"I have pledged my honor that I have abided by the Stevens Honor System."
Cindy Zhang

HOMEWORK 5

MA 222 Probability and Statistics

Problem One.

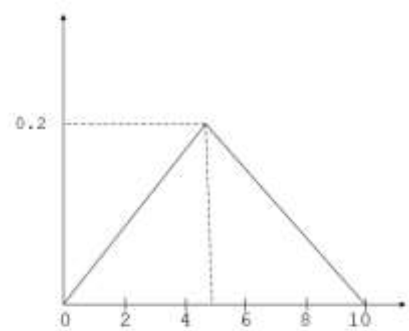
In commuting to work, a professor must first get on a bus near their house and then transfer to a second bus. If the waiting time (in minutes) at each stop has a uniform distribution with:

- A = 0
- B = 5

Then it can be shown that the total waiting time Y has the pdf.

$$f(y) = \begin{cases} \frac{1}{25}y & 0 \le y < 5 \\ \frac{2}{5} - \frac{1}{25}y & 5 \le y \le 10 \\ 0 & y < 0 \text{ or } y > 10 \end{cases}$$

a. Sketch a graph of the pdf of Y.



b. Verify that $\int_{-\infty}^{\infty} f(y)\,dy = 1$.

$$\int_{\infty}^{\infty} f(y)\,dy = area\ under\ graph = \frac{1}{2} * 0.2 * 10 = 1$$

$$\int_{\infty}^{\infty} f(y)dy = \int_{0}^{5} \frac{y}{25}dy + \int_{5}^{10} (\frac{2}{5} - \frac{y}{25})dy = \frac{1}{2} + 2 - \frac{3}{2} = 1$$

c. What is the probability that total waiting time is at most 3 min?

$$P(Y \le 3) = \int_{0}^{3} \frac{y}{25}dy = 0.18$$

d. What is the probability that total waiting time is at most 8 min?

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$$P(Y \leq 8) = \int_0^5 \frac{y}{25} dy + \int_5^8 (\frac{2}{5} - \frac{y}{25}) dy = 0.92$$

e. What is the probability that total waiting time is between 3 and 8 min?

$$P(3 \leq Y \leq 8) = P(Y \leq 8) - P(Y \leq 3) = 0.92 - 0.18 = 0.74$$

f. What is the probability that total waiting time is either less than 2 min or more than 6 min?

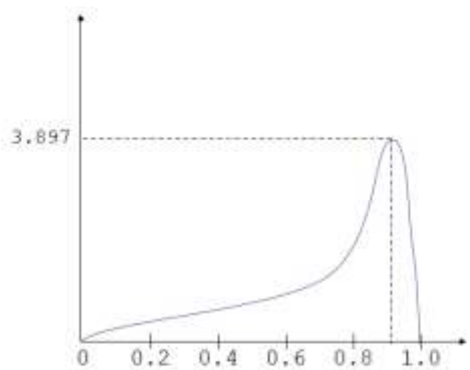
$$P(Y \leq 2 \text{ OR } 6 \leq Y) = \int_0^2 \frac{y}{25} dy + \int_6^{10} (\frac{2}{5} - \frac{y}{25}) dy = 0.4$$

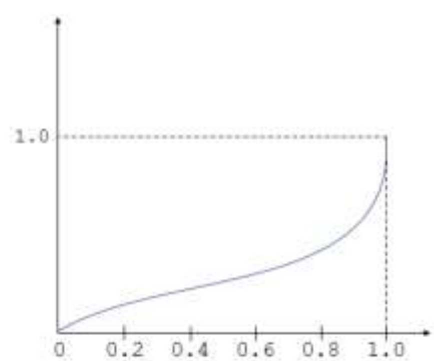
Problem Two.

Let X denote the amount of space occupied by an article placed in a 1-ft^3 packing container. The pdf of X is

$$f(x) = \begin{cases} 90x^8(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Graph the pdf. Then obtain the cdf of X and graph it.





Definition: Cumulative Distribution Function (cdf) F(x) for a continuous rv X is defined for every number x by:

$$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$$

$$F(X) = \int_0^x 90y^8(1-y)dy = x^9(10\text{-}9x)$$

$$F(x) = \begin{cases} 0, x \leq 0 \\ x^9(10-9x). 0 < x < 1 \\ 1, x \geq 1 \end{cases}$$

b. What is $P(X \leq 0.5)$?

$P(X \leq 0.5) = F(0.5) = 0.5^9(10 - 9(0.5)) = 0.01074$

c. Using the cdf from part a, what is $P(0.25 < X \leq 0.5)$? What is $P(0.25 \leq X \leq 0.5)$?

Proposition: $P(a \leq X \leq b) = F(b) - F(a)$

$$\begin{aligned} P(0.25 < X \leq 0.5) &= F(0.5) - F(0.25) \\ &= [0.5^9(10 - 9(0.5))] - [0.25^9(10 - 9(0.25))] \\ &= 0.01071 \end{aligned}$$

d. What is the 75th percentile of the distribution?

Definition: Let p be a number between 0 and 1. The 100pth percentile of the distribution of the continuous rv X, denoted by:

$$p = F(n_p) = \int_{-\infty}^{n_p} f(y)dy$$

$0.75 = F(n_{75}) = (n_{75})^9(10 - 9(n_{75})) = 0.9036$

e. Computer $E(X)$ and σ_X.

Proposition: $V(X) = E(X^2) - E(X)^2$

Definition: The expected or mean value of a continuous rv X with pdf is f(x) is:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(X) = \int_{-\infty}^{\infty} x\, 90x^8(1\text{-}x)dx = 0.8182$$

$$E(X^2) = \int_{-\infty}^{\infty} X^2\, 90x^8(1\text{-}x)dx = 0.6818$$

$$V(X) = 0.6818 - 0.8182^2 = 0.0123$$

$$\sigma_x = \sqrt{V(X)} = 0.11$$

f. What is the probability that X is more than 1 standard deviation from its mean value?

$$P(\mu - \sigma_x < X < \mu + \sigma_x) = P(0.7071 < X < 0.9293) = F(0.9293) - F(0.7071) = 0.6854$$

$$p = 1 - P(\mu - \sigma_x < X < \mu + \sigma_x) = 0.3146$$

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Problem Three.

Let X be a continuous rv with cdf

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x}{4}\left[1 + \ln\left(\frac{4}{x}\right)\right] & 0 < x \le 4 \\ 1 & x > 4 \end{cases}$$

a. $P(X \le 1)$?

$$P(X \le 1) = F(1) = \frac{1}{4}[1 + \ln 4] = 0.5966$$

b. $P(1 \le X \le 3)$?

$$P(1 \le X \le 3) = F(3) - F(1) = \frac{3}{4}\left[1 + \ln\frac{4}{3}\right] - \frac{1}{4}[1 + \ln 4] = 0.3692$$

c. The pdf of X?

$$f(x) = F'(x) = \frac{1}{4}\left[\ln\frac{4}{x}\right]$$

$$f(x) = \begin{cases} \frac{1}{4}\left[\ln\frac{4}{x}\right], 0 < x \le 4 \\ 0, \text{otherwise} \end{cases}$$

Problem Four.

Let X be the total medical expenses (in 1000s of dollars) incurred by a particular individual during a given year. Although X is a discrete random variable, suppose its distribution is quite well approximated by a continuous distribution with pdf f (x) = k(1 + x /2.5)-7for

a. What is the value of k?

$$f(x) = \begin{cases} k\left[1 + \frac{x}{2.5}\right]\text{-}7, x \ge 0 \\ 0, \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

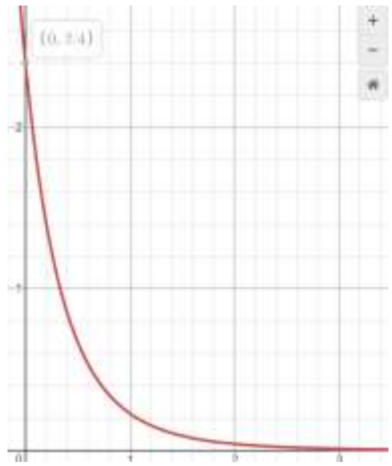
$$\int_{-\infty}^{\infty} k\left[1 + \frac{x}{2.5}\right]^{-7} dx = 1$$

$$t = 1 + \frac{x}{2.5}\ ;\ x = 2.5(t\text{-}1)\ ;\ dx = 2.5dt$$

$$\int_{1}^{\infty} kt^{-7}(2.5)\, dt = 1;\ k = 2.4$$

b. Graph the pdf of X.

$$f(x) = \begin{cases} 2.4\left[1+\dfrac{x}{2.5}\right]^{-7}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



c. What are the expected value and standard deviation of total medical expenses?

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(X) = \int_{0}^{\infty} x(2.4)\left[1+\frac{x}{2.5}\right]^{-7} dx$$

$$t = 1 + \frac{x}{2.5}\ ;\ x = 2.5(t\text{-}1)\ ;\ dx = 2.5dt$$

$$E(X) = 0.5$$

$$E(X^2) = \int_{0}^{\infty} x^2(2.4)\left[1+\frac{x}{2.5}\right]^{-7} dx$$

$$t = 1 + \frac{x}{2.5}\ ;\ x = 2.5(t\text{-}1)\ ;\ dx = 2.5dt$$

$$E(X^2) = 0.625$$

$$V(X) = E(X^2) - E(X)^2 = 0.625 - 0.5^2 = 0.375$$

$$\sigma_x = \sqrt{V(X)} = 0.6214$$

d. This individual is covered by an
- insurance plan that entails a \$500 deductible provision (so the first \$500 worth of expenses are paid by the individual). Then the
- plan will pay 80% of any additional expenses exceeding \$500, and the
- maximum payment by the individual (including the deductible amount) is \$2500.

- Let Y denote the amount of this individual's medical expenses paid by the insurance company.

What is the expected value of Y? [Hint: First figure out what value of X corresponds to the maximum out-of-pocket expense of $2500. Then write an expression for Y as a function of X (which involves several different pieces) and calculate the expected value of this function.]

Z = total medical expenses when individual pays 2500

0.2(1000 * Z – 500) = 200

Z = 10.5

$$y = \begin{cases} 0, & X < 0.5 \\ 0.8(1000^*X - 500), & 0.5 \le X \le 10.5 \\ 1000 * X - 2500, & X > 10.5 \end{cases}$$

$$E(y) = \int_{-\infty}^{\infty} y\, f(x)\, dx$$

$$= \int_{0.5}^{10.5} 0.8(1000^*x\text{-}500)(2.4)\left[1+\frac{x}{2.5}\right]^{-7} dx + \int_{10.5}^{\infty} (1000^*x\text{-}2500)(2.4)\left[1+\frac{x}{2.5}\right]^{-7} dx$$

$$t = 1 + \frac{x}{2.5}\ ;\ x = 2.5(t\text{-}1)\ ;\ dx = 2.5dt$$

$$E(y) = 160.78$$