1) Vectors in $\mathbb{R}^n$

V (집합하나) 에 $+$, 상수곱 정의되고 다음 조건 만족하면 vectorspace 됨.

임의의 벡터 $\mathbb{u}, \mathbb{v}, \mathbb{w} \in V$ 와 scalar c, d 에 대해

(i) $\mathbb{u}+\mathbb{v} = \mathbb{v}+\mathbb{u}$
(ii) $(\mathbb{u}+\mathbb{v})+\mathbb{w} = \mathbb{u}+(\mathbb{v}+\mathbb{w})$
(iii) $\mathbb{u}+\mathbb{0} = \mathbb{0}+\mathbb{u} = \mathbb{u}$
(iv) $\mathbb{u}+(-\mathbb{u}) = -\mathbb{u}+\mathbb{u} = 0 \quad (-\mathbb{u} = (-1)\mathbb{u})$
(v) $c(\mathbb{u}+\mathbb{v}) = c\mathbb{u}+c\mathbb{v}$
(vi) $(c+d)\mathbb{u} = c\mathbb{u}+d\mathbb{u}$
(vii) $c(d\mathbb{u}) = (cd)\mathbb{u}$
(viii) $1\mathbb{u} = \mathbb{u}$.

2) Linear Combination

vectors $\mathbb{v}_1, \mathbb{v}_2, \cdots, \mathbb{v}_p \in \mathbb{R}^n$, scalars $c_1, c_2, \cdots, c_p$ 에 대해

$\mathbb{y} = c_1\mathbb{v}_1 + c_2\mathbb{v}_2 + \cdots + c_p\mathbb{v}_p$ 를

linear combination of $\mathbb{v}_1, \cdots, \mathbb{v}_p$ 와 weights $c_1, \cdots, c_p$ 라고 함.

3) Linear Independent

$\mathbb{v}_1, \mathbb{v}_2, \cdots, \mathbb{v}_p \in V$,

$c_1\mathbb{v}_1 + c_2\mathbb{v}_2 + \cdots + c_p\mathbb{v}_p = 0 \iff c_1 = 0, c_2 = 0, \cdots, c_p = 0.$

이면 $\mathbb{v}_1, \mathbb{v}_2, \cdots, \mathbb{v}_p$ 가 linear independent 하다고 함.

* linear independent 가 아니면?

$c_1\mathbb{v}_1 + c_2\mathbb{v}_2 + \cdots + c_p\mathbb{v}_p = 0$ 에서 $c_1 \neq 0$ 일 때,

양 변을 c_1 으로 나눠서 $\mathbb{v}_2, \cdots, \mathbb{v}_p$ 로 $\mathbb{v}_1$ 표현가능. ($\mathbb{v}_1$ 불필요).

4) Basis

임의의 vectorspace V에 대해 $\mathbb{b}_1, \mathbb{b}_2, \cdots, \mathbb{b}_n \in V$라 하자.
vector들의 집합 $B = \{\mathbb{b}_1, \mathbb{b}_2, \cdots \mathbb{b}_n\}$이 다음 조건을 만족하면
B는 V의 basis이다.

i) B가 선형독립

ii) $V = \text{span}\{\mathbb{b}_1, \mathbb{b}_2, \cdots, \mathbb{b}_n\}$

(B의 linear combination으로 V의 모든 vector 생성가능)

5) Dimension

어떤 vectorspace V의 dimension은 V의 basis의 vector 개수.

6) Inner product & Dot product

- $\mathbb{v} = (v_1, v_2)$, $\mathbb{w} = (w_1, w_2) \Rightarrow \mathbb{v} \cdot \mathbb{w} = v_1 w_1 + v_2 w_2$

inner product

dot product

- dot product가 0 이면 두 vectors는 perpendicular.
- $\mathbb{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\mathbb{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ 에 대해

$$\mathbb{u} \cdot \mathbb{v} = \underbrace{[u_1 u_2 \cdots u_n]}_{\mathbb{u}^T} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbb{v}} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Inner product 기본성질 (이 성질 다 만족하면 그 연산은 inner product)

vectors $\mathbb{u}, \mathbb{v}, \mathbb{w} \in \mathbb{R}^n$, c is a scalar, then

a) $\mathbb{u} \cdot \mathbb{v} = \mathbb{v} \cdot \mathbb{u}$

b) $(\mathbb{u} + \mathbb{v}) \cdot \mathbb{w} = (\mathbb{u} \cdot \mathbb{w}) + (\mathbb{v} \cdot \mathbb{w})$

c) $(c\mathbb{u}) \cdot \mathbb{v} = c(\mathbb{u} \cdot \mathbb{v}) = \mathbb{u} \cdot (c\mathbb{v})$

d) $\left\{ \begin{array}{l} \mathbb{u} \cdot \mathbb{u} \geq 0 \\ \mathbb{u} \cdot \mathbb{u} = 0 \iff \mathbb{u} = \mathbb{0} \end{array} \right]$ positive definite

7) Length of a Vector

- length (= norm) of vector $\mathbf{v}$ is the scalar $\|\mathbf{v}\|$.

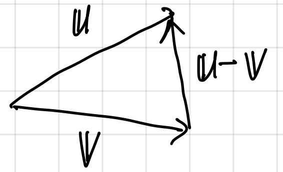
$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{\underbrace{v_1^2+v_2^2+\cdots+v_n^2}_{\text{entries of } \mathbf{v}\in\mathbb{R}^n}}, \quad \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$$

- unit vector : length 1인 vector

unit vector $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$

$\mathbf{v}$에 $\frac{1}{\|\mathbf{v}\|}$ 하는걸 normalization 이라고함.

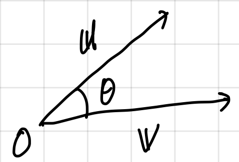
- distance between $\mathbf{u}$ and $\mathbf{v}$ $= \|\mathbf{u}-\mathbf{v}\|$



8) Orthogonal Basis

- vector $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$ (nonzero)

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$



$\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow \mathbf{u}, \mathbf{v}$ are orthogonal

- basis 가 orthogonal set 이면 orthogonal basis 라고 함.
- $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ 이 V의 orthogonal basis 라 하자.

$\mathbf{y}\in V$에 대해 linear combination

$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$ 의 weights는

$$c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j} \quad (j = 1, \cdots, n)$$ 이다.

pf) $\mathbf{y}\cdot\mathbf{u}_1 = c_1(\mathbf{u}_1\cdot\mathbf{u}_1) + c_2(\mathbf{u}_2\cdot\mathbf{u}_1) + \cdots + c_n(\mathbf{u}_n\cdot\mathbf{u}_1)$

$= c_1\|\mathbf{u}_1\|^2$

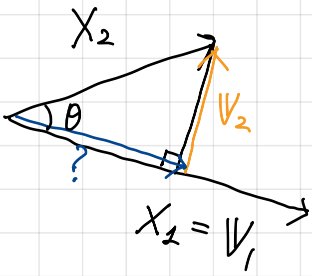
9) Gram-Schmidt Process

- basis를 orthogonal 또는 orthonormal 하게 만드는 방법

→ 정사영 부분을 구한 뒤 그걸 빼서 수직인 부분만 추려냄.

ex) $W = \text{span}\{x_1, x_2\}$ 인데

W의 orthogonal basis $\{v_1, v_2\}$ 얻고싶음.



X_2를 강제로 $X_1 = V_1$과 orthogonal 하게 만들어야 됨.

$$V_2 = X_2 - ?$$

$$\| ? \| = \|X_2\| \cos\theta$$

$$X_1 \cdot X_2 = \|X_1\| \|X_2\| \cos\theta \text{ 에서}$$

$$\|X_2\| \cos\theta = \frac{X_1 \cdot X_2}{\|X_1\|}$$

$$\| ? \| = \frac{X_1 \cdot X_2}{\|X_1\|}$$

$$? = \| ? \| \frac{X_1}{\|X_1\|} = \frac{X_1 \cdot X_2}{\|X_1\|^2} X_1$$

$$\therefore V_2 = X_2 - \frac{X_1 \cdot X_2}{\|X_1\|^2} X_1$$

W의 orthogonal basis $\left\{X_1, X_2 - \frac{X_1 \cdot X_2}{\|X_1\|^2} X_1\right\}$ 얻음!

10) Inner Product Spaces

- vector space V에서의 inner product는 다음 조건을 만족하며 $u, v \in V$에 대해 실수 $\langle u, v \rangle$를 대응시키는 함수.

$u, v, w \in V$, scalar c에 대해

$\langle 1 \rangle$ $\langle u, v \rangle = \langle v, u \rangle$

$\langle 2 \rangle$ $\langle u + v, w \rangle = \langle u, w \rangle + \langle v + w \rangle$

$\langle 3 \rangle$ $\langle cu + v \rangle = c \langle u, v \rangle$

$\langle 4 \rangle$ $[\langle u, u \rangle \geq 0]$ & $[\langle u, u \rangle = 0 \Leftrightarrow u = 0]$

- The Cauchy-Schwarz Inequality

$$\forall u, v \in V, \quad |\langle u, v\rangle| \le \|u\|\,\|v\|$$

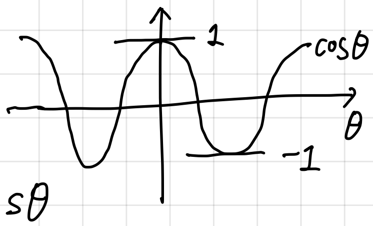
- The Triangle Inequality

$$\forall u, v \in V, \quad \|u+v\| \le \|u\| + \|v\|$$

- $\langle u, v\rangle = \|u\|\,\|v\| \cos\theta$ 에서 $\cos\theta$ 의미

$$-\|u\|\,\|v\| \le u\cdot v \le \|u\|\,\|v\|$$

$$-1 \le \underbrace{\frac{u\cdot v}{\|u\|\,\|v\|}}_{\cos\theta} \le 1$$



$$\frac{u\cdot v}{\|u\|\,\|v\|} = \cos\theta$$

$$\frac{u}{\|u\|}\cdot\frac{v}{\|v\|} = \begin{cases} \text{방향 같음} \to 1 \\ \text{방향 정반대} \to -1 \end{cases}$$

11) Matrix

$$\begin{cases} 4x_1 - 5x_2 = -13 \\ -2x_1 + 3x_2 = 9 \end{cases} \rightarrow \underbrace{\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -13 \\ 9 \end{bmatrix}}_{b}$$

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

$$\underbrace{A^{-1}A}_{I}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{A^{-1}}\begin{bmatrix} -13 \\ 9 \end{bmatrix} \quad (*A^{-1}A = AA^{-1} = I)$$

→ 기계적으로 구할 수 있음.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

① $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax + by$

$\underset{m \times n}{A} \; \underset{n \times l}{B}$ → 계산 결과는 $m \times l$ matrix

같아야 계산 가능

② $\underset{\boxed{2} \times 2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{2 \times \boxed{2}}{\begin{bmatrix} x & y \\ z & w \end{bmatrix}} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}$

1행 2열

③ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Identity matrix

$I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

12) Transpose of Matrix

- $a_{ij} \xrightarrow{T} a_{ji}$ ex) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow A^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow B^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

- transpose 관련 성질

<1> $(A^T)^T = A$

<2> $(AB)^T = B^T A^T$

<3> $(A+B)^T = A^T + B^T$

- Symmetric matrix

$A = A^t \Rightarrow$ A는 symmetric

ex) $A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$

13) Identity, Diagonal Matrices

- Identity matrix

$$\left.\begin{cases} ① \text{ square matrix} \\ ② \ I_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \end{cases}\right]$$ 모두 만족하는 I는 identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rightarrow IA = AI = A$

(계산 가능할 때)

- Diagonal matrix

non-diagonal elements 전부 0. ex) $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$

$$D_{ij} = \begin{cases} d_i & i=j \\ 0 & i \neq j \end{cases}$$

14) Subspace

- vector space V의 $\emptyset$이 아닌 subset H가 다음을 만족하면 subspace.

  a. closed under vector addition.

  b. closed under scalar multiplication.

- $v_1, \cdots v_n \in V$ 이면 $\text{Span}\{v_1, \cdots, v_n\}$은 V의 subspace.

- Null space of matrix A

  : $\text{Nul } A = \{x : x \in \mathbb{R}^n \text{ and } Ax = 0\}$

  * $Ax = 0$ 에서 x를 kurnel of A라고 함.

15) Linear Transformation

Linear transformation $T : V \to W$ 는 다음 조건을 만족하며
$x \in V$ 를 유일한 $T(x)$에 대응시키는 규칙.
(i) $T(u+v) = T(u) + T(v)$ $\quad (\forall u, v \in V)$
(ii) $T(cu) = cT(u)$ $\quad (\forall u \in V, \forall \text{scalars } c)$

- linear transformation $L : V \to W$에 대해

  $Lv = 0$ 이 되는 모든 vectors v의 집합을
  kernel of L 이라고 함.

  $\ker L = \{v \in V \mid Lv = 0\}$

16) Dimension Formula

linear transformation $L : V \to W$ 에 대해
V가 유한 차원의 vector space 이면
$\dim(V) = \dim(\ker(V)) + \dim(L(V))$

17) Eigen vector & Eigen values

$\underbrace{A}_{\text{matrix}} x = \underbrace{\lambda}_{\text{상수}} x$ 을 만족하는 모든 λ를 eigen values,
x를 eigen vector 라고 함.

18) Inverse of Matrix

$CA = I$ 이고 $AC = I$ 일 때
C는 A의 inverse 라고 한다.

19) Determinant

- $\begin{cases} \det(A) = 0 \rightarrow \text{깡} \\ \det(A) \neq 0 \rightarrow \exists A^{-1} \end{cases}$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A) = a_{11}a_{22} - a_{12}a_{21}$

- $\det(A) = 0 \iff A$ is not invertible

  $\iff \exists\, x_0 \neq \mathbb{0}$ s.t. $A x_0 = \mathbb{0}$

  $\iff \ker(A) \neq \{\mathbb{0}\}$