"I have pledged my honor that I have abided by the Stevens Honor System."
Cindy Zhang

HOMEWORK 7

MA 222 Probability and Statistics

Problem One. Annie and Alvie have agreed to meet between 5:00 p.m. and 6:00 p.m. for dinner at a local health-food restaurant. Let X = Annie's arrival time an d Y = Alvie's arrival time. Suppose X and Y are independent with each uniformly distributed on the interval [5, 6].

a. What is the joint pdf of X and Y?

Two random variables X and Y are independent if and only if:

1. $p(x, y) = p_x(x) * p_y(y)$ for every (x,y), and when X and Y discrete rv's
2. $f(x, y) = f_x(x) * f_y(y)$ for every (x,y), and when X and Y discrete rv's

Otherwise, they are dependent.

Since X and Y are independent and uniformly distributed. The joint pdf of X and Y is:

$f(x, y) = f_x(x) * f_y(y) = 1 * 1 = 1, \qquad 5 \le x \le 6, 5 \le y \le 6,$

$f(x, y) = 0$

That means:

$$f(x, y) = \begin{cases} 1, & 5 \le x \le 6, 5 \le y \le 6 \\ 0, & \text{otherwise.} \end{cases}$$

b. What is the probability that they both arrive between 5:15 and 5:45?

Multiplication Property: Two events A and B are independent if and only if:

P(A N B) = P(A) * P(B)

Converting 5:15 to 5.25 and 5:45 to 5.75. (Because 15 minutes is 0.25 of an hour)

$$\begin{aligned} P(5.25 \le x \le 5.75, 5.25 \le y \le 5.75) &= P(5.25 \le x \le 5.75) * P(5.25 \le y \le 5.75) \\ &= \int_{5.25}^{5.75} 1\,dx * \int_{5.25}^{5.75} 1\,dy \\ &= 0.5 * 0.5 \\ &= 0.25 \end{aligned}$$

c. If the first one to arrive will wait only 10 min before leaving to eat elsewhere, what is the probability that they have dinner at the health-food restaurant? [Hint: The event of interest is A={(x, y): |x – y| ≤ 1/6}.]

1/6 of an hour = 10 minutes.

$$P[(X, Y) \epsilon A] = \iint_A f(x,y)dxdy$$

$$P[(X, Y) \epsilon A] = \iint_A 1dxdy = \frac{11}{36} = 0.31$$

Problem Two. You have two lightbulbs for a particular lamp. Let X = the lifetime of the first bulb and Y= the lifetime of the second bulb (both in 1000s of hours). Suppose that X and Y are independent and that each has an exponential distribution with parameter

a. What is the joint pdf of X and Y?

Two random variables X and Y are independent if and only if:

1. $p(x, y) = p_x(x) * p_y(y)$ for every (x,y), and when X and Y discrete rv's
2. $f(x, y) = f_x(x) * f_y(y)$ for every (x,y), and when X and Y discrete rv's

Otherwise, they are dependent.

$$fx(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$fy(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

That means:

$$f(x,y) = fx(x) * fy(y) = \begin{cases} e^{-x-y}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

b. What is the probability that each bulb lasts at most1000 hours

Multiplication Property: Two events A and B are independent if and only if:

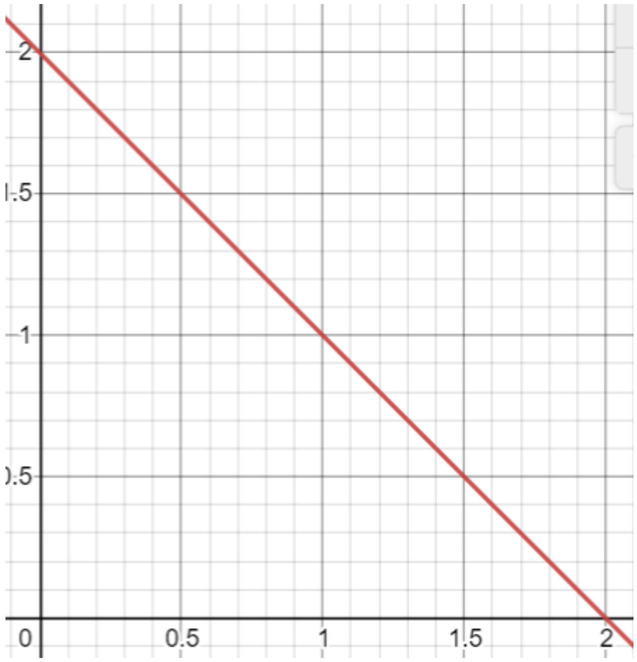
P(A N B) = P(A) * P(B)

Converting 5:15 to 5.25 and 5:45 to 5.75. (Because 15 minutes is 0.25 of an hour)

$P(X \leq 1 \text{ and } Y \leq 1)$ $= P(X \leq 1) * P(Y \leq 1)$
$= (1 - e^{-1}) * (1 - e^{-1})$
$= 0.4$

c. What is the probability that the total lifetime of the two bulbs is at most 2? [Hint: Draw a picture of the region A = {(x, y): integrating.]

$P(X + Y \leq 2) = \int_0^2 \int_0^{2-x} e^{-x-y} dydx = 1 - e^{-2} - 2e^{-2} = 0.594$



d. What is the probability that the total lifetime is between 1 and 2?

$P(X + Y \leq 2) = \int_0^2 \int_0^{2-x} e^{-x-y} dydx = 1 - e^{-2} - 2e^{-2} = 0.594$

$P(X + Y \leq 2) = \int_0^1 \int_0^{1-x} e^{-x-y} dydx = 1 - e^{-1} - e^{-1} = 0.264$

$P(1 \leq X + Y \leq 2) = P(X + Y \leq 2) - P(1 \leq X + Y) = 0.594 – 0.264 = 0.33$

Problem Three. Annie and Alvie have agreed to meet for lunch between noon (0:00 p.m.) and 1:00 p.m. Denote Annie's arrival time by X, Alvie's by Y, and suppose X and Y are independent with pdf's

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected amount of time that the one who arrives first must wait for the other person?

Two random variables X and Y are independent if and only if:

1. $p(x, y) = p_x(x) * p_y(y)$ for every (x,y), and when X and Y discrete rv's
2. $f(x, y) = f_x(x) * f_y(y)$ for every (x,y), and when X and Y discrete rv's

Otherwise, they are dependent.

Since X and Y are independent and uniformly distributed. The joint pdf of X and Y is:

$f(x, y) = f_x(x) * f_y(y) = 3x^2 * 2y = 6x^2y, \qquad 0 \le x, y \le 1$

$f(x, y) = 0$

$$E[g(X,Y)] = E[|X - Y|] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x - y| * f(x.y)dxdy$$

$$= = \int_0^1\int_0^1 |x - y| * 6x^2ydxdy = \frac{1}{6} + \frac{2}{3} - \frac{3}{4} + \frac{1}{6} = \frac{1}{4}$$

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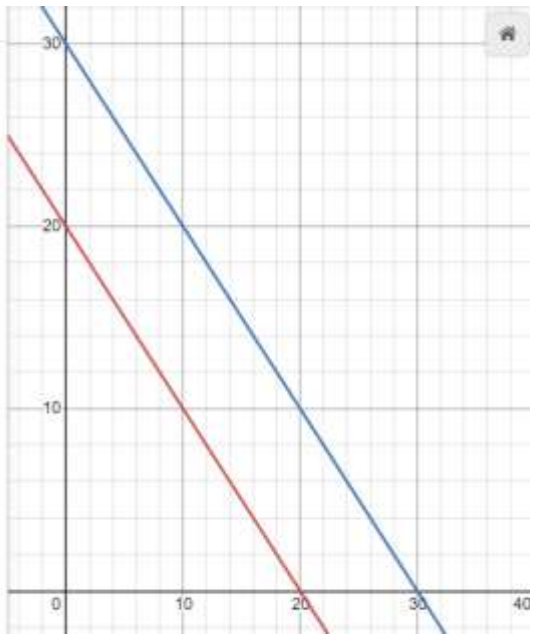
Problem Four. A health-food store stocks two different brands of a certain type of grain. Let X = the amount (lb) of brand A on hand and Y = the amount of brand B on hand. Suppose the joint pdf of X and Y is

a. Draw the region of positive density and determine the value of k.



$0 \le x \le 20 \rightarrow \int_0^{20}\int_{20-x}^{30-x} f(x,y)\,dydx = \frac{70000}{3}k$

$20 \le x \le 30 \rightarrow \int_0^{20}\int_{20-x}^{30-x} f(x,y)\,dydx = \frac{70000}{3}k$

$\frac{70000}{3}k + 3750k = \frac{81250}{3}k \rightarrow k = \frac{3}{81250}$

b. Are X and Y independent? Answer by first deriving the marginal pdf of each variable.

No, X and Y are not independent. You do not get the joint pdf when you multiple the marginal distributions.

$$f_x(x) = \int_{20-x}^{30-x} f(x,y)dy = \int_{20-x}^{30-x} kxydy = 250kx - 10kx^2$$

$$f_x(x) = \int_{0}^{30-x} f(x,y)dy = \int_{0}^{30-x} kxydy = 450kx - 30kx^2 + \frac{1}{2}kx^3$$

$$fx(x) = \begin{cases} 250kx - 10kx^2, & 0 \le x \le 20 \\ 450kx - 30kx^2 + \frac{1}{2}kx^3, & 20 < x \le 30 \end{cases}$$

$$fy(x) = \begin{cases} 250ky - 10ky^2, & 0 \le y \le 20 \\ 450ky - 30ky^2 + \frac{1}{2}ky^3, & 20 < y \le 30 \end{cases}$$

c. Compute $P(X + Y \le 25)$.

$$P(X + Y \le 25 | 0 \le X \le 20) = \int_0^{20} \int_{20-x}^{25-x} f(x,y)dydx$$

$$= \int_0^{20} \int_{20-x}^{25-x} kxydydx$$

$$= 125k(20)^2 - (10/3)k(20)^3 = (27500/3)k$$

$$P(X + Y \le 25 | 20 < X \le 30) = \int_{20}^{25} \int_{0}^{25-x} f(x,y)dydx$$

$$= \int_{20}^{25} \int_{0}^{25-x} kxydydx$$

$$= (10625/24)k$$

$$P(X + Y \le 25) = \frac{27500}{3}k + \frac{10625}{24}k = \frac{76875}{8}k = \frac{76875}{8}\frac{3}{81250} = \frac{369}{1040}$$

$$= 0.35$$

d. What is the expected total amount of this grain on hand?

Total amount of grain on this hand is X + Y.

$$0 \le X \le 20 = \int_0^{20}\int_{20-x}^{30-x}(x+y)f(x,y)dydx = \int_0^{20}\int_{20-x}^{30-x} kx^2ydy^2dx$$

$$= 125k(20)^2 - \frac{10}{3}k(20)^3 = 600000k$$

$$20 < X \le 30 = \int_{20}^{30}\int_{0}^{30-x}(x+y)f(x,y)dydx = \int_{20}^{30}\int_{0}^{30-x} kx^2ydy^2dx$$

$$= \frac{310000}{3}k$$

$$E(X+Y) = 600000k + \frac{310000k}{3} = \frac{2110000k}{3} = \frac{2110000}{3} * \frac{3}{81250} = \frac{1688}{65} = 25.96$$

e. Compute Cov(X, Y) and Corr(X, Y).

Calculating the mean:

$$\mu_X = E(X) = \int_0^{30} xf(x)dx$$

$$= \int_0^{20} 250kx^2 - 10kx^3dx$$

$$+ \int_{20}^{30} 450kx^2 - 30kx^3 + \frac{1}{2}kx^4dx = \frac{800000k}{3} + 85000k$$

$$= \frac{1055000}{30}k$$

Calculating expected value of XY:

$$E(XY) = \int_0^{20}\int_{20-x}^{30-x} xyf(x,y)dydx + \int_{20}^{30}\int_{0}^{30-x} xyf(x,y)dydx$$

$$= \int_0^{20}\int_{20-x}^{30-x} kx^2y^2dydx + \int_{20}^{30}\int_{0}^{30-x} kx^2y^2dydx$$

$$= \frac{29600000k}{9} + \frac{3650000k}{9} = \frac{33250000k}{9}$$

Calculating the Variance:

$$\sigma_x^2 = E((X - \mu_x)^2)$$
$$= \int_0^{30} (x - \frac{1055000}{3}k)^2 fx(x)dx$$
$$= \frac{400000k}{27}(243 - 12660000k + 194779375000k^2) + \frac{25000k}{3}(233 - 7174000k + 55651250000k^2)$$
$$= \frac{16625000k}{3} - \frac{2226050000000k^2}{9} + \frac{90433281250000000k^3}{27}$$

Calculating Covariance:

$$Cov(X,Y) = E(XY) - \mu_X\mu_Y = \frac{33250000k}{9} - \frac{1055000}{3}\frac{1055000}{3}k^2$$
$$= \frac{33250000}{9}\frac{3}{81250} - \frac{1055000}{3}\frac{1055000}{3}\frac{3^2}{81250^2} = -\frac{408008}{12675}$$
$$= -32.19$$

Calculating the Correlation:

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{-\frac{408008}{12675}}{\frac{16625000k}{3} - \frac{226050000000k^2}{9} + \frac{90433281250000000k^3}{27}}$$
$$= \frac{-\frac{408008}{12675}}{\frac{16625000}{3}\frac{3}{81250} - \frac{226050000000}{9}\frac{3^2}{81250^2} + \frac{90433281250000000}{27}\frac{3^3}{81250^3}}$$
$$= -\frac{102002}{114123} = -0.894$$

f. What is the variance of the total amount of grain on hand?

$$\mathrm{Var}(X+Y) = \int_0^{20}\int_{20\text{-}x}^{30\text{-}x}(x+y\text{-}25.97)^2 f(x,y)\,dydx + \int_{20}^{30}\int_0^{30\text{-}x}(x+y\text{-}25.97)^2 f(x,y)\,dydx$$

$$= \int_0^{20}\int_{20\text{-}x}^{30\text{-}x}(x+y\text{-}25.97)^2 kxy\,dydx + \int_{20}^{30}\int_0^{30\text{-}x}(x+y\text{-}25.97)^2 kxy\,dydx$$

$$= 184056k + 23141k = 207197k = 207197\frac{3}{81250} = \frac{621591}{81250}$$

$$= 7.6504$$

Problem Five. We have seen that if E(X1) = E(X2) = . . . = E(Xn) = , then E(X1 + . . . + Xn) = n . In some applications, the number of Xi's under consideration is not a fixed number n but instead is a random variable N. For example, let N= the number of components that are brought into a repair shop on a particular day, and let Xi denote the repair shop time for the ith component. Then the total repair time is X1+X2+ . . . +XN, the sum of a random number of random variables. When Nis independent of the Xi's, it can be shown that

a. If the expected number of components brought in on a particularly day is 10 and expected repair time for a randomly submitted component is 40 min, what is the expected total repair time for components submitted on any particular day?

   $E(X_1 + X_2 + \ldots + X_N) = 400$

b. Suppose components of a certain type come in for repair according to a Poisson process with a rate of 5 per hour. The expected number of defects per component is 3.5. What is the expected value of the total number of defects on components submitted for repair during a4-hour period? Be sure to indicate how your answer follows from the general result just given.

   Proposition: Number of events during a time interval of length t can be modeled using Poisson random variable with parameter $\mu = \alpha t$. This indicates that:

   $$P_k(t) = e^{-\alpha t} * \frac{(\alpha t)^k}{k!}$$

   $$\mu_N = \alpha * t = 5 * 4 = 20$$

   Proposition: For random variable X with Poisson Distribution with parameter $\mu > 0$, the following is true:

   $$E(X) = V(X) = \mu$$

   $$E(N) = \mu_N = 20$$

   $$E(N) * \mu = 20 * 3.5 = 70 \text{ (Given: } \mu = 3.5)$$