"I have pledged my honor that I have abided by the Stevens Honor System."
Cindy Zhang

FINAL EXAM

MA 222 Probability and Statistics

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Problem One. A chemical compound contains 3 particles of a reactant and 6 particles of a catalyst. Particles are removed from the compound at random, and after the removed particleis examined, it is returned back to the chemical compound together with an additional particle of the same type. We call this process of removal/return a "reaction".

a. Assume there are two consecutive reactions. Compute the probability that the first and the second removed particle is a reactant.

$$First\ Particle\ being\ a\ Reactant = P(R1) = \frac{x}{n} = \frac{3}{9} = 0.33$$

$$Second\ Particle\ being\ a\ Reactant = P(R2) = \frac{x+1}{n+1} = \frac{3+1}{9+1} = \frac{4}{10} = 0.40$$

$$P(R1) * P(R2) = 0.33 * 0.40 = 0.1333$$

There is a 13.33% chance that the 1st particle and the 2nd particle is a reactant.

b. Compute the probability that at least one of the two removed particles is a reactant.

$$First\ Particle\ being\ a\ Reactant = P(R1) = \frac{x}{n} = \frac{3}{9} = 0.33$$

$$Second\ Particle\ being\ a\ Reactant = P(R2) = \frac{x}{n+1} = \frac{3}{9+1} = \frac{3}{10} = 0.30$$

$$First\ Particle\ NOT\ being\ a\ Reactant = P(R1') = 1 - 0.33 = 0.66$$

$$Second\ Particle\ NOT\ being\ a\ Reactant = P(R2') = 1 - 0.30 = 0.70$$

P(R1') * P(R2') = 0.66 * 0.70 = 0.462

1 - 0.462 = 0.538

There is a 53.8% chance that the 1st particle or the 2nd particle is a reactant.

c. Assume now a third consecutive reaction occurs. Compute the probability that the third removed particle is a catalyst, given that exactly one of the first two is a reactant.

$$First\ Particle\ being\ a\ Reactant = P(R1) = \frac{x}{n} = \frac{3}{9} = 0.33$$

$$Second\ Particle\ being\ a\ Reactant = P(R2) = \frac{x}{n+1} = \frac{3}{9+1} = \frac{3}{10} = 0.30$$

$$First\ Particle\ NOT\ being\ a\ Reactant = P(R1') = 1 - 0.33 = 0.66$$

$$Second\ Particle\ NOT\ being\ a\ Reactant = P(R2') = 1 - 0.30 = 0.70$$

$$First\ Particle\ being\ a\ Catalyst = P(C1) = \frac{x}{n} = \frac{6}{9} = 0.66$$

$$Second\ Particle\ being\ a\ Catalyst = P(C2) = \frac{x}{n} = \frac{6}{10} = 0.60$$

$$Third\ Particle\ being\ a\ Catalyst = P(C3) = \frac{x}{n} = \frac{7}{11} = 0.63$$

P(C3 | R1 & C2) = (3/9 * 6/10 * 7/11) = 0.1272

P(C3 | C1 & R2) = (6/9 * 3/10 * 7/11) = 0.1272

0.1272 + 0.1272 = 0.2545

There is a 25.45% chance that exactly 1st particle or the 2nd particle is a reactant and the third is a catalyst.

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Problem Two. A seller stocks at the beginning of the day an item that perishes that same day. On any given day the demand is either 1, 2, 3 or 4 items with probabilities 0.02, 0.05,0.33 and 0.6. Each item costs to the seller $1 and he sells them for $1.16 each. If an item cannot be sold in the same day, the item perishes and it is a total loss for theseller. Compute the number of items that should be stocked at the beginning of the day to maximize seller's expected daily profit?

Given:

- 1 → 0.02
- 2 → 0.05
- 3 → 0.33
- 4 → 0.60
- Buys Items → $1.00
- Sells Items → $1.16
- If Item NOT sold → LOSS

Let's assume seller buys 1 item each day. The daily demand is 1 so their profit would be:

(1.16 – 1) * 1 = 0.16

They make this amount of money with a probability of 0.02. If the daily demand is 2, 3, or 4 the profit will still be 0.16 since she can only sell 1 item. This is what the probability distribution would look like for this case:

1 ITEM				
Y	0.16	0.16	0.16	0.16
P(Y)	0.02	0.05	0.33	0.60

E(1 ITEM) = (0.16*0.02) + (0.16*0.05) + (0.16*0.33) + (0.16*0.60)

= 0.16

Let's assume seller buys 2 item each day. The daily demand is 2 so their profit would be:

(1.16 – 1) * 2 = 0.32

They make this amount of money with a probability of 0.05.

If the daily demand is 1 she will lose a dollar because 1 item is left and she makes -0.68 with the probability of 0.02.

If the daily demand is 2, 3 or 4 the profit will still be 0.32 since she can only sell 2 item. This is what the probability distribution would look like for this case:

2 ITEMS				
Y	-0.68	0.32	0.32	0.32
P(Y)	0.02	0.05	0.33	0.60

E(2 ITEMS) = ((-0.68)*0.02) + (0.32*0.05) + (0.32*0.33) + (0.32*0.60)

= 0.30

Let's assume seller buys 3 item each day. The daily demand is 3 so their profit would be:

$$(1.16 - 1) * 3 = 0.48$$

They make this amount of money with a probability of 0.33.

If the daily demand is 1 she will lose 2 dollars because 2 item is left and she makes:

$$(1.16 - 1) * 1 = 0.16 - 2 = -1.84$$

with the probability of 0.02.

If the daily demand is 2 she will lose 1 dollar because 1 item is left and she makes:

$$(1.16 - 1) * 2 = 0.32 - 1 = -0.68$$

with the probability of 0.05.

If the daily demand is 3 or 4 the profit will still be 0.48 since she can only sell 3 item. This is what the probability distribution would look like for this case:

3 ITEMS				
Y	-1.84	-0.68	0.48	0.48
P(Y)	0.02	0.05	0.33	0.60

E(3 ITEMS) = ((-1.84)*0.02) + ((-0.68)*0.05) + (0.48*0.33) + (0.48*0.60)

= 0.38

Let's assume seller buys 4 item each day. The daily demand is 4 so their profit would be:

$$(1.16 - 1) * 4 = 0.64$$

They make this amount of money with a probability of 0.60.

If the daily demand is 1 she will lose 3 dollars because 3 item is left and she makes:

$$(1.16 - 1) * 1 = 0.16 - 3 = -2.84$$

with the probability of 0.02.

If the daily demand is 2 she will lose 2 dollars because 2 item is left and she makes:

(1.16 – 1) * 2 = 0.32 – 2 = -1.68

with the probability of 0.05.

If the daily demand is 3 she will lose 1 dollars because 1 item is left and she makes:

(1.16 – 1) * 3 = 0.48 - 1 = -0.52

with the probability of 0.33.

This is what the probability distribution would look like for this case:

4 ITEMS				
Y	-2.84	-1.68	-0.52	0.64
P(Y)	0.02	0.05	0.33	0.60

E(3 ITEMS) = ((-2.84)*0.02) + ((-1.68)*0.05) + ((-0.52)*0.33) + (0.64*0.60)

= 0.07

To maximize profit, you would want to stock up 3 items, with the profit of 0.38.

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Problem Three. If a sample of 100 people contains 15 left-handed:

a. Compute 95% confidence interval for the proportion of left-handers in the population.

Given:

- n = 100
- left-handed = 15
- right-handed = 85
- x = 0.15
- $\sigma = \sqrt{p(1-p)} = \sqrt{(0.15)(1-0.15)} = 0.357$
- $\alpha = 0.05$
- $Z_{a/2} = Z_{0.05/2} = Z_{0.025} = 1.96$

$$\left(x - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, x + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$$\left(0.15 - 1.96\frac{0.357}{\sqrt{100}}, 0.15 + 1.96\frac{0.357}{\sqrt{100}}\right)$$

$$(0.080028, 0.219972)$$

b. If we were to take a larger sample the next time we do this experiment, would we expect the confidence interval to be wider or narrower than the one in part(a)? Why?

Increasing the sample size decreases the width of the confidence intervals. This is due to the decrease in the standard error. The margin of error decreases with the increase of sample size.

c. If we compute a confidence interval with a higher (say 98%) confidence level forthe sample in part (a), would the new confidence interval be wider or narrower than the confidence interval computed in part a)?

Given:

- $\sigma = \sqrt{p(1-p)} = \sqrt{(0.15)(1-0.15)} = 0.357$
- $\alpha = 0.02$
- $Z_{a/2} = Z_{0.02/2} = Z_{0.01} = 2.33$

$$\left(x - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, x + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$$\left(0.15 - 2.33\frac{0.357}{\sqrt{100}}, 0.15 + 2.33\frac{0.357}{\sqrt{100}}\right)$$

$$(0.066819, 0.233181)$$

It is wider by 0.3, when you calculate the range between the two intervals the one with the 98% CI is larger.

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Problem Four. An old computer system has 2,000 components. The failure probability of one of th ecomponents, in a year of operation, is 0.001. For the sake of this calculation, assume each component fails independently of any other. What is the probability that at least two components will fail to operate during a year?

Given:

- 2000 Components
- P(Failure) = 0.001
- Components Fail Independently

We will represent 2 failures written as:

$$P(X \geq 2) = 1 - P(1) - P(0)$$

$$P(0) = \binom{2000}{0}(0.001)^0(1-0.001)^{2000-0} = 0.1352$$

$$P(1) = \binom{2000}{1}(0.001)^1(1-0.001)^{2000-1} = 0.2707$$

$$P(X \geq 2) = 1 - 0.2707 - 0.1352 = 0.5941$$

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Problem Five. A laser beam has 5 gates on its route to reach a receptor. If one gate is closed,the light cannot continue its path and therefore, cannot reach the receptor. The probability that a given gate is open is 1/3.

a. Write the pmf of the random variable X = "number of gates passed by the laserbeam before the first stop occurs".

Given:

- 5 Gates
- One Gate: Closed
- P(Open Gate) = 0.33

$$P_x(0) = P(X = 0) = P(Closed) = 0$$

$$P_x(1) = P(X = 1) = P(Open, Closed) = p$$

$$P_x(2) = P(X = 2) = P(Open, Open, Closed) = (1 - p)p$$

$$P_x(3) = P(X = 3) = P(Open, Open, Open, Closed) = (1 - p)^2 p$$

$$P_x(4) = P(X = 4) = P(Open, Open, Open, Open, Closed) = (1 - p)^3 p$$

$$P_x(5) = P(X = 5) = P(Open, Open, Open, Open, Open, Closed) = (1 - p)^4 p$$

$$P_x(x) = \begin{cases} (1 - p)^{x-1} p, & 0 < x \le 5 \\ 0, & otherwise \end{cases}$$

$$P_x(x) = \begin{cases} (1 - 0.33)^{x-1} 0.33, & 0 < x \le 5 \\ 0, & otherwise \end{cases}$$

b. Compute its expected value and variance.

Calculating the value:

$$E(X) = (0 * 0) + (0.33 * 0.33) + \big(0.33 * (0.66)(0.33)\big) + \big(0.33 * (0.66)^2(0.33)\big) + \big(0.33 * (0.66)^3(0.33)\big) + (0.33 * (0.66)^4(0.33)) = 0.28$$

Calculating the variance:

$$E(X^2) = (0 * 0) + (1^2 * 0.33) + \big(2^2 * (0.66)(0.33)\big) + \big(3^2 * (0.66)^3(0.33)\big) + \big(4^2 * (0.66)^3(0.33)\big) + (5^2 * (0.66)^4(0.33)) = 5.578$$

$$V(X) = 5.578 - (0.28)^2 = 5.4996$$

Problem Six. What is the probability that 1000 tosses of a fair coin will produce between 475 and 525 "Heads"? (A single numerical answer is required.)

Given:

- 1000 FAIR coin tosses
- Fair Coin → 50% chance of getting heads or tails
- P(475 < Heads < 525)

Let's have x represent the number of fair coin tosses and have it written like:

$$x \sim B(n = 1000, p = 0.5)$$

Now let's find the probability of number of between 475 and 525 and have it written like:

$$P(475 < x < 525)$$

We will use a central limit theorem:

$$E(X) = np = 1000 * 0.5 = 500$$

$$Var(x) = np(1-p) = 1000 * 0.5 * (1 - 0.5) = 250$$

Now:

$$P(475 < x < 525) = P(\frac{475 - 500}{\sqrt{250}} < \frac{x - np}{\sqrt{np(1-p)}} < \frac{525 - 500}{\sqrt{250}}$$

$$= P(-1.58 < Z < 1.58)$$

$$= 1 - P(Z < -1.58) - P(Z > 1.58)$$

$$= 1 - P(Z < -1.58) - P(Z < -1.58)$$

$$= 1 - 2(P(Z < -1.58)$$

$$= 1 - (2 * 0.0571)$$

$$= 0.8858$$

This means that the probability that 1000 with a fair coin to land approximately between 475 and 525 heads is about 90%.

Problem Seven. The final grades in a probability class that has been taught the same way for manyyears are known to be reasonably approximated by a Gaussian pdf with mean 76 and standard deviation 6. What is the probability that this year's class of 105 students will have a final grade average of 80 or higher?

Given:

- Mean = 76
- Standard Deviation = 6
- n = 105

$$z = \frac{x - \mu}{\sigma} = \frac{80 - 76}{6} = \frac{2}{3} = 0.667$$

$$P(X \geq 80) = P(0.667 \leq X) = (1 - \phi(0.666)) = 0.2546$$

However when given the size, the standard devation will change:

- New Standard Deviation = 6 / sqrt(105) = 0.58g

$$P\left(z > \frac{80 - 76}{0.5856}\right) = P(z > 6.83) = 1 - P(z < 6.83) = 0$$

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Problem Eight. Let X1 and X2 have the joint pdf given by

$$f(x_1, x_2) = \begin{cases} C(1-x_2), & 0 \le x_1 \le x_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a. Find the value C that makes this function a pdf.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f x_1, x_2 (x_1 x_2) dx_1 dx_2$$

$$= \int_0^{x_2} \int_{x_1}^{1} C(1-x_2) dx_2 dx_1$$

$$= C * \left[\frac{1}{2}x_1 - \frac{x_1^2}{2} + \frac{x_1^3}{6}\right] \Bigg| \begin{matrix} 0 \\ x_2 \end{matrix}$$

$$= C * \left[\frac{1}{2}x_1 - \frac{x_2^2}{2} + \frac{x_2^3}{6}\right] = 1$$

$$C = 0.9129$$

b. Compute P(X1≤ 3/4, X2≥1/2).

$$P\left(X1 \le \frac{3}{4} \middle| X2 \ge \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} \int_0^{\frac{3}{4}} 0.9129(1-x) dx2dx1$$

$$= \int_{0.5}^{1} 0.685(-x+1) dx1$$

$$= 0.085584$$

c. Compute both marginals.

$$f_{x1}(x) = \int_0^{x2} f(x_1, x_2) dx_2 = \int_0^{x2} 0.9129(1-x) dx_2$$

$$f_{x2}(x) = \int_{x2}^{1} f(x_1, x_2) dx_2 = \int_{x2}^{1} 0.9129(1-x) dx_2$$

$$f_{x1}(x) = \begin{cases} \int_{x1}^{1} 0.9129(1-x_2) dx_2, & 0 \le x_1 \le x_2 \\ \int_0^{x2} 0.9129(1-x_2) dx_2, & x_1 \le x_2 \le 1 \end{cases}$$

$$f_{x2}(x) = \begin{cases} \int_{x2}^{1} 0.9129(1 - x_1)dx_1 \ , & 0 \le x_1 \le x_2 \\ \int_{0}^{x2} 0.9129(1 - x_1)dx_1 \ , & x_1 \le x_2 \le 1 \end{cases}$$

d. Are X1 and X2 independent? Justify.

No, they are not independent. You do not the joint pdf when you multiply the marginal distributions.

e. Compute E(X1) and V(X1).

$$E(X_1) = \int_{0}^{1} 0.9129(1 - x_2)dx_2$$

$$= \int_{0}^{x_2} x_1 0.9129(1 - x_1)dx_1 + \int_{x2}^{1} 0.9129(1 - x_2)dx_2 = \frac{-3043{x_1}^2(2x_1 - 3)}{20000} + \frac{-3043{x_1}^2(2x_1 - 3)}{20000} = 0.038$$

$$E({X_1}^2) = \int_{0}^{1} \left(0.9129(1 - x_2)\right)^2 dx_2 = \int_{0}^{x_2} \left(0.9129(1 - x_2)\right)^2 dx_2 + \int_{x2}^{1} 0.9129(1 - x_2)^2 dx_2 = 0.0014$$

$$V(X_1) = 0.0014 - (0.038)^2 = 0.000044$$

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Problem Nine. Let the random variables (discrete) X1, X2 have the joint pmf

p(x1, x2) = 1/3, for (x1, x2) = (−1,0),(0,1),(1,0).

a. Compute Cov(X1, X2).

$$\mu_1 = E(X_1) = \sum_x x\frac{1}{3} = (-1)(0.33) + (0)(0.33) + (1)(0.33) = -0.33 + 0 + 0.33 = 0$$

$$\mu_2 = E(X_2) = \sum_y y\frac{1}{3} = (0)(0.33) + (1)(0.33) + (0)(0.33) = 0 + 0.33 + 0 = 0.33$$

$$\begin{aligned} E(X_1X_2) = &\sum_x (x_1x_2)\frac{1}{3} \\ &= (-1)(0)(0.33) + (-1)(1)(0.33) + (-1)(0)(0.33) + (0)(0)(0.33) + (0)(1)(0.33) \\ &+ (0)(0)(0.33) + (1)(0)(0.33) + (1)(1)(0.33) + (1)(0)(0.33) \\ &= -1 + 0 + 0 + 0 + 0 + 0 + 0 + 1 + 0 = 0 \end{aligned}$$

$$\rho = \frac{E((X-\mu_1)(Y-\mu_2))}{\sigma_1\sigma_2} = 0$$

b. Are X1 and X2 independent? Why?

Yes, they are independent. This is because when you multiple the two means will give you 0. They would form a joint pmf when multiplying the marginal pdfs.