curve by computing:

- Fair value ($B0$): Present value of future cash flows
- Dollar-durations ($S0, S1, S2, S3$): Sensitivities to changes in NSS parameters

- Parametric durations $(D0, D1, D2, D3)$: Normalized measures showing how bond values change with shifts in level, slope and curvature

In mathematical terms, the NSS Sensitivity analysis computes the risks measures as per below:

1. Compute discounted cash flows

For a bond with face value (par) = €100, annual coupon payments, and maturity (T) in years, we first determine the cash flows:

$$C_t = coupon\, rate \times 100$$

The bond pays C_t annually and redeems its face value at maturity:

$$C_T = C_T + 100$$

To compute the present value $(B0)$, we discount each cash flow using the NSS spot rates:

$$B0 = \sum_{t=1}^{T} C_t \times e^{-r(t)\cdot t}$$

where $r(t)$ is the NSS spot rate for maturity t. The discounting ensures that bond valuations reflect the current yield curve conditions.

2. Compute Dollar-Durations $(S0, S1, S2, S3)$

Dollar-durations measure how much a bond's price changes when NSS parameters shift. Each sensitivity is computed as:

$$S_i = -\sum_{t=1}^{T} C_t \times e^{-r(t)\cdot t} \cdot \frac{\partial r(t)}{\partial \beta i}$$

where $\frac{\partial r(t)}{\partial \beta i}$ represents how spot rates change when each NSS parameter $(\beta_0, \beta_1, \beta_2, \beta_3)$ shifts.

These sensitivities provide insights into different yield curve movements:

- S_0: Measures how bond prices change due to parallel shift in rates
- S_1: Measures how bond prices react to changes in the slope of the yield curve
- S_2, S_3: Measure how bond prices react to changes in medium-term and long-term curvature, respectively.

3. Compute Parametric Durations ($D0, D1, D2, D3$)

Parametric durations are normalized dollar-durations, showing how sensitive a bond's price is to yield curve shifts relative to its fair value:

$$D_i = \frac{S_i}{B0}$$

where D_i expresses risk exposure per unit of bond value, making comparisons between more meaningful.

- D_0: Percentage change in bond price due to a parallel shift in interest rates.
- D_1: Percentage change due to changes in the slope of the yield curve.
- D_2, D_3: Percentage changes due to shifts in medium- and long-term curvature.

Bond	B0	S0	S1	S2	S3	D0	D1	D2	D3
1	97,62	-281,58	-212,86	-56,41	-47,09	2,88	2,18	0,58	0,48
2	110,08	-397,30	-278,29	-91,92	-53,02	3,61	2,53	0,84	0,48
3	95,80	-442,59	-284,21	-114,51	-46,90	4,62	2,97	1,20	0,49
4	110,07	-566,33	-341,94	-153,45	-53,32	5,15	3,11	1,39	0,48
5	103,94	-621,07	-348,54	-175,31	-50,57	5,97	3,35	1,69	0,49
6	102,48	-686,09	-360,69	-197,55	-49,90	6,69	3,52	1,93	0,49
7	90,77	-756,19	-348,34	-220,13	-44,41	8,33	3,84	2,43	0,49
8	94,88	-935,95	-374,96	-261,80	-46,28	9,86	3,95	2,76	0,49
9	96,02	-658,72	-343,68	-190,55	-46,91	6,86	3,58	1,98	0,49
10	93,62	-1281,77	-378,85	-294,87	-45,59	13,69	4,05	3,15	0,49
11	85,25	-1297,64	-346,64	-273,97	-41,52	15,22	4,07	3,21	0,49
12	77,72	-1228,53	-317,86	-253,39	-37,88	15,81	4,09	3,26	0,49
13	91,96	-1404,95	-371,51	-291,65	-44,76	15,28	4,04	3,17	0,49

Table 5 – Portfolio Bonds Present Value and Risk Measures

After computing individual bond sensitivities, we aggregate them to determine portolio-level risk exposure.

$$w_i = \frac{Quantity \times B0}{\sum Quantity \times B0}$$

$$Porfolio\ Risk\ Measures = \sum w_i \times (S0, S1, S2, S3, D0, D1, D2, D3)$$

The results were:

S0	S1	S2	S3	D0	D1	D2	D3
-684.26	-325.96	-170.19	-49.58	6.99	3.23	1.73	0.49

Table 6 – Portfolio Aggregate Risk Measures

1. Impact of a Parallel Shift in Interest rates (S0, D0)

A 1% parallel increase in the yield curve results in a total portfolio value loss of €684.26.

This is reflected in:

- S0 = -684.26, meaning that if all interest rates increase by 1%, the portfolio will lose €684.26 in absolute value.
- D0 = 6.99, indicating that a 1% rate increase leads to a 6.99% portfolio decline.

This highlights the significant exposure of the portfolio to parallel yield curve shifts.

2. Impact of a Yield Curve Steepening/Flattening (S1, D1)

A 1% steepening of the yield curve (short-term rates fall, long-term rates rise) results in a €325.96 loss.

- S1 = -325.96, meaning a 1% steepening decreases the portfolio's value by €325.96.
- D1 = 3.23, indicating the portfolios loses 3.23% of its value when the yield curve steepens by 1%.

For example:

- If short-term rates drop by 0.50%, while long-term rates rise by 0.50%, the expected loss would be €162.98 (since 0.5% × -325.96 = -162.98€).
- Conversely, if short-term rates increase and long-term rates fall (flattening scenario), the portfolio would gain €325.96.

This suggests that the portfolio is moderately exposed to changes in the slope of the yield curve. A steepening negatively impacts the portfolio, whereas flattening benefits it.

3. Impact of Medium-Term Curvature changes (S2, D2)

A 1% increase in mid-term rates (5-10 years) results in a €170.19 loss.

- S2 = -170.19, meaning that a 1% increase in medium-term rates (5-10 years) decreases portfolio value by €170.19.
- D2 = 1.73, indicating that a 1% increase in medium-term rates leads to a 1.73% decline in portfolio value.

For example:

- If the 5-year rate increase by 0.75%, the portfolio would lose €127.64 (0.75% × -170.19 = -127.64).
- If the 5-year rate decreases by 1.00%, the portfolio would gain €170.19.

This indicates that the portfolio has moderate exposure to mid-term yield curve movements, requiring careful monitoring of 5-10 year rate fluctuations.

4. Impact of Long-Term Curvature changes (S3, D3)

A 1% shift in long-term rates (20-30 years) results in a €49.58 loss.

- S3 = -49.58, meaning that if long-term rates increase by 1%, the portfolio will lose €49.58.
- D3 = 0.49, indicating a 1% change in long-term rates leads to only a 0.49% decline in portfolio value.

Long-term yield curve distortions have minimal impact on the portfolio.

Based on the sensitivity parameters derived from this analysis, we can also conclude that bonds with longer maturities tend to be more sensitive to interest rate changes. This aligns with financial theory, as longer-term bonds are more exposed to the time value effect, meaning their cash flows are discounted more heavily. Because of this, even small changes in the discount rate lead to larger fluctuations in present value compared to shorter-term bonds.

Additionally, long-term bonds remain outstanding for a greater period, making them more exposed to interest rate movements over time. Since rates fluctuate due to economic conditions and monetary policy, these bonds experience greater cumulative price impacts than short-term securities. Their higher duration and convexity further amplify this effect, meaning they react strongly to rate shifts, with price declines being steeper when yields rise.

Overall, the analysis confirms that longer-term bonds exhibit greater price sensitivity, making them more vulnerable to rate increases but also more responsive when rates decline.

b) Hedging Assets

The hedging portfolio consists of five assets, each with varying maturities and coupon rates, selected to offset the interest rate risk of the target portfolio. The goal is to construct a hedging strategy that effectively mitigates level, slope and curvature risks.

Below are the hedging assets, along with their computed time to maturity, measured as the difference between the maturity date and the valuation date (09/02/2022).

Hedging asset	Coupon rate (%)	Maturity	Maturity in years
H1	4.5	12/04/2026	4.17
H2	5	28/12/2032	10.88
H3	6	06/05/2035	13.23
H4	6	10/10/2040	18.67
H5	6.5	10/10/2051	29.66

Table 7 – Hedging Assets

Following the same NSS Sensitivity methodology applied to the target portfolio, we computed the fair values (B0), dollar-durations (S0, S1, S2, S3) and parametric durations (D0, D1, D2, D3) for each hedging asset using the NSS parameters from the valuation date.

The purpose of this analysis is to determine how the hedging portfolio responds to different type of yield curve shifts.

Hedging Asset	B0	S0	S1	S2	S3	D0	D1	D2	D3
H1	98,52	-369,11	-256,87	-86,45	-48,04	3,75	2,61	0,88	0,49
H2	98,47	-795,27	-370,59	-230,62	-48,01	8,08	3,76	2,34	0,49
H3	106,51	-1008,87	-411,90	-281,84	-51,79	9,47	3,87	2,65	0,49
H4	106,68	-1238,20	-423,60	-315,15	-51,88	11,61	3,97	2,95	0,49
H5	113,82	-1634,70	-454,36	-350,69	-55,32	14,36	3,99	3,08	0,49

Table 8 – Hedging Assets Present Value and Risk Measures

1. Parallel Shift sensitivity (S0, D0)

The S0 values indicate how the hedging portfolio reacts to a uniform 1% increase of decrease in interest rates. Since all bonds exhibit negative S0 values, an increase in interest rates will lower their fair values, while a decrease in rates will increase their values.

The D0 values range from 3.75 to 14.36, confirming that longer-term hedging bonds are more sensitive to parallel shifts.

2. Slope sensitivity (S1, D1)

The S1 values measure how the hedging portfolio responds to changes in the yield curve slope (steepening or flattening).

- If short-term rates fall while long-term rates rise (steepening), the hedging bonds lose value
- If short-term rates rise while long-term rates fall (flattening), the hedging bonds gain value.

The D1 values (between 2.61 and 3.99) suggest that the hedging portfolio provides moderate control over slope risk.

A 1% steepening of the yield curve (where short-term rates decrease and long-term rates increase) will reduce the value of the hedging portfolio, with H5 alone losing €454.36.

3. <u>Medium-term Curvature sensitivity (S2, D2)</u>

The S2 values reflect the portfolio's sensitivity to medium-term rate fluctuations (5-10 years maturities).

- If medium-term rates increase, the fair values of the hedging bonds decrease.
- If medium-term rates decrease, the hedging bonds gain value.

The D2 values (0.88 to 3.08) indicate that the hedging bonds are not to exposed to mid-curve distortions.

4. <u>Long-term Curvature sensitivity (S3, D3)</u>

Since the S3 values are relatively small, this suggests that the hedging portfolio has minimal exposure to long-term curvature changes.

The D3 values (~0.49 across all bonds) confirm that long-term yield curve distortion are not a primary concern for the hedge.

3.1. c) Self-Financing Hedging Strategy

In this section, we compute the optimal weights of the hedging portfolio to offset the interest rate risks of the target portfolio while ensuring a self-financing constraint. The hedging strategy is structured to neutralize the level, slope and curvature risks using the risk measures computed in section 4.2 (b).

To construct the hedge, we adopt a hold-to-maturity strategy, meaning the investment horizon is set equal to the time to maturity of the longest bond in the target portfolio:

$$H = \max(\text{Maturity of Target Portfolio Bonds}) = 30.83 \text{ years}$$

This ensures that the hedge remains effective over the entire duration of the portfolio, mitigating exposure to interest rate shifts throughout the holding period.

To determine the quantity of each hedging instrument, we formulated the hedging problem using matrix notation.

This can be expressed as a system of equations:

$$Aw = b$$

where:

- A is the risk measure matrix of the hedging portfolio (including the bond fair values B_0).
- w is the vector of weights (quantities of each hedging instrument).
- b is the negative aggregated risk measures of the target portfolio, ensuring the hedge neutralizes the portfolio's interest rate sensitivities.

The matrix A is constructed as follows:

$$A = \begin{bmatrix} D_0^{H1} & D_0^{H2} & D_0^{H3} & D_0^{H4} & D_0^{H5} \\ D_1^{H1} & D_1^{H2} & D_1^{H3} & D_1^{H4} & D_1^{H5} \\ D_2^{H1} & D_2^{H2} & D_2^{H3} & D_2^{H4} & D_2^{H5} \\ D_3^{H1} & D_3^{H2} & D_3^{H3} & D_3^{H4} & D_3^{H5} \\ B_0^{H1} & B_0^{H2} & B_0^{H3} & B_0^{H4} & B_0^{H5} \end{bmatrix}$$

where:

- $\boldsymbol{D_0}, \boldsymbol{D_1}, \boldsymbol{D_2}, \boldsymbol{D_3}$ are the parametric durations (level, slope and curvature durations) of the hedging instruments.
- B_0 represents the present value of each hedging bond.

The vector b is defined as:

$$b = \begin{bmatrix} -D_0^T \\ -D_1^T \\ -D_2^T \\ -D_3^T \\ -B_0^T \end{bmatrix}$$

where:

- $D_0^T, D_1^T, D_2^T, D_3^T$ are the aggregate risk measures of the target portfolio.

- B_0^T is the total present value of the target portfolio.

Solving the equation:

$$w = A^{-1}b$$

yields the optimal quantities of each hedging bond:

$$w = \begin{bmatrix} -26{,}315.44 \\ 8{,}120{,}609.31 \\ -17{,}471{,}137.77 \\ 12{,}455{,}016.80 \\ -3{,}102{,}534.54 \end{bmatrix}$$

It is also important to mention that the self-financing requirement was satisfied, meaning the total market value of the hedging portfolio matches that of the target portfolio.

$$\sum wi\, B_0^{Hi} = B_0^T$$

Resulting in a total cost of the hedge:

$$\sum wi\, B_0^{Hi} = -88{,}262{,}838$$

$$B_0^T = 88{,}262{,}838$$

$$\sum wi\, B_0^{Hi} + B_0^T$$

Hedging Asset	Weight (quantity)
H1	-26,315.44
H2	8,120,609.31
H3	-17,471,137.77
H4	12,455,016.80
H5	-3,102,534.54

Table 9 – Hedge Weights

The computed hedge weights indicate that the optimal hedge involves both long and short positions, ensuring a net-zero cost structure while effectively neutralizing the level, slope and curvature risks of the target portfolio.

H2 and H4 are the core long positions, confirming that the Target Portfolio required additional mid- and long-duration exposure to properly hedge parallel and slope risks.

H3 is heavily shorted, suggesting that the Target Portfolio had excess exposure in mid to long maturities, require significant offsetting and ensuring the self-finance constraint.

H5's short position is present to fine-tune the long-term curvature hedge, ensuring that long-term rate fluctuations do not introduce unnecessary risk.

H1 plays a minimal role, confirming that short-term bonds have little impact on the overall hedge due to their low duration.

d) Impact of Yield Curve Shifts

Immediately after the hedging strategy was established, the yield curve changed and was then given by the following set of NSS parameters:

$$\beta_0 = 6.5\%, \quad \beta_1 = -1.0\%, \quad \beta_2 = 0.1\%, \quad \beta_3 = 2.0\%, \quad \tau_1 = 5, \quad \tau_2 = 0.5$$

Specifically, the shift in parameters consists of an increase in the level parameter β_0 from 5.9% to 6.5%, a reduction in the slope parameter β_1 from -1.6% to -1%, a shift in the curvature parameter β_2 from negative (-0.5%) to positive (0.1%), and an increase in the rate of curvature parameter β_3 from 1% to 2%. The scale parameters τ_1 and τ_2 remained unchanged.

i. Assuming no hedging strategy had been implemented

The immediate impact of these changes on the portfolio bonds is evident in their price behavior, as shown in the table below. All bond prices decreased following the parameter shift. This occurs because an increase in β_0 signifies a parallel upward movement in the yield curve, which raises rates and consequently reduces bond prices. The reduction in β_1 mitigates this effect slightly by flattening the yield curve, yet the overall impact remains negative due to the predominant level shift. Additionally, the increase in β_2 and β_3 enhances curvature effects, particularly for medium-term maturities, making the yield curve more convex.

Bond	B0	S0	S1	S2	S3	D0	D1	D2	D3
1	93.92	-270.69	-204.65	-54.21	-45.28	2.88	2.18	0.58	0.48
2	105.06	-378.24	-265.04	-87.44	-50.56	3.6	2.52	0.83	0.48
3	90.45	-416.79	-267.75	-107.76	-44.25	4.61	2.96	1.19	0.49
4	103.4	-529.11	-319.82	-143.18	-50.02	5.12	3.09	1.38	0.48
5	96.85	-574.92	-323.12	-162.07	-47.04	5.94	3.34	1.67	0.49
6	94.85	-629.87	-331.81	-181.1	-46.1	6.64	3.5	1.91	0.49
7	82.79	-682.79	-315.47	-198.5	-40.43	8.25	3.81	2.4	0.49
8	85.55	-829.53	-334.43	-231.75	-41.63	9.7	3.91	2.71	0.49
9	88.71	-604.24	-315.82	-174.57	-43.28	6.81	3.56	1.97	0.49
10	82.37	-1084.09	-327.27	-250.57	-39.98	13.16	3.97	3.04	0.49
11	74.33	-1076.55	-296.15	-229.57	-36.09	14.48	3.98	3.09	0.49
12	67.51	-1013.54	-270.46	-211.39	-32.79	15.01	4.01	3.13	0.49
13	80.2	-1161.08	-317.27	-244	-38.9	14.48	3.96	3.04	0.49

Table 10 – Portfolio Bonds Present Value and Risk Measures after shift

Examining the risk measures, the absolute values of dollar durations declined across all bonds. This decline can be attributed to the nature of duration as a measure of price sensitivity to yield changes. As the yield curve shifts, bond prices fall, reducing the present value of future cash flows, and thereby decreasing dollar durations. Furthermore, all parametric durations declined, with a more pronounced reduction, for level and curvature durations, at longer maturities. This reflects the fact that long-duration bonds are more sensitive to shifts in the level and curvature of the yield curve. The curvature duration, D3, exhibited only a minor decrease, as the increase in β_3 primarily affects intermediate maturities rather than the longer-term bonds in the portfolio.

S0	S1	S2	S3	D0	D1	D2	D3
-605.12	-298.8	-151.61	-46.15	6.72	3.19	1.67	0.48

Table 11 – Aggregate Portfolio Present Value and Risk Measures after shift

Aggregating these effects at the portfolio level, taking into account the new bond weights given their new prices, the portfolio's dollar durations decreased in absolute value, meaning that the overall sensitivity of the portfolio's value to yield curve movements was reduced. Similarly, the parametric durations also declined, indicating that the portfolio's exposure to level, slope, and curvature risks diminished. Without a hedging strategy in place, these adverse effects resulted in a loss of $6,411,190 in portfolio value, representing a -7.26% decrease. This significant reduction underscores the importance of considering hedging strategies when managing fixed-income portfolios under changing yield curve conditions.

ii. Assuming the hedging strategy had been implemented

The hedging assets used in the strategy were similarly affected by the yield curve shift, individually experiencing declines in value and changes in parametric risk measures. However, the hedging strategy was constructed to counteract these effects by taking offsetting positions in instruments that in this case benefit from the observed parameter shifts. As a result, the global portfolio value increased by $1,637,516, amounting to a 1.86% gain. This positive outcome arises because the hedging instruments were selected to mitigate any impact that changes in the NSS parameters might have, effectively offsetting the price declines in the target portfolio. By strategically positioning the hedge in assets with sensitivities opposed to those of the portfolio, the global portfolio benefited from the compensatory effects, leading to an overall increase in value despite the adverse movements in individual bond prices.

This analysis highlights the effectiveness of a well-designed NSS-based hedging strategy in protecting a fixed-income portfolio against systematic yield curve shifts. The observed results demonstrate that, while unhedged portfolios suffer significant losses under changing rate conditions, incorporating hedging instruments tailored to the NSS risk factors can not only mitigate these losses but even generate positive returns.

4. Appendix

A tibble: 197 × 6

	Date	Beta0	Beta1	Beta2	Lambda	MSE
	<date>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	2005-01-31	5.19	-2.74	-2.49	0.611	0.00364
2	2005-02-28	5.21	-2.72	-2.49	0.649	0.00397
3	2005-03-31	5.09	-2.57	-1.38	0.580	0.00400
4	2005-04-29	4.99	-2.55	-1.62	0.561	0.00434
5	2005-05-31	4.77	-2.33	-1.90	0.578	0.00389
6	2005-06-30	4.64	-2.14	-1.99	0.525	0.00350
7	2005-07-29	4.64	-1.98	-1.46	0.507	0.00389
8	2005-08-31	4.46	-1.69	-1.71	0.528	0.00314
9	2005-09-30	4.48	-1.52	-0.812	0.445	0.00305
10	2005-10-31	4.53	-1.38	-0.0652	0.394	0.00341
11	2005-11-30	4.27	-0.817	-0.000818	0.360	0.00376
12	2005-12-30	4.01	-0.533	0.000000791	1.20	0.00298
13	2006-01-31	4.28	-0.738	-0.000422	0.548	0.00190
14	2006-02-28	4.14	-0.393	-0.000691	0.735	0.00256
15	2006-03-31	4.28	-0.354	-0.500	1.15	0.00270
16	2006-04-28	4.56	-0.409	-0.960	1.05	0.00162
17	2006-05-31	4.52	-0.216	-0.902	1.03	0.00168
18	2006-06-30	4.62	-0.268	-0.325	0.725	0.00161
19	2006-07-31	4.40	-0.113	-0.819	0.767	0.00150
20	2006-08-31	4.24	-0.00559	-1.02	0.624	0.00130
21	2006-09-29	4.14	0.0858	-1.12	0.606	0.00133
22	2006-10-31	4.11	0.126	-0.916	0.616	0.00157
23	2006-11-30	4.04	0.216	-1.23	0.614	0.00126
24	2006-12-29	4.16	0.177	-1.00	0.762	0.00127
25	2007-01-31	4.23	0.0931	-0.816	0.765	0.00142
26	2007-02-28	4.11	0.208	-1.06	0.774	0.00140
27	2007-03-30	4.24	0.0353	-1.07	0.661	0.00133
28	2007-04-30	4.19	0.0400	-0.498	0.528	0.00134
29	2007-05-31	4.24	0.0309	0.878	0.588	0.000972
30	2007-06-29	1.27	3.23	3.56	0.0265	0.000662
31	2007-07-31	4.26	0.341	0.000233	0.122	0.00105
32	2007-08-31	4.39	-0.346	0.000145	2.13	0.00123
33	2007-09-28	4.51	-0.394	-0.686	0.529	0.000860
34	2007-10-31	4.42	-0.244	-0.431	0.421	0.000931
35	2007-11-30	4.29	-0.228	-1.71	0.630	0.00115
36	2007-12-31	4.17	-0.216	-1.02	0.691	0.000889
37	2008-01-31	4.45	-0.888	-2.64	0.696	0.00205
38	2008-02-29	4.50	-1.27	-3.34	0.612	0.00319
39	2008-03-31	4.46	-2.24	-1.30	0.363	0.00711
40	2008-04-30	4.57	-1.77	-2.71	0.506	0.00525
41	2008-05-30	4.55	-1.90	-1.13	0.395	0.00524
42	2008-06-30	4.38	-1.60	-0.00595	0.281	0.0105
43	2008-07-31	4.54	-1.99	-0.899	0.351	0.0111
44	2008-08-29	4.55	-1.97	-2.19	0.434	0.0104
45	2008-09-30	4.73	-2.68	-0.171	0.318	0.0193
46	2008-10-31	4.97	-3.07	-4.15	0.669	0.0232
47	2008-11-28	4.61	-2.83	-4.46	0.650	0.0248
48	2008-12-31	4.30	-3.47	-4.00	0.514	0.0259
49	2009-01-30	4.59	-3.75	-3.11	0.505	0.0301
50	2009-02-27	4.61	-4.03	-3.49	0.497	0.0368
51	2009-03-31	4.57	-4.24	-3.22	0.422	0.0317
52	2009-04-30	4.76	-4.64	-3.77	0.537	0.0264
53	2009-05-29	4.72	-4.68	-3.27	0.664	0.0223
54	2009-06-30	4.46	-4.31	-3.86	0.837	0.0225
55	2009-07-31	4.55	-4.49	-3.51	0.847	0.0210
56	2009-08-31	4.51	-4.46	-3.28	0.771	0.0248
57	2009-09-30	4.45	-4.46	-2.65	0.712	0.0225
58	2009-10-30	4.49	-4.38	-3.82	0.887	0.0204
59	2009-11-30	4.53	-4.42	-3.75	0.726	0.0234
60	2009-12-31	4.67	-4.78	-2.55	0.781	0.0191
61	2010-01-29	4.60	-4.65	-2.78	0.679	0.0157
62	2010-02-26	4.66	-4.62	-3.22	0.722	0.0129
63	2010-03-31	4.55	-4.61	-0.0546	0.550	0.00829
64	2010-04-30	4.38	-4.25	-0.00124	0.619	0.00633
65	2010-05-31	4.09	-3.81	-0.00128	0.562	0.00547
66	2010-06-30	4.20	-3.81	-1.38	0.506	0.00596
67	2010-07-30	4.29	-3.56	-2.66	0.576	0.00765
68	2010-08-31	4.10	-3.25	-3.45	0.550	0.00876
69	2010-09-30	3.98	-2.92	-3.08	0.517	0.00935
70	2010-10-29	4.10	-3.09	-3.16	0.500	0.00885
71	2010-11-30	4.01	-2.96	-2.56	0.623	0.00573
72	2010-12-31	4.02	-2.96	-2.39	0.635	0.00483
73	2011-01-31	4.26	-3.29	-2.99	0.696	0.00557
74	2011-02-28	4.15	-3.24	-1.91	0.665	0.00395
75	2011-03-31	4.20	-3.33	-1.98	0.699	0.00368
76	2011-04-29	4.19	-3.22	-2.07	0.596	0.00439
77	2011-05-31	3.98	-3.10	-1.98	0.567	0.00292
78	2011-06-30	4.07	-3.12	-2.48	0.597	0.00292
79	2011-07-29	3.85	-2.87	-2.65	0.526	0.00343
80	2011-08-31	3.77	-2.77	-3.59	0.510	0.00380
81	2011-09-30	3.43	-2.54	-3.62	0.498	0.00410
82	2011-10-31	3.59	-2.66	-3.46	0.489	0.00354
83	2011-11-30	3.27	-2.31	-3.10	0.508	0.00399
84	2011-12-30	3.08	-2.11	-3.07	0.478	0.00438
85	2012-01-31	3.14	-2.14	-3.12	0.439	0.00443
86	2012-02-29	3.22	-2.25	-2.53	0.400	0.00425
87	2012-03-30	3.19	-2.26	-2.00	0.418	0.00269
88	2012-04-30	3.19	-2.02	-1.88	0.346	0.00331
89	2012-05-31	2.92	-1.92	-2.24	0.331	0.00376
90	2012-06-29	2.99	-2.04	-2.26	0.325	0.00408
91	2012-07-31	2.91	-1.88	-2.17	0.308	0.00301
92	2012-08-31	2.95	-1.86	-2.14	0.326	0.00335
93	2012-09-28	2.94	-1.88	-2.41	0.343	0.00299
94	2012-10-31	2.99	-1.93	-2.41	0.357	0.00343
95	2012-11-30	2.90	-1.85	-2.32	0.340	0.00331
96	2012-12-31	2.93	-1.94	-1.83	0.325	0.00332
97	2013-01-31	3.12	-2.15	-1.93	0.371	0.00310
98	2013-02-28	3.16	-2.14	-2.96	0.403	0.00349
99	2013-03-29	3.16	-2.14	-2.96	0.403	0.00349
100	2013-04-30	3.11	-1.99	-3.22	0.342	0.00368
101	2013-05-31	3.19	-2.09	-2.74	0.431	0.00265
102	2013-06-28	3.33	-2.20	-2.91	0.589	0.00204
103	2013-07-31	3.47	-2.35	-3.23	0.581	0.00176
104	2013-08-30	3.55	-2.37	-3.60	0.689	0.00185
105	2013-09-30	3.58	-2.50	-3.32	0.595	0.00185
106	2013-10-31	3.56	-2.49	-3.58	0.559	0.00127
107	2013-11-29	3.73	-2.59	-4.15	0.599	0.00200
108	2013-12-31	3.74	-2.51	-4.57	0.758	0.00250
109	2014-01-31	3.48	-2.36	-4.05	0.587	0.00151
110	2014-02-28	3.53	-2.50	-3.68	0.531	0.00192
111	2014-03-31	3.54	-2.49	-3.55	0.553	0.00183
112	2014-04-30	3.49	-2.35	-3.71	0.560	0.00140
113	2014-05-30	3.35	-2.27	-3.38	0.508	0.00147
114	2014-06-30	3.36	-2.31	-3.07	0.484	0.00175
115	2014-07-31	3.23	-2.15	-2.96	0.487	0.00136
116	2014-08-29	3.11	-2.11	-2.38	0.402	0.00107
117	2014-09-30	3.15	-2.18	-2.42	0.470	0.000896
118	2014-10-31	3.06	-2.00	-2.84	0.510	0.000963
119	2014-11-28	2.94	-1.97	-2.50	0.423	0.00165
120	2014-12-31	2.84	-1.85	-2.37	0.418	0.00165
121	2015-01-30	2.46	-1.74	-3.68	0.425	0.00170
122	2015-02-27	2.59	-1.87	-3.64	0.407	0.00310
123	2015-03-31	2.64	-1.96	-3.53	0.417	0.00317
124	2015-04-30	2.80	-2.01	-3.30	0.436	0.00279
125	2015-05-29	2.83	-2.01	-3.88	0.490	0.00387
126	2015-06-30	2.98	-2.17	-4.49	0.527	0.00548
127	2015-07-31	2.81	-2.25	-4.05	0.467	0.00424
128	2015-08-31	2.98	-2.48	-3.80	0.426	0.00500
129	2015-09-30	2.98	-2.45	-3.46	0.390	0.00443
130	2015-10-30	3.06	-2.53	-3.40	0.404	0.00490
131	2015-11-30	2.99	-2.42	-3.19	0.409	0.00436
132	2015-12-31	2.87	-2.24	-3.79	0.428	0.00517
133	2016-01-29	2.81	-2.22	-3.86	0.392	0.00522
134	2016-02-29	2.74	-2.17	-3.37	0.357	0.00552
135	2016-03-31	2.77	-2.20	-3.31	0.360	0.00691
136	2016-04-29	2.71	-2.07	-2.90	0.404	0.00545
137	2016-05-31	2.68	-2.03	-3.05	0.351	0.00673
138	2016-06-30	2.44	-1.84	-3.05	0.330	0.00772
139	2016-07-29	2.31	-1.67	-2.83	0.330	0.00615
140	2016-08-31	2.29	-1.67	-2.58	0.314	0.00596
141	2016-09-30	2.35	-1.73	-2.88	0.327	0.00530
142	2016-10-31	2.51	-1.93	-2.85	0.358	0.00480

143	2016-11-30	2.74	-2.19	-2.44	0.416	0.00355
144	2016-12-30	2.90	-2.39	-2.23	0.408	0.00435
145	2017-01-31	3.06	-2.57	-2.39	0.397	0.00434
146	2017-02-28	3.05	-2.53	-2.40	0.356	0.00449
147	2017-03-31	2.95	-2.38	-2.38	0.377	0.00341
148	2017-04-28	2.83	-2.25	-2.34	0.336	0.00348
149	2017-05-31	2.76	-2.18	-2.26	0.301	0.00371
150	2017-06-30	2.53	-1.80	-0.267	0.252	0.00166
151	2017-07-31	2.86	-2.05	-0.00577	0.255	0.00362
152	2017-08-31	2.68	-1.81	-0.0262	0.205	0.00540
153	2017-09-29	2.83	-1.75	-0.00823	0.233	0.00379
154	2017-10-31	2.66	-1.66	-0.0208	0.228	0.00397
155	2017-11-30	2.50	-1.49	-0.000806	0.258	0.00472
156	2017-12-29	2.34	-1.23	-0.0000396	0.543	0.00535
157	2018-01-31	2.44	-1.32	-0.0000200	0.877	0.00119
158	2018-02-28	2.45	-1.32	-0.0000348	0.705	0.00286
159	2018-03-30	2.45	-1.32	-0.0000348	0.705	0.00286
160	2018-04-30	2.44	-1.35	-0.000175	1.02	0.00130
161	2018-05-31	2.30	-1.18	-0.0000155	1.45	0.000495
162	2018-06-29	2.22	-1.19	0.000466	1.99	0.000487
163	2018-07-31	2.37	-1.14	-0.0000692	1.85	0.000643
164	2018-08-31	2.25	-0.985	0.00000256	2.86	0.000682
165	2018-09-28	2.44	-1.19	0.000556	2.73	0.000586
166	2018-10-31	2.52	-1.10	0.000000842	3.05	0.000548
167	2018-11-30	2.33	-0.842	-0.00000538	2.95	0.00340
168	2018-12-31	2.45	-0.679	-0.379	0.164	0.00237
169	2019-01-31	2.44	-0.686	-0.852	0.227	0.00200
170	2019-02-28	2.45	-0.712	-0.728	0.256	0.00101
171	2019-03-29	2.18	-0.435	-1.44	0.311	0.000979
172	2019-04-30	2.28	-0.500	-1.59	0.346	0.00132
173	2019-05-31	2.04	-0.252	-1.83	0.354	0.00116
174	2019-06-28	1.91	-0.131	-1.48	0.333	0.00141
175	2019-07-31	1.95	-0.216	-1.26	0.262	0.000964
176	2019-08-30	1.74	-0.0182	-1.90	0.272	0.00126
177	2019-09-30	1.78	-0.00684	-1.32	0.231	0.00158
178	2019-10-31	1.78	-0.0211	-1.16	0.285	0.000955
179	2019-11-29	1.66	0.0730	-0.716	0.269	0.000771
180	2019-12-31	1.82	-0.0937	-0.270	0.324	0.000577
181	2020-01-31	1.59	0.135	-1.25	0.298	0.000488
182	2020-02-28	1.45	0.0885	-1.38	0.409	0.000469
183	2020-03-31	1.99	-1.69	-1.16	0.174	0.00389
184	2020-04-30	1.81	-1.52	-1.91	0.208	0.00277
185	2020-05-29	1.79	-1.53	-1.82	0.208	0.00257
186	2020-06-30	1.51	-1.25	-1.51	0.230	0.00140
187	2020-07-31	1.43	-1.23	-1.41	0.209	0.000653
188	2020-08-31	1.70	-1.52	-1.55	0.236	0.000749
189	2020-09-30	1.69	-1.53	-1.52	0.210	0.000482
190	2020-10-30	1.85	-1.73	-1.40	0.203	0.000441
191	2020-11-30	1.62	-1.48	-1.11	0.231	0.000217
192	2020-12-31	1.67	-1.57	-1.34	0.253	0.000306
193	2021-01-29	1.95	-1.85	-2.04	0.338	0.000392
194	2021-02-26	2.04	-1.85	-2.20	0.624	0.000474
195	2021-03-31	2.33	-2.13	-3.12	0.702	0.00102
196	2021-04-30	2.47	-2.29	-2.98	0.592	0.000836
197	2021-05-31	2.43	-2.30	-2.57	0.544	0.000343

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