Formalizing Bombieri Vinogradov

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0.1 Bombieri-Vinogradov Theorem

This module contains the formalization of the Bombieri-Vinogradov theorem, a fundamental result in analytic number theory.

0.1.1 Preliminaries

Decomposing the von Mangoldt function into type I and type II functions.

Definition 1.

$$\Delta_f(x;q,a) := \sum_{n\le x,\ n\equiv a \pmod q} f(n) \ - \frac{1}{\varphi(q)} \sum_{n\le x,(n,q)=1} f(n)$$

for $x \ge 1$, $q \in \mathbb{N}$

Theorem 2.

$$\Delta_f(y;q,a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q, \chi\neq\chi_0} \bar{\chi}(a) \sum_{n\le y} f(n)\chi(n)$$

Proof. □

Theorem 3. *Notice:*

$$\Delta_{1_P}(x;q,a) = \pi(x;q,a) - \frac{1}{\varphi(q)} \sum_{p\le x, p\nmid q} 1$$

Proof. □

Theorem 4.

$$\sum_{p\le x, p\nmid q} 1 = li(x) + O(xe^{-c\sqrt{\log x}} + \log q)$$

Proof. □

Theorem 5. *For $x \ge 2$, $q \in \mathbb{N}$ and $a \in (\mathbb{Z}/q\mathbb{Z})^*$ we have*

$$\max_{y\le x} \left|\Delta_{1_P}(y;q,a)\right| \ll \frac{1}{\log x} \left(\max_{\sqrt{x}\le y\le x} |\Delta_\Lambda(y;q,a)| + \sqrt{x} \right)$$

proof. See p. 279. Uses combinatorics, an estimate on $\sum_{p\le x} 1/\log p$ and partial summation.

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Wrapping up

Definition 6. $\Lambda^\sharp = \mu_{\le V} * \log - (\Lambda_{\le U} * \mu_{\le V}) * 1$

Definition 7. $\Lambda^\flat = (\Lambda_{>U} * 1) * \mu_{>V}$

Definition 8. $\Lambda_{\le U} = 1_{\le U} \cdot \Lambda$

Theorem 9. *Decompose $\Lambda = \Lambda^\sharp + \Lambda^\flat + \Lambda_{\le U}$*

Proof. □

Theorem 10 (Bombieri-Vinogradov)**.** *For each fixed $A \geq 0$,*

$$\sum_{q \leq Q} \max_{y \leq x} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \left| \pi(y; q, a) - \frac{li(y)}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^{A+1}}$$

uniformly for all $x \geq 2$ and $1 \leq Q \leq \frac{\sqrt{x}}{(\log x)^{A+3}}$.

Proof. Apply BV_Delta and absorb the error terms □