Formalizing Bombieri Vinogradov

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0.1 Assumptions

0.2 Basic Definitions

Definition 1.

$$\Delta_f(x;q,a) := \sum_{n\leq x,\ n\equiv a \pmod q} f(n) \ - \frac{1}{\varphi(q)} \sum_{n\leq x,(n,q)=1} f(n)$$

for $x \geq 1$, $q \in \mathbb{N}$

Theorem 2.

$$\Delta_f(y;q,a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q, \chi\neq\chi_0} \bar{\chi}(a) \sum_{n\leq y} f(n)\chi(n)$$

Proof. □

Definition 3. $\Lambda^\sharp = \mu_{\leq V} * \log - (\Lambda_{\leq U} * \mu_{\leq V}) * 1$

Definition 4. $\Lambda^\flat = (\Lambda_{>U} * 1) * \mu_{>V}$

Definition 5. $\Lambda_{\leq U} = 1_{\leq U} \cdot \Lambda$

Theorem 6. *Decompose* $\Lambda = \Lambda^\sharp + \Lambda^\flat + \Lambda_{\leq U}$

Proof. □

Definition 7. For $f : \mathbb{N} \to \mathbb{R}$ and $r : \mathbb{N}$ we use f_r to denote $n \mapsto f(n)1_{(n,r)=1}$

0.3 Final Derivations

Theorem 8.

Proof. □

Theorem 9.

Proof. □

Theorem 10.

Proof. See p. 279. Uses combinatorics, an estimate on $\sum_{p\leq x} 1/\log p$ and partial summation. □

Theorem 11. *For each fixed* $A \geq 0$ *we have*

$$\sum_{q\leq Q} \max_{\sqrt{x}\leq y\leq x} \max_{a\in(\mathbb{Z}/q\mathbb{Z})^*} |\Delta_\Lambda(y;q,a)| \ll_A \frac{x}{(\log x)^A}$$

uniformly for $x \geq 2$ *and* $1 \leq Q \leq \sqrt{x}/(\log(x))^{A+3}$

Proof. □

Theorem 12. *For each fixed* $A \geq 0$,

$$\sum_{q\leq Q} \max_{y\leq x} \max_{a\in(\mathbb{Z}/q\mathbb{Z})^*} \left|\pi(y;q,a) - \frac{li(y)}{\varphi(q)}\right| \ll_A \frac{x}{(\log x)^{A+1}}$$

uniformly for all $x \geq 2$ *and* $1 \leq Q \leq \frac{\sqrt{x}}{(\log x)^{A+3}}$.

Proof. Apply theorem 11 and absorb the error terms □