1) Norm

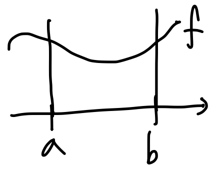
- norm 은 다음 4가지 성질 만족하는 어떤 함수 $f: \mathbb{R}^n \to \mathbb{R}$

$$\begin{cases} 1.\ \forall v \in \mathbb{V},\ \|v\| \geq 0 \\ 2.\ \|v\| = 0 \iff v = \mathbb{0} \\ 3.\ \|\alpha v\| = |\alpha|\,\|v\| \\ 4.\ \|v+w\| \leq \|v\| + \|w\| \end{cases}$$

- $f \in C[a,b]$, L^2: $\|f\|_{L^2} = \sqrt{\int_a^b (f(x))^2 dx} = \|f\|_2$

L^1: $\|f\|_{L^1} = \int_a^b |f(x)|\,dx = \|f\|_1$

L^∞: $\|f\|_{L^\infty} = \max_{a \leq x \leq b} |f(x)|$



$* \ \|f\|_{L^p} = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$

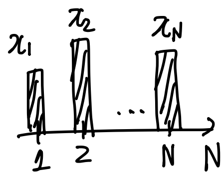
- $V \in \mathbb{R}^n$, $\|V\|_2 = \sqrt{\sum_{i=1}^{N} (x_i)^2}$

$\|V\|_1 = \sum_{i=1}^{N} |x_i| \rightarrow$ Taxicab Distance, Manhattan Distance

$\|V\|_\infty = \max_{i=1,\cdots,N} |x_i|$

$= \max\{|x_1|, \cdots, |x_N|\}$

$V = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$



- Inner Product

$u \cdot v$

$\langle u, v \rangle$

$\boxed{\|u\| = \sqrt{\langle u, u \rangle}}$

$\rightarrow$ norm의 4가지 성질 모두 만족! $\Rightarrow$ $\boxed{\text{Norm}}$

☆ 임의의 벡터 a, b에 대해 $\langle a, b \rangle$가 inner product의 조건을 만족하는 연산일 때, $\sqrt{\langle a, a \rangle}$는 norm의 조건을 만족함.

$\langle a, b \rangle$가 inner product $\Rightarrow$ $\sqrt{\langle a, a \rangle}$는 norm

2) Bilinear Function (= Bilinear Map)

- vector space V, W에 대해 bilinear function $L: V \times W \to \mathbb{F}$ (field)는 다음을 만족함.

  1. $L(\alpha u_1 + \beta u_2, w) = \alpha L(u_1, w) + \beta L(u_2, w)$

  2. $L(u, \alpha w_1 + \beta w_2) = \alpha L(u, w_1) + \beta L(u, w_2)$

- $\tilde{L}(x) = X^{-1}B$는 linear function.

  pf) ① $\tilde{L}(x+x') = (x+x')^T B = x^T B + (x')^T B = \tilde{L}(x) + \tilde{L}(x')$

  ② $\tilde{L}(\alpha x) = (\alpha x)^T B = \alpha x^T B = \alpha L(x)$

- $f(x, y) = x^T A y$는 bilinear function.

  pf) ① $f(\alpha x + \beta x', y) = (\alpha x + \beta x')^T A y = \alpha[x^T A y] + \beta[x'^T A y]$

  $= \alpha f(x, y) + \beta f(x', y)$

  ② $f(x, \alpha y + \beta y') = x^T A(\alpha y + \beta y') = \alpha[x^T A y] + \beta[x^T A y']$

  $= \alpha f(x, y) + \beta f(x, y')$

- Inner Product $f(x, y) = x \cdot y = x_1 y_1 + x_2 y_2$도 bilinear function.

  pf) $f(x, y) = x \cdot y = x_1 y_1 + x_2 y_2$

  $= x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y \quad \left(x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)$

  $\Rightarrow f(x, y) = x^T A y$ 꼴이므로 bilinear function.

- $L: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^m$일 때는

$$L\Big(\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{= \mathbb{X}}, \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{= \mathbb{Y}}\Big) = L(x_1 e_1 + x_2 e_2,\ y_1 e_1 + y_2 e_2) \quad \left(e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= x_1 y_1 L(e_1, e_1) + x_1 y_2 L(e_1, e_2) + x_2 y_1 L(e_2, e_1) + x_2 y_2 L(e_2, e_2)$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} L(e_1, e_1) & L(e_1, e_2) \\ L(e_2, e_1) & L(e_2, e_2) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbb{X}^T A \mathbb{Y}$$

- Inner product에서 ...

$\langle v,u\rangle = \langle u,v\rangle$

$v^T A u = u^T A v = (u^T A v)^T = v^T A^T u$

$\Rightarrow v^T A u = v^T A^T u \Rightarrow A = A^T$

$\therefore \langle u,v\rangle$는 $u^T A v$ 꼴로 표현되고 A는 symmetric matrix.

- Norm

$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u^T A u} \rightarrow$ ☆ Mahalanobis distance

<마할라노비스 거리>

→ ① $A = A^{-1}$

→ ② $u^T A u \geq 0$

3) Quadratic Forms and Semidefinite Matrix

- Quadratic form

$$X^T A X = [x \ \ y] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + dy^2 + (c+b)xy$$

$$= [x \ \ y] \begin{bmatrix} a & \frac{c+b}{2} \\ \frac{c+b}{2} & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = X^T \tilde{A} X$$

(A를 symmetric하게)

- Positive definite (PD)

$$\begin{bmatrix} ① \ X^T A X \geq 0 \\ ② \ X^T A X = 0 \iff X = 0 \end{bmatrix} \iff A \succ 0$$

* 내적을 위한 필요조건

- Positive semidefinite (PSD)

$$X^T A X \geq 0 \iff A \succeq 0$$

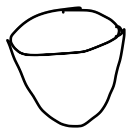
ex) $[x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq 0$, but $[0 \ 0 \ 1] A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$.

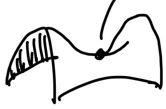
• $f(x,y) = x^2 + xy + 3y^2$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\downarrow$$

$$\frac{1}{2} \begin{bmatrix} \partial_{xx} f(0,0) & \partial_{xy} f(0,0) \\ \partial_{xy} f(0,0) & \partial_{yy} f(0,0) \end{bmatrix} \rightarrow \text{Hessian Matrix}$$

• $f(x,y) = x^2 + y^2 \rightsquigarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow$  ☆ : positive definite

• $f(x,y) = x^2 - y^2 \rightsquigarrow A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightsquigarrow$  미분 = 0 (극대 X 극소 X) $\rightsquigarrow$ 안장점 saddle point

• $f(x,y,z) = x^2 + 2y^2 + 0z^2 \rightsquigarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$: Semi-positive definition

4) Injective, Surjective, Bijective

function $T : V \to W$ 에 대해

- Injective (단사) :

  $T(v_1) = T(v_2) \Leftrightarrow v_1 = v_2$

- Surjective (전사) :

  $\forall w \in W, \exists v \in V$ s.t. $T(v) = w$

- Bijective (전단사) :

  Injective & Surjective 둘 다 만족. (일대일 대응)

5) Inverse 가 존재하기 위한 조건

- $B \in M_{n\times n}$, $x \in \mathbb{R}^n$, $L(x) = B$ 에 대해

  $\exists B^{-1} \Leftrightarrow L(x)$ is bijective.

- $L(x)$ 가 bijective 하기 위해서는 injective, surjective 둘 다 만족해야됨.

  그런데 Dimension theorem : $\dim(\mathrm{dom}(L)) = \dim(\mathrm{Im}(L)) + \dim(\ker(L))$

  에 의하면, injective 하기만 하면 $\dim(\ker(L)) = 0$ 이기 때문에 자동 surjective.

  pf) $L(0) = 0$ 이므로 $0 \in \ker(L)$.

  $L(x_1) = Y_1$, $\exists x_0 \in \ker(L)$ s.t. $x_0 \neq 0$ 이면

  $L(x_1 + x_0) = L(x_1) + L(x_0) = Y_1$,

  $L(x_1 + 2x_0) = L(x_1) + 2L(x_0) = Y_1$ 이므로 L은 injective 하지 않음.

  $\therefore \exists x_0 \in \ker L$ s.t. $x_0 \neq 0 \Leftrightarrow L$ is not injective.

- $\exists B^{-1} \Leftrightarrow \det(B) \neq 0$

  $\Leftrightarrow \ker(L) = \{0\}$

  $\exists x_0 \in \ker(L)$ s.t. $x_0 \neq 0$

  $\Leftrightarrow \det(B) = 0$

6) Eigen Vector

- $\exists x\ (x \neq 0),\ \exists \lambda$ with

$$Ax = \lambda X \Leftrightarrow Ax = \lambda IX \Leftrightarrow AX - \lambda IX = 0$$

$$\Leftrightarrow \underbrace{(A - \lambda I)}_{\substack{\parallel \\ B}} X = 0 \Leftrightarrow x \in \ker(\underbrace{BX}_{\substack{\parallel \\ L(x)}})\quad (x \neq 0)$$

$$\Leftrightarrow \det(A - \lambda I) = 0$$

Ex. $\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

$$\det\begin{pmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - \frac{1}{4} = 0$$

$$\Rightarrow \lambda = \frac{1}{2} \text{ or } \lambda = \frac{3}{2}$$

- Eigen value, Eigen vector 찾기

$$Ax = \lambda x \Leftrightarrow \det(A - \lambda I) = 0$$

1st λ 구하고 → 2nd λ에 대한 X 구하기.

but 5차식 이상 풀 수 없음 → $\underline{\text{QR Algorithm}}$

- $A = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 2 & & \\ & & & 2 & \\ 0 & & & & 3 \end{bmatrix}$

1, 1, 2, 2, 3

$X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow AX = X$

$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightsquigarrow AX = 2X$

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow AX = 3X$

서로 수직.

pf) A : symmetric

$X_1 \to \lambda_1$

$\quad\quad\ \ \#$

$X_2 \to \lambda_2$

$$X_2^T A X_1 = X_2^T(\lambda_1 X_1) = \lambda_1 X_2^T X_1 = \lambda X_2 X_1$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$X_2^T A^T X_1 = (AX_2)^T X_1 = \lambda_2 X_2^T X_1 = \lambda_2 X_2 X_1$$

$$(\lambda_1 - \lambda_2)(X_2 \cdot X_1) = 0 \Leftrightarrow X_2 \cdot X_1 = 0 \Longleftrightarrow X_1 \perp X_2$$

7) Rank

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \underbrace{[x_1+x_3]}_{=\tilde{x}_1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \underbrace{[x_2+x_3]}_{=\tilde{x}_2}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\dim(Ax \mid x \in \mathbb{R}^n) = \text{column rank}(A)$$

$$= 2 \qquad \| = \text{rank}(A) = 2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{matrix} = r_1 \\ = r_2 \\ = r_3 = r_1 - r_2 \end{matrix} \qquad \text{row rank}(A) = 2$$

8) SVD

A : $n \times n$ symmetric matrix $(A = A^T)$

$$U^{-1}AU = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix} \rightarrow A = UDU^{-1}$$

$$U = \underbrace{\begin{bmatrix} | & | & | & | \end{bmatrix}}_{\text{orthonormal}} \qquad U^T = U^{-1}$$

A : $m \times n$ matrix with rank r

$$\begin{bmatrix} \underbrace{|\;|\;|}_{r} & \sim \end{bmatrix} \qquad \underset{m\times n}{A} = \underset{m\times r}{U}\,\underset{r\times r}{\Sigma}\,\underset{r\times n}{V^T}$$

A^TA : symmetrix $n \times n$ matrix

$$= (U\Sigma V^T)^T U\Sigma V^T$$

$$= V\Sigma^T \underbrace{U^TU}_{I} \Sigma V^T = V\Sigma^T\Sigma V^T = V(\Sigma)^2 V^T$$

$$= UDU^T = \underbrace{[\;]}_{V} \; \underbrace{\begin{bmatrix} \alpha & \\ & \ddots \end{bmatrix}}_{\Sigma^2} \; \underbrace{[\;]}_{V^T} \qquad AA^T \rightarrow U$$

n   r   n   n