Theory of Statistics Likelihood Assigment

Sean Soutar STRSEA001[a], Fabio Fehr FHRFAB001[b]

[a] *UCT Statistics Honours, Cape Town, South Africa*

[b] *UCT Statistics Honours, Cape Town, South Africa*

Abstract

This project will explore the Accidents dataset. Various count models such as Poisson, Negative Binomial, Mixture of 2 Poissons and Zero Inflated Poisson models will be applied to the data. The model with the strongest support will be chosen and discussed. Profile likelihoods and confidence intervals for the parameters will be found and displayed for the chosen model.

Keywords: Likelihood, Overdispersion, Count data

JEL classification

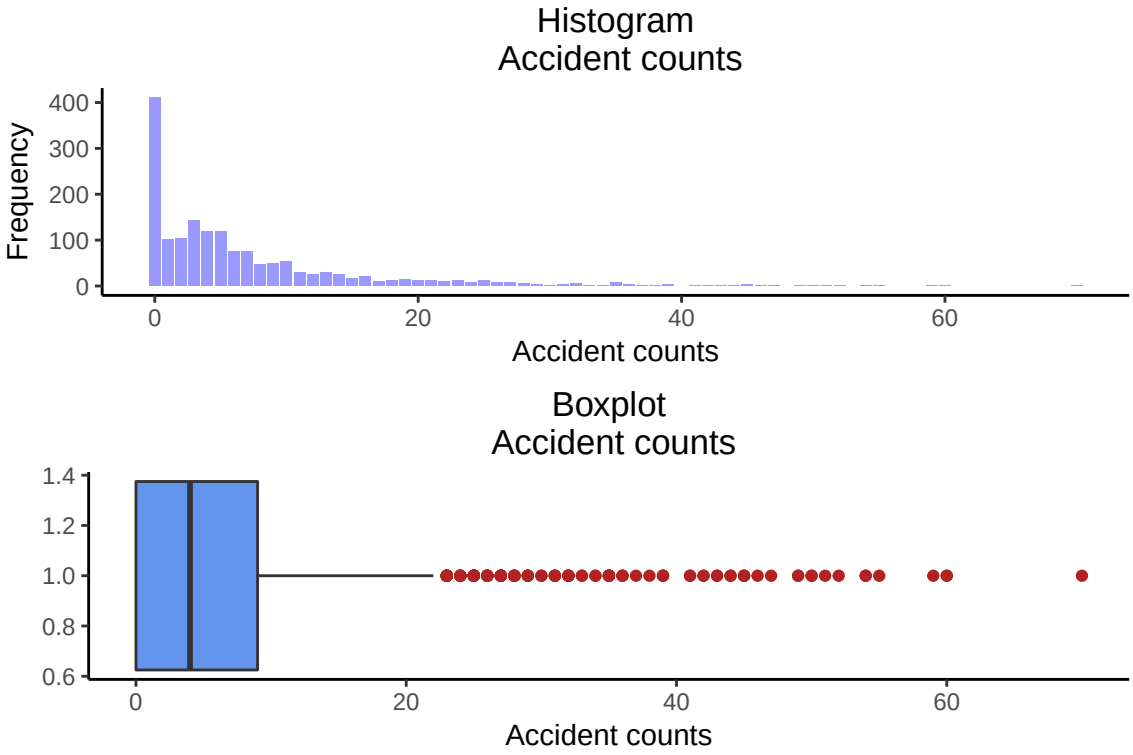
1. Introduction

This assignment is an explorative report on a dataset containing the number of accidents on two-lane (same direction) road segments in Cape Town over a five-year period. The segments differ in length between 0.2 and 7.2 km. The aim of the report is to find and fit a model which accurately describes the accident dataset. This report will first explore the data then fit different count distributions. The best fitting model will then be chosen. Once a model has been selected, the profile likelihood and confidence intervals for the model parameters will be calculated. The results will then be analysed critically and conclusions will be made whilst suggesting further areas of investigation.

1.1. Exploratory data analysis

To better understand our data this report shall explore the following properties; Firstly we examine the type of data within the accidents dataset and discuss whether our data is discrete ordinal or continuous. After the symmetry of the data and bounds will be discussed. This leads the exploration to outliers and extreme values.

Email addresses: sean.soutar@gmail.com (Sean Soutar STRSEA001), FHRFAB001@myuct.ac.za (Fabio Fehr FHRFAB001)



1.1.1. Data type

There are many instances where zero accidents were observed. This accounts for approximately 25.18% of the data. This suggests that the zero-inflated Poisson should be considered as this proportion is much higher than what would be expected of a regular Poisson distribution. The accident counts are discrete random variables. Specifically, they are discrete positive definite random variables on the interval $R \in \{0; +\infty\}$. Summary statistics of the data are shown below.

Table 1.1: Summary statisitics

Mean	Variance	Median
6.9179	85.0858	4

In the Poisson distribution, the mean should equal the variance. The sample variance far exceeds the sample mean. This indicates overdispersion if the Poisson distribution were to be used. This is when the observations are more variable than what would be expected. This suggests that alternative count models and mixture distributions should be used.

1.1.2. Symmetry

This property is visually seen in the histogram and boxplot. All counts are greater than zero with a median value of 4 accidents. The largest accident observed is 70 accidents. THe histogram shows that the data are non-symetrical and positively skewed which is usually expected of count data.

1.1.3. Outliers

From the boxplot it clear that many outliers exist. One common method of classifying a point as an extreme value or outlier is if it falls more than 1.5 times the inner-quartile range above the upper quartile. The proportion of outliers within our data set amount to 15.26%.

2. Methods

2.1. Model Formulation

The data is discrete, asymmetric, positive definite, contains many positive outliers and many zeros. This would suggest distributions such as Poisson, Negative Binomial, mixture distribution of 2 Poissons and a zero inflated Poisson. For all optimisation we shall constrain the bounds in order to ensure valid regions for our parameters. The best fitting distribution will be reparametised to aid interpretation and make the likelihood overal more quadratic.

2.2. Akiake Information Coefficient (AIC)

The AIC metric can be used to compare models from different families of distributions. They can be used to compare relative goodness of fit between models. Overfitting the model with too many parameter is penalised by an increased AIC, thus a lower AIC value indicates a better fitting model.

$$\begin{aligned} \text{AIC} &= -2l(\hat{\theta}) + 2\text{p} \\ p &= \text{Number of estimated parameters} \end{aligned}$$

2.3. Bayesian Information Criterion (BIC)

This metric, like AIC, also compares relative goodness of fit between models but penalises complex models when the sample size is large. BIC tends to produce simpler models than AIC.

$$\text{BIC} = -2l(\hat{\theta}) + log(n)$$
$$n = \text{Number of observations}$$

2.4. Model Distributions

2.4.1. Poisson

The Poisson is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time if these events occur with a known constant rate and independently of the time since the last event.This is typically used in count data where your mean and variance are equal.

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x \in \{0, 1, \ldots, \infty\}, \lambda > 0$$

$$L(\lambda|x) = \prod_{i=1}^{n} p(x_i)$$

$$L(\lambda|x) = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$l(\lambda|x) = -n\lambda + \left(\sum_{i=1}^{n} x_i\right) \ln \lambda - \sum_{i=1}^{n} \ln(x_i!)$$

The Poisson is characterised by the λ parameter which denotes the population average rate of event occurence. In this context it would be the average number of accidents per unit time frame. Due to the variance being far higher than our mean in our data we would expect that a Poisson distribution would not fit very well.

Table 2.1: Poisson MLE's & information metrics

$\hat{\lambda}$	AIC	BIC
6.9179	20263.64	20269.04

2.4.2. Negative Binomial

The Negative Binomial distribution is a discrete probability function of the number of successes in a sequence of independant and identitically distributed Bernoulli trials. The parameters p & r measure the probability of success in an individual trial and the number of successes until r failures occure. The mean in a negative binomial is defined as $m = \frac{pr}{1-p}$ thus we can reparamatize the distribution in terms of the mean parameter m and shape parameter r such that $p = \frac{m}{m+r}$ and $1 - p = \frac{r}{m+r}$. We can also manipulate the constant term as follows.

$$\binom{x+r-1}{x} = \frac{(x+r-1)(x+r-2)\cdots(r)}{x!} = \frac{\Gamma(x+r)}{x!\,\Gamma(r)}$$

This gives us the negative binomial in the form

$$p(x) = \frac{\Gamma(r+x)}{x!\,\Gamma(r)}\left(\frac{m}{r+m}\right)^x\left(\frac{r}{r+m}\right)^r \quad \text{for } x = 0, 1, 2, \ldots$$

$$L(m, r|x) = \prod_{i=1}^{n} p(x_i)$$

$$L(m, r|x) = [\frac{1}{\Gamma(r)}]^n \prod_{i=1}^{n} \frac{\Gamma(r+x_i)}{x_i!}(\frac{m}{r+m})^{\Sigma_{i=1}^n x_i}(\frac{r}{r+m})^{nr}$$

$$l(m, r|x) = -n\ln[\Gamma(r)] + \sum_{i=1}^{n}\ln(\Gamma(r+x_i)) - \sum_{i=1}^{n}\ln x_i! + \sum_{i=1}^{n} x_i \ln(\frac{m}{r+m}) + nr\ln(\frac{r}{r+m})$$

It is important to note that the variance of a Negative Binomial under this parameterisation is $m + \frac{m^2}{r}$ and always larger than our mean m. This would suggest a better fit than our Poisson model. Shape parameters are often regarded as nuisance parameters and do not play a meaningful role in maximising likelihood. Therefore, since we desire no under or over dispersion, we can express the shape parameter as a function of the mean parameter to be esimated and the sample variance.

$$Var(x) = m + \frac{m^2}{r}$$

$$r = \frac{m^2}{Var(x) - m}$$

$$\hat{r} = \frac{\hat{m}^2}{S^2 - \hat{m}}$$

Table 2.2: Negative Binomial MLE's & information metrics

Mean $\hat{m}$	Shape $\hat{r}$	AIC	BIC
6.8108	0.5926	9626.92	9630.31

2.4.3. Mixture of 2 Poissons

A finite mixture distributon of two Poisson variables will now be explored. A possible reason for the overdispersion is that the data are from two separate Poisson distributions. Since it is not known from which distribution that any given data point is from, presuming that the two distribution mixture is appropriate, an additional mixing proportion parameter p needs to be estimated.

$$p(x|\lambda_1, \lambda_2, p) = p\frac{e^{-\lambda_1}\lambda_1^x}{x!} + (1-p)\frac{e^{-\lambda_2}\lambda_2^x}{x!}, \quad x \in \{0, 1, \ldots, \infty\}, \lambda_1, \lambda_2, p > 0$$

$$L(\lambda_1, \lambda_2, p|x) = \prod_{i=1}^{n} p(x_i)$$

$$L(\lambda_1, \lambda_2, p|x) = \prod_{i=1}^{n} p\frac{e^{-\lambda_1}\lambda_1^{x_i}}{x_i!} + (1-p)\frac{e^{-\lambda_2}\lambda_2^{x_i}}{x_i!}$$

$$l(\lambda_1, \lambda_2, p|x) \sum_{i=1}^{n} \ln[p\frac{e^{-\lambda_1}\lambda_1^{x_i}}{x_i!} + (1-p)\frac{e^{-\lambda_2}\lambda_2^{x_i}}{x_i!}]$$

The parameters λ_1 and λ_2 refer to the average number of road accidents per road stretch for the first

and second distribution respectively. The parameter p is the proportion parameter. This represents the probability that a given observation belongs to distribution 1. Therefore, the probability that an observation belongs to distribution 2 is the $1 - p$.

Table 2.3: Poisson-Poisson MLE's & information metrics

$\hat{\lambda_1}$	$\hat{\lambda_2}$	Proportion p	AIC	BIC
19.3809	2.8407	0.2465	12075.12	12076.52

2.4.4. Zero inflated Poisson

The Zero Inflated Poisson is also a finite mixture distribution. This model supposes that the data can come from two distributions. The one is a Zero Process and the other is a Poisson process that can only take on non-zero values. This model is useful if there are many zeroes in the data. This was seen to be the case as discussed in 1.1.1. The Zero Inflated Poisson is a piecewise defined distribution with different mass functions for predicting the probability that a given observation will be zero rather than non-zero.

$$p(x_i = 0) = \pi + (1 - \pi)e^{-\lambda}$$

$$p(x_i \neq 0) = (1 - \pi)\frac{\lambda^{x_i}e^{-\lambda}}{x_i!}, \qquad x_i \geq 1$$

$$L(\lambda, \pi|x) = L(\lambda, \pi|x = 0)L(\lambda, \pi|x \neq 0)$$

An indicator variable I is defined.

$$I = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

$$L(\lambda, \pi|x) = \prod_{i=1}^{n} p(x_i = 0)^{1-I} p(x_i \neq 0)^I$$

$$L(\lambda, \pi|x) = \prod_{i=1}^{n} [\pi + (1-\pi)e^{-\lambda}]^{1-I} [(1-\pi)\frac{\lambda^{x_i} e^{-\lambda}}{x_i!}]^I$$

$$l(\lambda, \pi|x) = \sum_{i=1}^{n} \ln[(1-I)[\pi + (1-\pi)e^{-\lambda}] + I[(1-\pi)\frac{\lambda^{x_i} e^{-\lambda}}{x_i!}]]$$

The parameter λ is the average rate of accidents per road stretch. The parameter π is the probability of additional zeroes observed in our data. This mixture distribution has a mean of $(1-\pi)\lambda$ and variance $(1-\pi)\lambda(1+\pi\lambda)$. It is clear that the variation in this distribution is always greater than the average thus a good potential model to consider.

Table 2.4: Zero inflated Poisson MLE's & information metrics

$\hat{\lambda}$	$\hat{\pi}$	AIC	BIC
9.2456	0.2518	15556.16	15559.56

2.5. Model Selection

The AIC and BIC results for each model are summarised below.

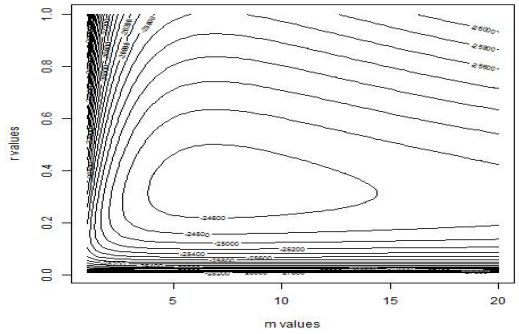
Table 2.5: Models fitted to accident data

Model	AIC	BIC
Poisson	20263.64	20269.04
Negative Binomial	**9626.92**	**9630.31**
Poisson Mixture	12075.12	12076.52
Zero Inflated Poisson	15556.16	15559.56

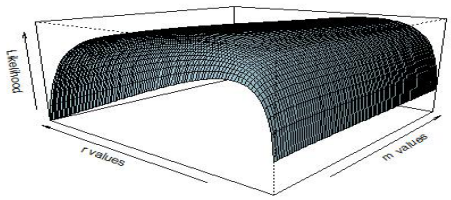
The Negative Binomial has the lowest AIC and BIC value at 9626.92 and 9630.31 respectively. This implies that the Negative Binomial is the best fitting model when compared to the others. The goodness of fit will now be assessed further. We start by examining the likelihood surface. Since the r shape parameter is a function of the m parameter for each m there is an exact r meaning that these parameters are highly correlated. We can see from the contour plots below that the surface of the

likelihood function peaks at the intersection of the MLE's of the parameters and dips off steeply at the zero values.

Negative Binomial likelihood surface

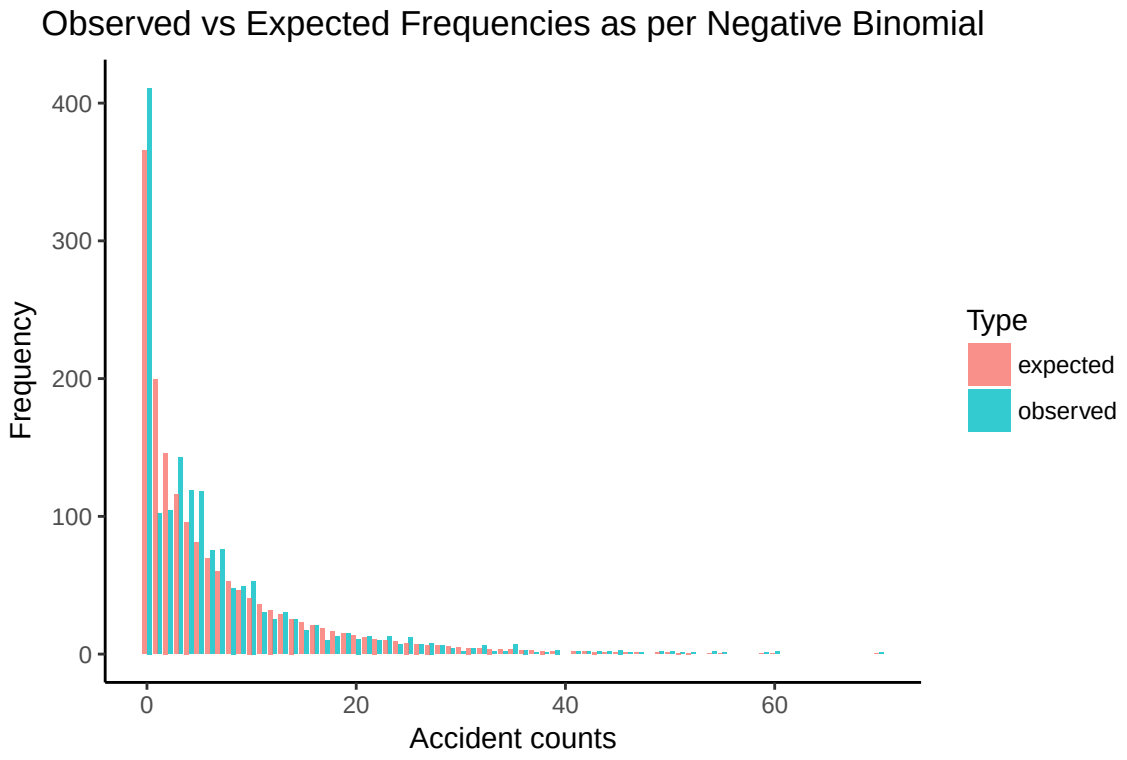


Negative Binomial likelihood surface



2.5.1. Goodness of Fit

We will now access the goodness of fit by first comparing the observed accident counts against the expected expected counts given a negative binomial distribution. This will then be used in the Pearson's Chi-Squared test and finally plotted using a quantile-quantile plot for the Negative Binomial.



As we can see the observed and expected accident counts seem to fit nicely except for first 5 counts where there is some dissanance. This could be due to the excess of zeros in the data. We will now construct a Pearson's Chi-Squared test with the following hypotheses.

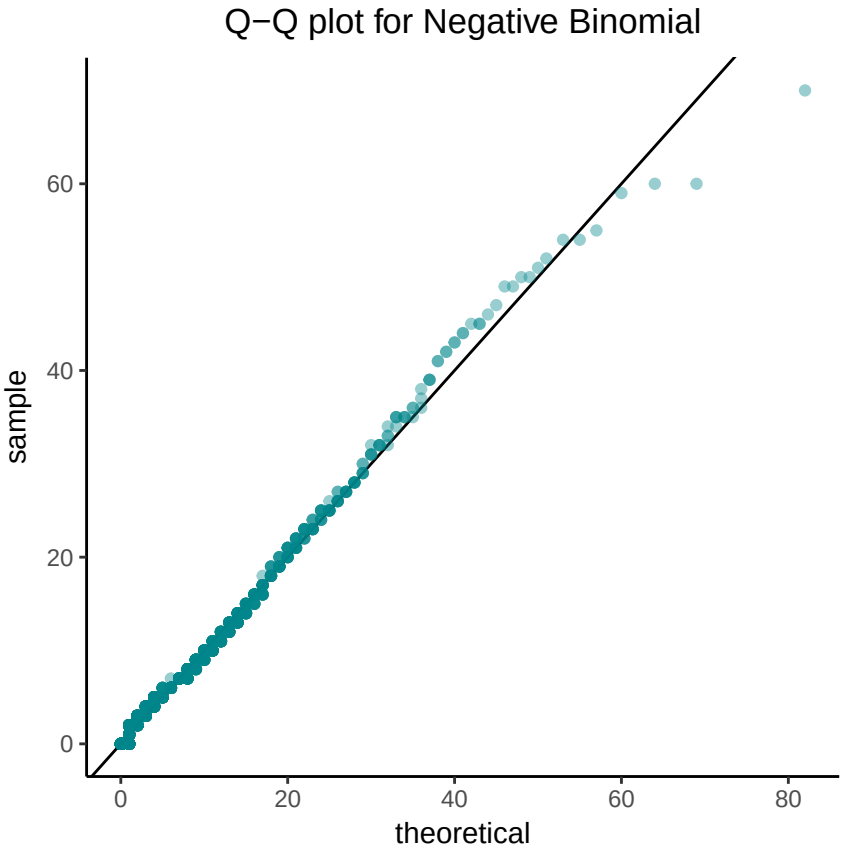
H_0 :The data is consistent with the Negative Binomial distribution.

H_1 :The data is NOT consistent with the Negative Binomial distribution.

Table 2.6: Pearson's χ^2 Goodness of Fit test

χ^2 Statistic	χ^2 DoF	P-Value
1456	1430	0.31

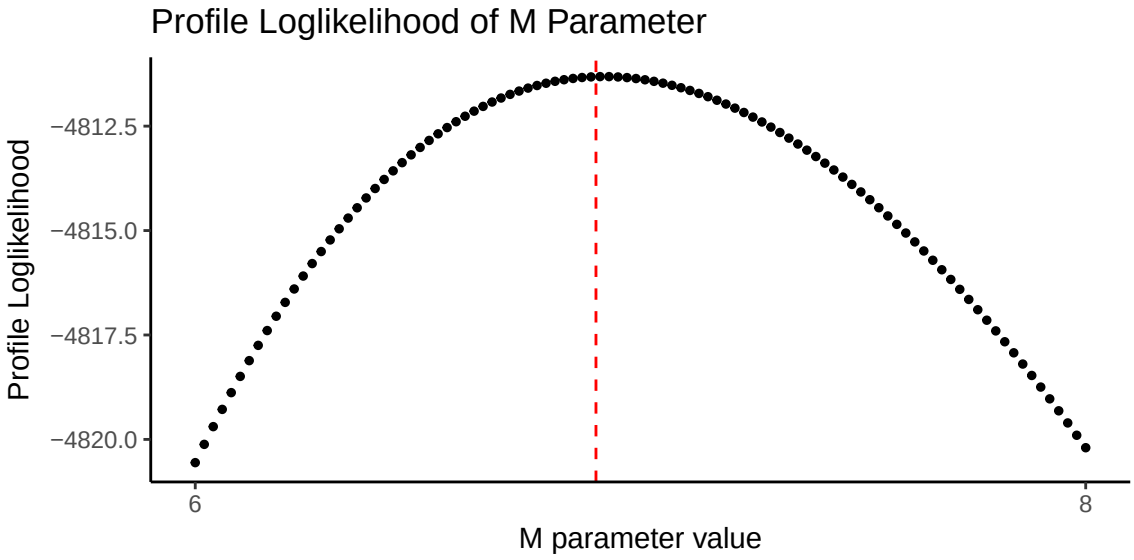
The goodness of fit test results in a significantly large p-value meaning that there is evidence to conclude that the data is consistant with the Negative Binomial distribution. We will now compare our sample distribution against a Negative Binomial in the form a a Quantile-Quantile plot.



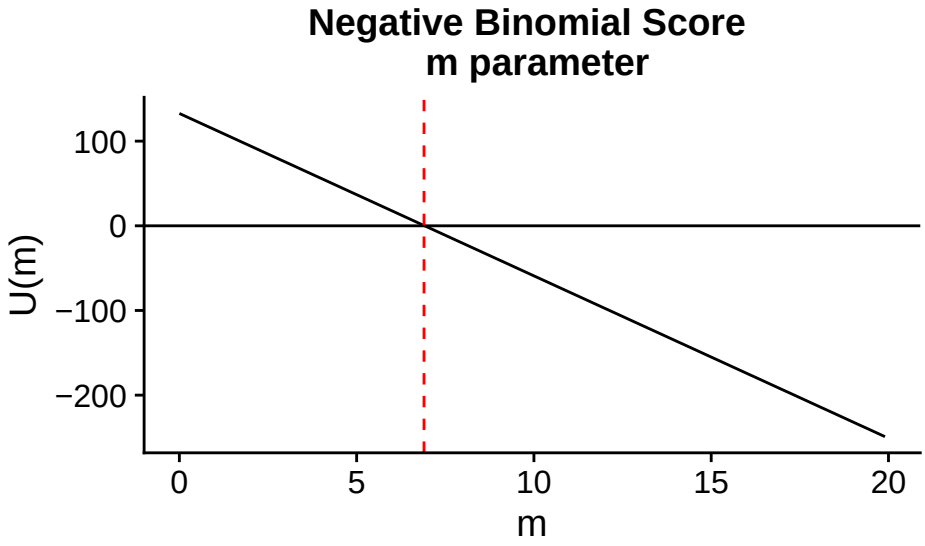
The plot is in agreement with the prior too goodness of fit evaluations thus it is reasonable to assume that our data follows a Negative Binomial distribution. We will now explore the parameters of our distribution.

2.6. Profile Likelihood & Confidence Intervals

The profile likelihood is a techinque used to estimate the likelihood function of a single parameter when multiple parameters are estimated simultaenously. The profile likelihoods for the m and r parameters are calculated as follows:

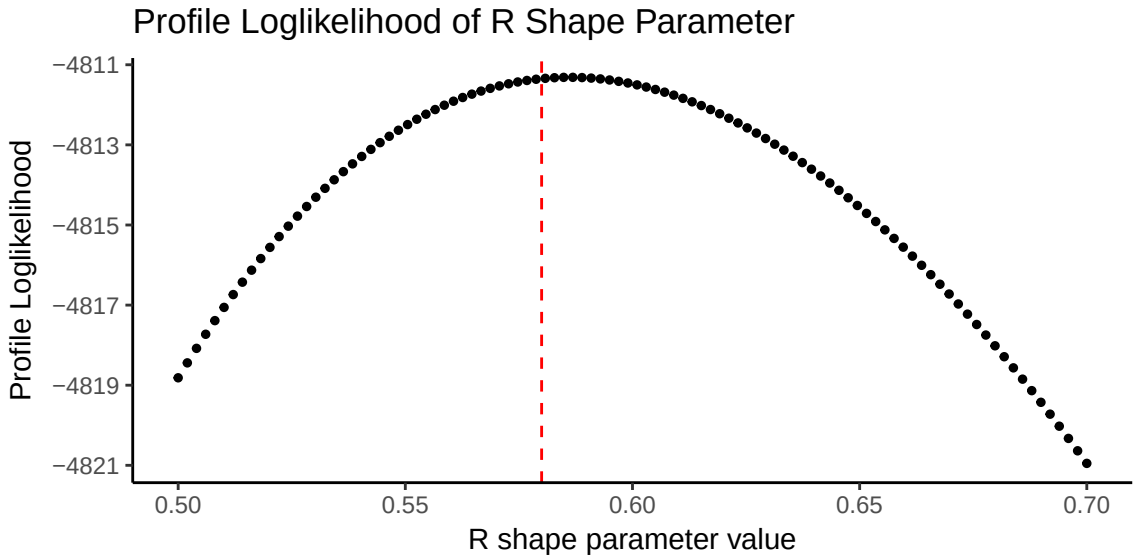


It can be seen that the profile likelihood for the m parameter is somewhat quadratic around the MLE. Since the score function can be solved for analytically we can plot the score function and examine it for linearity.



As we can see, our score function for m is linear meaning that our likelihood must be quadratic,

further implying that the asymptotic intervals relying on quadracity could be used.



We are unable to solve for the score function for r exactly, but from the profile likelihood around the MLE for the r shape parameter, it is clearly not quadratic. Therefore it would be inappropriate to use a Wilks Likelihood interval or Wald interval for the shape parameter. The direct likelihood interval should be used.

Since our shape parameter is a function of our mean paramter, $\hat{r} = \dfrac{\hat{m}^2}{S^2 - \hat{m}}$ we can solve for our intervals by using the invariance property (θ_L, θ_U) is a likelihood interval for θ such that $(g(\theta_L), g(\theta_U))$. This property does not hold for asymptotic intervals.

2.6.1. Wald Interval

We can attain an interval for out Maximum Likelihood Estimates if we assume they have asymptotic normal distribution. This means $n \to \infty$ the estimates will be approximately normal with the following parameters.

$$\hat{m} \dot{\sim} N(m, I(\hat{m})^{-1})$$
$$\hat{r} \dot{\sim} N(r, I(\hat{r})^{-1})$$

This means that we can form asymptotic confidence intervals for m and r know as a Wald interval. The advantages of this is it is simple to calculate provided you have a standard error estimate. The standard error can be calulated numerically by finding the square root of the inverse of the Hessian

matrix.

Assuming the asymptotic approximation holds we can obtain the following 95% confidence intervals for our parameters, meaning as we sampled infinity from our data we would expect to see our true parameter within 95% of our intervals obtained.

Table 2.7: Wald intervals

Parameter	Lower bound	Upper bound
m	6.5845624	7.0370376
r	0.5643434	0.6208566

2.6.2. Wilks Likelihood ratio

The Wilks Likelihood Ratio statistic is based on the deviance and is used to compare a certain parameter against the Maximum Likelihood estimate for that parameter. If the data come from a Normal distribution the following result is exactly true, but if the likelihood is quadratic the following is asymptotically true.

$$W = -2\log R(\theta) \sim \chi^2_p$$

Table 2.8: Wilks Likelihood intervals

Parameter	Lower bound	Upper bound
m	6.5861512	7.0385532
r	0.5405427	0.6350722

2.6.3. Pure Likelihood Interval

The likelihood interval can be found as $R(\theta) > \gamma^p$, where γ can be based on a χ^2 approximation and p is the dimension of θ. Equivalently we can use the deviance:

$$-2[\ell(\theta_p) - \ell(\hat{\theta_p})] \sim \chi^2_p.$$

We solve for points of θ where the deviance equals the 95th percentile of χ^2_p.

Otherwise, for the case when the loglikelihood is not quadratic, the γ value simply represents the lower bound for $\ell(\theta_p)$ values such that they are at leastγ times as likely as the MLE estimates in $\ell(\hat{\theta_p})$.

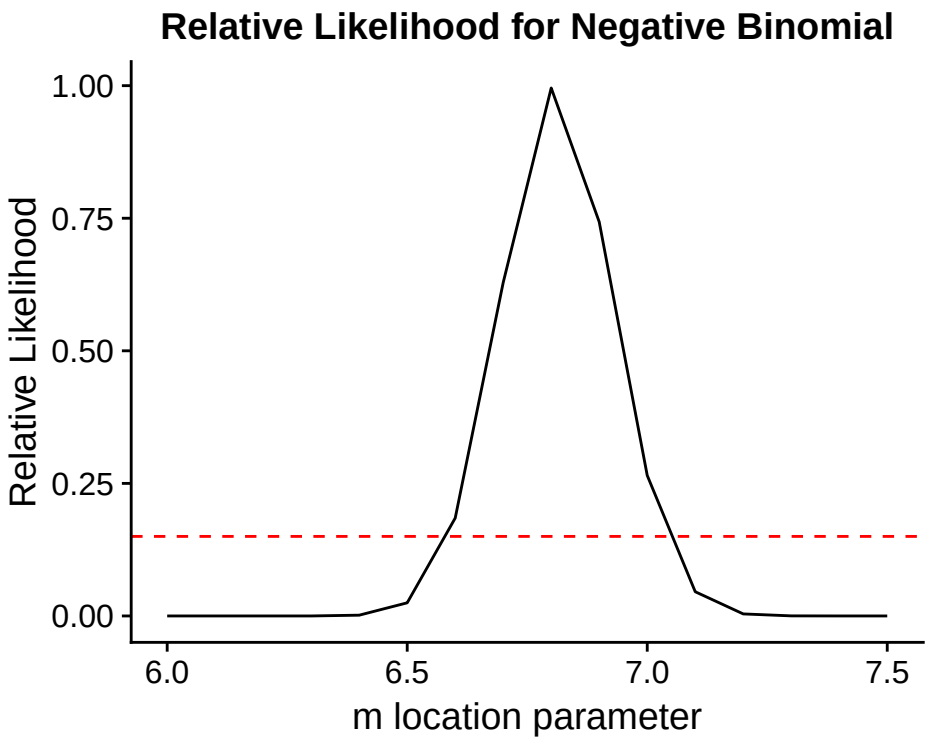


Table 2.9: interval for m parameter

Lower bound	Upper bound
6.587527	7.03714

As we can see the from the likelihood interval for m, all values within the interval [6.5875,7.0371] is at least 15% as likely as $\hat{m} = 6.8108$. Since this interval is not based on any underlying probability distribution of $\hat{m}$ thus it does not have any confidence level attached to it.

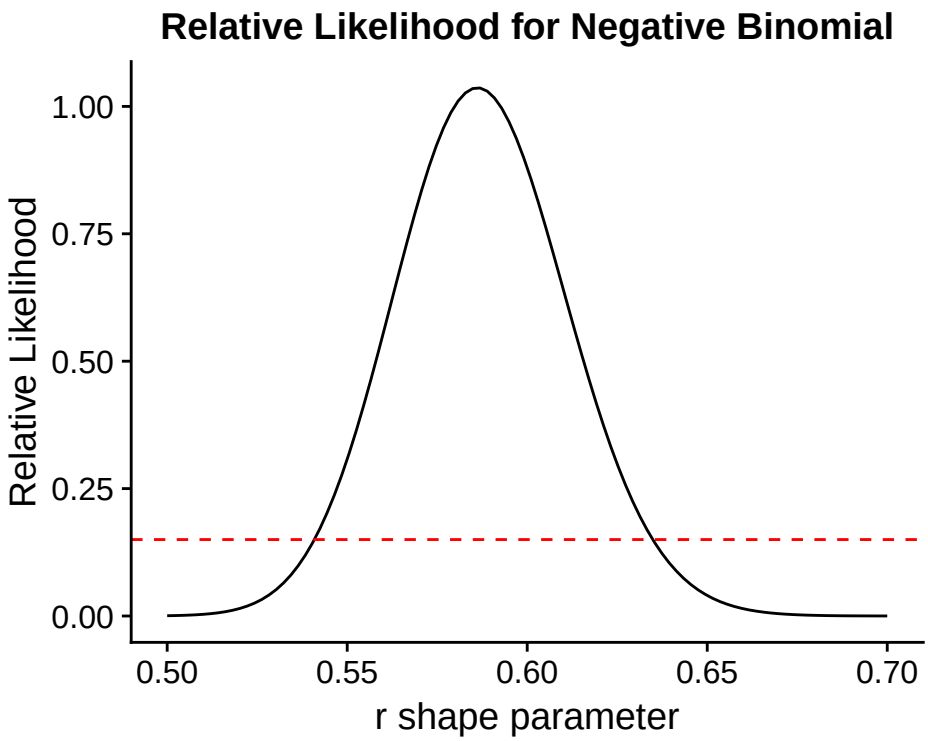


Table 2.10: interval for r parameter

Lower bound	Upper bound
0.5407778	0.6348027

The pure likelihood interval for r requires no asymptotic assumptions or underlying probability distribution, it is entirely based on the data and thus will be our best estimate for the shape parameter. All values within the interval [0.5408,0.6348] is at least 15% as likely as $\hat{r}$ =0.5926.

3. Results

We now compare all the results attained from our intervals and compare them. The direct intervals have been displayed on a relatively likelihood plot of each parameter respectively. The other intervals shall be ommited from the visualisation due to high similarity and thus not easily visible.

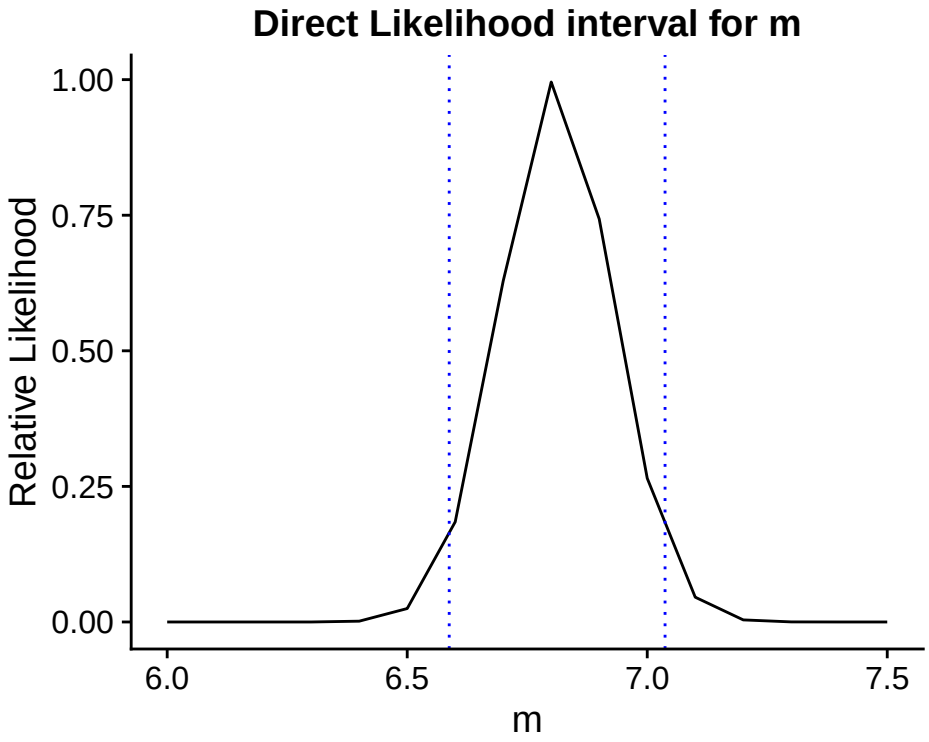


Table 3.1: Intervals for m parameter

Intervals	Lower Bound	Upper Bound
Wald Interval	6.584562	6.584562
Wilks Likelihood	6.586151	7.038553
Direct Likelihood 15%	6.587527	7.037140

Our intervals for the m parameter are very similar with the widest interval being the wald interval and narrowest being the pure likelihood interval. This means our asymptotic confidence interval for m is [6.5846,0.6209] meaning that if more data was collected and each sample a confidence interval was collected, 95% of these intevals would contain our true parameter m. However, this does not give any cetainty on our particular interval actually containing the true parameter.

The narrowest interval, marginally, is our direct likelihood interval [6.5875,0.6348] which is not based on any underlying probability distribution but only that the interval for m is 15% as likely as the MLE. The two intervals are in agreement.

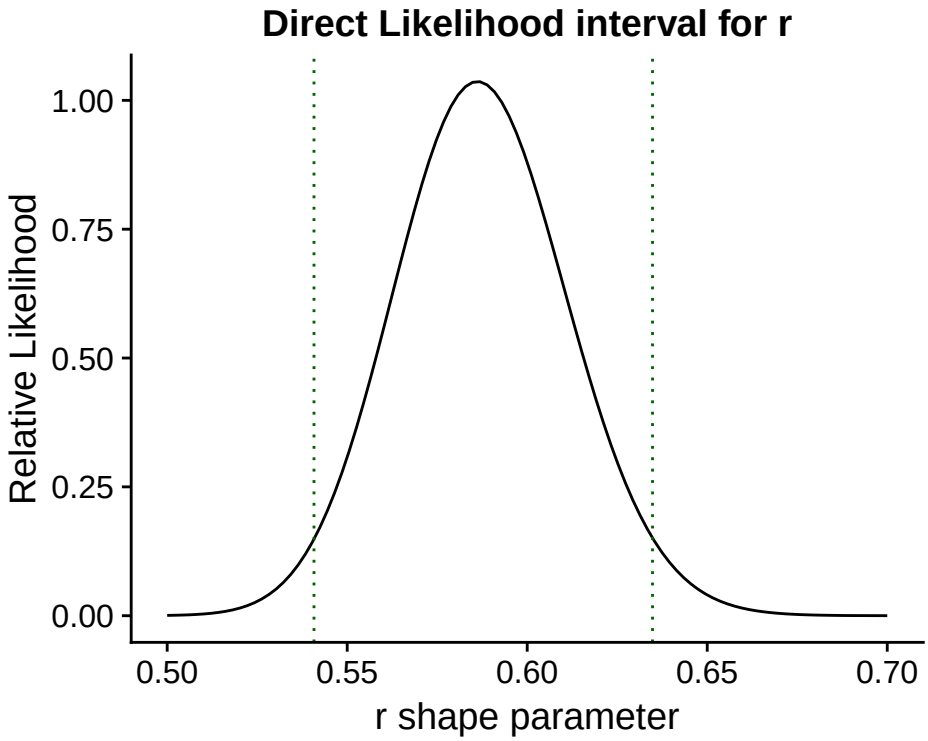


Table 3.2: Intervals for r parameter

Intervals	Lower Bound	Upper Bound
Wald Interval	0.5643434	0.6208566
Wilks Likelihood	0.5405427	0.6350722
Direct Likelihood 15%	0.5407778	0.6348027

Again we notice that the all intervals are very similar, but with marginal difference such that the Wald interval is the narrowest and the largest is the direct likelihood interval.

4. Conclusion

From our results we can see that all intervals for m are similar and the intervals for r are also very similar. In both instances the pure likelihood interval is appropriate and often the most robust as it requires the least assumptions, but since our parameters are quadratic around the MLEs, asymptotic intervals give good approximations.

To further improve these models one could gather more data such that regression models could be used. This may yield more accurate results than simply trying to fit a distribution to the data.

One could also explore different combinations of mixure models such as negative binomial, negative binomial or zero inflated negative binomial. These could potentially fit our data better, but The AIC, goodness of fit and asymptotic normality should be considered.

Another consideration regarding the data would be to identify the length of the roadway for each data point. The reason why the Poisson models do not perform as well, might be because it cannot accurately determine the rate per distance unit.