"I have pledged my honor that I have abided by the Stevens Honor System."
Cindy Zhang

HOMEWORK 6

MA 222 Probability and Statistics

Problem One.

Standard Deviation (σ) = 3432

Mean (μ) = 482

a. What is the probability that the birth weight of a randomly selected baby of this type exceeds 4000 grams? Is between 3000 and 4000 grams?

Proposition: Let Z be a continuous rv with cdf Φ(z). Then for any a:

$P(a < Z) = 1 - \Phi(a)$

$P(4000 < X)$:

$$\frac{4000 - 3432}{482} < \frac{X - 3432}{482} \rightarrow \frac{568}{482} < \frac{X - 3432}{482} \rightarrow 1.18 < Z$$

$P(4000 < X) = 1 - \Phi(1.18)$

$= 1 - 0.881$

$= 0.119$

Proposition: Let Z be a continuous rv with cdf Φ(z). Then for any a and b with a < b,

$P(a < Z < b) = \Phi(b) - \Phi(a)$

$P(3000 < X < 4000)$:

$$\frac{3000 - 3432}{482} < \frac{X - 3432}{482} < \frac{4000 - 3432}{482} \rightarrow \frac{-432}{482} < \frac{X - 3432}{482} < \frac{568}{482} \rightarrow$$

$-0.90 < Z < 1.18$

$P(3000 < X < 4000) = \Phi(1.18) - \Phi(0.90)$

$= 0.881 - 0.184$

$= 0.6969$

b. What is the probability that the birth weight of a randomly selected baby of this type is either less than 2000 grams or greater than 5000 grams?

Proposition: Let Z be a continuous rv with cdf Φ(z). Then for any a:

$P(Z < a) = \Phi(a)$

$P(a < Z) = 1 - \Phi(a)$

$P(X < 2000 \text{ or } X > 5000) \rightarrow P(X < 2000) + P(X > 5000)$:

$\frac{5000-3432}{482} < \frac{X-3432}{482} \rightarrow \frac{1568}{482} < \frac{X-3432}{482} \rightarrow 3.25 < Z$

$\frac{X-3432}{482} < \frac{2000-3432}{482} \rightarrow \frac{X-3432}{482} < \frac{-1432}{482} \rightarrow -2.97 > Z$

$P(X > 5000) = 1 - \Phi(3.25)$

$= 1 - 0.9994$

$= 0.0006$

$P(X < 2000) = \Phi(-2.97)$

$= 0.0014$

$P(X < 2000 \text{ or } X > 5000) = 0.0014 + 0.0006 =$ 0.002

c. What is the probability that the birth weight of randomly selected baby of this type exceeds 7 pounds?

7 pounds = 3175.15

$P(X > 3175.15)$:

$\frac{3175.15-3432}{482} < \frac{X-3432}{482} \rightarrow \frac{-256.85}{482} < \frac{X-3432}{482} \rightarrow -0.53 < Z$

$P(X > 3175.15) = 1 - \Phi(-0.53)$

$= 1 - 0.2981$

$=$ 0.7019

d. How would you characterize the most extreme 0.1% of all birth weights?

Greater than $Z_{0.0005}$ or Lower than $Z_{0.9995}$.

$\Phi(z_{0.0005}) = 0.9995$

$z_{0.0005} = 3.3 = \frac{X_{0.0005} - 3432}{482} \rightarrow X_{0.0005} = 3432 + (482)(3.3) = 5022.6$

$z_{0.9995} = -3.3 = \frac{X_{0.9995} - 3432}{482} \rightarrow X_{0.9995} = 3432 + (482)(-3.3) = 1841.4$

Extreme weights 0.1% weights are greater than 5022.6 grams or less than 1841 grams.

e. If X is a random variable with a normal distribution and a is a numerical constant (a is not equal to 0), then Y = aX also has a normal distribution. Use this to determine the distribution of birth weight expressed in pounds (shape, mean, and standard deviation), and then recalculate the probability from part (c). How does this compare to your previous answer?

<u>Proposition</u>: When h(x) = aX + b, then the expected value and the standard deviation of h(X) satisfy the following properties:

$E[h(x)] = aE[X] + b$

$\sigma_{h(x)} = a\sigma_X$

Expressed in Pounds:

$Y = \frac{X}{453.592}$

Mean = 7.566

Standard Deviation = 1.063

P(Y > 7):

$$\frac{7 - 7.566}{1.063} < \frac{X - 7.566}{1.063} \rightarrow \frac{-0.566}{1.063} < \frac{X - 7.566}{1.063} \rightarrow -0.53 < Z$$

$P(Y > 7) = 1 - \Phi(-0.53) = 1 - 0.2981 =$ 0.7019

Same calculation as in part c.

Problem Two.

Given:

- n = 200
- p = 0.1
- mean = 20
- standard deviation = 4.24

a. At most 30?

P(X ≤ 30):

$$\frac{X-20}{4.24} \leq \frac{30.5-20}{4.24} \rightarrow \frac{X-20}{4.24} \leq \frac{10.5}{4.24} \rightarrow Z \leq 2.48$$

P(X ≤ 30) = Φ(2.48) = 0.9934

b. Less than 30?

P(X ≤ 29.5):

$$\frac{X-20}{4.24} \leq \frac{29.5-20}{4.24} \rightarrow \frac{X-20}{4.24} \leq \frac{9.5}{4.24} \rightarrow Z \leq 2.24$$

P(X < 30) = Φ(2.24) = 0.9875

c. Between 15 and 25 (inclusive)?

P(14.5 ≤ X ≤ 25.5):

$$\frac{14.5-20}{4.24} \leq \frac{X-20}{4.24} \leq \frac{25.5-20}{4.24} \rightarrow \frac{-5.5}{4.24} \leq \frac{X-20}{4.24} \leq \frac{5.5}{4.24} \rightarrow -1.3 \leq Z \leq 1.3$$

P(14.5 ≤ X ≤ 25.5) = Φ(1.3) - Φ(-1.3) = 0.9032 – 0.0968 = 0.8064

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Problem Three.

a. The event X is greater than or equal to t is equivalent to what event involving A_1 to A_5?

E = $\{X \geq t\} = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$

b. Using the independence of the A_i's, compute $P(X \geq t)$. Ten obtain $F(t) = P(X \leq t)$ and the pdf of X. What type of distribution does X have?

$P(A_i) =$

$P(Y \geq t) = P(A_1) * P(A_2) * P(A_3) * P(A_4) * P(A_5)$

$= \left[e^{-(0.01)t}\right]^5$

$= e^{-(0.05)t}$

$F(x) = P(X \leq t)$

$= 1 - P(Y \geq t)$

$= 1- e^{-(0.05)t}$

$f(x) = F'(x)$

$= (0.05)e^{-(0.05)t}$

Lambda = 0.05

c. Suppose there are n components, each having exponential life time with parameter lambda. What type of distribution does X have?

Exponential Distribution from observation.

Problem Four.

Given:

- n = 250
- p = 0.05
- np = 12.5
- nq = 237.5
- mean = 12.5
- standard deviation = 3.446
- X distributed as N(12.5, 3.446)

a. What is the approximate probability that at least 10% of the boards in the batch are defective?

$P(X_{bin} \geq 25)$:

$$\frac{X - 12.5}{3.446} \geq \frac{24.5 - 12.5}{3.446} \rightarrow Z \geq 3.48$$

$P(X_{bin} \geq 25) = 1 - \Phi(3.48) = 1 - 0.9997 =$ 0.0003

b. What is the approximate probability that there are exactly 10 defectives in the batch?

$P(X_{bin} = 10)$:

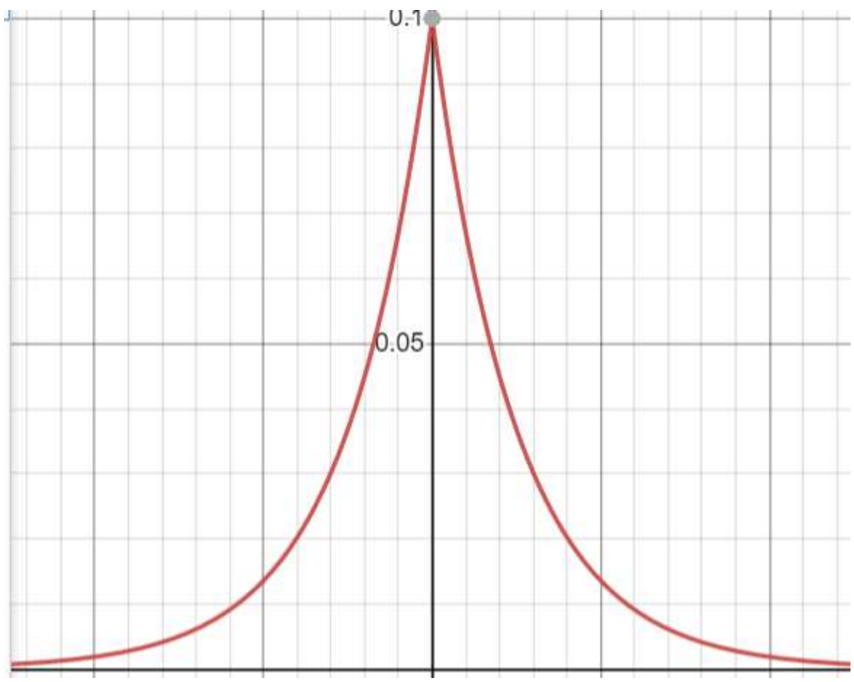
$$\frac{9.5 - 12.5}{3.446} \leq \frac{X - 12.5}{3.446} \leq \frac{10.5 - 12.5}{3.446} \rightarrow \frac{-3}{3.446} \leq \frac{X - 12.5}{3.446} \leq \frac{-2}{3.446}$$

$-0.87 \leq Z \leq -0.58$

$P(X_{bin} = 10) = \Phi(-0.58) - \Phi(-0.87) = 0.2810 - 0.1922 =$ 0.0888

Problem Five.

a. Sketch a graph of f(x) and verify that f(x) is legitimate pdf (show that it integrates to 1).



$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} (0.1)e^{0.2x}dx + \int_{0}^{\infty} (0.1)e^{-0.2x}dx$$

$$= 0.1\left[\frac{e^{0.2x}}{0.2}\right]_{-\infty}^{0} + 0.1\left[\frac{-e^{-0.2x}}{0.2}\right]_{0}^{\infty}$$

$$= \frac{0.1}{02} + \frac{0.1}{02} = 1$$

b. Obtain the cdf of the X and sketch it.

F(x) = P(X ≤ x) $= \int_{-\infty}^{x} f(y)dy$

$$f(x) = \begin{cases} (0.1)e^{0.2x}, & x < 0 \\ (0.1)e^{-0.2x}, & x \geq 0 \end{cases}$$

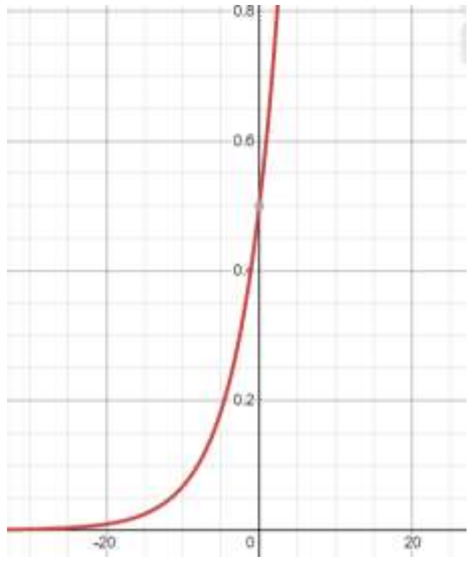
From -inf to 0:

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{x} (0.1)e^{0.2y}dy = (0.5)e^{0.2x}$$

From 0 to inf:

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{0} (0.1)e^{0.2y}dy + \int_{0}^{x} (0.1)e^{-0.2y}dy = 1 - (0.5)e^{-0.2x}$$

$$f(x) = \begin{cases} (0.5)e^{0.2x}, & x < 0 \\ 1 - (0.5)e^{-0.2x}, & x \geq 0 \end{cases}$$



c. Computer P(X ≤ 0), P(X ≤ 2), P(-1 ≤ X ≤ 2), and the probability that an error of more than 2 miles is made.

<u>Proposition</u>: Let X be continuous rv with pdf f(x) and cdf F(x). Then for any number a:

P(X ≤ a) = F(a)

$$P(X \leq 0) = F(0) = 1 - 0.5 = 0.5$$

$$P(X \leq 2) = F(2) = 1 - (0.5)e^{-0.4} = 0.6648$$

<u>Proposition</u>: Let X be continuous rv with pdf f(x) and cdf F(x). Then for any number a:

P(a ≤ X ≤ b) = F(b) – F(a)

$$P(-1 \leq X \leq 2) = F(2) - F(-1) = 1 - (0.5)e^{-0.4} - (0.5)e^{-0.2} = 0.2555$$

<u>Complement Rule:</u> P(A') = 1 – P(A)

P(x < -2 or X > 2) =

$$P(-2 \leq X \leq 2) = 1 - [F(2) - F(-1)] = 1 - [1 - (0.5)e^{-0.4} - (0.5)e^{-0.4}]$$

$$= 1 - [0.6648 - 0.3352] = 0.6704$$